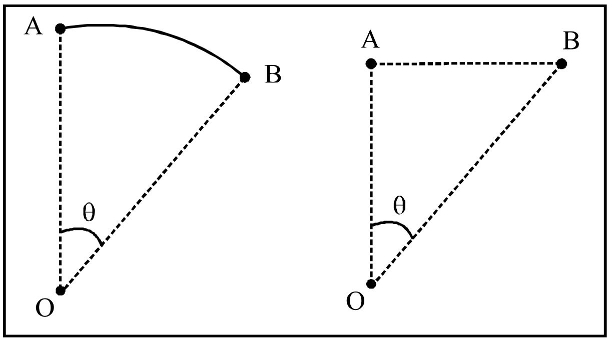
3. ROTATIONAL MOTION

1. KINEMATICS OF SYSTEM OF PARTICLES

1.1 System of particles can move in different ways as observed by us in daily life. To understand that we need to understand few new parameters.

(a) Angular Displacement

Consider a particle moves from A to B in the following figures.

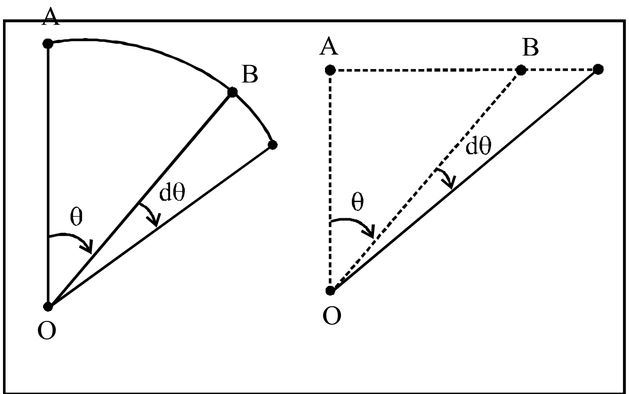


Angle is the angular displacement of particle about O.

Units → radian

(b) Angular Velocity

The rate of change of angular displacement is called as angular velocity.



$\omega = \dfrac{d\theta}{dt}$ $\omega = \dfrac{d\theta}{dt}$

Units → Rad/s

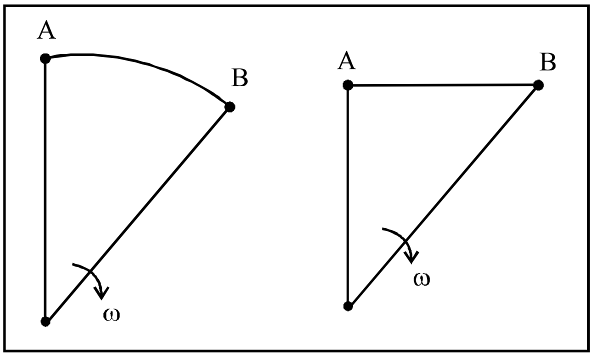
It is a vector quantity whose direction is given by right hand thumb rule.

According to right hand thumb rule, if we curl the fingers of right hand along with the body, then right hand thumb gives us the direction of angular velocity.

It is always along the axis of the motion.

(c) Angular Acceleration

Angular acceleration of an object about any point is rate of change of angular velocity about that point.



$\alpha = \dfrac{d\omega}{dt}$ $\alpha = \dfrac{d\omega}{dt}$

Units → Rad/s^2

It is a vector quantity. If α is constant then similarly to equation of motion (i.e.)

ω, α θ, t are related $\omega = \omega_0 + \alpha t$

$$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

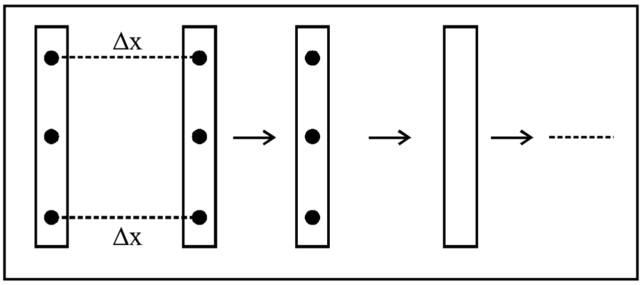
$$\omega_f^2 - \omega_0^2 = 2\alpha\theta$$

1.2 Various types of motion

(a) Translational Motion

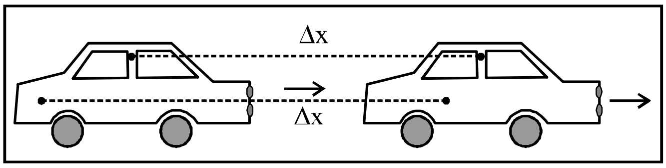
System is said to be in translational motion, if all the particles lying in the system have same linear velocity.

Example



Motion of a rod as shown.

Example

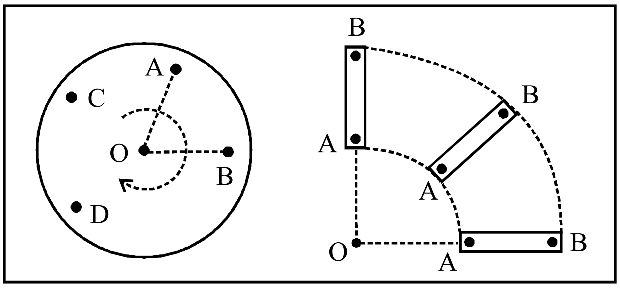


Motion of body of car on a straight rod.

In both the above examples, velocity of all the particles is same as they all have equal displacements in equal intervals of time.

(b) Rotational Motion

A system is said to be in pure rotational motion, when all the points lying on the system are in circular motion about one common fixed axis.



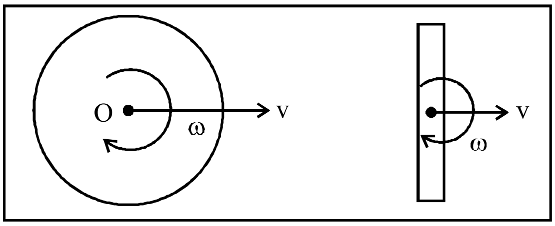
In pure rotational motion.

Angular velocity of all the points is same about the fixed axis.

(c) Rotational + Translational

A system is said to be in rotational + translational motion, when the particle is rotating with some angular velocity about a movable axis.

For example :



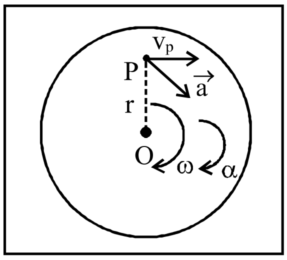
v = velocity of axis.

ω = Angular velocity of system about O.

1.3 Inter Relationship between kinematics variable

In general if a body is rotating about any axis (fixed or movable), with angular velocity ω and angular acceleration α then velocity of any point p with respect to axis is $\vec{v} = \vec{\omega} \times \vec{r}$ and $\vec{a} = \vec{\alpha} \times \vec{r} - \omega^2 \vec{r}$.

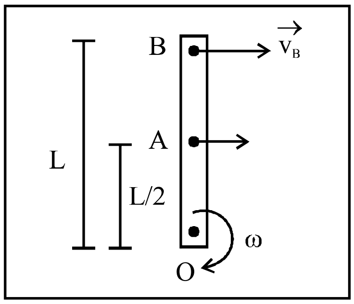
i.e.,



$\vec{v}_p = \vec{\omega} \times \vec{r}$

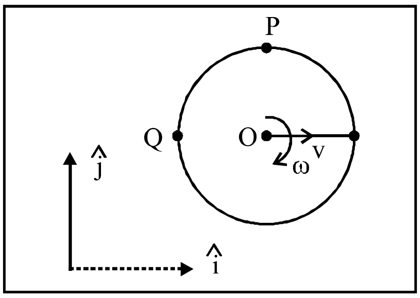
$\vec{a} = \vec{\alpha} \times \vec{r} - \omega^2 \vec{r}$

Example



$v_B = \omega L$ and $v_A = \dfrac{\omega L}{2}$, with directions as shown in figure.

Now in rotational + translational motion, we just superimpose velocity and acceleration of axis on the velocity and acceleration of any point about the axis. (i.e.)



$\vec{v}_{PO} = \omega R\hat{i}$

$\vec{v}_0 = v\hat{i}$

$\because \quad \vec{v}_P - \vec{v}_0 = \vec{v}_{PO}$

$\Rightarrow \quad \vec{v}_P = \vec{v}_{PO} + \vec{v}_O$

$\omega R + v\hat{i}$

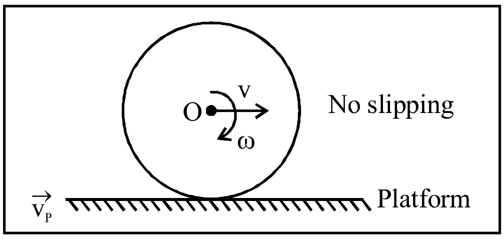
Similarly $\vec{v}_{QO} = \omega R\hat{j}$

$\vec{v}_0 = v\hat{i}$

$\therefore \quad \vec{v}_Q = v\hat{i} + \omega R\,\hat{j}$

Inter-relation between v of axis and ω or a of axis and α depends on certain constraints.

General we deal with the case of no slipping or pure rolling.



The constraint in the above case is that velocity of points of contact should be equal for both rolling body and playfrom.

(i.e.) $v - r\omega = v_P$

If platform is fixed then

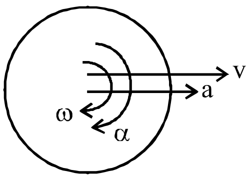
$v_P = 0 \Rightarrow \boxed{v = r\omega}$

An differentiating the above term we get

$$\frac{dv}{dt} = \frac{r\,d\omega}{dt}.$$

Now if $\frac{dv}{dt} = a$

$\frac{d\omega}{dt} = \alpha$



then $\boxed{a = r\alpha}$

Remember if acceleration is assumed opposite to velocity then $a = -\frac{dv}{dt}$ instead of $a = \frac{dv}{dt}$.

Similary : If α and ω are in opposite direction the $\alpha = -\frac{d\omega}{dt}$.

Accordingly the constraints can change depending upon the assumptions.

2. ROTATIONAL DYNAMICS

2.1 Torque

Similar to force, the cause of rotational motion is a physical quantity called a torque.

Torque incorporates the following factors.

→ Amount of force.

→ Point of application of force.

→ Direction of application of force.

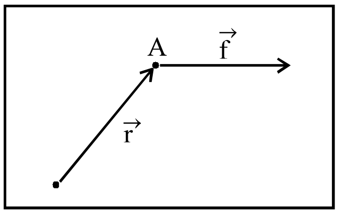
Combining all of the above.

Torque $\tau = rf \sin\theta$ about a point O.

Where r = distance from the point O to point of application of force.

f= force

θ = angle between $\vec{r}$ and $\vec{f}$



→ Torque about O.

→ A is point of application of force.

Magnitude of torque can also be rewritten as

$\tau = rf_\perp$ or $\tau = r_\perp f$ where

$f_\perp$ = component of force in the direction $\perp$ to $\vec{r}$.

$r_\perp$ = component of force in the direction $\perp$ to $\vec{f}$.

Direction :

Direction of torque is given by right hand thumb rule. If we curl the fingers of right hand from first vector $(\vec{r})$ to second vector $(\vec{f})$ then right hand thumb gives us direction of their cross product.

→ Torque is always defined about a point or about an axis.

→ When there are multiple forces, the net torque needs to be calculated, (i.e.)

$$\vec{\tau}_{net} = \vec{\tau}_{F_1} + \vec{\tau}_{F_2} + \ldots\ldots\ldots.\tau_{F_n}$$

All torque about same point/axis.

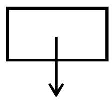
If $\sum\tau = 0$, then the body is in rotational equilibrium.

→ If $\sum F = 0$ along with $\sum\tau = 0$, then body is in mechanical equilibrium.

→ If equal and opp. force act to produce same torque then they constitutes a couple.

→ For calculating torque, it is very important to find the eff. point of application of force.

→ Mg → Acts at com/centre of gravity.



→ N → Point of application depends upon situation to situation.

2.2 Newtwon's Laws

$\sum \tau = I\alpha$.

→ I = moment of Inertia

→ α = Angular Acceleration.

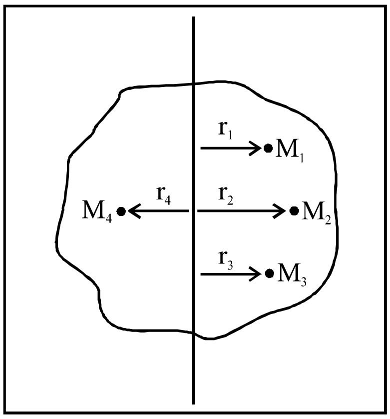
2.3 Moment of Inertia

→ Gives the measure of mass distribution about on axis.

→ $I = \sum m_i r_i^2$

$r_i = \perp$ distance of the i^{th} mass from axis.

→ Always defined about an axis.

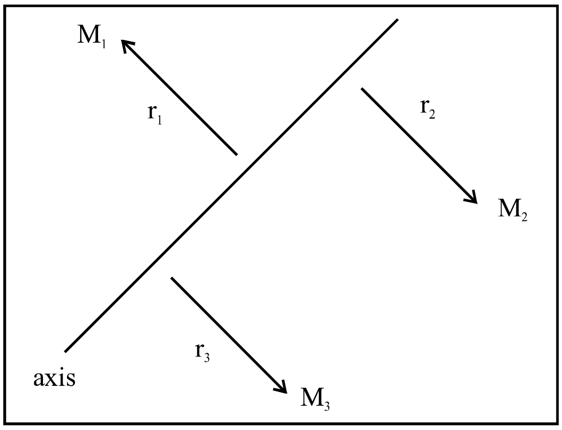


$I = M_1 r_1^2 + M_2 r_2^2 + M_3 r_3^2 + M_4 r_4^2$

→ SI units → kgm^2

→ Gives the measure of rotational inertia and is equavalent to mass.

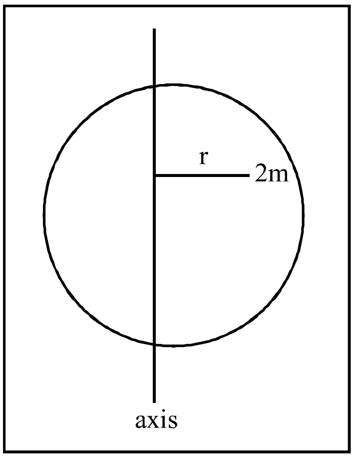
(a) Moment of Inertia of a discreet particle system :



$I = M_1 r_1^2 + M_2 r_2^2 + M_3 r_3^2$

(b) Continuous Mass Distribution

For continuous mass distribution, we need to take help of integration :



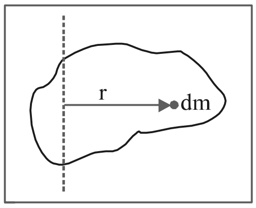
$I_{axis} = \int r^2 dm$

3. MOMENT OF INERTIA

3.1 Moment of inertia of Continuous Bodies

When the distribution of mass of a system of particle is continuous, the discrete sum $I = \sum m_i r_i^2$ is replaced by an integral. The moment of inertia of the whole body takes the form

$$I = \int r^2\, dm$$



Keep in mind that here the quantity r is the perpendicular distance to an axis, not the distance to an origin. To evaluate this integral, we must express m in terms of r.

Note...

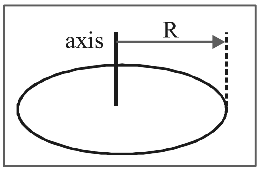
Comparing the expression of rotational kinetic energy with 1/2 mv^2, we can say that the role of moment of inertia (I) is same in rotational motion as that of mass in linear motion. It is a measure of the resistance offered by a body to a change in its rotational motion.

3.2 Moment of Inertia of some important bodies

1. Circular Ring

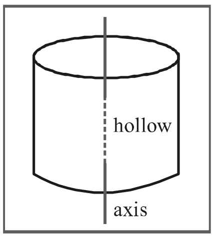
Axis passing through the centre and perpendicular to the plane of ring.

$I = MR^2$



2. Hollow Cylinder

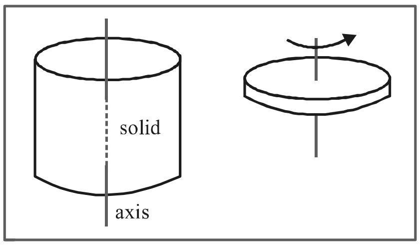
$I = MR^2$



3. Solid Cylinder and a Disc

About its geometrical axis :

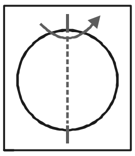
$$I = \frac{1}{2} MR^2$$



4. (a) Solid Sphere

Axis passing through the centre :

$I = 2/5\ MR^2$



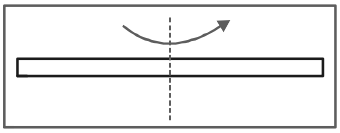
(b) Hollow Sphere

Axis passing through the centre :

$I = 2/3\ MR^2$

5. Thin Rod of length *l* :

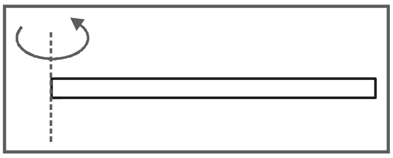
(a) Axis passing through mid point and perpendicular to the length :



$$I = \frac{M\ell^2}{12}$$

(b) Axis passing through an end and perpendicular to the rod:

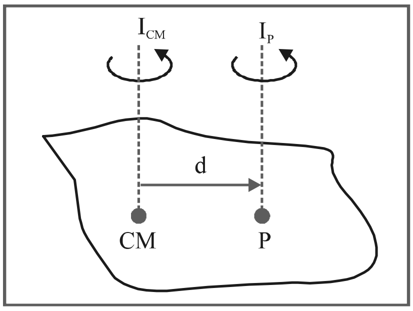
$$I = \frac{M\ell^2}{3}$$



3.3 Theorems on Moment of Inertia

1. Parallel Axis Theorem : Let I_{cm} be the moment of inertia of a body about an axis through its centre of mass and Let I_p be the moment of inertia of the same body about another axis which is parallel to the original one.

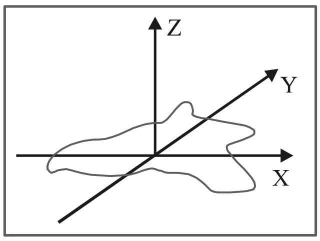
If d is the distance between these two parallel axes and M is the mass of the body then according to the parallel axis theorem :



$$I_p = I_{cm} + Md^2$$

2. Perpendicular Axis Theorem :

Consider a plane body (i.e., a plate of zero thickness) of mass M. Let X and Y axes be two mutually perpendicular lines in the plane of the body. The axes intersect at origin O.



Let I_x = moment of inertia of the body about X–axis.

Let I_y = moment of inertia of the body about Y–axis.

The moment of inertia of the body about Z–axis (passing through O and perpendicular to the plane of the body) is given by :

$$I_z = I_x + I_y$$

The above result is known as the perpendicular axis theorem.

3.4 Radius of Gyration

If M is the mass and I is the moment of inertia of a rigid body, then the radius of gyration (k) of a body is given by :

$$k = \sqrt{\frac{I}{M}}$$

4. ANGULAR MOMENTUM (L) AND IMPULSE

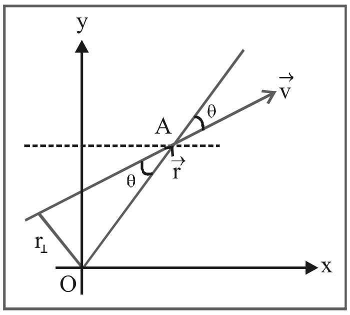
4.1 Angular Momentum

(a) For a particle

Angular momentum about origin (O) is given as :

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v})$$

where $\vec{r}$ = position vector of the particle ; $\vec{v}$ = velocity



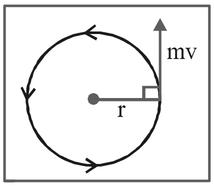
$\Rightarrow$ $L = mv\, r \sin\theta = mv\,(OA) \sin\theta = mvr_\perp$

where $r_\perp$ = perpendicular distance of velocity vector from O.

(b) For a particle moving in a circle

For a particle moving in a circle of radius r with a speed v, its linear momentum is mv, its angular momentum (L) is given as :

$$L = mvr_\perp = mvr$$



(c) For a rigid body (about a fixed axis)

L = sum of angular momentum of all particles

$$= m_1v_1r_1 + m_2v_2r_2 + m_3v_3r_3 + ...$$

$$= m_1r_1^2\omega + m_2r_2^2\omega + m_3r_3^2\omega + ... \qquad (v = r\omega)$$

$$= \left(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ...\right)\omega \Rightarrow L = I\omega$$

(compare with linear momentum p = mv in linear motion)

L is also a vector and its direction is same as that of ω (i.e. clockwise or anticlockwise)

We knows,

$$\vec{L} = I\vec{\omega}$$

$$\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\,\vec{\alpha} = \vec{\tau}_{net}$$

4.2 Conservation of angular momentum

If $\vec{\tau}_{net} = 0$

$$\Rightarrow \frac{d\vec{L}}{dt} = 0$$

$$\Rightarrow \vec{L} = \text{constant}$$

$$\Rightarrow \vec{L}_f = \vec{L}_i$$

4.3 Angular Impulse

$$\vec{J} = \int \vec{\tau}\, dt = \Delta\vec{L}$$

5. WORK AND ENERGY

5.1 Work done by a Torque

Consider a rigid body acted upon by a force F at perpendicular distance r from the axis of rotation. Suppose that under this force, the body rotates through an angle Δθ.

Work done = force × displacement

$$W = F\,r.\,\Delta\theta$$

$$W = \tau\,\Delta\theta$$

Work done = (torque) × (angular displacement)

$$\text{Power} = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\,\omega$$

5.2 Kinetic Energy

Rotational kinetic energy of the system

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + ...$$

$$= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + ...$$

$$= \frac{1}{2}\left(m_1r_1^2 + mr_2^2 + m_3r_3^2 + ...\right)\omega^2$$

Hence rotational kinetic energy of the system = $\frac{1}{2}\ I\omega^2$

The total kinetic energy of a body which is moving through space as well as rotating is given by :

$$K = K_{translational} + K_{rotational}$$

$$K = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}\ I_{CM}\omega^2$$

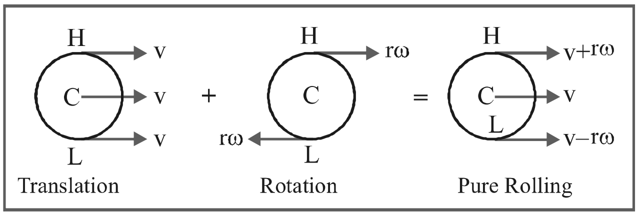
where V_{CM} = velocity of the centre of mass

I_{CM} = moment of inertia about CM

ω = angular velocity of rotation

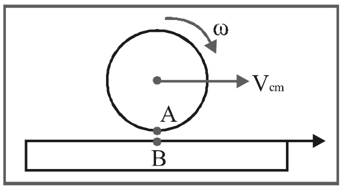
6. ROLLING

1. Friction is responsible for the motion but work done or dissipation of energy against friction is zero as there is no relative motion between body and surface at the point of contact.
2. In case of rolling all point of a rigid body have same angular speed but different linear speed. The linear speed is maximum for the point H while minimum for the point L.



3. Condition for pure rolling : (without slipping)

(i)



general (when surface is moving)

in terms of velocity : $V_{cm} - \omega R = V_B$

in terms of rotation : $a_{cm} - \alpha R = a_B$

special case (when $V_B = 0$)

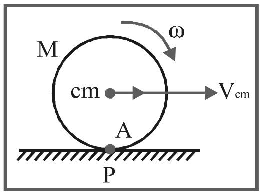
in terms of velocity : $V_{cm} = \omega R$

in terms of acceleration : $a_{cm} = \alpha\, R$

(ii) Total KE of Rolling body :

(i) $K = \frac{1}{2}I_P\omega^2$ **OR**

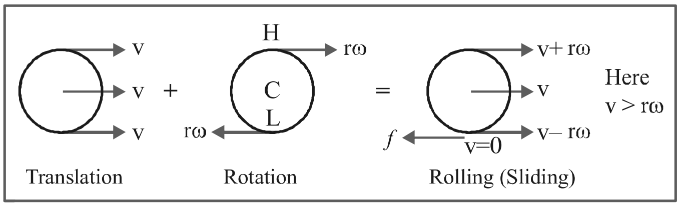
(ii) $K = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}MV_{cm}^2$



where (a) $I_P = I_{cm} + MR^2$ (parallel axes theorem)

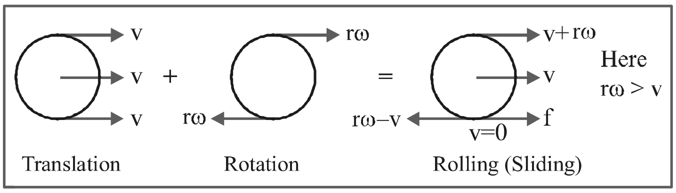
(b) $V_{cm} = \omega R$ [pure] rolling condition.

4. Forward Slipping



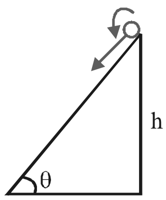
The bottom most point slides in the forward direction w.r.t. ground, so friction force acts opposite to velocity at lowest point i.e. opposite to direction of motion e.g. When sudden brakes are applied to car its 'v' remain same while 'ωr' decreases so its slides on the ground.

5. Backward Slipping

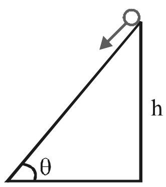


The bottom most point slides in the backward direction w.r.t. ground, so friction force acts opposite to velocity i.e. friction will act in the direction of motion e.g. When car starts on a slippery ground, its wheels has small 'v' but large 'ωr' so wheels slips on the ground and friction acts against slipping.

6.1 Rolling and sliding motion on an inclined plane



Pure Rolling



Sliding

Physical Quantity	Rolling	Sliding	Falling
Velocity	$V_R = \sqrt{(2gh)/\beta}$	$V_S = \sqrt{2gh}$	$V_F = \sqrt{2gh}$
Acceleration	$a_R = g \sin\theta/\beta$	$a_S = g\sin\theta$	$a_F = g$
Time of descend	$t_R = 1/\sin\theta\sqrt{\beta(2h/g)}$	$t_S = (1/\sin\theta)\sqrt{2h/g}$	$t_F = \sqrt{2h/g}$

(where $\beta = [1 + I/Mr^2]$)

- Velocity of falling and sliding bodies are equal and is more than rollings.
- Acceleration is maximum in case of falling and minimum in case of rolling.
- Falling body reaches the bottom first while rolling last.

SOLVED EXAMPLES

Example - 1

A flywheel of radius 30 cm starts from rest and accelerates with constant acceleration of 0.5 rad/s^2. Compute the tangential, radial and resultant accelerations of a point on its circumference :

(a) Initially at $\theta = 0°$

(b) After it has made one third of a revolution.

Sol. (a) At the start :

$\alpha = 0.5$ rad/s^2

$R = 0.3$ m

$\omega = \omega_i = 0$ rad/s

Radial acceleration = ar = $\omega^2 R = 0$ m/s

Tangential acceleration = $a_t = R\alpha = (0.3)\,(0.5) = 0.15$ m/s^2

Net acceleration = a_{net}

$$= \sqrt{a_r^2 + a_t^2} = \sqrt{0^2 + 0.15^2} = 0.15 \text{ m/s}^2$$

(b) After $\theta = 120°$ $(2\pi/3)$:

$$\omega_f^2 = \omega_i^2 + 2\alpha\theta = 0 + 2(0.5)(2\pi/3)$$

$$\Rightarrow \quad \omega_f = \sqrt{\frac{2\pi}{3}} \text{ rad/s}$$

$$a_r = \omega^2 R = 2\pi/3(0.3) = \pi/5 \text{ m/s}^2$$

$$a_t = R\alpha = (0.3)(0.5) = 0.15 \text{ m/s}^2$$

$$a_{net} = \sqrt{a_r^2 + a_t^2} = \sqrt{\frac{\pi^2}{25} + (0.15)^2} = 0.646 \text{ m/s}^2$$

Example - 2

A wheel mounted on a stationary axle starts at rest and is given the following angular acceleration :

$\alpha = 9 - 12\,t$ (in SI units)

where t is the time after the wheel begins to rotate. Find the number of revolutions that the wheel turns before it stops (and begins to turn in the opposite direction).

Sol. The kinematic equations do not apply because the angular acceleration α is not constant.

We start with the basic definition : $\alpha = d\omega/dt$ to write

$$\omega - \omega_0 = \int_0^t \alpha\, dt = \int_0^t (9 - 12\,t)dt = 9\,t - 6t^2 \text{ (in SI units)}$$

We find the elapsed time t between

$\omega_0 = 0$ and $\omega = 0$ by substituting these values :

$0 - 0 = 9t - 6t^2$

Solving for t, we obtain $t = 9/6 = 1.50$ s

From $\omega = d\theta/dt$, we have

$\theta - \theta_0$

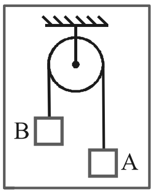
$$= \int_0^t \omega\, dt = \int_0^t \left(9t - 6t^2\right) dt = 4.5\,t^2 - 2t^3$$

Substituting $\theta_0 = 0$ and t = 1.5 s, we obtain

$$\theta - 0 = 4.5\,(1.5)^2 - 2\,(1.5)^3 = 3.375 \text{ rad}$$

Example - 3

In the given figure, calculate the linear acceleration of the blocks.



Mass of block A = 10 kg

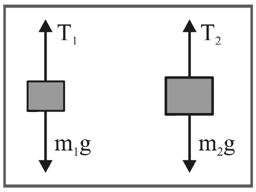
Mass of block B = 8 kg

Mass of disc shaped pulley = 2 kg (take g = 10 m/s^2)

Sol. Let R be the radius of the pulley and T_1 and T_2 be the tensions in the left and right portions of the string.

Let m_1 = 10 kg ; m_2 = 8 kg ; M = 2 kg.

Let a be the acceleration of blocks.



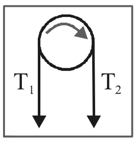
For the blocks (linear motion)

(i) $T_1 - m_1 g = m_1 a$

(ii) $m_2 g - T_2 = m_2 a$

For the pulley (rotation)

Net torque = Iα



(iii) $T_2 R - T_1 R = \frac{1}{2} M R^2 \alpha$

The linear acceleration of blocks is same as the tangential acceleration of any point on the circumference of the pulley which is Rα.

(iv) $a = R\alpha$

Dividing (iii) by R and adding to (i) and (ii),

$$m_2 g - m_1 g = m_2 a + m_1 a + \frac{M}{2} R\alpha$$

$$\Rightarrow \quad m_2 g - m_1 g = \left(m_2 + m_1 + \frac{M}{2}\right) a$$

$$a = \frac{m_2 - m_1}{m_2 + m_1 + \frac{M}{2}}\, g = \frac{(10-8)g}{10 + 8 + \frac{2}{2}} = \frac{20}{19}\ \text{m/s}^2$$

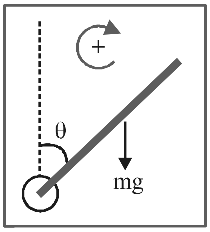
Example - 4

A uniform rod of length L and mass M is pivoted freely at one end.

(a) What is the angular acceleration of the rod when it is at angle θ to the vertical ?

(b) What is the tangential linear acceleration of the free end when the rod is horizontal ? The moment of inertia of a rod about one end is 1/3 ML^2.

Sol. The figure shows the rod at an angle θ to the vertical. If we take torques about the pivot we need not be connected with the force due to the pivot.



The torque due to the weight is mgL/2 sin θ, so the second law for the rotational motion is

$$\frac{mgL}{2}\sin\theta = \frac{ML^2}{3}\alpha \qquad \text{Thus } \alpha = \frac{3g\sin\theta}{2L}$$

When the rod is horizontal $\theta = \pi/2$ and $\alpha = 3g/2L$.

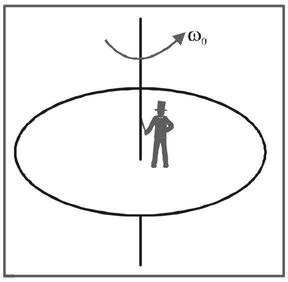
The tangential linear acceleration of the free end is

$$a_t = \alpha L = \frac{3g}{2}$$

Example - 5

A turntable rotates about a fixed vertical axis, making one revolution in 10 s. The moment of inertia of the turntable about the axis is 1200 kg m^2. A man of mass 80 kg initially standing at the centre of the turntable, runs out along a radius. What is the angular velocity of the turntable when the man is 2m from the centre ?

Sol.



I_0 = initial moment of inertia of the system

$I_0 = I_{man} + I_{table}$

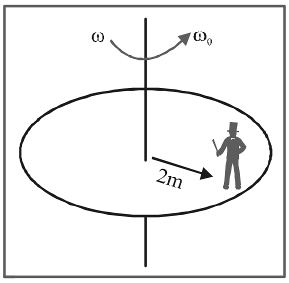
$I_0 = 0 + 1200 = 1200\ kg\ m^2$

($I_{man} = 0$ as the man is at the axis)

I = final moment of inertia of the system

$I = I_{man} + I_{table}$

$I = mr^2 + 1200$



$I = 80\,(2)^2 + 1200 = 1520\ kg\ m^2$

By conservation of angular momentum :

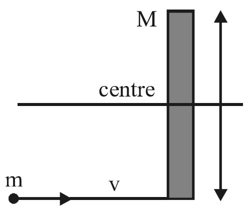
$I_0\ \omega_0 = I\omega$

Now $\omega_0 = 2\pi/T_0 = 2\pi/10 = \pi/5$ rad/s

$$\omega = \frac{I_0\omega_0}{I} = \frac{1200\times\pi}{1520\times5} = 0.51 \text{ rad/s}$$

Example - 6

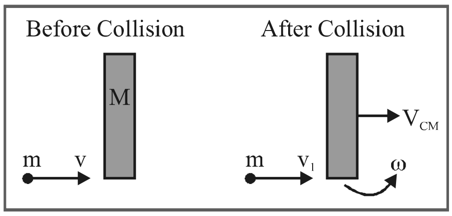
A meter stick lies on a frictionless horizontal table. It has a mass M and is free to move in any way on the table. A hockey puck m, moving as shown with speed v collide elastically with the stick.



(a) What is the velocity of the puck after impact ?

(b) What is the velocity of the CM and the angular velocity of the stick after impact ?

Sol.



There is no external impulse on the system.

∴ Linear momentum is conserved and Angular momentum about any point is conserved.

(i) $P_i = P_f$

$mv = mv_1 + MV_{CM}$ ...(i)

(ii) $(L_{CM})_i = (L_{CM})_f$ about CM of rod.

$$mv\frac{\ell}{2} + 0 = \frac{mv_1\,\ell}{2} + I_{CM}\,\omega \quad ...(ii)$$

(iii) At colliding points

$V_{sep} = eV_{app}$

$$\left(V_{CM} + \frac{\ell}{2}\omega - v_1\right) = ev \quad ...(iii)$$

e = 1 (Elastic collision)

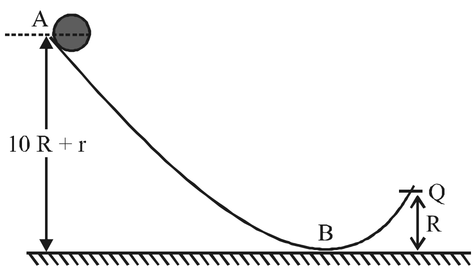
Solving (i), (ii) and (iii) we get :

$$v_1 = \left(\frac{4m - M}{4m + M}\right)v\ ;\quad V_{CM} = \frac{2m}{(4m + M)}\,v$$

$$\omega = \left(\frac{12m}{(4m + M)}\right)\frac{v}{\ell}$$

Example - 7

A solid sphere of radius r and mass m rolls without slipping down the track shown in the figure. At the end of its run at point Q its center–of–mass velocity is directed upward.



(a) Determine the force with which the sphere presses against the track at B.

(b) Upto what height does the CM rise after it leaves the track ?

Sol. (a) From A to B

Loss in GPE = gain in KE

$$mg(10\ R) = \frac{1}{2} m V_{cm_1}^2 + \frac{1}{2} I_{cm}\omega_1^2$$

For rolling without slipping on a fixed surface.

$V_{cm_1} = R\omega_1$

The CM follows a circular path of radius R – r

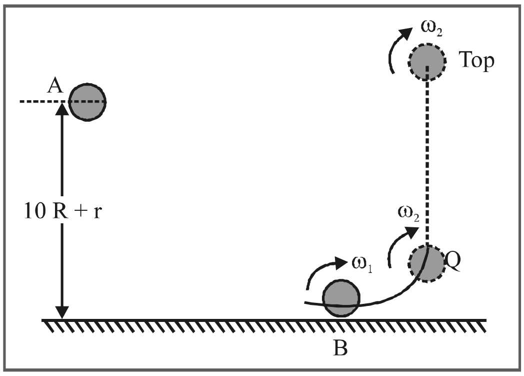
AT B, net force towards centre $= N - mg = \dfrac{mV_{cm}^2}{R-r}$

$\Rightarrow \quad N + mg = \dfrac{m(100\ gR)}{7(R-r)} = \dfrac{mg(107R - 7r)}{7(R-r)}$

(b) From A to Q, $mg(9R + r)$

$$= \frac{1}{2} mv_{cm_2}^2 + \frac{1}{2} I_{cm}\left(\frac{V_{cm_2}}{r}\right)^2$$

$(V_{cm_2} = r_{\omega_2} \text{ at } Q)$



From Q to P, ω does not change because about C.M torque is zero in air.

gain in GPE = loss in KE

$\Rightarrow \quad mg \times \text{gain in height} = \dfrac{1}{2} mV_{cm_2}^2$

$\Rightarrow \quad h = \dfrac{V_{cm_2}^2}{2g} = \dfrac{5}{7}(9R + r)$

$\Rightarrow \quad \text{height above the base} = R + h = \dfrac{52R}{7} + \dfrac{5r}{7}$

Example - 8

A rigid body of radius of gyration k and radius R rolls (without slipping) down a plane inclined at an angle θ with horizontal. Calculate its acceleration and the frictional force acting on it.

Sol. When the body is placed on the inclined plane, it tries to slip down and hence a static friction f acts upwards. This friction provides a torque which causes the body to rotate. Let A_{CM} be the linear acceleration of centre of mass and α be the angular acceleration of the body.

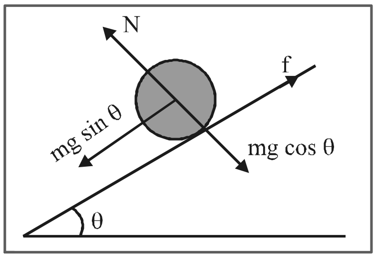
From force diagram :

For linear motion parallel to the plane

$mg \sin\theta - f = ma$

For rotation around the axis through centre of mass

Net torque $= I\alpha \Rightarrow fR = (mk^2)\alpha$



As there is no slipping, the point of contact of the body with plane is instantaneously at rest.

$\Rightarrow \quad v = R\omega$ and $A_{CM} = R\alpha$

Solve the following three equations for a and f :

$mg \sin\theta - f = m$

$fR = mk^2\alpha$

$A_{CM} = R\alpha$

$$A_{CM} = \frac{g \sin\theta}{1 + \frac{k^2}{R^2}} \quad \text{and} \quad f = \frac{mg \sin\theta}{1 + \frac{R^2}{k^2}}$$

We can also derive the condition for pure rolling (rolling without slipping) :

To avoid slipping, $f \le \mu_s N$

$$\frac{g \sin\theta}{1 + R^2/k^2} \le \mu_s\ mg \cos\theta$$

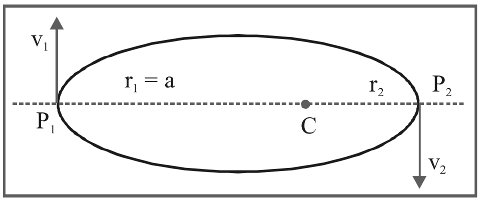
$$\Rightarrow \quad \mu_s \ge \frac{\tan\theta}{1 + \frac{R^2}{k^2}}$$

This is the condition on μ_s so that the body rolls without slipping.

Example - 9

A particle of mass m is subject to an attractive central force of magnitude k/r^2 where k is a constant. At the instant when the particle is at an extreme position in its closed elliptical orbit, its distance from the centre of force is 'a' and its speed is $\sqrt{\frac{k}{2ma}}$. Calculate its distance from force-centre when it is at the other extreme position.

Sol. Let P be the particle and C be the force–centre. P_1 and P_2 are its extreme positions at distance r_1 and r_2 from C.



We have r_1 = a and $v_1 = \sqrt{\frac{k}{2ma}}$

As the force is directed towards C, torque about C is zero.

Hence we will apply conservation of angular momentum about C and conservation of energy.

$F = k/r^2$

$\Rightarrow$ Potential energy (U) = $-k/r$

(Compare the expression of force with gravitational force)

From conservation of energy,

total energy at P_1 = total energy at P_2

$$\frac{1}{2}mv_1^2 + \left(\frac{-k}{r_1}\right) = \frac{1}{2}mv_2^2 + \left(\frac{-k}{r_2}\right)$$

From conervation of angular momentum about C,

$m\,v_1\,r_1 = m\,v_2\,r_2$

We have to find r_2. Hence we eliminate v_2.

$$\frac{1}{2}mv_1^2 - \frac{k}{r_1} = \frac{1}{2}m\left(\frac{v_1 r_1}{r_2}\right)^2 - \frac{k}{r_2}$$

Substituting $v_1 = \sqrt{\frac{k}{2ma}}$ and $r_1 = a$

$$\frac{1}{2}m\frac{k}{2ma} - \frac{k}{a} = \frac{1}{2}\frac{ma^2}{r_2^2}\frac{k}{2ma} - \frac{\ell}{r_2}$$

$\Rightarrow$ $3r_2^2 - 4ar_2 + a^2 = 0$

$\Rightarrow$ $r_2 = a, a/3$

The other extreme position is at a distance of a/3 from C.

EXERCISE - 1 BASIC OBJECTIVE QUESTIONS

Discreet Particles

1. The moment of inertia of a body does not depend on:

(a) the mass of the body
(b) the angular velocity of the body
(c) the axis of rotation of the body
(d) the distribution of the mass in the body

2. Three point masses m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of side 'a'. What is the moment of inertia of the system about an axis along the altitude of the triangle passing through m_1?

(a) $(m_1+m_2)\frac{a^2}{4}$ (b) $(m_2+m_3)\frac{a^2}{4}$

(c) $(m_1+m_3)\frac{a^2}{4}$ (d) $(m_1+m_2+m_3)\frac{a^2}{4}$

Continuous Body

3. A circular disc X of radius R is made from an iron plate of thickness t, and another disc Y of radius 4R is made from an iron plate of thickness t/4. Then the relation between the moment of inertia I_X and I_Y is :

(a) $I_Y = 32\, I_X$ (b) $I_Y = 16\, I_X$
(c) $I_Y = I_X$ (d) $I_Y = 64\, I_X$

4. The ratio of the squares of radii of gyration of a circular disc and a circular ring of the same radius about a tangential axis is :

(a) 1 : 2 (b) 5 : 6
(c) 2 : 3 (d) 2 : 1

5. Moment of inertia of a uniform annular disc of internal radius r and external radius R and mass M about an axis through its centre and perpendicular to its plane is:

(a) $\frac{1}{2}M(R^2-r^2)$ (b) $\frac{1}{2}M(R^2+r^2)$

(c) $\frac{M(R^4+r^4)}{2(R^2+r^2)}$ (d) $\frac{1}{2}\frac{M(R^4+r^4)}{(R^2-r^2)}$

6. For the same total mass which of the following will have the largest moment of inertia about an axis passing through the centre of gravity and perpendicular to the plane of the body?

(a) A disc of radius a (b) A ring of radius a
(c) A square lamina of side 2a
(d) Four roads forming square of side 2a

7. If the radius of a solid sphere is 35 cm, calculate the radius of gyration when the axis is along a tangent:

(a) $7\sqrt{10}$cm (b) $7\sqrt{35}$ cm

(c) $\frac{7}{5}$cm (d) $\frac{2}{5}$cm

8. The moment of inertia of a straight thin rod of mass M, length L about an axis perpendicular to its length and passing through its one end is:

(a) $\frac{1}{12}ML^2$ (b) $\frac{1}{3}ML^2$

(c) $\frac{1}{2}ML^2$ (d) ML^2

9. A closed tube partly filled with water lies in a horizontal plane. If the tube is rotated about perpendicular bisector, the moment of inertia of the system:

(a) increases (b) decreases
(c) remains constant (d) depends on sense of rotation

10. Two rings of same and mass are placed such that their centres are at a common point and their planes are perpendicular to each other. The moment of inertia of the system about an axis passing through the centre and perpendicular to the plane of one of the rings is (mass of the ring = m, radius = r)

(a) $\frac{1}{2}mr^2$ (b) mr^2

(c) $\frac{3}{2}mr^2$ (d) $2mr^2$

11. What is the moment of inertia I of a uniform solid sphere of mass M and radius R, privoted about an axis that is tangent to the surface of the sphere?

(a) $\frac{2}{3}MR^2$ (b) $\frac{3}{5}MR^2$

(c) $\frac{6}{5}MR^2$ (d) $\frac{7}{5}MR^2$

Parallel axis theorem

12. The moment of inertia of a solid cylinder of mass M, radius R and Length L about its axis

(a) ML^2 (b) MR^2

(c) $\frac{MR^2}{L}$ (d) $\frac{MR^2}{2}$

13. The moment of inertia of a metre stick of mass 300 gm, about an axis at right angles to the stick and located at 30 cm mark, is:

(a) $8.3\times10^5\,g-cm^2$ (b) $5.8\,g-cm^2$

(c) $3.7\times10^5\,g-cm^2$ (d) none of these

14. The moment of inertia of a solid sphere about an axis passing through centre of gravity is $\frac{2}{5}MR^2$; then its radius of gyration about a parallel axis at a distance 2R from first axis is:

(a) 5R (b) $\sqrt{22/5}\,R$

(c) $\frac{5}{2}R$ (d) $\sqrt{12/5}\,R$

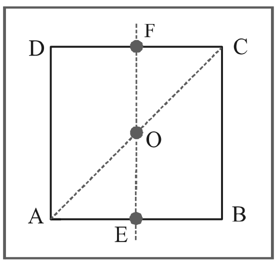
15. Four spheres of diameter 2a and mass M are placed with their centres on the four corners of a square of side b. Then the moment of inertia of the system about an axis along one of the sides of the square is:

(a) $\frac{4}{5}Ma^2+2Mb^2$ (b) $\frac{8}{5}Ma^2+2Mb^2$

(c) $\frac{8}{5}Ma^2$ (d) $\frac{4}{5}Ma^2+4Mb^2$

Perpendicular axis theorem

16. For the given uniform square lamina ABCD, whose centre is O



(a) $\sqrt{2}\,I_{AC}=I_{EF}$ (b) $I_{AD}=3I_{EF}$

(c) $I_{AC}=I_{EF}$ (d) $I_{AC}=\sqrt{2}\,I_{EF}$

17. Moment of inertia of a circular wire of mass M and radius R about its diameter is :

(a) $MR^2/2$ (b) MR^2

(c) $2\,MR^2$ (d) $MR^2/4$

18. One solid sphere A and another hollow sphere B are of same mass and same outer radii. Their monents of inertia about their diameters are respectively I_A and I_B such that

(a) $I_A=I_B$ (b) $I_A>I_B$

(c) $I_A<I_B$ (d) $\frac{I_A}{I_B}=\frac{d_A}{d_B}$

19. Three point masses, each of mass m, are placed at the corners of an equilateral triangle of side. l. Then the moment of inertia of this system about an axis along one side of the triangle is:

(a) $3ml^2$ (b) ml^2

(c) $\frac{3}{4}ml^2$ (d) $\frac{3}{2}ml^2$

20. Moment of inertia of a uniform circular disc about a diameter is I. Its moment of inertia about an axis perpendicular to its plane and passing through a point on its rim will be:

(a) $5I$ (b) $3I$

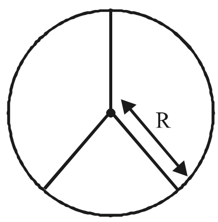
(c) $6I$ (d) $4I$

21. The moment of inertia of a circular ring of radius R and mass M about a tangent in its plane is:

(a) MR^2 (b) $(1/2)MR^2$

(c) $(3/2)MR^2$ (d) $2MR^2$

22. A wheel comprises of a ring of radius R and mass M and three spokes of mass m each. The moment of inertia of the wheel about its axis is :



(a) $\left(M+\frac{m}{4}\right)R^2$ (b) $(M+m)\,R^2$

(c) $(M+3m)\,R^2$ (d) $\left(\frac{M+m}{2}\right)R^2$

23. Four identical rods are joined end to end to form a square. The mass of each rod is M. The moment of inertia of the square about the median line is:

(a) $\frac{M\ell^2}{3}$ (b) $\frac{M\ell^2}{4}$

(c) $\frac{M\ell^2}{6}$ (d) $\frac{2M\ell^2}{3}$

Point of application

24. When a steady torque or couple acts on a body, the body:

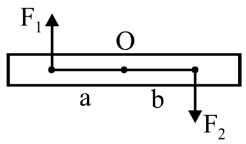
(a) continues in a state of rest or of uniform motion by Newton's 1st law

(b) gets linear acceleration by Newton's 2nd law

(c) gets an angular acceleration

(d) continues to rotate at a steady rate.

25. A uniform rod is kept on a frictionless horizontal table and two forces F_1 and F_2 are acted as shown in figure. The line of action of force F_{R_1} (which produces same torque) is at a perpendicular distance 'C' from O. Now F_1 and F_2 are interchanged and F_1 is reversed. The new forces F_{R_2} (which produces same torque in the present case) has its line of action at a distance $\frac{C}{2}$ from O. If the $F_{R_1} : F_{R_2}$ in the ratio 2:1, then a:b is (assume $F_2 a > F_1 b$):



(a) $\frac{2F_2 - F_1}{4F_3 - F_1}$ (b) $\frac{F_2 + 4F_1}{4F_2 - F_1}$

(c) $\frac{F_2 - 3F_1}{F_1 + F_2}$ (d) $\frac{F_2 + F_1}{2F_2 + 3F_1}$

26. What is the torque of force $\vec{F} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ acting at a point $\vec{r} = 3\hat{i} + 2\hat{j} + 3\hat{k}$ about the origin?

(a) $6\hat{i} - 6\hat{j} + 12\hat{k}$ (b) $-6\hat{i} + 6\hat{j} - 12\hat{k}$

(c) $17\hat{i} - 6\hat{j} - 13\hat{k}$ (d) $-17\hat{i} + 6\hat{j} + 13\hat{k}$

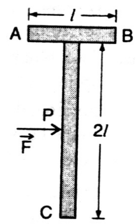
Rotational Equilibrium

27. A cubical block of mass M and edge a slides down a rough inclined plane of inclination θ with a uniform velocity. The torque of the normal force on the block about its centre has a magnitude:

(a) zero (b) Mga

(c) Mga sin θ (d) $\frac{Mga \sin\theta}{2}$

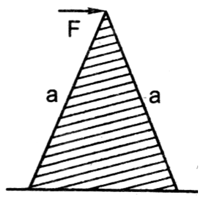
28. A T-shaped object with dimensions shown in the figure, is lying on a smooth floor. A force $\vec{F}$ is applied at the point P parallel to AB, such that the object has only the translational motion without rotation. Find the location of P with respect to C:



(a) $\frac{4l}{3}$ (b) l

(c) $\frac{2l}{3}$ (d) $\frac{3l}{2}$

29. An equilateral prism of mass m rests on a rough horizontal surface with coefficient of friction μ. A horizontal force F is applied on the prism as shown in the figure. If the coefficient of friction is sufficiently high so that the prism does not slide before toppling, then the minimum force required to topple the prism is:



(a) $\frac{mg}{\sqrt{3}}$ (b) $\frac{mg}{4}$

(c) $\frac{\mu mg}{\sqrt{3}}$ (d) $\frac{\mu mg}{4}$

Rotational Kinematics

30. The driving side belt has a tension of 1600 N and the slack side has 500 N tension. The belt turns a pulley 40 cm in radius at a rate of 300 rpm. This pulley drives a dynamo having 90% efficiency. How many kilowatts are being delivered by the dynamo ?

(a) 12.4 (b) 6.2

(c) 24.8 (d) 13.77

31. The angular velocity of a wheel increases from 100 rps to 300 rps in 10 s. The number of revolutions made during that time is:

(a) 600 (b) 1500

(c) 1000 (d) 2000

32. When a ceiling fan is switched off, its angular velocity falls to half while it makes 36 rotations. How many more rotations will it make before coming to rest?

(a) 24 (b) 36

(c) 18 (d) 12

33. A rigid body rotates about a fixed axis with variable angular velocity equal to $\alpha - \beta t$ at time t where α and β are constants. The angle through which it rotates before it comes to rest is:

(a) $\frac{\alpha^2}{2\beta}$ (b) $\frac{\alpha^2-\beta^2}{2\alpha}$

(c) $\frac{\alpha^2-\beta^2}{2\beta}$ (d) $\frac{\alpha(\alpha-\beta)}{2}$

34. A wheel is subjected to uniform angular acceleration about its axis. Initially, its angular velocity is zero. In the first 2s, it rotates through an angle θ_1, in the next 2s, it rotates through an angle θ_2. The ration of θ_2/θ_1 is:

(a) 1 (b) 2

(c) 3 (d) 5

35. The linear velocity of a particle on the equator is nearly (radius of the earth is 4000 miles):

(a) zero (b) 10 mile/hr

(c) 100 mile/hr (d) 1000 mile/hr

36. A stone of mass m is tied to a string of length L and rotated in a circle with a constant speed v; if the string is released the stone files:

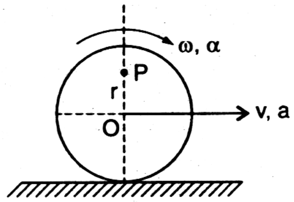
(a) Radially outward

(b) Radially inward

(c) Tangentially

(d) With an acceleration mv^2/L

37. A disc of radius r rolls on a horizontal ground with linear acceleration a and angular acceleration α as shown in figure. The magnitude of acceleration of point P shown in figure at an instant when its linear velocity is v and angular velocity is ω will be:



(a) $\sqrt{(a+r\alpha)^2+(r\omega^2)^2}$ (b) $\frac{ar}{R}$

(c) $\sqrt{r^2\alpha^2+r^2\omega^4}$ (d) $r\alpha$

38. An electric fan has blades of length 30 cm as measured from the axis of rotation. If the fan is rotating at 1200 rpm, the acceleration of a point on the tip of a blade is about:

(a) 4740 m/sec^2 (b) 5055 m/sec^2

(c) 1600 m/sec^2 (d) 2370 m/sec^2

Rotational Dynamics

39. A flywheel of mass 50 kg and radius of gyration about its axis of rotation of 0.5m is acted upon by a constant torque of 12.5 Nm. Its angular velocity at t = 5 sec is:

(a) 2.5 rad/sec (b) 5 rad/sec

(c) 7.5 rad/sec (d) 10 rad/sec

40. A uniform metre stick of mass M is hinged at one end and supported in a horizontal direction by a string attached to the other end. What should be the initial acceleration (in rad/sec^2) of the stick if the string is cut?

(a) $\frac{3}{2}g$ (b) g

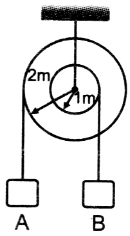
(c) 3g (d) 4g

41. A thin hollow cylinder is free to rotate about its geometrical axis. It has a mass of 8 kg and a radius of 20 cm. A rope is wrapped around the cylinder. What force must be exerted along the rope to produce an angular acceleration of 3 rad/sec^2?

(a) 8.4 N (b) 5.8 N

(c) 4.8 N (d) None of these

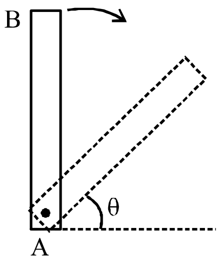
42. In the pulley system shown, if radii of the bigger and smaller pulley are 2 m and 1m respectively and the acceleration of block A is $5 m/s^2$ in the downward direction, then the acceleration of block B will be:



(a) $0\ m/s^2$ (b) $5\ m/s^2$

(c) $10\ m/s^2$ (d) $5/2\ m/s^2$

43. Figure shows a uniform rod of length ℓ and mass M which is pivoted at end A such that it can rotate in a vertical plane. The free end of the rod 'B' is initially vertically above the pivot and then released. As the rod rotates about A, its angular acceleration when it is inclined to horizontal at angle θ is



(a) $\frac{3g}{2\ell}\cos\theta$ (b) $\frac{g}{\ell}\tan\theta$

(c) $\frac{5g}{4\ell}\sin\theta$ (d) $\frac{g}{\ell}\sin\theta$

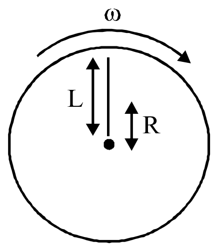
Rotational Energy

44. In the above quesiton, the end B of the rod will hit the ground with a linear speed :

(a) $\sqrt{2g\ell}$ (b) $\sqrt{5g\ell}$

(c) $\sqrt{3g\ell}$ (d) $\sqrt{\frac{2g}{\ell}}$

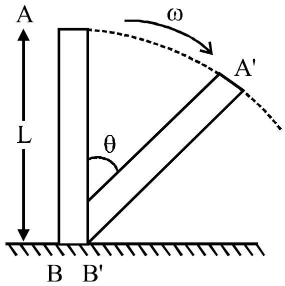
45. A uniform rod of mass M and length L lies radially on a disc rotating with angular speed ω in a horizontal plane about its axis. The rod does not slip on the disc and the centre of the rod is at a distance R from the centre of the disc. Then the kinetic energy of the rod is :



(a) $\frac{1}{2}m\omega^2\left(R^2+\frac{L^2}{12}\right)$ (b) $\frac{1}{2}m\omega^2R^2$

(c) $\frac{1}{24}m\omega^2L^2$ (d) none of these

46. A uniform rod of length L is free to rotate in a vertical plane about a fixed horizontal axis through B. The rod begins rotating from rest from its unstable equilibrium position. When it has turned through an angle θ its average angular velocity ω is given as :

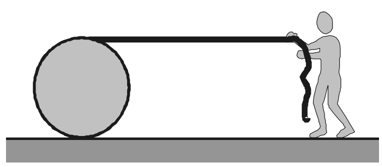


(a) $\sqrt{\frac{6g}{L}}\sin\theta$ (b) $\sqrt{\frac{6g}{L}}\sin\frac{\theta}{2}$

(c) $\sqrt{\frac{6g}{L}}\cos\frac{\theta}{2}$ (d) $\sqrt{\frac{6g}{L}}\cos\theta$

Kinematics (Rigid Body)

47. A string of negligible thickness is wrapped several times around a cylinder kept on a rough horizontal surface. A man standing at a distance from the cylinder holds one end of the string and pulls the cylinder towards him. There is no slipping anywhere. The length of the string passed through the hand of the man while the cylinder reaches his hands is:



(a) 1 (b) 2

(c) 3 (d) 4

48. A solid sphere of mass M and radius R is placed on a rough horizontal surface. It is pulled by a horizontal force F acting through its centre of mass as a result of which it begins to roll without slipping. Angular acceleration of the sphere can be expressed as:

(a) $\frac{3F}{4MR}$ (b) $\frac{5F}{7MR}$

(c) $\frac{7F}{11MR}$ (d) $\frac{5F}{2MR}$

49. A sphere cannot roll on :

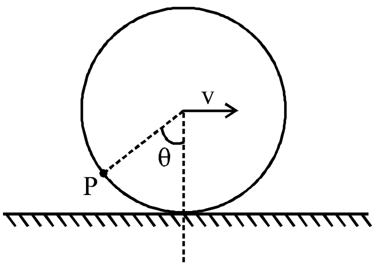
(a) a smooth horizontal surface

(b) a rough horizontal surface

(c) a smooth inclined surface

(a) a rough inclined surface

50. A hoop rolls on a horizontal ground without slipping with linear speed v. Speed of a particle P on the circumference of the hoop at angle θ is :



(a) $2v\sin\frac{\theta}{2}$ (b) $v\sin\theta$

(c) $2v\cos\frac{\theta}{2}$ (d) $v\cos\theta$

Dynamics

51. A sphere of mass m rolls without slipping on an inclined plane of inclination θ. The linear acceleration of the sphere is:

(a) $\frac{1}{7}g\sin\theta$ (b) $\frac{2}{7}g\sin\theta$

(c) $\frac{3}{7}g\sin\theta$ (d) $\frac{5}{7}g\sin\theta$

52. In the above question, the force of friction on the sphere is:

(a) $\frac{1}{7}Mg\sin\theta$ (b) $\frac{2}{7}Mg\sin\theta$

(c) $\frac{3}{7}Mg\sin\theta$ (d) $\frac{5}{7}Mg\sin\theta$

53. In the above question, the minimum value of coefficient of friction so that sphere may roll without slipping is :

(a) $\frac{2}{7}\sin\theta$ (b) $\frac{2}{7}\cos\theta$

(c) $\frac{2}{7}\tan\theta$ (d) $\frac{2}{7}\cot\theta$

54. A hoop rolls without slipping down an incline of slope 30°. Linear acceleration of its centre of mass is

(a) $\frac{g}{2}$ (b) $\frac{g}{3}$

(c) $\frac{g}{4}$ (d) $\frac{g}{6}$

Total Energy

55. A 6 kg ball starts from rest and rolls down a rough gradual slope until it reaches a point 80 cm lower than its starting point. Then the speed of the ball is :

(a) 1.95 ms^{-1} (b) 2.5 ms^{-1}

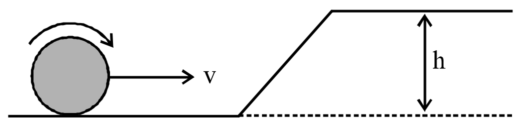
(c) 3.35 ms^{-1} (d) 4.8 ms^{-1}

56. A uniform solid sphere rolls on a horizonal surface at 20 ms^{-1}. It then rolls up an incline having an angle of inclination at 30° with the horizontal. If the friction losses are negligible, the value of height h above the ground where the ball stops is :

(a) 14.3 m (b) 28.6 m

(c) 57.2 m (d) 9.8 m

57. A solid sphere is rolling on a frictionless surface, shown in figure with a translational velocity v m/s. If it is to climb the inclined surface then v should be :



(a) $\geq\sqrt{\frac{10}{7}gh}$ (b) $\geq\sqrt{2gh}$

(c) 2 gh (d) $\frac{10}{7}gh$

58. A disc is rolling on an inclined plane. What is the ratio of its rotational K.E. to the total K. E. ?

(a) 1 : 3 (b) 3 : 1

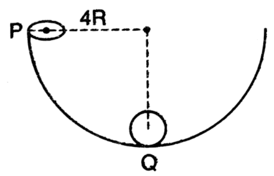
(c) 1 : 2 (d) 2 : 1

59. A spherical ball of mass 20 kg is stationary at the top of a hill of height 100 m. It rolls down a smooth surface to the ground, then climbs up another hill of height 30 m and finally rolls down to a horizontal base at a height of 20 m above the ground. The velocity attained by the ball is :

(a) 40 m/s (b) 20 m/s

(c) 10 m/s (d) $10\sqrt{30}$ m/s

60. Figure shows a hemisphere of radius 4R. A ball of radius R is released from position P. It rolls without slipping along the inner surface of the hemisphere. Linear speed of its centre of mass when the ball is at position Q is :



(a) $\sqrt{\frac{30\,gR}{7}}$ (b) $\sqrt{\frac{24\,gR}{5}}$

(c) $\sqrt{\frac{40\,gR}{9}}$ (d) $\sqrt{6gR}$

61. If a spherical ball rolls on a table without slipping, the fraction of its total energy associated with rotation is

(a) $\frac{3}{5}$ (b) $\frac{2}{7}$

(c) $\frac{2}{5}$ (d) $\frac{3}{7}$

Particle

62. A particle of mass m is projected with a velocity v making an angle of 45° with the horizontal. The magnitude of angular momentum of the projectile about an axis of projection when the particle is at maximum height h is :

(a) zero (b) $\frac{mv^3}{4\sqrt{2}\,g}$

(c) $\frac{mv^2}{\sqrt{2}\,g}$ (d) $m\sqrt{2gh^3}$

63. A particle of mass m = 5 units is moving with a uniform speed $v = 3\sqrt{2}\,m$ in the XOY plane along the line $Y = X + 4$. The magnitude of the angular momentum of the particle about the origin is :

(a) zero (b) 60 unit

(c) 7.5 unit (d) $40\sqrt{2}$ unit

64. A particle is moving along a straight line parallel to x–axis with constant velocity. Its angular momentum about the origin :

(a) decreases with time (b) increases with time

(c) remains constant (d) is zero

65. If a particle moves in the X–Y plane, the resultant angular momentum has :

(a) only x–component

(b) only y–component

(c) both x & y component

(d) only z–component

Torque relation and Angular Impulse

66. A constant torque acting on a uniform circular wheel changes its angular momentum from A_0 to $4A_0$ in 4 seconds. The magnitude of this torque is :

(a) $3A_0/4$ (b) A_0

(c) $4A_0$ (d) $12A_0$

67. A particle moves in a force field given by : $\vec{F} = \hat{r}F(r)$, where $\hat{r}$ is a unit vector along the position vector, $\vec{r}$, then which is true ?

(a) The torque acting on the particle is not zero

(b) The torque acting on the particle produces an angular acceleration in it

(c) The angular momentum of the particle is conserved

(d) The angular momentum of the particle increases

Rigid Body in fixed axis rotation

68. A rigid body rotates with an angular momentum L. If its rotational kinetic energy is made 4 times, its angular momentum will become :

(a) 4 L (b) 16 L

(c) $\sqrt{2}$ L (d) 2 L

69. The diameter of a flywheel is 1 m. It has a mass of 20 kg. It is rotating about its axis with a speed of 120 rotations is one minute. Its angular momentum (in kg–m^2/s) is :

(a) 13.4 (b) 31.4

(c) 41.4 (d) 43.4

70. The position of a particle is given by : $\vec{r} = \hat{i} + 2\hat{j} - \hat{k}$ and its linear momentum is given by : $\vec{P} = 3\hat{i} + 4\hat{j} - 2\hat{k}$. Then its angular momentum, about the origin is perpendicular to :

(a) YZ plane (b) z–axis

(c) y–axis (d) x–axis

Angular Momentum Conservation

71. If the radius of earth contracts $\frac{1}{n}$ of its present day value, the length of the day will be approximately :

(a) $\frac{24}{n}$h (b) $\frac{24}{n^2}$h

(c) 24n h (b) $24n^2$ h

72. A disc of moment of inertia I_1 is rotating freely with angular velocity ω_1 when a second, non-rotating disc with moment of inertia I_2 is dropped on it gently the two then rotate as a unit. Then the total angular speed is :

(a) $\frac{I_1\omega_1}{I_2}$ (b) $\frac{I_2\omega_1}{I_1}$

(c) $\frac{I_1\omega_1}{I_2 + I_1}$ (d) $\frac{(I_1 + I_2)\omega_1}{I_2}$

73. A thin circular ring of mass M and radius R is rotating about its axis with a constant angular velocity ω. Two objects, each of mass m, are attached gently to the opposite ends of a diameter of the ring. The ring rotates now with an angular velocity :

(a) $\frac{\omega M}{M + m}$ (b) $\frac{\omega(M - 2m)}{M + 2m}$

(c) $\frac{\omega M}{M + 2m}$ (d) $\frac{\omega(M + m)}{M}$

74. If a gymnast, sitting on a rotating stool with his arms outstretched, suddenly lowers his arms :

(a) the angular velocity increases

(b) his moment of inertia increases

(c) the angular velocity remains same

(d) the angular momentum increases

75. A thin uniform circular disc of mass M and radius R is rotating in a horizontal plane about an axis passing through its centre and perpendicular to the plane with angular velocity ω. Another disc of same mass but half the radius is gently placed over it coaxially. The angular speed of the composite disc will be :

(a) $\frac{5}{4}\omega$ (b) $\frac{4}{5}\omega$

(c) $\frac{2}{5}\omega$ (d) $\frac{5}{2}\omega$

EXERCISE - 2 PREVIOUS YEAR COMPETITION QUESTIONS

OBJECTIVE QUESTIONS (Only one correct answer)

1. A Couple produces : **(CBSE 1997)**

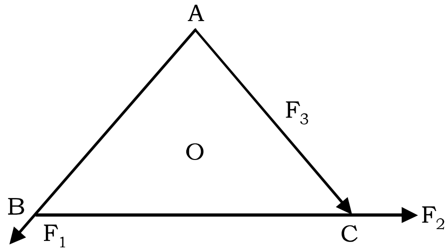
(a) no motion

(b) linear and rotational motion

(c) purely rotational motion

(d) purely linear motion

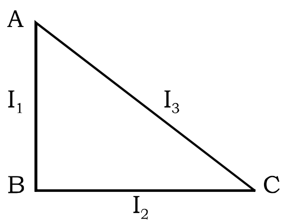
2. O is the centre of an equilateral ABC. F_1, F_2 and F_3 are three forces acting along the sides AB, BC and AC as shown in figure. What should be the magnitude of F_3 so that the total torque about O is zero ? **(CBSE 1998)**



(a) $\frac{(F_1 + F_2)}{2}$ (b) $(F_1 - F_2)$

(c) $(F_1 + F_2)$ (d) $2(F_1 + F_2)$

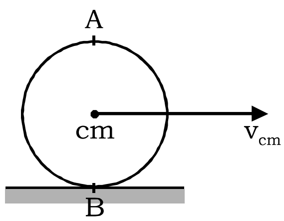
3. ABC is a right angled triangular plate of uniform thickness. The sides are such that AB > BC as shown in figure. I_1, I_2, I_3 are moments of inertia about AB, BC and AC respectively. Then which of the following relation is correct ? **(CBSE 2000)**



(a) $I_1 = I_2 = I_3$ (b) $I_2 > I_1 > I_3$

(c) $I_3 < I_2 < I_1$ (d) $I_3 > I_1 > I_2$

4. A wheel of bicycle is rolling without slipping on a level road. The velocity of the centre of mass is v_{cm}; then true statement is : **(CBSE 2001)**



(a) The velocity of point A is $2v_{cm}$ and velocity of point B is zero

(b) The velocity of point A is zero and velocity of point B is $2\ v_{cm}$

(c) The velocity of point A is $2v_{cm}$ and velocity of point B is $-v_{cm}$

(d) The velocities of both A and B are v_{cm}

5. A disc is rotating with angular velocity ω. If a child sits on it, what is conserved ? **(CBSE 2002)**

(a) Linear momentum (b) Angular momentum

(c) Kinetic energy (d) Moment of inertia

6. A circular disc is to be made using iron and aluminium. To keep its moment of inertia maximum about a geometrical axis, it should be so prepared that : **(CBSE 2002)**

(a) aluminium at interior and iron surround it

(b) iron at interior and aluminium surrounds it

(c) aluminium and iron layers in alternate order

(d) sheet of iron is used at both external surfaces and aluminium sheets as inner material

7. A solid sphere of radius R is placed on a smooth horizontal surface. A horizontal force F is applied at height h from the lowest point. For the maximum acceleration of the centre of mass : **(CBSE 2002)**

(a) h = R (b) h = 2 R

(c) h = 0

(d) the acceleration will be same whatever h may be

8. P is the point of contact of a wheel and the ground. The radius of wheel is 1 m. The wheel rolls on the ground without slipping. The displacement of point P when wheel completes half rotation is : **(CBSE 2002)**

(a) 2m (b) $\sqrt{\pi^2 + 4}$ m

(c) π m (d) $\sqrt{\pi^2 + 2}$ m

9. A ball rolls without slipping. The radius of gyration of the ball about an axis passing through its centre of mass is K. If radius of the ball be R, then the fraction of total energy associated with its rotational energy will be : **(CBSE 2003)**

(a) $\frac{K^2}{K^2 + R^2}$ (b) $\frac{R^2}{K^2 + R^2}$

(c) $\frac{K^2 + R^2}{R^2}$ (d) $\frac{K^2}{R^2}$

10. A solid cylinder of mass M and radius R rolls without slipping down an inclined plane of length L and height h. What is the speed of its centre of mass when the cylinder reaches its bottom ? **(CBSE 2003)**

(a) $\sqrt{\frac{4}{3}gh}$ (b) $\sqrt{4gh}$

(c) $\sqrt{2gh}$ (d) $\sqrt{\frac{3}{4}gh}$

11. The ratio of the radii of gyration of circular disc about a tangential axis in the plane of the disc and of a circular ring of the same radius about a tangential axis in the plane of the ring is : **(CBSE 2004)**

(a) 2 : 3 (b) 2 : 1

(c) $\sqrt{5}:\sqrt{6}$ (d) $1:\sqrt{2}$

12. A round disc of moment of inertia I_2 about its axis perpendicular to its plane and passing through its centre is placed over another disc of moment of inertia I_1 rotating with an angular velocity ω about the same axis. The final angular velocity of the combination of discs is : **(CBSE 2004)**

(a) $\frac{I_2\omega}{I_1+I_2}$ (b) ω

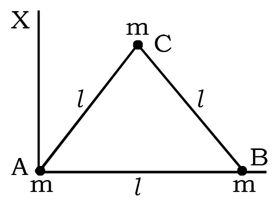
(c) $\frac{I_1\omega}{I_1+I_2}$ (d) $\frac{(I_1+I_2)\omega}{I_1}$

13. A wheel having moment of inertia 2 kg–m² about its vertical axis, rotates at the rate of 60 rev/min about its axis. The torque which can stop the wheel's rotation in 1 min would be : **(CBSE 2004)**

(a) $\frac{2\pi}{15}\text{N}-\text{m}$ (b) $\frac{\pi}{12}\text{N}-\text{m}$

(c) $\frac{\pi}{15}\text{N}-\text{m}$ (d) $\frac{\pi}{18}\text{N}-\text{m}$

14. Three particles, each of mass m grams situated at the vertices of an equilateral triangle ABC of side *l* cm (as shown in the figure). The moment of inertia of the system about a line AX perpendicular to AB and in the plane of ABC, in gram–cm² units will be : **(CBSE 2004)**



(a) $\left(\frac{3}{4}\right)m\ell^2$ (b) $2m\ell^2$

(c) $\left(\frac{5}{4}\right)m\ell^2$ (d) $\left(\frac{3}{2}\right)m\ell^2$

15. A drum of radius R and mass M, rolls down without slipping along an inclined plane of angle θ. The frictional force **(CBSE 2005)**

(a) converts part of potential energy to rotational energy

(b) dissipates energy as heat

(c) decreases the rotational motion

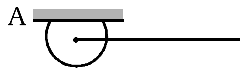
(d) decreases the rotational and translational motion

16. A tube of length L is filled completely with an incompressible liquid of mass M and closed at both the ends. The tube is then rotated in a horizontal plane about one of its ends with a uniform angular velocity ω. The force exerted by the liquid at the other end is : **(CBSE 2006)**

(a) $\frac{ML\omega^2}{2}$ (b) $\frac{ML^2\omega}{2}$

(c) $ML\omega^2$ (d) $\frac{ML^2\omega^2}{2}$

17. A uniform rod of length *l* and mass *m* is free to rotate in a vertical plane about A. The rod initially in horizontal position is released. The initial angular acceleration of the rod is : **(CBSE 2006)**



(a) $3g/2\ell$ (b) $2g/\ell$

(c) $g/2\ell$ (d) $3g/\ell$

18. If $\vec{F}$ is the force acting on a particle having postion vector $\vec{r}$ and $\vec{\tau}$ be the torque of this force the origin, then **(CBSE 2009)**

(a) $\vec{r}.\vec{\tau}\neq 0$ and $\vec{F}.\vec{\tau}=0$ (b) $\vec{r}.\vec{\tau}>0$ and $\vec{F}.\vec{\tau}<0$

(c) $\vec{r}.\vec{\tau}=0$ and $\vec{F}.\vec{\tau}=0$ (d) $\vec{F}.\vec{\tau}=0$ and $\vec{F}.\vec{\tau}\neq 0$

19. Four identical thin rods each of mass M and length *l*, form a square frame. Moment of inertia of this frame about an axis through the centre of the square and perpendicular to its plane is **(CBSE 2009)**

(a) $\frac{4}{3}M\ell^2$ (b) $\frac{2}{3}M\ell^2$

(c) $\frac{13}{3}M\ell^2$ (d) $\frac{1}{3}M\ell^2$

20. A thin circular ring of mass M and radius R is rotating in a horizontal plane about an axis vertical to its plane with a constant angular velocity ω. If two objects each of mass m be attached gently to the opposite ends of a diameter of the ring, the ring will then rotate with an angular velocity **(CBSE 2009)**

(a) $\frac{\omega(M-2m)}{M+2m}$ (b) $\frac{\omega M}{M+2m}$

(c) $\frac{\omega(M+2m)}{M}$ (d) $\frac{\omega M}{M+m}$

21. The ratio of the radii of gyration of a circular disc to that of a circular ring, each of same mass and radius, around their respective axis is **(CBSE 2008)**

(a) $\sqrt{3}:\sqrt{2}$ (b) $1:\sqrt{2}$

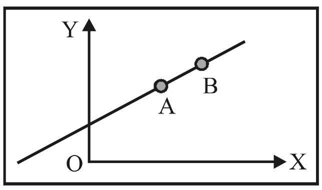
(c) $\sqrt{2}:1$ (d) $\sqrt{2}:\sqrt{3}$

22. A thin rod of length L and mass m is bent at its midpoint into two halves so that the angle between them is 90°. The moment of inertia of the bent rod about an axis passing through the bending point and perpendicular to the plane defined by the two halves of the rod is **(CBSE 2008)**

(a) $\frac{ML^2}{24}$ (b) $\frac{ML^2}{12}$

(c) $\frac{ML^2}{6}$ (d) $\frac{\sqrt{2}ML^2}{24}$

23. A particle of mass m moves in the XY plane with a velocity v along the straight line AB. If the angular momentum of the particle with respect to origin O is L_A when it is A and L_B when it is at B, then **(CBSE 2007)**



(a) $L_A > L_B$

(b) $L_A = L_B$

(c) the relationship between L_A and L_B depends upon the slope of the line AB

(d) $L_A < L_B$

24. The moment of inertia of a thin uniform rod of mass M and length L about an axis passing through its mid-point and perpendicular to its length is I_0. Its moment of inertia about an axis passing through one of its ends and perpendicular to its length is : **(CBSE 2010)**

(a) $I_0 + ML^2/4$ (b) $I_0 + 2ML^2$

(c) $I_0 + ML^2$ (d) $I_0 + ML^2/2$

25. When a mass is rotating in a plane about a fixed point, its angular momentum is directed along : **(CBSE 2012)**

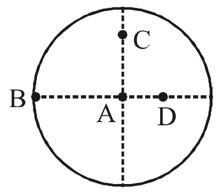
(a) the tangent to the orbit

(b) a line perpendicular to the plane of rotation

(c) the line making an angle of 45° to the plane of rotation

(d) the radius

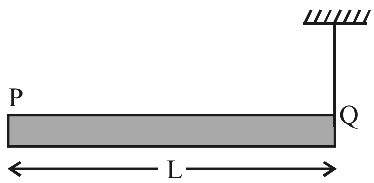
26. The moment of inertia of uniform circular disc is maximum about an axis perpendicular to the disc and passing through **(CBSE 2012)**



(a) A (b) B

(c) C (d) D

27. A rod PQ of mass M and Length L is hinged at end P. The rod is kept horizontal by a massless string tied to point Q as shown in figure. When string is cut, the initial angular acceleration of the rod is : **(CBSE 2013)**



(a) $\frac{2g}{3L}$ (b) $\frac{3g}{2L}$

(c) $\frac{g}{L}$ (d) $\frac{2g}{L}$

28. A small object of uniform density rolls up a curved surface with an initial velocity 'v'. It reaches upto a maximum height of $\frac{3v^2}{4g}$ with respect to the initial position. The object is **(CBSE 2013)**

(a) Disc (b) Ring

(c) Solid sphere (d) Hollow sphere

29. A body rolls down an inclined plane. If its kinetic energy of rotational motion is 40% of its kinetic energy of translation motion, then the body is a : **(AFMC 1996)**

(a) solid sphere (b) spherical shell

(c) cylinder (d) ring

30. A cylinder of 500 g and radius 10 cm has moment of inertia : (about its natural axis) **(AFMC 1997)**

(a) 3.5 kg–m^2 (b) 5×10^{-3} kg–m^2

(c) 2×10^{-3} kg–m^2 (d) 2.5×10^3 kg–m^2

31. A particle performs uniform circular motion with an angular momentum L. If the frequency of the particle motion is doubled and its kinetic energy is halved the angular momentum becomes : **(AFMC 1998)**

(a) L/4 (b) L/2

(c) 2 L (d) 4 L

32. A cord is wound round the circumference of the wheel of radius r. The axis of the wheel is horizontal and the moment of inertia about its centre is I. A weight mg is attached to the cord at the end. The weight falls from rest after falling through a distance h, the angular velocity of the wheel will be : **(AFMC 1998)**

(a) $\sqrt{2gh}$ (b) $\sqrt{\frac{2mgh}{I+mr^2}}$

(c) $\sqrt{\frac{2mgh}{I-2mr^2}}$ (d) $\sqrt{\frac{2gh}{I+2mr^2}}$

33. The moment of inertia of a regular circular disc of mass 0.4 kg and radius 100 cm about the axis perpendicular to the plane of the disc and passing through its centre is: **(AFMC 2000)**

(a) 0.002 kg–m^2 (b) 0.02 kg–m^2

(c) 2 kg–m^2 (d) 0.2 kg–m^2

34. A circular disc is rotating with angular velocity ω. If a man standing at the edge of the disc walks towards its centre, then the angular velocity of the disc : **(AFMC 2002)**

(a) is not changed (b) be halved

(c) decreases (d) increases

35. The moment of inertia of a rigid body depends upon : **(AFMC 2002)**

(a) distribution of mass from axis of rotation

(b) angular velocity of the body

(c) angular acceleration of the body

(d) all of the above

36. The radius of gyration of a body depends upon : **(AIIMS 1995)**

(a) translation motion (b) axis of rotation

(c) area of the body (d) all of these

ASSERTION REASON

(A) Statement–1 is True, Statement–2 is True; Statement–2 is a correct explanation for Statement–1.

(B) Statement–1 is True, Statement–2 is True; Statement–2 is NOT a correct explanation for Statement–1.

(C) Statement–1 is True, Statement–2 is False.

(D) Statement–1 is False, Statement–2 is True.

37. **Statement 1 :** A rigid disc rolls without slipping on a fixed rough horizontal surface with uniform angular velocity. Then the acceleration of lowest point on the disc is zero.

Statement 2 : For a rigid disc rolling without slipping on a fixed rough horizontal surface, the velocity of the lowest point on the disc is always zero.

(a) A (b) B

(c) C (d) D

38. **Statement 1 :** A uniform thin rod of length L is hinged about one of its end and is free to rotate about the hinge without friction. Neglect the effect of gravity. A force F is applied at a distance x from the hinge on the rod such that force always is perpendicular to the rod. As the value of x is increased from zero to L, the component of reaction by hinge on the rod perpendicular to length of rod increases.

Statement 2 : Under the conditions given in statement 1 as x is increased from zero to L, the angular acceleration of rod increases.

(a) A (b) B

(c) C (d) D

39. **Statement 1 :** If two different axes are at same distance from centre of mass of a rigid body, then moment of inertia of the given rigid body about both axis will always be same.

Statement 2 : From parallel axis is theorem $I = I_{cm} + md^2$, where all terms have usual meaning.

(a) A (b) B

(c) C (d) D

40. **Statement 1 :** Angular momentum of a system of particles is always conserved.

Statement 2 : Torque = time rate of change of angular momentum. **(AIIMS 1996)**

(a) A (b) B

(c) C (d) D

41. A constant torque of 31.4 Nm is exerted on a pivoted wheel. If the angular acceleration of the wheel is 4π rad/s^2, then the moment of inertia, will be : **(AIIMS 2001)**

(a) 4.8 kg–m^2 (b) 6.2 kg–m^2

(c) 5.6 kg–m^2 (d) 2.5 kg–m^2

42. If the equation for the displacement of a particle moving on a circular path is given as $\theta = 2t^3 + 0.5$ where θ is in radians and t is in second. Then the angular velocity of the particle after 2 s will be : **(AIIMS 1998)**

(a) 36 rad/s (b) 8 rad/s

(c) 48 rad/s (d) 24 rad/s

43. The angular speed of a body changes from ω_1 to ω_2 without applying a torque but due to change in its moment of inertia. The ratio of radii of gyration in the two cases is : **(BHU 2002)**

(a) $\omega_2 : \omega_1$ (b) $\omega_1 : \omega_2$

(c) $\sqrt{\omega_1} : \sqrt{\omega_2}$ (d) $\sqrt{\omega_2} : \sqrt{\omega_1}$

44. Five particles of mass 2 kg are attached to the rim of a circular disc of radius 0.1 m and negligible mass. Moment of inertia of the system about the axis passing through the centre of the disc and perpendicular to its plane is : **(BHU 2003)**

(a) 1 kg m^2 (b) 0.1 kg m^2

(c) 2 kg m^2 (d) 0.2 mg m^2

45. A sphere of diameter 0.2 m and mass 2 kg is rolling on an inclined plane with velocity v = 0.5 m/s. The kinetic energy of the sphere is : **(BHU 2003)**

(a) 0.10 J (b) 0.35 J

(c) 0.50 J (d) 0.42 J

46. A solid sphere and a spherical shell both of same radius and mass roll down from rest without slipping on an inclined plane from the same height h. The time taken to reach the bottom of the inclined plane is : **(CPMT 2000)**

(a) more for spherical shell

(b) more for solid sphere

(c) same for both

(d) depends on coefficient of friction

47. A body is rotating with angular velocity $\vec{\omega} = \left(3\hat{i} - 4\hat{j} + \hat{k}\right)$. The linear velocity of a point having position vector $\vec{r} = \left(5\hat{i} - 6\hat{j} + 6\hat{k}\right)$ is : **(CPMT 2002)**

(a) $6\hat{i} + 2\hat{j} - 3\hat{k}$ (b) $18\hat{i} + 3\hat{j} - 2\hat{k}$

(c) $-18\hat{i} - 3\hat{j} + 2\hat{k}$ (d) $6\hat{i} - 2\hat{j} + 8\hat{k}$

48. A coin, placed on a rotating turn–table slips, when it is placed at a distance of 9 cm from its centre. If the angular velocity of the turn table is tripped, it will just slip at a distance r from centre. The value of r is : **(CPMT 2001)**

(a) 1 cm (b) 3 cm

(c) 9 cm (d) 27 cm

49. A fan is moving around its axis. What will be its motion regarded as ? **(CPMT 2003)**

(a) pure rolling (b) rolling with slipping

(c) skidding (d) pure rotation

50. Angular momentum of a body with moment of inertia I and angular velocity ω is equal to : **(DPMT 2003)**

(a) I/ω (b) Iω^2

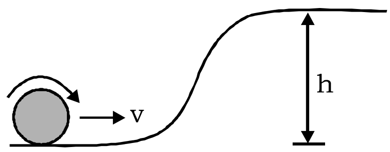
(c) Iω (d) none of these

51. A small disc of radius 2 cm is cut from a disc of radius 6 cm. If the distance between their centres is 3.2 cm, what is the shift in the centre of mass of the disc ? **(AFMC 2006)**

(a) 0.4 cm (b) 2.4 cm

(c) 1.8 cm (d) 1.2 cm

52. A solid sphere is rolling on a frictionless surface, shown in figure with a translational velocity v m/s. If it is to climb the inclined surface then v should be : **(AIIMS 2005)**



(a) $\geq \sqrt{\frac{10}{7}gh}$ (b) $\geq \sqrt{2gh}$

(c) 2 gh (d) $\frac{10}{7}$gh

53. A horizontal platform is rotating with uniform angular velocity around the vertical axis passing through its centre. At some instant of time a viscous fluid of mass m is dropped at the centre and is allowed to spread out and finally fall. The angular velocity during this period : **(AIIMS 2005)**

(a) decreases continuously

(b)decreases initially & increases again

(c) remain unaltered

(d) increases continuously

54. The moment of inertia of a rod about an axis through its centre and perpendicular to it is $\frac{1}{12}ML^2$ (where M is the mass and L the length of the rod). The rod is bent in the middle so that the two halves makes an angle of 60°. The moment of inertia of the bent rod about the same axis would be : **(AIIMS 2006)**

(a) $\frac{1}{48}ML^2$ (b) $\frac{1}{12}ML^2$

(c) $\frac{1}{24}ML^2$ (d) $\frac{ML^2}{8\sqrt{3}}$

55. The angular momentum of a rotating body changes from A_0 to 4 A_0 in 4 min. The torque acting on the body is : **(BHU 2005)**

(a) $\left(\frac{3}{4}\right)A_0$ (b) 4 A_0

(c) 3 A_0 (d) $\left(\frac{3}{2}\right)A_0$

56. Five particles of mass 2 kg are attached to the rim of a circular disc of radius 0.1 m and negligible mass. Moment of inertia of the system about the axis passing through the centre of the disc and perpendicular to its plane is : **(CPMT 2004)**

(a) 1 kg m^2 (b) 0.1 kg m^2

(c) 2 kg m^2 (d) 0.2 kg m^2

57. A solid sphere is rotating about a diameter at an angular velocity ω. If it cools so that its radius reduces to 1/n of its original value, its angular velocity becomes :**(CPMT 2005)**

(a) ω/n (b) ω/n^2

(c) nω (d) $n^2\omega$

58. A solid sphere rolls down two different inclined planes of same height, but of different inclinations. In both cases : **(CPMT 2006)**

(a) speed and time of descent will be same

(b) speed will be same, but time of descent will be different

(c) speed will be different, but time of descent will be same

(d) speed and time of descent both are different

59. The angular momentum of a system of particles is not conserved : **(DPMT 2004)**

(a) when a net external force acts upon the system

(b) when a net external torque is acting upon the system

(c) when a net external impulse is acting upon the system

(d) none of the above

60. A coin placed on a rotating turntable just slips if it is placed at a distance of 8 cm from the centre. If angular velocity of the turntable is doubled, it will just slip at a distance of : **(DPMT 2005)**

(a) 1 cm (b) 2 cm

(c) 4 cm (d) 8 cm

61. The distance between the sun and the earth be *r* then the angular momentum of the earth around the sun is proportional to : **(DPMT 2005)**

(a) $\sqrt{r}$ (b) $r^{3/2}$

(c) r (d) none of these

62. What is moment of inertia in terms of angular momentum (L) and kinetic energy (K) ? **(DPMT 2006)**

(a) $\frac{L^2}{K}$ (b) $\frac{L^2}{2K}$

(c) $\frac{L}{2K^2}$ (d) $\frac{L}{2K}$

63. A disc of mass 2 kg and radius 0.2 m is rotating with angular velocity 30 rad/s. What is angular velocity, if a mass of 0.25 kg is put on periphery of the disc ? **(DPMT 2006)**

(a) 24 rad/s (b) 36 rad/s

(c) 15 rad/s (d) 26 rad/s

64. A diver in a swimming pool bends his head before diving, because it : **(PPMT 2004)**

(a) decreases his moment of inertia

(b) decreases his angular velocity

(c) increases his moment of inertia

(d) increases his linear velocity

65. A disc revolves in a horizontal plane at a steady rate of 3 rev/s. A coin placed at a distance of 2 cm from the axis of rotation remains at rest on the disc. The minimum value of coefficient of friction between the coin and disc will be : ($g = 9.8\ m/s^2$) **(BHU 1997)**

(a) 0.43 (b) 0.26

(c) 0.72 (d) none of these

66. Under a constant torque the angular momentum of a body changes from A to 4 A in 4 s. The torque on the body will be : **(BHU 1998)**

(a) 1 A (b) $\frac{1}{4}$A

(c) $\frac{4}{3}$A (d) $\frac{3}{4}$A

67. A thin uniform circular ring is rolling down an inclined plane of inclination 30° without slipping. Its linear acceleration along the inclined plane is : **(BHU 1998)**

(a) $\frac{2g}{3}$ (b) $\frac{g}{4}$

(c) $\frac{g}{3}$ (d) $\frac{g}{2}$

68. A meter stick is held vertically with one end on the floor and is the other end is allowed to fall. Assuming that the end on the floor of the stick does not slip, the velocity of the other end when it hits the floor, will be : **(BHU 2000)**

(a) 10.8 m/s (b) 5.4 m/s

(c) 2.5 m/s (d) none of these

69. The moment of inertia of a body about a given axis is 1.2 $kg–m^2$. To produce a rotational kinetic energy of 1500 J an angular acceleration of 25 rad/s^2 must be applied for : **(BHU 2001)**

(a) 8.5 a (b) 5 s

(c) 2 s (d) 1 s

70. A sphere of diameter 0.2 and mass 2 kg is rolling on an inclined plane with velocity v = 0.5 m/s. The kinetic energy of the sphere is : **(BHU 2003)**

(a) 0.10 J (b) 0.35 J

(c) 0.50 J (d) 0.42 J

71. A body of moment of inertia 3 kg m^2 rotating with an angular velocity of 2 rad/s has the same kinetic energy as the mass of 12 kg moving with a speed of :**(CPMT 1997)**

(a) 1 m/s (b) 1.41 m/s

(c) 2 m/s (d) 4 m/s

72. A particle of mass 0.5 kg is moving in the X–Y plane with uniform speed of 3 m/s parallel of Y–axis and crosses the X–axis at 2 m from origin. The angular momentum about origin is : **(CPMT 2000)**

(a) zero (b) 3 kg m^2/s

(c) 1.5 kg m^2/s (d) changing with time

73. A long frictionless horizontal rod is set into rotation about a vertical axis passing through its centre. Two beads placed on the rod on either side of the axis, are released from rest. The angular speed of the rod : **(CPMT 2000)**

(a) decreases with time

(b) increases with time due to work done by the beads

(c) increases with time due to work done by centrifugal force

(d) remain unchanged

74. What is the angular momentum of a body whose rotational energy is 10 J, if the angular momentum vector coincides with the axis of rotation and its moment of inertia is 8 g–cm^2 : **(DMPT 1998)**

(a) 4×10^{-3} kg–m^2/s

(b) 8×10^{-3} kg–m^2/s

(c) 2×10^{-3} kg–m^2/s

(d) none of these

75. An automobile engine develops 100 kW when rotating at a speed of 1800 rev/min then the torque produced is : **(DPMT 1998)**

(a) 628 Nm (b) 531 Nm

(c) 330 Nm (d) none of these

76. The radius of gyration of a disc of mass 100 g and radius 5 cm about an axis passing through its centre of gravity and perpenedicular to plane is : **(DPMT 1998)**

(a) 6.54 cm (b) 3.54 cm

(c) 1.52 cm (d)2.57 cm

77. A wheel is rotating at the rate of 33 rev/min. If it comes to stop in 20 s. Then the angular retardation will be : **(DPMT 2000)**

(a) zero (b) 11 π rad/s^2

(c) $\frac{\pi}{200}$ rad / s^2 (d) $\frac{11\pi}{200}$ rad / s^2

78. The angular velocity of the wheel increases from 1200 to 4500 rev/min in 10 s with constant angular acceleration. Then the number of revolutions made during this time is : **(DPMT 2000)**

(a) 575 (b) 675

(c) 875 (d) 475

79. Three point masses, each of mass M are placed at the corners of an equilateral triangle of side L. The moment of inertia of this system about an axis along one side of the triangle is **(BHU 2009)**

(a) $\frac{1}{3}ML^2$ (b) $\frac{3}{2}ML^2$

(c) $\frac{3}{4}ML^2$ (d) ML^2

80. A thin and circular disc of mass m and radius R is rotating in a horizontal plane about an axis passing through its centre and perpendicular to its centre and perpendicular to its plane with an angular velocity ω. If another disc of same dimensions but of mass M/4 is placed gently on the first disc co–axially, then the new angular velocity of the system is **(BHU 2008)**

(a) $\frac{5}{4}\omega$ (b) $\frac{2}{3}\omega$

(c) $\frac{4}{5}\omega$ (d) $\frac{3}{2}\omega$

81. A door 1.6 m wide requires a force of 1 N to be applied at the free end to open or close it. The force that is required at a point 0.4 m distance from the hinges for opening or closing the door is **(KCET 2009)**

(a) 1.2 N (b) 3.6 N

(c) 2.4 N (d) 4 N

82. The moment of inertia of a circular ring of radius r and mass M about diameter is **(KCET 2009)**

(a) $\frac{2}{5}Mr^2$ (b) $\frac{Mr^2}{4}$

(c) $\frac{Mr^2}{2}$ (d) $\frac{Mr^2}{12}$

83. Radius of gyration of disc of mass 50 g and radius 2.5 cm about an axis passing through its centre of gravity and perpendicular to the plane is **(CPMT 2009)**

(a) 6.54 cm (b) 3.64 cm

(c) 1.77 cm (d) 0.88 cm

84. The radius of the rear wheel of bicycle is twice that of the front wheel. When the bicycle is moving, the angular speed of the rear wheel compared to that of the front is **(DUMET 2009)**

(a) greater (b) smaller

(c) same (d) exact double

85. A uniform rod of length L and mass 18 kg is made to rest on two measuring scale at its two ends. A uniform block of mass 2.7 kg is placed on the rod at a distance L/4 from the left end. The force experienced by the measuring scale on the right end is **(DUMET 2009)**

(a) 18 N (b) 96 N

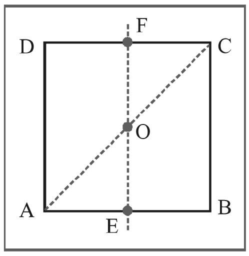
(c) 29 N (d) 45 N

86. Four point masses, each of value m, and placed at the corners of square ABCD of side l. The moment of inertia of this system about an axis passing through A and parallel to BD is **(AIIMS 20008)**

(a) $2\,ml^2$ (b) $\sqrt{3}\,ml^2$

(c) $3\,ml^2$ (d) ml^2

87. For the given uniform square lamina ABCD, whose centre is O **(AIIMS 2008**



(a) $\sqrt{2}\,I_{AC} = I_{EF}$ (b) $I_{AD} = 3\,I_{EF}$

(c) $I_{AC} = I_{EF}$ (d) $I_{AC} = \sqrt{2}\,I_{EF}$

Assertion and Reason

(A) If Statement–I is true, Statement–II is true; Statement–II is the correct explanation for Statement–I.

(B) If Statement–I is true, Statement–II is true; Statement–II is not a correct explanation for Statement–I.

(C) If Statement–I is true; Statement–II is false.

(D) If Statement–I is false; Statement–II is true.

88. **Assertion :** The velocity of a body at the bottom of an inclined plane of given height is more when it slides down the plane, compared to, when it rolling down the same plane.

Reason : In rolling down a body acquires both kinetic energy of translation and rotation. **(AIIMS 2008)**

(a) A (b) B

(c) C (d) D

89. **Assertion :** A ladder is more apt to slip, when you are high up on it than when you just begin to climb.

Reason : At the high up on a ladder, the torque is large and on climbing up the torque is small. **(AIIMS 2007)**

(a) A (b) B

(c) C (d) D

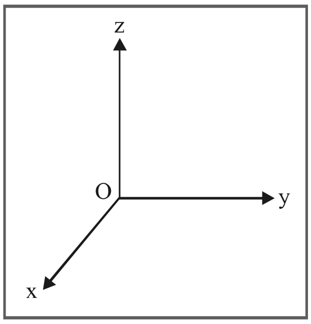
90. **Assertion :** The speed of whirlwind in a tornado is alarmingly high.

Reason : If no external torque acts on a body, its angular velocity remains conserved. **(AIIMS 2007)**

(a) A (b) B

(c) C (d) D

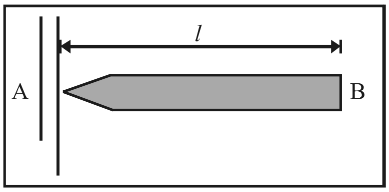
91. A force of $-F\hat{k}$ acts on O, the origin of the coordinate system. The torque about the point (1, –1) is **(CPMT 2008)**



(a) $F(\hat{i}-\hat{j})$ (b) $-F(\hat{i}+\hat{j})$

(c) $F(\hat{i}+\hat{j})$ (d) $-F(\hat{i}-\hat{j})$

92. A uniform rod AB of length l and mass m is free to rotate about point A. The rod is released from rest in the horizontal position. Given that the moment of inertia of the rod about A is $\frac{ml^2}{3}$, the initial angular acceleration of the rod will be **(CPMT 2008)**



(a) $\frac{2g}{3l}$ (b) $mg\frac{l}{2}$

(c) $\frac{3}{2}gl$ (d) $\frac{3g}{2l}$

93. If radius or earth is reduced **(DUMET 2008)**

(a) tide duration is reduced

(b) earth rotates slower

(c) time period of earth decreases

(d) duration of day increases

94. A cylinder is rolling down an inclined plane of inclination 60°. What is its acceleration ? **(DUMET 2008)**

(a) $g\sqrt{3}$ (b) $g/\sqrt{3}$

(c) $\sqrt{\frac{2g}{3}}$ (d) None of these

95. Two spheres of unequal mass but same radius are released on inclined plane. They rolls down with slipping. Which one will reach the ground first ? **(DUMET 2008)**

(a) Light sphere

(b) Heavier sphere

(c) Both will reach at the same time

(d) None of the above

96. Two identical concentric rings each of mass M and radius R are placed perpendicularly. What is the moment of inertia about axis of one of the rings ? **(DUMET 2008)**

(a) $\frac{3}{2}MR^2$ (b) $2\ MR^2$

(c) $3\ MR^2$ (d) $\frac{1}{4}MR^2$

97. A body is rolling down an inclined plane. If KE of rotation is 40% of KE in translatory state, then the body is **(DUMET 2008)**

(a) ring (b) cylinder

(c) hollow ball (d) solid ball

98. A ball rolls without slipping. The radius of gyration of the ball about an axis passing through its center of mass is K. If radius of the ball be R, then the fraction of total energy associated with its rotational energy will be **(BHU 2007)**

(a) $\frac{K^2}{K^2+R^2}$ (b) $\frac{R^2}{K^2+R^2}$

(c) $\frac{K^2+R^2}{R^2}$ (d) $\frac{K^2}{R^2}$

ANSWER KEY

EXERCISE - 1 : BASIC OBJECTIVE QUESTIONS

1. (b)	**2.** (b)	**3.** (d)	**4.** (b)	**5.** (b)	**6.** (d)	**7.** (b)	**8.** (b)	**9.** (a)	**10.** (c)
11. (d)	**12.** (d)	**13.** (c)	**14.** (b)	**15.** (b)	**16.** (c)	**17.** (a)	**18.** (c)	**19.** (c)	**20.** (c)
21. (c)	**22.** (b)	**23.** (d)	**24.** (c)	**25.** (b)	**26.** (c)	**27.** (d)	**28.** (a)	**29.** (a)	**30.** (a)
31. (d)	**32.** (d)	**33.** (a)	**34.** (c)	**35.** (d)	**36.** (c)	**37.** (a)	**38.** (a)	**39.** (b)	**40.** (a)
41. (c)	**42.** (d)	**43.** (a)	**44.** (c)	**45.** (a)	**46.** (b)	**47.** (b)	**48.** (b)	**49.** (c)	**50.** (a)
51. (d)	**52.** (b)	**53.** (c)	**54.** (c)	**55.** (c)	**56.** (b)	**57.** (a)	**58.** (c)	**59.** (a)	**60.** (a)
61. (b)	**62.** (b)	**63.** (b)	**64.** (c)	**65.** (d)	**66.** (a)	**67.** (c)	**68.** (d)	**69.** (b)	**70.** (d)
71. (b)	**72.** (c)	**73.** (c)	**74.** (a)	**75.** (b)					

EXERCISE - 2 : PREVIOUS YEAR COMPETITION QUESTIONS

1. (c)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (b)	**6.** (a)	**7.** (d)	**8.** (b)	**9.** (a)	**10.** (a)
11. (c)	**12.** (c)	**13.** (c)	**14.** (c)	**15.** (a)	**16.** (a)	**17.** (a)	**18.** (c)	**19.** (a)	**20.** (b)
21. (b)	**22.** (b)	**23.** (b)	**24.** (a)	**25.** (b)	**26.** (b)	**27.** (b)	**28.** (a)	**29.** (a)	**30.** (d)
31. (a)	**32.** (b)	**33.** (d)	**34.** (d)	**35.** (a)	**36.** (b)	**37.** (d)	**38.** (d)	**39.** (d)	**40.** (d)
41. (d)	**42.** (d)	**43.** (d)	**44.** (b)	**45.** (b)	**46.** (a)	**47.** (c)	**48.** (a)	**49.** (d)	**50.** (c)
51. (a)	**52.** (a)	**53.** (b)	**54.** (b)	**55.** (a)	**56.** (b)	**57.** (d)	**58.** (b)	**59.** (b)	**60.** (b)
61. (a)	**62.** (b)	**63.** (a)	**64.** (a)	**65.** (c)	**66.** (d)	**67.** (b)	**68.** (b)	**69.** (c)	**70.** (b)
71. (b)	**72.** (b)	**73.** (a)	**74.** (a)	**75.** (d)	**76.** (b)	**77.** (d)	**78.** (d)	**79.** (c)	**80.** (c)
81. (d)	**82.** (c)	**83.** (c)	**84.** (b)	**85.** (b)	**86.** (c)	**87.** (c)	**88.** (b)	**89.** (a)	**90.** (c)
91. (c)	**92.** (d)	**93.** (c)	**94.** (b)	**95.** (c)	**96.** (a)	**97.** (d)	**98.** (a)		

Dream on !!

Chapter 13 Rotational Dynamics

He sighed with the difficulty of talking mechanics to an unmechanical person.
"There's a torque," he said. "It ain't balanced ---"
Any mechanic would have understood his drift at once. If a three-bladed propeller loses a blade, there are two blades left on one-third of its circumference, and nothing on the other two-thirds. All the resistance to its rotation under water is consequently concentrated upon one small section of the shaft, and a smooth revolution would be rendered impossible ...[1]

C.S. Forester
The African Queen

Introduction

The physical objects that we encounter in the world consist of collections of atoms that are bound together to form systems of particles. When forces are applied, the shape of the body may be stretched or compressed like a spring, or sheared like jello. In some systems the constituent particles are very loosely bound to each other as in fluids and gasses, and the distances between the constituent particles will vary. We shall begin our study of extended objects by restricting ourselves to an ideal category of objects, rigid bodies, which do not stretch, compress, or shear.

A body is called a *rigid body* if the distance between any two points in the body does not change in time. Rigid bodies, unlike point masses, can have forces applied at different points in the body. For most objects, treating as a rigid body is an idealization, but a very good one. In addition to forces applied at points, forces may be distributed over the entire body. Distributed forces are difficult to analyze; however, for example, we regularly experience the effect of the gravitational force on bodies. Based on our experience observing the effect of the gravitational force on rigid bodies, we note that the gravitational force can be concentrated at a point in the rigid body called the *center of gravity*, which for small bodies (so that $\vec{\mathbf{g}}$ may be taken as constant within the body) is identical to the *center of mass* of the body (we shall prove this fact in Appendix 13.A).

Let's consider a rigid rod thrown in the air (Figure 13.1) so that the rod is spinning as its center of mass moves with velocity $\vec{\mathbf{v}}_{\text{cm}}$. Rigid bodies, unlike point-like objects, can have forces applied at different points in the body. We have explored the physics of translational motion; now, we wish to investigate the properties of rotation exhibited in the rod's motion, beginning with the notion that every particle is rotating about the center of mass with the same angular (rotational) velocity.

[1] The authors of these notes suspect either a math error on Mr. Forester's part or an oversight by his editors.

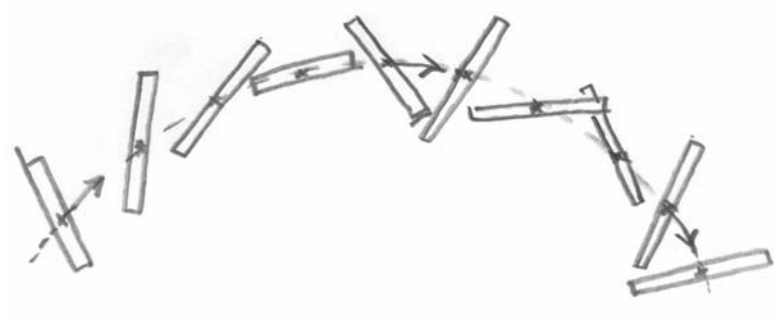


Figure 13.1 The center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.

We can use Newton's Second Law to predict how the center of mass will move. Since the only external force on the rod is the gravitational force (neglecting the action of air resistance), the center of mass of the body will move in a parabolic trajectory.

How was the rod induced to rotate? In order to spin the rod, we applied a torque with our fingers and wrist to one end of the rod as the rod was released. The applied torque is proportional to the angular acceleration. The constant of proportionality is called the *moment of inertia*. When external forces and torques are present, the motion of a rigid body can be extremely complicated while it is translating and rotating in space. We shall begin our study of rotating objects by considering the simplest example of rigid body motion, rotation about a fixed axis.

13.1 Fixed Axis Rotation: Rotational Kinematics

Fixed Axis Rotation

When we studied static equilibrium, we demonstrated the need for two conditions: The total force acting on an object is zero, as is the total torque acting on the object. If the total torque is non-zero, then the object will start to rotate.

A simple example of rotation about a fixed axis is the motion of a compact disc in a CD player, which is driven by a motor inside the player. In a simplified model of this motion, the motor produces angular acceleration, causing the disc to spin. As the disc is set in motion, resistive forces oppose the motion until the disc no longer has any angular acceleration, and the disc now spins at a constant angular velocity. Throughout this process, the CD rotates about an axis passing through the center of the disc, and is perpendicular to the plane of the disc (see Figure 13.2). This type of motion is called *fixed-axis rotation*.

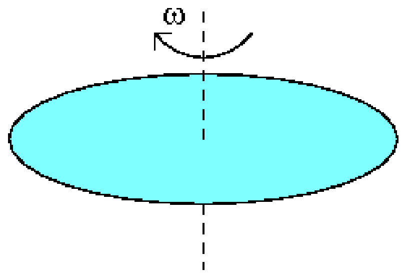


Figure 13.2 Rotation of a compact disc about a fixed axis.

When we ride a bicycle forward, the wheels rotate about an axis passing through the center of each wheel and perpendicular to the plane of the wheel (Figure 13.3). As long as the bicycle does not turn, this axis keeps pointing in the same direction. This motion is more complicated than our spinning CD because the wheel is both moving (translating) with some center of mass velocity, $\vec{\mathbf{v}}_{\text{cm}}$, and rotating.

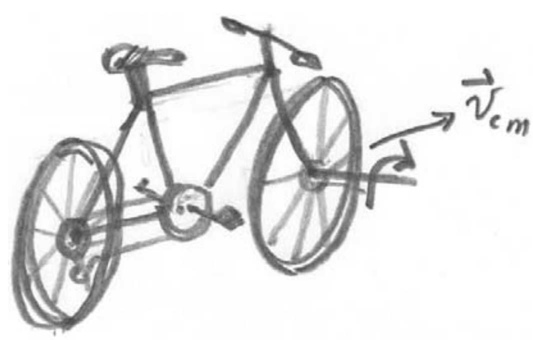


Figure 13.3 Fixed axis rotation and center of mass translation for a bicycle wheel.

When we turn the bicycle's handlebars, we change the bike's trajectory and the axis of rotation of each wheel changes direction. Other examples of non-fixed axis rotation are the motion of a spinning top, or a gyroscope, or even the change in the direction of the earth's rotation axis. This type of motion is much harder to analyze, so we will restrict ourselves in this chapter to considering fixed axis rotation, with or without translation.

Angular Velocity and Angular Acceleration

When we considered the rotational motion of a point-like object in Chapter 6, we introduced an angle coordinate θ, and then defined the angular velocity (Equation 6.2.7) as

$$\omega \equiv \frac{d\theta}{dt}, \tag{13.1.1}$$

and angular acceleration (Equation 6.3.4) as

$$\alpha \equiv \frac{d^2\theta}{dt^2}. \tag{13.1.2}$$

For a rigid body undergoing fixed-axis rotation, we can divide the body up into small volume elements with mass Δm_i. Each of these volume elements is moving in a circle of radius $r_{\perp,i}$ about the axis of rotation (Figure 13.4).

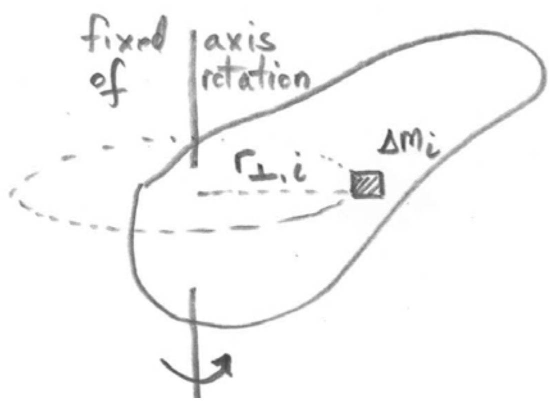


Figure 13.4 Coordinate system for fixed-axis rotation.

We will adopt the notation implied in Figure 13.4, and denote the vector from the axis to the point where the mass element is located as $\vec{\mathbf{r}}_{\perp,i}$, with $r_{\perp,i} = \left|\vec{\mathbf{r}}_{\perp,i}\right|$.

Because the body is rigid, all the volume elements will have the same angular velocity ω and hence the same angular acceleration α. If the bodies did not have the same angular velocity, the volume elements would "catch up to" or "pass" each other, precluded by the rigid-body assumption.

Sign Convention: **Angular Velocity and Angular Acceleration**

Suppose we choose θ to be increasing in the counterclockwise direction as shown in Figure 13.5.

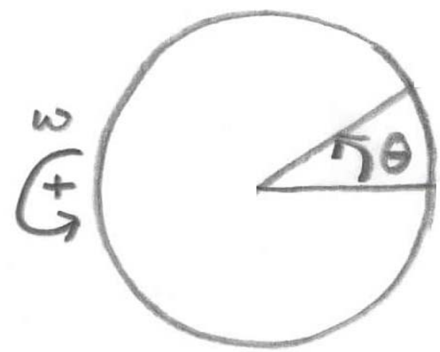


Figure 13.5 Sign conventions for rotational motion.

If the rigid body rotates in the counterclockwise direction, then the angular velocity is positive, $\omega \equiv d\theta/dt > 0$. If the rigid body rotates in the clockwise direction, then the angular velocity is negative, $\omega \equiv d\theta/dt < 0$.

- If the rigid body *increases* its rate of rotation in the counterclockwise (positive) direction then the angular acceleration is positive, $\alpha \equiv d^2\theta/dt^2 = d\omega/dt > 0$.
- If the rigid body *decreases* its rate of rotation in the counterclockwise (positive) direction then the angular acceleration is negative, $\alpha \equiv d^2\theta/dt^2 = d\omega/dt < 0$.
- If the rigid body *increases* its rate of rotation in the clockwise (negative) direction then the angular acceleration is negative, $\alpha \equiv d^2\theta/dt^2 = d\omega/dt < 0$.
- If the rigid body *decreases* its rate of rotation in the clockwise (negative) direction then the angular acceleration is positive, $\alpha \equiv d^2\theta/dt^2 = d\omega/dt > 0$.

To phrase this more generally, if α and ω have the same sign, the body is speeding up; if opposite signs, the body is slowing down. This general result is independent of the choice of positive direction of rotation.

Note that in Figure 13.2, the CD has a negative angular velocity as viewed from above; CDs do not operate the same way record player turntables do.

Tangential Velocity and Tangential Acceleration

Since the small volume Δm_i element of mass is moving in a circle of radius $r_{\perp,i} = |\vec{\mathbf{r}}_{\perp,i}|$ with angular velocity ω, the element has a tangential velocity component

$$v_{\tan,\,i} = r_{\perp,\,i}\,\omega\,. \tag{13.1.3}$$

If the magnitude of the tangential velocity is changing, the volume element undergoes a tangential acceleration given by

$$a_{\tan,\,i} = r_{\perp,\,i}\,\alpha\,. \qquad (13.1.4)$$

Recall from Chapter 6.3 Equation (6.3.14) that the volume element is always accelerating inward with magnitude

$$\left|a_{\mathrm{rad},\,i}\right| = \frac{v_{\tan,\,i}^2}{r_{\perp,\,i}} = r_{\perp,\,i}\,\omega^2\,. \qquad (13.1.5)$$

13.1.1 Example: Turntable, Part I

A turntable is a uniform disc of mass $1.2\ \mathrm{kg}$ and a radius $1.3\times10^{1}\ \mathrm{cm}$. The turntable is spinning initially at a constant rate of $f_0 = 33\,\mathrm{cycles}\cdot\mathrm{min}^{-1}$ ($33\,\mathrm{rpm}$). The motor is turned off and the turntable slows to a stop in $8.0\ \mathrm{s}$. Assume that the angular acceleration is constant.

a) What is the initial angular velocity of the turntable?

b) What is the angular acceleration of the turntable?

Answer:

Initially, the disc is spinning with a frequency

$$f_0 = \left(33\,\frac{\mathrm{cycles}}{\mathrm{min}}\right)\left(\frac{1\,\mathrm{min}}{60\ \mathrm{s}}\right) = 0.55\ \mathrm{cycles}\cdot\mathrm{s}^{-1} = 0.55\ \mathrm{Hz}\,, \qquad (13.1.6)$$

so the initial angular velocity is

$$\omega_0 = 2\pi\, f_0 = \left(2\pi\,\frac{\mathrm{radian}}{\mathrm{cycle}}\right)\left(0.55\,\frac{\mathrm{cycles}}{\mathrm{s}}\right) = 3.5\ \mathrm{rad}\cdot\mathrm{s}^{-1}\,. \qquad (13.1.7)$$

The final angular velocity is zero, so the angular acceleration is

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_0}{t_f - t_0} = \frac{-3.5\ \mathrm{rad}\cdot\mathrm{s}^{-1}}{8.0\ \mathrm{s}} = -4.3\times10^{-1}\ \mathrm{rad}\cdot\mathrm{s}^{-2}\,. \qquad (13.1.8)$$

The angular acceleration is negative, and the disc is slowing down.

13.2 Torque

In order to understand the rotation of a rigid body we must introduce a new quantity, the torque. Let a force $\vec{\mathbf{F}}_P$ with magnitude $F = |\vec{\mathbf{F}}_P|$ act at a point P. Let $\vec{\mathbf{r}}_{S,P}$ be the vector from the point S to a point P, with magnitude $r = |\vec{\mathbf{r}}_{S,P}|$. The angle between the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ is θ with $[0 \le \theta \le \pi]$ (Figure 13.6).

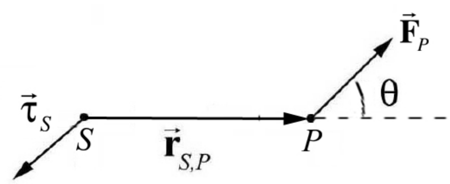


Figure 13.6 Torque about a point S due to a force acting at a point P

The torque about a point S due to force $\vec{\mathbf{F}}_P$ acting at P, is defined by

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P . \tag{13.2.1}$$

(See section 2.5 for a review of the definition of the cross product of two vectors). The magnitude of the torque about a point S due to force $\vec{\mathbf{F}}_P$ acting at P, is given by

$$\tau_S = r\,F\sin\theta . \tag{13.2.2}$$

The SI units for torque are $[\mathrm{N}\cdot\mathrm{m}]$. The direction of the torque is perpendicular to the plane formed by the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ (for $[0 < \theta < \pi]$), and by definition points in the direction of the unit normal vector to the plane $\hat{\mathbf{n}}_{RHR}$ as shown in Figure 13.7.

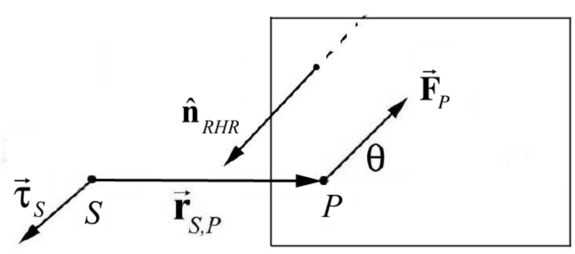


Figure 13.7 Vector direction for the torque

Recall that the magnitude of a cross product is the area of the parallelogram (the height times the base) defined by the two vectors. Figure 13.8 shows the two different ways of defining height and base for a parallelogram defined by the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$.

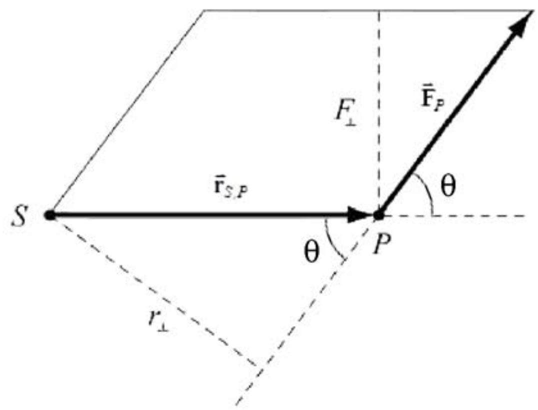


Figure 13.8 Area of the torque parallelogram.

Let $r_\perp = r\sin\theta$ and let $F_\perp = F\sin\theta$ be the component of the force $\vec{\mathbf{F}}_P$ that is perpendicular to the line passing from the point S to P. (Recall the angle θ has a range of values $0 \le \theta \le \pi$ so both $r_\perp \ge 0$ and $F_\perp \ge 0$.) Then the area of parallelogram defined by $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ is given by

$$\text{Area} = \tau_S = r_\perp F = r\,F_\perp = r\,F\sin\theta\,. \tag{13.2.3}$$

We can interpret the quantity $r_\perp$ as follows. We begin by drawing the *line of action of the force* $\vec{\mathbf{F}}_P$. This is a straight line passing through P, parallel to the direction of the force $\vec{\mathbf{F}}_P$. Draw a perpendicular to this line of action that passes through the point S (Figure 13.9). The length of this perpendicular, $r_\perp = r\sin\theta$, is called *the moment arm about the point S of the force* $\vec{\mathbf{F}}_P$.

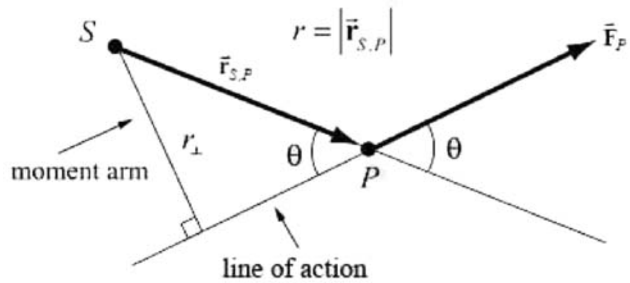


Figure 13.9 The moment arm about the point S associated with a force acting at the point P is the perpendicular distance from S to the line of action of the force passing through the point P

You should keep in mind three important properties of torque:

1. The torque is zero if the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ are parallel $(\theta = 0)$ or anti-parallel $(\theta = \pi)$.

2. Torque is a vector whose direction and magnitude depend on the choice of a point S about which the torque is calculated.

3. The direction of torque is perpendicular to the plane formed by the two vectors, $\vec{\mathbf{F}}_P$ and $r = \left|\vec{\mathbf{r}}_{S,P}\right|$ (the vector from the point S to a point P).

Alternative Approach to Assigning a Sign Convention for Torque

In the case where all of the forces $\vec{\mathbf{F}}_i$ and position vectors $\vec{\mathbf{r}}_{i,P}$ are coplanar (or zero), we can, instead of referring to the direction of torque, assign a purely algebraic positive or negative sign to torque according to the following convention. We note that the arc in Figure 13.10a circles in counterclockwise direction. (Figures 13.10a and 13.10b use the simplifying assumption, for the purpose of the figure only, that the two vectors in question, $\vec{\mathbf{F}}_P$ and $\vec{\mathbf{r}}_{S,P}$ are perpendicular. The point S about which torques are calculated is not shown.) We can associate with this counterclockwise orientation a unit normal vector $\hat{\mathbf{n}}_1$ according to the right-hand rule: curl your right hand fingers in the counterclockwise direction and your right thumb will then point in the $\hat{\mathbf{n}}_{RHR}$ direction. The arc in Figure 13.10b circles in the clockwise direction, and we associate this orientation with the unit normal $\hat{\mathbf{n}}_{LHR}$.

It's important to note that the terms "clockwise" and "counterclockwise" might be different for different observers. For instance, if the plane containing $\vec{\mathbf{F}}_P$ and $\vec{\mathbf{r}}_{S,P}$ is horizontal, an observer above the plane and an observer below the plane would assign disagree on the two terms. For a vertical plane, the directions that two observers on opposite sides of the plane would be mirror images of each other, and so again the observers would disagree.

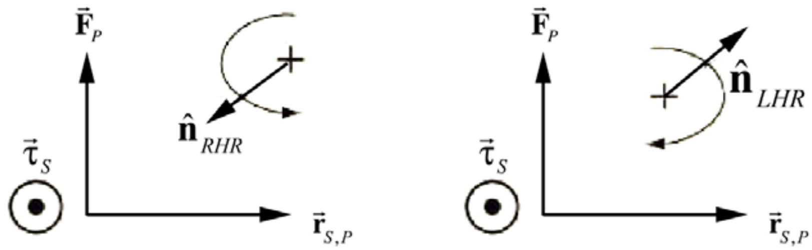


Figure 13.10a Positive torque **Figure 13.10b** Negative torque

1. Suppose we choose counterclockwise as positive. Then we assign a positive sign to the torque when the torque is in the same direction as the unit normal $\hat{\mathbf{n}}_{RHR}$, (Figure 13.10a).

2. Suppose we choose clockwise as positive. Then we assign a negative sign for the torque in Figure 13.10b since the torque is directed opposite to the unit normal $\hat{\mathbf{n}}_{LHR}$.

With rare exceptions, these notes will take the counterclockwise direction to be positive.

13.3 Torque, Angular Acceleration, and Moment of Inertia

For fixed-axis rotation, there is a direct relation between the component of the torque along the axis of rotation and angular acceleration.

Consider the forces that act on the rotating body. Most generally, the forces on different volume elements will be different, and so we will denote the force on the volume element of mass Δm_i by $\vec{\mathbf{F}}_i$.

Choose the z-axis to lie along the axis of rotation. As in Section 13.1, divide the body into volume elements of mass Δm_i. Let the point S denote a specific point along the axis of rotation (Figure 13.11). Each volume element undergoes a tangential acceleration as the volume element moves in a circular orbit of radius $r_{\perp,i} = |\vec{\mathbf{r}}_{\perp,i}|$ about the fixed axis.

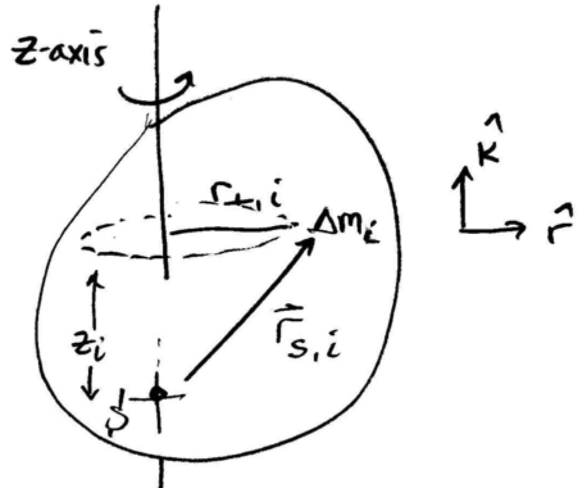


Figure 13.11 Volume element undergoing fixed-axis rotation about the z-axis.

The vector from the point S to the volume element is given by

$$\vec{\mathbf{r}}_{S,i} = z_i\,\hat{\mathbf{k}} + \vec{\mathbf{r}}_{\perp,i} = z_i\,\hat{\mathbf{k}} + r_{\perp,i}\,\hat{\mathbf{r}} \tag{13.3.1}$$

where z_i is the distance along the axis of rotation between the point S and the volume element. The torque about S due to the force $\vec{\mathbf{F}}_i$ acting on the volume element is given by

$$\vec{\boldsymbol{\tau}}_{S,i} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i\,. \tag{13.3.2}$$

Substituting Equation (13.3.1) into Equation (13.3.2) gives

$$\vec{\boldsymbol{\tau}}_{S,i} = \left(z_i\,\hat{\mathbf{k}} + r_{\perp,i}\,\hat{\mathbf{r}}\right) \times \vec{\mathbf{F}}_i\,. \tag{13.3.3}$$

For fixed-axis rotation, we are interested in the z-component of the torque, which must be the term

$$\left(\tau_{S,i}\right)_z = \left(r_{\perp,i}\,\hat{\mathbf{r}} \times \vec{\mathbf{F}}_i\right)_z \tag{13.3.4}$$

since the cross product $z_i\,\hat{\mathbf{k}} \times \vec{\mathbf{F}}_i$ must be directed perpendicular to the plane formed by the vectors $\hat{\mathbf{k}}$ and $\vec{\mathbf{F}}_i$, hence perpendicular to the z-axis.
The total force acting on the volume element has components

$$\vec{\mathbf{F}}_i = F_{\text{radial},i}\,\hat{\mathbf{r}} + F_{\tan,i}\,\hat{\boldsymbol{\theta}} + F_{z,i}\,\hat{\mathbf{k}}\,. \tag{13.3.5}$$

The z-component $F_{z,i}$ of the force cannot contribute a torque in the z-direction, and so substituting Equation (13.3.5) into Equation (13.3.4) yields

$$\left(\tau_{S,i}\right)_z = \left(r_{\perp,i}\,\hat{\mathbf{r}} \times \left(F_{\text{radial},i}\,\hat{\mathbf{r}} + F_{\tan,i}\,\hat{\boldsymbol{\theta}}\right)\right)_z . \tag{13.3.6}$$

The radial force does not contribute to the torque about the z-axis, since

$$r_{\perp,i}\,\hat{\mathbf{r}} \times F_{\text{radial},i}\,\hat{\mathbf{r}} = \vec{\mathbf{0}}\,. \tag{13.3.7}$$

So, we are interested in the contribution due to torque about the z-axis due to the tangential component of the force on the volume element (Figure 13.12). The component of the torque about the z-axis is given by

$$\left(\tau_{S,\,i}\right)_z = \left(r_{\perp,i}\,\hat{\mathbf{r}} \times F_{\tan,i}\,\hat{\boldsymbol{\theta}}\right)_z = r_{\perp,i}\,F_{\tan,i}\,. \tag{13.3.8}$$

The z-component of the torque is directed out of the page in Figure 13.12, where $F_{\tan,i}$ is positive (the tangential force is directed counterclockwise, as in the figure).

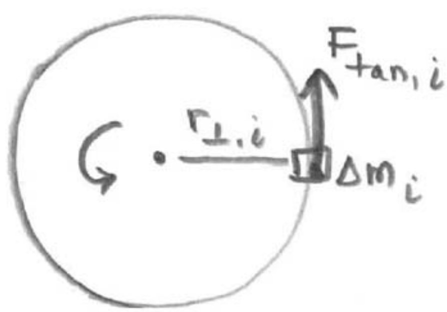


Figure 13.12 Tangential force acting on a volume element.

Applying Newton's Second Law in the tangential direction,

$$F_{\tan,\,i} = \Delta m_i\, a_{\tan,\,i}\,. \tag{13.3.9}$$

Using the expression in (13.1.4) for tangential acceleration, we have that

$$F_{\tan,\,i} = \Delta m_i\, r_{\perp,\,i}\,\alpha\,. \tag{13.3.10}$$

From Equation (13.3.8), the component of the torque about the z-axis is then given by

$$\left(\tau_{S,i}\right)_z = r_{\perp,\,i}\,F_{\tan,\,i} = \Delta m_i \left(r_{\perp,\,i}\right)^2 \alpha \,. \tag{13.3.11}$$

The total component of the torque about the z-axis is the summation of the torques on all the volume elements,

$$\begin{aligned}\left(\tau_S^{\text{total}}\right)_z &= \left(\tau_{S,1}\right)_z + \left(\tau_{S,2}\right)_z + \cdots = \sum_{i=1}^{i=N} \left(\tau_{S,i}\right)_z = \sum_{i=1}^{i=N} r_{\perp,\,i}\,F_{\tan,\,i} \\ &= \sum_{i=1}^{i=N} \Delta m_i \left(r_{\perp,\,i}\right)^2 \alpha.\end{aligned} \tag{13.3.12}$$

Since each element has the same angular acceleration, α, the summation becomes

$$\left(\tau_S^{\text{total}}\right)_z = \left(\sum_{i=1}^{i=N} \Delta m_i \left(r_{\perp,\,i}\right)^2\right)\alpha \,. \tag{13.3.13}$$

Definition: **Moment of Inertia about a Fixed Axis**

The quantity

$$I_S = \sum_{i=1}^{i=N} \Delta m_i \left(r_{\perp,i}\right)^2 \,. \tag{13.3.14}$$

is called the *moment of inertia* of the rigid body about a fixed axis passing through the point S, and is a physical property of the body. The SI units for moment of inertia are $\left[\mathrm{kg}\cdot\mathrm{m}^2\right]$.

Thus Equation (13.3.13) shows that the z-component of the torque is proportional to the angular acceleration,

$$\left(\tau_S^{\text{total}}\right)_z = I_S\,\alpha \,, \tag{13.3.15}$$

and the moment of inertia, I_S, is the constant of proportionality.

This is very similar to Newton’s Second Law: the total force is proportional to the acceleration,

$$\vec{\mathbf{F}}^{\text{total}} = m^{\text{total}}\vec{\mathbf{a}} \,. \tag{13.3.16}$$

where the total mass, m^{total}, is the constant of proportionality.

For a continuous mass distribution, the summation becomes an integral over the body

$$I_S = \int_{\text{body}} dm\left(r_\perp\right)^2 , \tag{13.3.17}$$

which will be explored in detail in the next section.

13.2.1 Example: Turntable, Part II

The turntable in Example 13.1.1, of mass 1.2 kg and radius $1.3\times10^1\text{ cm}$, has a moment of inertia $I_S = 1.01\times10^{-2}\text{ kg}\cdot\text{m}^2$ about an axis through the center of the disc and perpendicular to the disc. The turntable is spinning at an initial constant frequency of $f_0 = 33\,\text{cycles}\cdot\text{min}^{-1}$. The motor is turned off and the turntable slows to a stop in 8.0 s due to frictional torque. Assume that the angular acceleration is constant. What is the magnitude of the frictional torque acting on the disc?

Answer:

We have already calculated the angular acceleration of the disc in Example 13.1.1, where we found that the angular acceleration is

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_0}{t_f - t_0} = \frac{-3.5\text{ rad}\cdot\text{s}^{-1}}{8.0\text{ s}} = -4.3\times10^{-1}\text{ rad}\cdot\text{s}^{-2} \tag{13.3.18}$$

and so the magnitude of the frictional torque is

$$\begin{aligned} \left|\tau_{\text{friction}}^{\text{total}}\right| &= I_S\left|\alpha\right| = \left(1.01\times10^{-2}\text{ kg}\cdot\text{m}^2\right)\left(4.3\times10^{-1}\text{ rad}\cdot\text{s}^{-2}\right) \\ &= 4.3\times10^{-3}\text{ N}\cdot\text{m}. \end{aligned} \tag{13.3.19}$$

13.2.2 Example: Moment of Inertia of a Rod of Uniform Mass Density, Part I

Consider a thin uniform rod of length L and mass m. In this problem, we will calculate the moment of inertia about an axis perpendicular to the rod that passes through the center of mass of the rod. A sketch of the rod, volume element, and axis is shown in Figure 13.13.

Choose Cartesian coordinates, with the origin at the center of mass of the rod, which is midway between the endpoints since the rod is uniform. Choose the x-axis to lie along the length of the rod, with the positive x-direction to the right, as in the figure.

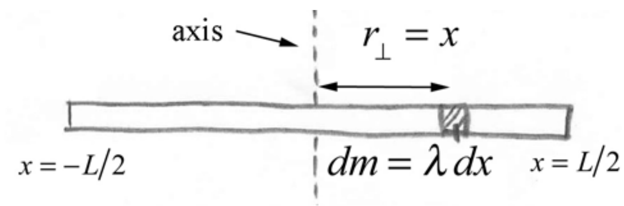


Figure 13.13 Moment of inertia of a uniform rod about center of mass.

Identify an infinitesimal mass element $dm = \lambda\, dx$, located at a displacement x from the center of the rod, where the mass per unit length $\lambda = m/L$ is a constant, as we have assumed the rod to be uniform.

When the rod rotates about an axis perpendicular to the rod that passes through the center of mass of the rod, the element traces out a circle of radius $r_\perp = x$.

We add together the contributions from each infinitesimal element as we go from $x = -L/2$ to $x = L/2$. The integral is then

$$\begin{aligned} I_{\text{cm}} &= \int_{\text{body}} (r_\perp)^2\, dm = \lambda \int_{-L/2}^{L/2} (x^2)\, dx = \lambda \left.\frac{x^3}{3}\right|_{-L/2}^{L/2} \\ &= \frac{m}{L}\frac{(L/2)^3}{3} - \frac{m}{L}\frac{(-L/2)^3}{3} = \frac{1}{12} m L^2. \end{aligned} \qquad (13.3.20)$$

By using a constant mass per unit length along the rod, we need not consider variations in the mass density in any direction other than the x- axis. We also assume that the width is the rod is negligible. (Technically we should treat the rod as a cylinder or a rectangle in the x-y plane if the axis is along the z- axis. The calculation of the moment of inertia in these cases would be more complicated.)

13.4 Parallel Axis Theorem

Consider a rigid body of mass m undergoing fixed-axis rotation. Consider two parallel axes. The first axis passes through the center of mass of the body, and the moment of inertia about this first axis is I_{cm}. The second axis passes through some other point S in the body. Let $d_{S,\text{cm}}$ denote the perpendicular distance between the two parallel axes (Figure 13.14). Then the moment of inertia I_S about an axis passing through a point S is related to I_{cm} by

$$I_S = I_{\text{cm}} + m\, d_{S,\text{cm}}^2 . \qquad (13.4.1)$$

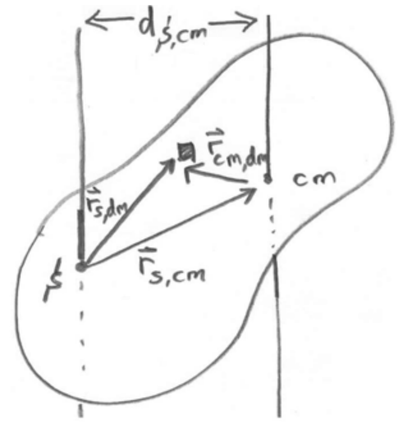


Figure 13.14 Geometry of the parallel axis theorem.

Proof of the Parallel Axis Theorem

Identify an infinitesimal volume element of mass dm. The vector from the point S to the mass element is $\vec{\mathbf{r}}_{S,dm}$, the vector from the center of mass to the mass element is $\vec{\mathbf{r}}_{\text{cm},dm}$, and the vector from the point S to the center of mass is $\vec{\mathbf{r}}_{S,\text{cm}}$. From Figure 13.14, we see that

$$\vec{\mathbf{r}}_{S,dm} = \vec{\mathbf{r}}_{S,\text{cm}} + \vec{\mathbf{r}}_{\text{cm},dm}\,. \tag{13.4.2}$$

The notation gets complicated at this point. We are interested in distances from the respective axes, so denote the following vectors as motivated in Section 13.2:

- As in Figure 13.14 and Equation (13.4.2), $\vec{\mathbf{r}}_{\text{cm},\,dm}$ is the vector from the center of mass to the position of the mass element of mass dm. This vector has a component vector $\vec{\mathbf{r}}_{\text{cm},\parallel,dm}$ parallel to the axis through the center of mass and a component vector $\vec{\mathbf{r}}_{\text{cm},\perp,dm}$ perpendicular to the axis through the center of mass. The magnitude of the perpendicular component vector is

$$\left|\vec{\mathbf{r}}_{\text{cm},\perp,dm}\right| = r_{\text{cm},\perp,dm}\,. \tag{13.4.3}$$

- As in Figure 13.9 and Equation (13.4.2), $\vec{\mathbf{r}}_{S,\,dm}$ is the vector from the point S to the position of the mass element of mass dm. This vector has a component vector $\vec{\mathbf{r}}_{S,\parallel,dm}$ parallel to the axis through the point S and a component vector

$\vec{\mathbf{r}}_{S,\perp,dm}$ perpendicular to the axis through the point S. The magnitude of the perpendicular component vector is

$$\left|\vec{\mathbf{r}}_{S,\perp,dm}\right| = r_{S,\perp,dm} . \tag{13.4.4}$$

- As in Figure 13.14 and Equation (13.4.2), $\vec{\mathbf{r}}_{S,\text{cm}}$ is the vector from the point S to the center of mass. This vector has a component vector $\vec{\mathbf{r}}_{S,\parallel,\text{cm}}$ parallel to *both* axes and a perpendicular component vector $\vec{\mathbf{r}}_{S,\perp,\text{cm}}$ of $\vec{\mathbf{r}}_{S,\text{cm}}$ perpendicular to *both* axes (the axes are parallel, of course). The magnitude of the perpendicular component vector is

$$\left|\vec{\mathbf{r}}_{S,\perp,\text{cm}}\right| = d_{S,\text{cm}} . \tag{13.4.5}$$

Equation (13.4.2) is now expressed as two equations,

$$\begin{aligned}\vec{\mathbf{r}}_{S,\perp,dm} &= \vec{\mathbf{r}}_{S,\perp,\text{cm}} + \vec{\mathbf{r}}_{\text{cm},\perp,dm}\\ \vec{\mathbf{r}}_{S,\parallel,dm} &= \vec{\mathbf{r}}_{S,\parallel,\text{cm}} + \vec{\mathbf{r}}_{\text{cm},\parallel,dm}.\end{aligned} \tag{13.4.6}$$

At this point, note that if we had simply decided that the two parallel axes are parallel to the z-direction, we could have saved some steps and perhaps spared some of the notation with the triple subscripts. However, we want a more general result, one valid for cases where the axes are not fixed, or when different objects in the same problem have different axes. For example, consider the turning bicycle, for which the two wheel axes will not be parallel, or a spinning top that *precesses* (wobbles). Such cases will be considered in Chapter 16, and we will show the general case of the parallel axis theorem in anticipation of use for more general situations.

The moment of inertia about the point S is

$$I_S = \int_{\text{body}} dm\left(r_{S,\perp,dm}\right)^2 . \tag{13.4.7}$$

From (13.4.6) we have

$$\begin{aligned}\left(r_{S,\perp,dm}\right)^2 &= \vec{\mathbf{r}}_{S,\perp,dm}\cdot\vec{\mathbf{r}}_{S,\perp,dm}\\ &= \left(\vec{\mathbf{r}}_{S,\perp,\text{cm}} + \vec{\mathbf{r}}_{\text{cm},\perp,dm}\right)\cdot\left(\vec{\mathbf{r}}_{S,\perp,\text{cm}} + \vec{\mathbf{r}}_{\text{cm},\perp,dm}\right)\\ &= d_{S,\text{cm}}^2 + \left(r_{\text{cm},\perp,dm}\right)^2 + 2\,\vec{\mathbf{r}}_{S,\perp,\text{cm}}\cdot\vec{\mathbf{r}}_{\text{cm},\perp,dm}.\end{aligned} \tag{13.4.8}$$

Thus we have for the moment of inertia about S,

$$I_S = \int_{\text{body}} dm\left(d_{S,\text{cm}}^2 \right) + \int_{\text{body}} dm\left(r_{\text{cm},\perp,dm} \right)^2 + 2\int_{\text{body}} dm\left(\vec{\mathbf{r}}_{S,\perp,\text{cm}} \cdot \vec{\mathbf{r}}_{\text{cm},\perp,dm} \right). \qquad (13.4.9)$$

In the first integral in Equation (13.4.9), $r_{S,\perp,\text{cm}} = d_{S,\text{cm}}$ is the distance between the parallel axes and is a constant and may be taken out of the integral, and

$$\int_{\text{body}} dm \left(d_{S,\text{cm}}^2 \right) = m\, d_{S,\text{cm}}^2 . \qquad (13.4.10)$$

The second term in Equation (13.4.9) is the moment of inertia about the axis through the center of mass,

$$I_{\text{cm}} = \int_{\text{body}} dm \left(r_{\text{cm},\perp,dm} \right)^2 . \qquad (13.4.11)$$

The third integral in Equation (13.4.9) is zero. To see this, note that the term $\vec{\mathbf{r}}_{S,\perp,\text{cm}}$ is a constant and may be taken out of the integral,

$$2\int_{\text{body}} dm \left(\vec{\mathbf{r}}_{S,\perp,\text{cm}} \cdot \vec{\mathbf{r}}_{\text{cm},\perp,dm} \right) = \vec{\mathbf{r}}_{S,\perp,\text{cm}} \cdot 2\int_{\text{body}} dm\left(\vec{\mathbf{r}}_{\text{cm},\perp,dm} \right) \qquad (13.4.12)$$

The integral $\int_{\text{body}} dm\left(\vec{\mathbf{r}}_{\text{cm},\perp,dm} \right)$ is the perpendicular component of the position of the center of mass with respect to the center of mass, and hence $\vec{\mathbf{0}}$, with the result that

$$2\int_{\text{body}} dm \left(\vec{\mathbf{r}}_{S,\perp\text{cm}} \cdot \vec{\mathbf{r}}_{\text{cm},\perp,dm} \right) = 0 . \qquad (13.4.13)$$

Thus, the moment of inertia about S is just the sum of the first two integrals in Equation (13.4.9),

$$I_S = I_{\text{cm}} + m\, d_{S,\text{cm}}^2 . \qquad (13.4.14)$$

13.3.1 Example: Uniform Rod, Part II

Let point S be the left end of the rod of Example 13.2.1 and Figure 13.13. Then the distance from the center of mass to the end of the rod is $d_{S,\text{cm}} = L/2$. The moment of inertia $I_S = I_{\text{end}}$ about an axis passing through the endpoint is related to the moment of inertia about an axis passing through the center of mass, $I_{\text{cm}} = (1/12)\,m L^2$, according to Equation (13.4.14),

$$I_S = \frac{1}{12} m L^2 + \frac{1}{4} m L^2 = \frac{1}{3} m L^2 . \tag{13.4.15}$$

In this case it's easy and useful to check by direct calculation. Use Equation (13.3.20) but with the limits changed to $x' = 0$ and $x' = L$, where $x' = x + L/2$;

$$\begin{aligned} I_{\text{end}} &= \int_{\text{body}} r_\perp{}^2\, dm = \lambda \int_0^L x'^2\, dx' \\ &= \lambda \left. \frac{x'^3}{3} \right|_0^L = \frac{m}{L}\frac{(L)^3}{3} - \frac{m}{L}\frac{(0)^3}{3} = \frac{1}{3} m L^2 . \end{aligned} \tag{13.4.16}$$

13.5 Simple Pendulum and Physical Pendulum

Simple Pendulum

A pendulum consists of an object hanging from the end of a string or rigid rod pivoted about the point S. The object is pulled to one side and allowed to oscillate. If the object has negligible size and the string or rod is massless, then the pendulum is called a *simple pendulum*. The force diagram for the simple pendulum is shown in Figure 13.15.

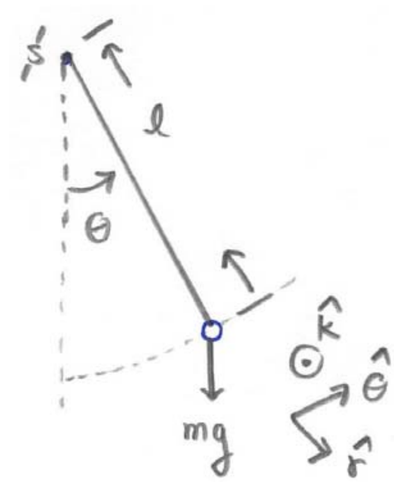


Figure 13.15 A simple pendulum.

The string or rod exerts no torque about the pivot point S. The weight of the object has radial $\hat{\mathbf{r}}$ - and $\hat{\boldsymbol{\theta}}$ - components given by

$$m\vec{\mathbf{g}} = mg\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right) \tag{13.5.1}$$

and the torque about the pivot point S is given by

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,m} \times m\vec{\mathbf{g}} = l\,\hat{\mathbf{r}} \times m\,g\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right) = -l\,m\,g\sin\theta\,\hat{\mathbf{k}} \tag{13.5.2}$$

and so the component of the torque in the z-direction (into the page in Figure 13.15 for θ positive, out of the page for θ negative) is

$$\left(\tau_S\right)_z = -mgl\sin\theta\,. \tag{13.5.3}$$

The moment of inertia of a point mass about the pivot point S is

$$I_S = ml^2\,. \tag{13.5.4}$$

From Equation (13.3.15) the rotational dynamical equation is

$$\begin{aligned}\left(\tau_S\right)_z &= I_S\alpha = I_S\frac{d^2\theta}{dt^2}\\ -mgl\sin\theta &= ml^2\frac{d^2\theta}{dt^2}.\end{aligned} \tag{13.5.5}$$

Thus we have the equation of motion for the simple pendulum,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta\,. \tag{13.5.6}$$

When the angle of oscillation is small, then we can use the small angle approximation

$$\sin\theta \cong \theta\,; \tag{13.5.7}$$

the rotational dynamical equation for the pendulum becomes

$$\frac{d^2\theta}{dt^2} \cong -\frac{g}{l}\theta\,. \tag{13.5.8}$$

This equation is similar to the object-spring simple harmonic oscillator differential equation from Chapter 10.2, Equation 10.2.3,

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x\,, \tag{13.5.9}$$

which describes the oscillation of a mass about the equilibrium point of a spring. Recall that in Chapter 10.2, Equation 10.2.7, the angular frequency of oscillation was given by

$$\omega_{\text{spring}} = \sqrt{\frac{k}{m}}\,. \tag{13.5.10}$$

By comparison, the frequency of oscillation for the pendulum is approximately

$$\omega_{\text{pendulum}} \cong \sqrt{\frac{g}{l}}\,, \tag{13.5.11}$$

with period

$$T = \frac{2\pi}{\omega_{\text{pendulum}}} \cong 2\pi\sqrt{\frac{l}{g}}\,. \tag{13.5.12}$$

A procedure for determining the period for larger angles is given in Appendix 13.B.

Physical Pendulum

A *physical pendulum* consists of a rigid body that undergoes fixed axis rotation about a fixed point S (Figure 13.16). The gravitational force acts at the center of mass of the physical pendulum (Appendix 13.A). Suppose the center of mass is a distance l_{cm} from the pivot point S.

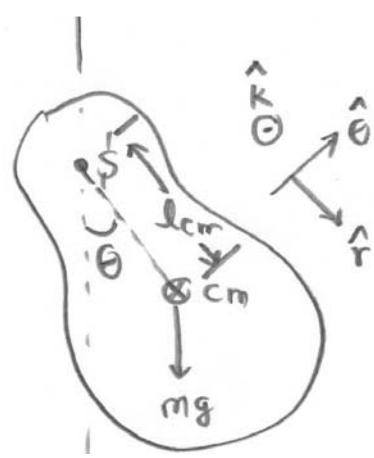


Figure 13.16 Physical pendulum.

The analysis is nearly identical to the simple pendulum. The torque about the pivot point is given by

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,\mathrm{cm}} \times m\vec{\mathbf{g}} = l_{\mathrm{cm}}\hat{\mathbf{r}} \times m\,g\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right) = -l_{\mathrm{cm}} m\,g\sin\theta\,\hat{\mathbf{k}}\,. \qquad (13.5.13)$$

Following the same steps that led from Equation (13.5.2) to Equation (13.5.6), the rotational dynamical equation for the physical pendulum is

$$\begin{aligned}(\tau_S)_z &= I_S\alpha = I_S\frac{d^2\theta}{dt^2}\\ -mgl_{\mathrm{cm}}\sin\theta &= I_S\frac{d^2\theta}{dt^2}.\end{aligned} \qquad (13.5.14)$$

Thus we have the equation of motion for the physical pendulum,

$$\frac{d^2\theta}{dt^2} = -\frac{mgl_{\mathrm{cm}}}{I_S}\sin\theta\,. \qquad (13.5.15)$$

As with the simple pendulum, for small angles $\sin\theta \approx \theta$ and Equation (13.5.15) reduces to the simple harmonic oscillator equation with angular frequency

$$\omega_{\mathrm{pendulum}} \approx \sqrt{\frac{mg\,l_{\mathrm{cm}}}{I_S}} \qquad (13.5.16)$$

and period

$$T_{\mathrm{physical}} = \frac{2\pi}{\omega_{\mathrm{pendulum}}} \cong 2\pi\sqrt{\frac{I_S}{m\,g\,l_{\mathrm{cm}}}}\,. \qquad (13.5.17)$$

It is sometimes convenient to express the moment of inertia about the pivot point in terms of l_{cm} and I_{cm} using the parallel axis theorem in Equation (13.4.14), with $d_{S,\mathrm{cm}} \equiv l_{\mathrm{cm}}$, $I_S = I_{\mathrm{cm}} + ml_{\mathrm{cm}}^2$, with the result

$$T_{\mathrm{physical}} \cong 2\pi\sqrt{\frac{l_{\mathrm{cm}}}{g} + \frac{I_{\mathrm{cm}}}{m\,g\,l_{\mathrm{cm}}}}\,. \qquad (13.5.18)$$

Thus, if the object is “small” in the sense that $I_{\text{cm}} \ll m l_{\text{cm}}^2$, the expressions for the physical pendulum reduce to those for the simple pendulum. Note that this is *not* the case shown in Figure 13.16.

13.6 Torque and Rotational Work

Introduction

When a constant torque τ_S is applied to an object, and the object rotates through an angle $\Delta\theta$ about an axis through the center of mass, then the torque does an amount of work $\Delta W = \tau_S\,\Delta\theta$ on the object. By extension of the linear work-energy theorem, the amount of work done is equal to the change in the rotational kinetic energy of the object,

$$W_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega_f^2 - \frac{1}{2} I_{\text{cm}} \omega_0^2 = K_{\text{rot},\,f} - K_{\text{rot},0}\,. \tag{13.6.1}$$

The rate of doing this work is the rotational power exerted by the torque,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta W_{\text{rot}}}{\Delta t} = \tau_S \frac{d\theta}{dt} = \tau_S\,\omega\,. \tag{13.6.2}$$

Rotational work

Consider a rigid body rotating about an axis. Each small element of mass Δm_i in the rigid body is moving in a circle of radius $\left(r_{S,\perp}\right)_i$ about the axis of rotation passing through the point S. Each mass element undergoes a small angular displacement $\Delta\theta$ under the action of a tangential force, $\vec{\mathbf{F}}_{\text{tan},\,i} = F_{\text{tan},\,i}\,\hat{\boldsymbol{\theta}}$, where $\hat{\boldsymbol{\theta}}$ is the unit vector pointing in the tangential direction (Figure 13.7). The element will then have an associated displacement vector for this motion, $\Delta\vec{\mathbf{r}}_{S,i} = \left(r_{S,\,i}\right)_\perp \Delta\theta\,\hat{\boldsymbol{\theta}}$ and the work done by the tangential force is

$$\Delta W_i = \vec{\mathbf{F}}_{\text{tan},i} \cdot \Delta\vec{\mathbf{r}}_{S,i} = \left(F_{\text{tan},i}\,\hat{\boldsymbol{\theta}}\right)\cdot\left(\left(r_{S,\,i}\right)_\perp \Delta\theta\,\hat{\boldsymbol{\theta}}\right) = F_{\text{tan},\,i}\left(r_{S,\,i}\right)_\perp \Delta\theta\,. \tag{13.6.3}$$

Applying Newton’s Second Law to the element Δm_i in the tangential direction,

$$F_{\text{tan},\,i} = \Delta m_i\, a_{\text{tan},\,i}\,. \tag{13.6.4}$$

Using the expression in Equation (13.1.4) for tangential acceleration we have that

$$F_{\tan,\,i} = \Delta m_i \left(r_{S,i}\right)_\perp \alpha \,. \qquad (13.6.5)$$

Thus the rotational work done on the mass element is

$$\Delta W_i = \Delta m_i \left(r_{S,i}\right)_\perp^2 \alpha \,\Delta\theta \,. \qquad (13.6.6)$$

Summing the rotational work done on all of the mass elements, we obtain

$$\Delta W = \sum_i \Delta W_i = \left(\sum_i \Delta m_i \left(r_{S,i}\right)_\perp^2\right)\alpha \,\Delta\theta \,. \qquad (13.6.7)$$

In the limit that the discrete mass elements become infinitesimal continuous mass elements, $\Delta m_i \to dm$, the summation becomes an integral over the body:

$$\Delta W = \left(\sum_i \Delta m_i \left(r_{S,i}\right)_\perp^2\right)\alpha \,\Delta\theta \;\;\to\;\; \left(\int_{\text{body}} dm \left(r_S\right)_\perp^2\right)\alpha \,\Delta\theta \,. \qquad (13.6.8)$$

Since the integral in this expression is just the moment of inertia about a fixed axis passing through the point S, we have for the rotational work

$$\Delta W = I_S \,\alpha \,\Delta\theta \,. \qquad (13.6.9)$$

Since the z-component of the torque (in the direction along the axis of rotation) about S is given by

$$\left(\tau_S\right)_z = I_S \,\alpha \,, \qquad (13.6.10)$$

the rotational work is the product of the torque and the angular displacement,

$$\Delta W = \left(\tau_S\right)_z \Delta\theta \,. \qquad (13.6.11)$$

Recall the result of Equation (13.3.8) that the component of the torque (in the direction along the axis of rotation) about S due to the tangential force, $\vec{\mathbf{F}}_{\tan,i}$, acting on the mass element Δm_i is

$$\left(\tau_{S,i}\right)_z = F_{\tan,i}\left(r_{S,i}\right)_\perp \,, \qquad (13.6.12)$$

and the total torque is the sum

$$\left(\tau_S\right)_z = \sum_i \left(\tau_{S,i}\right)_z = \sum_i F_{\tan,i} \left(r_{S,i}\right)_\perp \qquad (13.6.13)$$

and so the work done is

$$\Delta W = \sum_i \Delta W_i = \sum_i F_{\tan,i} \left(r_{S,i}\right)_\perp \Delta\theta = \left(\tau_S\right)_z \Delta\theta\,. \qquad (13.6.14)$$

In the limit of small angles, $\Delta\theta \to d\theta$, $\Delta W \to dW$ and the differential rotational work is

$$dW = \left(\tau_S\right)_z d\theta\,. \qquad (13.6.15)$$

We can integrate this amount of rotational work as the angle coordinate of the rigid body changes from some initial value $\theta = \theta_0$ to some final value $\theta = \theta_f$,

$$W = \int dW = \int_{\theta_0}^{\theta_f} \left(\tau_S\right)_z d\theta\,. \qquad (13.6.16)$$

Rotational Kinetic Energy

The general motion of a rigid body consists of a translation of the center of mass with velocity $\vec{\mathbf{v}}_{\text{cm}}$ and a rotation about the center of mass with angular velocity ω_{cm}.

Having defined translational kinetic energy in Chapter 7.2, Equation 7.2.1, we now define the rotational kinetic energy for a rigid body about its center of mass. Each individual mass element Δm_i undergoes circular motion about the center of mass with angular frequency ω_{cm} in a circle of radius $\left(r_{\text{cm},\,i}\right)_\perp$. Therefore the velocity of each element is given by $\vec{\mathbf{v}}_{\text{cm},i} = \left(r_{\text{cm},\,i}\right)_\perp \omega_{\text{cm}}\,\hat{\boldsymbol{\theta}}$. The rotational kinetic energy is then

$$K_{\text{cm},\,i} = \frac{1}{2}\Delta m_i\, v_{\text{cm},\,i}^2 = \frac{1}{2}\Delta m_i \left(r_{\text{cm},\,i}\right)_\perp^2 \omega_{\text{cm}}^2\,. \qquad (13.6.17)$$

We now add up the kinetic energy for all the mass elements,

$$\begin{aligned} K_{\text{cm}} &= \sum_i K_{\text{cm},\,i} = \left(\sum_i \frac{1}{2}\Delta m_i \left(r_{\text{cm},\,i}\right)_\perp^2\right)\omega_{\text{cm}}^2 = \left(\frac{1}{2}\int_{\text{body}} dm \left(r_{\text{cm}}\right)_\perp^2\right)\omega_{\text{cm}}^2 \\ &= \frac{1}{2} I_{\text{cm}}\, \omega_{\text{cm}}^2. \end{aligned} \qquad (13.6.18)$$

The total kinetic energy is the sum of the translational kinetic energy and the rotational kinetic energy,

$$K_{\text{total}} = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2} m v_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega_{\text{cm}}^2 . \qquad (13.6.19)$$

The above assertion that the total kinetic energy consists of two parts, which is certainly plausible, is derived in Appendix 13.C.

Rotational Work-Kinetic Energy Theorem

We will now show that the rotational work is equal to the change in rotational kinetic energy. We begin by substituting our result from Equation (13.6.10) into Equation (13.6.15) for the infinitesimal rotational work,

$$dW_{\text{rot}} = I_S \alpha \, d\theta . \qquad (13.6.20)$$

Recall that the rate of change of angular velocity is equal to the angular acceleration, $\alpha \equiv d\omega/dt$ and that the angular velocity is $\omega \equiv d\theta/dt$. Note that in the limit of small displacements,

$$\frac{d\omega}{dt} d\theta = d\omega \frac{d\theta}{dt} = d\omega \, \omega . \qquad (13.6.21)$$

Therefore the infinitesimal rotational work is

$$dW_{\text{rot}} = I_S \alpha \, d\theta = I_S \frac{d\omega}{dt} d\theta = I_S \, d\omega \frac{d\theta}{dt} = I_S \, d\omega \, \omega . \qquad (13.6.22)$$

We can integrate this amount of rotational work as the angular velocity of the rigid body changes from some initial value $\omega = \omega_0$ to some final value $\omega = \omega_f$,

$$W_{\text{rot}} = \int dW_{\text{rot}} = \int_{\omega_0}^{\omega_f} I_S \, d\omega \, \omega = \frac{1}{2} I_S \, \omega_f^2 - \frac{1}{2} I_S \, \omega_0^2 . \qquad (13.6.23)$$

When a rigid body is rotating about a fixed axis passing through a point S in the body, there is both rotation and translation about the center of mass unless S is the center of mass. If we choose the point S in the above equation for the rotational work to be the center of mass, then

$$W_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},f}^2 - \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},0}^2 = K_{\text{rot},f} - K_{\text{rot},0} \equiv \Delta K_{\text{rot}} . \qquad (13.6.24)$$

Recall the work-kinetic energy theorem stated that the total translational work done by all the forces is equal to the change in the translational kinetic energy of the center of mass,

$$W_{\text{trans}} = \frac{1}{2}mv_{\text{cm},f}^2 - \frac{1}{2}mv_{\text{cm},0}^2 \equiv \Delta K_{\text{trans}}\,. \tag{13.6.25}$$

Since we did not include the effect of rotational work in Equation (13.6.25), we now add the two contributions to the total work and find that the total work done on a rigid body is equal to the total change of the kinetic energy

$$\begin{aligned} W_{\text{total}} &= W_{\text{trans}} + W_{\text{rot}} \\ &= \left(\frac{1}{2}m\,v_{\text{cm},f}^2 - \frac{1}{2}mv_{\text{cm},0}^2\right) + \left(\frac{1}{2}I_{\text{cm}}\omega_f^2 - \frac{1}{2}I_{\text{cm}}\omega_0^2\right) \\ &= \Delta K_{\text{trans}} + \Delta K_{\text{rot}}. \end{aligned} \tag{13.6.26}$$

Rotational Power

The rotational power is defined as the rate of doing rotational work,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt}\,. \tag{13.6.27}$$

We can use our result for the infinitesimal work to find that the rotational power is the product of the applied torque with the angular velocity of the rigid body,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} = \left(\tau_S\right)_z \frac{d\theta}{dt} = \left(\tau_S\right)_z \omega\,. \tag{13.6.28}$$

Chapter 10

Rotational Motion

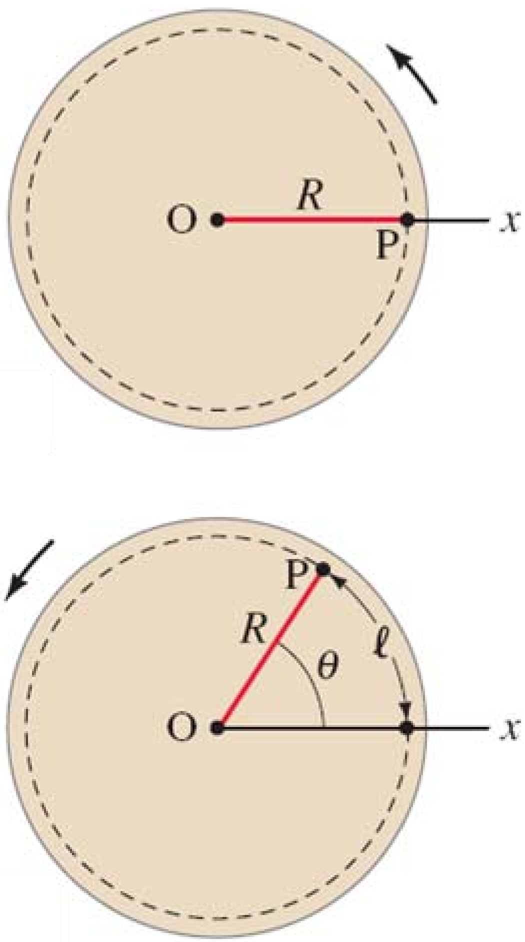
Units of Chapter 10

- Angular Quantities
- Vector Nature of Angular Quantities
- Constant Angular Acceleration
- Torque
- Rotational Dynamics; Torque and Rotational Inertia
- Solving Problems in Rotational Dynamics

Units of Chapter 10

- Determining Moments of Inertia
- Rotational Kinetic Energy
- Rotational Plus Translational Motion; Rolling
- Why Does a Rolling Sphere Slow Down?

10-1 Angular Quantities

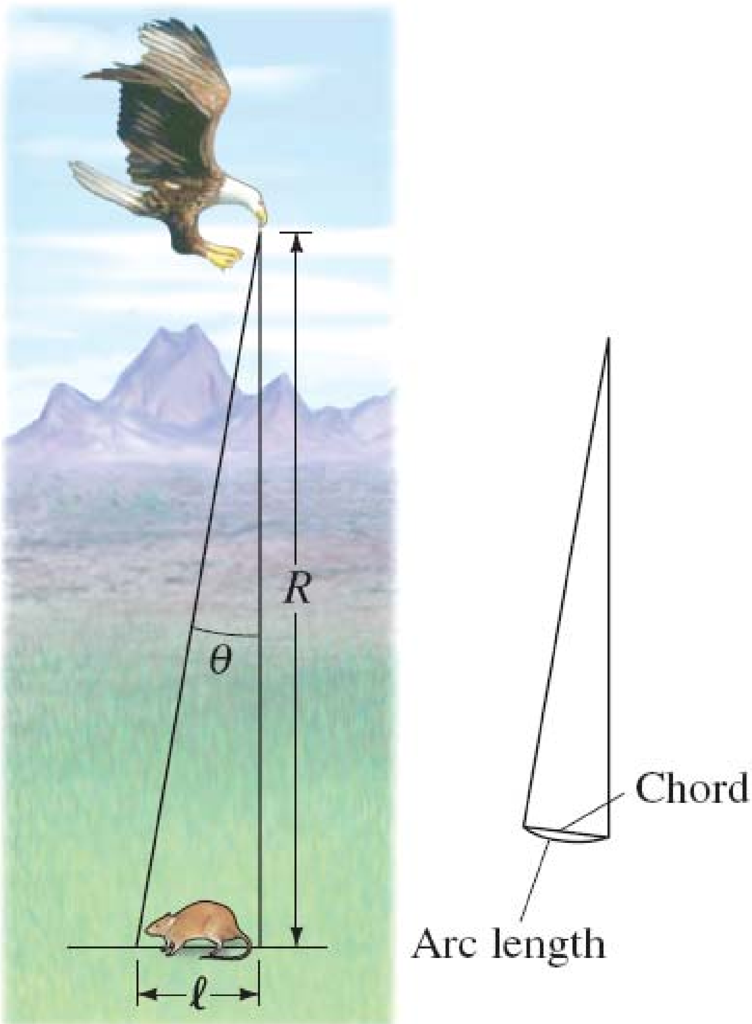


In purely **rotational** motion, all points on the object move in **circles** around the **axis** of rotation ("*O*"). The **radius** of the circle is *R*. All points on a straight line drawn through the axis move through the same **angle** in the same **time**. The angle *θ* in **radians** is defined:

$$\theta = \frac{l}{R},$$

where *l* is the **arc length**.

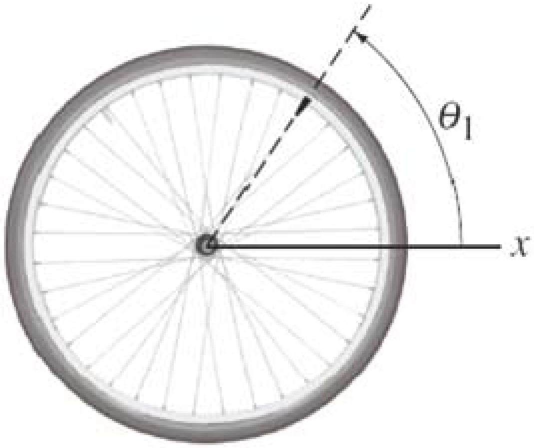
10-1 Angular Quantities

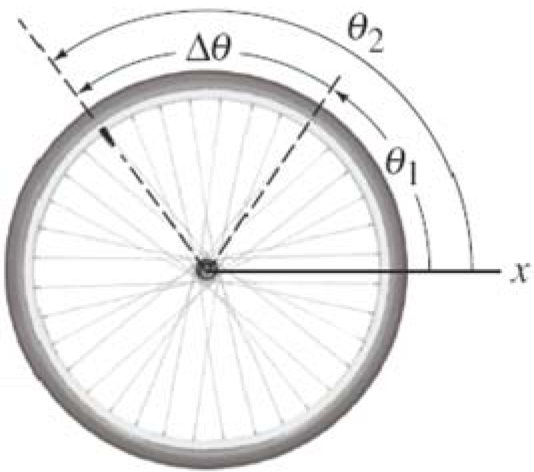


Example 10-1: Birds of prey—in radians.

A particular bird's eye can just distinguish objects that subtend an angle no smaller than about 3 x 10^{-4} rad. (a) How many degrees is this? (b) How small an object can the bird just distinguish when flying at a height of 100 m?

10-1 Angular Quantities





Angular displacement:

$$\Delta\theta = \theta_2 - \theta_1 .$$

The average angular velocity is defined as the total angular displacement divided by time:

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t} .$$

The instantaneous angular velocity:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} .$$

10-1 Angular Quantities

The angular **acceleration** is the rate at which the angular velocity changes with time:

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{\Delta t} = \frac{\Delta \omega}{\Delta t}.$$

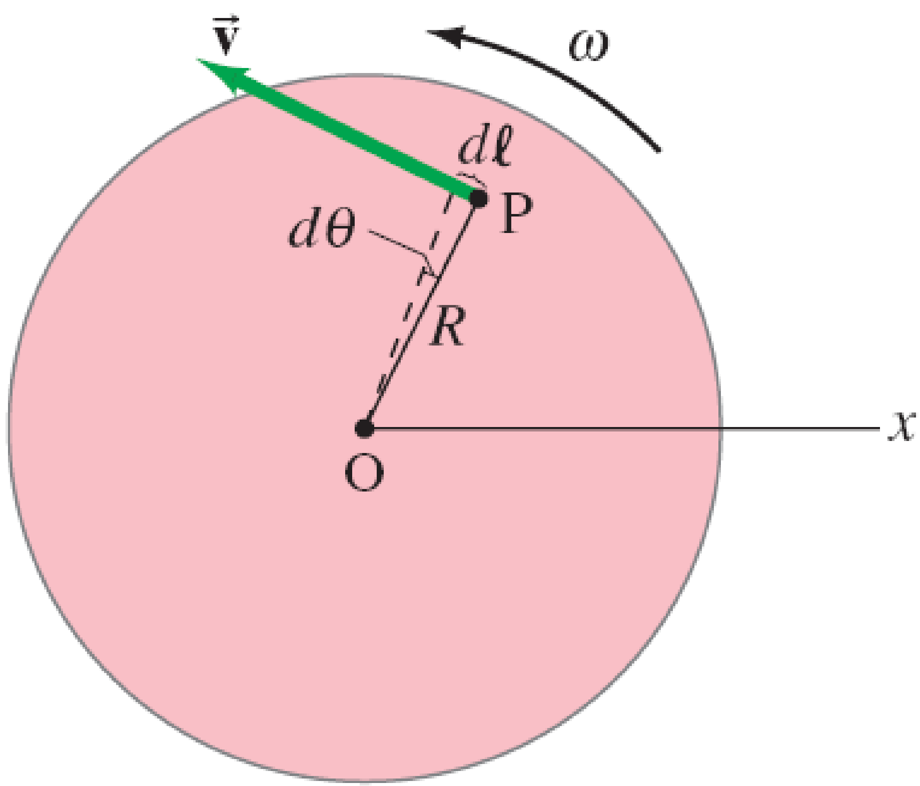
The **instantaneous** acceleration:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}.$$

10-1 Angular Quantities

Every point on a rotating body has an **angular velocity** ω and a **linear velocity** v.

They are related: $v = R\omega.$

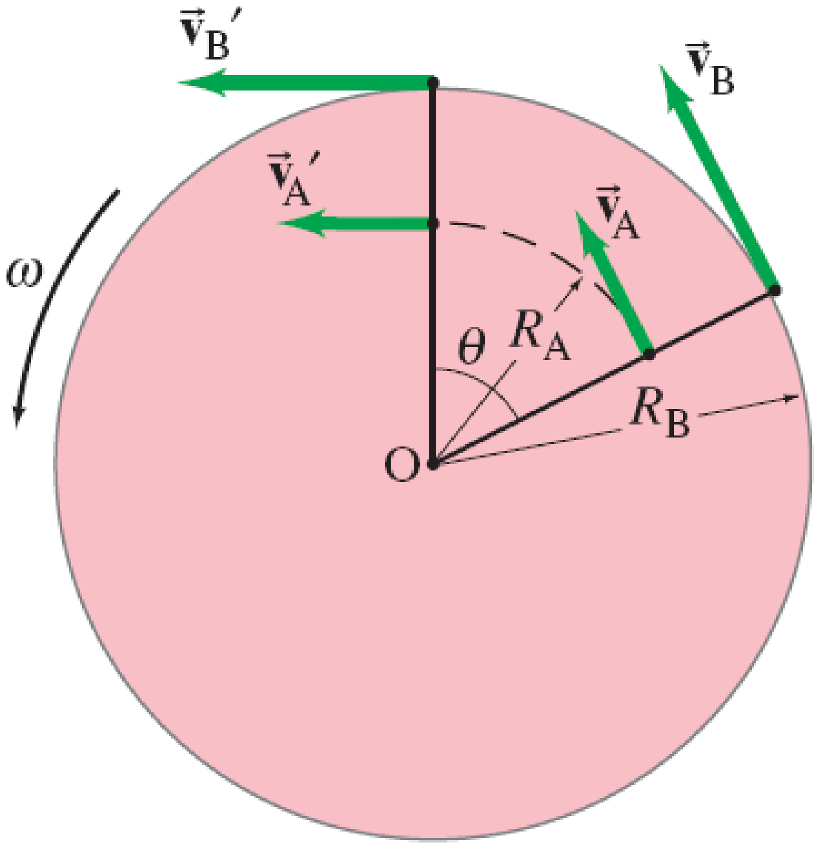


10-1 Angular Quantities

Conceptual Example 10-2: Is the lion faster than the horse?

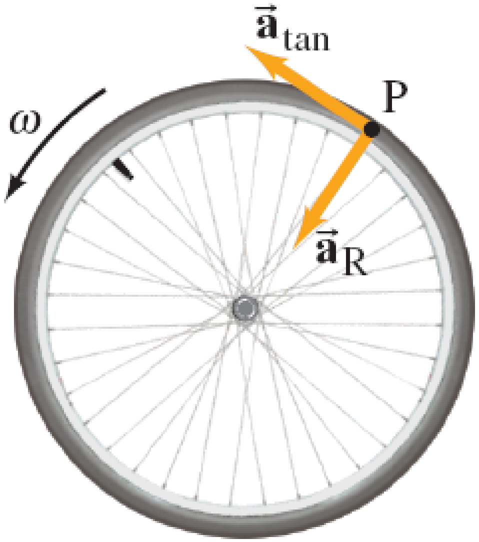
On a rotating carousel or merry-go-round, one child sits on a horse near the outer edge and another child sits on a lion halfway out from the center. (a) Which child has the greater linear velocity? (b) Which child has the greater angular velocity?

10-1 Angular Quantities



Objects farther from the axis of rotation will move faster.

10-1 Angular Quantities



If the angular velocity of a rotating object **changes**, it has a **tangential** acceleration:

$$a_{\tan} = \frac{dv}{dt} = R\frac{d\omega}{dt} = R\alpha.$$

Even if the angular velocity is constant, each point on the object has a **centripetal** acceleration:

$$a_R = \frac{v^2}{R} = \frac{(R\omega)^2}{R} = \omega^2 R.$$

10-1 Angular Quantities

Here is the **correspondence** between **linear** and **rotational** quantities:

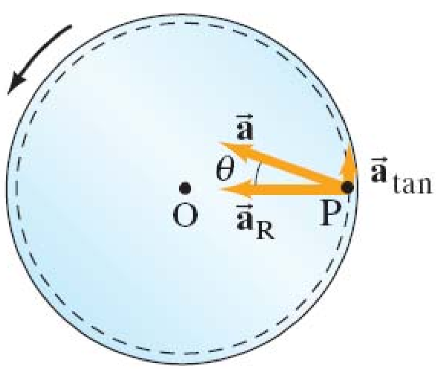
TABLE 10–1
Linear and Rotational Quantities

Linear	Type	Rotational	Relation (θ in radians)
x	displacement	θ	$x = R\theta$
v	velocity	ω	$v = R\omega$
$a_{\tan}$	acceleration	α	$a_{\tan} = R\alpha$

10-1 Angular Quantities

Example 10-3: Angular and linear velocities and accelerations.

A carousel is initially at rest. At $t = 0$ it is given a constant angular acceleration $\alpha = 0.060\ \mathrm{rad/s^2}$, which increases its angular velocity for 8.0 s. At $t = 8.0$ s, determine the magnitude of the following quantities: (a) the angular velocity of the carousel; (b) the linear velocity of a child located 2.5 m from the center; (c) the tangential (linear) acceleration of that child; (d) the centripetal acceleration of the child; and (e) the total linear acceleration of the child.



10-1 Angular Quantities

The **frequency** is the number of complete **revolutions** per second:

$$f = \frac{\omega}{2\pi}.$$

Frequencies are measured in **hertz**:

$$1\ \mathrm{Hz} = 1\ \mathrm{s}^{-1}.$$

The **period** is the time one revolution takes:

$$T = \frac{1}{f}.$$

10-1 Angular Quantities

Example 10-4: Hard drive.

The platter of the hard drive of a computer rotates at 7200 rpm (rpm = revolutions per minute = rev/min). (a) What is the angular velocity (rad/s) of the platter? (b) If the reading head of the drive is located 3.00 cm from the rotation axis, what is the linear speed of the point on the platter just below it? (c) If a single bit requires 0.50 μm of length along the direction of motion, how many bits per second can the writing head write when it is 3.00 cm from the axis?

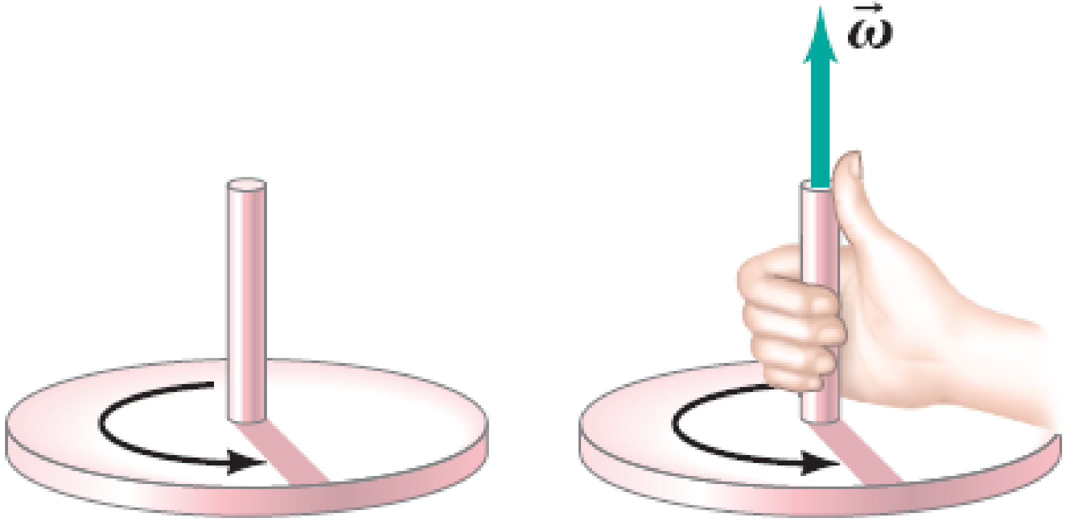
10-1 Angular Quantities

Example 10-5: Given ω as function of time.

A disk of radius R = 3.0 m rotates at an angular velocity ω = (1.6 + 1.2t) rad/s, where t is in seconds. At the instant t = 2.0 s, determine (a) the angular acceleration, and (b) the speed v and the components of the acceleration a of a point on the edge of the disk.

10-2 Vector Nature of Angular Quantities

The **angular velocity** vector points along the **axis** of rotation, with the direction given by the right-hand rule. If the direction of the rotation axis does not change, the **angular acceleration** vector points along it as well.



10-3 Constant Angular Acceleration

The equations of motion for **constant** angular acceleration are the same as those for **linear** motion, with the substitution of the **angular** quantities for the **linear** ones.

Angular	Linear
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$
$\overline{\omega} = \dfrac{\omega + \omega_0}{2}$	$\overline{v} = \dfrac{v + v_0}{2}$

10-3 Constant Angular Acceleration

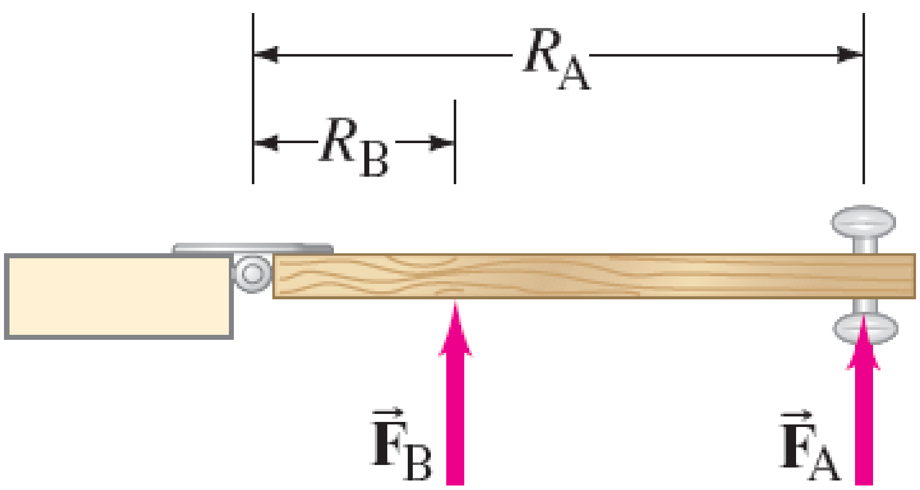
Example 10-6: Centrifuge acceleration.

A centrifuge rotor is accelerated from rest to 20,000 rpm in 30 s. (a) What is its average angular acceleration? (b) Through how many revolutions has the centrifuge rotor turned during its acceleration period, assuming constant angular acceleration?

10-4 Torque

To make an object **start** rotating, a **force** is needed; the **position** and **direction** of the force matter as well.

The **perpendicular** distance from the axis of rotation to the line along which the force acts is called the **lever arm**.

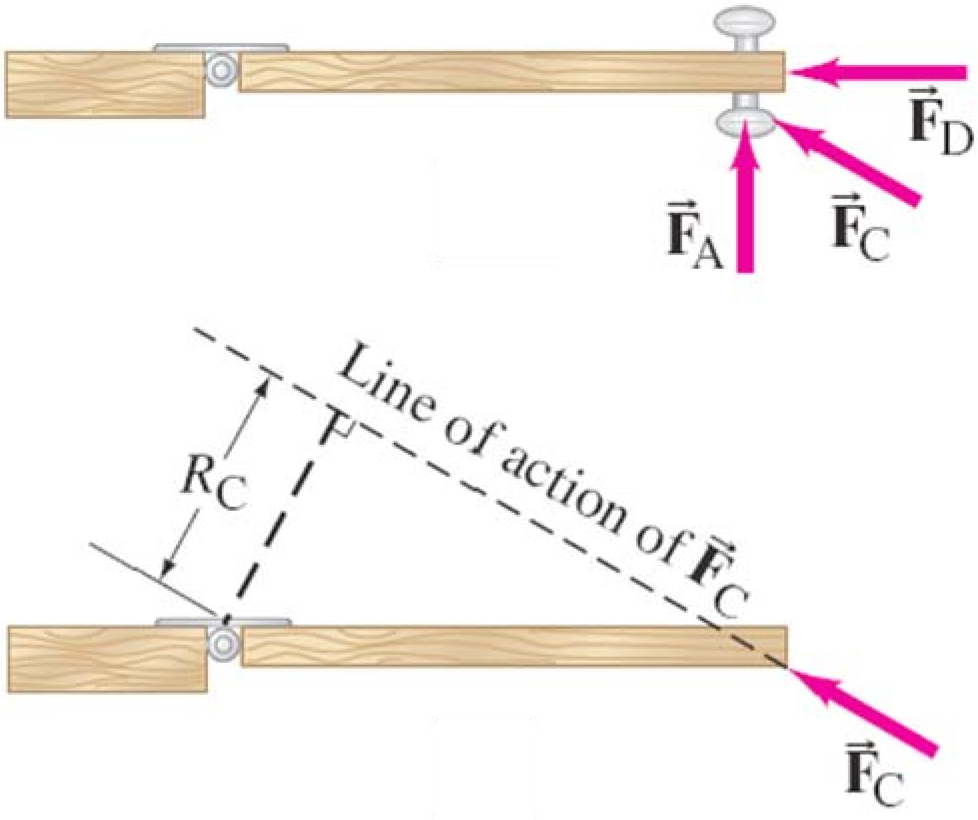


10-4 Torque

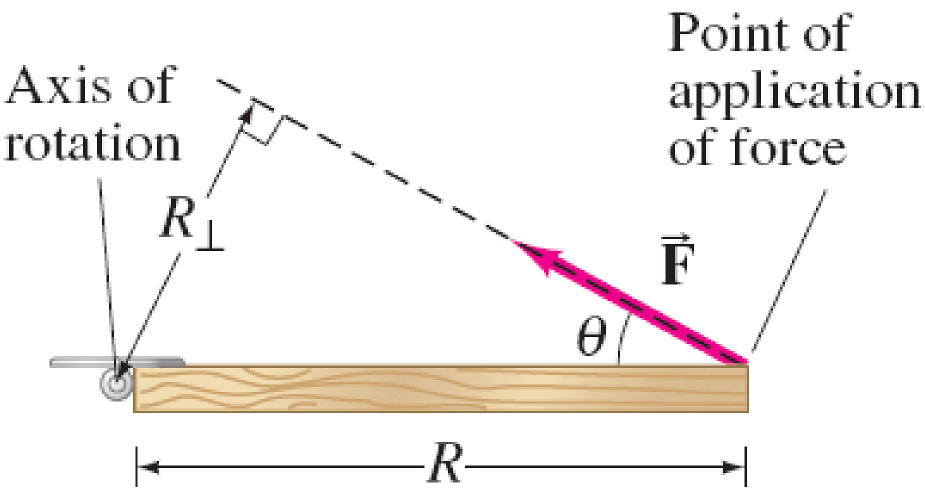
A **longer** lever arm is very helpful in rotating objects.

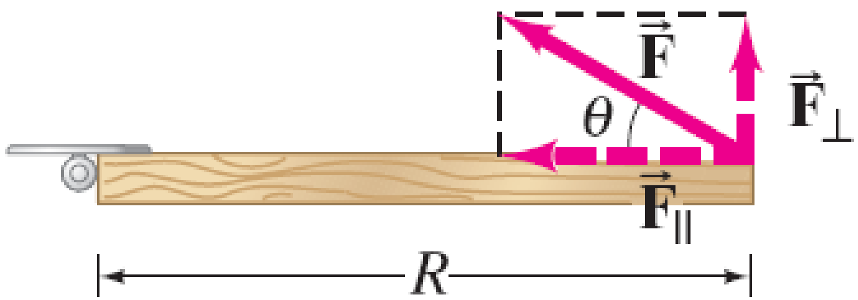
10-4 Torque

Here, the lever arm for F_A is the distance from the **knob** to the **hinge**; the lever arm for F_D is **zero**; and the lever arm for F_C is as shown.



10-4 Torque



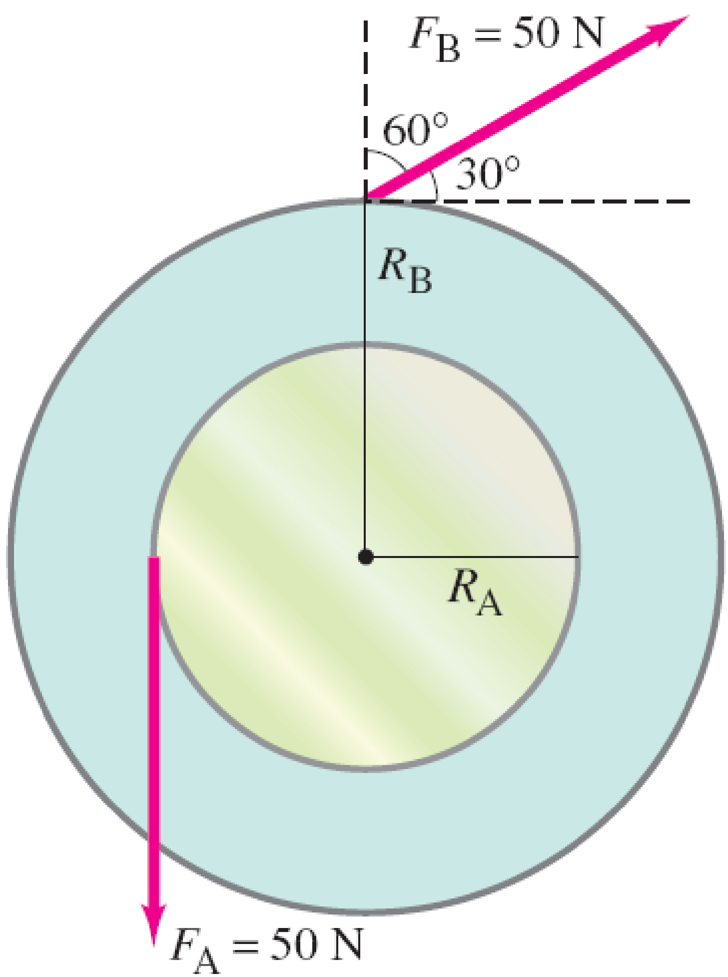


The torque is defined as:

$$\tau = R_\perp F.$$

10-4 Torque

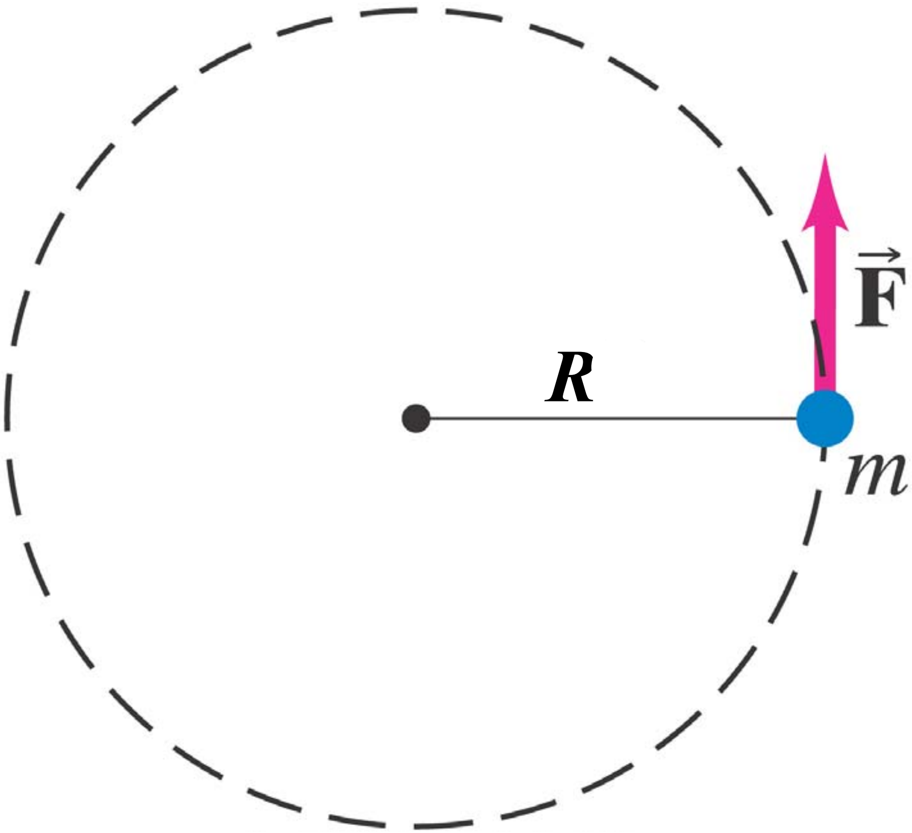
Example 10-7: Torque on a compound wheel.



Two thin disk-shaped wheels, of radii $R_A = 30$ cm and $R_B = 50$ cm, are attached to each other on an axle that passes through the center of each, as shown. Calculate the net torque on this compound wheel due to the two forces shown, each of magnitude 50 N.

10-5 Rotational Dynamics; Torque and Rotational Inertia

Knowing that $F = ma$, we see that $\tau = mR^2\alpha$.



This is for a single point mass; what about an extended object?

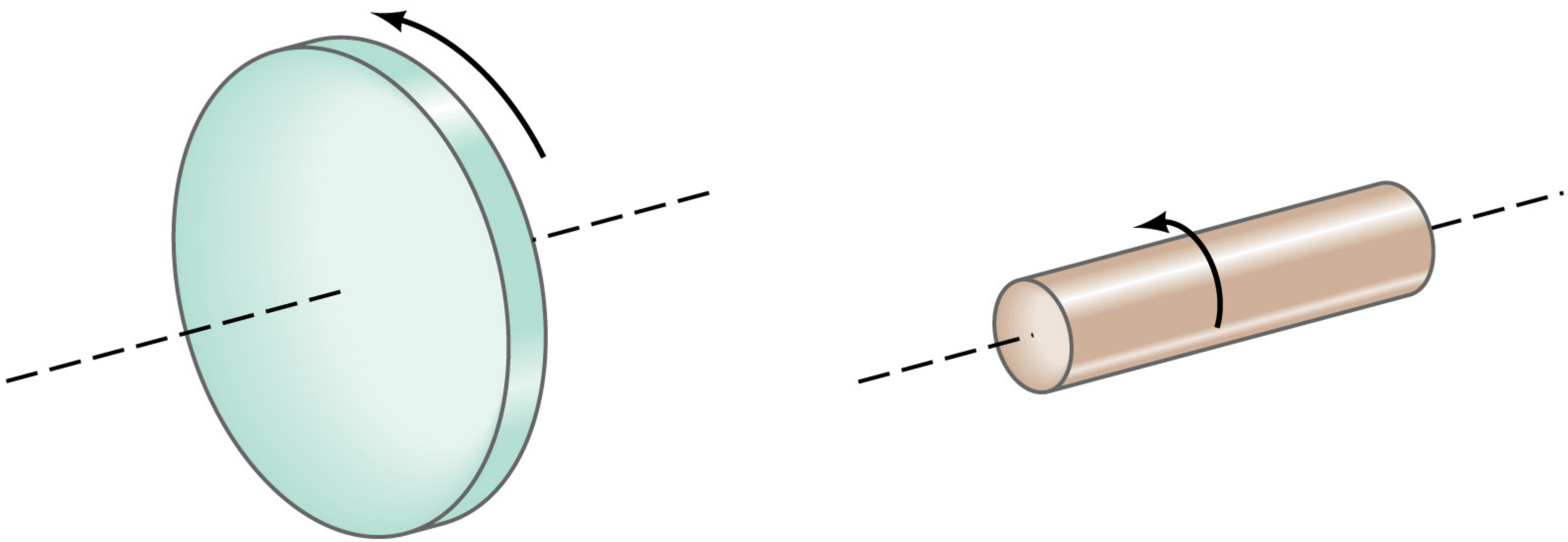
As the angular acceleration is the same for the whole object, we can write:

$$\Sigma\tau = \left(\Sigma mR^2\right)\alpha.$$

10-5 Rotational Dynamics; Torque and Rotational Inertia

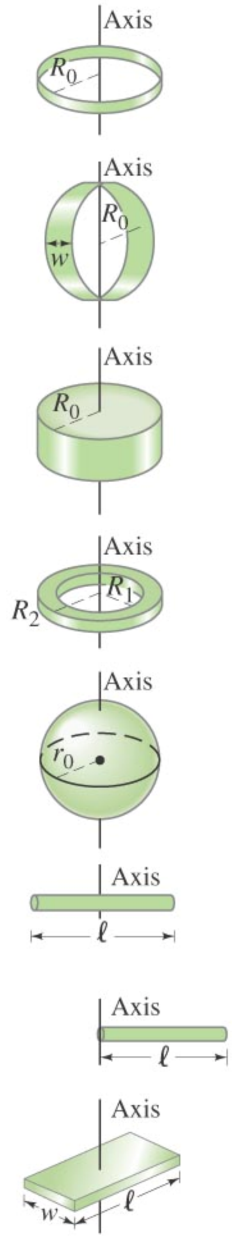
The quantity $I = \sum m_i R_i^2$ is called the **rotational inertia** of an object.

The **distribution** of mass matters here—these two objects have the same mass, but the one on the left has a greater **rotational inertia**, as so much of its mass is far from the axis of rotation.



10-5 Rotational Dynamics; Torque and Rotational Inertia

	Object	Location of axis	Moment of inertia
(a)	**Thin hoop,** radius R_0	Through center	MR_0^2
(b)	**Thin hoop,** radius R_0 width w	Through central diameter	$\frac{1}{2}MR_0^2 + \frac{1}{12}Mw^2$
(c)	**Solid cylinder,** radius R_0	Through center	$\frac{1}{2}MR_0^2$
(d)	**Hollow cylinder,** inner radius R_1 outer radius R_2	Through center	$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	**Uniform sphere,** radius r_0	Through center	$\frac{2}{5}Mr_0^2$
(f)	**Long uniform rod,** length ℓ	Through center	$\frac{1}{12}M\ell^2$
(g)	**Long uniform rod,** length ℓ	Through end	$\frac{1}{3}M\ell^2$
(h)	**Rectangular thin plate,** length ℓ, width w	Through center	$\frac{1}{12}M(\ell^2 + w^2)$



The **rotational inertia** of an object depends not only on its **mass distribution** but also the location of the **axis** of rotation—compare (f) and (g), for example.

10-6 Solving Problems in Rotational Dynamics

1. Draw a diagram.
2. Decide what the system comprises.
3. Draw a free-body diagram for each object under consideration, including all the forces acting on it and where they act.
4. Find the axis of rotation; calculate the torques around it.

10-6 Solving Problems in Rotational Dynamics

5. Apply Newton's second law for **rotation**. If the **rotational inertia** is not provided, you need to find it **before** proceeding with this step.

6. Apply Newton's second law for **translation** and other laws and principles as needed.

7. **Solve**.

8. **Check** your answer for units and correct order of magnitude.

10-7 Determining Moments of Inertia

If a physical object is available, the moment of inertia can be measured experimentally.

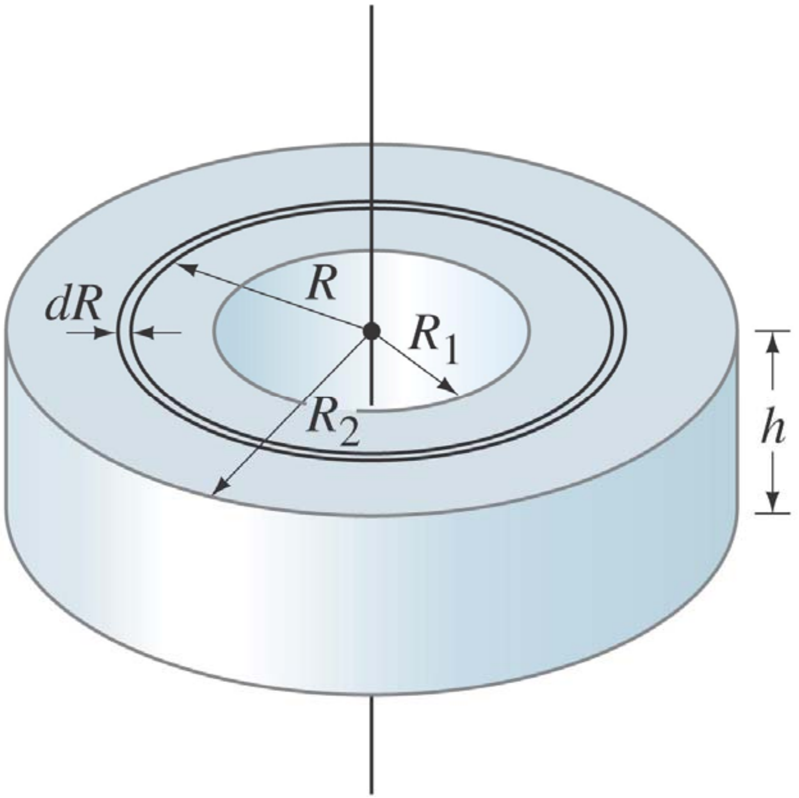
Otherwise, if the object can be considered to be a continuous distribution of mass, the moment of inertia may be calculated:

$$I = \int R^2 \, dm.$$

10-7 Determining Moments of Inertia

Example 10-12: Cylinder, solid or hollow.

(a) Show that the moment of inertia of a uniform hollow cylinder of inner radius R_1, outer radius R_2, and mass M, is $I = \frac{1}{2} M(R_1^2 + R_2^2)$, if the rotation axis is through the center along the axis of symmetry. (b) Obtain the moment of inertia for a solid cylinder.



10-7 Determining Moments of Inertia

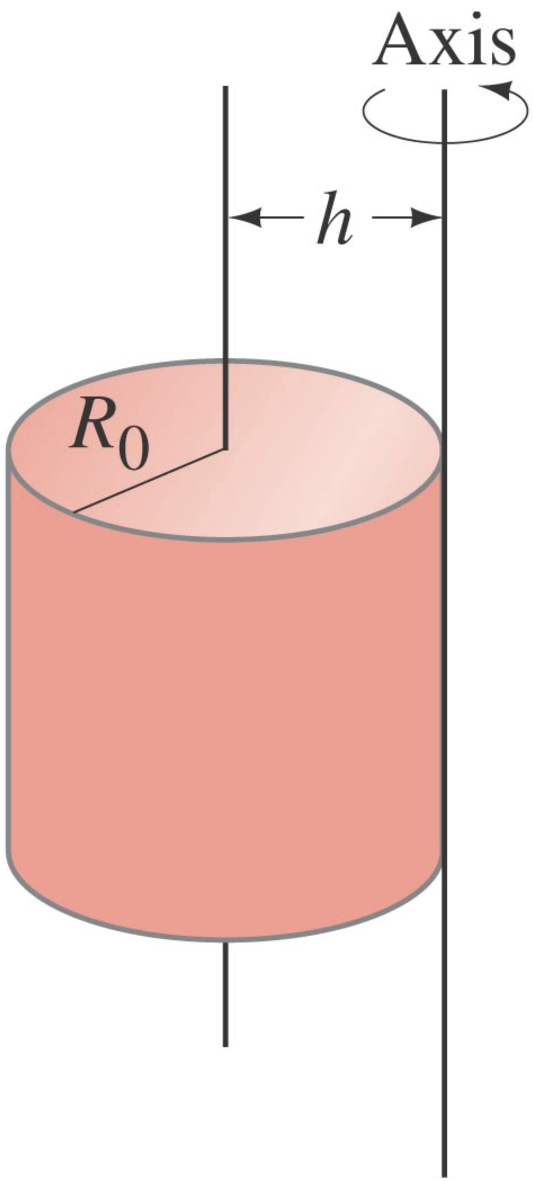
The parallel-axis theorem gives the moment of inertia about any axis parallel to an axis that goes through the center of mass of an object:

$$I = I_{CM} + Mh^2.$$

10-7 Determining Moments of Inertia

Example 10-13: Parallel axis.

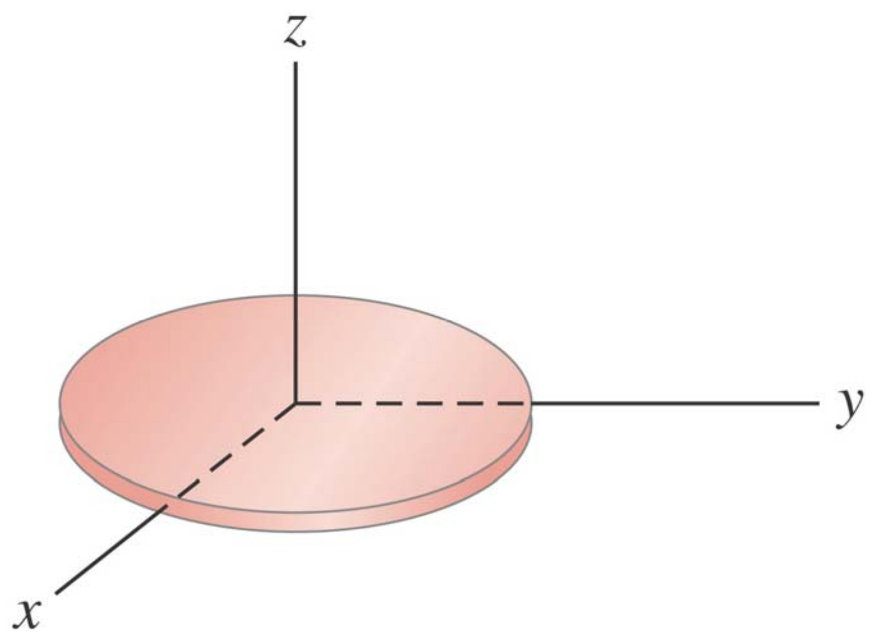
Determine the moment of inertia of a solid cylinder of radius R_0 and mass M about an axis tangent to its edge and parallel to its symmetry axis.



10-7 Determining Moments of Inertia

The perpendicular-axis theorem is valid only for flat objects.

$$I_z = I_x + I_y .$$



10-8 Rotational Kinetic Energy

The **kinetic energy** of a rotating object is given by

$$K = \Sigma\left(\tfrac{1}{2}mv^2\right).$$

By substituting the rotational quantities, we find that the rotational kinetic energy can be written:

$$\text{rotational } K = \tfrac{1}{2}I\omega^2.$$

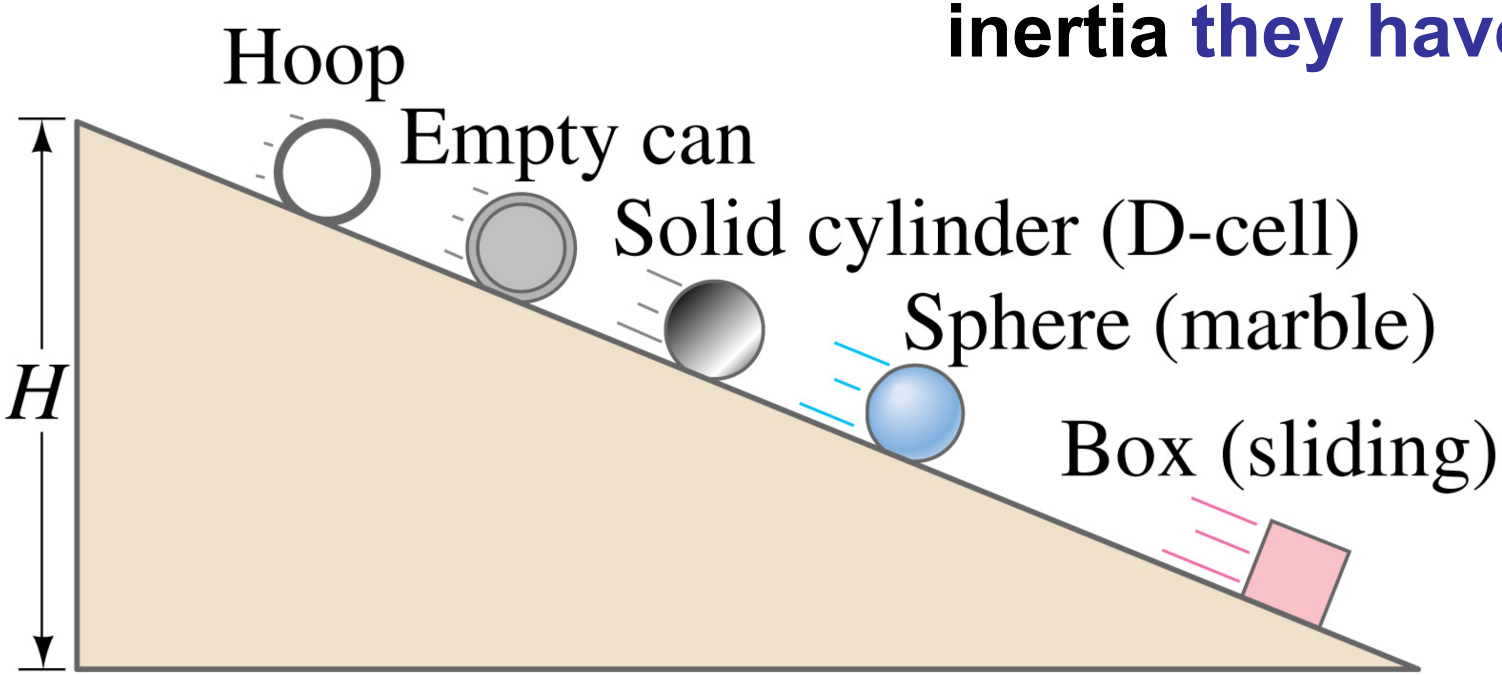
A object that both **translational** and **rotational** motion also has both translational and rotational **kinetic energy**:

$$K = \tfrac{1}{2}Mv_{\text{CM}}^2 + \tfrac{1}{2}I_{\text{CM}}\omega^2.$$

10-8 Rotational Kinetic Energy

When using **conservation** of energy, both **rotational** and **translational** kinetic energy must be taken into account.

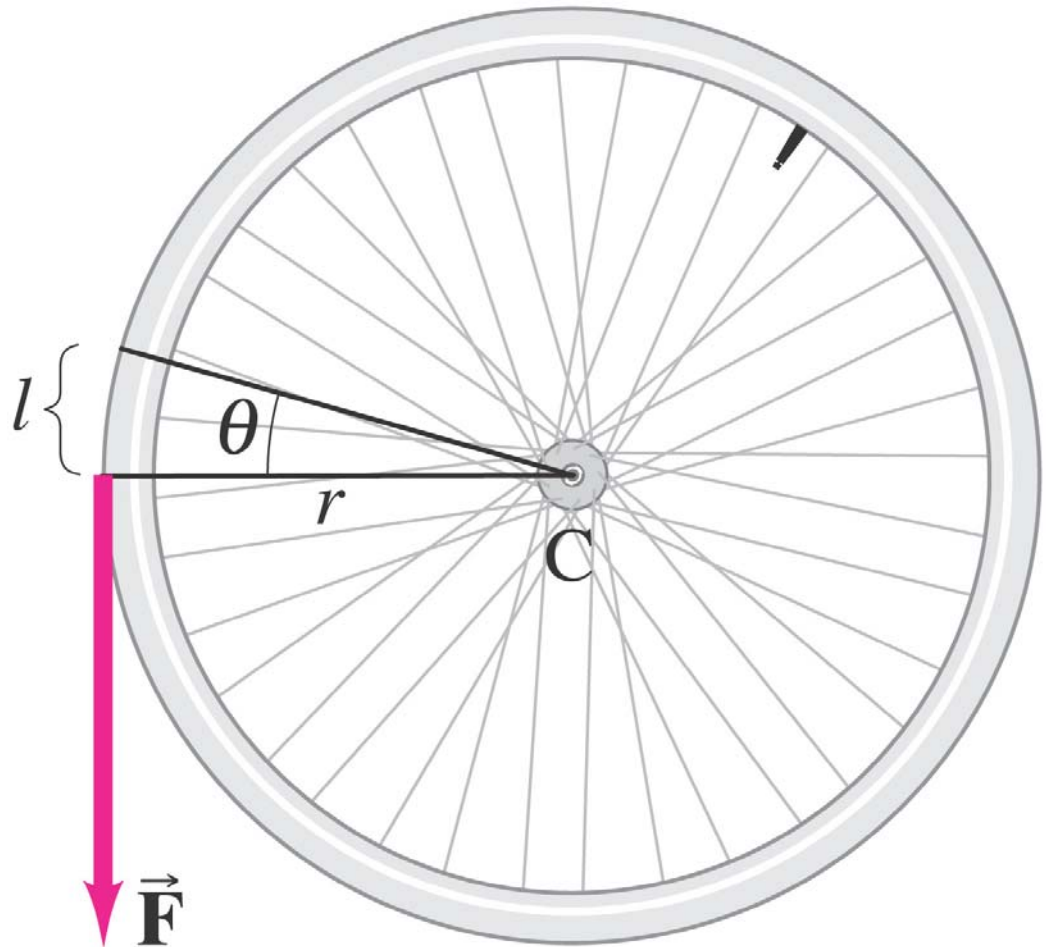
All these objects have the same **potential** energy at the top, but the time it takes them to get down the incline depends on how much **rotational inertia** they have.



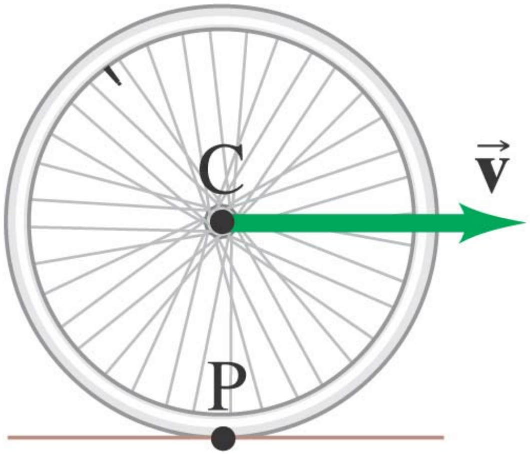
10-8 Rotational Kinetic Energy

The torque does **work** as it moves the wheel through an angle θ:

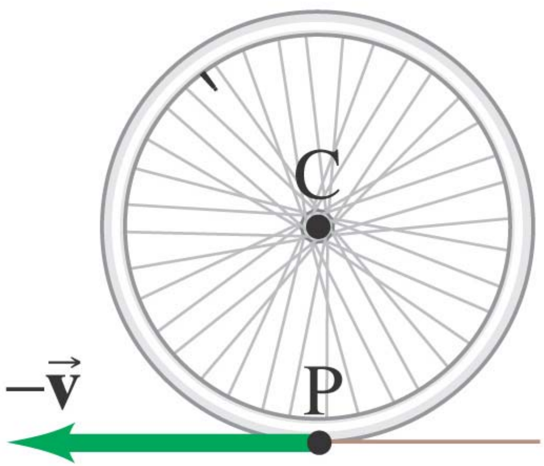
$$W = \tau \Delta \theta.$$



10-9 Rotational Plus Translational Motion; Rolling



In (a), a wheel is **rolling** without slipping. The point P, touching the ground, is instantaneously at **rest**, and the center moves with velocity $\vec{v}$.



In (b) the same wheel is seen from a **reference frame** where C is at rest. Now point P is moving with velocity $-\vec{v}$.

The **linear speed** of the wheel is related to its **angular speed**:

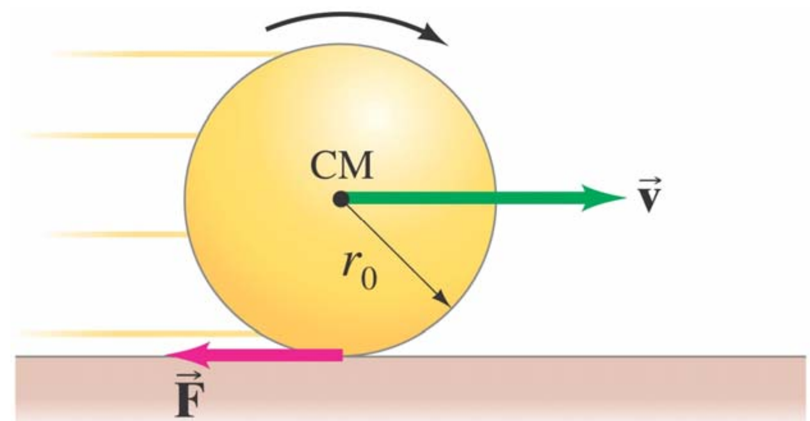
$$v = R\omega.$$

10-10 Why Does a Rolling Sphere Slow Down?

A rolling sphere will slow down and stop rather than roll forever. What force would cause this?

If we say "friction", there are problems:

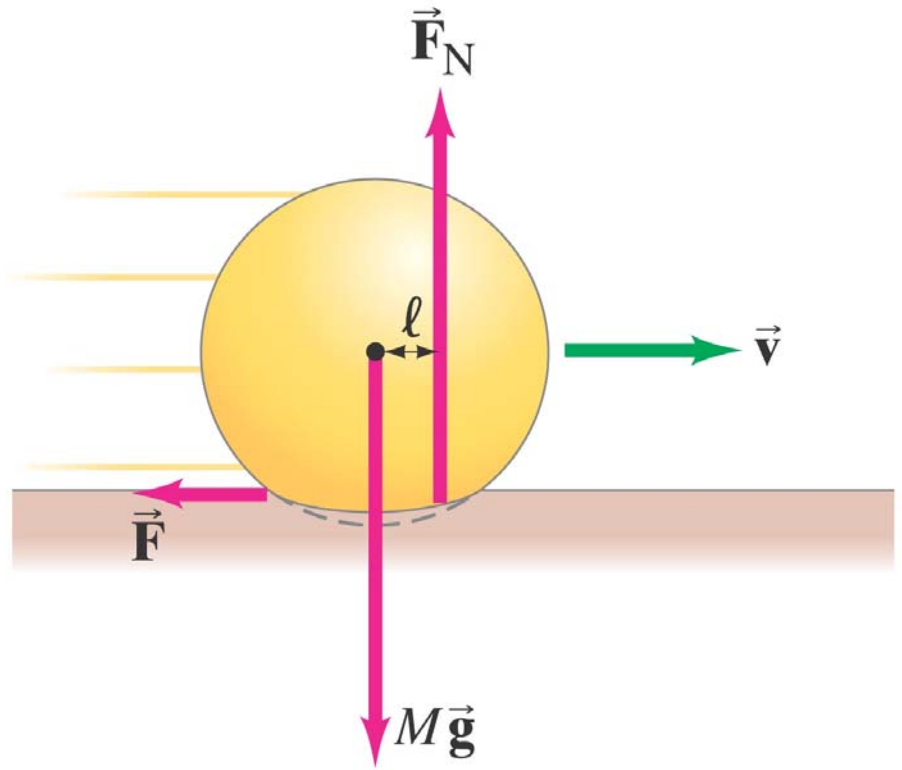
- The frictional force has to act at the point of contact; this means the angular speed of the sphere would increase.



- Gravity and the normal force both act through the center of mass, and cannot create a torque.

10-10 Why Does a Rolling Sphere Slow Down?

The solution: No real sphere is perfectly rigid. The bottom will deform, and the normal force will create a torque that slows the sphere.



Summary of Chapter 10

- Angles are measured in radians; a whole circle is 2π radians.
- Angular velocity is the rate of change of angular position.
- Angular acceleration is the rate of change of angular velocity.
- The angular velocity and acceleration can be related to the linear velocity and acceleration.
- The frequency is the number of full revolutions per second; the period is the inverse of the frequency.

Summary of Chapter 10, cont.

- The equations for rotational motion with constant angular acceleration have the same form as those for linear motion with constant acceleration.
- Torque is the product of force and lever arm.
- The rotational inertia depends not only on the mass of an object but also on the way its mass is distributed around the axis of rotation.
- The angular acceleration is proportional to the torque and inversely proportional to the rotational inertia.

Summary of Chapter 10, cont.

- An object that is rotating has rotational kinetic energy. If it is translating as well, the translational kinetic energy must be added to the rotational to find the total kinetic energy.
- Angular momentum is $L = I\omega$.
- If the net torque on an object is zero, its angular momentum does not change.

Formula Sheet

Charles Duan Aaron Lee

1 Kinematics

Velocity-distance relation under constant acceleration. Given an initial velocity v_0 and a distance d:

$$v^2 = {v_0}^2 + 2ad$$

Projectile motion distance:

$$d = \frac{v^2 \sin 2\theta}{g}$$

Center of mass:

$$\mathbf{r}_{\text{CM}} = \frac{1}{M} \sum_i \mathbf{r}_i m_i = \frac{1}{M} \int \mathbf{r} dm$$

2 Relativity

2.1 Kinematics

Let A be the "fixed" observer and B an observer moving with velocity v relative to A.

Loss of simultaneity. For two clocks a distance L apart in A's frame that are reading the same time in that frame, the "rear" clock from B's point of view will be faster by a factor of:

$$\Delta t = \frac{Lv}{c^2}$$

Beta and gamma factors:

$$\beta = \frac{v}{c},\ \gamma = \frac{1}{\sqrt{1-\beta^2}},\ \gamma > 1$$

Time dilation and length contraction:

$$t_B = \gamma t_A;\ L_B = \frac{L_A}{\gamma}$$

Velocity addition. Say A observes a motion of velocity v_A, then the velocity with respect to B is:

$$v_B = \frac{v_A + v}{1 + v v_A / c^2}$$

Lorentz transformations. Given that A has a coordinate system of (x, y, z, t), the coordinate system for B is (x', y, z, t') where:

$$\Delta x = \gamma(\Delta x' + v \Delta t'),\ \Delta t = \gamma \left(\Delta t' + \frac{v}{c^2} \Delta x' \right)$$

Time-space invariant. Given, for two events:

$$\Delta s^2 \equiv c^2 \Delta t^2 - \Delta x^2$$

The value Δs^2 is the same in any frame of reference.

2.2 Dynamics

We are given an observer and some system of mass m moving at a speed v.

Momentum and energy:

$$\mathbf{p} = \gamma m \mathbf{v},\ E = \gamma m c^2$$

Energy-momentum relations:

$$m^2 c^4 = E^2 - p^2 c^2,\ \frac{p}{E} = \frac{v}{c^2}$$

Energy/momentum for photons:

$$E = pc$$

Lorentz transformations for energy. Given a frame of reference moving with speed v, that measures for a system E' and p', we find in the nonmoving frame:

$$E = \gamma(E' + vp'),\ p = \gamma \left(p' + \frac{v}{c^2} E' \right)$$

Relativistic force:

$$F = \gamma^3 m a = \frac{dp}{dt} = \frac{dE}{dx}$$

3 Rotational Motion

Rolling without slipping:

$$\alpha = \frac{a}{r},\ \omega = \frac{v}{r}$$

Moment of inertia:

$$I = \sum m_i {r_i}^2 = \int r^2 dm$$

Parallel axis theorem. Given an axis of rotation parallel to an axis through the center of mass:

$$I_p = I_{\text{CM}} + M d^2$$

Perpendicular axis theorem. Given a *planar* object, with the $\mathbf{z}$ axis normal to it:

$$I_z = I_x + I_y$$

Definition of torque:

$$\vec{\tau} = \mathbf{r} \times \mathbf{F};\ \tau = rF\sin\theta$$

Torque and angular acceleration:

$$\sum \tau = I\alpha$$

Definition of angular momentum:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \int (\mathbf{r} \times \mathbf{v})dm;\ L = rp\sin\theta$$

Torque and angular momentum. Given a center of rotation that is fixed either in an inertial frame or on the center of mass:

$$\vec{\tau} = \frac{d\mathbf{L}}{dt}$$

Angular momentum and velocity. For a system that is only rotating about a single axis:

$$L = I\omega$$

Angular impulse. In a system where a force is applied at a constant distance r from the point of rotation:

$$\Delta L = r\Delta p$$

Translation and rotation. Given an object with angular momentum $\mathbf{L}'$ about its own center of mass, the angular momentum about any other center is:

$$L = M\mathbf{R}_{\text{CM}} \times \mathbf{v}_{\text{CM}} + \mathbf{L}'$$

4 Harmonic Motion

Given a differential equation of the form $y'' = -\omega^2 y$, the solution will be:

$$y = A\cos(\omega t + \phi)$$

The constant ω is the angular frequency. The period T and frequency ν are:

$$T = \frac{2\pi}{\omega},\ \nu = \frac{1}{T} = \frac{\omega}{2\pi}$$

For a spring, $\omega = \sqrt{k/m}$; for a pendulum, $\omega = \sqrt{g/l}$. For a physical pendulum with moment of inertia *at the pivot* a distance d from the CM, $\omega = \sqrt{mgd/I}$.

Damped motion. Consider a damping force $F = -bv$ and a harmonic force $F = kx$. Then define:

$${\omega_0}^2 = \frac{k}{m},\ \gamma = \frac{b}{2m};\ \omega'^2 = {\omega_0}^2 - \gamma^2,\ \Omega^2 = \gamma^2 - {\omega_0}^2$$

There are three possible cases:

Under	$\gamma < \omega_0$	$x(t) = Ae^{-\gamma t}\cos(\omega' t + \phi)$
Over	$\gamma > \omega_0$	$x(t) = Ae^{-(\gamma+\Omega)t} + Be^{-(\gamma-\Omega)t}$
Critical	$\gamma = \omega_0$	$x(t) = e^{-\gamma t}(A + Bt)$

Driven oscillation. In addition to the damping $-bv$ and harmonic kx forces, consider a driving force $F_d(t) = F\cos\omega_d t$:

$$x(t) = \frac{F}{mR}\cos(\omega_d t - \phi)$$

with the following constants:

$$R^2 = \left({\omega_0}^2 - {\omega_d}^2\right)^2 + \left(2\gamma\omega_d\right)^2,\ \tan\phi = \frac{2\gamma\omega_d}{{\omega_0}^2 - {\omega_d}^2}$$

5 Universal Gravitation

Newton's Law of gravitation:

$$F = -\frac{Gm_1m_2}{r^2} \text{ where } G = 6.6726 \times 10^{-11}\frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{kg}^2}$$

The units of G can also be $\mathrm{m}^2\mathrm{kg}^{-1}\mathrm{s}^{-2}$.

Gravitational potential:

$$U(r) = -\frac{Gm_1m_2}{r}$$

Kepler's Laws. The planets move in elliptical orbits, they sweep out equal areas over equal times, and for an orbit with semimajor axis a and period T:

$$T^2 = \frac{4\pi^2a^3}{GM_{\text{sun}}}$$

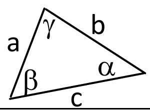
6 Fictitious Forces

In an accelerated reference frame $\mathbf{R}$ with rotation $\vec{\omega}$, the force on an object is the sum of the real forces on it and the following "fictitious forces":

$$\begin{aligned} \text{Translational:}&\ -m\frac{d^2\mathbf{R}}{dt^2} \\ \text{Centrifugal:}&\ -m\vec{\omega} \times (\vec{\omega} \times \mathbf{r}) \\ \text{Coriolis:}&\ -2m\vec{\omega} \times \mathbf{v} \\ \text{Azimuthal:}&\ -m\frac{d\vec{\omega}}{dt} \times \mathbf{r} \end{aligned}$$

Cascarano **Formula Sheet** Physics 4A

Math Stuff		**Constants**	
Scalar Product	$\vec{A}\cdot\vec{B} = \lvert\vec{A}\rvert\lvert\vec{B}\rvert\cos\theta$	Acceleration of gravity	g = 9.80 m/s^2
Vector Product	$\vec{A}\times\vec{B} = \lvert\vec{A}\rvert\lvert\vec{B}\rvert\sin\theta$	Universal gravitation	G = 6.67x10^{-11} Nm^2/kg^2
Quadratic formula (ax^2+bx+c=0)	$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$	Mass of Earth Radius of Earth	$M_E = 5.97x10^{24}kg$ $R_E = 6.38x10^6 m$

Kinematics	**Trig Identities**	**Newton's Laws / Friction / Circular motion**	
$v_x = v_{ox} + a_x t$	Law of cosines:	2nd Law	$\sum\vec{F} = m\vec{a}$ or $\sum\vec{F} = \frac{d\vec{p}}{dt}$
$x = x_o + v_{ox}t + \frac{1}{2}a_x t^2$	$a^2 = b^2 + c^2 - 2bc\cos\alpha$	Gravitation	$F = \dfrac{Gm_1m_2}{r^2}$
$v_x^2 = v_{ox}^2 + 2a_x(x-x_o)$	Law of sines:	Static friction	$f_s \le \mu_s n$
$x - x_o = \left(\dfrac{v_{ox}+v_x}{2}\right)t$	$\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$	Kinetic friction	$f_k = \mu_k n$
Velocity Addition		Centripetal acceleration	$a_c = \dfrac{v^2}{r}$
$\vec{v}_{AB} = \vec{v}_{AC} + \vec{v}_{CB}$			

Work & Power		**Energy**	
Work by constant force	$W = \vec{F}\cdot\vec{s}$	Kinetic energy	$K = \frac{1}{2}mv^2$
Work by varying force	$W = \int_{x_1}^{x_2} F_x\,dx$	Gravitational potential energy	$U_g = mgy = -\dfrac{GmM}{r}$
Conservative forces	$W = -\Delta U$	Elastic potential energy	$U_{el} = \frac{1}{2}kx^2$
Non-conservative forces	$\Delta U_{int} = f_k d$	Hooke's Law	$F = kx$
Power	$P = \dfrac{dW}{dt}$	Force from potential energy	$F_x = -\dfrac{\partial U}{\partial x}$

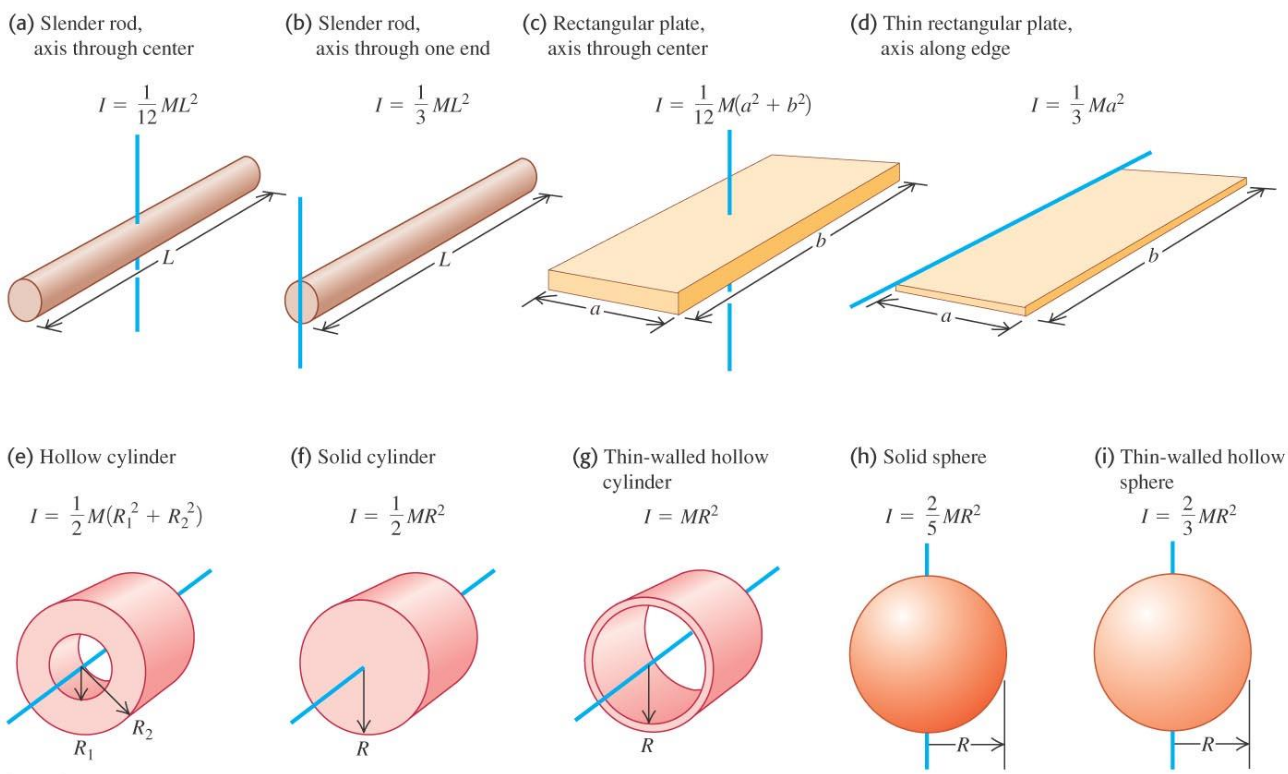
Impulse and Momentum		**Center of Mass**	
Linear momentum	$\vec{p} = m\vec{v}$	Center of mass (point objects)	$x_{CM} = \dfrac{\sum_n m_n x_n}{\sum_n m_n}$
Impulse	$\vec{J} = \int_{t_1}^{t_2}\sum\vec{F}dt = \left(\sum\vec{F}\right)_{ave}\Delta t$	Center of mass (solid objects)	$x_{CM} = \dfrac{\int xdm}{\int dm}$
Impulse - momentum	$\vec{J} = \Delta\vec{p}$	Total momentum	$\vec{p}_{total} = M_{total}\vec{v}_{CM}$ $\sum\vec{F}_{ext} = M_{total}\vec{a}_{CM}$

Rotational Kinematics	**Linear / Angular**	**Rotational Inertia and Energy**	
$\omega = \omega_o + \alpha t$	$s = r\theta$ $(\theta\ in\ radians)$	Rotational inertia (point objects)	$I = \sum_i m_i {r_i}^2$
$\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$	$v = r\omega$	Rotational inertia (solid objects)	$I = \int r^2\,dm$
$\omega^2 = \omega_o^2 + 2\alpha(\theta-\theta_o)$	$a_{tan} = r\alpha$	Parallel axis theorem	$I_P = I_{CM} + Md^2$
$\theta - \theta_o = \left(\dfrac{\omega_o+\omega}{2}\right)t$	$a_{rad} = \dfrac{v^2}{r}$	Rotational kinetic energy	$K = \frac{1}{2}I\omega^2$

Torque and Angular Momentum		**Rotational Motion**	
Torque	$\vec{\tau} = \vec{r}\times\vec{F} = rFsin\emptyset = Fl$	Newton's 2nd law	$\sum\vec{\tau} = I\vec{\alpha} = \frac{d\vec{L}}{dt}$
Angular momentum	$\vec{L} = \vec{r}\times\vec{p} = mvrsin\emptyset = I\vec{\omega}$	Work	$W = \int_{\theta_1}^{\theta_2}\tau d\theta = \Delta K_{rot}$
		Power	$P = \tau\omega$

Cascarano **Formula Sheet** Physics 4A

Simple Harmonic Motion		Angular Frequency	
Displacement	$x(t) = Acos(\omega t + \emptyset)$	Mass on a spring	$\omega = \sqrt{\frac{k}{m}}$
Velocity	$v(t) = -\omega Asin(\omega t + \emptyset)$	Simple pendulum	$\omega = \sqrt{\frac{g}{l}}$
Period	$T = \frac{2\pi}{\omega}$	Frequency	$f = \frac{1}{T} = \frac{\omega}{2\pi}$



(j) Thin hoop rotating on axis through any diameter of the hoop:

$$I = \frac{1}{2}MR^2$$