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SPACETIME CURVATURE

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Preface to the first edition

After some discussion, mostly via internet, I recognized that many searchers being interested in my papers do not have an easy access to my mathematical expositions. Gradually the amount of articles in my electronic journal 'Austrian Reports on Gravitation' has proliferated. For a better understanding of these articles, once more it seems to be necessary to present the application area clearly arranged and to treat the partly newly developed mathematics in greater detail. I hope to facilitate the access to my articles for the interested reader and to convey my own views to the gravitation theory.

Obritz, January 2009

Preface to the second edition

Since the completion of the first edition some problems have been expounded in more detail and further results have been added. They have been included in this new edition of the booklet.

Obritz, March 2011

Preface to the third edition

The booklet has been expanded to include a few sections on the free fall. A new addition is a section on the gravitational collapse.

Obritz, July 2013

Preface to the fourth edition

The mathematically related sections of the expanding universe and the gravitational collapse have been revised. The interior Schwarzschild solution has been enhanced with a collapsing version. Some sections have been complemented.

Obritz, November 2015

Preface to the fifth edition

Some supplements have been inserted. The section 'Cosmology' has been extended and quotations have been updated.

Obritz, February 2016, Rainer Burghardt

Contents

I. Introduction

I.1. What is the meaning of space-time curvature?

It was one of the greatest performances of the first decades of the last century to find a mathematical description of gravitational appearances that explains several observations in the universe in a satisfactory way and to make predictions that could be proved by measurement[1]. The mathematical formulation of the gravitation theory is founded on the principle of Riemannian geometry. It enhances the theory of curved surfaces to the theory of curved spaces. The meaning of the term curvature has changed during the development of gravitation theory.

In this booklet, we analyze what possibilities one has to interpret the space-time curvature and we show which consequences will have different interpretations for the formulation of the gravitation theory.

In the literature, one finds different views on the curvature of space-time.

I. The term curvature is used for any space with non-Euclidean geometry. A higher dimensional space as embedding space for our 4-dimensional world is not assumed to exist. This view has the advantage that solutions of the Einstein field equations can be discussed for which no embeddings are possible or for which no such embeddings have been found yet. This concept of curvature has been impressively formulated by Whittaker [W]: *"…, unhappily it has become important historically, for it has led to the creation of a terminology which is now so well established that we can never hope to change it, regrettable though it is, and which has been responsible for a great deal of popular misconception. The terminology in question is, that mathematicians apply the word 'curved' to any space whose geometry is not Euclidean. It is an unfortunate custom, because curvature, in the sense of bending, is a meaningless term except when the space is immersed in another space, whereas the property of being non-Euclidean is an intrinsic property which has nothing to do with immersion. However, nothing can be done but to utter a warning that what mathematicians understand by the term 'curvature' is not was the word connotes in ordinary speech: what the mathematician means is simply that the relations between the mutual distances of the points are different from the relations which obtain in Euclidean geometry. Curvature (in the mathematical sense) has nothing to do with the shape of the space – whether it is bent or not – but is defined solely by the metric, that is to say, the way in which 'distance' is defined. It is not the space that is curved, but the geometry of the space"*.

II. We can describe the geometry of a gravitational model by embedding a curved space into a higher dimensional flat space. However, such a procedure is not necessary, because the curvature can be described solely by the intrinsic properties of the embedded space.

III. The curvature is explained by embeddings of surfaces into a higher dimensional flat space. The advantage of this methodology is that all tools of the differential

[1] The perihelion shift of the Mercury and the deflection of light by the Sun.

geometry – the theorems of Gauss and Codazzi – can be used. Thus, one gains a new insight into the structure of the gravitational models.

The view described in (I) is almost minimalist. However, there are some arguments to define the term curvature literally. In the following, it will be judged how far one can go the way from (I) to (III). Moreover, it will be shown that the most important gravitational models admit the view (III), and that new solutions of the Einstein field equations can be found with this methodology of embedding. With the assumption that embeddings are possible, it is not presumed that a higher dimensional space exists in reality [2].

Our ability of imagination is limited to four dimensions and there will hardly be empirical methods that will tend to a detection of physically real extra dimensions. By the embedding procedure, we understand a mathematical method that facilitates the presentation of gravitation physics.

At the early time of gravitation theory one proceeded from the assumption that the curvature of space could be taken literally. Models of the world like the Einstein and the de Sitter universe were understood as a spherical space embedded into a 5-dimensional flat space. One tried to judge the circumference of this hyper-sphere. Once being convinced that we are living in a closed world, they claimed that light rays emitted by us would return to us after going all the way round the universe. Early searchers have believed in the possibility of observing a 'ghost sun', the other side of our sun.

As we try to give to a literally interpretation of curvature in this booklet, we firstly present cosmological models in terms of this view.

[2] In modern Kaluza-Klein and string theories, physical reality is assigned to more than one extra dimension. To make them invisible they are rolled up to the Planck length.

I.2. Mathematical basics

In essence, the gravitation theory is a theory of recording processes of measurements. In the environment of masses, space and time are measured by the use of rods and clocks. In a gravitational field the ratios of the radii and circumferences of a circle are not conform to the ones of the Euclidean geometry. Clocks approaching gravitating masses slow down. The results of measurements could be tabulated. This is a suitable method preceding a theory[3].

One can also try to present the data by formulae. The main idea of Einstein's gravitation theory has been and is to derive such formulae from the Riemannian geometry, a geometry founded on the curvature of space. The most basic process of measuring in a gravitational field is the measuring of distances s in space-time. To determine where and when something has to be measured, one has to cover the 4-dimensional space with co-ordinate lines. The points of intersection of the co-ordinate lines tag the positions where one measures. Because of the difficulties to measure finite distances in a curved space, one divides them into infinitesimal small segments ds and integrates the results. The independence of these results of different methods of measurement – the invariance of the line element ds – together with the 4-dimensional ansatz makes the gravitation theory a relativistic theory.

The theory of curved spaces will be demonstrated by a simple example. A co-ordinate system is set up in a 3-dimensional flat space. A vector x pointing from the origin of the co-ordinate system to an arbitrary point has the projections $x^{i'} = \left\{x^{1'}, x^{2'}, x^{3'}\right\}$ onto the three axes of the co-ordinate system. The joining vector from this point to an infinitesimally near neighboring point is $dx^{i'}$. The square of length is[4]

$$ds^2 = dx^{i'}dx^{i'} \ . \tag{I.2.1}$$

ds is the above-mentioned line element. For being prepared for later problems, it will be presented in spherical co-ordinates $\{r, \vartheta, \varphi\}$. The transformation

$$\begin{aligned} x^{1'} &= r\sin\vartheta\sin\varphi \\ x^{2'} &= r\sin\vartheta\cos\varphi \\ x^{3'} &= r\cos\vartheta \end{aligned} \tag{I.2.2}$$

is set up in such a way to be able to be extended to more dimensions later on. With the relation

$$x^{i'}x^{i'} = r^2 \tag{I.2.3}$$

a family of spheres with radii r is defined. Differentiating (I.2.2) and inserting into (I.2.1) one obtains the line element in spherical co-ordinates

$$ds^2 = dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta\, d\varphi^2 \ . \tag{I.2.4}$$

With the *co-ordinate differentials*

$$dx^i = \left\{dr, d\vartheta, d\varphi\right\} \tag{I.2.5}$$

the two-rank metric tensor g_{ik} is diagonal. Its matrix only contains the values

[3] Tycho Brahe has set up such tables for the motion of planets. It was a work of decades.

[4] It is summed over double indices

$$g_{11}=1, \quad g_{22}=r^2, \quad g_{33}=r^2\sin^2\vartheta \tag{I.2.6}$$

The line element

$$ds^2 = g_{ik}dx^i dx^k \tag{I.2.7}$$

arises in a general form. Defining differentials in the following way

$$dx^m = \left\{dr, rd\vartheta, r\sin\vartheta d\varphi\right\} \tag{I.2.8}$$

one gets for the line element the relation

$$ds^2 = g_{mn}dx^m dx^n \ , \tag{I.2.9}$$

whereby the matrix of the metric tensor is diagonal 1. There are no co-ordinates x^m from which the differentials dx^m could be derived[5]. In the following the sequence of indices {i, j, k, l} refer to a *co-ordinate system* while the {m, n,…, t} enumerate the components of a quantity with respect to a *reference system*. The geometrical meaning of the definitions is obvious. In both cases dr is an infinitesimal small step outwards into the radial direction. $d\vartheta$ and $d\varphi$ are increments on the unit circles of the spherical system. However, $rd\vartheta$ is an increment on a great circle of the surface of a sphere with the radius r = const. and $r\sin\vartheta d\varphi$ is an increment on a parallel of a sphere with the radius $r\sin\vartheta$. As the latter definition refers to the considered point and an infinitesimally near neighboring point, these definitions are chiefly used in this booklet. The increments on the circles just mentioned are the lengths of the arcs and because of their infinitesimal smallness also the lengths of the tangent vectors. The relation of both differentials can be to epitomized with the expression

$$dx^m = \overset{m}{e}_i dx^i \tag{I.2.10}$$

if the new quantities e take the already known values

$$\overset{1}{e}_1 = 1, \quad \overset{2}{e}_2 = r, \quad \overset{3}{e}_3 = r\sin\vartheta \ . \tag{I.2.11}$$

These are three vectors enumerated by the index[6] m. They are tangent to the co-ordinate lines. Because of the elementariness of the model, each of these vectors has only one component. They are mutual orthogonal and set up a 3-bein (triad)[7]. With the relation

$$\overset{m}{e}_i \underset{n}{e}^i = \delta^m_n, \quad \overset{m}{e}_i \underset{m}{e}^k = \delta^k_i \tag{I.2.12}$$

one can ascribe to the tangent vectors gradient vectors $\underset{n}{e}^i$, which are normal to the co-ordinate surfaces and also normal to the particular tangent vectors. The partial derivatives with respect to the co-ordinates have the form

$$\partial_i = \frac{\partial}{\partial x^i} = \left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\varphi}\right\} \ , \tag{I.2.13}$$

but with the respect to the reference systems

$$\partial_m = \frac{\partial}{\partial x^m} = \left\{\frac{\partial}{\partial r}, \frac{\partial}{r\partial\vartheta}, \frac{\partial}{r\sin\vartheta\partial\varphi}\right\}. \tag{I.2.14}$$

They are related by

[5] The dx^m are called anholonomic. Mostly, of the nonholonomicity of geometry it is made use by us.

[6] Such indices are called dead indices, the indices of the sequence i living indices.

[7] Later on 4-beins, (tetrads) and 5-beins (pentads) are introduced.

notation: $$\partial_m = \underset{m}{e}^{i}\partial_i, \quad \partial_i = \overset{m}{e}_i \partial_m \ . \tag{I.2.15}$$

With (I.2.12) one easily obtains

$$\underset{1}{e}^{1} = 1, \quad \underset{2}{e}^{2} = \frac{1}{r}, \quad \underset{3}{e}^{3} = \frac{1}{r\sin\vartheta} \ . \tag{I.2.16}$$

The operation with the bein vectors one can apply to arbitrary vectors

$$\Phi_m = \underset{m}{e}^{i}\Phi_i, \quad \Phi_i = \overset{m}{e}_i \Phi_m, \quad \Phi^m = \overset{m}{e}_i \Phi^i, \quad \Phi^i = \underset{m}{e}^{i}\Phi^m \ . \tag{I.2.17}$$

It has to be underlined that in the co-ordinate representation there is a difference between lower and upper indices[8]

$$\Phi_i = g_{ik}\Phi^k, \quad \Phi^i = g^{ik}\Phi_k . \tag{I.2.18}$$

In the bein representation both kinds of components coincide

$$\Phi_m = \Phi^m . \tag{I.2.19}$$

This considerably simplifies the calculations. We prefer this notation on other grounds, too. The components of a vector with respect to a reference system are the measured values of a quantity. We note that the metric can be decomposed in several ways into bein vectors

$$g_{ik} = \overset{m}{e}_i \overset{m}{e}_k = \overset{m'}{e}_i \overset{m'}{e}_k = \dots \ . \tag{I.2.20}$$

The various bein vectors can be converted by the transformation

$$\overset{m'}{e}_i = L^{m'}_{m} \overset{m}{e}_i \ . \tag{I.2.21}$$

Later on, we elaborately make use of this transformation. In the 4-dimensional space the quantities L will be interpreted as the matrix of a Lorentz transformation. In the curved space the transition from a tetrad system to another system corresponds to a rotation in the tangent space that is to say to a change to a usually accelerated system. Reference systems referring to a distinct state of motion of an observer are called preferred reference systems. In our simple model all bein vectors are aligned, they are tangent to the co-ordinate lines and unambiguously determined in the whole space. For several gravitational models this alignment is not demanded for all bein vectors. If one fully dismisses the connection with a co-ordinate system one has gained new degrees of freedom in dependence on the numbers of dimensions. The local rotation of a reference system can explain the action of a further force. In such a way Einstein [E] and Einstein [E] and Mayer had tried to unify the theory of gravitation and electromagnetism. This idea was taken up by several authors in modern theories:

Bokhari [B] and Qadir; Borzeszkowski [B] and Treder; Chao [C] and Kohler; Charap [C], Chee [C], Cheng [C], Chern and Nester; Chinea [C], Dadhich [D], Davis [D], Davis [D] and York; Deser [D] and Isham; Edgar [E], Estabrook [E], Gatha [G] and Dutt; Green [G], Henneaux [H], Hwang [H], Kasper [K], Kibble [K], Kämpfer [K], Lanczos [L], Lathrop [L], Maluf [M], Rocha-Neteo and Toribio; Marsh [M], Meyer [M], Mitskiewicz [M] and Nesterov; Møller [M], Müller [M] and Nitsch; Nester [N], Obukhov [O] and Pereira; Oliveira [O] and Teixeira; Papapetrou [P] and Stachel; Pirani [P], Saez [S], Saez [S] and Arnau; Saez [S] and Juan; Sauer [S], Schrödinger [S], Tamm [T], Thomas [T], Treder

[8] The change of the position of indices is called dragging of indices. The g^{ik} are reciprocal to the g_{ik} .

[T], Treder [T] and Borzeszkowski; Tupper [T], Tupper [T] and Phillips; Urani [U] and Kutchko; Weitzenböck [W], Wigner [W], Woodard [W], Zaycoff [Z], Zet [Z], Zhang [Z].

I.3. Covariant differentiation

The tensor calculus presented in the preceding Section and explained by a simple example will be the basic mathematical tool for the gravitational problems treated in this booklet and will be extended in more detail. The independency of the mathematical description of the basic quantities is an essential presupposition for a physically meaningful theory. If a vector Φ is represented in a co-ordinate system (i') and is transformed as

$$\Phi_{i'} = A^{i}_{i'}\Phi_{i}$$

the square of the quantity

$$\Phi^2 = \Phi_i\Phi^i = \Phi_{i'}\Phi^{i'}$$

has to have the same value after has been transformed, that is the quantity has to be invariant under this transformation.

If a vector is displaced along a curve, its increment has to vanish. By the use of Cartesian co-ordinates (i') this requirement reads as

$$d\Phi_{i'} = 0 \ .$$

One realizes that this simple relation is not valid in non-Cartesian co-ordinates (i):

$$A^{i'}_{i} d\Phi_{i'} = d\Phi_{i} - A^{j}_{i'} dA^{i'}_{i}\Phi_{j} \ .$$

The new term has to be compensated by an additional term. This is established by the introduction of the Christoffel symbols

$$\Gamma^{j}_{ki} = \frac{1}{2} g^{jl} g_{\{ik|l\}} \ , \tag{I.3.1}$$

wherein the strokes between the indices denote the partial derivatives

notation: $$\Phi_{|k} = \partial_k \Phi \ , \tag{I.3.2}$$

and the abbreviation

notation: $$\Phi_{\{ikl\}} = \Phi_{lik} + \Phi_{kli} - \Phi_{ikl} \tag{I.3.3}$$

is used. Bach's brackets symmetrize and antisymmetrize a tensor with respect to its indices

notation: $$\Phi_{(mn)} = \frac{1}{2}\left(\Phi_{mn} + \Phi_{nm}\right), \quad \Phi_{[mn]} = \frac{1}{2}\left(\Phi_{mn} - \Phi_{nm}\right), \quad \Phi_{(m \cdot s \cdot n)} = \frac{1}{2}\left(\Phi_{msn} + \Phi_{nsm}\right), \tag{I.3.4}$$

$$\Phi_{[msn]} = \frac{1}{6}\left(\Phi_{msn} + \Phi_{nms} + \Phi_{snm} - \Phi_{smn} - \Phi_{mns} - \Phi_{nsm}\right), \quad \Phi_{<msn>} = \left(\Phi_{msn} + \Phi_{nms} + \Phi_{snm}\right) \ . \tag{I.3.5}$$

The Christoffel symbols are symmetric with respect to their lower indices. They are not tensors because they transform inhomogeneously. That means their transformation law contains the above-mentioned additional term that corresponds to the additional term of the transformation equation of the partial derivatives. Operating with a co-ordinate transformation on (I.3.1) one obtains some further terms containing derivatives of the transformation matrix. Using the holonomic co-ordinates[9] (i) and (i') the transformation matrix can be written as

[9] If we dismiss the requirement for holonomicity the transformation law of the Christoffel symbols is enriched by the object of anholonomicity. This extension of differential geometry is not used here.

$$A^{i'}_{i} = x^{i'}_{|i} \ .$$

As a consequence of the commutability of the partial derivatives with respect to the holonomic co-ordinate system one obtains the transformation law for the Christoffel symbols

$$\Gamma^{j}_{ki} = A^{k'i'j}_{k\,i\,j'}\Gamma^{j'}_{k'i'} + A^{j}_{j'}A^{j'}_{i|k} \ . \qquad \text{(I.3.6)}$$

Thus, the condition for using arbitrary co-ordinate systems (i) for a parallel transport of a vector has the form

notation: $$\Phi_{i||k} = \Phi_{i|k} - \Gamma^{j}_{ki}\Phi_{j} = 0 \ , \qquad \text{(I.3.7)}$$

wherein the double stroke denotes the *covariant derivative* of a vector. We refer to the customary literature for a detailed study. Finally, we only are interested in the way how to formulate such a transport law by the use of a reference system instead of a co-ordinate system. For this purpose the relation (I.2.17) is applied to (I.3.7)

$$\Phi_{m||n} = \underset{m}{e}^{i}\,\underset{n}{e}^{k}\,\Phi_{i||k} = \Phi_{m|n} - \overset{s}{e}_{j}\,\underset{m|n}{e}^{j}\,\Phi_{j} - \underset{m}{e}^{i}\,\underset{n}{e}^{k}\,\overset{s}{e}_{j}\,\Gamma^{j}_{ki}\Phi_{s} \ . \qquad \text{(I.3.8)}$$

Inserting (I.2.20) into the Christoffel symbols (I.3.1) one obtains the law of transport

notation: $$\Phi_{m||n} = \Phi_{m|n} - A_{nm}{}^{s}\Phi_{s} \ . \qquad \text{(I.3.9)}$$

The quantities

$$A_{mn}{}^{s} = \overset{s}{e}_{j}\,\underset{[n|m]}{e}^{j} + g^{sr}g_{nt}\overset{t}{e}_{j}\,\underset{[m|r]}{e}^{j} - g^{sr}g_{mt}\overset{t}{e}_{j}\,\underset{[r|n]}{e}^{j} \qquad \text{(I.3.10)}$$

are called Ricci-rotation coefficients. They are antisymmetric in their last indices

$$A_{m(ns)} = 0 \ . \qquad \text{(I.3.11)}$$

With the help of (I.2.17) one finds the partial derivatives to be not commutative with respect to the reference system

$$\Phi_{|[mn]} = A_{[n\,m]}{}^{s}\Phi_{|s} \ . \qquad \text{(I.3.12)}$$

Use is repeatedly made by this rule. In addition, it follows from (I.3.9)

$$\underset{[m||n]}{e}^{i} = 0 \ . \qquad \text{(I.3.13)}$$

One reads the connection between the Christoffel symbols and the Ricci-rotation coefficients from the comparison from (I.3.8) and (I.3.9)

$$A_{nm}{}^{s} = \underset{n}{e}^{k}\underset{m}{e}^{i}\,\overset{s}{e}_{j}\,\Gamma^{j}_{ki} + \overset{s}{e}_{j}\,\underset{m|n}{e}^{j} \ . \qquad \text{(I.3.14)}$$

Γ and A differ by an additional term. Therefore, they are two different geometrical objects.

If we continue with the simple model of the surface of a sphere, it becomes clear that one can assign fundamental geometrical elements to the Ricci-rotation coefficients, but not to the Christoffel symbols. The Ricci-rotation coefficients supply directly the measured values of these elements, while the components of the Christoffel symbols have different values depending upon index position. The connection of the latter with the geometrical elements and finally with physical quantities is only intricately ascertainable[10].

[10] With the benefit of hindsight we can say it would have been more appropriate, if Einstein had not learned geometry with Grossmann, but with Levi-Civita and Ricci. Grossmann presented the geometry of curved

We apply to (I.3.10) a change of the reference system with the transformation

$$\overset{s'}{e}_i = L_s^{s'} \overset{s}{e}_i \ .$$

This leads to new quantities

$$\begin{gathered} 'A_{m'n'}{}^{s'} = \overset{s'}{e}_j \, e^j_{[n'|m']} + g^{s'r'} g_{n't'} \overset{t'}{e}_j \, e^j_{[m'|r']} - g^{s'r'} g_{m't'} \overset{t'}{e}_j \, e^j_{[r'|n']} \ , \\ A_{m'n'}{}^{s'} = L^{m\,n\,s'}_{m'n's} A_{mn}{}^{s}, \quad L_{m'n'}{}^{s'} = L^{s'}_{s} L^{s}_{n'|m'} \end{gathered} \tag{I.3.15}$$

whereby firstly

$$'A_{m'n's'} = A_{m'n's'} + L_{[m'n']s'} + L_{[s'm']n'} - L_{[n's']m'}$$

is obtained. If one limits oneself to transformations which again produce measurable quantities, that is if one transforms an orthogonal bein vector system into another orthogonal bein vector system, one obtains by use of $L_s^{s'} = L_{s'}^{s}$ the simpler expression

$$'A_{m'n'}{}^{s'} = A_{m'n'}{}^{s'} + L_{m'n'}{}^{s'} \ . \tag{I.3.16}$$

Our simple model is finally extended to four dimensions by inclusion of the time. Doing so, we will identify the transformation L with the Lorentz transformation and will treat in detail the transformation law again.

It is to be demonstrated, how the developed calculus can be applied to the introductory spherical model. From the Ricci-rotation coefficients (3.10) one computes with (I.2.11), (I.2.16), and (I.2.14)

$$\begin{aligned} A_{21}{}^{2} &= -A_{22}{}^{1} = -\overset{2}{e}_2 \, e^2_{2|1} = \frac{1}{r} \\ A_{31}{}^{3} &= -A_{33}{}^{1} = -\overset{3}{e}_3 \, e^3_{3|1} = \frac{1}{r} \\ A_{32}{}^{3} &= -A_{33}{}^{2} = -\overset{3}{e}_3 \, e^3_{3|2} = \frac{1}{r}\cot\vartheta \end{aligned} \ .$$

If one defines unit tangent vectors by dividing (I.2.11) by the associated arc-lengths of the curve, one obtains

$$m_m = \{1,0,0\}, \quad b_m = \{0,1,0\}, \quad c_m = \{0,0,1\} \ . \tag{I.3.17}$$

Decomposition of the Ricci-rotation coefficients into

$$\begin{gathered} A_{mn}{}^{s} = B_{mn}{}^{s} + C_{mn}{}^{s}, \quad B_{mn}{}^{s} = b_m B_n b^s - b_m b_n B^s, \quad C_{mn}{}^{s} = c_m C_n c^s - c_m c_n C^s \\ B_m = \left\{\frac{1}{r},0,0\right\}, \quad C_m = \left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0\right\} = \left\{\frac{1}{r\sin\vartheta}\sin\vartheta,\frac{1}{r\sin\vartheta}\cos\vartheta,0\right\} \end{gathered} \tag{I.3.18}$$

leads to a systematic representation of the spherical geometry. These quantities are made use of by the computation of the field equations of this geometry. The quantities B and C describe the curvatures of a family of spheres with the parameter r. r and $r\sin\vartheta$ are the curvature radii of the normal slices (great circles) and inclined slices (parallel circles) of the

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spaces in the co-ordinate way of writing, Levi-Civita and Ricci, however, recognized the importance of reference systems. Today both representations are common. This parallelism of the ways of writing makes the study and the advancement of the gravitation theory more difficult.

family of the spheres. The fact that the curvatures are invariant properties of the sphere will have to be examined in relation to the transformation law (**I**.3.15).

I.4. Curvature tensors

In this Section we presuppose knowledge of Riemannian geometry in a co-ordinate way of writing and we restrict ourselves to translating well-known formulae into the bein vector way of writing. That the partial derivatives of a quantity are not commutative with respect to a reference system we have underlined at the way to the formula (I.3.11). This non-commutability is valid for the covariant derivatives, as well. By the use of (I.3.9) and (I.3.12) one arrives at[11]

$$\Phi_{m||[nr]} = \frac{1}{2} R_{nrm}{}^{s} \Phi_{s}, \quad \Phi^{m}{}_{||[nr]} = -\frac{1}{2} R_{nrs}{}^{m} \Phi^{s} \ . \qquad \text{(I.4.1)}$$

If one transports a vector along a closed curve on a curved surface up to its starting point, then this vector does not coincide with its initial image. One computes the difference with the preceding formulae. From the Riemann curvature tensor

$$R_{smn}{}^{r} = 2\left[A_{[m\cdot n\cdot}{}^{r}{}_{|s]} + A_{[m\cdot n\cdot}{}^{t} A_{s]t}{}^{r} + A_{[ms]}{}^{t} A_{tn}{}^{r} \right] \qquad \text{(I.4.2)}$$

one obtains the Ricci-tensor by contraction

$$R_{mn} = A_{mn}{}^{s}{}_{|s} - A_{n|m} - A_{sm}{}^{r} A_{rn}{}^{s} + A_{mn}{}^{s} A_{s}, \quad A_{s} = A_{rs}{}^{r} \qquad \text{(I.4.3)}$$

and the curvature invariant

$$R = -2A^{s}{}_{|s} - A_{snr} A^{rns} - A_{s} A^{s} \ . \qquad \text{(I.4.4)}$$

The last term in (I.4.2) does not arise in the co-ordinate way of writing of the Riemann tensor. This term has its origin in the non-commutability (I.3.12) of the partial derivatives with respect to the reference system. The Riemann tensor has the following properties

$$R_{(sm)nr} = 0, \quad R_{sm(nr)} = 0, \quad R_{smnr} = R_{nrsm}, \quad R_{[smn]r} = 0, \quad R_{[smnr]} = 0 \ . \qquad \text{(I.4.5)}$$

Further, the Bianchi identity is applicable

$$R_{[sm}{}^{nr}{}_{||p]} = 0 \ . \qquad \text{(I.4.6)}$$

By contracting Eq. (I.4.6) twice one can show that the *Einstein tensor* is free from divergence

$$\left[R_{m}{}^{n} - \frac{1}{2} \delta_{m}^{n} R \right]_{||n} = 0 \ . \qquad \text{(I.4.7)}$$

This relation finally leads to the conservation law of matter. Since the introductory model is flat, the Riemann tensor vanishes identically and thus

$$R_{mn} \equiv 0$$

is satisfied. From this and from (I.3.10) and (I.3.18) the relations

$$B_{m\underset{2}{||}n} + B_{m} B_{n} = 0, \quad C_{m\underset{3}{||}n} + C_{m} C_{n} = 0, \quad B^{s}{}_{\underset{2}{||}s} + B^{s} B_{s} = 0, \quad C^{s}{}_{\underset{3}{||}s} + C^{s} C_{s} = 0 \qquad \text{(I.4.8)}$$

are deduced wherein the graded covariant derivatives [B] are used:

notation: $$B_{m\underset{2}{||}n} = B_{m|n}, \quad C_{m\underset{3}{||}n} = C_{m|n} - B_{nm}{}^{s} C_{s} \ . \qquad \text{(I.4.9)}$$

[11] The following equations can be deduced either from the analogous co-ordinate relation by means of (I.2.17) or straightforwardly by (I.3.9).

This guarantees that a quantity which is situated in a (n-m)-dimensional subspace, is situated again in this subspace after parallel transport. The graded covariant derivatives have the nice properties that the unit vectors satisfy the relations

$$b_{\underset{2}{m||n}} = 0, \quad c_{\underset{3}{m||n}} = 0 \ . \tag{I.4.10}$$

These properties will substantially simplify the computation of the Einstein field equations. The benefit of the graded derivatives seems to be minimal in the context of our introductory example and the way of writing is somewhat obscure. However, the use of these graded derivatives enables us to work out the structures (I.4.8) from the field equations in the framework of the higher-dimensional gravitation models. All those field equations are of the type

$$\frac{\partial}{\partial r}\frac{1}{r} + \frac{1}{r^2} = 0 \,. \tag{I.4.11}$$

They describe the curvatures of the model and above all, they support the view (III) of space curvature mentioned in Sec. I.1.

In this Section we have specified which mathematical methodology is used for the representation of gravitation problems. Now we are able to approach simple physical problems.

I.5. The time

Special relativity theory makes it imperative for us to formulate the transformation laws between moving observers by the use of four dimensions. Thus, one can easily refer to the invariance of the 4-dimensional line element with respect to Lorentz transformations. If one agrees that Latin indices have the range [1,2,3,4] but Greek indices the range [1,2,3], then one is able to perform a [3+1]-decomposition of the line element (I.2.9), a decomposition into space and time

$$ds^2 = g_{mn}dx^m dx^n = g_{\alpha\beta}dx^\alpha dx^\beta + g_{44}dx^4 dx^4 \quad . \tag{I.5.1}$$

There is still the question, how to find the connection of dx^4 with the time differential. In the literature two definitions are used[12]

$$\text{(I)} \qquad dx^4 = cdt, \qquad \text{(II)} \qquad dx^4 = icdt \ , \tag{I.5.2}$$

whereby the natural system of units with c = 1 is mostly used. In order to obtain for the first case the physically interpretable relationship

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta - c^2 dt^2 \tag{I.5.3}$$

one has to set g_{44}= - 1, for the second case g_{44}= + 1. For the shortness' sake (I) is to be called t-notation, (II) it-notation. First, it seems that both ways of writing are arbitrarily applicable and at any time exchangeable. The disadvantage of the t-notation is, however, that with dragging indices a change of signs of the time-like components is inevitable.

Misner, Thorne, and Wheeler [M] strongly recommend this notation in their textbook 'Gravitation': „*One sometime participant in special relativity will have to be put to the sword 'x^4=ict'. This imaginary coordinate was invented to make the geometry of spacetime look formally as little different as possible from the geometry of Euclidean space; to make a Lorentz transformation look on paper like a rotation; and to spare one the distinction that one otherwise is forced to make between quantities with upper indices (such as the components p^μ of the energy-momentum vector) and quantities with lower indices (such as the components p_μ of the energy-momentum 1-form). However, it is no kindness to be spared this latter distinction. Without it, one cannot know whether a vector is meant or the very different geometric object that is a 1-form. Moreover, there is a significant difference between an angle on which everything depends periodically (a rotation) and a parameter the increase of which gives rise to ever-growing momentum differences (the 'velocity parameter' of a Lorentz transformation). If the imaginary time-coordinate hides from view the character of the geometric object dealt with and the nature of a parameter in the transformation, it also does something even more serious: it hides the completely different metric structure of ++++ geometry and -+++ geometry. In Euclidean geometry, when the distance between two points is zero, the two points must be the same point. In Lorentz-Minkowski geometry when the interval between two events is zero, one event may be on Earth and the other on a supernova in the galaxy M31, but their separation must be a null ray (piece of a light cone). The backward-pointing light cone at a given event contains all the events by which that event can be influenced. The forward-pointing light cone contents all events that it can influence. The multitude of double light cones taking off from all the events of spacetime forms an interlocking causal structure. This structure makes the machinery of the physical world function as it does. If in a region where spacetime is flat, one can hide this structure from view by writing* $(\Delta s)^2 = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 + (\Delta x^4)^2$,

[12] In some text books both are concurrently in use.

with $x^4 = ict$*, no one has discovered a way to make an imaginary coordinate work in the general curved spacetime manifold. If '*$x^4 = ict$*' cannot be used there, it will not be used here."*

In order to approach the problem, three different geometrical objects are to be examined:

$$\text{(I)} \quad x^2 + t^2 = \mathbb{R}^2, \quad \text{(II)} \quad x^2 + (it)^2 = \mathbb{R}^2, \quad \text{(III)} \quad x^2 - t^2 = \mathbb{R}^2 \ . \qquad \text{(I.5.4)}$$

These are a circle, a pseudo-circle, and a hyperbola. While the circle and the hyperbola are real objects, a pseudo-circle is an object in a complex plane. Since the human imaginative power fails within complex spaces, a pseudo-circle is neither conceivable nor graphically representable. In order to make the properties of a pseudo-circle plausible nevertheless, one makes use of the *pseudo-real representation.* One replaces the pseudo-circle with a hyperbola. It consists of two open curves, which come from the infinite and go to the infinite. These are properties which pseudo-circles have as well. The pseudo-circle and the hyperbola do not agree, however, in their curvature properties. While the length of the radius of curvature of the hyperbola varies with the points of the curve, a pseudo-circle has a constant curvature everywhere, also at the points in the infinite. All points on a pseudo-circle are equivalent, however, they are not equivalent on the hyperbola. A pseudo-circle has with a real circle some properties in common, others with a hyperbola. Therefore, a pseudo-circle is also called a *hyperbola of constant curvature*.

The second equation (I.5.4) applied to the line element results in

$$ds^2 = dx^\alpha dx^\alpha + dx^4 dx^4, \quad dx^4 = icdt \ . \qquad \text{(I.5.5)}$$

The value dt read from a clock is the real accessory number of the imaginary variable dx^4. From the invariance of the line element demanded by the special relativity theory, one obtains for another reference system being in a relative motion

$$ds^2 = dx^{\alpha'} dx^{\alpha'} + dx^{4'} dx^{4'} \ .$$

For an observer who is in rest with respect to the primed system, we have $dx^{\alpha'} = 0$. For the remaining term $dx^{4'}$ one writes $icd\tau$ and τ is called the proper time of the observer. For such an observer the length of the 4-dimensional line element is the time, which is measured with his clock. For ds = 0 one obtains from (I.5.5) the equation of a light ray

$$\frac{dx^\alpha}{dt}\frac{dx^\alpha}{dt} - c^2 = 0, \quad \frac{dx}{dt} = c \ .$$

Therefore, the paths of light rays are called null-lines. If we have 4-dimensional null-distances this does not mean that these must also be 3-dimensional null-distances. They only state that for example the galaxy M31 is connected with us by a light ray. The measured values referring to a physical object gained by two relatively moving observers can be related by a Lorentz transformation:

$$\Phi_{m'} = L^m_{m'} \Phi_m \ .$$

If one reduces a motion of the reference system into the 1-direction one has with the relative velocity v[13] and with the Lorentz factor α

[13] From now on the natural units with c = 1 are used.

$$L^{1}_{1'} = \alpha, \quad L^{4}_{1'} = i\alpha v, \quad L^{1}_{4'} = -i\alpha v, \quad L^{4}_{4'} = \alpha, \quad \alpha = 1\Big/\sqrt{1-v^2} \ . \tag{I.5.6}$$

This transformation can be understood as a pseudo-rotation, thus as a rotation in the 4-dimensional complex space

$$L^{m}_{m'} = \begin{pmatrix} \cos i\chi & & & \sin i\chi \\ & 1 & & \\ & & 1 & \\ -\sin i\chi & & & \cos i\chi \end{pmatrix} . \tag{I.5.7}$$

A first look at the angular functions (I.5.7) gives us the impression of periodicity. That a rotation in complex space does not provide periodicity, however, is shown by the representation

$$L^{m}_{m'} = \begin{pmatrix} ch\chi & & & ish\chi \\ & 1 & & \\ & & 1 & \\ -ish\chi & & & ch\chi \end{pmatrix} . \tag{I.5.8}$$

Hyperbolic functions are not periodic, the equivalent trigonometric functions of imaginary angles are not periodic either. The angle χ has like the time variable the range $\left[-\infty, +\infty\right]$. With (I.5.6) arises

$$iv = \tan i\chi, \quad v = th\chi \ . \tag{I.5.9}$$

The Lorentz angle χ is also called *rapidity*. One recognizes that with infinitely high rapidity, one obtains $th\chi = 1$ and the relative velocity v reaches the speed of light.

In the pseudo-real representation one can by means of a circle represent a Lorentz rotation as ordinary rotation with the help of the it-notation, if one takes into account that the angle of rotation cannot go beyond ± 45°[14]. Thereby the Lorentz contraction and the time dilatation are not represented correctly. In the pseudo-real representation of the t-notation by means of a hyperbola, one obtains correct values for the contraction and dilatation, however, the transformed reference system appears to be oblique-angled.

Finally we again investigate the relationship (II) from (I.5.4). Using polar co-ordinates in the complex plane, parametrized by

$$x^{1'} = \Re \cos i\psi, \quad x^{4'} = \Re \sin i\psi \, , \tag{I.5.10}$$

one obtains the line element

$$ds^2 = d\Re^2 + \Re^2 di\psi^2 \ . \tag{I.5.11}$$

Thus, $\Re di\psi$ is the arc on a pseudo-circle with the radius $\Re$ and is related to the time by

$$idt = \Re di\psi \, . \tag{I.5.12}$$

In the following Section this representation is extended to pseudo-hyper spheres and is one of the basic notations for the embedding of gravitation models into higher-dimensional spaces. Thus, also the possibility of an imaginary co-ordinate in a curved space, missed by Misner, Thorne, and Wheeler, is satisfactorily explained.

[14] The null-lines have their position at $\pm 45°$. Infinitely high rapidity corresponds to the angle of ±45° on the drawing.

Before we examine gravitational models, we must go into the computational technology. To assist in the understanding of the physics of a model, a space-time decomposition ([3+1]-decomposition) of the quantities would be expedient.

If one puts a vector

$$u_m = \{0,0,0,1\}$$

into the local time direction such a decomposition can take place invariantly, i.e. co-ordinate independently:

$$\Phi_m = {}^3\Phi_m + u_m \Phi_s u^s \quad , \tag{I.5.13}$$

whereby the marker 3 at the kernel of the quantity refers to the space-like part of this quantity. In particular for the metric

$$g_{mn} = {}^3 g_{mn} + u_m u_n \quad . \tag{I.5.14}$$

is valid. u_m one obtains from the commonly not standardized bein vectors by

$$u_m = \overset{4}{e}_i \underset{m}{e}^i, \quad u^m = \overset{m}{e}_i \underset{4}{e}^i \quad .$$

One calls u_m observer field. It represents the 4-velocity of an observer field. With the help of a Lorentz transformation

$$u_{m'} = L^m_{m'} u_m$$

one obtains the components of the 4-velocity, experienced by a relative moving observer. If for the sake of simplicity one puts the motion into the 1-direction, one has

$$u_m = \{-i\alpha v, 0, 0, \alpha\} \quad .$$

Therein v is the relative velocity and

$$\alpha = 1\big/\sqrt{1-v^2}$$

the associated Lorentz factor. In their own system (rest system) the new observer fields have the components

$$'u_{m'} = \{0,0,0,1\} .$$

Use repeatedly is made of this procedure.

This procedure probably goes back to Pirani and was celebrated in the co- ordinate way of writing by Hönl [H] and Maue; Hönl [H] and Dehnen; Hönl [H] and Ruffer; Hönl [H] and Soergel-Fabricius and Dehnen [D], and was adopted by many authors or was invented again:

Abramowicz [A], Antonov [A], Efremov and Vladimirov; Asgekar [A] and Date; Audretsch [A], Barker [B] and O'Connell; Benvenuti [B], Bel [B], Bel [B] und Escard; Bini [B], Bini [B], de Felice, and Jantzen; Boersma [B] and Dray; Bonazzola [B], Bonnor [B], Cattaneo [C], Cattaneo-Gasparini [C], Chen [C], Chern and Nester; Cheng [C] and Nester; Chevalier [C], Coisson [C], Davidson [D], de Felice [D], DeFacio [D], Dennis and Retzloff; DeFacio [D] and Retzloff; Dehnen [D] and Obregon; Ehlers [E], Ellis [E], Ellis [E] and MacCallum; Elst [E] and Uggla; Ferrarese [F] and Antonelli; Friedman [F] and Scarr; Gharechahi [G], Nouri-Zonoz, and Tafanfar; Goodinson [G], Greene [G], Schucking and Vishveshwara; Grishchuk [G], Hafner [H], Herrera [H], Hofmann [H], Huei [H], Hyde [H], Ionescu-Pallas [I], Ivanitskaja [I], Ivanitskaja [I] and Vyblyj; Jordan [J], Jordan [J] , Ehlers and Kundt; Kohler [K], Krause [K], Krechet [K], Krumm [K] and Bedford; Kucina [K], Künzle [K] and Nester; Li [L] and Ni; Maartens [M] and Basset; Majernik [M], Manoff [M], Marsh [M] and Nissim-

Sabat; Mashhoon [M], Mashhoon [M], Hehl and Theiss; Mashhoon [M] and Theiss; Massa [M], Mitskievich [M], Mo [M], Muñoz [M], Nelson [N], Nester [N], Novotny [N] and Horsky; Oldroyd [O], Olmo [O] et al., Opat [O], Orwig [O], Palle [P], Pechlaner [P], Peng [P], Peres [P] and Rosen; Pessa [P], Pollock [P], Prakash [P] and Roy; Raigorodski [R], Ringermacher [R], Roditchev [R], Rosen [R], Ruffer [R], Rumer [R], Salié [S], Salzman [S] and Taub; Schmutzer [S], Schöpf [S], Sciama [S], Scorgie [S], Semón [S] and Schmieg; Singh [S] and Misra; Smarr [S] and York; Sonego [S] and Massar; Spieweck [S], Stachel [S], Stellmacher [S], Synge [S], Szekeres [S], Thorne [T] and MacDonald; Treder [T], Trencevski [T], Trümper [T], Tsagas [T], Urani [U] and Kemp; Zel'manov [Z].

Although we develop new mathematical methods and we often use the 5-dimensional representation as well, we remain in the framework of Einstein's theory of gravity and avoid alternative approaches.

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II. Cosmological models

II.1. The Einstein cosmos

After some fundamental matter has been clarified, we can turn towards the simplest gravitation model, towards the Einstein cosmos [E]. It is described by the extended Einstein field equations containing one additional term with the *cosmological constant*:

$$R_{mn}-\frac{1}{2}Rg_{mn}+\lambda g_{mn}=-\kappa T_{mn}\ . \tag{II.1.1}$$

The stress-energy tensor T_{mn} describes a cosmos free from pressure and stresses, uniformly filled with matter of constant density μ_0. The positive constant λ was introduced by Einstein into the field equations in order to compensate infinitely large forces between the masses. After Hubble had discovered that our cosmos expands, the cosmological term had been dropped and one studied expanding cosmological models.

Physicists, however, never wholly lost an interest in the extended Einstein field equations. In some papers of the last decades, the cosmological term was reappraised. In the equations

$$R_{mn}-\frac{1}{2}Rg_{mn}=-\kappa\left(T_{mn}^{mat}+T_{mn}^{vac}\right),\quad T_{mn}^{vac}=-\frac{\lambda}{\kappa}g_{mn} \tag{II.1.2}$$

T_{mn}^{vac} is interpreted as stress-energy tensor of a quantum-mechanically polarized vacuum.

In a 5-dimensional flat space a Cartesian co-ordinate system is connected with a spherical co-ordinate system $\left\{\mathfrak{R},\eta,\vartheta,\varphi\right\}$ with the help of

$$\begin{aligned}
x^{3'} &= \mathfrak{R}\sin\eta\sin\vartheta\sin\varphi\\
x^{2'} &= \mathfrak{R}\sin\eta\sin\vartheta\cos\varphi\\
x^{1'} &= \mathfrak{R}\sin\eta\cos\vartheta \qquad .\\
x^{0'} &= \mathfrak{R}\cos\eta\\
x^{4'} &= it
\end{aligned} \tag{II.1.3}$$

By

$$x^{a'}x^{a'}=\mathfrak{R}^2,\quad a'=0',1',2',3' \tag{II.1.4}$$

a family of hyper spheres with the radius $\mathfrak{R}$ is described. By differentiation of (II.1.3) one obtains the line element of the 5-dimensional space in spherical co-ordinates[15]

$$ds^2=d\mathfrak{R}^2+\mathfrak{R}^2d\eta^2+\mathfrak{R}^2\sin^2\eta\, d\vartheta^2+\mathfrak{R}^2\sin^2\eta\sin^2\vartheta\, d\varphi^2-dt^2\ . \tag{II.1.5}$$

With the embedding condition

$$\mathfrak{R}=\text{const.} \tag{II.1.6}$$

[15] The Einstein cosmos is called a spherical space or a cylindrical space. The first designation follows from the spatial spherical structure, the latter is derived from the straight-line time co-ordinate, which is orthogonal to the circular slices of the spatial hyper-sphere.

one selects from the family a hyper sphere. The remaining metric is the metric of the Einstein cosmos. With a new variable $r = \mathcal{R}\sin\eta$ one can write the metric of the Einstein cosmos in the form

$$ds^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 - dt^2 \ . \qquad \text{(II.1.7)}$$

From (II.1.5) one reads the bein vectors

$$\overset{0}{e}_0 = 1, \quad \overset{1}{e}_1 = \mathcal{R}, \quad \overset{2}{e}_2 = \mathcal{R}\sin\eta, \quad \overset{3}{e}_3 = \mathcal{R}\sin\eta\sin\vartheta, \quad \overset{4}{e}_4 = 1 \ . \qquad \text{(II.1.8)}$$

With the reciprocal gradient vectors one establishes the partial derivatives with respect to the spherical reference system

$$\partial_0 = \frac{\partial}{\partial\mathcal{R}}, \quad \partial_1 = \frac{\partial}{\mathcal{R}\partial\eta}, \quad \partial_2 = \frac{\partial}{\mathcal{R}\sin\eta\partial\vartheta}, \quad \partial_3 = \frac{\partial}{\mathcal{R}\sin\eta\sin\vartheta\partial\varphi}, \quad \partial_4 = \frac{\partial}{\partial it} \ . \qquad \text{(II.1.9)}$$

If we extend the definitions (I.3.17) of the unit tangent vectors to

notation:

$$n_a = \{1,0,0,0,0\}, \quad m_a = \{0,1,0,0,0\}, b_a = \{0,0,1,0,0\}, c_a = \{0,0,0,1,0\}, u_a = \{0,0,0,0,1\}, \qquad \text{(II.1.10)}$$

they span in the 5-dimensional space a 5-bein in such a way that the vector n is the normal vector of a 4-dimensional surface and the remaining vectors are 4-beine in the local tangent planes of the 4-dimensional surface. u is the unit vector of the time direction. The indices a, b, c,… run through the values {0, 1,… 4}. With the help of these vectors and with the decomposition of the Ricci-rotation coefficients into

$$A_{ab}{}^c = M_{ab}{}^c + B_{ab}{}^c + C_{ab}{}^c \qquad \text{(II.1.11)}$$

and by the use of the formula (I.3.10) one obtains

$$M_{ab}{}^c = m_a M_b m^c - m_a m_b M^c, \quad B_{ab}{}^c = b_a B_b b^c - b_a b_b B^c, \quad C_{ab}{}^c = c_a C_b c^c - c_a c_b C^c \ . \qquad \text{(II.1.12)}$$

Therein the quantities

$$M_a = \left\{\frac{1}{\mathcal{R}},0,0,0,0\right\}, \quad B_a = \left\{\frac{1}{\mathcal{R}},\frac{1}{\mathcal{R}}\cot\eta,0,0,0\right\}, \quad C_a = \left\{\frac{1}{\mathcal{R}},\frac{1}{\mathcal{R}}\cot\eta,\frac{1}{\mathcal{R}\sin\eta}\cot\vartheta,0,0\right\} \qquad \text{(II.1.13)}$$

describe the curvatures of the normal and inclined slices. The curvatures (the values of the above vectors)

$$M = \frac{1}{\mathcal{R}}, \quad B = \frac{1}{\mathcal{R}\sin\eta}, \quad C = \frac{1}{\mathcal{R}\sin\eta\sin\vartheta} \ . \qquad \text{(II.1.14)}$$

are the reciprocal radii of curvature. Since the time-like term in the metric does not have a space-dependent factor, no further expression results for (II.1.11) - (II.1.14).

In Section I.4 the usefulness of the graded covariant derivatives were suggested. Here they are extended to the 5-dimensional space.

notation:

$$\begin{aligned}
\Phi_{a\,\underset{1}{|||}b} &= \Phi_{a|b} \\
\Phi_{a\,\underset{2}{|||}b} &= \Phi_{a|b} - M_{ba}{}^c\Phi_c \\
\Phi_{a\,\underset{3}{|||}b} &= \Phi_{a|b} - M_{ba}{}^c\Phi_c - B_{ba}{}^c\Phi_c \\
\Phi_{a\,\underset{4}{|||}b} &= \Phi_{a|b} - M_{ba}{}^c\Phi_c - B_{ba}{}^c\Phi_c - C_{ba}{}^c\Phi_c
\end{aligned} \ . \qquad \text{(II.1.15)}$$

The threefold stroke connotes the covariant 5-dimensional derivative. Applied to the triads tangent to the spherical space one obtains

$$\underset{1}{m}_{a|||b}=0,\quad \underset{2}{b}_{a|||b}=0,\quad \underset{3}{c}_{a|||b}=0. \tag{II.1.16}$$

These relations will substantially simplify numerous computations. Investigation of the field equations shows that this preparing expenditure is worthwhile. For the flat space the Ricci tensor identical vanishes

$$R_{ab}=A_{ab}{}^{c}{}_{|c}-A_{b|a}-A_{da}{}^{c}A_{cb}{}^{d}+A_{ab}{}^{c}A_{c}\equiv 0. \tag{II.1.17}$$

If one considers (II.1.11) - (II.1.16) one obtains the curvature equations of the family of spheres:

$$\begin{aligned}
&\underset{1}{M}_{a|||b}+M_aM_b=0,\quad \underset{1}{M^c}_{|||c}+M^cM_c=0\\
&\underset{2}{B}_{a|||b}+B_aB_b=0,\quad \underset{2}{B^c}_{|||c}+B^cB_c=0\\
&\underset{3}{C}_{a|||b}+C_aC_b=0,\quad \underset{3}{C^c}_{|||c}+C^cC_c=0
\end{aligned}\;. \tag{II.1.18}$$

One gets the 4-dimensional graded covariant derivative of a quantity from the 5-dimensional one, by letting run the indices of the 5-dimensional expressions from 1 to 4 and if one isolates the terms with zero indices[16]

$$\begin{aligned}
&\underset{2}{B}_{m||n}+B_mB_n=-m_mm_nM_0B_0,\quad \underset{2}{B^s}_{||s}+B^sB_s=-M_0B_0\\
&\underset{3}{C}_{m||n}+C_mC_n=-m_mm_nM_0C_0-b_mb_nB_0C_0,\quad \underset{3}{C^s}_{||s}+C^sC_s=-M_0C_0-B_0C_0
\end{aligned}\;. \tag{II.1.19}$$

For the 4-dimensional Ricci tensor[17] one obtains

$$R_{mn}=-\left[\underset{2}{B}_{n||m}+B_nB_m\right]-b_mb_n\left[\underset{2}{B^s}_{||s}+B^sB_s\right]-\left[\underset{3}{C}_{n||m}+C_nC_m\right]-c_mc_n\left[\underset{3}{C^s}_{||s}+C^sC_s\right] \tag{II.1.20}$$

or

$$R_{mn}=m_mm_n\left[M_0B_0+M_0C_0\right]+b_mb_n\left[M_0B_0+B_0C_0\right]+c_mc_n\left[M_0C_0+B_0C_0\right]. \tag{II.1.21}$$

If one reads the 0-components from (II.1.13) one obtains from the last equation

$$R_{mn}=\frac{2}{\mathcal{R}^2}\,{}'g_{mn} \tag{II.1.22}$$

with the space-like part of the metric

$${}'g_{mn}=\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&0\end{pmatrix}.$$

By contraction of (II.1.22) one obtains

[16] To be more precise a covariant projection formalism would be necessary. However, with the special choice of the local reference systems (5-beine) a [0+4]–decomposition can be performed indexwise. A costlier formalism is superfluous.

[17] From (II.1.13) one recognizes that $B_{0|0}+B_0B_0=0,\ C_{0|0}+C_0C_0=0$.

$$R = \frac{6}{\mathcal{R}^2} \ . \tag{II.1.23}$$

If one adapts the cosmological constant by

$$\lambda = \frac{1}{\mathcal{R}^2} \ , \tag{II.1.24}$$

one gets

$$R_{mn} - \frac{1}{2} R g_{mn} + \lambda g_{mn} = -\frac{2}{\mathcal{R}^2} u_m u_n \, .$$

Since the right term of the Einstein field equation reads as $-\kappa T_{mn}$ one finally has

$$\kappa T_{mn} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \frac{2}{\mathcal{R}^2} \end{pmatrix} . \tag{II.1.25}$$

From this expression one can read the mass density of the universe

$$\mu_0 = \frac{1}{\kappa} \frac{2}{\mathcal{R}^2} \tag{II.1.26}$$

and the pressure p = 0. Since the mass density is constant, the conservation law

$$T^{mn}{}_{||n} = 0 \tag{II.1.27}$$

is reduced to $\dot{\mu}_0 = 0$.

Eddington [E] comments critically on the stability of the Einstein cosmos. He consults the basic equations of an expanding universe, and he considers the Einstein cosmos as a special case of this expanding universe. He shows that a small change in density of matter can cause an expansion or a contraction of the universe, hence a change in the radius of curvature of the universe

However, we are critical at this possibility. One would negate a basic requirement, namely that the constancy of the radius of the Einstein universe is said to have the character of a law of Nature. The problem of Eddington should be formulated differently: the probability is very small that a cosmos of type 'expanding/ contracting' can be in a stable phase. Today we know that Einstein's conditions, the constancy of the radius of curvature and the homogeneity of the universe, are too restrictive. Measurements of the redshift of the light spectra of distant galaxies have shown that most of them move away from us and the further they are away the faster. The cosmos is expanding as a whole. The homogeneity of the mass distribution is too simplistic. Stars cluster into galaxies, galaxies into galaxy clusters and superclusters, and these again into galaxy walls. In between it there is often the great void.

We will deal with expanding universes in the subsequent Sections. Inhomogeneous models we will mention only briefly.

The Einstein cosmos is a simple model, can be elegantly described with the methods of embedding, and the interpretation (III) of the space curvature is fulfilled. In the days of the publication of the model by Einstein, one had taken quite seriously the conception that the cosmos is embedded into a higher-dimensional flat space. One estimated the radius of the spherical space to be 10^{25} to 10^{27} m and believed to have found the chance for a natural unit of length. Since the universe must be regarded as closed, rays of light should principally be able to go around the world. One accepted to be able to observe a ‚ghost sun', i.e. our own sun, which appears after a circulation of the light once again.

However, we must be aware of the fact that mathematics is actually not the reality, but only an attempt to describe the reality with useful methods whereby observable facts settle in mathematical relations. If a cosmological model is to be described rather simply with the help of embedding procedures, nevertheless the existence of a higher-dimension embedding space is not presupposed. The usefulness of the mathematical methods, which are based on the surface theory, is sufficient motivation to use these methods in the sense of the view (III).

II.2. De Sitter cosmos, static version

In the year of 1916, de Sitter [D] suggested a cosmological model, which is free from matter. The field equations contain the cosmological constant. Searchers were caused to ponder about, what concerns the validity of Mach's principle. In accordance with Mach the gravitational effects should be determined by the entire mass of the space. However, in the empty de Sitter cosmos gravitation forces are present. The de Sitter model permits two interpretations, the static interpretation, and the expanding one. Firstly, the simpler static one will be treated. It is similar to the Einstein cosmos. By us [B] the model was revised in the sense of a strict embedding and a covariant decomposition of the field equations.

In a 5-dimensional flat space, with a Cartesian co-ordinate system $a' = 0',1',...4'$, a family of pseudo-hyper spheres with the radii $\mathcal{R}$ is defined by the equation

$$x^{a'}x^{a'} = \mathcal{R}^2. \tag{II.2.1}$$

In pseudo-spherical co-ordinates

$$\begin{aligned} x^{3'} &= \mathcal{R}\sin\eta\sin\vartheta\sin\varphi \\ x^{2'} &= \mathcal{R}\sin\eta\sin\vartheta\cos\varphi \\ x^{1'} &= \mathcal{R}\sin\eta\cos\vartheta \\ x^{4'} &= \mathcal{R}\cos\eta\sin i\psi \\ x^{0'} &= \mathcal{R}\cos\eta\cos i\psi \end{aligned} \tag{II.2.2}$$

the line element in the 5-dimensional space has the form

$$ds^2 = d\mathcal{R}^2 + \mathcal{R}^2 d\eta^2 + \mathcal{R}^2\sin^2\eta d\vartheta^2 + \mathcal{R}^2\sin^2\eta\sin^2\vartheta d\varphi^2 + \mathcal{R}^2\cos^2\eta di\psi^2 \ . \tag{II.2.3}$$

If one selects one of the pseudo-hyper spheres by means of the embedding condition

$$\mathcal{R} = \text{const.} \ , \tag{II.2.4}$$

one obtains from (II.2.3) the 4-dimensional metric with respect to one of these pseudo-hyper surfaces which is identified with de Sitter cosmos. The time interval is defined by

$$\mathcal{R}\, di\psi = i dt \tag{II.2.5}$$

and represents the arc element of an (open) pseudo-circle. With $r = \mathcal{R}\sin\eta$ one can bring the metric into the form

$$ds^2 = \frac{1}{\cos^2\eta}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\eta d\varphi^2 - \cos^2\eta dt^2 .$$

Furthermore, the metric can be written as

$$ds^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}^2}}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\eta d\varphi^2 - \left(1-\frac{r^2}{\mathcal{R}^2}\right)dt^2$$

In its formal structure it is similar to the Schwarzschild metric which will be discussed later.

From (II.2.3) one reads the components of the 5-bein

$$\overset{0}{e}_0 = 1, \quad \overset{1}{e}_1 = \mathcal{R}, \quad \overset{2}{e}_2 = \mathcal{R}\sin\eta, \quad \overset{3}{e}_3 = \mathcal{R}\sin\eta\sin\vartheta, \quad \overset{4}{e}_4 = \mathcal{R}\cos\eta \tag{II.2.6}$$

and can immediately calculate from it the reciprocal components. In contrast to the Einstein cosmos $\overset{4}{e}_4$ is space-dependent and leads to an additional field strength E

$$A_{ab}{}^{c} = M_{ab}{}^{c} + B_{ab}{}^{c} + C_{ab}{}^{c} + E_{ab}{}^{c}$$
$$M_{ab}{}^{c} = m_a M_b m^c - m_a m_b M^c, \quad B_{ab}{}^{c} = b_a B_b b^c - b_a b_b B^c \; . \tag{II.2.7}$$
$$C_{ab}{}^{c} = c_a C_b c^c - c_a c_b C^c, \quad E_{ab}{}^{c} = -u_a E_b u^c + u_a u_b E^c$$

The field strengths therein have been derived from the curvatures of the pseudo-hyper spheres

$$M_a = \left\{\frac{1}{\mathcal{R}},0,0,0,0\right\}, \quad B_a = \left\{\frac{1}{\mathcal{R}},\frac{1}{\mathcal{R}}\cot\eta,0,0,0\right\}$$
$$C_a = \left\{\frac{1}{\mathcal{R}},\frac{1}{\mathcal{R}}\cot\eta,\frac{1}{\mathcal{R}\sin\eta}\cot\vartheta,0,0\right\}, \quad E_a = \left\{-\frac{1}{\mathcal{R}},\frac{1}{\mathcal{R}}\tan\eta,0,0,0\right\} \; . \tag{II.2.8}$$

$\mathcal{R}$, $\mathcal{R}\sin\eta$, $\mathcal{R}\sin\eta\sin\vartheta$, $\mathcal{R}\cos\eta$ are the curvature radii of the normal and inclined slices of the pseudo-hyper surfaces. The 5-dimensional field equations

$$R_{ab} \equiv 0 \tag{II.2.9}$$

decouple to the individual curvature equations

$$\begin{aligned}
&M_{a\underset{1}{|||}b} + M_a M_b = 0, \quad M^c{}_{\underset{1}{|||}c} + M^c M_c = 0\\
&B_{a\underset{2}{|||}b} + B_a B_b = 0, \quad B^c{}_{\underset{2}{|||}c} + B^c B_c = 0\\
&C_{a\underset{3}{|||}b} + C_a C_b = 0, \quad C^c{}_{\underset{3}{|||}c} + C^c C_c = 0\\
&E_{a\underset{4}{|||}b} - E_a E_b = 0, \quad E^c{}_{\underset{4}{|||}c} - E^c E_c = 0
\end{aligned} \; , \tag{II.2.10}$$

whereby the graded derivatives were supplemented with a further grade

$$m_{a\underset{1}{|||}b} = m_{a|b} = 0, \quad b_{a\underset{2}{|||}b} = b_{a|b} - M_{ba}{}^{c} b_c = 0$$
$$c_{a\underset{3}{|||}b} = c_{a|b} - M_{ba}{}^{c} c_c - B_{ba}{}^{c} c_c = 0, \quad u_{a\underset{4}{|||}b} = u_{a|b} - M_{ba}{}^{c} u_c - B_{ba}{}^{c} u_c - C_{ba}{}^{c} u_c = 0 \; .$$

In four dimensions the curvature equations do not decouple any more. If one accomplishes a dimensional reduction by shifting all 0-components of the Ricci tensor to the right sides of the relations, one obtains with m = 1,2,3,4

$$\begin{aligned}
R_{mn} = &-\left[B_{n\underset{2}{||}m} + B_n B_m\right] - b_n b_m \left[B^s{}_{\underset{2}{||}s} + B^s B_s\right]\\
&-\left[C_{n\underset{3}{||}m} + C_n C_m\right] - c_n c_m \left[C^s{}_{\underset{3}{||}s} + C^s C_s\right]\\
&+\left[E_{n\underset{4}{||}m} - E_n E_m\right] + u_n u_m \left[E^s{}_{\underset{4}{||}s} - E^s E_s\right]\\
= &\; m_n m_m \left[M_0 B_0 + M_0 C_0 - M_0 E_0\right]\\
&+ b_n b_m \left[B_0 M_0 + B_0 C_0 - B_0 E_0\right]\\
&+ c_n c_m \left[C_0 M_0 + C_0 B_0 - C_0 E_0\right]\\
&- u_n u_m \left[E_0 M_0 + E_0 B_0 + E_0 C_0\right]
\end{aligned} \; . \tag{II.2.11}$$

The right side of (II.2.11) can be significantly simplified with the help of (II.2.8)

$$R_{mn} = \frac{3}{\mathcal{R}^2} g_{mn}, \quad R = \frac{12}{\mathcal{R}^2} \; . \tag{II.2.12}$$

Fitting the cosmological constant with

$$\lambda = \frac{3}{\mathcal{R}^2} \tag{II.2.13}$$

the Einstein field equations for de Sitter cosmos

$$R_{mn} - \frac{1}{2} R g_{mn} + \lambda g_{mn} = 0 \tag{II.2.14}$$

are satisfied. With these simple calculations we are not pleased. The 0-components of the curvature quantities can be interpreted as the second fundamental forms [18] of the surface theory

$$M_0 = A_{11}, \quad B_0 = A_{22}, \quad C_0 = A_{33}, \quad E_0 = -A_{44}. \tag{II.2.15}$$

If one uses the normal vector n_a of the surface[19], defined in (II.1.10), one can perform a [0+4]-decomposition of the Ricci-rotation coefficients

$$A_{ab}{}^{c} = {}'A_{ab}{}^{c} + n_b A_a{}^{c} - n^c A_{ab} \;, \tag{II.2.16}$$

whereby the prime at the kernel marks the 4-dimensional part of the quantity. If one inserts this into the Ricci tensor, one firstly has

$$R_{ab} = {}'R_{ab} + 2A_{a[c}A_{b]}{}^{c} \equiv 0, \quad R = {}'R + 2A^{d}{}_{[c}A_{d]}{}^{c} \;, \tag{II.2.17}$$

and with

$$2A_{m[s}A_{n]}{}^{s} - g_{mn}A_{[r}{}^{s}A_{s]}{}^{r} = \lambda g_{mn} \tag{II.2.18}$$

we lastly have explained the cosmological constant as an aggregate of the second fundamental forms of the surface theory. If one regards the model with respect to the view of embedding, the cosmological constant is not a quantity, which one obtains by adjustment. In contrast, it results from geometrical quantities. Thus, it is an element of the space structure[20]. We will hark back to this procedure in treating the interior solutions of the Einstein field equations.

It is still to be considered what is the meaning of the curvature of the time-like part of the surface. It is seen by an observer by means of the quantity E that has only one component in the local 1-direction of the surface. However, this direction depends on the choice of the reference system. Since on a three-dimensional hyper-surface all points and also all directions are equivalent, one can select the 1-direction on it at will. Thus, the force E can point into any direction. In this arbitrariness we recognize the weak point of the model. A correction of this lack inevitably leads to the expanding version of the de Sitter cosmos.

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[18] In the textbooks the second fundamental forms of the surface theory are usually indicated in the co-ordinate way of writing. With the Cartesian coordinates x^i, i=1,2,3 in the embedding space one obtains on a 2-dimensional surface with the help of curvilinear coordinates u, v

$$\frac{\partial^2 x^i}{\partial u \partial u} n^i = -\frac{\partial x^i}{\partial u}\frac{\partial n^i}{\partial u} = L, \quad \frac{\partial^2 x^i}{\partial u \partial v} n^i = -\frac{\partial x^i}{\partial u}\frac{\partial n^i}{\partial v} = M, \quad \frac{\partial^2 x^i}{\partial v \partial v} n^i = -\frac{\partial x^i}{\partial v}\frac{\partial n^i}{\partial v} = N \;.$$

By applying the rule (I.2.17) one obtains the components with respect to the reference system and a direct away reference to the curvatures. The second fundamental forms in the pentad representation are also the components of the higher-dimensional Ricci-rotation coefficients.

[19] The 4-dimensional surface in the 5-dimensional space.

[20] For the Einstein cosmos similar considerations could be carried out. There, however, the mass density is also included in this structure.

Frequently the t-notation is used in the literature for the description of the de Sitter cosmos, and an one-shell hyperboloid of revolution (Schrödinger [S]) for the illustration. At any rate, the necessary insight is not ensured by this representation, so we do without a visualization of this problem[21].

Weyl [W] assumed a singularity on the 3-dimensional equator sphere of the 4-dimensional hypersphere

$$x^a x^a = \mathcal{R}^2, \quad a = 0,1,\ldots,4$$

and believed that the velocity of light is zero at this location. Thus, the real world cannot come close to the equator. Since a massless space could not exist on the basis of Einstein's field equations, he required that the equator is to be surrounded by mass. He introduced there a zone with incompressible fluid with mass density μ_0. He calculated the problem and found that the pressure increases monotonically inside the fluid towards the equator and is positive everywhere.

If one shifts the cosmological term to the right side of the field equations, one reads from

$$R_{mn} - \frac{1}{2} R g_{mn} = -\kappa T_{mn}$$

the relation

$$\kappa T_{mn} = \lambda g_{mn} \tag{II.2.19}$$

Treating the de Sitter universe as a homogeneous fluid sphere

$$T_{mn} = \begin{pmatrix} -p & & & \\ & -p & & \\ & & -p & \\ & & & \mu_0 \end{pmatrix},$$

one finds

$$\kappa p = -\frac{3}{\mathcal{R}^2}, \quad \kappa \mu_0 = \frac{3}{\mathcal{R}^2} \ . \tag{II.2.20}$$

The equation of state of the 'cosmological fluid' is

$$p + \mu_0 = 0 \, . \tag{II.2.21}$$

One meets the expression for the energy density μ_0 in several gravity models which are based on a spherical geometry. The identification $\kappa\mu_0 = \lambda$ is used by some authors to bring the dark energy of the universe into relation with the cosmological constant.

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[21] Schrödinger on several pages tried hard to explain to the reader that elliptical slices on the hyperboloid are actually circles and that all points are equivalent on the surface. The circle at the waist of the hyperboloids has given rise to a mass horizon. One assumed that the mass of the otherwise empty cosmos is concentrated on the horizon, as sounded out by Eddington. Pseudo-hyper spheres do not possess circles at a waist. A discussion on horizons cannot arise from this. Moreover, the debate through many years among Einstein, de Sitter, Weyl, Eddington, and Klein might go back to a good part due to the t-notation.

II.3. De Sitter cosmos, expanding version

Lemaître [L] and Robertson [R] have found a co-ordinate system with the aid of which the metric of the de Sitter cosmos can be transformed in such a way that it is accessible to a new interpretation. With the co-ordinate transformation

$$r = \mathcal{K} r', \quad \mathcal{K} = e^{\psi'}, \quad \psi' = \psi + \ln\cos\eta, \quad e^{\psi'} = e^{\psi}\cos\eta, \quad t' = \mathcal{R}\psi' \qquad \text{(II.3.1)}$$

one can perform this conversion. Differentiating the above transformation equations and utilizing the relations $r = \mathcal{R}\sin\eta$ and $dt = \mathcal{R}d\psi$ as in the previous Section, one has first

$$e^{\psi'}dr' = \frac{1}{\cos\eta}\mathcal{R}d\eta - \sin\eta\mathcal{R}d\psi, \quad \mathcal{R}d\psi' = \mathcal{R}d\psi - \tan\eta\mathcal{R}d\eta .$$

Squaring both relations and subtracting them one obtains

$$ds^2 = \mathcal{K}^2\left[dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta d\varphi^2\right] - dt'^2 . \qquad \text{(II.3.2)}$$

One recognizes that in the time-like part of the line element no space-dependent factor arises, thus, no disturbing quantity of the type E is derivable. However, the three spatial elements of the line element have the same time-dependent factor. The spatial distance between two arbitrary points experiences a stretch independently of the position. It is the same in each direction. Thus, one can assume that the cosmos expands. The quantity $\mathcal{K}$ is called *scale factor*. It depends only on the time. The coordinate system $\{r', t'\}$ is comoving with the expansion. The coordinate system $\{r, t\}$ is the non-comoving one.

That an expanding cosmos can be quite realistic, we owe to the measurements of Hubble, which attribute the red shift in the light spectra of distant suns to the recession velocity of these stars. The de Sitter cosmos is matter-free and therefore can represent only approximately a model for our cosmos. After the discovery of the background radiation in the universe, it became certain that our cosmos expands and had its beginning in the Big Bang.

For the further processing of the metric (II.3.2) we again use the it-notation

$$ds^2 = \mathcal{K}^2\left[dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta d\varphi^2\right] + \mathcal{R}^2 di\psi'^2 \qquad \text{(II.3.3)}$$

and we bring the static dS metric into the form

$$\begin{gathered} ds^2 = \alpha^2 dr^2 + r^2 d\vartheta^2 + r^2\sin^2\eta d\varphi^2 + a^2\mathcal{R}^2 di\psi^2, \\ a = 1/\alpha = \cos\eta = \sqrt{1 - r^2/\mathcal{R}^2}, \quad v = r/\mathcal{R} \end{gathered} \qquad \text{(II.3.4)}$$

and we read off the 4-beine

$$\begin{gathered} \overset{1'}{e}_{1'} = \mathcal{K}, \quad \overset{2'}{e}_{2'} = r = e^{\psi'}r', \quad \overset{3'}{e}_{3'} = r\sin\vartheta = e^{\psi'}r'\sin\vartheta, \quad \overset{4'}{e}_{4'} = \mathcal{R} \\ \overset{1}{e}_{1} = \alpha, \quad \overset{2}{e}_{2} = r, \quad \overset{3}{e}_{3'} = r\sin\vartheta, \quad \overset{4}{e}_{4} = a\mathcal{R} \end{gathered} . \qquad \text{(II.3.5)}$$

In contrast to the preceding Sections the time dependence of the quantities has to be considered

$$\partial_{4'} = \frac{\partial}{\mathcal{R}\partial i\psi'} \quad . \tag{II.3.6}$$

Instead of $A_{41}{}^{4}$ a new quantity enters the theory

$$\begin{aligned} {}'U_{1'} &= {}'A_{4'1'}{}^{4'} = -\overset{4'}{e}_{4'} \underset{4'|1'}{e^{4'}} = \frac{1}{\mathcal{R}}\mathcal{R}_{|1'} = 0, \\ {}'U_{4'} &= {}'A_{1'4'}{}^{1'} = -\overset{1'}{e}_{1'} \underset{1'|4'}{e^{1'}} = e^{-\psi'} \frac{\partial}{\mathcal{R}\partial i\psi'} e^{\psi'} = -\frac{i}{\mathcal{R}}. \end{aligned} \tag{II.3.7}$$

In this derivation it has been considered that the curvature of the pseudo-hyper sphere is constant $\left(\mathcal{R}_{|m'} = 0\right)$. The old quantities get a time-like component. Thus, one has

$$\begin{aligned} {}'U_{m'} &= \left\{0,0,0,-\frac{i}{\mathcal{R}}\right\} \\ B_{m'} &= \left\{\frac{1}{r'}e^{-\psi'},0,0,-\frac{i}{\mathcal{R}}\right\} = \left\{\frac{1}{r},0,0,-\frac{i}{\mathcal{R}}\right\} \quad . \\ C_{m'} &= \left\{\frac{1}{r'}e^{-\bar{\psi}},\frac{1}{r'}e^{-\psi'}\cot\vartheta,0,-\frac{i}{\mathcal{R}}\right\} = \left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0,-\frac{i}{\mathcal{R}}\right\} \end{aligned} \tag{II.3.8}$$

According to (II.3.1), second relation, the scale factor is position-independent but dependent on time. Therefore, we note

$$\frac{1}{\mathcal{K}}\mathcal{K}_{|m'} = \left\{0,0,0,-\frac{i}{\mathcal{R}}\right\}. \tag{II.3.9}$$

To free ourselves from the tedious coordinate technology, we review the coordinate transformation which we have introduced at the beginning. We recall the relation $\mathcal{R}d\eta = \frac{1}{\cos\eta}dr$ and we obtain

$$\begin{aligned} dr' &= \frac{e^{-\psi'}}{\cos^2\eta}dr - \sin\eta e^{-\psi'}dt = \frac{1}{\mathcal{K}}\alpha^2 dr + \frac{1}{\mathcal{K}}iv\,idt \\ idt' &= idt - i\frac{\sin\eta}{\cos^2\eta}dr = idt - i\alpha^2 v dr \end{aligned} \quad .$$

From this we derive the transformation matrix

$$\Lambda_i^{i'} = x^{i'}{}_{|i} \tag{II.3.10}$$

for the Lemaître-coordinate transformation. Finally, one has

$$\Lambda_i^{i'} = \begin{pmatrix} \frac{\alpha^2}{\mathcal{K}} & & & \frac{iv}{\mathcal{K}} \\ & 1 & & \\ & & 1 & \\ -i\alpha^2 v & & & 1 \end{pmatrix}, \quad \Lambda_{i'}^{i} = \begin{pmatrix} \mathcal{K} & & & -iv \\ & 1 & & \\ & & 1 & \\ i\mathcal{K}\alpha^2 v & & & \alpha^2 \end{pmatrix}. \tag{II.3.11}$$

With the 4-bein system (II.3.5) and with

$$L^m_{m'} = \overset{m}{e}_i \, \Lambda^i_{i'} \, \underset{m'}{e}^{i'} , \tag{II.3.12}$$

we calculate the matrix of a pseudo-rotation

$$L^m_{m'} = \begin{pmatrix} \alpha & & & -i\alpha v \\ & 1 & & \\ & & 1 & \\ i\alpha v & & & \alpha \end{pmatrix} , \tag{II.3.13}$$

whereby it is to be considered wether this pseudo-rotation can be used as Lorentz transformation. The galaxies move apart as the consequence of the expansion of the universe. v is the velocity observed from any point of the pseudo-hypersphere and α is the associated Lorentz factor. Thus, v is the recession velocity of the galaxies and according to (II.3.4) a quantity determined by the geometry of the model.

For further calculations we present some auxiliary relations:

$$r_{|1} = \underset{1}{e}^1 \frac{\partial r}{\partial r} = a, \quad r_{|4} = 0,$$

$$r_{|1'} = \underset{1'}{e}^{1'} \frac{\partial \mathcal{K} r'}{\partial r'} = 1, \quad r_{|4'} = \left(\mathcal{K} r'\right)_{|4'} = r' \mathcal{K}_{|4'} = r \frac{1}{\mathcal{K}} \mathcal{K}_{|4'} = -i \frac{r}{\mathcal{R}} = -iv,$$

$$r_{|m} = \left\{a, 0, 0, 0\right\}, \quad r_{|m'} = \left\{1, 0, 0, -iv\right\} = \left\{\alpha, 0, 0, -i\alpha v\right\} a . \tag{II.3.14}$$

The second term in (II.3.14) could have been derived from the first with the Lorentz transformation (II.3.13). We repeatedly make use of such an operation. With (II.3.14) and (II.3.4), last relation, the change of the recession velocity can be calculated.

$$v_{|m} = \left(\frac{r}{\mathcal{R}}\right)_{|m} = \left\{a, 0, 0, 0\right\} \frac{1}{\mathcal{R}}, \quad v_{|m'} = \left\{\alpha, 0, 0, -i\alpha v\right\} \frac{a}{\mathcal{R}} . \tag{II.3.15}$$

The relation

$$v = \sin\eta = \frac{r}{\mathcal{R}} . \tag{II.3.16}$$

shows the linear dependency between the recession velocity v of the galaxies and the measure for the distance r of the stars[22] and leads to the Hubble parameter, which describes the expansion of the universe. With the simple equipment of the Lorentz transformation we are armed to deduce the expanding version of the de Sitter cosmos from the static version. Thereby the Robertson-Lemaître co-ordinate system proves to be redundant. A co-ordinate transformation could change the mathematical description of circumstances. However, it should not change the geometrical or physical background. The question of, why the properties of the cosmos change so drastically under such a co-ordinate transformation will be discussed critically later on.

The transition from one aspect to the other is caused by a pseudo rotation in the tangent space of the 4-dimensional surface. By this means, it is elucidated that the geometrical body of the theory does not undergo any change.

[22] The correct data on the distance of two points in the 1-direction would be the length of the curve $\mathcal{R}\eta$ on the hypersphere.

We highlight again the relative velocity obtained from the Lorentz transformation and make the connection to the usual definitions of the recession velocities in the cosmological models.

In the comoving system one has for the radial motion $dx^{1'}/dT' = 0$, whereby the proper time dT' of an observer coincides with the coordinate time dt'. For the non-comoving observer

$$v = \frac{dx^1}{dT}, \quad \frac{dT}{dT'} = \alpha, \quad dx^1 = \alpha dr$$

applies to the relative velocity. Thus, one has

$$r^{\bullet} = \frac{dr}{dT'} = v, \quad v = \frac{r}{\mathcal{R}} = \sin\eta . \tag{II.3.17}$$

The relative velocity can only have values in the interval $[0,1]$ and cannot exceed the speed of light. One obtains the same relation from

$$r = \mathcal{R}\sin\eta, \quad r = \mathcal{K}r', \quad \mathcal{K} = e^{\psi'}, \quad r^{\bullet} = \frac{1}{\mathcal{K}}\mathcal{K}^{\bullet}r, \quad r^{\bullet} = Hr, \quad H = \frac{1}{\mathcal{K}}\mathcal{K}^{\bullet}, \tag{II.3.18}$$

with H as Hubble parameter. With (II.3.9) again results (II.3.17). The horizon of the model is therefore at $r = \mathcal{R}$, ie at the equator of the spherical space which is assigned to any observer at $r = 0$.

In order to advance with the mathematical wording of the expanding version, one has to examine the transformation behavior of the Ricci-rotation coefficients. The transition from the comoving observer system m' to the non-comoving system m takes place with a pseudo-rotation L, which is at best a Lorentz transformation. The Ricci-rotation coefficients transform inhomogeneously

$$'A_{m'n'}{}^{s'} = A_{m'n'}{}^{s'} + 'L_{m'n'}{}^{s'}, \quad 'L_{m'n'}{}^{s'} = L_s^{s'} L^{s}_{n'|m'} . \tag{II.3.19}$$

The just-described operation has to be analyzed more exactly. In accordance with aspect (I) the Ricci-rotation coefficients $'A_{m'n'}{}^{s'}$ are responsible for the deviation from Euclidean geometry and therefore describe the curvature of the space. In this sense, one would have to attribute to the Lorentz transformation the ability to change the curvature of the space. In accordance with aspect (III) the term Riemann geometry is substantially more accurate. Riemannian is only what is attributed to a surface and describable by means of the surface theory. All the rest is an additional structure that is defined on the surface. In the case of the de Sitter cosmos this additional structure is a matter of reference systems, which are created from the static reference systems by a local pseudo rotation in the tangent spaces of the 4-dimensional surface. One can show that the structure is only attached to the Riemannian body, by proceeding from the Ricci tensor in the primed system. Equations that refer to the Lorentz terms can be decoupled from the Riemann tensor. The vanishing of these terms is a necessary and trivial condition for the Lorentz invariance of the Ricci tensor.

The first term on the right side of (II.3.19) is a tensor and is transformed according to

$$A_{n'm'}{}^{s'} = L^{m\,n\,s'}_{m'n's} A_{nm}{}^{s} . \tag{II.3.20}$$

There are good reasons to attribute tensorial properties to the Ricci-rotation coefficients. The Ricci-rotation coefficients are a subsumption of the curvatures of the 4-dimensional surface. A tangent-space transformation is not able to change these curvatures; it exhibits only a new aspect on these curvatures. The geometrical skeleton of the theory remains invariant under Lorentz transformations.

Since the pseudo-rotation takes place in the [1,4]-slice of the space the inhomogeneous term can be simplified as

$$'L_{m'n'}{}^{s'} = h_{m'}{}^{s'}\,'L_{n'} - h_{m'n'}\,'L^{s'}, \quad 'L_{n'} = 'L_{s'n'}{}^{s'} = \left\{'L_{4'1'}{}^{4'},\, 'L_{1'4'}{}^{1'}\right\}. \qquad \text{(II.3.21)}$$

With $L^{1}_{1'} = \alpha,\, L^{1}_{4'} = -i\alpha v$ one first obtains

$$'L_{1'} = i\alpha^2 v_{|4'}, \quad 'L_{4'} = -i\alpha^2 v_{|1'}$$

and from this with (II.3.15)

$$'L_{m'} = \left\{\alpha^2 v \frac{1}{\mathcal{R}}, 0, 0, -i\alpha^2 \frac{1}{\mathcal{R}}\right\}. \qquad \text{(II.3.22)}$$

The transformation law (II.3.19) is reduced to

$$'U_{m'n'}{}^{s'} = U_{m'n'}{}^{s'} + 'L_{m'n'}{}^{s'}, \qquad \text{(II.3.23)}$$

because the lateral field quantities B and C transform homogeneously. Since we now operate only in the [1,4]-slice of the surface, (II.3.23) is like (II.3.21) simplified to

$$'U_{m'} = U_{m'} + 'L_{m'}. \qquad \text{(II.3.24)}$$

In the previous Section we have calculated the quantity

$$E_m = \left\{\frac{1}{\mathcal{R}} \tan\eta, 0, 0, 0\right\}$$

for the static version from the metric coefficients. Here we use the geometric form $U_m = -E_m$. With (II.3.16) on obtains

$$U_m = \left\{-\alpha v \frac{1}{\mathcal{R}}, 0, 0, 0\right\}. \qquad \text{(II.3.25)}$$

After a Lorentz transformation one has

$$U_{m'} = L^{m}_{m'} U_m = \left\{-\alpha^2 v \frac{1}{\mathcal{R}}, 0, 0, i\alpha^2 v^2 \frac{1}{\mathcal{R}}\right\}.$$

Up to now, only half the path to the representation of the expanding cosmos has been covered. However, the equations $U_{m'}$ still describe the static cosmos, observed by the moving observer. The quantities in the brackets are the observer's measured values.

Finally, with (II.3.24) and (II.3.22) one has

$$'U_{m'} = \left\{0, 0, 0, -\frac{i}{\mathcal{R}}\right\}. \qquad \text{(II.3.26)}$$

This time the quantity $'U$ from (II.3.8) has been calculated by using a transformation of the reference system. The latter method is more pleasant. It can be used to calculate the field quantities for a system for which a coordinate system is not known, or for a system for which a coordinate system does not exist. We will consider the latter case later.

The Ricci transforms using (II.3.19) according to

$$R_{m'n'} = L^{m\ n}_{m'n'} R_{mn} + L_{m'n'} \ .$$

If the Ricci should be invariant with respect to the transformation the second term in the above relation must vanish. One can show that this term takes for spherically symmetric systems, after a pseudo-rotation in the [1,4]-slice and after some calculations the form

$$'L^{s'}{}_{|s'} + 'U_{s'}\, 'L^{s'} = 0, \quad 'U_{s'} = L^{s}_{s'} U_s + 'L_{s'}, \quad L_s = -L^{s'}_{s}\, 'L_{s'}, \quad L^{s}_{|s} + U_s L^s = 0 \ . \qquad \text{(II.3.27)}$$

Thus, (II.3.27) is the condition for invariance of the Ricci for the models we will treat and the guarantee that the transformations were prepared correctly. The remaining part of the Ricci tensor describes the curvature of the 4-dimensional surface in the stricter signification. With this we have the tools at hand with which we will examine cosmological models.

Before we turn to the field equations, we deal in detail with the reference systems. With the Lorentz transformation (II.3.13) one can observe the unit vectors from the comoving system

$$m_{n'} = \{\alpha, 0, 0, -i\alpha v\}, \quad b_{n'} \overset{*}{=} b_n, \quad c_{n'} \overset{*}{=} c_n, \quad u_{n'} = \{i\alpha v, 0, 0, \alpha\} \ . \qquad \text{(II.3.28)}$$

To obtain information about the physics of the expanding observer, one has to introduce a frame of reference

notation: $$'m_{m'} = \{1, 0, 0, 0\}, \quad 'u_{m'} = \{0, 0, 0, 1\} \qquad \text{(II.3.29)}$$

that is linked to the expanding observer, ie to its *rest frame*. The space-time decomposition is performed in the comoving system with respect to this reference system. If we compare (II.3.26) with (II.3.25) we notice that they are of quite different types. While the quantity (II.3.25) describes forces that are oriented into all directions at any point, these forces are no longer present in (II.3.26). Obviously, the comoving system follows these forces in such a way that an observer no longer perceives forces in this system. From the free fall in the gravity field of the earth one knows that a moving observer is exposed to other forces than one observer in rest is exposed. For such an observer the force of gravity is not to be experienced. However, tidal forces act on him. We will write about this subject later. On closer inspection of the relation (II.3.24) it can be seen that

$$'U_{1'} = U_{1'} + 'L_{1'} = 0 \, .$$

It is evident that the force U of the static model is canceled by the dynamic term $'L_{1'}$. Finally, still another fairly plausible explanation for the quantity E of the static version can be found. If one tries to stop the expanding motion in an arbitrary space point in an arbitrary direction, which one may call 1-direction, a force E offers resistance to this attempt.

But the fourth component of the quantity (II.3.26) is occupied. It describes the enlargement of a volume element in the radial direction. The fourth components of the quantities B and C of (II.3.8) are responsible for the two other spatial directions. The numerical identity

$$'U_{4'} \stackrel{*}{=} B_{4'} \stackrel{*}{=} C_{4'} = -\frac{i}{\mathcal{R}} \tag{II.3.30}$$

applies. Thus, the *expansion scalar*

$$'u^{s'}{}_{||s'} = 'A_{s'}u^{s'} \stackrel{*}{=} -3\frac{i}{\mathcal{R}} \tag{II.3.31}$$

describes the total expansion of a volume element. All this suggests that the metric (II.3.2) could be the metric of an expanding universe.

We will write down the field equations now in the geometric form. First, we decompose the Ricci-rotation coefficients according to

$$'A_{m'n'}{}^{s'} = 'U_{m'n'}{}^{s'} + B_{m'n'}{}^{s'} + C_{m'n'}{}^{s'}. \tag{II.3.32}$$

We decompose the above quantities once more

$$\begin{gathered} 'U_{m'n'}{}^{s'} = h_{m'}{}^{s'}\,'U_{n'} - h_{m'n'}\,'U^{s'}, \\ B_{m'n'}{}^{s'} = b_{m'}B_{n'}b^{s'} - b_{m'}b_{n'}B^{s'}, \quad C_{m'n'}{}^{s'} = c_{m'}C_{n'}c^{s'} - c_{m'}c_{n'}C^{s'}. \end{gathered} \tag{II.3.33}$$

Therein $h_{m'n'} = \mathrm{diag}(1,0,0,1)$ is a submatrix of the tetrad metric $g_{m'n'} = \mathrm{diag}(1,1,1,1)$.

With the quantities (II.3.8) the Ricci

$$R_{m'n'} = A_{m'n'}{}^{s'}{}_{|s'} - A_{n'|m'} - A_{r'm'}{}^{s'}A_{s'n'}{}^{r'} + A_{m'n'}{}^{s'}A_{s'}, \quad A_{n'} = A_{r'n'}{}^{'r} \tag{II.3.34}$$

takes the form

$$\begin{aligned} R_{m'n'} = &-\left['U^{s'}{}_{\underset{1}{||}s'} + 'U^{s}\,'U_{s'} \right] h_{m'n'} \\ &-\left[B_{n'\underset{2}{||}m'} + B_{n'}B_{m'} \right] - b_{n'}b_{m'}\left[B^{s'}{}_{\underset{2}{||}s'} + B^{s'}B_{s'} \right] \\ &-\left[C_{n'\underset{3}{||}m'} + C_{n'}C_{m'} \right] - c_{n'}c_{m'}\left[C^{s'}{}_{\underset{3}{||}s'} + C^{s'}C_{s'} \right] \\ -\frac{1}{2}R = &\left['U^{s'}{}_{\underset{1}{||}s'} + 'U^{s'}\,'U_{s'} \right] + \left[B^{s'}{}_{\underset{2}{||}s'} + B^{s'}B_{s'} \right] + \left[C^{s'}{}_{\underset{3}{||}s'} + C^{s'}C_{s'} \right] \end{aligned}, \tag{II.3.35}$$

whereby the use of the graded derivatives

$$'U_{n'\underset{1}{||}m'} = 'U_{n'|m'}, \quad B_{n'\underset{2}{||}m'} = B_{n'|m'} - 'U_{m'n'}{}^{s'}B_{s'}, \quad C_{n'\underset{3}{||}m'} = C_{n'|m'} - 'U_{m'n'}{}^{s'}C_{s'} - B_{m'n'}{}^{s'}C_{s'} \tag{II.3.36}$$

proves to be very advantageous. The structure (II.3.35) can be used for spherically symmetric systems, static, expanding, or collapsing systems, as well.

Now it is necessary to show that the quantities described in (II.3.8) satisfy the field equations

$$R_{m'n'} - \frac{1}{2}Rg_{m'n'} + \lambda g_{m'n'} = 0 \tag{II.3.37}$$

Since the radius of curvature of the pseudo-hypersphere is a constant, one has immediately

$$'U^{s'}{}_{\underset{1}{\|}s'} + 'U^{s}\,'U_{s'} = -\frac{1}{\mathcal{R}^2}\,. \tag{II.3.38}$$

The other subequations of the field equations in (II.3.35) are easy to calculate, if we use in (II.3.8) for the quantities B and C in each case the second parenthesized expressions, and if we also consider (II.3.14) and finally (II.3.36). One has

$$\begin{gathered} B_{n'\underset{2}{\|}m'} + B_{n'}B_{m'} = -h_{n'm'}\frac{1}{\mathcal{R}^2}, \quad C_{n'\underset{3}{\|}m'} + C_{n'}C_{m'} = -\left('m_{n'}\,'m_{m'} + b_{n'}b_{m'} + 'u_{n'}\,'u_{m'}\right)\frac{1}{\mathcal{R}^2} \\ B^{s'}{}_{\underset{2}{\|}s'} + B^{s'}B_{s'} = -\frac{2}{\mathcal{R}^2}, \quad C^{s'}{}_{\underset{3}{\|}s'} + C^{s'}C_{s'} = -\frac{3}{\mathcal{R}^2} \end{gathered}\,. \tag{II.3.39}$$

Summing up, one obtains

$$R_{m'n'} = \frac{3}{\mathcal{R}^2}g_{m'n'}, \quad R = \frac{12}{\mathcal{R}^2}, \quad G_{m'n'} = -\frac{3}{\mathcal{R}^2}g_{m'n'}, \quad \lambda = \frac{3}{\mathcal{R}^2} \tag{II.3.40}$$

and one notes that one has gained the same results as in the static case. This result will give us reason to think it over.

We make up for some details which have led us to the relation (II.3.24). If one introduces the Lorentz angle χ by

$$\cos i\chi = \alpha, \quad \sin i\chi = i\alpha v, \quad \tan i\chi = iv \tag{II.3.41}$$

one obtains from (II.3.13) and (II.3.19), second equation, the few non-zero components of the Lorentz term

$$L_{m'1'}{}^{4'} = i\chi_{|m'}\ . \tag{II.3.42}$$

This equation describes the change of the Lorentz angle, both in regard to the spatial, and the time-like change during the expansion. However, the Lorentz angle is not arbitrary, but is in accordance with the relation

$$\cos i\chi \cos\eta = 1 \tag{II.3.43}$$

closely attached to the underlying geometry, ie which pseudo-rotation has to be implemented in each position of the space is prescribed.

This also explains why the spatial components of the lateral field quantities in (II.3.8) appear to be flat. For the static version one has with (II.3.14)

$$A_{21}{}^{2} = -\overset{2}{e}_{2}\,\underset{2|1}{e^2} = \frac{1}{r}r_{|1} = \frac{a}{r}, \quad A_{21}{}^{2} = -\overset{3}{e}_{3}\,\underset{3|2}{e^3} = \frac{1}{r\sin\vartheta}\left(r\sin\vartheta\right)_{|2} = \frac{1}{r}\cot\vartheta, \quad a = \cos\eta$$

and thus,

$$B_m = \left\{\frac{1}{r}\cos\eta, 0, 0, 0\right\}, \quad C_m = \left\{\frac{1}{r}\cos\eta, \frac{1}{r}\cot\vartheta, 0, 0\right\}. \tag{II.3.44}$$

Having performed the Lorentz transformation with (II.3.41) and (II.3.43) one has with $\left(\alpha' = 1', 2', 3'\right)$

$$\begin{aligned} B_{\alpha'} &= \left\{\frac{1}{r}\cos\eta\cos i\chi, 0, 0\right\} = \left\{\frac{1}{r}, 0, 0\right\} \\ C_{\alpha'} &= \left\{\frac{1}{r}\cos\eta\cos i\chi, \frac{1}{r}\cot\vartheta, 0\right\} = \left\{\frac{1}{r}, \frac{1}{r}\cot\vartheta, 0\right\} \end{aligned}, \tag{II.3.45}$$

the typical expressions of a flat spherical geometry. A look at the first brackets shows that the typical factor for the curvature of the space $\cos\eta = \sqrt{1 - r^2/\mathbb{R}^2}$ is listed, but the curvature is hidden according to (II.3.43).

By treating the above quantities just as one is used to with the flat geometry, thus by using the second parenthetical expressions in each case, one has

$$\begin{aligned} & B_{\alpha'|\beta'} + B_{\alpha'}B_{\beta'} = 0, && B^{\gamma'}{}_{|\gamma'} + B^{\gamma'}B_{\gamma'} = 0 \\ & C_{\alpha'|\beta'} - B_{\beta'\alpha'}{}^{\gamma'}C_{\gamma'} + C_{\alpha'}C_{\beta'} = 0, && C^{\gamma'}{}_{|\gamma'} + B_{\gamma'}C^{\gamma'} + C^{\gamma'}C_{\gamma'} = 0 \end{aligned}. \tag{II.3.46}$$

Thus, the spatial Ricci vanishes

$$\begin{aligned} {}^*R_{\alpha'\beta'} = &-\left[B_{\beta'|\alpha'} + B_{\beta'}B_{\alpha'}\right] - b_{\alpha'}b_{\beta'}\left[B^{\gamma'}{}_{|\gamma'} + B^{\gamma'}B_{\gamma'}\right] \\ &-\left[C_{\alpha'|\beta'} - B_{\beta'\alpha'}{}^{\gamma'}C_{\gamma'} + C_{\alpha'}C_{\beta'}\right] - c_{\alpha'}c_{\beta'}\left[C^{\gamma'}{}_{|\gamma'} + B_{\gamma'}C^{\gamma'} + C^{\gamma'}C_{\gamma'}\right] = 0 \end{aligned}. \tag{II.3.47}$$

The spatial part of the expanding universe appears flat, which also the metric (II.3.2) suggests. Taking this literally, one follows the interpretation which has been described and illustrated by Schrödinger [S] in his textbook in detail.

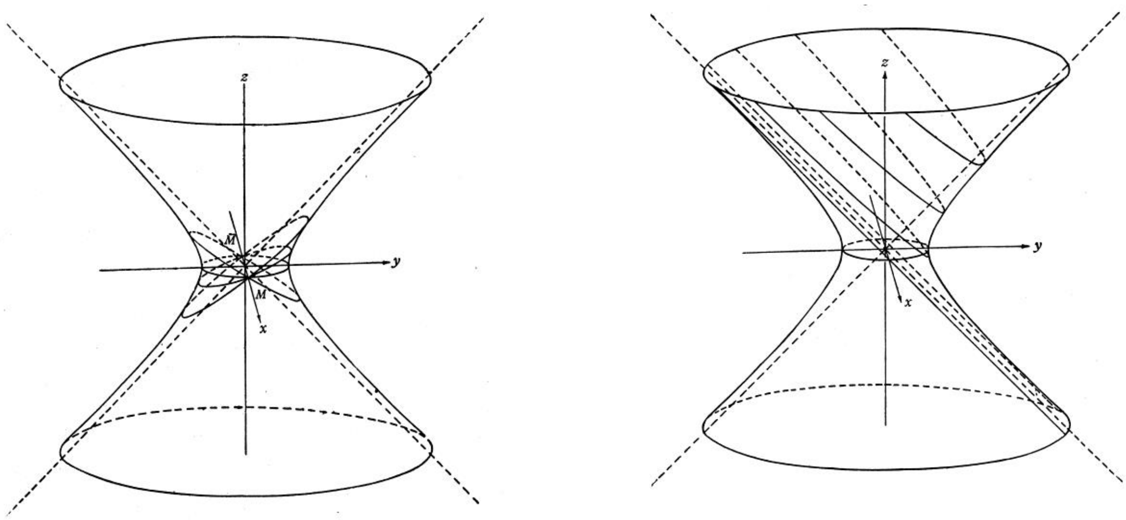

Fig. II.1

From Schrödinger's representation it can be seen that the two versions of the dS-cosmos are different sections of the pseudo-hyper sphere. We want to formulate the facts somewhat more strictly and say that there are two different models which are built on the same geometric framework.

Let us again turn to this problem. We will present the model mathematically in such a manner that the flatness of the space explicitly manifests itself. While retaining the 4-

dimensional notation, we split the Ricci-rotation coefficients in such a way that the 3-dimensional flat quantities are isolated. It turns out that the remaining terms are the second fundamental forms of the surface theory, which can be physically interpreted as tidal forces.

Since we now refer only to the expanding system, we now omit the primes at the kernels and indices. A space-time decomposition of the Ricci-rotation coefficients leads to

$$A_{mn}{}^{s} = {}^{*}B_{mn}{}^{s} + {}^{*}C_{mn}{}^{s} + D_{mn}{}^{s}, \quad D_{mn}{}^{s} = u_n D_m{}^{s} - u^s D_{mn} \ , \tag{II.3.48}$$

whereby

$${}^{*}B_m = \left\{ \frac{1}{\mathcal{R}\sin\eta}, 0,0,0 \right\}, \quad {}^{*}C_m = \left\{ \frac{1}{\mathcal{R}\sin\eta}, \frac{1}{\mathcal{R}\sin\eta}\cot\vartheta, 0,0 \right\} \tag{II.3.49}$$

are the constituents of the Ricci-rotation coefficients of a merely spatial transport

notation:
$$\Phi_{m\wedge n} = \Phi_{m|n} - {}^{*}B_{nm}{}^{s}\Phi_s - {}^{*}C_{nm}{}^{s}\Phi_s \ . \tag{II.3.50}$$

The time-like part of (II.3.48) contains, apart from the observer field u, the merely spatial quantities D_{mn} with

$$D_{11} = D_{22} = D_{33} = -\frac{i}{\mathcal{R}} \ . \tag{II.3.51}$$

One recognizes that all three components of this quantity have the same values. They are the tidal forces which act homogenously into all three directions of the hyper-surface. They cause expansions and thus, they are responsible for the expansion of the universe. Further, from (II.3.48) arises with

$$u_{m||n} = u_{(m||n)} = D_{mn}, \quad u_{[m||n]} = 0 \tag{II.3.52}$$

that the observer field u is subjected to deformations but not to rotations if transported to a neighboring point. The unit vector u is always normal to the 3-dimensional hyper-spherical surfaces and is geometrically understood as the *rigging vector* of these expanding surfaces. Thus, the equation (II.3.52) is the defining equation of the second fundamental forms of this surface. In Eq. (II.2.16) we had already harked back to these methods of the surface theory. There, however, in the course of the embedding of a 4-dimensional pseudo-hyper surface into a 5-dimensional flat space.

Splitting off the tidal forces the Ricci tensor gets the form

$$R_{mn} = {}^{*}R_{mn} - \left[D_{mn\wedge s}u^s + D_{mn}D_s{}^{s} \right] + 2u_n D_{[s}{}^{s}{}_{\wedge \underline{m}]} - u_m u_n \left[D_s{}^{s}{}_{\wedge r}u^r + D_{sr}D^{sr} \right] . \tag{II.3.53}$$

Since the second fundamental forms are constant quantities the relations[23]

$$D_{[s}{}^{s}{}_{\wedge \underline{m}]} = 0, \quad D_{mn\wedge s}u^s = 0 \tag{II.3.54}$$

contained in (II.3.53) are trivially satisfied, the space-time decomposition of the Ricci tensors, however, is not entirely settled. Yet, one can extract the relations

[23] $\underline{m} = 1,2,3$

$$^*B_{n|4} + {}^*B_{sn}{}^r D_r{}^s = 0, \quad {}^*C_{n|4} + {}^*C_{sn}{}^r D_r{}^s = 0 \ . \tag{II.3.55}$$

Therein

$$^*B_{n|4} = -i\alpha v \frac{\partial\, {}^*B_n}{\mathcal{R}\,\partial\eta}, \quad {}^*C_{n|4} = -i\alpha v \frac{\partial\, {}^*C_n}{\mathcal{R}\,\partial\eta} \tag{II.3.56}$$

has the meaning of the changes of the curvatures on the slices of the pseudo-hyper sphere that the expanding observer notices by changing his position in the space-time structure on the pseudo-hyper spheres[24]. Finally, from the Ricci tensor only remains

$$R_{mn} = {}^3R_{mn} - D_{mn}D_s{}^s - u_m u_n D_{sr}D^{sr} \ , \tag{II.3.57}$$

whereby it can easily be shown that all expressions in the brackets of the 3-dimensional Ricci tensor

$$\begin{aligned} {}^3R_{mn} = &-\left[{}^*B_{n|\underline{m}} + {}^*B_n{}^*B_m\right] - b_m b_n\left[{}^*B^s{}_{|s} + {}^*B^s{}^*B_s\right] \\ &-\left[{}^*C_{n|\underline{m}} - {}^*B_{mn}{}^s{}^*C_s + {}^*C_n{}^*C_m\right] - c_m c_n\left[{}^*C^s{}_{|s} + {}^*B_s{}^*C^s + {}^*C^s{}^*C_s\right] \end{aligned} \tag{II.3.58}$$

vanish and thus the 3-dimensional Ricci tensor vanishes as well. In the same simple way, it can be shown that the 3-dimensional Riemann tensor vanishes.

If one arranges, with the aid of the forgoing results, the extended Einstein field equations with the cosmological term by using (II.3.53), the contraction of this relation, and further by separating (II.3.54), (II.3.55), and (II.3.58), one gains another relation between the two kinds of the second fundamental forms

$$-\left[D_{mn}D_s{}^s + u_m u_n D_{sr}D^{sr} - \frac{1}{2}g_{mn}\left(D_s{}^s D_r{}^r + D_{sr}D^{sr}\right)\right] + 2A_{m[s}A_{n]}{}^s - g_{mn}A_{[r}{}^s A_{s]}{}^r = 0\,, \tag{II.3.59}$$

which could be checked with (II.2.15) and (II.3.51). The geometry and the physics of the model can be also described with the help of the equations (II.3.54), (II.3.55), and (II.3.58). Thus, the de Sitter cosmos as borderline case of a realistic expanding universe can be described exhaustively with the methods of the classical surface theory and supports the acceptance that nature can be characterized perfectly with the help of an actual curvature term.

The cosmological solution of de Sitter was soon regarded as unsatisfactory. The cosmological constant in the field equations seems rather artificial, and the cosmos is empty. Therefore, one has redefined the cosmological constant as pressure and energy density.

$$\kappa p = -\lambda, \quad \kappa\mu_0 = \lambda\,. \tag{II.3.60}$$

The stress-energy-momentum tensor

$$T_{mn} = -p\,g_{mn} + \left(p + \mu_0\right)u_m u_n$$

generally has with $u_m = \{-i\alpha v, 0, 0, \alpha\}$ the components

[24] It has to be pointed out that the expansion of the universe has to be understood as the inflation of the 3-dimensional hyper-sphere. However, the 4-dimensional pseudo-hyper sphere remains unchanged. A larger universe corresponds to a larger slice of the pseudo-hyper sphere.

$$\begin{aligned} T_{11} &= -p - \alpha^2 v^2 \left(p + \mu_0\right) \\ T_{22} &= T_{33} = -p \\ T_{14} &= -i\alpha^2 v\left(p + \mu_0\right) \\ T_{44} &= \mu_0 + \alpha^2 v^2 \left(p + \mu_0\right) \end{aligned} \tag{II.3.61}$$

and in the comoving system

$$T_{11} = T_{22} = T_{33} = -p, \quad T_{44} = \mu_0 \,. \tag{II.3.62}$$

Also for the non-moving observer the form (II.3.62) is valid for the stress-energy-momentum tensor as a consequence of the equation of state $p + \mu_0 = 0$. Such an observer cannot detect an energy flow. In the light of these considerations, the interpretation of (II.3.16) as the expansion rate of the universe is called to question. In a paper Mitra [M] has drawn attention to the fact that there exist conflicts, regarding the transformation from a comoving reference frame to a non-comoving one.

We note that the transformation of Lemaître still allows a different interpretation: The cosmos remains static, the coordinate system expands. Observers associated with this coordinate system, move in free fall away into all directions. In this observer system, the E forces which occur in the static system are no longer noticeable. This corresponds to the principle of Einstein's elevator. Observers moving in free fall in the gravitational field of the Earth, will not experience gravity. By this interpretation it is avoided that one has to explain how a global restructuring of the pseudo-hyper sphere by a local Lorentz transformation is to be performed. The question of the curvature of the dS-world has a simple answer.

In a simple way it can be shown that the 3-dimensional Riemann tensor vanishes. From this one could conclude that the curvature of the space depends to a very large degree on the observer. For a static observer the 3-dimensional space is curved, for the expanding observers flat. That does not contradict the initially mentioned aspect (I). However, it is in obvious contradiction to (II) and (III). There it has to be remembered that for the two latter aspects the term Riemannian geometry has to be understood in a more restrictive sense, than the supporters of (I) would permit. From the Riemann and Ricci tensors are to be extracted the dynamic terms which are created by the pseudo rotations in the tangent spaces. The only terms that remain describe, in the already well-known way, the curvatures of the pseudo-hyper spheres. Due to the motion on a pseudo-hyper sphere the curvature effects are hidden from the expanding observer, forces arising from the curvature of the space are compensated. In this sense, the curvature of the space must be looked upon immutably and the actual interpretation of the term curvature for de Sitter cosmos has worked out satisfactorily.

Some interesting, even very speculative ideas concerning the origin of the universe one could utter in connection with the model of the de Sitter cosmos which describes also some properties of our real cosmos. We follow the widely accepted opinion that the cosmos was created from a very small region of extremely high energy density by the Big Bang. Initially it underwent an inflationary development which slowed down to the expansion observed today. This early world was built up very simply. It was controlled by only one interaction which could possibly be derived from the energy object. This interaction has split up by spontaneous symmetry breaking into the four interactions of very different strengths which represent the basic forces of the physical phenomenon in today's world. Hawking brought a further idea into this scenario: The time has been initially imaginary and has become real shortly after the Big Bang.

If we take up this idea we have to re-formulate it. Since, with respect to our view the measurable time is the real associated number of an imaginary quantity which parameterizes the fourth dimension, it is only natural to accept this dimension as real in the beginning. That means that the early world is not to be regarded as a pseudo-hyper sphere but as a real hyper sphere which leads to a further simplification of the world model. Since there had existed only one interaction there had been only four equal dimensions as well which all had been closed. There had been no time and thus also no eternity. After the fourth dimension had unfolded it had gained the property of transience. The time-like lines are open, they are curves on a hyperboloid of revolution of constant curvature in the pseudo-real representation.

Again following the interpretation of the dS-cosmos shown in detail by Schrödinger, one can say that the spatial openness of the expanding de Sitter cosmos is forced by the pseudo-hyperbolic openness of the fourth dimension. A piece of the time-like openness is given to the spatial piece of the structure in performing a space-time rotation by means of a suitable slice of the pseudo-hyper sphere. The slices increasing in the flow of time give reason for the expansion of the cosmos. However, the essential geometrical structure remains stable.

The question of how to be understand the de Sitter model in its two versions, was widely discussed at that time. Finally, one contacted the famous mathematician Klein. His detailed answer terminated the discussion. It is not known whether the authority of Klein or the argumentative content of his paper was crucial. However, we cannot find a connection between the geometries of the hyper-spheres and the spacetime slices on one hand, and the statements of Klein on other hand. We have also found no papers which address this publication by Klein.

Physics could also give an answer to the origin of the early world. In our world particles are permanently created by vacuum fluctuation and they disappear again. Likewise, it is also conceivable that the early world resulted from emptiness by fluctuation and that it disappears again into emptiness. A procedure which cannot be sufficiently made understood with everyday-life terms. However, it is mathematically describable[25].

Physics does not permit further questions which go beyond the fluctuating emptiness. To think about causation of emptiness and its fluctuations, leads to questions empty of contents or to considerations which exceed the possibilities of linguistic matters. Moreover, the early world is constructed so simply that nothing can exist beyond the simplicity which could explain this simplicity. For further considerations one would need a super language which describes a super physics. In order to understand those, one would have to make a further step backwards, so that one gets into an infinite recursion.

The attempt to place an entity beyond the fluctuating emptiness and the early world, an entity which conceived our current universe with its whole development up to our current world with life and ourselves as reflecting human beings would presuppose a tremendous complexity of this entity which contradicts strongly the simplicity of the early world and also would entail a double track of the world affairs. In order to explain all further developments of the world, the acceptance is perfectly sufficient that the early world with its inherent simple laws, results from a fluctuation. Everything that goes beyond it, is not

[25] Some authors speculate on parallel universes and multi-universes. However, it makes little sense to talk about things that can never enter our experienced world.

science, but ideology. Everyone is left to formulate this at their own discretion. However, no one can get support from physics.

Literature: Du Val [D], Lemaître [L], Weyl [W] and quotations in Section II.7

II.4. The cosmological model of Lanczos

In a footnote in one of his papers Lanczos [L] mentioned an ansatz concerning an expanding cosmos, which has certain relation to the de Sitter cosmos, and in addition, to the Friedman cosmos that will be discussed later. Lanczos [L] harked back to this approach, but the ansatz has never been pursued by other authors, obviously because it does not comply with the expectations for a realistic model. Nevertheless, we deal in short with that model.

A 5-dimensional flat space is parameterized in the following way

$$\begin{aligned} x^1 &= \mathcal{R}_0 \cos i\psi \, \sin\eta \, \sin\vartheta \, \sin\varphi \\ x^2 &= \mathcal{R}_0 \cos i\psi \, \sin\eta \sin\vartheta \cos\varphi \\ x^3 &= \mathcal{R}_0 \cos i\psi \, \sin\eta \cos\vartheta \\ x^0 &= \mathcal{R}_0 \cos i\psi \, \cos\eta \\ x^4 &= \mathcal{R}_0 \sin i\psi \end{aligned} \quad . \tag{II.4.1}$$

The line element of this space reads as

$$ds^2 = d\mathcal{R}_0^2 + \cos^2 i\psi \left[\mathcal{R}_0^2 d\eta^2 + \mathcal{R}_0^2 \sin^2\eta d\vartheta^2 + \mathcal{R}_0^2 \sin^2\eta \sin^2\vartheta d\varphi^2 \right] + \mathcal{R}_0^2 di\psi^2 \ . \tag{II.4.2}$$

For $\mathcal{R}_0 = \text{const.}$ one obtains a specific sphere from the family of pseudo-hyper spheres. Its metric is

$$\begin{gathered} ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta d\varphi^2 + \mathcal{R}_0^2 di\psi^2 \\ \mathcal{R} = \mathcal{R}_0 \cos i\psi, \quad dx^4 = \mathcal{R}_0 di\psi = i dt \end{gathered} \quad . \tag{II.4.3}$$

It is noticeable that the 3-dimensional line element is time-dependent. The model expands or contracts. From the metric one reads

$$\begin{gathered} \overset{0}{e}_0 = 1, \quad \overset{1}{e}_1 = \mathcal{R}, \quad \overset{2}{e}_2 = \mathcal{R}\sin\eta, \quad \overset{3}{e}_3 = \mathcal{R}\sin\eta\sin\vartheta, \quad \overset{4}{e}_4 = \mathcal{R}_0 \\ \partial_0 = \frac{\partial}{\partial\mathcal{R}_0}, \quad \partial_1 = \frac{\partial}{\mathcal{R}\partial\eta}, \quad \partial_2 = \frac{\partial}{\mathcal{R}\sin\eta\partial\vartheta}, \quad \partial_3 = \frac{\partial}{\mathcal{R}\sin\eta\sin\vartheta\partial\varphi}, \quad \partial_4 = \frac{\partial}{\mathcal{R}_0\partial i\psi} \end{gathered} \ . \tag{II.4.4}$$

Therewith one computes the 5-dimensional Ricci-rotation coefficients. The lateral parts are

$$\begin{aligned} B_a &= \left\{ \frac{1}{\mathcal{R}_0}, \frac{1}{\mathcal{R}}\cot\eta, 0, 0, -\frac{1}{\mathcal{R}_0}\tan i\psi \right\} \\ C_a &= \left\{ \frac{1}{\mathcal{R}_0}, \frac{1}{\mathcal{R}}\cot\eta, \frac{1}{\mathcal{R}\sin\eta}\cot\vartheta, 0, -\frac{1}{\mathcal{R}_0}\tan i\psi \right\} \end{aligned} \ . \tag{II.4.5}$$

Therein the 0-components are the second fundamental forms of a 4-dimensional surface with constant radius $\mathcal{R}_0$

$$A_{10}{}^{1} = A_{20}{}^{2} = A_{30}{}^{3} = A_{40}{}^{4} = \frac{1}{\mathcal{R}_0} \tag{II.4.6}$$

and the 4-components the second fundamental forms of a 3-dimensional surface that expands or contracts

$$A_{14}{}^{1} = A_{24}{}^{2} = A_{34}{}^{3} = -\frac{1}{\mathcal{R}_0}\tan i\psi\,. \tag{II.4.7}$$

We want to limit the further development of the model to the 4-dimensional representation. We will discuss the model in two coordinate systems, in comoving $\{r',t'\}$ or $\{\eta',\psi'\}$ and in non-comoving $\{r,t\}$ or $\{\eta,\psi\}$. Between the two systems Florides [F] and Mitra [M] found a linking coordinate transformation which was revised by us [B] and supplemented with a Lorentz transformation. The metric was specified in comoving coordinates by Lanczos. Consequently, we re-write (II.4.3) as

$$\text{(A)} \qquad \begin{gathered} ds^2 = \mathcal{R}^2 d\eta'^2 + \mathcal{R}^2\sin^2\eta' d\vartheta^2 + \mathcal{R}^2\sin^2\eta'\sin^2\vartheta\, d\varphi^2 + \mathcal{R}_0^2 di\psi'^2 \\ \mathcal{R} = \mathcal{R}_0\cos i\psi', \quad dx^{4'} = \mathcal{R}_0 di\psi' = i\,dt' \end{gathered}\;. \tag{II.4.8}$$

and with

$$r' = \mathcal{R}_0\sin\eta', \quad \mathcal{K} = \cos i\psi' = \mathrm{ch}\psi', \quad \mathcal{R} = \mathcal{K}\mathcal{R}_0 \tag{II.4.9}$$

we get with $\mathcal{K}$ as the scale factor

$$\text{(A')} \qquad ds^2 = \mathcal{K}^2\left[\frac{1}{\cos^2\eta'}dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta\, d\varphi^2\right] - dt'^2\,, \tag{II.4.10}$$

or in canonical form

$$\text{(A'')} \qquad ds^2 = \mathcal{K}^2\left[\frac{1}{1-\frac{r'^2}{\mathcal{R}_0^2}}dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta\, d\varphi^2\right] - dt'^2\,. \tag{II.4.11}$$

The line element for cosmological models in comoving coordinates is written mostly as

$$ds^2 = \mathcal{K}^2\left[\frac{1}{1-k\frac{r'^2}{\mathcal{R}_0^2}}dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta\, d\varphi^2\right] - dt'^2\,. \tag{II.4.12}$$

Therein k is referred to as the curvature parameter. For $k=1$ the model is designated as positively curved and closed, for $k=0$ as flat and open, and for $k=-1$ as negatively curved and open. Later we will see that this definition is not quite reliable. In any case, the form (II.4.11) of the metric suggests that the Lanczos model is a positively curved and closed cosmos.

For the non-comoving System we expect

$$r = \mathcal{R}_0\sin\eta, \quad dt = \mathcal{R}_0 d\psi, \quad r = \mathcal{K}r'\,. \tag{II.4.13}$$

By comparing (II.4.9) and (II.4.13) we obtain the relation of the polar angles

$$\sin\eta = \sin\eta'\,\mathrm{ch}\psi' \tag{II.4.14}$$

and we write down the two auxiliary relations for later use

$$\cos^2\eta = \cos^2\eta' - \sin^2\eta'\,\mathrm{sh}^2\psi', \quad \cos^2\eta = \cos^2\eta'\,\mathrm{ch}^2\psi' - \mathrm{sh}^2\psi'. \tag{II.4.15}$$

We obtain the transformation matrix for the non-comoving coordinates with

$$r = \mathfrak{K}r', \quad \mathrm{th}\psi = \frac{1}{\cos\eta'}\mathrm{th}\psi'. \tag{II.4.16}$$

After differentiation and the use of the auxiliary relations (II.4.15) one has with $\Lambda^i_{i'} = x^i_{|i'}$

$$\Lambda^i_{i'} = \begin{pmatrix} \mathrm{ch}\psi' & & & -i\sin\eta'\,\mathrm{sh}\psi' \\ & 1 & & \\ & & 1 & \\ i\dfrac{\tan\eta}{\cos\eta}\dfrac{\mathrm{sh}\psi'}{\cos\eta'} & & & \dfrac{\cos\eta'}{\cos^2\eta} \end{pmatrix}$$
$$\Lambda^{i'}_i = \begin{pmatrix} \dfrac{\cos^2\eta'}{\cos^2\eta}\dfrac{1}{\mathrm{ch}\psi'} & & & i\sin\eta'\cos\eta'\,\mathrm{th}\psi' \\ & 1 & & \\ & & 1 & \\ -i\dfrac{\tan\eta}{\cos\eta}\mathrm{th}\psi' & & & \cos\eta' \end{pmatrix}. \tag{II.4.17}$$

This yields the line elements in the forms

$$\text{(II B)} \qquad ds^2 = \mathfrak{R}_0^2 d\eta^2 + \mathfrak{R}_0^2\sin^2\eta\, d\vartheta^2 + \mathfrak{R}_0^2\sin^2\eta\sin^2\vartheta\, d\varphi^2 + \mathfrak{R}_0^2\cos^2\eta\, di\psi^2, \tag{II.4.18}$$

$$\text{(II B')} \qquad ds^2 = \frac{1}{\cos^2\eta}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta\, d\varphi^2 - \cos^2\eta\, dt^2, \tag{II.4.19}$$

$$\text{(II B'')} \qquad ds^2 = \frac{1}{1-\dfrac{r^2}{\mathfrak{R}_0^2}}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta\, d\varphi^2 - \left(1-\frac{r^2}{\mathfrak{R}_0^2}\right)dt^2. \tag{II.4.20}$$

As with the de Sitter model, the line element relates to a pseudo-hyper sphere.

From the metrics A' and B' one reads the 4-beine for the comoving system A and the non-comoving System B

$$\overset{1'}{e}_{1'} = \frac{\mathrm{ch}\psi'}{\cos\eta'}, \quad \overset{2'}{e}_{2'} = \mathfrak{K}r', \quad \overset{3'}{e}_{3'} = \mathfrak{K}r'\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1$$
$$\overset{1}{e}_1 = \frac{1}{\cos\eta}, \quad \overset{2}{e}_2 = r, \quad \overset{3}{e}_3 = r\sin\vartheta, \quad \overset{4}{e}_4 = \cos\eta. \tag{II.4.21}$$

With

$$L^m_{m'} = \overset{m}{e}_i \Lambda^i_{i'} \underset{m'}{e}^{i'}, \tag{II.4.22}$$

we calculate the Lorentz transformation

$$L^{m}_{m'} = \begin{pmatrix} \dfrac{\cos\eta'}{\cos\eta} & & & -i\tan\eta\,\mathrm{th}\psi' \\ & 1 & & \\ & & 1 & \\ i\tan\eta\,\mathrm{th}\psi' & & & \dfrac{\cos\eta'}{\cos\eta} \end{pmatrix}. \tag{II.4.23}$$

Using (II.4.16), second equation, we borrow from (II.4.16) the relative velocity of the observers

$$v = \sin\eta\,\mathrm{th}\psi = \frac{r}{\mathcal{R}_0}\,\mathrm{th}\frac{t}{\mathcal{R}_0}. \tag{II.4.24}$$

The relative speed increases after an infinitely long time until it reaches the value of the velocity of light.

We want to see how this definition of velocity is related to the recession velocity of the galaxies commonly used in literature. The Hubble parameter H, which indicates the expansion behavior of the cosmos is with $\mathcal{K} = \mathrm{ch}\psi'$ for the present model

$$H = \frac{1}{\mathcal{K}}\mathcal{K}^{\bullet} = \frac{1}{\mathcal{R}_0}\mathrm{th}\psi'.$$

Thus, with (II.4.14) one has for the *coordinate velocity*

$$r^{\bullet} = Hr = \frac{r}{\mathcal{R}_0}\mathrm{th}\psi' = \sin\eta\;\mathrm{th}\psi' = \sin\eta'\,\mathrm{sh}\psi'. \tag{II.4.25}$$

The *relative velocity* in the expanding universe of a point calculated from the view of a non-comoving observer is

$$v = \frac{dx^1}{dT}, \quad dx^1 = \overset{1}{e}_1 dr = \frac{1}{\cos\eta}dr, \quad dT = \alpha dT' = \frac{\cos\eta'}{\cos\eta}dt'.$$

Thus, is

$$v = \frac{1}{\alpha\cos\eta}r^{\bullet} = \frac{1}{\cos\eta'}r^{\bullet}.$$

If we refer to the Lorentz factor $\alpha = \cos\eta'/\cos\eta$ of the matrix (II.4.23) we get with (II.4.14), (II.4.16) and (II.4.25)

$$v = \tan\eta'\,\mathrm{sh}\psi' = \sin\eta\;\mathrm{th}\psi, \tag{II.4.26}$$

an expression that matches (II.4.24). Converting the coordinate velocity common in the literature into the observer velocity, one obtains a relation which is contained in the Lorentz transformation between comoving and non-comoving observer systems. We will call this the physically relevant recession velocity of the galaxies and refer to similar considerations in the previous section on the dS-cosmos.

With the tetrads (II.4.21) the field quantities for both systems can be calculated:

$$
\begin{gathered}
B_m = \left\{\frac{1}{r}\cos\eta, 0,0,0\right\}, \quad C_m = \left\{\frac{1}{r}\cos\eta, \frac{1}{r}\cot\vartheta, 0,0\right\}, \quad U_m = \left\{-\frac{1}{\mathcal{R}_0}\tan\eta, 0,0,0\right\}, \\
B_{m'} = \left\{\frac{1}{r}\cos\eta', 0,0,-\frac{i}{\mathcal{R}_0}\mathrm{th}\psi'\right\}, \quad C_{m'} = \left\{\frac{1}{r}\cos\eta', \frac{1}{r}\cot\vartheta, 0, -\frac{i}{\mathcal{R}_0}\mathrm{th}\psi'\right\}, \\
{}'U_{m'} = \left\{0,0,0,-\frac{i}{\mathcal{R}_0}\mathrm{th}\psi'\right\}.
\end{gathered} \tag{II.4.27}
$$

With $r = \mathcal{R}_0 \sin\eta$, $\mathcal{R} = \mathcal{R}_0 \mathrm{ch}\psi'$ and $\sin\eta = \sin\eta' \mathrm{ch}\psi'$ we can write $B_1 = \frac{1}{\mathcal{R}_0}\cot\eta$ and $B_{1'} = \frac{1}{\mathcal{R}}\cot\eta'$. Also the Lanczos cosmos contains forces directed away from any point. The comoving observer is no longer exposed to these forces. The tidal forces $B_{4'}, C_{4'}, {}'U_{4'}$ act on it.

To convert these quantities into one another, one needs the Lorentz transformation from (II.4.23). With the relative velocity of (II.4.26) we can calculate

$$
{}'L_{m'} = \left\{\frac{\cos\eta'}{\cos\eta}\tan\eta\frac{1}{\mathcal{R}_0}, 0,0,-\frac{1}{\cos\eta}\mathrm{th}\psi'\frac{i}{\mathcal{R}_0}\right\} . \tag{II.4.28}
$$

The field quantities (II.4.27) transform correctly according to

$$
B_{m'} = {}'L^m_{m'} B_m, \quad C_{m'} = {}'L^m_{m'} C_m, \quad {}'U_{m'} = {}'L^m_{m'} U_m + {}'L_{m'} . \tag{II.4.29}
$$

It should also be noted that the spatial parts of the lateral field quantities B and C do not appear to be flat in the comoving system. This differentiates the Lanczos cosmos from the dS cosmos.

With the quantities (II.4.27) we are able to solve the field equations for both systems. By analogy to the calculations of the dS cosmos one obtains

$$
\begin{gathered}
B_{n\underset{2}{\|}m} + B_n B_m = -h_{nm}\frac{1}{\mathcal{R}_0^2}, \quad C_{n\underset{3}{\|}m} + C_n C_m = -\left(m_n m_m + b_n b_m + u_n u_m\right)\frac{1}{\mathcal{R}_0^2} \\
U^s{}_{\underset{1}{\|}s} + U^s U_s = -\frac{1}{\mathcal{R}_0^2}, \quad B^s{}_{\underset{2}{\|}s} + B^s B_s = -\frac{2}{\mathcal{R}_0^2}, \quad C^s{}_{\underset{3}{\|}s} + C^s C_s = -\frac{3}{\mathcal{R}_0^2}
\end{gathered} . \tag{II.4.30}
$$

The graded derivatives are defined in the same manner as in the dS model. For the comoving system we use the metric in the form (II.4.8) and first we get

$$
\begin{aligned}
&{}'U_{4'|4'} + {}'U_{4'}\,{}'U_{4'} = -\frac{1}{\mathcal{R}_0^2} \\
&B_{1'|1'} - {}'U_{1'1'}{}^{4'}B_{4'} + B_{1'}B_{1'} = -\frac{1}{\mathcal{R}_0^2} \\
&B_{1'|4'} + B_{1'}B_{4'} = 0 \\
&B_{4'|1'} - {}'U_{1'4'}{}^{1'}B_{1'} + B_{4'}B_{1'} = 0 \\
&B_{4'|4'} + B_{4'}B_{4'} = -\frac{1}{\mathcal{R}_0^2} \\
&C_{2'|2'} - B_{2'2'}{}^{1'}B_{1'} - B_{2'2'}{}^{4'}B_{4'} + C_{2'}C_{2'} = -\frac{1}{\mathcal{R}_0^2}
\end{aligned}
$$

With these values we get the same relations as in (II.4.30), but in primed form. For the Lanczos cosmos applies the same structure (II.3.35) for the Ricci

$$
R_{mn} = \frac{3}{\mathcal{R}_0^2} g_{mn}, \quad R = \frac{12}{\mathcal{R}_0^2}, \quad R_{mn} - \frac{1}{2} R\, g_{mn} = -\frac{3}{\mathcal{R}_0^2} g_{mn} \tag{II.4.31}
$$

and hence the stress-energy-momentum tensor is

$$
T_{mn} = \begin{pmatrix} -p & & & \\ & -p & & \\ & & -p & \\ & & & \mu_0 \end{pmatrix}, \quad \kappa p = -\frac{3}{\mathcal{R}_0^2}, \quad \kappa \mu_0 = \frac{3}{\mathcal{R}_0^2} \ . \tag{II.4.32}
$$

For the equation of state which contains the same problem as the dS cosmos results

$$
p + \mu_0 = 0
$$

and we see a close relationship to the de Sitter universe. Lemaître [L], apparently being unaware of the paper of Lanczos, has once more discovered the metric (II.4.3) and discussed it in detail.

II.5. The Lanczos-like model

Another model that is part of the de Sitter family will now be discussed. This model has the embedding

$$\begin{aligned} x^3 &= \mathcal{R}_0 \operatorname{sh}\psi \operatorname{sh}\eta \sin\vartheta \sin\varphi \\ x^2 &= \mathcal{R}_0 \operatorname{sh}\psi \operatorname{sh}\eta \sin\vartheta \cos\varphi \\ x^1 &= \mathcal{R}_0 \operatorname{sh}\psi \operatorname{sh}\eta \cos\vartheta \\ x^4 &= i\mathcal{R}_0 \operatorname{sh}\psi \operatorname{ch}\eta \\ x^0 &= \mathcal{R}_0 \operatorname{ch}\psi \end{aligned} \quad . \tag{II.5.1}$$

Again

$$x^a x^a = \mathcal{R}_0^2, \quad x^m x^m = \left(i\mathcal{R}\right)^2, \quad m = 1,2,...4, \quad \mathcal{R}\left(t\right) = \mathcal{R}_0 \operatorname{sh}\psi \tag{II.5.2}$$

applies to the spherical surfaces and

$$\text{(A)} \qquad ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \operatorname{sh}^2\eta\, d\vartheta^2 + \mathcal{R}^2 \operatorname{sh}^2\eta \sin^2\vartheta\, d\varphi^2 + \mathcal{R}_0^2\, di\psi^2 . \tag{II.5.3}$$

to the metric. With

$$\mathcal{R} = \mathcal{K}\mathcal{R}_0, \quad \mathcal{K} = \operatorname{sh}\psi, \quad r = \mathcal{R}_0 \operatorname{sh}\eta \tag{II.5.4}$$

one obtains from (II.5.3)

$$\text{(A')} \qquad ds^2 = \mathcal{K}^2 \left[\frac{1}{\operatorname{ch}^2\eta} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 \right] - dt^2 , \tag{II.5.5}$$

$$\text{(A'')} \qquad ds^2 = \mathcal{K}^2 \left[\frac{1}{1 + \frac{r^2}{\mathcal{R}_0}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 \right] - dt^2 . \tag{II.5.6}$$

The Lanczos-like model expands. One has $k = -1$. Furthermore, the cosmos is negatively curved, and infinite.

We assume that this model is formulated in the comoving coordinate system and we are writing the metric (II.5.6) now as

$$\text{(A'')} \qquad ds^2 = \mathcal{K}^2 \left[\frac{1}{1 + \frac{r'^2}{\mathcal{R}_0}} dr'^2 + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta\, d\varphi^2 \right] - dt'^2 . \tag{II.5.7}$$

The following relations hold

$$r' = \mathcal{R}_0 \operatorname{sh}\eta', \quad dt' = \mathcal{R}_0 d\psi', \quad \mathcal{R} = \mathcal{K}\mathcal{R}_0, \quad \mathcal{K} = \operatorname{sh}\psi' \tag{II.5.8}$$

and for the non-comoving system

$$r = \mathcal{R}_0 \operatorname{sh}\eta, \quad dt = \mathcal{R}_0 d\psi, \quad r = \mathcal{K} r' . \tag{II.5.9}$$

By comparing the last two relations we get

$$\mathrm{sh}\eta = \mathrm{sh}\eta'\mathrm{sh}\psi' \tag{II.5.10}$$

and from this the two auxiliary relations

$$\mathrm{ch}^2\psi' - \mathrm{ch}^2\eta'\mathrm{sh}^2\psi' = 1-\mathrm{sh}^2\eta, \quad \mathrm{ch}^2\eta' - \mathrm{sh}^2\eta'\mathrm{ch}^2\psi' = 1-\mathrm{sh}^2\eta. \tag{II.5.11}$$

For calculation of the transformation matrix we use

$$r = \mathcal{R}r', \quad \mathrm{th}\psi = \mathrm{ch}\eta'\mathrm{th}\psi' \tag{II.5.12}$$

and we get

$$\Lambda^{i}_{i'} = \begin{pmatrix} \mathrm{sh}\psi' & & & -i\,\mathrm{sh}\eta'\mathrm{ch}\psi' \\ & 1 & & \\ & & 1 & \\ \frac{i}{1-\mathrm{sh}^2\eta}\mathrm{th}\eta'\mathrm{sh}\psi'\mathrm{ch}\psi' & & & \frac{1}{1-\mathrm{sh}^2\eta}\mathrm{ch}\eta' \end{pmatrix}$$

$$\Lambda^{i'}_{i} = \begin{pmatrix} \frac{1}{1-\mathrm{sh}^2\eta}\frac{\mathrm{ch}^2\eta'}{\mathrm{sh}\psi'} & & & i\,\mathrm{sh}\eta'\mathrm{ch}\eta'\mathrm{cth}\psi' \\ & 1 & & \\ & & 1 & \\ -i\frac{1}{1-\mathrm{sh}^2\eta}\mathrm{sh}\eta'\mathrm{ch}\psi' & & & \mathrm{ch}\eta' \end{pmatrix}. \tag{II.5.13}$$

Thus, transforming to the non-comoving metric the line element takes the forms

(B') $$ds^2 = \frac{1}{1-\mathrm{sh}^2\eta}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 - \left(1-\mathrm{sh}^2\eta\right)dt^2, \tag{II.5.14}$$

(B'') $$ds^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}_0^2}}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 - \left(1-\frac{r^2}{\mathcal{R}_0^2}\right)dt^2. \tag{II.5.15}$$

For the Lanczos-like model we note

$$\overset{1'}{e}_{1'} = \frac{\mathrm{sh}\psi'}{\mathrm{ch}\eta'}, \quad \overset{4'}{e}_{4'} = 1; \quad \overset{1}{e}_{1} = \frac{1}{\sqrt{1-\mathrm{sh}^2\eta}}, \quad \overset{4}{e}_{4} = \sqrt{1-\mathrm{sh}^2\eta} \tag{II.5.16}$$

and with (II.5.13), the Lorentz transformation

$$L^{m}_{m'} = \frac{1}{\sqrt{1-\mathrm{sh}^2\eta}}\begin{pmatrix} \mathrm{ch}\eta' & & & -i\,\mathrm{sh}\eta'\mathrm{ch}\psi' \\ & 1 & & \\ & & 1 & \\ i\,\mathrm{sh}\eta'\mathrm{ch}\psi' & & & \mathrm{ch}\eta' \end{pmatrix}. \tag{II.5.17}$$

This implies the relative velocity

$$v = \mathrm{th}\eta'\mathrm{ch}\psi' = \mathrm{th}\eta'\mathrm{ch}\frac{t'}{\mathcal{R}_0}. \tag{II.5.18}$$

The relative speed increases infinitely after an infinitely long time.

The structure of the Lanczos-like cosmos is even less close to the universe we are living in than the ones discussed earlier. Nevertheless, we will find that the quantities derived for this cosmos satisfy the field equations. After all, Einstein's field equations are nonlinear differential equations of 2nd order, the solutions for which no reference need be made to their physical usability.

Thus, for the components of the Ricci-rotation coefficients one has

$$B_m = \left\{\frac{1}{r}\sqrt{1-\mathrm{sh}^2\eta},0,0,0\right\},\quad C_m = \left\{\frac{1}{r}\sqrt{1-\mathrm{sh}^2\eta},\ \frac{1}{r}\cot\vartheta,0,0\right\},$$

$$U_m = \left\{-\frac{1}{\mathcal{R}_0}\frac{\mathrm{sh}\eta}{\sqrt{1-\mathrm{sh}^2\eta}},0,0,0\right\},$$

$$B_{m'} = \left\{\frac{1}{r}\cos\eta',0,0,-\frac{i}{\mathcal{R}_0}\mathrm{cth}\psi'\right\},\quad C_{m'} = \left\{\frac{1}{r}\cos\eta',\ \frac{1}{r}\cot\vartheta,0,-\frac{i}{\mathcal{R}_0}\mathrm{cth}\psi'\right\}, \tag{II.5.19}$$

$$'U_{m'} = \left\{0,0,0,-\frac{i}{\mathcal{R}_0}\mathrm{cth}\psi'\right\}.$$

With the Lorentz transformation from (II.5.17) and

$$'L_{m'} = \frac{1}{1-\mathrm{sh}^2\eta}\left\{\frac{1}{\mathcal{R}_0}\mathrm{sh}\eta\,\mathrm{ch}\eta',0,0,-\frac{i}{\mathcal{R}_0}\mathrm{cth}\psi'\right\} \tag{II.5.20}$$

the field quantities U transform inhomogeneously and all quantities of (II.5.19) satisfy the Einstein field equations.

II.6. The anti-de Sitter-Model

The line element of the anti-de Sitter model which is to complement the de Sitter model in the comoving coordinate system is reported in the literature as

$$\text{(A)} \qquad ds^2 = \mathcal{R}^2 d\eta'^2 + \mathcal{R}^2\,\mathrm{sh}^2\eta'\,d\vartheta^2 + \mathcal{R}^2\,\mathrm{sh}^2\eta'\sin^2\vartheta\, d\varphi^2 - dt'^2. \qquad \text{(II.6.1)}$$

With

$$r' = \mathcal{R}_0\,\mathrm{sh}\eta', \quad \mathcal{R} = \mathcal{K}\mathcal{R}_0, \quad \mathcal{K} = \sin\psi' \qquad \text{(II.6.2)}$$

it can be written as

$$\text{(A')} \qquad ds^2 = \mathcal{K}^2\left[\frac{1}{\mathrm{ch}\eta'^2}dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta\, d\varphi^2\right] - dt'^2, \qquad \text{(II.6.3)}$$

$$\text{(A'')} \qquad ds^2 = \mathcal{K}^2\left[\frac{1}{1+\dfrac{r'^2}{\mathcal{R}_0^2}}dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta\, d\varphi^2\right] - dt'^2. \qquad \text{(II.6.4)}$$

Therefore, we need only to show that the metrics of types A and B mutually transform by a suitable coordinate transformation.

For the non-comoving System we expect

$$r = \mathcal{R}_0\,\mathrm{sh}\eta, \quad r = \mathcal{K}r' \qquad \text{(II.6.5)}$$

and by comparing this with (II.6.2) we get

$$\mathrm{sh}\eta = \mathrm{sh}\eta'\sin\psi' \qquad \text{(II.6.6)}$$

and the auxiliary relations

$$\mathrm{ch}^2\eta = \cos^2\psi' + \mathrm{ch}^2\eta'\sin^2\psi', \quad \mathrm{ch}^2\eta = \mathrm{ch}^2\eta' - \mathrm{sh}^2\eta'\cos^2\psi'. \qquad \text{(II.6.7)}$$

Deriving the coordinate transformation we start with

$$r = \mathcal{K}r', \quad \tan\psi = \mathrm{ch}\eta'\tan\psi'. \qquad \text{(II.6.8)}$$

and we obtain the matrices

$$\Lambda^{i}_{i'} = \begin{pmatrix} \sin\psi' & & & -i\,\mathrm{sh}\eta'\cos\psi' \\ & 1 & & \\ & & 1 & \\ \dfrac{i}{\mathrm{ch}^2\eta}\mathrm{th}\eta'\sin\psi'\cos\psi' & & & \dfrac{\mathrm{ch}\eta'}{\mathrm{ch}^2\eta} \end{pmatrix}. \qquad \text{(II.6.9)}$$

$$\Lambda^{i'}_{i'} = \begin{pmatrix} \dfrac{\mathrm{ch}^2\eta'}{\mathrm{ch}^2\eta\sin\psi'} & & & i\,\mathrm{sh}\eta'\mathrm{ch}\eta'\cot\psi' \\ & 1 & & \\ & & 1 & \\ -i\dfrac{\mathrm{sh}\eta'}{\mathrm{ch}^2\eta}\cos\psi' & & & \mathrm{ch}\eta' \end{pmatrix}$$

The coordinate transformations derived in this Section are useful, but only a precursor to the Lorentz transformations.

The anti-de Sitter model has the embedding

$$\begin{aligned} x^3 &= \mathcal{R}_0 \mathrm{sh}\eta \sin\vartheta \sin\varphi \\ x^2 &= \mathcal{R}_0 \mathrm{sh}\eta \sin\vartheta \cos\varphi \\ x^1 &= \mathcal{R}_0 \,\mathrm{sh}\eta \cos\vartheta \\ x^4 &= i\mathcal{R}_0 \,\mathrm{ch}\eta \sin\psi \\ x^0 &= i\mathcal{R}_0 \mathrm{ch}\eta \cos\psi \end{aligned} \quad . \tag{II.6.10}$$

It applies

$$x^a x^a = \left(i\mathcal{R}_0\right)^2, \quad x^\alpha x^\alpha = \mathcal{R}_0^2 \mathrm{sh}^2\eta, \quad \alpha = 1,2,3, \quad x^4 x^4 + x^0 x^0 = \left(i\mathcal{R}_0 \mathrm{ch}\eta\right)^2 . \tag{II.6.11}$$

The AdS universe is spatially infinite and periodic in the time $dt = \mathcal{R}_0 d\psi$. With

$$r = \mathcal{R}_0 \mathrm{sh}\eta \tag{II.6.12}$$

the line elements can be written as

$$\text{(B)} \qquad ds^2 = \mathcal{R}_0^2 d\eta^2 + \mathcal{R}_0^2 \,\mathrm{sh}^2\eta d\vartheta^2 + \mathcal{R}_0^2 \,\mathrm{sh}^2\eta \sin^2\vartheta d\varphi^2 + \mathcal{R}_0^2 \,\mathrm{ch}^2\eta di\psi^2 , \tag{II.6.13}$$

$$\text{(B')} \qquad ds^2 = \frac{1}{\mathrm{ch}^2\eta} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \mathrm{ch}^2\eta dt^2 , \tag{II.6.14}$$

$$\text{(B'')} \qquad ds^2 = \frac{1}{1+\dfrac{r^2}{\mathcal{R}_0^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left(1+\frac{r^2}{\mathcal{R}_0^2}\right) dt^2 \tag{II.6.15}$$

and in the same way as in the de Sitter model they are given in non-comoving coordinate systems.

For the anti-de Sitter model one gathers from the metrics (II.6.3) and (II.6.14) the tetrads

$$\overset{1'}{e}_{1'} = \frac{\sin\psi'}{\mathrm{ch}\eta'}, \quad \overset{4'}{e}_{4'} = 1; \quad \overset{1}{e}_{1} = \frac{1}{\mathrm{ch}\eta}, \quad \overset{4}{e}_{4} = \mathrm{ch}\eta \tag{II.6.16}$$

and from (II.6.9) with the help of

$$L^m_{m'} = \overset{m}{e}_i \Lambda^i_{i'} \underset{m'}{e}^{i'} , \tag{II.6.17}$$

one establishes the pseudo-rotation

$$L^m_{m'} = \begin{pmatrix} \dfrac{\mathrm{ch}\eta'}{\mathrm{ch}\eta} & & & -i\dfrac{\mathrm{th}\eta}{\tan\psi'} \\ & 1 & & \\ & & 1 & \\ i\dfrac{\mathrm{th}\eta}{\tan\psi'} & & & \dfrac{\mathrm{ch}\eta'}{\mathrm{ch}\eta} \end{pmatrix} . \tag{II.6.18}$$

From the relative speeds of the systems A and B we gather

$$v = \mathrm{th}\eta'\cos\psi' = \mathrm{sh}\eta\cot\psi. \tag{II.6.19}$$

The relative velocity is infinitely high at infinity and is periodic in time. (II.6.18) is not a Lorentz transformation.

For the anti-de Sitter universe counts

$$\mathcal{K} = \sin\psi', \quad r = \mathcal{K} r', \quad r = \mathcal{R}_0\,\mathrm{sh}\eta, \quad H = \frac{1}{\mathcal{R}_0}\cot\psi'$$

and therefore for the *coordinate* velocity

$$r^{\cdot} = Hr = \mathrm{sh}\eta\cot\psi' = \mathrm{sh}\eta'\cos\psi'. \tag{II.6.20}$$

It is infinitely high at infinity of the open universe. With

$$\overset{1}{e}_1 = \frac{1}{\mathrm{ch}\eta}, \quad \alpha = \frac{\mathrm{ch}\eta'}{\mathrm{ch}\eta}, \quad v = \frac{1}{\mathrm{ch}\eta} r^{\cdot}$$

the *relative velocity* is

$$v = \mathrm{th}\eta'\cos\psi' = \mathrm{sh}\eta\cot\psi \tag{II.6.21}$$

in accordance with (II.6.19). Its highest value lies at infinity and is infinitely high.

For the *anti-de Sitter cosmos* the basic relations

$$\begin{gathered}
\overset{1'}{e}_{1'} = \frac{\sin\psi'}{\mathrm{ch}\eta'}, \quad \overset{2'}{e}_{2'} = r, \quad \overset{3'}{e}_{3'} = r\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1, \\
\overset{1}{e}_1 = \frac{1}{\mathrm{ch}\eta}, \quad \overset{2}{e}_{2'} = r, \quad \overset{3}{e}_3 = r\sin\vartheta, \quad \overset{4}{e}_4 = \mathrm{ch}\eta, \\
\mathcal{K} = \sin\psi', \quad r = \mathcal{K} r'
\end{gathered} \tag{II.6.22}$$

are valid. Therewith one calculates the field quantities

$$\begin{gathered}
B_m = \left\{\frac{1}{r}\mathrm{ch}\eta, 0,0,0\right\}, \quad C_m = \left\{\frac{1}{r}\mathrm{ch}\eta,\ \frac{1}{r}\cot\vartheta, 0,0\right\}, \\
U_m = \left\{\frac{1}{\mathcal{R}_0}\mathrm{th}\eta, 0,0,0\right\}, \\
B_{m'} = \left\{\frac{1}{r}\mathrm{ch}\eta', 0,0,-\frac{i}{\mathcal{R}_0}\cot\psi'\right\}, \quad C_{m'} = \left\{\frac{1}{r}\mathrm{ch}\eta', \frac{1}{r}\cot\vartheta, 0, -\frac{i}{\mathcal{R}_0}\cot\psi'\right\}, \\
'U_{m'} = \left\{0,0,0,-\frac{i}{\mathcal{R}_0}\cot\psi'\right\}.
\end{gathered} \tag{II.6.23}$$

With

$$'L_{m'} = -\left\{\frac{1}{\mathcal{R}_0}\frac{\mathrm{ch}'}{\mathrm{ch}\eta}\mathrm{th}\eta, 0, 0\, \frac{i}{\mathrm{ch}^2\eta}\mathrm{cth}\psi'\right\} \tag{II.6.24}$$

one establishes the connection of the quantities (II.6.23). With these expressions the Einstein field equations are satisfied. However, pressure and mass density have the sign opposite to the corresponding quantities of the de Sitter universe. This result and also the periodic lapse of time pose the AdS cosmos far away from reality.

II.7. The cosmological model of Friedman

Friedman [F] suggested a time-dependent cosmological model which can be regarded as a generalization of the Einstein cosmos. We recall a 5-dimensional embedding. However, we will recognize that the ansatz

$$\mathcal{R} = \mathcal{R}(t) \tag{II.7.1}$$

cannot be implemented parameterizing the embedding in the same way that was possible with the Lanczos cosmos. The 5-dimensional embedding space is parameterized analogous to the Einstein cosmos (II.1.3). The space of the model is represented by a hyper-sphere with the radius $\mathcal{R}$. The time is the cosmic time, the proper time of an observer who participates in the expansion/contraction of the cosmos, but otherwise performs no movement of his own. The time-like metric factor is 1, the universe is expanding in 'free fall'. A spherical co-ordinate system is connected with the Cartesian co-ordinate system of the 5-dimensional flat embedding space by

$$\begin{aligned}
x^{3'} &= \mathcal{R}\sin\eta\sin\vartheta\sin\varphi \\
x^{2'} &= \mathcal{R}\sin\eta\sin\vartheta\cos\varphi \\
x^{1'} &= \mathcal{R}\sin\eta\cos\vartheta \\
x^{0'} &= \mathcal{R}\cos\eta \\
x^{4'} &= it
\end{aligned}$$

This results in the equation for a hypersphere

$$x^{a'}x^{a'} = \mathcal{R}^2, \quad a' = 0',1',...,3'$$

and the metric on it is

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta\, d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta\, d\varphi^2 - dt^2 \tag{II.7.2}$$

differs from that of the Einstein cosmos by the ansatz (II.7.1). In the theory we have new time-dependent variables. Since the analytic form of $\mathcal{R}(t)$ cannot be deduced from the embedding, the time-dependent variables remain at first undetermined. Values will be assigned later by solving the Friedman differential equation. It becomes recognizable that the ansatz (II.7.2) is not the only possible one, in order to gain the solution set of the Friedman cosmos. Because of the time dependence (II.7.1) of $\mathcal{R}$ also the pseudo-hyper sphere with the radius $\mathcal{R}$ only plays the part of one snapshot of the Friedman cosmos. It is substantially different from the before-regarded de Sitter model. The time-dependence is invoked into the metric of the de Sitter model by a suitable slice of the pseudo-hyper sphere, while the geometrical basic structure of the theory remains unchanged. For the Friedman cosmos one must admit that the expansion alters the pseudo-hyper sphere.

The bein vectors read from (II.7.2)

$$\overset{1}{e}_1 = \mathcal{R}, \quad \overset{2}{e}_2 = \mathcal{R}\sin\eta, \quad \overset{3}{e}_3 = \mathcal{R}\sin\eta\sin\vartheta, \quad \overset{4}{e}_4 = 1 \tag{II.7.3}$$

are processed in sequence to the field quantities

$$U_m = \{0,0,0,U_4\}$$
$$B_m = \left\{\frac{1}{\mathcal{R}}\cot\eta, 0, 0, B_4\right\} \quad . \qquad \text{(II.7.4)}$$
$$C_m = \left\{\frac{1}{\mathcal{R}}\cot\eta, \frac{1}{\mathcal{R}\sin\eta}\cot\vartheta, 0, C_4\right\}$$

The cosmos is homogeneous, its radius of curvature is independent of the three spatial co-ordinates. To simplify the later calculations, we summarize the three time-like quantities

$$U_4 \stackrel{*}{=} B_4 \stackrel{*}{=} C_4 = \frac{1}{\mathcal{R}}\mathcal{R}_{|4} \quad . \qquad \text{(II.7.5)}$$

We decompose the Ricci-rotation coefficients into

$$A_{mn}{}^s = U_{mn}{}^s + B_{mn}{}^s + C_{mn}{}^s \, . \qquad \text{(II.7.6)}$$

Together with the unit vectors defined earlier one has

$$U_{mn}{}^s = m_m U_n m^s - m_m m_n U^s, \quad B_{mn}{}^s = b_m B_n b^s - b_m b_n B^s, \quad C_{mn}{}^s = c_m C_n c^s - c_m c_n C^s \, . \qquad \text{(II.7.7)}$$

With the graded derivatives

$$U_{m\underset{1}{||}n} = U_{m|n}, \quad B_{m\underset{2}{||}n} = B_{m|n} - U_{mn}{}^s B_s, \quad C_{m\underset{3}{||}n} = C_{m|n} - U_{mn}{}^s C_s - B_{mn}{}^s C_s \qquad \text{(II.7.8)}$$

the Ricci can be brought into the form

$$\begin{aligned} R_{mn} = &-\left[U^s{}_{\underset{1}{||}s} + U^s U_s\right] h_{mn} \\ &-\left[B_{n\underset{2}{||}m} + B_n B_m\right] - b_n b_m \left[B^s{}_{\underset{2}{||}s} + B^s B_s\right] \\ &-\left[C_{n\underset{3}{||}m} + C_n C_m\right] - c_n c_m \left[C^s{}_{\underset{3}{||}s} + C^s C_s\right] \end{aligned} \quad . \qquad \text{(II.7.9)}$$

Therein

$$h_{mn} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \qquad \text{(II.7.10)}$$

is a submatrix of the metric $g_{mn} = \delta_{mn}$.

It is expedient to unravel the field equations so that the variables which describe the geometric skeleton and those which stand for the expansion will be outsourced into separate blocks. The skeleton quantities and their field equations are well known from other models. Thus, there remains only the task of determining the time-dependent variables.

If one extracts the time-dependent quantities (II.7.5) from the Ricci-rotation coefficients[26]

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[26] $'g_{mn}$ is the spatial part of the metric. An asterisk at a quantity indicates a 3-dimensional element of the body geometry.

$$A_{mn}{}^{s} = {}^{*}A_{mn}{}^{s} + {}'\delta^{s}_{m}U_{n} - {}'g_{mn}U^{s}, \quad A_{m} = {}^{*}A_{m} + 3U_{m} \tag{II.7.11}$$

one obtains for the Ricci tensor

$$R_{mn} = \left[{}^{*}R_{mn} + {}^{*}A_{m}U_{n}\right] - 2U_{n|m} - g_{mn}\left[U^{s}{}_{|s} + 3U^{s}U_{s}\right] \tag{II.7.12}$$

which by virtue of

$$B_{\alpha|4} + B_{\alpha}U_{4} = 0, \qquad C_{\alpha|4} + C_{\alpha}U_{4} = 0 \tag{II.7.13}$$

gives $R_{\alpha 4} = 0$. This means that no currents occur in the stress-energy-momentum tensor. This is a necessary condition for a reference system that participates in the expansion movement. Thus, one is left with

$$R_{\alpha\beta} = {}^{*}R_{\alpha\beta} - g_{\alpha\beta}\left[U^{s}{}_{|s} + 3U^{s}U_{s}\right], \quad R_{44} = -3\left[U_{4|4} + U_{4}U_{4}\right] . \tag{II.7.14}$$

From

$$\begin{aligned} {}^{*}R_{\alpha\beta} = &-\left[B_{\beta\underset{2}{||}\alpha} + B_{\beta}B_{\alpha}\right] - b_{\alpha}b_{\beta}\left[B^{\gamma}{}_{\underset{2}{||}\gamma} + B^{\gamma}B_{\gamma}\right] \\ &-\left[C_{\beta\underset{3}{||}\alpha} + C_{\beta}C_{\alpha}\right] - c_{\alpha}c_{\beta}\left[C^{\gamma}{}_{\underset{3}{||}\gamma} + C^{\gamma}C_{\gamma}\right] \end{aligned} \tag{II.7.15}$$

results with the field quantities (II.7.4)

$${}^{*}R_{\alpha\beta} = \frac{2}{\mathcal{R}^{2}}g_{\alpha\beta} \tag{II.7.16}$$

a spherical space with positive curvature. This allows to bring the Einstein tensor into the form

$$G_{\alpha\beta} = -g_{\alpha\beta}\frac{1}{\mathcal{R}^{2}} + g_{\alpha\beta}\left[2U^{s}{}_{\underset{1}{||}s} + 3U^{s}U_{s}\right], \quad G_{44} = -\frac{3}{\mathcal{R}^{2}} + 3U^{s}U_{s}, \quad G_{\alpha 4} = G_{4\alpha} = 0 \; . \tag{II.7.17}$$

Einstein's field equations we firstly place with the cosmological term

$$G_{mn} + \lambda g_{mn} = -\kappa T_{mn} \; . \tag{II.7.18}$$

To start with, we assume that the stress-energy tensor has as the only component the time-dependent matter density $T_{44} = \mu_{0}(t)$. Later we will also integrate the pressure. From the conservation law and the simple structure of the stress-energy-momentum tensor one obtains

$$T^{4n}{}_{||n} = \mu_{0|4} + 3\mu_{0}U_{4} = 0, \quad \frac{1}{\mu_{0}}\mu_{0}^{\cdot} = -3\frac{1}{\mathcal{R}}\mathcal{R}^{\cdot} \; . \tag{II.7.19}$$

After integration one has

$$\kappa\mu_{0} = \frac{a}{\mathcal{R}^{3}} \tag{II.7.20}$$

with a as an integration constant and the still undetermined time-dependent quantity $\mathcal{R}$. From the second equation (II.7.17) we obtain the Friedman differential equation

$$\left(\mathcal{R}^{\cdot}\right)^{2} = \frac{a}{3\mathcal{R}} + \frac{\lambda}{3}\mathcal{R}^{2} - 1 \; . \tag{II.7.21}$$

Simplifying the model by discarding the cosmological constant and writing for the integration constant

$$\mathcal{R}_0 = \frac{a}{3} \tag{II.7.22}$$

one has for the Friedman differential equation

$$\left(\mathcal{R}^{\bullet}\right)^2 = \frac{\mathcal{R}_0}{\mathcal{R}} - 1 \, .$$

If one further introduces with

$$\mathcal{R} = \mathcal{K}\mathcal{R}_0 \tag{II.7.23}$$

the *scale factor* $\mathcal{K}$, one finally arrives at

$$\mathcal{K}^{\bullet} = \pm \frac{1}{\mathcal{R}_0}\sqrt{\frac{1}{\mathcal{K}} - 1} \tag{II.7.24}$$

and notes that this solution is also compatible with the first equations in (II.5.17).

If one performs the substitution $\mathcal{R} \to i\mathcal{R}, \eta \to i\eta$ in the Friedman metric one obtains

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \operatorname{sh}^2\eta\, d\vartheta^2 + \mathcal{R}^2 \operatorname{sh}^2\eta \sin^2\vartheta\, d\varphi^2 - dt^2 . \tag{II.7.25}$$

Now the extra dimension of the embedding space is imaginary and the embedded space is open. Its curvature is again time-dependent. However, the curvature in each position of the 3-dimensional space is the same all the time[27]. The relation between the Cartesian co-ordinates and hyperspherical co-ordinates is

$$\begin{aligned}
x^{3'} &= i\mathcal{R} \sin i\eta \sin\vartheta \sin\varphi \\
x^{2'} &= i\mathcal{R} \sin i\eta \sin\vartheta \cos\varphi \\
x^{1'} &= i\mathcal{R} \sin i\eta \cos\vartheta \\
x^{0'} &= i\mathcal{R} \cos i\eta \\
x^{4'} &= it
\end{aligned}$$

and the equation of the pseudo-hyper sphere with imaginary radius is

$$x^{a'}x^{a'} = -\mathcal{R}^2, \quad a' = 0', 1', \ldots, 3' .$$

Restricted to the [0 ', 1']-slice

$$\begin{aligned}
x^{1'} &= i\mathcal{R} \sin i\eta = -\mathcal{R}\operatorname{sh}\eta \\
x^{0'} &= i\mathcal{R} \cos i\eta = i\mathcal{R}\operatorname{ch}\eta
\end{aligned} .$$

Therefore the extra dimension in the embedding space is imaginary. By differentiating one obtains

$$\begin{aligned}
dx^{1'} &= -\mathcal{R}\operatorname{ch}\eta\, d\eta - \operatorname{sh}\eta\, d\mathcal{R} \\
dx^{0'} &= i\mathcal{R}\operatorname{sh}\eta\, d\eta + i\operatorname{ch}\eta\, d\mathcal{R}
\end{aligned}$$

and

[27] The space is usually referred to as hyperbolic. Using this phraseology one negates that the space has a constant curvature. The throat of a hyperboloid has preferred points. At this location the hyperboloid has the highest curvature. Outwardly the curvature decreases. The spatial constant quantity $\mathcal{R}$ would be the radius of the circle at the throat.

$$\left(dx^{0'}\right)^2 + \left(dx^{1'}\right)^2 = -d\mathcal{R}^2 + \mathcal{R}^2 d\eta^2 .$$

If we restrict ourselves to the surface of the pseudo-hyper sphere and if we extend the prolem to the remaining dimensions, we finally have the metric (II.7.25).

Instead of (II.7.16) one obtains the equation for a space of negative curvature

$$^*R_{\alpha\beta} = -\frac{2}{\mathcal{R}^2} g_{\alpha\beta} . \qquad \text{(II.7.26)}$$

In this case $\mathcal{R}$ is the time-dependent radius of a *hyperboloid of constant curvature*. After the above substitution the U-equations remain unchanged, so that instead of (II.7.17) one has

$$G_{\alpha\beta} = g_{\alpha\beta}\frac{1}{\mathcal{R}^2} + g_{\alpha\beta}\left[2U^s{}_{\underset{1}{||}s} + 3U^sU_s\right], \quad G_{44} = \frac{3}{\mathcal{R}^2} + 3U^sU_s, \quad G_{\alpha 4} = G_{4\alpha} = 0 . \qquad \text{(II.7.27)}$$

The solution of the second equation leads to

$$\dot{\mathcal{K}} = \pm\frac{1}{\mathcal{R}_0}\sqrt{\frac{1}{\mathcal{K}} + 1} . \qquad \text{(II.7.28)}$$

A flat geometry has also been considered

$$ds^2 = \mathcal{R}(t)^2\left[dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta\, d\varphi^2\right] - dt^2 . \qquad \text{(II.7.29)}$$

Without calculation we immediately conclude that

$$^*R_{\alpha\beta} = 0 . \qquad \text{(II.7.30)}$$

From the [44]-component of the field equations only remains

$$3U^sU_s = -3\frac{\mathcal{R}_0}{\mathcal{R}^3}$$

and finally

$$\dot{\mathcal{K}} = \pm\frac{1}{\mathcal{R}_0}\frac{1}{\sqrt{\mathcal{K}}} . \qquad \text{(II.7.31)}$$

By some reshaping the line element of the Friedman cosmos can be brought into a form in which the scale factor is already used in the ansatz. We have defined $\mathcal{R} = \mathcal{K}\mathcal{R}_0$, where $\mathcal{R}_0$ is a constant. Thus, one gains from (II.7.2)

$$ds^2 = \mathcal{K}^2\left[\mathcal{R}_0^2 d\eta^2 + \mathcal{R}_0^2\sin^2\eta\, d\vartheta^2 + \mathcal{R}_0^2\sin^2\eta\sin^2\vartheta\, d\varphi^2\right] - dt'^2 . \qquad \text{(II.7.32)}$$

If we further put

$$r' = \mathcal{R}_0\sin\eta, \quad r = \mathcal{K}r', \qquad \text{(II.7.33)}$$

we introduce with $\{r', \vartheta, \varphi, t'\}$ a quasipolar co-ordinate system, carried along with the expansion of the cosmos. $\partial r'/\partial t' = 0$ applies. By differentiating (II.7.33) one obtains $dr' = \mathcal{R}_0\cos\eta\, d\eta$. Thus one gets from (II.7.32)

$$ds^2 = \mathcal{K}^2 \left[\frac{1}{1 - \frac{r'^2}{\mathcal{R}_0^2}} + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta \, d\varphi^2 \right] - dt'^2 . \qquad \text{(II.7.34)}$$

From this one reads the tetrad system

$$\overset{1'}{e}_{1'} = \mathcal{K}\alpha_I, \quad \overset{2'}{e}_{2'} = r', \quad \overset{3'}{e}_{3'} = r' \sin\vartheta, \quad \overset{4'}{e}_{4'} = 1, \quad \alpha_I = 1 \Big/ \sqrt{1 - r'^2 / \mathcal{R}_0^2} \, . \qquad \text{(II.7.35)}$$

α_I is time-independent. Therefore, we obtain instead of (II.7.5)

$$U_{4'} \stackrel{*}{=} B_{4'} \stackrel{*}{=} C_{4'} = \frac{1}{\mathcal{K}} \mathcal{K}_{|4'} = -\frac{i}{\mathcal{K}} \dot{\mathcal{K}} . \qquad \text{(II.7.36)}$$

If we extend the metric with the *curvature parameter* k, which can take the values $k = \{1, 0, -1\}$ we obtain

$$ds^2 = \mathcal{K}^2 \left[\frac{1}{1 - k \frac{r'^2}{\mathcal{R}_0^2}} + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta \, d\varphi^2 \right] - dt'^2 . \qquad \text{(II.7.37)}$$

Thus, the three metrics (II.7.2), (II.7.25), and (II.7.29) can be represented conjointly. For $k = -1$ one has to use in (II.7.25)

$$r' = \mathcal{R}_0 \, \mathrm{sh}\eta, \qquad \mathcal{R}_0 d\eta = \frac{1}{\mathrm{ch}\eta} dr', \qquad \mathrm{ch}^2\eta = 1 + \frac{r'^2}{\mathcal{R}_0^2}$$

For $k = 0$ the universe is flat.

For the basic geometry one has

$$^*R_{\alpha\beta} = k g_{\alpha\beta} \frac{1}{\mathcal{R}^2} \qquad \text{(II.7.38)}$$

and for the Friedman differential equation

$$\dot{\mathcal{R}}^2 = \frac{a}{3\mathcal{R}} + \frac{\lambda}{3} \mathcal{R}^2 - k \, . \qquad \text{(II.7.39)}$$

As a solution of the equation with $\lambda = 0$ one obtains

$$\dot{\mathcal{K}} = \pm \frac{1}{\mathcal{R}_0} \sqrt{\frac{1}{\mathcal{K}} - k} \, . \qquad \text{(II.7.40)}$$

The type of curvature of space is determined by the curvature parameter. At present, observations in space do not provide any determination of the parameter.

For the three cosmological scenarios one obtains by integration of (II.7.40) up to a constant of integration (from now on the primes of the variables are omitted)

$$k=1 \qquad t=\mathcal{R}_0\begin{cases}\frac{1}{2}\arccos(1-2\mathcal{K})-\sqrt{\mathcal{K}(1-\mathcal{K})}\\ \frac{1}{2}\arccos(2\mathcal{K}-1)+\sqrt{\mathcal{K}(1-\mathcal{K})}\end{cases}$$

$$k=0 \qquad t=\pm\mathcal{R}_0\frac{3}{2}\sqrt{\mathcal{K}^3} \quad . \tag{II.7.41}$$

$$k=-1 \qquad t=\pm\mathcal{R}_0\left[\sqrt{\mathcal{K}^2+\mathcal{K}}-\frac{1}{2}\ln\frac{\sqrt{1+\mathcal{K}}+\sqrt{\mathcal{K}}}{\sqrt{1+\mathcal{K}}-\sqrt{\mathcal{K}}}\right]$$

t is in each case the time which elapses during the expansion of the universe depending on the scale factor. The two open universes ($k=0$, $k=-1$) expand eternally, while the closed Friedman universe ($k=1$) reaches its maximum expansion after a finite time and then starts to contract. In this case $\mathcal{K}$ has the range $[0,...,1]$ and $\mathcal{R}$ has the range $[0,...,\mathcal{R}_0]$. Thus, the constant $\mathcal{R}_0$ has a specific meaning. It is the maximum radius of curvature which the spherical space takes in its expansion. In Fig. II.2 the three functions (II.7.41) are depicted.

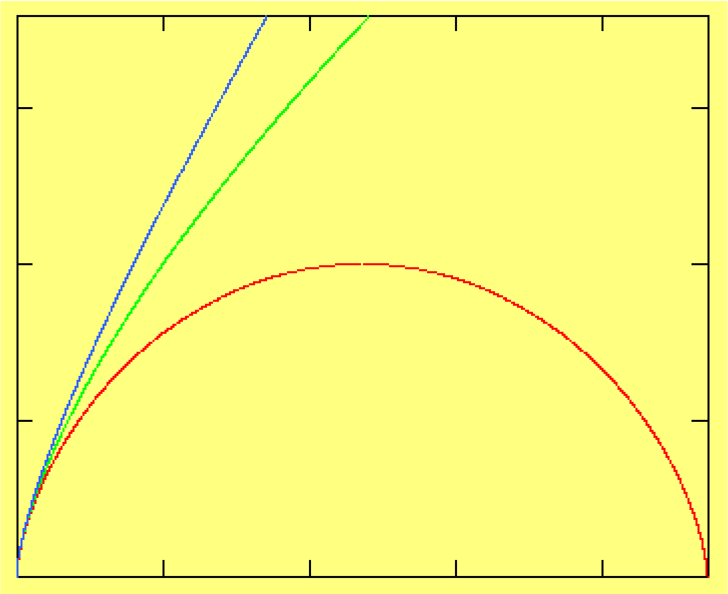

Fig. II.2

The function

$$H_0=\frac{\mathcal{R}^{\bullet}(t_0)}{\mathcal{R}(t_0)} \tag{II.7.42}$$

is referred to as Hubble parameter, whereby is t_0 the present time. H_0 describes the current expansion of the universe and enters into the field equations by means of the quantity U_4. If $l(t)$ is the distance of a point in the universe from $\eta=0$ to η by constant ϑ and φ one has

$$l^{\bullet}=\mathcal{R}^{\bullet}\eta=H\mathcal{R}\eta$$

and thus one can calculate the escape velocity (Hubble velocity) of the galaxy

$$v=Hl$$

and compare it with measurements.

This simple calculation implies a Galilean addition law of velocities. Facing a star which moves away from us with the velocity v_1 and another star which moves away from the first with the speed v_2, then this star should move away from us with the speed $v_3 = v_1 + v_2$. It is often argued that the violation of the laws of special relativity is admissible, because the relative motion of the stars is due to the expansion of the space. Thus, it is also possible that distant stars move away from us with a velocity faster than light and are invisible to us. The *Hubble horizon* is the limit at which the recession velocity has reached the speed of light and is to be for us the border of the visible region.

Several author have treated the velocity problem:

Bolós [B], Bolós [B], Havens, and Klein; Bolós [B] and Klein; Carney [C] and Fischler; Klein [K], Klein [K] and Collins; Klein [K] and Randles; Klein [K] and Reschke; Lindegren [L] and Dravins.

Since the density of galaxies in the universe is relatively small, one can assume that the pressure-free Friedman cosmos describes Nature in a good approximation. In the early stages of the genesis of the world, matter, however, was so highly compressed and so hot that one cannot work with a pressure-free model. Neither do we share the opinion that the universe could have evolved from a point-like world with the radius of curvature $\mathcal{R} = 0$ and infinitely high density. At the origin of the world, instead of the four interactions that determine our physical events there existed only a single universal interaction and therefore no gravity. At this time, Einstein's field equations were not valid and no conclusions can be made of the primordial state of matter in terms of gravitational theory.

Solutions of Einstein's field equations describing matter with pressure are hard to find, because the field equations are generally underdetermining. However, there exist simple solutions for expanding universes. These are the radiation dominated universes.

But there are simple solutions for the Friedman cosmos: the radiation cosmos and the extension of the Friedman cosmos to matter with pressure. The latter has already been envisaged by Lemaître [L] in the early days of cosmology. According to Eddington [E] pressure in the cosmos can be interpreted by random motions of the stars relative to the comoving co-ordinate system. The expansion of the universe is not caused by pressure, it also takes place in a pressure-free universe. If the individual motions of galaxies become smaller, the pressure decreases. The average random speed then changes proportionally to $\mathcal{R}$. The pressure varies adiabatically, that is, without energy exchange. The rest mass is maintained. The contribution to the energy density by the kinetic energy of the random motion decreases during the expansion.

Hönl [H] has dealt with the individual motion of galaxies in detail. Facing a motion for $\vartheta = \text{const.}$, $\varphi = \text{const.}$ from the line element of the Friedman cosmos remains

$$ds^2 = \mathcal{R}^2 d\eta^2 - dt^2 . \tag{II.7.43}$$

For a geodesic motion one has

$$\delta\int ds = \delta\int \sqrt{1-\mathcal{R}^2\dot{\eta}^2}\,dt = 0 .$$

From the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\eta}}\sqrt{1-\mathcal{R}^2\dot{\eta}^2}\right) = 0$$

we conclude that

$$\frac{\mathcal{R}^2\eta^\bullet}{\sqrt{1-\mathcal{R}^2\eta^{\bullet 2}}} = \text{const.}.$$

The speed of an individual motion is $v = \mathcal{R}\eta^\bullet$ and the momentum

$$p = \frac{mv}{\sqrt{1-v^2}} = \frac{m\mathcal{R}\eta^\bullet}{\sqrt{1-\mathcal{R}^2\eta^{\bullet 2}}}.$$

Multiplying the relation with $\mathcal{R}$, one has

$$p\mathcal{R} = m\frac{\mathcal{R}^2\eta^\bullet}{\sqrt{1-\mathcal{R}^2\eta^{\bullet 2}}} = \text{const.}, \quad p\mathcal{R} = \text{const.}.$$

The momentum of a freely moving galaxy decreases with the expansion of the universe.

However, for light rays one has

$$ds^2 = \mathcal{R}^2 d\eta^2 - c^2 dt^2 = 0. \tag{II.7.44}$$

With $dl = \mathcal{R}d\eta$ one obtains

$$\frac{dl}{dt} = c.$$

The propagation of light takes place at a constant speed, even if the curvature of the world changes.

The model of Friedman can be extended in such a way that the stress-energy momentum tensor takes the form

$$T_{mn} = -p\,g_{mn} + (p+\mu_0)u_m u_n. \tag{II.7.45}$$

The pressure and density of matter are position independent, but functions of time. From the field equations (II.7.17) the relations

$$\kappa p = -\frac{2}{\mathcal{R}}\mathcal{R}^{\bullet\bullet} - \frac{2}{\mathcal{R}^2}\mathcal{R}^{\bullet 2} - \frac{1}{\mathcal{R}^2}, \quad \kappa\mu_0 = \frac{3}{\mathcal{R}^2}\left(\mathcal{R}^{\bullet 2} - 1\right) \tag{II.7.46}$$

are obtained, relations which have been discussed extensively in the literature.

For the radiation cosmos the equation of state is

$$\mu_0 = 3p. \tag{II.7.47}$$

The stress-energy-momentum tensor has the form

$$T_{mn} = \begin{pmatrix} -p & & & \\ & -p & & \\ & & -p & \\ & & & \mu_0 \end{pmatrix} \tag{II.7.48}$$

and for its contraction one obtains with (II.7.47) $T = \mu_o - 3p = 0$.

From the conservation law one gets

$$p_{|\alpha} = 0, \quad \mu_{0|4} + 3(p+\mu_0)U_4 = 0. \tag{II.7.49}$$

The pressure in the universe is spatially constant. The change of the mass density provides with (II.7.47) the relation

$$\mu_0^\bullet + 4\mu_0 \frac{1}{\mathcal{R}}\mathcal{R}^\bullet = 0\,,$$

from which we conclude $\mu_o \mathcal{R}^4 = \text{const}$. We put for later use

$$\kappa\mu_0 = \frac{3\mathcal{R}_0^2}{\mathcal{R}^4}\ . \tag{II.7.50}$$

Again, three cases are possible. One gains from

$$G_{44} = -k\frac{3}{\mathcal{R}^2} + 3U_4U_4 = -\kappa\mu_0$$

with (II.7.50) for $k = \{1, 0, -1\}$

$$\mathcal{K}^\bullet = \pm\frac{1}{\mathcal{R}_0}\sqrt{\frac{1}{\mathcal{K}^2} - 1},\quad \mathcal{K}^\bullet = \pm\frac{1}{\mathcal{R}_0}\frac{1}{\mathcal{K}},\quad \mathcal{K}^\bullet = \pm\frac{1}{\mathcal{R}_0}\sqrt{\frac{1}{\mathcal{K}^2} + 1}\ . \tag{II.7.51}$$

Integration yields

$$t = \pm\mathcal{R}_0\sqrt{1-\mathcal{K}^2},\quad t = \mathcal{R}_0\frac{\mathcal{K}^2}{2},\quad t = \mathcal{R}_0\sqrt{1+\mathcal{K}^2}\ . \tag{II.7.52}$$

For $k = 1$ one obtains a closed sphere which expands to the radius $\mathcal{R}_0$ and then contracts again. For the other two cases one has an open space that expands eternally. The time functions behave similarly to Fig. II.2. For $k = 1$ one obtains a semi-circle.

Frequently the representation of Robertson and Walker is used. If we define the Gaussian radial co-ordinate with

$$r = \sin\eta\,, \tag{II.7.53}$$

we derive from it

$$d\eta = \frac{1}{\cos\eta}dr = \frac{1}{\sqrt{1-r^2}}dr\ .$$

From $\mathcal{R}\sin\eta$ in (II.7.2) is made $\mathcal{R}r$. Generalizing to the three cases discussed above, one finally has the Robertson-Walker metric

$$ds^2 = \mathcal{R}^2\left[\frac{1}{1-kr^2}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2\right] - dt^2. \tag{II.7.54}$$

The line element is sometimes expressed in coordinates which can be attributed to a stereographic projection. If is $\mathcal{R}$ the radius of the positively curved universe, then is according to Fig. II.3.

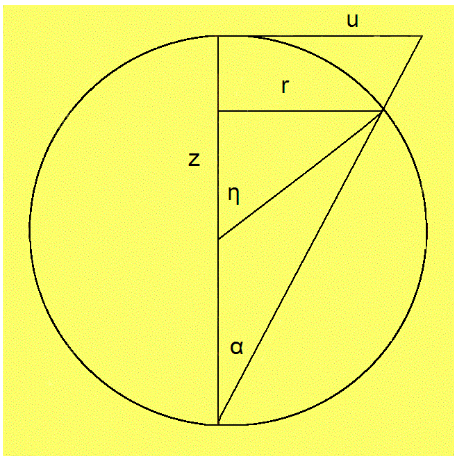

Fig. II.3

$$r = \mathfrak{R}\sin\eta, \quad z = \mathfrak{R}\cos\eta, \quad \tan\alpha = \frac{r}{\mathfrak{R}+z} = \frac{\sin\eta}{1+\cos\eta} = \tan\eta/2 \quad . \tag{II.7.55}$$
$$u = 2\mathfrak{R}\tan\alpha$$

From this one finally gets

$$u = 2\mathfrak{R}\tan\eta/2$$

and

$$du = \mathfrak{R}\left(1+\tan^2\eta/2\right)d\eta = \mathfrak{R}\left(1+\frac{u^2}{4\mathfrak{R}^2}\right)d\eta$$

can be used in the line element. Now one has to calculate

$$\mathfrak{R}\sin\eta = \mathfrak{R}\frac{2\tan\eta/2}{1+\tan^2\eta/2} = \frac{u}{1+\frac{u^2}{4\mathfrak{R}^2}}$$

and finally has written down the line element

$$ds^2 = \frac{1}{\left(1+\frac{u^2}{4\mathfrak{R}^2}\right)^2}\left(du^2 + u^2 d\Omega^2\right) - dt^2 \tag{II.7.56}$$

in isotropic form.

The positive cosmological constant as a force with high range was introduced by Einstein, so that his proposed static universe does not collapse due to its own gravitational force. The cosmological constant holds in balance all the particles of the universe. In an expanding universe it is not necessary to maintain stability. The cosmological constant is dispensable. For this reason we can accept the above simplification without hesitating.

Altaie [A] is of opinion that the cosmological constant may play a role in the vacuum fluctuation in the universe. The cosmological constant has been brought to the right side of the field equations and added to the stress-energy-momentum tensor by him. With the definitions

$$\mu = \mu_0 + \frac{\lambda}{\kappa} \quad p = -\frac{\lambda}{\kappa}$$

we get the stress-energy-momentum tensor[28]

$$T_{mn} = \begin{pmatrix} -p & & & \\ & -p & & \\ & & -p & \\ & & & \mu \end{pmatrix} \tag{II.7.57}$$

and from the conservation law $T^{4n}{}_{||n} = 0$ the equation

$$\mu^{\bullet} = 3\frac{\mathcal{R}^{\bullet}}{\mathcal{R}}(\mu + p) \ , \tag{II.7.58}$$

in analogy to (II.7.49). Such a universe consists of matter of time-dependent density, but of constant pressure.

Perlmutter [P] et al and Riess [R] et al concluded after extensive measurement of the brightness of distant supernovae that the expansion of the universe is accelerating. Krauss [K] and Starkman; Krauss [K] and Scherrer and Krauss [K] and Turner provide a model in which this acceleration is taken into account. They maintain the cosmological constant of the Friedman model and they shift this term to the right side of Einstein's field equations. The term is referred to as dark energy and is equally associated with the dark matter of the model. The presence of non-visible matter has been presumed by astrophysicists for a long time in order to explain the 'weight' of the galaxies. The deflection of light by distant galaxies can only be understood in terms of magnitude if one assumes that in this there is more matter than can be explained by visible stars. Several authors admit that the space can expand faster than light, which should not be in conflict with special relativity. Accordingly, the galaxies are moving away from each other also with superluminal velocity, so that light from distant galaxies can no longer reach our star system in many millions of years. However, according to the special theory of relativity, it is considered that despite the acceleration, the speed can only asymptotically reach the speed of light after an infinitely long time. Therefore, we believe that an expanding model is conceivable that expands with higher and higher speed, however, reaches the speed of light only after an infinitely long time.

Several authors require that the curvature parameter is $k = 0$ ie universe is flat and infinite. This simplifies the solution of the differential equation of Friedman, which in this case could incorporate the cosmological constant. An infinitely large spatially uniform universe contains infinitely many stars which emit an infinite amount of light (Olbers' paradox). It should have to be explained when and how an endless number of stars could be created. Moreover, in the literature the cases $k = 0$ and $k = -1$ are sometimes referred to as unsolved problems. They are not mentioned in the original Friedman paper. It is likely that Friedman was aware of the full set of solutions of his differential equation, but that he has considered the two infinite cosms as unphysical.

Krasiński [K] has examined critically the possibility that the expansion of the universe is accelerated. He notes that the accelerated expansion is not an observed phenomenon, but

[28] Differences to Altai concerning the sign result from our it-notation.

the result of an interpretation of observations. Whether the universe is accelerating or not is dependent on the model that one puts on concerning the observations. The Friedman model in the Robertson-Walker form (II.7.54) suggests an acceleration. The model of Lemaître [L] (1933) and Tolman [T] (1934) does not allow for acceleration. It takes into account the inhomogeneity of the universe.

An inhomogeneous model does not require acceleration of the expansion of the universe. The expansion velocity does not need to be proportional to the distance from an observer. The universe is inhomogeneous in its temporal extension. In some regions the velocity may have different values. Krasiński illustrated this with Fig. II.4.

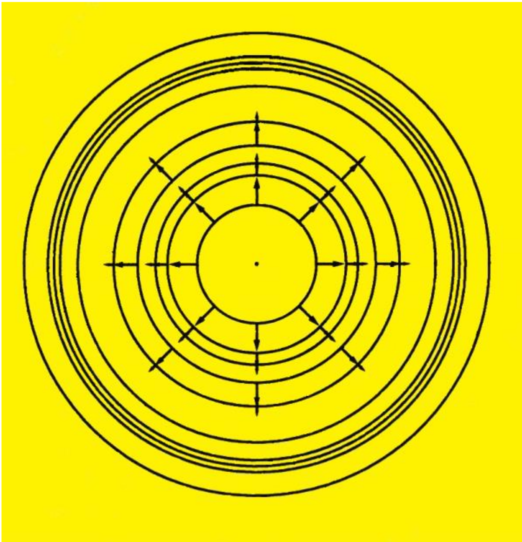

Fig. II.4

The redshifts measured by Perlmutter and Riess could stem from epochs of higher expansion velocities. The ansatz for the metric of an inhomogeneous universe is

$$ds^2 = \frac{\partial r/\partial r'}{1+f(r')}dr'^2 + r^2 d\Omega^2 - dt'^2 .$$

Instead of the Friedman differential equation one obtains

$$\dot{r}^2 = f\left(r'\right) + \frac{2M\left(r'\right)}{r} - \frac{\lambda r^2}{3} ,$$

where M and f are arbitrary functions. It is evident that the model contains indefinite quantities, It cannot be read from the geometric structure of the theory that inhomogeneities occur at a particular cosmic time.

II.8. The model of Milne

In 1934 Milne [M] has published a cosmological model based on the principles of special relativity. In a flat space there is first concentrated matter, which spreads into all directions after an initial process. The redshift of the light emanating from the receding stars has a kinematic origin and is explained by the Doppler effect. Although the model of Milne has been rejected for physical reasons, some cosmologists still have some interest in it: McCrea [M], McCrea [M] and Milne, Layzer [L], Chodorowski [C], Macleod [M] und Rindler [R]. The Milne cosmos is also regarded as a special case of the Friedman cosmos with the curvature parameter being $k=-1$. With a special coordinate transformation one wants to bring the metric into a flat form in order to obtain the original description of the Milne cosmos. We want to highlight this process critically.

We carry on from the metric of the Friedman cosmos in case of negative constant spatial curvature

$$ds^2 = \mathcal{R}^2\left[d\eta^2 + \mathrm{sh}^2\eta d\vartheta^2 + \mathrm{sh}^2\eta \sin^2\vartheta d\varphi^2\right] - dt^2, \quad \mathcal{R} = \mathcal{R}(t). \tag{II.8.1}$$

With the radial coordinate $r = \mathcal{R}\,\mathrm{sh}\eta$ we obtain the metric in canonical form

$$ds^2 = \frac{1}{1+\dfrac{r^2}{\mathcal{R}^2}} dr'^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - dt^2 \tag{II.8.2}$$

and read from it $k=-1$. From (II.8.1) we borrow the 4-bein system:

$$\overset{1}{e}_1 = \mathcal{R}, \quad \overset{2}{e}_2 = \mathcal{R}\mathrm{sh}\eta, \quad \overset{3}{e}_3 = \mathcal{R}\mathrm{sh}\eta\sin\vartheta, \quad \overset{4}{e}_4 = 1. \tag{II.8.3}$$

From this, we calculate the field strengths from the Ricci-rotation coefficients

$$A_{mn}{}^{s} = \overset{s}{e}_i \underset{[n|m]}{e^i} + g^{sr} g_{mt} \overset{t}{e}_i \underset{[n|r]}{e^i} + g^{sr} g_{nt} \overset{t}{e}_i \underset{[m|r]}{e^i}. \tag{II.8.4}$$

We separate them into

$$A_{mn}{}^{s} = B_{mn}{}^{s} + C_{mn}{}^{s} + U_{mn}{}^{s} \tag{II.8.5}$$

and with the unit vectors

$$m_m = \{1,0,0,0\}, \quad b_m = \{0,1,0,0\}, \quad c_m = \{0,0,1,0\}, \quad u_m = \{0,0,0,1\} \tag{II.8.6}$$

we split up further into

$$B_{mn}{}^{s} = b_m B_n b^s - b_m b_n B^s, \quad C_{mn}{}^{s} = c_m C_n c^s - c_m c_n C^s, \quad U_{mn}{}^{s} = u_m U_n u^s - u_m u_n U^s. \tag{II.8.7}$$

B and C are the lateral field quantities and U is the radial field quantity. For the 4th components of all field quantities we get

$$U_4 \overset{*}{=} B_4 \overset{*}{=} C_4 = \frac{1}{\mathcal{R}} \mathcal{R}_{|4}. \tag{II.8.8}$$

This means that the expansions in all three spatial directions are the same. Therefore, the curvature scalar has the form

$$u^s{}_{||s} = \frac{3}{\mathcal{R}} \mathcal{R}_{|4}.$$

The model expands in free fall because $g_{44}=1$ is the metric coefficient in (II.8.1). Thus, the proper time T is equal to the coordinate time t. The right-hand side of (II.8.8) can also be written as

$$\frac{i}{\mathcal{R}}\frac{\partial\mathcal{R}}{\partial T}=\frac{i}{\mathcal{R}}\mathcal{R}^{\bullet}. \tag{II.8.9}$$

Furthermore, we put with regard to Milne's ansatz

$$\mathcal{R}^{\bullet}=1, \tag{II.8.10}$$

where 1 is the velocity of light in the natural measuring system. Finally, with (II.8.4)-(II.8.8) one has

$$\begin{gathered}
U_m = A_{1m}{}^{1} = \{0,0,0,1\}\left(-\frac{i}{\mathcal{R}}\right)\\
B_m = A_{2m}{}^{2} = \left\{\frac{1}{\mathcal{R}}\mathrm{cth}\eta,0,0,-\frac{i}{\mathcal{R}}\right\} = \{\mathrm{ch}\eta,0,0,-i\,\mathrm{sh}\eta\}\left(\frac{1}{\mathcal{R}\,\mathrm{sh}\eta}\right)\\
C_m == A_{3m}{}^{3} = \left\{\frac{1}{\mathcal{R}}\mathrm{cth}\eta,\frac{1}{\mathcal{R}\mathrm{sh}\eta}\cot\vartheta,0,-\frac{i}{\mathcal{R}}\right\}\\
=\{\mathrm{ch}\eta\sin\vartheta,\cos\vartheta,0,-i\,\mathrm{sh}\eta\sin\vartheta\}\left(\frac{1}{\mathcal{R}\,\mathrm{sh}\eta\sin\vartheta}\right)
\end{gathered}. \tag{II.8.11}$$

Therein $\frac{i}{\mathcal{R}},\frac{1}{\mathcal{R}\,\mathrm{sh}\eta},\frac{1}{\mathcal{R}\,\mathrm{sh}\eta\sin\vartheta}$ are the curvatures of normal and oblique slices of the surface with the metric (II.8.1), which is the base of the model. The relations

$$U^{s}{}_{\underset{1}{||}s}+U^{s}U_{s}=0,\quad B_{m\underset{2}{||}n}+B_{m}B_{n}=0,\quad C_{m\underset{3}{||}n}+C_{m}C_{n}=0 \tag{II.8.12}$$

are formulated with the unit vectors (II.8.6) and the graded derivatives

$$\begin{gathered}
U_{m\underset{1}{||}n}=U_{m|n},\quad B_{m\underset{2}{||}n}=B_{m|n}-U_{nm}{}^{s}B_{s},\quad C_{m\underset{2}{||}n}=C_{m|n}-U_{nm}{}^{s}C_{s}-B_{nm}{}^{s}C_{s}\\
U_{nm}{}^{s}=u_{n}U_{m}u^{s}-u_{n}u_{m}U^{s},\quad B_{nm}{}^{s}=b_{n}B_{m}b^{s}-b_{n}b_{m}B^{s}
\end{gathered}. \tag{II.8.13}$$

They are subequations of the Ricci [11]. Since all the expressions in (II.8.12) vanish one has $R_{mn}=0$. Therefore the stress-energy-momentum tensor is $T_{mn}=0$. The universe is empty.

The special feature of the Milne model is the simple dependency of the scale factor on time:

$$\mathcal{R}(t)=t\,. \tag{II.8.14}$$

Instead of (II.8.9) one has

$$\frac{i}{t}\frac{\partial t}{\partial T}=\frac{i}{t}\frac{\partial t}{\partial t}=\frac{i}{t}$$

and instead of (II.8.10) the trivial relation $t^{\bullet}=1$ which justifies the approach (II.8.10) in retrospect.

With this and with $r = \Re \,\mathrm{sh}\eta$ the simple expressions for the field quantities are obtained

$$U_m = \{0,0,0,1\}\left(-\frac{i}{t}\right), \quad B_m = \left\{\frac{1}{r}\mathrm{ch}\eta, 0, 0, -\frac{i}{t}\right\}, \quad C_m = \left\{\frac{1}{r}\mathrm{ch}\eta, \frac{1}{r}\cot\vartheta, 0, -\frac{i}{t}\right\}. \qquad \text{(II.8.15)}$$

If we also identify the quantity t introduced by (II.8.14) with the time variable, it is possible to apply the coordinate transformation

$$r' = t\,\mathrm{sh}\eta, \quad t' = t\,\mathrm{ch}\eta. \qquad \text{(II.8.16)}$$

Thus, one first obtains

$$dr'^2 - dt'^2 = t^2 d\eta^2 - dt^2.$$

Now the metric (II.8.1) can be written as

$$ds^2 = t^2\left[d\eta^2 + \mathrm{sh}^2\eta\, d\vartheta^2 + \mathrm{sh}^2\eta \sin^2\vartheta\, d\varphi^2\right] - dt^2 \qquad \text{(II.8.17)}$$

and with the above relation as

$$\begin{gathered} ds^2 = dr'^2 + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta\, d\varphi^2 - dt'^2 \\ \overset{1'}{e}_{1'} = 1, \quad \overset{2'}{e}_{2'} = r', \quad \overset{3'}{e}_{3'} = r'\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1 \end{gathered}. \qquad \text{(II.8.18)}$$

This is obviously the metric in the flat Minkowski space. It seems that with a coordinate transformation

$$\Lambda_i^{i'} = \begin{pmatrix} t\,\mathrm{ch}\eta & -i\,\mathrm{sh}\eta \\ t\,i\,\mathrm{sh}\eta & \mathrm{ch}\eta \end{pmatrix}$$

and the associated pseudo rotation

$$L_m^{m'} = \begin{pmatrix} \mathrm{ch}\eta & -i\,\mathrm{sh}\eta \\ i\,\mathrm{sh}\eta & \mathrm{ch}\eta \end{pmatrix} = \begin{pmatrix} \cos i\eta & -\sin i\eta \\ \sin i\eta & \cos i\eta \end{pmatrix}, \quad L_m^{m'} = \overset{m'}{e}_{i'} \Lambda_i^{i'} \underset{m}{e}^{i} \qquad \text{(II.8.19)}$$

a negatively curved space $(k = -1)$ can be made from a flat space $(k = 0)$.

With the help of the pseudo rotation (II.8.19) the field quantities (II.8.11) can be converted into those of a static system. The Ricci-rotation coefficients transform according to

$$'A_{m'n'}{}^{s'} = L_{m'n's}^{m\,n\,s'} A_{mn}{}^{s} + {}'L_{m'n'}{}^{s'}.$$

The quantity U is the very component of the Ricci-rotation coefficients, which transforms inhomogeneously

$$'U_{m'n'}{}^{s'} = U_{m'n'}{}^{s'} + {}'L_{m'n'}{}^{s'}.$$

With

$$\begin{gathered} U_{m'n'}{}^{s'} = h_m{}^{s'} U_{n'} - h_{m'n'} U^{s'}, \quad h_{m'n'} = \mathrm{diag}\{1,0,0,1\} \\ 'L_{m'n'}{}^{s'} = h_{m'}{}^{s'}\,{}'L_{n'} - h_{m'n'}\,{}'L^{s'}, \quad 'L_{n'} = {}'L_{s'n'}{}^{s'} = \left\{{}'L_{4'1'}{}^{4'}, {}'L_{1'4'}{}^{1'}\right\} \end{gathered} \qquad \text{(II.8.20)}$$

one finally obtains the simple relations

$$'U_{m'} = U_{m'} + {}'L_{m'}\,.$$

With

$$'L_{m'n'}{}^{s'} = L_s^{s'} L^s_{n'|m'}, \quad 'L_{1'} = 'L_{4'1'}{}^{4'} = -i\eta_{|4'}, \quad 'L_{4'} = 'L_{1'4'}{}^{1'} = i\eta_{|1'}, \tag{II.8.21}$$

as well as with $\eta_{|m'} = L^m_{m'} \underset{m}{e}^i \eta_{|i}$ and (II.8.3) one first has

$$'L_{m'} = \{-i\,sh\eta, 0, 0, ch\eta\}\frac{i}{t} \tag{II.8.22}$$

and from the inhomogeneous transformation law of the Ricci-rotation coefficients

$$U_{m'} = L^m_{m'} U_m = \{-i\,sh\eta, 0, 0, ch\eta\}\left(-\frac{i}{t}\right), \quad 'U_{m'} = U_{m'} + 'L_{m'} = 0\,. \tag{II.8.23}$$

In the transformed system one does not experience expansion, the space would be not only flat but also static.

In order to clarify this contradiction, we once again deal with the transformation (II.8.16) and we remember Eq. (II.8.14). Thus, we obtain

$$r' = \mathcal{R}\,sh\eta, \quad t' = \mathcal{R}\,ch\eta \tag{II.8.24}$$

and we recognize that t' is an inappropriate identification with the time variable. Instead we write

$$x^{1'} = \mathcal{R}\,sh\eta, \quad x^{0'} = i\mathcal{R}\,ch\eta \tag{II.8.25}$$

and we obtain with

$$dx^{1'} = sh\eta\, d\mathcal{R} + \mathcal{R}\,ch\eta\, d\eta, \quad dx^{0'} = i\,ch\eta\, d\mathcal{R} + i\mathcal{R}\,sh\eta\, d\eta \tag{II.8.26}$$

after all

$$dx^{0'2} + dx^{1'2} = (i\,d\mathcal{R})^2 + \mathcal{R}^2 d\eta^2\,. \tag{II.8.27}$$

Now one can see that the angle $i\eta$ describes a rotation in the [0',1']-plane of the embedding space and not a rotation in the [1',4']-plane of the physical space. Therefore (II.8.19) cannot be a Lorentz transformation which performs the transition from the comoving to the non-comoving system. (II.8.25) describes the relation between the hyperspherical coordinates of the Milne model and the Cartesian coordinates of the flat 5-dimensional embedding space. The complete transformation reads as

$$\begin{aligned}
x^{3'} &= i\mathcal{R}\,\sin i\eta \sin\vartheta \sin\varphi = \mathcal{R}\,sh\eta \sin\vartheta \sin\varphi \\
x^{2'} &= i\mathcal{R}\,\sin i\eta \sin\vartheta \cos\varphi = \mathcal{R}\,sh\eta \sin\vartheta \cos\varphi \\
x^{1'} &= i\mathcal{R}\,\sin i\eta \cos\vartheta = \mathcal{R}\,sh\eta \cos\vartheta \\
x^{0'} &= i\mathcal{R}\,\cos i\eta = i\mathcal{R}\,ch\eta \\
x^{4'} &= it
\end{aligned}\,. \tag{II.8.28}$$

The Milne model with the metric (II.8.1) can be expressed geometrically as the surface of a pseudo-sphere

$$x^{a'} x^{a'} = (i\mathcal{R})^2, \quad a' = 0', 1', \ldots, 3'$$

with an imaginary radius. This corresponds to the tradition of treating cosmological models. Recalling the above-mentioned considerations we can see that one has obtained the flat metric (II.8.18). One has performed a transformation into the flat embedding space. The misleading approach (II.8.16) has possibly been used first by Walker [W] and has been adopted by other authors.

The attempt to transform the comoving coordinate system of the Milne universe into a non-comoving coordinate system has failed because the radius of curvature of the imaginary hypersphere defining the geometry has been confused with the time variable. However, we can reject the assertion that the curvature of the space may depend on the representation by pointing out these circumstances.

II.9. Einstein's elevator in cosmology

One of the main features of general relativity is the identification of the gravitational forces with the effect of the curvature of space on observers, but also the possibility to 'transform away' the gravitational effect, that is to keep an observer without force by an appropriate choice of a reference system. This does not mean that one can eliminate the curvature of space by an observer transformation, but only that one can annul the effect of the curvature of space for certain observers. This is the case for a freely falling observer in the Schwarzschild field of a stellar object. Such an effect can be expected also for cosmological models which expand in free fall.

The effect has become known as Einstein's elevator in the literature. We will recall this effect for the case of the Schwarzschild field, but also we will show that it is useful to consider it for cosmological models, whereby the cosmological principle should apply. The elevator principle is discussed referring to the de Sitter cosmos and in a later Section to a subluminal model. We resort to earlier results [B] and discuss the problems from the perspective of Einstein's elevator principle outlined in detail therein. In modern research Einstein's elevator is denoted as 'Weak Einstein Equivalence Principle' (WEEP)[29]. The introduction of the elevator principle will be crucial for the structure of the universe. It determines whether a metric with the curvature parameter $k = 0$ is flat or positively curved and whether the universe is expanding in free fall.

We discuss the problem of Einstein's elevator using the Schwarzschild model, since the effect is particularly obvious in this model and then we will transfer the knowledge obtained to the de Sitter model. Thus we have paved the way for cosmology.

1. Schwarzschild model

The line element of the Schwarzschild field in the standard form is

$$ds^2 = \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\Omega^2 - \left(1-\frac{2M}{r}\right)dt^2 \tag{II.9.1}$$

with r as the radial coordinate and Ω as the solid angle. M is, except for factors, the mass of the field-producing stellar object. A detailed description of the Schwarzschild problem can be found in later Sections. The Schwarzschild geometry is parabolically curved into the radial direction. We want to bring this line element into the *canonical form*, in which the FRW[30]-curvature parameter k occurs. It is usually said that to k are attributed the curvature properties of the model. This method was first used in cosmology. Later, it was introduced by McVittie [M] for collapsing gravity models. Based on the Schwarzschild model we will examine what the FRW method is able to accomplish and we will use this knowledge for cosmology.

We should remember that the radius of curvature ρ of the so-called Schwarzschild parabola, which describes the specific spatial curvature of the model, is twice as long as its extension $\mathbb{R} = \mathbb{R}(r)$ to the directrix of the Schwarzschild parabola.

[29] Kopeikin [K] has recently treated the problem of WEEP in connection with cosmological FRW metrics. He has shown that the cosmological expansion could be detected in local gravitational experiments.

[30] Friedman-Robertson-Walker

$$\rho = 2\mathcal{R} = \sqrt{\frac{2r^3}{M}}, \quad \mathcal{R} = r\sqrt{\frac{r}{2M}} \tag{II.9.2}$$

Thus we have

$$\sqrt{\frac{2M}{r}} = \frac{r}{\mathcal{R}} \tag{II.9.3}$$

and the Schwarzschild metric in canonical form is written down as

$$ds^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\Omega^2 - \left(1-\frac{r^2}{\mathcal{R}^2}\right) dt^2 . \tag{II.9.4}$$

It is now similar to the de Sitter metric which we discuss in the next Section. By comparison with the FRW standard form

$$ds^2 = \frac{1}{1-k\frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\Omega^2 - dT^2 \tag{II.9.5}$$

we find $k = 1$. Flamm's paraboloid, which is created by a rotation around the directrix of the Schwarzschild parabola, appears positively curved in the canonical representation of the metric (II.9.4). But in contrast to the FWR definition Flamm's paraboloid is open and infinite. Since already occur deviations from the usual interpretation of the quantity k in the Schwarzschild model, let from now on denote k as the *form parameter* of the model.

From (II.9.1) or (II.9.4) we calculate the components of the Ricci-rotation coefficients[31]. The radial and the two lateral components are

$$U_m = \left\{-\alpha v\frac{1}{\rho},0,0,0\right\}, \quad B_m = \left\{\frac{a}{r},0,0,0\right\}, \quad C_m = \left\{\frac{a}{r},\frac{1}{r}\cot\vartheta,0,0\right\} . \tag{II.9.6}$$
$$v = -\sqrt{2M/r} = -r/\mathcal{R}, \quad a = 1/\alpha = \sqrt{1-2M/r} = \sqrt{1-r^2/\mathcal{R}^2}$$

The geometric quantity U is the negative of the physical quantity, the force of gravity.

Lemaître has found a coordinate transformation associated with a freely falling observer. We will treat this coordinate transformation in detail later on. The metric in these coordinates is

$$ds^2 = \mathcal{K}^2\left[dr'^2 + \mathcal{R}^2 d\Omega^2\right] - dt'^2, \quad \mathcal{K} = \frac{r}{\mathcal{R}} . \tag{II.9.7}$$

Herein $\mathcal{K}$ is referred to as *scale factor* how it is done in the cosmological models. Thus, the line element is of type $k = 0$. According to the FRW-classification the model would be referred to as flat. If one calculates from this metric the Ricci-rotation coefficients one has

$$'U_{m'} = \left\{0,0,0,-\frac{i}{\rho}\right\}, \quad B_{m'} = \left\{\frac{1}{r},0,0,\frac{i}{\mathcal{R}}\right\}, \quad C_{m'} = \left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0,\frac{i}{\mathcal{R}}\right\} . \tag{II.9.8}$$

Is noteworthy that the space-like components of the lateral field quantities

[31] Details for the calculation with the tetrad method can be found in papers published about 1900 by Ricci, Bianchi, Levi-Cività, furthermore by Treder [19], Liebscher and Treder [20] and also in our paper [4].

$$B_{\alpha'} = \left\{\frac{1}{r},0,0\right\}, \quad C_{\alpha'} = \left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0\right\}, \quad \alpha' = 1',2',3'. \tag{II.9.9}$$

are precisely those that one would expect for a flat geometry in polar coordinates. But it would be premature to call the geometry flat. The basic geometric structure of a model cannot be modified by a coordinate transformation. We want to get to the bottom of the matter.

From the coordinate transformation $x^i \to x^{i'}$ of Lemaître can be determined the matrix of the coordinate transformation with $\Lambda_i^{i'} = x^{i'}{}_{|i}$. Since for the two systems, the tetrads can be read from (II.9.1) and (II.9.7) one can calculate the associated Lorentz transformation of this coordinate transformation with $L_m^{m'} = \overset{m'}{e}{}_{i'} \Lambda_i^{i'} \underset{m}{e}{}^i$:

$$L_m^{m'} = \begin{pmatrix} \alpha & & & i\alpha v \\ & 1 & & \\ & & 1 & \\ -i\alpha v & & & \alpha \end{pmatrix}, \quad \alpha = \frac{1}{\sqrt{1-\frac{2M}{r}}}, \quad v = -\sqrt{\frac{2M}{r}}. \tag{II.9.10}$$

For the lateral field quantities one obtains with

$$B_{m'} = L_{m'}^{m} B_m, \quad C_{m'} = L_{m'}^{m} C_m \tag{II.9.11}$$

from (II.9.6) first the components

$$B_{m'} = \left\{\alpha a \frac{1}{r},0,0,\frac{i}{\mathcal{R}}\right\}, \quad C_{m'} = \left\{\alpha a \frac{1}{r},\frac{1}{r}\cot\vartheta,0,\frac{i}{\mathcal{R}}\right\}, \tag{II.9.12}$$

which a free falling observer would measure. However, since the velocity of a freely falling object in the Schwarzschild field is coupled to the angle of ascent η of the Schwarzschild parabola via

$$r = \mathcal{R}\sin\eta, \quad v = -\frac{r}{\mathcal{R}} = -\sin\eta$$

one has for the Lorentz factor and the metric factor according to (II.9.6)

$$a\alpha = 1. \tag{II.9.13}$$

This means that the geometric quantity a is supplemented with the kinematic quantity α to 1. Thus, the parameter a for the curvature of space in (II.9.8) is just *hidden*.

Special attention should be paid to the conversion $U_m \to {}'U_{m'}$. The radial field quantities are also components of the Ricci-rotation coefficients, but they transform inhomogeneously. Since the free fall takes place in the [1,4]-slice of the space the inhomogeneous transformation law is limited to the radial field quantities [1] and the transformation formula is reduced to

$${}'U_{m'} = L_{m'}^{m} U_m + {}'L_{m'}, \quad {}'L_{1'} = i\alpha^2 v_{|4'}, \quad {}'L_{4'} = -i\alpha^2 v_{|1'}. \tag{II.9.14}$$

The ${}'L$-terms are calculated using $v = -\sqrt{2M/r}$. Finally, one obtains

$$U_m = \left\{-\alpha v \frac{1}{\rho},0,0,0\right\} \quad \to \quad {}'U_{m'} = \left\{0,0,0,-\frac{i}{\rho}\right\} \tag{II.9.15}$$

in accordance with (II.9.8). We recognize that in the freely falling system the spatial components of the U-field are all zero. In detail, one has

$$'U_{1'} = U_{1'} + 'L_{1'} = 0\,. \tag{II.9.16}$$

This means that the gravitational force $U_{1'}$ is canceled by the dynamic term $'L_{1'}$, ie by the counter force. This is the principle of *Einstein's elevator*. Observers who are in a free-falling elevator are not subjected to gravitational forces, they hover. Since these observers do not feel any gravitational forces, they might think that space is flat. Thus

$$'U_{m'} = L^{m}_{m'} U_m + 'L_m$$

is *Einstein's elevator equation* and

$$'U^{s'}{}_{|s'} + 'U^{s'}\,'U_{s'} = -\frac{1}{\mathcal{R}^2} \tag{II.9.17}$$

the *Friedman equations* for free fall in the Schwarzschild field in tetrad form.

Although the metric which relates to the free fall is of the type $k = 0$ and the space-like part of this metric appears flat and the field quantities appear flat as well, the 3-dimensional space is nevertheless curved and is represented by Flamm's paraboloid. The apparent flatness of the space is due to the effect of Einstein's elevator. The quantity k, common in cosmology, cannot be related to the curvature of space, it describes the form of the metric. $k = 0$ indicates that the metric relates to a freely falling system.

We have dealt with the problem in as much detail because in cosmological models the problem is the same and the formal treatment does not differ much from what has just been put forward.

2. De Sitter model

By de Sitter [S] was proposed a cosmological model with the line element

$$ds^2 = \frac{1}{\cos^2\eta} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \cos^2\eta dt^2 \tag{II.9.18}$$

It is the line element on a pseudo-hyper sphere with the time-independent radius $\mathcal{R}$. We have discussed the model in the previous Sections in detail and we will examine it again with respect to the curvature properties. The space is positively curved and closed. With

$$r = \mathcal{R}\sin\eta \tag{II.9.19}$$

it can be brought into the canonical form

$$ds^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left(1-\frac{r^2}{\mathcal{R}^2}\right) dt^2\,. \tag{II.9.20}$$

We read from this metric $k = 1$. We calculate the Ricci-rotation coefficients from (II.9.20) and we find the following field quantities

$$U_m = \left\{-\alpha v \frac{1}{\mathcal{R}_0}, 0,0,0\right\}, \quad B_m = \left\{\frac{a}{r}, 0,0,0\right\}, \quad C_m = \left\{\frac{a}{r}, \frac{1}{r}\cot\vartheta, 0,0\right\} \tag{II.9.21}$$

with the definitions

$$a = \cos\eta = \sqrt{1 - r^2/\mathcal{R}^2} = 1/\alpha, \quad v = r/\mathcal{R} = \sin\eta\,. \tag{II.9.22}$$

B and C are the typical lateral field quantities for a spherical geometry. The quantity U_1 is a force acting at any point and wants to pull apart neighboring points.

By Lemaître [L] a coordinate transformation has been found which transforms the metric (II.9.20) into the expanding metric

$$ds^2 = e^{2\psi'}\left[dr'^2 + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta\, d\varphi^2\right] - dt'^2 . \tag{II.9.23}$$

Therein $e^{\psi'} = \mathcal{K}$ is the position-independent but time-dependent scale factor, and $\{r', \vartheta, \varphi, t'\}$ the coordinates comoving with the expansion. The following relations

$$r = \mathcal{K}\, r', \quad t' = \mathcal{R}\psi', \quad \frac{1}{\mathcal{K}}\mathcal{K}_{|m'} = \left\{0,0,0,-\frac{i}{\mathcal{R}}\right\}. \tag{II.9.24}$$

apply.

From the metric (II.9.23) we derive the field quantities

$$'U_{m'} = \left\{0,0,0,-\frac{i}{\mathcal{R}_0}\right\}, \quad B_{m'} = \left\{\frac{1}{r},0,0,-\frac{i}{\mathcal{R}_0}\right\}, \quad C_{m'} = \left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0,-\frac{i}{\mathcal{R}_0}\right\} \tag{II.9.25}$$

which differ from those of the freely falling Schwarzschild system only in the time-like components, ie the tidal forces. The Schwarzschild geometry is parabolic, the dS geometry spherical. The Schwarzschild reference system contracts, the dS reference system expands. The geometry (II.9.23) is only *apparently* flat. The same arguments as in the Schwarzschild geometry speak against the flatness of space, if the metric is written in the expanding form (II.9.23).

The field quantities in (II.9.25) could also have been derived with the Lorentz transformation

$$L^{m}_{m'} = \begin{pmatrix} \alpha & & & -i\alpha v \\ & 1 & & \\ & & 1 & \\ i\alpha v & & & \alpha \end{pmatrix}, \tag{II.9.26}$$

whereby for the variables U the inhomogeneous transformation law

$$'U_{m'} = L^{m}_{m'} U_m + 'L_m, \quad 'U_{1'} = U_{1'} + 'L_{1'} = 0 \tag{II.9.27}$$

applies. We also find just like in (II.9.15)

$$U_m = \left\{-\alpha v \frac{1}{\mathcal{R}_0},0,0,0\right\} \quad \rightarrow \quad 'U_{m'} = \left\{0,0,0,-\frac{i}{\mathcal{R}_0}\right\}. \tag{II.9.28}$$

(II.9.27) is the *Einstein elevator equation* for the dS universe. The reference system established with the Lorentz transformation (II.9.26) expands in freefall. The form parameter read from the metric (II.9.23) is $k = 0$. Here k stands again for the form of a metric which is determined by a freely falling coordinate system and does not make any statements about the curvature of space.

In this Section we have shown by means of two models how the quantity, generally referred to as the curvature parameter, can be interpreted in a different way. This new assessment has far-reaching implications for cosmological models. Universes that were previously considered to be infinite with infinite matter, now present themselves to be finite and thus are plausible. It also prevents models from seeming to be curved and closed in the non-comoving system. However, in the moving system they are flat and open. We will often make use of this fact.

II.10. Literature to cosmology

In the preceding Sections some few cosmological models have been treated, essentially with the intention of stating the embedding methods which are used for further solutions of the Einstein field equations. In his textbook Tolman [T] has laid the foundations of cosmology. Many authors refer to the models discussed in this book. Several time-dependent models stem from him and were continued by other authors. It has to be mentioned that in the last decades about three hundred different cosmological models have been found and also such models which are used for the explanation of the origin of the universe. Some examples are mentioned.

Textbooks:

Bondi H., *Cosmology*, Cambridge, At the University Press, 1961
Carmeli M., *Relativity: Modern large-scale spacetime structure of the cosmos*. World Scientific, Singapore 2008
Ehrensperger J., *Energie im Kosmos*. Verlag W. Vogel. Winterthur 1988
Hoyle F., Burbridge G., and Narlikar J. V., *A different approach to cosmology.* Cambridge University Press, 2005
Islam J. N., *An introduction* to *mathematical cosmology*. Cambridge University Press, London 2002
Krasiński A., *Inhomogeneous cosmological models.* Cambridge University Press 1975
Liebscher D-E., *Kosmologie.* Johann Ambrosius Barth. Leipzig, Heidelberg, 1994
McVittie G. C., *Fact and theory in cosmology.* The Macmillan company. New York 1961
McVittie G. C., *General relativity and cosmology*. The University of Illinois Press, Urbana 1965
Rich J., *Fundamentals of Cosmology*. Springer 2009
Robinson I., Schild A., and Schucking E. L., *Quasi-stellar sources and gravitational collapse.* The University of Chicago Press
Ryan M. P. Jr. and Shepley L. C., *Homogeneous relativistic cosmologies.* Princeton University Press, New Jersey 1975
Schramm D. M., Editor. *Inner space/outer space*. The University Chicago Press, Chicago 1986
Weinberg S., *Cosmology.* Oxford University Press 2008

Heckmann [H] has thoroughly discussed all possible combinations for the values of $k = \{1, 0, -1\}$ and λ (positive, 0 negative).

Behr [B] has extended the Friedman model with respect to spatial anisotropy.

Banerji [B] has examined rotating cosmological models with pressure, but without shears.

Belinskiĭ [B] and Khalatnikov have treated models with viscosity.

Hönl [H] has derived the formulae for calculation of the world age. Hughston [H] and Jacobs have examined vector mesons in Bianchi cosmologies.

Klimek [K] has extended the Friedman cosmos with viscosity. Krasiński [K] sets the curvature parameter as a function of time. It can change from +1 to -1, the model can be open or closed alternatively. The solution contains some arbitrary functions of time. The paths of matter are not geodesics.

Laue [L] deals with the propagation of light in expanding spaces.

Ludwig [L] and Müller have treated a model for star formation based on the projective theory of relativity by Jordan.

Misner [M] and Taub have included in their model effects of the viscosity with radiation and anisotropy of pressure.

Pachner [P] has calculated the gravitational field of a mass point in an expanding universe. In another paper he has generalized the stress-energy-momentum tensor.

Robertson [R] has presented an overview of the problem of cosmology of his time.

Sen Gupta [S] has calculated a pulsating universe with geodesics and light paths. Szekeres [S] and Rankin have examined models that go beyond homogeneity and isotropy.

The question was raised by Bondi [B], Price [P], and Romano whether the atoms whose components are bound by electric attraction take part in the expansion or at least partially take part in the expansion of the universe.

Ahluwalia [A] and Goldman; Adkins [A], McDonnell and Fell; Alabareti [A], Aldrovandi [A] and Pereira; Alhendi [A], Lashin and Nashed; Alles [A] et al; Anderson [A], Ank [A], Balcerzak [B] and Dąbrowski; Baykal and Cahk; Arcidiacono [A], Asada [A], Aróstica and Bahamonde; Astefanesei [A], Mann and Radu; Atkatz [A], Baker [B], Balbinot [B], Bergamini, and Comastri; Bamba [B], Yesmakhanova, Yerzhanov and Myzakulov; Bandyopadhyay [B] and Chakraborty; Banerjee [B] and Chakravarty; Bargueño [B], Barnes [B], Baryshev [B], Barnich [B], Brandt and Claes; Barrow [B], Juszkiewicz and Sonada; Batakis [B] and Cohen; Bayin [B] and Cooperstock; Belinski [B], Belinskii [B], Khalatnikov and Lifshitz; Belserne [B], Bergamini [B] and Giampieri; Bergmann [B], Bisnovatyi-Kogan [B] and Thorne; Blome [B] and Priester; Bochicchio [B] and Laserra; Böhmer [B] and Harko; Boero [B] and Moreschi; Bolejko [B], Brevik [B] and Grøn; Caldwell [C], Dave and Steinhardt; Capoziello [C] et al., Carrera [C] and Giulini; Célérier and Krasiński; Bolejko [B], Bolotin [B] et al; Bona [B] and Stela; Bonnor [B], Bonnor [B] and Knutsen; Borzeszkowski [B] and Treder; Bradley [B] and Sviestins; Brevik [B], Brill [B] and Flaherty; Brill [B], Buchert [B], Buchert [B] and Carfora; Buchert [B] and Ehlers, Buchert [B] and Domiíngues; Buchert [B], Ellis, and van Elst; Caldwell [C], Caldwell [C], Kamionkowski and Weinberg; Callan [C], Carroll [C] and Hoffman; Cavallo and Isenberg; Bull [B], Castaldo [C], Aróstica and Bahamonde; Canuto [C], Hsieh and Owen; Canuto [C] and Hsieh; Canuto [C] and Owen; Carmeli [C], Carrera [C] and Giulini; Carvalho [G], Lima and Waga; Chen [C], Gibbons and Yang; Célérier [C] and Szekeres; Célérier [C], Bolejko and Krasiński; Cembranos, de la Cruz-Dombriz and Dombado; Chandrasekhar [C], Chang [C] and Shuang; Chung [C], Clarkson [C], Ananda, and Larena; Clifton [C], Coble [C], Dodelson und Frieman; Cohen [C] and Gautreau; Coley [C], Coley [C] and Tupper; Chen [C] and Wu; Clarke [C], Coble [C], Dodelson and Frieman; Collins [C], Collins [C] and Wainwright; Collins [C] and Szafron; Cooperstock [C], Cooperstock [C] and Israelit; Coquereaux [C] and Grossmann; Cox [C], Critchley [C] and Dowker; Dąbrowski [D], Das [D] et al, Davies [D], Davis [D] and Lineweaver Dechant [D], Lasenby and Hobson; Del Campo [D], Cárdenas and Herrera; DeDeo [D] and Gott III; De Lorenci [D], Klippert, Novello and Salim; Demianski [D], Deser [D], Deser [D] and Levin; Domínguez [D] and Gaite; Doroshkevich [D], Zel'dovich and Novikov; Dzuhunushaliev [D], Eddington [E], Eisenstein [E], Ellis [E], Ellis [E] and MacCallum; Buchert [B] and Ehlers; Ellis [E], Hwag and Bruni; Ellis [E], Bruni and Hwag; Eardley [E], Liang and Sachs; Eisenstaedt [E], Ellis [E], Ellis [E] , Maartens and Nel; Ellis [E] and Stoeger; Eriksen [E] and Grøn; Evans [E], Fahr [F] and Siewert; Fanizza [F] and Tedesco; Faraoni [F] and Jacques; Fennelly [F], Florides [F], Fonseca-Neto [F] and Romero; Francis [F], Barnes, James and Lewis; Freese [F], Adams, Frieman and Mottola; Füzfa [F], Garecki [G], Garfinkle [G], Garfinkle [G], Garriga [G], Liang [L], Livio and Vilenkin; Gautreau [G], Giacommazzo [G], Giambo [G], Goode [G] and Wainwright; Goode [G] and Wainwrihgt; Goode [G], Cooley and Wainwright; Godłowski [G], Stelmach and Sydłowski; Goldman [G] and Rosen; Gomide [G], Gorkavyi [G] and Vasilikov; Gorini [G], Grillo and Pelizza; Goswami [G] and Trivedi; Grøn [G], Grøn [G] and Soleng; Grøn [G] and Elgarøy; Guendelman [G] et al; Guendelman [G], Guth [G], Guth [G] and Weinberg; Harada [H], Maeda and Carr; Hamilton [H], Harrison [H], Hartnett [H], Heckmann [H] and Schücking; Hashemi [H], Jalazadeh and Riazi; Hawking [H], Hawking [H] and Ellis; Helbig [H], Hellaby [H], Hellaby [H] and Lake; Henneaux [H], Herrera [H], Di Prisco and Ibáñez; Heyl [H], Henriksen [H], Emslie and Wesson; Hirata [H] and Seljak; Hogg [H], Hogan [H], Hönl [H] and Dehnen; Hoyle [H], Hoyle [H] and Narlikar; Hoyle [H] and Wickramashinghe; Hu [H], Turner and Weinberg; Hurd [H], Ibison [I], Iguchi [I], Nakamura and Nakao; Ishibashi [I] and Wald; Jacobs [J], Jaime [J] and Salgado; Javanović [J], Jensen [J] and Stein-Schabes; Jafarizadeh [J], Darabi, Rezaei-Aghdam and Rastegar; John [J], John [J] and Joseph; Just [J], Kaiser [K], Kalam [K] et al; Kaloper [K], Kalitzin [K], Katanaev [K], Kehagias [K] and Riotto; Kiang [K], Kim [K], Lasenby and Hobson; Kleban and Martin; King [K] and Ellis; Klein [K], Koijam [K], Kokus

[K], Kolb [K], Matarrese and Riotto; Kopeikin [K], Kopeikin [K] and Petrov; Korotkii [K] and Obukhov; Korotkiĭ [K] and Obukhov; Korotky [K] and Obukhov; Krasiński, Krasiński [K] and Bolejko; Krasiński [K], Boleijko and Célérier; Krauss [K] and Turner; Krechet [K], Krechet [K] and Panov; Kristian [K] and Sachs; Krori [K], Sarmah and Goswami; Kroupa [K], Pawlowski and Milgrom; Kundt [K],, Labraña [L], Laciana [L] and Las Haras; Lahav [L] et al; Lake [L], Lanczos [L], Laserra [L], Lang [L] and Collins; Lanius [L], Laviolette [L], Lemaître [L], Leontovski [L], Lewis [L] et al, Liddle [L], Linde [L],Lima [L]and Trodden, Lineweaver [L] and Davis; Lemaître [L], Lessner [L], Lieu [L], Lifschitz [L] and Khalatnikov; Lima [L] and Trodden, Linde [L], Linder [L], Linder and Wagoner; Liu [L], Loeb [L], Lopez [L] and Nanopoulos; Lorentz-Petzold [L], Lu [L] and Hellaby; Lyttleton [L] and Bondi; MacCallum [M] and Taub; Makino [M], Maia [M] and Silveira; Maia [M] and Silva; Maitra [M], Mena [M] and Tavakol; Mena [M], Tavakol and Vera; Mersini-Houghton [M], Mashhoon [M] and Partovi; Maiti [M], Matzner [M], McClure [M] and Dyer; McCrea [M], McCrea [M] and McVittie; McVittie [M], Melia [M], Melia [M] and Abdelquader; Melia [M] and Shevchuk; Misner [M], Mitra [M], Modak [M], Moffat [M] and Vincent; Monjo [M], Morgan [M], Müller [M], Fagundeh and Opher; Murdoch [M], Murphy [M], Nakamura [N] and Sato; Nachtmann [N], Nandra [N], Lasenby and Hobson; Nariai [N], Nariai [N] and Ueno; Nariai [N] and Kimura; Nariai [N], Narlikar [N], Newman [N], Newman [N] and McVittie; Noerdlinger [N] and Pertrosian; Nolan [N]and and Odinstsov; Obukhov [O] and Piskareva; Obukhov [O], Omer [O], Onuchukwu [O] and Ezeribe; Ostriker [O] and Steinhardt, Övgün [Ö], Özdemir [Ö], Özer [Ö] and Taha; Ozsvath [O], Ozsváth [O] and Schücking; Pachner [P], Padmanabhan [P], Padmanabhan [P] and Padmanabhan; Page [P], Paiva [P], Patel [P] and Vaidya; Pavlov [P], Podolsky [P], Pössel [P], Quevedo [Q] and Sussman; Rácz [R] et al.; Radu [R] and Astefandi; Räsänen [R], Ratra [R] and Peebles; Ray [R], Ray [R] and Zimmermann; Räsänen [R], Räsänen [R], Bolejko, and Finoguenov; Ratnaprabha [R] and Singh; Raychaudhuri [R], Raychaudhuri [R] and Modak; Rebouças [R], Rebouças [R] and Åman; Rebouças [R] and Ademir; Rebouças and Teixeira; Rebouças [R] and d'Olival; Rindler [R], Robertson [R], Rosen [R], Roukema [R], Roy [R] and Prasad; Ruben [R], Ruzmaĭkina [R] and Ruzmaĭkin; Ryan Jr. [R], Sakoto [S], Sanz [S], Sasse [S], Sharma [S], Mukherjee, Dey and Dey; Sharma [S] and Mukherjee; Soares and Tiomno; Saulder [S], Mieske and Zeilinger; Raychaudhuri [R], Schäfer [S], Schleich [S] and Witt; Schmidt [S], Schröder [S] and Treder; Schücking [S], Sciama [S], Serret [S], Shepley [S], Silveira [S] and Waga; Singh [S] and Yadav; Singh [S], Singh [S] and Agrawal; Singh [S] and Singh; Singh [S] and Singh; Stabell [S] and Refsdal; Stelmach [S] and Jakacka; Soares [S], Soleng [S], Sushkov [S] and Kim; Sussman [S], Sussman [S] and Bolejko; Szafron [S], Tauber [T], Takéuchi [T], Tauber [T], Tewari [T], Tipler [T], Tolman [T], Tomita [T]and Hayashi; Thakurta [T], Tomonaga [T], Treder [T], Trevese [T], Tsagas [T] , Challinor, and Maartens; Tsoubelis [T], Triginer [T] and Pavón; Tsagas [T], Turner [T], Tzihong [T] and Xiao-Gang; Upadhyay [U], Vaidya [V], Vaidya [V] and Patel; Vanderfeld [V], Flanagan and Wassermann; van Elst [V] and Ellis; Vilenkin [V], Voracek [V], Wainwright [W], Wainwright [W] and Hsu; Wald [W], Walker [W], Wainwright [W] and Anderson; Wanas [W], Wang [W] and Deng; Weinberg [W], Wells [W], Wesson [W], Wesson [W] and Ponce de Leon; Weyl [W], Vilenkin [V], Wiltshire [W], Wyman [W], Wu [W], Yoo [Y], Kai and Nakao; Zel'dovich [Z].

III. Subluminal cosmology

The standard model (StM) is based on the pressure-free Friedman model which has been extended with relations involving pressure. Although this model is largely based on astrophysical considerations, it must be mentioned that it has some deficiencies that limit the usability of the model. First, we will list these deficiencies and then we will investigate whether some of these deficiencies can be avoided by improving the model or by applying new strategies.

1. Pressure is inserted by hand into the Friedman model. The StM is not an exact solution to Einstein's field equations.
2. Since one cannot determine all the variables with Einstein’s field equations, five parameters have to be adapted in such a way that a good agreement with data collected by observation is possible.
3. The method described above shows that our universe is almost flat, but that it is nevertheless homogeneous and closed and therefore must contain a very large number of galaxies.
4. The theory needs a deceleration parameter that contains second derivatives of the scale factor with respect to time.
5. This means that accelerations are possible. The time-like metric coefficient of the metric of the StM given in comoving coordinates is $g_{44} = 1$. This means that the cosmos expands in free fall. In a free-falling system no accelerations can be observed according to the theory of general relativity. Thus, the StM violates the principles of general relativity.
6. By a motion with superluminal velocity galactic island formation can occur. Galaxies are causally separated from each other. No information exchange can take place between them.
7. The Hubble law suggests a Galilean velocity addition.
8. In order to reconcile the StM with the data, it must be assumed that soon after the Big Bang an inflation, ie a temporary expansion with approximately 500 times the light speed must have taken place.

In Sec. III.1 we consider some of the topics mentioned. In Sec. III.2, we examine the problem of Einstein's elevator in the context of free-falling observers in the expanding cosmos. In Sec. III.3 we describe a *subluminal model* based on a previous paper [B], a model which is an exact solution to Einstein’s field equations, which contains pressure, and which does not allow superluminal velocities. In Sec. III.4 we treat the field equations of the model in the comoving system, in Sec. III.4 in the non-comoving system. In Sec. III.5 we discuss the special properties of the model, the recession velocities, the cosmic horizon and the relation to the model of Melia.

III.1. Preliminary remarks

We believe that Nature is not so cruel as to describe the world by an incomplete system of equations. In [B] we have proposed a model which is based on an exact solution of Einstein's field equations and which in addition describes the pressure of the cosmic fluid. First we have presented our model only as a possibility of thinking. The model of Melia [Melia [M], Melia [M] and Shevchuk; Melia [M] and Maier; Wei [W] and Wu; and Bikwa [B], Melia and Shevchuk] is actually flat, but in some cases it is consistent with our spatially positively curved model. Thus, we are encouraged to present our proposal as a realistic model. Melia and his co-workers have shown that the data observed fit better into their model than the StM or other FRW models and without any adjustment. If one reinterprets Melia's model (MeM), the identity with our subluminal model is ensured. All the statements made by Melia about observed data and their fitting apply equally to the subluminal model (SuM).

Although it is emphasized according to previous observations that one cannot say whether the universe is positively curved, flat, or negatively curved, we assume that the SuM is spatially positively curved. In addition, we do not propose that it is almost flat, as is deduced from the StM.

Melia calls his model $R_h = ct$-model. It has been questioned by some authors [Bilicki [B] and Seikel; Lewis [L], Lewis [L] and van Oirschot; Kim [K], Lasenby and Hobson; van Oirschot [V], Kwan and Lewis; Mitra [M], Yu [Y] and Wang], chiefly concerning the cosmic horizon. In one of his papers Melia [M] has presented convincing arguments against these objections.

With the subluminal model, we also avoid the conception that, as is the case with the version of Melia, the flat nevertheless homogeneous universe contains infinitely many stars. We do not need to answer the question as to how and when these infinite stars have been created. A closed universe can only contain a finite number of stars.

The MeM as well as the SuM does not contain a deceleration parameter, thus no second derivatives of the scale factor with respect to the time. The expansion of the cosmos is uniform, no acceleration occurs. Most cosmological models assume that expansion comes about in free. This is reflected in the form of the metric. The timelike metric coefficient in comoving coordinates is $g_{44} = 1$ or at least position-independent. From it no accelerations can be derived. The fact that a deceleration parameter is still contained in such models follows from the supplement of the Friedman model by pressure. This extension takes place by hand and thus the model is not an exact solution to Einstein's field equations.

One of the most remarkable features of the StM is the fact that it allows superluminal velocities. The recession velocity of the galaxies reaches the speed of light at the Hubble horizon, at this location the red shift of the light is infinitely high, and signal transmission is no longer possible. Beyond this horizon the galaxies move faster than light. This causes galaxies and galaxy clusters to separate. They do not have any connection to each other and a galactic island formation will take place. Thus, the StM allows violating the basic principles of the special theory of relativity. This interpretation is favored by Davis [D] and Davis [D], Lineweaver, and Webb. In these considerations it is important whether everything is expanding in the universe or whether local areas such as our solar system or atomic areas are excluded. With this problem the following researchers have been concerned: Davis [D] and Lineweaver, Anderson [A], Blau, Callender [C] and Weingard, Carrera [C], Coooperstock [C], Faraoni, and Vollick , Dicke [D] and Peebles, Irvine [I], Mizony [M] and Lachièze-Rey and also Sereno [S] and Jetzer. Remarks on the recession velocity can be found in Chodorowski [C], Cook [C], Endean [E], Harrison [H], Kiang [K], Lewis [L] et al, Murdoch [M], Silverman [S], Stuckey [S] and Liebscher [S]. On the horizons the following researchers have made contributions: Rindler [R], Barnes [B], Francis, James and Lewis, Ellis [E] and Rothman, Ellis [E] and Stoeger, Harrison [H], Shi [S] and Turner. Some authors claim that the redshift is not due to the extension of the space, but to the relative motion of the stars, ie to the associated Doppler effect:

Whiting [W], Harvey [H], Schucking. and Surowitz, Bunn [B] and Hogg, Chodorowski [C], Faraoni [F] and Narlikar [N]. The gravitational effect of the stars could also influence the wavelength of the light: Bondi [B], Endean [E], Infeld [I] and Schild and Querella [Q]. Thus, it is not necessary to introduce the expansion of the cosmos in order to explain the redshift: In a static cosmos, the stars move away from each other according to the ideas of several authors.

Abramowicz [A], Bajtlik, Lasota, and Moudens have decisively opposed this point of view. They argue that it can be decided by observation whether the cosmos expands or the redshift can be explained by kinematic effects. Nevertheless, some authors are inclined to this view: Aspden [A], Chodorowski [C], Epstein [E], Felten [F] and Isaacman, Harvey [H], Peacock [P] and Stuckey [S].

Olbers' paradox gets little attention in the literature. In an infinite universe with an infinite number of stars an infinite amount of light is emitted. But in this case only a finite amount arrives at us, still so much that the night sky would be as bright as our sun. Expanding cosmological models which have a horizon prevent the radiation of light from this region and thus solve Olbers [O], paradox: Harrison [H], Whitrow [W] and Yallop; Wesson [W], Valle, and Strabell and Wesson [W] argue that the intensity of extragalactic background light (EBL) is low, because it is reduced not only by the expansion of the universe, but also by the lifetime of the galaxies. The latter effect reduces the number of photons emitted into space. Wesson and his staff produce their numerical results using the standard model.

These often very detailed discussions obviously arise from the discomfort which causes the StM among astrophysicists. In the next Section, we will examine the influence on the form of the metric of a cosmos which expands in free fall. For the purpose of understanding, it is also necessary to recognize the relation between the curvature parameter k and the curvature of the space.

III.2. Expansion in free fall

Most expanding models, among them the StM, expand in free fall. The metrics of the models are generally written in comoving coordinates. The metric in this representation is quite simple and can easily be further processed

$$ds^2 = \mathcal{K}^2 \left[\frac{1}{1-k\frac{r'^2}{\mathcal{R}_0^2}} dr'^2 + r'^2 d\Omega^2 \right] - dt'^2 . \qquad \text{(III.2.1)}$$

Therein $\mathcal{K}$ is the position-independent but time-dependent scale factor, Ω the solid angle, and k the curvature parameter, which can take the values 1, 0, -1 according to the FRW classification. For $k=1$ the spatial curvature of the cosmos is positive and the cosmos is closed. A space with $k=0$ is called a flat open space and with $k=-1$ a negatively curved and open cosmos. $\mathcal{R}_0$ is a constant that can be absorbed by r'. $\{r',t'\}$ are comoving coordinates, in particular t' is the cosmic time which applies equally to all observers. Since the metric factor is $g_{4'4'}=1$, the coordinate time coincides with the proper time of comoving observers. The representation in comoving coordinates has the advantage that the field equation system substantially simplifies. The form (III.2.1) of the metric is called a *canonical form.*

We are critical concerning the interpretation of the quantity k as the curvature parameter, and have communicated this in some papers [B]. Above all, we want to doubt that $k=0$ stands necessarily for a flat space. One of the fundamentals of the general theory of relativity is that a gravitational attraction cannot be experienced in a freely falling system. Observers in a freely falling elevator *hover*. Since there seem to be no gravitational forces, they could opine the space to be flat, thus, that $k=0$ is valid. Therefore, they could assess the space to be flat, although the curvature of the space did not change due to the falling motion. Exactly this reasoning has been lost to cosmology. In a cosmos that expands in free fall, no gravitational forces are experienced. If its metric in the comoving coordinate system reads as

$$\text{(A)} \qquad ds^2 = \mathcal{K}^2 \left[dr'^2 + r'^2 d\Omega^2 \right] - dt'^2 , \qquad \text{(III.2.2)}$$

it is of the type $k=0$ and the cosmos is assumed to be flat and spatially infinite. The fact that a metric of type A can describe a cosmos that is positively curved and spatially closed is shown by the de Sitter cosmos and by our subluminal model. The metric of the static dS cosmos has the canonical form

$$\text{(B)} \qquad ds^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}_0^2}} dr^2 + r^2 d\Omega^2 - \left(1-\frac{r^2}{\mathcal{R}_0^2}\right) dt^2 \qquad \text{(III.2.3)}$$

and thus is of type $k=1$. The cosmos is spatially positively curved. The coordinate system $\{r,t\}$ is the static one. The positive force $U_m = \{U_1,0,0,0\}$ is derived from the metric coefficient g_{44}. It acts at any point of the universe and tries to push away neighboring points. With the transformation of Lemaître, the metric B can be transformed into the form A, whereby the structure of the space is not changed. To this coordinate transformation a Lorentz transformation can be assigned which leads to a reference system which gives way to the forces U. The new system is expanding. Mathematically, this is reflected in the relation

$$'U_{m'} = L^{m}_{m'} U_m + 'L_{m'} . \tag{III.2.4}$$

We call it *Einstein's elevator equation.* $L^{m}_{m'}$ is the matrix of a Lorentz transformation, $'L_{m'}$ the Lorentz term. It arises from the inhomogeneous transformation law of the Ricci-rotation coefficients which determine the geometry. Particularly reasonable is the first component of Eq. (III.2.4)

$$'U_{1'} = U_{1'} + 'L_{1'} = 0 . \tag{III.2.5}$$

The driving force U of the static system is canceled by the dynamic term 'L. The system expanding in free fall in the dS cosmos is free of forces. Although the quantity 'U does not have a radial component, it gets a time-like part by (III.2.4). All in all one has

$$\{U_1, 0, 0, 0\} \rightarrow \{0, 0, 0, 'U_{4'}\} . \tag{III.2.6}$$

Thus, the same considerations as for the free fall in the Schwarzschild field apply to the dS cosmos. By no means a metric of type (A) need not to describe a flat space. It stands for a reference system which is in free fall.

Similar mechanisms apply to other cosmological models, also for our subluminal model. In the latter case, the situation is somewhat more complicated and it will be dealt with in more detail in the next Section.

III.3. The subluminal model

The model which we [B] have proposed earlier, we will here present in a lucid form and we also will elaborate further details. We start from the static dS model which is based on a pseudo-hyper sphere, embedded in a flat 5-dimensional space. We generalize the model by dropping the condition that the radius of the pseudo-hyper sphere is constant and we put

$$\mathcal{R} = \mathcal{R}(t). \tag{III.3.1}$$

This leads to a genuine expanding cosmos, whose stress-energy-momentum tensor contains pressure and mass density. At any time of the expansion the geometry of our model is the geometry of the dS model. Therefore, we will not make straightaway use of the definition (III.3.1), but we will first explain the fundamentals of the dS cosmos.

The above-mentioned pseudo-hyper sphere has the embedding in a flat 5-dimensional space with the Cartesian coordinates x^a, $a = 0,1,\ldots,4$

$$\begin{aligned} x^3 &= \mathcal{R}\sin\eta\sin\vartheta\sin\varphi \\ x^2 &= \mathcal{R}\sin\eta\sin\vartheta\cos\varphi \\ x^1 &= \mathcal{R}\sin\eta\cos\vartheta \qquad , \\ x^4 &= \mathcal{R}\cos\eta\sin i\psi \\ x^0 &= \mathcal{R}\cos\eta\cos i\psi \end{aligned} \tag{III.3.2}$$

wherein $\{\mathcal{R},\eta,\vartheta,\varphi,i\psi\}$ are the quasi-spherical coordinates. On the surface of the pseudo-hyper sphere the metric in these coordinates is given by

$$\text{(B)} \quad ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2\sin^2\eta d\vartheta^2 + \mathcal{R}^2\sin^2\eta\sin^2\vartheta d\varphi^2 + \mathcal{R}^2\cos^2\eta di\psi^2 . \tag{III.3.3}$$

With the radial coordinate and the coordinate time

$$r = \mathcal{R}\sin\eta, \quad dx^4 = idt = \mathcal{R}di\psi \tag{III.3.4}$$

and $\mathcal{R}d\eta = dr/\cos\eta$ first

$$\text{(B')} \quad ds^2 = \frac{1}{\cos^2\eta}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 - \cos^2\eta dt^2 \tag{III.3.5}$$

is derived from it. Finally, one obtains the metric in canonical form

$$\text{(B'')} \quad ds^2 = \frac{1}{1 - r^2/\mathcal{R}^2}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 - \left(1 - r^2/\mathcal{R}^2\right)dt^2 . \tag{III.3.6}$$

Performing the Lemaître transformation the metric in the comoving system obtains the form (III.2.2). However, considering (4.1), the scale factor of the subluminal model has a form different from that of the expanding dS model. We will elaborate this expression step by step.

First we will work with the metric (III.2.2) in the simpler comoving system and we will return later to the non-comoving system. From the metric (A) we read the 4-bein system

$$\overset{1'}{e}_{1'} = \mathcal{K}, \quad \overset{2'}{e}_{2'} = \mathcal{K}r', \quad \overset{3'}{e}_{3'} = \mathcal{K}r'\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1. \tag{III.3.7}$$

Furthermore, the non-comoving radial coordinate r is connected with the comoving r' b

$$r = \mathcal{K} r' . \tag{III.3.8}$$

From (III.3.7) we calculate the Ricci-rotation coefficients using the general formula

$$A_{mn}{}^{s} = \overset{s}{e}_{i}\, e^{i}_{[n|m]} + g^{sr} g_{mt}\, \overset{t}{e}_{i}\, e^{i}_{[n|r]} + g^{sr} g_{nt}\, \overset{t}{e}_{i}\, e^{i}_{[m|r]} \ , \tag{III.3.9}$$

however, for the primed system (III.3.7). We split the Ricci-rotation coefficients into the following components

$$'A_{m'n'}{}^{s'} = B_{m'n'}{}^{s'} + C_{m'n'}{}^{s'} + {'U}_{m'n'}{}^{s'} \ . \tag{III.3.10}$$

With the orthogonal unit vectors of the comoving system

$$'m_{m'} = \{1,0,0,0\}, \quad b_{m'} = \{0,1,0,0\}, \quad c_{m'} = \{0,0,1,0\}, \quad 'u_{m'} = \{0,0,0,1\} \tag{III.3.11}$$

we can further disassemble

$$\begin{gathered} B_{m'n'}{}^{s'} = b_{m'} B_{n'} b^{s'} - b_{m'} b_{n'} B^{s'}, \quad C_{m'n'}{}^{s'} = c_{m'} C_{n'} c^{s'} - c_{m'} c_{n'} C^{s'}, \\ 'U_{m'n'}{}^{s'} = {'m}_{m'}\, {'U}_{n'}\, {'m}^{s'} - {'m}_{m'}\, {'m}_{n'}\, {'U}^{s'}. \end{gathered} \tag{III.3.12}$$

B and C are the lateral field quantities of the model and 'U is a timelike quantity still to be discussed.

A look at the metric (A) and the 4-beine (III.3.7) shows that we are essentially concerned with a spherically symmetric problem, the treatment of which does not pose a particular problem concerning the field equations. Only the scale factor is a time-dependent variable which describes the change of the pseudo-hyper sphere and cannot be determined from the properties of the pseudo-hyper sphere. For the time-like parts of the Ricci-rotation coefficients one obtains after a short calculation

$$\begin{aligned} 'U_{4'} &= {'A}_{1'4'}{}^{1'} = -\overset{1'}{e}_{1'}\, e^{1'}_{1'|4'} &&= \frac{1}{\mathcal{K}} \mathcal{K}_{|4'} \\ B_{4'} &= {'A}_{2'4'}{}^{2'} = -\overset{2'}{e}_{2'}\, e^{2'}_{2'|4'} &&= \frac{1}{\mathcal{K} r'} (\mathcal{K} r')_{|4'} = \frac{1}{\mathcal{K}} \mathcal{K}_{|4'} \qquad . \\ C_{4'} &= {'A}_{3'4'}{}^{3'} = -\overset{3'}{e}_{3'}\, e^{3'}_{3'|4'} &&= \frac{1}{\mathcal{K} r' \sin\vartheta} (\mathcal{K} r' \sin\vartheta)_{|4'} = \frac{1}{\mathcal{K}} \mathcal{K}_{|4'} \end{aligned} \tag{III.3.13}$$

To approach the problem, let us assume that the radius of the pseudo-hyper sphere as well as the radial coordinate in (III.3.8) are subject to expansion and so we write

$$\mathcal{R} = \mathcal{K} \mathcal{R}_0 , \tag{III.3.14}$$

where $\mathcal{R}_0$ is a constant. Furthermore, we can see that in the calculation of the Ricci-rotation coefficients there occurs the time derivative of the scale factor and therefore we define a quantity

$$\mathcal{F}_{m'} = \frac{1}{\mathcal{R}} \mathcal{R}_{|m'} = \frac{1}{\mathcal{K}} \mathcal{K}_{|m'} = \left\{ 0,0,0, \frac{1}{\mathcal{K}} \mathcal{K}_{|4'} \right\} . \tag{III.3.15}$$

This shows that the expansion scalar

$$'u^{s'}{}_{||s'} = {'A}_{r's'}{}^{r'}\, {'u}^{s'} = {'A}_{r'4'}{}^{r'} = {'A}_{4'}$$

is composed of three equal contributions

$$'U_{4'} \overset{*}{=} B_{4'} \overset{*}{=} C_{4'} = \frac{1}{\mathcal{K}} \mathcal{K}_{|4'} ,$$

and that the expansion of the cosmos in the three spatial directions is equal.

With

$$\partial_{1'} = \frac{1}{K}\frac{\partial}{\partial r'}, \quad \partial_{2'} = \frac{1}{K r'}\frac{\partial}{\partial \vartheta} = \frac{1}{r}\frac{\partial}{\partial \vartheta}$$

the following relations

$$B_{\alpha'} = \left\{\frac{1}{r}, 0, 0\right\}, \quad C_{\alpha'} = \left\{\frac{1}{r}, \frac{1}{r}\cot\vartheta, 0\right\}, \quad \alpha' = 1', 2', 3' \qquad \text{(III.3.16)}$$

result for the spatial components of the Ricci-rotation coefficients, relations which are known from a flat geometry with a polar reference system. As already indicated in Sec III.2, this does not mean that the space is flat, but that the reference system is in free fall. We will discuss this more closely later.

III.4. The field equations in the comoving system

We use the Ricci rotation coefficients to calculate the field equations in the tetrad calculus. Most of the cosmological models are based on spherical symmetric spaces, or at least on spaces with a symmetry that is parameterized with spherical coordinates. Since the mathematical treatment of these spaces is simple, the cosmologists mostly concentrate only on the Friedman equation because it makes the essential statements about the temporal change of the universe. However, we want to work through the field equations system completely, because the spherical-symmetric part of the field equations also provides information, in particular in regard to the curvature of the space.

By means of the relations (III.3.10) and (IX.2.9) the Ricci

$$R_{m'n'} = {}'A_{m'n'}{}^{s'}{}_{|s'} - {}'A_{n'|m'} - {}'A_{r'm'}{}^{s'}\,{}'A_{s'n'}{}^{r'} + {}'A_{m'n'}{}^{s'}\,{}'A_{s'}\,, \quad {}'A_{n'} = {}'A_{r'n'}{}^{r'}$$

$$\begin{aligned} R_{m'n'} = &-\left[{}'U^{s'}{}_{\underset{1}{||}s'} + {}'U^{s'}\,{}'U_{s'}\right]h_{m'n'} \\ &-\left[B_{n'\underset{2}{||}m'} + B_{n'}B_{m'}\right] - b_{n'}b_{m'}\left[B^{s'}{}_{\underset{2}{||}s'} + B^{s'}B_{s'}\right] \\ &-\left[C_{n'\underset{3}{||}m'} + C_{n'}C_{m'}\right] - c_{n'}c_{m'}\left[C^{s'}{}_{\underset{3}{||}s'} + C^{s'}C_{s'}\right] \end{aligned} \qquad . \qquad \text{(III.4.1)}$$

$$-\frac{1}{2}R = \left[{}'U^{s'}{}_{\underset{1}{||}s'} + {}'U^{s'}\,{}'U_{s'}\right] + \left[B^{s'}{}_{\underset{2}{||}s'} + B^{s'}B_{s'}\right] + \left[C^{s'}{}_{\underset{3}{||}s'} + C^{s'}C_{s'}\right]$$

can be decomposed into subequations for the three variables 'U, B, and C. Therein $h_{m'n'} = \mathrm{diag}(1,0,0,1)$ is a sub-matrix of the tetrad metric $g_{m'n'} = \mathrm{diag}(1,1,1,1)$. We are using the graded derivatives as in earlier Sections

$${}'U_{n'\underset{1}{||}m'} = {}'U_{n'|m'}, \quad B_{n'\underset{2}{||}m'} = B_{n'|m'} - {}'U_{m'n'}{}^{s'}B_{s'}, \quad C_{n'\underset{3}{||}m'} = C_{n'|m'} - {}'U_{m'n'}{}^{s'}C_{s'} - B_{m'n'}{}^{s'}C_{s'} \qquad \text{(III.4.2)}$$

with which the field equations can be clearly exposed. The field equations are solved with the ansatz

$$\frac{1}{\mathcal{R}}\mathcal{R}_{|4'} = -\frac{i}{\mathcal{R}}, \quad \mathcal{F}_{4'} = -\frac{i}{\mathcal{R}}, \qquad \text{(III.4.3)}$$

which, as we shall see later on, is confirmed by the conservation law. That means that all field quantities of the system are known

$${}'U_{m'} = \left\{0,0,0,-\frac{i}{\mathcal{R}}\right\}, \quad B_{m'} = \left\{\frac{1}{r},0,0,-\frac{i}{\mathcal{R}}\right\}, \quad C_{m'} = \left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0,-\frac{i}{\mathcal{R}}\right\}. \qquad \text{(III.4.4)}$$

The quantity $'U$ does not contain a radial component. According to (III.3.7) one has

$${}'U_{1'} = {}'A_{4'1'}{}^{4'} = -\overset{4'}{e}{}_{4'}e^{4'}_{4'|1'} = 0\,.$$

This means that the universe is in free fall and that no radial forces occur.

With the help of (II.3.36) the subequations of Einstein's field equations (III.4.1) can be calculated

$$'U^{s'}{}_{\underset{1}{\|s'}} + 'U^{s'}\,'U_{s'} = 0$$

$$B_{m'\underset{2}{\|}n'} + B_{m'}B_{n'} = -'m_{m'}\,'m_{n'}\frac{1}{\mathcal{R}^2}, \qquad B^{s'}{}_{\underset{2}{\|s'}} + B^{s'}B_{s'} = -\frac{1}{\mathcal{R}^2} \quad . \tag{III.4.5}$$

$$C_{m'\underset{3}{\|}n'} + C_{m'}C_{n'} = -\left('m_{m'}\,'m_{n'} + b_{m'}b_{n'}\right)\frac{1}{\mathcal{R}^2}, \quad C^{s'}{}_{\underset{3}{\|s'}} + C^{s'}C_{s'} = -\frac{2}{\mathcal{R}^2}$$

The first relation in it is the *Friedman equation*. With $\partial_{4'} = \partial/i\partial T'$, where T' corresponds to the proper time of the comoving observer, it can be converted into the form familiar to cosmologists

$$\frac{1}{\mathcal{R}}\mathcal{R}^{\bullet} - \frac{1}{\mathcal{R}} = 0, \quad \mathcal{R}^{\bullet} = 1, \quad \mathcal{R}^{\bullet\bullet} = 0. \tag{III.4.6}$$

The rate of expansion of the subluminal model is constant. Recent measurements of the redshift of supernovae indicate an acceleration of the expansion of the cosmos. However, the data have been adapted to the less convincing FRW model. The discussion on this topic, however, is by no means completed[32].

As stated at the outset, conceptual difficulties arise, if an accelerated expansion is to be considered on the basis of a model which expands in free fall. In the comoving system one has $g_{4'4'} = 1$. No accelerations can be derived from such a metric. The principle of Einstein's elevator applies. This principle is called 'Weak Einstein equivalence Principle' (WEEP) [Kopeikin [K]] in modern literature. We have illuminated the problem on the basis of the de Sitter cosmos in Sec. 3. For a model with accelerated expansion a metric with $g_{4'4'} = g_{4'4'}(r)$ would be expected. Such an ansatz is rarely found in cosmological models. The result (III.4.6) is a consequence of the free-falling expanding cosmos. In the next Section we will treat the recession velocities of galaxies and we will return to this ansatz.

If one constructs the Einstein field equations with (III.4.1), a stress-energy tensor arises on the right side of the form

$$T_{m'n'} = -pg_{m'n'} + (p + \mu_0)\,'u_{m'}\,'u_{n'}. \tag{III.4.7}$$

Therein, the pressure and the energy density are

$$\kappa p = -\frac{1}{\mathcal{R}^2}, \quad \kappa\mu_0 = \frac{3}{\mathcal{R}^2}, \tag{III.4.8}$$

from which the equation of state for the cosmic fluid is obtained

$$p = -\frac{1}{3}\mu_0. \tag{III.4.9}$$

One gets from the conservation law $T^{m'n'}{}_{\|n'} = 0$ the two subequations

$$p_{|\alpha'} = 0, \quad \mu_{0|4'} = -3(p + \mu_0)\mathcal{F}_{4'}, \quad \alpha' = 1', 2', 3' \tag{III.4.10}$$

It can be seen that the pressure is position-independent. If the second relation of (III.4.8) is used for μ_0 in (III.4.10) the ansatz (5.3) is confirmed.

[32] Padmanabhan [P] is critical of Friedman's cosmos in some respects.

In this Section we have established the field equations for observers who comove with the expansion of the cosmos, and we have also solved these equations and thereby obtained quite simple results. We have already made statements about the curvature of space. However, for their affirmation it is also necessary to treat the problem for non-comoving observers and to make statements about the relative motion of the two observer systems. Only then can we say how the model behaves in free fall and what curvature of space can be read from the metric.

III.5. The non-comoving system

The theory of expanding cosmological models is carried out almost exclusively in comoving coordinates. The question arises whether a representation in the non-comoving system is possible. Such a system is usually referred to as static. It was Florides [F], who succeeded in bringing six FRW models into the static form. Mitra [M] has revised the procedure and we have accomplished the coordinate transformations with Lorentz transformations. In the latter are included the physically relevant values for the relative velocities between the two observer systems, ie the comoving one and the non-comoving one. Only with them one can make correct statements about the motions in the cosmos and also clarify the question whether superluminal velocities are possible.

We have put up the subluminal model on the static version of the de Sitter model, have passed to the comoving system by a Lemaître transformation and have abandoned the condition $\mathcal{R} = \mathrm{const.}$. If we transform the field equations of this system back into the static form, we cannot assume that we recover the dS universe. We recall that the subluminal model contains the condition $\mathcal{R} = \mathcal{R}(t)$. Thus, we will only get the dS model, if we subsequently put $\mathcal{R} = \mathrm{const.}$. We will work out this in detail.

In order to present the model in a static system, it is not sufficient, or even necessary, to refer to a static coordinate system. A reference system in rest is required, ie, a system of four orthogonal vectors in which the quantities of the model can be represented.

The quantities of the comoving system transform with a Lorentz transformation into the non-comoving system. Since the expansion-related relative motion of the galaxies takes place in the radial direction, ie in the 1-direction, only the radial and timelike components are occupied in the matrix of the Lorentz transformation

$$L_m^{m'} = \begin{pmatrix} \alpha & & & -i\alpha v \\ & 1 & & \\ & & 1 & \\ i\alpha v & & & \alpha \end{pmatrix}. \tag{III.5.1}$$

Therein α is the Lorentz factor and v is the relative velocity of the observers. The relative velocity and the Lorentz factor are taken from the de Sitter model on which our subluminal model is based

$$v = \frac{r}{\mathcal{R}}, \quad \alpha = \frac{1}{\sqrt{1 - r^2/\mathcal{R}^2}}, \quad a = \frac{1}{\alpha}. \tag{III.5.2}$$

Alternatively, the Lorentz transformation can be derived via the Lemaître coordinate transformation. The matrix of this coordinate transformation is $\Lambda_i^{i'} = x^{i'}{}_{|i}$, where the i,i' are coordinate indices relating to the non-comoving and to the comoving system. From this we obtain with

$$L_m^{m'} = \overset{m'}{e}{}_{i'} \Lambda_i^{i'} \underset{m}{e}{}^{i} \tag{III.5.3}$$

the matrix of the Lorentz transformation, the 4-beine (tetrads) being read from the metrics (A) and (B), respectively. We have extensively discussed these procedures with the models of the dS family in preceding Sections.

Therefore, we limit ourselves directly to operations in the local tetrad spaces in order to remain as close as possible to the physically relevant variables, in addition, in expectation that we will not be successful with the coordinate method.

The Ricci-rotation coefficients transform inhomogeneously from the comoving to the non-comoving system

$$A_{mn}{}^{s} = L^{m'n's}_{m\,n\,s'}\,{}'A_{m'n'}{}^{s'} + L_{mn}{}^{s}, \quad L_{mn}{}^{s} = L^{s}_{s'}\,L^{s'}_{n|m}. \tag{III.5.4}$$

The second term in this relation is the Lorentz term. Since the Lorentz transformation is a pseudo-rotation in the [1,4]-subspace, the above relation can be simplified to

$$U_{mn}{}^{s} = {}'U_{mn}{}^{s} + L_{mn}{}^{s}. \tag{III.5.5}$$

The 3-rank quantities U and L can be reduced to 1-rank ones. With

$$\begin{gathered} U_{mn}{}^{s} = h_{m}{}^{s} U_{n} - h_{mn} U^{s}, \\ L_{mn}{}^{s} = h_{m}{}^{s} L_{n} - h_{mn} L^{s}, \quad L_{n} = L_{sn}{}^{s} = \left\{ L_{41}{}^{4}, L_{14}{}^{1} \right\} \end{gathered} \tag{III.5.6}$$

one gains the simple relation

$$U_{m} = {}'U_{m} + L_{m}. \tag{III.5.7}$$

Considering (III.5.4) and (III.5.1), one first has

$$L_{1} = -i\alpha^{2} v_{|4}, \quad L_{4} = i\alpha^{2} v_{|1}.$$

Defining the relative velocity (III.5.2) we obtain the auxiliary relations

$$v_{|m} = \{1,0,0,0\}\frac{a}{\mathcal{R}} - v\mathcal{F}_{m}, \qquad \mathcal{F}_{m} = L^{m'}_{m}\,\mathcal{F}_{m'} = \left\{ -\alpha v\frac{1}{\mathcal{R}}, 0, 0, -\alpha\frac{i}{\mathcal{R}} \right\} \tag{III.5.8}$$

and finally

$$L_{m} = \left\{ \alpha^{3} v\frac{1}{\mathcal{R}}, 0, 0, i\alpha^{3}\frac{1}{\mathcal{R}} \right\} = \{-i\alpha v, 0, 0, \alpha\}\, i\alpha^{2}\frac{1}{\mathcal{R}}. \tag{III.5.9}$$

From (II.9.14) one obtains

$$U_{m} = \left\{ \alpha^{3} v^{3}\frac{1}{\mathcal{R}}, 0, 0, i\alpha^{3} v^{2}\frac{1}{\mathcal{R}} \right\}, \tag{III.5.10}$$

a quantity which clearly differs from the static dS expression. However, it can be decomposed according to

$$U_{m} = \hat{U}_{m} + f_{m}, \quad \hat{U}_{m} = \left\{ -\alpha v\frac{1}{\mathcal{R}}, 0, 0, 0 \right\}, \quad f_{m} = \left\{ i\alpha^{2} v\mathcal{F}_{4}, 0, 0, -i\alpha^{2} v\mathcal{F}_{1} \right\},$$

so that only the dS-expression for the radial force remains after the expansion has been switched off $(\mathcal{F} = 0)$. At this point it can be discussed whether the radial field quantity in the non-comoving system can be derived from a metric coefficient. If this is not the case, there will be no non-comoving coordinate system. It is easy to find $f_{m} = (\ln\alpha)_{|m}$. However, the dS piece $\hat{U}$ of the quantity U is only a gradient if $\mathcal{R} = \text{const.}$, ie if the subluminal model is reduced to the dS model. We recognize that a Lorentz transformation of the reference system is not always accompanied by a transformation of the coordinate system.

We obtain the lateral field quantities B and C directly from the dS ansatz. Lastly, the subluminal geometry is a snapshot of the dS geometry at any stage of expansion. From (III.3.6) we obtain, with the usual techniques of the tetrad method,

$$B_m = \left\{\frac{a}{r}, 0,0,0\right\}, \quad C_m = \left\{\frac{a}{r}, \frac{1}{r}\cot\vartheta, 0,0\right\}, \quad a = \cos\eta = \sqrt{1 - r^2/\mathcal{R}^2}\,. \tag{III.5.11}$$

Alternatively, we can determine these quantities from those of the comoving system with the Lorentz transformation (III.5.1).

Differentiating (III.5.10) one obtains the relation

$$U^s{}_{\underset{1}{\|}s} + U^s U_s = 0\,, \tag{III.5.12}$$

the Friedman equation of the model. A comparison with (III.4.5) shows that the U-equations are form-invariant under a Lorentz transformation. For the treatment of the B- and C-equations one has to consider again (III.5.8)

$$B_{m\underset{2}{\|}n} + B_m B_n = -m_m m_n \frac{1}{\mathcal{R}^2} + \begin{pmatrix} -\alpha^2 v^2 \frac{1}{\mathcal{R}^2} & & & -i\alpha^2 v \frac{1}{\mathcal{R}^2} \\ & 0 & & \\ & & 0 & \\ -i\alpha^2 v \frac{1}{\mathcal{R}^2} & & & \alpha^2 v^2 \frac{1}{\mathcal{R}^2} \end{pmatrix},$$

$$C_{m\underset{3}{\|}n} + C_m C_n = -\left(m_m m_n + b_m b_n\right)\frac{1}{\mathcal{R}^2} + \begin{pmatrix} -\alpha^2 v^2 \frac{1}{\mathcal{R}^2} & & & -i\alpha^2 v \frac{1}{\mathcal{R}^2} \\ & 0 & & \\ & & 0 & \\ -i\alpha^2 v \frac{1}{\mathcal{R}^2} & & & \alpha^2 v^2 \frac{1}{\mathcal{R}^2} \end{pmatrix}, \tag{III.5.13}$$

$$B^s{}_{\underset{2}{\|}s} + B^s B_s = -\frac{1}{\mathcal{R}^2}, \quad C^s{}_{\underset{3}{\|}s} + C^s C_s = -\frac{2}{\mathcal{R}^2}.$$

The relations (III.4.1) can be used for the field equations. For the stress-energy tensor we expect

$$\begin{aligned} &T_{11} = -p - \alpha^2 v^2\left(p + \mu_0\right), \quad T_{22} = -p, \quad T_{33} = -p, \\ &T_{41} = -i\alpha^2 v\left(p + \mu_0\right), \quad T_{44} = \mu_0 + \alpha^2 v^2\left(p + \mu_0\right). \end{aligned} \tag{III.5.14}$$

With the values from (III.4.7) the relations (III.5.12) and (III.5.13) yield the above expressions.

III.6. Discussion of the model

In the Sec. III.2 haben, we have shown that a metric of the type $k=0$ does not necessarily describe a flat model, but a spatially curved, closed model that expands in free fall. According to the principle of Einstein's elevator observers cannot experience gravitational forces in a free-falling system nor perceive the space as flat.

With our previous results we want to show that these considerations are appropriate for the subluminal model. We put the focus on Sec. III.5 and we take the opposite approach. We transform the field quantities of the non-comoving system into those of the comoving system. We proceed in a step-by-step manner and show that quantities which actually have the properties of curvature can be brought into an apparently flat form.

We use the property of the lateral field quantities to transform homogeneously into the comoving system

$$B_{m'}=L^{m}_{m'}B_{m},\quad C_{m'}=L^{m}_{m'}C_{m} \tag{III.6.1}$$

and we get, with the Lorentz transformation (III.5.1), from (III.5.11)

$$\begin{aligned}B_{m'}&=\left\{\alpha a\frac{1}{r},0,0,-i\alpha va\frac{1}{r}\right\}=\left\{\frac{1}{r},0,0,-\frac{i}{\mathcal{R}}\right\}\\ C_{m'}&=\left\{\alpha a\frac{1}{r},\frac{1}{r}\cot\vartheta,0,-i\alpha va\frac{1}{r}\right\}=\left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0,-\frac{i}{\mathcal{R}}\right\}\end{aligned}, \tag{III.6.2}$$

whereby the expressions known from (III.3.16) are produced in the second step of the calculation. These expressions correspond to a flat geometry which is parameterized with spherical coordinates. Here we have used the relation

$$\alpha a=1. \tag{III.6.3}$$

In this, α is the Lorentz factor, thus a kinematic quantity. On the other hand, a is a geometric quantity which is related to the curvature of space. However, since the relative velocity is linked to the structure of space, the quantities α and a are also linked via (III.6.3) and simplify the components of the lateral field quantities. Those, however, still contain information about the curvature of space which is hidden by the Einstein elevator effect.

However, the radial field quantities transform inhomogeneously. From the transformation law of the Ricci-rotation coefficients, we obtain the elevator equation

$$'U_{m'}=L^{m}_{m'}U_{m}+'L_{m'}. \tag{III.6.4}$$

Its recalculation is quite simple. From (III.5.10) one obtains with the Lorentz transformation

$$U_{m'}=L^{m}_{m'}U_{m}=\{0,0,0,1\}\,i\,\alpha^{2}v^{2}\frac{1}{\mathcal{R}} \tag{III.6.5}$$

and with

$$'L_{m'}=-L^{m}_{m'}L_{m}=\{0,0,0,1\}\left(-i\,\alpha^{2}\frac{1}{\mathcal{R}}\right) \tag{III.6.6}$$

finally

$$'U_{m'}=\left\{0,0,0,-\frac{i}{\mathcal{R}}\right\},$$

a relation which we have deduced with (III.4.4) directly from the geometry and which misleadingly suggests a flat space.

It is clear that the use of Einstein's elevator principle is significant in cosmology. It decides whether we live in an infinite cosmos with infinite many stars or in a finite cosmos with a limited number of stars.

In the following, we hark back to the relations found in (III.4.6). From $\mathcal{R}^{\bullet} = 1, \mathcal{R}^{\bullet\bullet} = 0$ we have concluded that the expansion of the cosmos is constant. It is now to be investigated how the structure of the cosmos influences the recession velocities of the galaxies, furthermore the significance of this influence for the cosmic horizon, and whether superluminal velocities can occur in the universe.

We start from the relation $r = \mathcal{R}\sin\eta$ with the polar angle η of the pseudo-sphere which is the basis of the model. If an observer does not perform an individual motion then one has $\eta = \text{const.}$. Differentiation of $r = \mathcal{R}\sin\eta$ leads to the Hubble law with the Hubble parameter H

$$r^{\bullet} = \frac{1}{\mathcal{R}}\mathcal{R}^{\bullet} r = Hr. \tag{III.6.7}$$

On the other hand, one has, with $\mathcal{R}^{\bullet} = 1$, according to (III.5.2)

$$r^{\bullet} = \frac{r}{\mathcal{R}} = \sin\eta = v,$$

where v has been introduced as a relative velocity in the preceding Sections.

At the equator $(r = \mathcal{R})$ of the pseudo-sphere one has $r^{\bullet} = \mathcal{R}^{\bullet} = 1$ or in physical units

$$v_H = r_H^{\bullet} = c. \tag{III.6.8}$$

The expansion-induced recession velocity of the galaxies has the highest attainable value, the speed of light, at the equator. A galactic island formation in this model is not possible. The model has a horizon

$$r_H = cT'. \tag{III.6.9}$$

No signal beyond the horizon can reach an observer at $\eta = 0$. Since all points on the hyper surface are equivalent, each observer at any position of the universe has his individual horizon.

We also want to survey whether the definition of the velocity

$$v = r^{\bullet} = \frac{dr}{dT'} \tag{III.6.10}$$

(with the proper time T' of the comoving observer) conforms to the velocity definition of the theory of relativity. An observer in the non-comoving system detects the radial velocity of a receding galaxy as

$$v = \frac{dx^1}{dT}. \tag{III.6.11}$$

Therein T is its proper time. At each time of expansion, the radial arc element is determined by the dS metric

$$dx^1 = \alpha dr, \quad \alpha = 1/a = 1/\cos\eta = 1/\sqrt{1 - r^2/\mathcal{R}^2}. \tag{III.6.12}$$

The proper times of the observers are linked by the Lorentz relations

$$\frac{dT}{dT'} = \alpha \,. \tag{III.6.13}$$

The universe expands in free fall, the Lorentz factor α and the metric factor α are identical according to (III.6.3) and (III.6.12) so that

$$v = \frac{\alpha dr}{\alpha dT'} = r^{\bullet} \tag{III.6.14}$$

has been proven as the relative velocity of the observers and thus as the recession velocity of the galaxies.

Remarkably, these results are identical to those derived by Melia [M] from a model he calls $R_h = ct$ model[33]. However, Melia gains his relations from a flat FRW ansatz. In contrast, our subluminal model is positively curved and closed. This has the advantage that Olbers' paradox is neither concealed nor discussed away by expansion effects. The question remains whether both models are identical. We start with our considerations from the dS model, which is based on a pseudo-hyper sphere, ie, positively curved and closed. According to the FRW classification, however, the expanding version of the dS model is referred to as flat $(k = 0)$. In earlier Sections we have found that the principle of Einstein's elevator plays an important role in cosmology, and we have used it repeatedly in the preceding Sections. In the light of this method the contradiction between the model of Melia and our subluminal model is resolved. $k = 0$ means that the model is based on a positively curved space which expands in free fall and that no gravitational forces are experienced in the comoving system. We can therefore assume that, despite some formal differences, the model of Melia and our subluminal model are identical.

We have shown that an exact solution to Einstein's field equations exists, a solution which describes an expanding cosmological model which respects the fundamental laws of special and general relativity. In this model, the recession velocity of galaxies cannot exceed the velocity of light. Since the universe is expanding in free fall, no acceleration of the expansion occurs.

[33] Melia's expression agrees with (III.6.9). Melia's coordinate time t corresponds in the free-falling, comoving system to the proper time of this system. This time is designated by us with T '.

III.7. New interpretation of the dS model

In Sec. II.3 we have presented two possible interpretations for the expanding dS cosmos: the historical interpretation, which requires to be understood expansion as a genuine expansion of space, and our critical interpretation, which envisages space as steady, but allows the coordinate grid to expand and permits the assigned reference systems to drift apart. We also recall that the curvature parameter $k=1$ can be read from the static metric, but that the metric takes on the structure of $k=0$ after having performed the Lemaître transformation. It is generally believed that this means that the static dS cosmos is positively curved and finite, the expanding one is flat and infinite. We have removed this contradiction by showing that $k=0$ does not refer to a flat space, but to a freely falling frame of reference in which forces do not act. We have called this *Einstein's elevator principle in cosmology*.

We now assume that the dS cosmos can be represented by a pseudo-hyper sphere with a constant radius $\mathcal{R}=\text{const.}$. On the pseudo-hyper sphere, forces start from each point, a fact which leads to a motion of the observers, whereby these observers give way to these forces. Thus, the expanding version of the dS cosmos is considered to be Milne-like.

In this sense, the nonlinear expansion law

$$r^{\bullet}=Hr, \quad H=\frac{1}{\mathcal{K}}\mathcal{K}^{\bullet}$$

with H as the Hubble parameter and $\mathcal{K}$ as the scale factor is to interpreted as the expansion law of the coordinate net.

The above relation suggests that r can accept arbitrarily high values. That may be correct for a flat or a negatively curved, thus open cosmos. Both models are infinite and contain an infinite amount of matter. Apart from the fact that it is hard to imagine the infinite, the question of how an infinite amount of matter may have been created by the Big Bang remains unanswered. Therefore we believe that only closed universes are physically meaningful. Thus, the range of r is limited, namely to the range $[0,\mathcal{R}]$ and thus the recession velocity is also limited. In Sec. II.3 we have determined the value of the recession velocity

$$v_R=r^{\bullet}=\frac{r}{\mathcal{R}} \tag{III.7.1}$$

from the Lorentz transformation. At the equator of the pseudo-hyper sphere is $r=\mathcal{R}$, and thus $v_R=1$ is the velocity of light in the natural measuring system.

However, we doubt that a drifting galaxy in the cosmos can really reach the speed of light. This would violate the laws of special relativity. Therefore, we examine the behavior of an observer when approaching the cosmic horizon more closely.

Now we write (III.7.1) as

$$\frac{dr}{dT'}=\frac{r}{\mathcal{R}}. \tag{III.7.2}$$

Therein T' is the proper time of the comoving observer. We face the region in front of the cosmic horizon. We calculate the time that passes if an observer approaches the horizon or if he possibly reaches the horizon.

Thus, we integrate (III.7.2) in the interval mentioned

$$T'(r) = \int_{\mathcal{R}-r}^{\mathcal{R}} \frac{1}{r} dr = \mathcal{R}\left[\ln\mathcal{R} - \ln(\mathcal{R}-r)\right].$$

The time function obtained we have plotted in Fig. III.1.

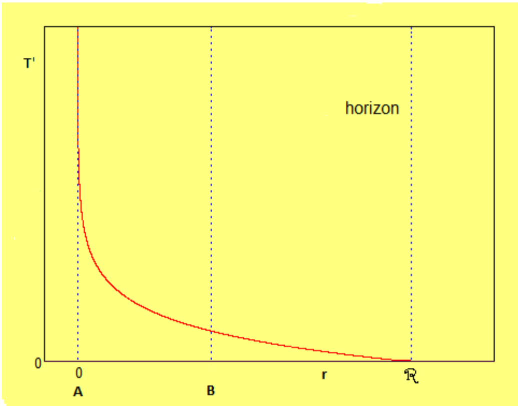

Fig. III.1. Time function for the dS cosmos

It can be seen that an observer A, who starts at $r = 0$ and travels the whole distance from $r = 0$ to $r = \mathcal{R}$ needs an infinite amount of time to get infinitely close to the cosmic horizon. Drifting observers in the cosmos neither exceed the velocity of light nor do they even reach it.

A second observer B, released anywhere in the considered interval, reaches the horizon of A in finite time, but at a speed that is respectively lower than that of A. Since all points on a sphere are equal, any observer will call his starting point a pole. When the observer B leaves his pole, he travels through the horizon of A with subluminal speed and he reaches his individual equator, ie, he reaches his horizon only asymptotically. The equator of B lies beyond the equator of A.

In Sec. III we have extended the dS cosmos to a subluminal expanding model by giving up the condition $\mathcal{R} = \text{const.}$, ie allowing an expansion of the pseudo-hyper sphere. In this case, the observer's motion described above is related to expansion and is responsible for the recession velocity of the galaxies. In the subluminal model we have proposed this velocity is also geometrically defined. We can access the same formulae as in the above section and we get the results discussed. Even in a generalized expanding model, the speed of light is the unattainable barrier to any motion of galaxies.

If A sends a beam of light to a galaxy B and is reflected back to A, both runtimes are equal in accordance with the special theory of relativity, because B can define his position as a pole on the hypersurface. Both distances have the same length and the motion of the light source has no influence on the behavior of a light beam. This applies likewise to motions in the static model as well as to motions caused by expansion in the subluminal model.

It should also be mentioned that Chodorowski [C] has dealt with the question of whether the recession velocity of the galaxies is due to an expansion of space or whether is a motion in the static space in the sense of Milne.

III.8. The generalized dS model

We raise the question of whether the expansion of a cosmos can take place at a lower speed than that of free fall. We simplify the problem by studying it in the dS cosmos. We start with its static form. We define a motion by *relativistically* subtracting a second velocity from the velocity of free fall.

This *double-velocity model* will by no means describe Nature. It only serves to elaborate the mathematical methods. But it can also be stimulating to think about a genuine expanding model that expands more slowly and that predicts to have a lower recession velocity for its galaxies. This can affect the interpretation of the redshift.

To formulate the problem we use the formula apparatus of the special relativity theory for relative velocities. We use Lorentz the transformations

$$\begin{array}{llll} L_1^{1'} = \alpha, & L_1^{4'} = -i\alpha v, & L_4^{1'} = i\alpha v, & L_4^{4'} = \alpha \\ L_{1'}^{1''} = \alpha_E, & L_{1'}^{4''} = -i\alpha_E v_E, & L_{4'}^{1''} = i\alpha_E v_E, & L_{4'}^{4''} = \alpha_E\,. \\ L_1^{1''} = \alpha_R, & L_1^{4''} = -i\alpha_R v_R, & L_4^{1''} = i\alpha_R v_R, & L_4^{4''} = \alpha_R \end{array} \tag{III.8.1}$$

Therein v_R is the speed of a fictional observer driven by the dS forces. This velocity is reduced by a second speed v_E. This gives the actual recession velocity v. In the formulae m'' tags the fictitious system, m' the physical comoving system, and m the static system. We illustrate this with the figure below:

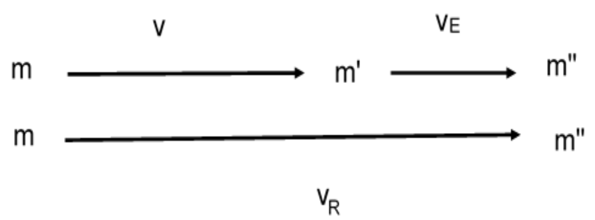

Fig. III.2. The composition of the velocities

Furthermore, we need the formulae for the relativistic relation of the velocities

$$v = \frac{v_R - v_E}{1 - v_R v_E}, \quad v_R = \frac{v + v_E}{1 + v v_E}, \quad v_E = \frac{v_R - v}{1 - v_R v} \tag{III.8.2}$$

together with the Lorentz relations

$$\alpha = 1\Big/\sqrt{1 - v^2}, \quad \alpha_R = 1\Big/\sqrt{1 - v_R^2}, \quad \alpha_E = 1\Big/\sqrt{1 - v_E^2}, \tag{III.8.3}$$

$$\alpha = \alpha_R \alpha_E \left(1 - v_R v_E\right), \quad \alpha_R = \alpha \alpha_E \left(1 + v v_E\right), \quad \alpha_E = \alpha_R \alpha \left(1 - v_R v\right), \tag{III.8.4}$$

$$\alpha v = \alpha_R \alpha_E \left(v_R - v_E\right), \quad \alpha_R v_R = \alpha \alpha_E \left(v + v_E\right), \quad \alpha_E v_E = \alpha_R \alpha \left(v_R - v\right). \tag{III.8.5}$$

The analytic form of the new speed cannot be chosen arbitrarily. All quantities which contain v_E must satisfy Einstein's field equations and the conservation law. Furthermore, we expect that in case of reduced recession velocity radial repulsive forces in the comoving system still occur, but with lower strength than those in the freely falling system.

Furthermore, we expect tidal forces. The latter are easy to derive and they give first useful hints for the ansatz of the velocity v_E.

In addition, we convince ourselves that the recession velocity is relativistically defined. For the dS coordinate r applies

$$r_{|1} = \frac{\partial r}{\alpha_R \partial r} = a_R, \quad r_{|4} = 0\,.$$

Thus, one gets

$$r_{|m} = \{1,0,0,0\} a_R, \quad r_{|m'} = L^m_{m'} r_{|m} = \{\alpha, 0, 0, -i\alpha v\} a_R\,. \tag{III.8.6}$$

With $dx^{4'} = idT'$ one has $\alpha_R dr/dT' = \alpha v$. In the dS cosmos is $dx^1 = \alpha_R dr$ and for the comoving system one has $dx^{1'} = 0,\ dT/dT' = \alpha$. This finally results in

$$\frac{dx^1}{dT} = v < v_R, \quad x^{1'} = \text{const.}. \tag{III.8.7}$$

On the other hand, one obtains from $dx^1 = 0,\ dT'/dT = \alpha$

$$\frac{dx^{1'}}{dT'} = -v, \quad x^1 = \text{const.}. \tag{III.8.8}$$

The recession velocity fits the basic structure of the special theory of relativity.

To calculate the new field quantities, we need the inhomogeneous transformation law of the Ricci-rotation coefficients. Since the motions take place in the [1,4]-slice of the space, the [2,3]-quantities transform homogeneously, just like vectors. Thus,

$$B_m = \frac{1}{r} r_{|m} = \{1,0,0,0\} \frac{a_R}{r}, \quad C_m = \frac{1}{r\sin\vartheta}\left(r\sin\vartheta\right)_{|m} = \left\{\frac{a_R}{r}, \frac{1}{r}\cot\vartheta, 0, 0\right\}$$

$$B_{m'} = L^m_{m'} B_m, \quad C_{m'} = L^m_{m'} C_m$$

applies for the lateral field quantities. With (III.8.1) they attain the form

$$B_{m'} = \left\{\alpha\frac{a_R}{r}, 0, 0, -i\alpha v\frac{a_R}{r}\right\}, \quad C_{m'} = \left\{\alpha\frac{a_R}{r}, \frac{1}{r}\cot\vartheta, 0, -i\alpha v\frac{a_R}{r}\right\}. \tag{III.8.9}$$

With this we recognize that the radial part and the timelike part of these quantities no longer appear to be flat, as it was the case for the freely falling frame of reference of the dS Model with

$$B_{m'} = \left\{\frac{1}{r}, 0, 0, -\frac{i}{\mathcal{R}}\right\}, \quad C_{m'} = \left\{\frac{1}{r}, \frac{1}{r}\cot\vartheta, 0, -\frac{i}{\mathcal{R}}\right\}.$$

The formula for free fall is obtained only if $\alpha = \alpha_R$, ie, if $v_E = 0$. If one now demands that the comoving volume element expands equally in all three directions one has

$$B_{4'} \overset{*}{=} C_{4'} \overset{*}{=} {}'U_{4'} = -i\alpha v\frac{a_R}{r} \tag{III.8.10}$$

and thus one has derived the timelike component of the radial quantity ${}'U_{m'}$ of the comoving system. Since one must obtain the same expression with the inhomogeneous transformation law of the Ricci-rotation coefficients, one can draw conclusions about the properties of the quantity v_E.

The inhomogeneous transformation law of the Ricci-rotation coefficients

$$'A_{m'n'}{}^{s'} = A_{m'n'}{}^{s'} + 'L_{m'n'}{}^{s'}, \quad 'L_{m'n'}{}^{s'} = L_s^{s'} L^s_{n'|m'} \tag{III.8.11}$$

can be simplified, since in the present case the Lorentz transformation is a pseudorotation in the [1,4]-slice of the surface. The inhomogeneous term which we call the Lorentz term, can be expressed with $h_{m'n'} = \mathrm{diag}\{1,0,0,1\}$ and can be brought into vector form with

$$'L_{m'n'}{}^{s'} = h_{m'}{}^{s'}\,'L_{n'} - h_{m'n'}\,'L^{s'}, \quad 'L_{n'} = 'L_{s'n'}{}^{s'} = \left\{'L_{4'1'}{}^{4'}, 'L_{1'4'}{}^{1'}\right\}. \tag{III.8.12}$$

From (III.8.11) remains only *Einstein's elevator law*

$$'U_{m'} = U_{m'} + 'L_{m'}, \quad U_{m'} = \{\alpha,0,0,-i\alpha v\}\left(-\alpha_R v_R \frac{1}{\mathcal{R}}\right). \tag{III.8.13}$$

With (III.8.11), (III.8.12), and (III.8.1) we calculate

$$'L_{1'} = i\alpha^2 v_{|4'}, \quad 'L_{4'} = -i\alpha^2 v_{|1'}.$$

With the help of the relation

$$\alpha^2 dv = \alpha_R^2 dv_R - \alpha_E^2 dv_E$$

we evaluate

$$'L_{1'} = i\alpha_R^2 v_{R|4'} - i\alpha_E^2 v_{E|4'}, \quad 'L_{4'} = -i\alpha_R^2 v_{R|1'} + i\alpha_E^2 v_{E|1'}.$$

First, we determine the first terms of these relations

$$v_{R|m} = \{1,0,0,0\} a_R \frac{1}{\mathcal{R}}, \quad v_{R|m'} = \{\alpha,0,0,-i\alpha v\} a_R \frac{1}{\mathcal{R}}. \tag{III.8.14}$$

With (III.8.10) and (III.8.13) on hand we obtain

$$'U_{4'} = i\alpha v \alpha_R v_R \frac{1}{\mathcal{R}} - \alpha_R^2 \cdot i\alpha\alpha_R \frac{1}{\mathcal{R}} + i\alpha_E^2 v_{E|1'} = -i\alpha\alpha_R(1 - v v_R) + i\alpha_E^2 v_{R|1'} = -i\alpha_E \frac{1}{\mathcal{R}} + i\alpha_E^2 v_{E|1'}.$$

We take the value for $'U_{4'}$ from (III.8.10) and after some calculation we get

$$v_{E|1'} = a_E v_E \frac{1}{r}. \tag{III.8.15}$$

Furthermore, we require that the quantity v_E in the comoving system is time independent, so that we finally obtain

$$v_{E|m'} = \{1,0,0,0\} a_E v_E \frac{1}{r}, \quad v_{E|m} = \{\alpha,0,0,i\alpha v\} a_E v_E \frac{1}{r}. \tag{III.8.16}$$

Now we are also able to clearly present the Lorentz term

$$'L_{m'} = -G_{m'} + l_{m'}, \quad G_{m'} = \{i\alpha v,0,0,\alpha\} i\alpha_R \frac{1}{\mathcal{R}}, \quad l_{m'} = \{0,0,0,1\} i\alpha_E v_E \frac{1}{r} \tag{III.8.17}$$

and also with $L_m = -L_m^{m'}\,'L_{m'}$ the inverse transformation

$$L_m = G_m - I_m, \quad G_m = \{0,0,0,1\} i\alpha_R \frac{1}{\mathcal{R}}, \quad I_m = \{-i\alpha v, 0, 0, \alpha\} i\alpha_E v_E \frac{1}{r}. \qquad \text{(III.8.18)}$$

The components G and I are assigned to the changes of v_R and v_E, respectively. Now we can continue with (III.8.13)

$$'U_{1'} = -\alpha\alpha_R v_R \frac{1}{\mathcal{R}} + \alpha\alpha_R v \frac{1}{\mathcal{R}}.$$

Finally, we have with (III.8.5)

$$'U_{m'} = \left\{-\alpha_E v_E \frac{1}{\mathcal{R}}, 0, 0, -i\alpha v a_R \frac{1}{r}\right\}. \qquad \text{(III.8.19)}$$

It can be seen that, in contrast to the 'free falling' observer, radial forces act on a less rapidly comoving observer

$$'E_{1'} = -'U_{1'} = \alpha_E v_E \frac{1}{\mathcal{R}},$$

acting repulsively. At the same time tidal forces occur.

With (III.8.11) we have obtained this quantity from the non-comoving system by an inhomogeneous transformation law. But since the field quantities of the freely falling system are also known, we can derive (III.8.19) from this system as well.

The inhomogeneous transformation law for $m'' \to m'$ is

$$'A_{m'n'}{}^{s'} = L_{m'n's''}^{m''n''s'}\, ''A_{m''n''}{}^{s''} + I_{m'n'}{}^{s'}, \quad I_{m'n'}{}^{s'} = L_{s''}^{s'} L_{n'|m'}^{s''}. \qquad \text{(III.8.20)}$$

The Lorentz term

$$I_{m'} = \left\{-i\alpha_E^2 v_{E|4'}, 0, 0, i\alpha_E^2 v_{E|1'}\right\} \qquad \text{(III.8.21)}$$

leads to the simple expression

$$I_{m'} = \{0,0,0,1\} i\alpha_E v_E \frac{1}{r} \qquad \text{(III.8.22)}$$

which we have already worked out on the way to (III.8.17). But now we can give a better justification. Now we write

$$''U_{m''} = \left\{0,0,0,-\frac{i}{\mathcal{R}}\right\}, \quad ''U_{m'} = \{-i\alpha_E v_E, 0, 0, \alpha_E\}\left(-\frac{i}{\mathcal{R}}\right)$$

and finally we have recovered the quantity (III.8.19) with

$$'U_{m'} = ''U_{m'} + I_{m'}. \qquad \text{(III.8.23)}$$

We have deduced all field quantities which we need for the generalized version of the dS model.

III.9. The field equations of the generalized dS model

The ansatz introduced in the last Section for a double-velocity model as a generalization of the dS model can only be justified if the field quantities obtained satisfy Einstein's field equations. For verification, we have to process the quantities

$$B_{m'}=\left\{\alpha\frac{a_R}{r},0,0,-i\alpha v\frac{a_R}{r}\right\},\quad C_{m'}=\left\{\alpha\frac{a_R}{r},\frac{1}{r}\cot\vartheta,0,-i\alpha v\frac{a_R}{r}\right\}$$
$$'U_{m'}=\left\{-\alpha_E v_E\frac{1}{\mathcal{R}},0,0,-i\alpha v a_R\frac{1}{r}\right\}\quad. \tag{III.9.1}$$

With these quantities and the unit vectors

$$m_m=\{1,0,0,0\},\quad b_m=\{0,1,0,0\},\quad c_m=\{0,0,1,0\},\quad u_m=\{0,0,0,1\} \tag{III.9.2}$$

the Riccitensor and Ricci scalar take the form

$$\begin{aligned}R_{m'n'}=&-\left[{}'U^{s'}{}_{\underset{1}{||}s'}+{}'U^{s'}{}'U_{s'}\right]h_{m'n'}\\&-\left[B_{n'\underset{2}{||}m'}+B_{n'}B_{m'}\right]-b_{n'}b_{m'}\left[B^{s'}{}_{\underset{2}{||}s'}+B^{s'}B_{s'}\right]\\&-\left[C_{n'\underset{3}{||}m'}+C_{n'}C_{m'}\right]-c_{n'}c_{m'}\left[C^{s'}{}_{\underset{3}{||}s'}+C^{s'}C_{s'}\right]\quad.\end{aligned} \tag{III.9.3}$$

$$-\frac{1}{2}R=\left[{}'U^{s'}{}_{\underset{1}{||}s'}+{}'U^{s'}{}'U_{s'}\right]+\left[B^{s'}{}_{\underset{2}{||}s'}+B^{s'}B_{s'}\right]+\left[C^{s'}{}_{\underset{3}{||}s'}+C^{s'}C_{s'}\right]$$

Therein we have used the graded derivatives

$$'U_{n'\underset{1}{||}m'}={}'U_{n'|m'},\quad B_{n'\underset{2}{||}m'}=B_{n'|m'}-{}'U_{m'n'}{}^{s'}B_{s'},\quad C_{n'\underset{3}{||}m'}=C_{n'|m'}-{}'U_{m'n'}{}^{s'}C_{s'}-B_{m'n'}{}^{s'}C_{s'} \tag{III.9.4}$$

which allow a clear representation of the field equations. The subequations of (III.9.3) describe the curvatures in the radial and lateral slices of the dS space, as viewed by the comoving observers. Therefore we solve the subequations of Einstein's field equations separately

$$'U^{s'}{}_{|s'}+{}'U^{s'}{}'U_{s'}=-\frac{1}{\mathcal{R}^2}$$
$$B_{m'\underset{2}{||}n'}+B_{m'}B_{n'}=-h_{m'n'}\frac{1}{\mathcal{R}^2},\quad B^{s'}{}_{\underset{2}{||}s'}+B^{s'}B_{s'}=-\frac{2}{\mathcal{R}^2}\quad. \tag{III.9.5}$$
$$C_{m'\underset{3}{||}n'}+C_{m'}C_{n'}=-\left(h_{m'n'}+b_{m'}b_{n'}\right)\frac{1}{\mathcal{R}^2},\quad C^{s'}{}_{\underset{3}{||}s'}+C^{s'}C_{s'}=-\frac{3}{\mathcal{R}^2}$$

With this we get for the Ricci-quantities

$$R_{m'n'}=g_{m'n'}\frac{3}{\mathcal{R}^2},\quad R=\frac{12}{\mathcal{R}^2},\quad G_{m'n'}=-g_{m'n'}\frac{3}{\mathcal{R}^2}=-\kappa T_{m'n'}\,. \tag{III.9.6}$$

From the last relation we get for the components of the stress-energy-momentum tensor and the equation of state

$$\kappa p = -\frac{3}{\mathcal{R}^2}, \quad \kappa\mu_0 = \frac{3}{\mathcal{R}^2}, \quad p + \mu_0 = 0\,. \tag{III.9.7}$$

The values obtained are equal to those of the static dS cosmos. We note that matter transport cannot be observed in the comoving system. The cause is the form of the equation of state. For all models with $p + \mu_0 = 0$ one has $T_{1'4'} = 0$, as we can easily convince ourselves with the transformation $T_{m'n'} = L^{m\ n}_{m'n'} T_{mn}$.

III.10. More about the velocities

While the velocity $v_R = r/\mathcal{R}$ is geometrically determined, we do not so far have knowledge of the second part of the recession velocity. We only know the change from v_E which we have determined as a general feature of the model. We have succeeded in establishing plausible field quantities which fulfill Einstein's field equations and which lead to the familiar expressions for the pressure and the density of matter in the cosmos.

Now we will examine whether an analytic expression can be found which is compatible with all previous relation and provides a deeper insight into the geometric structures of the model.

We put

$$v_E = \frac{r}{'\mathcal{R}}, \quad '\mathcal{R} = '\mathcal{R}(T'). \tag{III.10.1}$$

$'\mathcal{R}$ is a new time-dependent parameter and we notice the analogy with the definition $v_R = r/\mathcal{R}$. Thus, we can assume that v_E has a similar geometrical meaning in a fictive cosmos, which is preliminary to the dS cosmos, where v_R is well defined.

For the recession velocity of the galaxies we then have

$$v = \frac{\dfrac{r}{\mathcal{R}} - \dfrac{r}{'\mathcal{R}}}{1 - \dfrac{r^2}{\mathcal{R}\,'\mathcal{R}}} \tag{III.10.2}$$

according to (III.8.2). For $'\mathcal{R} = \mathcal{R}$ the recession velocity is $v = 0$. For $'\mathcal{R} \to \infty$ is $v_E = 0$ and v takes its maximum value, namely the dS velocity. At the cosmic horizon, the recession velocity asymptotically reaches the velocity of light regardless of the value of v_E. Thus, the ranges of $'\mathcal{R}$ and r are

$$\mathcal{R} \le {'\mathcal{R}} \le \infty, \quad 0 \le r \le \mathcal{R} \tag{III.10.3}$$

and $'\mathcal{R}$ is a parameter with which one can manipulate the recession velocity v. If this technique can be applied to a model that is closer to Nature, the calculable redshift values may be better adapted to the values observed. In Fig. 5.3 the recession velocity is plotted in the range $0 \le r \le \mathcal{R}$ for different $'\mathcal{R}$. It can be seen that a deviation from the linear velocity law ($'\mathcal{R} = \infty$) by an appropriate choice of $'\mathcal{R}$ is possible.

Since v depends on time, as shown in (III.10.1) and (III.10.2), the generalized dS model allows accelerations of the drifting systems.

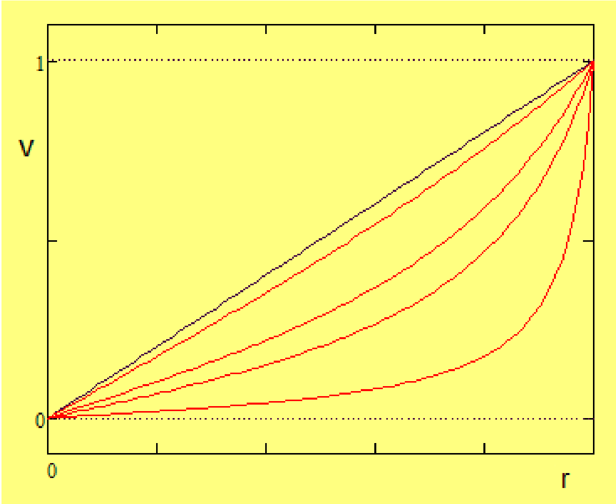

Fig. 5.3. Recession velocity

Now all we have to do is to show that the approach (III.10.1) meets the requirements made in the previous sections. In particular, we have to check if

$$v_{E|m'} = \{1,0,0,0\} a_E v_E \frac{1}{r} \tag{III.10.4}$$

is well-matched with Eq. (III.10.1). Following (III.10.1) we write

$$v_{E|m'} = v_E \left(\frac{1}{r} r_{|m'} - \frac{1}{'\mathcal{R}} {}'\mathcal{R}_{|m'} \right). \tag{III.10.5}$$

Since $r_{|m'}$ has already been calculated with (III.8.14), we only need to know more about the quantity

$${}'\mathcal{R}_{m'} = \frac{1}{'\mathcal{R}} {}'\mathcal{R}_{|m'} \tag{III.10.6}$$

which we obtain by equating (III.10.4) with (III.10.5). After some calculation and repeated use of (III.8.3) - (III.8.5) we get

$${}'\mathcal{R}_{m'} = \left\{ -\alpha v a_R \frac{1}{'\mathcal{R}}, 0, 0, -i\alpha v a_R \frac{1}{r} \right\}.$$

We recognize that the 4th component of this quantity has already arisen as $'U_{4'}$. Applying again the Lorentz relations leads to

$${}'\mathcal{R}_{m} = \{-i\alpha_E v_E, 0, 0, \alpha_E\} \left(\frac{i}{'\mathcal{R}} - \frac{i}{\mathcal{R}} \right), \quad {}'\mathcal{R}_{m''} = \{0,0,0,1\} \left(\frac{i}{'\mathcal{R}} - \frac{i}{\mathcal{R}} \right). \tag{III.10.7}$$

The second relation (III.10.7) contains the already known expression $''U_{4''} = -i/\mathcal{R}$ which results from the temporal change of the scale factor $\mathcal{K}$.

Therefore the ansatz (III.10.1) is satisfactory and thus (III.10.4) can be written as

$$v_{E|m'} = \{1,0,0,0\} \frac{a_E}{'\mathcal{R}}, \quad a_E = \sqrt{1 - \frac{r^2}{'\mathcal{R}^2}}, \tag{III.10.8}$$

ie, in the same way as in (III.8.14). The inhomogeneous transformation law of the Ricci-rotation coefficients can be simplified with the quantities (III.8.17) and (III.8.18), if one considers (III.10.1)

$$\begin{aligned} & G_m = \{0,0,0,1\}\, i\alpha_R \frac{1}{\mathcal{R}}, \quad I_m = \{-i\alpha v,0,0,\alpha\}\, i\alpha_E \frac{1}{'\mathcal{R}} \\ & G_{m'} = \{i\alpha v,0,0,\alpha\}\, i\alpha_R \frac{1}{\mathcal{R}}, \quad I_{m'} = \{0,0,0,1\}\, i\alpha_E \frac{1}{'\mathcal{R}} \;. \\ & 'U_{m'} = \{i\alpha_E v_E,0,0,\alpha_E\}\left(-\frac{i}{\mathcal{R}}\right) + \{0,0,0,1\}\,\alpha_E \frac{i}{'\mathcal{R}} \end{aligned} \tag{III.10.9}$$

One should note the analogy of the quantities G and I.

III.11. Coordinate transformations

The model described above was carried out in the tetrad calculus. A resort to a coordinate system was only necessary when basic mathematical operations had to be performed. For this, the static dS coordinate system was sufficient. Cosmologists, however, are trying to find coordinate systems for both the comoving and the non-comoving frames of reference. We want to investigate whether coordinate systems exist for all states of motion and also transformations between them. The reader who is only interested in the general structure of the model, can skip this section.

Coordinate systems have been known for the static and the fictitious comoving systems since the Sitter and Lemaître. The question is, however, whether there is not only a frame of reference for the physical comoving system, but also a coordinate system.

To get closer to the problem, let us start with the static dS system

$$\overset{1}{e}_1=\alpha_R,\quad \overset{2}{e}_2=r,\quad \overset{3}{e}_3=r\sin\vartheta,\quad \overset{4}{e}_4=a_R,\quad a_R=1/\alpha_R=\sqrt{1-r^2/\mathbb{R}^2} \tag{III.11.1}$$

and the expanding one

$$\overset{1''}{e}_{1''}=\mathcal{K},\quad \overset{2''}{e}_{2''}=r,\quad \overset{3''}{e}_{3''}=r\sin\vartheta,\quad \overset{4''}{e}_{4''}=1,\quad \mathcal{K}=\frac{r}{r''}, \tag{III.11.2}$$

where r is the non-comoving and r'' is the comoving radial coordinate and $\mathcal{K}$ he scale factor. From (III.11.2) it can be seen that the coordinate time and the proper time of the drifting observers coincide ($t''=T''$). T'' is the cosmic time common to all drifting observers.

The coordinate transformation between the two systems and the associated Lorentz transformation has been given in Sec. III.3. Since the Lorentz transformation to the physical system is known as (III.8.1), we first transform the static system (III.11.1) into the physically comoving system while maintaining the static coordinates

$$\begin{array}{llll} \overset{1'}{e}_1=\alpha\alpha_R, & \overset{4'}{e}_1=-i\alpha v\alpha_R, & \overset{1'}{e}_4=i\alpha v a_R, & \overset{4'}{e}_4=i\alpha a_R \\ \underset{1'}{e}^1=\alpha a_R, & \underset{4'}{e}^1=-i\alpha v a_R, & \underset{1'}{e}^4=i\alpha v\alpha_R, & \underset{4'}{e}^4=\alpha\alpha_R \end{array}. \tag{III.11.3}$$

The Ricci-rotation coefficients for this system provide the values obtained in the previous Section. With the help of

$$\overset{m'}{e}_{i'}=\Lambda^i_{i'}\overset{m'}{e}_i \tag{III.11.4}$$

the system can be diagonalized. For the matrix of the coordinate transformation we get

$$\Lambda^i_{i'}=\begin{pmatrix} \frac{\alpha^2 v}{\alpha_R^2} & & & -i\frac{\alpha^2 v}{\alpha_R^2} \\ & 1 & & \\ & & 1 & \\ i\alpha^2 v^2 & & & \alpha^2 \end{pmatrix},\quad \Lambda^{i'}_i=\begin{pmatrix} \frac{\alpha_R^2}{v} & & & i \\ & 1 & & \\ & & 1 & \\ -i\alpha_R^2 v & & & 1 \end{pmatrix}. \tag{III.11.5}$$

Alternatively, one can start with the freely falling dS system

$$\overset{m'}{e}_{i''}=L^{m'}_{m''}\overset{m''}{e}_{i''},\quad \overset{m'}{e}_{i'}=\overset{m'}{e}_{i''}\Lambda^{i''}_{i'}\ . \tag{III.11.6}$$

With

$$\Lambda^{i''}_{i'} = \begin{pmatrix} \frac{1}{\mathcal{K}}\alpha_E \frac{\alpha v}{\alpha_R} & & & i\frac{1}{\mathcal{K}}\alpha_E v_E \frac{\alpha}{\alpha_R} \\ & 1 & & \\ & & 1 & \\ -i\alpha_E v_E \frac{\alpha v}{\alpha_R} & & & \alpha_E \frac{\alpha}{\alpha_R} \end{pmatrix}, \quad \Lambda^{i'}_{i''} = \begin{pmatrix} \mathcal{K}\alpha_E \frac{\alpha_R}{\alpha v} & & & -i\alpha_E v_E \frac{\alpha_R}{\alpha v} \\ & 1 & & \\ & & 1 & \\ -i\mathcal{K}\alpha_E v_E \frac{\alpha_R}{\alpha} & & & \alpha_E \frac{\alpha_R}{\alpha} \end{pmatrix} \qquad \text{(III.11.7)}$$

one also arrives at the diagonalized system for the physical comoving observer:

$$\overset{1'}{e}_{1'} = \frac{\alpha v}{\alpha_R}, \quad \overset{4'}{e}_{4'} = \frac{\alpha}{\alpha_R}, \quad \underset{1'}{e}^{1'} = \frac{\alpha_R}{\alpha v}, \quad \underset{4'}{e}^{4'} = \frac{\alpha_R}{\alpha}. \qquad \text{(III.11.8)}$$

Herewith the line element takes the form

$$ds^2 = \frac{\alpha^2 v^2}{\alpha_R^2} dr'^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 - \frac{\alpha^2}{\alpha_R^2} dt'^2. \qquad \text{(III.11.9)}$$

The two observer transformations (III.11.5) and (III.11.7) are related to the holonomic Lemaître transformation in Sec. II.3 by means of

$$\Lambda^{i''}_{i} = \Lambda^{i''}_{i'} \Lambda^{i'}_{i}, \quad \Lambda^{i''}_{i} = x^{i''}{}_{|i}\,.$$

It can be seen that there is no cosmic universal time for the comoving physical observer system. Rather, one has for the proper time of the observers

$$dT' = \frac{\alpha}{\alpha_R} dt' \qquad \text{(III.11.10)}$$

in accordance with the general theory of relativity.

However, the Ricci-rotation coefficients cannot be derived from the 4-bein system (III.11.8), as we have learned in the previous sections. The reason is the anholonomicity of the two coordinate transformations (III.11.5) and (III.11.7). One easily convinces oneself that

$$\Lambda^{j'}_{[i|k]} \neq 0 \;\Rightarrow\; \Lambda^{j'}_{i} \neq x^{j'}{}_{|i}, \quad \Lambda^{j'}_{[i''|k'']} \neq 0 \;\Rightarrow\; \Lambda^{j'}_{i''} \neq x^{j'}{}_{|i''}. \qquad \text{(III.11.11)}$$

Thus, there exists no associated global coordinate mesh for the comoving observer system, but only anholonomic coordinates. These are mathematical artifacts described in detail by Schouten [S].

If one proceeds in the usual way and derives from (III.11.8) the expression

$$A_{m'n'}{}^{s'} = \overset{s'}{e}_{j'}\, e^{j'}_{[n'|m']} + g^{s'r'} g_{n't'} \overset{t'}{e}_{j'}\, e^{j'}_{[r'|m']} - g^{s'r'} g_{m't'} \overset{t'}{e}_{j'}\, e^{j'}_{[r'|n']}$$

and if one then complements the *object of anholonomity*

$$\Lambda_{m'n'}{}^{s'} = \underset{m'}{e}^{i'} \underset{n'}{e}^{k'} \overset{s'}{e}_{j'} \Lambda^{j'}_{j} \Lambda^{j}_{[k'|i']} \qquad \text{(III.11.12)}$$

one finally gains the Ricci-rotation coefficients

$$'A_{m'n'}{}^{s'} = A_{m'n'}{}^{s'} + \Lambda_{m'n'}{}^{s'} + \Lambda^{s'}{}_{m'n'} + \Lambda^{s'}{}_{n'm'} \qquad \text{(III.11.13)}$$

with the values known of Sec. III.8.

It is clear that the search for coordinate systems which accompany the families of observers is not necessary and often not practical.

We have proposed a model which we do not assume to be realized by Nature. However, it includes useful mathematical techniques which can be transferred to more sophisticated models. In particular, it is possible to reduce the recession velocity of the galaxies compared to those of the 'free fall' and thus to manipulate the calculated values for the redshift. This could allow a better adjustment to the observed values.

IV. The Schwarzschild metric

IV.1. Schwarzschild model, basics

Soon after the formulation of the general relativity theory by Einstein, Schwarzschild [S] published an exact solution of the Einstein field equations for a spherical, nonrotating, and uncharged stellar object. It belongs to those models, which have been discussed in great detail by the physicist community, particularly since it not only supplies useful results for the gravitation physics of our solar system, but also provides the possibility of describing black holes. The metric of this solution, differing from the originally published one, is indicated in the following form

$$ds^2 = \alpha^2 dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 + a^2 dit^2, \tag{IV.1.1}$$

wherein

$$a = \sqrt{1 - 2M/r}, \quad \alpha = a^{-1} \tag{IV.1.2}$$

is called mostly redshift factor. The co-ordinates in use are based on spherical co-ordinates, are adapted to the spherical symmetry of the problem, and are called standard Schwarzschild co-ordinates. From the metric we read the 4-bein components

$$\overset{1}{e}_1 = \alpha, \quad \overset{2}{e}_2 = r, \quad \overset{3}{e}_3 = r\sin\vartheta, \quad \overset{4}{e}_4 = a \tag{IV.1.3}$$

and from these the reciprocal ones. Thus, one obtains the partial derivatives with respect to the 4-bein

$$\partial_1 = \frac{\partial}{\alpha\partial r}, \quad \partial_2 = \frac{\partial}{r\partial\vartheta}, \quad \partial_3 = \frac{\partial}{r\sin\vartheta\partial\varphi}, \quad \partial_4 = \frac{\partial}{a\partial it} \ . \tag{IV.1.4}$$

With the methods used in the above Sections the Ricci-rotation coefficients can be computed

$$\begin{gathered} A_{mn}{}^{s} = B_{mn}{}^{s} + C_{mn}{}^{s} + E_{mn}{}^{s} \\ B_{mn}{}^{s} = b_m B_n b^s - b_m b_n B^s, \quad C_{mn}{}^{s} = c_m C_n c^s - c_m c_n C^s, \quad E_{mn}{}^{s} = -\left[u_m E_n u^s - u_m u_n E^s\right] \ . \\ B_m = \left\{\frac{a}{r}, 0,0,0\right\}, \quad C_m = \left\{\frac{a}{r}, \frac{1}{r}\cot\vartheta, 0,0\right\}, \quad E_m = \left\{-\alpha\frac{M}{r^2}, 0,0,0\right\} \end{gathered} \tag{IV.1.5}$$

The vector E has only one component in the radial direction is negative and pointing inwards, i.e. to the central mass. It is accepted for the latter that it produces the Schwarzschild field. The quantity E differs from Newton's force of gravity only by the factor α, which in turn differs a little from 1 at larger distances from the gravitation center. It is the gravitational field strength, the force acting on the unit mass of a test body. The first components of the quantities B and C differ from the Euclidean values by the factor a. These two quantities enter into the geodesic equations. Performing the integration of the equations of motion the factor a provides paths deviating from Newton's theory[34].

Utilization of the quantities (IV.1.5) in the Ricci tensor specifies

[34] The Schwarzschild model stands the test concerning the computation of the light deviation at the sun and the perihelion motion of Mercury.

$$
\begin{aligned}
R_{mn} = &-\left[B_{m\underset{2}{\|}n} + B_m B_n \right] - b_m b_n \left[B^r{}_{\underset{2}{\|}r} + B^r B_r \right] \\
&-\left[C_{m\underset{3}{\|}n} + C_m C_n \right] - c_m c_n \left[C^r{}_{\underset{3}{\|}r} + C^r C_r \right] . \\
&+\left[E_{m\underset{4}{\|}n} - E_m E_n \right] + u_m u_n \left[E^r{}_{\underset{4}{\|}r} - E^r E_r \right]
\end{aligned}
\tag{IV.1.6}
$$

The expansion of the graded derivatives in (IV.1.6) confirms that the vacuum field equations

$$R_{mn} = 0 \tag{IV.1.7}$$

are satisfied. That means that the stress-energy tensor vanishes and the field equations are valid only outside the central mass. In addition, the Maxwell-like equations are valid

$$B_{[m\underset{2}{\|}n]} = 0, \quad C_{[m\underset{3}{\|}n]} = 0, \quad E_{[m\underset{4}{\|}n]} = 0 \tag{IV.1.8}$$

and from (IV.1.6) the relation

$$E^r{}_{\underset{4}{\|}r} - E^r E_r = 0 \tag{IV.1.9}$$

can be extracted. It can also be written as

$$\mathrm{div}E = E^2 \tag{IV.1.10}$$

with the operator div as 3-dimensional covariant divergence. By this equation one recognizes the effects of the non-linearity of the Einstein field equations. The force of gravity is coupled to itself, the gravitational field itself is gravitating.

We again hark back to the metric (IV.1.1). At the position r = 2M the factors turn out to be a = 0, $\alpha = \infty$, and the radial arc αdr to be singular. In this case r is called gravitation radius, Schwarzschild radius, or event horizon. One has discussed for a long time whether this locus is a genuine singularity or a co-ordinate singularity[35].

The range r < 2M is mathematically obvious if one re-writes the metric as

$$\frac{1}{1-\frac{2M}{r}} dr^2 \to -\frac{1}{\frac{2M}{r}-1} dr^2, \qquad -\left(1-\frac{2M}{r}\right) dt^2 \to \left(\frac{2M}{r}-1\right) dt^2 . \tag{IV.1.11}$$

The change of the sign implies that the initial space-like element dr is time-like in this region, the time-like element of the metric becomes space-like. Space and time exchange their meanings beneath the event horizon. The inner region is identified as a black hole. The theory of the black holes takes up much space in the literature. However, this will not be a subject of this booklet. Rather we will take a critical view on this interpretation of the inner region of the Schwarzschild metric.

1. The exchange of space and time, mentioned above, is not only a question of accustoming but also incomplete. In $r d\vartheta$ and $r \sin\vartheta d\varphi$ r is connected with the radii of circles and is purely spatial.

[35] By an unfavorable choice of the co-ordinate system one can effect that the mathematical representation of a regular object fails at certain positions. In his textbook Rindler [R] shows a simple example, how a metric can obtain a co-ordinate singularity by a co-ordinate transformation. With $x^3 = 3\xi$ the line element $ds^2 = dx^2 + dy^2$ receives the form $ds^2 = (3\xi)^{-4/3} d\xi^2 + dy^2$ and with $\xi = 0$ a singularity.

2. The position at r = 2M appears as singularity in the standard Schwarzschild representation. A co-ordinate system was found by Kruskal [K] and Szekeres [S] which avoids this singularity. Later on we will deal with it critically.
3. The speed of a freely falling body reaches the speed of light at the event horizon. That is why an intrusion beneath the Schwarzschild radius appears problematic. Thereinafter an approach to this problem is discussed.
4. The force of gravity and the red shift turn out to be infinite at the event horizon.
5. Collapsing stars end up in a relativistic fireball.
6. At stellar objects which one would like to regard as black holes, one has found emerging magnetic fields. This led to a maceration of the term black hole.
7. A loss of information quantum-mechanically not justifiable arises, if the matter dissolves by falling into a black hole.

The possibility for freely falling bodies to go through the event horizon of the Schwarzschild field into hypothetical black holes is crucial for their existence. One obtains the information necessary for this possibility by the integration of the radial equation of motion. However, the managing of this task seems to be a challenge for the physicists. In the last decades some attempts at quite contradictory solutions were made. While some authors agree that an object coming from infinity reaches the speed of light at the Schwarzschild radius r = 2M, this should not be valid for bodies which are released at another arbitrary position.

Gautreau [G] assumes that bodies passing the event horizon continue their motion with superluminous velocity in the inner region r < 2M. De Sabbata, Pavšič, and Recami [D] examine in detail this tachyonic (motion with velocity higher than light) and bradyonic behavior (designating a motion with velocity lower than light).

Janis [J] introduces a new reference system for measuring the speed of a freely falling body and shows that in this system the speed is less than the speed of light at the event horizon r = 2M. Cavalleri [C] and Spinelli recognize as the cause a special co-ordinate choice and a misinterpretation of the particle speed. In an answer, Janis [J] tries to support his point of view by a comparison with the Minkowski space.

From Jaffe's [J] and Shapiro's computations it becomes clear that particles falling to a stellar object accelerate at first then, however, become slower again. Their transition speed at the event horizon would then be smaller than the speed of light. McGruder III [M] recognizes a repulsive effect in the environment of the event horizon. Baierlein [B] contradicts Jaffe and Shapiro and points out that the incorrect choice of the co-ordinate time in place of the physical time has led to these results.

Tereno [T] integrates the radial equation of motion in a likewise way and shows that radial geodesics do not become null-lines at the event horizon. This means that material particles can fall into a black hole with a velocity less than the velocity of light. Mitra [M] attributes this view to an error concerning the limit operation in Tereno's calculations[36]. In

[36] After integration of the radial equation of motion Tereno arrived at an expression $\lim\limits_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon}$ at the event horizon and implements the limit operation to this fraction, so that the indefinite form 0/0 develops. Due to this result he argues that the speed of a particle at the event horizon can take any value. However, since Bernoulli and

order to confirm his point of view Tereno performs some more calculations in Kruskal co-ordinates. Mitra answers with a detailed analysis in Kruskal co-ordinates. Since Tereno [T] contradicts again, Mitra [M] consults the Shapiro-Teukolsky [S] [r,t]-relation and by the use of cycloidal co-ordinates arrives again at the result v = c at the event horizon.

Unaffected by this criticism Crawford [C] and Tereno returned to this problem in a later paper. After a detailed discussion on reference systems in the Schwarzschild field they specify a formula from which the speed is calculable in such a way that a freely falling body could enter a black hole. Krori [K] and Paul, Lynden-Bell and Katz [L], and Salzmann [S] and Salzmann were concerned with similar problems. We refer to the papers of Logunov [L] and Loinger [L]. De Sabbata [D] and Shah showed that the red shift at the event horizon becomes infinite. Another aspect has been considered by Royzen [R].

All these quite contradictory papers are practically a short history of gravitational physics. It remains to be seen whether the physicist community can convey suitable arguments in order to decide this question.

In the next Section we ourselves bring a proposal derived from the basic laws of relativity, which definitively excludes the possibility to exceed the event horizon, but also shows that the event horizon is unreachable for an infalling observer within a finite proper time.

In further sections the Schwarzschild metric will be based on a geometry with respect to view III which supports our point of view concerning the crossing of the event horizon. There we will discuss the possibility of embedding the geometry into a 5-dimensional flat space. The embedded space is created by the rotation of the Schwarzschild parabola

$$R^2 = 8M(r-2M), \quad r = \frac{R^2 + 16M^2}{8M} \tag{IV.1.12}$$

and is called Flamm's paraboloid. The circle at the throat (or in the 3-dimensional space the sphere at the throat) is the event horizon. R is the co-ordinate of the extra dimension in the embedding space. Since no statements can be made inside the throat with respect to the geometry, no black holes can be described with this geometry. The possibility of embedding is discussed in several subsequent sections in detail whereby the time-like dimension is also treated which has not been elaborately characterized in this Section.

It remains to be clarified how matters stand with the singularity at $r = 2M$. In our view III it is a co-ordinate singularity, i.e. a description error which stems from the commonly used co-ordinates $\{r, \vartheta, \varphi, t\}$. These co-ordinates are called standard Schwarzschild co-ordinates. We will show that by using other co-ordinates the singularity can be eliminated. Particularly useful are the co-ordinates introduced by Einstein [E] and Rosen which use the vertical variable R instead of the horizontal standard variable r. Both the standard variable r and the Einstein-Rosen variable R have advantages and disadvantages.

At the location $r = 2M$ is the vertex of the (lying) Schwarzschild parabola. The parabola is at this position perpendicular to its axis of symmetry, in which the co-ordinate r lies. Therefore, the Schwarzschild parabola cannot be described at the event horizon,

---

de l'Hospital one knows that ε can be arbitrarily small, even infinitely small, but still not 0. Therefore, the fraction has to be reduced. It remains $\lim_{\varepsilon \to 0} 1 = 1$. At the event horizon one obtains ε = 0 and for the speed of light the value 1.

which leads to the occurrence of a supposed singularity. The r-system is well suited to describe the parabola at infinity. There the parabola is parallel to the r-co-ordinate and at $r=\infty$ the metric (IV.1.11) gets flat, as one can see from the factors (IV.1.2). However, substituting (IV.1.12) into (IV.1.1) we obtain

$$ds^2 = \frac{R^2+16M^2}{16M^2}dR^2 + \left(\frac{R^2+16M^2}{8M}\right)^2 d\Omega^2 - \frac{R^2}{R^2+16M^2}dt^2, \qquad \text{(IV.1.13)}$$

wherein $d\Omega^2$ is the abbreviation for the lateral differentials.

Since according to (IV.1.12) $R=0$ denotes the vertex $r=2M$ of the Schwarzschild parabola, the radial arc element has no singularity at this location. Thus, it is shown that the Schwarzschild singularity is a co-ordinate singularity. Furthermore, it is clear that in this co-ordinate system the region $r<2M$ cannot be described. Therefore, there is no room for speculations about black holes.

Less suitable is the R-co-ordinate system for the description of the metric at infinity. The Schwarzschild parabola is perpendicular to the R-co-ordinate at this location, and the metric in this co-ordinate system cannot be brought into a flat form at infinity.

In the early time of gravitational physics one was of opinion that the velocity of light is variable due to the action of the gravitational field. The position-dependent gravitation factor in the time-like arc element and the application of the commonly used co-ordinate method has led to this opinion. If one measures the velocity of light with respect to a reference system, it proves to be a physical constant. Horedt [H] has referred to this problem. He rests upon the analyses on radar measurements by McVittie [M].

IV.2. Free fall

In the preceding Section we have explained in detail the problem that causes the attempt to follow the radial motion of a test particle beneath the event horizon. However, it has to be accepted that r = 2M is the boundary of the geometry which is described by the metric (IV.1.1). The region beneath r = 2M will be discussed later. Formulating the theory of the free fall we will make use of the Lorentz transformation. Once more the question concerning the interpretation of the space curvature will arise and we will find already-known results of the Section describing the de Sitter cosmos. In the static Schwarzschild reference system the velocity of an infalling observer has the form

$$v_m = \left\{ -\sqrt{\frac{2M}{r}}, 0, 0, 0 \right\} \ , \qquad \text{(IV.2.1)}$$

whereby for the sake of the shortness we call the only nonvanishing component of (IV.2.1)

$$v = -\sqrt{\frac{2M}{r}}$$

the velocity of a freely falling object. With the help of this relation the components of a generalized Lorentz transformation can be written down as

$$L^{1}_{1'} = \alpha, \quad L^{1}_{4'} = -i\alpha v, \quad L^{4}_{1'} = i\alpha v, \quad L^{4}_{4'} = \alpha, \quad \alpha = 1 / \sqrt{1 - v^2} \ . \qquad \text{(IV.2.2)}$$

A generalized Lorentz transformation has position-dependent velocity parameters. This kind of transformation refers to an accelerated motion. In order to study the effects of such a transformation on the results of the observers' measurements, we firstly interpret (IV.2.2) as a passive transformation, i.e. we calculate the forces which involve the static observer, as they could be measured from the falling system. One obtains the components of the 4-velocity of the static observer with respect to the falling system

$$u_{m'} = L^{m}_{m'} u_m = L^{4}_{m'} = \{ i\alpha v, 0, 0, \alpha \} \ . \qquad \text{(IV.2.3)}$$

We bear in mind that the Schwarzschild solution can be represented in the context of the surface theory. We require that in accordance with view (III) the curvature of the underlying surface remains unchanged under a Lorentz transformation. Furthermore, the Ricci-rotation coefficients must behave like tensors. Transforming the covariant derivative of a vector (or tensor) derivatives of the transformation matrix arise. The cause is the position-dependence of the components of the transformation matrix (IV.2.2). These new contributions, however, are allocated to the partial derivatives of the considered vector. Thus, we have the same construction which we used for the expanding de Sitter cosmos

$$\Phi_{m'||n'} = L^{m\,n}_{m'n'} \Phi_{m||n} = \left[\Phi_{m'|n'} - L^{s'}_{s} L^{s}_{m'|n'} \Phi_{s'} \right] - A_{n'm'}{}^{s'} \Phi_{s'} \,, \quad A_{n'm'}{}^{s'} = L^{n\,m\,s'}_{n'm's} A_{nm}{}^{s} \ , \qquad \text{(IV.2.4)}$$

whereby the emergent Lorentz term is interposed into the graded covariant derivative

$$\Phi_{m'\underset{1}{||}n'} = \Phi_{m'|n'} - L_{n'm'}{}^{s'} \Phi_{s'} \,, \quad L_{n'm'}{}^{s'} = L^{s'}_{s} L^{s}_{m'|n'} \ . \qquad \text{(IV.2.5)}$$

The graded derivatives of the static 4-bein system vanish as well if measured by the falling observer

$$m_{m'\underset{1}{\|}n'} = m_{m'|n'} - L_{n'm'}{}^{s'}m_{s'} = 0, \quad b_{m'\underset{2}{\|}n'} = b_{m'|n'} - L_{n'm'}{}^{s'}b_{s'} = 0$$

$$c_{m'\underset{3}{\|}n'} = c_{m'|n'} - \left(L_{n'm'}{}^{s'} + B_{n'm'}{}^{s'}\right)c_{s'} = 0 \qquad .$$

$$u_{m'\underset{4}{\|}n'} = u_{m'|n'} - \left(L_{n'm'}{}^{s'} + B_{n'm'}{}^{s'} + C_{n'm'}{}^{s'}\right)u_{s'} = u_{m'|n'} - L_{n'm'}{}^{s'}u_{s'} = 0$$

In the following, the quantity

$$\rho = \sqrt{\frac{2r^3}{M}} \tag{IV.2.6}$$

has to be used in the expression for the force of gravity. It will be interpreted in a later Section as the radius of curvature of a curve on a surface. A Lorentz transformation of the quantities

$$B_n = \left\{\frac{a}{r},0,0,0\right\}, \quad C_n = \left\{\frac{a}{r},\frac{1}{r}\cot\vartheta,0,0\right\}, \quad E_n = \left\{\frac{1}{\rho}\frac{v}{a},0,0,0\right\} \tag{IV.2.7}$$

leads to

$$B_{n'} = \left\{\frac{1}{r},0,0,-iv\frac{1}{r}\right\}, \quad C_{n'} = \left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0,-iv\frac{1}{r}\right\}, \quad E_{n'} = \left\{\alpha\frac{1}{\rho}\frac{v}{a},0,0,-i\alpha v\frac{1}{\rho}\frac{v}{a}\right\}. \tag{IV.2.8}$$

These are the expressions for the curvatures of the space, as they can be measured by the accelerated system. Since the Ricci tensor is invariant under Lorentz transformations $R_{m'n'} = L^{m\,n}_{m'n'}R_{mn} = 0$ one obtains after a short calculation

$$R_{m'n'} = A_{m'n'}{}^{s'}{}_{\underset{1}{\|}s'} - A_{n'\underset{1}{\|}m'} - A_{r'm'}{}^{s'}A_{s'n'}{}^{r'} + A_{m'n'}{}^{s'}A_{s'} \tag{IV.2.9}$$

and from this

$$\begin{aligned} R_{n'm'} = &-\left[B_{m'\underset{2}{\|}n'} + B_{m'}B_{n'}\right] - b_{m'}b_{n'}\left[B^{r'}{}_{\underset{2}{\|}r'} + B^{r'}B_{r'}\right] - \left[C_{m'\underset{3}{\|}n'} + C_{m'}C_{n'}\right] \\ &- c_{m'}c_{n'}\left[C^{r'}{}_{\underset{3}{\|}r'} + C^{r'}C_{r'}\right] + \left[E_{m'\underset{4}{\|}n'} - E_{m'}E_{n'}\right] + u_{m'}u_{n'}\left[E^{r'}{}_{\underset{4}{\|}r'} - E^{r'}E_{r'}\right] = 0 \ , \end{aligned} \tag{IV.2.10}$$

$$B_{[m'\underset{2}{\|}n']} = 0, \quad C_{[m'\underset{3}{\|}n']} = 0, \quad E_{[m'\underset{4}{\|}n']} = 0$$

relations which we already know from (IV.1.6) and (IV.1.8). In particular is

$$E^{r'}{}_{\underset{4}{\|}r'} - E^{r'}E_{r'} = 0 \ . \tag{IV.2.11}$$

It is shown that not only the Einstein field equations are invariant under Lorentz transformations but also that the subequations are form invariant. That can be understood quite well, if one bears in mind that a Lorentz transformation is a pseudo-rotation in a tangent space of a 4-dimensional surface (which will be discussed in a subsequent Section). Such a rotation has no influence on the curvature of this surface.

Next we are concerned with an active Lorentz transformation. We want to determine, which forces affect the falling system and which statements can be made by the falling observer with respect to his own system. We oppose to the static system

$$m_{s'} = \{\alpha,0,0,-i\alpha v\}, \quad b_{s'} \stackrel{*}{=} b_s, \quad c_{s'} \stackrel{*}{=} c_s, \quad u_{s'} = \{i\alpha v,0,0,\alpha\} \tag{IV.2.12}$$

the accelerated system

$$'m_{s'} = \{1,0,0,0\}, \quad 'b_{s'} = b_{s'}, \quad 'c_{s'} = c_{s'}, \quad 'u_{s'} = \{0,0,0,1\} \tag{IV.2.13}$$

and thus, we finally perform a [3+1]-decomposition with respect to the freely falling system. In addition, the Lorentz term has not played an important role within the above considerations, because it was absorbed by the graded derivatives, it is necessary to compute explicitly this term. By differentiating (IV.2.2) one firstly obtains

$$L_{n'm'}{}^{s'} = h_{n'}{}^{s'} L_{m'} - h_{n'm'} L^{s'} \tag{IV.2.14}$$

where h is the unit matrix in the local [1', 4']-subspace and

$$L_{m'} = L_{s'm'}{}^{s'} = \left\{ \alpha^2 v \frac{1}{\rho}, 0, 0, -i\alpha^2 \frac{1}{\rho} \right\} . \tag{IV.2.15}$$

For an active Lorentz transformation one has to reorder (IV.2.4)

$$\Phi_{m'||n'} = L^{m\ n}_{m'n'} \Phi_{m||n} = \Phi_{m'|n'} - \left[A_{n'm'}{}^{s'} + L_{n'm'}{}^{s'} \right] \Phi_{s'} = \Phi_{m'|n'} - 'A_{n'm'}{}^{s'} \Phi_{s'} . \tag{IV.2.16}$$

The new quantity 'A consists of a gravitational and a kinematic term, the latter arises from a Lorentz transformation. This rearrangement affects mainly the time-like part of the Ricci-rotation coefficients. One has

$$\begin{gathered} 'E_{n'm'}{}^{s'} = E_{n'm'}{}^{s'} + L_{n'm'}{}^{s'} \\ E_{n'm'}{}^{s'} = L^{n\,m\,s'}_{n'm's} E_{nm}{}^{s} = -L^{n\,m\,s'}_{n'm's} \left[u_n E_m u^s - u_n u_m E^s \right] \end{gathered} . \tag{IV.2.17}$$

Calculating the last expression we get

$$E_{n'm'}{}^{s'} = -\left[h_{n'}{}^{s'} E_{m'} - h_{n'm'} E^{s'} \right] , \tag{IV.2.18}$$

where $E_{m'}$ are the values in (IV.2.8). With this and with (IV.2.14) the expression (IV.2.17) can be calculated easily. One has

$$'E_{n'm'}{}^{s'} = -\left[h_{n'}{}^{s'}\, 'E_{m'} - h_{n'm'}\, 'E^{s'} \right], \quad 'E_{m'} = \left\{ 0,0,0,\frac{i}{\rho} \right\} . \tag{IV.2.19}$$

The new quantity $'E_{m'}$ has no radial component, the freely falling observer is weightless. The new fourth component we will interpret as tidal force. It is striking that the new quantity $L_{1'}$ in (IV.2.15) has the same value as the force of gravity $E_{1'}$ in (IV.2.8). However, the two forces have different sources. The force of gravity E has been deduced from the fourth vector of the tetrad field

$$E_{1'} = -\frac{1}{a} a_{|1'} .$$

Thus, E is a property of the space, while the new quantity L arises from the Lorentz factor of the motion of free fall

$$L_{1'} = \frac{1}{\alpha} \alpha_{|1'} .$$

It is the result of a pseudo-rotation in the tangent space and the consequence of a structure *on* a 4-dimensional surface which we will consult for the explanation of the Schwarzschild geometry. Although the Lorentz transformation does not have any influence on the space curvature it is nevertheless determined by it. The Lorentz factor is *numerically equivalent* to the reciprocal of the redshift factor

$$\alpha = 1/a .$$

Thus, the speed of a freely falling object is determined in each point of the space. The laws of the free fall are regulated by the space curvature. From[37]

$$'u_{m'\|s'}\,'u^{s'} = -\left(E_{s'm'}{}^{r'} - L_{s'm'}{}^{r'}\right)'u_{r'}\,'u^{s'} = E_{\underline{m}'} - L_{\underline{m}'} = 0 \qquad \text{(IV.2.20)}$$

it is evident that a freely falling observer is not exposed to acceleration forces. Following the Einstein gedankenexperiment one can say that an observer, who is in a closed freely falling elevator cannot notice the effect of the force of gravity. The force of gravity E survives, but is *nullified* by the quantity L, because E and L enter into the theory with the opposite sign. The gravitation effects coming from the space curvature remain intact under a Lorentz transformation. This can be verified by Lorentz-transforming the Ricci tensor and extracting the emergent Lorentz terms. They decouple from the field equations and they vanish separately

$$L_{m'n'}{}^{s'}{}_{|s'} - L_{s'n'}{}^{s'}{}_{|m'} - L_{s'm'}{}^{r'}L_{r'n'}{}^{s'} + L_{m'n'}{}^{s'}L_{r's'}{}^{r'} + 2A_{[m's']}{}^{r'}L_{r'n'}{}^{s'} = 0\,. \qquad \text{(IV.2.21)}$$

Applied to our problem, we see that the last term of this relation vanishes and we are left with

$$L^{s'}{}_{|s'} + L^{s'}L_{s'} = 0\ , \qquad \text{(IV.2.22)}$$

a relation which is satisfied with (IV.2.15).

Only the Ricci tensor of the static geometry remains and still describes the same curvature. Liebscher [L] and Bräuer [B] treated in detail Lorentzian and non-Lorentzian transformations and their significance for the gravitation theory.

In addition, the force of gravity cannot be experienced by a freely falling observer coming from the infinite, it would be too simple to assume that no forces act in a spherically symmetrical field, as is represented by the Schwarzschild model. Tidal forces act perceptibly on a freely falling observer if he is exposed to a sufficiently strong spherically symmetrical gravitational field. An astronaut, who jumps off to a massive neutron star, ends his day a long time earlier than the impact occurs. Because of the potential difference of the gravity field his legs are more attracted than his head. Thus, he is stretched. Since the gravitational field lines converge radially and run to the gravitation center, his legs are also more strongly squeezed than his head.

In order to formulate this descriptive illustration also mathematically, we consult the new components of the quantities which result from the Lorentz transformation and which have not been treated yet. The quantities

$$B_{4'} = -\frac{iv}{r}, \quad C_{4'} = -\frac{iv}{r}, \quad Q_{4'} = -\frac{i}{\rho} \qquad \text{(IV.2.23)}$$

are the components of the Ricci-rotation coefficients. After renaming them and omitting the primes at the indices, these quantities are summarized to a two-rank tensor

$$Q_{11} = -\frac{i}{\rho}, \quad Q_{22} = -\frac{iv}{r}, \quad Q_{33} = -\frac{iv}{r}\ . \qquad \text{(IV.2.24)}$$

The new Qs are the second fundamental forms of a three-dimensional space-like surface, which shrinks to the gravitation center and which takes part in the motion of the freely falling observers. They are physically interpreted as tidal forces and they satisfy the relations

[37] $\underline{m}'$=1',2',3'

$$Q_{mn} = {}'u_{m||n}, \quad Q_{mn}{}'u^m = 0, \quad Q_{mn}{}'u^n = 0, \quad Q_{[mn]} = 0 \ . \tag{IV.2.25}$$

The 4-velocities of the freely falling observers are the rigging vectors (normal vectors) of this shrinking surface. To work out the effects of these tidal forces ^{B}, they are isolated from the Ricci-rotation coefficients. What is left is the purely space-like contribution *A

$$A_{mn}{}^s = {}^*A_{mn}{}^s + Q_m{}^s{}'u_n - Q_{mn}{}'u^s \ . \tag{IV.2.26}$$

Defining therewith a space-like 3-dimensional covariant derivative[38]

notation:

$$\begin{gathered}\Phi_{\alpha\wedge\beta} = \Phi_{\alpha|\beta} - {}^*A_{\beta\alpha}{}^\gamma\Phi_\gamma, \quad {}^*A_{\beta\alpha}{}^\gamma = B_{\beta\alpha}{}^\gamma + C_{\beta\alpha}{}^\gamma \\ B_{\beta\alpha}{}^\gamma = b_\beta B_\alpha b^\gamma - b_\beta b_\alpha B^\gamma, \quad C_{\beta\alpha}{}^\gamma = c_\beta C_\alpha c^\gamma - c_\beta c_\alpha C^\gamma\end{gathered} \tag{IV.2.27}$$

with the 3-dimensional components of the field strengths

$$B_\alpha = \left\{\frac{1}{r}, 0, 0\right\}, \quad C_\alpha = \left\{\frac{1}{r}, \frac{1}{r}\cot\vartheta, 0\right\} \tag{IV.2.28}$$

one recognizes that these components are quantities of the flat geometry, as they have already been described in the introduction (I.3.18). However, the flatness of the space is only shammed. One can write the first components of quantities (IV.2.28) in more detail as

$$B_1 = \alpha a\frac{1}{r}, \quad C_1 = \alpha a\frac{1}{r} \ . \tag{IV.2.29}$$

Again, the Lorentz factor α compensates the space curvature a in accordance with (IV.1.2). From the Einstein field equations one obtains the relations

$$\begin{gathered}B_{\alpha|\beta} + B_\alpha B_\beta = 0, \quad B^\gamma{}_{|\gamma} + B^\gamma B_\gamma = 0 \\ C_{\alpha|\beta} - B_{\beta\alpha}{}^\gamma C_\gamma + C_\alpha C_\beta = 0, \quad C^\gamma{}_{|\gamma} + B_\gamma C^\gamma + C_\gamma C^\gamma = 0 \ . \\ B_{\beta|4} + Q_{22}B_\beta = 0, \quad C_{\beta|4} + Q_{33}C_\beta = 0\end{gathered} \tag{IV.2.30}$$

For the rest of the 4-dimensional Ricci tensor only remains

$$\begin{aligned} R_{\underline{m}\underline{n}} &= -\left[Q_{mn\wedge s}{}'u^s + Q_{mn}Q_s{}^s\right] = 0 \\ R_{\underline{m}n}{}'u^n &= -\left[Q_s{}^s{}_{\wedge\underline{m}} - Q_m{}^s{}_{\wedge s}\right] \qquad = 0 \ . \\ R_{mn}{}'u^m{}'u^n &= -\left[Q_s{}^s{}_{\wedge m}{}'u^m + Q_{rs}Q^{rs}\right] = 0 \end{aligned} \tag{IV.2.31}$$

These are the equations which describe the effects of the tidal forces Q. In particular, the second equation (IV.2.31) is the contracted Codazzi equation for the second fundamental forms of the shrinking surface. It is easy to show that not only the spatial Ricci tensor $^*R_{\alpha\beta}$ but also the assigned 3-dimensional Riemann tensor $^*R_{\alpha\beta\gamma}{}^\delta$ vanishes. In accordance with view (I) the 3-dimensional space could be regarded as being flat and a Lorentz transformation would be able to change the curvature of the space. However, we have explained in detail that the Lorentz transformation does not change the space curvature, but that the curvature is hidden to a freely falling observer.

In the next sections we again deal with the free fall. We investigate an observer infalling from an arbitrary position. We deal in detail with the behavior of the observer at

[38] α=1,2,3

the event horizon and we discuss the controversial question of the possibility of penetration of the event horizon.

IV.3. Free fall from infinity

In recent decades, the problem of free fall has repeatedly been treated in the literature. While general agreement exists that an observer who comes from infinity would reach the speed of light at the event horizon, there are controversial views regarding the speed of an infalling observer who falls from a finite position towards the center of gravitation. We attack the problem and we will show that an observer coming from infinity or from any other position would attain the speed of light at the event horizon and therefore will take an infinitely long proper time. To manage this problem one has to apply Einstein's addition law of velocities and the velocity formula for the free fall in the Schwarzschild field. We [B] have investigated this problem in some previous papers.

We are confronted with the actual problem that initially only the velocity of an observer B'' who comes from infinity is known. It is determined by the Schwarzschild geometry by

$$v = v(r) = -\sqrt{\frac{2M}{r}} \tag{IV.3.1}$$

and therefore is a geometric quantity. To simplify the following consideration, we reduce the problem to a 2-dimensional one. We suppress the ϑ- and φ-dimensions and we denote the radial co-ordinate with x.

The observer B'' coming from the infinite does not change his position in the comoving system. Therefore, one has

$$\frac{dx''}{dT''} = 0, \qquad x'' = \text{const.} \; , \tag{IV.3.2}$$

where T'' is the proper time of B''. In view of the system B, which is in rest, his velocity is

$$\frac{dx}{dT} = v \; , \tag{IV.3.3}$$

where we now have used the proper time T of the static system.

If we include the well-known relation

$$\frac{dT}{dT''} = \alpha \tag{IV.3.4}$$

with α as the Lorentz factor of the transformation $x \leftrightarrow x''$ and if we take into account the relation $dx = \alpha dr$, we have

$$v = \frac{dr}{dT''}, \quad dT'' = \frac{1}{v} dr \; . \tag{IV.3.5}$$

The integral of dT''

$$T''(r) = -\int \sqrt{\frac{r}{2M}} dr = -\frac{1}{3}\sqrt{\frac{2r^3}{M}} \tag{IV.3.6}$$

is a well-known expression in the literature which graphically shows a curve being zero at $r = 0$ and increasing to infinity for $r \to \infty$. It determines the time the observer has to take in order to reach a point r, starting at $r = 0$. Since there is invariance under time reversal, one obtains for the fall time a function that is infinitely high at $r = 0$.

It is noticeable that the curve $T''(r)$ crosses the event horizon, although any incoming object would reach the speed of light at this location. The fall velocity $v(r) = -\sqrt{2M/r}$ is mathematically continued into the inner region $0 < r < 2M$ of the Schwarzschild solution. Thus, it is mathematically quite correct that the integral (IV.3.6) also covers the inner region. Due to our geometrical interpretation of the gravitation theory a penetration of the event horizon is not possible. The circle at the throat of Flamm's paraboloid is the boundary of the geometry and beneath it no statements can be made.

This raises the question whether the integral (IV.3.6) can be corrected in such a way that an observer incoming from infinity requires an infinitely long proper time to reach the event horizon. In fact, the problem can be solved easily, if one calculates the integral (IV.3.6) within limits. From[39]

$$T''(r) = \int_{2M}^{r} \sqrt{\frac{r}{2M}}\, dr = \frac{1}{3}\sqrt{\frac{2r^3}{M}} - \frac{1}{3} 4M \qquad \text{(IV.3.7)}$$

one obtains the rise time for an observer from 2M to infinity, whereas

$$T''(2M) = 0, \quad \lim_{r\to\infty} T''(r) = \infty \qquad \text{(IV.3.8)}$$

is valid. Due to the invariance under time reversal, an observer who comes from the infinite reaches the event horizon only after infinite proper time. The two functions (IV.3.6) and (IV.3.7) are depicted in Fig. IV.1.

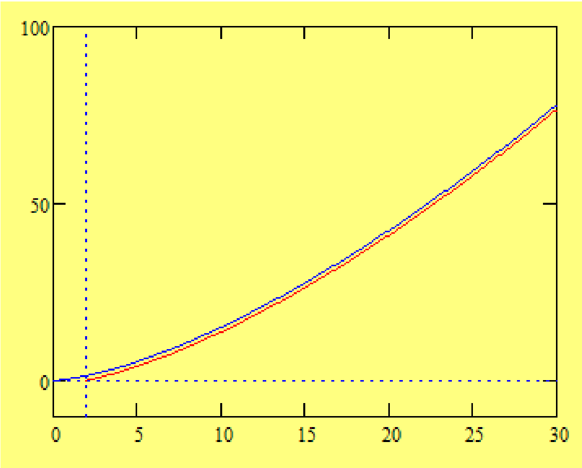

Fig. IV.1

Since the event horizon is considered to be unreachable, neither for a falling observer nor for the surface of a collapsing stellar object, the existence of black holes is questioned. One might argue that the restriction of the integral (IV.3.7) to the range $[2M \le r \le \infty]$ is arbitrary. To justify this practice, we calculate the problem once again with other variables viz with those variables that can easily be taken from the Schwarzschild geometry, but which are defined from the beginning above the event horizon. Furthermore, we extend the investigation to the case where the incoming object does not come from infinity, but from any finite position.

39 For reasons of clarity we use here the value of the fall velocity.

The fall velocity of an object coming from infinity is a quantity closely associated with the Schwarzschild geometry. It is related to the angle of ascent of the Schwarzschild parabola by

$$v = \sin\varepsilon = -\sqrt{2M/r} \ . \tag{IV.3.9}$$

By rearranging and differentiating this relation we obtain

$$r = \frac{2M}{\sin^2\varepsilon}, \quad dr = -\frac{4M}{\sin^3\varepsilon}\cos\varepsilon\, d\varepsilon \ . \tag{IV.3.10}$$

The geometry at infinity is flat, because of $\varepsilon = 0$ at this position. The tangent at the vertex of the Schwarzschild parabola is normal to r-axis, i.e. $\varepsilon = -\pi/2$. From

$$dT'' = \frac{1}{v}dr = -\frac{4M}{\sin^4\varepsilon}\cos\varepsilon\, d\varepsilon \tag{IV.3.11}$$

one obtains with the lower limit $-\pi/2$ the function[40]

$$T''(\varepsilon) = \frac{4M}{3\sin^3\varepsilon} + \frac{4M}{3} \ , \tag{IV.3.12}$$

which describes the rise time. Due to the invariance under time reversal, one obtains the curve for the fall time.

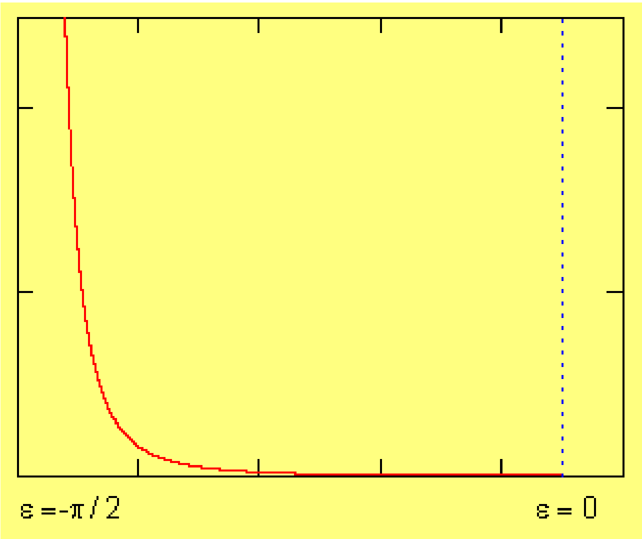

Fig. IV.2

If the fall velocity is parameterized with the angle of ascent of the Schwarzschild parabola one obtains for the time function a curve starting with zero value at $\varepsilon = 0$ and increasing to the infinite at the event horizon. The choice of this variable has the advantage that the spatial infinity can be represented graphically. The same applies to isotropic co-ordinates, which will be treated in the Section ‘Wormholes’ in more detail.

From the nonlinear transformation

[40] The first term is negative because ε is negative.

$$r=\left(1+\frac{M}{2\bar{r}}\right)^2\bar{r} \tag{IV.3.13}$$

one obtains

$$dr=\left(1-\frac{M^2}{4\bar{r}^2}\right)d\bar{r}\ . \tag{IV.3.14}$$

The new isotropic co-ordinate $\bar{r}$ describes both branches of the Schwarzschild parabola, by running through them. It starts at $\bar{r}=0$ at infinity on the lower branch of the parabola and runs into the infinite of the upper branch ($\bar{r}=\infty$). The co-ordinate has a minimum at $\bar{r}=M/2$ which corresponds to $r=2M$, the event horizon. Therefore isotropic co-ordinates describe only the outer region $r\geq 2M$ of the Schwarzschild geometry, but in a twofold manner. The fall velocity is

$$v(\bar{r})=-\frac{1}{1+\frac{M}{2\bar{r}}}\sqrt{\frac{2M}{\bar{r}}}\ . \tag{IV.3.15}$$

Taking into account (IV.3.5) together with (IV.3.14) and (IV.3.15) one arrives at

$$dT''=-\frac{8\bar{r}^3+4M\bar{r}^2-2M^2\bar{r}-M^3}{8\bar{r}^3}\sqrt{\frac{\bar{r}}{2M}}d\bar{r}\ . \tag{IV.3.16}$$

If we take the positive value of v for the sake of simplicity, the integral of (IV.3.16) provides the function

$$f(r)=\frac{1}{4\sqrt{2M}}\left[\frac{8}{3}\sqrt{\bar{r}^3}+4M\sqrt{\bar{r}}+\frac{2M^2}{\sqrt{\bar{r}}}+\frac{M^3}{3\sqrt{\bar{r}^3}}\right]. \tag{IV.3.17}$$

Within the limits of the lower branch one has

$$T''(\bar{r})=f(\bar{r})-f\left(\frac{M}{2}\right) \tag{IV.3.18}$$

with $\lim\limits_{\bar{r}\to 0}T''(\bar{r})=\infty$ and $T''\left(\frac{M}{2}\right)=0$. Time symmetry attributes to the function for the fall time, which can be seen from Fig. IV.3 ($M=2$).

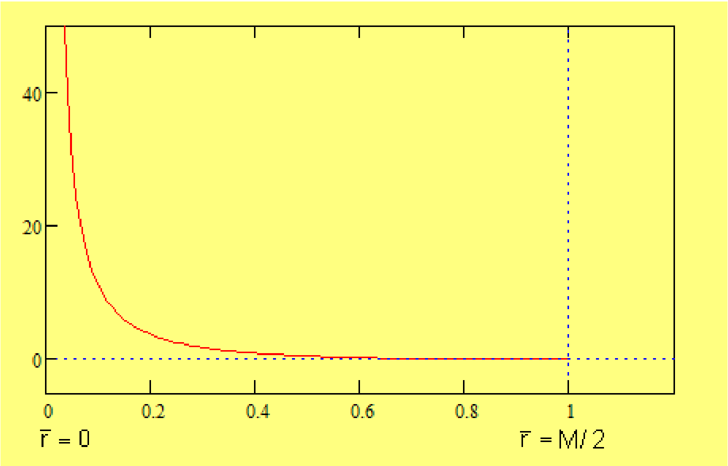

Fig. IV.3

One obtains the same results with Einstein-Rosen co-ordinates and also with the Lorentz angle as parameter. The latter problem will be executed in one of the next sections.

By using different variables, we have shown in this Section that an observer infalling from the infinite can reach the event horizon only after infinite proper time. Whether this also applies to objects coming from any finite position, will be examined in the following.

Since no massive object can reach the speed of light according to the special theory of relativity, the event horizon for an observer is inaccessible and therefore not traversable, as mentioned earlier. If there could be an observer at the event horizon, an infinite acceleration would be necessary to take off from there. In addition to these physical arguments, there is a logical argument, which forbids an observer to start from the event horizon. His future would be unlimited undetermined. He would be able to stop at any point r_0 with the same (i.e. infinitely high) initial acceleration.

With these considerations we have dismissed the possibility that infalling objects can cross the Schwarzschild radius and we are far away from the hypothesis of black holes. The constantly growing literature in which alternatives are discussed shows that several other authors face the existence of black holes critically. We refer to the models for Q-stars, gravastars, and holostars. In the electronic journal of Andreas Müller [M] one can find a simple representation of these ideas. Narlikar [N] and Padmanabhan mention conceptual difficulties concerning black holes. One of the most convincing alternatives stems from Mitra [M]. His models, the ECOs (eternally collapsing objects) and MECOs (magnetospheric eternally collapsing objects) are closely related to the black-hole scenario. Stars contract and their surfaces shrink asymptotically to the event horizon and reach it after an infinitely long time. In this way stellar objects develop which guide light into a course around the gravitation center due to the large gravitational attraction. Thus, they have a negligible luminosity. ECOs and MECOs are in certain respects similar to black holes, however, they do not possess their contradictory properties. The two astronomers Leiter [L] and Robertson [R] showed that one of the oldest candidates for a black hole has the very properties of an ECO computed by Mitra[41].

[41] While Mitra's first paper was published discreditable arguments were raised by the physicist community. Mitra was personally attacked and also abandoned by his teammates. It is unfortunate that some scientists miss objectivity in discussions.

IV.4. Free fall from an arbitrary position

The velocity of an observer who is infalling from an arbitrary position can be determined only circuitously. For this purpose, we perform the following considerations:

An object coming from infinity has at an arbitrary position r_0 the velocity[42] $v_0 = -\sqrt{2M/r_0}$. Another object is released from r_0 at the very moment the first object is passing the point r_0. In this moment the difference of their fall velocity is simply v_0. However, the difference decreases according to Einstein's composition law of velocities. With regard to the static Schwarzschild system the speed of the second object is calculated with respect to the relative velocity of the first object

$$v' = v(r, r_0) = \frac{-\sqrt{\frac{2M}{r}} - \left(-\sqrt{\frac{2M}{r_0}}\right)}{1 - \sqrt{\frac{2M}{r}}\sqrt{\frac{2M}{r_0}}} \ . \qquad \text{(IV.4.1)}$$

The latter is at the starting position $v(r_0, r_0) = 0$, at the event horizon $v(2M, r_0) = -1$. Fig. IV.4 shows some examples. The top curve corresponds to the observer who comes from infinity.

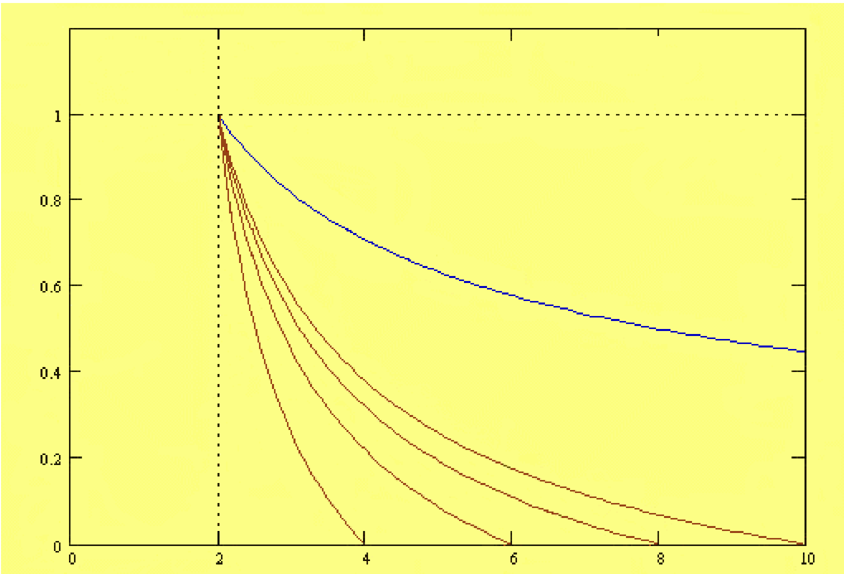

Fig. IV.4

To handle the velocity relations we introduce three reference frames. The first is called B being in rest in the Schwarzschild field, the second is the reference system B' which accompanies an observer with the velocity v', and the third B'' is coming in free fall from infinity with the velocity v. The systems are connected by the Lorentz relations

[42] One obtains the expression for the speed of an infalling particle by the integration of the radial equation of motion (Mitra [M]). The minus sign suggests that the motion is directed inwards.

$$v' = \frac{v - v_0}{1 - v v_0}, \qquad v = \frac{v' + v_0}{1 + v' v_0}, \qquad v_0 = \frac{v - v'}{1 - v v'} \ , \tag{IV.4.2}$$

$$\alpha' = \alpha\alpha_0 \left(1 - v v_0\right), \quad \alpha = \alpha' \alpha_0 \left(1 + v' v_0\right), \quad \alpha_0 = \alpha' \alpha \left(1 - v' v\right) \ , \tag{IV.4.3}$$

$$\alpha' v' = \alpha\alpha_0 \left(v - v_0\right), \quad \alpha v = \alpha' \alpha_0 \left(v' + v_0\right), \quad \alpha_0 v_0 = \alpha\alpha' \left(v - v'\right) \ . \tag{IV.4.4}$$

The quantities α are the Lorentz factors associated with the relative velocities. With the help of a table beneath we provide other useful relations.

The Lorentz transformations have the form

$$\begin{array}{llll} L_1^{1'} = \alpha', & L_1^{4'} = -i\alpha' v', & L_4^{1'} = i\alpha' v', & L_4^{4'} = \alpha' \\ L_{1'}^{1''} = \alpha_0, & L_{1'}^{4''} = -i\alpha_0 v_0, & L_{4'}^{1''} = i\alpha_0 v_0, & L_{4'}^{4''} = \alpha_0 \ . \\ L_1^{1''} = \alpha, & L_1^{4''} = -i\alpha v, & L_4^{1''} = i\alpha v, & L_4^{4''} = \alpha \end{array} \tag{IV.4.5}$$

For the table, we calculate the relative velocities and proper times of the above-mentioned three observers. The position of an observer coming from infinity does not change with respect to the comoving observer system. For B" one has $x'' = \text{const.}$, $dx'' = 0$.

Next, we write $dx^{4''} = idT''$, $dx^{4'} = idT'$, where dT'' and dT' are the proper times of the observers with respect to B" and B'. From the Lorentz transformation

$$dx^{1''} = L_{1'}^{1''} dx^{1'} + L_{4'}^{1''} dx^{4'}, \quad dx^{4''} = L_{1'}^{4''} dx^{1'} + L_{4'}^{4''} dx^{4'}$$

we infer the relation $0 = \alpha_0 dx' + i\alpha_0 v_0 dT'$ which leads to

$$\frac{dx'}{dT'} = v_0 \, .$$

In addition, one has with $idT'' = -i\alpha_0 v_0 dx' + \alpha_0 idT'$ and with the result obtained above

$$dT'' = \alpha_0 \left(-v_0^2 + 1\right) dT'$$

and finally

$$\frac{dT'}{dT''} = \alpha_0 \, .$$

Similar considerations can be made for all rows of the table and we obtain the elementary relations listed in it of which we frequently make use. With these relations and the velocity definition (IV.4.1) we are prepared to study the free fall of objects falling from a finite position.

<table>
<tr><th colspan="7">I. $x''=\text{const.}$</th></tr>
<tr><td>systems</td><td>L</td><td>transformations</td><td>rel. velocities</td><td>phys. time</td><td>rel. vel. of</td><td>meas. in</td></tr>
<tr><td>1. B'' || B'</td><td>$L(v_0)$</td><td>$dx^{m''}=L^{m''}_{m'}dx^{m'}$</td><td>$\frac{dx'}{dT'}=v_0$</td><td>$\frac{dT'}{dT''}=\alpha_0$</td><td>B'' a. B'</td><td>B'</td></tr>
<tr><td>2. B'' || B</td><td>$L(v)$</td><td>$dx^{m''}=L^{m''}_{m}dx^{m}$</td><td>$\frac{dx}{dT}=v$</td><td>$\frac{dT}{dT''}=\alpha$</td><td>B'' a. B</td><td>B</td></tr>
<tr><td>3. B || B'</td><td>$L(v')$</td><td>$dx^{m}=L^{m}_{m'}dx^{m'}$</td><td>$\frac{dx''}{dT''}=0$</td><td>$\frac{dT}{dT'}=\frac{\alpha}{\alpha_0}$</td><td></td><td></td></tr>
<tr><th colspan="7">II. $x'=\text{const.}$</th></tr>
<tr><td>systems</td><td>L</td><td>transformations</td><td>rel. velocities</td><td>phys. time</td><td>rel. vel. of</td><td>meas. in</td></tr>
<tr><td>1. B' || B</td><td>$L(v')$</td><td>$dx^{m'}=L^{m'}_{m}dx^{m}$</td><td>$\frac{dx}{dT}=v'$</td><td>$\frac{dT}{dT'}=\alpha'$</td><td>B' a. B</td><td>B</td></tr>
<tr><td>2. B' || B''</td><td>$L(v_0)$</td><td>$dx^{m'}=L^{m'}_{m''}dx^{m''}$</td><td>$\frac{dx''}{dT''}=-v_0$</td><td>$\frac{dT''}{dT'}=\alpha_0$</td><td>B' a. B''</td><td>B''</td></tr>
<tr><td>3. B'' || B</td><td>$L(v)$</td><td>$dx^{m''}=L^{m''}_{m}dx^{m}$</td><td>$\frac{dx'}{dT'}=0$</td><td>$\frac{dT''}{dT}=\frac{\alpha_0}{\alpha'}$</td><td></td><td></td></tr>
<tr><th colspan="7">III. $x=\text{const.}$</th></tr>
<tr><td>systems</td><td>L</td><td>transformations</td><td>rel. velocities</td><td>phys. time</td><td>rel. vel. of</td><td>meas. in</td></tr>
<tr><td>1. B || B'</td><td>$L(v')$</td><td>$dx^{m}=L^{m}_{m'}dx^{m'}$</td><td>$\frac{dx'}{dT'}=-v'$</td><td>$\frac{dT'}{dT}=\alpha'$</td><td>B a. B'</td><td>B'</td></tr>
<tr><td>2. B || B''</td><td>$L(v)$</td><td>$dx^{m}=L^{m}_{m''}dx^{m''}$</td><td>$\frac{dx''}{dT''}=-v$</td><td>$\frac{dT''}{dT}=\alpha$</td><td>B a. B''</td><td>B''</td></tr>
<tr><td>3. B' || B''</td><td>$L(v_0)$</td><td>$dx^{m'}=L^{m'}_{m''}dx^{m''}$</td><td>$\frac{dx}{dT}=0$</td><td>$\frac{dT'}{dT''}=\frac{\alpha'}{\alpha}$</td><td></td><td></td></tr>
</table>

Table 1

IV.5. Schwarzschild-standard co-ordinates

In Section IV.3 we have examined the free fall from infinity in several respects. Let us now consider the more complicated case in which an observer can fall away from any position. In the last Section we have determined the required velocity definitions and we have prepared the essential mathematics.

With the help of the table of Section IV.4, we refer to an observer B' who falls in from an arbitrary position r_0, and we gather the relations

$$\frac{dx}{dT} = v', \quad \frac{dT}{dT'} = \alpha', \quad x' = \text{const.} \;. \tag{IV.5.1}$$

With $dx = \alpha dr$ we write

$$\frac{\alpha dr}{dT'} = \alpha' v' \;,$$

and we bear in mind that the metric coefficient α is identical with the Lorentz factor of an observer incoming from the infinite. With (IV.4.4) one has

$$dT' = \frac{\alpha}{\alpha' v'} dr = \frac{1}{\alpha_0 \left(v - v_0\right)} dr \;. \tag{IV.5.2}$$

The integration of this expression leads to an integral of the type

$$\int \frac{\sqrt{x}}{\sqrt{x} - 1} dx = x + 2\sqrt{x} + 2\ln(1 - \sqrt{x}), \quad x < 1, \quad \lim_{x \to 1} \int \frac{\sqrt{x}}{\sqrt{x} - 1} dx = \infty \tag{IV.5.3}$$

and with $x = \frac{r}{r_0}$ to the function

$$f\left(r, r_0\right) = -\sqrt{\frac{r_0}{2M} - 1}\left[r - 2\sqrt{r_0 r} + 2 r_0 \ln\left(1 - \sqrt{\frac{r}{r_0}}\right)\right] \tag{IV.5.4}$$

which describes the rise time, and which is infinite at r_0. As an observer ascending from the event horizon cannot be realized physically, we will mirror the function. Substituting $r \to r_0 - r$ into the function (IV.5.4), we measure the instantaneous fall distance beginning at r_0. With $r \to r - 2M$, $\bar{r}_0 \to r_0 - 2M$ we restrict the integration to the range $\left[2M, r_0\right]$. Thus, we get from (IV.5.4) a function which is plotted in Fig. IV.11. One can see that for the starting point and end point of the downward motion the relations

$$T'\left(r_0 - r, r_0 - 2M\right)_{|r = r_0} = 0, \quad \lim_{r \to 2M} T'\left(r_0 - r, r_0 - 2M\right) = \infty \tag{IV.5.5}$$

are valid. A freely falling observer can reach the event horizon only after an infinite proper time.

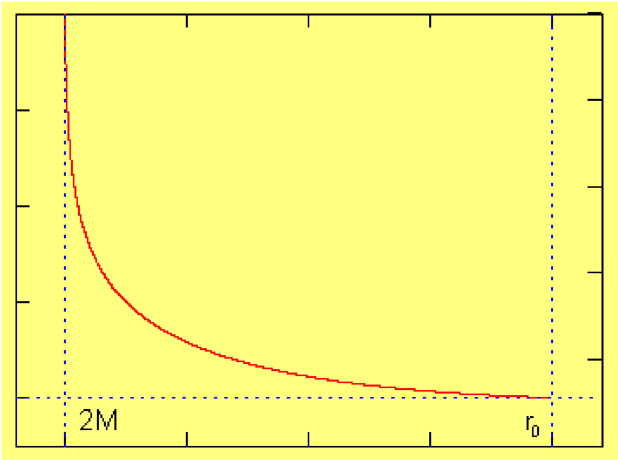

Fig. IV.5

In order to free ourselves from the suspicion that we have restricted the variable r arbitrarily in order to exclude the region $r < 2M$, we carry out the same calculations again with other variables. We use variables which in principle cannot describe the inner region of the Schwarzschild geometry.

IV.6. Einstein-Rosen-coordinates

Since physics forces us to exclude the inner region from the integration process, we use instead the standard Schwarzschild co-ordinates, the Einstein-Rosen co-ordinates[43] [E]. From the equation of the Schwarzschild parabola

$$R^2 = 8M(r-2M)$$

results

$$r = \frac{R^2 + 16M^2}{8M}, \tag{IV.6.1}$$

wherewith we obtain the singularity-free line element

$$ds^2 = \frac{R^2 + 16M^2}{16M^2} dR^2 + \left(\frac{R^2+16M^2}{8M}\right)^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) - \frac{R^2}{R^2+16M^2} dt^2 \ . \tag{IV.6.2}$$

$R = 0$ is the vertex of the Schwarzschild parabola and corresponds to $r = 2M$. For $R = 0$ one obtains $dx = dR$, the tangent of the Schwarzschild parabola at the vertex. On principle R cannot take values in the inner region of the Schwarzschild metric.

It should be noted that neither r nor R corresponds to a physical distance. The physical distance from the horizon we get from

$$r^* = \int_{2M}^{r} dx = \int_{2M}^{r} \alpha dr = r\sqrt{1-\frac{2M}{r}} + M \ln\left(\frac{1+\sqrt{1-\frac{2M}{r}}}{1-\sqrt{1-\frac{2M}{r}}}\right) . \tag{IV.6.3}$$

This is the rectification formula for the Schwarzschild parabola. r and R are both auxiliary variables which define Cartesian co-ordinates in the higher dimensional embedding space. Both can be used to formulate the theory of Schwarzschild. For the velocity of free fall results

$$v(R) = -\frac{4M}{\sqrt{R^2+16M^2}} \tag{IV.6.4}$$

and has the value -1 for $R = 0$. With

$$dr = \frac{R}{4M} dR$$

one first obtains by integration of $dT'' = \frac{1}{v} dr$

$$f(R) = \int \frac{R}{16M^2}\sqrt{R^2+16M^2}\, dR = \frac{1}{48M^2}\sqrt{\left(R^2+16M^2\right)^3} + C \ . \tag{IV.6.5}$$

If we choose the integration constant in such a way that we obtain for $R = 0$ also $T'' = 0$ we finally have

$$T''(R) = f(R) - f(0) \ , \tag{IV.6.6}$$

[43] The co-ordinates we use differ from the original Einstein-Rosen co-ordinates by the factor $\sqrt{8M}$.

a curve, starting at $R = 0$, $(r = 2M)$ and growing to the infinite. Time reversal results in the image that an observer incoming with velocity v(R) from infinity, takes an infinit time to reach the event horizon which he can never cross.

It is more difficult to compute the time function for observers that are incoming from an arbitrary position r_0. We have to use the relations

$$\frac{dx}{dT} = v', \quad \frac{dT}{dT'} = \alpha', \quad x' = \text{const.} \ . \tag{IV.6.7}$$

With

$$\frac{\alpha dr}{dT'} = \alpha' v'$$

one has

$$dT' = \frac{\alpha}{\alpha' v'} dr = \frac{1}{\alpha_0 \left(v - v_0 \right)} dr \ . \tag{IV.6.8}$$

Integrating this expression one cannot prevent that r runs beneath the event horizon. Therefore, we recall the Einstein-Rosen co-ordinates. One has

$$dT' = \frac{R_0}{\sqrt{R_0^2 + 16M^2}} \frac{1}{\dfrac{4M}{\sqrt{R_0^2 + 16M^2}} - \dfrac{4M}{\sqrt{R^2 + 16M^2}}} \frac{R}{4M} dR \ .$$

After some rearrangement one obtains an integral of the type

$$\int \frac{\sqrt{x}}{\sqrt{x} - 1} dx = x + 2\sqrt{x} + 2\ln(1 - \sqrt{x}), \quad x < 1, \quad \lim_{x \to 1} \int \frac{\sqrt{x}}{\sqrt{x} - 1} dx = \infty \ .$$

For $x = \dfrac{R^2 + 16M^2}{R_0^2 + 16M^2}$ one gets the rise time $f(R)$. Starting up from $R = 0$ and ending at R_0 it increases to infinity. In this case it is simple to mirror the function in such a way that an observer starting from R_0 reaches the event horizon at $R = 0$ after an infinitely long time. Replacing in the time function the variable R by $R_0 - R$ the observer starts at $R = R_0$ and passes through the fall distance at $R = 0$. One has

$$\begin{aligned} f(R, R_0) &= \frac{R_0}{32M^2} \left(R_0^2 + 16M^2 \right) \times \\ &\times \left[\frac{(R_0 - R)^2 + 16M^2}{R_0^2 + 16M^2} + 2\sqrt{\frac{(R_0 - R)^2 + 16M^2}{R_0^2 + 16M^2}} + 2\ln\left(1 - \sqrt{\frac{(R_0 - R)^2 + 16M^2}{R_0^2 + 16M^2}} \right) \right] + C \ . \end{aligned} \tag{IV.6.9}$$

After a suitable choice of the integration constant

$$T'(R, R_0) = f(R, R_0) - f(R_0, R_0) \tag{IV.6.10}$$

one gains the following plot

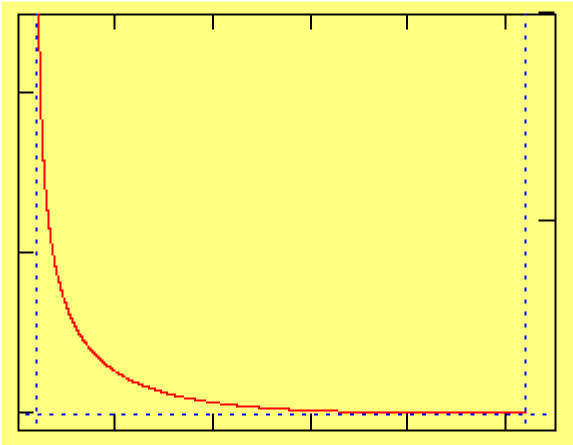

Fig. IV.6

In Fig. IV.6 marks the left vertical line the event horizon. It turns out that no object infalling from a finite or infinite position can reach the event horizon in finite proper time.

IV.7. Calculation with the angle of ascent

We extend the method discussed in Section IV.3, where we have used the angle of ascent of the Schwarzschild parabola as parameter. For an observer coming from the infinite, we use for the fall velocity the quantity (IV.4.1). For the velocity of an observer who comes from infinity and has reached the position r_0 we find the expression

$$v_0 = \sin\varepsilon_0 = -\sqrt{2M/r_0} \ , \tag{IV.7.1}$$

where ε_0 is a negative angle.

If we insert this into (IV.4.1) we have

$$v' = v(\varepsilon,\varepsilon_0) = \frac{\sin\varepsilon - \sin\varepsilon_0}{1-\sin\varepsilon\sin\varepsilon_0} \ . \tag{IV.7.2}$$

T' is the proper time of a freely falling observer who comes from a finite position. The Lorentz factor for the constant velocity v_0 is

$$\alpha_0 = \frac{1}{\sqrt{1-\sin^2\varepsilon_0}} = \frac{1}{\cos\varepsilon_0} \ . \tag{IV.7.3}$$

Then (IV.5.2) becomes with (IV.3.10)

$$dT' = \frac{\cos\varepsilon_0}{\sin\varepsilon - \sin\varepsilon_0}\left(-\frac{4M}{\sin^3\varepsilon}\cos\varepsilon\, d\varepsilon\right) . \tag{IV.7.4}$$

To make the calculations clearer, we take the angle to be positive. In the region $\varepsilon = \left[\frac{\pi}{2},\varepsilon_0\right]$ we have $\varepsilon > \varepsilon_0$. Integration leads to

$$f(\varepsilon) = -4M\cos\varepsilon_0\left[\frac{1}{2\sin^2\varepsilon\sin\varepsilon_0} + \frac{1}{\sin\varepsilon\sin^2\varepsilon_0} - \frac{1}{\sin^3\varepsilon_0}\ln\frac{\sin\varepsilon}{\sin\varepsilon - \sin\varepsilon_0}\right] . \tag{IV.7.5}$$

Within the limits one has

$$T'(\varepsilon) = f(\varepsilon) - f\left(\frac{\pi}{2}\right), \quad T'\left(\frac{\pi}{2}\right) = 0, \quad \lim_{\varepsilon\to\varepsilon_0} T'(\varepsilon) = \infty \tag{IV.7.6}$$

Time symmetry leads back to the previously developed results as shown in Fig. IV.7.

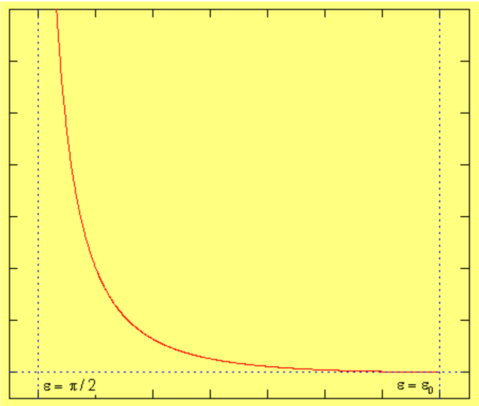

Fig. IV.7

IV.8. Calculation with the Lorentz angle

The velocity of an observer falling freely from infinity can be written as

$$v = -\mathrm{th}\chi \qquad \text{(IV.8.1)}$$

as well. The Lorentz angle χ is also called rapidity. For $\chi \to \infty$ one gets $v = -1$. For the Lorentz factor one has $\alpha = \mathrm{ch}\chi$. With

$$i\alpha v = -i\,\mathrm{th}\chi\,\mathrm{ch}\chi = -i\,\mathrm{sh}\chi = -\sin i\chi, \quad \alpha = \cos i\chi$$

we notice that with (IV.8.1) a rotation through the angle $i\chi$ is finally described, i.e. a pseudo-rotation is applied. With

$$v' = \frac{v - v_0}{1 - vv_0} = \frac{\mathrm{th}\chi - \mathrm{th}\chi_0}{1 - \mathrm{th}\chi\,\mathrm{th}\chi_0} = \mathrm{th}\left(\chi - \chi_0\right) \qquad \text{(IV.8.2)}$$

one obtains the simple relation

$$\mathrm{th}\chi' = \mathrm{th}\left(\chi - \chi_0\right), \quad \chi > \chi_0 . \qquad \text{(IV.8.3)}$$

In addition, one has from (IV.3.1)

$$dr = -\frac{4M}{v^3}dv$$

and with (IV.8.1)

$$dv = -\frac{1}{\mathrm{ch}^2\chi}d\chi \; .$$

One finally gets

$$dr = -\frac{4M}{\mathrm{sh}^2\chi\,\mathrm{th}\chi}d\chi \; .$$

With

$$dT' = \frac{\alpha}{\alpha' v'}dr$$

one obtains

$$dT' = \frac{4M}{\mathrm{sh}\left(\chi - \chi_0\right)}\frac{1}{\mathrm{sh}\chi\,\mathrm{th}^2\chi}d\chi \; . \qquad \text{(IV.8.4)}$$

The integral yields

$$f\left(\chi\right) = \frac{2M}{\mathrm{sh}\chi_0}\left[2\,\mathrm{cth}\chi\,\mathrm{cth}\chi_0 + \frac{1}{\mathrm{sh}^2\chi} + 2\,\mathrm{cth}^2\chi_0 \ln\frac{\mathrm{sh}\left(\chi - \chi_0\right)}{\mathrm{sh}\chi}\right] . \qquad \text{(IV.8.5)}$$

For $\chi \to \infty$, i.e. at the location $r = 2M$, we obtain

$$g = 4M\frac{\mathrm{cth}\chi_0}{\mathrm{sh}\chi_0}\left(1 - \chi_0\mathrm{cth}\chi_0\right) ,$$

so that we finally gain the time function

$$T'\left(\chi\right) = f\left(\chi\right) - g, \quad \lim_{\chi\to\infty} T'\left(\chi\right) = 0, \quad \lim_{\chi\to\chi_0} T'\left(\chi\right) = \infty \qquad \text{(IV.8.6)}$$

which is depicted in Fig. IV.8. Due to time reversal we obtain the fall time again.

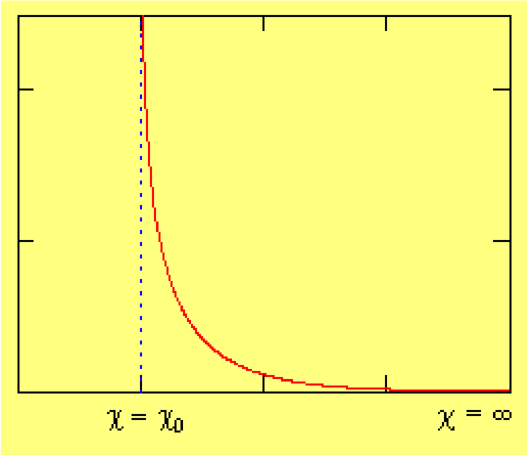

Fig. IV.8

From the previous considerations conclusions can be drawn for the collapse of a star. If the surface of a stellar object is located at the position r_0 , and even if one assumes the highest possible speed of contraction, namely the free fall, the object can never contract to the event horizon, or even exceed it. This has far-reaching consequences for the theory of stellar collapse. Models that satisfy this condition have been proposed by Mitra.

We have shown that an observer in the Schwarzschild field infalling from infinity or from any other position can reach the event horizon only after infinite proper time. We have taken up a position in contrast to the current literature and we will show in the next Section how it can lead to different interpretations concerning the velocities and fall times of free fall.

IV.9. Comparisons with the literature

Misner [M], Thorne, and Wheeler (MTW) have extensively discussed the problem of free fall in their textbook ‘Gravitation’. Their proposed solution has entered into some more textbooks and into more than a hundred publications. The derivation of the free fall from an arbitrary position takes several pages of a few sub-sections of the textbook.

In addition to the proper time τ the variables $\lambda = \tau/\mu$ with μ as the rest mass, the energy E at infinity, and the local energy E_{local} are used by MTW. Furthermore, the vectors are represented in the co-ordinate notation and as 1-forms as well. If a quantity is formulated with the help of a reference system, the indices are suppressed. Likewise the t-notation ($x^0 = t$) is used instead of the more convenient it-notation ($x^4 = it$). The t-notation requires a careful treatment of the time-like components of 4-vectors.

In MTW p 663 we find the relation

$$r_0 = \frac{2M}{1-\tilde{E}^2} \tag{IV.9.1}$$

which they have educed after lengthy deliberations. Therein r_0 is the position where the observer has zero velocity (apastron), $\tilde{E} = E/m$ the energy per rest mass and E is the energy at infinity. Since we want to free ourselves from terms like 'energy at infinity', we have to rewrite Eq. (IV.9.1) in such a way that it contains basic variables which are directly related to the free fall.

Since r_0 is the very position from which the observer (we will call him B') is released for the free fall, he has at this location the initial velocity $v' = 0$. A second observer (we will call him B''), who is in free fall coming from infinity has, at the moment he passes this position, the speed

$$v_0 = -\sqrt{\frac{2M}{r_0}} \ . \tag{IV.9.2}$$

By rearranging the equation (IV.9.1) one gains

$$\tilde{E}^2 = 1 - v_0^2 = \frac{1}{\alpha_0^2} \ .$$

Thus, one has

$$\tilde{E} = \frac{1}{\alpha_0} \ , \tag{IV.9.3}$$

wherein α_0 is the Lorentz factor for the velocity of the observer B'' who comes from infinity and is just passing the position r_0.

MTW start with their considerations from the equation (p 656)

$$g_{\alpha\beta}p^\alpha p^\beta + m^2 = 0 \ , \tag{IV.9.4}$$

wherein p^α is the 4-momentum. Going on with the tetrad representation and with the it-notation, and by dividing by m^2 and by multiplying by the proper time, we arrive at a relation that can be derived from the invariance of the line element with respect to Lorentz transformation. Reduced to two dimensions, this is

$$ds^2 = dx'^2 - dT'^2 = dx^2 - dT^2 \ , \tag{IV.9.5}$$

wherein dx' and dT' are the physical radial and time-like arc elements of the metric in terms of an observer B' who is infalling from r_0. dx and dT refer to the static observer in the Schwarzschild field. For this observer the relations

$$dx = \alpha dr, \quad dT = \frac{1}{\alpha} dt, \quad \alpha = \frac{1}{\sqrt{1-\frac{2M}{r}}}$$

are to be applied, where the metric coefficient α is identical with the Lorentz factor of an observer B" incoming from the infinite.

In a reference system that is linked to an observer B' falling down from r_0, one has

$$x' = \text{const.}, \quad dx' = 0 \ . \tag{IV.9.6}$$

If one utilizes these definitions one obtains from (IV.9.5)

$$-dT'^2 = dx^2 - dT^2 \ , \tag{IV.9.7}$$

which corresponds to the MTW relation (IV.9.4), if one undoes all the changes in notation. Clearly, (IV.9.7) is more insightful than (IV.9.4). We write (IV.9.7) in the form

$$dx^2 = \left(\frac{dT^2}{dT'^2} - 1 \right) dT'^2 \ . \tag{IV.9.8}$$

From now on we perform the calculations in such a way that we get the result of MTW. In (IV.9.8) we are using

$$\frac{dT}{dT'} = \frac{\alpha}{\alpha_0}, \quad \alpha_0 = \frac{1}{\sqrt{1-v_0^2}}, \quad v_0 = -\sqrt{\frac{2M}{r_0}} \ , \tag{IV.9.9}$$

and we will reflect the first relation therein later on.

Substituting the first relation (IV.9.9) into (IV.9.8) one obtains

$$dx^2 = \left(\frac{\alpha^2}{\alpha_0^2} - 1 \right) dT'^2 = \alpha^2 \left(\frac{1}{\alpha_0^2} - \frac{1}{\alpha^2} \right) dT'^2 \ .$$

$$dr^2 = \left[\left(1 - v_0^2\right) - \left(1 - v^2\right) \right] dT'^2 = \left(v^2 - v_0^2\right) dT'^2$$

Finally, we have for the proper time of an observer B' who falls down from r_0

$$dT' = \frac{1}{\sqrt{v^2 - v_0^2}} dr \ . \tag{IV.9.10}$$

By using the standard Schwarzschild metric coefficients as in $dT = \frac{1}{\alpha} dt$ we get with

$$dT' = \frac{\alpha_0}{\alpha} dT = \frac{\alpha_0}{\alpha^2} dt \tag{IV.9.11}$$

an expression for the co-ordinate time related to the static observer

$$dt = \sqrt{\frac{1-v_0^2}{v^2 - v_0^2}} \frac{1}{1-v^2} dr \ . \tag{IV.9.12}$$

For the integral of (IV.9.12) MTW provide a solution with cycloid parameters.

Until now we have carried out a method, how to get, under simplified conditions, the result of MTW and we will be able refer to these statements hereafter.

Previously, we have noted that we are reserved concerning the use of the first equation (IV.9.9). We will now examine this relation in more detail. From the table of Section IV.4 we see that this relation is valid for $x'' = \text{const.}$, i.e. for an observer, who is associated with a system that comes from infinity. We want to reconsider this once more in detail.

With the help of (IV.4.5) one obtains

$$dx^{4'} = -i\alpha' v' dx^1 + \alpha' dx^4 ,$$

and with $dx^{4'} = idT'$, $dx^4 = idT$

$$dT' = \alpha' \left[1 - v' \frac{dx}{dT} \right] dT \ .$$

Since the velocity of B" with respect to the static system B is defined by $v = \frac{dx}{dT}$, one has finally derived

$$\frac{dT}{dT'} = \frac{\alpha}{\alpha_0}$$

with the help of (IV.4.3), last relation. We still have to show that this applies to $x'' = \text{const.}$. With the Lorentz transformation(IV.4.5) holds for $x'' = \text{const.}$ the relation

$$dx^{1''} = \alpha dx^1 + i\alpha v dx^4 = 0 \ .$$

This yields the aforementioned expression for the relative velocity

$$v = \frac{dx}{dT} \ ,$$

as stipulated in Table 1. We have performed quite elementary calculations to show which formulae are valid if one associates a reference system with a moving observer.

We now recognize the problems of the MTW method. The relation (IV.9.7) was derived by applying $x' = \text{const.}$. Then an expression was used which refers to a relation with $x'' = \text{const.}$. That means that two excluding conditions are used in the same equation.

If we take from Table 1 the appropriate expression for $x' = \text{const.}$, namely

$$\frac{dT}{dT'} = \alpha' \qquad \text{(IV.9.13)}$$

we refer to a single reference system, namely to the one which is associated with the observer who is released from r_0. Substituting in (IV.9.8) leads after a short calculation to

$$dT' = \frac{\alpha}{\alpha' v'} dr \ ,$$

which we have written down with (IV.6.8) after elementary considerations.

MTW do not explicitly provide a formula for the fall velocity of an observer who is released from r_0. But it can be found in the textbook by Raine [R] and Thomas and can be derived with

$$\frac{dx}{dt} = v' ,$$

where we have taken the above expression from the table for $x' = \text{const.}$. With $dx = \alpha dr,\ dT = \frac{1}{\alpha} dt$ we finally obtain with the MTW formula (IV.9.10)

$$v' = \sqrt{\frac{v^2 - v_0^2}{1 - v_0^2}} . \tag{IV.9.14}$$

We recognize that this formula contradicts the Einstein addition law of velocities. If one represents this function graphically for some values of r_0, one can see that the velocity of freely falling objects v' reaches the speed of light at the event horizon for all r_0. However, the curves alter the curvatures at certain points. It is not very plausible that Nature provides for the free fall preferred points, in which the velocity changes significantly its behavior. We illustrate this in Fig. IV.9.

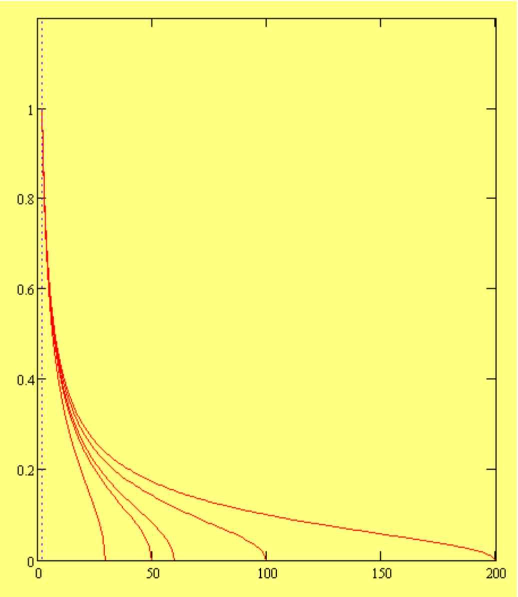

Fig. IV.9

Compare with Fig. IV.4.

For the question which of the two approaches is correct, there exists a decisive criterion. According to Einstein no gravity acts on objects which are inside a freely falling elevator. They have to hover in the elevator, independently of the position from which the elevator starts. In the next Section we will show that this is the case for our approach, and therefore the method of MTW has to be dismissed.

IV.10. Einstein's elevator

It was Einstein's idea that in a curved space a co-ordinate system can be chosen in such a way that the space appears flat at any point or along a curve, i.e. that an observer can experience no forces in this system. In modern language this means that a family of local reference systems can be chosen which accompany such an observer and that the observer does not experience any forces with respect to these systems.

For the free fall from the infinite, the problem is easy to manage, as the velocity of fall is a quantity which can be determined from the geometry. With the aid of a Lorentz transformation to the comoving system it can be shown that the force of gravity in this system is dynamically nullified.

In case the elevator is released at an arbitrary position, we have to rely on the system which comes from infinity and has reached the speed v_0 at the location r_0. Furthermore, the counter forces which compensate the gravity of the stellar object are to be calculated with a Lorentz transformation.

First, we derive some useful relations. Differentiating the second equation (IV.4.3) and (IV.4.4) gives

$$\alpha^2 dv = \alpha'^2 dv' + \alpha_0{}^2 dv_0 \ , \tag{IV.10.1}$$

$$\frac{1}{\alpha} d\alpha v = \frac{1}{\alpha'} d\alpha' v' + \frac{1}{\alpha_0} d\alpha_0 v_0 \ , \tag{IV.10.2}$$

wherein each of the last terms vanishes, since v_0 is a constant value at r_0. Using the corresponding proper times we obtain the accelerations of the observers in the radial direction

$$\frac{1}{\alpha}\frac{d\alpha v}{dT''} = G_{1''} \qquad \frac{1}{\alpha'}\frac{d\alpha' v'}{dT'} = G_{1'} \ . \tag{IV.10.3}$$

The indices with two and one primes denote the components of the variables in the systems that comove with the observers B'' and B'. In addition, multiplying with the rest mass m_0 of the comoving objects, one obtains the Lorentz forces acting on these objects

$$\frac{1}{\alpha}\frac{dmv}{dT''} = K_{1''}, \quad \frac{1}{\alpha'}\frac{dm'v'}{dT'} = K_{1'}, \quad m = m_0\alpha, \quad m' = m_0\alpha' \tag{IV.10.4}$$

in accordance with the definitions of the electrodynamics of moving media.

From the above table we take the relations between the proper times

$$\frac{dT}{dT''} = \alpha, \ x^{1''} = \text{const.}, \quad \frac{dT}{dT'} = \alpha', \ x^{1'} = \text{const.} \tag{IV.10.5}$$

and thus we can substitute the proper time of the static observer B into Eq. (IV.10.4). The accelerations (IV.10.3) can be calculated if one starts with the Lorentz transformation

$$\Phi_{|4''} = L^{m}_{4''}\Phi_{|m}, \quad \Phi_{|4'} = L^{m}_{4'}\Phi_{|m}$$

and resolves these as

$$\frac{d\Phi}{dT''} = \alpha\left(\frac{\partial\Phi}{\partial T} + v\Phi_{|1}\right), \quad \frac{d\Phi}{dT'} = \alpha'\left(\frac{\partial\Phi}{\partial T} + v'\Phi_{|1}\right)$$

and then if one relies on geometric relationships of the Schwarzschild geometry. In the last relation the partial derivatives with respect to the time of the static observer vanish, since the quantities of the Schwarzschild geometry do not explicitly depend on time. We use the auxiliary relations

$$d\alpha v = \alpha^3 dv, \quad d\alpha = \alpha^3 v dv, \quad dv' = \frac{\alpha^2}{\alpha'^2} dv \tag{IV.10.6}$$

and furthermore

$$v_{|1} = \frac{\partial v}{\alpha \partial r} = \frac{1}{\alpha \rho}, \quad \rho = \sqrt{\frac{2r^3}{M}} \ ,$$

ρ being the radius of curvature of the Schwarzschild parabola. We finally obtain the forces acting on the unit mass

$$G_{1''} = \alpha^2 v \frac{1}{\rho}, \quad G_{1'} = \alpha\alpha' v' \frac{1}{\rho} \ . \tag{IV.10.7}$$

With the auxiliaries formulae (IV.10.6) one obtains

$$\frac{d\alpha}{dT''} = \alpha v G_{1''}, \quad \frac{d\alpha'}{dT'} = \alpha' v' G_{1'} \tag{IV.10.8}$$

or

$$\frac{dm}{dT} = m v G_{1''} = v K_{1''}, \quad \frac{dm'}{dT} = m' v' G_{1'} = v' K_{1'} \ . \tag{IV.10.9}$$

The last two equations describe the work which is done on accelerating objects per time unit.

Following the idea of Einstein's elevator we first refer to the gravity-free space. If we accelerate the elevator downwards by external forces, objects in the elevator are accelerated towards the ceiling of the elevator with the counter force -G because of their inertia. If we interpret the external force as the gravitational force, it acts likewise on the objects inside the elevator and nullifies the counterforce. The objects in the elevator are hovering during the free fall.

We know from the special theory of relativity that the accelerations between relatively moving systems do not transform as vectors. In the context of the gravitation theory this is expressed by the inhomogeneous transformation law of the Ricci-rotation coefficients. To incorporate gravity into the problem in the above relations the partial derivatives have to be replaced by covariant ones. We arrive at the Ricci-rotation coefficients and the forces contained in them.

Since it is sufficient for our considerations to consider only the radial and time-like part of the metric, only

$$E_{nm}{}^{s} = h_{nm} E^{s} - h_{n}{}^{s} E_{m} \tag{IV.10.10}$$

remains for the Ricci-rotation coefficients with the unit matrix in the 2-dimensional [1,4] subspace

$$h_{mn} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} . \tag{IV.10.11}$$

The covariant derivative of a 4-vector is then reduced to

$$\Phi_{m||n} = \Phi_{m|n} - E_{nm}{}^{s}\Phi_{s} \ .$$

The force of gravity has the components

$$E_m = \left\{ \alpha v \frac{1}{\rho}, 0, 0, 0 \right\} \tag{IV.10.12}$$

with the negative velocity v of an observer who comes from infinity and the associated Lorentz factor α. Therefore, E is also negative and is directed inward.

First we transform the gravitational field strength (IV.10.12) with (IV.4.5) into the accelerated systems

$$E_{n"m"}{}^{s"} = h_{n"m"}E^{s"} - h_{n"}{}^{s"}E_{m"}, \quad E_{n'm'}{}^{s'} = h_{n'm'}E^{s'} - h_{n'}{}^{s'}E_{m'} \ , \tag{IV.10.13}$$

$$E_{m"} = \left\{ \alpha E_1, 0, 0, -i\alpha v E_1 \right\}, \quad E_{m'} = \left\{ \alpha' E_1, 0, 0, -i\alpha' v' E_1 \right\} \ . \tag{IV.10.14}$$

However, we note that the quantities (IV.10.14) are the field strength of the static system measured by the accelerated observers, but are not the forces acting in this system. By means of the Lorentz invariance of the covariant derivative

$$L^{n\,m}_{n'm'}\Phi_{n||m} = \Phi_{m'|n'} - \left[E_{n'm'}{}^{s'} + L^{s'}_{s} L^{s}_{m'|n'} \right] \Phi_{s'} \tag{IV.10.15}$$

we get the following relations

$$"E_{n"m"}{}^{s"} = E_{n"m"}{}^{s"} + L^{s"}_{s} L^{s}_{m"|n"}, \quad 'E_{n'm'}{}^{s'} = E_{n'm'}{}^{s'} + L^{s'}_{s} L^{s}_{m'|n'} \ . \tag{IV.10.16}$$

They express the inhomogeneous transformation law of the Ricci-rotation coefficients. First, we exploit the Lorentz terms

$$L^{s"}_{s}\, L^{s}_{m"|n"} = h_{n"}{}^{s"}G_{m"} - h_{m"n"}G^{s"}, \quad L^{s'}_{s}\, L^{s}_{m'|n'} = h_{n'}{}^{s'}G_{m'} - h_{m'n'}G^{s'}$$
$$G_{m"} = \left\{ \alpha^2 v \frac{1}{\rho}, 0, 0, -i\alpha^2 \frac{1}{\rho} \right\}, \quad G_{m'} = \left\{ \alpha\alpha' v' \frac{1}{\rho}, 0, 0, -i\alpha\alpha' \frac{1}{\rho} \right\} \ . \tag{IV.10.17}$$

The new variables describe the forces acting on the two types of observers. We have gotten to know the first components of G with (IV.10.3). The relevant fourth components of the quantities G arise from the covariant approach (IV.10.16). The two quantities are connected by

$$G_{m"} = L^{m'}_{m"}G_{m'} \tag{IV.10.18}$$

and transform like vectors. Therewith one can calculate the effective field strengths

$$"E_{n"m"}{}^{s"} = h_{n"m"}\,"E^{s"} - h_{n"}{}^{s"}\,"E_{m"}, \quad 'E_{n'm'}{}^{s'} = h_{n'm'}\,'E^{s'} - h_{n'}{}^{s'}\,'E_{m'}$$
$$"E_{m"} = \left\{ 0, 0, 0, -\frac{i}{\rho} \right\}, \quad 'E_{m'} = \left\{ \alpha_0 v_0 \frac{1}{\rho}, 0, 0, -i\alpha_0 \frac{1}{\rho} \right\} \tag{IV.10.19}$$

which act in the observer systems. For further investigations, we note the components of the unit vectors in the radial (m) and the time-like (u) directions for all observers B, B', B" as they are measured in their own systems, and in the respective other systems. We indicate the coordination by primes on the kernels, the measured components by primes on the indices respectively, where we have to use the Lorentz transformations from (IV.4.5) for the calculation.

$$m_n = \{1,0,0,0\}, \qquad {}'m_n = \{\alpha',0,0,i\alpha'v'\}, \qquad {}''m_n = \{\alpha,0,0,i\alpha v\}$$
$$m_{n'} = \{\alpha',0,0,-i\alpha'v'\}, \qquad {}'m_{n'} = \{1,0,0,0\}, \qquad {}''m_{n'} = \{\alpha_0,0,0,i\alpha_0 v_0\} \quad \text{, (IV.10.20)}$$
$$m_{n''} = \{\alpha,0,0,-i\alpha v\}, \qquad {}'m_{n''} = \{\alpha_0,0,0,-i\alpha_0 v_0\}, \qquad {}''m_{n''} = \{1,0,0,0\}$$

$$u_n = \{0,0,0,1\}, \qquad {}'u_n = \{-i\alpha'v',0,0,\alpha'\}, \qquad {}''u_n = \{-i\alpha v,0,0,\alpha\}$$
$$u_{n'} = \{i\alpha'v',0,0,\alpha'\}, \qquad {}'u_{n'} = \{0,0,0,1\}, \qquad {}''u_{n'} = \{-i\alpha_0 v_0,0,0,\alpha_0\} \quad \text{. (IV.10.21)}$$
$$u_{n''} = \{i\alpha v,0,0,\alpha\}, \qquad {}'u_{n''} = \{i\alpha_0 v_0,0,0,\alpha_0\}, \qquad {}''u_{n''} = \{0,0,0,1\}$$

In addition, we list the components of the force of gravity and of the counterforce in the three systems for later use. The effective forces are calculated with (IV.10.16). Because of the constancy of v_0 'E and ''E coincide.

$$E_m = \left\{\alpha v\frac{1}{\rho},0,0,0\right\}, \qquad E_{m'} = \{\alpha'E_1,0,0,-i\alpha'v'E_1\}, \qquad E_{m''} = \{\alpha E_1,0,0,-i\alpha v E_1\}$$
$$G_m = \left\{0,0,0,-i\alpha\frac{1}{\rho}\right\}, \quad G_{m'} = \left\{\alpha\alpha'v'\frac{1}{\rho},0,0,-i\alpha\alpha'\frac{1}{\rho}\right\} \quad G_{m''} = \left\{\alpha^2 v\frac{1}{\rho},0,0,-i\alpha^2\frac{1}{\rho}\right\}. \quad \text{(IV.10.22)}$$
$${}''E_m = \left\{\alpha v\frac{1}{\rho},0,0,i\alpha\frac{1}{\rho}\right\}, \; {}''E_{m'} = \left\{\alpha_0 v_0\frac{1}{\rho},0,0,i\alpha_0\frac{1}{\rho}\right\}, \qquad {}''E_{m''} = \left\{0,0,0,i\frac{1}{\rho}\right\}$$

Thus, we are ready to calculate the equations of motion of the different observers.

IV.11. Equations of motion

For a freely falling observer who comes from infinity a relation is used written down with tetrads

$$"u_{m||n}"u^n = 0 \; . \qquad \text{(IV.11.1)}$$

However, we put it into a more stringent covariant form[44]

$$"m^m "u_{m||n}"u^n = 0 . \qquad \text{(IV.11.2)}$$

Written in more detail we obtain

$$"m^1 "u_{1|n}"u^n + "m^4 "u_{4|n}"u^n - E_{nm}{}^s "u^n "u_s "m^m = 0 \; . \qquad \text{(IV.11.3)}$$

With (IV.10.20), (IV.10.21) and taking into account $x^{1"} = \text{const.}$ one has

$$"u^n \partial_n = "u^{n"} \partial_{n"} = \frac{d}{idT"}$$

the derivative with respect to the proper time of the freely falling observer coming from infinity. With (IV.10.13) and (IV.10.14) one finally gains

$$-\frac{1}{\alpha}\frac{d\alpha v}{dT"} - E_{4"1"}{}^{4"} = 0, \quad -\frac{1}{\alpha}\frac{d\alpha v}{dT"} + E_{1"} = 0 \; . \qquad \text{(IV.11.4)}$$

The first term in the second relation is the counter-force which tries to pull the objects in the elevator to the ceiling. With (IV.10.3) one has

$$-G_{1"} + E_{1"} = 0 \; , \qquad \text{(IV.11.5)}$$

a relation which is satisfied with (IV.10.7), (IV.10.12), and (IV.10.14). The force of gravity is balanced by the counter-force. Objects hover in Einstein's elevator. With (IV.11.2) we have written down the equations of motion with the help of the static system (m) and have taken a way which is based on the electrodynamics of moving systems. Our static system corresponds to the laboratory system of electrodynamics. Instead of using (IV.11.2) we could have started with the equations of motion

$$"m^{m"} "u_{m"||n"}"u^{n"} = "m^{m"} "u_{m"|n"}"u^{n"} - "E_{n"m"}{}^{s"} "u_{s"} "m^{m"} "u^{n"} = 0 \qquad \text{(IV.11.6)}$$

by using the comoving system (m"). (IV.11.6) is satisfied, because the first term on the right side of the above relation vanishes trivially and the first component of the effective field strength "E vanishes in accordance with (IV.10.19) as well. The nullification of the forces can be understood via (IV.10.7).

The calculation of the equations of motion for an observer who comes from an arbitrary position is considerably more difficult. We must rely on the free fall from infinity and on the observer's values at the position r_0. If we apply the Lorentz transformation to the tetrads

$$\overset{m"}{e}_n = L^{m"}_{m'} \overset{m'}{e}_n$$

and if we insert m and u, we obtain

---

[44] Eq. (IV.11.1) leads with $"u_{1||n}"u^n = 0$ to the Minkowski force, which differs from the Lorentz force by the Lorentz factor. Accurate measurements of the electron have shown that the Lorentz force is the correct definition of the force in the theory of moving media in electrodynamics. We believe that one has to use the Lorentz force in the theory of gravity as well.

$$''m^m = \alpha_0\,'m^m + i\alpha_0 v_0\,'u^m, \quad ''u_m = -i\alpha_0 v_0\,'m_m + \alpha_0\,'u_m \ , \qquad \text{(IV.11.7)}$$

thus,

$$\left[\alpha_0\,'m^m + i\alpha_0 v_0\,'u^m\right]\left[-i\,\alpha_0 v_0\,'m_m + \alpha_0\,'u_m\right]_{||n}{''u^n} = 0 \ .$$

Since m and u are unit vectors one has

$$'m^m\,'m_{m||n} = 0, \quad 'u^m\,'u_{m||n} = 0 \qquad \text{(IV.11.8)}$$

and because of the orthogonality of the two vectors also

$$'u^m\,'m_{m||n} = -'m^m\,'u_{m||n} \ . \qquad \text{(IV.11.9)}$$

This simplifies the above expression to

$$'m^m\,'u_{m||n}\,''u^n = 0 \ . \qquad \text{(IV.11.10)}$$

Using once again the second Eq. (IV.11.7) leads to

$$-i\alpha_0 v_0\,'m^m\,'u_{m||n}\,'m^n + \alpha_0\,'m^m\,'u_{m||n}\,'u^n = 0 \ .$$

If we use the Lorentz invariance of the covariant derivative, we get for the first term

$$'m^{m'}\,'u_{m'||n'}\,'m^{n'} = -'E_{n'm'}{}^{s'}\,'u_{s'}\,'m^{m'}\,'m^{n'} = -'E_{4'} = -i\alpha_0\frac{1}{\rho} \ .$$

The second term is straight resolved

$$\begin{aligned} 'm^m\,'u_{m||n}\,'u^n &= 'm^m\,'u_{m|n}\,'u^{n'} - E_{nm}{}^s\,'u_s\,'m^m\,'u^n \\ &= 'm^1\,'u_{1|n'}\,'u^{n'} + 'm^4\,'u_{4|n'}\,'u^{n'} - E_{4'1'}{}^{4'} \ . \\ &= i\alpha^2 v'_{|n'}\,u^{n'} + E_{1'} \end{aligned}$$

Dividing the output equation through α_0 and merging, one finally has the equations of motion of an observer who is released from r_0

$$-\frac{1}{\alpha'}\frac{d\alpha' v'}{dT'} + E_{1'} - ''E_{1'} = 0 \ . \qquad \text{(IV.11.11)}$$

With (IV.10.3) we recognize the first term as a counter-force which nullifies the difference of two gravity components. $E_{1'}$ is the gravitational force of the static system B, measured in system B', which accompanies the observer who starts from r_0. $''E_{1'}$ is the gravity of the freely falling system B'', which comes from infinity, also measured in the system B', which accompanies the observer falling away from r_0. The difference between the two quantities is the gravity acting on the observer B'. Objects in an elevator coming in free fall from r_0 are hovering. Fig. IV.10 illustrates this behavior. $E_{1'}$ is the topmost curve, $''E_{1'}$ is beneath located. The counter-force $G_{1'}$ is the lowest curve. The differences of the curves in the sense above yield the zero line.

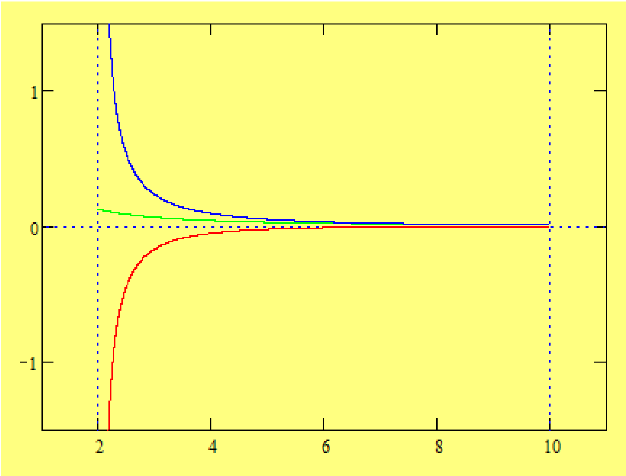

Fig. IV.10

The relation (IV.11.11) derived from the equations of motion can be directly reviewed with the formulae (IV.4.2) - (IV.4.4). Substituting (IV.10.3) into the left side of (IV.11.11) one has

$$-G_{1'} + E_{1'} - {''}E_{1'} = 0 \qquad \text{(IV.11.12)}$$

or

$$-\alpha\alpha' v' \frac{1}{\rho} + \alpha' \alpha v \frac{1}{\rho} - \alpha_0 v_0 \frac{1}{\rho} = 0 \ ,$$

which one can verify with the Einstein addition law of velocities. Compare (IV.11.12) with (IV.11.5).

If one performs in (IV.11.12) the Newtonian approximation by putting the Lorentz factors to 1 and if one recognizes that the proper time passes over into the absolute time T, one has for the counter-force in the elevator

$$-\frac{dv'}{dT} = -\frac{M}{r^2}\left(1 - \sqrt{\frac{r}{r_0}}\right) . \qquad \text{(IV.11.13)}$$

We see that the counter-force for $r = r_0$ vanishes i.e. if the elevator is still in rest. If the elevator comes from the infinite, then is $v' = v$. Further, the root in the brackets vanishes for $r_0 \to \infty$. Thus,

$$m\frac{d(-v)}{dT} = -\frac{mM}{r^2} \qquad \text{(IV.11.14)}$$

remains, the Newtonian formula for the force in geometric units.

IV.12. Lemaître co-ordinates

A co-ordinate transformation of the Schwarzschild metric introduced by Lemaître [L] will be treated with respect to related problems raised by us. The substitution of Lemaître

$$r=\sqrt[3]{2M}\left[\frac{3}{2}\left(r''-t''\right)\right]^{\frac{2}{3}},\quad t''=t+2\sqrt{2Mr}+2M\ln\frac{1-\sqrt{\frac{2M}{r}}}{1+\sqrt{\frac{2M}{r}}} \tag{IV.12.1}$$

can be simplified with[45]

$$\mathcal{R}\left(r'',t''\right)=\frac{3}{2}\left(r''-t''\right)$$

to

$$r^3=2M\mathcal{R}^2\ .$$

From the Schwarzschild definition $v=-\sqrt{2M/r}$ for the velocity of observers falling in from the infinite one obtains

$$v=-\frac{r}{\mathcal{R}}=-\sqrt[3]{\frac{2M}{\mathcal{R}}}\ . \tag{IV.12.2}$$

Differentiating (IV.12.1) and substituting the result into the standard form of the Schwarzschild metric, we get the Lemaître metric

$$ds^2=v^2\left(r'',t''\right)\left[dr''^2+\mathcal{R}^2d\vartheta^2+\mathcal{R}^2\sin^2\vartheta d\varphi^2\right]-dt''^2\ . \tag{IV.12.3}$$

The arc elements are functions of the time t'' and express their temporal change originated by the motion of fall. The time-like arc element appears to be pseudo-Euclidean. No forces can be derived from it, so that for an observer who is associated with the Lemaître co-ordinate system no gravity can be experienced. The Lemaître metric can be obtained by the transformation matrix

$$\begin{array}{llll}\Lambda^{1}_{1''}=-v, & \Lambda^{1}_{4''}=-iv, & \Lambda^{4}_{1''}=-i\alpha^2v^2, & \Lambda^{4}_{4''}=\alpha^2\\ \Lambda^{1''}_{1}=-\dfrac{\alpha^2}{v}, & \Lambda^{1''}_{4}=-i, & \Lambda^{4''}_{1}=-i\alpha^2v, & \Lambda^{4''}_{4}=1\end{array}\ , \tag{IV.12.4}$$

where the indices are co-ordinate indices. However, a co-ordinate transformation cannot change the geometric and physical background of the model, because the co-ordinate invariance is a basic requirement for any physical theory.

Therefore it is essential to question the significance of the Lemaître transformation. If we calculate from (IV.12.1) the co-ordinate differentials

$$dr=-v\left(dr''-dt''\right),\quad dt=dt''+\alpha^2vdr$$

and if we multiply by the static bein vectors, we arrive at the arc elements

$$dx^1=\alpha dx^{1''}-i\alpha vdx^{4''},\quad dx^4=\alpha dx^{4''}+i\alpha vdx^{1''}\ ,$$

---

[45] The new quantity $\mathcal{R}$ can be interpreted geometrically. If one extends the curvature vector of the Schwarzschild parabola still to be discussed to its guideline, the distance between the guideline and the Schwarzschild parabola is cut out.

from which one reads the already known Lorentz transformation (IV.4.5). The Lemaître transformation which relates to the co-ordinates of the Schwarzschild metric, can thus be replaced by a Lorentz transformation which is based on a reference system. According to (IV.4.5) it is that very reference system which is linked to an observer who is in free fall from infinity. The Lemaître co-ordinate system is comoving, it shrinks toward the center of gravity.

With the 2-bein which we are facing and which is adjusted to the Lemaître co-ordinate system in the 2-dimensional subspace one obtains with the help of

$$\overset{m''}{e}_{i''} = L^{m''}_{m} \overset{m}{e}_{i} \Lambda^{i}_{i''} \tag{IV.12.5}$$

the values

$$\overset{1''}{e}_{1''} = -v, \quad \overset{4''}{e}_{4''} = 1, \quad \underset{1''}{e}^{1''} = -\frac{1}{v}, \quad \underset{4''}{e}^{4''} = 1 \ , \tag{IV.12.6}$$

on the condition that the $\overset{m}{e}_{i}$ are Schwarzschild-standard vectors. We can immediately write down the metric in the form of Lemaître. Since the redshift factor in this system is equal to 1, the Ricci-rotation coefficients do not contain gravitational field strengths. A freely falling observer is weightless, as already noted in previous sections. From the 1-component of the bein vectors the radial tidal force can be determined as we will show later on.

Using Lemaître co-ordinates the time can be calculate which a free-falling observer, coming from infinity needs to reach a position r > 2M. For an observer comoving with the shrinking co-ordinate system applies

$$\frac{dr''}{dt''} = 0 \ . \tag{IV.12.7}$$

In Lemaître co-ordinates the co-ordinate time t'' matches the proper time of the observer. After a simple calculation one gets from (IV.12.1)[46]

$$\frac{dr}{dt''} = v, \quad dt'' = \frac{1}{v} dr = -\sqrt{\frac{r}{2M}} dr \ . \tag{IV.12.8}$$

Integration yields

$$t'' = -\frac{\rho}{3}, \quad \rho = \sqrt{\frac{2r^3}{M}} \ . \tag{IV.12.9}$$

If we again insert this into (IV.12.1) we have

$$t(r) = -\frac{\rho}{3} - 2\sqrt{2Mr} - 2M\ln\frac{1+v}{1-v} \ . \tag{IV.12.10}$$

This is the very formula which Misner [M], Thorne, and Wheeler have derived in another way. t is the Schwarzschild-coordinate time which coincides with the physical time at infinity. Therefore, we can say that $t(r)$ is the time which an observer at infinity can

[46] The following formula is in conformity with those of the table, if one considers the following: For $x'' = \text{const.}$ applies $\frac{dx}{dT} = v$, $\frac{dT}{dT''} = \alpha$. Thus, one obtains $\frac{dx}{dT''} = \alpha v$, and finally the co-ordinate expression $\frac{dr}{dT''} = \frac{dr}{dt''} = v$.

specify for a freely falling observer who comes from infinity and has reached the position r. Since the logarithmic expression in (IV.12.10) will be infinite for r = 2M, a freely falling observer who comes from infinity needs an infinitely long co-ordinate time t to reach the event horizon.

Since the considerations just made apply only to observers who come from infinity, the question arises whether there are corresponding co-ordinates which apply to observers who are released from an arbitrary position. To answer this, we proceed from a relation analogous to (IV.12.5)

$$\Lambda_i^{i'} = \underset{m'}{e}^{i'} L_m^{m'} \overset{m}{e}_i \tag{IV.12.11}$$

in order to find a suitable matrix for a co-ordinate transformation.

The relation, however, contains more unknown quantities than equations. Thus, we need additional conditions:

1. The transformation Λ must be holonomic.
2. It must be orthogonal.
3. It should not explicitly depend on the Schwarzschild-standard time co-ordinate.
4. Λ has to coincide with the Lemaître transformation (IV.12.4) for $r_0 \to \infty$.
5. The new 2-bein vectors must be orthogonal, they must not contain 1' and 4' mixed indices, so that no cross terms emerge in the metric.

To comply with condition 1 $\Lambda^{i'}_{[i|k]} = 0$ must hold, which leads immediately to $\Lambda^{i'}_{4|1} - \Lambda^{i'}_{1|4} = 0$ and is reduced to $\Lambda^{1'}_{4|1} = 0$ and $\Lambda^{4'}_{4|1} = 0$ due to condition 3. This means that $\Lambda_4^{4'}$ must be a real constant that one can put to 1 and $\Lambda_4^{1'}$ should be an imaginary constant which we put i, both in accordance with the fourth condition. Since both the Lorentz transformation and the static bein vectors

$$\overset{1}{e}_1 = \alpha, \quad \overset{4}{e}_4 = \frac{1}{\alpha}, \quad \underset{1}{e}^1 = \frac{1}{\alpha}, \quad \underset{4}{e}^4 = \alpha \tag{IV.12.12}$$

are known, it is easy to calculate the remaining components of $\Lambda_i^{i'}$ with the help of condition 5, and at the same time the new 2-bein. With the conditions 2 and 5 one gets the reciprocal values of the two quantities. They are

$$\begin{aligned} &\Lambda^1_{1'} = -\frac{\alpha'^2}{\alpha^2} v', \quad \Lambda^1_{4'} = -i\frac{\alpha'^2}{\alpha^2} v', \quad \Lambda^4_{1'} = -i\alpha'^2 v'^2, \quad \Lambda^4_{4'} = \alpha'^2 \\ &\Lambda^{1'}_1 = -\frac{\alpha^2}{v'}, \qquad \Lambda^{1'}_4 = -i, \qquad \Lambda^{4'}_1 = -i\alpha^2 v', \qquad \Lambda^{4'}_4 = 1 \end{aligned} , \tag{IV.12.13}$$

$$\overset{1'}{e}_{1'} = -\frac{\alpha'}{\alpha} v', \quad \overset{4'}{e}_{4'} = \frac{\alpha'}{\alpha}, \quad \underset{1'}{e}^{1'} = -\frac{\alpha}{\alpha'}\frac{1}{v'}, \quad \underset{4'}{e}^{4'} = \frac{\alpha}{\alpha'} \ . \tag{IV.12.14}$$

This leads to the metric

$$ds^2 = \frac{\alpha'^2}{\alpha^2}\left(v'^2 dr'^2 - dt'^2\right) \ . \tag{IV.12.15}$$

For $v_0 = 0$ and thus for $\alpha' = \alpha$ one immediately obtains the Lemaître metric for freely falling observers coming from the infinite

$$ds^2 = v^2 dr''^2 - dt''^2 \ . \tag{IV.12.16}$$

It can be seen that only for observers who come from infinity, the co-ordinate time t" coincides with the proper time T". The metric (IV.12.15) can alternatively be written as

$$ds^2 = \alpha_0^2\left[\left(v - v_0\right)^2 dr'^2 - \left(1 - vv_0\right)^2 dt'^2\right] . \qquad \text{(IV.12.17)}$$

The Ricci-rotation coefficients calculated from the 2-bein (IV.12.14) are merely the already-known primed quantities of (IV.10.14).

In addition, we give a co-ordinate transformation which transforms the co-ordinates of the freely falling system B" which comes from infinity, into the co-ordinate system comoving with respect to B'. With the transformation coefficients

$$\begin{array}{llll} \Lambda^{1'}_{1''} = \alpha_0 \dfrac{\alpha v}{\alpha' v'}, & \Lambda^{1'}_{4''} = -i\alpha_0 v_0 \dfrac{\alpha}{\alpha' v'}, & \Lambda^{4'}_{1''} = -i\,\alpha_0 v_0 \dfrac{\alpha v}{\alpha'}, & \Lambda^{4'}_{4''} = \alpha_0 \dfrac{\alpha}{\alpha'} \\ \Lambda^{1''}_{1'} = \alpha_0 \dfrac{\alpha' v'}{\alpha v}, & \Lambda^{1''}_{4'} = -i\alpha_0 v_0 \dfrac{\alpha'}{\alpha v}, & \Lambda^{4''}_{1'} = i\alpha_0 v_0 \dfrac{\alpha' v'}{\alpha}, & \Lambda^{4''}_{4'} = \alpha_0 \dfrac{\alpha'}{\alpha} \end{array} \qquad \text{(IV.12.18)}$$

is associated the Lorentz transformation (IV.4.5), second line.

Finally, let us consider the various line elements that emerge by applying a Lemaître transformation. For the time-like arc elements we have for the static observer B, the observer B' who comes in free fall from r_0, and for the observer B" who comes from infinity, respectively, the relations

$$dx^4 = \frac{1}{\alpha} i dt, \quad dx^{4'} = \frac{\alpha'}{\alpha} i dt', \quad dx^{4''} = i dt'' \ . \qquad \text{(IV.12.19)}$$

It turns out that the gravitational factor in the time-like arc element is equal to 1 only in the very system that comes from infinity. This should be taken into account if one considers comoving co-ordinate systems with respect to a collapsing star.

IV.13. Field equations for Einstein's elevator

In the preceding Sections we have extensively studied the free fall in the Schwarzschild field with particular attention to the case of an arbitrary distant position with respect to the central mass. Since we have significantly deviated from the standard treatment in the literature, we want to ensure our results by constructing the field equations for the free-falling system.

With (IV.12.14) we have determined the 4-bein system for an observer B' who falls away from the position r_0

$$\overset{1'}{e}_{1'} = -\frac{\alpha' v'}{\alpha}, \quad \overset{2'}{e}_{2'} = r, \quad \overset{3'}{e}_{3'} = r\sin\vartheta, \quad \overset{4'}{e}_{4'} = \frac{\alpha'}{\alpha}. \tag{IV.13.1}$$

The co-ordinates $\{r', t'\}$ are comoving co-ordinates. In the non-moving system B with $\{r, t\}$ the usual Schwarzschild values

$$\overset{1}{e}_{1} = \alpha, \quad \overset{2}{e}_{2} = r, \quad \overset{3}{e}_{3} = r\sin\vartheta, \quad \overset{4}{e}_{4} = a, \quad a = 1/\alpha = \sqrt{1-2M/r} \tag{IV.13.2}$$

are valid. The Lorentz transformation mediates between the two systems

$$L^{1}_{1'} = \alpha', \quad L^{1}_{4'} = -i\alpha' v', \quad L^{4}_{1'} = i\alpha' v', \quad L^{4}_{4'} = \alpha' \ , \tag{IV.13.3}$$

whereby the fall velocity v' of B' is according to (IV.4.1)composed of the two velocities $v = -\sqrt{2M/r}$ and $v_0 = -\sqrt{2M/r_0}$, taking into account Einstein's addition law of velocities. Between the co-ordinates the Lemaître transformation specified in (IV.12.13) mediates.

For further calculations we need the derivatives of r in the tetrad representation

$$r_{|m} = \{1,0,0,0\}a, \quad r_{|m'} = \{\alpha', 0, 0, -i\alpha' v'\}a, \tag{IV.13.4}$$

wherein the first relation is calculated with (IV.13.2) and the second is obtained with (IV.13.3) from the first. We note that $r_{|4} = 0$. v is the velocity of a freely falling observer B'' who comes from infinity. We have

$$v_{|m} = \{1,0,0,0\}\frac{a}{\rho}, \quad v_{|m'} = \{\alpha', 0, 0, -i\alpha' v'\}\frac{a}{\rho}. \tag{IV.13.5}$$

We have derived the second term with the Lorentz transformation (IV.13.3). The quantity

$$\rho = \sqrt{\frac{2r^3}{M}} \tag{IV.13.6}$$

is the radius of curvature of the Schwarzschild parabola. v_0 is the velocity of a freely falling observer who comes from infinity, measured at the position r_0 ie at the position the observer whose laws we want to investigate, falls off. v_0 is used as a reference velocity and is a constant.

From the addition law of velocities

$$v' = \frac{v - v_0}{1 - v v_o} \tag{IV.13.7}$$

we obtain with (IV.13.5)

$$v'_{|m} = \{1,0,0,0\} \frac{\alpha}{\alpha'^2} \frac{1}{\rho}, \quad v'_{|m'} = \{\alpha',0,0,-i\alpha' v'\} \frac{\alpha}{\alpha'^2} \frac{1}{\rho}. \tag{IV.13.8}$$

The $\alpha, \alpha_0, \alpha'$ are the Lorentz factors associated with the velocities v, v_0, v'. With

$$\frac{1}{\alpha} d\alpha = \alpha^2 v dv$$

we obtain

$$\begin{aligned} \frac{1}{\alpha}\alpha_{|m} &= \{1,0,0,0\}\alpha v \frac{1}{\rho}, \quad \frac{1}{\alpha}\alpha_{|m'} = \{\alpha',0,0,-i\alpha' v'\}\alpha v \frac{1}{\rho} \\ \frac{1}{\alpha'}\alpha'_{|m} &= \{1,0,0,0\}\alpha v' \frac{1}{\rho}, \quad \frac{1}{\alpha'}\alpha'_{|m'} = \{\alpha',0,0,-i\alpha' v'\}\alpha v' \frac{1}{\rho} \end{aligned}. \tag{IV.13.9}$$

With this set of equations we are able to calculate the field quantities for the free-falling observer B' and the observer at rest B. To configure compact field equations for these observers, we apply here in deviation from previous Sections the geometrical quantities U which have the opposite sign in contrast to the physical quantities E: $U = -E$.

From

$$\begin{aligned} U_1 &= U_{41}{}^4 = -\overset{4}{e}_4 \underset{4|1}{e}{}^4, \quad U_4 = U_{14}{}^1 = -\overset{1}{e}_1 \underset{1|4}{e}{}^1 \\ 'U_{1'} &= 'U_{4'1'}{}^{4'} = -\overset{4'}{e}_{4'} \underset{4'|1}{e}{}^{4'}, \quad 'U_{4'} = 'U_{1'4'}{}^{1'} = -\overset{1'}{e}_{1'} \underset{1'|4}{e}{}^{1'} \end{aligned} \tag{IV.13.10}$$

we calculate the field quantities

$$U_m = \{1,0,0,0\}\left(-\alpha v \frac{1}{\rho}\right), \quad 'U_{m'} = \{-i\alpha_0 v_0, 0, 0, \alpha_0\}\left(-\frac{i}{\rho}\right). \tag{IV.13.11}$$

The relationship of these quantities is obtained with the inhomogeneous transformation law of the Ricci-rotation coefficients

$$'A_{m'n'}{}^{s'} = L^{m\ n\ s'}_{m'n's} A_{mn}{}^s + L^{s'}_s L^s_{n'|m'}, \quad 'L_{n'} = 'L_{s'n'}{}^{s'}. \tag{IV.13.12}$$

The last term we call Lorentz term[47] and we write it as

$$'L_{m'n'}{}^{s'} = L^{s'}_s L^s_{n'|m'} = h_{m'}{}^{s'}\,'L_{n'} - h_{m'n'}\,'L^{s'}. \tag{IV.13.13}$$

Therein

$$h_{m'n'} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \tag{IV.13.14}$$

[47] In the Section on the equations of motion the dynamic quantity L has been called counterforce G.

is a submatrix of the metric in tetrad representation. For the inverse transformation we form analogous to (IV.13.13)

$$L_{mn}{}^{s} = L^{s}_{s'} L^{s'}_{n|m} = h_m{}^{s} L_n - h_{mn} L^{s}, \quad L_n = L_{sn}{}^{s} \ . \qquad \text{(IV.13.15)}$$

For the Lorentz terms one obtains

$$'L_{1'} = i\alpha'^2 v'_{|4'}, \quad 'L_{4'} = -i\alpha'^2 v'_{|1'}, \quad L_1 = -i\alpha'^2 v'_{|4}, \quad L_4 = i\alpha'^2 v'_{|1} \ . \qquad \text{(IV.13.16)}$$

With (IV.13.8) this gives

$$'L_{1'} = \alpha' v' \alpha \frac{1}{\rho}, \quad 'L_{4'} = -i\alpha'\alpha \frac{1}{\rho}, \quad L_1 = 0, \quad L_4 = i\alpha \frac{1}{\rho} \ . \qquad \text{(IV.13.17)}$$

We also recognize that

$$'L_{m'} = -L^{m}_{m'} L_m \qquad \text{(IV.13.18)}$$

and we find with (IV.13.11)

$$\begin{gathered} 'U_{m'} = U_{m'} + 'L_{m'} \quad U_m = 'U_m + L_m, \quad U_{m'} = L^{m}_{m'} U_m \quad 'U_m = L^{m'}_{m}\, 'U_{m'} \\ U_m = \{1,0,0,0\}\left(-\alpha v \frac{1}{\rho}\right) \end{gathered} \ . \qquad \text{(IV.13.19)}$$

$E_m = -U_m$ is the force of gravity in the static Schwarzschild field.

We calculate the lateral field quantities B and C for the free-falling system with

$$\begin{aligned} B_{m'} &= 'A_{2'm'}{}^{2'} = -\overset{2'}{e}_{2'} e^{2'}_{2'|m'} = \{\alpha', 0, 0, -i\alpha' v'\} \frac{a}{r} \\ C_{m'} &= 'A_{3'm'}{}^{3'} = -\overset{3'}{e}_{3'} e^{3'}_{3'|m'} = \left\{\alpha' \frac{a}{r}, \frac{1}{r}\cot\vartheta, 0, -i\alpha' v' \frac{a}{r}\right\} \end{aligned} \qquad \text{(IV.13.20)}$$

and we also recognize that the lateral field quantities transform as vectors

$$B_{m'} = L^{m}_{m'} B_m, \quad C_{m'} = L^{m}_{m'} C_m \ , \qquad \text{(IV.13.21)}$$

whereby are the

$$B_m = \left\{\frac{a}{r}, 0, 0, 0\right\}, \quad C_m = \left\{\frac{a}{r}, \frac{1}{r}\cot\vartheta, 0, 0\right\} \qquad \text{(IV.13.22)}$$

the well-known Schwarzschild values. With the unit vectors in the 2- and 3-direction

$$b_{m'} = \{0,1,0,0\}, \quad c_{m'} = \{0,0,1,0\} \qquad \text{(IV.13.23)}$$

we finally have

$$B_{m'n'}{}^{s'} = b_{m'} B_{n'} b^{n'} - b_{m'} b_{n'} B^{n'}, \quad C_{m'n'}{}^{s'} = c_{m'} C_{n'} c^{n'} - c_{m'} c_{n'} C^{n'} \qquad \text{(IV.13.24)}$$

and we can write down the complete Ricci-rotation coefficients for the freely falling system as

$$'A_{m'n'}{}^{s'} = 'U_{m'n'}{}^{s'} + B_{m'n'}{}^{s'} + C_{m'n'}{}^{s'} \ . \qquad \text{(IV.13.25)}$$

If we further define the graded derivatives as

$$'U_{\underset{1}{n||m}} = 'U_{n|m}, \quad B_{\underset{2}{n'||m'}} = B_{n'|m'} - 'U_{m'n'}{}^{s'} B_{s'}, \quad C_{\underset{3}{n'||m'}} = C_{n'|m'} - B_{m'n'}{}^{s'} C_{s'} - 'U_{m'n'}{}^{s'} C_{s'} ,$$

(IV.13.26)

we obtain the Ricci in a compact form

$$\begin{aligned} R_{m'n'} = &-\left[{}'U^{s'}{}_{\underset{1}{\|}s'} + {}'U^{s'}\,{}'U_{s'} \right] h_{m'n'} \\ &-\left[B_{n'\underset{2}{\|}m'} + B_{n'}B_{m'} \right] - b_{n'}b_{m'}\left[B^{s'}{}_{\underset{2}{\|}s'} + B^{s'}B_{s'} \right] . \\ &-\left[C_{n'\underset{3}{\|}m'} + C_{n'}C_{m'} \right] - c_{n'}c_{m'}\left[C^{s'}{}_{\underset{3}{\|}s'} + C^{s'}C_{s'} \right] \end{aligned} \qquad \text{(IV.13.27)}$$

If one applies this to the particular subequations one has

$$\begin{aligned} &{}'U^{s'}{}_{\underset{1}{\|}s'} + {}'U^{s'}\,{}'U_{s'} = -\frac{4}{\rho^2} \\ &B_{n'\underset{2}{\|}m'} + B_{n'}B_{m'} = h_{m'n'}\frac{2}{\rho^2}, \quad B^{s'}{}_{\underset{2}{\|}s'} + B^{s'}B_{s'} = \frac{4}{\rho^2} \quad . \\ &C_{n'\underset{3}{\|}m'} + C_{n'}C_{m'} = h_{m'n'}\frac{2}{\rho^2} - b_{n'}b_{m'}\frac{4}{\rho^2}, \quad C^{s'}{}_{\underset{3}{\|}s'} + C^{s'}C_{s'} = 0 \end{aligned} \qquad \text{(IV.13.28)}$$

Thus one finds $R_{m'n'} = 0$ the vacuum field equations of the Schwarzschild theory in a reference system connected with an observer freely falling from an arbitrary position.

Einstein's field equations are satisfied with the field quantities derived by us from the tetrad system (IV.13.1). The Ricci is Lorentz invariant as expected. The field equations of the freely falling system can be derived by a Lorentz transformation from those of the static system.

IV.14. Schwarzschild metric, historical review

It should be recalled that the metric which we assign to Schwarzschild [S] does not stem from him but from Hilbert [H]. Although the two solutions are formally identical if one writes them down in a suitable way, they differ in the definition of the radial co-ordinate and they are topologically different. While the original Schwarzschild solution describes a particle, the Hilbert solution represents the field of an extended object. In 1917 Droste [D] found a spherically symmetric solution which one has to allocate to the Schwarzschild theory. Casadio [C], Garrantini, and Scardigli have applied point masses to quantum gravity.

Both metrics can be written in the form

$$ds^2 = \frac{1}{1-\frac{\alpha}{r}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left(1-\frac{\alpha}{r}\right)dt^2 \qquad \text{(IV.14.1)}$$

$\alpha = 2M$ is commonly used for the Hilbert form and the radial co-ordinate is defined by

$$r = \sqrt{x^{\alpha'}x^{\alpha'}} \ . \qquad \text{(IV.14.2)}$$

$r = 0$ is the origin of the co-ordinate system and the position of the field-producing mass. The $x^{\alpha'}, \alpha' = 1,2,3$ are Cartesian co-ordinates. Evidently, the form of Hilbert has a co-ordinate singularity at $r = \alpha$.

However, in the *original* form of the Schwarzschild metric (IV.14.1) r has the meaning

$$r^3 = r_*^3 + \alpha^3 \ , \qquad \text{(IV.14.3)}$$

whereby the radial co-ordinate is defined by

$$r_* = \sqrt{x^{\alpha'}x^{\alpha'}} \qquad \text{(IV.14.4)}$$

and $r_* = 0$ determines the origin of the co-ordinate system. At this location $r = \alpha$ and the radial arc element is singular. Thus, the singularity occurs at the origin of the co-ordinate system and (IV.14.1) is interpreted as metric of a mass point. Antoci [A], Antoci [A] and Liebscher, and Corda [C] have taken up this problem in several papers.

The original Schwarzschild metric can also be brought into the form

$$ds^2 = \frac{1}{1-\frac{\alpha}{r}} r'^2 dr_*^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left(1-\frac{\alpha}{r}\right)dt^2 \ . \qquad \text{(IV.14.5)}$$

If one calculates r' and if one substitutes also (IV.14.3), one has written the original Schwarzschild metric complete with centric variables. This possibility has been discussed in several papers by Abrams [A] . Abrams has ascribed to Hilbert the mistake to have shifted the singularity of the metric from the origin to 2M. We in contrast value this effort of Hilbert's highly.

Facing the lateral arc element after this substitution

$$\sqrt[3]{r_*^3 + \alpha^3}\, d\vartheta \ ,$$

it is notable that at the mass point $r_* = 0$ $\alpha d\vartheta$ is valid for the arc element. Therefore, the the radius α is assigned to the mass point. Mitra [M] has pointed out this aspect. This discrepancy is corrected by the Hilbert form of the metric.

However, if we perform the substitution (IV.14.3) in the Hilbert form we have introduced non-centric co-ordinates. Facing (IV.14.3) it can be seen that the relation $r = 2M$ applies for $r_* = 0$. In this case the co-ordinate r_* starts at the event horizon. Also Brillouin [B] was engaged with off-center co-ordinates.

Since we do not want to deal with mass points, but with extended objects, we exclusively use in our paper the Hilbert form of the metric.

In addition to Schwarzschild and Droste also Gullstrand [G] has found a solution of Einstein's field equations for a spherically symmetric field. It has the structure

$$ds^2 = Ndr^2 + Or^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) + 2Pdrdt - Ldt^2 \ . \qquad \text{(IV.14.6)}$$

Therein is

$$O = \sqrt[3]{\left(1+\frac{\beta}{r^3}\right)^2}, \quad L = 1 - \frac{\alpha}{\sqrt{O}r}, \quad O^2\left(LN + P^2\right) - 1 \ , \qquad \text{(IV.14.7)}$$

where α and β are arbitrary positive constants. For $\beta = 0$ is $O = 1$. If one also puts $L = 1/N$, one obtains $P = 0$. Thus one has eliminated the cross term from the metric (IV.14.6) and has obtained the Schwarzschild metric. We want to show that with fewer restrictions, the metric (IV.14.6) is the Schwarzschild metric. The presence of a cross term refers to a non-orthogonal co-ordinate system, which makes it hard to work out the physical content of the metric.

If we put $\alpha = 2M$, $\beta = 0$ and $N = 1$ we get from (IV.14.7) $L + P^2 = 1, P = \sqrt{2M/r}$. Now the metric has the form

$$ds^2 = dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) + 2\sqrt{2M/r}\, dr\, dt - \left(1 - 2M/r\right)dt^2 \, . \qquad \text{(IV.14.8)}$$

If we reference-write it using

$$v = -\sqrt{2M/r}, \quad a = \sqrt{1-2M/r} = 1/\alpha$$

we get the somewhat practical form of the metric

$$ds^2 = dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) + 2iv\, dr\, idt - a^2dt^2 \, .$$

From this one can read two 4-bein systems

$$\text{(A)} \quad \begin{array}{lllll} \overset{1}{e}_1 = \alpha, & \overset{2}{e}_2 = r, & \overset{3}{e}_3 = r\sin\vartheta, & \overset{4}{e}_1 = -i\alpha v, & \overset{4}{e}_4 = a \\ \underset{1}{e}^1 = a, & \underset{2}{e}^2 = \dfrac{1}{r}, & \underset{3}{e}^3 = \dfrac{1}{r\sin\vartheta}, & \underset{1}{e}^4 = i\alpha v, & \underset{4}{e}^4 = \alpha \end{array} , \qquad \text{(IV.14.9)}$$

$$\text{(B)} \quad \begin{array}{lllll} \overset{1}{e}_1 = 1, & \overset{2}{e}_2 = r, & \overset{3}{e}_3 = r\sin\vartheta, & \overset{1}{e}_4 = -iv, & \overset{4}{e}_4 = 1 \\ \underset{1}{e}^1 = 1, & \underset{2}{e}^2 = \dfrac{1}{r}, & \underset{3}{e}^3 = \dfrac{1}{r\sin\vartheta}, & \underset{4}{e}^1 = iv, & \underset{4}{e}^4 = 1 \end{array} . \qquad \text{(IV.14.10)}$$

With the transformation

$$dt = dt' - \alpha^2 vdr \qquad \text{(IV.14.11)}$$

we obtain the Schwarzschild metric and thus we have verified that (IV.14.8) is the

Schwarzschild metric in an oblique-angled co-ordinate system. Integration of (IV.14.11) yields

$$t = t' + 2\sqrt{2Mr} + 2M\ln\frac{1+v}{1-v}\ . \qquad \text{(IV.14.12)}$$

We know the expression from (IV.12.1)It is part of the Lemaître co-ordinate transformation.

Taking into account that all the variables of the static system are time-independent and considering $\partial_1 = a\frac{\partial}{\partial r}$ we can derive from the oblique system (A) the well-known Schwarzschild field strenghts

$$\begin{aligned} B_m &= \left\{\frac{a}{r},0,0,0\right\}, \\ C_m &= \left\{\frac{a}{r},\frac{1}{r}\cot\vartheta,0,0\right\}, \qquad . \\ E_m &= \left\{\alpha v\frac{1}{\rho},0,0,0\right\}, \quad \rho = \sqrt{\frac{2r^3}{M}} \end{aligned} \qquad \text{(IV.14.13)}$$

Considering $\partial_1 = \frac{\partial}{\partial r},\ \partial_4 = iv\frac{\partial}{\partial r}$ we can derive from the oblique system (B) the field strengths

$$\begin{aligned} B_m &= \left\{\frac{1}{r},0,0,\frac{i}{\mathcal{R}}\right\}, \\ C_m &= \left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0,\frac{i}{\mathcal{R}}\right\}, \ . \\ E_m &= \left\{0,0,0,\frac{i}{\rho}\right\} \end{aligned} \qquad \text{(IV.14.14)}$$

Those are the very field strengths resulting from the freely falling Lemaître system.

IV.15. Tidal forces

Tidal forces which act on each body which is located in the static Schwarzschild field radially stretch and laterally squeeze this body. Misner [M], Thorne, and Wheeler (p. 861) have brought a calculation for this effect for the case of an extended body and Sharan [S] has drawn the field lines for the tidal forces.

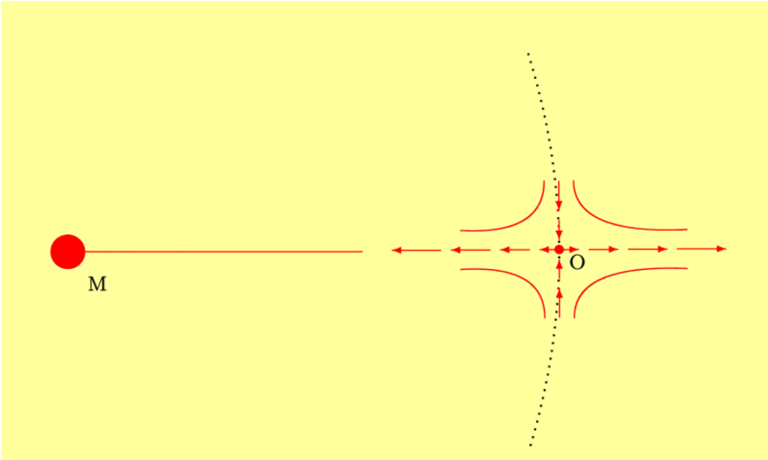

Fig. IV.11

For the free fall we have made extensive use of the tidal forces. Reviewing the static field equations no tidal forces can be detected. We want to show that the tidal forces occur in the static field after decomposing Einstein's field equations into subequations. After a transformation to a freely falling system they are replaced by those tidal forces already-discussed which are acting on a freely falling system.

In the literature the tidal forces are also known for the static case. They are not accessible from the Einstein field equations, but calculated with the Riemann tensor. First we follow the historic route and follow the exposition of Synge [S]. Restricting ourselves to two dimensions, a surface is covered with two geodesic curves and these curves are parameterized by u and v. We calculate the varying distances between the curves. This leads to a relative acceleration of the observers who are situated on these curves. If

$$u^i = \frac{dx^i}{du}, \quad v^i = \frac{dx^i}{dv} ,$$

are the tangent vectors of the curves then will apply

$$\frac{dv^i}{du} = \frac{du^i}{dv} .$$

The reason is the commutativity of the derivatives. The last equation can also be written as

$$v^i{}_{|k} u^k = u^i{}_{|k} v^k .$$

If we use reference systems instead of co-ordinates we obtain after a short calculation

$$v^s{}_{||m} u^m = u^s{}_{||m} v^m .$$

Differentiating again

$$\left(v^s{}_{||m}u^m\right)_{||n}u^n = \left(u^s{}_{||m}v^m\right)_{||n}u^n$$

and processing the brackets and taking into account that the curves are geodesics

$$u^m{}_{||n}u^n = 0$$

one finally arrives at

$$v^s{}_{||mn}u^m u^n = -R_{rmn}{}^s u^m u^n v^r \ . \qquad \text{(IV.15.1)}$$

If we define the deviation vector

$$\eta^s = \overset{s}{e}_i \frac{dx^i}{dv} dv = v^s dv \ ,$$

wherein dv is a constant infinitesimal quantity, one has

$$\eta^s{}_{||mn}u^m u^n = -R_{rmn}{}^s u^m u^n \eta^r \ , \qquad \text{(IV.15.2)}$$

which can be written as

$$\frac{D^2\eta^s}{D\tau^2} = -R_{rmn}{}^s u^m u^n \eta^r \ .$$

This relation describes the relative acceleration between two observers, sitting on two adjacent u-curves and having the distance η. The relation can be extended to four dimensions and the acceleration can be calculated for more curves. The forces that are associated with this acceleration are called tidal forces and can be educed from the Riemann tensor.

In the literature[48] they are mostly specified in the form

$$R_{2112} = R_{3113} = R_{4224} = R_{4334} = -\frac{M}{r^3}, \quad R_{4114} = R_{3223} = \frac{2M}{r^3} \ . \qquad \text{(IV.15.3)}$$

All other components result from the symmetry properties of the Riemann tensor or they vanish. However, this representation conceals the geometric background. From the Riemann tensor in tetrad representation

$$R_{smn}{}^r = A_{mn}{}^r{}_{|s} - A_{sn}{}^r{}_{|m} + A_{mn}{}^h A_{sh}{}^r - A_{sn}{}^h A_{mh}{}^r + A_{ms}{}^h A_{hn}{}^r - A_{sm}{}^h A_{hn}{}^r \qquad \text{(IV.15.4)}$$

follows

$$R_{2112} = R_{3113} = R_{4224} = R_{4334} = \frac{1}{\rho}\frac{1}{r}\sin\varepsilon, \quad R_{4114} = \frac{1}{\rho^2} - E_1\frac{1}{\rho}\rho_{|1}, \quad R_{3223} = \frac{1}{r^2}\sin^2\varepsilon \ . \qquad \text{(IV.15.5)}$$

Therein is

$$\sin\varepsilon = v = -\sqrt{\frac{2M}{r}}, \quad \rho = \sqrt{\frac{2r^3}{M}}, \quad E_1 = \alpha v\frac{1}{\rho}, \quad E_1\frac{1}{\rho}\rho_{|1} = -\frac{3}{\rho^2} \qquad \text{(IV.15.6)}$$

with the angle of ascent ε and the curvature vector ρ of the so-called Schwarzschild parabola. By rotation about the directrix of the parabola a surface of the 4th order is created. Except for the second term in the second relation (IV.15.5) (which will be discussed in detail later on) these are the products of the curvatures of the normal and oblique slices of this surface. This indicates the possibility of an embedding of the

[48] Differences of the signs occurring in the literature result from different definitions of the Riemann tensor and from the time-like components.

Schwarzschild metric into a higher dimensional flat space. Concerning the spatial part of the Schwarzschild metric Flamm [F] has recognized this possibility and has found as the basis of the Schwarzschild geometry the paraboloid named after him. We will treat this problem in detail later on. The quantities

$$A_{11} = \frac{1}{\rho}, \quad A_{22} = \frac{v}{r}, \quad A_{33} = \frac{v}{r}, \quad A_{44} = \frac{1}{\rho} \tag{IV.15.7}$$

we will interpret as second fundamental forms of the surface theory. Here they are understood as tidal forces.

If we define the unit matrix in the 2-dimensional [1,4]-subspace by

$$h_{mn} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \tag{IV.15.8}$$

we obtain for the Riemann tensor and Ricci tensor

$$R_{rmn}{}^{s} = 2A_{m[n}A_{r]}{}^{s} + 2h_{m[n}h_{r]}{}^{s}\frac{3}{\rho^2}, \quad R_{mn} = 2A_{m[n}A_{s]}{}^{s} + 2h_{m[n}h_{s]}{}^{s}\frac{3}{\rho^2}\ . \tag{IV.15.9}$$

These are up to the second terms on the right sides the Gaussian and contracted Gaussian equations. Using (IV.15.7) we see that the Ricci vanishes. Thus, we have a vacuum solution. It is obvious that the tidal forces which occur in the Ricci of the static system, are hidden by the very term which we will derive from the embedding of the Schwarzschild model later on.

The Einstein tensor has the form

$$G_{mn} = \left[2A_{m[n}A_{s]}{}^{s} - g_{mn}A^{r}{}_{[r}A_{s]}{}^{s}\right] + \left[2h_{m[n}h_{s]}{}^{s} - g_{mn}h^{r}{}_{[r}h_{s]}{}^{s}\right]\frac{3}{\rho^2}\ , \tag{IV.15.10}$$

wherein the second brackets can be simplified to $h_{mn} - g_{mn}$. Forming the divergence of the Einstein tensor

$$G_{m}{}^{n}{}_{||n} = \left[2A_{m}{}^{[n}A_{s}{}^{s]} - \delta^{n}_{m}A^{r}{}_{[r}A_{s]}{}^{s}\right]_{||n} + \left[\left(h_{m}{}^{n}{}_{||n} - \delta^{n}_{m||n}\right)\frac{3}{\rho^2} + \left(h_{m}{}^{n} - \delta^{n}_{m}\right)\left(\frac{3}{\rho^2}\right)_{||n}\right] \tag{IV.15.11}$$

of the second square brackets only remain

$$h_{m}{}^{n}{}_{||n} = \frac{4}{\rho^2}\rho_{|m}, \quad \rho_{|1} = -3\frac{a}{v}\ , \tag{IV.15.12}$$

while the terms in the first brackets can be written as

$$A_{<n}{}^{n}A_{m}{}^{s}{}_{||s>}\ .$$

The contracted Codazzi equations are included in (IV.15.11), but due to

$$2A_{[n}{}^{s}{}_{||m]} = \frac{2}{\rho^2}\rho_{||[n}u_{m]}u^{s}, \quad 2A_{[m}{}^{n}{}_{||n]} = \frac{1}{\rho^2}\rho_{|m} \tag{IV.15.13}$$

they do not vanish. This is derived from the fact that the Schwarzschild geometry is not a V_4 which can be embedded into a 5-dimensional flat space, as we will see. Inserting (IV.15.12) and (IV.15.13) into (IV.15.11) we obtain the trivial result that the divergence of the Einstein tensor vanishes.

Tidal forces will come to light explicitly if we calculate the subequations of Einstein's field equations. In Section IV.1 we have decomposed the Ricci into its subequations which are the equations of the curvatures of a surface still to be discussed:

$$\begin{aligned} R_{mn} = &-\left[B_{m\underset{2}{||}n} + B_m B_n\right] - b_m b_n\left[B^s{}_{\underset{2}{||}s} + B^s B_s\right] \\ &-\left[C_{m\underset{3}{||}n} + C_m C_n\right] - c_m c_n\left[C^s{}_{\underset{3}{||}s} + C^s C_s\right] \\ &+\left[E_{m\underset{4}{||}n} - E_m E_n\right] + u_m u_n\left[E^s{}_{\underset{4}{||}s} - E^s E_s\right] \end{aligned} . \qquad \text{(IV.15.14)}$$

For the individual brackets one has:

$$\begin{aligned} &B_{m\underset{2}{||}n} + B_m B_n = A_{mn}A_{22}, && B^s{}_{\underset{2}{||}s} + B^r B_r = A_s{}^s A_{22}, && m,n,s = 1 \\ &C_{m\underset{3}{||}n} + C_m C_n = A_{mn}A_{33}, && C^s{}_{\underset{3}{||}s} + C^s C_s = A_s{}^s A_{33}, && m,n,s = 1,2 \\ &E_{m\underset{4}{||}n} - E_m E_n = A_{mn}A_{44} + \frac{3}{\rho^2}h_{mn}, && E^s{}_{\underset{4}{||}s} - E^s E_s = A_s{}^s A_{44} + \frac{3}{\rho^2}, && m,n,s = 1,2,3 \end{aligned} \qquad \text{(IV.15.15)}$$

Summing up these relations we get again (IV.15.9). The tidal forces which occur in the freely falling system we have already discussed in detail. The next step is to formulate the relations between the tidal forces of the static system and the ones of the freely falling system. We apply to the static quantities the Lorentz transformation to the free-falling system and the Lorentz term

$$L_{m'n'}{}^{s'} = L_s^{s'} L^s_{n'|m'} = h_{m'}{}^{s'} L_{n'} - h_{m'n'} L^{s'}, \quad L_{n'} = L_{s'n'}{}^{s'} = \left\{\alpha^2 v\frac{1}{\rho}, 0, 0, -i\alpha^2\frac{1}{\rho}\right\} . \qquad \text{(IV.15.16)}$$

The field strengths B and C transform as vectors

$$B_{n'} = \left\{\frac{1}{r}, 0, 0, -iv\frac{1}{r}\right\}, \quad C_{n'} = \left\{\frac{1}{r}, \frac{1}{r}\cot\vartheta, 0, -iv\frac{1}{r}\right\} , \qquad \text{(IV.15.17)}$$

but the time-like part of the Ricci-rotation coefficients require the inhomogeneous transformation law

$$'E_{n'm'}{}^{s'} = E_{n'm'}{}^{s'} + L_{n'm'}{}^{s'}, \quad E_{n'm'}{}^{s'} = L^{n\,m\,s'}_{n'm's} E_{nm}{}^{s} . \qquad \text{(IV.15.18)}$$

Since this can be written as

$$E_{m'n'}{}^{s'} = -\left[h_{m'}{}^{s'} E_{n'} - h_{m'n'} E^{s'}\right] , \qquad \text{(IV.15.19)}$$

with (IV.15.16) only

$$'E_{m'n'}{}^{s'} = -\left[h_{m'}{}^{s'}\,{}'E_{n'} - h_{m'n'}\,{}'E^{s'}\right], \quad 'E_{n'} = \left\{0, 0, 0, \frac{i}{\rho}\right\} \qquad \text{(IV.15.20)}$$

remains. The three-rank inhomogeneous transformation law (IV.15.18) was attributed to the vector addition law of forces

$$'E_{n'} = E_{n'} + L_{n'} . \qquad \text{(IV.15.21)}$$

The force of gravity was dynamically nullified. The fourth remaining component of 'E is the radial tidal force of the freely falling system. The fourth components in (IV.15.17) are the lateral tidal forces. To see how the tidal forces of the static and freely falling systems are related, we maintain the vector notation for the tidal forces, and we define the graded derivatives

$$\begin{gathered}\Phi_{m'\underset{1}{\|}n'} = \Phi_{m'|n'} - L_{n'm'}{}^{s'}\Phi_{s'}, \qquad \Phi_{m'\underset{2}{\|}n'} = \Phi_{m'|n'} - L_{n'm'}{}^{s'}\Phi_{s'} \\ \Phi_{m'\underset{3}{\|}n'} = \Phi_{m'|n'} - \left(B_{n'm'}{}^{s'} + L_{n'm'}{}^{s'}\right)\Phi_{s'} \qquad . \\ \Phi_{m'\underset{4}{\|}n'} = \Phi_{m'|n'} - \left(B_{n'm'}{}^{s'} + C_{n'm'}{}^{s'} + L_{n'm'}{}^{s'}\right)\Phi_{s'}\end{gathered} \tag{IV.15.22}$$

Omitting the primes on the indices we get for the Ricci the auxiliary form

$$\begin{aligned}{}'R_{mn} = h_{mn}\left[{}'E^{s}{}_{\underset{1}{\|}s} - {}'E^{s}\,{}'E_{s}\right] - \left[B_{n\underset{2}{\|}m} + B_{n}B_{m}\right] - b_{m}b_{n}\left[B^{s}{}_{\underset{2}{\|}s} + B^{s}B_{s}\right] \\ - \left[C_{n\underset{3}{\|}m} + C_{n}C_{m}\right] - c_{m}c_{n}\left[C^{s}{}_{\underset{3}{\|}s} + C^{s}C_{s}\right],\end{aligned} \tag{IV.15.23}$$

which we evaluate with (IV.15.17) and (IV.15.19). If we define further auxiliary quantities the tidal forces

$$\tilde{Q}_{11} = Q_{11}, \quad \tilde{Q}_{22} = Q_{22}, \quad \tilde{Q}_{33} = Q_{33}, \quad \tilde{Q}_{44} = Q_{11}\ , \tag{IV.15.24}$$

the desired relationship can be established with

$${}'R_{mn} = -2\tilde{Q}_{m[n}\tilde{Q}_{s]}{}^{s} + h_{mn}\frac{3}{\rho^{2}}\,. \tag{IV.15.25}$$

From the definitions of the two types of tidal force can be seen that

$$A_{mn}A_{rs} = -\tilde{Q}_{mn}\tilde{Q}_{rs}\ . \tag{IV.15.26}$$

Taking into account the invariance of the Ricci under Lorentz transformations

$$R_{m'n'} = L^{m\ n}_{m'n'}R_{mn} \tag{IV.15.27}$$

and in addition that the right side of (IV.15.9)(2) is form invariant under the Lorentz transformation to the freely falling system, i.e. that this relation is also valid for primed indices, it follows from (IV.15.9) and (IV.15.25) that the tidal forces are exchanged in the course of the transition from the static system to the freely falling system. The tidal forces which a freely falling observer experiences are geometrically understood as the second fundamental forms of the surface theory, referring to an observer associated with the shrinking surface.

With the problem of tidal forces have dealt in detail Mashhoon [M], Mashhoon [M], and Theiss; Costa [C] and Herdeiro; Chicone [C] and Mashhoon; and Mashhoon [M] and McClune and also Campbell [C] and Morgan. However, most authors do not start with Einstein's field equations to study the tidal forces, but with the Riemann tensor or with the Weyl tensor, the latter being deduced from the Riemann tensor. Decomposing these tensors they get quantities which are comparable with the quantities of the theory of electromagnetism. In a later Section we will show that the Einstein field equations supply that correspondence and recourse to the Riemann tensor or the Weyl tensor is not necessary.

IV.16. The Kruskal metric

Kruskal [K] and Szekeres [S] have transformed the Schwarzschild line element into a form that makes it possible to avoid the singularity of the metric in the standard form at r = 2M and to describe the region r < 2M. Geodesics can be tracked beneath the event horizon. A singularity remains only at r = 0 which is regarded as a genuine singularity. In Kruskal co-ordinates the Schwarzschild metric takes the form

$$ds^2 = \gamma^2\left(du^{1^2} + du^{4^2}\right) + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 \ , \tag{IV.16.1}$$

$$\gamma^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} \ . \tag{IV.16.2}$$

If the statements which are derivable from the Kruskal metric have a physical meaning, the decision on the aspects of the space curvature will be in favor of view (I). An interpretation of the Kruskal metric in the sense of the surface theory is not possible any longer.

We will show by a simple shaping that all statements on the interior region can be eliminated from the theory and that a geometrical interpretation of the space curvature in the sense of (III) is possible [B]. Thereby the transition from the t-notation to the it-notation plays a substantial role. Treating this problem the close relation between it-notation and the interpretation of the spacetime curvature in accordance with (III) becomes known. By a careful mathematical development of this description the geometrically determined acceleration of the Kruskal observers becomes conspicuous as well.

In the form (IV.16.1) of the metric the u-co-ordinate[49] system describes a wider range than the standard Schwarzschild co-ordinate system. Parameterizing the metric by

$$Y(r) = \frac{\sqrt{1-2M/r}}{\sqrt{2M/r}} e^{\frac{r}{4M}}, \quad Y(2M) = 0, \quad \chi = \frac{t}{4M} \tag{IV.16.3}$$

four sectors result

$$\mathrm{I}\ \begin{matrix} u^1 = Y\cos i\chi \\ u^4 = Y\sin i\chi \end{matrix}, \quad \mathrm{II}\ \begin{matrix} u^1 = \ \ Y\sin i\chi \\ u^4 = -Y\cos i\chi \end{matrix}, \quad \mathrm{III}\ \begin{matrix} u^1 = -Y\cos i\chi \\ u^4 = -\,Y\sin i\chi \end{matrix}, \quad \mathrm{IV}\ \begin{matrix} u^1 = -Y\sin i\chi \\ u^4 = \ \ Y\cos i\chi \end{matrix} . \tag{IV.16.4}$$

Because of this multitudinousness one speaks of the fourfold truth of the Kruskal system[50]. The equation

$$u^{1^2} + u^{4^2} = Y^2$$

stands for the families of pseudo circles or for the four families of the branches of hyperbolae of constant curvature. The above relations are illustrated by the Kruskal diagram. In order to find a pseudo-real representation one has to use the t-notation and to set $u^4 = iu^0$ in the above equation[51].

The sectors I and III explain the acceleration of an observer, the regions II and IV refer to the interior region of the Schwarzschild metric. Region II is quoted for elucidating a

[49] The co-ordinate system has here been adapted to the it-notation contrary to the original Kruskal way of writing.

[50] We will reduce these to a plain truth.

[51] Since we reject the t-notation, we do not use the Kruskal diagram but replace it by another representation.

black hole, the region IV for a white hole. The latter is a hypothetical stellar object which ejects matter. The straight lines r = 2M in the Kruskal diagram Fig. IV.12 are null-lines.

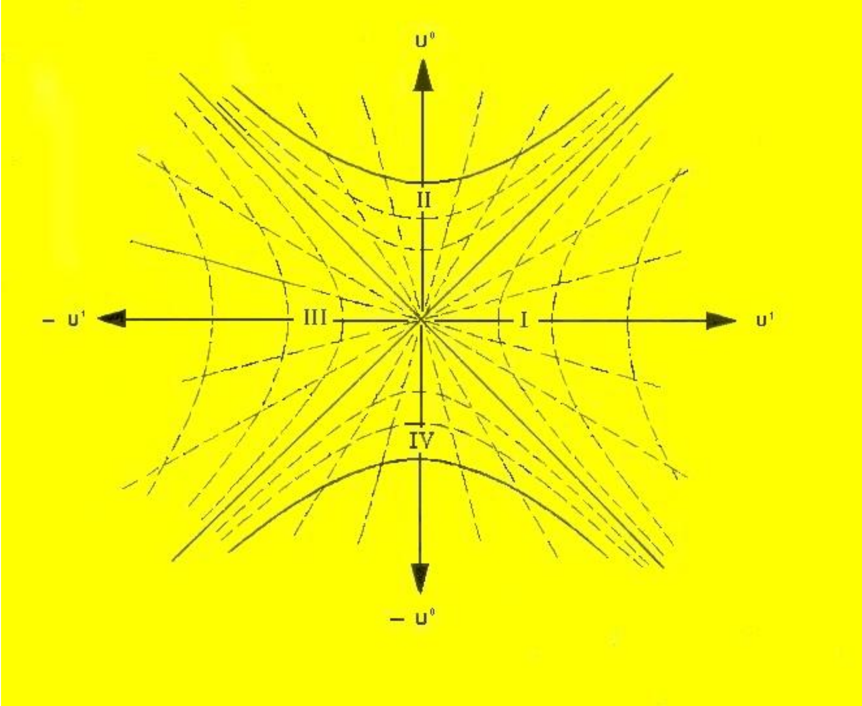

Fig. IV.12

The relations (IV.16.4) have been set in such a way that a subsequent handling is immediately possible. By differentiating these relations one obtains

$$du^1 = \cos i\chi dY - Y \sin i\chi di\chi, \quad du^4 = \sin i\chi dY + Y \cos i\chi di\chi \,. \tag{IV.16.5}$$

Calculating dY from (IV.16.3) and multiplying by γ one has

$$dx^{1'} = \cos i\chi dx^1 - \sin i\chi dx^4, \quad dx^{4'} = \sin i\chi dx^1 + \cos i\chi dx^4 \,. \tag{IV.16.6}$$

From this one can read the Lorentz transformation

$$L^{1'}_1 = \cos i\chi, \quad L^{1'}_4 = -\sin i\chi, \quad L^{4'}_1 = \sin i\chi, \quad L^{4'}_4 = \cos i\chi \;. \tag{IV.16.7}$$

In (IV.16.6) the expressions for the static Schwarzschild system

$$dx^1 = \alpha dr, \quad dx^4 = aidt \tag{IV.16.8}$$

are used. The Lorentz transformation describes a transformation to an accelerated observer, for example to a rocket which is to leave the Earth. From the Lorentz transformation one extracts the relative velocities

$$v_K = \mathrm{th}\chi \tag{IV.16.9}$$

of the observers that depend on the rescaled co-ordinate time $\chi = t/4M$. With infinitely high rapidity $\chi \to \infty$, which means after an infinitely long time, the observers reach the velocity of light. If one makes up $dx^{1'}/dx^{4'}$ from (IV.16.6) and if one uses the proper times by means of $dx^{4'} = id\tau'$, $dx^4 = id\tau$, one obtains the Einstein addition law for velocities

$$\frac{dx^{1'}}{d\tau'} = \frac{v_R + v_K}{1 + v_R v_K} \;, \tag{IV.16.10}$$

wherein $v_R = dx^1/d\tau$ is the unspecified velocity of an observer in radial motion. For time reversal one obtains the speed of an incoming observer and a similar addition law. For the sectors II and IV one gets an expression which corresponds to (IV.16.10), with the relative velocity

$$v_K = \operatorname{cth}\chi \,. \qquad \text{(IV.16.11)}$$

This motion is tachyonic[52] (De Sabbata [D], Pavšič, Ricami). A rocket would have an infinite high speed at r = 2M and would deboost to the speed of light if it reaches infinity. With time reversal the rocket has the speed of light at infinity and reaches infinite high speed on its way to the gravitation center. Fig. IV.13 illustrates the circumstances.

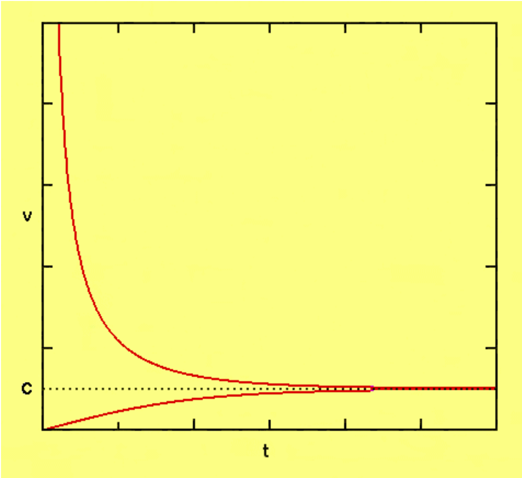

Fig. IV.13

We point out that all these considerations refer to the exterior region of the Schwarzschild solution and that the above operations take place in the tangent spaces of the Schwarzschild geometry. The two tachyonic sectors are recognized as unphysical and are excluded from the theory. Incoming observers reach the velocity of light at r = 2M independently of whether they fall freely or are Kruskal accelerated. r = 2M represents the boundary of the geometry. Since the sectors I and III state the same, we have reduced the fourfold truth to a plain one. This is a consequence of the transition of the t-notation to the it-notation.

Since the Lorentz transformation (IV.16.7) contains the non-constant parameter χ, one can conclude that the reference system to which one turns to by this Lorentz transformation must be accelerated. This aspect has now to be worked out. In the usual way we apply the transformation law

$$\Phi_{m'\|n'} = L^{m\ n}_{m'n'}\Phi_{m\|n} = \left[\Phi_{m'|n'} - L^{s'}_{s}L^{s}_{m'|n'}\Phi_{s'}\right] - A_{n'm'}{}^{s'}\,\Phi_{s'} \qquad \text{(IV.16.12)}$$

and again we allocate the Lorentz term to the graded derivative

$$\Phi_{m'\underset{1}{\|}n'} = \Phi_{m'|n'} - L_{n'm'}{}^{s'}\,\Phi_{s'}\,, \quad L_{n'm'}{}^{s'} = L^{s'}_{s}L^{s}_{m'|n'} \;. \qquad \text{(IV.16.13)}$$

After performing the Lorentz transformation the first and fourth unit vectors of the static system have the form

$$m_{n'} = \{\cos i\chi, 0, 0, \sin i\chi\} \quad u_{m'} = \{-\sin i\chi, 0, 0, \cos i\chi\} \;. \qquad \text{(IV.16.14)}$$

The vectors connected with the Kruskal system are

[52] Superluminous motion

$$'m_{n'} = \{1,0,0,0\}, \quad 'u_{m'} = \{0,0,0,1\} \ . \tag{IV.16.15}$$

If one computes thereby the Lorentz term

$$L_{m'n'}{}^{s'} = h_{m'n'}K^{s'} - h_{m'}{}^{s'}K_{n'}, \quad h_{m'n'} = 'm_{m'}'m_{n'} + 'u_{m'}'u_{n'} = m_{m'}m_{n'} + u_{m'}u_{n'} \tag{IV.16.16}$$

one obtains the Kruskal acceleration

$$K_{n'} = \frac{1}{4Ma}m_{n'} = \frac{1}{4Ma}\{\cos i\chi, 0, 0, \sin i\chi\} \tag{IV.16.17}$$

that has the form

$$K_n = \frac{1}{4Ma}m_n = \frac{1}{4Ma}\{1,0,0,0\} \ , \tag{IV.16.18}$$

experienced by an observer of the static system. With Fig. IV.14 one can compare the Kruskal acceleration (left figure) with the gravitational acceleration (right figure)[53].

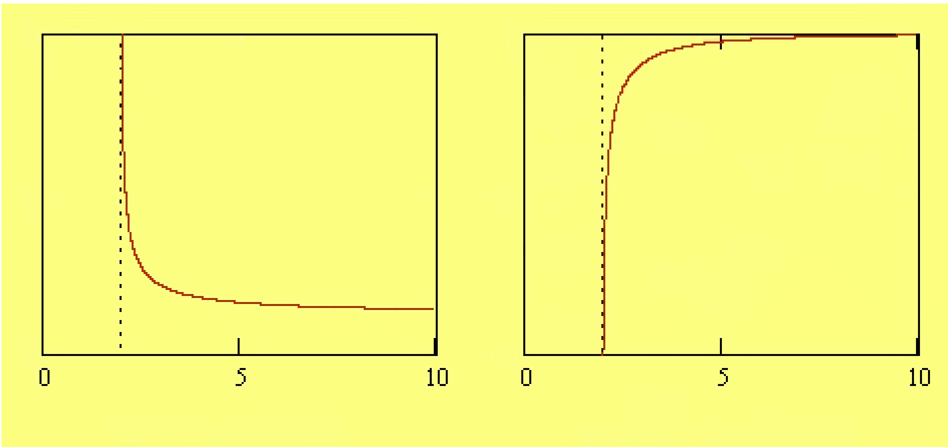

Fig. IV.14

One computes the effective acceleration with

$$'u_{\alpha'\|n'}{}'u^{n'} = -\left[A_{4'\alpha'}{}^{4'} + L_{4'\alpha'}{}^{4'}\right] = E_{\alpha'} + K_{\alpha'}, \quad \alpha = 1,2,3 \ , \tag{IV.16.19}$$

where the gravitational acceleration E is negative. For negative K (incoming observers) and positive K (running out observers) the effective acceleration is depicted in Fig. IV.15.

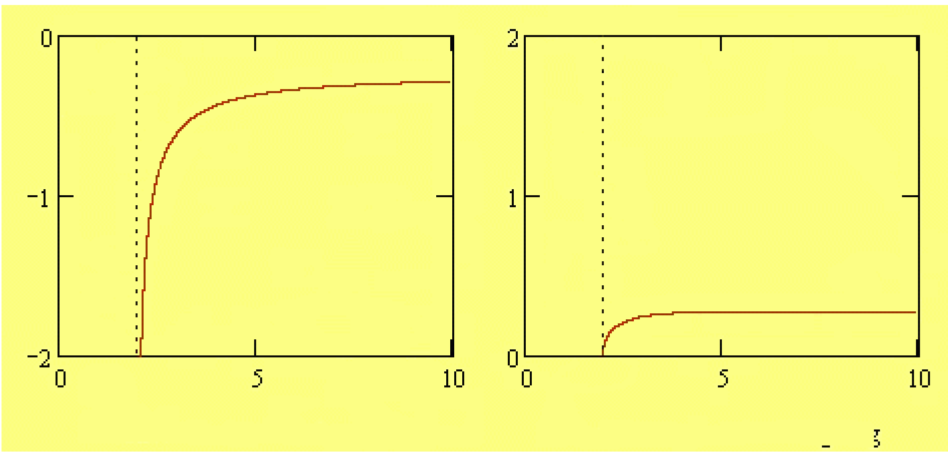

Fig. IV.15

[53] The vertical dashed line marks the event horizon at r = 2M.

The equations for the force of gravity, for the free fall, and for the Kruskal acceleration are composed of the geometrical quantities M, a, and v. In the course of the derivation of the Schwarzschild model from the surface theory the close connection of the forces with the geometry will become evident.

In the Kruskal system the field strengths take the already well-known form

$$\begin{aligned} B_{m'} &= \left\{ \frac{a}{r}\cos i\chi, 0, 0, \frac{a}{r}\sin i\chi \right\} \\ C_{m'} &= \left\{ \frac{a}{r}\cos i\chi, \frac{1}{r}\cot\vartheta, 0, \frac{a}{r}\sin i\chi \right\} \; . \\ E_{m'} &= \left\{ \frac{1}{\rho}\frac{v}{a}\cos i\chi, 0, 0, \frac{1}{\rho}\frac{v}{a}\sin i\chi \right\} \end{aligned} \qquad \text{(IV.16.20)}$$

From the Lorentz transformation of the Ricci tensor one obtains

$$R_{m'n'} = A_{m'n'}{}^{s'}{}_{\underset{1}{\|} s'} - A_{n' \underset{1}{\|} m'} - A_{r'm'}{}^{s'} A_{s'n'}{}^{r'} + A_{m'n'}{}^{s'} A_{s'} \qquad \text{(IV.16.21)}$$

and from this once more the field equations for the static system

$$\begin{aligned} R_{n'm'} = &-\left[B_{m' \underset{2}{\|} n'} + B_{m'}B_{n'} \right] - b_{m'}b_{n'}\left[B^{r'}{}_{\underset{2}{\|} r'} + B^{r'}B_{r'} \right] \\ &-\left[C_{m' \underset{3}{\|} n'} + C_{m'}C_{n'} \right] - c_{m'}c_{n'}\left[C^{r'}{}_{\underset{3}{\|} r'} + C^{r'}C_{r'} \right] \\ &+\left[E_{m' \underset{4}{\|} n'} - E_{m'}E_{n'} \right] + u_{m'}u_{n'}\left[E^{r'}{}_{\underset{4}{\|} r'} - E^{r'}E_{r'} \right] = 0 \\ & B_{[m' \underset{2}{\|} n']} = 0, \quad C_{[m' \underset{3}{\|} n']} = 0, \quad E_{[m' \underset{4}{\|} n']} = 0 \end{aligned} \; , \qquad \text{(IV.16.22)}$$

this time, however, with the measured values of the Kruskal accelerated observer. The invariance of the subequations of the field equations under Lorentz transformations manifests the invariance of the physical statements which are made on the static system.

However, if one wants to make statements which concern directly the Kruskal accelerated observer, one has to interpret actively the Lorentz transformation and one has to perform a [3+1]-decomposition with respect to $'m_{m'}, 'u_{m'}$. The Ricci tensor takes the familiar form under the Lorentz transformation

$$\begin{aligned} R_{m'n'} &= 'A_{m'n'}{}^{s'}{}_{| s'} - 'A_{n' | m'} - 'A_{r'm'}{}^{s'}\, 'A_{s'n'}{}^{r'} + 'A_{m'n'}{}^{s'}\, 'A_{s'} \\ 'A_{n'm'}{}^{s'} &= L^{n\,m\,s'}_{n'm's} A_{nm}{}^{s} + L_{n'm'}{}^{s'} \end{aligned} \; , \qquad \text{(IV.16.23)}$$

whereby the Ricci-rotation coefficients 'A contain in addition to the curvature contributions A the Lorentz term L. The invariance of the Ricci tensor is assured by decoupling an equation for the Lorentz term

$$L_{m'n'}{}^{s'}{}_{| s'} - L_{s'n'}{}^{s'}{}_{| m'} - L_{r'm'}{}^{s'} L_{s'n'}{}^{r'} + L_{m'n'}{}^{s'} L_{r's'}{}^{r'} + 2A_{[m'r']}{}^{s'} L_{s'n'}{}^{r'} = 0 \qquad \text{(IV.16.24)}$$

If one processes therein (IV.16.16) one obtains the simple field equations

$$K^{s'}{}_{\underset{1}{\|} s'} - K^{s'} E_{s'} = 0 \qquad \text{(IV.16.25)}$$

for the Kruskal acceleration, which contains a coupling term with the force of gravity as source of the acceleration field. In order to make transparent the physics of the Kruskal accelerated observer, we replace the representation with the one of the second fundamental forms of the surface theory, as we have done for the free fall. If one decomposes the Ricci-rotation coefficients according to the schema (IV.1.5), first line, one can represent the force of gravity similar to (IV.16.16)

$$E_{m'n'}{}^{s'} = h_{m'n'} E^{s'} - h_{m'}{}^{s'} E_{n'} \; . \tag{IV.16.26}$$

We summarize the Kruskal acceleration and the gravitational acceleration to the effective Kruskal acceleration

$$K^{e}_{m'} = K_{m'} + E_{m'} \tag{IV.16.27}$$

as it was presented in Fig. IV.15. Then the quantity

$$Q_{1'}{}^{1'} = A_{1'4'}{}^{1'} + L_{1'4'}{}^{1'} = -K^{e}_{4'} \tag{IV.16.28}$$

is separated from (IV.16.16) and (IV.16.26). It contains as curvature piece the second fundamental form $A_{1'4'}{}^{1'} = -E_{4'}$ and the dynamic piece $L_{1'4'}{}^{1'} = -K_{4'}$. In addition, if one supplements with the curvature equations

$$Q_{2'2'} = B_{4'} = \frac{a}{r} \sin i\chi, \quad Q_{3'3'} = C_{4'} = \frac{a}{r} \sin i\chi \;, \tag{IV.16.29}$$

one can finally split off the tidal forces from the Ricci-rotation coefficients

$$A_{mn}{}^{s} = {}^{*}A_{mn}{}^{s} + Q_{m}{}^{s}\,{}'u_{n} - Q_{mn}\,{}'u^{s} \, . \tag{IV.16.30}$$

They act on the Kruskal-accelerated observer. From now on the primes at the indices are omitted. The subsequent treatment of this expression becomes somewhat more difficult than the treatment of the free fall, because the field strengths derived from the Lorentz transformation do not nullify the force of gravity. If one implements the space-time decomposition with respect to the Kruskal-accelerated reference system using (IV.16.30) one obtains the set of equations[54]

$$\begin{aligned}
R_{mn} = & -\left[{}^{*}B_{n\underset{2}{\wedge}\underline{m}} + {}^{*}B_{n}{}^{*}B_{m} \right] - b_{m}b_{n}\left[{}^{*}B^{s}{}_{\underset{2}{\wedge}s} + {}^{*}B^{s}{}^{*}B_{s} \right] \\
& -\left[{}^{*}C_{n\underset{3}{\wedge}\underline{m}} + {}^{*}C_{n}{}^{*}C_{m} \right] - c_{m}c_{n}\left[{}^{*}C^{s}{}_{\underset{3}{\wedge}s} + {}^{*}C^{s}{}^{*}C_{s} \right] \\
& +\left[{}^{*}K^{e}_{n\underset{4}{\wedge}\underline{m}} + {}^{*}K^{e}_{n}{}^{*}K^{e}_{m} \right] + {}'u_{m}{}'u_{n}\left[{}^{*}K^{s}_{e\underset{4}{\wedge}s} + {}^{*}K^{s}_{e}{}^{*}K^{e}_{s} \right] \\
& -\left[Q_{mn|s}\,{}'u^{s} + Q_{mn}Q_{r}{}^{r} \right] \\
& -{}'u_{m}\left[{}^{*}B_{n|s}\,{}'u^{s} + {}^{*}B_{sn}{}^{r}Q_{r}{}^{s} + {}^{*}K^{e}_{n}Q_{22} \right] - {}'u_{m}\left[{}^{*}C_{n|s}\,{}'u^{s} + {}^{*}C_{sn}{}^{r}Q_{r}{}^{s} + {}^{*}K^{e}_{n}Q_{33} \right] \\
& -{}'u_{n}\left[Q_{s}{}^{s}{}_{\underset{4}{\wedge}\underline{m}} - Q_{m}{}^{s}{}_{\underset{4}{\wedge}s} \right] \\
& -{}'u_{m}{}'u_{n}\left[Q_{s}{}^{s}{}_{|r}\,{}'u^{r} + Q^{rs}Q_{rs} \right]
\end{aligned} \quad .\text{(IV.16.31)}$$

Further work about the Kruskal metric stem from Antoci [A] and Liebscher, Beck [B], Beig [B], Buchdahl [B], Espinosa [E], Ferraris [F], Gautreau [G], Hawking [H] and Ellis, Mitra [M] and Rindler [R]. Sassi [S] deduces more generally an embedding found by Fronsdal.

In this Section a picture has been drawn deviating from the literature concerning the Kruskal metric. We have found our way to it by applying view (III) of the space curvature and consequently abiding by the it-notation. This strategy will also be maintained in the following Sections. It enables us to describe the gravitation by means of surfaces. That

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[54] Quantities with * are purely spatial, so is an underlined index. The graded derivatives with ^ are constructed similarly to the graded derivatives with ||, however, with the quantities caring asterisks.

comes to meet our imagination and will bring in techniques which facilitate the description of gravitational models.

IV.17. The Eddington – Finkelstein – co-ordinate system

In the previous Sections, the Schwarzschild metric has been examined and some problems have been treated in detail. The standard Schwarzschild co-ordinate r will be considered by us as a co-ordinate of the flat embedding space later on and is to be understood as an auxiliary variable in the 4-dimensional space of imagination. It has the limited range of values $2M \le r \le \infty$. The Schwarzschild radius defines the boundary of the geometry. Not all authors share this view. About attempts to go beneath the Schwarzschild radius has already been written here.

Eddington [E] and later on Finkelstein [F] have introduced a new co-ordinate system which is free from singularities. The position $r = 2M$ does not exhibit any special features of the metric. This apparently opens the possibility to describe the inner region $0 < r < 2M$. That this is not the case we will show by a careful analysis.

The transformation

$$dt = dt' + \frac{2M}{r-2M} dr', \quad dt = dt' - \frac{2M}{r-2M} dr', \quad r = r' \tag{IV.17.1}$$

leads, with the Eddington co-ordinates $\{r, t\}$ and the standard Schwarzschild co-ordinates $\{r', t'\}$, to the singularity-free form of the metric

$$ds^2 = dr^2 - dt^2 + v^2 (dr + dt)^2 + d\Omega^2, \quad d\Omega^2 = r^2 d\vartheta^2 + r^2 \sin^2\vartheta \, d\varphi^2 \; . \tag{IV.17.2}$$

Therein $v = -\sqrt{2M/r}$, in turn, has the meaning of the velocity of a freely falling observer. It is not surprising that a metric free from singularities is formed by the transformation (IV.17.1). Finally, the transformation itself has a singularity at $r = 2M$ which is not mentioned in the literature. Integration of (IV.17.1) provides

$$t = t' \pm 2M \ln(r - 2M) \; . \tag{IV.17.3}$$

This result requires $r > 2M$ and excludes the inner region of the Schwarzschild metric. To prevent just this, most authors have used the more general solution[55]

$$t = t' \pm 2M \ln|r - 2M| \tag{IV.17.4}$$

This approach has been criticized by Mitra [M]. The metric can also be written in the form

$$ds^2 = (1 + v^2) dr^2 + 2v^2 dr dt - a^2 dt^2 \tag{IV.17.5}$$

whereby the polar terms have been suppressed. It is clear that this form of the metric is not symmetric under time reversal, although the Einstein field equations possess this symmetry. That the problems at the event horizon cannot be removed by using a co-ordinate system for the metric free from singularities will be shown by a more detailed analysis. From (IV.17.5) we notice that the metric has a cross term which indicates an oblique-angled co-ordinate system. If we restrict ourselves to the positive sign in (IV.17.1) we can bring the metric into the form

$$ds^2 = \alpha^2 dr^2 + (iadt - i\alpha v^2 dr)^2, \quad \alpha = 1/\sqrt{1 - 2M/r}, \quad a = \alpha^{-1} \; . \tag{IV.17.6}$$

[55] Eddington co-ordinates are usually called incoming co-ordinates.

With this notation one does not have insight concerning the singularity-free representation of the metric, but a direct reading of the 4-bein is possible

$$\begin{array}{lll} \overset{1}{e}_1 = \alpha, & \overset{4}{e}_1 = -i\alpha v^2, & \overset{4}{e}_4 = a \\ \underset{1}{e}^1 = a, & \underset{1}{e}^4 = i\alpha v^2, & \underset{4}{e}^4 = \alpha \end{array} . \tag{IV.17.7}$$

If we calculate with these quantities the Ricci-rotation coefficients we obtain the same results that we would obtain by using the Schwarzschild standard co-ordinates, but circumstantially. Thus, there exists an event horizon, even if one takes the first steps using Eddington co-ordinates. An incoming observer reaches the velocity of light at this position as well. This is less surprising, because the physics of a model should be independent of the co-ordinate system. Having determined the Ricci-rotation coefficients by using a particular co-ordinate system, one can continue the calculations without co-ordinates.

The co-ordinate transformation can be formulated with the help of a matrix

$$\begin{array}{llll} \Lambda_1^{1'} = 1, & \Lambda_4^{1'} = 0, & \Lambda_1^{4'} = -i\alpha^2 v^2, & \Lambda_4^{4'} = 1 \\ \Lambda_{1'}^{1} = 1, & \Lambda_{4'}^{1} = 0, & \Lambda_{1'}^{4} = i\alpha^2 v^2, & \Lambda_{4'}^{4} = 1 \end{array} . \tag{IV.17.8}$$

No Lorentz transformation can be associated with it. Therefore the Eddington co-ordinates are not related to the new state of motion of an observer.

With the help of the above matrix the 4-bein system of Eddington and Finkelstein can be derived from the standard 4-bein system of the Schwarzschild theory

$$\overset{m}{e}_i = \Lambda_i^{i'} \overset{m}{e}_{i'} .$$

However, the Ricci-rotation coefficients are generally invariant under such transformations

$$A_{mn}{}^{s} = \overset{s}{e}_i \, e^i_{[n|m]} + g^{sr} g_{mt} \overset{t}{e}_i \, e^i_{[n|r]} + g^{sr} g_{nt} \overset{t}{e}_i \, e^i_{[m|r]} = \overset{s}{e}_{i'} \, e^{i'}_{[n|m]} + g^{sr} g_{mt} \overset{t}{e}_{i'} \, e^{i'}_{[n|r]} + g^{sr} g_{nt} \overset{t}{e}_{i'} \, e^{i'}_{[m|r]} ,$$

provided that the Λ are holonomic so that

$$\Lambda^{i'}_{[i|k]} = x^{i'}_{|[ik]} = 0 . \tag{IV.17.9}$$

is satisfied.

If this relation is not fulfilled, the Ricci-rotation coefficients are modified and they describe a new geometrical or physical fact. The method can be used to make a more complicated but physically viable theory from a simple and geometrically comprehensible auxiliary metric (seed metric). The method proves to be advantageous when building the Kerr metric and all the members of the Kerr family.

For completeness' sake we specify the metric in Eddington co-ordinates

$$\begin{array}{lll} g_{11} = 1 + v^2, & g_{14} = -iv^2, & g_{44} = a^2 \\ g^{11} = a^2, & g^{14} = iv^2, & g^{44} = 1 + v^2 \end{array} . \tag{IV.17.10}$$

The Christoffel symbols can be calculated by it. This is rather difficult. The Christoffel symbols are only vaguely related to the physical quantities. The field equations which are formulated with them are less informative.

Unruh [U] deals extensively with the metric of Gulstrand, Kasner, Kruskal, Eddington, Finkelstein and Lemaître. He discusses the transformations to these different forms of the Schwarzschild metric. But he insists that at $r = 2M$ is a coordinate singularity.

IV.18. Schwarzschild metric, 5-dimensional embedding

In the preceding Section we have shown, how the Schwarzschild metric could be embedded into a higher-dimensional flat space with at least 6 dimensions. All these embeddings have the peculiarity that the surfaces do not possess a pointer, i.e. a finite vector, which is fixed at a point and travels along the surface with its tip. In the preceding Section we have shown that the Schwarzschild metric can be described completely with the curvatures of a surface and that one can arrive intuitively at a 5-dimensional ansatz for the field strengths. We will continue to follow the 5-dimensional way. However, we stick to the method that we do not intend to disprove the theorems of Kasner and Eisenhart, but to circumvent them. Our representation needs only 5 dimensions, but 6 variables, whereby one variable remains hidden.

We start with a 4-dimensional pseudo-hyper sphere which is embedded into a 5-dimensional flat space. For this surface we set up the field equations. These already contain the formal skeleton for the field equations of the Schwarzschild model. Finally, the geometry of the pseudo-hyper sphere is deformed in such a way that it obtains the properties of Schwarzschild geometry. The ansatz for the pseudo-hyper sphere is well-known from the de Sitter model

$$\begin{aligned} X^{3'} &= X\sin\varepsilon\sin\vartheta\sin\varphi \\ X^{2'} &= X\sin\varepsilon\sin\vartheta\cos\varphi \\ X^{1'} &= X\sin\varepsilon\cos\vartheta \qquad . \\ X^{0'} &= X\cos\varepsilon\cos i\psi \\ X^{4'} &= X\cos\varepsilon\,\sin i\psi \end{aligned} \tag{IV.18.1}$$

However, it differs from it by the orientation of the angle, here designated with ε. In contrast to the conventions, the orientation of the angle ε is chosen in the clockwise direction. That has the consequence that the quantity E_m arising from the geometry is pointing inward and can be identified as the force of gravity without a change of sign. Since on a surface of a sphere all directions are equivalent, the direction of the quantity E of the static de Sitter model was still arbitrary. After the deformation of the pseudo-hyper sphere to the Schwarzschild geometry, the direction of E becomes unambiguous. It is radial and points to the gravitation center. The pointer directed from the origin of the Cartesian co-ordinate system to the surface of the sphere satisfies the relation

$$X_{a'}X^{a'} = X^2, \quad a' = 0',1',...,4' \ . \tag{IV.18.2}$$

In the pseudo-spherical co-ordinate system a = 0,1,…, 4 has the vector X^a only one component $X^0 = X$. It corresponds to the radius of curvature of a pseudo-hyper-spherical surface which one selects from a family of surfaces with X = const.. By differentiating (IV.18.1) the tangent vectors of the pseudo-spherical co-ordinate system

$$dX^a = \{dX^3, dX^2, dX^1, dX^0, dX^4\} = \{X\sin\varepsilon\sin\vartheta d\varphi\,, X\sin\varepsilon d\vartheta, Xd\varepsilon, dX, X\cos\varepsilon di\psi\} \tag{IV.18.3}$$

arise.

From these one can read the 5-dimensional tangent bein vectors and the reciprocal gradient bein vectors. For the distinction of the Schwarzschild quantities discussed later on a hat is set over the symbols, if they refer to the pseudo-hyper surface. Thus, we have

$$\hat\partial_0 = \frac{\partial}{\partial X}, \quad \hat\partial_1 = \frac{\partial}{X\partial\varepsilon}, \quad \hat\partial_2 = \frac{\partial}{X\sin\varepsilon\partial\vartheta}, \quad \hat\partial_3 = \frac{\partial}{X\sin\varepsilon\sin\vartheta\partial\varphi}, \quad \hat\partial_4 = \frac{\partial}{X\cos\varepsilon\partial i\psi} \ . \tag{IV.18.4}$$

The line element, parameterized in pseudo-spherical co-ordinates, has the form

$$dS^2 = X^2\sin^2\varepsilon\sin^2\vartheta d\varphi^2 + X^2\sin^2\varepsilon d\vartheta^2 + X^2 d\varepsilon^2 + dX^2 + X^2\cos^2\varepsilon di\psi^2 \qquad \text{(IV.18.5)}$$

and is reduced for X = const. to the line element of a 4-dimensional pseudo-hyper sphere. The 5-dimensional covariant derivative is defined by

notation: $$\Phi_{a;b} = \Phi_{a,b} - X_{ba}{}^{c}\Phi_{c} \ , \qquad \text{(IV.18.6)}$$

wherein the semicolon denotes the 5-dimensional covariant derivative with respect to the Ricci-rotation coefficients X. The comma denotes the partial derivative

notation: $$\hat{\partial}_a\Phi = \Phi_{,a} \ . \qquad \text{(IV.18.7)}$$

The Ricci-rotation coefficients are computed with the help of the bein vectors according to the method described in the introduction, and they have the components

$$\begin{aligned} &X_{10}{}^{1} = \frac{1}{X},\ X_{20}{}^{2} = \frac{1}{X},\ X_{21}{}^{2} = \frac{1}{X}\cot\varepsilon \\ &X_{30}{}^{3} = \frac{1}{X},\ X_{31}{}^{3} = \frac{1}{X}\cot\varepsilon,\ X_{32}{}^{3} = \frac{1}{X\sin\varepsilon}\cot\vartheta \ . \\ &X_{40}{}^{4} = \frac{1}{X},\ X_{41}{}^{4} = -\frac{1}{X}\tan\varepsilon \end{aligned} \qquad \text{(IV.18.8)}$$

With the four orthogonal unit vectors m, b, c, u these relations can be written more clearly as

$$\begin{aligned} &X_{ab}{}^{c} = \hat{M}_{ab}{}^{c} + \hat{B}_{ab}{}^{c} + \hat{C}_{ab}{}^{c} + \hat{E}_{ab}{}^{c} \\ &\hat{M}_{ab}{}^{c} = m_a\left[\hat{M}_b m^c - m_b\hat{M}^c\right],\ \hat{B}_{ab}{}^{c} = b_a\left[\hat{B}_b b^c - b_b\hat{B}^c\right] \ , \\ &\hat{C}_{ab}{}^{c} = c_a\left[\hat{C}_b c^c - c_b\hat{C}^c\right] \ ,\ \hat{E}_{ab}{}^{c} = -u_a\left[\hat{E}_b u^c - u_b\hat{E}^c\right] \end{aligned} \qquad \text{(IV.18.9)}$$

$$\begin{aligned} &\hat{M}_b = \left\{\frac{1}{X},0,0,0,0\right\},\quad \hat{B}_b = \left\{\frac{1}{X},\frac{1}{X}\cot\varepsilon,0,0,0\right\} \\ &\hat{C}_b = \left\{\frac{1}{X},\frac{1}{X}\cot\varepsilon,\frac{1}{X\sin\varepsilon}\cot\vartheta,0,0\right\},\quad \hat{E}_b = \left\{-\frac{1}{X},\frac{1}{X}\tan\varepsilon,0,0,0\right\} \end{aligned} \ , \qquad \text{(IV.18.10)}$$

whereby a full set of graded covariant derivatives

$$\begin{aligned} &\hat{M}_{\underset{1}{a;b}} = \hat{M}_{a,b},\quad \hat{B}_{\underset{2}{a;b}} = \hat{B}_{a,b} - \hat{M}_{ba}{}^{c}\hat{B}_{c},\quad \hat{C}_{\underset{3}{a;b}} = \hat{C}_{a,b} - \hat{M}_{ba}{}^{c}\hat{C}_{c} - \hat{B}_{ba}{}^{c}\hat{C}_{c} \\ &\hat{E}_{\underset{4}{a;b}} = \hat{E}_{a,b} - \hat{M}_{ba}{}^{c}\hat{E}_{c} - \hat{B}_{ba}{}^{c}\hat{E}_{c} - \hat{C}_{ba}{}^{c}\hat{E}_{c} \end{aligned} \qquad \text{(IV.18.11)}$$

are used. The relations

$$m_{\underset{1}{a;b}} = 0,\quad b_{\underset{2}{a;b}} = 0,\quad c_{\underset{3}{a;b}} = 0,\quad u_{\underset{4}{a;b}} = 0 \qquad \text{(IV.18.12)}$$

substantially facilitate the calculations. Since the 5-dimensional embedding space is flat, its Riemann tensor and likewise its Ricci tensor

$$R_{ab} = X_{ab}{}^{c}{}_{,c} - X_{cb}{}^{c}{}_{,a} - X_{da}{}^{c}X_{cb}{}^{d} + X_{ab}{}^{c}X_{dc}{}^{d} \equiv 0 \qquad \text{(IV.18.13)}$$

vanish identically. If one uses in it the quantities (IV.18.9), the subequations for the individual field quantities decouple

$$\begin{aligned}&\hat{M}_{\underset{1}{b;a}}+\hat{M}_b\hat{M}_a=0,\quad \hat{B}_{\underset{2}{b;a}}+\hat{B}_b\hat{B}_a=0,\quad \hat{C}_{\underset{3}{b;a}}+\hat{C}_b\hat{C}_a=0,\quad \hat{E}_{\underset{4}{b;a}}-\hat{E}_b\hat{E}_a=0\\ &\hat{M}^c{}_{\underset{1}{;c}}+\hat{M}^c\hat{M}_c=0,\quad \hat{B}^c{}_{\underset{2}{;c}}+\hat{B}^c\hat{B}_c=0,\quad \hat{C}^c{}_{\underset{3}{;c}}+\hat{C}^c\hat{C}_c=0,\quad \hat{E}^c{}_{\underset{4}{;c}}-\hat{E}^c\hat{E}_c=0\end{aligned}\ . \tag{IV.18.14}$$

All equations are of the type

$$\frac{\partial}{\partial r}\frac{1}{r}+\frac{1}{r^2}=0$$

and are the equations for the curvatures[56] of the normal and inclined slices

$$\hat{M}=\frac{1}{X},\quad ,\ \hat{B}=\frac{1}{X\sin\varepsilon},\quad \hat{C}=\frac{1}{X\sin\varepsilon\sin\vartheta},\quad \hat{E}=\frac{1}{X\cos\varepsilon} \tag{IV.18.15}$$

on the pseudo-hyper sphere. With the equation (IV.18.14) we have compiled a general structure for the Schwarzschild equations which could be applied to more complicated models, as we will see later on. Thus, we have made clear the meaning of the Einstein field equations as well: the curvatures of a surface are described by the above equations. The 5-dimensional Einstein tensor

$$G_{ab}=R_{ab}-\frac{1}{2}g_{ab}R \tag{IV.18.16}$$

has the form

$$\begin{aligned}G_{ab}=&-\left[\hat{M}_{\underset{1}{b;a}}+(m_am_b-g_{ab})\hat{M}^c{}_{\underset{1}{;c}}+{}^1t_{ab}\right]-\left[\hat{B}_{\underset{2}{b;a}}+(b_ab_b-g_{ab})\hat{B}^c_{\underset{2}{;c}}+{}^2t_{ab}\right]\\ &-\left[\hat{C}_{\underset{3}{b;a}}+(c_ac_b-g_{ab})\hat{C}^c{}_{\underset{3}{;c}}+{}^3t_{ab}\right]+\left[\hat{E}_{\underset{4}{b;a}}+(u_au_b-g_{ab})\hat{E}^c{}_{\underset{4}{;c}}-{}^4t_{ab}\right]\equiv 0\end{aligned}\ , \tag{IV.18.17}$$

whereby the quantities t are formally similar to a possible stress-energy tensor of the gravitational field and conserved

$$\begin{aligned}&{}^1t_{ab}=\hat{M}_a\hat{M}_b+(m_am_b-g_{ab})\hat{M}^c\hat{M}_c,\quad {}^1t_a{}^b{}_{\underset{1}{;b}}=0\\ &{}^2t_{ab}=\hat{B}_a\hat{B}_b+(b_ab_b-g_{ab})\hat{B}^c\hat{B}_c,\quad {}^2t_a{}^b{}_{\underset{2}{;b}}=0\\ &{}^3t_{ab}=\hat{C}_a\hat{C}_b+(c_ac_b-g_{ab})\hat{C}^c\hat{C}_c,\quad {}^3t_a{}^b{}_{\underset{3}{;b}}=0\\ &{}^4t_{ab}=\hat{E}_a\hat{E}_b+(u_au_b-g_{ab})\hat{E}^c\hat{E}_c,\quad {}^4t_a{}^b{}_{\underset{4}{;b}}=0\end{aligned}\ , \tag{IV.18.18}$$

if one uses the graded derivatives. However, a connection with a common conservation law[57] of matter and field energy cannot be deduced from it.

With the embedding of the Schwarzschild geometry in higher-dimensional flat space, several authors have been engaged:

Abolghasem [A], Abramowicz [A] and Prasanna; Abramowicz [A], Arraut [A], Batic, and Nowakowski; Bach [B], Biswas [B], Bolós [B], Coley and Mcmanus; Barnes [B], Anderson [A] and Lidsey; Bel [B] and Hamoui; Burstin [B], Caltenco, Linares y M., López-Bonilla, and Montiel-Pérez J;Cartan [C], Caltenco [C], Cattaneo [C], Cattaneo Gasparini [C].

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[56] The curvatures are the reciprocal values of the radii of curvature.

[57] On the problem of the conservation of energy of the gravitational and matter fields physicists have been working for nearly a century. However, the problem is still unsolved. Therefore, we will not deal with it.

Chyba [C] describes a method for embedding Schwarzschild-like models with Kruskal-like co-ordinates.
Collinson [C], Dahia [D] and Romero; Davidson [D] and Paz; de Felice [D], Ehlers [E] and Krasiński; Eiesland [E], Ferraris [F] and Francaviglia; Fialkow [F], Florides [F] and Jones; Friedman [F], Fromholz [F], Poisson, and Will; Gallone [G], Gião [G], Gibli [G], Goedecke [G], Goenner [G], Goląb [G], Guendelman [G] et al., Gupta [G] and Pandey; Gupta [G] and Gupta; Gupta [G], Sharma and Gupta; Gupta [G] and Goel; Haesen [H] and Verstraelen; Hogan [H], Hodgkinson [H], Ikeda [I], Janis [J], Newman and Winicour; Kitamuro, and Matsumoto; Janet [J], Joseph [J], Karmarkar [K], Kasner [K], Kerner [K], Kerner [K] and Vitale; Kitamura [K], Kohler [K] and Chao; Korkina [K] and Buts; Kottler [K], Krasiński [K] and Plebański; Krause [K], Lam-Estrada [L], López-Bonilla and López-Vázquez; Lense [L], Lidsey [L], Romero, Tavako and Rippl; López-Bonilla [L] and Núñez-Yépez; López-Bonilla [L], Monte [M], Montez-Peralta, and Islas-Gamboa; Luna [L], Bonilla, Rivas and Zúñiga; Matsumoto [M], Mayer [M], Meskhishvili [M], Ne'eman [N], Misra [M], Pandey [P] and Sharma; Painlevé [P], Pandey [P] and Kansal; Paston [P], Paston [P] and Franke; Pavšič [P], Pervushin [P] and Smirichinski; Pfeifer [P] and York; Płazowski [P].

Paston [P] and Sheykin discuss some known and less well-known embeddings of the Schwarzschild metric into a flat space of dimension $N = 6$, and in a further paper Paston [P] formulates embeddings for any N.
Plebanski [P], Pokhariyal [P], Rastall [R], Rosen [R], Roy [R] and Prasad, Schouten [S], Stephani [S], Tapia [T], Tiwari [T], Reinhart [R], Sassi [S], Sinha [S], Sonego [S] and Massar; Stuchlik [S], Szekeres [S], Taub [T], Tikekar [T], Thomas [T], Van den Bergh [V], Villaseñor [V], Bonilla, Zuñiga and Matos; Weyl [W].

Other models stem from:

Abreu [A], Akbar [A], Antola [A] and Ferrari; Azreg-Ainou [A], Bhar [B] et al; Banerjee [B], Banerjee, Hansraj, and Ovgun; Banerjee [B] and Santos; Barreto [B], Barreto [B], Rodrígues, Rosales, and Serrano; Bhaskaran [B] and Prasanna; Böhmer [B] and Harko; Bondi [B], Bondi [B], Bonnor [B] Bonnor [B], Boyer [B] and Plebański; Bowers [B], Bowers [B] and Liang; Chatterjee [C], Chavoya, and López-Bonilla; Chermyanin [C], Cosenza [C], Herrera and Witten; Corchero [C], Das [D], Das [D], Ray, Radinschi, and Rahman; Deb [D], Paul, and Tikekar; de Parga [D], Debever [D], De Felice [D], Yu, and Fang; Dev [D] and Gleiser; Deser [D], Di Prisco [D], Herrera, and Varela; Durgapal [D] and Fuloria; Dwivedi [D] and Joshi; Ehlers [E], Esculpi [E], Malaver and Aloma;Estabrook [E] and Wahlquist; Faridi [F] et al., Ferarese [F], Gautreau [G], Geroch [G], Geyer [G], Glass [G] and Goldman; Goldman [G], Goswami [G] and Joshi; Gutsunaev [G] and Manko; Grammenos [G], Grøn [G], Harada [H] and Maeda; Harrison [H], Harko [H], Harness [H], Harrison [H], Herrera [H], Santos, and Wang; Herrera [H], Herrera [H], Barreto and Hernández; Herrera [H], Herrera [H] et al; Di Prisco, and Martínez; Herrera [H], Di Prisco, and Ospino; Herrera [H], Ospino, and Di Prisco; Herrera [H] et al; Herrera [H] and Barreto; Hill [H], Hiscock [H], Hossein [H] et al.; Humi [H] and Mansour; Israel [I], Ivanov I, Jantzen [J], Kalam [K] et al; Karlhede [K], Lindström, and Åman; Keres [K], Kolassis [K] , Santos, and Tsoubelis; Kinnerley [K], Kramer [K] and Neugebauer; Krisch [K] and Glass; Knutsen [K], Kuang [K] and Liang; Kuchowicz [K], Kyriakopoulos [K], Kyriakopoulos [K], Letelier [L], López-Bonilla [L] , Ovando, and Peña; López-Bonilla [L], Lynden-Bell [L], McGruder III [M] and Van der Meer; Martin [M] and Pritchett; Medina [M], Núnez, Rago, and Patino; Mehra [M], Mitra [M], Morales, and Ovando; Maharaj [M] and Leach; Mak [M], Dobson, and Harko; Mak [M] and Harko; Maluf [M], Marklund [M] and Bradley; Maurya [M], Maurya [M], Gupta, Ray and Chatterjee; Misthry [M], Maharaj, and Leach; Martínez [M], Rojas and Cuesta; Sharma [S] and Maharaj; Misra [M], Mitra [M], Pandey, Srivastava and Tripathi; Mora [M], López-Bonilla, and López-Vázquez; Mukherjee [M], Paul, and Dadhich; Nathanail [N] , Most, and Rezzolla; Nogueira [N] and Chan; Ori [O], Pandey [P] and Sharma; Patel [P] and Singh; Paul [P] and Deb; Perlick [P], Poisson [P] and Israel; Podurets [P], Rahaman [R] et al; Ray [R] and Das; Ray [R] et al; Robertson [R] and Leitner; Rosseland [S], Schaudt [S] and

Pfister; Singh [S], Singh [S] and Kotambkar; Thirukkanesh [T] and Maharaj; Tikekar [T], Tiwari [T] and Ray; Tooper [T], Varela [V] et al; Viaggiu [V], Waugh [W] and Lake.

IV.19. Schwarzschild metric, dimensional reduction

The structures in the last Sections are discussed so generally that the de Sitter cosmos can easily be deduced from them and that they can also serve as starting points for other models. The following steps will elucidate how one can gain the Schwarzschild model from these structures. Moreover, it can be shown that a 5-dimensional embedding of the Schwarzschild model is possible by a long way round.

We start again from the equation of the family of pseudo-hyper spheres and we shift the center of the family of spheres from the origin of the co-ordinate system to the co-ordinates $\bar{x}^{a'}$. The tips of the pointers to the spheres are situated at $x^{a'}$, and from $X_{a'}X^{a'} = X^2$ we obtain

$$(x_{a'} - \bar{x}_{a'})(x^{a'} - \bar{x}^{a'}) = X^2 \ . \tag{IV.19.1}$$

For the further investigations it is sufficient to regard a radial slice of a pseudo-hyper sphere and to reduce the dimensions to two. By renaming the variables $\{x^{0'}, x^{1'}\} = \{R, r\}$, $\{\bar{x}^{0'}, \bar{x}^{1'}\} = \{\bar{R}, \bar{r}\}$ one arrives at a more familiar way of writing. In the following well-known geometrical relations are used which refer to the Schwarzschild parabola. If one establishes the curvature vectors, which are normal to the parabola in all the points of the Schwarzschild parabola, then the base points of the curvature vectors are situated on the Neil parabola, which is the evolute of the Schwarzschild parabola. The curvature vectors are tangent to the Neil parabola. From both the Schwarzschild parabola and the Neil parabola

$$R^2 = 8M(r - 2M), \quad \bar{R}^2 = \frac{2}{M}\left(\frac{\bar{r}}{3} - 2M\right)^3 \tag{IV.19.2}$$

only the positive or negative branch respectively, is quoted for the physical interpretation.

Furthermore, there exists the elementary relation

$$\bar{r} = 3r \tag{IV.19.3}$$

between the values of the abscissae of the two parabolae. We will recognize $\bar{r}$ as that hidden variable which is included in the model and which is the missing quantity for the successful embedding corresponding to the laws of Kasner and Eisenhart. Furthermore, $X = \sqrt{2r^3/M}$ is the radius of curvature of the Schwarzschild parabola (the only remaining component of the curvature vector X^a in the polar co-ordinate system). The relation (IV.19.1) is fulfilled if one inserts into (IV.19.1) this value for X, and in addition the relation (IV.19.3) for $\bar{r}$, further, from (IV.19.2) the positive values of R, and the negative ones of $\bar{R}$. This is the fundamental idea of deforming the geometry of a pseudo-hyper sphere to the geometry of the Schwarzschild model.

By the new interpretation of the quantities contained in (IV.19.1) a mapping of the spherical geometry onto a parabolic geometry is established. In addition, the polar co-ordinate system gets a new meaning. The radial co-ordinate lines of the spherical geometry which start from the center of the pseudo-hyper spheres now are straight lines which start from the Neil parabola and are tangent to it. The circles become parallel curves which cut perpendicularly all the just-mentioned straight lines. All these parallel curves are involutes of the Neil parabola. Only one of these curves is a parabola. It is that curve which we select for the basis of the Schwarzschild geometry Fig. IV.16.

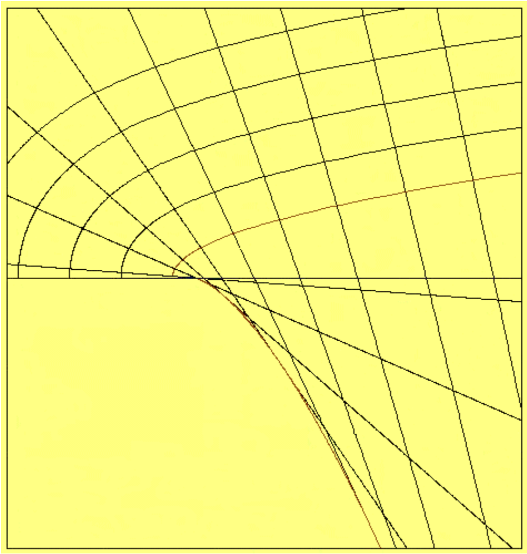

Fig. IV.16

The center of the spherical geometry mutates to the base points of the curvature vectors. This co-ordinate system covers only one part of the radial slice viz that part which we want to consider. The vector X^a can reach any point of this part of the slice. Since the points outside the Schwarzschild parabola are of less interest, we work only with those curvature vectors which point to the Schwarzschild parabola. We designate them with ρ, as we have already done before. We want to point out that the quantities X and ρ have a different functional dependence. While X is permitted to be directed to any point on the upper part of the slice ρ is *constrained* to the parabola. If X moves on the slice the four variables which describe its starting and end points change, while ρ is constrained in such a way that its starting and end points slide on the Schwarzschild parabola and on the Neil parabola, respectively. Finally, ρ only depends on one variable r:

$$X = X(r, \bar{r}, R, \bar{R}), \quad \rho(r) = X\big|_{\text{parabola}} \ . \tag{IV.19.4}$$

We make a start with these considerations and we will set up a methodology as simple as possible which makes possible the transition from the spherical geometry to the Schwarzschild geometry and which ensures a simple dimensional reduction. One should bear in mind that by constraining X → ρ with the use of (IV.19.3) the variable $\bar{r}$ is invisible for the 4-dimensional theory. That is in accordance with the insight that curved spaces with their intrinsic properties can be described without resort to higher dimensions and their embedding spaces. At the beginning of this booklet this was expressed with view (II).

r has the range of values [2M, ∞] in accordance with the relation (IV.19.2). Values for $r < 2M$ supply imaginary R and prevent a geometrical interpretation in the sense of view (III). r = 2M is the boundary of the graphic geometry.

One arrives at the complete model, if one rotates both curves, the Schwarzschild parabola and the Neil parabola around the directrix of the Schwarzschild parabola ($x^{0'}$ = R-axis) through the angles ϑ and φ and once again around the symmetry axis of the Schwarzschild parabola ($x^{1'}$ = r-axis) through the angle $i\psi$. Thereby the two surfaces are connected by the curvature vector of the Schwarzschild parabola. Thus, the two surfaces are correlated. In total one has the variables $\left\{r, R, \bar{r}, \bar{R}, \vartheta, \varphi, i\psi\right\}$ which can by means of (IV.19.2) be reduced to four. If one rotates the two curves two correlated surfaces result. In such a way the emergent system is described by a double-surface theory which will be developed in the following. Putting dX = 0 in (IV.18.5) one obtains the metric on a pseudo-hyper spherical surface. If the above new interpretation is applied one has likewise the metric on the double surface. Thus, one obtains from (IV.18.3) with (IV.20.4) and with the equation below (IV.2.1)

$$X\sin\varepsilon \to \rho v = \sqrt{\frac{2r^3}{M}}\left(-\sqrt{\frac{2M}{r}}\right) = -2r = r - \bar{r}$$

and finally

$$dX^2 \to \rho v d\vartheta = r d\vartheta - \bar{r} d\vartheta, \quad dX^3 \to \rho v \sin\vartheta d\varphi = r\sin\vartheta d\varphi - \bar{r}\sin\vartheta d\varphi \ .$$

From (IV.19.1) follows

$$\rho^2 = \left(r - \bar{r}\right)^2 + \left(R - \bar{R}\right)^2 \qquad \text{(IV.19.5)}$$

and from this also

$$X\cos\varepsilon \to \rho a = \sqrt{\frac{2r^3}{M}}\sqrt{1 - \frac{2M}{r}} = R - \bar{R} \ ,$$

and finally,

$$dX^4 \to \rho a d i\psi = \left(R - \bar{R}\right) d i\psi \ .$$

A dimensional reduction consists in removing everything from the theory which is not necessary for the description of the 4-dimensional space of observation. One immediately recognizes from some of the above relations that the terms with $\bar{r}$ refer to the Neil surface and do not contribute to the Schwarzschild metric. However, it would be wrong to assume that in the last relation we can do without the contribution with $\bar{R}$ as well. Then one could identify $idt = Rdi\psi$ and would have abandoned the Schwarzschild theory[58]. The meaning of a rotation through an imaginary angle is well known from the de Sitter cosmos. For r = const. one obtains pseudo circles, also called hyperbolae of constant curvature. They are open, i.e. the range of values of ψ is $\left[-\infty, +\infty\right]$ and ψ is the parameter for the time. With increasing r a family of pseudo circles arises which are laminated perpendicularly to the r-axis. From the rotation of the Schwarzschild parabola results a 2-dimensional piece of the 4-dimensional Schwarzschild surface in the flat $\{x^{1'}, x^{0'}, x^{4'}\}$-space which can be represented in the pseudo-real representation as a parabolic-hyperbolic surface (saddle). Rotation of the Neil parabola creates a similar surface. The patches remaining after dimensional reduction are summarized under the term *physical surface*. By introducing a double-surface the view (III) is understood less severely. The conception that we live on our planet Earth with its constantly flowing time on a curved surface, is not to be sustained. Instead, we must consult a more complicated mathematical construction which is less graphic. However, it will be shown that the double-surface can be treated quite simply with the methods of the surface theory and the physical surface permits all procedures one is acquainted with from the Riemannian geometry as well.

In order to gain more clearness, one can proceed as follows: on the way to the dimensional reduction we can do without Neil's parabola after it has supplied an explanation for the hidden variable. One can shrink the remaining curvature vectors which run out from the Neil parabola to the Schwarzschild parabola, so that in every point there

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[58] The ansatz would correspond to an embedding of a V_4 into an E_5. Such an embedding does not violate the laws of Kasner and Eisenhart. However, it does not lead to a vacuum solution.

remains only one definite function $\rho(r)$ that depends on the auxiliary variable r. Indeed, one has merely a V_4 with an *additional function* which makes up our space of observation. This almost approximates Gauss' conception. That means that the curvature of a surface is to be understood as an intrinsic property of the surface and that it is superfluous to assume an embedding space. That again is in accordance to view (II). These constructions can be described with the Riemannian calculus. Constructions which are not in the strict sense objects of the classical surface theory we will make out for other models. Until now, our examinations on the derivation of the Schwarzschild metric from a 5-dimensional theory with the methods of a double-surface theory have been merely descriptive. It will be shown in the following Section that all this can be formulated strictly mathematically.

IV.20. Schwarzschild metric, embedding

After numerous problems were discussed concerning the Schwarzschild geometry, the basic problem initially outlined has to be examined. It will be shown, how the Schwarzschild metric can be represented as a metric of an actual surface. In the same year, in which the model was published by Schwarzschild, Flamm [F] has interpreted the space-like part of the Schwarzschild metric geometrically. Its ϑ-part and φ-part do not deviate from the Euclidean metric written in polar co-ordinates. The radial part was interpreted by Flamm as a line element on the parabola[59]

$$R^2 = 8M\left(r - 2M\right) \ . \tag{IV.20.1}$$

The plane, in which the parabola lies, is parameterized by the Cartesian co-ordinates {R, r}. R is the extra dimension. The symmetry axis of the parabola coincides with r. Thus, r is situated outside the parabola and finally outside our experience. r is an auxiliary variable and does not represent an accessible way. One obtains the radial physical distance of a point from the event horizon with the rectification formula for the parabola

$$r^* = \int_{2M}^{r} \alpha dr = ar + M \ln\frac{1+a}{1-a} \ . \tag{IV.20.2}$$

Since r is an implicit function of r*, the auxiliary variable r cannot be replaced by the physical distance r* in the formulae of the Schwarzschild geometry. The vertex of the (lying) parabola is at r = 2M, R = 0 and is a definite point on the Schwarzschild parabola. A singularity at the event horizon r = 2M is not present. The seeming singularity of the line element is due to a lack of representation of the Schwarzschild standard co-ordinate system. In contrast, r = 2M is the boundary of the geometry. For r < 2M one cannot make any statements due to the ansatz (IV.20.1). One has to resign from black holes in the framework of this strict geometrical treatment.

To obtain a graphic geometrical representation one has to suppress a dimension in the spatial line element and has to rotate the parabola around the directrix through the angle ϑ. The emergent surface of the fourth order one calls Flamm's paraboloid (Fig. IV.17). Obviously, one can identify the event horizon on the boundary of the geometry.

[59] Only the positive (upper) branch of the lying parabola enters into further considerations.

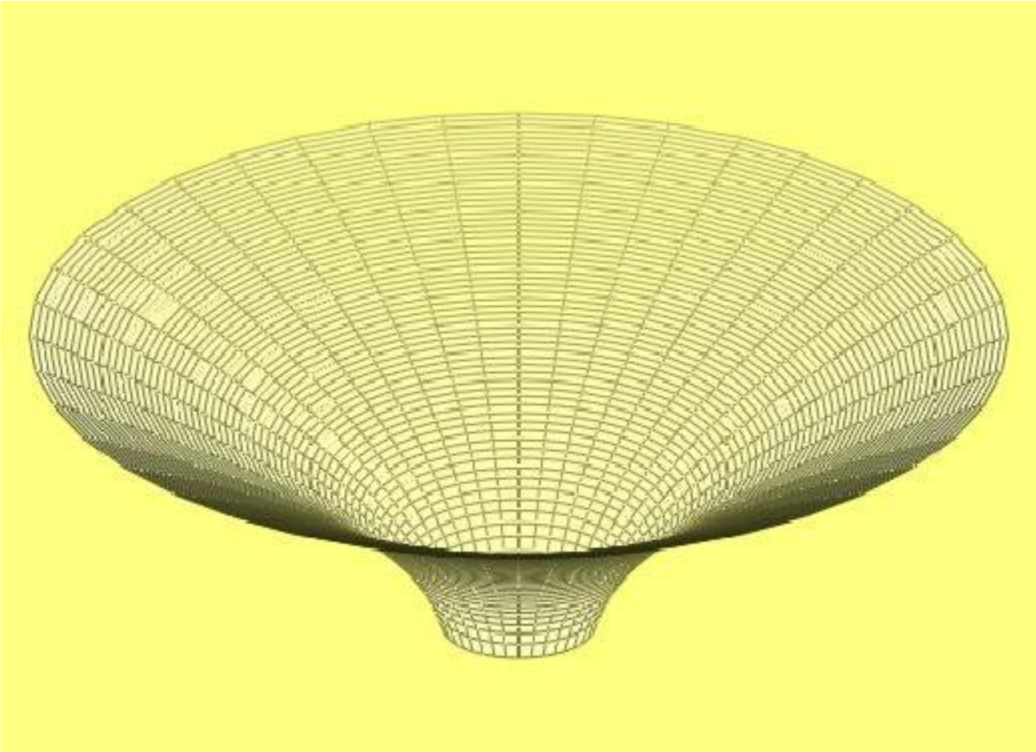

Fig. IV.17

By differentiation of (IV.20.1) one obtains the ascent of the parabola

$$\frac{dR}{dr} = \frac{4M}{R} = -\frac{v}{a} = -\tan\varepsilon, \tag{IV.20.3}$$

whereby the angle of ascent ε of the parabola is chosen to be negative. Thus, one obtains a simple geometrical interpretation of the quantities[60,61]

$$v = \sin\varepsilon, \quad a = \cos\varepsilon. \tag{IV.20.4}$$

From (IV.20.1) the radius of curvature of the Schwarzschild parabola can be determined as

$$\rho = \sqrt{\frac{2r^3}{M}}. \tag{IV.20.5}$$

It is a quantity which was already used in the preceding Sections. In addition, defining

$$idt = \rho d i\psi, \tag{IV.20.6}$$

one can interpret the time-like part of the metric as the arc element of a local pseudo circle, an interpretation, which is familiar from the de Sitter cosmos. We must postpone the deeper reason of this ansatz to later Sections.

If we perform a Lorentz transformation to a freely falling system the angle ψ receives a new value which can easily be calculated. In Section IV.2 we have investigated the free

[60] This consideration is helpful for models whose embeddings would be possible, but one does not know them yet. From the factor of the radial arc element one can judge the ascent of a curve that is still to be computed.

[61] Since the direction of ε was chosen clockwise, i.e. negatively, $\sin\varepsilon$ and v are negative quantities as well and inward directed. v is the first and single component of a vector in the standard Schwarzschild coordinate system.

fall and also the co-ordinate system introduced by Lemaître which is comoving with the freely falling system. With the approach of (III.6.6) we can assign the angle ψ' to the freely falling system. In the comoving system the gravitational factor is a = 1, thus

$$dx^{4'} = idt' = \rho di\psi' \ . \qquad \text{(IV.20.7)}$$

From the Lorentz transformation we have obtained

$$dt' = dt - \alpha^2 vdr = dt - \alpha vdx^1 \ .$$

Using (IV.20.6), $dx^1 = \rho d\varepsilon$ one has

$$\rho d\psi' = \rho d\psi - \alpha v\rho d\varepsilon, \quad d\psi' = d\psi - \tan\varepsilon d\varepsilon$$

or

$$\psi' = \psi + \ln\cos\varepsilon, \quad \psi' = \psi - \ln\alpha \ . \qquad \text{(IV.20.8)}$$

The two equivalent formulas show that the angle ψ' depends on the geometry (ε) and on the state of motion of the observer, respectively, thus, on the Lorentz factor (α).

The Schwarzschild metric can fully be formulated with the curvatures of curves on a surface

$$ds^2 = \rho_i \rho_k d\varepsilon^i d\varepsilon^k, \quad i = k \ , \qquad \text{(IV.20.9)}$$

with the radii of curvature and angles

$$\begin{gathered} \rho_1 = \rho = \sqrt{\frac{2r^3}{M}}, \quad \rho_2 = r, \quad \rho_3 = r\sin\vartheta, \quad \rho_4 = \rho\cos\varepsilon \\ \varepsilon^1 = \varepsilon, \quad \varepsilon^2 = \vartheta, \quad \varepsilon^3 = \varphi, \quad \varepsilon^4 = i\psi \end{gathered} \ . \qquad \text{(IV.20.10)}$$

Written in full, (IV.20.9) has the form

$$ds^2 = \rho^2 d\varepsilon^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 + \rho^2\cos^2\varepsilon di\psi^2 \ . \qquad \text{(IV.20.11)}$$

The coefficients of the co-ordinate differentials are the radii of curvature of the normal and inclined slices of a surface, a subject that is still to be discussed. If one uses the relations (IV.1.5) one is tempted to supplement these by a further component (it will be called the 0-component)

$$\begin{aligned} B_a &= \left\{\frac{\sin\varepsilon}{r}, \frac{\cos\varepsilon}{r}, 0, 0, 0\right\} \\ C_a &= \left\{\frac{\sin\varepsilon}{r}, \frac{\cos\varepsilon}{r}, \frac{1}{r}\cot\vartheta, 0, 0\right\} \\ E_a &= \left\{-\frac{1}{\rho\cos\varepsilon}\cos\varepsilon, \frac{1}{\rho\cos\varepsilon}\sin\varepsilon, 0, 0, 0\right\} \end{aligned} \ . \qquad \text{(IV.20.12)}$$

Thus, one has outlined the theory of a 4-dimensional surface embedded into a 5-dimensional flat space, which one would call V_4 in R_5. We will immediately accomplish that this is not possible.

Eisenhart [E] proved that any 4-dimensional Ricci-flat[62] metric could only be embedded into a 5-dimensional flat space, if the 4-dimensional space is flat as well[63]. Kasner [K]

[62] $R_{mn} = 0$.

[63] If in addition $R_{rmns} = 0$ is valid.

presented the proof in detail and showed that one can embed the Schwarzschild model into a 6-dimensional flat space. His ansatz

$$ds^2 = dx^2 + dy^2 + dz^2 - dX^2 - dY^2 + dZ^2 \qquad \text{(IV.20.13)}$$

with

$$X = \frac{R\sin t}{\sqrt{R^2+16M^2}}, \quad Y = \frac{R\cos t}{\sqrt{R^2+16M^2}}, \quad Z = \int \sqrt{1+\frac{256M^4}{\left(R^2+16M^2\right)^3}}\,dR \qquad \text{(IV.20.14)}$$

has some discrepancy, since into the trigonometric functions enters the time instead of a dimensionless angle and also the radicand of Z consists of two addends of different dimensions[64]. However, that can be corrected merely by replacing t with the dimensionless variable

$$\chi = \frac{t}{4M} \ . \qquad \text{(IV.20.15)}$$

The radial part of the Schwarzschild line element can be explained using (IV.20.3) and (IV.20.4) with the law of Pythagoras

$$\alpha^2 dr^2 = \frac{1}{\cos^2\varepsilon} dr^2 = \left(1+\tan^2\varepsilon\right) dr^2 = dr^2 + dR^2, \quad dR = -\tan\varepsilon\, dr \ . \qquad \text{(IV.20.16)}$$

Thus, the line element has a 5-dimensional form

$$ds^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 + dR^2 - \frac{R^2}{R^2+16M^2} dt^2 \ . \qquad \text{(IV.20.17)}$$

The last term was transformed with (IV.1.2) and (IV.20.1). The first three terms determine the line element of a flat space in polar co-ordinates. This can be described with a simple transformation by means of Cartesian co-ordinates. If one uses $dt = 4M d\chi$, one has for the new variables in a 3-dimensional auxiliary space

$$X = 4M\cos\varepsilon\sin\chi, \quad Y = 4M\cos\varepsilon\cos\chi, \quad Z = \int \sqrt{1+\sin^6\varepsilon}\, dR \ . \qquad \text{(IV.20.18)}$$

For the processing of these variables one uses together with (IV.1.2) and (IV.20.4) the relation

$$\cos\varepsilon = \frac{R}{\sqrt{r^2+16M^2}}, \quad \sin\varepsilon = -\frac{4M}{\sqrt{r^2+16M^2}} \qquad \text{(IV.20.19)}$$

and thus obtains the relation (IV.20.13). Z is a hyperelliptic integral, whose solution does not have a closed form. dZ can be integrated numerically, and supplies in the 3-dimensional (XYZ)-space a surface of revolution, where X and Y put up the horizontals. Due to the use of the t-notation X and Y are periodical functions of the time, so that different points of the original space are identified with one another.

By considering the dimension problem of these quantities, Plebański [P] has started with a more general spherically symmetrical ansatz, then he has specialized it to the

[64] If one uses the natural system of units and the ‘geometrical' units respectively, the fundamental values of the theory have the dimension of the length, and the time as well. That can be read from the definition $x^4 = ict$ if one uses the physical system of units. The mass parameter M is the form parameter of the Schwarzschild parabola, 2M is the distance of the vertex to the directrix, the focus lies at 4M. M has the dimension of the length as well. The quantity M is related to the central mass m by factors like the gravitational constant and the velocity of light.

Schwarzschild metric, and extended it to the inner region and thus approached the Kruskal metric. Basically, he also has treated the Reissner-Nordström metric[65].

Fronsdal [F] has used only dimensionless quantities (r → r/2M, t → t/2M) and has replaced in Kasner's ansatz the trigonometric functions by hyperbolic ones. That comes close to the it-notation favored by us. Thus, the periodic behavior of the time is avoided. Fronsdal finally arrived at

$$\begin{gathered} ds^2 = dx^2 + dy^2 + dz^2 - dX^2 + dY^2 + dZ^2 \\ X = 2\sqrt{1-\frac{1}{r}}\,\mathrm{sh}\frac{t}{2}, \quad Y = 2\sqrt{1-\frac{1}{r}}\,\mathrm{ch}\frac{t}{2}, \quad Z = \int \frac{r^2+r+1}{\sqrt{r^3}}dr \end{gathered} \quad . \tag{IV.20.20}$$

The surface, correlated to the metric, is defined in the full region $0 < r \le \infty$ because Z is a monotonic function of r. The ansatz also includes the inner regions described by the Kruskal metric.

Hong [H] and Hong [H] and Kim have somewhat modified the ansatz of Fronsdal and have discussed the 6-dimensional embedding of the Reissner-Nordström metric in detail.

Harrison [H] has used a 7-dimensional ansatz. We refer to a recent paper of Fronsdal [F], and also to papers of Fonseca-Neto [F] and Romero, and Jonsson [J].

Some authors have been engaged with embeddings before: Schäfli [S], Cartan [C] and Burstin [B]. However, in most cases embeddings of Riemannian spaces into higher dimensional flat spaces have been treated: Chen [C] and Tian; Chen [C], Tian, Gao and Song, Collinson [C], Maia [M] and Mecklenburg; Ikeda [I], Kitamuro and Matsumoto; Pogorelov [P], Nash [N], Friedman [F], Fujitani [F] and Matsumoto; Stephani [S], Kitamuro and Matsumoto. Maia [M], have embedded Riemannian spaces into curved spaces.

An interesting way to derive Flamm's paraboloid should be noted. The velocity of a freely falling observer who comes from the infinite is $v = -\sqrt{2M/r}$, where r is a co-ordinate of the embedding space which cannot be replaced by the physical distance r* from the circle of the throat of Flamm's paraboloid to a point according to (IV.20.2). Strictly speaking, v is a quantity of the flat embedding space. All further considerations are based on this fact.

Let be

$$dS = \sqrt{dx^{0'2} + dx^{1'2} + dx^{2'2} + dx^{3'2}}$$

the pure spatial line element of the embedding space in Cartesian co-ordinates.

Working ahead we will call the extra dimension $R = x^{0'}$, further $X = x^{1'}$, and if we subsume all other co-ordinates by $P = \left\{x^{2'}, x^{3'}\right\}$, then we have

$$dS = \sqrt{dR^2 + dX^2 + dP^2}$$

which simplifies all further calculations. We also assume that the velocity of an object depends only on one co-ordinate, namely X. Then the question arises for the shortest time the object takes to travel on a path between two points in this space and thus, also for the shortest path between these two points.

From

[65] It will be examined in a later Section.

$$v = \frac{dS}{dT}$$

one has

$$\int_{x_1}^{x_2} \frac{1}{v(X)} dS = \text{Min!} \ . \qquad \text{(IV.20.21)}$$

If we choose X as a parameter, then we get the above integral in the form

$$\int \frac{\sqrt{R'^2 + P'^2 + 1}}{v(X)} dX, \quad \{R', P'\} = \left\{ \frac{dR}{dX}, \frac{dP}{dX} \right\} \ .$$

The integrand written as $L(R', P')$ depends only on the derivatives of the variables R and P. To solve the variation problem we need

$$\left\{ \frac{dL}{dR'}, \frac{dL}{dP'} \right\} = \frac{1}{v} \frac{1}{\sqrt{R'^2 + P'^2 + 1}} \{R', P'\} \ .$$

From the Lagrange equation only remains

$$\frac{d}{dX} \left\{ \frac{dL}{dR'}, \frac{dL}{dP'} \right\} = 0 \ ,$$

which is to be solved by integration. This is

$$\frac{1}{v} \frac{1}{\sqrt{R'^2 + P'^2 + 1}} \{R', P'\} = \{C_1, C_2\}$$

or

$$\{R', P'\} = \{C_1, C_2\} v \sqrt{R'^2 + P'^2 + 1} \ .$$

From this it is clear that

$$\frac{R'}{P'} = \frac{C_1}{C_2} = \text{const.}, \quad \frac{dR}{dP} = \text{const.} \ .$$

The solution is

$$R = aP + b \, .$$

These are planes parallel to the X-axis depending on the quantities a and b. These planes are all extremals. All these planes are sufficiently equivalent to be treated as one. We select the plane P = 0. We have put the plane without loss of generality through the X-axis. Thus, we have

$$R' = C_1 v \sqrt{R'^2 + 1} \ .$$

Solving for R' results in

$$R' = C_1 \frac{v}{\sqrt{1 - C_1^2 v^2}}$$

and

$$R + C_2 = C_1 \int \frac{v(X)}{\sqrt{1 - C_1^2 v(X)^2}} dX \ .$$

Substituting $C_1 = 1$ and $C_2 = 0$ and specializing

$$v(X) = -\sqrt{\frac{2M}{X}}$$

one finally has a solution of the integral

$$R = \sqrt{8M\left(X - 2M\right)} \ . \tag{IV.20.22}$$

This is the equation of a parabola, whose vertex is located at a distance 2M from the origin and which has the axis R as directrix. The equation applies to all planes through the symmetry axis X of the parabola. All the parabolae in these planes are extremals and form Flamm's paraboloid. In polar co-ordinates with r = X is (IV.20.22) the simplest representation of Flamm's paraboloid. However, adding the suppressed Cartesian co-ordinates, we obtain

$$R^2 = 8M\sqrt{x^{1'^2} + x^{2'^2} + x^{3'^2}} - 16M^2 \ , \tag{IV.20.23}$$

from which it is obvious that a fourth order surface exists. For objects that move with the velocity $v = -\sqrt{2M/r}$ in the embedding space, the shortest distance between two points is located on the radial curves of Flamm's paraboloid. The fact that it is possible to make with appropriate considerations already statements about the embedded geometry in a higher dimensional embedding space encouraged us to treat the problem in greater detail.

IV.21. Schwarzschild metric, calculations with the double-surface

In the preceding Section the theory of double-surfaces has been treated entirely generally. Now it has to be demonstrated in more detail, how one can obtain a double surface by a projection procedure from the pseudo-hyper sphere, which has been treated in Sec. IV.7. Finally, the double-surface is reduced to the physical surface which represents the basis of the Schwarzschild geometry. Something will be worked out in more detail than it would be necessary for the later use, because it provides a deeper understanding.

Firstly, we again return to the pseudo-hyper sphere and we make some remarks which are necessary to establish the projection formalism. From the pseudo-spherical parameterization of the pseudo-hyper sphere (IV.18.1) we deduce by means of

$$D_a^{a'} = X^{a'}{}_{,a} \,, \qquad \text{(IV.21.1)}$$

and by consideration of (IV.18.4) and (IV.18.7) the transformation matrix

$$D_a^{a'} = \begin{pmatrix} \sin\varepsilon\sin\vartheta\sin\varphi & \cos\varepsilon\sin\vartheta\sin\varphi & \cos\vartheta\sin\varphi & \cos\varphi & 0 \\ \sin\varepsilon\sin\vartheta\cos\varphi & \cos\varepsilon\sin\vartheta\cos\varphi & \cos\vartheta\cos\varphi & -\sin\varphi & 0 \\ \sin\varepsilon\cos\vartheta & \cos\varepsilon\cos\vartheta & -\sin\vartheta & 0 & 0 \\ \cos\varepsilon\cos i\psi & -\sin\varepsilon\cos i\psi & 0 & 0 & -\sin i\psi \\ \cos\varepsilon\sin i\psi & -\sin\varepsilon\sin i\psi & 0 & 0 & \cos i\psi \end{pmatrix}. \qquad \text{(IV.21.2)}$$

The sequence of indices is {3,2,1,0,4} in agreement with (IV.18.1)[66]. With the inverse matrix the pointer X can be brought to the simple form

$$X^a = \{0,0,0,1,0\}. \qquad \text{(IV.21.3)}$$

With (IV.18.6) one gets

$$X^a{}_{;b} = \delta^a_b \,. \qquad \text{(IV.21.4)}$$

The same matrix can be applied to the double-surface, in order to rotate the local Cartesian co-ordinate system in any point of the surface in such a way that the 0^{th} component is normal to the surface. The 0^{th} bein vector lies in the extra dimension. In order to describe the transition from the pseudo-hyper sphere to the double-surface, it is sufficient to regard the procedures in one of the cutting planes of the pseudo-hyper sphere. The rotation matrix in such a plane is

$$D_a^{a'} = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix} = \begin{pmatrix} a & -v \\ v & a \end{pmatrix} \qquad \text{(IV.21.5)}$$

with the index sequence {0,1}. The partial derivatives in the local system are

$$\hat{\partial}_0 = \frac{\partial}{\partial X}, \quad \hat{\partial}_1 = \frac{\partial}{X\partial\varepsilon}. \qquad \text{(IV.21.6)}$$

[66] The sequence of indices is a result of the order of the rotations through the angles $\{\varepsilon, \eta, \vartheta, i\psi\}$. Apart from that the arrangement of the values of the indices is arbitrary.

If one transcribes the geometry of the pseudo-hyper sphere in such a way that one is able to explain a double-surface, then one has for a function f which depends on all the variables of the radial cutting plane

notation:
$$df\left(r,R,\bar{r},\bar{R}\right)=\frac{\partial f}{\partial r}dr+\frac{\partial f}{\partial R}dR+\frac{\partial f}{\partial \bar{r}}d\bar{r}+\frac{\partial f}{\partial \bar{R}}d\bar{R}\ . \tag{IV.21.7}$$

In particular, for the pointer X, which was the radius of curvature of the pseudo-hyper sphere, has to be written

$$dX\left(r,R,\bar{r},\bar{R}\right)=\frac{\partial X}{\partial r}dr+\frac{\partial X}{\partial R}dR+\frac{\partial X}{\partial \bar{r}}d\bar{r}+\frac{\partial X}{\partial \bar{R}}d\bar{R}\ . \tag{IV.21.8}$$

Now X connects Neil's parabola with the Schwarzschild parabola or with one of the other curves which are parallel to the Schwarzschild parabola. The co-ordinates of the tip of X are $\left\{r,R\right\}$, those from the tail $\left\{\bar{r},\bar{R}\right\}$. From

$$X^2=\left(r-\bar{r}\right)^2+\left(R-\bar{R}\right)^2 \tag{IV.21.9}$$

one obtains

$$dX=\frac{r-\bar{r}}{X}dr+\frac{R-\bar{R}}{X}dR-\frac{r-\bar{r}}{X}d\bar{r}-\frac{R-\bar{R}}{X}d\bar{R} \tag{IV.21.10}$$

and reads from it with the help of (IV.21.5)

$$\begin{aligned}&X^{1'}=X\sin\varepsilon=Xv,\quad X^{1'}=x^{1'}-\bar{x}^{1'}=r-\bar{r},\quad v=\frac{r-\bar{r}}{X}\\&X^{0'}=X\cos\varepsilon=Xa,\quad X^{0'}=x^{0'}-\bar{x}^{0'}=R-\bar{R},\quad a=\frac{R-\bar{R}}{X}\end{aligned}\ . \tag{IV.21.11}$$

If one harks back to (IV.21.10) one gets

$$\begin{aligned}&dX=\left[vdr+adR\right]-\left[vd\bar{r}+ad\bar{R}\right]=dx^0-d\bar{x}^0\\&\frac{\partial X}{\partial r}=v,\quad \frac{\partial X}{\partial R}=a,\quad \frac{\partial X}{\partial \bar{r}}=-v,\quad \frac{\partial X}{\partial \bar{R}}=-a\end{aligned} \tag{IV.21.12}$$

and one recognizes that the vector X can experience a change of two kinds. On the one hand with the contribution dx^0, by being prolongated from a curve to a parallel curve, on the other hand, if the tip of X additionally moves along at one of the curves. In this case the tail of the vector X is forced to slide along at the Neil parabola and yields $d\bar{x}^0$. If one constrains the motion of X in such a way that the tip of the vector remains on the Schwarzschild parabola and if we designate the vector with ρ as before, then one obtains by differentiation of (IV.19.2) first of all

$$\frac{\partial R}{\partial r}=-\frac{v}{a},\quad \frac{\partial \bar{R}}{\partial \bar{r}}=\frac{a}{v} \tag{IV.21.13}$$

and lastly

$$d\rho=\left[v-a\frac{v}{a}\right]dr-\left[v+a\frac{a}{v}\right]d\bar{r}\ . \tag{IV.21.14}$$

As expected one has

$$dx^0 = 0, \quad d\rho = -d\bar{x}^0 = -\frac{1}{v} d\bar{r} \ . \tag{IV.21.15}$$

If one considers, in addition, (IV.19.3) and the definition of the arc element of the Schwarzschild parabola, one finally has

$$d\rho = -3\frac{a}{v}dx^1, \quad \rho_{|1} = -3\frac{a}{v} \tag{IV.21.16}$$

and thus completely has withdrawn to the intrinsic properties of the Schwarzschild parabola in the sense of Gauss, and again hidden the variable $\bar{r}$, and has obnubilated the relevancy of $d\rho$.

For the further development of the theory it is also necessary to regard the extrinsic geometry, i.e. the behavior of the quantities outside the physical surface. The extra components of the quantities which are to be deduced directly from the view (III) will play an important role. With de Sitter cosmos we have already made use of this method and as a result we have found a geometrical explanation for the cosmological constant.

In order to get beyond the physical surface the *prolongation* of the 4-dimensional quantities in the direction of the extra dimension must be regarded. Thereby an infinitesimal prolongation is sufficient. That is no problem in the present case because the 4-dimensional geometry has been deduced from a 5-dimensional theory and the 0th components of all quantities are well known.

Nevertheless, we once again turn to the radius of curvature of the Schwarzschild parabola and we examine its infinitesimal prolongation beyond the Schwarzschild parabola. In this case

$$\bar{r} = \text{const.}, \quad \bar{R} = \text{const.}, \quad d\bar{x}^0 = 0 \tag{IV.21.17}$$

is valid. If one differentiates ρ in using polar co-ordinates one has

$$d\rho = \frac{\partial\rho}{\partial x^0}dx^0 + \frac{\partial\rho}{\partial x^1}dx^1 \ . \tag{IV.21.18}$$

Since the tip of ρ on the Schwarzschild parabola does not move on now

$$dx^1 = 0$$

is also valid and it remains

$$d\rho = dx^0, \quad \rho_{|0} = 1, \quad \rho_{|1} = \frac{\partial}{\rho\partial\varepsilon}\rho = 0 \ . \tag{IV.21.19}$$

Thus, for the quantities of the extrinsic 1-dimensional geometry one has to use

$$\partial_0 = \frac{\partial}{\partial\rho} \ . \tag{IV.21.20}$$

It has still to be examined in which way the angle of ascent ε changes on the parallel curves if one proceeds on these or moves perpendicularly to these. From (IV.21.10) and (IV.21.11) one first obtains by differentiation

$$\frac{\partial v}{\partial r} = \frac{a^2}{X}, \quad \frac{\partial v}{\partial R} = -\frac{av}{X}, \quad \frac{\partial a}{\partial r} = -\frac{av}{X}, \quad \frac{\partial a}{\partial R} = \frac{v^2}{X} \ .$$

With the transformation matrix (IV.21.5) one immediately obtains the operators for the partial derivatives from the Cartesian co-ordinate system by a local rotation of the polar co-ordinate system

$$\partial_0 = a\frac{\partial}{\partial R} + v\frac{\partial}{\partial r}, \quad \partial_1 = -v\frac{\partial}{\partial R} + a\frac{\partial}{\partial r} \quad . \tag{IV.21.21}$$

Applied to the trigonometric functions a and v one obtains

$$a_{|0} = 0, \quad a_{|1} = -\frac{v}{X}, \quad v_{|0} = 0, \quad v_{|1} = \frac{a}{X} . \tag{IV.21.22}$$

The fact that a and v do not change in the direction of the extra dimension is less surprising. A comparison of the angles along the straight lines normal to the parallel curves[67] takes place. All these curves have in these loci per definitionem the same angle of ascent, as one can read from Fig. IV.16. Constraining to the Schwarzschild parabola with $X\big|_{\text{parabola}} = \rho$ one obtains from (IV.21.22)

$$a_{|0} = 0, \quad a_{|1} = -\frac{v}{\rho}, \quad v_{|0} = 0, \quad v_{|1} = \frac{a}{\rho} \ .$$

With the help of (IV.19.5) one gets with

$$\rho^2{}_{|0} = \left[v\frac{\partial}{\partial r} + a\frac{\partial}{\partial R} \right]\left[\left(r - \bar{r}\right)^2 + \left(R - \bar{R}\right)^2 \right]$$

the relation

$$\rho_{|0} = \frac{r - \bar{r}}{\rho} v + \frac{R - \bar{R}}{\rho} a \ .$$

If one confines (IV.21.11) to the Schwarzschild parabola, one gets in the end $\rho_{|0} = v^2 + a^2 = 1$ and has recovered (IV.21.19). These simple calculations demonstrate the connection of the different kinds of differentiations in the Schwarzschild system.

The arc element of a curve generally has the form $\rho d\varepsilon$, if ρ is the radius of curvature and ε the polar angle of the curve. It remains to be shown that this expression can be transformed into the well-known radial Schwarzschild arc element. From

$$d\cos\varepsilon = -\sin\varepsilon d\varepsilon, \quad a_{|1}dx^1 = -\frac{v}{\rho}dx^1$$

one immediately obtains

$$\rho d\varepsilon = dx^1 = \alpha dr = \frac{1}{\sqrt{1 - \frac{2M}{r}}} dr \ . \tag{IV.21.23}$$

In order to clarify once more the different kinds of differentiation on the double-surface, the derivatives of the force of gravity are examined. Since the force of gravity on the physical surface has only one component in the radial direction, the view of the radial cutting plane is sufficient for the completion of this task. One computes the changes of

$$E_1 = \frac{1}{\rho}\frac{v}{a}$$

[67] With (IV.20.4) and (IV.21.6) one can show that similar relations are valid for the pseudo-hyper sphere.

along the Schwarzschild parabola in two steps. The changes of the trigonometric functions take place on the Schwarzschild parabola, the changes of ρ takes place on the Neil parabola. With the just-calculated values (IV.21.22) one obtains on the Schwarzschild parabola

$$d\left(\frac{v}{a}\right)=\frac{1}{\rho}\left(1+\frac{v^2}{a^2}\right)dx^1$$

and from (IV.21.15)

$$d\frac{1}{\rho}=-\frac{1}{\rho^2}d\rho=\frac{1}{\rho^2}d\bar{x}^0\ ,$$

so that

$$\left(E_{1|1}-E_1E_1\right)dx^1=\frac{1}{\rho^2}dx^1+E_1\frac{1}{\rho}d\bar{x}^0$$

and in the context of the dimensional reduction

$$d\bar{x}^0=\frac{1}{v}d\bar{r}=3\frac{1}{v}dr=3\frac{a}{v}dx^1$$

we can withdraw to the physical surface. Finally, we have

$$E_{1|1}-E_1E_1=\frac{4}{\rho^2}=\frac{2M}{r^3}\ .$$

The same result would be obtained by operating directly on the physical surface with the traditional expression for the force of gravity

$$E_1=-\alpha\frac{M}{r^2}\ .$$

By this derivation we again go a step backwards. We had

$$dE_1=\left(E_1E_1+\frac{1}{\rho^2}\right)dx^1-E_1\frac{1}{\rho}d\rho\ .$$

We also refer to the physical surface with

$$d\rho=\rho_{|1}dx^1\ ,$$

and we finally obtain with the help of

$$E_{1|1}-E_1E_1-\frac{1}{\rho^2}+E_1\frac{1}{\rho}\rho_{|1}=0\ , \qquad \text{(IV.21.24)}$$

an expression which we will notice for later use. For the term $1/\rho^2$ a tensorial form will be found in the context of the 5-dimensional theory and the last term of (IV.21.24) plays a substantial role in the structure of the theory.

Heretofore has been operated in detail on the double-surface what is admittedly quite laborious, because by every step of calculation has to be considered, where any quantity is based in and which functional dependence it has. We also have not taken into consideration the direct transition from the pseudo-hyper sphere to the Schwarzschild geometry. Now this has to be complemented and demonstrated that this transition leads, very easily and directly, to the operations on the physical surface and prepares the dimensional reduction to a large extent. The quantities

notation: $$\mathfrak{p}^3_3 = \mathfrak{p}^2_2 = \frac{Xv}{r}, \quad \mathfrak{p} = \mathfrak{p}^0_0 = \mathfrak{p}^1_1 = \mathfrak{p}^4_4 = \frac{X}{\rho}, \qquad \mathfrak{p}^2_2\Big|_{\text{parabola}} = -2, \quad \mathfrak{p}\Big|_{\text{parabola}} = 1 \quad \text{(IV.21.25)}$$

are called projectors because they map the geometry of the pseudo-hyper sphere onto the Schwarzschild geometry. They operate on the pentad differentials, on the partial derivatives, and on the derivative indices of the connexion coefficients of the pseudo-hyper sphere

notation: $$\partial_a = \mathfrak{p}^b_a \hat{\partial}_b, \quad dx^a = \left(\mathfrak{p}^{-1}\right)^a_b dX^b, \quad A_{ab}{}^c = \mathfrak{p}^d_a X_{db}{}^c \,. \qquad \text{(IV.21.26)}$$

Their purpose consists in exchanging the functional dependences of the quantities and in accomplishing the constraining onto the Schwarzschild parabola. With

$$\mathfrak{p} = \frac{X(r,R,\bar{r},\bar{R})}{\rho(r)} \qquad \text{(IV.21.27)}$$

the dependence of four variables is reduced to only one by use of the restricting relations (IV.19.2). With this procedure the variable $\bar{r}$ is hidden as well. For the motion on a straight line of the extrinsic geometry one firstly has with $X^0 = X$ and (IV.21.19)

$$\frac{\partial X^0}{\partial x^0} = \frac{\partial X}{\partial \rho}$$

and with (IV.21.26)

$$\partial_0 X^0 = \mathfrak{p}^0_0 \hat{\partial}_0 X = \mathfrak{p}^0_0$$

and with the ansatz (IV.21.25)

$$\mathfrak{p}^0_0 = \frac{\partial X}{\partial \rho} = \frac{X}{\rho}, \quad dX = \mathfrak{p}^0_0 d\rho \,. \qquad \text{(IV.21.28)}$$

With these projectors one obtains for the differentials

$$\begin{aligned}
dx^0 &= \left(\mathfrak{p}^{-1}\right)^0_0 dX^0 = \frac{\rho}{X} dX = d\rho \\
dx^1 &= \left(\mathfrak{p}^{-1}\right)^1_1 dX^1 = \frac{\rho}{X} X d\varepsilon = \rho d\varepsilon \\
dx^2 &= \left(\mathfrak{p}^{-1}\right)^2_2 dX^2 = \frac{r}{Xv} X \sin\varepsilon d\vartheta = r d\vartheta \qquad , \qquad \text{(IV.21.29)} \\
dx^3 &= \left(\mathfrak{p}^{-1}\right)^3_3 dX^3 = \frac{r}{Xv} X \sin\varepsilon \sin\vartheta d\varphi = r \sin\vartheta d\varphi \\
dx^4 &= \left(\mathfrak{p}^{-1}\right)^4_4 dX^3 = \frac{\rho}{X} X \cos\varepsilon di\psi = \rho a di\psi = iadt
\end{aligned}$$

and thus, from the metric of the pseudo-hyper sphere the Schwarzschild metric

$$ds^2 = g_{mn} (\mathfrak{p}^{-1})^m_a (\mathfrak{p}^{-1})^n_b dX^a dX^b = \rho^2 d\varepsilon^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta \, d\varphi^2 + \rho^2 a^2 di\psi^2 \;, \quad \text{(IV.21.30)}$$

which can be converted with (IV.21.23) and

$$dt = \rho d\psi \qquad \text{(IV.21.31)}$$

into the familiar form

$$ds^2 = a^{-2}dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 - a^2 dt^2, \quad a = \sqrt{1-2M/r}\ . \tag{IV.21.32}$$

Likewise, one obtains from (IV.18.4) the partial tetrad derivatives

notation: $$\partial_1 = a\frac{\partial}{\partial r}, \quad \partial_2 = \frac{\partial}{r\partial\vartheta}, \quad \partial_3 = \frac{\partial}{r\sin\vartheta\partial\varphi}, \quad \partial_4 = \frac{\partial}{\rho\cos\varepsilon\partial i\psi} = \frac{\partial}{ia\partial t}\ , \tag{IV.21.33}$$

which have to be supplemented with

notation: $$\partial_0 = v\frac{\partial}{\partial r}, \quad \partial_0 = \frac{\partial}{\partial\rho} \tag{IV.21.34}$$

depending on the kind of the functions to be treated.

The connexion coefficients for the pseudo-hyper sphere have already been indicated by the relations (IV.18.8) - (IV.18.10). From these we compute the connexion coefficients of the 5-dimensional Schwarzschild geometry

$$\begin{aligned}
&M_{10}{}^{1} = \mathfrak{p}_1^1 \hat{M}_{10}{}^{1} = \frac{1}{\rho} \\
&B_{20}{}^{2} = \mathfrak{p}_2^2 \hat{B}_{20}{}^{2} = \frac{v}{r}, \quad B_{21}{}^{2} = \mathfrak{p}_2^2 \hat{B}_{21}{}^{2} = \frac{a}{r} \\
&C_{30}{}^{3} = \mathfrak{p}_3^3 \hat{C}_{30}{}^{3} = \frac{v}{r}, \quad C_{31}{}^{3} = \mathfrak{p}_3^3 \hat{C}_{31}{}^{3} = \frac{a}{r} \\
&E_{40}{}^{4} = \mathfrak{p}_4^4 \hat{E}_{40}{}^{4} = \frac{1}{\rho}, \quad E_{41}{}^{4} = \mathfrak{p}_4^4 \hat{E}_{41}{}^{4} = -\frac{1}{\rho}\frac{v}{a}
\end{aligned}\ . \tag{IV.21.35}$$

Similarly to (IV.18.9) we can write

$$\begin{gathered}
A_{ab}{}^{c} = M_{ab}{}^{c} + B_{ab}{}^{c} + C_{ab}{}^{c} + E_{ab}{}^{c} \\
M_{ab}{}^{c} = m_a\left[M_b m^c - m_b M^c\right], \ B_{ab}{}^{c} = b_a\left[B_b b^c - b_b B^c\right]\ . \\
C_{ab}{}^{c} = c_a\left[C_b c^c - c_b C^c\right]\ , \ E_{ab}{}^{c} = -u_a\left[E_b u^c - u_b E^c\right]
\end{gathered} \tag{IV.21.36}$$

Thus, we define the 5-dimensional covariant derivative

$$\Phi_{a|||b} = \Phi_{a|b} - A_{ba}{}^{c}\Phi_c\ , \tag{IV.21.37}$$

which can be broken up by a [0+4]-decomposition into equations for the extrinsic and intrinsic geometry. If the

$$\begin{aligned}
B_{a'} &= \left\{0, \frac{1}{r}, 0, 0, 0\right\} \\
C_{a'} &= \left\{0, \frac{1}{r\sin\vartheta}\sin\vartheta, \frac{1}{r\sin\vartheta}\cos\vartheta, 0, 0\right\} \\
E_{a'} &= \left\{-\frac{1}{\rho\cos\varepsilon}, 0, 0, 0, 0\right\}
\end{aligned} \tag{IV.21.38}$$

are the components of the curvature quantities of the physical surface in the Cartesian co-ordinate system of the embedding space, then one obtains in local pseudo-polar co-ordinates by applying the rotation matrix (IV.21.5)

$$\begin{aligned} B_a &= \{B_{1'} \sin\varepsilon, B_{1'} \cos\varepsilon, 0,0,0\} \\ C_a &= \{C_{1'} \sin\varepsilon, C_{1'} \cos\varepsilon, C_{2'}, 0,0\} \\ E_{a'} &= \{E_{0'} \cos\varepsilon, -E_{0'} \sin\varepsilon, 0,0,0\} \end{aligned} \qquad \text{(IV.21.39)}$$

and the result (IV.21.35) is regained. Evidently the relations

$$B_a E^a = 0, \quad C_a E^a = 0 \qquad \text{(IV.21.40)}$$

are satisfied. The 5-dimensional quantities B_a and C_a are situated in the parallels of Flamm's paraboloid, the force of gravity E_a is vertical to them. With

$$\mathfrak{p}^a_b = X^a{}_{|||b} \qquad \text{(IV.21.41)}$$

one finds an additional substantiation for the projectors.

IV.22. Schwarzschild metric, 5-dimensional field equations

With the introduction of the projector technology we are well prepared to set up the 5-dimensional field equations which take a very compact and geometrically understandable form. Firstly, one defines a curvature tensor with the quantities A

notation: $$R_{abc}{}^{d}(A) = 2\left[A_{[b\cdot c\cdot}{}^{d}{}_{|a]} + A_{[b\cdot c\cdot}{}^{f}A_{a]f}{}^{d} + A_{[ba]}{}^{f}A_{fc}{}^{d}\right] \tag{IV.22.1}$$

and the identically vanishing Riemann tensor of the flat embedding space

notation: $$R_{abc}{}^{d}(X) = 2\left[X_{[b\cdot c\cdot}{}^{d}{}_{,a]} + X_{[b\cdot c\cdot}{}^{f}X_{a]f}{}^{d} + X_{[ba]}{}^{f}X_{fc}{}^{d}\right] . \tag{IV.22.2}$$

If the projectors act on (IV.22.2), one obtains

$$\mathfrak{p}_{a}^{g}\mathfrak{p}_{b}^{h}R_{ghc}{}^{d}(X) = R_{abc}{}^{d}(A) + 2X_{fc}{}^{d}\mathfrak{p}^{f}_{[a||b]} \equiv 0 \ , \tag{IV.22.3}$$

whereby the last term arises from the first term of (IV.22.2) by differentiation of the projectors and can also be written as

$$2A_{hc}{}^{d}\left(\mathfrak{p}^{-1}\right)^{h}_{f}\mathfrak{p}^{f}_{[a||b]} \ . \tag{IV.22.4}$$

The expression (IV.22.1) does not have the necessary Riemannian properties like (IV.22.2). However, the 4-dimensional part $R_{rmn}{}^{s}(A)$ will possess these properties, i.e. the same symmetry properties that the Riemann tensor of a V_4 possesses which is embedded into an E_5.

We expressly state that the 4-dimensional Riemann tensor is not the curvature tensor of a 4-dimensional surface in the sense of the classical surface theory, but the curvature tensor of the physical surface which has substantial pieces of the double-surface. Just because of these properties there exists the possibility of circumventing the laws of Eisenhart and Kasner.

The expression (IV.22.4) is only weakly occupied with regard to its components despite its voluminous structure and essentially supplies the last term of (IV.21.24) which thereby obtains a further geometrical substantiation. Some partial derivatives are noted, in order to be able to continue work on the expression (IV.22.4). One can expect that $\mathfrak{p}_{0}^{0}, \mathfrak{p}_{1}^{1}$ and $\mathfrak{p}_{4}^{4}$ do not change along the straight lines of the co-ordinate system as shown in Fig. IV.7

$$\mathfrak{p}^{0}_{0|0} = \mathfrak{p}^{1}_{1|0} = \mathfrak{p}^{4}_{4|0} = \left(\frac{X}{\rho}\right)_{|0} = \frac{X_{|0}}{\rho} - \frac{X}{\rho^2}\rho_{|0} = \frac{1}{\rho}\left(\mathfrak{p}X_{,0} - \frac{X}{\rho}\right) = 0 \ . \tag{IV.22.5}$$

However, because of $X_{|1} = \dfrac{\partial X}{\rho\partial\varepsilon} = 0$ and (IV.21.16) one has along the parabola

$$\begin{aligned}
&\mathfrak{p}_{|1} = \mathfrak{p}^{0}_{0|1} = \mathfrak{p}^{1}_{1|1} = \mathfrak{p}^{4}_{4|1} = -\frac{X}{\rho^2}\rho_{|1} = 3\frac{X}{\rho^2}\frac{a}{v} \\
&\mathfrak{p}^{2}_{2|0} = \mathfrak{p}^{3}_{3|0} = \left[\frac{Xv}{r}\right]_{|0} = X_{|0}\frac{v}{r} - \frac{Xv}{r^2}r_{|0} = \mathfrak{p}^{1}_{1}B_0 - \mathfrak{p}^{2}_{2}B_0 \ . \\
&\mathfrak{p}^{2}_{2|1} = \mathfrak{p}^{3}_{3|1} = \left[\frac{Xv}{r}\right]_{|1} = \frac{X}{r}v_{|1} - \frac{Xv}{r^2}r_{|1} = \mathfrak{p}^{1}_{1}B_1 - \mathfrak{p}^{2}_{2}B_1
\end{aligned} \tag{IV.22.6}$$

With (IV.21.35) and (IV.21.36) one can compute (IV.22.4). Only the two expressions

$$2\mathfrak{p}^{0}_{[0|||1]} = 2\mathfrak{p}^{4}_{[4|||1]} = \mathfrak{p}_{|1}, \quad \mathfrak{p}_{|1} = 3\mathfrak{p}\frac{1}{\rho}\frac{a}{v} \qquad \text{(IV.22.7)}$$

remain.

As a first step towards a [0+4]-decomposition of (IV.22.3) one has to extract the terms with the 0-indices in the summation of the double indices[68] in (IV.22.1)

$$'R_{abc}{}^{d} + Z_{abc}{}^{d} = 0, \quad Z_{abc}{}^{d} = 2\left[A_{[b\cdot c\cdot}{}^{0}A_{a]0}{}^{d} + A_{cb}{}^{d}\left(\mathfrak{p}^{-1}\right)^{c}_{f}\mathfrak{p}^{f}_{[d|||a]}\right]. \qquad \text{(IV.22.8)}$$

The first expression in Z is the already well-known product of the second fundamental forms of the surface theory and Eq. (IV.22.8) is Gauss' equation generalized to the double-surface theory. The next step will be to let run the indices only from 1 to 4

$$'R_{rmn}{}^{s} + Z_{rmn}{}^{s} = 0, \quad Z_{rmn}{}^{s} = 2\left[A_{[m\cdot n\cdot}{}^{0}A_{r]0}{}^{s} + A_{cn}{}^{s}\left(\mathfrak{p}^{-1}\right)^{c}_{f}\mathfrak{p}^{f}_{[r|||m]}\right]. \qquad \text{(IV.22.9)}$$

Since the connexion coefficients for c = 0 are empty, the last expression in the brackets reduce to

$$A_{pn}{}^{s}\left(\mathfrak{p}^{-1}\right)^{p}_{q}\mathfrak{p}^{q}_{[r|||m]}. \qquad \text{(IV.22.10)}$$

Since this term has only one component in accordance with (IV.22.7), we explicitly calculate only the appropriate Z-term

$$Z_{411}{}^{4} = A_{11}{}^{0}A_{40}{}^{4} + 2A_{p1}{}^{4}\left(\mathfrak{p}^{-1}\right)^{p}_{q}\mathfrak{p}^{q}_{[4|||1]} = M_0E_0 + 2A_{41}{}^{4}\left(\mathfrak{p}^{-1}\right)^{4}_{4}\mathfrak{p}^{4}_{[4|||1]} = -\frac{4}{\rho^2}. \qquad \text{(IV.22.11)}$$

For the components of Z one obtains by the use of (IV.21.35) and

$$\rho v = -2r \qquad \text{(IV.22.12)}$$

$$Z_{211}{}^{2} = Z_{311}{}^{3} = Z_{422}{}^{4} = Z_{433}{}^{4} = \frac{2}{\rho^2}, \quad Z_{411}{}^{4} = Z_{322}{}^{3} = -\frac{4}{\rho^2} \qquad \text{(IV.22.13)}$$

and for the components of the 4-dimensional Riemann tensors with (IV.22.8)

$$'R_{211}{}^{2} = 'R_{311}{}^{3} = 'R_{422}{}^{4} = 'R_{433}{}^{4} = -\frac{M}{r^3}, \quad 'R_{411}{}^{4} = 'R_{322}{}^{3} = \frac{2M}{r^3}, \qquad \text{(IV.22.14)}$$

the values indicated in the literature for the Schwarzschild geometry. All other components of 'R and Z vanish, up to those which one obtains with the symmetry properties of the Riemann tensor. All components behave regularly on the boundary of the geometry r = 2M. That the Riemann tensor beneath the boundary of the geometry is not pursuable, one recognizes with (IV.22.13), because the radius of curvature of the surface and the surface as well do not exist in this region. On the other hand the range of validity of r has been limited from the beginning to $\left[2M,\infty\right]$[69]. It is also simple to show that the contraction of (IV.22.3) to

[68] A prime ahead the kernel of a tensor refers to the 4-dimensional part of this quantity.

[69] It is no special property of Flamm's paraboloid that the curvature tensor seemingly takes reasonable values also outside the surface by a suitable co-ordinate choice. This behavior and also the behavior of invariants which can be formed from squares of the Riemann tensors are often consulted by the supporters of the view (I), in order to support the validity of the metric beneath the event horizon. Schimming [S] and

$$R_{ab} + 2A_{cb}{}^{d}\left(\mathfrak{p}^{-1}\right)^{c}_{f}\,\mathfrak{p}^{f}_{[d|||a]} = 0 \qquad \text{(IV.22.15)}$$

finally leads to

$$Z_{mn} = 0 \qquad \text{(IV.22.16)}$$

which also ensures the Ricci-flatness of geometry

$$'R_{mn}\left(A\right) = 0 \ . \qquad \text{(IV.22.17)}$$

The Schwarzschild geometry is a vacuum solution of the Einstein field equations. After the mode of action of the projectors has been discussed in detail, one can set up the 5-dimensional field equations of the Schwarzschild geometry by means of a recourse to the equations (IV.21.33) to (IV.21.37), whereby again the graded derivatives are applied. One gets a set of relations[70]

$$\begin{aligned}
&M_{\underset{1}{a|||b}} + M_a M_b = 0, \quad M^{c}{}_{\underset{1}{|||c}} + M^{c} M_c = 0 \\
&B_{\underset{2}{a|||b}} + B_a B_b = 0, \quad B^{c}{}_{\underset{2}{|||c}} + B^{c} B_c = 0 \\
&C_{\underset{3}{a|||b}} + C_a C_b = 0, \quad C^{c}{}_{\underset{3}{|||c}} + C^{c} C_c = 0 \\
&E_{\underset{4}{a|||b}} - E_a E_b + E_b\, \mathfrak{p}^{-1}\mathfrak{p}_{|a} = 0, \quad E^{c}{}_{\underset{4}{|||c}} - E^{c} E_c + E^{c}\, \mathfrak{p}^{-1}\mathfrak{p}_{|c} = 0
\end{aligned} \quad , \qquad \text{(IV.22.18)}$$

which deviate from those of the de Sitter cosmos (II.2.10) only by the E-equations which contain elements of the double-surface theory as has already been discussed in detail. The E-equations written in full read as

$$\begin{aligned}
&E_{0|0} - E_0 E_0 = 0, \quad E_{0|1} - M_{10}{}^{1} E_1 - E_0 E_1 + E_0 \frac{1}{\rho}\rho_{|1} = 0 \\
&E_{1|0} - E_1 E_0 = 0, \quad E_{1|1} - M_{11}{}^{0} E_0 - E_1 E_1 + E_1 \frac{1}{\rho}\rho_{|1} = 0
\end{aligned} \quad . \qquad \text{(IV.22.19)}$$

With

$$-M_{11}{}^{0} E_0 = -\frac{1}{\rho^2}$$

one has geometrically interpreted the still unidentified term in the relation (IV.21.24). The equations (IV.22.18) describe the curvatures of the normal and inclined slices of the physical surface. One could have obtained the same relations with the help of the projectors from the equations of the pseudo-hyper sphere (IV.18.14) which exhibits the close relationship of a single surface with a double-surface. With the above formulae and the definitions in former Sections it is not difficult to convert the 5-dimensional equations (IV.22.18) into the 4-dimensional ones. Again one obtains the relations (IV.1.6) to (IV.1.8). Thus, one has to accept that the equations for the curvatures do not decouple from the Einstein field equations.

Fiedler [F] and Schimming established with the help of such invariants a modified gravitation theory which admits singularity-free solutions.

[70] It has to be emphasized that M_a is a quantity of the extrinsic geometry. The differential operator (IV.21.20) has to be used for that quantity.

In the last Sections it has been shown how gravitation models can be constructed in a flat higher-dimensional space by means of suitable transformations. Animated by the large successes of the elementary particle theories some years ago one tried to implement gravitation in flat spaces by gauge transformations whereby group transformations played a substantial role. Fields, generally called Yang-Mills fields, are produced by such transformations. These transformations form covariant derivatives together with the partial derivatives, as they are known from the gravitation theory. Utiyama [U] developed the fundamental gauge structures for the gravitation theory and suggested a gauging with a group extending the Lorentz group. Liebscher [L] and Treder pointed out that one can ‘switch on and off’ gravitation only with non-Lorentzian transformations. In our booklet Lorentz transformations play an important role. However, they only change the point of view to an already existing gravitation, while the curvature structure of the space remains absolutely unchanged. Kibble [K] supplemented the Lorentz group with a group of translations. Numerous further approaches by other authors followed. Hehl [H] and his coworkers extended Einstein’s gravitation theory including the torsion of the space. On this basis they deduced a very detailed gauge theory of gravitation. In recent time one has tried to affiliate the gravitation to the other three interaction fields of nature with the help of supersymmetrical theories.

The question, whether the embedding procedure outlined here gives a lead for a gauge theory of gravitation, must be answered in the negative. A transformation of the type (IV.21.2) is an element of a group and would be sufficient to explain the de Sitter cosmos. However, the projectors which are necessary for the derivation of the Schwarzschild geometry do not form a group.

Throughout the preceding investigations on the Schwarzschild field we have limited ourselves to the range $r = \left[2M, \infty\right]$ and we have left open, how the gap beneath this range can be filled. Since we are highly skeptical about the theory of the black holes we turn to the simplest possibility. This is the completion of the model by the Schwarzschild interior solution which establishes the complete Schwarzschild solution together with the exterior solution.

IV.23. Schwarzschild interior solution

One wants to describe the region of the space which is filled with matter, commonly with a stellar object, with the Einstein field equations as well. In this case the solution is called interior solution. The field which is produced by such a source is described by the exterior solution. For the interior solution the Einstein field equations read in the most general form as

$$R_{mn} - \frac{1}{2} g_{mn} R = -\kappa T_{mn} . \tag{IV.23.1}$$

It is required that not all components of the stress-energy tensor vanish and that the remaining components are accessible to a physical interpretation. The stress-energy tensor has to satisfy the conservation law

$$T^{mn}{}_{||n} = 0 \ . \tag{IV.23.2}$$

In addition, it has to fulfill certain conditions on the surface of the stellar object. The interior solution has to match the exterior, i.e. the metric and the first derivatives of the metric of the two solutions must coincide on the boundary surface. By means of the geometrical view, in the sense of an embedding procedure, the tangent planes of the two surfaces which represent these solutions have to coincide. We will show that the interior Schwarzschild solution, just as the exterior, can be embedded into a 5-dimensional flat space and fulfills the just-mentioned requirements. The standard form of the interior Schwarzschild metric [S] is

$$ds^2 = \frac{1}{1 - \frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta \, d\varphi^2 - \frac{1}{4}\left[3\cos\eta_g - \cos\eta\right]^2 dt^2 \ , \tag{IV.23.3}$$

where η_g is a constant which still has to be discussed. It is easy to see that the spatial part of the metric takes with the substitution

$$r = \mathcal{R} \sin\eta \tag{IV.23.4}$$

the well-known form of the Einstein cosmos and de Sitter cosmos

$$'ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta \, d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta \, d\varphi^2 \tag{IV.23.5}$$

and can be interpreted as the metric of a 3-dimensional hyper-sphere with the radius $\mathcal{R}$ which can be embedded into a 4-dimensional flat space. In spherical co-ordinates the metric of this flat space has the form

$$ds^2 = d\mathcal{R}^2 + \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta \, d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta \, d\varphi^2 . \tag{IV.23.6}$$

The embedding condition

$$\mathcal{R} = \mathcal{R}_g, \quad \mathcal{R}_g = \text{const.} \tag{IV.23.7}$$

selects a spherical surface from the family of spherical surfaces described by (IV.23.6). For the Schwarzschild interior solution not the whole hyper surface is used, but only a cap which is taken from downside and put onto a suitable position on Flamm's paraboloid. As mentioned at the start the tangent planes of the hyper sphere must coincide with the tangent planes of Flamm's paraboloid. The aperture angle of the cap of the sphere is η_g, it plays a substantial role in the time-like part of the metric.

Attempts to enunciate the interior Schwarzschild geometry as embedding theory including the time-like part seem to have never been tackled. Therefore we present a solution [B] which closely follows the methods used for the exterior solution. A trivial way to clarify the problem is the geometrical interpretation of the factor 3 in the time-like part of the metric. From the exterior solution we know that the motion of the curvature vector[71] ρ along the Schwarzschild parabola determines the properties of the geometry to a large extent. The curvature vector on the boundary between the exterior and interior solutions has the value ρ_g. If one extends this curvature vector to the directrix of the parabola ($x^{0'}$-axis), then the center of the sphere is situated in the emerging point of the intersection. The cap of the sphere matches the boundary of Flamm's paraboloid. Thus, the distance of the center of the sphere to the boundary is $\Re_g$. Since also the angles of ascent of the radial cutting curves on the two surfaces must agree, one has[72]

$$\eta_g = -\varepsilon_g \ . \tag{IV.23.8}$$

Therefore

$$\sin\eta_g = -\sin\varepsilon_g = \sqrt{\frac{2M}{r_g}}, \quad \rho_g = \sqrt{\frac{2r_g^{\,3}}{M}} = \frac{2r_g}{\sin\eta_g}$$

is valid and with (IV.23.4) one finally recognizes that

$$\rho_g = 2\Re_g . \tag{IV.23.9}$$

For the entire distance, from the base point of the curvature vector up to the directrix one has

$$\rho_g + \Re_g = 3\Re_g \tag{IV.23.10}$$

and for the projection of this line segment onto the Cartesian co-ordinate axes of the embedding space

$$\left\{x^{0'}, x^{1'}\right\} = \left\{3\Re_g \cos\eta_g, 3\Re_g \sin\eta_g\right\} \ . \tag{IV.23.11}$$

Since these considerations are not only valid for the curves and vectors on the boundary, but are valid for any positions on the Schwarzschild parabola

$$x^{1'} = 3\Re\sin\eta \ , \quad \overline{r} = 3r \tag{IV.23.12}$$

is generally valid if one respects (IV.23.4), whereby subsequently a fundamental property of the Schwarzschild parabola was found.

If one puts in (IV.23.3)

$$-idt = 2\Re\, di\psi \ , \tag{IV.23.13}$$

one has for the time-like arc element

$$dx^4 = \left[3\Re_g \cos\eta_g - \Re\cos\eta\right] di\psi \tag{IV.23.14}$$

and the metric of the interior solution reads as

---

[71] More exactly, the only nonvanishing component of the curvature vector in the applied local reference system.

[72] We give to ε the somewhat unorthodox orientation cw. The marker g refers to the value of a quantity on the boundary surface.

$$ds^2 = \mathfrak{R}^2 d\eta^2 + \mathfrak{R}^2 \sin^2\eta\, d\vartheta^2 + \mathfrak{R}^2 \sin^2\eta \sin^2\vartheta\, d\varphi^2 + \left[3\mathfrak{R}_g \cos\eta_g - \mathfrak{R}\cos\eta\right]^2 di\psi^2 \ . \quad \text{(IV.23.15)}$$

Now (IV.23.14) can be interpreted geometrically. The radius $\mathfrak{R}$ of the hyper sphere has the projection $\mathfrak{R}\cos\eta$ onto the global extra dimension, the line segment discussed above has the projection $3\mathfrak{R}_g\cos\eta_g$ onto the directrix. If one rotates these two line segments in the $\left[x^{0'},x^{4'}\right]$-plane of the 5-dimensional flat embedding space through the imaginary angle $i\psi$, one obtains two (open) pseudo circles (hyperbolae of constant curvature). Together they form a ring sector of the pseudo circles, whose area is associated with the time that has past.

With a given form parameter the Schwarzschild parabola and thus, with a given mass of the field-creating stellar object, special adjustments can be made for the exterior geometry by shifting the center of the hyper sphere on the directrix of the parabola. The radius $\mathfrak{R}$ of the hyper sphere and the aperture angle η_g take different values with respect the cap of the sphere. We will see later that the definition of the parameters of the interior solution is not completely arbitrary, because the physics of the stellar object imposes conditions upon these parameters. It should be mentioned that for the two solutions different Cartesian and local co-ordinate systems are used. The origin O' of the Cartesian co-ordinate system of the interior solution is situated in the center of the hyper sphere, the origin O of the co-ordinate system of the exterior solution is situated in the intersecting point of the directrix and the symmetry axis of the parabola. The angles η and ε have opposite orientations, the curvature vectors $\vec{\mathfrak{R}}$ and $\vec{\rho}$ point with the tips to each other. Accordingly, the Cartesian 0'-axis and the local 0-axes have opposite orientations as well. This can be seen from Fig. IV.18. This choice simplifies the calculations. However, the mutual adjustment of the two solutions does not make difficulties.

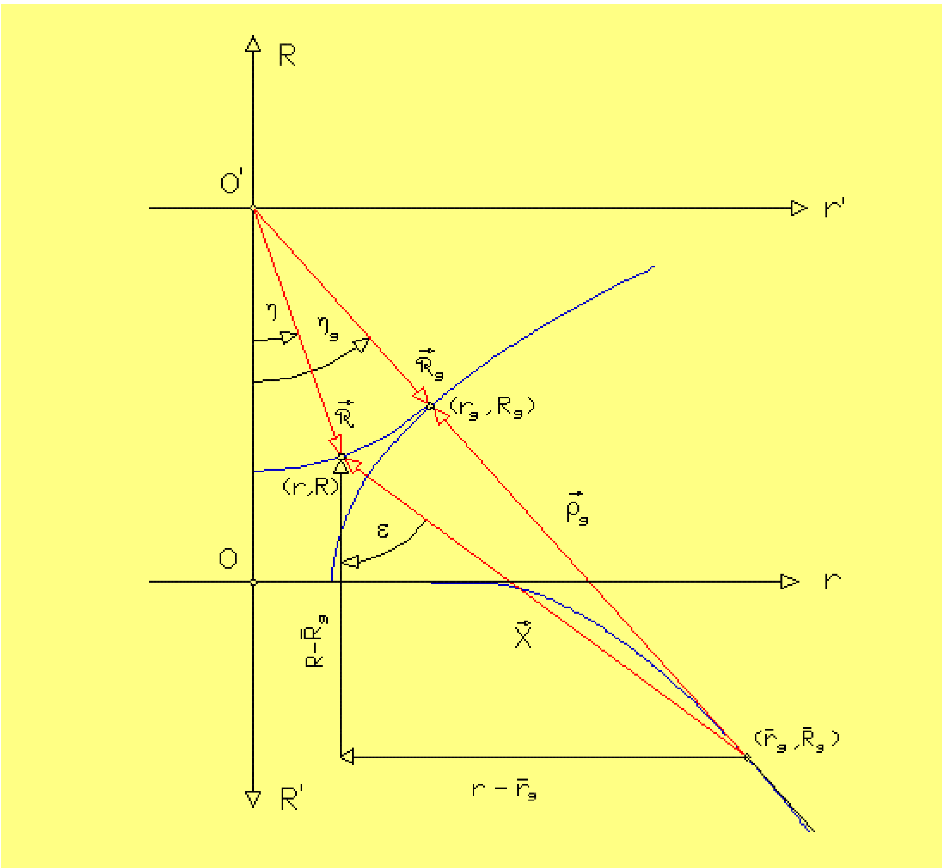

Fig. IV.18

With the last considerations we came closer to the possibility of an embedding of the interior Schwarzschild solution into a 5-dimensional flat space. While the purely spatial part of the metric can be explained by the cap of a sphere in the context of the classical surface theory, one again needs the metric of the two surfaces in order to embed the time-like part just as we needed for the exterior solution. The curvature vector on the boundary of the exterior and the interior solutions has been designated by ρ_g, its tail has the co-ordinates $\left\{\bar{r}_g, \bar{R}_g\right\}$ with respect to the exterior co-ordinate system. Now we fix the tail on the Neil parabola. Then we move inwards the tip of the vector, which now we designate by $\vec{X}$, on the circular arc of a radial slice of the cap of the sphere. Thus, we already have described to a large extent the interior geometry by the movement of this vector and we have found the connection to the double-surface. On the radial cutting planes of the surfaces[73] one has for X

$$X^2 = \left(r - \bar{r}_g\right)^2 + \left(R - \bar{R}_g\right)^2, \quad X \sin\varepsilon = r - \bar{r}_g, \quad X \cos\varepsilon = R - \bar{R}_g \ . \qquad \text{(IV.23.16)}$$

In the 5-dimensional space the pointer X can be represented by

$$\begin{aligned} X^{3'} &= X \sin\varepsilon \sin\vartheta \sin\varphi \\ X^{2'} &= X \sin\varepsilon \sin\vartheta \cos\varphi \\ X^{1'} &= X \sin\varepsilon \cos\vartheta \\ X^{0'} &= X \cos\varepsilon \cos i\psi \\ X^{4'} &= X \cos\varepsilon \sin i\psi \end{aligned} \qquad \text{(IV.23.17)}$$

and these relations can be interpreted as the parameterizing of a single surface.

After discussing the properties of the geometry which supplies the exterior solution we go over to the co-ordinate system with the origin O' of the interior solution, which partly entails a change of sign by parameterizing the single surface

$$\begin{aligned} X^{3'} &= X \sin\varepsilon \sin\vartheta \sin\varphi \\ X^{2'} &= X \sin\varepsilon \sin\vartheta \cos\varphi \\ X^{1'} &= X \sin\varepsilon \cos\vartheta \\ X^{0'} &= -X \cos\varepsilon \cos i\psi \\ X^{4'} &= -X \cos\varepsilon \sin i\psi \end{aligned} \quad . \qquad \text{(IV.23.18)}$$

For a point on the cap of the sphere one has

$$r = \mathcal{R} \sin\eta, \qquad R = \mathcal{R} \cos\eta \ , \qquad \text{(IV.23.19)}$$

for the fixed point on Neil's parabola

$$\bar{r}_g = 3\mathcal{R}_g \sin\eta_g, \qquad \bar{R}_g = 3\mathcal{R}_g \cos\eta_g \qquad \text{(IV.23.20)}$$

measured from O'[74]. From (IV.23.16) we read

[73] It has to be emphasized that $r - \bar{r}_g$ has a negative value and sin ε is negative as well. However, in $R - \bar{R}_g$ the quantity $\bar{R}_g$ is negative. Thus, cos ε has a positive value.

[74] $\bar{R}_g$ is a positive quantity, measured from O'.

$$\begin{aligned} X\sin\varepsilon &= r-\bar{r}_g = \mathfrak{R}\sin\eta - 3\mathfrak{R}_g\sin\eta_g \\ -X\cos\varepsilon &= R-\bar{R}_g = \mathfrak{R}\cos\eta - 3\mathfrak{R}_g\cos\eta_g \end{aligned}\ , \tag{IV.23.21}$$

and we obtain the parameterization of the double-surface

$$\begin{aligned} X^{3'} &= \mathfrak{R}\sin\eta\sin\vartheta\sin\varphi - 3\mathfrak{R}_g\sin\eta_g\sin\vartheta\sin\varphi \\ X^{2'} &= \mathfrak{R}\sin\eta\sin\vartheta\cos\varphi - 3\mathfrak{R}_g\sin\eta_g\sin\vartheta\cos\varphi \\ X^{1'} &= \mathfrak{R}\sin\eta\cos\vartheta \qquad - 3\mathfrak{R}_g\sin\eta_g\cos\vartheta \\ X^{0'} &= \mathfrak{R}\cos\eta\cos i\psi \qquad - 3\mathfrak{R}_g\cos\eta_g\cos i\psi \\ X^{4'} &= \mathfrak{R}\cos\eta\sin i\psi \qquad - 3\mathfrak{R}_g\cos\eta_g\sin i\psi \end{aligned}\ . \tag{IV.23.22}$$

The dimensional reduction would consist in eliminating the first three lines in the right column of (IV.23.22) and in isolating the local 0-components in the field equations by projectors. Since the calculations turn out to be somewhat laborious, we try another way this time. Instead of projecting away the superfluous elements in (IV.23.22) we use as an alternative of (IV.23.18) the ansatz

$$\begin{aligned} \mathfrak{R}^{3'} &= \mathfrak{R}\sin\eta\sin\vartheta\sin\varphi \\ \mathfrak{R}^{2'} &= \mathfrak{R}\sin\eta\sin\vartheta\cos\varphi \\ \mathfrak{R}^{1'} &= \mathfrak{R}\sin\eta\cos\vartheta \\ \mathfrak{R}^{0'} &= \mathfrak{R}\cos\eta\cos i\psi \\ \mathfrak{R}^{4'} &= \mathfrak{R}\cos\eta\sin i\psi \end{aligned} \tag{IV.23.23}$$

for a family of pseudo-hyper spheres which contains the spatial hyper sphere of the interior solution. However, this ansatz cannot describe the time-like part of the metric. The line element of the 5-dimensional space has the well-known form of the de Sitter cosmos

$$ds^2 = d\mathfrak{R}^2 + \mathfrak{R}^2 d\eta^2 + \mathfrak{R}^2\sin^2\eta\, d\vartheta^2 + \mathfrak{R}^2\sin^2\eta\sin^2\vartheta\, d\varphi^2 + \mathfrak{R}^2\cos^2\eta\, di\psi^2\ , \tag{IV.23.24}$$

whereby a pseudo-hyper sphere is selected from this family of pseudo-hyper spheres by $\mathfrak{R} = \text{const.}$ and the line element (IV.23.24) is reduced to the line element of a pseudo-hyperspherical surface. For the representation of the merely spatial geometry the parameterizations (IV.23.22) and (IV.23.23) are equivalent after performing the dimensional reduction. For the description of a curve by means of a pole and a polar angle the position of the pole is not important. It can also be shown that

$$dX^2 + X^2 d\varepsilon^2 = d\mathfrak{R}^2 + \mathfrak{R}^2 d\eta^2 \tag{IV.23.25}$$

is valid, if one knows the somewhat complicated relationship between ε and η. The relation states that both the tips of the vectors $\vec{\mathfrak{R}}$ and $\vec{X}$ move on the circular arc of the hyper sphere. In the polar co-ordinates belonging to O' the pointer to the hyper sphere has the components

$$\mathfrak{R}^a = \{\mathfrak{R}, 0, 0, 0, 0\}\ , \tag{IV.23.26}$$

whereby the index sequence is now set to $\{0,1,\ldots,4\}$. From (IV.23.23) one computes the arc elements

$$d\mathfrak{R}^a = \{d\mathfrak{R},\ \mathfrak{R}d\eta,\ \mathfrak{R}\sin\eta\, d\vartheta,\ \mathfrak{R}\sin\eta\sin\vartheta\, d\varphi,\ \mathfrak{R}\cos\eta\, di\psi\}\ . \tag{IV.23.27}$$

The conversion of the arc elements from the single surface to the double-surface takes place with

notation:
$$d\mathcal{R}^a = \mathfrak{p}^a_b dx^b \ . \tag{IV.23.28}$$

One simply deduces the projectors $\mathfrak{p}$ from the preceding considerations. Since in the course of the dimensional reduction we have restricted ourselves to the spatial part of the 3-dimensional hyper sphere, the associated arc elements can be read unmodified from (IV.23.24). Therefore, the associated projectors have the value 1. In contrast, the time-like arc element can only be understood with the double-surface theory. Thus, we have

$$dx^4 = dX^4 \ .$$

From

$$d\mathcal{R}^4 = \mathcal{R}\cos\eta \, d\,i\psi, \quad dX^4 = -X\cos\varepsilon \, d i\psi$$

one deduces with the help of

$$d\mathcal{R}^4 = \mathfrak{p}^4_4 dx^4$$

the negative quantity

$$\mathfrak{p} = \mathfrak{p}^4_4 = \frac{\mathcal{R}\cos\eta}{-X\cos\varepsilon} = \frac{\mathcal{R}\cos\eta}{\mathcal{R}\cos\eta - 3\mathcal{R}_g\cos\eta_g} \tag{IV.23.29}$$

so that

$$\mathfrak{p}^0_0 = \mathfrak{p}^1_1 = \mathfrak{p}^2_2 = \mathfrak{p}^3_3 = 1, \quad \mathfrak{p}^4_4 = \mathfrak{p} \tag{IV.23.30}$$

contains only one substantial component. For later use we note

$$\mathfrak{p}_{|a} = (1-\mathfrak{p})\frac{\mathfrak{p}}{\mathcal{R}}\{1, -\tan\eta, 0, 0, 0\} \ . \tag{IV.23.31}$$

From the connexion coefficients $\hat{A}$ which refer to the pseudo-hyper sphere (IV.23.23) one deduces by means of

$$A_{ab}{}^c = \mathfrak{p}^d_a \hat{A}_{db}{}^c \tag{IV.23.32}$$

the connexion coefficients A of the double-surface and from these with a [0+4]-decomposition the connexion coefficients of the physical surface.

Thus, most has been said about the motivation and the use of the projectors. The application of this technology to field quantities and field equations is carried out similarly to the technology which was developed for the exterior Schwarzschild solution and will come across repeatedly. In the following Sections it will be presented only in a compact form.

IV.24. Schwarzschild interior solution, field equations

The field equations which describe the interior of a stellar object are more complicated than those of a vacuum solution are. Since the model of the interior Schwarzschild solution is static and spherically symmetric, a substantial simplification takes place nevertheless. Thus, we can hark back to preceding Sections concerning the treatment of the field equations. We will only take care of the most important matters. We will discuss the stress-energy tensor in more detail and we will show that the structure of this tensor can be inferred from the 5-dimensional theory. With regard to an interpretation of the Einstein field equations, being the equations of the curvatures for the physical surface, the connection coefficients are decomposed in the usual way

$$\hat{A}_{ab}{}^{c} = \hat{M}_{ab}{}^{c} + \hat{B}_{ab}{}^{c} + \hat{C}_{ab}{}^{c} + \hat{U}_{ab}{}^{c} \,, \tag{IV.24.1}$$

whereby the different components have the meaning

$$\begin{aligned} \hat{M}_{ab}{}^{c} &= m_a \hat{M}_b m^c - m_a m_b \hat{M}^c, \quad \hat{B}_{ab}{}^{c} = b_a \hat{B}_b b^c - b_a b_b \hat{B}^c \\ \hat{C}_{ab}{}^{c} &= c_a \hat{C}_b c^c - c_a c_b \hat{C}^c, \quad \hat{U}_{ab}{}^{c} = u_a \hat{U}_b u^c - u_a u_b \hat{U}^c \end{aligned} \tag{IV.24.2}$$

and the curvature quantities take the forms

$$\begin{aligned} &\hat{M}_a = \left\{ \frac{1}{\mathcal{R}}, 0,0,0,0 \right\}, && \hat{B}_a = \left\{ \frac{1}{\mathcal{R}}, \frac{1}{\mathcal{R}} \cot\eta, 0,0,0 \right\} \\ &\hat{C}_a = \left\{ \frac{1}{\mathcal{R}}, \frac{1}{\mathcal{R}} \cot\eta, \frac{1}{\mathcal{R}\sin\eta} \cot\vartheta, 0,0 \right\}, && \hat{U}_a = \left\{ \frac{1}{\mathcal{R}}, -\frac{1}{\mathcal{R}} \tan\eta, 0,0,0 \right\} \end{aligned} \,. \tag{IV.24.3}$$

Their values

$$\left|\hat{M}_a\right| = \frac{1}{\mathcal{R}}, \quad \left|\hat{B}_a\right| = \frac{1}{\mathcal{R}\sin\eta}, \quad \left|\hat{C}_a\right| = \frac{1}{\mathcal{R}\sin\eta\sin\vartheta}, \quad \left|\hat{U}_a\right| = \frac{1}{\mathcal{R}\cos\eta} \tag{IV.24.4}$$

are the curvatures of the pseudo-hyper sphere. The first three quantities are the curvatures of the purely spatial hyper sphere and are adopted unmodified, while the fourth quantity must be replaced with (IV.23.32). With

$$A_{4a}{}^{4} = \mathfrak{p}_4^4 \hat{A}_{4a}{}^{4}$$

one obtains

$$\begin{aligned} M_{ab}{}^{c} &= m_a M_b m^c - m_a m_b M^c, \quad B_{ab}{}^{c} = b_a B_b b^c - b_a b_b B^c \\ C_{ab}{}^{c} &= c_a C_b c^c - c_a c_b C^c, \quad E_{ab}{}^{c} = -\left(u_a E_b u^c - u_a u_b E^c \right) \end{aligned} \,, \tag{IV.24.5}$$

if one replaces in addition the geometrical quantity $\hat{U}$ by the physical quantity E. The gravitational field strength has the components

$$E_a = \left\{ -\frac{\mathfrak{p}}{\mathcal{R}}, \frac{\mathfrak{p}}{\mathcal{R}} \tan\eta, 0,0,0 \right\} \,. \tag{IV.24.6}$$

Since the quantity $\mathfrak{p}$ is negative in its range of validity[75], E_1 is likewise negative and oriented inwards. Some preceding formulae among other things contain the expression

[75] This range of validity will be discussed later.

$$\mathcal{R}\cos\eta - 3\mathcal{R}_g \cos\eta_g \ .$$

We want to point out that we consider the quantity $\mathcal{R}$ to be a variable, even though the embedding condition (IV.23.7) has been introduced. The matching of the interior solution to the exterior solution already has been performed by the choice of the constants $\mathcal{R}_g$ and η_g. Nevertheless it should be possible to make statements on the exterior geometry, i.e. outside the selected cap of the sphere. It is necessary to envisage the prolongation of the quantities away from the cap of the sphere. The change of the quantities normal to the cap of the sphere are computed with

$$\partial_0 = \frac{\partial}{\partial\mathcal{R}} \ .$$

However, if one operates on the cap, i.e. on the physical surface, the embedding condition has an effect.

Finally, it has to be examined whether the model is correct, i.e. whether the field strengths of the interior solution on the boundary surface match with the field strengths of the exterior solution. With (IV.23.7) and (IV.23.8) one gains conformability for

$$B_0^g = \frac{1}{\mathcal{R}} = \frac{1}{\mathcal{R}\sin\eta_g}\sin\eta_g = -\frac{1}{r_g}\sin\varepsilon_g , \quad B_1^g = \frac{1}{\mathcal{R}}\cot\eta_g = \frac{1}{\mathcal{R}\sin\eta_g}\cos\eta_g = \frac{1}{r_g}\cos\varepsilon_g \ ,$$

by bearing in mind that the local 0-directions of the two systems have opposite orientations. Since from (IV.23.29) one obtains $\mathfrak{p}_g = -1/2$ one can deduce from (IV.24.6) and (IV.23.9) for the force of gravity

$$E_0^g = \frac{1}{2\mathcal{R}_g} = \frac{1}{\rho_g}, \quad E_1^g = -\frac{1}{2\mathcal{R}_g}\left(-\tan\varepsilon_g\right) = \frac{1}{\rho_g}\tan\varepsilon_g ,$$

which matches the values of the exterior solution, if one again considers the orientations of the co-ordinate systems. Fig. IV.19 shows the complete Schwarzschild solution.

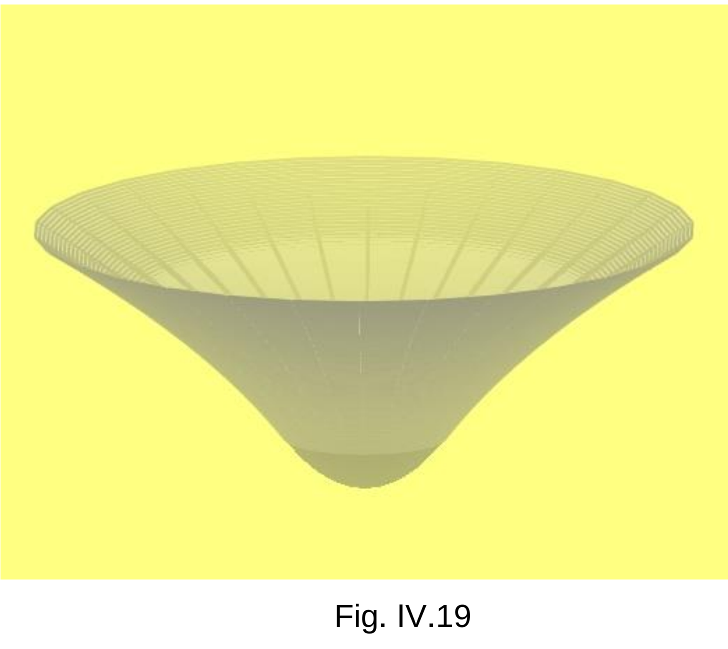

Fig. IV.19

For the computation of the field equations we utilize the relation

$$\mathfrak{p}^g_a \mathfrak{p}^h_b R_{ghc}{}^d(\hat{A}) = R_{abc}{}^d(A) + 2\hat{A}_{fc}{}^d \mathfrak{p}^f_{[a|||b]} \equiv 0 \ . \qquad \text{(IV.24.7)}$$

However, we affirm that $\mathfrak{p}$-term therein does not supply a contribution. We convince ourselves by calculating this term with the help of (IV.23.29) to (IV.23.31). In the 5-dimensional flat space the relation

$$\begin{aligned} &R_{ab} = 2\Big[A_{[a\cdot b\cdot}{}^c{}_{|||c]} - A_{[a\cdot b\cdot}{}^d A_{c]d}{}^c\Big] \equiv 0 \\ &A_{ab}{}^c = M_{ab}{}^c + B_{ab}{}^c + C_{ab}{}^c + E_{ab}{}^c \\ &\Phi_{a|||b} = \Phi_{a|b} - A_{ba}{}^c \Phi_c \end{aligned} \qquad \text{(IV.24.8)}$$

is valid. Utilizing the graded derivatives it decouples to

$$M_{\underset{1}{b|||a}} + M_b M_a = 0, \quad B_{\underset{2}{b|||a}} + B_b B_a = 0, \quad C_{\underset{3}{b|||a}} + C_b C_a = 0, \quad E_{\underset{4}{b|||a}} - E_b E_a = 0 \ . \qquad \text{(IV.24.9)}$$

The 5-dimensional Einstein field equations

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \equiv 0 \qquad \text{(IV.24.10)}$$

split into a set of equations

$$\begin{aligned} &M_{\underset{1}{b|||a}} - (g_{ab} - m_a m_b) M^c{}_{\underset{1}{|||c}} + {}^1t_{ab} = 0, \quad B_{\underset{2}{b|||a}} - (g_{ab} - b_a b_b) B^c{}_{\underset{2}{|||c}} + {}^2t_{ab} = 0 \\ &C_{\underset{3}{b|||a}} - (g_{ab} - c_a c_b)\, C^c{}_{\underset{3}{|||c}} + {}^3t_{ab} = 0, \quad E_{\underset{4}{b|||a}} - (g_{ab} - u_a u_b) E^c{}_{\underset{4}{|||c}} + {}^4t_{ab} = 0 \end{aligned} \qquad \text{(IV.24.11)}$$

$$\begin{aligned} {}^1t_{ab} &= \quad M_a M_b - (g_{ab} - m_a m_b) M^c M_c, \quad {}^1t_a{}^b{}_{\underset{1}{|||b}} = 0 \\ {}^2t_{ab} &= \quad B_a B_b - (g_{ab} - b_a\, b_b\,) B^c B_c, \quad {}^2t_a{}^b{}_{\underset{2}{|||b}} = 0 \\ {}^3t_{ab} &= \quad C_a C_b - (g_{ab} - c_a\, c_b\,) C^c C_c, \quad {}^3t_a{}^b{}_{\underset{3}{|||b}} = 0 \\ {}^4t_{ab} &= -\Big[E_a E_b - (g_{ab} - u_a\, u_b) E^c E_c\Big], \quad {}^4t_a{}^b{}_{\underset{4}{|||b}} = 0 \end{aligned} \ . \qquad \text{(IV.24.12)}$$

For the present the [0+4]-decomposition of the field equations (IV.24.10) is formally performed in more detail than the exterior solution. The surface theory will offer new insights.

$$n^a = \{1,0,0,0,0\} \qquad \text{(IV.24.13)}$$

is the normal vector of the surface, it points into the local 0-direction. With

$$A_{ab} = n_{a|||b} = -A_{ba}{}^0, \quad A_{[ab]} = 0 \qquad \text{(IV.24.14)}$$

the generalized second fundamental forms of the surface theory are defined. They have only 4-dimensional components and they are symmetric. From

$$2n_{a|||[bc]} = R_{bca}{}^d n_d \equiv 0 \qquad \text{(IV.24.15)}$$

one derives the Codazzi equation

$$A_{a[b|||c]} = 0 \ . \qquad \text{(IV.24.16)}$$

We can split the connexion coefficients into

$$A_{ab}{}^c = {}'A_{ab}{}^c + A_a{}^c n_b - A_{ab} n^c, \quad A_a{}^c = A_{ab}{}^c n^b \ , \qquad \text{(IV.24.17)}$$

whereby the 'A are the 4-dimensional connexion coefficient. From (IV.24.4), (IV.24.5), and (IV.24.6) one infers the values

$$A_{11} = A_{22} = A_{33} = \frac{1}{\mathcal{R}}, \quad A_{44} = \frac{\mathcal{P}}{\mathcal{R}}. \qquad \text{(IV.24.18)}$$

The 5-dimensional Ricci tensor is represented by

$$R_{ab} = \left[{}'R_{ab} + 2A_{[a}{}^{c}A_{c]b} \right] + 2\left[A_{[a}{}^{c}{}_{|||c]}\, n_b - A_{[a\cdot b\cdot|||c]}\, n^c \right] + 2n_{[c}\left[{}'A_{a]b}{}^{c}{}_{|||d}\, n^d + A_{a]}{}^{d}\, {}'A_{db}{}^{c} \right] \equiv 0\,. \qquad \text{(IV.24.19)}$$

Therein all brackets vanish separately. The last bracketed term describes the change of geometry into the 0-direction. It belongs to the extrinsic geometry and is less interesting for the physical part of the theory. For the examination of the relation contained in (IV.24.19)

$$A_{[a\cdot b\cdot|||c]}\, n^c = 0 \quad \Rightarrow \quad A_{ab|0} + A_a{}^c A_{bc} = 0 \qquad \text{(IV.24.20)}$$

(IV.23.31) is useful. (IV.24.20) describes the development of the second fundamental forms into the 0-direction[76]. The 4-dimensional Ricci tensor is defined by

$$\begin{aligned} \Phi_{a||\underline{b}} &= \Phi_{a|\underline{b}} - {}'A_{ba}{}^{c}\Phi_c, \quad \Phi_{a|\underline{b}} = \Phi_{a|b} - \Phi_{a|c}\, n^c n_b \\ {}'R_{ab} &= 2\left[{}'A_{[a\cdot b\cdot}{}^{c}{}_{||\underline{c}]} - {}'A_{[a\cdot b\cdot}{}^{d}\, {}'A_{c]d}{}^{c} \right] \end{aligned}\,. \qquad \text{(IV.24.21)}$$

The underlined indices refer to the index sequence {1,2,3,4}. If we calculate the Einstein tensor using (IV.24.19)

$${}'R_{mn} - \frac{1}{2} g_{mn}\, {}'R = -\kappa T_{mn}, \quad \kappa T_{mn} = 2A_{[m}{}^{s}A_{s]n} - g_{mn}\, A_{[r}{}^{s}A_{s]}{}^{r}\,, \qquad \text{(IV.24.22)}$$

we recognize that the stress-energy tensor is constructed by the second fundamental forms of the surface theory and that the matter is described by these fields. The Einstein field equations are the contracted Gauss equations. The field equations of the matter field A_{mn} are the Codazzi equations

$$A_{[m}{}^{n}{}_{||n]} = 0\,, \qquad \text{(IV.24.23)}$$

which one reads from the second bracketed term of (IV.24.19). One can regard the Schwarzschild interior solution as a simple ansatz for a geometrization of the matter, an aspect, which has not been considered yet. The cosmological constant of the Einstein and de Sitter cosmos was explained in a similar way. The cosmological terms in the Einstein field equations of these models describe the cosmic background energy and supplement or represent the stress-energy tensor.

It remains to show that the stress-energy tensor can be rewritten in (IV.24.22) in such a way that physically interpretable quantities arise. Evaluating the quantities (IV.24.18) one obtains

$$\begin{aligned} T_{mn} &= -p\, {}^3g_{mn} + \mu_0\, u_m u_n, \quad {}^3g_{mn} = g_{mn} - u_m u_n \\ \kappa p &= -\frac{1}{\mathcal{R}^2}\left(1 + 2\mathcal{P}\right), \quad \kappa\mu_0 = \frac{3}{\mathcal{R}^2}, \quad \mathcal{R} = \mathcal{R}_g = \text{const.} \end{aligned}\,. \qquad \text{(IV.24.24)}$$

---

[76] The equations are identical with $M_{0|0} + M_0 M_0 = 0,\ B_{0|0} + B_0 B_0 = 0,\ C_{0|0} + C_0 C_0 = 0,\ E_{0|0} - E_0 E_0 = 0$.

p and μ_0 are the pressure and the energy density (mass density) of the stellar object. The mass density is constant in the whole region which the object fills and has the same value on the surface of the object. The pressure in the radial direction is pointing inwards and increases inwards. It vanishes on the boundary surface between interior and exterior solution. This is the condition for the stability of the object. One can recognize this, if one inserts $\mathfrak{p}_g = -1/2$ into the above formula for the pressure. The stellar object which is described by the interior solution can be interpreted as a homogeneous fluid sphere with the hydrostatic pressure p and the mass density μ_0. Thus, the interior solution can only approximately stand for a model describing a star or a planet.

That the Einstein tensor

$$G_{mn} = {}'R_{mn} - \frac{1}{2} g_{mn} {}'R \qquad \text{(IV.24.25)}$$

is free from divergence which can be proven with the Bianchi identity. However, from it and from the field equations (IV.24.22) it follows that also the right side of the field equations must be free from divergence. Therefore the conservation law

$$T_m{}^n{}_{||n} = 0 \qquad \text{(IV.24.26)}$$

is valid.

It is remarkable that in the context of the 5-dimensional theory one does not have to demand in particular the stress-energy tensor to be divergence-free at the outset. It can be deduced from the geometrical structure. One obtains

$$\kappa T_m{}^n{}_{||n} = A_{<n}{}^s A^n{}_{m||s>} = 0 \ , \qquad \text{(IV.24.27)}$$

a relation which is fulfilled with the help of the Codazzi equation (IV.24.23). Using the values for the stress-energy tensor already computed

$$p_{||m} = (p + \mu_0) E_m, \quad \dot{p} = 0, \quad \dot{\mu}_0 = 0 \qquad \text{(IV.24.28)}$$

arises.

The Einstein field equations (IV.24.25) can also be written in the form

$${}'R_{mn} = -\kappa \left(T_{mn} - \frac{1}{2} g_{mn} T \right), \quad T = \mu_0 - 3p \ . \qquad \text{(IV.24.29)}$$

The [44]-component of this equation is

$$\mathrm{div}\vec{E} = \vec{E}^2 - \frac{\kappa}{2}(\mu_0 + 3p) \ . \qquad \text{(IV.24.30)}$$

div has the meaning of the 3-dimensional covariant differential operator. The gravitational field strength is coupled not only to the mass and pressure-energy density, but also to the field-energy density. The latter is a consequence of the nonlinearity of the Einstein field equations. Gravitation is self-gravitating.

The above relation can be written with (IV.24.24), third equation, as

$$\mathrm{div}\vec{E} = \vec{E}^2 - \kappa \mathfrak{p} \mu_0 \ .$$

It should be noted that $\mathfrak{p}$ is a negative quantity.

As mentioned above $\mathcal{R}_g$ and η_g are not quite arbitrary. The expressions (IV.24.6) and (IV.24.24) show with (IV.23.29) that in the center of the star ($\eta = 0$) both the force of gravity and the pressure become infinite for

$$\cos\eta_g = \frac{1}{3} \ . \qquad \text{(IV.24.31)}$$

For still larger aperture angles of the cap of the sphere these quantities change the sign[77]. Therefore the parameters of the interior solution have a reduced range of values. One also realizes that for the critical case

$$\mathcal{R}\cos\eta - 3\mathcal{R}_g\cos\eta_g = 0$$

the projections of the quantities $\mathcal{R}$ and $3\mathcal{R}_g$ onto the $x^{0'}$ axis coincide. The cap of the sphere contacts the parallel planes at the fixed base point of the Neil parabola. Beneath this plane there is no coherency with the physics of the model. With

$$\sin\eta_g = \sqrt{\frac{2M}{r_g}}$$

and (IV.24.31) one obtains the lower limit

$$r_g^m = 2.25M, \quad r_g > r_g^m \qquad \text{(IV.24.32)}$$

for the r-position, at which the cap of the sphere can join Flamm's paraboloid. In any case the value has to be higher than the event horizon r = 2M, in order that the interior solution avoids inconsistencies which could result at the event horizon.

On the other hand, one can interpret the relation (IV.24.32) in the following way: Near the smallest junction r_g^m between interior solution, and exterior solution the mass of the stellar object is limited by

$$M = \frac{4}{9} r_g^m \ . \qquad \text{(IV.24.33)}$$

This aspect has been discussed by Chandrasekhar [C] in the light of the dynamical instability of a spherically symmetric mass distribution. The author has stated that the problem of the event horizon does not occur if the Schwarzschild field is induced by a mass. The competent gravitational physicist Chandrasekhar was obviously an early critic of black holes.

In the permissible range at r = 0 the pressure is finite and the gravitational force vanishes. This distinguishes this model from other field theories in which the field strengths are of the type $1/r^n$ and are infinite in the center of the source. One has also attempted to explain elementary particles by means of a gravity model.

To recognize this, we take from (IV.24.6) and (IV.23.29)

$$E_1 = \frac{\mathfrak{p}}{\mathcal{R}}\tan\eta = \frac{\sin\eta}{\cos\eta - 3\cos\eta_g}\frac{1}{\mathcal{R}} \ .$$

[77] Sometimes in the literature one finds figures in which a hemisphere is put onto the waist of Flamm's paraboloid. That does not seem thought out well.

At the center of the star the polar angle is $\eta = 0$ and thus $E_1 = 0$. In a later Section concerning the free fall inside the stellar object, the freedom of singularity of gravity will prove to be essential. The singularities of the quantities B and C are the usual singularities of the polar coordinate system. Since r and $r \sin\vartheta$ are the radii of the circles of latitude on the cap of the sphere, $1/r$ and $1/r\sin\vartheta$ are the curvatures of these circles. In the center of the star these circles shrink to a point. The curvature is infinite; one cannot draw a circle at this location[78]. By shifting the coordinate origin this singularity can be moved on a sphere anywhere. It is therefore a co-ordinate singularity and not a physical one. If we consider this, the inner Schwarzschild solution is a solution of Einstein's field equations free from singularities.

For the exterior solution we have found that the form parameter M of the Schwarzschild parabola must match up to factors the mass of the field generating object. If the theory is consistent, we have to find the parameter M in the interior solution and also we have to gain it from the geometry. We rely on the relation (IV.23.9). The radius of the cap of the sphere is related to the radius of curvature of the Schwarzschild parabola on the boundary surface via

$$2\mathbb{R} = \rho_g = \sqrt{\frac{2r_g^3}{M}} \,.$$

From this one reads the desired relation between the mass parameter

$$M = \frac{r_g^3}{2\mathbb{R}^2} \qquad \text{(IV.24.34)}$$

and the geometrical quantity $\mathbb{R}$.

With the help of $\kappa\mu_0 = 3/\mathbb{R}^2$ one can bring M into relation with μ_0. In addition, if one replaces Einstein's gravitational constant by the Newtonian by

$$\kappa = \frac{8\pi k}{c^4} \quad ,$$

one has

$$M = \frac{k}{c^4}\frac{4\pi}{3} r_g^3 \mu_0 \ .$$

Dividing by the constants one finally has for the mass of the stellar object

$$m = \frac{4\pi}{3} r_g^3 \mu_0 \ , \qquad \text{(IV.24.35)}$$

wherein the factor before μ_0 is the volume of the sphere with the radius r_g at the boundary surface between the interior solution and exterior solution.

With a sufficiently large aperture angle the pressure becomes so large inside the stellar object that the atomic structure or even the elementary structure of the object could break down. This is the result of the steep slope of the pressure function. The Schwarzschild interior solution suggests the possibility of a neutron star or rather an object which is similar to the hypothetical black holes. For a star which goes through a development towards such objects, a collapse should be accepted, to whose computation,

[78] The same applies to the poles of the Earth. At the North Pole one can only go south.

however, a time-dependent model would be necessary. The Schwarzschild interior solution could be understood as snapshot of such a procedure. It would be of some interest to determine, how the interior solution relates to the model of 'eternally collapsing stars' (ECOs) proposed by Mitra [M]. Fig. IV.20 shows the pressure profile for some aperture angles of the cap of the sphere within the critical range. The p-scale is chosen to be logarithmical.

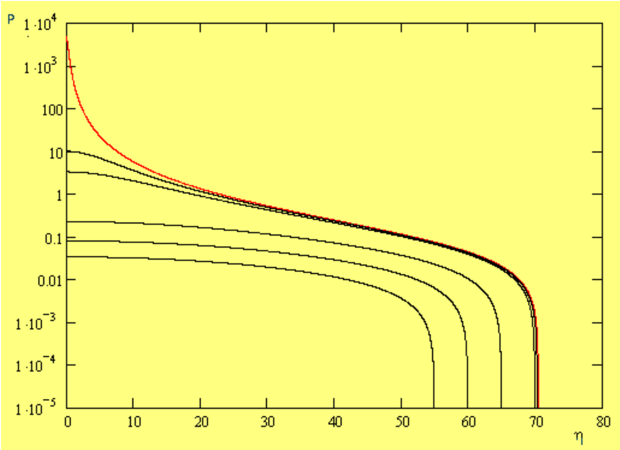

Fig. IV.20

While the exterior solution since its discovery by Schwarzschild has been the subject of extensive discussions, less work has been done for the interior solution. That is understandable, because there is hardly a stellar object which can be described only in a rough approximation by a homogeneous fluid sphere. We mention some of these publications:

Buchdahl [B] states in his paper that due to the constant density of the fluid sphere the speed of sound must be infinite and thereby the causality is violated. He represents the interior metric in an isotropic form. In a second paper he examines other ansätze for fluid spheres.

Adler [A] extends the interior solution by assuming the mass of the fluid spheres to be position-dependent.

Lake [L] finds with a similar consideration an infinitely large class of spherically symmetric solutions for an ideal fluid.

Nariai [N] represents the metric of the interior solution in elliptical co-ordinates and shows that the interior solution is conformal to the Einstein cosmos.

Boonserm [B], Visser, and Weinfurtner find methods which transform static solutions of the Einstein field equations into those which describe an ideal fluid sphere.

Rahman [R] and Visser examine static fluid spheres in a most general form.

Culetu [C] examines a compressible fluid with time-dependent viscosity coefficients.

Delgaty [D] and Lake mention 127 models which could replace the Schwarzschild interior solution. Only 16 of them fulfill the following conditions:

- Pressure and energy density are regular and positive definite in the origin.
- The pressure is isotropic and becomes zero at a certain finite radius.
- Pressure and energy density decrease monotonously with increasing radius.
- The speed of sound is lower than the speed of light[79]. These ideas are appropriate for creating a model which approximately satisfies astrophysical requirements.

[79] In this paper are numerous quotations.

All these studies are based on the idea to establish a model which is closer to the astrophysical requirements.

Tangherlini [T] has treated the Schwarzschild problem in higher dimensions.
Felber [F] and Ohanian [O] claim that they have found a reverse acceleration in the Schwarzschild field.
Wyman [W] has dealt with the interior Schwarzschild solution in isotropic co-ordinates.
Raychaudhuri [R] and Maiti have considered the Schwarzschild interior solution in the light of conformal flatness of the metric.
Vaidya [V] and Tikekar have made an approach to a metric for a superdense star. By a suitable choice of the parameters this ansatz is reduced to the interior Schwarzschild solution, the Einstein, and the de Sitter cosmos.
Jex [J] has shown that the inner Schwarzschild solution can be embedded into a 5-dimensional flat space.

Now it can be questioned whether the interior solution proposed by Florides [F] can be matched to the exterior Schwarzschild solution and wether it corresponds to physical requirements. Florides puts the line element in the form

$$ds^2 = \alpha^2 dr^2 + r^2 d\Omega^2 - a_T^2 dt^2, \quad \alpha = 1\Big/\sqrt{1 - r^2/\mathcal{R}^2}$$
$$a_T^2 = \left(1 - 2M/r_g\right)\exp\left(\int_{r_g}^{r} \frac{2\mu}{r^2\left(1 - 2\mu/r\right)} dr\right) dt^2 \quad . \tag{IV.24.36}$$

In it

$$\mu = \mu(r) = 4\pi \int_0^r \mu_0\left(r\right) r^2 dr \tag{IV.24.37}$$

is the mass of the field generating object, $\mu_0\left(r\right)$ the mass density, wherein at the surface of the star ($r = r_g$)

$$\mu\left(r_g\right) = M$$

is valid. M is the Schwarzschild mass. An ansatz for a mass density which depends on the radial distance is interesting, but in the present case $\mu_0\left(r\right)$ remains undetermined.

A specific form of (IV.24.36) is obtained with

$$a_T^2 = \frac{\left(1 - r_g^2/\mathcal{R}^2\right)^{\frac{3}{2}}}{\left(1 - r^2/\mathcal{R}^2\right)^{\frac{1}{2}}} = \frac{a_g^2}{a}, \quad a_T = \frac{a_g^{3/2}}{a^{1/2}} \ . \tag{IV.24.38}$$

The term coincides with the metric coefficient of the exterior field at the surface of the star:

$$a_T^g = a_g = \sqrt{1 - r_g^2/\mathcal{R}^2} = \sqrt{1 - 2M/r_g} \ . \tag{IV.24.39}$$

From this and with the help of

$$A_{41}{}^4 = -E_1 = \frac{1}{a_T} a_{T|1} = \alpha v \frac{1}{\rho_g}, \quad v = -\frac{r}{\mathcal{R}}, \quad \rho_g = 2\mathcal{R} \tag{IV.24.40}$$

one can calculate the force of gravity in the interior of the star. This force coincides with the gravitational force of the exterior field at the surface of the star.

Since the space-like part of the metric (IV.24.36) is identical with the one of the interior Schwarzschild solution, the linking condition is satisfied. Thus, the metrics and the first derivatives of the metrics of the two solutions at the boundary surface of the geometries match. The linking conditions have been extensively studied by Kofinti [K].

For the above considerations was used

$$\mathcal{R} = \sqrt{\frac{r_g^3}{2M}} \tag{IV.24.41}$$

and thus $v_g = -\sqrt{2M/r_g}$. For the quantity α one has on the boundary surface

$$\alpha_g = 1\Big/\sqrt{1-2M/r_g}\ . \tag{IV.24.42}$$

This shows that the extension of a star described by (IV.24.36) can approach arbitrarily close to $r = 2M$. However, at the event horizon of the exterior solution one has $\alpha_g \to \infty$ and the force of gravity (IV.24.40) is also infinite. For $r < 2M$ the model with (IV.24.36) does not exist, the metric cannot describe a black hole.

The gravity has the form $E_1 = \alpha v/2\mathcal{R}$ and differs significantly from the analogous expression of the interior Schwarzschild solution.

The relation

$$E_{1|1} - E_1 E_1 = -\frac{1}{2}\alpha^2 \frac{1}{\mathcal{R}^2} - \frac{1}{4}\alpha^2 v^2 \frac{1}{\mathcal{R}^2}$$

enters Einstein's field equations. As a consequence one has $G_{11} = 0$, ie the radial pressure, which is responsible for the consistency of matter vanishes. For the tangential pressure in the two lateral directions one obtains $\kappa p = 3E^2$, ie positive values that do not vanish on the boundary surface of the geometries. Only for the matter density one obtains the familiar and convincing expression $\kappa\mu_0 = 3/\mathcal{R}^2$.

It is evident that the proposed model does not satisfy the physical requirements.

Some more works about this topic are to be listed: Abramowicz [A], Carter and Lasota; Abramowicz [A] and Lasota; Abrams [A], Alvarez [A], Antoci [A], Antoci [A] and Liebscher; Arnowitt [A], Deser and Misner; Bartnik [B], Beig [B] and Muchadha; Bel [B], Belinfante [B]; Bicak [B], Biederbeck [B], Blinnikoc [B], Okun and Vysotsky; Bondi [B] , Bonnor [B] and Wickramasuriya; Boulware [B], Brecher [B] and Wasserman; Chicone [C] and Mashhoon; Celetu [C], Chen [C], Cohen [C] and Gautreau; Cooperstock [C] and Junevicus; Cooperstock [C] and Lim; Cooperstock [C] and Sarracino; Cooperstock [C], Sarracino and Bayin; Culetu [C], Curir [C] and Demianski; Czerniawski [C], Davidson [D] and Efinger; Debney [D] and Farnsworth; Deser [D] and Franklin; Dinculescu [D], Doran [D], Lobo and Crawford; Farhoosh [F] and Zimmermann; Finkelstein [F], Francis [F] and Kosowsky; Fodor [F], Fonseca-Neto [F] and Romero; Fronsdal [F] and Romero; Gautreau [G], Gertsenshtein [G] and Melkumova; Gershtein [G], Logunov and Mestvirishvili; Giblin [G], Marolf and Garvey; Goldoni [G], Gonzalez-Diaz [G], Grøn [G], Gruber [G], Price and Matthews; Gürses [G], Harrison [H], Hilton [H], Iriondo [I] , Israel [I], Israelit [I] and Rosen; Itin [I], Karageorgis [K] and Stalker; Kaufmann III [K], Kazakov [K] and Solodukhin; Kocinski [K] and Wierzbicki; Kodama [K], Kroon [K], Lanczos [L], Lindhard [L], Lindquist [L], Schwartz and Misner; Lindquist [L] and Wheeler; Lundgren [L] and Schmekel; Lynden-Bell [L] and Katz; Magli [M], Malec [M] and Murchadha; Markley [M], Mars [M], Marolf [M], Martin [M] and Visser; McCrea [M], Mehra [M], Vaidya and Kushwaha; Misner [M], Mitra [M], Mitra [M] and Gendennig; Mociutchi [M], Murchadha [M] and Roszkowski; Namsrai [N], Narayan [N], Narlikar [N] and Padmanabhan; Nyack [N], MacCallum and Vishveshwara; Obukhov [O], Palatni [P], Pechlaner [P], Quevedo [Q], Ray [R], Ramachandra [R] and Vishveshwara; Recami [R], Raychaudhuri [R], Rosen [R], Rosen [R], Rosquist [R], Rylov [R], Sakina [S], Sakina [S] and Ohiba; Sato [S], Sasaki and Kodama; Sai [S], Schmidt [S], Semiz [S], Siemieniec-Ozieblo [S] and Klimek; Singh [S] and Yadav; Stewart [S], Papadopoulos and Witten; Stuchlik [S], Hledik and Soltes; Szabo [S], Tangherlini [T]; Tian [T], Tilbrook [T], Treder [T], Treder [T] and Yourgrau; Treibergs [T], Vacaru [V], Vaidya [V], Van den Bergh [V] and Wils; Visser [V], Whitman [W], Wiechert [W], Yilmaz [Y], Zimmermann [Z] and Farhoosh.

Concerning black holes the authors Abramowicz [A], Kluzniak and Lasota; Bronikov [B], Dymnikova [D] and Mazur [M] are reserved.

There are more studies that come with the Schwarzschild problem: Abramowicz [A], Bonnor [B], Buchdahl [B], Eune [E], Gim and Kim; Faridi [F], Guven [G] and Murchadha; Israel [I], López-Bonilla [L], Montez-Peralta and Islas-Gamboa; Mahajan [Q], Qadir and Valanju; Martel [M] and Matese [M] and Whitman; Poisson; Prasanna [P] and Iyer; Qadir [Q] and Zafahrullah; Rothman [R], Ellis and Murugan; Sonego [S] and Lanza; Stettner [S] and Tooper [T]; Vishveshwara [V], Zel'dovich [Z] and Novikov.

IV.25. Schwarzschild, complete solution

In the following we will show that the interior and exterior Schwarzschild solutions can be represented by a common formal system, if one applies the 5-dimensional embedding methods. This representation excludes the inner region $0 \le r < 2M$ of the exterior Schwarzschild solution from the outset. An approach to the event horizon will be prevented as well. The part of the complete solution which refers to the interior of the mass covers the critical region of the exterior solution. Speculations on black holes are not possible in this version of the model.

The modified representation with respect to former Sections needs getting used to. The orientations of the co-ordinate axes and angles are adjusted to the interior solution in order to ensure a common co-ordinate system for both parts of the solution. This leads to a somewhat unusual choice of signs for the exterior part of the model and a moving origin of the co-ordinate system.

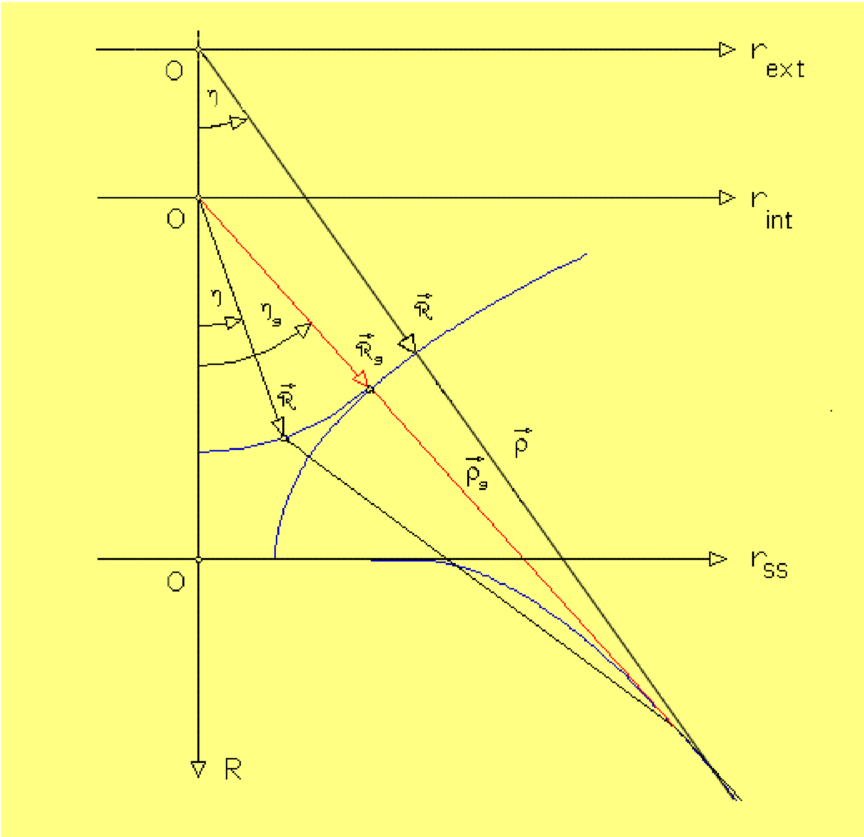

Fig. IV.21

Fig. IV.21 shows the essential elements of the approach. If we join a point on the Schwarzschild parabola on its way from the exterior to the interior, then moves downwards the origin of the local co-ordinate system on the R-axis (0' co-ordinate line of the embedding space) until this point has reached the junction point of the interior part of the solution. From then on, it moves on an arc until it reaches the R-axis. The curvature vector of the parabola accompanies the point. Its extension cuts the R-axis and thus defines the moving origin of the new co-ordinate system. Having reached the junction point the base point of the curvature vector is fixed on Neil's parabola. An auxiliary vector follows the curvature vector $\vec{\mathbb{R}}$ of the circle to the R-axis. We have explained the mechanism of the interior geometry in the previous Section in detail. Concerning the new co-ordinate system we now have to take for the Schwarzschild parabola the negative and for the Neil parabola the positive root

$$R = -\sqrt{8M(r-2M)}, \quad \bar{R} = +\sqrt{\frac{2}{M}\left(\frac{\bar{r}}{3}-2M\right)^3} \ . \tag{IV.25.1}$$

The polar angle η is positive in contrast to the previous angle ε and is also the angle of ascent of the Schwarzschild parabola. From (IV.25.1) one gets

$$dR = -\tan\eta dr, \quad d\bar{R} = \cot\eta d\bar{r} \ , \tag{IV.25.2}$$

where $\{x^{0'}, x^{1'}\} = \{R, r\}$ and $\{\bar{x}^{0'}, \bar{x}^{1'}\} = \{\bar{R}, \bar{r}\}$ are the Cartesian co-ordinates of the points on the Schwarzschild parabola and Neil's parabola respectively. With this sign convention the velocity of a freely falling observer is

$$v = -\sin\eta = -\sqrt{2M/r} \ . \tag{IV.25.3}$$

The redshift factor

$$a = \cos\eta = \sqrt{1-2M/r} \tag{IV.25.4}$$

is not subjected to a change of sign. The addition of a second surface, based on the Neil's parabola has proved to be necessary to ensure the embedding of the Schwarzschild model in five dimensions without violating the theorems of Kasner and Eisenhart. The curvature vector of the Schwarzschild parabola and its motion are significant. To have insight into the orientation of the vectors of the constructive geometry and its projections onto the axis, we visualize them in Fig. IV.22. As a consequence the motion of the curvature vector differentials arise, both on the Schwarzschild parabola and on the Neil parabola. These are shown in Fig. IV.23.

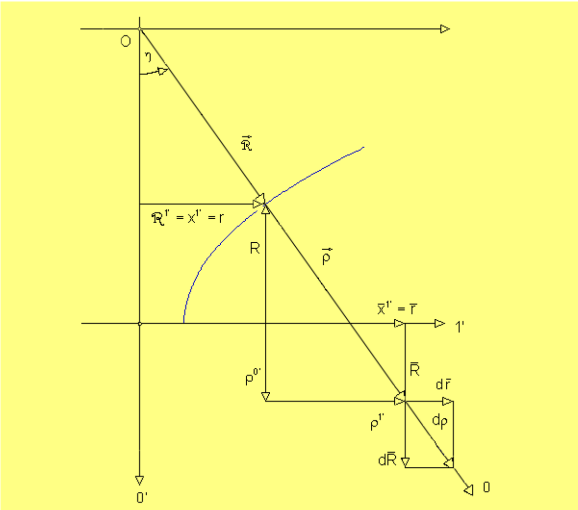

Fig. IV.22

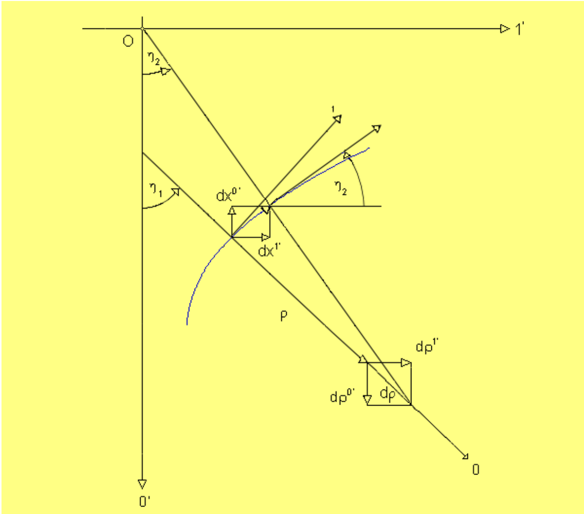

Fig. IV.23

If

$$\left\{dx^{0'},dx^{1'}\right\}=\left\{dR,dr\right\},\quad \left\{d\bar{x}^{0'},d\bar{x}^{1'}\right\}=\left\{d\bar{R},d\bar{r}\right\} \tag{IV.25.5}$$

are the changes of the curvature vector in the Cartesian co-ordinates of the embedding space at the Schwarzschild parabola and at the Neil parabola respectively, then we obtain from them the rotation matrix

$$D^{a}_{a'}=\begin{pmatrix}\cos\eta & \sin\eta\\ -\sin\eta & \cos\eta\end{pmatrix},\quad a=0,1 \tag{IV.25.6}$$

with the help of (IV.25.2) the components on the two curves in terms of the local system

$$dx^{a}=\left\{0,\frac{1}{\cos\eta}dr\right\},\quad d\bar{x}^{a}=\left\{\frac{1}{\sin\eta}d\bar{r},0\right\}\ . \tag{IV.25.7}$$

The change of the curvature vector throughout its motion outwards consists of two parts. One contribution is due to the movement of the base point along the Schwarzschild parabola, the other by the movement of the tip on the Neil parabola. Thus, one has

$$d\rho^{a}=\left\{d\bar{x}^{a}-dx^{a}\right\}=\left\{\frac{1}{\sin\eta}d\bar{r},-\frac{1}{\cos\eta}dr\right\}\ . \tag{IV.25.8}$$

Since the curvature vector of the Schwarzschild parabola has the value $\rho=\sqrt{2r^3/M}$ one finds by differentiating this expression and using (IV.25.3), in turn, $d\rho=d\rho^{0}=d\bar{r}/\sin\eta$, whereby the well-known relation $\bar{r}=3r$ has been used. Besides one gets with (IV.25.3) $d\sin\eta=-dr/\rho=\cos\eta d\eta$ so that

$$dx^{1}=\frac{1}{\cos\eta}dr=-\rho d\eta\ . \tag{IV.25.9}$$

dx^{1} is the positive tangent vector of the Schwarzschild parabola. The right side of (IV.25.9)is also positive as $d\eta=\eta_2-\eta_1$ is negative, because the angle decreases throughout a movement outwards. If one uses the two relations just-derived, one has

$$d\rho^{a}=\left\{d\rho,\ \rho d\eta\right\}\ . \tag{IV.25.10}$$

The first component thereof is the change on the Neil parabola, the second on the Schwarzschild parabola. As a byproduct of the calculation we also note

$$\eta_{|1} = -\frac{1}{\rho} \ . \tag{IV.25.11}$$

This relation is frequently applied. Transforming (IV.25.8) with the inverse matrix of (IV.25.6) to the Cartesian co-ordinates of the embedding space, one gains with the use of (IV.25.2)

$$d\rho^{a'} = \left\{ d\bar{R} - dR, d\bar{r} - dr \right\} \ , \tag{IV.25.12}$$

a relation which can be seen in Fig. IV.22. Taking into account that R is negative in the new co-ordinate system, one can write the curvature vector in Cartesian co-ordinates of the embedding space as

$$\rho^2 = \left(\bar{R} - R\right)^2 + \left(\bar{r} - r\right)^2 \ . \tag{IV.25.13}$$

From this straightforwardly results (III.13.12). Differentiating this, the relation

$$d\rho^0 = d\rho = \frac{\bar{R} - R}{\rho} d\rho^{0'} + \frac{\bar{r} - r}{\rho} d\rho^{1'} \tag{IV.25.14}$$

emerges and by comparison with the rotation matrix (IV.25.6)

$$\cos\eta = \frac{\bar{R} - R}{\rho}, \quad \sin\eta = \frac{\bar{r} - r}{\rho} \ . \tag{IV.25.15}$$

Operating on the Schwarzschild parabola we obtain with the rotation matrix

$$\partial_0 = \cos\eta \frac{\partial}{dR} + \sin\eta \frac{\partial}{dr}, \quad \partial_1 = -\sin\eta \frac{\partial}{dR} + \cos\eta \frac{\partial}{dr} \ . \tag{IV.25.16}$$

Horizontal quantities are those which lie in the horizontal plane of the surfaces, i.e. are independent of R. For such quantities from (IV.25.16) only remains

$$\partial_0 = \sin\eta \frac{\partial}{dr}, \quad \partial_1 = \cos\eta \frac{\partial}{dr} \ . \tag{IV.25.17}$$

In particular, one has

$$r_{|a} = \left\{ \sin\eta, \cos\eta \right\} \ . \tag{IV.25.18}$$

Until now, only changes of ρ have been envisaged, where the curvature vector is constrained to the Schwarzschild parabola. To apply embeddings, one must also take a look at the environment of the embedded space and examine the exterior geometry. It is also required to investigate the *prolongation of* the quantities of the embedded space. Although an infinitesimal prolongation would be sufficient, the Schwarzschild model provides a co-ordinate system (Fig. IV.16) which enables a global prolongation. A progression upwards to an adjacent curve which is parallel to the Schwarzschild parabola extends the curvature vector by $d\rho$. However, as a consequence of the orientation of the co-ordinate system one has

$$dx^0 = -d\rho, \quad \partial_0 = -\frac{\partial}{\partial\rho} \ . \tag{IV.25.19}$$

In the last paragraphs we have again discussed the techniques necessary for the Schwarzschild model in relation to the new co-ordinate system, also taking into account the unusual sign. We have not discussed in detail the new co-ordinate system with the variable origin[80] which enables the common representation of the interior and exterior solutions.

The vector $\vec{\mathcal{R}}$ points from the 0'-axis of the embedding space perpendicular to the Schwarzschild parabola and has the curvature vector as prolongation at this position. The values of the two vectors have the ratio 1:2. The vector $\vec{\mathcal{R}}$ specifies the local 0'-axis. For this vector one has

$$\mathcal{R}^{a'} = \left\{\mathcal{R}\cos\eta, \mathcal{R}\sin\eta\right\}, \quad \mathcal{R}^{a} = \left\{\mathcal{R}, 0\right\}, \quad d\mathcal{R}^{a} = \left\{d\mathcal{R}, \mathcal{R}d\eta\right\} \; . \tag{IV.25.20}$$

If another vector $\bar{\mathcal{R}}^a$ points from the local origin to the Neil parabola we have

$$\bar{\mathcal{R}}^a = \mathcal{R}^a + \rho^a \; , \tag{IV.25.21}$$

wherein the values $\mathcal{R}$ and $\bar{\mathcal{R}}$ have the ratio 1:3. Thus, we find the relation

$$d\rho^a = \left\{d\bar{\mathcal{R}} - d\mathcal{R}\,, \left(\bar{\mathcal{R}} - \mathcal{R}\right)d\eta\right\}. \tag{IV.25.22}$$

The ansatz (IV.25.20) can be used for the interior solution as well. If the vector in consideration is leaving the Schwarzschild parabola then $\mathcal{R}$ moves along the arc of a circle. The curvature vector of the Schwarzschild parabola residues at the junction point and supplies the factor 3 for the time-like arc element of the interior solution which has just been discussed. It is possible to start with the pseudo-hyper sphere

$$\begin{aligned}
\mathcal{R}^{3'} &= \mathcal{R}\sin\eta\sin\vartheta\,\sin\varphi \\
\mathcal{R}^{2'} &= \mathcal{R}\sin\eta\sin\vartheta\cos\varphi \\
\mathcal{R}^{1'} &= \mathcal{R}\sin\eta\cos\vartheta \\
\mathcal{R}^{0'} &= \mathcal{R}\cos\eta\cos i\psi \\
\mathcal{R}^{4'} &= \mathcal{R}\cos\eta\,\sin i\psi
\end{aligned} \tag{IV.25.23}$$

for both parts of the solution. The relations represent a part of a double-surface. Co-ordinate differentials, field strengths, and field equations of this simple geometry have been discussed in previous Sections. We only have to note that the field strengths and field equations for both parts of the solution can be generated with appropriate projectors, referring to the double-surface theory already described, whereby we follow the same path to the field equations for the two parts. First, we note the projectors

$$\begin{aligned}
&A: \; \mathcal{p}^0_0 = \mathcal{p}^1_1 = \mathcal{p}^4_4 = -\frac{\mathcal{R}}{\rho}, \quad \mathcal{p}^2_2 = \mathcal{p}^3_3 = \frac{\mathcal{R}\sin\eta}{r} \\
&I\;: \; \mathcal{p}^0_0 = \mathcal{p}^1_1 = \mathcal{p}^2_2 = \mathcal{p}^3_3 = 1, \quad \mathcal{p}^4_4 = -\frac{\mathcal{R}\cos\eta}{3\mathcal{R}_g\cos\eta_g - \mathcal{R}\cos\eta}
\end{aligned} \; . \tag{IV.25.24}$$

The gravitational field strength E_a of the two parts is obtained from the Ricci-rotation coefficients

$$A_{4a}{}^{4} = \mathcal{p}^4_4\,\mathcal{R}_{4a}{}^{4} = -E_a \; . \tag{IV.25.25}$$

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[80] The variable origin of the new co-ordinate system has on the 0'-axis the distance $d(r) = \left(r + 4M\right)\sqrt{\dfrac{r}{2M} - 1}$ to the standard origin.

If ρ_g is again the radius of curvature at the junction surface of the two parts of the solution, one has

$$\mathrm{A}: \quad E_a = \left\{ \frac{1}{\rho} \frac{\cos\eta}{a_T}, -\frac{1}{\rho} \frac{\sin\eta}{a_T} \right\}, \quad \mathrm{I}: \quad E_a = \left\{ \frac{1}{\rho_g} \frac{\cos\eta}{a_T}, -\frac{1}{\rho_g} \frac{\sin\eta}{a_T} \right\} . \qquad \text{(IV.25.26)}$$

Therein is

$$\mathrm{A}: \quad a_T\left(x^1\right) = \cos\eta, \quad \mathrm{I}: \quad a_T\left(x^0, x^1\right) = \left[3\Re_g \cos\eta_g - \Re\cos\eta\right] \frac{1}{\rho_g} . \qquad \text{(IV.25.27)}$$

However, constraining this function a_T for the interior solution onto the surface and applying the embedding condition $\Re = \Re_g = \text{const.}$ and $\rho_g = 2\Re_g$ remains

$$a_T\left(x^1\right) = \frac{1}{2}\left[3\cos\eta_g - \cos\eta\right] , \qquad \text{(IV.25.28)}$$

the well-known expression for the factor of the time-like arc element of the interior Schwarzschild geometry. In (IV.25.26) a strong similarity between the structures of the two parts of the solution can be noticed. On can establish a closer correlation between the relations by using the time-like line element

$$dx^4 = a_\psi di\psi, \quad a_\psi = \bar{\Re}\cos\gamma - \Re\cos\eta \qquad \text{(IV.25.29)}$$

for both solutions. Specifying therein to

$$\begin{aligned} \mathrm{A}: \quad & \bar{\Re} = 3\Re, \quad \gamma = \eta, \quad \Rightarrow a_\psi = 2\Re\cos\eta \\ & dx^4 = a_T dit, \quad dit = \rho di\psi, \quad a_\psi = \rho a_T \\ & \hspace{6cm} , \\ \mathrm{I}: \quad & \bar{\Re} = 3\Re_g, \quad \gamma = \eta_g, \Rightarrow a_\psi = 3\Re_g \cos\eta_g - \Re\cos\eta \\ & dx^4 = a_T dit, \quad dit = \rho_g di\psi, \quad a_\psi = \rho_g a_T \end{aligned} \qquad \text{(IV.25.30)}$$

one has noted the features of the two parts of the solution. The quantity a_ψ in (IV.25.29) is an element of the double-surface theory and also includes the contributions of the evolute of the geometry. Looking at the change of a_ψ in the direction of, and normal to, the Schwarzschild parabola and writing the force of gravity as

$$E_a = -\wp_a^b \frac{1}{a_\psi} a_{\psi,b} = -\wp_a^b \frac{\bar{\Re}_{,b}\cos\gamma - \Re_{,b}\cos\eta - \bar{\Re}\sin\gamma\,\gamma_{,b} + \Re\sin\eta\,\eta_{,b}}{\bar{\Re}\cos\gamma - \Re\cos\eta} , \qquad \text{(IV.25.31)}$$

one obtains from the rather complicated expression

$$\begin{aligned} E_0 &= -\wp_0^0 \frac{\bar{\Re}_{,0}\cos\gamma - \Re_{,0}\cos\eta}{\bar{\Re}\cos\gamma - \Re\cos\eta} \\ E_1 &= -\wp_1^1 \frac{-\bar{\Re}\sin\gamma\,\gamma_{,1} + \Re\sin\eta\,\eta_{,1}}{\bar{\Re}\cos\gamma - \Re\cos\eta} \end{aligned} \qquad \text{(IV.25.32)}$$

and with the specific values (IV.25.24) and (IV.25.30) again the quantities (IV.25.26). For the further development of the model, the derivatives of the projectors are to be determined. In particular, $\mathfrak{p}^4_4$ is formed with the ratio of the time-like metric factors a_H of the spherical geometry and a_ψ of the Schwarzschild geometry

$$\mathfrak{p}^4_4 = \frac{a_H}{a_\psi}, \quad a_H = \mathfrak{R}\cos\eta \ . \tag{IV.25.33}$$

It should be pointed out that the two variables in a_H are independent, whereas concerning the exterior part of the solution the quantities $\bar{\mathfrak{R}}$ and $\mathfrak{R}$ in a_ψ are also dependent on the quantity r, because both the starting point and the endpoint of the curvature vector $\vec{\rho} = \vec{\bar{\mathfrak{R}}} - \vec{\mathfrak{R}}$ move on the Schwarzschild parabola and on the Neil parabola. First one has

$$\mathfrak{p}^4_{4|a} = -\frac{1}{a_H} a_{H|a} - \mathfrak{p}^4_4 \frac{1}{a_\psi} a_{\psi|a} \ .$$

With

$$a_{H,a} = \{\cos\eta, -\sin\eta\}$$

one obtains

$$\frac{1}{a_H} a_{H|a} = \mathfrak{p}^b_a E_b \ .$$

Further

$$\frac{1}{a_\psi} a_{\psi|a} = \frac{\bar{\mathfrak{R}}_{|a}\cos\gamma - \mathfrak{R}_{|a}\cos\eta - \bar{\mathfrak{R}}\sin\gamma\gamma_{|a} + \mathfrak{R}\sin\eta\eta_{|a}}{\bar{\mathfrak{R}}\cos\gamma - \mathfrak{R}\cos\eta} \ .$$

This expression has been calculated above up to the term

$$\Lambda_a = \delta^1_a \frac{\bar{\mathfrak{R}}_{|1}\cos\gamma - \mathfrak{R}_{|1}\cos\eta}{\bar{\mathfrak{R}}\cos\gamma - \mathfrak{R}\cos\eta}$$

The Λ-term

$$\text{A}: \quad \Lambda_1 = \frac{1}{\rho}\rho_{|1}, \quad \text{I}: \quad \Lambda_1 = 0$$

provides, for the exterior solution, the important contribution Λ_1, while it disappears for the interior solution. In sum one has

$$\mathfrak{p}^4_{4|a} = -\mathfrak{p}^b_a E_b + \mathfrak{p}^4_4 E_a + \Lambda_a \ . \tag{IV.25.34}$$

Finally, this gives

$$\begin{array}{ll} \text{A}: & \mathfrak{p}^0_0 = \mathfrak{p}^1_1 = \mathfrak{p}^4_4, \quad \mathfrak{p}^4_{4|0} = 0, \quad \mathfrak{p}^4_{4|1} = \dfrac{1}{\rho}\rho_{|1} = \dfrac{3}{\rho}\cot\eta \\ \text{I}: & \mathfrak{p}^0_0 = \mathfrak{p}^1_1 = 1, \qquad \mathfrak{p}^4_{4|a} = -\left(1 - \mathfrak{p}^4_4\right)E_a \end{array} \ . \tag{IV.25.35}$$

The further projectors mentioned in (IV.25.24) and in addition their derivatives can also be calculated with the ratios of the metric factors of the two geometries, but considerably more easily. They also satisfy the simple relation

$$\mathfrak{p}^a_b = \mathfrak{R}^a{}_{|||b} \ . \tag{IV.25.36}$$

To determine the field equations one needs the auxiliary variable

$$\mathfrak{p}_{[ba]}{}^{g} = \left(\mathfrak{p}^{-1}\right)_{f}^{g} \mathfrak{p}^{f}_{[a||b]} \ , \tag{IV.25.37}$$

which has only a few occupied components concerning the exterior solution and vanishes for the interior solution

$$\text{A}: \quad 2\mathfrak{p}_{[01]}{}^{0} = \mathfrak{p}_{[41]}{}^{4} = \frac{1}{\rho}\rho_{|1} = \frac{3}{\rho}\cot\eta, \quad \text{I}: \quad \mathfrak{p}_{[ba]}{}^{g} = 0 . \tag{IV.25.38}$$

Using the projection of the identical vanishing curvature tensor of the embedding space

$$\mathfrak{p}^{gh}_{ab}\mathcal{R}_{ghc}{}^{d} = {}^{5}\text{R}_{abc}{}^{d} + 2\mathfrak{p}_{[ba]}{}^{g}\text{A}_{gc}{}^{d} \equiv 0 \tag{IV.25.39}$$

one finds the 5-dimensional Ricci

$${}^{5}\text{R}_{ab} + 2\mathfrak{p}_{[ac]}{}^{d}\text{A}_{db}{}^{c} \equiv 0 \ . \tag{IV.25.40}$$

In the course of the dimensional reduction this relation at first is simplified to

$${}^{5}\text{R}_{mn} + 2\mathfrak{p}_{[mr]}{}^{s}\text{A}_{sn}{}^{r} = 0 \ . \tag{IV.25.41}$$

To get the four-dimensional Ricci one has hived off from the 5-dimensional Ricci. We summarize the 0-terms to get the expression

$$\text{Z}_{mn} = 2\text{A}_{[m}{}^{s}\text{A}_{s]n} \tag{IV.25.42}$$

The A_{mn} are the generalized second fundamental forms of the surface-theory. They differ in part from those of the past Sections in the new choice of the signs. This gives

$${}^{4}\text{R}_{mn} + \text{Z}_{mn} + 2\mathfrak{p}_{[mr]}{}^{s}\text{A}_{sn}{}^{r} = 0 \tag{IV.25.43}$$

Contracting this expression and inserting it into the Einstein field equations, we obtain on the right side

$$\kappa\text{T}_{mn} = \left[\text{Z}_{mn} - \frac{1}{2}\text{Z}\,\text{g}_{mn}\right] + 2\left[\mathfrak{p}_{[mr]}{}^{s}\text{A}_{sn}{}^{r} - \frac{1}{2}\text{g}_{mn}\mathfrak{p}_{[tr]}{}^{s}\text{A}_{s}{}^{tr}\right], \tag{IV.25.44}$$

the common stress-energy-momentum tensor of the model. It vanishes for the exterior part of the solution and provides for the interior part the well-known expressions for the hydrostatic pressure and energy density. Therefore the conservation law is to be investigated only for the interior part which is simplified due to the second equation of (IV.25.38) to

$$\left[\text{Z}_{m}{}^{n} - \frac{1}{2}\text{Z}\,\delta^{n}_{m}\right]_{||n} = 2\text{A}_{<m}{}^{s}\text{A}_{[s}{}^{n}{}_{||n]>} \ . \tag{IV.25.45}$$

The right side of this relation corresponds to the contracted Codazzi equation and vanishes. Thus, the conservation law is satisfied in the most general form.

The pressure function contained in the stress-energy-momentum tensor of the interior solution has a pole, as noted earlier. At a particular angle of the cap of the sphere, which represents the spatial part of the interior solution, the hydrostatic pressure is infinitely high. For this reason, the junction surface of the solutions cannot be placed anywhere, but in any case it is situated beyond the event horizon. The Schwarzschild radius, occurring in the exterior solution is therefore outside the physically achievable range. No observer will ever be exposed to infinitely high forces and no infalling observer can reach speeds which are close to the speed of light.

The exterior Schwarzschild solution has been extended for the case that the source of the fields has an electrical charge. In addition, also charged interior solutions have been found. The charged solutions will be treated in the later sections. A further interior solution will be added by us.

Rosen [R] found a complete Schwarzschild solution using isotropic co-ordinates. He points out that in this representation an event horizon does not exist.

IV.26. Free fall into the interior

In the following we will face a motion in the interior of a stellar object which formally corresponds to the free fall in the exterior. This is definitely a gedankenexperiment, because no free movement is possible in the interior of matter. We imagine a tube drilled through the center of the stellar object through which a test particle can fall from any position from the exterior. Since we allow no forces other than gravitational forces, the test particle will accelerate to the center of the stellar object, and will come to rest at the opposite side of the object, symmetrical to the starting point. Its motion will reverse, the body will oscillate back and forth. In the framework of Newtonian theory calculations have been made in this regard. We have no information concerning similar considerations which affect the interior Schwarzschild field. An ansatz has to be found for the speed of a test particle in the interior of the matter.

First, we turn to the simpler problem, namely that a test particle is freely falling from infinity. Its speed at the surface of the stellar object is

$$v_g = -\sqrt{\frac{2M}{r_g}} \tag{IV.26.1}$$

and must match the initial velocity in the interior. Furthermore, we know that a freely falling observer cannot measure gravity, because this force is nullified by a force that can be derived from the acceleration of the free fall. Such a force G, we will gain from the Lorentz transformation that connects the system of a static observer with that of a freely falling observer. For the latter, must apply G - E = 0, where E is the force of gravity inside the stellar object. Thus, we have already found a way of determining the speed of a freely falling observer falling through the tube. From the exterior Schwarzschild solution we know that the speed of a free-falling observer is determined by the geometry. The redshift factor is the reciprocal of the Lorentz factor of the motion of a freely falling observer and has for the interior solution the form

$$a_T = \frac{1}{2}\left[3\cos\eta_g - \cos\eta\right] \ . \tag{IV.26.2}$$

If one wants to replace the trigonometric functions by the standard Schwarzschild coordinate r, one has first to apply the relation

$$\cos^2\eta = 1 - \frac{r^2}{\mathcal{R}^2} \tag{IV.26.3}$$

and then to apply the values at the boundary surface

$$\rho_g = 2\mathcal{R} = \sqrt{\frac{2r_g^3}{M}}, \quad \mathcal{R} = \sqrt{\frac{r_g^3}{2M}} \ . \tag{IV.26.4}$$

Finally, we have for the Lorentz factor

$$\alpha_T(r) = \frac{2}{3\sqrt{1-\frac{2M}{r_g}} - \sqrt{1-\frac{2Mr^2}{r_g^3}}} \tag{IV.26.5}$$

and for the velocity of the free fall

$$v_T(r) = -\sqrt{1 - \frac{1}{4}\left(3\sqrt{1-\frac{2M}{r_g}} - \sqrt{1-\frac{2Mr^2}{r_g^3}}\right)^2} \quad . \qquad \text{(IV.26.6)}$$

One can see immediately that this relation coincides with (IV.26.1) for $r = r_g$. In the Section IV.1 we have established a formula for a freely falling object that does not come from infinity, but starts with its motion from any position r_0 in the exterior Schwarzschild field. This formula is easy to transfer to the case in which the test particle continues its movement in the interior. One has

$$v_I(r,r_0) = \frac{v_T - \left(-\sqrt{\frac{2M}{r_0}}\right)}{1 - v_T\left(-\sqrt{\frac{2M}{r_0}}\right)} \qquad \text{(IV.26.7)}$$

For $r_0 = \infty$ one retrieves the relation (IV.26.6)). If the body starts on the surface of the object there is $v_I(v_g, v_g) = 0$. In Fig. IV.24 we offer some examples. The surface of the stellar object is indicated by the dashed lines.

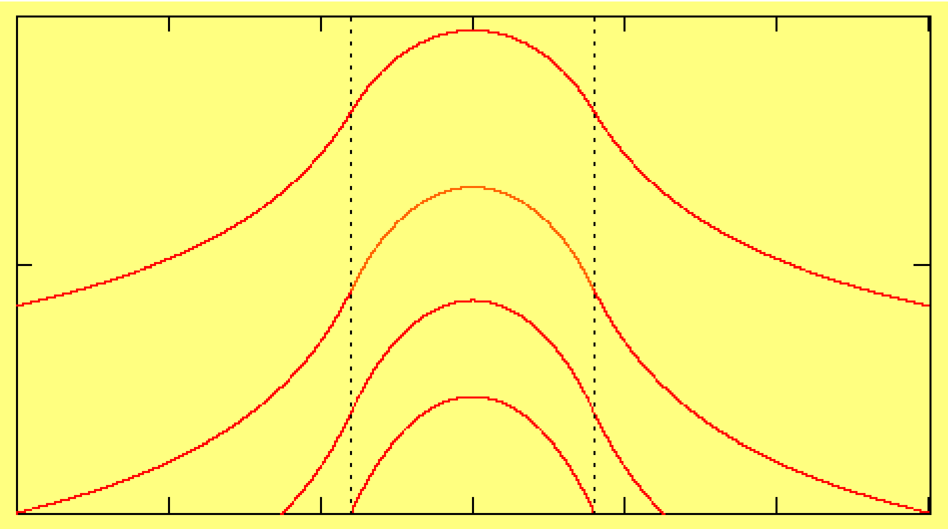

Fig. IV.24

To undertake further considerations concerning the freely falling observer we restrict ourselves to the simpler case. The test particles begin their fall at infinity, but the fall is examined only from the surface of the stellar object. We consult the formula (IV.26.2) and

$$v = -\sqrt{1 - \frac{1}{4}\left(3\cos\eta_g - \cos\eta\right)^2}, \quad \alpha = a_T^{-1} \qquad \text{(IV.26.8)}$$

whereby we omit the marker $_T$ at α from now on. Thus, we already have developed the parameters of a Lorentz transformation

$$L^{1}_{1'} = \alpha, \quad L^{1}_{4'} = -i\alpha v, \quad L^{4}_{1'} = i\alpha v, \quad L^{4}_{4'} = \alpha \qquad \text{(IV.26.9)}$$

which provides the connection between the static and the falling observer. The unit vectors of the 1- and 4-direction of the static system

$$m_m = \{1,0,0,0\}, \quad u_m = \{0,0,0,1\} \qquad \text{(IV.26.10)}$$

obtain the form

$$m_{m'} = \{\alpha, 0, 0, -i\alpha v\}, \quad u_{m'} = \{i\alpha v, 0, 0, \alpha\} \tag{IV.26.11}$$

being viewed by the falling observer The freely falling observer has for his own system

$$'m_{m'} = \{1, 0, 0, 0\}, \quad 'u_{m'} = \{0, 0, 0, 1\} \ , \tag{IV.26.12}$$

but his unit vectors are measured by the static observer as

$$'m_{m} = \{\alpha, 0, 0, i\alpha v\}, \quad 'u_{m} = \{-i\alpha v, 0, 0, \alpha\} \ . \tag{IV.26.13}$$

The last four formulas include the basic laws of the relativity theory.

To learn more about the forces of the freely falling system we start with the static metric of the interior Schwarzschild solution

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta \, d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta \, d\varphi^2 + a_T^2 dit^2 \ . \tag{IV.26.14}$$

We read off the bein vectors in the usual manner and we calculate the field strengths of the static system, which are transformed with the help of (IV.26.9) to the falling system. In particular, it applies to the partial derivatives

$$\partial_{1'} = \alpha \frac{\partial}{\mathcal{R}\partial\eta}, \quad \partial_{4'} = -i\alpha v \frac{\partial}{\mathcal{R}\partial\eta} \ . \tag{IV.26.15}$$

The Ricci-rotation coefficients, which describe the curvatures of the surface, retain their geometric properties under the Lorentz transformation, i.e. they transform as tensors. Subjecting the covariant derivatives to a Lorentz transformation the Lorentz term

$$\Phi_{m'||n'} = L^{m\ n}_{m'n'} \Phi_{m||n} = \left[\Phi_{m'|n'} - L^{s'}_{s} L^{s}_{m'|n'} \Phi_{s'}\right] - A_{n'm'}{}^{s'} \Phi_{s'}, \quad A_{n'm'}{}^{s'} = L^{n\ m\ s'}_{n'm's} A_{nm}{}^{s} \ , \tag{IV.26.16}$$

occurs, which has its origin in the non-constant parameters of the Lorentz transformation. Thus, the basics of a free fall through the interior of a stellar object are outlined. For the further treatment of the problem we refer to the results of the free fall in the exterior field. However, we have to explain the differences.

Having calculated the field quantities

$$\begin{aligned}
B_{m'} &= \left\{\alpha \frac{1}{\mathcal{R}} \cot\eta, 0, 0, -i\alpha v \frac{1}{\mathcal{R}} \cot\eta\right\} \\
C_{m'} &= \left\{\alpha \frac{1}{\mathcal{R}} \cot\eta, \frac{1}{\mathcal{R}\sin\eta} \cot\vartheta, 0, -i\alpha v \frac{1}{\mathcal{R}} \cot\eta\right\} \\
E_{m'} &= \left\{-\alpha \frac{1}{\rho_g} \frac{\sin\eta}{a_T}, 0, 0, i\alpha v \frac{1}{\rho_g} \frac{\sin\eta}{a_T}\right\}
\end{aligned} \tag{IV.26.17}$$

from the static system with the help of the Lorentz transformation, we only need to deal with the Lorentz term

$$L_{n'm'}{}^{s'} = L^{s'}_{s} L^{s}_{m'|n'} \ . \tag{IV.26.18}$$

It has the components

$$L_{4'1'}{}^{4'} = G_{1'}, \quad L_{1'4'}{}^{1'} = -i\frac{1}{v} G_{1'}, \quad G_{1'} = -\alpha \frac{1}{\rho_g} \frac{\sin\eta}{a_T} \ . \tag{IV.26.19}$$

It can be seen from (IV.26.16) that the Lorentz term joins the connexion coefficients. Thus, one has

$$L_{4'1'}{}^{4'} + A_{4'1'}{}^{4'} = G_{1'} - E_{1'} = 0 \; . \tag{IV.26.20}$$

The two quantities E and G have different sources, G is a kinematic quantity and E a geometric one. However, they are formally identical and nullify each other. In the interior of a stellar object a freely falling observer experiences no gravity, like a freely falling observer in the exterior field. We have met a requirement laid down at the start.

Since the Ricci tensor and hence the Einstein tensor are Lorentz-invariant we obtain for the field equations, well-known structures which need not be repeated. Thus, the stress-energy-momentum tensor has the form

$$T_{m'n'} = -m_{n'}m_{m'}p - b_{n'}b_{m'}p - c_{n'}c_{m'}p + u_{n'}u_{m'}\mu_0 \tag{IV.26.21}$$

whereby the values (IV.26.11) are to be considered. Written in greater detail

$$T_{m'n'} = \begin{pmatrix} -p & & & \\ & -p & & \\ & & -p & \\ & & & \mu_0 \end{pmatrix} + \begin{pmatrix} -\alpha^2 v^2 & & & -i\alpha^2 v \\ & 0 & & \\ & & 0 & \\ -i\alpha^2 v & & & \alpha^2 v^2 \end{pmatrix} (p+\mu_0) \tag{IV.26.22}$$

it demonstrates that the stress-energy-momentum tensor is separated into a static and a kinematic component. p is the hydrostatic pressure of the fluid sphere, which is taken in a rough approximation for the stellar objects, and μ_0 its energy density. $\mu_0 + p$ is the total energy density, resulting from the hydrostatic pressure energy and energy density derived from the mass of the stellar object.

After some calculation one gets with the help of the expressions (IV.26.2), (IV.26.8), and with the relation $E_{m'} = \alpha^{-1}\alpha_{|m'}$ known from previous Sections the conservation law

$$T_{m'}{}^{n'}{}_{||n'} = -p_{|m'} + (p+\mu_0)E_{m'} = 0 \; . \tag{IV.26.23}$$

It corresponds to the conservation law in the static system

$$p_{|\alpha} = (p+\mu_0)E_\alpha, \quad \dot{p} = 0, \quad \dot{\mu} = 0 \; . \tag{IV.26.24}$$

Until now our investigations have been closely based on the static system, by regarding the components of the static field strengths from the falling system. The first new result is that we can observe that no gravity acts on the freely falling observer. Analogous to the theory of the exterior field, we construct the formulae in such a way that we obtain relations for the tidal forces acting on the freely falling observer in place of gravity. The first component of the tidal forces we gain from the interplay of the gravity term and the Lorentz term.

With

$$\begin{aligned} G_{n'm'}{}^{s'} &= u_{n'}G_{m'}u^{s'} - u_{n'}u_{m'}G^{s'} = {}'u_{n'}G_{m'}{}'u^{s'} - {}'u_{n'}{}'u_{m'}G^{s'} \\ E_{n'm'}{}^{s'} &= -u_{n'}E_{m'}u^{s'} + u_{n'}u_{m'}E^{s'} = -{}'u_{n'}E_{m'}{}'u^{s'} + {}'u_{n'}{}'u_{m'}E^{s'} \end{aligned} \tag{IV.26.25}$$

the relation (IV.26.20) can be formulated more generally. One has

$$L_{n'm'}{}^{s'} = Q_{n'm'}{}^{s'} + G_{n'm'}{}^{s'} \; . \tag{IV.26.26}$$

In the Ricci-rotation coefficients one has as a consequence of

$$G_{n'm'}{}^{s'} + E_{n'm'}{}^{s'} = 0 \tag{IV.26.27}$$

only

$$L_{n'm'}{}^{s'} + E_{n'm'}{}^{s'} = Q_{n'm'}{}^{s'} \tag{IV.26.28}$$

wherein $Q_{n'm'}{}^{s'}$ has only one component

$$Q_{1'4'}{}^{1'} = Q_{4'} = \frac{i}{\rho_g}\frac{\sin\eta}{v} \tag{IV.26.29}$$

which can be deduced with (IV.26.19). On the boundary surface one has

$$Q^g_{4'} = \frac{i}{\rho_g}\frac{\sin\eta_g}{v_g} = -\frac{i}{\rho_g}\ . \tag{IV.26.30}$$

This quantity coincides with a component of the second fundamental forms of the surface theory, which describes the shrinking space that accompanies a freely falling observer in the exterior field. Since we are now dealing only with the freely falling system, we omit the primes on the indices and the kernels. We write the equations (IV.26.29) in a short form as $Q_{11} = Q_4$ and complement

$$Q_{11} = Q_4,\quad Q_{22} = B_4,\quad Q_{33} = C_4,\quad Q_{[mn]} = 0\ . \tag{IV.26.31}$$

Thus we have got the whole set of the second fundamental forms of the shrinking surface and have also inferred the tidal forces. The Ricci-rotation coefficients we split into

$$A_{mn}{}^s = {}^*A_{mn}{}^s + Q_m{}^s u_n - Q_{mn} u^s,\quad A_n = {}^*A_n + u_n Q_s{}^s \tag{IV.26.32}$$

and thereby decompose the Ricci in a purely spatial part and a part that describes the mechanism of the tidal field strength

$$\begin{aligned} R_{mn} = {}&{}^*R_{mn} \\ &-\left[Q_{mn|s}u^s + Q_{mn}Q_s{}^s\right] \\ &-u_n\left[Q_s{}^s{}_{\wedge\underline{m}} - Q_m{}^s{}_{\wedge s}\right] \\ &-u_m\left[{}^*A_{n|s}u^s + {}^*A_{sn}{}^r Q_r{}^s\right] \\ &-u_m u_n\left[Q_s{}^s{}_{|m}u^m + Q_{rs}Q^{rs}\right] \end{aligned}\ . \tag{IV.26.33}$$

Therein an index with an underline designates the spatial component of the quantity and the hat the corresponding three-dimensional covariant derivative. *R is the 3-dimensional Ricci. Its structure corresponds to the 4-dimensional Ricci. It includes only 3-dimensional quantities with the corresponding graded derivatives

$$\begin{aligned} {}^*R_{mn} = &-\left[{}^*B_{n\underset{2}{\wedge}\underline{m}} + {}^*B_n{}^*B_m\right] - \left[{}^*C_{n\underset{3}{\wedge}\underline{m}} + {}^*C_n{}^*C_m\right] \\ &-b_m b_n\left[{}^*B^s{}_{\underset{2}{\wedge}s} + {}^*B^s{}^*B_s\right] - c_m c_n\left[{}^*C^s{}_{\underset{3}{\wedge}s} + {}^*C^s{}^*C_s\right] \end{aligned}\ . \tag{IV.26.34}$$

A further treatment of Einstein's field equations is more difficult than that of the exterior field. The square brackets in (IV.26.33) describing the mechanism of field tidal forces decouple only partly from the field equations. Thus, one has to calculate the Einstein and the resulting stress-energy-momentum tensor. Solving the Q-equations leads to the already-calculated expression (IV.26.22). From (IV.26.33) one can isolate the somewhat Maxwell-like Q-relations

$$2Q_{[s}{}^{s}{}_{\wedge\underline{m}]} = \kappa T_{\underline{m}4} \ . \tag{IV.26.35}$$

On the right side of the equation is the energy-current density of the matter which is coupled to the tidal forces. Besides the differences for the free fall in the exterior field just described is the geometry in the interior of the matter for a freely falling observer by no means flat. This was the case for the exterior field, because the factor of the space-like field quantities due to the curvature and the Lorentz factor were nullified.

Benish [B] has treated the question as to whether the theoretically discussed process of free fall in the interior can be supported by an experiment. He proposed a torsion balance, similar to that used in the Cavendish experiment. Two massive spheres can move through greater spheres on approximately radial orbits and thus could perform an oscillatory motion. The velocity of this motion could be detected on the axis of the suspension by optical devices. The design of Benish is shown in Fig. IV.25.

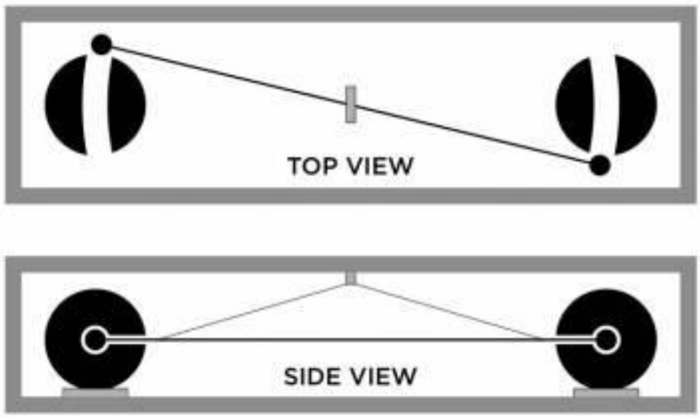

Fig. IV.25

Taylor [T] calculated oscillations. He assumed an approximate expression for the Schwarzschild interior metric. Martin [M] interpreted gravitation as a flow of the space and examined the motion in hollow spheres under this aspect.

IV.27. The Reissner – Nordström solution

Reissner [R] and Nordström [N] extended the exterior Schwarzschild solution assuming that electromagnetic fields have the same influence on gravitation as matter. Their modification leads to a stress-energy tensor of Maxwellian structure which is composed of the electrical field strengths and refers to a charge distribution being in rest. Since the fields are regarded to be outside the charge and matter distribution, the model belongs to the electrovac solutions.

This solution has to be examined with regard to the geometrical structure. As expected one obtains a surface which is similar to Flamm's paraboloid for the space-like part of the model. For the time-like part the methods of the double-surface theory have again to be used. This procedure provides the way for supplementing the Reissner-Nordström solution with an interior solution which is reduced to the Schwarzschild interior solution, if one puts the charge zero.

The charge of the source is invoked by the metric

$$\begin{gathered} ds^2 = \alpha^2 dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta \, d\varphi^2 + a^2 dit^2 \\ a^2 = 1 - \frac{2M}{r} + \frac{e^2}{r^2}, \quad \alpha = a^{-1} \end{gathered} \quad . \tag{IV.27.1}$$

The further processing of the calculations is carried out in a similar way to that of the Schwarzschild solution. However, from the form of the factor a results a two-part expression for the gravitational field strength

$$E_m = \bar{E}_m + \mathcal{E}_m \ . \tag{IV.27.2}$$

The first component is defined in a similar way to the Schwarzschild force of gravity

$$\bar{E}_m = \left\{ -\alpha \frac{M}{r^2}, 0, 0, 0 \right\} \ . \tag{IV.27.3}$$

In addition, there occurs a repulsive force, quadratic in the charge

$$\mathcal{E}_m = \left\{ \alpha \frac{e^2}{r^3}, 0, 0, 0 \right\} \tag{IV.27.4}$$

which decreases with third order of r. Its origin is due to the charge of the central mass. In addition, it affects uncharged test particles in the exterior field. Thus, the charge contributes to the gravitational action. Fig. IV.26 shows[81] that with a suitable choice of the parameters M and e the repulsive force has only little influence on the action of gravity.

[81] The black curve represents $\bar{E}$, the green $\mathcal{E}$ and the red represent the total field strength E.

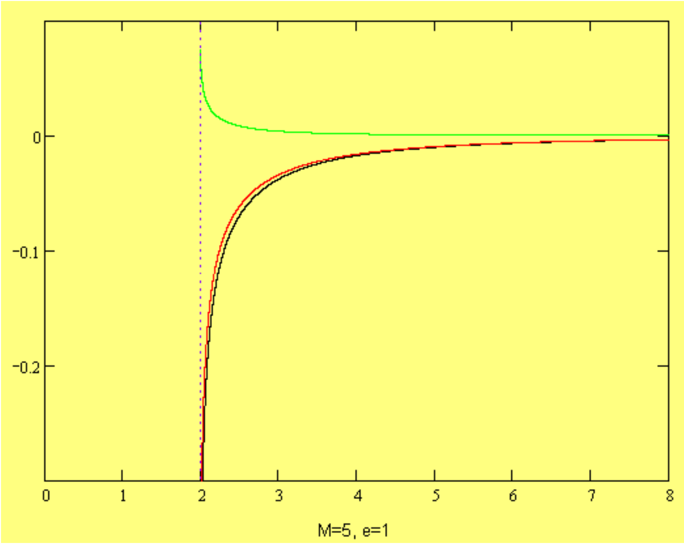

Fig. IV.26

The velocity of a freely falling observer is

$$v = -\sqrt{\frac{2M}{r} - \frac{e^2}{r^2}} \tag{IV.27.5}$$

and is related to the angle ε of the radial ascent by

$$\sin\varepsilon = v \ . \tag{IV.27.6}$$

The gravitational force can be written in the form

$$E_1 = \frac{1}{\rho}\tan\varepsilon \ . \tag{IV.27.7}$$

From this one obtains the radii of the curvature of the radial curves on the surface

$$\rho = r^2\,\frac{\sqrt{2Mr - e^2}}{Mr - e^2} \ . \tag{IV.27.8}$$

With these quantities the physical surface can be fully described in the context of the generalized surface theory. The 5-dimensional field equations derived from it reduce for $e = 0$ to the equations of the Schwarzschild theory. Here only the 4-dimensional field equations are to be discussed. With the quantities

$$B_m = \left\{\frac{a}{r},0,0,0\right\},\quad C_m = \left\{\frac{a}{r},\frac{1}{r}\cot\vartheta,0,0\right\},\quad \bar{E}_m = \left\{-\alpha\frac{M}{r^2},0,0,0\right\},\quad \mathcal{E}_m = \left\{\alpha\frac{e^2}{r^3},0,0,0\right\} \tag{IV.27.9}$$

one arrives at

$$\begin{aligned} R_{mn} = &-\left[B_{n\underset{2}{\|}m} + B_nB_m\right] - b_nb_m\left[B^s{}_{\underset{2}{\|}s} + B^sB_s\right] \\ &-\left[C_{n\underset{3}{\|}m} + C_nC_m\right] - c_nc_m\left[C^s{}_{\underset{3}{\|}s} + C^sC_s\right] \ . \\ &+\left[E_{n\underset{4}{\|}m} - E_nE_m\right] + u_nu_m\left[E^s{}_{\underset{4}{\|}s} - E^sE_s\right] \end{aligned} \tag{IV.27.10}$$

The Einstein field equations are simplified by the relation

$$R = 0 \tag{IV.27.11}$$

which one can derive from (IV.27.10). Rescaling temporarily the electrical part by

$$e^2 \to \frac{\kappa}{2} e^2 \ , \qquad \text{(IV.27.12)}$$

it turns out that the field equations

$$\begin{aligned} R_{mn} - \frac{1}{2} g_{mn} R &= -\kappa T_{mn} \\ T_{mn} &= \mathcal{F}_m{}^s \mathcal{F}_{ns} - \frac{1}{4} g_{mn} \mathcal{F}^{rs} \mathcal{F}_{rs} \end{aligned} \qquad \text{(IV.27.13)}$$

are satisfied. The tensor of the electrical field strength has only one component in the radial direction

$$\mathcal{F}_{14} = \mathcal{F}_1 = \frac{e}{r^2} \qquad \text{(IV.27.14)}$$

and satisfies the Maxwell equations

$$\mathcal{F}^{mn}{}_{||n} = 0, \quad \mathcal{F}_{<mn||s>} = 0 \ . \qquad \text{(IV.27.15)}$$

The electrical stresses and the electrical energy are conserved

$$T^{mn}{}_{||n} = 0 \ , \qquad \text{(IV.27.16)}$$

which can be verified by the use of Maxwell's equations (IV.27.15). From the last brackets of (IV.27.10) and (IV.27.13) one obtains

$$E^s{}_{||s} = 0, \quad \mathcal{E}^s{}_{||s} = -\frac{\kappa}{2} \mathcal{F}^s \mathcal{F}_s \ , \qquad \text{(IV.27.17)}$$

from which it is evident that the repulsive force

$$\mathcal{E}_s = \left\{ \alpha \frac{\kappa}{2} \frac{e^2}{r^3}, 0, 0, 0 \right\} \qquad \text{(IV.27.18)}$$

is coupled to the electrical energy. Since the total force of gravity can be written as

$$E_1 = -\frac{\alpha}{r^2} \left(M - \frac{e^2}{r} \right) \qquad \text{(IV.27.19)}$$

in accordance with (IV.27.9), some authors designate

$$M_{eff} = M - \frac{e^2}{r} \qquad \text{(IV.27.20)}$$

as effective mass. Since this mass is repulsive for $r < e^2 / M$, there exists a critical radius $r = e^2 / M$ which gives rise to a spherical surface around the source on which freely moving mass would accumulate from both sides (Marsh [M]). Herrera [H] and Varela have treated point-like sources. Som [S], Santos, and Teixeira have stated that the mass parameter M and the charge parameter e are inseparably connected and that for M = 0 inevitably follows e = 0. The Reissner-Nordström solution cannot describe mass-free charges. In addition, charged massless particles are not observed in nature. Barbachoux [B], Gariel, Marcilhacy, and Santos have related g_{44} to the electrical potentials. Gao [G] and Lemos have been concerned with charged dust shells which collapse to a Reissner-Nordström black hole. Paolino [P] and Pizzi have described two Reissner-Nordström sources. The one is a naked singularity, the other one is a black hole. Shah [S] has examined the energy of a charged particle in the context of Møller's tetrad theory. Karade [K] and Rao have described the free fall in the RN-field. Bonnor [B] and Vaidya have been concerned with the radiation of charges. Das [D] has extended the Birkhoff theorem to models with electromagnetic fields. Krori [K] and Barua and Krori [K] and Paul have treated inelastic and elastic charged fluid spheres. Kuchar [K] has treated the collapse of charged shells, Ghezzi [G] the collapse of charged neutron stars. Leaute [L] and Linet have described the electrostatics in the Reissner-Nordström model. Nashed [N] has made an attempt at the stability of the solution. Dokuchaev [D] and Chernov have described a shell whose inner and outer regions are electrically charged and have assigned these regions to a black hole. Greenwood [G] has described the effect of an RN-collapse on the Hawking radiation and has treated additional scalar fields. Novikov [N] has examined the collapse of a charged stellar object, whereby the matter does not get contracted to infinitely high density. In this case the object has to expand again. Pugliese [P], Quevedo, and Ruffini have discussed a circular motion of an observer in the RN field. Further problems one can find in papers of Kim [K], Kim [K] , Park and Soh; and Rosen [R]. Pekeris [P] has found it inconvenient that in the RN-model gravity has a repulsive component which is due to the charge. Therefore, he has used the original form of the Schwarzschild metric, derived by Schwarzschild himself, which describes a mass point. Pekeris has implemented electrical energy, which then acts attractive everywhere. The authors Pugliese [P], Quevedo, and Ruffini have calculated circular motion of neutral particles in the RN field. They have found stable and unstable orbits. Qadir [Q] and Siddiqui have investigated the geodesic of free fall in the RN field. Ray [R], Usami, Kalam, and Chakraborty investigate if the mass of a charged sphere can be derived from the electric field of the source.

Further: Andrèasson [A], Bini [B], Gemelli and Ruffini; Mammadov [M], Qadir [Q], Qadir [Q] and Siddiqui; Stoica [S].

Even though this extension of the Schwarzschild model is quite simple we have, however, doubts concerning its usefulness. The occurrence of a repulsive force of electrical origin which also affects uncharged test particles is rather unknown[82]. Furthermore, one could understand this model as *semi-unified* theory of gravitation and electromagnetism. The effect of the charge, together with the effect of the masses, is introduced in a geometrical way. However, the Maxwell equations do not have any geometrical interpretation. For the complete unification of these two physical phenomena one could hark back to the methods of the Kaluza-Klein theories.

Although we are of opinion that the Reissner-Nordström model is unsatisfactory we add a new interior solution [B] to the model. We want to show that in the context of the Schwarzschild interior solution the elaborated mathematical methods can be consulted for the search of new interior solutions.

[82] The existence of this force has not been clearly worked out in the literature.

IV.28. Reissner - Nordström, interior solution

In the last decades some attempts have been made to supplement the Reissner-Nordström solution with an interior solution. Tiwari [T], Rao, and Kanakamedala have found an interior solution with the condition $g_{11}g_{44}=1$ which obviously is valid for the exterior solution. For e = 0 it does not lead to the Schwarzschild solution. Kyle [K] and Martin have found a rather complicated solution. They have discussed the self-energy of charged matter. Wilson [W] has modified this solution, by taking another expression for the total charge. Cohen [C] and Cohen have used this solution for the special case of a charged thin shell and showed that the energy density of the electromagnetic field contributes to the mass of the stellar object. Boulware [B] has studied the temporal evolution of thin shells. Graves [G] and Brill have regarded possible oscillating properties of the Reissner Nordström metric using Kruskal-like co-ordinates. Fronsdal [F] has found an application for a stellar photon gas. Ftaclas [F] and Cohen have continued a paper of Gautreau [G] and Hoffman. They have used a more general co-ordinate system free from singularities. Bekenstein [B] has studied an ansatz for charged matter with the stress-energy tensor

$$T_{mn}=-p'g_{mn}+\mu_0 u_m u_n+\mathcal{F}_m{}^s\mathcal{F}_{ns}-\frac{1}{4}g_{mn}\mathcal{F}^{rs}\mathcal{F}_{rs} \qquad \text{(IV.28.1)}$$

and has tried to adjust p and μ_0 in such a way that the field equations are satisfied. Krori [K] and Jayantimala have used the same ansatz and have found a singularity-free solution. Bonnor [B] has taken a more general start to the matter part. Gautreau [G] and Hoffmann have studied the sources of electrovac solutions of the Weyl type. They have obtained parameters for the source with junction conditions for the exterior solution. Efinger [E] has deduced the stability of a charged particle from the self-energy of the gravitational field.

We construct another interior solution, by embedding it into a 5-dimensional flat space. This solution has the advantage of reducing the model to the Schwarzschild interior solution, if one puts the charge zero. However, a decomposition of the stress-energy tensor into a matter part and into an electrical part in accordance with (IV.28.1) is not possible. Inside the matter the force of gravity and the electrical force are inseparably associated.

We refer the space-like part of the interior metric to a cap of a sphere

$$ds^2=\mathcal{R}^2 d\eta^2+\mathcal{R}^2\sin^2\eta\, d\vartheta^2+\mathcal{R}^2\sin^2\eta\sin^2\vartheta\, d\varphi^2\ , \qquad \text{(IV.28.2)}$$

wherein $\mathcal{R}$ is the radius and η the polar angle of the sphere. η_g designates the aperture angle of the cap of the sphere. The spherical co-ordinates $\{\mathcal{R},\eta,\vartheta,\varphi\}$ of the geometry are connected with the Cartesian co-ordinates of the embedding space by

$$\begin{aligned}\mathcal{R}^{3'}&=\mathcal{R}\sin\eta\sin\vartheta\sin\varphi\\ \mathcal{R}^{2'}&=\mathcal{R}\sin\eta\sin\vartheta\cos\varphi\\ \mathcal{R}^{1'}&=\mathcal{R}\sin\eta\cos\vartheta\\ \mathcal{R}^{0'}&=\mathcal{R}\cos\eta\end{aligned}\ . \qquad \text{(IV.28.3)}$$

With the standard Schwarzschild co-ordinate

$$r=\mathcal{R}\sin\eta \qquad \text{(IV.28.4)}$$

the line element on the 3-dimensional surface of the sphere can also be written as

$$ds^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta \, d\varphi^2 \ . \qquad \text{(IV.28.5)}$$

The *embedding condition* is

$$\mathcal{R} = \mathcal{R}_g = \text{const.} \ . \qquad \text{(IV.28.6)}$$

On the boundary surface of the exterior and interior solutions the angle of ascent[83] of the radial curves must be $\eta_g = -\varepsilon_g$. From (IV.27.5), (IV.27.6), and (IV.28.4) one obtains the *junction condition*

$$\mathcal{R}_g = \frac{r_g^2}{\sqrt{2Mr_g - e^2}} \ . \qquad \text{(IV.28.7)}$$

In order to obtain the time-like part of the interior line element we proceed in a similar way as we have done with the Schwarzschild interior solution. The relation (IV.28.2) is supplemented with the projections of the arc elements of two concentric pseudo circles in the extra dimension. Thus, one obtains

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta \, d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta \, d\varphi^2 + a_T^2 dit^2$$

$$a_T = \left[\left(\mathcal{R}_g + \rho_g\right)\cos\eta_g - \mathcal{R}\cos\eta\right]\frac{1}{\rho_g} = \frac{1}{2}\left[\left(1 + 2\Phi_g^2\right)\cos\eta_g - \cos\eta\right]\Phi_g^{-2} \ . \qquad \text{(IV.28.8)}$$

$$\rho_g = 2\mathcal{R}_g\Phi_g^2 = r_g^2 \frac{\sqrt{2Mr_g - e^2}}{Mr_g - e^2} = \mathcal{R}_g \frac{2Mr_g - e^2}{Mr_g - e^2}, \quad 2\Phi_g^2 = \frac{2Mr_g - e^2}{Mr_g - e^2}$$

ρ_g one reads from (IV.27.8). ρ_g is the radius of curvature of the radial curves of the exterior surface on the boundary. $\mathcal{R}_g + \rho_g$ is the distance of the base point of the curvature vector up to the symmetry axis of the surface of revolution ($\mathcal{R}^{0'}$-axis). $\left(\mathcal{R}_g + \rho_g\right)\cos\eta_g$ is the projection of this line segment onto $\mathcal{R}^{0'}$ and $\mathcal{R}\cos\eta$ the projection of the radius vector to any point of the cap of the sphere onto the $\mathcal{R}^{0'}$-line. If these projections are rotated through the imaginary angles $i\psi$, one obtains two pseudo circles. With the embedding condition[84] (IV.28.6) the area of the enclosed sector of the annulus is associated with the flow of time

$$dt = \rho_g d\psi \ . \qquad \text{(IV.28.9)}$$

The field strengths which can be deduced from the geometry have the already well-known structure of the Schwarzschild solution

$$B_m = \left\{\frac{1}{\mathcal{R}}\cot\eta, 0,0,0\right\}, \ C_m = \left\{\frac{1}{\mathcal{R}}\cot\eta, \frac{1}{\mathcal{R}\sin\eta}\cot\vartheta, 0,0\right\}, \ E_m = \left\{-\frac{1}{\rho_g a_T}\sin\eta, 0,0,0\right\}. \ \text{(IV.28.10)}$$

If one defines components for the extra dimension with

[83] The orientations of the angles η and ε are counter-clockwise and clockwise, respectively.

[84] On the surface a_T is only a function of η.

$$M_0=\frac{1}{\mathcal{R}},\quad B_0=\frac{1}{\mathcal{R}},\quad C_0=\frac{1}{\mathcal{R}},\quad E_0=\frac{1}{\rho_g a_T}\cos\eta\ , \tag{IV.28.11}$$

which will be treated later on, one gets for the 4-dimensional Ricci tensor

$$\begin{aligned} R_{mn}=&m_m m_n\left(M_0B_0+M_0C_0-M_0E_0\right)\\ &+b_m b_n\left(B_0M_0+B_0C_0-B_0E_0\right)\\ &+c_m c_n\left(C_0M_0+C_0B_0-C_0E_0\right)\\ &-u_m u_n\left(E_0M_0+E_0B_0+E_0C_0\right) \end{aligned}\ . \tag{IV.28.12}$$

With the quantity

$$\mathcal{p}=-\frac{\mathcal{R}\cos\eta}{\rho_g a_T}=-\frac{1}{\left(\frac{\mathcal{R}_g}{\mathcal{R}}+\frac{\rho_g}{\mathcal{R}}\right)\frac{\cos\eta_g}{\cos\eta}-1} \tag{IV.28.13}$$

and the Einstein field equations

$$R_{mn}-\frac{1}{2}g_{mn}R=-\kappa T_{mn}$$

one can compute the stress-energy tensor

$$T_{mn}=\begin{pmatrix} -p & & & \\ & -p & & \\ & & -p & \\ & & & \mu_0 \end{pmatrix}. \tag{IV.28.14}$$

Therein are

$$\kappa p=-\left(1+2\mathcal{p}\right)\frac{1}{\mathcal{R}^2},\quad \kappa\mu_0=\frac{3}{\mathcal{R}^2} \tag{IV.28.15}$$

the hydrostatic pressure and the energy density.

Both formally agree with the expressions of the interior Schwarzschild solution. However, for $\mathcal{R}$ and for $\mathcal{p}$ the more complicated values (IV.28.7) and (IV.28.13) are used which also have charged contributions. On the boundary surface one has the same electrical stresses

$$\kappa p_g=-\mathcal{F}^{\,g}_{1}\mathcal{F}^{\,g}_{1} \tag{IV.28.16}$$

that supplies the exterior solution. The conservation laws

$$p_{\|m}=\left(\mu_0+p\right)E_m,\quad \dot{p}=0,\quad \dot{\mu}_0=0 \tag{IV.28.17}$$

are valid.

Finally, following our preceding remarks it has to be demonstrated how this solution can be embedded into a 5-dimensional flat space. However, it has to be mentioned that originally we have gone in the opposite direction and that we have found this solution by the method of embedding. For the pseudo-hyper sphere we put

$$\begin{aligned}
X^{3'} &= \left(\mathcal{R}_g + \rho_g\right)\sin\eta_g \sin\vartheta \sin\varphi - \mathcal{R}\sin\eta \sin\vartheta \sin\varphi \\
X^{2'} &= \left(\mathcal{R}_g + \rho_g\right)\sin\eta_g \sin\vartheta \cos\varphi - \mathcal{R}\sin\eta \sin\vartheta \cos\varphi \\
X^{1'} &= \left(\mathcal{R}_g + \rho_g\right)\sin\eta_g \cos\vartheta \qquad - \mathcal{R}\sin\eta \cos\vartheta \\
X^{0'} &= \left(\mathcal{R}_g + \rho_g\right)\cos\eta_g \cos i\psi \qquad - \mathcal{R}\cos\eta \cos i\psi \\
X^{4'} &= \left(\mathcal{R}_g + \rho_g\right)\cos\eta_g \sin i\psi \qquad - \mathcal{R}\cos\eta \sin i\psi
\end{aligned} \tag{IV.28.18}$$

and in the following we interpret this as a double-surface. After the first step of the dimensional reduction we arrive at the cap of a sphere by cutting off pieces from the double surface (IV.28.3). Only the time-like pieces of the double-surface remain. After the application of the customary procedures one obtains the 5-dimensional field strengths

$$\begin{gathered}
A_{ab}{}^{c} = M_{ab}{}^{c} + B_{ab}{}^{c} + C_{ab}{}^{c} + E_{ab}{}^{c} \\
M_{ab}{}^{c} = m_a M_b m^c - m_a m_b M^c, \quad B_{ab}{}^{c} = b_a B_b b^c - b_a b_b B^c \\
C_{ab}{}^{c} = c_a C_b c^c - c_a c_b C^c, \quad E_{ab}{}^{c} = -\left[u_a E_b b^c - u_a u_b E^c\right]
\end{gathered} \tag{IV.28.19}$$

and by means of $R_{ab} \equiv 0$ one arrives at

$$\begin{gathered}
M_{\underset{1}{a|||b}} + M_a M_b = 0, \quad B_{\underset{2}{a|||b}} + B_a B_b = 0, \quad C_{\underset{3}{a|||b}} + C_a C_b = 0, \quad E_{\underset{4}{a|||b}} - E_a E_b = 0 \\
M^{c}{}_{\underset{1}{|||c}} + M^c M_c = 0, \quad B^{c}{}_{\underset{2}{|||c}} + B^c B_c = 0, \quad C^{c}{}_{\underset{3}{|||c}} + C^c C_c = 0, \quad E^{c}{}_{\underset{4}{|||c}} - E^c E_c = 0
\end{gathered} \;. \tag{IV.28.20}$$

With the help of the graded derivatives

$$\begin{gathered}
M_{\underset{1}{a|||b}} = M_{a|b}, \quad B_{\underset{2}{a|||b}} = B_{a|b} - M_{ba}{}^{c} B_c, \quad C_{\underset{3}{a|||b}} = C_{a|b} - M_{ba}{}^{c} C_c - B_{ba}{}^{c} C_c \\
E_{\underset{4}{a|||b}} = E_{a|b} - M_{ba}{}^{c} E_c - B_{ba}{}^{c} E_c - C_{ba}{}^{c} E_c
\end{gathered} \tag{IV.28.21}$$

and (IV.28.10), (IV.28.11) one can verify these relations. After having renamed the quantities (IV.28.11) in

$$\begin{gathered}
A_{11} = M_0, \quad A_{22} = B_0, \quad A_{33} = C_0, \quad A_{44} = -E_0 \\
A_{11} = A_{22} = A_{33} = \frac{1}{\mathcal{R}}, \quad A_{44} = \frac{\mathcal{P}}{\mathcal{R}}
\end{gathered} \tag{IV.28.22}$$

they are interpreted as generalized second fundamental forms of the surface theory. They are components of the 5-dimensional connexion coefficients. If one isolates these quantities on the right side of the field equations, one can write the stress-energy tensor in the form

$$\kappa T_{mn} = A_m{}^{s} A_{ns} - A_{mn} A_s{}^{s} - \frac{1}{2} g_{mn}\left[A^{rs} A_{rs} - A_r{}^{r} A_s{}^{s}\right] \tag{IV.28.23}$$

and the conservation law as

$$\kappa T_m{}^{n}{}_{||n} = A_{<m}{}^{s} A_{[s}{}^{n}{}_{||n]>} \;. \tag{IV.28.24}$$

It is satisfied with the help of the Codazzi equations

$$A_{[m}{}^{n}{}_{||n]} = 0 \;. \tag{IV.28.25}$$

The matter endowed with charge can be described with geometrical quantities and can be integrated into the theory as field. Geometrical relations correspond to physical laws.

The interest in charged interior solutions which relate to the works of Reissner and Nordström still remains. Israelit [I] and Rosen have applied a charged model to elementary particles. Özdemir [Ö] has extended the Reissner-Nordström solution by the cosmological constant and has introduced time-dependent parameters. This model can be applied to an expanding or contracting universe with charged dust. Models with collapsing charged matter are the subject of numerous recent investigations. Kristiansson [K], Sonego, and Abramowicz have examined embeddings for different values for M and e, however, they have used the optical metric[85] as starting point. Florides [F] has examined static, spherically symmetric, charged distributions in more general form with the ansatz (IV.28.1) . He has used a function $\mathcal{e}(r)$ for the charge density which is confined to a sphere with the radius r. Paul [P] has used a similar ansatz with three constants which lead to a physically acceptable model if a suitable choice is applied to these constants. Stewart [S] and Ellis have treated the problem of charged fluids with rotational symmetry and have examined cosmological applications as well. Ponce de Leon [P] has designed a model for particles which obtain their mass from the Reissner-Nordström ansatz. Jacob [J] and Piran have embedded the space-like part of the solution into a flat space by using Kruskal-like co-ordinates. Bonnor [B] has deduced an interior solution for the RN-model from the equilibrium configuration of a charged gas sphere of large mass. Lopez [L] has regarded the distribution of a charged gas and dust on the equator disk and at the singular rim in the context of a rigidly rotating Kerr interior and has interpreted the interior solution as an electron. Grøn [G] has deduced this solution once more but under different criteria. Lukacs [L], Newman, Spaling, and Winicourhave found an interior solution with a fluid. Nduka [N] has found a homogeneously charged static spherical interior solution which, however, does not match the exterior RN-solution. Singh [S] and Yadav have examined charged fluid spheres. Papapetrou [P] has generalized the solution of Reissner and Nordström for several charge distributions. Paranjape [P] and Dadhich have embedded patches of the Reissner-Nordström solution into a flat space, including those regions which lie beneath the event horizon and which are excluded by us. Paolino [P] and Pizzi have described two Reissner-Nordström sources. The one is a naked singularity, the other one a black hole. Krori [K] and Barua and Krori [K] and Paul have treated inelastic and elastic charged fluid spheres. Dokuchaev [D] and Chernov have described a shell whose inner region and outer region are electrically charged and have assigned these regions to a black hole. Bailyn [B] has examined a charged shell and its oscillations. e(R) and m(R) are the charge integral and mass integral, quantities which remain undetermined. Bailyn [B] and Eimerl have found a charged interior solution which connects to the exterior RN-solution, wherein the energy density of a negative and a variable remains undetermined. Barceló [B] has examined the self-energy of charged shells. De [D] and Raychaudhuri have described a charged dust model without referring to the RN solution. Efinger [E] has embedded the metric of a charged particle into a 6-dimensional space with two time-like dimensions. The embedded surface has the shape of a cone whose apex is a real singularity. Junevicus [J] has dealt with the model of Krori and Barua. Kramer [K] and Neugebauer have used an ansatz for the stress-energy-momentum tensor similar to the one of Bekenstein. They start with a charge-free solution whereby the charged solution is produced by an invariance transformation. The charge is obtained from the Schwarzschild mass, a free real parameter occurs. The procedure is carried out in isotropic co-ordinates. These have been also written down for the interior solution. Patel [P] and Pandya have obtained a charged interior solution from a specific ansatz for a metric. They have determined the constants of the solution by connecting them to the exterior RN metric. They have tabulated the

[85] The optical metric is used by some authors. One obtains it by dividing the metric by g_{44} . The optical metric is conformal to the original metric.

solution for different values of the mass, of the charge, and of the radius of the star and a further free parameter. Raychaudhuri [R] has examined a class of electrically charged mass distributions which can be static, oscillating, or collapsing. Shi-Chang [S] has found conformally plane solutions for charged, spherically symmetric objects with pressure and mass density which are finite in the entire region. The metric is regular everywhere. Tikekar [T] and Singh have put on a metric in elliptic co-ordinates. The time-like part of the metric is characterized by the shape of the Schwarzschild interior metric. The stress-energy-momentum tensor of an ideal fluid is extended with the electromagnetic stress-energy-momentum tensor. Wang [W] has discovered new solutions for the interior RN field. One of them corresponds, in structure, to the interior Schwarzschild solution. Whitman [W] and Burch have found some charged solutions and have investigated their stability. Heusler [H], Kiefer, and Straumann have calculated the self-energy of a charged shell with the help of the Gauss-Codazzi and Lanczos equations.

Further: Bonnor [B] and Cooperstock; Chandrasekhar [C], Cooperstock [C] and De la Cruz; Hajj-Boutros [H] and Sfeila; Mehra [M], Mehra [M] and Bohra; Mukherjee and Maharaj; Patil [P], Joshi, Nakao and Kimura; Pant [P] and Sah; Patel [P], Tikekar and Sabu; Patel [P] and Koppar; Sharma [S], Tikekar [T].

IV.29. Wormholes

Einstein [E] and Rosen have introduced a new co-ordinate system using

$$u^2 = r - 2M \tag{IV.29.1}$$

and have brought the Schwarzschild metric into a form free from singularities. The value $u = 0$ corresponds to r = 2M, the definition of the event horizon of the Schwarzschild geometry. With this choice of the co-ordinate r is limited to the region $2M \le r \le \infty$. The inner region $0 \le r < 2M$ which is used for the explanation of black holes is thus excluded from the theory from the outset. To recognize the importance of the variable u, we rescale it with $u = R/\sqrt{8M}$. This gives the familiar formula for the Schwarzschild parabola

$$R^2 = 8M(r - 2M) \; . \tag{IV.29.2}$$

Rotating the parabola one gets Flamm's paraboloid. R and r are the Cartesian co-ordinates of the embedding space. Einstein and Rosen admit both roots of (IV.29.2)

$$R = \pm\sqrt{8M(r - 2M)} \; . \tag{IV.29.3}$$

The range of R is $-\infty \le R \le \infty$. Thus, the geometry is mirrored into the 'lower' half-space, Flamm's paraboloid is doubled. The construction has been interpreted as a wormhole. Two distant regions of the universe, or two separate universes are connected by a wormhole. In particular, the two half-spaces are to be concatenated by the Einstein-Rosen bridge. Geometrically interpreted, the bridge is the spherical surface with the radius r = 2M of the throat of Flamm's paraboloid.

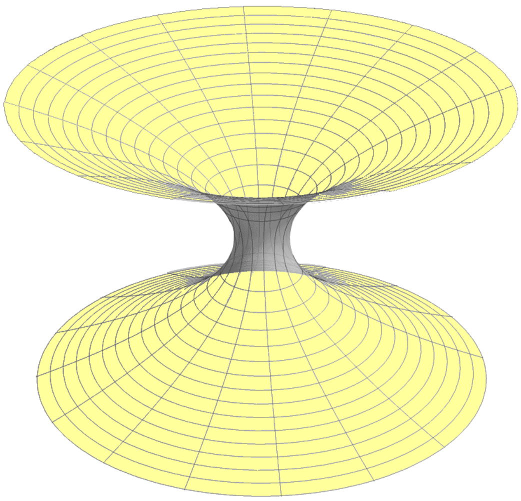

Fig. IV.27

Einstein and Rosen have interpreted their model as a neutral elementary particle theory, whereby the mass of the particle should be concentrated on the surface of the throat. The advantage of the approach should be that an object could be described without the use of a stress-energy-momentum tensor. Einstein and Rosen have extended the model to charged particles, whereby they have obviously fallen back to the already well-known theory of the charged metric of Reissner and Nordström. Since the sign of the electric stress-energy-momentum tensor did not seem to them to be correct, they have briefly referred to an approach in which they have changed the sign of the charge. If they had continued to pursue this approach, they would have found the auxiliary metric (seed metric) of the NUT-model which has been discovered some decades later[86].

From (IV.29.2) one can infer the substitutions for the metric

$$ds^2 = \frac{R^2+16M^2}{16M^2}dR^2 + \left(\frac{R^2+16M^2}{8M}\right)^2 d\Omega^2 - \frac{R^2}{R^2+16M^2}dt^2 \ . \qquad \text{(IV.29.4)}$$

$d\Omega$ is the abbreviation for the polar terms of the metric. R = 0 corresponds to the vertex of the Schwarzschild parabola. dR determines the tangent vector of the parabola at this point, commonly for both branches of the parabola. At this position the redshift factor vanishes. The general expression for the radial tangent vector is

$$dx^1 = \sqrt{\frac{R^2+16M^2}{16M^2}}dR = \frac{1}{\sin\eta}dR, \quad \eta = \eta(R) \ . \qquad \text{(IV.29.5)}$$

For the trigonometric functions we note

$$\sin\eta = \frac{4M}{\sqrt{R^2+16M^2}}, \quad \cos\eta = \frac{R}{\sqrt{R^2+16M^2}}, \quad \tan\eta = \frac{4M}{R} \ . \qquad \text{(IV.29.6)}$$

In contrast to previous Sections we use the angle η with the orientation ccw instead of the angle of ascent ε of the Schwarzschild parabola. From the above relations one gets for the velocity of a freely falling observer

$$v = \sin\varepsilon = -\sin\eta \ . \qquad \text{(IV.29.7)}$$

For R = 0 follows v = -1, the velocity of light. We also note

$$\rho = \frac{\sqrt{\left(R^2+16M^2\right)^3}}{16M^2}, \quad E_1 = -\left(\frac{16M^2}{R^2+16M^2}\right)^{\frac{3}{2}}\frac{1}{R} = -\frac{1}{R}\sin^3\eta \ , \qquad \text{(IV.29.8)}$$

the curvature radius of the Schwarzschild parabola and force of gravity. The latter becomes infinite at the event horizon. These are not new results. They confirm again that the event horizon can only be reached asymptotically. A journey through a wormhole is not possible.

Einstein and Rosen start with modified field equations $g^2R_{ik} = 0$, where g is the determinant of the metric, an approach which does not affect further calculations and has not been pursued in the literature. Treder [T] has faced this approach critically.

Since the quantity M is generally interpreted as a mass, one has not put aside the idea that a wormhole can have a mass. We do not join those who say that the mass can be distributed on a spherical surface. We interpret the quantity M only as a parameter which determines the shape of the wormhole. The assumption that a wormhole could be unstable or even pulsating, is probably due to the fact that one still takes into consideration the inner region of the construction. In this region the radial and time co-ordinate would have changed their meaning. We recall that the co-ordinates of Einstein and Rosen prevent making any statements on this region. Putting the form parameter M of the Schwarzschild parabola in connection with the mass is useful to add to the exterior solution an interior solution, which has a stress-energy-momentum tensor which contains the mass density. The junction condition which matches both solutions justifies the relationship of the quantity M with the mass. If we supplement the wormhole with a

[86] The NUT model will be discussed in later Sections.

(double) interior solution the bridge will be disconnected. Thus, the basic idea of the model gets lost. Therefore we can assume that wormholes are interesting mathematical models, but do not have any significance for our world.

In order to be able to discuss wormholes with mass, several authors have modified the Schwarzschild metric in such a way that the new metric is no longer a vacuum solution. The resulting stress-energy-momentum tensor gives rise to problems because the mass described by it must be located somewhere and the energy conditions should be fulfilled. The latter presupposition has not been satisfied in most of the treated cases. Hence, exotic mass has been assigned to the stress-energy-momentum tensor. Some authors have tried to overcome this problem with quantum mechanical considerations. In all these models a new strategy is applied, concerning the description of the mass. Not an interior solution is added to an exterior, but a geometric structure based on a single metric is used for the regions with mass and without mass. The question, if wormholes are traversible is an important part of recent research.

All these models have an event horizon and a stress-energy-momentum tensor as well. Such structures also occur in models which we discuss later on. The static metrics of these models are used as auxiliary metrics for rotating systems, but could be set into relation to the metrics of the wormholes.

The ansätze describe several types of wormhole:

- Inter-universe wormholes: they combine two universes.
- Intra-universe wormholes: they connect two distant regions of our universe. For open universes one uses a topological identification of the spatial infinite.
- Multi-universes: several universes connected by wormholes.
- Baby universes: grow out of the parent universe. If they have moved away from it, they are connected with the parent universe by an umbilical cord.
- Charged wormholes.
- Rotating wormholes.

This raises the question of what reality may have extra dimensions and how plausible the assumption is that besides our own universe there exist even others. For this purpose we consider gravitational models which have proven accurate and which describe the Nature quite well.

The Schwarzschild model was very extensively treated by us, describes a non-rotating star and is verified by observations. It explains the light deflection by the sun and the perihelion motion of Mercury. The geometry of the Schwarzschild field may, concerning the space-like part, be explained by Flamm's paraboloid. For the sake of clarity reduced to two dimensions, it is also depicted as a 'funnel' in popular science. The upper part is getting flatter and flatter and is running into infinity, but the bottom ends with the circle at the waist which is called event horizon. It connects with the region which is called black hole by many authors. If one wants to involve the time in the embedding theory, one needs a sixth variable. In this simple model it is already apparent that an attempt is difficult to represent the physics as a geometry of a 4-dimensional curved surface.

A rotating star is described by the model of Kerr. We will treat this model in more detail later on. The space-like part of the geometry is again based on a "funnel“, which, however, has an elliptical slice. The physical parameters are not presented in the tangent planes of the surface as it is in the Schwarzschild theory, but in planes which are rotated out of the surface and thus are not tangent to the surface. They are called anholonomic hyperplanes, and they are connected by transport laws. At every step we make, we jump from one hyperplane to another. Our space of experience is described by non-contiguous local planes. Thus, the embedding process is reduced to a mathematical procedure which can be applied very advantageously because its methods are well known and independent of physics. Mathematical statements correspond to physical methods. The mathematics by its content may recommend to physicists new statements for examination. Even if one definitely excludes the possibility of a higher-dimensional embedding space, it is advantageous to maintain the mathematical methods of embedding.

From this perspective the envisaged embedding regarding our universe looks strange. The insight that an embedding of these very models which allow for the possibility of an experimental verification is not possible, devaluates considerably all attempts to understand the origin of our world with the help of a higher-dimensional space. Analyzing this speculative model and considering the space in which there are multiple universes and the cosms created in time and vanishing again, one must be aware that here is being operated with terms such as space and time, which are borrowed from our intuitive space. One should not transfer concepts from our own detectable world to areas which are beyond this world.

We therefore believe that our universe is the only one that was created by vacuum fluctuation from nothing and has expanded rapidly after the Big Bang. If one does not try to bend terms of our intuitive world, one remains in a featureless void that does not allow coexistence of worlds.

Morris [M] and Thorne have provided further models for wormholes and they have investigated the Einstein-Rosen bridge. They have shown that the tidal forces are so strong in the vicinity of the bridge that they would destroy every material object. Brill [B] and Lindquist have studied the Einstein-Rosen bridge of the Reissner-Nordström solution. They have considered several related multiple connected spaces, using isotropic co-ordinates. They have extended the metric in such a way that two particles of equal mass can be described.

We quote papers of some other authors on this topic:

Agnese [A] and Camera; Anchordoqui [A], Torres and Trobo; Anchordoqui [A], Torres, Romero and Andruchow; Armendáriz-Picón [A], Barceló [B], Barceló [B] and Visser; Bochicchio [B] and Faraoni; Borde [B], Trodden and Vachaspati; Cataldo [C], Cataldo [C], Del Campo, Minning and Salgado; Culetu [C],Collas [C] and Klein; Coule [C], Cox [C] and Harms; DeBenedictis [D] and Das; Delgaty [D] and Mann; Ebrahimi [E] and Riazzi; Egorov [E], Kashargin, and Sushkova; Faraoni [F], Faraoni [F] and Israel; Fewster [F] and Roman; Folomeev [F] and Dzuhunuushaliev; Ford [F] and Roman; Frolov [F], Frolov [F] and Novikov; Fuller [F] and Wheeler; Ghoroku [G] and Soma; Guendelman [G] et al, Hochberg [H] , González-Díaz [G], Grib [G] and Pavlov; Hayward [H], Hochberg [H], Hochberg [H] and Visser; James [J], Tunzelman, von Franklin, and Thorne; Kar [K], Kar [K] and Dahdev; Kim [K], Kim [K] and Thorne; Kuhfittig [K], Lemos [L], Lobo and De Oliveira; Lobo [L] and Crawford; Maeda [M], Harada and Carr; Matos [M], Ureña-López and Miranda; Morris [M] , Thorne and Yurtsever; Novikov [N] and Shatinsky, Roman [R], Simeone [S], Sultana [S] and Dryer; Sushkov [S] and Zhang; Tanaka [T] and Hiscock; Torres [T], Anchordoqui and Romero; Torres [T], Romero and Anchordoqui; Visser [V], Visser [V], Kar and Dadhich; Wang [W] and Letelier; Zangeneh [Z] and Riazi.

Having mentioned some fascinating ways how our universe could be designed, we fall back to the wormhole problem. Einstein and Rosen have not been the first to admit both branches of the Schwarzschild parabola for a physical interpretation. It was Weyl [W].

He has formulated the model with isotropic Schwarzschild co-ordinates. Isotropic means that the space-like components of the line element have all the same factors. This causes the non-linear transformation

$$r = \left(1 + \frac{M}{2\bar{r}}\right)^2 \bar{r} \ , \tag{IV.29.9}$$

which brings the Schwarzschild line element into the form

$$ds^2 = \left(1 + \frac{M}{2\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2 d\vartheta^2 + \bar{r}^2 \sin^2\vartheta d\varphi^2\right) - \left(\frac{1 - \frac{M}{2\bar{r}}}{1 + \frac{M}{2\bar{r}}}\right)^2 dt^2 . \tag{IV.29.10}$$

In these relations r is the radial standard Schwarzschild co-ordinate and $\bar{r}$ the new isotropic co-ordinate. The space-like part of the line element is conformal to the Euclidean line element. The factor in (IV.29.9) is the linear enlargement ratio. The circumference at the position $\bar{r}$ is

$$2\pi\left(1 + \frac{M}{2\bar{r}}\right)^2 \bar{r}$$

Weyl has introduced this co-ordinate system, to support his view on the field of a point mass. He has noted that the normal projections of the points of Flamm's paraboloid[87]

$$R = \pm\sqrt{8M(r - 2M)}$$

cover twice the plane R = 0, namely outside the circle $r = 2M$. Within this circle no projections are possible. Weyl did not care for this region. In his day, speculations about black holes were not familiar. He has correctly supposed that the Schwarzschild geometry is defined only up to $r = 2M$. Nevertheless, Weyl was speculative. He assumed that the mass of a 'mass point' is located on the sphere $r = 2M$ and the two halves of the geometry describe the exterior and interior of this mass point. Whether Einstein and Rosen have fallen back to the paper of Weyl cannot be ascertained. Anyhow they have not cited Weyl[88].

The properties of the isotropic co-ordinates should be examined in greater detail. It is clear from the outset that they do not cover the interior Schwarzschild region $0 \le r < 2M$. Thus, a discussion on black holes is excluded from the beginning. This is in accordance with the fact that every physical theory should be independent of the choice of the co-ordinates. From the isotropic co-ordinates one can draw conclusions with respect to the Schwarzschild co-ordinates, viz that one cannot make any statements on the model within the range $0 \le r < 2M$ using these co-ordinates. This supports our view that the circle of the waist of Flamm's paraboloid is the boundary of the geometry.

A look at the line element (IV.29.10) shows that the metric is regular in the whole range $0 < \bar{r} \le \infty$. However, at

$$\bar{r}_H = \frac{M}{2} \tag{IV.29.11}$$

[87] Weyl obviously forgot the second (negative) sign ahead of the root.

[88] Nor have they cited the results of Reissner and Nordström, although they have made use of the charged metric.

the redshift factor is zero. For even smaller $\bar{r}$ the sign of dx^4 changes. Weyl stated that in this region the cosmic time and the proper time have opposite directions.

Inserting (IV.29.11) into (IV.29.9), we see that this value corresponds exactly to the event horizon $r_H = 2M$, viz to the radius of the circle of the waist of Flamm's paraboloid. From (IV.29.9) one gets

$$dr = \left(1 - \frac{M^2}{4\bar{r}^2}\right) d\bar{r}\,, \tag{IV.29.12}$$

the relation between the standard Schwarzschild radial co-ordinate and the isotropic differential. With $dr/d\bar{r} = 0$ it is also apparent that the function $r(\bar{r})$ has at a minimum at $\bar{r}_H$. Within the range $0 \le \bar{r} < \bar{r}_H$ the variable r maintains the values $r > 2M$. If $\bar{r}$ decreases in this region, then the standard variable r increases. In particular, the value $r = \infty$ corresponds to $\bar{r} = 0$. In the range $\bar{r}_H < \bar{r} \le \infty$ both variables r and $\bar{r}$ grow simultaneously. The two regions of the isotropic co-ordinates

$$0 \le \bar{r} < \bar{r}_H, \quad \bar{r}_H < \bar{r} \le \infty \tag{IV.29.13}$$

are described by the metric (IV.29.10). They correspond to the two branches of the Schwarzschild parabola. The isotropic co-ordinates continuously parameterize both branches of the Schwarzschild parabola. They start at the lower branch with the value 0 at infinity, run through the vertex of this parabola at $\bar{r}_H$ and take the value ∞ on the upper branch at infinity. On the lower branch is not only the time retrograde, but also $\bar{r}$. There the tangent vectors of the parabola points inwards, but on the upper branch points outwards. Accordingly, opposite signs occur in the calculation of the field quantities on the lower branch.

Isotropic co-ordinates have displeasing property. They are not able to specify faithful distances. The reason can be found in the non-linear transformation (IV.29.9). If $r(\bar{r})$ is calculated with it one obtains the two roots

$$\bar{r}_+ = \frac{1}{2}\left(r - M + \sqrt{r^2 - 2Mr}\right), \quad \bar{r}_- = \frac{1}{2}\left(r - M - \sqrt{r^2 - 2Mr}\right)\,, \tag{IV.29.14}$$

from which it is apparent that there exist different scales for the two branches. If one carelessly uses isotropic co-ordinates one obtains obscure results for the progression of the field quantities. To correct this deficiency, the scales must be locally gauged [89]. This means more or less falling back on the standard-Schwarzschild co-ordinates. Astrophysicists sometimes use isotropic co-ordinates for their calculations. One should hope that they take the specific features of the isotropic co-ordinates into account.

Fig. IV.28 shows the relations[90] of the co-ordinates.

[89] In drawings which show the progression of the field quantities nonlinear scales have to be applied on the $\bar{r}$ axis, but different ones on both branches.

[90] The Schwarzschild parabola $R^2 = 8M(r - 2M)$ is invariant under the transformation

$$M = 4\bar{M}, \quad r = \left(1 + \frac{2\bar{M}}{\bar{r}}\right)^2 \bar{r}, \quad R = \frac{2\bar{R}^2}{\sqrt{\bar{R}^2 + 16\bar{M}^2}}\ .$$

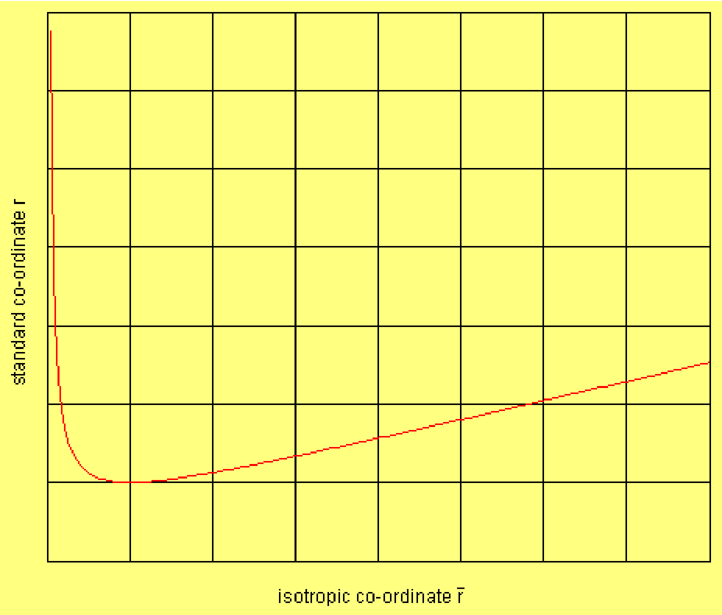

Fig. IV.28

The value of the velocity of a freely falling observer in isotropic co-ordinates

$$v(\bar{r}) = \frac{1}{1+\frac{M}{2\bar{r}}}\sqrt{\frac{2M}{\bar{r}}} \ . \qquad \text{(IV.29.15)}$$

takes at the event horizon the value $v(\bar{r}) = 1$, as expected. That is the highest value which the function $v(\bar{r})$ can obtain. Having passed the position $\bar{r}_H$ it again decreases. In isotropic co-ordinates the field strengths are constructed a little more complicatedly than in standard co-ordinates

$$\begin{aligned} &A_{21}{}^{2} = A_{31}{}^{3} = \frac{1-\frac{M}{2\bar{r}}}{\left(1+\frac{M}{2\bar{r}}\right)^{3}}\frac{1}{\bar{r}}, \quad A_{32}{}^{3} = \frac{1}{\left(1+\frac{M}{2\bar{r}}\right)^{2}}\frac{1}{\bar{r}}\cot\vartheta, \\ &A_{41}{}^{4} = \frac{1}{\left(1+\frac{M}{2\bar{r}}\right)^{3}\left(1-\frac{M}{2\bar{r}}\right)}\frac{M}{\bar{r}^{2}} = -E_{1} \end{aligned} \ . \qquad \text{(IV.29.16)}$$

They satisfy the vacuum field equations. However, they can be simply converted into the well-known expressions of the Schwarzschild theory. In particular, we obtain the curvature radius in isotropic co-ordinates

$$\rho = \left(1+\frac{M}{2\bar{r}}\right)^{3}\sqrt{\frac{2\bar{r}^{3}}{M}} \qquad \text{(IV.29.17)}$$

The $\{R, r\}$ are the Cartesian co-ordinates of the embedding space. The $\{\bar{R}, \bar{r}\}$ can also be interpreted as Cartesian co-ordinates, if a position-dependent gauge of the scales is used in order to measure geometric quantities. The event horizon is at all times $\bar{r}_H = 2\bar{M}$.

and the force of gravity in the well-known form

$$E_1 = -\frac{1}{\rho}\tan\eta, \quad \eta = \eta(\bar{r}) \ , \tag{IV.29.18}$$

wherein $\tan\eta(\bar{r})$ is calculated from $\sin\eta(\bar{r}) = v(\bar{r})$. In addition, the Kretschmann scalar should be discussed. It is used by many authors to ensure that the geometry is well behaved in a certain region and that it has no singularity. In particular, the Kretschmann scalar is calculated in the inner region of the Schwarzschild geometry. Except at r = 0 it is regular. Thus, one concludes that the inner region of the Schwarzschild model is physically interpretable.

With respect to our view (III) we do not accept such arguments. It can be shown that the Kretschmann scalar also provides values where no surface exists, if one uses less appropriate variables or if one ignores the range of such variables. If one representatively takes a non-vanishing component of the Riemann tensor

$$R^{12}{}_{12} = \frac{M}{r^3} \ , \tag{IV.29.19}$$

it seems to be valid in the whole region $0 < r \le \infty$. However, if one replaces r by the curvature radius of the Schwarzschild parabola

$$R^{12}{}_{12} = \frac{2}{\rho^2} \ . \tag{IV.29.20}$$

it becomes clear that the Riemann tensor can only be valid up to the event horizon. There the curvature radius has its smallest value, namely 4M. Thus, one has for Kretschmann scalar

$$K = R^{mnsr}R_{mnsr} = \frac{48M^2}{r^6} = \frac{192}{\rho^4} \ , \tag{IV.29.21}$$

which has its highest value at the event horizon. A strong support for this argument is obtained by use of isotropic co-ordinates. Thus, the Kretschmann scalar has the form

$$K = R^{mnsr}R_{mnsr} = \frac{48M^2}{\bar{r}^6}\frac{1}{\left(1+\frac{M}{2\bar{r}}\right)^{12}} \ . \tag{IV.29.22}$$

It cannot describe the inner region of the Schwarzschild model, because the co-ordinate $\bar{r}$ is not defined in this region. Now, if one understandably requires that a co-ordinate transformation cannot change the geometrical and physical content of a model, it must be concluded that the Kretschmann scalar does not exist in the inner region in standard co-ordinates as well.

The isotropic co-ordinates make it quite easy to describe the Einstein-Rosen bridge. Brill [B] and Lindquist have discussed multiply connected spaces. Using isotropic co-ordinates they have written down line elements which describe a space with several Einstein-Rosen bridges. For a solution with two bridges they have specified the spatial part of the metric by

$$ds^2 = \left(1 + \frac{M}{2\bar{r}_1} + \frac{M}{2\bar{r}_2}\right)\left(d\bar{r}^2 + \bar{r}^2 d\vartheta^2 + \bar{r}^2\sin^2\vartheta d\varphi^2\right) \ . \tag{IV.29.23}$$

Popławski [P] has once more renewed the idea of Weyl. He has identified the region $\bar{r}_H < \bar{r} \leq \infty$ with the exterior field of a black hole and the region $0 \leq \bar{r} < \bar{r}_H$ with the inside of the black hole. With the co-ordinate transformation

$$\bar{r}' = \frac{M}{4\bar{r}} \ , \tag{IV.29.24}$$

which leaves invariant the metric (IV.29.10), he has repaired the retrograde behavior of the radial co-ordinate and has identified the lower part of Flamm's paraboloid with a white hole[91]. Concerning these considerations it is overlooked that the 'inner region' $0 \leq \bar{r} < \bar{r}_H$, despite its suggestive form, is an infinitely large region, because $\bar{r} = 0$ is situated at infinity. Such a region cannot be identified with a stellar object. In one of our papers [B] we have treated this problem.

Appended are a few articles in which the isotropic co-ordinate system is used.

Abrams [A], Brügmann [B], Buchdahl [B]. Crothers [C] has dealt extensively with the length measurement. FuJun Wang [F], YuanXing Gui, and ChunRui Ma; Hajj-Boutros [H], Hayashi [H], Ibison [I] and Shirafuji; Kuchowicz [K], Linet [L], O'Connell [O], Pappas [P] and Apostolatos, Petrich [P], Shapiro and Teukolsky; Rudmin [R].

[91] A stellar object that continuously emits mass.

V. Rotating models

V.1. Rotation

The problem of a rotating system occupies an important position in both the classical and relativistic physics, and in gravitation physics. Practically all stellar objects perform an axial rotation which leads to the occurrence of forces. Concerning their interpretation in the context of the relativity principle there were detailed discussions in the past century which have continued into the present epoch. A contribution has to be made by us as well, beginning with fundamental considerations. Starting with the classical rotation problem a first step is to be made towards rotating gravitational models. Some models are dealt only briefly with. However, the rotating Gödel cosmos and the Kerr metric are discussed in detail.

The concept of relative velocities and the associated consequences have caused profound progress in the field of physics. A further generalization of the theory of relativity with respect to non-constant velocities would call into question the measurability of absolute acceleration. In the conventional formulation such a principle is not distinctly included in the general relativity theory. Foremost Treder [T] has shown that a measuring convention for the acceleration can be articulated in the context of the general relativity theory by means of a careful construction. Treder's concept is mathematically reflected by the use of tetrads which represent rods and clocks.

Two observations essentially seem to contradict the relativity principle, generalized in the above way: Newton's bucket experiment and the Sagnac experiment. In the rotating bucket a peripheral rise of the water level arises, caused by centrifugal forces which are a consequence of the rotation of the bucket. Thus, the rotation would have to be an absolute motion, because such an effect cannot be observed in non-rotating systems. In contrast, a more general view is possible. The acceleration arising during the rotational motion and the associated forces would raise the same forces if the bucket were in rest and the space rotated itself around the bucket. Thirring [T] showed that all the stars rotating around the bucket can cause the centrifugal and Coriolis forces. Without doubt, these forces are too weak to be responsible for those effects at the bucket. However, a representation of the Einstein field equations by tetrads shows that nonlinear field terms are present which have to be interpreted physically as gravitation energy and energy current and have to be contrasted equivalently to the rotating masses of the universe. Those energies exercise forces of the necessary strength on the system in the center of the rotation. In the course of the following mathematical investigations we will generally put aside the stars of the universe. If we use tetrads the field equations of the gravitation take a Maxwell-like form, as is already familiar from the above-examined models. The physical principles which are described by them, remind us of the theory of electromagnetism[92]: The gravitation energy corresponds to a negative field mass which acts repulsively and causes centrifugal forces. The flow of the gravitational energy (Poynting vector) produces the Coriolis dipole field. The non-linearity of the Einstein field equations is responsible for the back-reacting field mechanism and leads to a generalization of Mach's principle (Hönl [H] and Dehnen). The distribution of the energy and momentum density of the gravitational field alone is sufficient to determine the forces of inertia in any reference system.

The optical experiment of Sagnac [S], the pedant to the Michelson experiment, apparently speaks in favor of the absoluteness of rotation: Rays of light orbiting in the

[92] Therefore some authors call it gravito-magnetism.

opposite directions on a platform produce interferences, if a mirror system is adjusted in such a way that the rays meet after a circulation. If the platform is set in rotation, the interference fringes are shifted. The shift is a function of the angular velocity of the platform. As the reason one has accepted that the speeds of light are different in the opposite directions (Langevin [L], Post [P]) and, thus, refer to the absoluteness of the rotating motion. Only in the rest system both light rays have the same speed.

Apart from this special experimental arrangement one could argue that an observer, in constant motion who deviates occasionally from its straight-line course changes his physical principles permanently: The principle of constancy of the velocity of light is only valid for the linear sections, any deviation from them leads to a violation of the constancy principle. This possible consequence of the Sagnac experiment is not only unsatisfactory, but contradicts the general relativity principles as well. Therefore we look for another interpretation of the shift of the fringes observed in the Sagnac interferometer: The experimental arrangement on the rotating platform can be considered to be in relative rest, however, gravitational energy orbits the platform. The arising forces lead to an extension and to a shortening of the optical paths and to changes in the physical flow of time. Light rays moving in opposite directions cover different optical distances, whereby the measurable fringe shifts emerge. However, they require a different time. The quotients of these distances, the velocities of the light rays are equal. Therefore the constancy principle is also valid for accelerated reference systems and one can show that it are the very gravitational forces which affect Newton's bucket and the optical paths of the Sagnac experiment. In opposition to the popular opinion at his times it was pointed out by Reissner [R] already in 1914 that the acceleration could have no absolute meaning. He followed Mach's principle.

In the first decades after the discovery of the Sagnac effect one was convinced that the constancy principle is not valid for rotating systems. Today, most authors are of the opposite opinion whereby that problem area is treated in rather different ways. Malykin [M] summarized all the proposals for the explanation of the Sagnac effect and listed 290 quotations[93]. He arranged them according to correct, conditionally correct, and incorrect explanations. Because of this broad literature it is very difficult to determine who was the first to bring in the idea that the constancy principle is valid for rotating systems as well. Corum [C] might have been first to approach the problem. With strictly relativistic methods he consequently has used those anholonomic systems which we likewise prefer. However, he did not mention the constancy principle although it directly follows from his ansatz. We made up this in one of our articles [B].

In order to mathematically analyze the before-mentioned mechanisms it would be expedient to examine a rotating solution of the Einstein field equations. We withdraw from the variety of the well-known rotating models by taking in hand the simplest system: a rotating observer field in the flat space. Because of the missing central mass the Riemann curvature tensor vanishes. This model cannot be added either to the special relativity theory, because it permits non-inertial observer systems, or to the general relativity theory, because fields are closely linked with the space-time curvature in this framework. The flat problem is the field-theoretical formulation of the classical rotation problem and shows the substantial structures of a rotating gravitational model. Because of its simplicity the effects of the forces on an experimental assembly can be described without special mathematical expenditure. Some genuine gravitation models reduce for $M = 0$ to this system. Thus, this simplification is quite justified. At first, however, we take the historical road and we base our investigations on the flat line element in spherical co-ordinates:

[93] Therefore a listing in the bibliography can be avoided.

$$ds^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 + \left(dx^4\right)^2. \tag{V.1.1}$$

Having performed the transformation

$$\varphi \to \varphi + \omega t \tag{V.1.2}$$

the line element takes the form

$$ds^2 = dr^2 + r^2 d\vartheta^2 + \sigma^2 d\varphi^2 - 2i\omega\sigma^2 d\varphi\, dx^4 + \left(1-\omega^2\sigma^2\right)\left(dx^4\right)^2 \tag{V.1.3}$$

with the cross term $2g_{34}dx^3dx^4$ which is characteristic for a rotating model. For the radius of a circle parallel with respect to the equatorial plane the abbreviation

$$\sigma = r\sin\vartheta \tag{V.1.4}$$

is used by us. The metric (V.1.3) will describe a *rigid* rotator, the angular velocity ω is constant. This has as a consequence that the orbital velocity of outer parts of the system is faster than the one of the inner parts. At a certain radius, the *cut-off radius* reaches the orbital speed

$$v_{orb} = \omega\sigma\ , \tag{V.1.5}$$

the velocity of light[94]. One would arrive at a more realistic model if one introduced a *differential rotation law* with outward decreasing angular velocity, so that the orbital speed never exceeds the speed of light. One has to abandon the conception of a rotating compact object. Attempts to improve rotating models have been made. However, a functional model is only supplied by the Kerr metric.

Ehrenfest [E] has criticized this ansatz in a short note 1909. In accordance with the relativistic length-contraction at the edge of a rotating disk the ratio between distance and radius is not any more 2π. It was conjectured that a rotating disk has to be understood only with the methods of Riemannian geometry. Also stresses have been computed which could arise at the material of a rotating object.

If one wanted to be unfriendly, one could regard the metric (V.1.3) as metric of the static system (V.1.1) in rotating co-ordinates. The metric (V.1.3) emerges from the static metric by the co-ordinate transformation (V.1.2). The cross term shows that the new co-ordinate system is oblique-angled. Such co-ordinate transformations can only change the way of description, but cannot change the physical circumstances. We will follow the historical way by assuming that physical statements can be obtained if they are related to a *reference system*. Such a system can be assigned to the metric (V.1.3) in two ways

$$\left.\begin{array}{lllll} \overset{1}{e}_1 = 1, & \overset{2}{e}_2 = r, & \overset{3}{e}_3 = \alpha\sigma, & \overset{4}{e}_3 = -i\alpha\omega\sigma^2, & \overset{4}{e}_4 = \dfrac{1}{\alpha} \\ \underset{1}{e}^1 = 1, & \underset{2}{e}^2 = \dfrac{1}{r}, & \underset{3}{e}^3 = \dfrac{1}{\alpha\sigma}, & \underset{3}{e}^4 = i\alpha\omega\sigma, & \underset{4}{e}^4 = \alpha \end{array}\right\}\ \mathrm{A}\ , \tag{V.1.6}$$

$$\left.\begin{array}{lllll} \overset{1}{e}_1 = 1, & \overset{2}{e}_2 = r, & \overset{3}{e}_3 = \sigma, & \overset{3}{e}_4 = -i\omega\sigma, & \overset{4}{e}_4 = 1 \\ \underset{1}{e}^1 = 1, & \underset{2}{e}^2 = \dfrac{1}{r}, & \underset{3}{e}^3 = \dfrac{1}{\sigma}, & \underset{4}{e}^3 = i\omega, & \underset{4}{e}^4 = 1 \end{array}\right\}\ \mathrm{B} \tag{V.1.7}$$

[94] With the same problem one is confronted with the rotating Gödel cosmos, as will be discussed later on.

wherein

$$\alpha=\frac{1}{\sqrt{1-\omega^2\sigma^2}}\,. \tag{V.1.8}$$

The associated metrics

$$\begin{aligned}
ds^2 &= dr^2+r^2d\vartheta^2+\alpha^2\sigma^2d\varphi^2+\left(-i\alpha\omega\sigma^2d\varphi+\frac{1}{\alpha}dx^4\right)^2 && \text{(A)}\\
ds^2 &= dr^2+r^2d\vartheta^2+\left(\sigma d\varphi-i\alpha\omega\sigma dx^4\right)^2+\left(dx^4\right)^2 && \text{(B)}
\end{aligned} \tag{V.1.9}$$

lead after a short calculation again to (V.1.3).

With the methods of the preceding Sections one can derive forces from the system (A) which correspond to the Coriolis and the centrifugal forces of classical mechanics. However, the system (B) is free from forces. Thus, (A) can be regarded as a rotating system and (B) as a system in rest. We continue to compute the forces from both systems. We recognize that the system (A) can be derived from the system (B) by a generalized Lorentz transformation[95] with the Lorentz factor (V.1.8)

$$L_3^{3'}=\alpha,\quad L_4^{3'}=i\alpha\omega\sigma,\quad L_3^{4'}=-i\alpha\omega\sigma,\quad L_4^{4'}=\alpha\ . \tag{V.1.10}$$

The primed indices mark the system (A). Hence it follows that our first suspicion has been confirmed and that the co-ordinate transformation (V.1.2) is redundant. Moreover, Ehrenfest's paradox is solved by this relativistic treatment. For the rotating observer the same considerations are valid as for an observer who is in a translational motion. The different views on the lengths are based on different space-time world lines of the observers. Nobody would assume that a moving rod obtains Riemannian properties or that stresses arise due to the Lorentz contraction. The just-addressed world line is a pseudo helix with respect to the rotating observer. The baseline of this helix is the circular path of the observer, experienced by an observer in rest. After a circulation the starting and ending points of the motion are dislocated. The offset (the pitch of the helix) is proportional to the time elapsed measured by the observer in rest. The length of the path[96] on the pseudo helix differs from the one of the base circle. The time intervals are not integrable, i.e. no global time-like lines can be found. The local (infinitesimal) time vector is orthogonal to the helix. These are exactly the considerations which can be consulted for the interpretation of the Sagnac effect as well. Examining the Gödel cosmos we will deal in greater detail with the geometrical basics.

The metric (V.1.1) refers to the static system and the Lorentz transformation (V.1.10) leads to the rotating reference system

$$\left.\begin{aligned}
&\overset{1}{e}_1=1,\quad \overset{2}{e}_2=r,\quad \overset{3}{e}_3=\alpha\sigma,\quad \overset{4}{e}_3=-i\alpha\omega\sigma^2,\quad \overset{3}{e}_4=\quad i\alpha\omega\sigma,\ \overset{4}{e}_4=\alpha\\
&\underset{1}{e}^1=1,\quad \underset{2}{e}^2=\frac{1}{r},\quad \underset{3}{e}^3=\frac{\alpha}{\sigma},\quad \underset{3}{e}^4=\ i\alpha\omega\sigma,\quad \underset{4}{e}^3=\ -i\alpha\omega,\quad \underset{4}{e}^4=\alpha
\end{aligned}\right\}\ A\ , \tag{V.1.11}$$

while the static starting system has the components

[95] Generalized in the sense that the parameters of the Lorentz transformation are position-dependent. Here the orbital speed $\omega r\sin\vartheta$ depends on the radius σ, finally on r and ϑ

[96] A visualization of the circumstances by means of a pseudo-real helix may be helpful, however, it does not show up the correct lengths ratios.

$$\left.\begin{array}{llll} \overset{1}{e}_1 = 1, & \overset{2}{e}_2 = r, & \overset{3}{e}_3 = \sigma, & \overset{4}{e}_4 = 1 \\ \underset{1}{e}^1 = 1, & \underset{2}{e}^2 = \dfrac{1}{r}, & \underset{3}{e}^3 = \dfrac{1}{\sigma}, & \underset{4}{e}^4 = 1 \end{array}\right\} \quad B\ . \tag{V.1.12}$$

The rotating system (A) which we obtain from (B) we call *attached*. The rotational effects are produced by a Lorentz transformation and can be removed by such a transformation. Thus, this model differs from a genuine gravitation model. For the latter the gravitational effects can only be produced by a non-Lorentzian transformation from a static auxiliary metric (seed metric).

First the static system is examined. The connexion coefficients

$$A_{mn}{}^{s} = B_{mn}{}^{s} + C_{mn}{}^{s}\ , \tag{V.1.13}$$

which can be derived from the system (B) enable a parallel transport which is adapted to the spherical[97] properties of the system. With the auxiliary variables

$$\sigma_m = \sigma_{|m} = \{\sin\vartheta, \cos\vartheta, 0, 0\} \tag{V.1.14}$$

one finally has

$$B_{mn}{}^{s} = b_m B_n b^s - b_m b_n B^s, \quad C_{mn}{}^{s} = c_m C_n c^s - c_m c_n C^s \tag{V.1.15}$$

by using

$$B_m = \left\{\frac{1}{r}, 0, 0, 0\right\}, \quad C_m = \left\{\frac{1}{\sigma}\sigma_1, \frac{1}{\sigma}\sigma_2, 0, 0\right\}\ . \tag{V.1.16}$$

The ‘field equations’

$$B_{\underset{2}{m||n}} + B_m B_n = 0, \quad C_{\underset{3}{m||n}} + C_m C_n = 0 \tag{V.1.17}$$

entirely describe the geometrical properties of the system. They are the curvature equations of the spherical system. With the help of the Lorentz transformation (V.1.10), the transformation law

$$A_{m'n'}{}^{s'} = L^{m\,n\,s'}_{m'n's} A_{mn}{}^{s} + L^{s'}_{s} L^{s}_{n'|m'}\ , \tag{V.1.18}$$

and with the above equations one obtains the connexion coefficients for the rotating system.

Likewise, one could deduce them from the reference system (A) with the help of the already-known methods. Since the Lorentz transformation does not have any influence on the geometry, the equations (V.1.15) and (V.1.16) remain unchanged. Dynamic terms occur in

$$\begin{array}{c} F_{mn}{}^{s} = c_m F_n c^s - c_m c_n F^s, \quad E_{mn}{}^{s} = -\left[u_m F_n u^s - u_m u_n F^s\right] \\ H_{mn}{}^{s} = H_{mn} u^s + H^s{}_m u_n + H^s{}_n u_m \end{array}\ . \tag{V.1.19}$$

Therein are

$$H_{mn} = 2i\alpha^2\omega\sigma_{[m}c_{n]}, \quad F_m = \alpha^2\omega^2\sigma\sigma_m \tag{V.1.20}$$

[97] Here it would be more appropriate to use a cylindrical system. However, with respect to the gravitational models to be treated, a spherical co-ordinate system is already used.

the relativistic generalizations of the Coriolis and centrifugal forces. As an auxiliary formula we use

$$\alpha^2 C_m = C_m + F_m \ . \tag{V.1.21}$$

The Ricci-rotation coefficients have the form

$$A_{mn}{}^s = B_{mn}{}^s + C_{mn}{}^s + F_{mn}{}^s + H_{mn}{}^s + E_{mn}{}^s \ . \tag{V.1.22}$$

The quantities (V.1.20) are summarized to

$$F_{mn} = H_{mn} + F_{[m}u_{n]} \ . \tag{V.1.23}$$

By the use of the auxiliary formulae

$$H_{m3}H_{n3} = -\left(C_m + F_m\right)F_n, \quad H^{sr}H_{sr} = 2H^{s3}H_{s3} \tag{V.1.24}$$

one obtains the field equations

$$\begin{aligned} R_{mn} = &-\left[B_{n\underset{2}{||}m} + B_nB_m\right] - b_nb_m\left[B^s{}_{\underset{2}{||}s} + B^sB_s\right] \\ &-\left[C_{n\underset{3}{||}m} + C_nC_m\right] - c_nc_m\left[C^s{}_{\underset{3}{||}s} + C^sC_s\right] \\ &-c_nc_m\left[F^s{}_{\underset{4}{||}s} - F^sF_s + H^{sr}H_{sr}\right] + u_nu_m\left[F^s{}_{\underset{4}{||}s} - F^sF_s + H^{sr}H_{sr}\right] \\ &+2u_{(n}\left(H^s{}_{m)\underset{4}{||}s} - 2H^s{}_{m)}F_s\right) \end{aligned} \ . \tag{V.1.25}$$

The 3-dimensional purely spatial covariant derivative is defined by

$$\Phi_{m\underset{4}{||}n} = \Phi_{m|n} - \left(B_{nm}{}^s + C_{nm}{}^s + F_{nm}{}^s\right)\Phi_s \ . \tag{V.1.26}$$

The first two lines of (V.1.25) are the geometrical contributions of the spherical reference system in the flat space and are satisfied with (V.1.17). The dynamical parts in the field equations

$$R_{m'n'} = L^{m\,n}_{m'n'}R_{mn} \equiv 0 \tag{V.1.27}$$

decouple due to the Lorentz invariance of the Ricci tensor. For the centrifugal and the Coriolis force one has

$$F^s{}_{\underset{4}{||}s} - F^sF_s + H^{sr}H_{sr} = 0, \quad H^s{}_{m\underset{4}{||}s} - 2H^s{}_mF_s = 0 \ . \tag{V.1.28}$$

With

$$A_{[mn]}{}^4 = F_{mn} \tag{V.1.29}$$

one obtains from the symmetry properties of the Riemann tensor

$$R_{[mns]}{}^4 = 0 \tag{V.1.30}$$

the relation

$$F_{[mn||s]} = 0 \tag{V.1.31}$$

which splits into the two equations

$$H_{[mn\underset{4}{||}s]} = 0, \quad F_{[m\underset{4}{||}n]} = 0 \ . \tag{V.1.32}$$

With the quantity

$$H_s = -\frac{i}{2}\varepsilon_s{}^{mn}H_{mn} = \{H\cos\vartheta, -H\sin\vartheta\}, \quad H = \alpha^2\omega \tag{V.1.33}$$

dual to the Coriolis field strength the above-derived equations can be expressed in a symbolic way of writing so that the similarity to the electrodynamics emerges more clearly

$$\begin{aligned} &\mathrm{div}\vec{F} = \vec{F}^2 + 2\vec{H}^2, \quad \mathrm{rot}\vec{F} = 0 \\ &\mathrm{div}\vec{H} = 0, \qquad \mathrm{rot}\vec{H} = 2\vec{H}\times\vec{F} \end{aligned} . \tag{V.1.34}$$

Hund [H] has set up the equations for a slowly rotating system in an innovative work. For $\alpha \to 1$ one has

$$\vec{F} = \omega^2\sigma\vec{\sigma}, \quad \vec{H} = \vec{\omega}, \quad \mathrm{div}\vec{F} = 2\omega^2, \quad \mathrm{rot}\vec{H} = 0 \ . \tag{V.1.35}$$

In Newton's gravitation law

$$\mathrm{div}\,\vec{g} = -4\pi k\mu \ ,$$

with k as Newton's constant of gravitation is μ the density of that mass distribution which causes the force of gravity $\vec{g}$. If one compares this with the third relation of (V.1.35), then $2\omega^2$ corresponds to a negative mass density

$$\mu = -\frac{\omega^2}{2\pi k} \ , \tag{V.1.36}$$

acting repulsively. It is the cause of the centrifugal force. With a revolution time of 10 seconds it corresponds to the mass density of a compact white dwarf. The third equation (V.1.35) shows that the Coriolis field is the source of the centrifugal force. In the relativistic case (V.1.34) the centrifugal force itself joins the source. The centrifugal force is coupled to itself. This is a direct consequence of the non-linearity of the Einstein field equation. The mass density and also the energy density are up to a factor c^2. It only arises in the rotating system and, thus, depends on the motion of the observer.

The expression $2\vec{H}\times\vec{F}$ corresponds to the Poynting vector of the electrodynamics. The Poynting vector and the energy density of the field are conserved

$$\mathrm{div}\left(2\vec{H}\times\vec{F}\right) = 0, \quad \left(\vec{F}^2 + 2\vec{H}^2\right)^{\cdot} = 0 \ . \tag{V.1.37}$$

One verifies the first formula with (V.1.34). The second one is trivially satisfied, since a stationary[98] system was considered. Due to this analogy to the electrodynamics one could be tempted to establish a stress-energy tensor which is similar to the electromagnetic one. After all the Coriolis and centrifugal forces correspond to the magnetic and electrical field strengths. If one reads the quadratic terms from the physical contributions of the field equations (V.1.25) one arrives at the trace-free tensor

$$t_{mn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & F^sF_s - H^{rs}H_{rs} & -2H^s{}_3F_s \\ 0 & 0 & -2H^s{}_3F_s & H^{rs}H_{rs} - F^sF_s \end{pmatrix} . \tag{V.1.38}$$

A covariant conservation law would read as

[98] Rotating, however, time-independent

$$t^{mn}{}_{||n} = 0 \ . \qquad \text{(V.1.39)}$$

With

$$s^m = u_n t^{mn} \qquad \text{(V.1.40)}$$

one forms the 4-energy-current density. Since the second term of the right site of

$$u_m t^{mn}{}_{||n} = s^n{}_{||n} - t^{mn} u_{m||n} = s^n{}_{||n} + t^{mn} H_{nm} - t^{\alpha 4} F_\alpha$$

vanishes due to the symmetry of t and due to $t^{\alpha 4} F_\alpha = 0$ (α =1,2)

$$s^n{}_{||n} = 0 \qquad \text{(V.1.41)}$$

remains, a relation which covers both formulae (V.1.37). The stresses in the φ-direction (t_{33}), for which the two fields H and F are responsible, do not satisfy the conservation law (V.1.39). Despite a certain analogy to the electrodynamics its principles are not fully adaptable to gravitational models or, as in this case, to gravitation-like models. In contrast, this is not to be expected. Finally, the Maxwell equations are linear, the Einstein field equations are quadratic. The electromagnetic fields are coupled to charges and currents, the fields of this model and also the fields of genuine gravitational models are coupled to themselves. Thus, back-reactions result. The situation becomes still more difficult because the interior solutions or the charged solutions in addition contain the matter tensor and the electromagnetic stress-energy tensor, respectively. Latter quantities are covariantly conserved. A conservation law which also includes the field contributions of gravitation could not be found to date. The Einstein stress-energy complex and Møller's expression and some proposals of other authors are regarded to be unphysical and not to be consistent. The definition of stress, energy, and momentum of the gravitational field by a co-ordinate invariant expression and their conservation is an unsolved problem of gravitational physics.

At last, the equations of motion of a particle with the proper mass m_0, the relativistic mass $m = m_0 \big/ \sqrt{1-v^2}$, and the 4-velocity

$$w_m = \frac{1}{\sqrt{1-v^2}} \{-iv_\alpha, 1\}, \quad v^2 = v^\alpha v_\alpha \qquad \text{(V.1.42)}$$

are examined. If one exposes such a particle or a particle field to the rotating system, and if one splits the equation of motion

$$w^n w_{m||n} = 0 \qquad \text{(V.1.43)}$$

into space-like and time-like components

$$w^\beta w_{\underset{4}{m||n}} + w^4 w_{\underset{4}{m||4}} = A_{\beta m}{}^4 w^\beta w_4 + A_{4m}{}^\beta w^4 w_\beta + A_{4m}{}^4 w^4 w_4 \ ,$$

and multiplies them by the proper mass m_0, and if one divides by the Lorenz factor, one recognizes with

$$\begin{aligned} (\vec{v}\,\mathrm{grad})\vec{p} + p^\bullet &= 2m\left(\vec{v} \times \vec{H}\right) + m\vec{F} \\ (\vec{v}\,\mathrm{grad})m + m^\bullet &= m\vec{F}\vec{v}, \quad \vec{p} = m\vec{v} \end{aligned} \qquad \text{(V.1.44)}$$

the effect of the forces on the particle. Relations like these have been deduced by Dehnen [D] in more general form.

Finally, we address Sagnac's experiment. For the interpretation of Sagnac's experiment we must make use of the nonholonomicity of the observer system A, Eq. (V.1.6). The non-commutativity of the partial derivatives in the Lorentz space results in the relation

$$\Phi_{|[mn]} = S_{mn}{}^{r}\Phi_{|r}, \quad S_{mn}{}^{r} = -A_{[mn]}{}^{r}. \tag{V.1.45}$$

The Lorentz components of the position vector are not integrable:

$$\Delta x^{r} = \iint d \wedge dx^{r} = 2\iint S_{mn}{}^{r} dx^{m} \wedge dx^{n}. \tag{V.1.46}$$

It is sufficient to specify the components of S to a first approximation, whereby we temporarily abandon the natural unit system and replace ω with ω/c. Then the Lorentz factor is $\alpha \approx 1$ and

$$2S_{13}{}^{3} = S_1 + F_1 \approx \frac{1}{r}, \quad 2S_{14}{}^{3} = 0, \quad 2S_{14}{}^{4} = -E_1, \quad 2S_{13}{}^{4} = -2H_{13} \approx -2i\omega/c. \tag{V.1.47}$$

Following the path around the Sagnac disc in the direction of rotation, one has

$$\Delta x_1^3 = 2\iint \left[S_{13}{}^{3} dx^1 dx^3 + S_{14}{}^{3} dx^1 dx^4 \right].$$

Going back to the system at rest and finally using for integration the holonomic coordinates (i) of the Einstein space one obtains

$$\Delta x_1^3 = 2\pi r - \omega r t$$

and for the path in the opposite direction of the rotation

$$\Delta x_2^3 = 2\pi r + \omega r t.$$

The difference between the paths with different lengths is

$$\Delta s = -2\omega r t. \tag{V.1.48}$$

In the course of time

$$\Delta x_1^4 = 2\iint \left[S_{13}{}^{4} dx^1 dx^3 + S_{14}{}^{4} dx^1 dx^4 \right] = -2i \iint \frac{\omega r}{c} dr \left(d\varphi - \omega t \right)$$

different intervals arise depending on the sense of rotation (A is the circulated area)

$$\Delta t_1 = -\frac{\omega}{c^2}\int r^2 d\varphi = -2\frac{\omega A}{c^2}, \quad \Delta t_2 = 2\frac{\omega A}{c^2},$$

if one returns to the starting point and performs a time comparison with a residual clock, so that one has

$$\Delta t = \frac{4\omega A}{c^2} \tag{V.1.49}$$

in compliance to Corum. But we do not follow Corum, who reads from (V.1.49) a rotation-related change in the frequency of light. Frequency distortions would lead to disintegration of the fringe pattern on the interferometer.

A light beam of the wavelength λ and the oscillation period T covers the path $n\lambda$ on the platform at rest and needs the time $nT = t$, so that

$$n\lambda = cnT = ct. \tag{V.1.50}$$

On the rotating plate

$$\Delta n\lambda = c\Delta t$$

applies to the two counter-rotating beams with constant c, and if one takes advantage of (V.1.49) Sagnac's formula for the phase shift reads as

$$\Delta n = \frac{4\omega A}{\lambda c}. \tag{V.1.51}$$

It has no effect on the experiment whether we consider the platform as rotating or at rest. In both cases, it is the forces (V.1.47) that result from a relative acceleration and cause the fringe shift. The principle of constancy of the velocity of light is fully applicable. From the metric (V.1.11) one takes for $ds^2 = 0$

$$\left[\overset{3}{e}_3\, dx^3\right]^2 + \left[\overset{4}{e}_3\, dx^3 + \overset{4}{e}_4\, dx^4\right]^2 = 0$$

while maintaining the system A. If one defines the velocity term stricly in the Lorentz space, then the coordinate invariant relation

$$\frac{dx^{\hat{3}}}{dx^{\hat{4}}} = \pm i, \quad \frac{dx^{\hat{3}}}{d\tau} = \pm c \tag{V.1.52}$$

applies to the circulating light beam with

$$dx^m = \overset{m}{e}_i dx^i$$

in accordance with the principle of constancy.

We add some quotations concerning the Sagnac effect:

Anandan [A], Anderson [A], Arditty [A] and Lefèvre; Ashtekar [A] and Magnon; Bashkov [B] and Malakhaltsev; Bergia [B] and Guidone; Bini [B], Jantzen and Mashhoon; Browne [B], Burton [B], Cantoni [C], Cheo [C] and Heer; Chiu [C], Chow [C], Gea-Banacloche, and Pedrotti; Ciufolini [C], Cohen [C] and Moses; Croca [C], Davies [D] , Davies [D] and Ashworth; Davies [D] and Jennison; Dresden [D] and Yang; Franklin [F], Forder [F], Gogberashvili [G], Goy [G] and Selleri; Grøn [G], Gustavson [G], Landragin, and Kasevich; Hasselbach [H] and Nicklaus; Henriksen [H] and Nelson; Hill [H], Kajari [K], Walser, Schleich and Delgado; Kajari [K], Buser, Feiler and Schleich; Kerr [K], Klauber [K], Kursunoglu [K], Laue [L], Leeb [L], Schiffner, and Scheiterer; Lichtenberg [L] and Newton; Lichtenegger [L], Longhi [L], Logunov [L] and Chugreev; Ludwin [L], Mashhoon [M] and Muench; Mashhoon [M], Gronwald, Hehl and Theiss; Matolcsi [M], McCrea [M], Metz [M], Michel [M], Nikolic [N], Minguzzi [M], Ori [O] and Avron; Pascual-Sanches [P], San Miguel and Vicente; Petkov [P], Rizzi [R], Rizzi [R] and Tartaglia; Rizzi [R] and Ruggiero; Rodrigues [R] and Sharif; Ruggiero [R] and Tartaglia; Rumpf [R], Sagnac [S], Sakurai [S], Selleri [S], Scully [S], Scorgie [S], Tartaglia [T], Tartaglia T and Ruggiero; Trocheris [T], Stedman [S], Vall [V], Shorthill, and Berg; Vigier [V], Wang [W], Zheng, and Yao; Weber [W], Weihofen [W], Wucknitz [W].

Hönl [H] and Dehnen have shifted out Maxwell-like relations from the Einstein field equations, have made use of the observer field u and have admitted also shears[99] of this field. The idea of the observer field goes back to Uhlmann [U], Komar [K] and Pirani [P]. Hönl [H] and Dehnen have been concerned in detail with the relativity of rotational motion. They have introduced a rotating observer field to the Einstein cosmos and have regarded the Coriolis force as an inductive effect of the world matter rotating in the opposite sense. The quadratic terms have not been incorporated by them. In the second paper they have extended the computations to other models and they have brought the relativity of acceleration into connection with Mach's principle. Dehnen [D] has computed the centrifugal

[99] The symmetrical part of $u_{\alpha\|\beta} = u_{[\alpha\|\beta]} + u_{(\alpha\|\beta)}$ describes the shears.

and Coriolis force in the Friedman spherical space for rotating observers. In a further paper he has represented this problem most generally and has tried to include the field terms into a common stress-energy tensor for matter and field. In all treated cases the rotation is *attached* to non-rotating metrics. Rindler [R] has referred to the Lense-Thirring effect. Bass [B] and Pirani have refined the considerations of Lense and Thirring by introducing a tensor for elastic interaction. The rotating shell no longer consists of incoherent matter which is exposed to destructive centrifugal forces but is kept together by the elastic interaction.

In the preceding no gravitational model has been treated, but mathematical procedures have been developed which will be used in the following. The metric of a genuine rotating gravitational model differs from the metric of the flat rotators by the fact that their metric possesses a cross term. This cross term can neither be removed by a *holonomic* co-ordinate transformation nor by a *Lorentzian* transformation of the reference systems. In order to avoid that the orbital speed reaches the velocity of light at a certain radius, the metric should be constructed in such a manner that the angular velocity decreases away from the center of rotation, whereby it takes, if at all possible, the value zero in the infinite.

This Section has highlighted the similarity between the field equations of a rotating system and the field equations of electrodynamics, and the corresponding relations for the gravity of Einstein's field equations have been obtained. Many authors have observed such a similarity to electrodynamics, having performed an appropriate decomposition of the Riemann tensor. Tidal forces are also significant. The method has entered into the literature under the name gravitoelectromagnetism or gravitomagnetism. However, it must be noted that for this case, further relations must be evaluated in addition to Einstein's field equations. We refer to:

Bel [B], Bini [B], Bini [B], Carini, and Jantzen; Bini [B], Merloni, and Jantzen; Bonnor [B], Bonnor [B] and Steadman; Bouda [B] and Belabas; Bunchaft [B] and Caneiro; Carstiou [C], Cherubini, Jantzen, and Mashhoon; Braginsky [B], Caltenco [C], López-Bonilla, and Peña-Rivero; Caves and Thorne; Chicone [C] and Mashhoon, Ciufolini [C], Ciufolini [C], Chieppa, Lucchesi, and Vespe; Cohen [C] and Mashhoon; Costa [C] and Herdeiro; Costa [C] and Natário; Ciufolini [C], Kopeikin and Mashhoon; Dadhich [D], Ellis [E] and Hogan; Haddow [H]; Harris [H], Hawking [H]; Jantzen [J], Carini, and Bini; Ferrando [F] and Sáez; Lesame [L], Lesame [L], Dunsby, and Ellis; Lesame [L], Ellis and Dunsby; Leiby [L], Li [L], Li [L] and Torr; Lichtenegger [L], Gronwald and Mashhoon; Maartens [M], Maartens [M] and Basset; Maartens [M] and Lesame; Maartens [M], Ellis and Siklos; Majerník [M], Mashhoon [M] and Theiss; Mashhoon [M], Mashhoon [M], Gronwald, and Lichtenegger; Mashhoon [M], Gronwald, Hehl, and Theiss; Mashhoon [M], McClune and Quevedo; McIntosh [M], Arianrhod, Wade and Hoenselaers; Natário [N], Nordtvedt [N], Nouri-Zonoz [M], Pfister [P], Pascual-Sánches [P], Ruggiero [R] and Tartaglia; Schwebel [S], Sciama [S], Semerák [S], Soffel [S], Klioner, Müller, and Biskupek; Sopuerta [S], Maartens, Ellis, and Lesame; Swain [S], Tartaglia [T] and Ruggiero; Tartaglia [T], Trümper [T], Van den Bergh [V], Zee [Z].

Numerous authors have been working with rotating metrics:

Antoci [A] and Liebscher; Bayin [B], Bel [B], Bergmann [B], Bonnor [B], Chakrabarti [C], Chandrasekhar [C] and Miller, Cohen [C], Tiomno and Wald; Collins [C], Coke [C], Dalakashvili [D], Defacio [D], de Felice [D], Dennis and Retzlov; Farup [F] and Grøn; Florides [F] and Synge; Friedman [F], Isper, and Parker; Gafto [G], Greene [G] and Schucking; Grøn [G], Gupta [G], Iyer and Prasanna; Gürlebeck [G], Heller [H], Herrera [H], Santos and Carot; Herrera [H] and Hernández-Pastora; Herrera [H] et al., Jusufi [J], Kichenassamy [K] and Krikorian; Krasiński [K], Lämmerzahl [L] and Neugebauer; Li [L] and Ni; Marklund [M], Marochnik [M], Neugebauer [N], Ni [N] and Zimmermann; Nouri-Zonoz [N], Ozsváth [O], Pachner [P], Paschali [P] and Chrysostomou;

Perjes [P], Fodor, Gergely and Marklund; Radu [R], Mashhoon [M] and Santos; Papakostas [P], Pietronero [P], Quevedo [Q], Rosen [R], Ruggiero [R], Scherfner [S], Schiff [S], Schiff [S], Skrotskii [S], Sorge [S], Bini, and De Felice; Steward [S] and Ellis; Sviestins [S], Tartaglia [T], Vaidya [V], Williams [W].

In the following Sections we will be engaged with 4-bein systems which are not inevitably tangent to any co-ordinate lines. They are accompanied by non-exact differentials and are called anholonomic. Moreover, we will hark back to anholonomic hyper planes. These are locally defined (higher dimensional) planes which are defined in the points of a (higher dimensional) surface, and which are, however, not tangent. Schouten [S] and a team of mathematicians have developed the anholonomic geometry. We refer to some fundamental papers:

Cartan [C], Dienes [D], Bortolotti [B], Land [L], Hlavaty [H], Horak [H], Schouten [S] and Kampen; Schouten [S], Vranceanu [V], Synge [S], Takasu [T], Yano [Y].

Anholonomic elements have repeatedly been used by physicists though this is not explicitly enunciated in their papers. Furthermore, the long disputed Sagnac effect experienced a simple explanation by the use of anholonomic reference systems. Modern authors derive new physical perspectives from nonholonomicity:

Aldrovandi [A], Barros and Perreira; Treat [T], Vacaru [V], Vacaru [V] and Tintareanu-Mircea; Vacaru [V] and Dehnen; Dehnen [D] and Vacaru.

V.2. Oblique-angled systems

In the last Section a rotation has been *attached* to a metric by means of a Lorentz transformation. Now, the general mechanism has to be examined, how rotational effects can be implemented geometrically in both the flat and curved spaces. For such a rotation it is typical that a cross term arises in the metric which cannot be eliminated by a holonomic co-ordinate transformation or a Lorentz transformation. If a line element has the form

$$ds^2 = g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3 + 2g_{34}dx^3dx^4 + g_{44}dx^4dx^4 \qquad \text{(V.2.1)}$$

then in accordance with view (I) one has no difficulties to assign the cross term to the curvature of the space. However, if one favors view (II) or (III), one will hopelessly look for a surface which is constructed in such a way that a Gaussian co-ordinate system can be drawn on it and that a cross term emerges. Rather one has to look for a surface based on a metric which does not have a cross term. Its curvature properties have to be deduced from the remaining components of the metric tensor. The cross term gives rise to an oblique-angled co-ordinate system. It is not appropriate to support geometrical statements by the properties of any co-ordinate system.

It is required of the laws of nature and the geometrical laws which describe the laws of nature that they are valid independently of their representation. Above all, they have to be independent of the choice of the co-ordinates. Hence we draw our attention to such vectors which accompany an oblique-angled co-ordinate system. We show that rotational effects result from the implementation of such vectors. A rotating metric does not only describe the curvature of a surface, but describes another supplementary structure on the surface as well. Thus, it is appropriate to examine oblique-angled systems in order to be prepared for understanding a rotating metric.

For the addressed problems it is sufficient to limit ourselves to two dimensions because the twist takes place in the local [3,4]-planes. The space (here reduced to a two-dimensional curved surface) is parameterized by spherical co-ordinates[100] which are rectangular but not rectilinear. In each tangent plane of the generally curved space one can find a local Cartesian co-ordinate system which matches the shape of the global spherical geometry. In the origin of this local co-ordinate system two vectors which enclose an inclined angle are put into position. They establish an oblique-angled co-ordinate system that need not be necessarily global. For further considerations we can leave the family of tangent planes which covers the whole curved surface and we can restrict ourselves to one plane, in which a Cartesian and an oblique-angled co-ordinate system are spanned. It has to be examined, how vectors in these two co-ordinate systems are represented, and how one converts the two representations into one another.

To each curvilinear and generally oblique-angled co-ordinate system can be designated local vectors in two kinds: the tangent and the gradient vectors. For the special case that both co-ordinate systems are Cartesian, the two kinds of vectors coincide. The first are tangent to the co-ordinate lines. The latter are vertical to the co-ordinate lines and point to a neighboring line. One can use both systems for the representation of vectors. We will deal with the difficulties to measure vectors in these systems.

Three possibilities are distinguished, how one can deduce an inclined system from the orthonormal one. (A): one rotates the 3rd co-ordinate line of the Cartesian system and maintains the 4th. (B): one rotates the 4th co-ordinate line of the Cartesian system and

[100] The use of cylindrical or elliptical co-ordinates does not impair the fundamental considerations.

maintains the 3rd. (C): Both co-ordinate lines are rotated[101]. Firstly, the subject will be formulated as co-ordinate transformation, subsequently bein vectors will be attributed to this transformation.

The 3-axis is rotated through the angle $i\chi$ and then it will be called 3'-axis. The 4'-axis is identical with the 4-axis. The system m = {3,4} is the rectangular system, the system i= {3',4'} is the oblique-angled system. Any radius vector[102] $\vec{x}$ has to be measured in both systems. A co-ordinate transformation which has to be interpreted more accurately is

$$\begin{aligned} x^{3'} &= x^3, \quad x^{4'} = x^4 - x^3 \tan i\chi \\ x^3 &= x^{3'}, \quad x^4 = x^{4'} + x^{3'} \tan i\chi \end{aligned} . \tag{V.2.2}$$

To the transformation coefficients

$$\Lambda^{i}_{m} = x^{i}_{|m} \quad \Lambda^{m}_{i} = x^{m}_{|i} , \tag{V.2.3}$$

which connect the two systems can be assigned the above-mentioned tangent and gradient vectors

$$\overset{m}{e}_{i} \overset{*}{=} \Lambda^{m}_{i} \quad \underset{m}{e}^{i} \overset{*}{=} \Lambda^{i}_{m} . \tag{V.2.4}$$

The index m enumerates the vectors, the index i designates the components with respect to the oblique-angled system. Likewise, the tangent and gradient vectors of the oblique-angled system can be formed in the same way

$$\underset{i}{h}^{m} \overset{*}{=} \Lambda^{m}_{i} \quad \overset{i}{h}_{m} \overset{*}{=} \Lambda^{i}_{m} . \tag{V.2.5}$$

Now the index i enumerates the oblique-angled vectors, the index m designates the components with regard to the Cartesian system. The components of the tangent vectors of the system are *numerically* equal to the components of the gradient vectors of the other system and vice versa. In particular one has mutually reciprocal vectors

$$\begin{aligned} &\overset{3}{e}_{3'} = 1, \quad \overset{4}{e}_{3'} = \tan i\chi, \quad \overset{4}{e}_{4'} = 1 \\ &\underset{3}{e}^{3'} = 1, \quad \underset{3}{e}^{4'} = -\tan i\chi, \quad \underset{4}{e}^{4'} = 1 \end{aligned} . \tag{V.2.6}$$

If the dx^i are the co-ordinate differentials of the oblique-angled system, then one obtains with

$$dx^m = \overset{m}{e}_{i} dx^i \tag{V.2.7}$$

the components dx^m with respect to the Cartesian system, written more exactly as

$$dx^3 = dx^{3'}, \quad dx^4 = dx^{4'} + \tan i\chi dx^{3'} \tag{V.2.8}$$

and for their squares

$$ds^2 = g_{mn} dx^m dx^n = dx^m dx^m = g_{3'3'} dx^{3'} dx^{3'} + 2 g_{3'4'} dx^{3'} dx^{4'} + g_{4'4'} dx^{4'} dx^{4'} , \tag{V.2.9}$$

[101] This will be the case with the Kerr metric.

[102] In the local tangent planes only those vectors are meaningful, whose starting points lie in the points of contact with the surface.

a line element with a cross term. The metrical tensor has been built with the help of

$$g_{ik} = g_{mn} \overset{m}{e}_i \overset{n}{e}_k = \overset{m}{e}_i \overset{m}{e}_k \ . \tag{V.2.10}$$

Its components and those of the reciprocal tensor are

$$\begin{aligned} &g_{3'3'} = \frac{1}{\cos^2 i\chi}, \quad g_{3'4'} = \tan i\chi, \quad g_{4'4'} = 1 \\ &g^{3'3'} = 1, \quad g^{3'4'} = -\tan i\chi, \quad g^{4'4'} = \frac{1}{\cos^2 i\chi} \end{aligned} \tag{V.2.11}$$

and the line element is

$$ds^2 = \frac{1}{\cos^2 i\chi} dx^{3'2} + 2\tan i\chi dx^{3'} dx^{4'} + dx^{4'2} \ . \tag{V.2.12}$$

These Cartesian vectors e are normalized (are 1-vectors), as can easily be proved with

$$\overset{m}{e}_i \overset{n}{e}_k g^{ik} = g^{mn} = \delta^{mn} \ . \tag{V.2.13}$$

However, the vectors of the oblique-angled system are not normalized. In our example their lengths are

$$\begin{aligned} &\tau_3 = \sqrt{\underset{3'}{h^m} \underset{3'}{h^m}} \overset{*}{=} \sqrt{g_{3'3'}} = \frac{1}{\cos i\chi}, \quad \tau_4 = \sqrt{\underset{4'}{h^m} \underset{4'}{h^m}} \overset{*}{=} \sqrt{g_{4'4'}} = 1 \\ &\gamma^3 = \sqrt{\overset{3'}{h_m} \overset{3'}{h_m}} \overset{*}{=} \sqrt{g^{3'3'}} = 1, \quad \gamma^4 = \sqrt{\overset{4'}{h_m} \overset{4'}{h_m}} \overset{*}{=} \sqrt{g^{4'4'}} = \frac{1}{\cos i\chi} \end{aligned} \ . \tag{V.2.14}$$

The angle which is enclosed by each of the two vectors is

$$\begin{aligned} &\frac{\underset{3'}{h^m} \underset{4'}{h^m}}{\tau_3 \tau_4} \overset{*}{=} \frac{g_{3'4'}}{\sqrt{g_{3'3'} g_{4'4'}}} = \sin i\chi \\ &\frac{\overset{3'}{h_m} \overset{4'}{h_m}}{\gamma^3 \gamma^4} \overset{*}{=} \frac{g^{3'4'}}{\sqrt{g^{3'3'} g^{4'4'}}} = -\sin i\chi \end{aligned} \ . \tag{V.2.15}$$

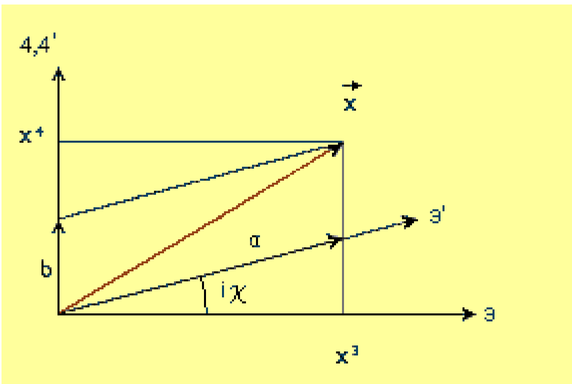

Fig. V.1

From Fig. V.1 one reads the parallel projections {a, b} of the radius vector $\vec{x}$ onto the tangent vectors of the oblique-angled system. To obtain the co-ordinates of $\vec{x}$ in the contravariant representation x^i, the parallel projections onto the tangent vectors must be divided by the value of the corresponding tangent vector:

$$x^{3'} = \frac{a}{\tau_3} = x^3 = \underset{3}{e}^{3'} x^3, \quad x^{4'} = \frac{b}{\tau_4} = x^4 - x^3 \tan i\chi = \underset{3}{e}^{4'} x^3 + \underset{4}{e}^{4'} x^4 . \tag{V.2.16}$$

Thus, one has explained the co-ordinate transformation (V.2.2) and has simplified the tedious derivation of

$$x^i = \underset{m}{e}^i x^m, \quad x^m = \overset{m}{e}_i x^i . \tag{V.2.17}$$

Moreover, one is able to pick out from a metric with a cross term which refers to oblique-angled co-ordinates, orthogonal and oblique-angled bein vectors and we can do without the co-ordinate system as a mathematical tool for further calculations. By working with oblique-angled co-ordinate systems one is confronted with the inconvenience that the covariant and the contravariant representations of tensors supply different values which can only with difficulty be assigned to physical quantities. The covariant representation of the radius vector $\vec{x}$ with

$$x_i = g_{ik} x^k \tag{V.2.18}$$

one can deduce in a similar way by dividing the parallel projections onto the gradient vectors by the value of the corresponding gradient vector. Then one obtains

$$x_i = \overset{m}{e}_i x_m, \quad x_m = \underset{m}{e}^i x_i , \tag{V.2.19}$$

no difference existing between the two representations in the local Cartesian system

$$x^m = x_m, \quad g_{mn} = \delta_{mn} . \tag{V.2.20}$$

The Cartesian vectors are mutually orthogonal. This means for the physics that space and time are orthogonal.

The second possibility to implement an oblique-angled system is to rotate the local 4-axis through the angle $i\chi$, then to call this axis 4'-axis, and to maintain the 3-axis. With the help of the co-ordinate transformation

$$\begin{aligned} x^{3'} &= x^3 + \tan i\chi \, x^4, \quad x^{4'} = x^4 \\ x^3 &= x^{3'} - \tan i\chi \, x^{4'}, \quad x^4 = x^{4'} \end{aligned} \tag{V.2.21}$$

the transformation coefficients can be computed and likewise the bein vectors in accordance with (V.2.3) to (V.2.5)

$$\begin{aligned} &\overset{3}{e}_{3'} = 1, \quad \overset{3}{e}_{4'} = -\tan i\chi, \quad \overset{4}{e}_{4'} = 1 \\ &\underset{3}{e}^{3'} = 1, \quad \underset{4}{e}^{3'} = \tan i\chi, \quad \underset{4}{e}^{4'} = 1 \end{aligned} . \tag{V.2.22}$$

The metric has again a cross term

$$\begin{aligned} &g_{3'3'} = 1, \quad g_{3'4'} = -\tan i\chi, \quad g_{4'4'} = \frac{1}{\cos^2 i\chi} \\ &g^{3'3'} = \frac{1}{\cos^2 i\chi}, \quad g^{3'4'} = \tan i\chi, \quad g^{4'4'} = 1 \end{aligned} \tag{V.2.23}$$

and the lengths of the bein vectors are

$$\tau_3 = 1, \quad \tau_4 = \frac{1}{\cos i\chi}, \quad \gamma^3 = \frac{1}{\cos i\chi}, \quad \gamma^4 = 1 \ . \tag{V.2.24}$$

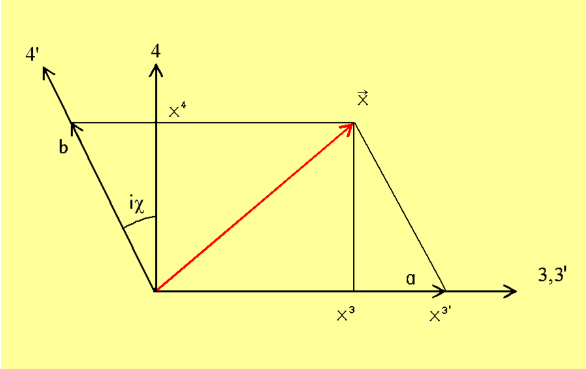

Fig. V.2

The parallel projections {a, b} onto the tangent vectors must be divided by the values of the corresponding tangent vectors in order to obtain the contravariant representation of the radius vector $\vec{x}$, namely x^i. With

$$a = x^3 + x^4 \tan i\chi, \quad b = \frac{1}{\cos i\chi} \tag{V.2.25}$$

one is in conformity with the co-ordinate transformation set before

$$x^{3'} = \frac{a}{\tau_3} = x^3 + \tan i\chi \, x^4, \quad x^{4'} = \frac{b}{\tau_4} = x^4 \tag{V.2.26}$$

and with (V.2.17). These possibility of choosing oblique-angled systems we have already got to know by investigating the flat rotator, and we have called the systems as (A) and as (B).

If one puts

$$dx^3 = \sigma d\varphi, \quad \tan i\chi = \omega\sigma \ , \tag{V.2.27}$$

one can explain

$$dx^3 = dx^{3'} - i\omega\sigma dx^{4'}, \quad dx^4 = dx^{4'} = idt \tag{V.2.28}$$

with the co-ordinate transformation

$$\varphi \to \varphi + \omega t, \quad t \to t \ . \tag{V.2.29}$$

Thus,

$$g_{4'4'} = 1 - \omega^2\sigma^2$$

is valid as well. The transformation of the type (B) can also be interpreted as Galilean transformation. Lastly, we want to follow more closely the way of writing of the preceding Sections. For the two systems one has

$$\begin{aligned} &dx^3 = c_i dx^i, && dx^4 = u_i dx^i \\ &c_i = \{1,0\}, && u_i = \{\tan i\chi, 1\} \quad (A) \ . \\ &c_i = \{1, -\tan i\chi\}, && u_i = \{0,1\} \quad\quad (B) \end{aligned} \tag{V.2.30}$$

On the other hand one obtains for the system (A) in a nostalgic way of writing

$$c_{3'} = 1, \quad u_{3'} = \frac{g_{3'4'}}{\sqrt{g_{4'4'}}}, \quad u_{4'} = \sqrt{g_{4'4'}} \; .$$

The spatial part of the metric in the inclined system is

$$'g_{ik} = g_{ik} - u_i u_k \, . \tag{V.2.31}$$

The space-like distance of two points in the 3-directions leads to the Landau-Lifshiz-formula [L]

$$dl^2 = {'g}_{3'3'} dx^{3'} dx^{3'} = \left[g_{3'3'} - \frac{g_{3'4'} g_{3'4'}}{g_{4'4'}} \right] dx^{3'} dx^{3'} = dx^3 dx^3 \, , \tag{V.2.32}$$

which has played a role with the problem of the time synchronization. Points in the space which do not move are specified by $dx^3 = 0$, simultaneous events in two neighboring points are specified by $dx^4 = 0$. For the time-like part of the metric of the system (B), there exists a similar formula related to (V.2.32)

$$''g_{ik} = g_{ik} - c_i c_k \tag{V.2.33}$$

$$\left(i\,dt \right)^2 = {''g}_{4'4'} dx^{4'} dx^{4'} = \left[g_{4'4'} - \frac{g_{3'4'} g_{3'4'}}{g_{3'3'}} \right] dx^{4'} dx^{4'} = dx^4 dx^4 \, . \tag{V.2.34}$$

However, this formula is not much in use in the literature. In the newer literature the space-time decompositions according to the methods (A) and (B) are called threading and slicing (Jantzen [J], Bini and Carini; Bini [B], Cherubini, Jantzen, and Mashhoon; Bini [B], Carini and Jantzen; Bini [B], Jantzen and Merloni).

If one has a metric with a cross term, one needs local orthogonal tetrads to calculate the Ricci-rotation coefficients and finally to analyze the field equations. For this purpose a simple method exists which is in close relation to the above two formulae. First, one has to put

Notation:

$$X = g_{33}, \quad W = g_{34}, \quad V = g_{44} \, . \tag{V.2.35}$$

The rotation arm σ is defined by

$$\sigma^2 = XV - W^2 \, . \tag{V.2.36}$$

In Newtonian approximation it is the distance from the axis of rotation. If one wants to, one can calculate the contravariant metric coefficients

$$g^{33} = \frac{V}{\sigma^2}, \quad g^{34} = -\frac{W}{\sigma^2}, \quad g^{44} = \frac{X}{\sigma^2} \, . \tag{V.2.37}$$

With equal ease one obtains the 4-bein vectors for two systems which are connected by a Lorentz transformation:

$$\begin{array}{ll} \text{(A)} & \overset{3}{e}_3 = \dfrac{\sigma}{\sqrt{V}}, \quad \overset{4}{e}_3 = \dfrac{W}{\sqrt{V}}, \quad \overset{4}{e}_4 = \sqrt{V}, \quad \underset{3}{e}^3 = \dfrac{\sqrt{V}}{\sigma}, \quad \underset{3}{e}^4 = -\dfrac{W}{\sigma\sqrt{V}}, \quad \underset{4}{e}^4 = \dfrac{1}{\sqrt{V}} \\ \text{(B)} & \overset{3}{e}_3 = \sqrt{X}, \quad \overset{3}{e}_4 = \dfrac{W}{\sqrt{X}}, \quad \overset{4}{e}_4 = \dfrac{\sigma}{\sqrt{X}}, \quad \underset{3}{e}^3 = \dfrac{1}{\sqrt{X}}, \quad \underset{4}{e}^3 = -\dfrac{W}{\sigma\sqrt{X}}, \quad \underset{4}{e}^4 = \dfrac{\sqrt{X}}{\sigma} \end{array} \; . \tag{V.2.38}$$

With

$$L^{m'}_{m} = \overset{m'}{e}_{i} \underset{m}{e}^{i} \tag{V.2.39}$$

can easily be derived the associated Lorentz transformation from the two 4-beine of the systems A and B. The index m refers to the system A, m ' to the system B. One has

$$\alpha = \frac{\sigma}{\sqrt{XV}} = \frac{1}{\sqrt{1+\frac{W^2}{\sigma^2}}}, \quad iv = \frac{W}{\sigma}, \quad i\alpha v = \frac{W}{\sqrt{XV}} . \tag{V.2.40}$$

$$L^{3'}_{3} = \alpha, \quad L^{3'}_{4} = i\alpha v, \quad L^{4'}_{3} = -i\alpha v, \quad L^{4'}_{4} = \alpha$$

With the auxiliary quantities

$$\underline{g}_{33} = g_{33}/\sigma^2, \quad \underline{g}_{34} = g_{34}/\sigma, \quad \underline{g}_{44} = g_{44}$$

and the associated bein vectors

$$\underline{g}_{ik} = \overset{m}{\underline{e}}_{i} \overset{m}{\underline{e}}_{k} = \mathbf{e}_i \mathbf{e}_k$$

one obtains the two relations

$$(\mathbf{e}_3)^2 (\mathbf{e}_4)^2 - (\mathbf{e}_3 \cdot \mathbf{e}_4)^2 = 1, \quad (\mathbf{e}_3 \times \mathbf{e}_4)^2 = 1.$$

We will successfully apply the system (V.2.38) to the Gödel universe and to rotating gravity models.

In addition to the above-discussed system (A) and (B), another is also applied

$$ds^2 = dx^{\underline{1}^2} + dx^{\underline{2}^2} + \left(\sqrt{X_1} dx^{\underline{3}} + \sqrt{V_1} dx^{\underline{4}}\right)^2 + \left(-\sqrt{X_2} dx^{\underline{3}} + \sqrt{V_2} dx^{\underline{4}}\right)^2 . \tag{V.2.41}$$

The bared indices are coordinate indices and X_2 and V_1 are negative variables. It follows

$$X = X_1 + X_2, \quad W = \sqrt{X_1 V_1} - \sqrt{X_2 V_2}, \quad V = V_1 + V_2 . \tag{V.2.42}$$

Thus, one first obtains

$$\sigma^2 = XV - W^2, \quad \sigma = \sqrt{X_1 V_2} + \sqrt{X_2 V_1} \tag{V.2.43}$$

and finally the 4-bein system

$$\text{(C)} \quad \begin{array}{llll} \overset{3}{e}_{3} = \sqrt{X_1}, & \overset{3}{e}_{4} = \sqrt{V_1}, & \overset{4}{e}_{3} = -\sqrt{X_2}, & \overset{4}{e}_{4} = \sqrt{V_2} \\ \underset{3}{e}^{3} = \frac{\sqrt{V_2}}{\sigma}, & \underset{4}{e}^{3} = -\frac{\sqrt{V_1}}{\sigma}, & \underset{3}{e}^{4} = \frac{\sqrt{X_2}}{\sigma}, & \underset{4}{e}^{4} = \frac{\sqrt{X_1}}{\sigma} \end{array} . \tag{V.2.44}$$

This system was used by Carter for the metric of rotating stellar objects.

In the next Section the rotating Gödel cosmos will be examined. The rotational effect can be extracted from the metric with the method (A). In several Sections the Kerr metric and the geometry derived from it will be analyzed. We will see that in order to understand these models the two inclined systems (A) and (B) are necessary.

V.3. The Gödel cosmos

In 1949 Gödel [G] has published a model for a rotating cosmos filled with matter dust. It came to the attention of the physicist community and supplied the science fiction literature with suggestions. According to Gödel the model admits closed time-like world lines; time journeys would be possible. Due to the causality violation this model is called unphysical by most of the gravitation physicists. We [B] do not only face critically the Gödel cosmos for the reason of the physical consequences, we also make out problems which concern the mathematical treatment.

The acceptance of a rotating cosmological model is problematic from the outset, on the one hand because nature does not supply an indication that our universe could rotate, on the other hand, because a question like ‘With respect to which system is the universe in relative rotation?’ is senseless indeed. If one likes, one could interpret the arising forces which correspond to the Coriolis forces as the result of a rotation of the universe. Ruben [R] and Treder discussed the possibility of a measurement of the rotation by aberration effects of radio waves which are emitted by quasars. They conclude that no indication for a rotation of the universe is to be picked out from the available data.

After group-theoretical considerations Gödel has initially written the metric[103] with the constant a as

$$ds^2 = a^2\left[dx^{1^2} + \frac{e^{2x^1}}{2} dx^{2^2} + dx^{3^2} - \left(e^{x^1} dx^2 + dx^0\right)^2 \right] . \tag{V.3.1}$$

From this he has computed (with the co-ordinate method) the Ricci tensor and the invariant curvature scalar

$$R_{ik} = \frac{1}{a^2} u_i u_k, \quad R = \frac{1}{a^2}, \quad u_i = \left\{0,0,0,\frac{1}{a}\right\} . \tag{V.3.2}$$

The extended Einstein field equations with the cosmological constant λ

$$R_{ik} - \frac{1}{2} g_{ik} R = 8\pi\kappa\mu u_i u_k + \lambda g_{ik} \tag{V.3.3}$$

are satisfied with

$$\frac{1}{a^2} = 8\pi\kappa\mu, \quad \lambda = -\frac{R}{2} = -\frac{1}{2a^2} = -4\pi\kappa\mu . \tag{V.3.4}$$

Therein μ is the average mass density of the cosmos filled uniformly with matter. The constant a is related to the speed of rotation. Kundt [K] has computed the geodesics for this metric. Raval [R] and Vaidya have extended the model for perfect and non-perfect fluids.

The transformation to cylindrical co-ordinates {r, φ, y, t}

$$\begin{gathered} e^{x^1} = \mathrm{ch}2r + \cos\varphi\, \mathrm{sh}2r, \quad x^2 e^{x^1} = \sqrt{2}\sin\varphi\, \mathrm{sh}2r \\ \tan\left(\frac{\varphi}{2} + \frac{x^0 - 2t}{2\sqrt{2}}\right) = e^{-2r}\tan\frac{\varphi}{2}, \quad \left|\frac{x^0 - 2t}{2\sqrt{2}}\right| < \frac{\pi}{2}, \quad x^3 = 2y \end{gathered} \tag{V.3.5}$$

leads to the axially symmetric line element

[103] Here Gödel’s t-notation is maintained.

$$ds^2 = 4a^2\left[dr^2 + dy^2 + \left(\mathrm{sh}^2 r - \mathrm{sh}^4 r\right)d\varphi^2 - 2\sqrt{2}\,\mathrm{sh}^2 r\, d\varphi\, dt - dt^2\right] \qquad \text{(V.3.6)}$$

that can be better understood. From this it is immediately evident that for

$$\mathrm{sh}^2 r - \mathrm{sh}^4 r < 0 \qquad \text{(V.3.7)}$$

the relevant part of the metric becomes negative and the arc element on the φ-curves imaginary. Gödel derives from this fact that in this case the φ-curves are time-like and the universe admits closed time-like curves. Other authors examined further models which admit such curves as well. They have entered into the literature with the designation CTC (closed time-like curves).

However, it is exactly this point we just take up our criticism. It remained unuttered that in the model two time-like dimensions would arise but only two space-like dimensions. If we use the simplifying t-notation, then the metric would have the structure

$$ds^2 = dx^{1^2} + dx^{2^2} - dx^{3^2} - dx^{4^2} \qquad \text{(V.3.8)}$$

and the index 0 instead of 2. Further, we want to underline that if an expression becomes imaginary this does not necessarily mean that it becomes time-like. There are many simple geometrical examples[104].

For a better understanding of the metric (V.3.6) some variables are renamed or rescaled. This does not change anything concerning the relevancy of the metric. With the replacement

$$a \to \mathcal{R}, \quad 2ay \to z, \quad 2at \to t$$

the metric takes the form

$$ds^2 = \mathcal{R}^2 (d2r)^2 + dz^2 + 4\mathcal{R}^2\left(\mathrm{sh}^2 r - \mathrm{sh}^4 r\right)d\varphi^2 - 4\sqrt{2}\,\mathrm{sh}^2 r\,\mathcal{R}\, d\varphi\, dt - dt^2 \;.$$

Now we shift the φ-term

$$4\left(1 + \mathrm{sh}^2 r - 2\mathrm{sh}^2 r\right)\mathcal{R}^2 \mathrm{sh}^2 r\, d\varphi^2 = \mathcal{R}^2 4\,\mathrm{sh}^2 r\,\mathrm{ch}^2 r\, d\varphi^2 - 8\mathcal{R}^2 \mathrm{sh}^4 r\, d\varphi^2 \;.$$

Thus, one obtains

$$ds^2 = \mathcal{R}^2 (d2r)^2 + dz^2 + \mathcal{R}^2 \mathrm{sh}^2 2r\, d\varphi^2 - \left[8\mathcal{R}^2 \mathrm{sh}^4 r\, d\varphi^2 + 4\sqrt{2}\,\mathcal{R}\,\mathrm{sh}^2 r\, d\varphi\, dt + dt^2\right] \;.$$

It is somewhat unusual that the variable r, which should have the dimension of a length and which is interpreted as a radial co-ordinate of the cylindrical system, is an argument of a hyperbolic function. Here one would expect an angle. With the further replacement

$$2r \to \eta, \quad r \to \chi$$

the metric appears in more familiar form

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \mathrm{sh}^2 \eta\, d\varphi^2 + dz^2 - \left[2\sqrt{2}\,\mathcal{R}\,\mathrm{sh}^2 \chi\, d\varphi + dt\right]^2 \qquad \text{(V.3.9)}$$

which has to be examined step by step. The first two terms refer to a hyperbolic structure. If one uses Cartesian co-ordinates with the extra dimension $x^{0'}$

[104] If one cuts a circle with a straight line, there can be two real solutions (secant), one solution (tangent), or two imaginary solutions (passant). One will hardly call the imaginary intersections time-like points.

$$x^{0'} = \mathcal{R}\,\mathrm{ch}\eta$$
$$x^{1'} = \mathcal{R}\,\mathrm{sh}\eta\cos\varphi \ , \qquad (V.3.10)$$
$$x^{2'} = \mathcal{R}\,\mathrm{sh}\eta\sin\varphi$$

one obtains the two-shell hyperboloid

$$x^{0'2} - x^{1'2} - x^{2'2} = \mathcal{R}^2 \qquad (V.3.11)$$

and for the metric on one of the shells

$$dx^{1'2} + dx^{2'2} - dx^{0'2} = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \mathrm{sh}^2\eta\, d\varphi^2 \ . \qquad (V.3.12)$$

With this assumption one has to interpret $\mathcal{R}$ as the distance of the vertices of the hyperbolae to the center of the hyperbolae. With

$$\overset{1}{e}_1 = \mathcal{R}, \quad \overset{2}{e}_2 = \mathcal{R}\,\mathrm{sh}\eta, \quad \partial_1 = \frac{\partial}{\mathcal{R}\partial\eta}, \quad \partial_2 = \frac{\partial}{\mathcal{R}\,\mathrm{sh}\eta\,\partial\varphi} \qquad (V.3.13)$$

one obtains the only field quantity

$$B_\alpha = \left\{\frac{1}{\mathcal{R}}\mathrm{cth}\eta, 0\right\}, \quad \alpha = 1,2 \qquad (V.3.14)$$

and the field equations

$$^{*}R_{\alpha\beta} = -\left[B_{\beta|\alpha} + B_\beta B_\alpha\right] - b_\beta b_\alpha \left[B^\gamma{}_{|\gamma} + B^\gamma B_\gamma\right] = -\frac{1}{\mathcal{R}^2}\,{}^{*}g_{\alpha\beta} \ . \qquad (V.3.15)$$

The structure of the right side of this equation refers to a 2-dimensional space of constant negative curvature. It is easy to understand this with respect to view (I). Due to this view the geometry is curved but is not a surface. Thus, the curvature of this space is constant. Since it is also negative the space is open. However, if one interprets the metric with regard to view (II) or (III) one has to take the surface graphically. It is then a hyperboloid, whose curvature decreases outwards. Thus, there will be a conflict with the definition of the curvature term. It is demanded that the Gödel cosmos is spatially homogeneous. This means that the universe looks the same everywhere. The hyperbolic interpretation does not fulfill this requirement. If we nevertheless do not want to limit ourselves to the view (I), we must formulate the problem in a different way. With the ansatz for an imaginary pseudo sphere[105]

$$x^{0'} = i\mathcal{R}\cos i\eta$$
$$x^{1'} = i\mathcal{R}\sin i\eta\cos\varphi \qquad (V.3.16)$$
$$x^{2'} = i\mathcal{R}\sin i\eta\sin\varphi$$

one obtains

$$x^{a'}x^{a'} = (i\mathcal{R})^2 \qquad (V.3.17)$$

and the metric on the pseudo sphere

$$dx^{a'}dx^{a'} = (i\mathcal{R})^2 (di\eta)^2 + (i\mathcal{R})^2 \sin^2 i\eta\, d\varphi^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \mathrm{sh}^2\eta\, d\varphi^2 \ . \qquad (V.3.18)$$

[105] More exactly: with that part of the pseudo sphere in which the pseudo radius takes imaginary values. Then $\mathcal{R}$ is the real accessory number of the constant pseudo radius in this range.

This embedding deviates from the discussed one so far, because in this case the extra dimension is imaginary, but the two other dimensions are real[106]. The embedding fulfills the homogeneity requirements of the model, satisfies the proponents of all views (I) to (III), and does not lead to a conflict of definitions concerning the constant curvature of space.

With the definition of the radial distance in the base plane[107]

$$r = \mathcal{R}\mathrm{sh}\eta \tag{V.3.19}$$

one obtains for the line element

$$dx^{a'}dx^{a'} = \frac{1}{1+\frac{r^2}{\mathcal{R}^2}}dr^2 + r^2 d\varphi^2 \ , \tag{V.3.20}$$

whereby one has to take notice of the plus sign in the denominator of the first term. It points out that the model has a structure different from that of former models.

Concerning z the metric is cylindrical, whereby the z-axis is the rotation axis of the universe. Due to the homogeneity of the universe z can be put in an arbitrary position. In accordance with Gödel any observer experiences the universe rotating around himself and defines himself as the rotation axis. This leads again to conflicts concerning the statements of two neighboring observers.

The last two terms of the metric (V.3.9) have already been presented by us as a full square. In addition, if one turns to the it-notation the bracketed term has the structure[108]

$$dx^4 = \overset{4}{e}_{2'}dx^{2'} + \overset{4}{e}_{4'}dx^{4'} = 2\sqrt{2}\,i\mathcal{R}\mathrm{sh}^2\chi d\varphi + dx^{4'} \tag{V.3.21}$$

which is typical for an oblique-angled co-ordinate system. The case (A) is at hand, which was discussed in the Section 'Oblique-angled systems'. The second axis of the co-ordinates was rotated through an imaginary angle to the 4-axis. To come closer to the geometrical meaning of this expression one has to work out

$$\sigma d\varphi = \mathcal{R}\mathrm{sh}\eta\, d\varphi \tag{V.3.22}$$

from it. With

$$2\sqrt{2}\mathcal{R}\mathrm{sh}^2\chi = \frac{\sqrt{2}\mathrm{sh}\chi}{\mathrm{ch}\chi}\mathcal{R}\,2\mathrm{sh}\chi\mathrm{ch}\chi = \sqrt{2}\mathrm{th}\chi\mathcal{R}\mathrm{sh}\eta$$

first one has

$$dx^4 = i\sqrt{2}\,\mathrm{th}\chi\sigma d\varphi + dx^{4'} = \sqrt{2}\,\mathrm{tan}i\chi\sigma d\varphi + dx^{4'} .$$

In addition, if one puts

$$\mathrm{tan}i\theta = \sqrt{2}\,\mathrm{tan}i\chi, \quad dx^4 = \mathrm{tan}i\theta\,\sigma d\varphi + dx^{4'} \ ,$$

one has found the connection to the last Section. Now one also recognizes how the Gödel metric can be derived from the assigned seed metric

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2\mathrm{sh}^2\eta\, d\varphi^2 + dz^2 + dx^{4'2} \ . \tag{V.3.23}$$

[106] $i\mathcal{R}\sin i\eta = -\mathcal{R}\mathrm{sh}\eta$

[107] The quantity r is newly defined and is not identical with the r from Eq. (V.3.6).

[108] For the inclined system primed indices are used temporarily.

If one uses primed indices for the seed metric, then the transformation

$$\overset{m}{e}_{i'} = \Lambda^{m}_{m'} \overset{m'}{e}_{i'}, \quad \Lambda^{2}_{2'} = 1, \quad \Lambda^{4}_{2'} = \tan i\theta, \quad \Lambda^{4}_{4'} = 1 \tag{V.3.24}$$

leads us from the seed metric to the Gödel metric. The transformation describes the rotation of the tangent vectors of the 2-direction in the local [2,4]-planes. The 2-lines of the seed metric are circles with the radius $\mathcal{R}\,\mathrm{sh}\eta$. The family of all circles with η = const. are laminated along the axis $x^4 = it$. Thus, the circles form a pseudo cylinder, the rotation takes place in the tangent planes of this cylinder. If one follows the rotated vectors on the surface of the cylinder, one recognizes that they are tangent to a pseudo helix. If one supports the conception with a pseudo-real representation, one can say that the world line of an observer co-rotating with the Gödel cosmos loops helically the time axis. This statement has to be examined mathematically.

A family of coaxial pseudo helices can be parameterized with

$$\begin{aligned} x &= \sigma\cos\varphi \\ y &= \sigma\sin\varphi \quad , \\ w &= b\varphi \end{aligned} \tag{V.3.25}$$

taking also into account (V.3.22). Therein is

$$b = \tan i\theta\,\sigma \tag{V.3.26}$$

and $2\pi b$ is the pitch of the helix. The quantity lies in the direction of the co-ordinate time $x^{4'}$. A helix is selected from the family with $\eta = \text{const.}$. It has a constant pitch, since χ and θ are kept constant with $\eta = \text{const.}$. The line element of the helix is

$$dl^2 = dx^2 + dy^2 + dw^2 = \left(\sigma^2 + b^2\right)d\varphi^2 = \left(1 + \tan^2 i\theta\right)\sigma^2 d\varphi^2 \ .$$

Therefore one has

$$dl = \frac{1}{\cos i\theta}\sigma\,d\varphi$$

and the length of the associated bein vector is

$$\tau_3 = \frac{1}{\cos i\theta} \ .$$

Since the parallel projection of a covariant vector has to be divided by the value of the associated tangent vector, one obtains

$$dx^3 = \frac{dl}{\tau_3} = \mathcal{R}\,\mathrm{sh}\eta\,d\varphi = \sigma d\varphi = dx^{3'} \ .$$

The φ-term in the Gödel metric is the arc element of the helix. One obtains the same value for the arc element of the helix and for the assigned arc element of the base circle if one applies for the helix an appropriate gauge and for the base circle as well. Finally, one obtains

$$dx^3 = dx^{3'}, \quad dx^4 = \tan i\theta\,dx^{3'} + dx^{4'} \ . \tag{V.3.27}$$

Since the Gödel system is inclined, the co-ordinate time dt is not perpendicular to the world lines on the helix of the co-rotating observers. The local time axis must be constructed foremost by establishing an orthogonal vector field $dx^{4''} = i d\tau$ in all points of the helix. Again we hark back to the last Section and regard a local 4''-axis which we obtain from the 4'-axis by rotation. If one applies a suitable gauge $dx^{4''}=dx^{4'}$ is valid and one can read from (V.3.27) the proper time of the rotating observer

$$d\tau = dt + th\theta\,\sigma d\varphi \ . \tag{V.3.28}$$

$d\tau$ is anholonomic, i.e. no global lines can be found to which $d\tau$ is tangent. After this detailed geometrical investigation of the metric we are prepared to make physical statements. If one puts

$$th\theta = \omega\sigma \tag{V.3.29}$$

for the orbital velocity of the rotating observer, θ can be interpreted as the rapidity of the observer. The metric reads as

$$ds^2 = \mathcal{R}^2 d\eta^2 + dz^2 + \sigma^2 d\varphi^2 - \left[\omega\sigma\cdot\sigma d\varphi + dt\right]^2 \tag{V.3.30}$$

or following the original form of Gödel

$$ds^2 = \mathcal{R}^2 d\eta^2 + dz^2 + \left[1-\omega^2\sigma^2\right]\sigma^2 d\varphi^2 - 2\omega\sigma^2 d\varphi\, dt - dt^2 \ . \tag{V.3.31}$$

Using this representation one recognizes the actual problem of Gödel's theory: The range of validity of the metric is limited. For $\omega\sigma > 1$ the φ-term becomes negative which can tempt searchers to the acceptance of time-like φ-lines. However, $\omega\sigma > 1$ means that the orbital velocity exceeds the speed of light which must be excluded for physical reasons. Thus, the Gödel cosmos has a *cut-off radius*

$$\sigma_c = \frac{1}{\omega_c} \ . \tag{V.3.32}$$

Beyond σ_c of the metric is no longer physically be interpreted. Thus, the whole Gödel cosmos is contradictory. If one writes the φ-term by means of (V.3.29) as

$$\left[1-th^2\theta\right]\sigma^2 d\varphi^2 \tag{V.3.33}$$

one recognizes that for mathematical reasons the cut-off radius cannot be exceeded at all, because for $\theta\to\infty$ the hyperbolic function $th\theta$ takes its highest value, i.e. 1. Those authors, who do not bring into connection the critical radius with the speed of light admit CTCs beyond σ_c. The range below σ_c is called a chronologically safe cylinder. Cooperstock [C] and Tieu came closer to this view. They examined the CTCs in the Gödel universe. They showed that the light cones of the rotating observers are tangent to the φ-lines at a critical radius.

After some rearrangements of the Gödel metric geometrical quantities have been identified with physical quantities by means of (V.3.29). One can work out which properties the Gödel cosmos could present below the cut-off radius. From (V.3.29) can be deduced that the angular velocity

$$\omega(\chi) = \frac{\sqrt{2}th\chi}{\mathcal{R}sh\eta} = \frac{1}{\sqrt{2}\mathcal{R}ch^2\chi} \tag{V.3.34}$$

is dependent on the distance from the rotation axis. $ch\chi$ becomes larger for increasing χ and thus for increasing distance from the rotation axis. For the same reason $\omega(\chi)$ becomes smaller, but the orbital velocity increases outwards due to $\omega\sigma = \sqrt{2}th\chi$ and exceeds the speed of light at the cut-off radius.

A rotating model which is based on a position-dependent angular velocity includes a *differential rotation law*. Such a law is compelling for any physically interpretable rotating model. In addition, it must be required that the orbital velocity decreases outwards fast enough in order to avoid a cut-off radius. Ideally, the orbital velocity becomes zero in the infinite.

In a physically adapted way of writing the 4-bein system has the form

$$\begin{array}{lllll} \overset{1}{e}_1 = 1, & \overset{2}{e}_2 = \sigma, & \overset{3}{e}_3 = 1, & \overset{4}{e}_2 = i\omega\sigma^2, & \overset{4}{e}_4 = 1 \\ \underset{1}{e}^1 = 1, & \underset{2}{e}^2 = \dfrac{1}{\sigma}, & \underset{3}{e}^3 = 1, & \underset{2}{e}^4 = -i\omega\sigma, & \underset{4}{e}^4 = 1 \end{array} . \tag{V.3.35}$$

The term

$$H_{mn}{}^s = H_{mn}u^s + H^s{}_m u_n + H^s{}_n u_m \tag{V.3.36}$$

joins the Ricci-rotation coefficients of the above-discussed rotation-free seed metric. H_{mn} is an antisymmetric quantity which has only the two components

$$H_{12} = -H_{21} = -i\left[\omega\sigma_{|1} + \frac{1}{2}\omega_{|1}\sigma\right] ,$$

whereby one has

$$\sigma_{|1} = \mathrm{ch}\eta = \mathrm{ch}^2\chi + \mathrm{sh}^2\chi \ .$$

The first part in the bracketed term above corresponds to the classical Coriolis force, the second part is a contribution from the differential rotation law. Both expressions are to be examined in more detail. From (V.3.34) one obtains

$$\omega\sigma_{|1} = \omega\left(\mathrm{ch}^2\chi + \mathrm{sh}^2\chi\right), \quad \frac{1}{2}\omega_{|1}\sigma = \frac{\partial}{2\mathcal{R}\partial\chi}\left(\frac{1}{\sqrt{2}\mathcal{R}\mathrm{ch}^2\chi}\right)\mathcal{R}\,\mathrm{sh}2\chi = -\omega\,\mathrm{sh}^2\chi$$

and thus, the constant quantity

$$H_{12} = -i\omega\ \mathrm{ch}^2\chi = -i\frac{1}{\sqrt{2}\mathcal{R}}, \quad H^{mn}H_{mn} = -\frac{1}{\mathcal{R}^2}\ , \tag{V.3.37}$$

whereby its dual vector lies in the 3-direction.

The entire force of rotation has the same value in every position of the Gödel cosmos, the rotation of the cosmos does not impair its homogeneity. With less detailed calculations it does not become known that the force of rotation H consists of two parts both of which are position-dependent and are summed up in such a way that they both result in a constant value. Thus, many authors regard H as the constant angular velocity and call the rotation *rigid*.

The Ricci tensor supplies together with the Ricci tensor of the seed metric (V.3.15)

$$R_{mn} = {}^*R_{mn} + 2u_{(m}H^s{}_{n)|s} - 2H_m{}^sH_{ns} + u_m u_m H^{sr}H_{sr}\,. \tag{V.3.38}$$

With (V.3.37) one has

$$H^s{}_{n|s} = 0 \tag{V.3.39}$$

a subequation of the Einstein field equations. With the contraction

$$R = {}^{*}R - H^{sr} H_{sr} = -\frac{1}{\mathcal{R}^2} \ . \tag{V.3.40}$$

one finally obtains the Einstein tensor

$$R_{mn} - \frac{1}{2} R g_{mn} = {}^{*}R_{mn} - \frac{1}{2} {}^{*}R g_{mn} - 2 H_m{}^s H_{ns} + u_m u_n H^{sr} H_{sr} + \frac{1}{2} g_{mn} H^{sr} H_{sr} \ .$$

After one has substituted the Einstein tensor into the field equations with the cosmological term

$$R_{mn} - \frac{1}{2} R g_{mn} - \lambda g_{mn} = -\kappa T_{mn}$$

the field equations reduce to the simple relation

$$\frac{1}{2\mathcal{R}^2} g_{mn} - \frac{1}{\mathcal{R}^2} u_m u_n - \lambda g_{mn} = -\kappa \mu_0 u_m u_n \ .$$

From the first three components of the above system one reads that the cosmological constant can receive only the value

$$\lambda = \frac{1}{2\mathcal{R}^2} \ . \tag{V.3.41}$$

Thus, the constant energy density of the universe

$$\kappa \mu_0 = \frac{1}{\mathcal{R}^2} \tag{V.3.42}$$

results from the fourth component.

Since it has been shown that the field equations of the Gödel cosmos are also fulfilled if one introduces an alternative view, we hark back a step. We are concerned again with the embedding, in order to see, what this embedding contributes to the field equations. First we examine the imaginary pseudo sphere with the radius $i\mathcal{R}$, parameterized by (V.3.16). Since, deviating from earlier models, the local 0-direction is imaginary[109], one has for the extrinsic geometry

$$M_{ab}{}^c = m_a M_b m^c - m_a m_b M^c, \quad M_b = \left\{ \frac{1}{i\mathcal{R}}, 0, 0 \right\}, \quad a = \{0,1,2\} \,. \tag{V.3.43}$$

For the quantity B one finds the 0-component simply by supplementing

$$B_0 = \frac{1}{i\mathcal{R} \sin i\eta} \sin i\eta, \quad B_1 = -\frac{1}{i\mathcal{R} \sin i\eta} \cos i\eta \ ,$$

or alternatively, by computation from the metric. Thus, one has

$$B_a = \left\{ \frac{1}{i\mathcal{R}}, \frac{1}{\mathcal{R}} \mathrm{cth}\eta, 0 \right\} \ . \tag{V.3.44}$$

If one considers $\partial_0 = \dfrac{\partial}{i\partial\mathcal{R}}, \partial_1 = \dfrac{\partial}{\mathcal{R}\partial\eta}$, the geometry of the pseudo sphere is completely described by the two equations

[109] But under no circumstances time-like.

$$M_{a\underset{1}{|||}b}+M_aM_b=0,\quad B_{a\underset{2}{|||}b}+B_aB_b=0\ . \tag{V.3.45}$$

In accordance with earlier investigations the representation is being changed now. Using the normal vector of the pseudo-spherical surface

$$n_a=\{1,0,0\} \tag{V.3.46}$$

and the second fundamental forms

$$Q_{11}=Q_{22}=\frac{1}{i\mathcal{R}} \tag{V.3.47}$$

one can write the Ricci-rotation coefficients

$$A_{ab}{}^{c}=M_{ab}{}^{c}+B_{ab}{}^{c} \tag{V.3.48}$$

in the form

$$A_{ab}{}^{c}={}^*B_{ab}{}^{c}+Q_a{}^{c}n_b-Q_{ab}n^c \tag{V.3.49}$$

as well, whereby the quantity *B refers to the 2-dimensional embedded surface which was already mentioned by (V.3.14). A [0+2]-decomposition of the Ricci tensors specifies

$$\begin{aligned} R_{ab}={}^*R_{ab}&-\left[Q_{ab|c}n^c+Q_{ab}Q_c{}^{c}\right]\\ &-n_a\left[{}^*B_{b|c}n^c+Q_d{}^{c}\,{}^*B_{cb}{}^{d}\right]\\ &-n_b\left[Q_c{}^{c}{}_{\wedge\underline{a}}-Q_a{}^{c}{}_{\wedge c}\right]\\ &-n_an_b\left[Q_c{}^{c}{}_{|d}n^d+Q_{cd}Q^{cd}\right] \end{aligned}\ . \tag{V.3.50}$$

Therein the index $\underline{a}$ takes only the values 1 and 2. If one calculates the first brackets, one finds

$$R_{\alpha\beta}={}^*R_{\alpha\beta}+\frac{1}{\mathcal{R}^2}g_{\alpha\beta}=0,\quad \alpha,\beta=1,2 \tag{V.3.51}$$

which is already well known from the relation (V.3.15). They are the Gauss equations. The term with $1/\mathcal{R}^2$ is attributed to the second fundamental forms. The relations in the second and fourth brackets of (V.3.50) describe the changes of the quantities, if one proceeds into the local 0-direction. These relations are of less interest for the 4-dimensional theory. The third brackets contain the Codazzi equation

$$Q_{[\alpha}{}^{\beta}{}_{\wedge\beta]}=0\ . \tag{V.3.52}$$

Therein the covariant differential operator is defined by

$$Q_\alpha{}^{\beta}{}_{\wedge\beta}=Q_\alpha{}^{\beta}{}_{|\beta}-{}^*B_{\beta\alpha}{}^{\gamma}Q_\gamma{}^{\beta}+{}^*B_\beta Q_\alpha{}^{\beta}\ . \tag{V.3.53}$$

From (V.3.51) one obtains

$${}^*R=\frac{2}{\mathcal{R}^2}\ , \tag{V.3.54}$$

and thus, the 2-dimensional Einstein tensor

$${}^*R_{\alpha\beta}-\frac{1}{2}{}^*R\,g_{\alpha\beta}=0 \tag{V.3.55}$$

vanishes. The two-dimensional world is curved, but empty. No direct contributions from the pseudo sphere are to be expected for the matter tensor. Since the rest of the space is flat, the curvature contributions of geometry are exhaustively treated. Now the contributions of the attached inclined system are still to be considered. The cosmological constant has no geometrical substantiation in this model. However, one can manipulate the Einstein field equations in such a way that on the right side emerges a working stress-energy tensor. Apart from the interpretation of the cosmological constant as background energy, the substantial task seems to consist in supplementing the Einstein field equations in such a way that they describe a reasonable model.

V.4. More on the Gödel cosmos

In this Section some supplementations have to be added to the preceding topic. Maybe it is rather new for a reader acquainted with Gödel's theory that the Gödel cosmos does not rotate rigidly but rotates differentially. With this problem we [B] have dealt in more detail in one of our papers.

Still another 4-bein structure can be read from Gödel metric (V.3.31), as it has been indicated in (V.3.35)

$$\begin{array}{l} \overset{1}{e}_1 = 1, \quad \overset{2}{e}_2 = \dfrac{\sigma}{\alpha}, \quad \overset{3}{e}_3 = 1, \quad \overset{2}{e}_4 = i\alpha\omega\sigma, \quad \overset{4}{e}_4 = \alpha \\ \underset{1}{e}^1 = 1, \quad \underset{2}{e}^2 = \dfrac{\alpha}{\sigma}, \quad \underset{3}{e}^3 = 1, \quad \underset{4}{e}^2 = -i\alpha\omega\,, \quad \underset{4}{e}^4 = \dfrac{1}{\alpha} \end{array} . \tag{V.4.1}$$

Both structures (V.3.35) and (V.4.1) are connected by the Lorentz transformation

$$\begin{array}{c} L_2^{2'} = \cos i\theta, \quad L_4^{2'} = \sin i\theta, \quad L_2^{4'} = -\sin i\theta, \quad L_4^{4'} = \cos i\theta \\ \cos i\theta = \mathrm{ch}\theta = \alpha, \quad \sin i\theta = i\,\mathrm{sh}\theta = i\alpha\omega\sigma \end{array} \tag{V.4.2}$$

The primed indices are temporarily used for the new system. The speed of the matter relative to this observer is

$$u_{m'} = \{0, i\alpha\,\omega\,\sigma, 0, \alpha\} = \{0, \sin i\theta, 0, \cos i\theta\}\,. \tag{V.4.3}$$

A new observer $'u_{m'} = \{0,0,0,1\}$ defines himself to be in rest. The differential rotation law proves that this is a most individual view: Observers move past on neighboring orbits. The ones forwards, the others apparently backwards. This is the reason of the shears which remain to be discussed later.

(V.4.1) describes a system of the type B. It emerges, if one rotates the fourth vector of the static seed metric through the angle $i\theta$.

From

$$dx^{2'} = \frac{1}{\alpha}\sigma d\varphi + i\alpha\omega\sigma dx^4, \quad dx^{4'} = dx^4 \tag{V.4.4}$$

one obtains the nonrelativistic approximation

$$d\varphi' = d\varphi - \omega dt\ , \tag{V.4.5}$$

the historical ansatz for a rotating system. If one computes the Ricci-rotation coefficients from (V.4.1), then the circular curvature quantity B of the system A is joined by another centripetal force

$$F_m = \alpha^2\omega^2\sigma\sigma_{|m} \tag{V.4.6}$$

and in addition by a force

$$D_m = \alpha^2\omega\,\omega_{|m}\sigma^2\ , \tag{V.4.7}$$

so that the only nonvanishing component is

$$\bar{B}_1 = B_1 - F_1 - D_1\ . \tag{V.4.8}$$

Thus, the new quantities are contained in

$$A_{4m}{}^{4} = G_m = F_m + D_m \ . \tag{V.4.9}$$

Together they result in

$$F_1 + D_1 = \omega\,\mathrm{sh}\theta\,\mathrm{ch}\theta \ . \tag{V.4.10}$$

Further, by using the re-defined observer field, the expression

$$D_{mns} = 2\left[D_{(mn)}u_s - D_{(ms)}u_n + D_{[ns]}u_m\right] \tag{V.4.11}$$

has to be considered, whereby D_{mn} is an asymmetrical quantity

$$D_{12} = i\omega\left[\mathrm{sh}^2\theta - \mathrm{sh}^2\chi\right], \quad D_{21} = 0 \ . \tag{V.4.12}$$

One finds with the above relations

$$u_{m||n}u^n = -\left(F_m + D_m\right), \quad u_{[\alpha||\beta]} = 0, \quad u_{(\alpha||\beta)} = -2D_{(\alpha\beta)} \ . \tag{V.4.13}$$

The first relation is the equation of motion of the matter. A test particle that is comoving with the system B is exposed to the forces F and D. The last two expressions show that no Coriolis forces arise in this system, but symmetrical field strengths which lead to shears of the volume elements which surround the observers of the type B. With the Lorentz transformation (V.4.2) the stress-energy tensor for the system B can be computed

$$T_{m'n'} = L^{m\ n}_{m'n'}\,\mu_0 u_m u_n = \mu_0 L^{4\ 4}_{m'n'} \ , \tag{V.4.14}$$

$$T_{m'n'} = \mu_0\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\alpha^2\omega^2\sigma^2 & 0 & -i\alpha^2\omega\sigma \\ 0 & 0 & 0 & 0 \\ 0 & -i\alpha^2\omega\sigma & 0 & \alpha^2 \end{pmatrix} = \mu_0\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\mathrm{sh}^2\theta & 0 & -i\,\mathrm{sh}\theta\mathrm{ch}\theta \\ 0 & 0 & 0 & 0 \\ 0 & -i\,\mathrm{sh}\theta\mathrm{ch}\theta & 0 & \mathrm{ch}^2\theta \end{pmatrix} . \tag{V.4.15}$$

Without primed indices this results in the relation

$$R_{mn} = -\kappa T_{mn} \ . \tag{V.4.16}$$

We have taken advantage of the fact that the R- and λ-terms in the field equations nullify due to Eq. (V.3.40) and (V.3.41). Thus, one has only to solve the above equation with (V.4.15). Although the field equations have a simple structure the calculations are somewhat laborious, because one has to do with the three angles η, θ, and χ, which are closely connected with each other[110]. With the abbreviations defined above one has

[110] In order to facilitate the examination of the field equations, we note some helpful formulae.

$$\mathrm{sh}^2\theta = \frac{2\,\mathrm{sh}^2\chi}{1-\mathrm{sh}^2\chi}, \quad \mathrm{ch}^2\theta = \frac{\mathrm{ch}^2\chi}{1-\mathrm{sh}^2\chi}, \quad \mathrm{th}^2\theta = 2\,\mathrm{th}^2\chi, \quad \sigma = 2\mathcal{R}\mathrm{sh}\chi\mathrm{ch}\chi$$

$$B_1 = \frac{1}{\mathcal{R}}\frac{1+2\,\mathrm{sh}^2\chi}{2\,\mathrm{sh}\chi\mathrm{ch}\chi}, \quad F_1 + D_1 = \frac{1}{\mathcal{R}}\frac{\mathrm{th}\chi}{1-\mathrm{sh}^2\chi}, \quad D_{12} = \frac{i}{\sqrt{2}\,\mathcal{R}}\frac{\mathrm{sh}^2\chi}{1-\mathrm{sh}^2\chi}$$

$$\begin{aligned} R_{mn} = &-\left[\bar{B}_{n\underset{2}{\|}m} + \bar{B}_n\bar{B}_m\right] - \left[G_{n\underset{3}{\|}m} + G_nG_m + 2D_n{}^sD_{ms}\right] \\ &-b_mb_n\left[\left(B^s{}_{\underset{2}{\|}s} + B^sB_s\right) - \left(G^s{}_{\underset{3}{\|}s} + G^sG_s + 2D^{sr}D_{sr}\right)\right] \\ &-u_mu_n\left[G^s{}_{\underset{3}{\|}s} + G^sG_s + 2D^{sr}D_{sr}\right] \\ &-2u_{(m}\left[D^s{}_{n)\underset{3}{\|}s} + D^s{}_{n)}\bar{B}_s\right] \end{aligned} \quad . \qquad \text{(V.4.17)}$$

In this case the graded derivatives read as

$$\bar{B}_{n\underset{2}{\|}m} = \bar{B}_{n|m}, \quad G_{n\underset{3}{\|}m} = G_{n|m} - \bar{B}_{mn}{}^sG_s \ , \qquad \text{(V.4.18)}$$

whereby the third graded derivative coincides here with the purely spatial derivative, because the z-dimension is cylindrical and supplies no contribution to the Ricci-rotation coefficients. In particular, the [44]-equation shows how the radial forces are coupled to the field energy and to the energy density of the matter

$$G^s{}_{\underset{3}{\|}s} = \kappa\mu_0\alpha^2 - G^sG_s - 2D^{sr}D_{sr} \qquad \text{(V.4.19)}$$

and the [24]-equation shows how the shears

$$D^s{}_{n\underset{3}{\|}s} = -\kappa\mu_0 i\alpha^2\omega\sigma b_n - D^s{}_n\bar{B}_s \qquad \text{(V.4.20)}$$

have as source the matter and field current. It was our actual goal to obtain the latter equation, in order to confirm our view on the Gödel cosmos. In contrast, a rigid rotation is attributed to the Gödel cosmos in the literature. However, we have shown that a differential rotation law occurs. If that were correct, the Einstein field equations would have to exhibit, with the choice of a suitable reference system, a field mechanism which contains stresses and shears. That we have shown with the help of the locally nonrotating system (system B). The set of equations (V.4.17) will not be discussed, because we have essential doubts concerning the coherence of the Gödel cosmos:

- In our view the Gödel metric has a cut-off radius. Beyond this radius superluminal velocities and acausalities are to be expected for the rotation.
- With regard to the time axis the model is flat. Hence, the force of gravity which keeps the matter together cannot be deduced from the metric. However, the cosmological constant as background energy can undertake this task.
- The model does not provide suitable mathematical equipment in order to describe those physical effects which one generally expects of a rotating system. For co-rotating observers there do not exist centrifugal forces, however, in the locally non-rotating system such forces of opposite sign emerge.
- No anisotropies can be observed in our material cosmos. Hence, a rotating cosmos is impossible. Obviously, Nature prefers simple solutions. The Gödel metric is one of the many solutions of nonlinear differential equations of second order, the Einstein equations. Many of these solutions cannot be applied to physics, but the Gödel solution though is mathematically interesting, but physically useless.

Finally some articles are indicated which are concerned with the Gödel cosmos.

Accioly [A] and Gonçalves consider universes of Gödel type in generalized theories of gravity. Bampi [B] and Zordan have shown that a class of metrics for perfect fluids is isometric to the Gödel cosmos. Banerjee [B] and Banerii have extended Gödel's solution to a cosmological model which is filled not only with matter but also with dust endowed with electromagnetic fields as well. Barrow [B] and Dąbrowski have found Gödel-like universes in string theories. Barrow [B] and Tsagas have examined the stability of the Gödel universe, have given an overview to articles on the Gödel cosmos, and have examined the effects of perturbations. Bonnor [B], Santos, and McCallum have cut out a cylindrical patch of the Gödel universe and have used a suitable interior solution. Bonnor [B] has been concerned in detail with CTCs. Bray [B] has examined a model of the Gödel type with additional magnetic fields and has used the methods of hydrodynamics. Caldarelli [C] and Klemm have found a 4-dimensional universe of the Gödel type. Calvão [C], Soares, and Tiomno investigate geodesics in spaces of Gödel type. Carneiro [C] has extended the Gödel cosmos by applying Friedman properties. Chakraborty [C] shows that the solution by Hoenselaers and Vishveshwara can be attributed to the solution of Gödel by a transformation from a non-comoving system to a comoving system. Chicone [C] and Mashhoon have used Fermi co-ordinates for the Gödel cosmos and have examined tidal effects. Chandrasekhar [C] and Wright have computed the geodesics for the Gödel cosmos. With circular orbits they have obtained null-geodesics at a 'maximum radius'. Clifton [C] and Barrow have deduced the Gödel cosmos from a Lagrange function. Dąbrowski [D] and Garecki have discussed Einstein's stress-energy complex for the Gödel cosmos. Das [D] and Gegenberg have regarded more general solutions with magnetic-like fields. The Gödel solution is a special case of these models. De [D] investigates cosmological models with closed time-like lines. Drukker [D], Fiol, and Simón have recognized Gödel-like properties in supergravitation and string models. They have also noticed a close relationship between the Gödel cosmos and its extended versions with charged particles which are coupled to a magnetic field. Dryuma [D] has found a 'Riemannian extension' of the Gödel metric. Figueiredo [F] has transferred the ansatz of Gödel to the Einstein-Cartan theory. It includes electromagnetic fields and charged particles. Fonseca-Neto [F], Romero, and Dahia have found a 5-dimensional local embedding for the Gödel metric. Garcia-Olivo [G], López-Bonilla and Vidal-Beltran; Gürses [G], Karasu, and Sarıoğlu have extended the model to more dimensions and have described sources with charged dust. Kanti [K] and Vayonakis have described Gödel-like universes with string-like and charged gravitation. Laurent [L], Rosquist, and Sviestins have examined the null geodesics for the Ozsvath class III. That class contains the Gödel cosmos as a special case. They have discovered that in these models the rotational velocity of objects increases relative to an observer, decreases, or gets infinitely large, and can change the orientation. Novello [N] and Rebouças have also included shears, temperature, and heat flows into a rotating model which emerges from the Gödel model. Investigating the geodesics, Novello [N], Soares, and Tiomno have found out that the geodesics are consistent only up to a maximum radius. Hence possible particle trajectories are enclosed by a cylindrical surface whose radius is identical with our cut-off radius. Maitra [M] has treated rotating universes; however, he has avoided closed time-like curves. Ozsvath [O] and Ozsvath and Schücking have treated rotating world models in a general form. The Gödel cosmos is contained in these models as a special case. In a further paper [O] they have again dealt with Gödel's problem. However, they have stuck to the rigid rotator. Panov [P] has constructed a Gödel-like model for an anisotropic fluid with a heat flow, an additional scalar field, and a radiation field. Paiva [P], Rebouças, and Teixeira have been occupied with closed time-like world lines. Patel [P] and Trivedi have added electromagnetic fields to the Gödel cosmos. Pfarr [P] has dealt in detail with paths in the Gödel cosmos: time-like geodesics, closed time-like world lines, closed null-lines, and orbits with superluminal velocity. Radu [R] has expanded the two-parameter solution of Rebouças and Tiomno and also has used higher-dimensional local embeddings. Raval [R] and Vaidya have generalized to a universe with charged incoherent matter. Raychaudhuri [R] and Thakurta have regained the Gödel solution in a general class of homogeneous world models.

Rebouças [R] has made a start with a rotating universe with a perfect fluid and an electromagnetic field. Rebouças [R], Åman, and Teixeira have set up a classification of possible solutions of the Gödel type and have performed calculations in the Newman-Penrose formalism. Rebouças [R] and Åman have reported on computer-aided analyses. Rebouças [R] and Tiomno have examined a two-parameter extension of the Gödel metric by the use of Killing vectors. Romano [R] and Romano [R] and Goebel have been concerned in detail with the group-theoretical derivation of the Gödel cosmos. Rooman [R] and Spindel have generalized Gödel, have investigated embeddings into higher dimensions, and have interpreted the cosmos as slices of surfaces in a higher-dimensional space. The stability of the CTCs has been computed by Rosa [R] and Letelier. Sahdev [S], Sundararaman, and Modgil have computed and drawn numerous possibilities for world lines and their light cones. Sharif [S] has deduced the stress-energy complex of Gödel's gravitational field in Einstein and Papapetrou form. Silk [S] has computed the local irregularities which arise by perturbations of the metric. Stein [S] has tried to present the Gödel cosmos as a realistic model. Teixeira [T], Rebouças, and Åman have been concerned with isometries of homogeneous Gödel-like worlds. Vaidya [V] has found a transformation from the Einstein cosmos to the Gödel cosmos and in the end a common representation of these models. Wright [W] has extended Gödel's metric in such a way that the matter density can be position-dependent. Novello [N], Svaiter, and Guimaraes have found synchronized reference systems. Wilkins [W] and Jacobs have obtained relations on the torque-free motion of gyroscopes and have compared the problem with charged particles in a homogeneous magnetic field. García-Olivo [G], López-Bonilla and Vidal-Beltrán have tried an embedding of the Gödel cosmos. Agakov [A] has treated an extension to the case of non-stationary perfect fluid. Carrion [C], Rebouças, and Teixeira have found a 5-dimensional extension of Gödel's approach. Chakraborty [C] and Bandyopadhay have expanded the model to a model with perfect fluid and with a scalar field. Chandrasekhar [C] and Wright have investigated the geodesics of the Gödel universe and have realized that the geodesics are null geodesics at a certain radius. That radius is our cut-off radius. Cohen [C], Vishveshwara, and Dhurandhar have generalized to electromagnetic fields. Dunn [D] has described a Gödel-like solution for two perfect fluids, one of which is afflicted with radiation, the other with matter. He has also discussed a solution with electromagnetic fields. Figueiredo [F] has treated the motion of charged particles in the field of the Gödel universe. Vaidya [V], Bedran, and Som have generalized the Gödel universe by introducing the cosmological constant instead of an electromagnetic stress-energy-momentum tensor. They have got a solution which does not violate causality. Yavuz [Y] and Baysal extend the model of Gödel to a perfect fluid with heat transport.

Further: Abramowicz [A] and Fragile; Ahsan [A], Bardeen [B] and Wagoner; Boyda [B], Ganguli, Hořava and Varadarajan; Caltenco, Linares and López-Bonilla; Camci [C], Camci [C] and Sharif; Calváo [C], Chakraborty [C] and Prasanna; Clark [C], Dautcourt [D], De Oliveira [D], Teixeira and Tiomno; Furtado [F], Nascimento, Petrov and Santos; Garcia-Olivo [G], López-Bonilla and Vidal-Beltran; Gleiser [G], Gürses, Karasu, and Sanıoğlu; Grave [G] et al, Hiscock [H], Hoffman [H], Huang [H], Kajari [K], Walser, Schleich, and Delgado; Lathrop [L]; Leahy [L], Melfo [M], Núnez, Percoco, and Villalba; Navez [N], Nouri-Zonoz [N] and Parvici; Novello [N] and Motta da Silva; Novello [N], Svaiter and Guimarães; Patel [P] and Koppar; Patel [P] and Vaidya; Pitanga [P], Radu [R], Ray [R], Rebouças [R] and Teixeira; Rebouças [R] and Tiomno; Rosquist [R], Saibatlov [S], Santaló [S], Silk [S], Smaranda [S], Soleng [S], Warner [W] and Buchdahl; Thakurta [T].

VI. Axially-symmetric models

There are numerous studies on the axially symmetric solutions of Einstein's field equations. However, little is available in textbooks. In recent years, the physicist community has lost interest in these models. Nevertheless, it seems attractive to re-examine some of the classic solutions. There are static models that describe the field of rods, of infinite strings, or of two masses. Furthermore, there are stationary but time-independent solutions which treat the field of rotating objects. Among these is the Kerr solution, which is the only one with good reason to be given physical significance. We will discuss it separately in some Sections.

A more modern view also makes it possible to separate the presumably physical quantities from those that enter the theory by the choice of the reference system. The use of cylindrical or spherical coordinates leads to coordinate surfaces whose curvatures are also described by Einstein's field equations. If the description scheme is changed, other curvatures will occur. The physical curvatures, however, remain unchanged.

The application of the tetrad method to rotating models is particularly advantageous. With it, a proper space-time splitting can be carried out, Coriolis and centrifugal forces can easily be worked out. Stationary axially symmetrical models can be tested for physical relevance by comparison with the relativistic rotator described in Sec. VI.1. It will be seen that most models do not meet the requirements of a physically usable theory.

Dautcourt [D] has published a good overview of the historical development of axisymmetric systems. Restricting himself to axial symmetry, he discussed the development of solution methods of Einstein's field equations.

VI.1. The model of Levi-Civita

Weyl [W] described the first axisymmetric solution. The line element from which the axially symmetrical models can be derived has according to him the form

$$ds^2 = e^{2(k-u)}\left(dR^2 + dZ^2\right) + e^{-2u} R^2 d\varphi^2 - e^{2u} dt^2, \quad u = u(R,Z), \quad k = k(R,Z). \quad \text{(VI.1.1)}$$

The coordinates $\{R, Z, \varphi\}$ used therein are referred to as Weyl's canonical cylindrical coordinates. As the first model based on this line element we treat the quite simple model of Levi-Civita, in which the quantities u and k depend only on R. Thus, the fields of this model have mere cylindrical symmetry. If there exists an interior solution to this model, it would represent an infinite cylinder.

If one puts

$$e^u = R^M, \quad e^k = R^{M^2}, \quad M = \text{const.} \quad \text{(VI.1.2)}$$

and if one reads from (VI.1.1) the 4-beine

$$\overset{1}{e}_1 = \overset{2}{e}_2 = e^{k-u} = \alpha, \quad \overset{3}{e}_3 = e^{-u} R, \quad \overset{4}{e}_4 = e^u \quad \text{(VI.1.3)}$$

one is able to calculate with

$$\partial_1 = \frac{1}{\alpha}\frac{\partial}{\partial R} \tag{VI.1.4}$$

the components

$$U_1 = u_{|1} = \frac{1}{\alpha}\frac{M}{R}, \quad K_1 = k_{|1} = \frac{1}{\alpha}\frac{M^2}{R}, \quad B_1 = \frac{1}{\alpha}\alpha_{|1} = K_1 - U_1, \quad {}^*C_1 = K_1 - U_1, \quad {}^*C_1 = \frac{1}{\alpha R}$$

from the Ricci-rotation coefficients. Finally, one gets the quantities

$$U_m, \quad B_m = K_m - U_m, \quad C_m = {}^*C_m - U_m$$

in which only the first components are occupied. U is a geometric quantity which leads with

$$E_1 = -U_1 = -e^{u-k}\frac{M}{R}$$

in certain circumstances to a physically interpretable quantity. It is attractive, ie it is directed to the cylinder axis and becomes infinite at this location. However, the quantity K is repulsive. $*C$ is a geometric quantity. It describes the curvatures of the cylindrical coordinate surfaces. However, since the factor α depends on R these surfaces are settled in such a way that their relative distances decrease inwards. $*C$ obeys the relation

$$*C^s{}_{||s} = 0 \tag{VI.1.5}$$

and thus drops out from the field equations. Since also

$$U^s{}_{||s} = 0, \quad B^s{}_{||s} = 0, \quad K^s{}_{||s} = 0 \tag{VI.1.6}$$

the calculation of Einstein's field equations is very simple. The vacuum field equations $R_{mn} = 0$ are obtained as a solution. However, they cannot describe a physical situation.

VI.2. The model of Wilson and Marder

The axial-symmetric metric of Wilson [W] and Marder [M] is, like the one of Levi-Civita, independent of the coordinate Z. The metric[111]

$$ds^2 = R^{2C^2/(1-C)}dR^2 + R^{-2C}dZ^2 + R^2 d\varphi^2 - R^{2C/(1-C)}dt^2 \tag{VI.2.1}$$

is not isotropic in R and Z. With

$$\alpha = R^{C^2/(1-C)}$$

one obtains for the tetrads

$$\overset{1}{e}_1 = \alpha, \quad \overset{2}{e}_2 = R^{-C}, \quad \overset{3}{e}_3 = R, \quad \overset{4}{e}_4 = R^{C/(1-C)} \tag{VI.2.2}$$

and the field quantities

$$\begin{aligned} &B_\nu = \left\{-\frac{C}{\alpha}\frac{1}{R},0\right\} = \left\{-CR^N,0\right\}, \quad N = \frac{C^2 - C + 1}{C-1}, \quad \frac{1}{\alpha R} = R^N, \quad \nu = 1,2 \\ &C_\nu = \left\{\frac{1}{\alpha}\frac{1}{R},0\right\} = \left\{R^N,0\right\}, \quad U_\nu = \left\{\frac{1}{\alpha}\frac{C}{1-C}\frac{1}{R},0\right\} = \left\{\frac{C}{1-C}R^N,0\right\} \end{aligned} . \tag{VI.2.3}$$

For the three field quantities result the simple relations

$$B^s{}_{\|s} = 0, \quad C^s{}_{\|s} = 0, \quad U^s{}_{\|s} = 0 . \tag{VI.2.4}$$

The vacuum field equations are satisfied with them. Marder takes another look at an interior solution. But we are of the opinion that an infinitely long cylinder with mass cannot be realized in Nature.

[111] The approach of Wilson (1920) differs from that of Marder (1958) by choice C = 2M.

VI.3. The model of Curzon

Curzon [C] has investigated a model that describes the field of two masses and therefore has cylindrical symmetry. Two-body models were also envisaged by Trefftz [T], Palatini [P], and Chazy [C]. A simple model builds on the results of Curzon. This is obtained by bringing the two masses into coincidence. In the literature this version is often also referred to as Curzon.

For this model, we have to abandon the restrictions made in the previously discussed Section. Now one has

$$u = u(R,Z), \quad k = k(R,Z).$$

The 4-beine again have the form (VI.1.3). The coordinate distance of an incident point from the origin of the system and the distance from the axis are defined by

$$r^2 = R^2 + Z^2, \quad \sigma = R. \tag{VI.3.1}$$

We will see that the ansatz for the solutions

$$u = -\frac{M}{r}, \quad k = -\frac{M^2 R^2}{2r^4}, \quad M = \text{const.} \tag{VI.3.2}$$

results in vacuum field equations. We define auxiliary variables

$$\begin{gathered} r_\nu = r_{|\nu} = \frac{1}{\alpha r}\{R,Z\}, \quad r^\nu r_\nu = \frac{1}{\alpha^2}, \quad \sigma_\nu = \sigma_{|\nu} = \frac{1}{\alpha}\{1,0\} \\ \sigma^\nu \sigma_\nu = \frac{1}{\alpha^2}, \quad \alpha = e^{k-u}, \quad \nu = 1,2 \end{gathered}. \tag{VI.3.3}$$

After a short calculation we obtain the field quantities

$$U_\nu = u_{|\nu} = \frac{M}{r^2} r_\nu, \quad K_\nu = k_{|\nu} = \frac{M^2}{\alpha r^6}\{R^2 - Z^2, 2RZ\}, \quad {}^*C_\nu = \frac{1}{R}\sigma_\nu = \left\{\frac{1}{\alpha R}, 0\right\}. \tag{VI.3.4}$$

The Ricci-rotation coefficients have the following components

$$\begin{gathered} A_{21}{}^2 = B_1 = K_1 - U_1, \quad A_{12}{}^1 = B_2 = K_2 - U_2 \\ A_{31}{}^3 = C_1 = {}^*C_1 - U_1, \quad A_{32}{}^3 = C_2 = -U_2 \end{gathered}.$$

We write for short

$$\begin{gathered} B_{\alpha\beta}{}^\gamma = \delta^\gamma_\alpha B_\beta - g_{\alpha\beta} B^\gamma \\ C_{3\beta}{}^3 = C_\beta = {}^*C_\beta - U_\beta, \quad U_{4\beta}{}^4 = U_\beta \end{gathered}. \tag{VI.3.5}$$

Together, this gives the full expression for the Ricci-rotation coefficients

$$A_{mn}{}^s = B_{mn}{}^s + C_{mn}{}^s + U_{mn}{}^s.$$

$^*C$ is also a geometric quantity which describes the curvature of the cylindrical surfaces. Owing to

$$^*C^s{}_{||s} = 0 \tag{VI.3.6}$$

this subequation of Einstein's field equations drops out of the system of equations. As one can easily show that

$$U^s{}_{||s} = 0\,, \tag{VI.3.7}$$

with the result that two relations of the Ricci are already fixed

$$\begin{aligned} R_{33} &= -C^s{}_{||s} = -{}^*C^s{}_{||s} + U^s{}_{||s} = 0 \\ R_{44} &= -U^s{}_{||s} = 0 \end{aligned} \;. \tag{VI.3.8}$$

The calculation of the remaining field equations is considerably more difficult. After some algebra we find the auxiliary relations

$$B^\gamma{}_{|\gamma} + B^\gamma B_\gamma = K^\gamma{}_{|\gamma} + B_\gamma K^\gamma + {}^*C_\gamma U^\gamma, \quad K^\gamma{}_{|\gamma} + B_\gamma K^\gamma + U_\gamma U^\gamma = 0\,,$$

with which it can be shown that $R_{\alpha\beta} = 0$, thus, the vacuum field equations for this axisymmetric model are also fulfilled. In the whole system there are only two field quantities which could be physically interpreted. The quantity U, which is the negative of the gravitational force is $E_1 = -U_1$. It is directed to the center of the system and becomes infinite at this location. Its norm is

$$E = \frac{M}{\alpha r^2}$$

and corresponds to Schwarzschild's gravitational force. A space-time splitting of (VI.3.7) results in

$$\left(E^s{}_{|s} + B_s E^s + C_s E^s\right) - E_s E^s = 0$$

or in

$$\operatorname{div}\vec{E} = E^2\,.$$

As a consequence of the non-linearity of Einstein's field equations the gravitational force is coupled to itself. It is well-known that the partial derivatives in the tetrad system are not commutative. One has

$$\Phi_{|[mn]} = A_{[nm]}{}^s \Phi_{|s}\,.$$

Applied to the quantity $U = u_{|m}$ one obtains with the spatial differential operator

$$U_{[\alpha\wedge\beta]} = U_{[\alpha|\beta]} - A_{[\beta\alpha]}{}^\gamma U_\gamma = 0\,.$$

Thus, the Maxwell-like relations

$$\operatorname{div}\vec{E} = \rho, \quad \rho = E^2, \quad \operatorname{rot}\vec{E} = 0 \qquad \text{(VI.3.9)}$$

apply to the gravitational force of the system. A similar relation holds for the K-field strength

$$\operatorname{div}\vec{K} = E^2 .$$

where now div is the 2-dimensional covariant divergence. K is also coupled to the gravitational energy. The K-field lines have an interesting course. With

$$\frac{K_2}{K_1} = \frac{2RZ}{R^2 - Z^2} = Z'$$

we calculate the ascent of the K-vectors. If we integrate this directional field, we obtain

$$R^2 \quad + \left(Z - \frac{C}{2} \right)^2 = \frac{C^2}{4}, \quad k\left(0, \frac{C}{2} \,||\, \frac{|C|}{2} \right) ,$$

ie, a family of circles with the origins on $\pm\frac{C}{2}$ and the radii $\frac{|C|}{2}$ (Fig. VI.1)

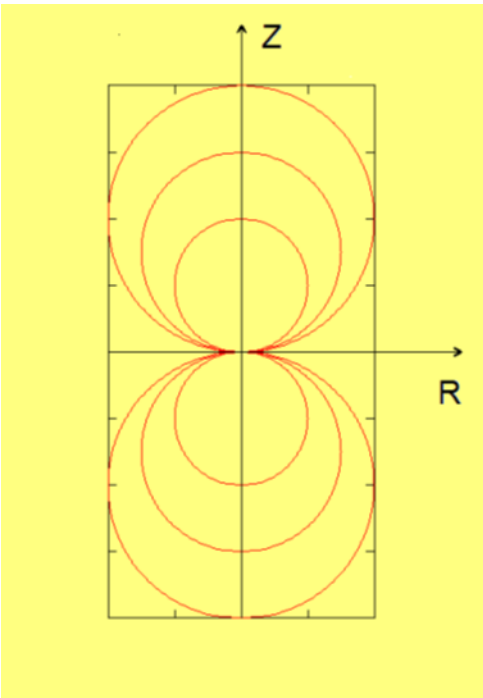

Fig. VI.1. The field lines of the quantity K

If one introduces spherical coordinates with

$$R = r\sin\vartheta, \quad Z = r\cos\vartheta, \quad r^2 = R^2 + Z^2 \qquad \text{(VI.3.10)}$$

one obtains the metric

$$ds^2 = e^{2(k-u)}\left(dr^2 + r^2 d\vartheta^2\right) + e^{-2u} r^2 \sin^2\vartheta d\varphi^2 - e^{2u} dt^2, \quad u = u(r), \quad k = k(r,\vartheta) \qquad \text{(VI.3.11)}$$

and the 4-bein system

$$\overset{1}{e}_1 = e^{k-u}, \quad \overset{2}{e}_2 = e^{k-u} r, \quad \overset{3}{e}_3 = e^{-u} r\sin\vartheta, \quad \overset{4}{e}_4 = e^{u}. \qquad \text{(VI.3.12)}$$

With

$$u = -\frac{M}{r}, \quad k = -\frac{M^2 \sin^2\vartheta}{2r^2}, \quad M = \text{const.}, \quad \alpha = e^{k-u} \qquad \text{(VI.3.13)}$$

and

$$\partial_1 = \frac{1}{\alpha}\frac{\partial}{\partial r}, \quad \partial_2 = \frac{1}{\alpha}\frac{\partial}{r\partial\vartheta}, \quad r_\nu = r_{|\nu} = \left\{\frac{1}{\alpha}, 0\right\}, \quad \nu = 1,2$$

the B content of the Ricci-rotation coefficients is calculated

$$B_{\alpha\beta}{}^{\gamma} = \delta_\alpha^\gamma B_\beta - g_{\alpha\beta} B^\gamma, \quad B_\nu = {}^*B_\nu + K_\nu - U_\nu, \quad K_\nu - U_\nu = \frac{1}{\alpha}\alpha_{|\nu}. \qquad \text{(VI.3.14)}$$

Therein is

$${}^*B_\nu = \frac{1}{r} r_{|\nu} = \left\{\frac{1}{\alpha r}, 0\right\}, \quad K_\nu = \frac{M^2}{\alpha r^3}\sin\vartheta\{\sin\vartheta, -\cos\vartheta\}, \quad U_\nu = \left\{\frac{M}{\alpha r^2}, 0\right\} \qquad \text{(VI.3.15)}$$

Furthermore,

$$C_\nu = {}^*C_\nu - U_\nu, \quad {}^*C_\nu = \frac{1}{\alpha}\left\{\frac{1}{r}, \frac{1}{r}\cot\vartheta\right\}. \qquad \text{(VI.3.16)}$$

In spherical coordinates, the radial quantity U has only one component, as expected. The quantity *C is, except for the factor $1/\alpha$, the expression known from the spherical coordinates. It describes the curvature of the parallels on the spherical coordinate surfaces

$${}^*C_\nu = \frac{1}{\alpha}\frac{1}{r\sin\vartheta}\{\sin\vartheta, \cos\vartheta\}. \qquad \text{(VI.3.17)}$$

$R = r\sin\vartheta$ is the radius of curvature of these circles. On the other hand, there is the quantity $^*B$. It describes the curvature $1/r$ of the great circles of the spherical surfaces.

Let us examine why and how the components of these two geometric variables change in the transition from the one reference system to the other. We recall the inhomogeneous transformation law of the Ricci-rotation coefficients and now we relate this to the geometric quantity $^*B$. The cylindrical system is now marked by primed indices

$$x^{1'} = R = r\sin\vartheta, \quad x^{2'} = Z = r\cos\vartheta. \qquad \text{(VI.3.18)}$$

For the B content of the Ricci-rotation coefficients the transformation equation is

$$B_{\alpha\beta}{}^{\gamma} = A_{\alpha\ \beta\ \gamma'}^{\alpha'\beta'\gamma}{}'B_{\alpha'\beta'}{}^{\gamma'} - A_{\beta}^{\gamma'} A_{\gamma'|\alpha}^{\gamma} \tag{VI.3.19}$$

with the transformation matrix

$$A_{\alpha}^{\alpha'} = \begin{pmatrix} \sin\vartheta & \cos\vartheta \\ \cos\vartheta & -\sin\vartheta \end{pmatrix}. \tag{VI.3.20}$$

While for $B_1 = B_{21}{}^2$ the first term in (VI.3.19) is easy to calculate, one has for the inhomogeneous term

$$-A_1^{1'} A_{1'|2}^{2} - A_1^{2'} A_{2'|2}^{2} = -\sin\vartheta \frac{1}{\alpha}\frac{\partial}{r\partial\vartheta}\cos\vartheta + \cos\vartheta \frac{1}{\alpha}\frac{\partial}{r\partial\vartheta}\sin\vartheta = \frac{1}{\alpha r} = {}^*B_1 \quad .$$

Thus, the curvature of the great circles of the spherical coordinates is obtained from the inhomogeneous term. An analogous expression for cylindrical coordinates does not exist: ${}^*B_{1'} = 0$.

The development of a model in the tetrad calculus enables us to separate the geometric auxiliary variables from the physical ones and to explain them as representational variables, which change correspondingly during a change of representation.

The for the new quantity ${}^*B$ results the relation

$${}^*B^{\alpha}{}_{|\alpha} + B_{\alpha}{}^*B^{\alpha} = 0 .$$

The form of the subequations of the field equations remains unaffected compared to those formulated in cylindrical coordinates.

VI.4. The model of van Stockum

An axially symmetric stationary solution, which describes the inside of a rotating cylinder, is given by van Stockum [V]. The metric

$$ds^2 = e^{-\omega^2 R^2}\left(dR^2 + dZ^2\right) + \left(1-\omega^2 R^2\right)R^2 d\varphi^2 + 2\omega R^2 d\varphi\, dt - dt^2 \tag{VI.4.1}$$

is isotropic in R and Z and contains the constant rotational velocity ω. Using the methods (V.2.35) - (V.2.38) we read from the metric two types of 4-bein systems

$$\text{(A)}\quad \begin{aligned} &\overset{1}{e}_1 = \overset{2}{e}_2 = e^{-\frac{1}{2}\omega^2 R^2}, \quad \overset{3}{e}_3 = R, \quad \overset{4}{e}_3 = -i\omega R^2, \quad \overset{4}{e}_4 = 1 \\ &\underset{1}{e}^1 = \underset{2}{e}^2 = e^{\frac{1}{2}\omega^2 R^2}, \quad \overset{3}{e}_3 = \frac{1}{R}, \quad \underset{3}{e}^4 = i\omega R, \quad \underset{4}{e}^4 = 1 \end{aligned} \tag{VI.4.2}$$

$$\text{(B)}\quad \begin{aligned} &\overset{1}{e}_1 = \overset{2}{e}_2 = e^{-\frac{1}{2}\omega^2 R^2}, \quad \overset{3}{e}_3 = \frac{R}{\alpha}, \quad \overset{4}{e}_3 = -i\omega R, \quad \overset{4}{e}_4 = \alpha \\ &\underset{1}{e}^1 = \underset{2}{e}^2 = e^{\frac{1}{2}\omega^2 R^2}, \quad \underset{3}{e}^3 = \frac{\alpha}{R}, \quad \underset{3}{e}^4 = i\omega, \quad \underset{4}{e}^4 = \frac{1}{\alpha} \end{aligned} \tag{VI.4.3}$$

$$\alpha = \frac{1}{\sqrt{1-\omega^2 R^2}}, \quad e^k = e^{-\frac{1}{2}\omega^2 R^2}. \tag{VI.4.4}$$

For the system (A) we calculate the field quantities

$$B_\nu = \left\{-e^{-k}\omega^2 R, 0\right\}, \quad C_\nu = \left\{e^{-k}\frac{1}{R}, 0\right\}, \quad \nu = 1,2. \tag{VI.4.5}$$

Therein B is a quantity to which physical significance is to be assigned, but C is a geometric quantity which is due to the cylindrical representation. The stationary approach is followed by a new component of the Ricci-rotation coefficients

$$H_{mn}{}^s = H_{mn}u^s + H^s{}_m u_n + H^s{}_n u_m, \quad u_m = \{0,0,0,1\}. \tag{VI.4.6}$$

The rotational quantity H has only one component

$$H_{13} = \frac{1}{2}\left(\overset{4}{e}_3 \underset{3|1}{e}^3 + \overset{4}{e}_4 \underset{3|1}{e}^4\right) = ie^{-k}\omega. \tag{VI.4.7}$$

It describes the rotation about the Z axis:

$$H_\alpha = \frac{i}{2}\varepsilon_\alpha{}^{\beta\gamma}H_{\beta\gamma}, \quad H_2 = e^{\frac{1}{2}\omega^2 R^2}\omega.$$

The rotational vector has the constant value ω on the Z axis and increases outwards. To avoid the orbital velocity reaching the speed of light must be $\omega R < 1$. Therefore, the diameter of the cylinder is not arbitrary.

From the field equations one has

$$R_{34} = H^{\alpha}{}_{3\|\alpha} = 0 \tag{VI.4.8}$$

or the Maxwell-like relation

$$\mathrm{rot}\,\vec{H} = 0\,, \tag{VI.4.9}$$

with rot as a covariant differential operator. The quantity C is a quantity which results from the cylindrical representation and drops out from the field equations on the basis of

$$C^{s}{}_{\|s} = 0\ . \tag{VI.4.10}$$

There remains

$$B^{s}{}_{\|s} = H^2, \quad \mathrm{div}\vec{B} = H^2, \quad H^2 = H^{\alpha\beta}H_{\alpha\beta} = 2H^{13}H_{13}\,. \tag{VI.4.11}$$

B is a quantity directed inwards. Its value on the Z axis is 0 and it increases outwards. B has the structure of a centrifugal force, but has the wrong sign. It is coupled to the energy density H^2 of the field as expected from a rotating model.

So far, we have already discussed parts of Einstein's field equations. The complete representation of the system is still missing. Using (VI.4.10) and (VI.4.11) we obtain for the Ricci

$$R_{mn} = \begin{pmatrix} -H^2 & & & \\ & -H^2 & & \\ & & -H^2 & \\ & & & H^2 \end{pmatrix}, \quad R = -2H^2 \tag{VI.4.12}$$

and thus for the Einstein tensor

$$G_{mn} = 2H^2 u_m u_n = -\kappa\mu_0 u_m u_n\,. \tag{VI.4.13}$$

The mass density $\mu_0 = \mu_0(R)$ depends on the distance from the cylinder axis

$$\kappa\mu_0 = 4e^{\omega^2 R^2}\omega^2 \tag{VI.4.14}$$

and therefore has the value $\kappa\mu_0 = 4\omega^2$ on the axis of rotation. However, (VI.4.14) has the structure of the energy density of a field. We have depicted this function in Fig. VI.2Fig. VI.2

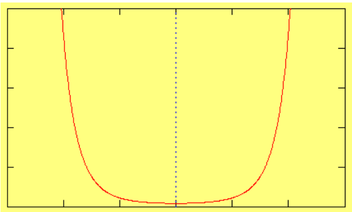

Fig. VI.2. The mass density inside the cylinder.

The simple relation

$$\mu_{0|\alpha} = 2\mu_0 B_\alpha \tag{VI.4.15}$$

satisfies the conservation law.

At a glance at the previous relations we recognize that the model does not provide the relations for a *relativistic rotator*. For such a rotator we expect as Coriolis and centrifugal forces the expressions

$$\Omega_{\alpha 3} = i\alpha^2\omega\sigma_\alpha, \quad F_\alpha = \alpha^2\omega^2\sigma\sigma_\alpha, \quad \alpha = 1\big/\sqrt{1-\omega^2\sigma^2}\ . \tag{VI.4.16}$$

If we refer to the methods of Sec. V.1 we have

$$\sigma^2 = XV - W^2, \quad \sigma_\alpha = \sigma_{|\alpha}\,.$$

For the model of vS is

$$\sigma = R, \quad \sigma_\alpha = R_{|\alpha} = \left\{e^{-k}, 0\right\}. \tag{VI.4.17}$$

After a short calculation, one obtains the values (VI.4.16) from

$$\Omega_{\nu 3} = \frac{1}{\alpha}\left(\alpha\omega\sigma\right)_{|\nu}, \quad F_\nu = \frac{1}{\alpha}\alpha_{|\nu}\,. \tag{VI.4.18}$$

Similar to the electrodynamics of moving media one has

$$\Omega_{\alpha\beta} = \alpha^2 H_{\alpha\beta} = H_{\alpha\beta} + 2F_{[\alpha}v_{\beta]}, \quad v_\beta = \left\{0,0,\omega\sigma\right\}, \quad \alpha = 1,2,3$$

and with

$$\Omega_\alpha = \frac{i}{2}\varepsilon_\alpha{}^{\beta\gamma}\Omega_{\beta\gamma}$$

also

$$\vec{\Omega} = \vec{H} + \vec{v}\times\vec{F}\,. \tag{VI.4.19}$$

The necessary quantities Ω and F can be introduced into theory with $\Omega_{\alpha\beta} = \alpha^2 H_{\alpha\beta}$ and with the help the following method

$$R_{34} = \frac{1}{\alpha^2}\left[\Omega^\nu{}_{3||\nu} - 2\Omega^\nu{}_3 F_\nu\right] = 0$$

or

$$\Omega^{\alpha\beta}{}_{||\alpha} = 2\Omega^{\alpha\beta}F_\alpha\,. \tag{VI.4.20}$$

With

$$\varepsilon^{\alpha\beta\gamma}\Omega_{\gamma||\alpha} = 2\varepsilon^{\alpha\beta\gamma}\Omega_\gamma F_\alpha$$

one finally has

$$\mathrm{rot}\vec{\Omega} = 2\vec{F}\times\vec{\Omega}\,. \tag{VI.4.21}$$

The structure of the model is next to the theory of the electrodynamics of moving media. However, the method is artificial. For the centrifugal force one has

$$F^{\alpha}{}_{\|\alpha} = -\Omega^{\alpha\beta}\Omega_{\alpha\beta} = +\Omega^{\alpha}\Omega_{\alpha}\,,\quad \mathrm{div}\vec{F} = \vec{\Omega}^2\,.$$

The repulsive centrifugal force has as a source the positive rotational energy. However, the last relation is not part of Einstein's field equations and must be added manually. This is a flaw of the model.

Although the results are so far unsatisfactory, we further want to elucidate the structure of the vS model. We still have a second reference system (VI.4.3) which we will bring into a form adapted to the rotation problem

$$\begin{array}{llll} \overset{1}{e}_1 = \overset{2}{e}_2 = e^{k}, & \overset{3}{e}_3 = \dfrac{\sigma}{\alpha}, & \overset{4}{e}_3 = -i\omega\sigma, & \overset{4}{e}_4 = \alpha \\ \underset{1}{e}^1 = \underset{2}{e}^2 = e^{-k}, & \overset{3}{e}_3 = \dfrac{\alpha}{\sigma}, & \underset{3}{e}^4 = i\omega, & \underset{4}{e}^4 = \dfrac{1}{\alpha} \end{array}\,. \tag{VI.4.22}$$

This system (B) is calculated from the system (A) with a Lorentz transformation

$$L^{3'}_{3} = \alpha,\quad L^{3'}_{4} = -i\alpha\omega\sigma,\quad L^{4'}_{3} = i\alpha\omega\sigma,\quad L^{4'}_{4} = \alpha\,, \tag{VI.4.23}$$

which connects the relatively rotating systems

$$\overset{m'}{e}_i = L^{m'}_{m}\overset{m}{e}_i\,,$$

or by reordering the metric (VI.4.1). The associated field quantities are calculated from (VI.4.22) or with the inhomogeneous transformation law of the Ricci-rotation coefficients from the field quantities of the system (A). If we drop the primes at the indices, we have

$$\begin{array}{c} B_m = \omega^2\sigma^2\sigma_m,\quad {}^*C_m = \dfrac{1}{\sigma}\sigma_m,\quad A_{3m}{}^{3} = {}^*C_m - F_m,\quad A_{4m}{}^{4} = F_m \\ H_{mns} = 2i\left[F_{[m}u_{n]}v_s + F_{[s}u_{m]}v_n + F_{[s}u_{n]}v_m\right],\quad v_m = \{0,0,\omega\sigma,0\},\quad u_m = \{0,0,0,1\} \end{array}\,. \tag{VI.4.24}$$

Remarkably, the centrifugal forces missing in the system (A) occur in the system (B), but no Coriolis forces occur. Einstein's field equations now also provide the above-established relationship for the centrifugal force. The B content of Einstein's field equations is not affected by the Lorentz transformation and remains unchanged. It is only to consider

$${}^*C^{s}{}_{\|s} = 0,\quad F^{s}{}_{\|s} = -\Omega^2\,. \tag{VI.4.25}$$

The stress-energy tensor transforms with (VI.4.23) and takes the form of

$$T_{33} = -\alpha^2\omega^2\sigma^2\mu_0,\quad T_{34} = -i\alpha^2\omega\sigma\mu_0,\quad T_{44} = \alpha^2\mu_0\,. \tag{VI.4.26}$$

With (VI.4.25) and Einstein's field equations one obtains these expressions.

vS also discusses three exterior solutions, which are adapted to the interior vS solution. Frehland [F] shows that these metrics can be diagonalized and therefore are static. Bonnor [B] corrects this observation and shows that only one of the three solutions is static. Tipler [T] examines whether CTCs (closed timelike lines) occur on the vS cylinder. Bonnor [B] examines all these solutions with the help of the Weyl tensor.

VI.5. The model of Levy

Levy [L], starts with the metric

$$ds^2 = e^{2(k-u)}\left(dR^2 + dZ^2\right) + e^{-2u} R^2 d\varphi^2 - e^{2u}\left(ad\varphi + dt\right)^2, \tag{VI.5.1}$$

from which the 4-beine for the system (A) can directly be read

$$(A) \quad \begin{array}{llll} \overset{1}{e}_1 = \overset{2}{e}_2 = e^{k-u}, & \overset{3}{e}_3 = e^{-u}R, & \overset{4}{e}_3 = \ iae^{u}, & \overset{4}{e}_4 = e^{u} \\ \underset{1}{e}^1 = \underset{2}{e}^2 = e^{u-k}, & \underset{3}{e}^3 = e^{u}\dfrac{1}{R}, & \underset{3}{e}^4 = -iae^{u}\dfrac{1}{R}, & \underset{4}{e}^4 = e^{-u} \end{array}. \tag{VI.5.2}$$

Therein $a = a(R,Z)$ is a quantity which indicates a rotating system. If one multiplies the brackets in (VI.5.1) and if one reassembles them by using a method described above, one obtains quantities for later use

$$X = e^{-2u}R^2 - e^{2u}a^2, \quad W = ie^{2u}a, \quad V = e^{2u}. \tag{VI.5.3}$$

Furthermore, according to a brief calculation one finds $\sigma = R$, a relation which is typical for axially symmetric systems. This also provides the reference system (B)

$$(B) \quad \begin{array}{llll} \overset{1}{e}_1 = \overset{2}{e}_2 = e^{k-u}, & \overset{3}{e}_3 = \sqrt{X}, & \overset{3}{e}_4 = \dfrac{iae^{2u}}{\sqrt{X}}, & \overset{4}{e}_4 = \dfrac{\sigma}{\sqrt{X}} \\ \underset{1}{e}^1 = \underset{2}{e}^2 = e^{u-k}, & \underset{3}{e}^3 = \dfrac{1}{\sqrt{X}}, & \underset{4}{e}^3 = -iae^{2u}\dfrac{1}{\sigma\sqrt{X}}, & \underset{4}{e}^4 = \dfrac{\sqrt{X}}{\sigma} \end{array}. \tag{VI.5.4}$$

First we calculate the Ricci-rotation coefficients for the system (A). With

$$K_\alpha = k_{|\alpha}, \quad U_\alpha = u_{|\alpha}, \quad \alpha = 1,2$$

the components

$$\begin{gathered} B_\alpha = K_\alpha - U_\alpha, \quad B_{\alpha\beta}{}^{\gamma} = \delta^{\gamma}_{\alpha}B_\beta - g_{\alpha\beta}B^{\gamma} \\ A_{3\alpha}{}^{3} = C_\alpha = {}^*C_\alpha - U_\alpha, \quad {}^*C_\alpha = \frac{1}{\sigma}\sigma_\alpha, \quad \sigma_\alpha = \sigma_{|\alpha}. \\ A_{4\alpha}{}^{4} = U_\alpha, \quad H_{\alpha 3} = -\frac{i}{2}\frac{1}{\sigma}e^{2u}a_{|\alpha} \end{gathered} \tag{VI.5.5}$$

are made up. The quantity H is a member of the collection which we have discussed with (VI.4.6). Furthermore, it can be seen that the transvection of the Ricci-rotation coefficients

$$A_m = A_{sm}{}^{s} = B_m + {}^*C_m = {}^*A_m \tag{VI.5.6}$$

takes a simple form. The quantity $^*C$ is due to the cylindrical structure of the model and has only a geometrical meaning. It satisfies the relation

$${}^*C^{s}{}_{||s} = 0. \tag{VI.5.7}$$

This simplifies the field equations. For the Ricci, we get the relations

$$
\begin{gathered}
R_{\alpha\beta} = -g_{\alpha\beta} B^{\gamma}{}_{\wedge\gamma} + 2M_{\alpha\beta}, \quad M_{\alpha\beta} = {}^*C_{(\alpha}K_{\beta)} - U_\alpha U_\beta - H_{\alpha 3}H_{\beta 3} \\
R_{33} = U^{\gamma}{}_{\wedge\gamma} - H^2, \quad R_{34} = H^{\gamma}{}_{3\wedge\gamma} + 2H^{\gamma}{}_3 U_\gamma, \quad R_{44} = -\left(U^{\gamma}{}_{\wedge\gamma} - H^2\right)
\end{gathered}. \tag{VI.5.8}
$$

Therein the 3-dimensional covariant derivatives are

$$
\begin{gathered}
B^{\gamma}{}_{\wedge\gamma} = B^{\gamma}{}_{|\gamma} + {}^*A_\gamma B^\gamma, \quad U^{\gamma}{}_{\wedge\gamma} = U^{\gamma}{}_{|\gamma} + {}^*A_\gamma U^\gamma, \quad {}^*A_\gamma = B_\gamma + {}^*C_\gamma \\
H^{\gamma}{}_{3\wedge\gamma} = H^{\gamma}{}_{3|\gamma} + {}^*A_\gamma H^{\gamma}{}_3 - {}^*A_{\gamma 3}{}^{\nu} H^{\gamma}{}_{\nu}, \quad {}^*A_{\gamma\beta}{}^{\nu} = B_{\gamma\beta}{}^{\nu} + {}^*C_{\gamma\beta}{}^{\nu} \quad , \\
H^2 = H_{\alpha\beta}H^{\alpha\beta} = 2H_{\alpha 3}H^{\alpha 3}
\end{gathered} \tag{VI.5.9}
$$

whereby we resort to the structures (VI.3.5). The Ricci scalar is

$$
R = {}^*R = -2\left(B^{\gamma}{}_{\wedge\gamma} - M^{\gamma}{}_{\gamma}\right). \tag{VI.5.10}
$$

Thus, the 2-dimensional Einstein tensor takes the simple form

$$
G_{\alpha\beta} = 2\left[M_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} M^{\gamma}{}_{\gamma}\right]. \tag{VI.5.11}
$$

The field equations (VI.5.8) have a structure similar to the field equations of Maxwell's theory of electrodynamics. Before we go into the solutions mentioned by Levy, we propose a simple solution. We assume that the equation system (VI.5.8)) is that of a vacuum solution, so that the relations

$$
U^{\gamma}{}_{\wedge\gamma} - H^2 = 0, \quad H^{\gamma}{}_{\beta\wedge\gamma} + 2H^{\gamma}{}_{\beta} U_\gamma = 0 \tag{VI.5.12}
$$

are valid. If we further require that the parameter u depends only on R, then

$$
U_\gamma = \left\{U_1, 0, 0, 0\right\}
$$

is a quantity which is normal to the cylinder axis. For the quantity a we make the ansatz

$$
a = \omega e^{-2u} \sigma^2, \quad \omega = \text{const.}. \tag{VI.5.13}
$$

Thus, a is only dependent on R. The quantity X from (VI.5.3) takes the form

$$
X = \frac{\sigma^2}{\alpha^2} e^{-2u}, \quad \alpha = \sqrt{1 - \omega^2 \sigma^2}\,. \tag{VI.5.14}
$$

This allows to recalculate the two 4-beine

$$
\text{(A)} \quad
\begin{gathered}
\overset{1}{e}_1 = \overset{2}{e}_2 = e^{k-u}, \quad \overset{3}{e}_3 = \sigma e^{-u}, \quad \overset{4}{e}_3 = i\omega\sigma^2 e^{-u}, \quad \overset{4}{e}_4 = e^{u} \\
\underset{1}{e}^1 = \underset{2}{e}^2 = e^{u-k}, \quad \underset{3}{e}^3 = \frac{1}{\sigma} e^{u}, \quad \underset{3}{e}^4 = -i\omega\sigma e^{-u}, \quad \underset{4}{e}^4 = e^{-u}
\end{gathered}, \tag{VI.5.15}
$$

$$
\text{(B)} \quad
\begin{gathered}
\overset{1'}{e}_1 = \overset{2'}{e}_2 = e^{k-u}, \quad \overset{3'}{e}_3 = \frac{\sigma}{\alpha} e^{-u}, \quad \overset{3'}{e}_4 = i\alpha\omega\sigma e^{u}, \quad \overset{4'}{e}_4 = \alpha e^{u} \\
\underset{1'}{e}^1 = \underset{2'}{e}^2 = e^{u-k}, \quad \underset{3'}{e}^3 = \frac{\alpha}{\sigma} e^{u}, \quad \underset{4'}{e}^3 = -i\alpha\omega e^{u}, \quad \underset{4'}{e}^4 = \frac{1}{\alpha} e^{-u}
\end{gathered}. \tag{VI.5.16}
$$

It can be seen that the two systems are related via the Lorentz transformation

$$L^{3'}_{3} = \alpha, \quad L^{3'}_{4} = i\alpha\omega\sigma, \quad L^{4'}_{3} = -i\alpha\omega\sigma, \quad L^{4'}_{4} = \alpha\,. \tag{VI.5.17}$$

Now the 4-bein indices of the system (B) are primed. Since ω is the rotational speed of an observer and α the corresponding Lorentz factor, the system (B) is to be regarded as a system in rest in rotating coordinates. In fact, the coordinate transformation

$$\begin{aligned} &\Lambda^{3}_{3'} = \alpha, \quad \Lambda^{4}_{3'} = 0, \quad \Lambda^{3}_{4'} = -i\alpha\omega e^{2u}, \quad \Lambda^{4}_{4'} = \frac{1}{\alpha} \\ &\Lambda^{3'}_{3} = \frac{1}{\alpha}, \quad \Lambda^{4'}_{3} = 0, \quad \Lambda^{3'}_{4} = \ i\alpha\omega e^{2u}, \quad \Lambda^{4'}_{4} = \alpha \end{aligned} \tag{VI.5.18}$$

converts the system (B) with the help of

$$\overset{m'}{e}_{i'} = \Lambda^{i}_{i'}\overset{m'}{e}_{i} \tag{VI.5.19}$$

into the static Weyl metric

$$(S) \qquad \begin{aligned} &\overset{3'}{e}_{3'} = \sigma e^{-u}, \quad \overset{3'}{e}_{4'} = 0, \quad \overset{4'}{e}_{3'} = 0, \quad \overset{4'}{e}_{4'} = e^{u} \\ &\underset{3'}{e}^{3'} = \frac{1}{\sigma}e^{u}, \quad \underset{4'}{e}^{3'} = 0, \quad \underset{3'}{e}^{4'} = 0, \quad \underset{4'}{e}^{4'} = e^{-u} \end{aligned}\ . \tag{VI.5.20}$$

Further, we transform the system (A) into static coordinates

$$\overset{m}{e}_{i'} = \Lambda^{i}_{i'}\overset{m}{e}_{i}, \quad \overset{m}{e}_{i'} = L^{m}_{m'}\overset{m'}{e}_{i'}\,. \tag{VI.5.21}$$

One obtains

$$(C) \qquad \begin{aligned} &\overset{3}{e}_{3'} = \alpha\sigma e^{-u}, \quad \overset{3}{e}_{4'} = -i\alpha\omega\sigma e^{u}, \quad \overset{4}{e}_{3'} = i\alpha\omega\sigma\cdot\sigma e^{-u}, \quad \overset{4}{e}_{4'} = \alpha e^{u} \\ &\underset{3}{e}^{3'} = \alpha\frac{1}{\sigma}e^{u}, \quad \underset{4}{e}^{3'} = i\alpha\omega\sigma\frac{1}{\sigma}e^{u}, \quad \underset{3}{e}^{4'} = -i\alpha\omega\sigma e^{-u}, \quad \underset{4}{e}^{4'} = \alpha e^{-u} \end{aligned}, \tag{VI.5.22}$$

ie the structure of a system of type (C). Applying this to the line element one immediately gets

$$ds^{2} = e^{2(k-u)}\left(dR^{2} + dZ^{2}\right) + e^{-2u}\sigma^{2}d\varphi^{2} - e^{2u}dt^{2}\,, \tag{VI.5.23}$$

the cylindrically static Weyl structure.

We turn to the system (B). It is the system of static observers described in rotating coordinates. Calculating the field equations the quantity

$$F_{\nu} = \frac{1}{\alpha}\alpha_{|\nu} = \alpha^{2}\omega^{2}\sigma\sigma_{\nu} \tag{VI.5.24}$$

occurs. Without doubt, this is the relativistic generalization of the classical centrifugal force. Indeed we get

$$A_{4\nu}{}^{4} = \frac{1}{\alpha}\alpha_{|\nu} + u_{|\nu}\,, \rightarrow {}'U_{\nu} = F_{\nu} + U_{\nu}\,. \tag{VI.5.25}$$

Considering the quantity

$$D_{\alpha 3} = \frac{1}{2}\left[\overset{3}{e}_{3}\underset{4|\alpha}{e^{3}} + \overset{3}{e}_{4}\underset{4|\alpha}{e^{4}}\right] = -i\omega\sigma\left(U_{\alpha} + F_{\alpha}\right) \tag{VI.5.26}$$

we must demand that

$$D_{\alpha 3} = 0, \quad U_{\alpha} = -F_{\alpha}\,, \tag{VI.5.27}$$

in order to avoid the occurrence of rotational forces in the static system. As the consequence of (VI.5.25) one has

$${}'U_{\alpha} = 0\,. \tag{VI.5.28}$$

From (VI.5.25) and (VI.5.27) one obtains

$$u_{|\nu} = -\left(\ln\alpha\right)_{|\nu}$$

or

$$e^{2u} = 1 - \omega^{2}\sigma^{2} = \frac{1}{\alpha^{2}} \tag{VI.5.29}$$

and with (VI.5.13)

$$a = \alpha^{2}\omega\sigma^{2}\,. \tag{VI.5.30}$$

If we insert these values into (VI.5.1) we first get

$$ds^{2} = e^{2k}\alpha^{2}\left(dR^{2} + dZ^{2}\right) + \alpha^{2}\sigma^{2}d\varphi^{2} - \frac{1}{\alpha^{2}}\left(\alpha^{2}\omega\sigma^{2}d\varphi + dt\right)^{2}, \tag{VI.5.31}$$

and after reordering

$$ds^{2} = e^{2k}\alpha^{2}\left(dR^{2} + dZ^{2}\right) + \sigma^{2}\left(d\varphi - \omega dt\right)^{2} - dt^{2}\,. \tag{VI.5.32}$$

We immediately recognize that the substitution

$$\varphi \rightarrow \varphi + \omega t$$

leads to the static line element. From (VI.5.32) one obtains the 4-beine, again of the type (B)

$$(B)\qquad \begin{array}{l} \overset{1'}{e}_{1} = \overset{2'}{e}_{2} = e^{k}\alpha, \quad \overset{3'}{e}_{3} = \sigma, \quad \overset{3'}{e}_{4} = i\omega\sigma, \quad \overset{4'}{e}_{4} = 1 \\ \underset{1'}{e^{1}} = \underset{2'}{e^{2}} = e^{k}\dfrac{1}{\alpha}, \quad \underset{3'}{e^{3}} = \dfrac{1}{\sigma}, \quad \underset{4'}{e^{3}} = -i\omega, \quad \underset{4'}{e^{4}} = 1 \end{array} \tag{VI.5.33}$$

With these results we get directly from (VI.5.26)

$$D_{\alpha\beta} = 0, \quad \alpha = 1,2,3\,.$$

In addition, the coordinate transformation is simplified

$$\begin{aligned}&\Lambda^3_{3'} = \alpha, \quad \Lambda^3_{4'} = -\frac{1}{\alpha} i\omega, \quad \Lambda^4_{4'} = \frac{1}{\alpha} \\ &\Lambda^{3'}_3 = \frac{1}{\alpha}, \quad \Lambda^{3'}_4 = \frac{1}{\alpha} i\omega, \quad \Lambda^{4'}_4 = \frac{1}{\alpha}\end{aligned}\ . \tag{VI.5.34}$$

With (VI.5.19) and with $\alpha = e^{-u}$ one again receives the static Weyl system

$$(S) \qquad \begin{aligned}&\overset{1'}{e}_{1'} = \overset{2'}{e}_{2'} = e^k \alpha, \quad \overset{3'}{e}_{3'} = \alpha\sigma, \quad \overset{4'}{e}_{4'} = \frac{1}{\alpha} \\ &\underset{1'}{e}^{1'} = \underset{2'}{e}^{2'} = e^k \frac{1}{\alpha}, \quad \underset{3'}{e}^{3} = \frac{1}{\alpha\sigma}, \quad \underset{4'}{e}^{4} = \alpha\end{aligned}\ , \tag{VI.5.35}$$

whereby only the quantity k is be to determined by using the field equations.

Let us turn to the system (A). Instead (V.5.15) we get with our ansatz

$$(A) \qquad \begin{aligned}&\overset{1}{e}_1 = \overset{2}{e}_2 = e^k \alpha, \quad \overset{3}{e}_3 = \alpha\sigma, \quad \overset{4}{e}_3 = i\alpha\omega\sigma^2, \quad \overset{4}{e}_4 = \frac{1}{\alpha} \\ &\underset{1}{e}^1 = \underset{2}{e}^2 = e^{-k} \frac{1}{\alpha}, \quad \underset{3}{e}^3 = \frac{1}{\alpha\sigma}, \quad \underset{3}{e}^4 = -i\alpha\omega\sigma, \quad \underset{4}{e}^4 = \alpha\end{aligned}\ . \tag{VI.5.36}$$

We calculate the field quantities. We borrow from (VI.5.5)

$$H_{\alpha 3} = -\frac{i}{2}\frac{1}{\sigma} e^{2u} a_{|\alpha} = -\frac{i}{2}\frac{1}{\sigma}\frac{1}{\alpha^2}\left(\alpha^2 \omega \sigma^2\right)_{|\alpha}$$

and we get the expression of the Coriolis force for the relativistic rotator in cylindrical coordinates

$$H_{\alpha 3} = -i\alpha^2 \omega \sigma_{|\alpha}\,. \tag{VI.5.37}$$

Further from $R_{34} = 0$

$$H^{\alpha}{}_{\beta\wedge\alpha} = 2H^{\alpha}{}_{\beta} F_{\alpha}, \quad \alpha,\beta = 1,2,3\,, \tag{VI.5.38}$$

a relation which is significant for electrodynamics. The centrifugal force F has already been defined in (VI.5.24). It fulfills the condition

$$F^{\alpha}{}_{\wedge\alpha} = -H^2\,. \tag{VI.5.39}$$

Thus we have only to evaluate the relation

$$G_{\alpha\beta} = 2\left[M_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} M^{\gamma}{}_{\gamma}\right] = 0 \tag{VI.5.40}$$

according to (V.5.11). Since

$$H_{\alpha 3}H_{\beta 3} + F_\alpha F_\beta = -\alpha^2\omega^2\sigma_\alpha\sigma_\beta$$

applies, one has

$$G_{11} = \left[2{}^*C_1K_1 + 2\alpha^2\omega^2\sigma_1\sigma_1\right] - \left[{}^*C_1K_1 + \alpha^2\omega^2\sigma_1\sigma_1\right] = 0$$

or with $*C_\alpha = \frac{1}{\sigma}\sigma_\alpha$

$$K_1 = -\alpha^2\omega^2\sigma\sigma_1 = -F_1 .$$

Thus, we have $K_1 = k_{|1} = -\frac{1}{\alpha}\alpha_{|1} = \left(\ln\frac{1}{\alpha}\right)_{|1}$, $K_2 = 0$ and

$$e^k = \frac{1}{\alpha} . \tag{VI.5.41}$$

Further results $B_1 = K_1 - U_1 = -F_1 + F_1 = 0$. For the model only remain the field quantity $*C_\alpha$ describing the cylindrical structure and the two dynamic quantities F_α and $H_{\alpha\beta}$ which, in analogy to classical mechanics, describe the forces acting on a rotating observer. The line element is therefore the line element of the flat space

$$ds^2 = dR^2 + dZ^2 + R^2 d\varphi^2 - dt^2 . \tag{VI.5.42}$$

The rotational properties of the model are not genuine because in this case a rotating observer is introduced. Instead of the extensive calculations and analyses which we have made here, one can start from the flat metric (VI.5.42) and from the only, non-vanishing component of the Ricci-rotation coefficients

$$A_{31}{}^3 = *C_1 = \frac{1}{\sigma}\sigma_{|1} = \frac{1}{R} .$$

Further, one generates the dynamic variables F and H by means of a Lorentz transformation of the type (VI.5.17). We will examine how the solutions proposed by Levy behave.

Levy offers three solutions for his ansatz, but we are investigating only one of them. In all proposals, the quantity u depends on the two coordinates R and Z. The assigned field variable U therefore does not point to the cylinder axis, but to the origin of the coordinate system. Therefore one could interpret the field quantity U as a gravitational force. Levy did not give a value for u for all three solutions, but he pointed out that u satisfies Weyl's solution equation. Levy also introduces some constants in his approach. We simplify the model by putting one of these to 0. Thereby the formal structure of the theory is not altered. We have

$$k = \frac{1}{2}\ln\left(\frac{1}{\sqrt{R}}e^{2u}\right), \quad a = -Re^{-2u} + \beta, \quad e^k = R^{-\frac{1}{4}}e^u, \quad e^{k-u} = R^{-\frac{1}{4}} . \tag{VI.5.43}$$

With these values one obtains the following 4-bein system

$$\text{(A)} \quad \begin{array}{l} \overset{1}{e}_1 = \overset{2}{e}_2 = R^{-\frac{1}{4}}, \quad \overset{3}{e}_3 = Re^{-u}, \quad \overset{4}{e}_3 = i\left(\beta e^u - R e^{-u}\right), \qquad \overset{4}{e}_4 = e^u \\ \underset{1}{e}^1 = \underset{2}{e}^2 = R^{\frac{1}{4}}, \quad \underset{3}{e}^3 = \frac{1}{R}e^u, \quad \underset{3}{e}^4 = -i\left(\beta e^u - R e^{-u}\right)\frac{1}{R}, \quad \underset{4}{e}^4 = e^{-u} \end{array} , \tag{VI.5.44}$$

which only contains the unknown parameter u. For the line element (VI.5.1) the values (VI.5.43) are used to determine the quantities

$$\begin{aligned} X &= e^{-2u}R^2 - e^{-2u}a^2 = 2\beta R - \beta^2 e^{2u} \\ W &= ie^{2u}a = i\left(\beta e^{2u} - R\right), \quad V = e^{2u} \end{aligned} . \tag{VI.5.45}$$

For the geometric quantity $*C$ we obtain

$$*C_m = \left\{R^{-3/4}, 0, 0, 0\right\}. \tag{VI.5.46}$$

It fulfills the relation

$$*C^s{}_{||s} = 0 \tag{VI.5.47}$$

and does not provide a significant contribution to the solution system. The field quantity for the parameter u is contained in

$$A_{3\alpha}{}^3 = C_\alpha = *C_\alpha - U_\alpha . \tag{VI.5.48}$$

The parameter u is no longer present in the quantity

$$B_\alpha = -\frac{1}{4}*C_\alpha . \tag{VI.5.49}$$

(VI.5.49) can be treated in analogy to (VI.5.47). One has

$$B^s{}_{||s} = 0 . \tag{VI.5.50}$$

For the Coriolis force we get

$$H_{\alpha 3} = i\left(U_\alpha - 2B_\alpha\right), \tag{VI.5.51}$$

a physically not very convincing expression. With the above values one also has $M_{\alpha\beta} = 0$ and thus

$$G_{\alpha\beta} = 0 \tag{VI.5.52}$$

is satisfied.

For the remaining components of the Ricci the Maxwell-like relations

$$\begin{gathered} R_{33} = U^s{}_{||s} - H^2, \quad R_{34} = H^\alpha{}_{\beta\wedge\alpha} + 2H^\alpha{}_\beta U_\alpha, \quad R_{44} = -\left(U^s{}_{||s} - H^2\right) \\ R_{34} = -i\left(U^s{}_{||s} - H^2\right) \end{gathered} \tag{VI.5.53}$$

are obtained. As the consequence of $R = 0$ vacuum field equations exist, provided that (VI.5.53) can be solved for U with $R_{mn} = 0$. Since

$$D_{\alpha 3} = \frac{1}{2\sigma}\left[\frac{W}{X}X_{|\alpha} - W_{|\alpha}\right]$$

does not vanish for this ansatz of Levy the system cannot be globally transformed to a static one. The ansatz describes a genuine, but little realistic, rotating system.

Bardeen [B] has worked on the problem of local rotation in some of his papers. If rotation effects can be transformed away only locally but not globally, a *differential rotation law* is present. Then the angular speed $\omega = \omega(R, Z)$ depends on the distance. If one has performed a transformation into the system (B) and if the quantity D defined above does not vanish, (B) is a locally non-rotating system (LNR). An arbitrary observer of the system (B) in the distance $\{R, Z\}$ can define himself as in rest. Coriolis and centrifugal forces do not act on him. However, since the points lying more inwards rotate with a higher speed than the observers in a position more outwards, deformations of the volume element occur around the observer and thus shear forces D take place. Supremely, the differential rotation law requires the angular velocity to be reduced outwards and to vanish at infinity. As a result, there are no forces at infinity which are due to the rotation. If the rotational speed does not slow down fast enough, there is a possibility that the points rotating at a certain distance will reach or exceed the speed of the light. Thus, the model would have a cut-off radius. The region outside this radius is unphysical and so the whole model[112].

Rotating exterior fields are usually assigned to rotating stellar objects. The fact that a rotating source produces an exterior field that exhibits rotational effects such as Coriolis force and centrifugal force is called *frame dragging*. One of the most important models, which probably describes Nature well, is the model of Kerr.

[112] The fact that the rotating Gödel cosmos is based on a differential rotation law has been overlooked for years. Thus, it was believed that closed timelike curves (CTCs) exist in the Gödel cosmos. However, these contradictory phenomena lie beyond the cut-off radius.

VI.6. The model of Lewis

Lewis [L] has described an axially symmetric model. From the metric of this model the quantities

$$
\begin{aligned}
&X = \gamma^2\left[\beta_1^2 R^2 e^{-2u} - \omega^2\beta_2^{-2} e^{2u}\right] \\
&W = i\gamma^2\omega\left[\beta_1\beta_2 R^2 e^{-2u} - \beta_1^{-1}\beta_2^{-1} e^{2u}\right], \quad \gamma = \frac{1}{\sqrt{1-\omega^2}}, \quad \omega = \text{const.} \qquad (\text{VI.6.1}) \\
&V = \gamma^2\left[\beta_1^{-2} e^{2u} - \omega^2\beta_2^2 R^2 e^{-2u}\right]
\end{aligned}
$$

can be read. As expected, the relation $\sigma = R$ results from $\sigma^2 = XV - W^2$. Lewis notes that with his approach he describes observers who rotate in a static Weyl field. Therefore the model is not a *genuine* rotating model. We will deal extensively with the question of the transformation to a static system and the kind of rotation.

To calculate the 4-beine we decompose the quantities (VI.6.1) into

$$
\begin{aligned}
&X_1 = \gamma^2\beta_1^2 R^2 e^{-2u}, \quad X_2 = -\gamma^2\omega^2\beta_2^{-2} e^{2u} \\
&V_2 = \gamma^2\beta_1^{-2} e^{2u}, \quad V_1 = -\gamma^2\omega^2\beta_2^2 R^2 e^{-2u}
\end{aligned} \qquad (\text{VI.6.2})
$$

and we finally have a mixed-type system (C)

$$
(C) \quad
\begin{aligned}
&\overset{3}{e}_3 = \gamma R\beta_1 e^{-u}, \quad \overset{3}{e}_4 = i\gamma\omega R\beta_2 e^{-u}, \quad \overset{4}{e}_3 = -i\gamma\omega\beta_2^{-1} e^{u}, \quad \overset{4}{e}_4 = \gamma\omega\beta_1^{-1} e^{u} \\
&\underset{3}{e}^3 = \gamma\frac{1}{R\beta_1 e^{-u}}, \quad \underset{4}{e}^3 = -i\gamma\omega\beta_2 e^{-u}, \quad \underset{3}{e}^4 = i\gamma\omega\frac{1}{R\beta_2 e^{-u}}, \quad \underset{4}{e}^4 = \gamma\frac{1}{\beta_1^{-1} e^{u}}
\end{aligned} . \qquad (\text{VI.6.3})
$$

The tetrad differentials cohere with the coordinate differentials via

$$
dx^3 = \gamma\left[\beta_1 d\varphi - \omega\beta_2 dt\right] R e^{-u}, \quad dx^4 = i\gamma\left[\beta_1^{-1} dt - \omega\beta_2^{-1} d\varphi\right] e^{u}.
$$

The fact that the model of Lewis is the static Weyl metric in rotating coordinates can be shown most quickly with a coordinate transformation that contains only constant parameters

$$
\begin{aligned}
&\Lambda_{3'}^{3} = \gamma\beta_1^{-1}, \quad \Lambda_{3'}^{4} = i\gamma\omega\beta_2^{-1}, \quad \Lambda_{4'}^{3} = -i\gamma\omega\beta_2, \quad \Lambda_{4'}^{4} = \gamma\beta_1 \\
&\Lambda_{3}^{3'} = \gamma\beta_1, \quad \Lambda_{4}^{3'} = i\gamma\omega\beta_2, \quad \Lambda_{3}^{4'} = -i\gamma\omega\beta_2^{-1}, \quad \Lambda_{4}^{4'} = \gamma\beta_1^{-1}
\end{aligned} . \qquad (\text{VI.6.4})
$$

With

$$
\overset{m}{e}_{i'} = \Lambda_{i'}^{i}\overset{m}{e}_i, \quad \underset{m}{e}^{i'} = \Lambda_i^{i'}\underset{m}{e}^i, \quad \Lambda_{i'}^{i}\Lambda_k^{i'} = \delta_k^i
$$

the static Weyl system is immediately obtained

$$
(S) \quad
\begin{aligned}
&\overset{3}{e}_{3'} = R e^{-u}, \quad \overset{3}{e}_{4'} = 0, \quad \overset{4}{e}_{3'} = 0, \quad \overset{4}{e}_{4'} = e^{u} \\
&\underset{3}{e}^{3'} = \frac{1}{R} e^{u}, \quad \underset{4}{e}^{3'} = 0, \quad \underset{3}{e}^{4'} = 0, \quad \underset{4}{e}^{4'} = e^{-u}
\end{aligned} . \qquad (\text{VI.6.5})
$$

We wonder whether a representation is possible in systems (A) and (B). To get closer to this problem, we define the orbital velocities of the rotating observers as

$$v = \omega\rho, \quad \rho = R\beta_1\beta_2 e^{-2u}, \quad \alpha = \frac{1}{\sqrt{1-\omega^2\rho^2}}. \tag{VI.6.6}$$

The structure variables can be exposed more easily with the new quantity ρ

$$X = \gamma^2\left(\rho^2 - \omega^2\right)\beta_2^{-2}e^{2u}, \quad W = i\gamma^2\omega R\left(\rho - \rho^{-1}\right), \quad V = \alpha^{-2}\gamma^2\beta_1^{-2}e^{2u}. \tag{VI.6.7}$$

With these expressions we obtain the system (A)

$$\begin{array}{lll} \overset{3'}{e}_{3} = \alpha R\dfrac{1}{\gamma}\beta_1 e^{-u}, & \overset{4'}{e}_{3} = i\alpha\gamma\omega R\beta_1\left(\rho-\rho^{-1}\right)e^{-u}, & \overset{4'}{e}_{4} = \dfrac{1}{\alpha}\gamma\beta_1^{-1}e^{u} \\ \underset{3'}{e}^{3} = \dfrac{1}{\alpha R}\gamma\beta_1^{-1}e^{u}, & \underset{3'}{e}^{4} = -i\alpha\gamma\omega\beta_1\left(\rho-\rho^{-1}\right)e^{-u}, & \underset{4'}{e}^{4} = \alpha\dfrac{1}{\gamma}\beta_1 e^{-u} \end{array}. \tag{VI.6.8}$$

The same relations are obtained by means of the Lorentz transformation

$$L^{3'}_{3} = \alpha, \quad L^{3'}_{4} = -i\alpha v, \quad L^{4'}_{2} = i\alpha v, \quad L^{4'}_{4} = \alpha \tag{VI.6.9}$$

from the system (C) by

$$\overset{m'}{e}_{i} = L^{m'}_{m}\overset{m}{e}_{i}.$$

The system (A) again can be transformed into static coordinates by means of the coordinate transformation (VI.6.4). With

$$\overset{m'}{e}_{i'} = \Lambda^{i}_{i'}\overset{m'}{e}_{i} \tag{VI.6.10}$$

one obtains a representation of the type (C)

$$(C')\quad \begin{array}{llll} \overset{3'}{e}_{3'} = \alpha R e^{-u}, & \overset{3'}{e}_{4'} = -i\alpha v e^{u}, & \overset{4'}{e}_{3'} = i\alpha v R e^{-u}, & \overset{4'}{e}_{4'} = \alpha e^{u} \\ \underset{3'}{e}^{3'} = \alpha\dfrac{1}{R}e^{u}, & \underset{4'}{e}^{3'} = i\alpha v\dfrac{1}{R}e^{u}, & \underset{3'}{e}^{4'} = -i\alpha v e^{-u}, & \underset{4'}{e}^{4'} = \alpha e^{-u} \end{array}. \tag{VI.6.11}$$

With

$$\overset{m}{e}_{i'} = L^{m}_{m'}\overset{m'}{e}_{i'}$$

however, the static system (VI.6.5) is restored. We have carried out numerous transformations on the reference systems of the metric of Lewis. We have illustrated rotating reference systems, which, however, all can be reduced to the static Weyl metric. We have, however, renounced the representation in the system (B) because the complicated expression X in (VI.6.1) or in (VI.6.7) makes the calculations very tedious. We limit ourselves to the observation that is

$$\frac{1}{X}X_{|\alpha} \neq \frac{1}{W}W_{|\alpha}$$

and therefore a differential rotation law is present.

Without going into detail, we note the field quantities for the system (C ')

$$
\begin{gathered}
{}'C_{\alpha} = {}'A_{3\alpha}{}^{3} = C_{\alpha} + ivH_{\alpha 3}, \quad C_{\alpha} = {}^{*}C_{\alpha} - U_{\alpha}, \quad H_{\alpha 3} = i\alpha^{2}v\left(C_{\alpha} - U_{\alpha}\right) \\
{}'U_{\alpha} = {}'A_{4\alpha}{}^{4} = U_{\alpha} - ivH_{\alpha 3}
\end{gathered}, \qquad \text{(VI.6.12)}
$$

where we have omitted the primes on the indices. It can be realized that there exists a certain similarity to the electrodynamics of moving media. For the Ricci one obtains

$$
\begin{aligned}
R_{33} &= -\left[{}'C^{\alpha}{}_{\wedge\alpha} + H^{2}\right] \\
R_{44} &= -\left[{}'U^{\alpha}{}_{\wedge\alpha} + H^{2}\right]. \\
R_{34} &= H^{\alpha}{}_{3\wedge\alpha} + {}'U_{\alpha}H^{\alpha}{}_{3}
\end{aligned} \qquad \text{(VI.6.13)}
$$

However, the expressions are very complicated, since the quantities $'C$ and $'U$ contain the quantity H. For the representation as solution equations we refer to the original paper of Lewis.

VI.7. The model of Som, Teixeira, and Wolk

Som [S] Teixeira, and Wolk have found a solution that is related to the solution of Lewis and also describes rotating observers in the Weyl field. First they put

$$\begin{aligned} X &= R\left(\mathrm{ch}\varepsilon + \mathrm{sh}\varepsilon\,\mathrm{ch}2\psi\right) \\ V &= R\left(\mathrm{ch}\varepsilon - \mathrm{sh}\varepsilon\,\mathrm{ch}2\psi\right). \\ W &= -iR\,\mathrm{sh}\varepsilon\,\mathrm{sh}2\psi \end{aligned} \tag{VI.7.1}$$

Then the hyperbolic functions are eliminated with

$$\omega = \mathrm{th}\psi, \quad \mathrm{sh}\psi = \mathrm{th}\psi \Big/ \sqrt{1-\mathrm{th}^2\psi} = \gamma\omega, \quad \gamma = 1\Big/\sqrt{1-\omega^2}\ . \tag{VI.7.2}$$

After a short calculation one obtains

$$X = R\gamma^2\left(e^{\varepsilon} - \omega^2 e^{-\varepsilon}\right), \quad V = R\gamma^2\left(e^{-\varepsilon} - \omega^2 e^{\varepsilon}\right), \quad W = -i\gamma^2\omega R\left(e^{\varepsilon} - e^{-\varepsilon}\right). \tag{VI.7.3}$$

With

$$\varepsilon = \ln R + 2u, \quad u = \frac{M}{r}, \quad r^2 = R^2 + Z^2, \quad e^{\varepsilon} = Re^{2u} \tag{VI.7.4}$$

one finally has

$$X = \gamma^2\left(R^2 e^{2u} - \omega^2 e^{-2u}\right), \quad V = \gamma^2\left(e^{-2u} - \omega^2 R^2 e^{2u}\right), \quad W = -i\gamma^2\omega\left(R^2 e^{2u} - e^{-2u}\right). \tag{VI.7.5}$$

The metric with the above variables is obviously of type (C). We therefore decompose into

$$\sqrt{X_1} = \gamma Re^{u}, \quad \sqrt{X_2} = -i\gamma\omega e^{-u}, \quad \sqrt{V_1} = -i\gamma\omega R e^{u}, \quad \sqrt{V_2} = \gamma e^{-u}\ . \tag{VI.7.6}$$

This makes it possible to write the 4-bein system for the system (C) as

$$(C)\qquad \begin{matrix} \overset{3}{e}_3 = \gamma Re^{u}, & \overset{3}{e}_4 = -i\gamma\omega Re^{u}, & \overset{4}{e}_3 = i\gamma\,\omega e^{-u}, & \overset{4}{e}_4 = \gamma e^{u} \\ \underset{3}{e}^3 = \gamma\dfrac{1}{R}e^{-u}, & \underset{3}{e}^4 = -i\gamma\omega\dfrac{1}{R}e^{-u}, & \underset{4}{e}^3 = i\gamma\omega e^{u}, & \underset{4}{e}^4 = \gamma e^{-u} \end{matrix}\ . \tag{VI.7.7}$$

Then one has

$$dx^3 = \gamma R\left(d\varphi + \omega dt\right)e^{u}, \quad dx^4 = i\gamma\left(\omega d\varphi + dt\right)e^{-u}.$$

Furthermore, it is evident that with a coordinate transformation with constant parameters

$$\begin{gathered} \overset{m}{e}_{i'} = \Lambda^{i}_{i'}\overset{m}{e}_i \\ \Lambda^{3}_{3'} = \gamma, \quad \Lambda^{3}_{4'} = -i\gamma\omega, \quad \Lambda^{4}_{3'} = i\gamma\omega, \quad \Lambda^{4}_{4'} = \gamma \end{gathered} \tag{VI.7.8}$$

the system (C) can be transformed into the static system (S). With

$$u = \frac{M}{r}, \quad k = -\frac{MR^2}{2r^4} \tag{VI.7.9}$$

one obtains the Curzon metric[113]. With (VI.7.7) one can make shure that no rotational effects occur in the system (C). One has

$$H_{mn}{}^{s} = 0,$$

where H is defined in (VI.4.6). Also no centrifugal forces are contained in the quantities

$$A_{3\alpha}{}^{3} = C_{\alpha} = {}^{*}C_{\alpha} + U_{\alpha}, \quad A_{4\alpha}{}^{4} = -U_{\alpha}. \tag{VI.7.10}$$

For the system (C) (VI.7.7) provides the expressions known by the Curzon metric. The STW metric describes a static system in rotating coordinates. The latter have no influence on the geometric structure of the theory.

STW provide a second, much more complex solution. For this applies

$$\begin{aligned} X &= \frac{RR_0}{1-\omega^2}\left(\beta_1^{-2} e^{\Phi} - \omega^2 \beta_2^{-2} e^{-\Phi}\right) \\ W &= -i\frac{\omega R}{1-\omega^2}\left(\beta_1^{-1}\beta_2 e^{\Phi} - \beta_1\beta_2^{-1} e^{-\Phi}\right), \\ V &= \frac{R/R_0}{1-\omega^2}\left(\beta_1^{2} e^{-\Phi} - \omega^2\beta_2^{2} e^{\Phi}\right) \end{aligned} \tag{VI.7.11}$$

the constants being defined by

$$\begin{gathered} \beta_1^2 = (1-K)^2 \frac{(1+K)e^{u_0} - e^{-u_0}}{(1-K)e^{u_0} + e^{-u_0}}, \quad \beta_2^2 = (1-K)^2 \frac{(1-K)e^{u_0} - e^{-u_0}}{(1+K)e^{u_0} + e^{-u_0}} \\ \omega^2 = \frac{(1-K^2)e^{2u_0} - e^{-2u_0} - 2K}{(1-K^2)e^{2u_0} - e^{-2u_0} + 2K} \end{gathered}. \tag{VI.7.12}$$

K and u_0 are also constants. Further, one has

$$e^{\Phi} = \frac{R}{R_0} e^{\frac{2M}{r}} = \frac{R}{R_0} e^{2u}. \tag{VI.7.13}$$

The metric factors of the R- and Z-terms are $e^{2\lambda}$ and given by

$$\lambda = \frac{1+\omega^2}{1-\omega^2}\left(\frac{M}{r} - \frac{M^2 R^2}{r^3}\right). \tag{VI.7.14}$$

The factors differ from the metric factors of the Curzon metric only by the constant factor in front the brackets.

STW specify a coordinate transformation with which the rotating system can be transformed into the static one

$$\begin{aligned} R_0\varphi &\to \gamma\beta_1 R_0\varphi + i\gamma\omega\beta_2 x^4 \\ x^4 &\to \gamma\beta_1^{-1} x^4 - i\gamma\omega\beta_2^{-1} R_0\varphi \end{aligned}. \tag{VI.7.15}$$

---

[113] STW take the opposite sign for u. Accordingly, the quantity U_m also has the opposite sign.

With this and (VI.7.13) and (VI.7.14) results

$$ds^2 = e^{2\lambda}\left(dR^2 + dZ^2\right) + e^{2u}R^2 d\varphi^2 - e^{-2u}dt^2, \qquad \text{(VI.7.16)}$$

a metric that is little different from the Curzon metric. The rotation of the model is not genuine.

We also take a look at the field quantities which can be calculated in the rotating coordinate system. We decompose (VI.7.11) and we use (VI.7.13)

$$\begin{aligned} X_1 &= R^2\gamma^2\beta_1^{-2}e^{2u}, & X_2 &= -R_0^2\gamma^2\omega^2\beta_2^{-2}e^{-2u} \\ V_1 &= -\frac{R^2}{R_0^2}\gamma^2\omega^2\beta_2^2 e^{2u}, & V_2 &= \gamma^2\beta_1^2 e^{-2u} \end{aligned} . \qquad \text{(VI.7.17)}$$

These quantities we use for the 4-bein system of type (C)

$$\text{(C)} \quad \begin{aligned} &\overset{3}{e}{}_3 = R\gamma\beta_1^{-1}e^u, \quad \overset{3}{e}{}_4 = i\frac{R}{R_0}\gamma\omega\beta_2 e^u, \quad \overset{4}{e}{}_3 = -iR_0\gamma\omega\beta_2^{-1}e^{-u}, \quad \overset{4}{e}{}_4 = \gamma\beta_1 e^u \\ &\underset{3}{e}{}^3 = \frac{1}{R}\gamma\beta_1 e^{-u}, \quad \underset{3}{e}{}^4 = -i\frac{R_0}{R}\gamma\omega\beta_2^{-1}e^{-u}, \quad \underset{4}{e}{}^3 = -i\frac{1}{R_0}\gamma\omega\beta_2 e^u, \quad \underset{4}{e}{}^4 = \gamma\beta_1^{-1}e^{-u} \end{aligned} . \qquad \text{(VI.7.18)}$$

The resulting field strengths

$$A_{3\alpha}{}^3 = {}^*C_\alpha + U_\alpha, \quad A_{4\alpha}{}^4 = -U_\alpha \qquad \text{(VI.7.19)}$$

contain no centrifugal forces, and the rotational part

$$H_{mn}{}^s = 0 \qquad \text{(VI.7.20)}$$

vanishes at all. Thus, the rotating coordinate system has no influence on the physical structure of the model. It was not to be expected that a coordinate transformation could alter the physics. It becomes clear that the formulation of the problem with local reference systems is capable of eliminating virtual effects.

Finally, one can read from (VI.7.15) the constant coordinate transformation

$$\Lambda^3_{3'} = \gamma\beta_1, \quad \Lambda^3_{4'} = -i\gamma\omega\frac{1}{R_0}\beta_2^{-1}, \quad \Lambda^4_{3'} = i\gamma\omega R_0\beta_2^{-1}, \quad \Lambda^4_4 = \gamma\beta_1^{-1} \qquad \text{(VI.7.21)}$$

which indicates that the coordinate system rotates rigidly with ωR_0 around the cylinder axis.

STW also consider the case for a gravitational field coupled to a massive scalar field. Thus, Einstein's field equations have the form

$$R_{mn} - \frac{1}{2}g_{mn}R = -\kappa\left(V_{|m}V_{|n} - \frac{1}{2}g_{mn}V^{|s}V_{|s}\right)$$

and can be simplified to

$$R_{mn} = V_{|m}V_{|n} \qquad \text{(VI.7.22)}$$

wherein only the first two components of V_{lm} are occupied. For the scalar field one puts

$$V = -\frac{BM}{r}, \tag{VI.7.23}$$

where B is a constant. Further one has

$$\psi = \frac{AM}{r} - \frac{M^2 R^2}{2r^4} \tag{VI.7.24}$$

and the new constant is defined with

$$A^2 = 1 + \frac{\kappa}{2} B^2. \tag{VI.7.25}$$

Finally, one gets with $u = AM/r$

$$\begin{aligned} X &= \gamma^2 \left(R^2 e^{2u} - \omega^2 R_0^2 e^{-2u} \right) \\ W &= -i\gamma^2 \omega \frac{1}{R_0} \left(R^2 e^{2u} - R_0^2 e^{-2u} \right). \\ V &= \gamma^2 \left(e^{-2u} - \omega^2 \frac{R^2}{R_0^2} e^{2u} \right) \end{aligned} \tag{VI.7.26}$$

This extended approach does not lead to a true rotating model either. It should also be noted that for $\omega = 0$ this model is reduced to a static model found by Gautreau [G]. Gautreau also considers the case of a finite rod and discusses in detail the singularities of this solution.

VI.8. The solution of Som and Raychaudhuri

Som [S] and Raychaudhuri have found a solution for electrically charged pressure-free dust. The cylindrical object should rotate rigidly and refers to the metric

$$ds^2 = e^{2\lambda}\left(dR^2 + dZ^2\right) + Xd\varphi^2 + 2Wd\varphi dx^4 + Vdx^{4^2}. \quad \text{(VI.8.1)}$$

The stress-energy-momentum tensor has a matter component and an electro-magnetic component

$$T_{mn} = \mu_0 u_m u_n + \frac{2}{\kappa}\left(F_m{}^s F_{ns} - \frac{1}{4} g_{mn} F^2\right). \quad \text{(VI.8.2)}$$

In addition to Einstein's field equations with q as charge density

$$F^{mn}{}_{||n} = \frac{\kappa}{2} j^m = \frac{\kappa}{2} q u^m \quad \text{(VI.8.3)}$$

must be satisfied.

SR put

$$\begin{gathered}\lambda = \frac{1}{2}\left(A^2 - \omega^2\right)R^2 \\ X = \left(1 - \omega^2 R^2\right)R^2, \quad W = i\omega R^2, \quad V = 1\end{gathered} \quad \text{(VI.8.4)}$$

with the constants A and ω, with which $XV - W^2 = R^2 = \sigma^2$ also results. The 4-bein system is of type (A)

$$\text{(A)} \quad \begin{matrix} \overset{3}{e}_3 = R, & \overset{4}{e}_3 = -i\omega R^2, & \overset{4}{e}_4 = 1 \\ \underset{3}{e}^3 = \dfrac{1}{R}, & \underset{3}{e}^4 = i\omega R, & \underset{4}{e}^4 = 1 \end{matrix}. \quad \text{(VI.8.5)}$$

This results in

$$dt \to dt + \omega\sigma \cdot Rd\varphi.$$

The metric coefficients are only dependent on R, which significantly simplifies the model. It is a generalization of van Stockum's model. For the field strengths one obtains

$$B_1 = \lambda_{|1} = \left(A^2 - \omega^2\right)Re^{-\lambda}, \quad C_1 = {}^*C_1 = \frac{1}{R}e^{-\lambda}, \quad H_{13} = -i\omega e^{-\lambda}. \quad \text{(VI.8.6)}$$

H_{13} is the Coriolis field strength of the system. There is no centrifugal force. The electromagnetic stress-energy tensor (VI.8.2) contains only the magnetic field strength

$$F_{13} = -iAe^{-\lambda}, \quad F_{14} = 0, \quad F^2 = -2A^2e^{-2\lambda}, \quad F^\gamma = -\frac{i}{2}\varepsilon^{\gamma\alpha\beta}F_{\alpha\beta}, \quad \mathbf{F}^2 = 2A^2e^{-2\lambda}. \quad \text{(VI.8.7)}$$

Electrical forces are not present[114].

For the Ricci one gets with (VI.8.6) and (VI.8.7)

$$R_{11} = -H^2, \quad R_{22} = -2B_1C_1 = H^2 + F^2, \quad R_{33} = -H^2, \quad R_{44} = H^2 \atop R = -2H^2 + F^2 \qquad . \tag{VI.8.8}$$

If one puts in addition

$$\kappa\mu_0 = -2H^2 + F^2, \tag{VI.8.9}$$

the field equations are satisfied with (VI.8.2). It turns out that the mass density has a rotational and magnetic origin.

In order to fit the above quantities into the general form of rotation, we write

$$B_\alpha = \frac{1}{2}\left(H^2 - F^2\right) {}^*C_\alpha, \quad {}^*C_\alpha = \frac{1}{\sigma}\sigma_\alpha, \quad H_{\alpha 3} = -i\omega\sigma_\alpha, \quad F_{\alpha 3} = -iA\,\sigma_\alpha \atop \sigma_\alpha = \sigma_{|\alpha} = R_{|\alpha} \qquad . \tag{VI.8.10}$$

From the field equation $R_{34} = 0$ one obtains

$$H^{\alpha\beta}{}_{\wedge\beta} = 0, \quad \mathrm{rot}\,\mathbf{H} = 0\,. \tag{VI.8.11}$$

It is now necessary to check with which value for the charge density q the equation (VI.8.3) is satisfied. It follows in analogy to (VI.8.3)

$$F^{\alpha\beta}{}_{\wedge\beta} = 0, \quad \mathrm{rot}\,\mathbf{F} = 0\,. \tag{VI.8.12}$$

From it one can see that the charge density must vanish[115].

In addition, we process the model using the system (B)

$$\text{(B)} \qquad \begin{gathered} \overset{3'}{e}_3 = \frac{\sigma}{\alpha}, \quad \overset{3'}{e}_4 = i\alpha\omega\sigma, \quad \overset{4'}{e}_4 = \alpha, \\ \underset{3'}{e}^3 = \frac{\alpha}{\sigma}, \quad \underset{4'}{e}^3 = -i\alpha\omega, \quad \underset{4'}{e}^4 = \frac{1}{\alpha}, \\ \alpha = 1\big/\sqrt{1-\omega^2\sigma^2}\,. \end{gathered} \tag{VI.8.13}$$

From the relation

$$L^m_{m'} = \overset{m}{e}_i \underset{m'}{e}^i$$

we determine the components of the Lorentz transformation

$$L^3_{3'} = \alpha, \quad L^3_{4'} = -i\alpha\omega\sigma, \quad L^4_{3'} = i\alpha\omega\sigma, \quad L^4_{4'} = \alpha \tag{VI.8.14}$$

and read from it the relativistic rotation velocity

[114] SR process their model in the coordinate representation. In the oblique-angled coordinate system (VI.8.4) apparently electrical forces occur.

[115] This is not the case in the SR article. We refer to the ambiguous use of indices 0 and 4 in this paper.

$$v_{m'} = \{0, 0, i\alpha\omega\sigma, \alpha\}. \tag{VI.8.15}$$

SR, however, use the coordinate speed in their calculations

$$v_i = \{0, 0, i\omega\sigma^2, 1\},$$

which has little meaning, but can be converted to (VI.8.15) with (VI.8.13).

To calculate the field strengths in the system (B), we arrange the relation

$$h_\nu = \frac{1}{\alpha}\alpha_{|\nu} = \alpha^2\omega^2\sigma\sigma_\nu \tag{VI.8.16}$$

omitting the bars at the indices. The quantity h_ν is the expression for the relativistic centrifugal force. From the Ricci-rotation coefficients we obtain

$$D_{\alpha 3} = \frac{1}{2}\overset{3}{e}_3 e^3_{4|\alpha} = -i\omega\sigma h_\alpha. \tag{VI.8.17}$$

We recognize that the quantity $D_{\alpha\beta}$ does not vanish in the LNR system. In addition, the centrifugal force h_ν missing in the rotating system does occur in this system. Thus, the physical usability of this model is questioned.

Since the quantity $D_{\alpha\beta}$ does not vanish, our special interest is the R_{34}-component of the field equations. We note that, in addition to the magnetic field strength, an electric field strength

$$F_{\alpha'3'} = \alpha F_{\alpha 3}, \quad F_{\alpha'4'} = -i\alpha\omega\sigma F_{\alpha 3} \tag{VI.8.18}$$

is also present in the system (B) as is required by the theory of the electrodynamics of moving media. However, we do not revert to these expressions, but we basically calculate the stress-energy-momentum tensor of the system (B) with a Lorentz transformation. From

$$\kappa T_{33} = \frac{1}{2}F^2, \quad \kappa T_{44} = \kappa\mu_0 - \frac{1}{2}F^2 = -2H^2 + \frac{1}{2}F^2$$

one gains with

$$T_{3'4'} = L^{3}_{3'}L^{3}_{4'}T_{33} + L^{4}_{3'}L^{4}_{4'}T_{44}$$

the desired component

$$T_{3'4'} = 4i\alpha^2\omega^3\sigma\sigma^\alpha\sigma_\alpha = -4H_{\alpha'3'}H^{\alpha'}{}_{4'}.$$

We omit the primes at the indices and we get for the [3,4]-component of the Ricci

$$R_{34} = D^{\alpha 3}{}_{|\alpha} + B_\alpha D^{\alpha 3} + 2C_\alpha D^{\alpha 3}.$$

Therein B_α is already known from the system (A). The centrifugal force h_α is included in

$$A_{3\alpha}{}^3 = C_\alpha = {}^*C_\alpha - h_\alpha, \quad {}^*C_\alpha = \frac{1}{\sigma}\sigma_{|\alpha}, \quad A_{4\alpha}{}^4 = h_\alpha. \tag{VI.8.19}$$

However, the two contributions drop out. The task remains to calculate the derivative of D

$$D^{\alpha 3}{}_{|\alpha} = -B_\alpha D^{\alpha 3} + 2\,{}^*C_\alpha D^{\alpha 3} + 2h_\alpha D^{\alpha 3}.$$

With it we obtain the expression given above for T_{34}.

Now we have to calculate R_{33} and R_{44}. One has

$$R_{44} = -\left[h^\alpha{}_{\wedge\alpha} + h^\alpha h_\alpha + 2D^{\alpha 3}D_{\alpha 3}\right], \quad h^\alpha{}_{\wedge\alpha} = h^\alpha{}_{|\alpha} + B_\alpha h^\alpha + {}^*C_\alpha h^\alpha \qquad \text{(VI.8.20)}$$

and finally

$$G_{44} = 2\alpha^2 H^2 - \frac{1}{2}F^2.$$

If one computes the associated components of the stress-energy-momentum tensor, it is clear that this part of Einstein's field equations is satisfied with the above relation. Furthermore one has

$$R_{33} = -{}^*C^\alpha{}_{\wedge\alpha} - R_{44}, \quad {}^*C^\alpha{}_{\wedge\alpha} = {}^*C^\alpha{}_{|\alpha} + B_\alpha {}^*C^\alpha + {}^*C_\alpha {}^*C^\alpha. \qquad \text{(VI.8.21)}$$

Since ${}^*C_\alpha$ is a purely geometric quantity, it drops out of the field equations with

$${}^*C^\alpha{}_{\wedge\alpha} = 0 \qquad \text{(VI.8.22)}$$

It is easy to see that Einstein's field equations are satisfied with the above relations.

We split the electromagnetic field tensor F_{mn} of the system (B) into the magnetic tensor $F_{\alpha\beta}$ and the electric vector f_α. Thus, the Maxwell equations have the form

$$\begin{aligned} F^\alpha{}_{\beta\wedge\alpha} &= h_\alpha F^\alpha{}_\beta, && \mathrm{rot}\,\mathbf{F} = \mathbf{h}\times\mathbf{F}, \\ f^\alpha{}_{\wedge\alpha} &= h_\alpha f^\alpha + H_{\alpha\beta}F^{\alpha\beta}, && \mathrm{div}\,\mathbf{f} = \mathbf{h}\cdot\mathbf{f} + \mathbf{H}\cdot\mathbf{F}. \end{aligned} \qquad \text{(VI.8.23)}$$

Therein $H_{\alpha\beta}$ is the rotational force of the system (A) and h_α the centrifugal force of the system (B). The relations contain the interaction terms of the gravitational field with the electromagnetic field.

In this model we cannot recognize the electrical charge given by SR.

We add some further references concerning axially symmetric models:

Bonnor [B], Buchdahl [B], Cooperstock [C] and Junevicus; Carot [C], Dautcourt [D], Davies [D], Davies and Caplan; De Felice [D], Fu [F], Gautreau [G] and Anderson; Herrera [H], Herrera [H], Di Prisco and Ibanez; Herrera [H], Di Prisco and Ospino; Herrera [H], Gonzáles, Pachón and Ruede; Herrera [H], Maglio and Malafarina; Hernandez [H], Hernadez-Pastora [H], Herrera and Martin; Hernadez-Pastora [H], Herrera and Martin; Hoffman [H], Kinnersley [K], Levy [L] and Robinson; Misra [M], Misra [M] und Radhakrishna; Montero-Camacho [M], Montero-Camacho and Frutos-Alfaro; Palatini [P], Papapetrou [P], Perjés [P], Rácz [R], Rinne [R] und Stewart;Stachel [S], Thorne [T], Sharif [S] and Zaeem Ul Haq Bhatti, Weyl [W], Winicour [W].

VII. The Kerr metric

VII.1. Kerr metric, preliminaries

In the year 1963 Kerr [K] has found a solution to the Einstein field equations which can be regarded as the field of a rotating stellar object. It is counted among the few solutions, to which a physical meaning can be assigned. The Kerr solution is reduced to the Schwarzschild solution, if one puts zero the specific parameter a for the rotation. The model of Kerr has been extended by other authors. Charged solutions have been considered and solutions with gravimagnetic monopoles. For the interior of the object trial solutions have been suggested. All these models can be combined to form the Kerr family.

Kerr has deduced the metric of the model with the help of complex null-tetrads. Boyer [B] and Lindquist have presented the metric by means of a suitable transformation into elliptical co-ordinates. This form of the metric has found general acceptance, the co-ordinate system (BL system) is named after the two authors. Certain notions and symbols have come into use. Most of them are not applied by us, because they do not describe geometrically or physically interpretable quantities.

It turned out to be difficult to work out the physical background of the Kerr metric. Hence, there are a large number of publications concerned with this subject. In addition, several preferred reference systems can be read from the metric, from which we treat the most important ones. We take great care of the geometrical background which leaves room for the possibility of an embedding. We [B] approach the model in several steps:

- In a flat space an elliptical system is examined.
- With a Lorentz transformation a flat rotator is obtained in this system.
- Preferred reference systems are used for the actual Kerr metric.
- A surface is assigned to a static seed metric which will be embedded.
- The rotation is implemented on this surface by an intrinsic transformation.
- An interior solution is added.
- Extended models are discussed.

This concept is worked out in several Sections, whereby we must concede that some questions will remain open.

VII.2. Kerr metric, the elliptical system

Before the actual Kerr metric is examined, we are concerned in detail with the elliptic-hyperbolic geometry, which forms the basis of the model. According Enderlein [E] it can be parameterized with families of elliptic surfaces and hyperbolic surfaces of revolution, the latter being orthogonal with respect to the elliptic ones, and are represented with the help of the BL co-ordinates.

The minor axes of the elliptical slices of these surfaces will be designated with r for historical reasons, the major axes with A, the eccentricities of the ellipses with a, and the radii of the parallel circles with σ. The ellipses of the slices are confocal and there exist the relations

$$A^2 = r^2 + a^2, \quad \sigma = A\sin\vartheta \, . \tag{VII.2.1}$$

Fig. VII.1 shows the construction of the ellipses with the just-defined quantities and the quasi-polar angle ϑ.

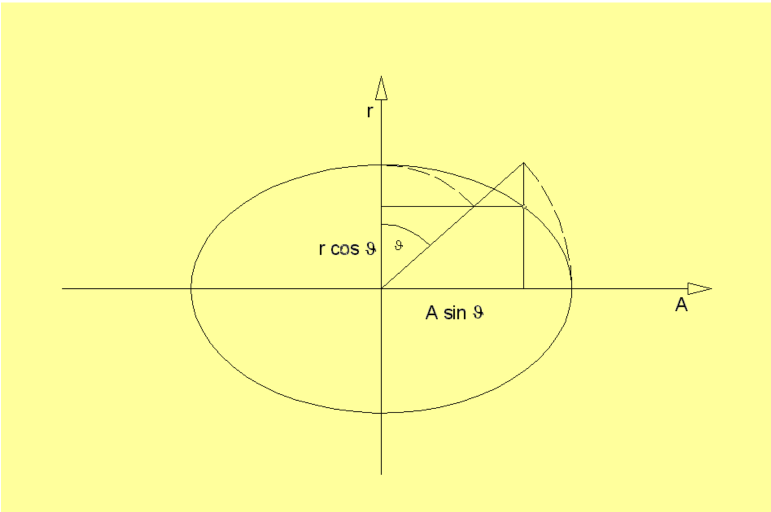

Fig. VII.1

A rotation of the ellipse around the r-axis through the angle φ generates oblate ellipsoids of revolution. The connection between Cartesian co-ordinates and BL co-ordinates are given by the relations

$$\begin{aligned} x^{3'} &= a\,\mathrm{ch}\eta\sin\vartheta\cos\varphi = A\sin\vartheta\cos\varphi \\ x^{2'} &= a\,\mathrm{ch}\eta\sin\vartheta\sin\varphi \;= A\sin\vartheta\sin\varphi \\ x^{1'} &= a\,\mathrm{sh}\eta\cos\vartheta \qquad\;\; = r\cos\vartheta \\ r &= a\,\mathrm{sh}\eta, \; A = a\,\mathrm{ch}\eta \end{aligned} \, . \tag{VII.2.2}$$

The ellipsoid of revolution and hyperboloid of revolution are given by

$$\frac{(x^{2'})^2 + (x^{3'})^2}{a^2\mathrm{ch}^2\eta} + \frac{(x^{1'})^2}{a^2\mathrm{sh}^2\eta} = 1, \quad \frac{(x^{2'})^2 + (x^{3'})^2}{a^2\sin^2\vartheta} - \frac{(x^{1'})^2}{a^2\cos^2\vartheta} = 1 \, . \tag{VII.2.3}$$

Zippoy [Z] and Bonnor [B] have been concerned in detail with elliptic-hyperbolic co-ordinates. Vaidya [V] has used elliptical co-ordinates for Kerr-like solutions and for the Robertson-Walker model. Voorhees [V] has presented axisymmetric solutions of the Einstein field equations with elliptical co-ordinates.

Slices through the center of the ellipsoids of revolution in the [r, A]-plane can be seen from Fig. VII.2

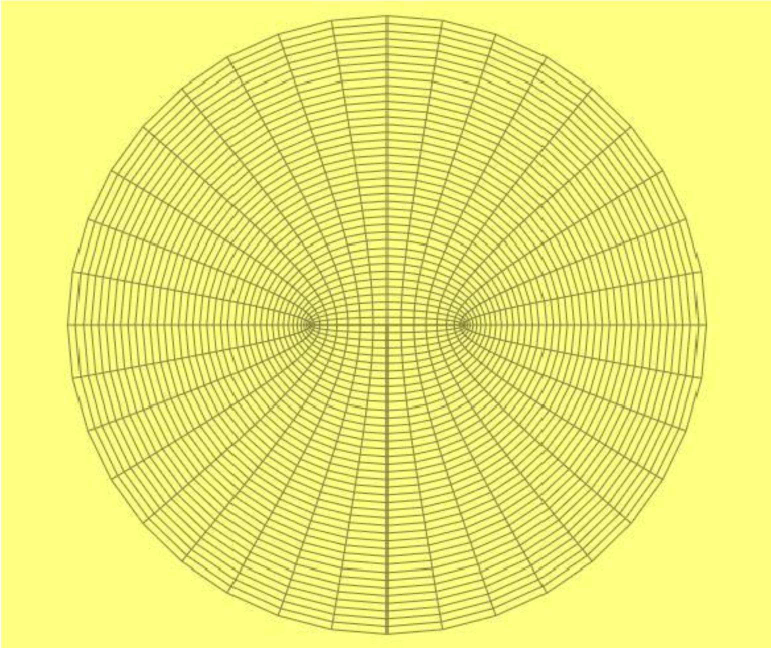

Fig. VII.2

From (VII.2.2) one obtains the line element

$$(dx^{1'})^2+(dx^{2'})^2+(dx^{3'})^2=a^2\left(\mathrm{sh}^2\eta+\cos^2\vartheta\right)\left(d\eta^2+d\vartheta^2\right)+a^2\mathrm{ch}^2\eta\sin^2\vartheta d\varphi^2 . \quad \text{(VII.2.4)}$$

Suppressing in (VII.2.2) the third dimension

$$\begin{aligned} x^{1'} &= r\cos\vartheta \\ x^{2'} &= A\sin\vartheta \end{aligned},$$

one can make accessible further properties of the ellipses. If one draws the focal rays s_1 and s_2 from the foci of an ellipse to any point of the ellipse, one finds the relation

$$s_1^2=x^{1'2}+\left(a+x^{2'}\right)^2, \quad s_2^2=x^{1'2}+\left(a-x^{2'}\right)^2$$

and from this

$$\begin{aligned} s_1 &= A+a\sin\vartheta, \quad s_2= A-a\sin\vartheta \\ s_1 s_2 &= A^2-a^2\sin^2\vartheta=r^2+a^2\cos^2\vartheta \end{aligned} .$$

Fundamental quantities of the ellipses can be represented as the geometric and the arithmetic means of the focal rays

$$\begin{aligned} \Lambda=\sqrt{s_1 s_2}, \quad A=\frac{s_1+s_2}{2} \\ \Lambda^2=r^2+a^2\cos^2\vartheta \end{aligned} . \quad \text{(VII.2.5)}$$

After some calculation one obtains the line element in the form

$$ds^2=\frac{\Lambda^2}{A^2}dr^2+\Lambda^2 d\vartheta^2+A^2\sin^2\vartheta d\varphi^2 , \quad \text{(VII.2.6)}$$

which one can simply supplement to a 4-dimensional one with $\left(dx^{4'}\right)^2=-dt^2$. The factor ahead of dr^2 in (VII.2.6) has to be examined in more detail. At $\vartheta=0$, i.e. at the minor axes of the ellipses, this factor is equal to 1, so that the radial part of the metric is reduced to dr^2. At $\vartheta=\pi/2$ the factor is r^2/A^2 . Together with (VII.2.1) it follows that the radial arc element has the value dA at this position. Thus, the factor describes the varying distance of two neighboring confocal ellipses, whereby at the minor axis this distance is dr. It proves that several quantities of the geometry show this behavior, so that it is sufficient to compute these quantities at one of the minor axes, and then to multiply by this factor which we call *elliptical factor*. However, these quantities offer a further surprise. With

$$a_R^2=\frac{\Lambda^2}{A^2}=1-\frac{a^2}{A^2}\sin^2\vartheta=1-\omega^2\sigma^2, \quad \omega=\frac{a}{A^2} \tag{VII.2.7}$$

one recognizes that in these relations are buried the angular velocity ω, and the orbital speed ωσ, respectively, of a potentially rotating model. The BL system already makes available the fundamental quantities for a rotating system. Not surprisingly, the Kerr geometry is built up on an elliptical system.

From the equations of the ellipses and hyperbolae one gets the nonvanishing components of the curvature vectors

$$\{\rho_E,0,0\}, \quad \{0,\rho_H,0\}, \quad \rho_E=\frac{\Lambda^3}{rA}, \quad \rho_H=-\frac{\Lambda^3}{a^2\sin\vartheta\cos\vartheta}\ . \tag{VII.2.8}$$

The two vectors are at right angles and are in each case tangent to the other family of curves. Thus, also the term radial is defined in this geometry. By radial one understands the directions which are specified by the tangents of the hyperbolae.

As has been shown in the preceding Section, the 'field quantities' of the geometry are due to the curvatures. With the previous models it has been presumed that the radial curves do not have a second curvature. That means that their projections onto the zero-plane of the geometry-determining surface are outgoing straight lines from the origin of the co-ordinate system. That is no longer the case with the elliptical system Fig. V.2 is the representation of the zero-plane, on which the surface of the Kerr geometry is built up later on. The 'radial' curves in this plane already have a curvature which is denoted by ρ_H. In this model a new quantity N_m arises which has its origin in that curvature.

From the metric (VII.2.6) one reads the bein vectors

$$\begin{aligned}\overset{1}{e}_1&=a_R, \quad \overset{2}{e}_2=\Lambda, \quad \overset{3}{e}_3=\sigma\\ \underset{1}{e}^1&=\frac{1}{a_R}, \quad \underset{2}{e}^2=\frac{1}{\Lambda}, \quad \underset{3}{e}^3=\frac{1}{\sigma}\end{aligned} \tag{VII.2.9}$$

and computes with them the Ricci-rotation coefficients

$$\begin{gathered}A_{mn}{}^s=B_{mn}{}^s+N_{mn}{}^s+C_{mn}{}^s\\ B_{mn}{}^s=b_m B_n b^s-b_m b_n B^s, \; N_{mn}{}^s=m_m N_n m^s-m_m m_n N^s, \; C_{mn}{}^s=c_m C_n c^s-c_m c_n C^s\\ B_1=\frac{1}{\Lambda}\Lambda_{|1}=\frac{1}{\rho_E}, \quad N_2=\frac{1}{\Lambda}\Lambda_{|2}=\frac{1}{\rho_H}, \quad C_1=\frac{1}{\sigma}\sigma_1=\frac{r}{A\Lambda}, \quad C_2=\frac{1}{\sigma}\sigma_2=\frac{1}{\Lambda}\cot\vartheta \quad .\\ \sigma_1=\sigma_{|1}=\frac{r}{\Lambda}\sin\vartheta, \quad \sigma_2=\sigma_{|2}=\frac{A}{\Lambda}\cos\vartheta\end{gathered} \tag{VII.2.10}$$

As a consequence of the flatness of geometry one has

$$R_{mn} \equiv 0 .$$

Therein the equations

$$\begin{gathered}\left[N_{n|m} - m_m N_{n|s} m^s + N_n N_m\right] + \left[B_{n|m} - b_m B_{n|s} b^s + B_n B_m\right] \\ + m_n m_n \left[N^s{}_{|s} + N^s N_s\right] + b_n b_n \left[B^s{}_{|s} + B^s B_s\right] = 0 \\ C_{n\underset{3}{||}m} + C_n C_m = 0\end{gathered} \tag{VII.2.11}$$

and

$$\left[N^s{}_{\underset{2}{||}s} + N^s N_s\right] + \left[B^s{}_{\underset{2}{||}s} + B^s B_s\right] = 0, \quad \left[C^s{}_{\underset{3}{||}s} + C^s C_s\right] = 0 \ , \tag{VII.2.12}$$

are contained, whereby the graded derivatives of the elliptical system

$$m_{m\underset{1}{||}n} = m_{m|n}, \quad b_{m\underset{2}{||}n} = b_{m|n}, \quad c_{m\underset{3}{||}n} = c_{m|n} - N_{nm}{}^s c_s - B_{nm}{}^s c_s \tag{VII.2.13}$$

are somewhat richer. Nevertheless,

$$m_{m\underset{1}{||}n} = 0, \quad b_{m\underset{2}{||}n} = 0, \quad c_{m\underset{3}{||}n} = 0 \tag{VII.2.14}$$

is valid. It should be noted that the B- and the N-equations do not decouple. A careful investigation of the elliptical system is necessary in order to determine what results in each of these equations. That will be done in the next Section.

VII.3. Kerr metric, inclusion of the evolute

The ellipse is the affine image of the circle. Therefore one can assume that between these two objects there exists a connection which is applicable to the actual problem. The method is already obvious as a consequence of the investigations of the last Sections. In the same way as the parabolic-hyperbolic Schwarzschild geometry was deduced from a pseudo-hyper spherical geometry, we derive an ellipse from the circle, and an ellipsoid of revolution from a sphere, respectively.

First we parameterize the family of spheres with

$$\begin{aligned} X^{3'} &= \rho_E \sin\theta\cos\varphi \\ X^{2'} &= \rho_E \sin\theta\sin\varphi \ , \qquad X_{\alpha'}X^{\alpha'} = {\rho_E}^2 \ , \\ X^{1'} &= \rho_E \cos\theta \end{aligned} \tag{VII.3.1}$$

whereby the radius of the sphere is already designated by ρ_E with regard to future use. The $X^{\alpha'}$ are the Cartesian co-ordinates of the radius vector of a spherical surface with $\rho_E = \text{const.}$ as well. The metric of the flat space written in polar co-ordinates $\{\rho_E, \theta, \varphi\}$ is

$$ds^2 = d{\rho_E}^2 + {\rho_E}^2 d\theta^2 + {\rho_E}^2 \sin^2\theta \, d\varphi^2 \ . \tag{VII.3.2}$$

The arc elements and the partial derivatives in these co-ordinates read as

$$\begin{aligned} &dX^1 = d\rho_E, \quad dX^2 = \rho_E d\theta, \quad dX^3 = \rho_E \sin\theta\, d\varphi \\ &\Phi_{,\alpha} = \hat{\partial}_\alpha \Phi = \frac{\partial\Phi}{\partial X^\alpha} = \left\{ \frac{\partial}{\partial\rho_E}, \frac{\partial}{\rho_E\partial\theta}, \frac{\partial}{\rho_E \sin\theta\partial\varphi} \right\} \ . \end{aligned} \tag{VII.3.3}$$

One arrives at an elliptical geometry by demanding that the ρ_E are not constant and that they also do not describe spherical surfaces any longer. In this case the angle θ is not a free parameter. Hence, ρ_E and θ are position-dependent functions

$$\rho_E = \rho_E(r, \vartheta), \ \ \theta = \theta(r, \vartheta) \ . \tag{VII.3.4}$$

We have explained the angle ϑ as quasi-polar angle by means of Fig. V.1. The angle θ which points from the pole to the radius ρ_E of the sphere is the polar angle of the sphere in accordance with (VII.3.1). Having mapped the sphere onto the ellipsoid, θ has the meaning of the aperture angle of the curvature radius ρ_E of the ellipse. With

$$\sigma_1 = \frac{r}{\Lambda}\sin\vartheta = \sin\theta, \ \ \sigma_2 = \frac{A}{\Lambda}\cos\vartheta = \cos\theta \tag{VII.3.5}$$

one gets the relation of the two angles ϑ and θ. The evolute of the Schwarzschild parabola has played an important role in the context of the Schwarzschild geometry. We will see that the evolute of the ellipse is of importance as well. Temporarily we suppress the φ-dimension. The evolute of the ellipse

$$\left(r\,\overline{x}^{1'}\right)^{\frac{2}{3}} + \left(A\overline{x}^{2'}\right)^{\frac{2}{3}} = \left(a^2\right)^{\frac{2}{3}} \tag{VII.3.6}$$

can be parameterized with quasi-polar co-ordinates

$$\overline{x}^{1'} = -\frac{a^2}{r}\cos^3\vartheta, \quad \overline{x}^{2'} = \frac{a^2}{A}\sin^3\vartheta \tag{VII.3.7}$$

as well. If one differentiates these relations and if one converts the angles by (VII.3.5) one first obtains a relation between the Cartesian co-ordinate differentials on the evolute and the elliptical co-ordinate differentials

$$\begin{aligned} d\overline{x}^{1'} &= -a^2\frac{\Lambda^2}{A^2r^2}\cos^2\theta\left[\cos\theta dx^1 - 3\sin\theta dx^2\right] \\ d\overline{x}^{2'} &= \ \ a^2\frac{\Lambda^2}{A^2r^2}\sin^2\theta\left[-\sin\theta dx^1 + 3\cos\theta dx^2\right] \end{aligned} .$$

Since θ is the azimuthal angle of the curvature vector of the ellipse and since this very curvature vector is also tangent to the evolute, it is obvious to rotate the Cartesian co-ordinate differentials through θ. In this way one arrives at a local orthogonal quasi-polar reference system. With

$$d\overline{x}^1 = \cos\theta d\overline{x}^{1'} + \sin\theta d\overline{x}^{2'}, \quad d\overline{x}^2 = -\sin\theta d\overline{x}^{1'} + \cos\theta d\overline{x}^{2'} \tag{VII.3.8}$$

one gets the correlation between the co-ordinate differentials on the evolute and involute

$$\begin{aligned} d\overline{x}^1 &= a^2\frac{\Lambda^2}{A^2r^2}\left[\left(\cos^2\theta - \sin^2\theta\right)dx^1 + 3\sin\theta\cos\theta\, dx^2\right] \\ d\overline{x}^2 &= -a^2\frac{\Lambda^2}{A^2r^2}\sin\theta\cos\theta\, dx^1 \end{aligned} ,$$

which is still worth reconsidering. By the use of the relations (VII.2.7) and (VII.2.8) an interpretation of the above expressions is possible. The reciprocal of the elliptical factor a_R is the Lorentz factor of a rotation

$$\alpha_R = 1\big/\sqrt{1-\omega^2\sigma^2} \ . \tag{VII.3.9}$$

With this relation and (VII.3.5) one obtains for the first term in the co-ordinate differentials

$$a^2\frac{\Lambda^2}{A^2r^2}\left(\cos^2\theta - \sin^2\theta\right) = -\rho_E^2\alpha_R^4\omega^2\left(\sigma_1\sigma_1 - \sigma_2\sigma_2\right) \ ,$$

while for the second equation

$$a^2\frac{\Lambda^2}{A^2r^2}\sin\theta\cos\theta = -\frac{\rho_E}{\rho_H}$$

can be used.

In the Section ‘Rotation’ has been found an expression for the Coriolis field strength which obviously can be recognized in the last but one relation. However, the properties of the rigid rotators cannot be transferred to that elliptical rotator without modifications which will be discussed soon. In accordance with (VII.2.7) $\omega = a/A^2$ depends on the distance to the axis of rotation. The angular speed decreases outwards and vanishes in the infinite. Thus, a *differential rotation law* is implemented. This is similar to the corresponding rotation law of the Gödel cosmos. However, the difference is that no cut-off radius is present, which is certainly an achievement.

Although still a static system is treated here, the hidden rotational quantities must be examined in order to be well prepared for the later introduction of the rotation. The change of the orbital speed from one position to another position has two contributions

$$(\omega\sigma)_{|\alpha}=\omega\sigma_{\alpha}+\omega_{|\alpha}\sigma\ .$$

If one moves outwards and if the orbital radius increases, the first contribution[116] emerges similarly to the rigid rotator. The second contribution is due to the change of the angular speed. The latter decreases outwards and is responsible for the good quality of this kind of rotating system. From the relations just-discussed two field quantities can be deduced

$$H_{\alpha\beta}=2\,i\alpha_R^2\omega\sigma_{[\alpha}c_{\beta]},\ \ D_{\alpha\beta}=i\alpha_R^2\omega_{|\alpha}\sigma c_{\beta}\ . \tag{VII.3.10}$$

H is antisymmetric and is the relativistic generalization of the Coriolis field strength. The quantity D is asymmetric ($D_{31}=0$). Both quantities are summarized to

$$\tilde{\Omega}_{\beta\alpha}=-H_{\alpha\beta}-D_{\alpha\beta}\ . \tag{VII.3.11}$$

By the decomposition of $\tilde{\Omega}$ into an antisymmetric and a symmetric part

$$\tilde{\Omega}_{\beta\alpha}=-\left[H_{\alpha\beta}+D_{[\alpha\beta]}\right]-D_{(\alpha\beta)} \tag{VII.3.12}$$

one obtains the total rotational field strength and the deformation field strength. More on this subject will be treated in the next Section. Since the relation

$$\omega_{|1}\sigma=-2\omega\sigma_1,\ \ \omega_{|2}\sigma=0 \tag{VII.3.13}$$

is easy to examine one obtains the auxiliary relation

$$D_{13}\overset{*}{=}-2H_{13}\ . \tag{VII.3.14}$$

If one uses this in (VII.3.11) and if one considers the result after Eq. (VII.3.9) one finally has

$$-\rho_E^2\tilde{\Omega}^{\alpha3}\tilde{\Omega}_{3\alpha}\ .$$

After some explanations which already anticipate the rotating system, we have been successful in indicating the co-ordinate differentials on the evolute in a compact way of writing

$$d\bar{x}^1=\rho_E{}^2\,\tilde{\Omega}^{m3}\tilde{\Omega}_{3m}dx^1-3\frac{\rho_E}{\rho_H}dx^2,\quad d\bar{x}^2=\frac{\rho_E}{\rho_H}dx^1\ . \tag{VII.3.15}$$

One calculates the curvature vector X of the ellipse, the value of which is designated by ρ_E, with the difference of the co-ordinates at the starting and ending points. With $X^\alpha=x^\alpha-\bar{x}^\alpha$ one gets

$$\begin{aligned}dX^1&=d\rho_E=dx^1-d\bar{x}^1=\left(1-\rho_E^2\tilde{\Omega}^{m3}\tilde{\Omega}_{3m}\right)a_R dr+3\frac{\rho_E}{\rho_H}\Lambda d\vartheta\\ dX^2&=\rho_E d\theta=dx^2-d\bar{x}^2=\Lambda d\vartheta-\frac{\rho_E}{\rho_H}a_R dr\end{aligned}\ . \tag{VII.3.16}$$

[116] One should note that ‘outwards’ means that not only the radial co-ordinate changes, but also the quasi-polar angle ϑ. This should be imagined at the ellipsoid of revolution.

If one adds the 3rd dimension one can easily supplement the above equations taking a glance at (VII.3.3)

$$dX^3 = \hat{\sigma} d\varphi = dx^3 - d\overline{x}^3$$
$$\hat{\sigma} = \rho_E \sin\theta = a_R^2 \sigma = \sigma - \overline{\sigma}, \quad \overline{\sigma} = \omega^2 \sigma^2 \cdot \sigma = \frac{a^2}{A} \sin^3 \vartheta \quad . \qquad \text{(VII.3.17)}$$

Now it also makes sense to separate the B- and N-equations in (VII.2.11) and in (VII.2.12). Calculating the two blocks results in

$$B_{1|1} + B_1 B_1 = \tilde{\Omega}^{\alpha 3} \tilde{\Omega}_{3\alpha}, \quad N_{2|2} + N_2 N_2 = -\tilde{\Omega}^{\alpha 3} \tilde{\Omega}_{3\alpha}$$

and in a covariant way of writing

$$B_{n|m} - b_m B_{n|s} b^s + B_n B_m = m_m m_n \tilde{\Omega}^{s3} \tilde{\Omega}_{3s}$$
$$N_{n|m} - m_m N_{n|s} m^s + N_n N_m = -b_m b_n \tilde{\Omega}^{s3} \tilde{\Omega}_{3s} \quad . \qquad \text{(VII.3.18)}$$

It can be shown that the contributions of the evolute of the ellipse $\tilde{\Omega}$ enter into the elliptical curvature equations. This will have to be considered by evaluating the actual Kerr metric as well.

VII.4. Kerr metric, elliptic projectors

In the last Section it has been suggested that the geometry of an ellipsoid could be deduced from the geometry of a sphere, if one uses the possibility of mutual images of these objects. We have got to know a similar methodology in the context of the Schwarzschild geometry. The curvature equations of the one geometry have been converted into the curvature equations of the other geometry by the projectors. Here the projector technology will be likewise useful and will serve as a preparation for the actual Kerr geometry. From the ansatz for the family of spheres (VII.3.1) one obtains the curvature quantities

$$\hat{A}_{mn}{}^{s} = \hat{B}_{mn}{}^{s} + \hat{C}_{mn}{}^{s}$$
$$\hat{B}_{mn}{}^{s} = b_m \hat{B}_n b^s - b_m b_n \hat{B}^s, \quad \hat{C}_{mn}{}^{s} = c_m \hat{C}_n c^s - c_m c_n \hat{C}^s \tag{VII.4.1}$$
$$\hat{B}_n = \left\{\frac{1}{\rho_E}, 0, 0\right\}, \quad \hat{C}_n = \left\{\frac{1}{\rho_E}, \frac{1}{\rho_E}\cot\theta, 0\right\}$$

and the curvature equations

$$\hat{B}_{n\underset{2}{;}m} + \hat{B}_n \hat{B}_m = 0, \quad \hat{C}_{n\underset{3}{;}m} + \hat{C}_n \hat{C}_m = 0, \quad \hat{B}_{n\underset{2}{;}m} = \hat{B}_{n,m}, \ \hat{C}_{n\underset{3}{;}m} = \hat{C}_{n,m} - \hat{B}_{mn}{}^{s}\hat{C}_s \ . \tag{VII.4.2}$$

Defining the projectors after a glance at (VII.3.16) with

$$dX^m = \mathfrak{p}^m_n dx^n, \quad \partial_n = \mathfrak{p}^m_n \hat{\partial}_m$$
$$\mathfrak{p}^1_1 = 1 - \rho_E{}^2 \Omega^{m3}\Omega_{3m}, \quad \mathfrak{p}^1_2 = 3\frac{\rho_E}{\rho_H}, \quad \mathfrak{p}^2_1 = -\frac{\rho_E}{\rho_H}, \ \mathfrak{p}^2_2 = 1, \quad \mathfrak{p}^3_3 = a_R^2 \ , \tag{VII.4.3}$$

the elliptical curvature quantities can be deduced from the spherical curvature quantities by

$$A_{mn}{}^{s} = \mathfrak{p}^r_m \hat{A}_{rn}{}^{s}$$
$$B_1 = A_{21}{}^{2} = \mathfrak{p}^2_2 \hat{A}_{21}{}^{2} = \frac{1}{\rho_E}, \quad N_2 = A_{12}{}^{1} = \mathfrak{p}^2_1 \hat{A}_{22}{}^{1} = \frac{1}{\rho_H} \ . \tag{VII.4.4}$$
$$C_1 = A_{31}{}^{3} = \mathfrak{p}^3_3 \hat{A}_{31}{}^{3} = \frac{1}{\rho_E} a_R^2 = \frac{r}{\Lambda A}, \quad C_2 = A_{32}{}^{3} = \mathfrak{p}^3_3 \hat{A}_{32}{}^{3} = \frac{1}{\rho_E} a_R^2 \cot\theta = \frac{1}{\Lambda}\cot\vartheta$$

How to proceed in the case of the curvature equations we show with the example

$$B_{1|1} = \mathfrak{p}^1_1\left(\mathfrak{p}^2_1 \hat{B}_1\right)_{,1} = -B_1 B_1 + \Omega^{m3}\Omega_{3m} \ .$$

For the Riemann tensor one has

$$\mathfrak{p}^p_r \mathfrak{p}^q_m \hat{R}_{pqn}{}^{s} = R_{rmn}{}^{s} + 2\hat{A}_{pn}{}^{s} \mathfrak{p}^{\ p}_{[r||m]} \equiv 0 \ . \tag{VII.4.5}$$

We draw our attention to the projector term on the right side of the relation. It has played an important role concerning the Schwarzschild geometry and it also will be important for the Kerr geometry. Since it has become clear that the elliptical geometry supplies rotational terms we note for later use

$$\frac{1}{\alpha_R}\alpha_{R|m} = F_m + D_m, \quad F_m = \alpha_R{}^2 \omega^2 \sigma\sigma_m, \quad D_m = \alpha_R{}^2 \omega\, \omega_{\ |m} \sigma^2 \ . \tag{VII.4.6}$$

Therein F is the relativistic generalization of the centrifugal force and D is a field strength which arises from the differential rotation law. It has only the radial component D_1 and points inwards.

VII.5. Kerr metric, the flat rotating system

In order to describe an elliptical rotating system one proceeds in the same manner as we have done with the rigid relativistic rotator. With the help of (VII.2.7) and (VII.3.9) the parameters of a Lorentz transformation have been exploited from the geometry. If one operates with this Lorentz transformation

$$L_3^{3'}=\alpha_R,\quad L_4^{3'}=i\alpha_R\omega\sigma,\quad L_3^{4'}=-i\alpha_R\omega\sigma,\quad L_4^{4'}=\alpha_R \tag{VII.5.1}$$

onto the 4-beins of the metric (VII.2.6) which is extended by the term $dx^{4^2}=-dt^2$, one obtains

$$\begin{array}{llllll} \overset{1}{e}_1=a_R, & \overset{2}{e}_2=\Lambda, & \overset{3}{e}_3=\alpha_R\sigma, & \overset{4}{e}_3=-i\alpha_R\omega\sigma^2, & \overset{3}{e}_4=i\alpha_R\omega\sigma, & \overset{4}{e}_4=\alpha_R \\ \underset{1}{e}^1=\dfrac{1}{a_R}, & \underset{2}{e}^2=\dfrac{1}{\Lambda}, & \underset{3}{e}^3=\dfrac{\alpha_R}{\sigma}, & \underset{4}{e}^3=-i\alpha_R\omega, & \underset{3}{e}^4=i\alpha_R\omega\sigma, & \underset{4}{e}^4=\alpha_R \end{array}. \tag{VII.5.2}$$

With $dx^3=\sigma d\varphi$ the metric takes the form

$$ds^2=a_R^2dr^2+\Lambda^2d\vartheta^2+\left[\alpha_R dx^3+i\alpha_R\omega\sigma dx^4\right]^2+\left[-i\alpha_R\omega\sigma dx^3+\alpha_R dx^4\right]^2 \tag{VII.5.3}$$

which is already very similar to the actual Kerr metric and identical with the static metric

$$ds^2=a_R^2dr^2+\Lambda^2d\vartheta^2+A^2\sin^2\vartheta\, d\varphi^2+dx^{4^2} \tag{VII.5.4}$$

as one can determine by squaring the brackets. The metric is Lorentz invariant and the field equations will be Lorentz invariant as well. The Ricci-rotation coefficients are now computed by using (VII.5.2)

$$\begin{gathered} A_{mn}{}^s=B_{mn}{}^s+N_{mn}{}^s+C_{mn}{}^s+F_{mn}{}^s+H_{mn}{}^s+E_{mn}{}^s \\ F_{mn}{}^s=c_mF_nc^s-c_mc_nF^s,\quad E_{mn}{}^s=-\left[u_mF_nu^s-u_mu_nF^s\right]\ . \\ H_{mn}{}^s=H_{mn}u^s+H^s{}_mu_n+H^s{}_nu_m+D_{mn}u^s-D_m{}^su_n \end{gathered} \tag{VII.5.5}$$

Therein still arise the centrifugal force F, the Coriolis force H, and the force D, but no force of gravity. The system is flat after all, the rotation is not an *intrinsic* property of the system, but is *attached*. The geometry is not changed by the Lorentz transformation, but the static geometry is equipped with an additional structure. One recognizes that the rotating observers are exposed to centrifugal forces

$$u_{m||n}u^n=F_m \tag{VII.5.6}$$

and that with

$$u_{\alpha||\beta}=\tilde{\Omega}_{\alpha\beta} \tag{VII.5.7}$$

the ansatz (VII.3.11) is justified. The total force of rotation which affects the observer is

$$u_{[\alpha||\beta]}=H_{\alpha\beta}+D_{[\alpha\beta]}\ , \tag{VII.5.8}$$

while the shears

$$u_{(\alpha||\beta)}=-D_{(\alpha\beta)} \tag{VII.5.9}$$

are to be understood in such a way that the observers slide past on account of the different speeds on neighboring circular paths, whereby shears of the surrounding volume elements arise.

While in contrast to the static system nothing new occurs in the B- and N-parts of the field equations of the rotating system, the rest of the field equations must still be considered. The 4th graded derivative has now an additional term

$$F_{m\underset{4}{||}n} = F_{m|n} - \left(N_{nm}{}^{s} + B_{nm}{}^{s} + C_{nm}{}^{s} + F_{nm}{}^{s}\right)F_{s} \tag{VII.5.10}$$

and the new quantities supply new subequations of the Einstein field equations. From $R_{mn} \equiv 0$ one obtains equations of the Maxwell type

$$\begin{gathered} F^{m}{}_{\underset{4}{||}m} - F^{m}F_{m} - \tilde{\Omega}^{mn}\tilde{\Omega}_{nm} = 0 \\ \tilde{\Omega}^{mn}{}_{\underset{4}{||}m} + 2\tilde{\Omega}^{[nm]}F_{m} = 0 \end{gathered} \quad . \tag{VII.5.11}$$

The centrifugal force is coupled to the field energy which is composed of quadratic terms. The quantity $\tilde{\Omega}$ is coupled to the Poynting vector $2\tilde{\Omega}^{[nm]}F_{m}$. Further, the relations

$$\begin{gathered} F_{[m\underset{4}{||}n]} + D_{[m\underset{4}{||}n]} = 0, \quad F_{[m\underset{4}{||}n]} + 2\tilde{\Omega}_{3[m}\tilde{\Omega}_{n]3} = 0 \\ \tilde{\Omega}_{[mn\underset{4}{||}s]} + \tilde{\Omega}_{[mn}F_{s]} = 0 \end{gathered} \tag{VII.5.12}$$

are satisfied. They are Maxwell-like as well. In contrast to the rigid rotator nothing new arises, apart from the deformation quantities. The conservation laws

$$\frac{\partial}{\partial t}\left(F^{m}F_{m} + \tilde{\Omega}^{mn}\tilde{\Omega}_{nm}\right) = 0, \quad \left(2\tilde{\Omega}^{[nm]}F_{m}\right)_{\underset{4}{||}n} = 0 \ . \tag{VII.5.13}$$

are valid.

Before the actual Kerr metric is treated we make still another intermediary step. A static, but elliptical model endowed with the force of gravity will be examined.

VII.6. Kerr metric, metric on a surface

Due to the close relationship of the Kerr geometry to the Schwarzschild geometry we expect that the radial factor which is responsible for the deviation of the Kerr geometry from the flat geometry leads us to an embedding of the Kerr geometry in a correct way. In analogy to Flamm's paraboloid we suppose a surface which is similar to Flamm's paraboloid, but is deformed elliptically. An event horizon is to be expected which is the boundary of the geometry and is explained as an ellipse at the waist of the surface. In order to follow this way we take a step backwards and start again with the static elliptical system. We insert factors into the metric of the static elliptical system which lead to the curvature of the geometry.

This *seed metric* does not satisfy the vacuum field equations any longer, but supplies a stress-energy tensor. Although this model is an exact solution of the Einstein field equations, no physical meaning is attributed to it. It has the structure of an interior solution, but has nevertheless an event horizon.

We will show that the radial factor of the Kerr metric can be decomposed into the already well-known elliptical factor a_R and the *gravitational factor*

$$\alpha_S = \frac{A}{\delta}, \quad a_S = \alpha_S^{-1}, \quad A^2 = r^2 + a^2, \quad \delta^2 = r^2 + a^2 - 2Mr \ . \tag{VII.6.1}$$

Thus, the ansatz

$$dx^1 = \alpha_S a_R dr, \quad dx^4 = a_S i dt \tag{VII.6.2}$$

is promising with respect to the seed metric. The seed metric has the form

$$ds^2 = \alpha_S^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 + a_S^2 (i dt)^2 \ . \tag{VII.6.3}$$

The gravitation factor can be interpreted in two respects. Firstly, as Lorentz factor of a freely falling observer

$$\alpha_S = \frac{1}{\sqrt{1 - v_S^2}}, \quad v_S = -\frac{r}{A}\sqrt{\frac{2M}{r}} \ , \tag{VII.6.4}$$

whereby the velocity v_S of the fall differs from the Schwarzschild velocity $v = -\sqrt{2M/r}$ of a freely falling object by the ratio of the main axes of the ellipses. Secondly, the gravitational factor can be brought into connection with the angle of ascent of a surface from which we hope that it is of importance for the actual Kerr model.

Therefore one has

$$v_S = \sin\varepsilon, \quad a_S = \cos\varepsilon \ , \tag{VII.6.5}$$

whereby the angle of ascent ε of the surface has the orientation cw. Thus, the minus sign of the speed[117] of the fall is taken into account. The ascent of the surface

$$\tan\varepsilon = -\frac{\sqrt{2Mr}}{\delta} \tag{VII.6.6}$$

[117] More exactly, v_S is the first component of the vector of the speed of fall pointing to the gravitation center. All other components vanish in the co-ordinate system in use.

can be computed[118] from (VII.6.1), (VII.6.4), and (VII.6.5). It becomes infinitely large at $\delta = 0$. If one solves this relation with (VII.6.1) with respect to r one obtains

$$r_H = M + \sqrt{M^2 - a^2} \quad , \tag{VII.6.7}$$

the lowest value for r. $\delta = 0$ is the definition of the ellipse at the waist of the surface. It marks the limit of the validity of the geometry which is to be described. For values smaller than r_H no points are defined on the surface. The surface looks like a ‘funnel’ similar to Flamm's paraboloid, as we will see later on. If one denotes the distances of the points of the surface from the elliptically parameterized base plane with $x^{0'}$, these points can be computed with the integral

$$x^{0'} = -\int_{r_0}^{r_1} \tan\varepsilon \, dr \quad . \tag{VII.6.8}$$

The solution does not have a closed form, however, the integral can be evaluated numerically. The set of all of these points forms the surface considered by us. The ansatz in Cartesian co-ordinates

$$\begin{aligned} x^{0'} &= -\int \tan\varepsilon \, dr \\ x^{1'} &= r\cos\vartheta \\ x^{2'} &= A\sin\vartheta\cos\varphi \\ x^{3'} &= A\sin\vartheta\sin\varphi \end{aligned} \tag{VII.6.9}$$

describes the points of this surface. If one suppresses the φ-dimension the surface appears as shown in Fig. VII.3.

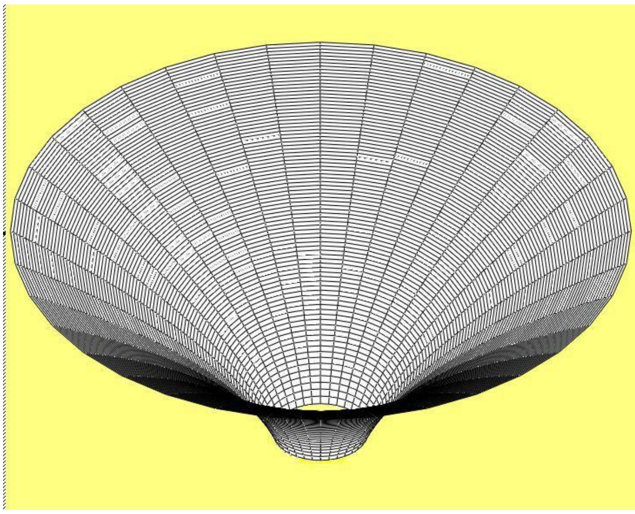

Fig. VII.3

Here we must bear in mind that dr is the increase of the minor axes of the ellipses on the base plane

$$dx^{\underline{1}} = a_R dr \quad , \tag{VII.6.10}$$

$dx^{\underline{1}}$ depends on the angle ϑ and is the distance of two neighboring ellipses. If one pulls up elliptical cylinders on two of such ellipses, one can see that their cutting curves with the surface do not lie on the horizontal slices of the surface. During a circulation on the surface

[118] It is positive, since ε has been taken negatively.

the points of the cutting curve oscillate. From (VII.6.9) one gets the augment of the height of the surface into the extra dimension

$$dx^{0'}_{hol} = -\tan\varepsilon\, dr = -\tan\varepsilon\, \alpha_R(r,\vartheta)dx^{\underline{1}} = -\tan\chi\, dx^{\underline{1}} \ . \qquad \text{(VII.6.11)}$$

It is evident that the tangent of the auxiliary angle χ which points to the next higher point of the surface depends on the angle ϑ. A short calculation would show that one does not arrive at the desired seed metric by the use of (VII.6.11), either. For our model only the horizontal elliptical slices of the surface are of importance. If one follows the normal vector of the surface along an ellipse, one will discover that this vector also oscillates on its way, because the walls of the surface are round about differently scarped. In order to be able to use the surface after all, it is to be equipped with an additional structure. On the minor axes of the ellipses the elliptical factor is $a_R = 1$, the geometry is Schwarzschild-like. There we make a start and we define a *rigging vector* in such a way that it coincides with the normal vector at this position, and that it always encloses the same angle with the base plane during its circulation. Then this rigging vector is no longer vertical to the surface and its vertical planes are no longer tangent to the surface. The family of all of these planes - and if one adds the φ-dimension, the family of the 3-dimensional hyperplanes - represents our graphic space, if we assume that we live in such a world. Those hyperplanes are anholonomic, as will be justified.

The problem of such hyperplanes is not new in gravitation physics. Some models make use of such hyperplanes. For the 5-dimensional Kaluza-Klein theories which unify electromagnetism and gravitation, and also for modern versions of these theories, which include further interactions, such anholonomities represent the ultimate solution for a compact description of physics[119]. Many authors hesitate to accept that the world mathematically is not described by a continuum, but by locally defined hyperplanes which do not form a hypersurface. These authors occasionally have invented a quite complex formula apparatus in order to project the quantities which actually lie in the hyperplanes onto the tangent planes of a coherent Riemannian space. Although we have decided at the outset to prefer a quite graphical embedding as the basic structure of gravitation we do not go so far to require from mathematics to refer entirely to coherent surfaces. The formula apparatus used here does not describe figuratively the graphical world, but describes the forces which are active in this world.

Now we replace the holonomic differential (VII.6.11) by an anholonomic one

$$dx^{0'}_{anh} = -\tan\varepsilon\ a_R(r,\vartheta)\, dr = -\tan\varepsilon\, dx^{\underline{1}} \qquad \text{(VII.6.12)}$$

which is not integrable any longer. The family of hyperplanes which are orthogonal to these differentials and which are no longer V_3-forming we call *physical surface*. It is the area of all possible physical observations.

However, the structure of the [r, φ]-part of the surface is simple. On this patch no elliptical properties and hence no anholonomities arise. It corresponds to Flamm's paraboloid of the Schwarzschild geometry. Sharp [S] has searched for such a surface for $\vartheta = \pi/2$. However, he did not start from the seed metric, but has incorporated rotating parts of the Kerr metric into his computations. Hence, his result differs substantially from ours. From

$$ds^2 = dx^{0'^2} + dx^{1'^2} + dx^{2'^2} + dx^{3'^2}$$

[119] The mathematical structure of the Kaluza-Klein theories has only been well understood after recognizing the anholonomic properties of these theories.

one indeed obtains with

$$ds^2 = \left(\tan^2\varepsilon + 1\right)\frac{\Lambda^2}{A^2}dr^2 + \Lambda^2 d\vartheta^2 + A^2\sin^2\vartheta\, d\varphi^2$$

the desired seed metric, which serves as a basis for the actual Kerr metric.

In (VII.6.5) $\cos\varepsilon$ was identified with the gravitation factor a_S. Therewith and with the definition of the elliptical factor in (VII.2.7) one finally obtains (VII.6.3). The anholonomic construction is to be examined in more detail. Fig. VII.4 shows the constellation of the holonomic and anholonomic vectors, namely, the two 'radial' vectors, i.e. the vector[120] dx^1_{hol} in the tangent planes of the surface and dx^1_{anh}, the associated vector in the anholonomic hyperplanes which have the same projections onto the basic plane.

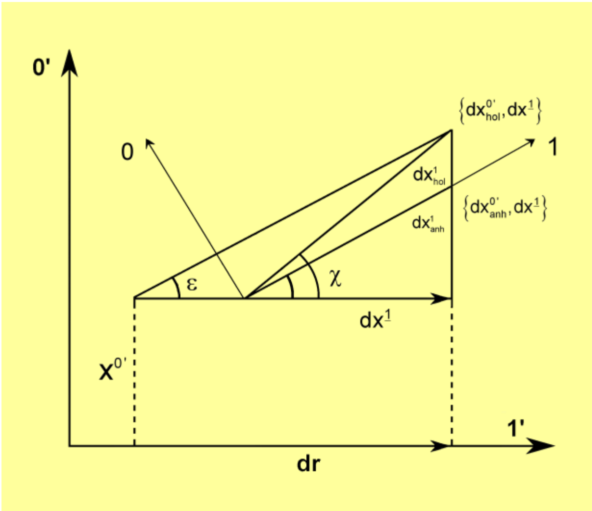

Fig. VII.4

With (VII.6.11) we are able to compute the curvature radius of the holonomic surface. With the help of the classical formula[121] one obtains first

$$\rho = \frac{1}{\cos\varepsilon\,\varepsilon'} \ ,$$

whereby the prime has the meaning of the derivative with respect to r. From (VII.6.5) one obtains $v_S' = \cos\varepsilon\,\varepsilon'$. If one uses (VII.6.4) one gets the curvature radius of the surface

$$\rho = A\sqrt{\frac{2r}{M}}\Phi^2, \quad \Phi^2 = \frac{r^2 + a^2}{r^2 - a^2} \ . \qquad \text{(VII.6.13)}$$

The auxiliary variable Φ will turn out to be useful later on. For the physical surface – the family of the anholonomic hyperplanes – one can define the quantity

$$\rho_S\left(r,\vartheta\right) = a_R\rho = a_R A\sqrt{\frac{2r}{M}}\Phi^2 \qquad \text{(VII.6.14)}$$

[120] All the other components of the vector except the mentioned one vanish in the local reference system.

[121] $\rho = -\frac{\left(1 + y'^2\right)^{3/2}}{y''}$

by analogy with the holonomic surface. ρ_S is called the curvature radius of the physical surface. As a by-product one obtains with the above intermediary formulae the useful relations

$$v_{S|1} = \frac{a_S}{\rho_S}, \quad a_{S|1} = -\frac{v_S}{\rho_S} \tag{VII.6.15}$$

and also the relation

$$dx^1 = \rho_S d\varepsilon \ , \tag{VII.6.16}$$

which obviously reminds us of an arc element in polar co-ordinates, but has local significance.

The definition of the physical time interval is better understood, if one puts

$$idt = \rho_S di\psi \ . \tag{VII.6.17}$$

In this case one has for the time-like element of the metric

$$dx^4 = a_S\rho_S di\psi = \rho_S \cos\varepsilon\, di\psi \ . \tag{VII.6.18}$$

$\rho\cos\varepsilon$ is the projection of the curvature radius of the radial curves into the direction of the global extra dimension $x^{0'}$. This straight line is rotated through the imaginary angle $i\psi$. Both, the starting and the end points of the straight line describe a flat (open) pseudo circle (hyperbola of constant curvature). The time interval is associated wtih the area of a ring sector of the pseudo circles. This consideration is already known from the Schwarzschild geometry and finally leads to the theory of a double-surface. We make use of it in later Sections.

Since (VII.6.14) can also be written as

$$\rho = -2r\Phi^2 \frac{1}{\sin\varepsilon}$$

and since $\rho\sin\varepsilon$ is the projection of ρ onto the r-axis, the r-co-ordinate of the base point of the curvature vector can be computed therewith as

$$\overline{r} = r - \rho\sin\varepsilon = r\left(1 + 2\Phi^2\right) \ . \tag{VII.6.19}$$

The base point is situated on the evolute of the radial curve of the surface. In the case of Schwarzschild the formula shortens to the well-known expression $\overline{r} = 3r$ and the evolute is Neil's parabola.

In addition, we note

$$d\overline{r} = \left[1 + 2\Phi^2 + \frac{4r^2}{r^2 - a^2}\left(1 - \Phi^2\right)\right] dr \ , \tag{VII.6.20}$$

$$\frac{d\rho}{d\overline{r}} = -\frac{1}{v_S} \ . \tag{VII.6.21}$$

Thus, we have provided all tools for the development of a 5-dimensional formulation of the Kerr model.

VII.7. Kerr metric, field quantities of the seed metric

The seed metric (VII.6.3) differs from the flat elliptical metric discussed in the last but one Section only by a factor in both dx^1 and in dx^4. Due to the first new factor the radial components of the curvature quantities differ by that factor, too. The new additional factor in the time-like part of the metric leads to the occurrence of the force of gravity. Now a genuine gravitation model is present which must be supplemented with rotational effects in order to obtain a physical interpretation. From the metric (VII.6.3) one reads the 4-bein field

$$\begin{gathered}\overset{1}{e}_1=\alpha_S a_R,\quad \overset{2}{e}_2=\Lambda,\quad \overset{3}{e}_3=\sigma,\quad \overset{4}{e}_4=a_S\\ \underset{1}{e}^1=\frac{1}{\alpha_S a_R},\quad \underset{2}{e}^2=\frac{1}{\Lambda},\quad \underset{3}{e}^3=\frac{1}{\sigma}\quad \underset{4}{e}^4=\alpha_S\end{gathered}\,. \tag{VII.7.1}$$

From it one computes the Ricci-rotation coefficients with the decomposition

$$A_{mn}{}^s=B_{mn}{}^s+N_{mn}{}^s+C_{mn}{}^s+E_{mn}{}^s\ . \tag{VII.7.2}$$

The quantities of the individual members have the meaning

$$\begin{gathered}B_{mn}{}^s=b_m B_n b^s-b_m b_n B^s,\quad N_{mn}{}^s=m_m N_n m^s-m_m m_n N^s\\ C_{mn}{}^s=c_m C_n c^s-c_m c_n C^s,\quad E_{mn}{}^s=-\left[u_m E_n u^s-u_m u_n E^s\right]\\ B_1=a_S\frac{1}{\Lambda}\Lambda_{|1}=\frac{a_S}{\rho_E},\quad N_2=\frac{1}{\Lambda}\Lambda_{|2}=\frac{1}{\rho_H},\quad C_1=\frac{1}{\sigma}\sigma_1=a_S\frac{r}{A\Lambda},\quad C_2=\frac{1}{\sigma}\sigma_2=\frac{1}{\Lambda}\cot\vartheta\,.\\ \sigma_1=\sigma_{|1}=a_S\frac{r}{\Lambda}\sin\vartheta,\ \sigma_2=\sigma_{|2}=\frac{A}{\Lambda}\cos\vartheta\end{gathered} \tag{VII.7.3}$$

The force of gravity has only one component in the radial direction. It is computed by

$$E_1=\frac{1}{\alpha_S}\alpha_{S|1}=-\alpha_S\alpha_R\frac{M}{A^2}\Phi^{-2}\ . \tag{VII.7.4}$$

Using (VII.6.13) and (VII.6.14) one obtains the geometrically simply interpretable expression[122]

$$E_1=\frac{1}{\rho_S}\frac{v_S}{a_S}=\frac{1}{\rho_S}\tan\varepsilon \tag{VII.7.5}$$

which we will consider later on. At this stage of development one can recognize that the quantities treated here and the equations connected with them can be represented more simply in the context of a 5-dimensional theory. The radial components of the curvature quantities can be written down with the help of the trigonometric functions defined in (VII.6.5) as well

$$B_1=\frac{1}{\rho_E}\cos\varepsilon,\quad C_1=\frac{1}{\rho_E}\alpha_R^2\cos\varepsilon,\quad E_1=\frac{1}{\rho_S\cos\varepsilon}\sin\varepsilon \tag{VII.7.6}$$

and can immediately be supplemented by local 0-components

$$B_0=\frac{1}{\rho_E}\sin\varepsilon,\quad C_0=\frac{1}{\rho_E}\alpha_R^2\sin\varepsilon,\quad E_0=-\frac{1}{\rho_S}\ . \tag{VII.7.7}$$

[122] The force of gravity E is negative, i.e. oriented inwards because the angle ε is negative.

It will be shown that with the help of these quantities the field equations are more compact and can be formulated in a more understandable way. The 5-dimensional subequations for these quantities decouple from the Einstein field equations, in the same way as the flat equations of the elliptical system have done. For this reason the 5-dimensional equations are initially set up intuitively. In a later Section they will undergo a deeper substantiation.

VII.8. Kerr metric, a first step towards five dimensions

In the preceding Section the curvature quantities have been supplemented by components which lie in the local extra dimension. From (VII.7.6) and (VII.7.7) one makes out that the model contains two types of quantities, the *horizontal* ones B and C, and a *vertical* one, namely E. By the term horizontal we imply that $B_a = \{B_0, B_1, 0,0,0\}$ and $C_a = \{C_0, C_1, C_2, 0,0\}$, a=0,1,…,4 are the 5-dimensional components of quantities which lie in the horizontal cutting planes of the surface and are oriented outwards. In principle they are identical with the above-defined flat quantities. The new quantity $E_a = \{E_0, E_1, 0,0,0\}$ is vertical to the horizontal slices. This has consequences for the ansatz for a pseudo-hyper sphere from which the Kerr theory is finally deduced. This ansatz still has to be discussed. We will see that rotations about two different orthogonal axes will take place. The circumstances of these quantities are illustrated in Fig. VII.5 and Fig. VII.6.

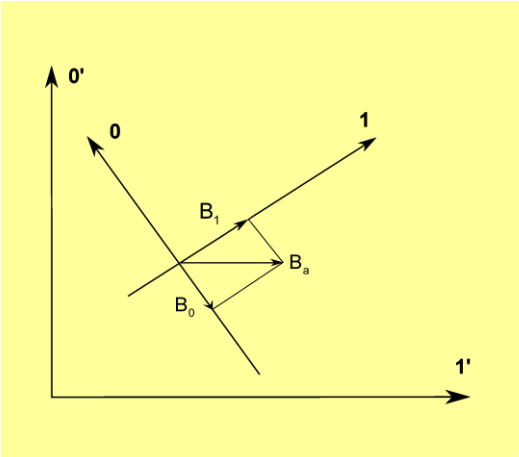

Fig. VII.5

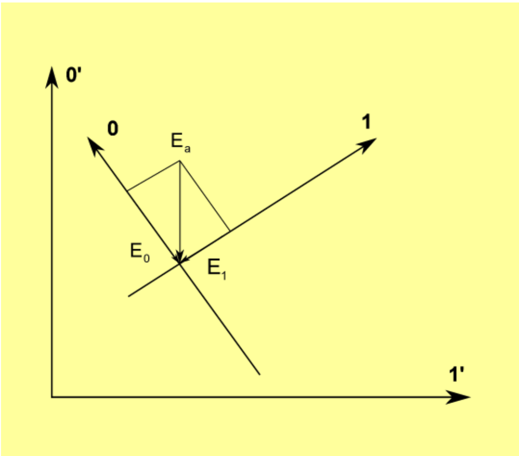

Fig. VII.6

B_a and C_a are perpendicular to each other

$$B_a E^a = 0, \quad C_a E^a = 0 \ . \qquad \text{(VII.8.1)}$$

These two relations will be used occasionally. In order to complete the field equations the quantities

$$M_{ab}{}^{c} = m_a M_b m^c - m_a m_b M^c, \quad M_b = \left\{ \frac{1}{\rho_S}, 0, 0, 0, 0 \right\} \tag{VII.8.2}$$

will be introduced and justified more exactly later on. The vector M points into the local 0-direction. It is derived from the first curvature of the radial integral curves of the surface and is vertical to the anholonomic hyperplanes. It is an element of the extrinsic geometry. The subequations of the Einstein field equations for the above-described surface can be computed once more with it

$$B_{1\|1} = a_S^2 B_{\underline{1}|\underline{1}} + B_1 \frac{1}{a_S} a_{S|1}, \quad B_1 = a_S B_{\underline{1}}.$$

The bar under the indices refers to the components of the quantities on a horizontal slice. These components are identical with the corresponding quantities of the flat geometry. If one harks back to the first relation before (VII.3.18) (the indices are thought to be underlined) and further to the second relation in Eq. (VII.6.15) one obtains

$$B_{1\|1} = a_S^2 \left[\tilde{\Omega}^{s3} \tilde{\Omega}_{3s} - B_{\underline{1}} B_{\underline{1}} \right] - B_1 \frac{1}{\rho_S} \frac{v_S}{a_S} . \tag{VII.8.3}$$

With (VII.7.5) the last term reads as $-B_1 E_1$ or with (VII.8.1) as $B_0 E_0$. However, since in accordance with (VII.7.7) and (VII.8.2) one has $M_0 = -E_0$. One finally obtains

$$B_{1\|1} + B_1 B_1 = a_S^2 \tilde{\Omega}^{s3} \tilde{\Omega}_{3s} - M_0 B_0 \tag{VII.8.4}$$

or more generally written as

$$B_{n\underset{2}{\|}m} - B_{n\underset{2}{\|}s} b^s b_m + B_n B_m = m_m m_n \left[a_S^2 \tilde{\Omega}^{s3} \tilde{\Omega}_{3s} - M_0 B_0 \right]. \tag{VII.8.5}$$

We have spoken about the quantity N just a little. Since it lies in the horizontal cutting planes its curvature equation does not change

$$N_{n\underset{2}{\|}m} - N_{n\underset{2}{\|}s} m^s m_s + N_n N_m = -b_m b_n \tilde{\Omega}^{s3} \tilde{\Omega}_{3s} . \tag{VII.8.6}$$

For the circular curvature quantity one obtains according to a similar calculation

$$C_{n\underset{3}{\|}m} + C_n C_m = m_m m_n \left[v_S^2 N_2 C_2 - M_0 C_0 \right] - b_m b_n B_0 C_0 , \tag{VII.8.7}$$

whereby again the graded covariant derivatives

$$\begin{gathered} m_{m\underset{1}{\|}n} = m_{m|n} = 0, \quad b_{m\underset{2}{\|}n} = b_{m|n} = 0, \quad c_{m\underset{3}{\|}n} = c_{m|n} - B_{nm}{}^{s} c_s - N_{nm}{}^{s} c_s = 0 \\ u_{m\underset{4}{\|}n} = u_{m|n} - B_{nm}{}^{s} u_s - N_{nm}{}^{s} u_s - C_{nm}{}^{s} u_s = 0 \end{gathered} \tag{VII.8.8}$$

have been used. One should compare these relations with (VII.2.11) and (VII.3.18). The computation of the E-equation will be somewhat more difficult. From (VII.7.5) one obtains

$$E_{1\|1} - E_1 E_1 = -M_0 E_0 - E_1 \frac{1}{\rho_S} \rho_{S|1} . \tag{VII.8.9}$$

The last term will play an important role by the embedding procedure. Different representations of this term and useful conversions of other expressions are listed among other things in the mathematical appendix (A1.1). Since the elliptical factor a_R is contained in ρ_S which also depends on ϑ, one has to consider

$$E_{1\underset{4}{||}2} = -E_1 N_2 .$$

Likewise one has

$$E_{2\underset{4}{||}1} = E_{2|1} - A_{12}{}^1 E_1 = -N_2 E_1, \quad E_{2\underset{4}{||}2} = B_1 E_1, \quad E_{3\underset{4}{||}3} = C_1 E_1 .$$

This relation gains a covariant form by consideration of (VII.8.1)

$$E_{n\underset{4}{||}m} - E_n E_m = -m_m m_n \left[M_0 E_0 + E_1 \frac{1}{\rho_S} \rho_{S|1} \right] - b_m b_n B_0 E_0 - c_m c_n C_0 E_0 - 2E_{(m}N_{n)} \quad \text{(VII.8.10)}$$

as well. Now the contractions of the above equations are to be determined

$$\begin{aligned}
&B^s{}_{\underset{2}{||}s} + B^s B_s = a_S^2 \tilde{\Omega}^{s3} \tilde{\Omega}_{3s} - M_0 B_0 \\
&N^s{}_{\underset{2}{||}s} + N^s N_s = -\tilde{\Omega}^{s3} \tilde{\Omega}_{3s} \\
&C^s{}_{\underset{3}{||}s} + C^s C_s = v_S^2 N_2 C_2 - M_0 C_0 - B_0 C_0 \\
&E^s{}_{\underset{4}{||}s} - E^s E_s = -M_0 E_0 - B_0 E_0 - C_0 E_0 - E_1 \frac{1}{\rho_S} \rho_{S|1}
\end{aligned} \quad . \qquad \text{(VII.8.11)}$$

If one composes now

$$\begin{aligned}
&-\left[B_{n\underset{2}{||}m} - B_{n\underset{2}{||}s} b^s b_m + B_n B_m \right] - b_n b_m \left[B^s{}_{\underset{2}{||}s} + B^s B_s \right] \\
&-\left[N_{n\underset{2}{||}m} - N_{n\underset{2}{||}s} m^s m_m + N_n N_m \right] - m_n m_m \left[N^s{}_{\underset{2}{||}s} + N^s N_s \right] \\
&-\left[C_{n\underset{3}{||}m} + C_n C_m \right] - c_n c_m \left[C^s{}_{\underset{3}{||}s} + C^s C_s \right] \\
&+\left[E_{n\underset{4}{||}m} - E_n E_m \right] + u_n u_m \left[E^s{}_{\underset{4}{||}s} - E^s E_s \right] = -\kappa \left(T_{mn} - \frac{1}{2} g_{mn} T \right)
\end{aligned} \qquad \text{(VII.8.12)}$$

and if the one summarizes the results of (VII.8.11) one obtains

$$\begin{aligned}
\kappa \left(T_{mn} - \frac{1}{2} g_{mn} T \right) = &-m_m m_n \left(M_0 B_0 + M_0 C_0 - M_0 E_0 \right) \\
&- b_m b_n \left(B_0 M_0 + B_0 C_0 - B_0 E_0 \right) \\
&- c_m c_n \left(C_0 M_0 + C_0 B_0 - C_0 E_0 \right) \\
&+ u_m u_n \left(E_0 M_0 + E_0 B_0 + E_0 C_0 \right) \\
&-\left(m_m m_n + b_m b_n \right) v_S^2 \tilde{\Omega}^{s3} \tilde{\Omega}_{3s} \\
&+\left(m_m m_n + c_m c_n \right) v_S^2 N_2 C_2 + 2N_{(m}E_{n)} \\
&+\left(m_m m_n + u_m u_n \right) E_1 \frac{1}{\rho_S} \rho_{S|1}
\end{aligned} \quad . \qquad \text{(VII.8.13)}$$

The block built by the first four lines is well known from earlier investigations. It contains the second fundamental forms of the surface theory which must correctly be designated as generalized fundamental forms. They refer to a surface which is additionally equipped with an anholonomic structure. The just-derived expression can be substantially simplified. For the [11]-component one reads off

$$-M_0B_0 - M_0C_0 + M_0E_0 - v_S^2\,\tilde{\Omega}^{s3}\tilde{\Omega}_{3s} + v_S^2\,N_2C_2 + E_1\frac{1}{\rho_S}\rho_{S|1}.$$

The $\tilde{\Omega}$-term was indicated in (VII.3.11). With the relation (VII.3.14) this can be written in a simplified way as

$$\tilde{\Omega}^{s3}\tilde{\Omega}_{3s} \overset{*}{=} \tilde{H}_{13}\tilde{H}_{13} - \tilde{H}_{23}\tilde{H}_{23}, \qquad \text{(VII.8.14)}$$

whereby the tildes suggest again that the flat quantities are intended. $\tilde{H}_{23}$ does not differ from H_{23}, $H_{13} = a_S\tilde{H}_{13}$, however, differs from the flat quantities by the gravitational factor. Although the static model is still treated, quantities are already used which arise from the rotation of the system later on. It has already been mentioned that the elliptical geometry makes available quantities which become important only after the implementation of the rotation has been performed. Now we are ready to discover a further relationship.

In the flat system the centrifugal force

$$\tilde{F}_m = \alpha_R^2\omega^2\sigma\tilde{\sigma}_m$$

has been discussed which can be written in accordance with (VII.2.10) as

$$\tilde{F}_m = \alpha_R^2\omega^2\sigma^2\tilde{C}_m\ . \qquad \text{(VII.8.15)}$$

Hence one has

$$\tilde{C}_m + \tilde{F}_m = \left(1 + \alpha_R^2\omega^2\sigma^2\right)\tilde{C}_m = \alpha_R^2\tilde{C}_m.$$

If one multiplies by $\tilde{F}_n$

$$\left(\tilde{C}_m + \tilde{F}_m\right)\tilde{F}_n = \alpha_R^2\tilde{C}_m\tilde{F}_n = \alpha_R^4\omega^2\tilde{\sigma}_m\tilde{\sigma}_n,$$

one obtains the useful relation

$$\tilde{H}_{m3}\tilde{H}_{n3} = -\left(\tilde{C}_m + \tilde{F}_m\right)\tilde{F}_n \qquad \text{(VII.8.16)}$$

which is also valid for ‘curved’ quantities. That means that the above relations can be written without tildes as well. In the next Sections use is made of it. From the definitions (VII.2.8) and (VII.2.10) one infers

$$N_2 = -\frac{a^2}{\Lambda^3}\sin\vartheta\cos\vartheta. \qquad \text{(VII.8.17)}$$

On the other hand one has

$$F_2 = \alpha_R^2\omega^2\sigma\sigma_2 = \frac{A^2}{\Lambda^2}\frac{a^2}{A^4}A\sin\vartheta\frac{A}{\Lambda}\cos\vartheta = \frac{a^2}{\Lambda^3}\sin\vartheta\cos\vartheta$$

and thus[123] we can use in our calculation

$$N_2 = -F_2. \qquad \text{(VII.8.18)}$$

If one starts with (A1.1) and if one considers that one has

$$M_0 = -E_0 \qquad \text{(VII.8.19)}$$

[123] The tilde is only necessary for the first components of all quantities. The gravitational factor arises only in the ‘curved’ quantities.

due to (VII.7.7) and (VII.8.2), and further if one converts the second components of the term (VII.8.14) with (VII.8.16) then the expression considered at first has the form

$$-2\left[2C_0E_0 - C_0C_0\right] + v_S^2\left[-\tilde{H}_{13}\tilde{H}_{13} - 2C_2F_2 - F_2F_2\right].$$

If one inserts (A1.2) into this relation and once more (A1.3), one finally obtains

$$-2\left[v_S^2\tilde{\Omega}_{13}\tilde{\Omega}_{13} + F_1E_1\right] .$$

However, the [22]-components of the relations one can easily convert with (VII.8.19), (VII.8.14), and (A1.4). The [33]- and [44]-components differ only by the sign. Thus, one finally obtains

$$\kappa\left(T_{mn} - \frac{1}{2}g_{mn}T\right) = v_S^2\begin{pmatrix} -2\tilde{\Omega}_{13}\tilde{\Omega}_{31} & & & \\ & -2\tilde{\Omega}_{23}\tilde{\Omega}_{32} & & \\ & & \tilde{\Omega}_{rs}\tilde{\Omega}^{rs} + 2\tilde{F}_s\tilde{F}^s & \\ & & & -\tilde{\Omega}_{rs}\tilde{\Omega}^{rs} - 2\tilde{F}_s\tilde{F}^s \end{pmatrix} - 2F_{(m}E_{n)} . \quad \text{(VII.8.20)}$$

The aggregate is composed almost only of the contribution of the evolutes of the elliptical geometry. From this result the stress-energy tensor of the model can be computed. In accordance with the conditions it is covariantly conserved, but it is less interesting because no physical meaning is attributed to this auxiliary model.

We have seen that for the examination of the static model numerous quantities have been necessary which have their origin in the elliptical geometry. However, the physical meaning can only be deepened in the context of the rotating model. With the above-derived formulae we are already well prepared for the analysis of the actual Kerr metric.

VII.9. Kerr metric, rotating systems

The actual Kerr metric can be derived from the seed metric with an anholonomic intrinsic transformation. On the one hand this means that the rotational effects are not dynamically attached by means of a Lorentz transformation, on the other hand that the geometry is changed by the intrinsic transformation. This change does not lead to a deformation of the surface discussed in the preceding Section, but leads to the implementation of an additional, generally position-dependent structure on the surface. In the actual case it results in a local definition of 4-beins. The 3rd and the 4th bein of the initially orthogonal local system of the seed metric are rotated in such a way that they enclose an inclined angle. If one lets act the anholonomic transformation

$$\begin{aligned}&\Lambda^{3'}_{3}=\alpha_R,\quad \Lambda^{3'}_{4}=\ i\alpha_R\omega\,,\quad \Lambda^{4'}_{3}=-i\alpha_R\omega\sigma^2,\quad \Lambda^{4'}_{4}=\alpha_R\\ &\Lambda^{3}_{3'}=\alpha_R,\quad \Lambda^{3}_{4'}=-i\alpha_R\omega\,,\quad \Lambda^{4}_{3'}=\ i\alpha_R\omega\sigma^2\,,\quad \Lambda^{4}_{4'}=\alpha_R\end{aligned} \tag{VII.9.1}$$

onto the co-ordinates of the 4-beins and the metric

$$\overset{m}{e}_i=\Lambda^{i'}_{i}\overset{m}{e}_{i'}\,,\quad \underset{m}{e}^{i}=\Lambda^{i}_{i'}\underset{m}{e}^{i'}\,,\quad g_{ik}=\Lambda^{i'k'}_{ik}g_{i'k'}\ , \tag{VII.9.2}$$

one obtains the 4-bein field in oblique-angled co-ordinates

$$\begin{aligned}&\overset{1}{e}_1=\alpha_S a_R,\quad \overset{2}{e}_2=\Lambda,\quad \overset{3}{e}_3=\alpha_R\sigma,\quad \overset{3}{e}_4=i\alpha_R\omega\sigma,\quad \overset{4}{e}_3=-ia_S\alpha_R\omega\sigma^2,\quad \overset{4}{e}_4=a_S\alpha_R\\ &\underset{1}{e}^1=a_S\alpha_R,\quad \underset{2}{e}^2=\frac{1}{\Lambda},\quad \underset{3}{e}^3=\frac{\alpha_R}{\sigma},\quad \underset{3}{e}^4=i\alpha_R\omega\sigma,\quad \underset{4}{e}^3=-i\alpha_S\alpha_R\omega\,,\quad \underset{4}{e}^4=\alpha_S\alpha_R\end{aligned} \tag{VII.9.3}$$

which leads to the line element

$$ds^2={dx^1}^2+{dx^2}^2+\left[\alpha_R dx^3+i\alpha_R\omega\sigma dx^4\right]^2+a_S^2\left[-i\alpha_R\omega\sigma dx^3+\alpha_R dx^4\right]^2\ . \tag{VII.9.4}$$

The anholonomic differentials are defined as

$$dx^1=\alpha_S a_R dr,\quad dx^2=\Lambda d\vartheta,\quad dx^3=\sigma d\varphi,\quad dx^4=\rho_S di\psi=idt\ . \tag{VII.9.5}$$

Inside the brackets of the line element one identifies the characteristic expressions for a Lorentz transformation. However, the gravitational factor before the last brackets makes clear that the Kerr line element cannot be created by a Lorentz transformation from the line element of the seed metric. The gravitational factor is located in the 'wrong' place[124]. The 4-bein system obtained in such a way is called after Carter [C] system C and is one of the preferred reference systems which are to be treated. The systems A and B discussed in earlier Sections emerge from the Carter system by the use of a Lorentz transformation. The components of the 4-bein fields written down in (VII.9.3) fulfill the conditions

$$g^{ik}\overset{m}{e}_i\overset{n}{e}_k=g^{mn}=\delta^{mn},\quad g_{ik}\underset{m}{e}^i\underset{n}{e}^k=g_{mn}=\delta_{mn} \tag{VII.9.6}$$

and they are the components of the 4-beins defined for the seed metric which are represented in an oblique-angled co-ordinate system. The use of the co-ordinate transformation (VII.9.1) is a summary proceeding, in order to gain the actual Kerr metric without making great efforts. In that way, the Carter bein has a simple interpretation. Since the use of co-ordinate methods is not accepted by us as a suitable procedure for describing physical effects, we again repeat the whole story, whereby we exclusively refer to reference systems.

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[124] The last term of a Lorentz-transformed line element would have the form $\left[-i\alpha_R\omega\sigma dx^3+\alpha_R a_S dx^4\right]^2$.

The tangent vectors of the oblique-angled co-ordinate system will be designated by $\underset{i}{h}^m$, the gradient vectors by $\overset{i}{h}_m$. The indices i enumerate these vectors, the indices m designate the components with regard to the local Cartesian co-ordinate system of the seed metric. However, the components of these vectors are *numerically* equivalent to the above-defined bein vectors

$$\overset{i}{h}_m \triangleq \underset{m}{e}^i, \quad \underset{i}{h}^m \triangleq \overset{m}{e}_i \ , \tag{VII.9.7}$$

however, they are not unit vectors. Their lengths[125] are computed as

$$\begin{array}{ll} \tau_3 = \sqrt{\underset{3}{h}^m \underset{3}{h}^m} = a_{BC}\alpha_R\sigma, & \tau_4 = \sqrt{\underset{4}{h}^m \underset{4}{h}^m} = a_{AC}\alpha_R a_S \\ \gamma^3 = \sqrt{\overset{3}{h}_m \overset{3}{h}_m} = a_{AC}\alpha_R \dfrac{1}{\sigma}, & \gamma^4 = \sqrt{\overset{4}{h}_m \overset{4}{h}_m} = a_{BC}\alpha_R\alpha_S \end{array} , \tag{VII.9.8}$$

whereby new quantities

$$\begin{array}{lll} a_{AC}^2 = 1-\omega_{AC}^2\sigma^2, & \alpha_{AC} = 1/a_{AC}, & \omega_{AC} = \alpha_S\omega \\ a_{BC}^2 = 1-\omega_{BC}^2\sigma^2, & \alpha_{BC} = 1/a_{BC}, & \omega_{BC} = a_S\omega \end{array} \tag{VII.9.9}$$

arise which will be substantial for the above-mentioned preferred reference systems. The angle which is enclosed by the 3rd and the 4th bein vectors, results in

$$\cos i\beta'_{AB} = \frac{\underset{3}{h}^3 \underset{4}{h}^3 + \underset{3}{h}^4 \underset{4}{h}^4}{\tau_3\tau_4} = i\alpha_{AC}\alpha_{BC}\left(\omega_{AC}-\omega_{BC}\right)\sigma = i\alpha_{AB}\omega_{AB}\sigma \ , \tag{VII.9.10}$$

whereby the relations

$$\begin{aligned} \alpha_{AB} &= \frac{1}{\sqrt{1-\omega_{AB}^2\sigma^2}} = \alpha_{AC}\alpha_{BC}\left(1-\omega_{AC}\omega_{BC}\sigma^2\right) = \alpha_{AC}\alpha_{BC}a_R^2 \\ \omega_{AB} &= \frac{\omega_{AC}-\omega_{BC}}{1-\omega_{AC}\omega_{BC}\sigma^2} = \alpha_R^2\left(\omega_{AC}-\omega_{BC}\right) \end{aligned} \tag{VII.9.11}$$

show that the recently introduced expressions are ingredients of Lorentz transformations which mediate between the systems A, B, and C. Further, one should try to illustrate the just-suggested relations. At first one must remember the fact that one operates in a complex space which is beyond any imagination. Thus, one can hark back only to a pseudoreal representation. However, if all this is taken too literally, it can be misleading. It ends up in the fact that imaginary angles are replaced by real ones and one operates with trigonometric functions, although hyperbolic functions would correspond to the functions of imaginary angles. In addition, terms such as complementary angle and supplementary angle are merely analogies. They can be understood by a sketch, but they are only definitions of expressions.

In this sense the construction of the 'complementary angle' β_{AB} from β'_{AB} should be understood. Fig. VII.7 illustrates the relationship. Anyhow, the definitions

$$\cos i\beta_{AB} = \alpha_{AB}, \quad \sin i\beta_{AB} = i\alpha_{AB}\omega_{AB}\sigma \tag{VII.9.12}$$

are free of any attempt of an exemplification.

[125] The position of the index m is irrelevant because m refers to Cartesian co-ordinates.

In Fig. VII.7 the orthogonal (black) system $\{3^C, 4^C\}$ corresponds to the local system of the seed metric. The rotational effects are implemented into the metric by means of this oblique-angled (red) system {3, 4}, the vectors of which enclose the angle $i\beta_{AB}$.

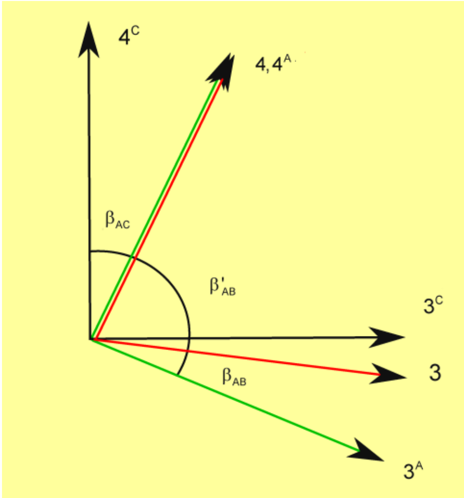

Fig. VII.7

Since oblique-angled systems are not suitable for the representation of physical quantities, one must use orthogonal systems. Such systems can be arbitrarily selected within certain limits. Since reference systems are connected with a condition for a motion of an observer, it is advisable to select reference systems which describe a simple state of motion for an observer. Such a system would be the one of Carter which maintains the system of the seed metric. The (green) system $\{3^A, 4^A\}$ of Iyer [I] and Kumar, is called system A. It is based on the 4th vector of the inclined system and in addition the 3rd vector of the bein is drawn orthogonally to the given 4th vector.

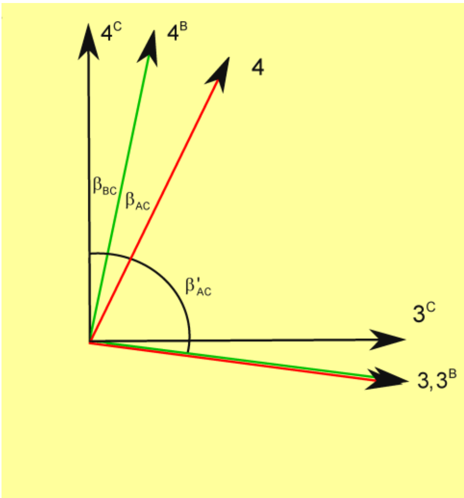

Fig. VII.8

From Fig. VII.8 one infers the positions of the (green) vectors of the system $\{3^B, 4^B\}$ of Bardeen [B]. It is the locally nonrotating system (LNR) which will be discussed later on. Here the 3rd bein vector is merged with the 3rd vector of the oblique-angled system, and the 4th vector is put up vertically to the 3rd vector.

One obtains the oblique-angled system from the orthogonal unit vector system of the seed metric by a rotation and dilatation

$$\overset{m}{e}_3 \stackrel{*}{=} \underset{3}{h}^m = \{\cos i\beta_{BC}, -\sin i\beta_{BC}\}\tau_3, \quad \overset{m}{e}_4 \stackrel{*}{=} \underset{4}{h}^m = \{\sin i\beta_{AC}, \cos i\beta_{AC}\}\tau_4$$

$$\underset{m}{e}^3 \stackrel{*}{=} \overset{3}{h}_m = \{\cos i\beta_{AC}, -\sin i\beta_{AC}\}\gamma^3, \quad \underset{m}{e}^4 \stackrel{*}{=} \overset{4}{h}_m = \{\sin i\beta_{BC}, \cos i\beta_{BC}\}\gamma^4 \quad . \qquad \text{(VII.9.13)}$$

$$\cos i\beta_{AC} = \alpha_{AC}, \quad \sin i\beta_{AC} = i\alpha_{AC}\omega_{AC}\sigma$$

$$\cos i\beta_{BC} = \alpha_{BC}, \quad \sin i\beta_{BC} = i\alpha_{BC}\omega_{BC}\sigma$$

In each case the unit vectors are rotated through the angles $i\beta_{AC}$ and $i\beta_{BC}$ and also stretched by τ and γ, respectively. τ and γ are the local units for the oblique-angled system. With this procedure one gets for the metric tensor the simple form

$$\begin{aligned} &g_{33} = \tau_3\tau_3, \quad g_{34} = i\alpha_{AB}\omega_{AB}\sigma\tau_3\tau_4, \quad g_{44} = \tau_4\tau_4 \\ &g^{33} = \gamma^3\gamma^3, \quad g^{34} = -i\alpha_{AB}\omega_{AB}\sigma\gamma^3\gamma^4, \quad g^{44} = \gamma^4\gamma^4 \end{aligned} \quad . \qquad \text{(VII.9.14)}$$

If one continues with the pseudoreal representation (for the sake of serving the illustration) and if one takes a further step, then the rotational part of the metric can be formulated with the cosine law

$$ds^2 = \left(\tau_3 dx^3\right)^2 - 2\cos i\beta''_{AB}\left(\tau_3 dx^3\right)\left(\tau_4 dx^4\right) + \left(\tau_4 dx^4\right)^2, \quad dx^3 = d\varphi, \quad dx^4 = idt \text{ .(VII.9.15)}$$

Again the ‘supplementary angle’ is only a definition for visualizing the problem.

From this Section it becomes clear that the curvature of the space described by the Kerr metric is not articulated in a narrower sense. In accordance with view (III) the curvature is sufficiently described by the surface of the seed metric. The cross term in the Kerr metric, which is substantially responsible for the rotation effects, is due to an additional structure on the surface. In other words, the metric with its cross term does not describe a surface which would be constituted in such a manner that only a Gauss co-ordinate system can be drawn on it in this way that this co-ordinate system must necessarily require a cross term.

We again reconsider this Section. From the seed metric

$$ds^2 = \alpha_S^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 + a_S^2 dx^{4^2}, \; dx^4 = idt$$

one reads the components of the bein vectors

$$\overset{3}{e}_{3'} = \sigma, \quad \overset{4}{e}_{4'} = a_S, \quad \underset{3}{e}^{3'} = \frac{1}{\sigma}, \quad \underset{4}{e}^{4'} = \frac{1}{a_S} \; .$$

Their lengths are

$$\tau_{3'} = \sigma, \quad \tau_{4'} = a_S, \quad \gamma^{3'} = \frac{1}{\sigma}, \quad \gamma^{4'} = \frac{1}{a_S} \; .$$

The combined rotational dilatation indicated in (VII.9.1) can be separated into a rotation and into a dilatation (without summation over k and k')

$$\underset{i}{h}^m = \Lambda_i^{i'} \underset{i'}{h}^m, \quad \overset{i}{h}_m = \Lambda_{i'}^{i} \overset{i'}{h}_m, \quad \Lambda_k^{i'} = \bar{\Lambda}_k^{i'}\tau_k\gamma^{i'}, \quad \Lambda_{k'}^{i} = \bar{\Lambda}_{k'}^{i}\tau_{k'}\gamma^{i'} \; , \qquad \text{(VII.9.16)}$$

whereby it is evident from the rotation matrix

$$\bar{\Lambda}_k^{i'} = \begin{pmatrix} \alpha_{BC} & i\alpha_{AC}\omega_{AC}\sigma \\ -i\alpha_{BC}\omega_{BC}\sigma & \alpha_{AC} \end{pmatrix}, \quad \bar{\Lambda}_{k'}^{i} = \begin{pmatrix} \alpha_{AC} & -i\alpha_{AC}\omega_{AC}\sigma \\ i\alpha_{BC}\omega_{BC}\sigma & \alpha_{BC} \end{pmatrix} \qquad \text{(VII.9.17)}$$

that the 3rd and the 4th vectors achieve different rotations. After it has been revealed with which geometrical methods the seed metric can be equipped with rotational effects, these effects can be examined in the light of some preferred reference systems.

After this detailed analysis of the Kerr metric we return to the LNR system. It has been introduced by Bardeen [B] into the literature. He intended to remove the dragging effects of the Kerr field from the theory as far as possible by a suitable choice of the reference system. He rearranged the metric in such a way that it has the form

$$\begin{gathered} ds^2 = \frac{\Lambda^2}{\delta^2} dr^2 + \Lambda^2 d\vartheta^2 + \frac{\mathcal{A}}{\Lambda^2} \sin^2\vartheta \left(d\varphi - \Omega dt\right)^2 - \frac{\Lambda^2\delta^2}{\mathcal{A}} dt^2 \\ \mathcal{A} = A^4 - a^2\delta^2 \sin^2\vartheta, \quad \Omega = \frac{2Mra}{\mathcal{A}} \end{gathered} \quad . \tag{VII.9.18}$$

In it one can already recognize the form B as classified by us. An observer, who co-rotates with the geometry - as Bardeen says - is specified by the conditions

$$r = \text{const.}, \quad \vartheta = \text{const.}, \quad \varphi = \Omega t = \text{const.}. \tag{VII.9.19}$$

We do not accept the last requirement of Bardeen. In particular, in the above form of the metric

$$d\varphi - \Omega(r) dt$$

a nonintegrable expression is contained which arises from the differential rotation law. In order to find out the cause we re-formulate the metric (VII.9.18) step-by-step in such a way that it takes the structure preferred by us. First we eliminate the quantity $\mathcal{A}$, which has neither a clear physical nor a geometrical meaning

$$\mathcal{A} = A^4\left(1 - \delta^2\omega^2\sin^2\vartheta\right) = A^4\left(1 - a_S^2\omega^2\sigma^2\right) = A^4\left(1 - \omega_{BC}^2\sigma^2\right) .$$

With

$$\mathcal{A} = A^4 a_{BC}^2 \tag{VII.9.20}$$

we introduce the reciprocal Lorentz factor of a relative motion into the theory. Thus, the metrical factors in (VII.9.18) can be manipulated:

$$\begin{gathered} \frac{\mathcal{A}}{\Lambda^2}\sin^2\vartheta = \frac{A^4 a_{BC}^2}{\Lambda^2}\sin^2\vartheta = a_{BC}^2\alpha_R^2\sigma^2 = \tau_3^2 \\ \frac{\Lambda^2\delta^2}{\mathcal{A}} = \frac{\Lambda^2\delta^2}{A^4 a_{BC}^2} = \alpha_{BC}^2 \frac{\delta^2}{A^2}\frac{\Lambda^2}{A^2} = \alpha_{BC}^2 a_S^2 a_R^2 \end{gathered} \quad . \tag{VII.9.21}$$

The two terms already contain exclusively expressions which are used in our representation. However, the last term is still worthy of shaping. From the formula of a composed Lorentz transformation

$$\alpha_{AB} = \alpha_{AC}\alpha_{BC}\left(1 - \omega_{AC}\omega_{BC}\sigma^2\right) \tag{VII.9.22}$$

one obtains after a short calculation

$$1 - \omega_{AC}\omega_{BC}\sigma^2 = 1 - \alpha_S\omega a_S\omega\sigma^2 = 1 - \omega^2\sigma^2 = a_R^2$$

the useful auxiliary relations

$$\alpha_{AB} = \alpha_{AC}\alpha_{BC}a_R^2, \quad \alpha_{BC} = \alpha_{AB}a_{AC}\alpha_R^2 , \tag{VII.9.23}$$

with which one again can write the above result with the help of

$$\alpha_{BC}a_S a_R = \alpha_{AB}a_{AC}\alpha_R a_S = \alpha_{AB}\tau_4$$

in the concise form

$$\frac{\Lambda^2\delta^2}{\mathcal{A}} = \alpha_{AB}^2\tau_4^2 \ . \tag{VII.9.24}$$

Now Bardeen's expression Ω is to be tackled. Since one has

$$v_S^2 = \frac{2Mr}{A^2} = 1 - a_S^2$$

the relation

$$\Omega = \alpha_{BC}^2 v_S^2 \omega = \alpha_{BC}^2 a_S \left(\omega_{AC} - \omega_{BC}\right)$$

is valid. With Einstein's composition law of velocities

$$\omega_{AB} = \frac{\omega_{AC} - \omega_{BC}}{1 - \omega_{AC}\omega_{BC}\sigma^2} = \alpha_R^2\left(\omega_{AC} - \omega_{BC}\right)$$

one has

$$\Omega = \alpha_{BC}^2 a_S a_R^2 \omega_{AB}$$

and with (VII.9.23)

$$\Omega = \alpha_{BC} a_{AC} a_S \alpha_{AB} \omega_{AB}$$

which is still imprecise. More light is shed on the problem by the relation

$$\frac{\tau_4}{\tau_3} = \alpha_{BC} a_{AC} a_S \frac{1}{\sigma}$$

so that finally one has

$$\Omega = \alpha_{AB}\omega_{AB}\sigma\frac{\tau_4}{\tau_3} \ . \tag{VII.9.25}$$

If one inserts all these complex conversions into the metric (VII.9.18)

$$ds^2 = \alpha_S^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \left[\tau_3 d\varphi - \alpha_{AB}\omega_{AB}\sigma\tau_4 dt\right]^2 - \alpha_{AB}^2\left(\tau_4 dt\right)^2 \ , \tag{VII.9.26}$$

one has recovered the metric of the system B with the use the dilatation factors τ_3 and τ_4.

With this one recognizes that the conditions

$$r = \text{const.}, \quad \vartheta = \text{const.}, \quad \tau_3 d\varphi = \alpha_{AB}\omega_{AB}\sigma\tau_4 dt \tag{VII.9.27}$$

replace the relation (VII.9.19) and that they specify the locally nonrotating reference systems. In the third relation all factors on the right side depend on the radial variable r. They are not integrable. The formula describes the relative velocity of the reference-defined observer B in relation to the observer A. A comparison of ω_{AB} with Ω makes clear that we define differently the angular velocity of a LNR-observer from Bardeen. For the system B we derive the relations

$$u_{[\alpha||\beta]} = 0, \quad u_{(\alpha||\beta)} \neq 0 \ . \tag{VII.9.28}$$

The first relation corresponds to Bardeen's demand to switch off the dragging effects to a very large extent. Coriolis-like forces do not act on the observer B. However, neighboring observers moving past are exposed to shears. This states the second relation. In a differential rotating system (VII.9.28) is a co-ordinate-independent supposition for the smallest amount of dragging of the observer system.

VII.10. Kerr metric, preferred reference systems

It has been shown in the preceding Section that the necessity to introduce local orthogonal reference systems has resulted from the geometrical analysis of the Kerr metric. With the aid of these systems the physical quantities of the theory can be represented. One of these systems, the system of Carter, has been provided by the seed metric. The other two systems have resulted from the adjustments of the oblique-angled system. All these three systems refer to rotations with different angular velocities. The angular speed of the Carter systems has been settled with

$$\omega = \frac{a}{A^2} \;, \tag{VII.10.1}$$

the relative speeds of the system A and the system B, respectively relative to the Carter system with

$$\omega_{AC} = \alpha_S \omega, \quad \omega_{BC} = a_S \omega \;. \tag{VII.10.2}$$

Hence, the relativistic orbital velocities of the system A, B, and C at the distance $\sigma = A \sin\vartheta$ of the rotation axis are

$$\alpha_{AC}\omega_{AC}\sigma, \quad \alpha_{BC}\omega_{BC}\sigma, \quad \alpha_R\omega\sigma \tag{VII.10.3}$$

and the associated Lorentz factors are

$$\alpha_{AC} = 1\Big/\sqrt{1-\omega_{AC}^2\sigma^2}, \quad \alpha_{BC} = 1\Big/\sqrt{1-\omega_{BC}^2\sigma^2}, \quad \alpha_R = 1\Big/\sqrt{1-\omega^2\sigma^2} \;. \tag{VII.10.4}$$

In particular $\omega\sigma$ is that speed with which an observer in the exterior field of a rotating source is carried along by the forces of the field. The effect is well known in the literature under the name of *frame dragging*. Finally, the relative velocity between the system A and the system B can be computed with the composition law of the velocities

$$\omega_{AB} = \frac{\omega_{AC} + \omega_{CB}}{1+\omega_{AC}\omega_{CB}\sigma^2}, \; \alpha_{AB} = 1\Big/\sqrt{1-\omega_{AB}^2\sigma^2} = \alpha_{AC}\alpha_{CB}\left(1+\omega_{AC}\omega_{CB}\sigma^2\right), \; \omega_{CB} = -\omega_{BC} \,. \tag{VII.10.5}$$

Since the coefficients of the Lorentz transformation have already resulted from geometrical views, the transition between the three systems can be established with

$$L_3^{3'} = \alpha_X, \quad L_3^{4'} = i\alpha_X\omega_X\sigma, \quad L_4^{3'} = -i\alpha_X\omega_X\sigma, \quad L_4^{4'} = \alpha_X, \qquad X = AC \text{ or } X = BC \;. \tag{VII.10.6}$$

In these systems the 4-beins take the form

(A):

$$\begin{gathered}
\overset{1}{e}_1 = \alpha_S a_R, \; \overset{2}{e}_2 = \Lambda, \; \overset{3}{e}_3 = \alpha_{AC} a_R \sigma, \; \overset{4}{e}_3 = i\alpha_{AC}\left(\omega_{AC}-\omega_{BC}\right)\alpha_R\sigma^2, \; \overset{3}{e}_4 = 0, \; \overset{4}{e}_4 = a_{AC} a_S \alpha_R \\
\underset{1}{e}^1 = a_S\alpha_R, \; \underset{2}{e}^2 = \frac{1}{\Lambda}, \; \underset{3}{e}^3 = \frac{1}{\alpha_{AC} a_R \sigma}, \; \underset{3}{e}^4 = -i\alpha_{AC}\left(\omega_{AC}-\omega_{BC}\right)\alpha_S\alpha_R\sigma, \; \underset{4}{e}^3 = 0, \; \underset{4}{e}^4 = \alpha_{AC}\alpha_S a_R
\end{gathered} \tag{VII.10.7}$$

(B):

$$\begin{gathered}
\overset{1}{e}_1 = \alpha_S a_R, \; \overset{2}{e}_2 = \Lambda, \; \overset{3}{e}_3 = a_{BC}\alpha_R\sigma, \; \overset{4}{e}_3 = 0, \; \overset{3}{e}_4 = i\alpha_{BC} a_S\left(\omega_{AC}-\omega_{BC}\right)\alpha_R\sigma, \; \overset{4}{e}_4 = \alpha_{BC} a_S a_R \\
\underset{1}{e}^1 = a_S\alpha_R, \; \underset{2}{e}^2 = \frac{1}{\Lambda}, \; \underset{3}{e}^3 = \frac{1}{a_{BC}\alpha_R\sigma}, \; \underset{3}{e}^4 = 0, \; \underset{4}{e}^3 = -i\alpha_{BC}\left(\omega_{AC}-\omega_{BC}\right)\alpha_R, \; \underset{4}{e}^4 = a_{BC}\alpha_S\alpha_R
\end{gathered} \tag{VII.10.8}$$

(C):

$$\overset{1}{e}_1 = \alpha_S a_R,\ \overset{2}{e}_2 = \Lambda,\ \overset{3}{e}_3 = \alpha_R \sigma,\ \overset{3}{e}_4 = i\alpha_R \omega\sigma,\ \overset{4}{e}_3 = -i a_S \alpha_R \omega \sigma^2,\ \overset{4}{e}_4 = a_S \alpha_R$$
$$\underset{1}{e}^1 = a_S \alpha_R,\ \underset{2}{e}^2 = \frac{1}{\Lambda},\ \underset{3}{e}^3 = \frac{\alpha_R}{\sigma},\ \underset{3}{e}^4 = i\alpha_R \omega\sigma,\ \underset{4}{e}^3 = -i\alpha_S \alpha_R \omega,\quad \underset{4}{e}^4 = \alpha_S \alpha_R \tag{VII.10.9}$$

With their help the Ricci-rotation coefficients can be computed. First, the system C is to be elaborated. The first two dimensions do not exhibit anything new with regard to the seed metric. The results can be applied. Only the connexion coefficients for the rotational part must be computed once again. From

$$A_{3m}{}^3 = -\overset{3}{e}_i \underset{3|m}{e}{}^i = C^C_m = C_m + F_m$$

one obtains

$$C_{mn}{}^s = c_m\left(C_n c^s - c_n C^s\right) + c_m\left(F_n c^s - c_n F^s\right)\ , \tag{VII.10.10}$$

whereby the circular field quantity

$$C_m = \frac{1}{\sigma}\sigma_{|m} \tag{VII.10.11}$$

and the centrifugal force

$$F_m = \alpha_R^2 \omega^2 \sigma \sigma_m \tag{VII.10.12}$$

differ from the flat quantities (VII.4.1) and (VII.4.6) by the gravitational factor a_S with respect to the first components. The factor enters into the formulae in accordance with (VII.10.9)

$$\sigma_1 = \sigma_{|1} = \underset{1}{e}^1 \frac{\partial}{\partial r}\sigma = a_S \alpha_R \frac{\partial\sigma}{\partial r}\ . \tag{VII.10.13}$$

Likewise

$$A_{4m}{}^4 = -\overset{4}{e}_i \underset{4|m}{e}{}^i = -E^C_m = -E_m - F_m$$

and

$$E_{mn}{}^s = -u_m\left(E_n u^s - u_n E^s\right) - u_m\left(F_n u^s - u_n F^s\right) \tag{VII.10.14}$$

are extended.

We represent the still missing 'mixed' terms which are specific for the cross term of the metric in two kinds. The first kind strongly takes into account the circumstances of the geometry. It takes in consideration that the oblique-angled system results from two rotations of the bein of the seed metric. From this are derived two rotational quantities, whereby the second is less useful for the physical treatment. If one defines

$$H^C_{mn} = \overset{4}{e}_i \underset{[n|m]}{e}{}^i = 2i a_S \alpha_R^2 \omega \sigma_{[m} c_{n]} + i a_S \alpha_R^2 \sigma \omega_{|[m} c_{n]}\,,\quad m,n = 1,2,3 \tag{VII.10.15}$$

one obtains from the Ricci-rotation coefficients

$$H_{mns} = H^C_{mn} u_s + H^C_{sm} u_n - H^C_{ns} u_m = H^C_{\{mn} u_{s\}}\ . \tag{VII.10.16}$$

In a similar way one also obtains with

$$K_{mn} = \overset{3}{e}_i \underset{[n|m]}{e}{}^i = 2i\alpha_S \alpha_R^2 \sigma\omega_{\,|[m} u_{n]},\quad m,n = 1,2,4 \tag{VII.10.17}$$

the quantity

$$K_{mns} = K_{mn}c_s + K_{sm}c_n - K_{ns}c_m = K_{\{mn}c_{s\}} \ . \tag{VII.10.18}$$

The two quantities H_{mns} and K_{mns} have Christoffel symmetry. As a consequence of $\omega_{|1}\sigma = -2\omega\,\sigma_{|1}$, $\omega_{|2} = 0$ one has

$$H^C_{13} = 0, \quad K_{24} = 0 \ .$$

In order to get a better arrangement of (VII.10.15) the auxiliary variables

$$H_{mn} = 2i\alpha_R^2\omega\sigma_{[m}c_{n]}, \quad D_{mn} = i\alpha_R^2\sigma\omega_{|m}c_n, \quad D_{31} = 0 \tag{VII.10.19}$$

are introduced. $H^m = -i\frac{1}{2}a_s\varepsilon^{mnr}H_{nr}$ is the analog of the classical Coriolis field strength. With (VII.10.19) one constructs the antisymmetric and symmetric quantities

$$H^C_{mn} = a_s\left(H_{mn} + D_{[mn]}\right), \quad D^C_{mn} = \alpha_S D_{(mn)} \ . \tag{VII.10.20}$$

The symmetric quantity D^C_{mn} describes the shears to which the observer field is subjected. Due to the differential rotation law neighboring observers who move on circular paths around the gravitation center have different speeds. A volume element which accompanies an observer is distorted. The points closer to the source of gravitation run ahead, farther ones hang behind.

Since the physical statements of the theory refer to the observer field u_m, the quantity K can be less well integrated into those equations which have to describe the physics of the Kerr geometry. Again we do without the symmetry of the relations (VII.10.16) and (VII.10.18) and we process the terms (VII.10.17) together with (VII.10.15). If we completely elaborate the observer field u_m then a re-calculation of the K-term firstly results in

$$D^C_{mn}u_s - D^C_{sm}u_n + \alpha_S D_{[ns]}u_m \ .$$

With the simple relations

$$D_{mn} = D_{(mn)} + D_{[mn]} \tag{VII.10.21}$$

one obtains

$$D^C_{mn}u_s - D^C_{sm}u_n - D^C_{ns}u_m + \alpha_S D_{ns}u_m \ .$$

If one defines the quantity

$$\Omega^C_{\beta\alpha} = -\left[H^C_{\alpha\beta} + D^C_{\alpha\beta}\right], \quad \alpha,\beta = 1,2,3 \tag{VII.10.22}$$

which will be fundamental for all further investigations and if one re-defines also H_{mns}, then one can describe the entire rotational contribution with

$$H_{mns} = -\Omega^C_{nm}u_S + \Omega^C_{sm}u_n + \Omega^C_{sn}u_m + \alpha_S D_{ns}u_m \ . \tag{VII.10.23}$$

In the mathematical appendix the derivation is indicated in more detail. The physical usefulness of Ω^C is shown by the calculation

$$u_{\alpha||\beta} = -A_{\beta\alpha}{}^4 = \Omega^C_{\alpha\beta}$$

and by symmetrization and antisymmetrization

$$u_{(\alpha||\beta)} = -D^C_{\alpha\beta}, \quad u_{[\alpha||\beta]} = H^C_{\alpha\beta} \ . \tag{VII.10.24}$$

The deformation field strength and the extended Coriolis field strength enter separately into the relations. The Section will be completed with a recapitulation of all field quantities of the Kerr geometry which refer to the observer system C:

$$A_{mn}{}^{s} = B_{mn}{}^{s} + N_{mn}{}^{s} + C_{mn}{}^{s} + H_{mn}{}^{s} + E_{mn}{}^{s} \qquad \text{(VII.10.25)}$$

$$\begin{gathered}
B_{mn}{}^{s} = b_m B_n b^s - b_m b_n B^s, \quad N_{mn}{}^{s} = m_m N_n m^s - m_m m_n N^s \\
C_{mn}{}^{s} = c_m C^C_n c^s - c_m c_n C^s_C, \quad C^C_n = C_n + F_n \\
E_{mn}{}^{s} = -u_m \left(E^C_n u^s - u_n E^s_C \right), \quad E^C_n = E_n + F_n \\
B_1 = \frac{a_S}{\rho_E}, \quad N_2 = \frac{1}{\rho_H}, \quad C_1 = \frac{a_S r}{A\Lambda}, \quad C_2 = \frac{1}{\Lambda}\cot\vartheta \\
F_n = \alpha_R^2 \omega^2 \sigma \sigma_n, \quad E_n = \frac{1}{\rho_S}\tan\varepsilon \\
H_{mns} = -\Omega^C_{nm} u_S + \Omega^C_{sm} u_n + \Omega^C_{sn} u_m + \alpha_S D_{ns} u_m \\
\Omega^C_{nm} = -\left[H^C_{mn} + D^C_{mn} \right], \quad H^C_{mn} = a_s \left(H_{mn} + D_{[mn]} \right), \quad D^C_{mn} = \alpha_S D_{(mn)} \\
H_{mn} = 2i\alpha_R^2 \omega \sigma_{[m} c_{n]}, \quad D_{mn} = i\alpha_R^2 \sigma \omega_{|m} c_n
\end{gathered} \quad . \qquad \text{(VII.10.26)}$$

One also should compare these equations with the similar relations (VII.5.5) - (VII.5.9) and with the equations of the seed metric (VII.7.3). With the complete set of the field strengths we are prepared to decompose the Einstein field equations in such a way that statements can be read from the individual curvatures, and the rotational effects can be brought into a form which is akin to the classical mechanics.

VII.11. Kerr metric, the field equations of the system C

The Kerr metric describes the exterior field of a rotating stellar object. Apart from the force of gravity there are still several effects which are due to the rotation of the object. A frame dragging of the reference systems takes place. This entails Coriolis forces, shears, and centrifugal forces. For the vacuum field equations

$$R_{mn} = A_{mn}{}^{s}{}_{|s} - A_{n|m} - A_{rm}{}^{s}A_{sn}{}^{r} + A_{mn}{}^{s}A_{s} = 0 \qquad \text{(VII.11.1)}$$

one finds with the use of the field strengths (VII.10.26) and the graded covariant derivatives the Ricci tensor

$$\begin{aligned} R_{mn} = &-\left[N_{n\underset{2}{\|}m} - N_{n\underset{2}{\|}s}m^{s}m_{m} + N_{n}N_{m}\right] - m_{n}m_{m}\left[N^{s}{}_{\underset{2}{\|}s} + N^{s}N_{s}\right] \\ &-\left[B_{n\underset{2}{\|}m} - B_{n\underset{2}{\|}s}b^{s}b_{m} + B_{n}B_{m}\right] - b_{n}b_{m}\left[B^{s}{}_{\underset{2}{\|}s} + B^{s}B_{s}\right] \\ &-\left[C^{C}_{n\underset{3}{\|}m} + C^{C}_{n}C^{C}_{m}\right] - c_{n}c_{m}\left[C^{s}_{C\underset{3}{\|}s} + C^{s}_{C}C^{C}_{s} - \Omega^{rs}_{C}\Omega^{C}_{sr}\right] \\ &+\left[E^{C}_{n\underset{4}{\|}m} - E^{C}_{n}E^{C}_{m}\right] + u_{n}u_{m}\left[E^{s}_{C\underset{4}{\|}s} - E^{s}_{C}E^{C}_{s} - \Omega^{rs}_{C}\Omega^{C}_{sr}\right] \\ &+ 2u_{(n}\left[\Omega^{s}_{Cm)\underset{4}{\|}s} - 2H^{s}_{Cm)}F_{s}\right] - 2\Omega^{C}_{n3}\Omega^{C}_{m3} \end{aligned} \qquad . \qquad \text{(VII.11.2)}$$

If one considers

$$E^{C}_{n\underset{4}{\|}m} = E^{C}_{n\underset{3}{\|}m} + c_{n}c_{m}C^{s}_{C}E^{C}_{s}, \quad \Omega^{[ms]}_{C} = H^{ms}_{C}, \quad E^{C}_{s} = F_{s} + E_{s} \qquad \text{(VII.11.3)}$$

some subequations can be decoupled from the field equations

$$\begin{aligned} &\left[N_{n\underset{2}{\|}m} - N_{n\underset{2}{\|}s}m^{s}m_{m} + N_{n}N_{m}\right] + b_{n}b_{m}\left[B^{s}{}_{\underset{2}{\|}s} + B^{s}B_{s}\right] + \\ &+\left[B_{n\underset{2}{\|}m} - B_{n\underset{2}{\|}s}b^{s}b_{m} + B_{n}B_{m}\right] + m_{n}m_{m}\left[N^{s}{}_{\underset{2}{\|}s} + N^{s}N_{s}\right] + \\ &+\left[C^{C}_{n\underset{3}{\|}m} + C^{C}_{n}C^{C}_{m}\right] - \left[E^{C}_{n\underset{3}{\|}m} - E^{C}_{n}E^{C}_{m}\right] + 2\Omega^{C}_{n3}\Omega^{C}_{m3} = 0 \\ &\left[C^{s}_{C\underset{3}{\|}s} + C^{s}_{C}C^{C}_{s} - C^{s}_{C}E^{C}_{s} - \Omega^{rs}_{C}\Omega^{C}_{sr}\right] = 0 \\ &\left[E^{s}_{C\underset{4}{\|}s} - E^{s}_{C}E^{C}_{s} - \Omega^{rs}_{C}\Omega^{C}_{sr}\right] = 0 \\ &\left[\Omega^{sm}_{C}{}_{\underset{4}{\|}s} + 2\Omega^{[ms]}_{C}E^{C}_{s}\right] = 0 \end{aligned} \qquad . \qquad \text{(VII.11.4)}$$

For the individual blocks one finds

$$\begin{aligned} &N_{n\underset{2}{\|}m} - N_{n\underset{2}{\|}s}m^{s}m_{m} + N_{n}N_{m} = -\,b_{n}b_{m}\tilde{\Omega}^{s3}\tilde{\Omega}_{3s} \\ &B_{n\underset{2}{\|}m} - B_{n\underset{2}{\|}s}b^{s}b_{m} + B_{n}B_{m} = m_{n}m_{m}a^{2}_{S}\tilde{\Omega}^{s3}\tilde{\Omega}_{3s} - E_{n}B_{m} \\ &N^{s}{}_{\underset{2}{\|}s} + N^{s}N_{s} = -\tilde{\Omega}^{s3}\tilde{\Omega}_{3s}, \quad B^{s}{}_{\underset{2}{\|}s} + B^{s}B_{s} = a^{2}_{S}\tilde{\Omega}^{s3}\tilde{\Omega}_{3s} - B^{s}E_{s} \\ &\left[C^{C}_{n\underset{3}{\|}m} + C^{C}_{n}C^{C}_{m}\right] - \left[E^{C}_{n\underset{3}{\|}m} - E^{C}_{n}E^{C}_{m}\right] + 2\Omega^{C}_{n3}\Omega^{C}_{m3} = \left(m_{n}m_{m} + b_{n}b_{m}\right)\left[\left(1 - a^{2}_{S}\right)\tilde{\Omega}^{s3}\tilde{\Omega}_{3s} + B^{s}E_{s}\right] \end{aligned} \qquad .\text{(VII.11.5)}$$

The quantities $\tilde{\Omega}$ are the contributions of the evolute, as has been elucidated in former Sections, and are closely related to the quantities Ω^{C}. In an abridged form one also obtains

$$C^{s}_{C||s} - \Omega^{rs}_{C}\Omega^{C}_{sr} = 0, \quad E^{s}_{C||s} - \Omega^{rs}_{C}\Omega^{C}_{sr} = 0 \ , \qquad \text{(VII.11.6)}$$

relations which one can write as

$$C^{s}{}_{||s} + F^{s}{}_{||s} - \Omega^{rs}_{C}\Omega^{C}_{sr} = 0, \quad E^{s}{}_{||s} + F^{s}{}_{||s} - \Omega^{rs}_{C}\Omega^{C}_{sr} = 0 \ . \qquad \text{(VII.11.7)}$$

The centrifugal force enters into the field equations always together with the circular field strength or with the force of gravity. Some of the subequations (VII.11.4) are reminiscent of the equations of the classical mechanics (one should compare them with the flat rotating model) or with the Maxwell equations of electrodynamics. Here we are reminded of the derivation of the quantity D_{m} from Eq. (VII.4.6). In this case this quantity is extended by the gravitational factor and is equally valid for the full Kerr metric. With the above quantities the relations

$$F_{\underset{4}{[m||n]}} + D_{\underset{4}{[m||n]}} = 0, \quad E_{\underset{4}{[m||n]}} = 0, \quad \Omega^{C}_{\underset{4}{[mn||s]}} = \Omega^{C}_{[mn}D_{s]} - \Omega^{C}_{[mn}E_{s]} = 0 \qquad \text{(VII.11.8)}$$

can be deduced quite simply. These relations have again the structure of the Maxwell equations. According to the Maxwell theory one can pick out the field energy and the Poynting vector from (VII.11.4), quantities which satisfy the conservation laws

$$\left[E^{s}_{C}E^{C}_{s} + \Omega^{sr}_{C}\Omega^{C}_{rs}\right]_{|4} = 0, \quad \left[2\Omega^{[ms]}_{C}E^{C}_{s}\right]_{\underset{4}{||m}} = 0 \ . \qquad \text{(VII.11.9)}$$

For the first relation in (VII.11.8) one can also write

$$F_{\underset{4}{[m||n]}} = 2\Omega_{3[n}\Omega_{n]3}, \quad \Omega_{n3} = \{H_{13}, H_{23}\}, \quad \Omega_{3n} = \{H_{13}, -H_{23}\} \ . \qquad \text{(VII.11.10)}$$

For the conversions used here some formulae can be found in the mathematical appendix 3. The equations (VII.11.5) have a fairly unclear structure. Hence, we will apply the 5-dimensional formulation, and thus we will obtain a substantial simplification of the field equations.

VII.12. Kerr metric, the physical surface

In a former Section the 4-dimensional quantities of the seed metric have been supplemented in an intuitive way by the components (VII.7.7) of an extra dimension and their meaning has been graphically described in Fig. VII.7 and Fig. VII.8. A further quantity M was introduced by (VII.8.2). It describes the extrinsic curvature of the geometry. Firstly, the intuitive way will be continued and some subequations of the Einstein field equations will be formulated 5-dimensionally and thus represented more compactly. The Kerr geometry will be deduced from a pseudo-hyper sphere later on.

The spatial part of the Kerr metric presents itself similarly to Flamm's paraboloid. If the φ-dimension is suppressed, one obtains a surface, as one can see in Fig. VII.3. The radial integral curves, the curvature vectors, and the evolutes have been already discussed. The local extra dimension is parameterized by straight lines, on which the rigging vectors of the physical surface and the associated curvature vectors of this surface are situated.

The angle of ascent on any of these straight lines is constant and differs from the angle of ascent of the radial hyperplanes, which forms the physical surface, only by $\pi/2$. Thus, one has $\varepsilon_{|0}=0$. Furthermore, one infers with

$$v_S = \sin\varepsilon, \quad a_S = \cos\varepsilon$$

the important relations

$$v_{S|0} = 0, \quad a_{S|0} = 0 \ . \tag{VII.12.1}$$

With their aid one performs the computations of the 5-dimensional curvature equations. Thus, one arrives with

$$B_{0|0} = \left(v_S \frac{1}{\rho_E}\right)_{|0} = -v_S \frac{1}{\rho_E^2} \cdot v_S \rho_{E|\underline{1}} = -v_S^2 \frac{1}{\rho_E^2}\left[1-\rho_E^2 \tilde{\Omega}^{3s}\tilde{\Omega}_{s3}\right]$$

at

$$B_{0|0} + B_0 B_0 = v_S^2 \tilde{\Omega}^{3s}\tilde{\Omega}_{s3} \ ,$$

whereby we have considered (VII.3.16). However, evaluating $B_{0|1}$ and $B_{\underline{1}|1}$ one has to apply (VII.6.15). This motivates us to define a 5-dimensional covariant derivative armed with the quantity M from (VII.8.2), as we have done with earlier models

$$B_{b\underset{2}{|||}a} - B_{b\underset{2}{|||}c} b^c b_a + B_b B_a = \tilde{m}_b \tilde{m}_a \tilde{\Omega}^{3s}\tilde{\Omega}_{s3}, \quad B_{b\underset{2}{|||}a} = B_{b|a} - M_{ab}{}^c B_c \ .$$

The new quantity

notation: $$\tilde{m}_a = \{\sin\varepsilon, \cos\varepsilon, 0, 0, 0,\} = \{v_S, a_S, 0, 0, 0,\} \tag{VII.12.2}$$

is a unit vector which lies in a plane parallel to the base plane of the Kerr surface. However,

$$m_a = \{0, 1, 0, 0, 0\} \tag{VII.12.3}$$

is a unit vector which lies in the anholonomic hyperplanes of the Kerr surface and points into the radial direction. Both vectors can be merged by a rotation through the angle ε. In the same way other curvatures can be treated. Finally, one obtains a set of 5-dimensional equations

$$N_{b|||a \atop 2} - N_{b|||c \atop 2} n^c n_a - N_{b|||c \atop 2} m^c m_a + N_b N_a = -b_b b_a \tilde{\Omega}^{3s} \tilde{\Omega}_{s3}\,, \quad N^c_{\ |||c \atop 2} + N^c N_c = -\Omega^{3s} \tilde{\Omega}_{s3}$$

$$B_{b|||a \atop 2} - B_{b|||c \atop 2} b^c b_a + B_b B_a = \tilde{m}_b \tilde{m}_a \tilde{\Omega}^{3s} \tilde{\Omega}_{s3}\,, \qquad B^c_{\ |||c \atop 2} + B^c B_c = \tilde{\Omega}^{3s} \tilde{\Omega}_{s3} \quad ,\text{(VII.12.4)}$$

$$C_{b|||a \atop 3} + C_b C_a = 0\,, \qquad C^c_{\ |||c \atop 3} + C^c C_c = 0$$

which refer to the seed metric.

The 5-dimensional graded covariant derivatives have the same structure as the 4-dimensional ones. In addition, the quantity M emerges from (VII.8.2) as already has been suggested above. With the peculiarities of the 5-dimensional connexion coefficients which are contained in the relations (VII.12.4) we will deal later on.

The rigging vector

$$n_a = \{1,0,0,0,0\} \tag{VII.12.5}$$

is vertical to the anholonomic hyperplanes of the Kerr surface, but is not vertical to the surface. This has already been mentioned previously.

The treatment of the above quantities in a 5-dimensional form was rather simple, because the 5-dimensional quantities are straightly related to the flat quantities as Fig. VII.5 and Fig. VII.6 show. They are quantities which do not change with respect to the direction of the extra dimension $0'$ of the embedding space.

From

$$\partial_0 = \cos\varepsilon\,\partial_{0'} + \sin\varepsilon\,\partial_{1'}\,, \quad \partial_1 = -\sin\varepsilon\,\partial_{0'} + \cos\varepsilon\,\partial_{1'}$$

only

$$\partial_0 = \sin\varepsilon\,\partial_{1'}\,, \quad \partial_1 = \cos\varepsilon\,\partial_{1'}$$

applies, in contrast to the quantities M and E which contain the curvature vector ρ_S. Both quantities refer definitely to the radial curvature of the surface and are not elements of the flat geometry. The change of the curvature vector is caused also by the alteration into the local 0-direction

$$d\rho_S = \rho_{S|0} dx^0 + \rho_{S|1} dx^1 + \rho_{S|2} dx^2\ .$$

The dependence of ϑ is attributed to the factor a_R in ρ_S. It indicates that the curvature of elliptically deformed surfaces changes by a circulation on a horizontal cutting plane. First, we are again concerned with the surface of the seed metric with the curvature vector ρ. The fact that this vector changes along the radial curves of the surface can be directly understood. Indeed, the base point of the curvature vector moves along the evolute together with the motion of this curvature vector on one of the radial curves. However, if one fixes ρ at one point of the surface and if one examines its change in the local 0-direction, one is concerned with the extrinsic curvature of the geometry. One has to imagine the flat 5-dimensional space, or more simply, a vertical slice of this space, to be covered with a co-ordinate system[126]. This co-ordinate system consists of curves which are parallel to the radial Kerr curves and have a common evolute. If one travels from a surface along the above-discussed straight lines to a neighboring surface, then ρ changes. If the distance of two surfaces is dx^0_{hol}, then ρ is extended by dx^0_{hol}. If one transfers the train of thought to the physical surface with the anholonomic hyperplanes, one obtains from the above equation

[126] For the Schwarzschild geometry similar considerations have been presented in Fig. IV.7.

$$d\rho_S = dx^0, \quad \rho_{S|0} = 1, \quad \partial_0 = \frac{\partial}{\partial \rho_S} \ . \qquad \text{(VII.12.6)}$$

Thus, one has all necessary tools for treating the 0-components of the 5-dimensional curvature equations. From the third equation (VII.12.6) follows

$$M_{0|0} + M_0 M_0 = 0, \quad E_{0|0} - E_0 E_0 = 0 \ . \qquad \text{(VII.12.7)}$$

Both equations are based on the first relations of

$$\frac{\partial}{\partial \rho_S}\frac{1}{\rho_S} + \frac{1}{\rho_S}\frac{1}{\rho_S} = 0, \quad \frac{\partial}{\Lambda \partial \vartheta}\frac{1}{\rho_S} + \frac{1}{\rho_S}\frac{1}{\rho_H} = 0 \ . \qquad \text{(VII.12.8)}$$

The second relation can be verified with $\rho_S = \rho a_R$, the expressions (VII.4.6), (VII.8.18), (VII.8.17), and (VII.2.8). It will be used later on, when further components of the 5-dimensional curvature equations have been computed.

One has seen that one is successful with the construction of 5-dimensional field equations in an intuitive way. However, the constructed equations are uncertain as long as the structure of the 5-dimensional connexion coefficients is not completely indicated and justified.

VII.13. Kerr metric, more on 5 dimensions

In order to deduce for the Kerr model justifiable 5-dimensional field equations we start with a pseudo-hyper sphere which is embedded into a 5-dimensional flat space, as we have done with earlier models. With a projection formalism we get from it the Kerr-specific curvature structures. We are reminded of the fact that the rotational effects have been obtained by an intrinsic transformation on the Kerr surface. Hence, they have only 4-dimensional properties and they are not the subject of an embedding. Thus, our considerations refer only to the rotation-free seed metric. The following remarks rest upon the techniques used in the Schwarzschild model and in the Section concerning elliptic-hyperbolic systems.

If a' = 0 ', 1 ',…, 4 ' are the Cartesian co-ordinates in a 5-dimensional flat space, then a family of pseudo-hyper spheres with the radius X is parameterized by

$$\begin{aligned} X^{3'} &= X\sin\varepsilon\sin\theta\sin\varphi \\ X^{2'} &= X\sin\varepsilon\sin\theta\cos\varphi \\ X^{1'} &= X\sin\varepsilon\cos\theta \qquad . \\ X^{0'} &= X\cos\varepsilon\cos i\psi \\ X^{4'} &= X\cos\varepsilon\,\sin i\psi \end{aligned} \qquad \text{(VII.13.1)}$$

The vectors $X^{a'}$ point from the center of the pseudo-hyper sphere to any point on the surfaces. The slices of the pseudo-hyper sphere are circles and a hyperbola of constant curvature. The line element in pseudo-polar co-ordinates $\{X,\varepsilon,\theta,\varphi,i\psi\}$ in the 5-dimensional flat space is

$$ds^2 = dX^2 + X^2 d\varepsilon^2 + X^2\sin^2\varepsilon\, d\theta^2 + X^2\sin^2\varepsilon\sin^2\theta\, d\varphi^2 + X^2\cos^2\varepsilon\, di\psi^2 \ . \qquad \text{(VII.13.2)}$$

The anholonomic differentials are

$$dX^a = \{dX, Xd\varepsilon, X\sin\varepsilon d\theta, X\sin\varepsilon\sin\theta d\varphi, X\cos\varepsilon di\psi\}, \qquad \text{(VII.13.3)}$$

the partial derivatives are

$$\hat{\partial}_a = \frac{\partial}{\partial X^a} = \left\{\frac{\partial}{\partial X}, \frac{\partial}{X\partial\varepsilon}, \frac{\partial}{X\sin\varepsilon\partial\theta}, \frac{\partial}{X\sin\varepsilon\sin\theta\partial\varphi}, \frac{\partial}{X\cos\varepsilon\partial i\psi}\right\} . \qquad \text{(VII.13.4)}$$

The rotation matrix (with the index sequence 3,2,1,0,4) which transforms the Cartesian components into the local components of the vectors is

$$D_a^{a'} = \begin{pmatrix} \cos\varphi & \cos\theta\sin\varphi & \cos\varepsilon\,\sin\theta\sin\varphi & \sin\varepsilon\sin\theta\sin\varphi & 0 \\ -\sin\varphi & \cos\theta\cos\varphi & \cos\varepsilon\sin\theta\cos\varphi & \sin\varepsilon\sin\theta\cos\varphi & 0 \\ 0 & -\sin\theta & \cos\varepsilon\cos\theta & \sin\varepsilon\cos\theta & 0 \\ 0 & 0 & -\sin\varepsilon\cos i\psi & \cos\varepsilon\cos i\psi & -\sin i\psi \\ 0 & 0 & -\sin\varepsilon\sin i\psi & \cos\varepsilon\sin i\psi & \cos i\psi \end{pmatrix}, \text{(VII.13.5)}$$

and one has to use the inverse rotation matrix, respectively.

For X = const. one of the surfaces is selected from the family of surfaces. Then the line element on this surface is

$$ds^2 = X^2 d\varepsilon^2 + X^2\sin^2\varepsilon\, d\theta^2 + X^2\sin^2\varepsilon\sin^2\theta\, d\varphi^2 + X^2\cos^2\varepsilon\, di\psi^2 \ . \qquad \text{(VII.13.6)}$$

The connexion coefficients of the pseudo-hyper sphere are well known from earlier treated models

$$X_{10}{}^{1}=\frac{1}{X}\,,\quad X_{20}{}^{2}=\frac{1}{X},\qquad X_{21}{}^{2}=\frac{1}{X}\cot\varepsilon$$
$$X_{30}{}^{3}=\frac{1}{X}\,,\quad X_{31}{}^{3}=\frac{1}{X}\cot\varepsilon,\quad X_{32}{}^{3}=\frac{1}{X\sin\varepsilon}\cot\theta\ . \tag{VII.13.7}$$
$$X_{40}{}^{4}=\frac{1}{X},\quad X_{41}{}^{4}=-\frac{1}{X}\tan\varepsilon$$

If one defines a set of projectors

$$\begin{gathered}\mathfrak{p}_{0}^{0}=\frac{X}{\rho_S},\quad \mathfrak{p}_{1}^{1}=\frac{X}{\rho_S},\quad \mathfrak{p}_{2}^{2}=\frac{Xv_S}{\rho_E},\quad \mathfrak{p}_{3}^{3}=\frac{Xv_S}{\rho_E}a_R^2\\ \mathfrak{p}_{0}^{2}=-\frac{X}{\rho_H}v_S^2,\quad \mathfrak{p}_{1}^{2}=-\frac{X}{\rho_H}a_Sv_S,\quad \mathfrak{p}_{4}^{4}=\frac{X}{\rho_S}\end{gathered}\ , \tag{VII.13.8}$$

one has the option of bringing the geometry of the pseudo-hyper sphere into connection to the Kerr geometry. With

$$\partial_a=\mathfrak{p}^{\,b}_{\,a}\hat{\partial}_b,\quad dX^b=\mathfrak{p}^{\,b}_{\,a}dx^a,\quad \Phi_{,b}=\hat{\partial}_b\Phi=\frac{\partial\Phi}{\partial X^b} \tag{VII.13.9}$$

the basic relations are specified.

The projection procedure is to be understood in such a way that the surfaces of the pseudo-hyper spheres from the family of pseudo-hyper spheres are deformed into a Kerr-like surface family[127], where one of these surfaces is the Kerr surface. The center of the pseudo-hyper spheres becomes the common evolute surface of the Kerr-like surface family. After the projection the radius vector X of a pseudo-hyper sphere becomes the curvature vector of a Kerr-like surface. From the pre-selected pseudo-hyper sphere one obtains the radius of curvature of the Kerr surface regarded by us which we also designate by ρ. At the minor axes of the elliptical slices of the surface one has

$$\left.\frac{X}{\rho}\right|_{Kerr}=1\ . \tag{VII.13.10}$$

At any other position on the horizontals of the Kerr surface the physical curvature radius is $\rho_S=\rho a_R$. First, the second relation (VII.13.9) is to be examined for its usefulness. Written in full it results with (VII.13.8) in

$$dX^0=\mathfrak{p}_0^0dx^0=\frac{X}{\rho_S}dx^0$$
$$dX^1=\mathfrak{p}_1^1dx^1=\frac{X}{\rho_S}dx^1$$
$$dX^2=\mathfrak{p}_0^2\,dx^0+\mathfrak{p}_1^2\,dx^1+\mathfrak{p}_2^2\,dx^2=-\frac{X}{\rho_H}v_S^2dx^0-\frac{X}{\rho_H}a_Sv_Sdx^1+\frac{X}{\rho_E}v_Sdx^2\ .$$
$$dX^3=\mathfrak{p}_3^3\,dx^3=\frac{X}{\rho_E}v_Sa_R^2dx^3$$
$$dX^4=\mathfrak{p}_4^4dx^4=\frac{X}{\rho_S}dx^4$$

[127] There are surfaces parallel to the Kerr surface which have all of them a common evolute surface. The latter results from the rotation of the base points of the curvature vectors of the radial curves of these parallel Kerr surfaces.

The first relation results with (VII.13.10) and (VII.12.6) in $dX^0 = dX$, the second with (VII.6.16) $dX^1 = Xd\varepsilon$. From the third one we extract $Xv_S = X\sin\varepsilon$

$$dX^2 = X\sin\varepsilon\left[-\frac{1}{\rho_H}\left(\sin\varepsilon dx^0 + \cos\varepsilon dx^1\right) + \frac{1}{\rho_E}dx^2\right].$$

The term in the brackets can be reduced to the flat[128] geometry

$$dX^2 = X\sin\varepsilon\left[-\frac{1}{\rho_H}dx^{\underline{1}} + \frac{1}{\rho_E}dx^2\right].$$

It has to be remembered that the angle θ is the angle of ascent of the curvature vectors of the ellipses and is a function $\theta = \theta(r, \vartheta)$. If one differentiates the first relation (VII.3.5), one has

$$\cos\theta\, d\theta = \left[\frac{1}{\Lambda} - \frac{r}{\Lambda^2}\frac{\partial\Lambda}{\partial r}\right]\sin\vartheta\, dr + \left[\frac{r}{\Lambda}\cos\vartheta - \frac{r}{\Lambda^2}\frac{\partial\Lambda}{\partial\vartheta}\sin\vartheta\right]d\vartheta\ .$$

If one uses the second relation (VII.3.5)

$$\frac{A}{\Lambda}\cos\vartheta\, d\theta = \left[\frac{1}{\Lambda} - \frac{r}{A}\frac{1}{\Lambda}\Lambda_{|\underline{1}}\right]\sin\vartheta\, dr + \left[\frac{r}{\Lambda}\cos\vartheta - \frac{r}{\Lambda}\Lambda_{|2}\sin\vartheta\right]d\vartheta\ ,$$

one obtains with regard to (VII.2.8)

$$\begin{aligned}\frac{A}{\Lambda}\cos\vartheta d\theta &= \left[1 - \frac{r^2}{\Lambda^2}\right]\sin\vartheta\frac{A}{\Lambda^2}dx^{\underline{1}} + \left[1 + \frac{a^2}{\Lambda^2}\sin\vartheta\right]\cos\vartheta\frac{r}{\Lambda^2}dx^2 \\ &= \frac{a^2\cos^2\vartheta\sin\vartheta}{\Lambda^3}\frac{A}{\Lambda}dx^{\underline{1}} + \frac{Ar}{\Lambda^3}\frac{A}{\Lambda}\cos\vartheta dx^2\end{aligned},$$

and with (VII.2.10)

$$d\theta = -\frac{1}{\rho_H}dx^{\underline{1}} + \frac{1}{\rho_E}dx^2, \quad \rho_E d\theta = \Lambda d\vartheta - \frac{\rho_E}{\rho_H}dx^{\underline{1}}\ . \tag{VII.13.11}$$

Finally, $dX^2 = X\sin\varepsilon d\theta$ is in agreement with (VII.13.3). We note for the horizontal quantity θ

$$\theta_{|\underline{1}} = -\frac{1}{\rho_H}, \quad \theta_{|2} = \frac{1}{\rho_E}\ , \tag{VII.13.12}$$

or in the local system of the surface

$$\theta_{|0} = -\frac{v_S}{\rho_H}, \quad \theta_{|1} = -\frac{a_S}{\rho_H}, \quad \theta_{|2} = \frac{1}{\rho_E}\ . \tag{VII.13.13}$$

For

$$dX^3 = Xv_S\frac{1}{\rho_E}a_R^2 A\sin\vartheta d\varphi$$

one obtains with (VII.2.8) and (VII.3.5) $dX^3 = X\sin\theta d\varphi$. $dx^4 = \rho_S\cos\varepsilon di\psi$ leads to $dX^4 = X\cos\varepsilon di\psi$. After making ourselves familiar with the use of the projectors, we can compute the 5-dimensional Ricci-rotation coefficients. They are substantially richer than

[128] The bar under the index marks the flat components of the quantity. $dx^{\underline{1}} = a_R dr$, $dx^{\underline{2}} = dx^2 = \Lambda d\vartheta$.

the expressions of models treated earlier. We must name them Y in order to be able to distinguish them from the 4-dimensional quantities A. These are obtained by dimensional reduction from Y. One can compute them with the help of

$$Y_{abc} = D^{a'}_{c} D^{a'}_{[b|a]} + D^{a'}_{a} D^{a'}_{[b|c]} + D^{a'}_{b} D^{a'}_{[a|c]} \ , \qquad \text{(VII.13.14)}$$

if one regards in (VII.13.5) the parameters as such of an already deformed geometry and if one uses the above arithmetic rules of differentiating. However, more simply one obtains Y, by letting the projectors operate onto the first index of X

$$Y_{ab}{}^{c} = \mathfrak{p}^{d}_{a} X_{db}{}^{c} \ . \qquad \text{(VII.13.15)}$$

With (VII.13.7) and (VII.13.8) it is not difficult to compute

$$Y_{ab}{}^{c} = M_{ab}{}^{c} + \tilde{N}_{ab}{}^{c} + B_{ab}{}^{c} + C_{ab}{}^{c} + E_{ab}{}^{c} \ . \qquad \text{(VII.13.16)}$$

The individual components of Y are defined by

$$\begin{aligned} M_{ab}{}^{c} &= m_a M_b m^c - m_a m_b M^c \\ \tilde{N}_{ab}{}^{c} &= \tilde{m}_a N_b \tilde{m}^c - \tilde{m}_a \tilde{m}_b N^c \\ B_{ab}{}^{c} &= b_a B_b b^c - b_a b_b B^c \qquad . \\ C_{ab}{}^{c} &= c_a C_b c^c - c_a c_b C^c \\ E_{ab}{}^{c} &= -\left[u_a E_b u^c - u_a u_b E^c \right] \end{aligned} \qquad \text{(VII.13.17)}$$

The curvature quantities occurring therein are again listed

$$\begin{aligned} &M_b = \left\{ \frac{1}{\rho_S}, 0,0,0,0 \right\}, \quad N_b = \left\{ 0,0, \frac{1}{\rho_H}, 0,0 \right\}, \quad B_b = \left\{ \frac{v_S}{\rho_E}, \frac{a_S}{\rho_E}, 0,0,0 \right\} \\ &C_b = \left\{ v_S \frac{r}{\Lambda A}, a_S \frac{r}{\Lambda A}, \frac{1}{\Lambda} \cot\vartheta, 0,0 \right\}, \quad E_b = \left\{ -\frac{1}{\rho_S}, \frac{1}{\rho_S} \frac{v_S}{a_S}, 0,0,0 \right\} \end{aligned} \ . \qquad \text{(VII.13.18)}$$

One should pay attention to the fact that the quantity $\tilde{N}$ is built up with the vector $\tilde{m}$ of (VII.12.2) which lies in the horizontal cutting planes. This can easily be understood, because the Kerr metric is to be described by the curvatures of the slices of the surface and the $N_2 = 1/\rho_H$ are the curvatures of the hyperbolae in the horizontals. However, the Kerr model requires for a quantity

$$N_{ab}{}^{c} = m_a N_b m^c - m_a m_b N^c$$

which should be worked out by means of the dimensional reduction. From what has been said so far one can see that the relation

$$\mathfrak{p}^{b}_{a} = X^{b}{}_{|||a} = X^{b}{}_{|a} + Y_{a0}{}^{b} X, \qquad X^{a} = \left\{ X,0,0,0,0 \right\} \qquad \text{(VII.13.19)}$$

becomes also obvious.

From the Riemann tensor of the 5-dimensional flat space which is parameterized by a family of pseudo-hyper spheres

$$R_{abc}{}^{d}(X) = 2\left[X_{[b\cdot c\cdot}{}^{d}{}_{,a]} + X_{[b\cdot c\cdot}{}^{f} X_{a]f}{}^{d} + X_{[ba]}{}^{f} X_{fc}{}^{d} \right] \equiv 0 \ , \qquad \text{(VII.13.20)}$$

one obtains by projection of the Riemann tensor of the 5-dimensional flat space which is parameterized by a family of Kerr-like surfaces

$$\begin{aligned} &\mathfrak{p}^{g}_{a} \mathfrak{p}^{h}_{b} R_{ghc}{}^{d}(X) = R_{abc}{}^{d}(Y) \\ &R_{abc}{}^{d}(Y) = 2\left[Y_{[b\cdot c\cdot}{}^{d}{}_{|a]} + Y_{[b\cdot c\cdot}{}^{f} Y_{a]f}{}^{d} + Y_{[ba]}{}^{f} Y_{fc}{}^{d} + X_{fc}{}^{d} \mathfrak{p}^{f}_{[a|||b]} \right] \equiv 0 \end{aligned} \ . \qquad \text{(VII.13.21)}$$

The last term in the brackets which can also be written as $Y_{gc}{}^{d}\left(\mathcal{p}^{-1}\right)^{g}_{f}\mathcal{p}^{f}_{[a|||b]}$. It must be examined more exactly. The components

$$2\left(\mathcal{p}^{-1}\right)^{0}_{0}\mathcal{p}^{\ 0}_{[0|||1]}=-\frac{1}{\rho_S}\rho_{S|1},\quad 2\left(\mathcal{p}^{-1}\right)^{0}_{0}\mathcal{p}^{0}_{[0|||2]}=-a_S^2N_2$$
$$2\left(\mathcal{p}^{-1}\right)^{0}_{0}\mathcal{p}^{0}_{[2|||1]}=-a_Sv_SN_2,\quad 2\left(\mathcal{p}^{-1}\right)^{1}_{1}\mathcal{p}^{1}_{[2|||0]}=-a_Sv_SN_2,\quad 2\left(\mathcal{p}^{-1}\right)^{1}_{1}\mathcal{p}^{1}_{[1|||2]}=-v_S^2N_2 \qquad \text{(VII.13.22)}$$

are of no importance with regard to the dimensional reduction of the field equations. In contrast, the first expression

$$2\left(\mathcal{p}^{-1}\right)^{4}_{4}\mathcal{p}^{\ 4}_{[4|||1]}=-\frac{1}{\rho_S}\rho_{S|1},\quad 2\left(\mathcal{p}^{-1}\right)^{4}_{4}\mathcal{p}^{\ 4}_{[4|||2]}=-\frac{1}{\rho_S}\rho_{S|2} \qquad \text{(VII.13.23)}$$

is well known from the Schwarzschild geometry. It is substantial for the completion of the double-surface theory. All these considerations are conferrable to the Kerr geometry. However, one has to add a second expression in (VII.13.23) which can be simplified to

$$2\left(\mathcal{p}^{-1}\right)^{4}_{4}\mathcal{p}^{\ 4}_{[4|||2]}=-\frac{1}{a_R}a_{R|2}=-N_2 \qquad \text{(VII.13.24)}$$

and will be used in this way.

VII.14. Kerr metric, the 5-dimensional field equations

Having made the necessary arrangements, we can hark back to the contracted form of (VII.13.21) and decompose the 5-dimensional Ricci tensor in such a way that the equations for the curvatures involved in the Kerr surface emerge, whereby we get the equations found intuitively in the last Section. The Ricci tensor

$$R_{ab}\left(Y\right)=Y_{ab}{}^{c}{}_{|c}-Y_{b|a}-Y_{ca}{}^{d}Y_{db}{}^{c}+Y_{ab}{}^{c}Y_{c}+2Y_{gb}{}^{d}\left(\mathcal{P}^{-1}\right)^{g}_{f}\mathcal{P}^{f}_{[d|||a]}\equiv 0 \qquad \text{(VII.14.1)}$$

is to be treated with (VII.13.17), (VII.13.22), and (VII.13.23). We take a glance at the [11]-component of the field equations on the basis of sampling. The contribution of the extrinsic curvature

$$M_{0\underset{1}{|||}0}+M_{0}M_{0}=0,\quad M^{c}{}_{\underset{1}{|||}c}+M^{c}M_{c}=0,\quad M_{b\underset{1}{|||}a}=M_{b|a} \qquad \text{(VII.14.2)}$$

decouples with regard to (VII.12.7) from the field equations.

The N-contribution

$$\tilde{N}_{11}{}^{2}{}_{|2}+\tilde{N}_{11}{}^{2}\tilde{N}_{2},\quad \tilde{N}_{c2}{}^{c}=\tilde{N}_{2}=N_{2}$$

supplies with (VII.12.2) and (VII.13.17)

$$-a_{S}^{2}\left[N^{2}{}_{|2}+N^{2}N_{2}\right]=-a_{S}^{2}\left[N^{2}{}_{\underset{2}{|||}2}+N^{2}N_{2}\right].$$

After a look at (VII.12.4) this results together with the B-equation in

$$\left[B_{1\underset{2}{|||}1}+B_{1}B_{1}\right]-a_{S}^{2}\left[N^{2}{}_{\underset{2}{|||}2}+N^{2}N_{2}\right]=0 .$$

For [22]-component we note

$$\tilde{N}_{c2}{}^{c}{}_{|2}+\tilde{N}_{c2}{}^{d}\tilde{N}_{d2}{}^{c}=N_{2|2}+N_{2}N_{2},\quad \tilde{N}_{c2}{}^{d}\tilde{N}_{d2}{}^{c}=N_{2}N_{2} .$$

In the [12]-component of the Ricci tensors one finds for the N-parts

$$\begin{aligned}&\tilde{N}_{12}{}^{0}{}_{|0}+\tilde{N}_{12}{}^{1}{}_{|1}-\tilde{N}_{2|1}-M_{11}{}^{0}\tilde{N}_{02}{}^{1}+\tilde{N}_{12}{}^{0}M_{0}=\\&=\left(a_{S}v_{S}N_{2}\right)_{|0}+\left(a_{S}^{2}N_{2}\right)_{|1}-N_{2|1}+2a_{S}v_{S}N_{2}M_{0}=\\&=v_{S}^{2}N_{2|1}+a_{S}^{2}N_{2|1}+2N_{2}a_{S}a_{S|1}-N_{2|1}+2a_{S}v_{S}N_{2}M_{0}=0\end{aligned} .$$

As a consequence of (VII.6.15) the whole expression vanishes. The [12]-component does not have an effective N-content. The computation of the [00]-, [01]-, and [02]-components take place in a similar way. As a consequence of these pieces of information and the preliminary work in earlier Sections concerning the B-content it is to be recognized that relations found in (VII.12.4) are present in the 5-dimensional field equations. In a covariant way of writing one has

$$\begin{aligned}&\left[B_{b\underset{2}{|||}a}-B_{b\underset{2}{|||}c}b^{c}b_{a}+B_{b}B_{a}\right]+\tilde{m}_{b}\tilde{m}_{a}\left[N^{c}{}_{\underset{2}{|||}c}+N^{c}N_{c}\right]=0\\&\left[N_{b\underset{2}{|||}a}-N_{b\underset{2}{|||}c}n^{c}n_{a}-N_{b\underset{2}{|||}c}m^{c}m_{a}+N_{b}N_{a}\right]+b_{b}b_{a}\left[B^{c}{}_{\underset{2}{|||}c}+B^{c}B_{c}\right]=0\end{aligned} . \qquad \text{(VII.14.3)}$$

The terms $B_{b|2},N_{b|0},N_{b|1}$ are to be filtered out with the help of $B_{b\underset{2}{|||}c}b^{c},N_{b\underset{2}{|||}c}n^{c},N_{b\underset{2}{|||}c}m^{c}$. They are not contained in the field equations. One sees that the N-content of the equations contains only the quantity N_b. The dimensional reduction substantially simplifies the relations. The B and the N-contents decouple together. The C-part of the Ricci tensor has already been deduced with (VII.12.4) and is confirmed after calculating (VII.12.4)

$$C_{b\underset{3}{|||}a} + C_b C_a = 0 \ . \tag{VII.14.4}$$

From the [44]-component one obtains without much effort

$$E^c{}_{\underset{4}{|||}c} - E^c E_c + 2E_1 \left(p^{-1}\right)_4^4 p^4_{[1|||4]} = 0 \ . \tag{VII.14.5}$$

The third term therein explains the occurrence of $E_1 \frac{1}{\rho_s} \rho_{s|1}$ in the E-equation. It is a consequence of the projection procedure and an element of the double-surface theory. A further E-content is to be found in the space-like part of the Ricci tensor

$$E_{b\underset{4}{|||}a} - E_b E_a - 2E_b \left(p^{-1}\right)_4^4 p^4_{[4|||a]} = 0 \ . \tag{VII.14.6}$$

The contraction of this equation leads to (VII.14.5). The [11]-component leads to the already well-known relation (VII.8.9). For the remaining components one has

$$\begin{aligned}
&E_{0\underset{4}{|||}0} - E_0 E_0 = E_{0|0} - E_0 E_0 = -\frac{\partial}{\partial \rho_S}\frac{1}{\rho_S} - \frac{1}{\rho_S^2} = 0 \\
&E_{0\underset{4}{|||}1} - E_0 E_1 - 2E_0 \left(p^{-1}\right)_4^4 p^4_{[4|||1]} = E_{0|1} - M_{10}{}^1 E_1 - E_0 E_0 + E_0 \frac{1}{\rho_S} \rho_{S|1} = 0 \\
&E_{1\underset{4}{|||}0} - E_1 E_0 = E_{1|0} - E_1 E_0 = 0 \\
&E_{0\underset{4}{|||}2} - E_0 E_2 - 2E_0 \left(p^{-1}\right)_4^4 p^4_{[4|||2]} = \frac{1}{\rho_S^2} \rho_{S|1} + 2E_0 \frac{1}{\rho_S} \rho_{S|2} \\
&E_{2\underset{4}{|||}0} - E_2 E_0 = -\tilde{N}_{02}{}^0 E_0 - \tilde{N}_{02}{}^1 E_1 = 0 \\
&E_{1\underset{4}{|||}2} - E_1 E_2 - 2E_1 \left(p^{-1}\right)_4^4 p^4_{[4|||2]} = E_{1|2} + E_1 N_2 = 0 \\
&E_{2\underset{4}{|||}2} - E_2 E_2 = -B_{22}{}^0 E_0 - B_{22}{}^1 E_1 = B^c E_c = 0 \\
&E_{3\underset{4}{|||}3} - E_3 E_3 = -C_{33}{}^0 E_0 - C_{33}{}^1 E_1 = C^c E_c = 0
\end{aligned} \quad .$$

The usage of the projectors has been treated here in detail. The method has been introduced by us into the gravitation theory and deserves special attention due to its novelty.

After the individual blocks of the field equations have been explored, they are assembled again to the Ricci tensor of the seed metric

$$\begin{aligned}
R_{ab}(Y) = &-\left[M_{b\underset{1}{|||}c} n^c n_a + M_b M_a \right] - m_b m_a \left[M^c{}_{\underset{1}{|||}c} + M^c M_c \right] \\
&-\left[B_{b\underset{2}{|||}a} - B_{b\underset{2}{|||}c} b^c b_a + B_b B_a \right] - \tilde{m}_b \tilde{m}_a \left[N^c{}_{\underset{2}{|||}c} + N^c N_c \right] \\
&-\left[N_{b\underset{2}{|||}a} - N_{b\underset{2}{|||}c} n^c n_a - N_{b\underset{2}{|||}c} m^c m_a + N_b N_a \right] - b_b b_a \left[B^c{}_{\underset{2}{|||}c} + B^c B_c \right] \\
&-\left[C_{b\underset{3}{|||}a} + C_b C_a \right] - c_b c_a \left[C^c{}_{\underset{3}{|||}c} + C^c C_c \right] \\
&+\left[E_{b\underset{4}{|||}a} - E_b E_a - 2E_b \left(p^{-1}\right)^4_4 p^4_{[4|||a]} \right] + u_b u_a \left[E^c{}_{\underset{4}{|||}c} - E^c E_c - 2E_1 \left(p^{-1}\right)^4_4 p^4_{[4|||1]} \right] \\
= &\; 0
\end{aligned} \tag{VII.14.7}$$

as the result of the equation (VII.14.1). The first graded covariant derivative in the first line is identical with the partial derivative in accordance with (VII.14.2). A dimensional reduction of this relation will again supply the above-discussed 4-dimensional field equations.

VII.15. Kerr metric, dimensional reduction

A dimensional reduction can directly be implemented into the equations (VII.14.7). Since the Kerr metric and also its seed metric can be treated only with substantial expenditure, we carry out the derivation of the field equations in four steps starting with those of the pseudo-hyper sphere families. The 5-dimensional Ricci-rotation coefficients Y are derived by projection from the connexion coefficients X of the pseudo-hyper sphere families. From these are derived the 4-dimensional Ricci-rotation coefficients A of the seed metric by a dimensional reduction, and finally the 4-beins of the actual Kerr metric by an intrinsic rotation. The somewhat voluminous quantity $\tilde{N}$ has already been reduced to the 4-dimensional quantity N in the course of the transformations of some equations. In order to arrive at a dimensional reduction, only the components $R_{mn}(Y)$ will be taken from (VII.14.7). The 5-dimensional covariant graded derivatives are converted by expansion into the 4-dimensional ones. By summation over double-indices the 0-terms are isolated together with the $\mathfrak{p}$-terms, and are shifted to the right side of the field equations. There they form the base of the stress-energy tensor of the auxiliary model. Although the seed metric is an exact solution of Einstein's field equations, one cannot and one need not assign to it a physical meaning. The presence of a stress-energy tensor refers to an interior solution which has however, an event horizon, in the same way as most of the exterior solutions. The intrinsic transformation removes the stress-energy tensor of the seed metric and makes the solution a vacuum solution, viz to the actual Kerr solution.

The relations in the first line of (VII.14.7) describe the extrinsic geometry and decouple due to the first equation in (VII.12.7). In the next line one converts for the second bracketed term

$$\tilde{m}_n\tilde{m}_m = a_S^2 m_n m_m = m_n m_m - v_S^2 m_n m_m \ .$$

The second term, emerging in such a way, refers to the extra dimension, is resolved in accordance with the second line of (VII.8.11), and is provided for the right side of the field equations. If one expands the B-term the whole second line of (VII.14.7) reads as

$$-\left[B_{\underset{2}{n||m}} - B_{\underset{2}{n||s}} b^s b_m + B_n B_m + m_n m_m M_0 B_0\right] - m_n m_m \left[N^s{}_{\underset{2}{||s}} + N^s N_s\right] - m_n m_m v_S^2 \tilde{\Omega}^{s3}\tilde{\Omega}_{3s} \ .$$

The second term in the first brackets of the third line of (VII.14.7) vanishes with regard to $n_m = 0$. The rest can be written down immediately in a 4-dimensional way, but one has to expand the second bracketed term

$$B^c{}_{\underset{2}{|||c}} + B^c B_c = \left[B^s{}_{\underset{2}{||s}} + B^s B_s\right] + \left[B^0{}_{|0} + M_0 B^0 + B^0 B_0\right] \ .$$

With a relation which has been mentioned before (VII.12.2) one obtains

$$\left[B^s{}_{\underset{2}{||s}} + B^s B_s\right] + M_0 B_0 + v_S^2 \tilde{\Omega}^{s3}\tilde{\Omega}_{3s} \ .$$

In the first C-term of (VII.14.7) the transition to the 4-dimensional form requires some care

$$\begin{aligned} C_{\underset{3}{n|||m}} &= C_{n|m} - M_{mn}{}^c C_c - B_{mn}{}^c C_c - \tilde{N}_{mn}{}^c C_c \\ &= \left[C_{n|m} - B_{mn}{}^s C_c - N_{mn}{}^s C_s\right] + m_n m_m M_0 C_0 + b_n b_m B_0 C_0 + \left[N_{mn}{}^s C_s - \tilde{N}_{mn}{}^c C_c\right] \ . \end{aligned}$$

The last term is evaluated as

$$\left(m_n m_m - \tilde{m}_n \tilde{m}_m\right) N^s C_s = m_n m_m \left(1 - a_S^2\right) N_2 C_2 = m_n m_m v_S^2 N_2 C_2, \quad \left(m_m m^s C_s - \tilde{m}_m \tilde{m}^c C_c\right) N_n = 0 ,$$

so that

$$C_{n\underset{3}{|||}m} = C_{n\underset{3}{||}m} + m_n m_m M_0 C_0 + b_n b_m B_0 C_0 - m_n m_m v_S^2 N_2 C_2 \ .$$

The second C-equation is resolved in accordance with

$$C^c{}_{\underset{3}{|||}c} + C^c C_c = \left[C^s{}_{|s} + B_s C^s + N_s B^s + C^s C_s \right] + \left[C^0{}_{|0} + M_0 C^0 + B_0 C^0 + C^0 C_0 \right] .$$

Therein is

$$C^0{}_{|0} + C^0 C_0 = \left(v_S C^{\underline{1}} \right)_{|0} + v_S^2 C^{\underline{1}} C_{\underline{1}} \ ,$$

whereby the underlined indices refer to the flat geometry of the horizontal slices of the surface. With (VII.12.1) and $\partial_0 = v_S \partial_{\underline{1}}$ the expression is simplified to

$$v_S^2 \left[C_{\underline{1}|\underline{1}} + C_{\underline{1}} C_{\underline{1}} \right] = v_S^2 \left[C_{\underline{1}\underset{3}{||}\underline{1}} + C_{\underline{1}} C_{\underline{1}} + N_{11}{}^2 C_2 \right] = -v_S^2 N_2 C_2 \ ,$$

if one still picks out the [11]-component of the last line of (VII.2.11). Thus, one finally has

$$C^c{}_{\underset{3}{|||}c} + C^c C_c = \left[C^s{}_{\underset{3}{||}s} + C^s C_s \right] + M_0 C_0 + B_0 C_0 - v_S^2 N_2 C_2 \ .$$

With the first E-term one recognizes that

$$\tilde{N}_{mn}{}^c E_c = \tilde{m}_m N_n \tilde{m}^c E_c = 0 \ ,$$

because E^c is a vertical quantity in accordance with $\tilde{m}^c E_c = 0$. Therefore

$$N_{mn}{}^s E_s = N_n E_m$$

has to be added to the 4-dimensional derivative, a term which one again finds on the right side. From

$$E^c{}_{\underset{4}{|||}c} - E^c E_c = \left[E^s{}_{\underset{4}{||}s} - E^s E_s \right] + \left[E^0{}_{|0} + M_0 E^0 + B_0 E^0 + C_0 E^0 - E_0 E^0 \right]$$

one obtains with (VII.12.7)

$$\left[E^s{}_{\underset{4}{||}s} - E^s E_s \right] + M_0 E_0 + B_0 E_0 + C_0 E_0 \ .$$

By using (VII.13.23) and (VII.13.24) we have already prepared the re-formulation of the $\mathfrak{p}$-terms. For the first E-equation one obtains

$$E_n \left(m_m \frac{1}{\rho_S} \rho_{S|1} + b_m \frac{1}{\rho_S} \rho_{S|2} \right) = m_n m_m E^s \frac{1}{\rho_S} \rho_{S|s} + E_n N_m \ ,$$

in the second E-equation one obtains alike

$$u_n u_m E^s \frac{1}{\rho_S} \rho_{S|s} \ ,$$

whereby the ρ-terms can be taken from the appendix.

If one gathers the quadratic 0-terms and the other components which are typical for the extra dimension, one firstly obtains

$$\begin{aligned}
R_{mn}(Y) = &-\left[\underset{2}{B_{n||m}} - \underset{2}{B_{n||s}} b^s b_m + B_n B_m\right] - m_n m_m \left[\underset{2}{N^s{}_{||s}} + N^s N_s\right] \\
&-\left[\underset{2}{N_{n||m}} - \underset{2}{N_{n||s}} m^s m_m + N_n N_m\right] - b_n b_m \left[\underset{2}{B^s{}_{||s}} + B^s B_s\right] \\
&-\left[\underset{3}{C_{n||m}} + C_m C_m\right] - c_n c_m \left[\underset{3}{C^s{}_{||s}} + C^s C_s\right] \\
&+\left[\underset{4}{E_{n||m}} - E_n E_m\right] + u_n u_m \left[\underset{4}{E^s{}_{||s}} - E^s E_s\right] \\
&-m_n m_m \left[M_0 B_0 + M_0 C_0 - M_0 E_0\right] \\
&-b_n b_m \left[B_0 M_0 + B_0 C_0 - B_0 E_0\right] \\
&-c_n c_m \left[C_0 M_0 + C_0 B_0 - C_0 E_0\right] \\
&+u_n u_m \left[E_0 M_0 + E_0 B_0 + E_0 C_0\right] \\
&-(m_n m_m + b_n b_m) v_S^2 \tilde{\Omega}^{s3} \tilde{\Omega}_{3s} + (m_n m_m + c_n c_m) v_S^2 N_2 C_2 \\
&+(m_n m_m + u_n u_m) E^s \frac{1}{\rho_S} \rho_{S|s} + 2E_{(n} N_{m)} \\
= 0 &
\end{aligned} \quad . \qquad \text{(VII.15.1)}$$

By comparison with (VII.8.12) one recognizes that the 4-dimensional Ricci tensor corresponds to the first block, the second corresponds to the base part of the stress-energy tensor (VII.8.13) of the seed metric. Thus, the conversion

$$R_{mn}(Y) = R_{mn}(A) + \kappa\left(T_{mn} - \frac{1}{2} g_{mn} T\right) = 0 \qquad \text{(VII.15.2)}$$

is implemented. Moreover, this relation can be transformed into (VII.8.20). The splitting off the extra components from the Ricci tensor has been implemented block-by-block. It would have been more correct to put the quantities Y into relation to the quantities A in tensorial form and to carry out the transformation with this relation. However, this way would have been substantially more complicated. In addition, we note that the

$$M_0 = A_{11}, \quad B_0 = A_{22}, \quad C_0 = A_{33}, \quad E_0 = -A_{44} \qquad \text{(VII.15.3)}$$

are the components of the generalized second fundamental forms of the surface theory. The block with the quadratic 0-terms can be written in a familiar way as

$$2 A_{m[s} A_{n]}{}^s \ . \qquad \text{(VII.15.4)}$$

The relation deviates from the usual Gauss equation as a consequence of the additional terms in (VII.15.2) which refer to the double-surface theory.

VII.16. Kerr metric, more on the geometry

In the preceding Section we have again referred to the generalized second fundamental forms of the surface theory. Although the classical surface theory is not directly applicable due to the anholonomicity of the embedding and also due to the underlying double-surface, there are nevertheless parallelisms which one can use for understanding the Kerr geometry. We want to deal with some of these problems in greater detail.

With the rigging vector of the anholonomic hyper planes one makes up

$$n_{a|||b} = n_{a|b} - Y_{ba}{}^{c} n_c = -Y_{ba}{}^{0} \ . \qquad \text{(VII.16.1)}$$

With that the asymmetry of $n_{[a|||b]} \neq 0$, the anholonomicity of the embedding is confirmed. However, with

$$A_{mn} = n_{m|||n}, \quad m = n \qquad \text{(VII.16.2)}$$

one can extract the generalized second fundamental forms of the surface theory (VII.15.3). These can also be written as

$$A_{11} = \frac{1}{\rho_S}, \quad A_{22} = \frac{1}{\rho_E}\sin\varepsilon, \quad A_{33} = \frac{1}{\sigma}\sin\varepsilon\sin\theta, \quad A_{44} = \frac{1}{\rho_S} \ . \qquad \text{(VII.16.3)}$$

In this form one better makes out the curvatures of the normal and inclined slices of the surface. The 5-dimensional quantity Y combines the properties of the intrinsic and the extrinsic geometries. We only have looked at the latter. Now we will deal with outstanding problems in greater detail. The construction of the holonomic surface of the seed metric has essentially been based on the fact that the points on the surface have been computed with (VII.6.8). If one does not want to regard all points of the 5-dimensional embedding space, but only the points on the surface, one has to consider the *constraint* $\delta x^0 = 0$. Thereby δx^0 is the distance of two neighboring surfaces which result from the projection and from the elliptical deformation from the pseudo-hyper spheres.

However, one has

$$\delta x^0 = d\rho \ , \qquad \text{(VII.16.4)}$$

whereby $d\rho$ is the change of the curvature vector of the radial curves, if one moves from a surface to a neighboring one. If one is interested in the extrinsic geometry, one has to introduce another constraint which does not permit a radial movement on the surface, i.e.

$$dx^1 = 0 \ .$$

The extrinsic geometry is parameterized by a family of straight lines, whose ascent is the negative reciprocal of the ascent of the radial Kerr curves. One has for the straight lines

$$dx^{0'}_{hol} = \cot\varepsilon \, dr \ , \qquad \text{(VII.16.5)}$$

whereby the cotangent of ε is constant along a straight line due to $\varepsilon_{|0} = 0$. However, the cotangent changes along the radial curves of the Kerr surface. The straight lines defined by (VII.16.5) lie in the direction of the normal vectors of the Kerr surface. We remember that such a construction is not sufficient for the explanation of the Kerr metric. The rigging vectors must be deduced from the normal vectors with the help of a rotation. These rigging vectors are no longer perpendicular to the Kerr surface except at the minor axes of the BL ellipses. However, if we perform a circulation on such an ellipse the rigging vectors maintain their angles with respect to the horizontals. These are the very angles the normal vectors of the surface enclose with the minor axes of the BL ellipses, i.e. ε. If one defines in a similar way with respect to the former problem the nonintegrable expression

$$dx^{0'}_{anh} = \cot\varepsilon\, dx^{\underline{1}} = \cot\varepsilon\, a_R dr\,, \tag{VII.16.6}$$

than the arc elements of those straight lines which are specified by the rigging vector can be computed by

$$dx^{0^2} = dx^{0'^2} + dx^{1'^2} = \left[\cot^2\varepsilon + 1\right] dx^{\underline{1}^2} = \frac{1}{\sin^2\varepsilon} a_R^2 dr^2\ . \tag{VII.16.7}$$

Thus,

$$dx^0 = \frac{1}{\sin\varepsilon} a_R dr = \frac{1}{v_S} dx^{\underline{1}}, \quad \partial_0 = \sin\varepsilon \frac{\partial}{a_R \partial r} = v_S \partial_{\underline{1}} \tag{VII.16.8}$$

is valid for operations on the straight lines and we remember of $v_{S|0} = 0,\ a_{S|0} = 0$. Suppressing the φ and the time dimension we put with the help of (VII.16.6)

$$\begin{aligned} dx^{0'} &= \cot\varepsilon\, a_R dr = \cos\varepsilon\, dx^0 \\ x^{1'} &= r\cos\vartheta \qquad\qquad . \\ x^{2'} &= A\sin\vartheta \end{aligned} \tag{VII.16.9}$$

By the use of the second relation of (VII.16.8) one obtains from (VII.16.9)

$$n^{a'} = x^{a'}{}_{|0} = \left\{\cos\varepsilon,\ \sin\varepsilon\cos\theta,\ \sin\varepsilon\sin\theta\right\}\ ,$$

whereby also (VII.3.5) has been used. These are the components of the rigging vector in the Cartesian co-ordinates of the embedding space. Following the classical surface theory we deduce the components of further unit vectors. Since these refer to the surface and belong to the intrinsic geometry, we have to hark back to (VII.6.12) for $dx^{0'}$.

With

$$m^{a'} = x^{a'}{}_{|1}, \quad b^{a'} = x^{a'}{}_{|2}$$

one gets the components of an orthogonal 3-bein

$$\begin{aligned} n^{a'} &= \{\ \cos\varepsilon,\ \sin\varepsilon\cos\theta, \sin\varepsilon\sin\theta\} \\ m^{a'} &= \{-\sin\varepsilon, \cos\varepsilon\cos\theta, \cos\varepsilon\sin\theta\}\ , \\ b^{a'} &= \{\quad 0\quad ,\quad -\sin\theta,\qquad \cos\theta\} \end{aligned} \tag{VII.16.10}$$

which accompanies the radial curves of the Kerr surface, but in contrast to the accompanying 3-bein of the conventional theory of surfaces and the theory of space curves it is subjected to the twist repeatedly discussed. In the holonomic case these would be the components of the normal vectors, the tangent vectors, and the binormal vectors of the radial curves on the surface. One finds these vectors in the columns of (VII.13.5), if the φ and the time dimension are suppressed. Finally, the trivial relations

$$n^{a'} = D^{a'}_a n^a = D^{a'}_0, \quad m^{a'} = D^{a'}_a m^a = D^{a'}_1, \quad b^{a'} = D^{a'}_a b^a = D^{a'}_2$$

are valid with $D^{a'}_a$ as rotation matrix.

For these vectors result the orthogonality relations

$$n^{a'} = \varepsilon^{a'}{}_{b'c'} m^{b'} b^{c'}, \quad m^{a'} = \varepsilon^{a'}{}_{b'c'} b^{b'} n^{c'}, \quad b^{a'} = \varepsilon^{a'}{}_{b'c'} n^{b'} m^{c'}$$

as well.

One has to consider that in

$$x^{a'}{}_{|||bc} = 0 \tag{VII.16.11}$$

the co-ordinate $x^{0'}$ is not an analytic function, and $dx^{0'}$ is only given as an incomplete differential (VII.6.12) or (VII.16.6), respectively. Hence this simple equation represents most of the properties of the geometry. It can be examined with (VII.16.10) and the definitions of the connexion coefficients Y. If δs is the length of an anholonomic arc element in the local 1-direction, and if 'Y is the spatial part of the connexion coefficients of Y, viz those components of Y which do not contain 0-indices, one can define the following derivative[129]

$$\frac{\delta\Phi^{a'}{}_{b}}{\delta s} = \Phi^{a'}{}_{b|1} - {}'Y_{1b}{}^{s}\Phi^{a'}{}_{s} \ . \tag{VII.16.12}$$

If one inserts

$$\Phi^{a'}{}_{b} = x^{a'}{}_{|b}$$

into (VII.16.12), and if one expands (VII.16.11) it in such a way that the 0-components therein are isolated from Y one has

$$\begin{aligned}
\frac{\delta n^{a'}}{\delta s} &= x^{a'}{}_{|01} = M_{10}{}^{1}x^{a'}{}_{|1} + N_{10}{}^{2}x^{a'}{}_{|2} \\
\frac{\delta m^{a'}}{\delta s} &= x^{a'}{}_{|11} - N_{11}{}^{2}x^{a'}{}_{|2} = M_{11}{}^{0}x^{a'}{}_{|0} \\
\frac{\delta b^{a'}}{\delta s} &= x^{a'}{}_{|21} - N_{12}{}^{1}x^{a'}{}_{|1} = N_{12}{}^{0}x^{a'}{}_{|0} \ .
\end{aligned}$$

With the first and second curvatures

$$\kappa = -\frac{1}{\rho_S}, \quad \tau = -\frac{1}{\rho_H}\sin\varepsilon\cos\varepsilon \tag{VII.16.13}$$

one obtains the Frenet formulae

$$\begin{aligned}
\frac{\delta m^{a'}}{\delta s} &= \kappa n^{a'} \\
\frac{\delta n^{a'}}{\delta s} &= -\kappa m^{a'} + \tau b^{a'} \ . \\
\frac{\delta b^{a'}}{\delta s} &= -\tau n^{a'}
\end{aligned} \tag{VII.16.14}$$

Since the vectors n, m, b are not the normal vectors, tangent vectors, and binormal vectors of the radial space curve, the κ and τ are not the first and second curvatures of this curve, but they are thereto the anholonomic analogue. They are called *generalized curvatures*. In order to compare this procedure with the classical theory we note the classical second fundamental forms of the surface theory

$$L = n^{a'}x^{a'}{}_{|\underset{\sim}{1}\underset{\sim}{1}}, \quad M = n^{a'}x^{a'}{}_{|\underset{\sim}{1}\underset{\sim}{2}}, \quad N = n^{a'}x^{a'}{}_{|\underset{\sim}{2}\underset{\sim}{2}} \ . \tag{VII.16.15}$$

[129] The index a' refers to the Cartesian co-ordinate system of the embedding space and does not entail connexion coefficients.

The tildes under the indices refer to the co-ordinate representation. The procedure is not applicable to the Kerr geometry for two different reasons. First of all the dx^{0} are not exact differentials. Secondly, r is not a Gaussian co-ordinate on the surface as the conventional surface theory presupposes. r is an artificial variable which can be interpreted geometrically only in the embedding space. Our suggested procedure represents a generalization of the conventional surface theory for anholonomic systems, a generalization with respect to the physical surface.

VII.17. Kerr metric, some more preferred reference systems

Having discussed the 5-dimensional representation of the seed metric and its embedding in preceding Sections in detail, we return to the 4-dimensional Kerr model and treat the reference systems A and B already-mentioned. Both systems can be deduced by Lorentz transformations from the system C. The conversions are somewhat laborious, however, they require no new techniques. Therefore we mention only the substantial results.

The Lorentz transformations with the help of which the two systems can be deduced from Carter's system have already been treated in (VII.10.2) to (VII.10.6), the Lorentz transformed 4-beins in (VII.10.7) and (VII.10.8). The system A is characterized by the relations

$$u_{m||n} = \Omega^{A}_{mn} + E^{A}_{m} u_{n}, \; \Omega^{A}_{(mn)} = 0 \; , \qquad \text{(VII.17.1)}$$

from which a shear-free motion of the observers can be inferred

$$u_{(\alpha||\beta)} = 0 \; . \qquad \text{(VII.17.2)}$$

To the forces of the system C come relative forces, which are equipped with the double markers AC. Thus, the relative centrifugal force and the radial deformation field strength

$$F^{AC}_{n} = \alpha^{2}_{AC}\omega^{2}_{AC}\sigma\sigma_{n}, \quad D^{AC}_{n} = \alpha^{2}_{AC}\omega_{AC}\omega_{AC|n}\sigma^{2} \qquad \text{(VII.17.3)}$$

enter into

$$C_{mn}{}^{s} = c_{m}C^{A}_{n}c^{s} - c_{m}c_{n}C^{s}_{A}, \quad C^{A}_{n} = C_{n} + \left(F^{AC}_{n} - F_{n}\right) + \left(D^{AC}_{n} - D_{n}\right) \; , \qquad \text{(VII.17.4)}$$

likewise into the term of the force of gravity

$$E_{mn}{}^{s} = -\left(u_{m}E^{A}_{n}u^{s} - u_{m}u_{n}E^{s}_{A}\right), \quad E^{A}_{n} = E_{n} + \left(F^{AC}_{n} - F_{n}\right) + \left(D^{AC}_{n} - D_{n}\right) \; . \qquad \text{(VII.17.5)}$$

The mixed components of the connexion coefficients, i.e. those which have both time-like and space-like indices, can easily be compared with those of the system C. They do not contain symmetrical parts which correspond to deformations

$$H_{mns} = \Omega^{A}_{mn}u_{s} + \Omega^{A}_{sm}u_{n} + \Omega^{A}_{sn}u_{m}$$

$$\Omega^{A}_{nm} = \Omega^{AC}_{mn} - \Omega_{mn}, \; \Omega^{AC}_{mn} = H^{AC}_{mn} + D^{AC}_{mn}, \; H^{AC}_{mn} = 2i\alpha^{2}_{AC}\omega_{AC}\sigma_{[m}c_{n]}, \; D^{AC}_{mn} = 2i\alpha^{2}_{AC}\omega_{AC|[m}c_{n]}\sigma \; . \qquad \text{(VII.17.6)}$$

$$\Omega_{3n} = -\Omega_{n3} \stackrel{*}{=} \Omega^{C}_{3n}$$

The dual vector of the Coriolis field strength H^{AC}_{mn} is parallel to the symmetry axis of the ellipsoids of revolution which are typical for the structure of spatial Kerr geometry. The field equations of the system of A are similar to those of the system of C. The relations

$$E^{s}{}_{||s} + \left[F^{s}_{AC} - F^{s}\right]_{||s} + \left[D^{s}_{AC} - D^{s}\right]_{||s} - \Omega^{rs}_{A}\Omega^{A}_{sr} = 0, \quad \Omega^{sm}_{A}{}_{\underset{4}{||s}} - 2\Omega^{sm}_{A}E^{A}_{s} = 0 \qquad \text{(VII.17.7)}$$

are valid and are again Maxwell-like. From the definitions of the field quantities further Maxwell-like equations

$$F^{AC}_{\underset{4}{[m||n]}} + D^{AC}_{\underset{4}{[m||n]}} = 0, \quad \Omega^{AC}_{\underset{3}{[mn||s]}} = 0 \qquad \text{(VII.17.8)}$$

can be deduced. For the field energy and for the Poynting vector the conservation laws

$$\left[E^{s}_{A}E^{A}_{s} + \Omega^{sr}_{A}\Omega^{A}_{rs}\right]_{|4} = 0, \quad \left[2\Omega^{sm}_{A}E^{A}_{s}\right]_{\underset{4}{||m}} = 0 \qquad \text{(VII.17.9)}$$

are valid.

In the system B relative centrifugal and deformation forces appear as well which are built up in a way similar to that of the system A

$$C_n^B = C_n - \left(F_n^{BC} - F_n\right) - \left(D_n^{BC} - D_n\right), \quad E_n^B = E_n - \left(F_n^{BC} - F_n\right) - \left(D_n^{BC} - D_n\right)$$
$$F_n^{BC} = \alpha^2_{BC}\omega^2_{BC}\sigma\sigma_n, \quad D_n^{BC} = \alpha^2_{BC}\omega_{BC}\omega_{BC|n}\sigma^2 \quad . \qquad \text{(VII.17.10)}$$

However, the mixed components of the connexion coefficients have a more complicated structure

$$\Omega^B_{mns} = 2\left[\Omega^B_{(sm)}u_n - \Omega^B_{(mn)}u_s + \Omega^B_{[ns]}u_m\right]$$
$$\Omega^B_{3\alpha} = H^{BC}_{\alpha 3} + D^{BC}_{\alpha 3} + \Omega^C_{3\alpha} = \left[H^{BC}_{\alpha 3} - H^C_{\alpha 3}\right] + \left[D^{BC}_{\alpha 3} - D^C_{\alpha 3}\right]. \qquad \text{(VII.17.11)}$$
$$H^{BC}_{\alpha 3} = i\alpha^2_{BC}\omega_{BC}\,\sigma_\alpha, \quad D^{BC}_{\alpha 3} = i\alpha^2_{BC}\omega_{BC|\alpha}\sigma, \quad \Omega^B_{\alpha 3} = 0$$

From

$$u_{m||n} = 2\Omega^B_{mn} + E^B_m u_n \qquad \text{(VII.17.12)}$$

one infers that the observers in this system are only exposed to the shears Ω^B_{mn} and because of

$$u_{[\alpha||\beta]} = 0, \quad u_{(\alpha||\beta)} = 2\Omega^B_{(\alpha\beta)} \qquad \text{(VII.17.13)}$$

no rotational forces act on them. The above field quantities fulfill the relations

$$E^s_{\;||s} - \left[F^s_{BC} - F^s\right]_{||s} - \left[D^s_{BC} - D^s\right]_{||s} - D^{rs}_B D^B_{sr} = 0, \qquad D^{sm}_{B\;\underset{4}{||s}} = 0 \ . \qquad \text{(VII.17.14)}$$

In addition, the Maxwell-like equation

$$F^{BC}_{\underset{4}{[m||n]}} + D^{BC}_{\underset{4}{[m||n]}} = 0 \ . \qquad \text{(VII.17.15)}$$

emerges. In accordance with the second equation (VII.17.14) a Poynting vector does not exist, i.e. no transport of field energy occurs in the locally nonrotating system. The field energy is conserved

$$\left[E^s_B E^B_s + D^{sr}_B D^B_{rs}\right]_{|4} = 0 \ . \qquad \text{(VII.17.16)}$$

Having established the field physics of different rotating systems in the last Sections, we supplement the investigations by a further observer system, the freely falling system.

VII.18. Kerr metric, free fall

By use of a Lorentz transformation

$$L_{1}^{1'}=\alpha_S,\quad L_{4}^{1'}=i\alpha_S v_S,\quad L_{1}^{4'}=-i\alpha_S v_S,\quad L_{4}^{4'}=\alpha_S \tag{VII.18.1}$$

one arrives at a freely falling system, whereby the parameters of the transformation

$$\alpha_S=\frac{1}{\sqrt{1-v_S^2}},\quad v_S=-\frac{r}{A}\sqrt{\frac{2M}{r}} \tag{VII.18.2}$$

are already known and

$$v_S=\sin\varepsilon,\quad \alpha_S=1/\cos\varepsilon\ . \tag{VII.18.3}$$

result from the geometry.

The falling velocity v_S of an observer who comes from the infinite is just like in the Schwarzschild geometry correlated to the angle of ascent of the physical surface and independent of the angle ϑ. Punsly [P] has grappled with this problem[130]. In an earlier Section we have already mentioned that the ascent of the surface

$$\tan\varepsilon=-\frac{\sqrt{2Mr}}{\delta}$$

becomes infinite for $\delta=0$. The tangent to the surface in the radial direction is perpendicular to the base plane. A look at Fig. VII.3 shows that the ellipse at the waist of the Kerr surface is described by $\delta=0$. It is the lower boundary of geometry. If one resolves

$$\delta^2=r^2+a^2-2Mr=0 \tag{VII.18.4}$$

with respect to r one gets

$$r_H=M+\sqrt{M^2-a^2}\ , \tag{VII.18.5}$$

the smallest value for r which the geometry permits. It defines the *event horizon*. If an infalling body were not be subjected to dragging effects caused by the rotation, it would reach the velocity of light at the event horizon. (VII.18.4) can also be written as

$$A_H^2=2Mr_H$$

and with (VII.18.2) one verifies $v_S^H=-1$. A further condition

$$\delta_0=a\sin\vartheta \tag{VII.18.6}$$

determines the *static horizon* which leads with (44.4) to

$$r_0=M+\sqrt{M^2-a^2\cos^2\vartheta}\ . \tag{VII.18.7}$$

If one substitutes the expression for ω into (VII.18.6) one first has $\delta_0/A_0=\omega\sigma$ and $\omega_{AC}=\alpha_S\omega$, and finally also

$$\omega_{AC}^0\sigma^0=1\ . \tag{VII.18.8}$$

The rotating observer A reaches the speed of light at the static horizon. An observer who is not exposed to forces other than the gravitation forces in the Kerr field, moves

[130] For the distance of a rotating observer from the rotation axis we have chosen $\sigma=A\sin\vartheta$. Thus, we obtain results different from those of Punsly.

towards the central mass, however, is moved around the gravitation center at the same time by the rotational effects of the field. In summa the object moves helically into the central mass. This provides the relation[131]

$$v_S^2 + \frac{1}{\alpha_S^2}\omega_{AC}^2\sigma^2 = 1 \tag{VII.18.9}$$

which can be deduced with (VII.18.7) and (VII.18.8). It corresponds to the relativistic addition law of two perpendicular velocities. The static horizon specifies the boundary of all physical events. The range between static horizon and event horizon is called ergoregion. With respect to our preferred view it should be covered by an interior solution and not be offered for discussion.

In order to learn something about the physics of a freely falling object in the Kerr field we consider the Lorentz transformation

$$L_1^{1'} = \alpha_S, \quad L_4^{1'} = i\alpha_S v_S, \quad L_1^{4'} = -i\alpha_S v_S, \quad L_4^{4'} = \alpha_S \ . \tag{VII.18.10}$$

If we want to describe what the freely falling observer can state on an observer being in relative rest with respect to the central mass, we require again that the Ricci-rotation coefficients behave as tensors under this transformation. One has to allocate the inhomogeneous term which emerges from the transformation to the partial derivatives

$$\Phi_{m'\|n'} = L_{m'n'}^{m\,n}\Phi_{m\|n} = \left[\Phi_{m'|n'} - L_s^{s'}L_{m'|n'}^{s}\Phi_{s'}\right] - A_{n'm'}{}^{s'}\Phi_{s'}, \quad A_{n'm'}{}^{s'} = L_{n'm's}^{n\,m\,s'}A_{nm}{}^{s} \tag{VII.18.11}$$

and thus to enrich the graded covariant derivatives by a further grade

$$\Phi_{m'\underset{1}{\|}n'} = \Phi_{m'|n'} - L_{n'm'}{}^{s'}\Phi_{s'}, \quad L_{n'm'}{}^{s'} = L_s^{s'}L_{m'|n'}^{s} \ . \tag{VII.18.12}$$

Then the Ricci tensor appears in the form

$$R_{m'n'} = A_{m'n'}{}^{s'}{}_{\underset{1}{\|}s'} - A_{n'\underset{1}{\|}m'} - A_{r'm'}{}^{s'}A_{s'n'}{}^{r'} + A_{m'n'}{}^{s'}A_{s'} \ . \tag{VII.18.13}$$

Under this condition not only the Einstein field equations are form invariant but also their subequations which describe the curvatures of the slices of the physical surface. In the context of our geometrical interpretation of the gravitation it is also to be expected that a change of a reference system *on* the surface does not change the shape of the surface.

In the freely falling system the subequations of the Einstein field equations will contain the graded derivatives extended by the Lorentz term. The curvature quantities will include a further time-like component. Solving (VII.18.13) with respect to the subequations leads again to the same subequations with the same physical statements we have made for the stationary system, but with the measured values of the freely falling observer.

The Lorentz transformation generates new unit vectors

$$\begin{aligned} {}'m_{n'} &= L_1^{1'}m_{n'} + L_4^{1'}u_{n'} = \{1,0,0,0\}, \quad & {}'u_{n'} &= L_1^{4'}m_{n'} + L_4^{4'}u_{n'} = \{0,0,0,1\} \\ {}'b_{n'} &= b_n = \{0,1,0,0\}, & {}'c_{n'} &= c_n = \{0,0,1,0\} \end{aligned} \tag{VII.18.14}$$

which are consulted for the determination of the physics of the freely falling observer.

Now we interpret the Lorentz transformation as an active transformation. The inhomogeneous term is to be allocated in a conventional way to the Ricci-rotation coefficients. Thus, the objects described by the Ricci-rotation coefficients do not change, but a new object is formed from them with the help of the additional Lorentz terms which presents to the freely falling observer some physics different from that which the stationary observer has experienced. In the following we proceed in a similar way as we have done

[131] Here the 0-tags are suppressed.

with the similar problem of the Schwarzschild geometry and we decompose the Lorentz term into two components by the consideration of

$$L_{4'1'}{}^{4'} = G_{1'}, \quad L_{1'4'}{}^{1'} = Q_{4'} + G_{4'}$$

$$G_{m'} = \left\{ \alpha_S \frac{1}{\rho_S} \frac{v_S}{a_S}, 0, 0, -i\alpha_S v_s \frac{1}{\rho_S} \frac{v_S}{a_S} \right\}, \quad Q_{m'} = \left\{ 0, 0, 0, -\frac{i}{\rho_S} \right\} \ . \tag{VII.18.15}$$

The force of gravity still exists for the freely falling observer because it describes the action of the central mass. It has a time-like component

$$E_{4'1'}{}^{4'} = -E_{1'}, \quad E_{1'4'}{}^{1'} = -E_{4'}, \quad E_{m'} = \left\{ \alpha_S \frac{1}{\rho_S} \frac{v_S}{a_S}, 0, 0, -i\alpha_S v_s \frac{1}{\rho_S} \frac{v_S}{a_S} \right\} \ . \tag{VII.18.16}$$

We can right away note that acceleration G which follows from the Lorentz transformation and the force of gravity E being supplied by the Ricci-rotation coefficients of the stationary observer are formally identical, but enter into the geometry with opposite signs. The quantities nullify one another in such a way that the freely falling observer cannot experience any gravity effect. In order to work out the difference of both quantities more clearly, we emphasize that a Lorentz transformation can be represented with the help of imaginary angles

$$\alpha_S = \cos i\chi, \quad i\alpha_S v_S = \sin i\chi \ .$$

On the other hand the two quantities α_S and v_S are in relation to the radial angle of ascent ε of the physical surface as we can see from (VII.18.3). The two angles ε and χ have different origins, but χ is adapted to ε in such a manner that the velocity of the falling observer is determined by the geometry. Most of that has already been discussed in the context of the Schwarzschild geometry, but it remains to be mentioned that the above relations can also be written as

$$L_{mn}{}^{s} + E_{mn}{}^{s} = {}'m_m Q_n {}'m^s - {}'m_m {}'m_n Q^s \ , \tag{VII.18.17}$$

whereby the primes at the indices have been omitted. Having nullified the force of gravity only the quantity Q of (VII.18.15) remains. It can be interpreted as a component of the second fundamental forms of the surface theory and can be supplemented by further components of geometry, so that all the

$$Q_{11} = Q_4, \quad Q_{22} = B_4, \quad Q_{33} = C_4 \tag{VII.18.18}$$

can be interpreted as tidal forces which affect the freely falling observer. Geometrically, they represent the complete set of the second fundamental forms of a surface shrinking toward the gravitation center, a surface which accompanies the falling observer. Thus, the most important mechanisms and interactions between physics and geometry of the freely falling observers have been explained. Still pending are the calculations for the whole rotating system. These are left to further research.

VII.19. Kerr interior solution, the space-like part

From the numerous solutions which provide the Einstein field equations and which try to describe the field of a stellar object, probably only two solutions are physically reasonable: the Schwarzschild solution which describes the field of a static stellar object and the Kerr solution which illuminates the gravitational effects of a rotating stellar object. Both models have been examined by measurements or, concerning the dragging effects, are at least examinable.

It is natural to continue the field of the exterior of the gravitating body into its interior and to arrive in such a way at a *complete solution* of the Einstein field equations. It is subsumed that it is almost impossible to get a realistic description of the inside of a stellar object with geometrical methods. Each star or each planet has a rather complex structure with layers of different density, with thermodynamic effects not to be neglected, and internal inhomogeneities and currents not easily to be described.

From the outset any interior solution will be a very rough approximation of the physical requirements. Thus, the interior Schwarzschild solution represents a homogeneous fluid ball with constant matter density. As a consequence of the latter the speed of sound would be infinitely high in such an object. The interior Kerr solutions which have been adapted to the exterior Kerr field endowed with rotational effects deviate still more from reality. Their sources should rotate as well. If one prolongs the differential rotation law inwards, extremely high rotation velocities emerge in the vicinity of the rotation axis which cannot be realized by nature. In addition, the well-known ring singularity of Kerr geometry seems unavoidable. Many gravitation physicists regard singularities as an element of the physical world. McManus [M] has developed an interior Kerr solution by means of this singularity. However, we are of opinion that singularities are regions within which mathematics breaks down and which have to be avoided. In contrast to the workers on quantum mechanics the gravitation physicists have made little useful attempts to create singularity-free models.

The interior problem of the Kerr model is said to be unsolved. There exist numerous ansätze which are modestly called trial solutions. We [B] want to add a further proposal which has the advantage reducing to the interior Schwarzschild solution if one puts the rotation parameter zero. However, it has the disadvantages outlined above. The new solution can be found quite simply with the help of the embedding methods developed by us. However, it is rather laborious to calculate the field equations.

The application of these very embedding methods and the 5-dimensional representation associated with them is motivated by finding interior solutions, not only for the Kerr geometry, but for the whole Kerr family, as will be shown.

Before we turn to the new solution, we will quote the solutions already suggested and the critical literature referring to it.

Collas [C] and Lawrence, and Collas [C] have tested the possibility of the existence of an interior Kerr solution by means of the red shift. Further proposals are due to Abramowicz [A], Ali [A], Bicak [B] and Ledvinka; Boyer [B], Cohen [C], Cox [C], De la Cruz [D], De Felice [D], Nobili and Clavani; Drake [D] and Turolla; Florides [F], Haggag [H] and Marek; Hamity [H], Hamity [H] and Lamberti; Hernandez [H], Hernadez-Pastora [H] and Herrera; Herrera [H] and Hernandez-PastoraHogan [H], Israel [I], Krasiński [K], Lopez [L], Magli [M], Majidi [M], Marek [M] and Sloane; Roos [R], Viaggiu [V], Wolf [W] and Neugebauer.

In order to come closer to the interior Kerr metric, we again make a start with a static seed metric which we transform into the actual interior by an intrinsic transformation. Let us take a look at the exterior Kerr surface. We will cut it off at a suitable position and we will connect this elliptical 'funnel'' from below by a closing surface. Hence it is certain from the very beginning that the interior surface must exhibit elliptical parallels. These ellipses are confocal just like the exterior ones. The differential rotation law is prolonged inwards and the Kerr ring singularity occurs. For the seed metric we put, in formal regard, in a similar way to the Schwarzschild interior

$$ds^2 = \alpha_I^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 + a_T^2 \left(dit\right)^2 \; . \qquad \text{(VII.19.1)}$$

The first factor in the radial arc element

$$\alpha_I = \frac{1}{a_I}, \quad a_I^2 = 1 - \frac{r^2}{\mathcal{R}^2} \qquad \text{(VII.19.2)}$$

reminds us of the Schwarzschild cap of the sphere. Furthermore, it contains the elliptical factor a_R. If one makes a start with the 3-dimensional surface with

$$\begin{aligned} x^{0'} &= \mathcal{R}\cos\eta \\ x^{1'} &= \mathcal{R}\sin\eta\cos\vartheta \\ x^{2'} &= {}^*\mathcal{R}\sin\eta\sin\vartheta\cos\varphi \\ x^{3'} &= {}^*\mathcal{R}\sin\eta\sin\vartheta\sin\varphi \end{aligned} \qquad \text{(VII.19.3)}$$

and

$$\sin\eta = \frac{r}{\mathcal{R}} = \frac{A}{{}^*\mathcal{R}}, \quad \cos\eta = \sqrt{1 - \frac{r^2}{\mathcal{R}^2}}, \quad {}^*\mathcal{R} = \frac{A}{r}\mathcal{R} \qquad \text{(VII.19.4)}$$

one recognizes with

$$\frac{\left(x^{0'}\right)^2 + \left(x^{1'}\right)^2}{\mathcal{R}^2} + \frac{\left(x^{2'}\right)^2 + \left(x^{3'}\right)^2}{{}^*\mathcal{R}^2} = 1 \qquad \text{(VII.19.5)}$$

the deviation from the spherical interior Schwarzschild geometry. Having eliminated the angle η from the relations one obtains

$$\begin{aligned} x^{0'} &= \pm\sqrt{\mathcal{R}^2 - r^2} \\ x^{1'} &= r\cos\vartheta \\ x^{2'} &= A\sin\vartheta\cos\varphi \\ x^{3'} &= A\sin\vartheta\sin\varphi \end{aligned} \; . \qquad \text{(VII.19.6)}$$

Suppressing the third dimension one obtains for $\mathcal{R} = \text{const.}$ an elliptical surface. Its lower half is represented in Fig. VII.9.

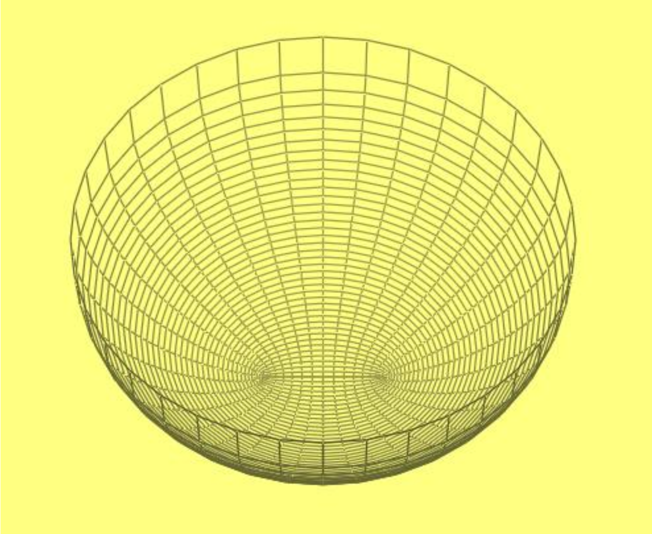

Fig. VII.9

If one varies $\mathcal{R}$, one obtains a family of such surfaces, from which one surface is selected with the embedding condition

$$\mathcal{R} = \mathcal{R}_g = \text{const.} \qquad \text{(VII.19.7)}$$

From this surface a band has to be cut off and the remaining surface has to be matched horizontally to the auxiliary surface of the Kerr metric. $\mathcal{R}$ is the radius of the circular arc at the minor axes of the ellipses. All individual 'radial' curves have hyperbolic contributions in their properties. For r = 0, A = a the horizontal ellipses reduce to a distance which is clamped by the common foci of the ellipses. If one now adds the third dimension, these points rotate through φ. For $\vartheta = \pi/2$ emerges a circle. The radius of curvature of the ellipses is zero on this circle and the assigned field strengths are infinitely large. This is the Kerr ring singularity.

If one differentiates (VII.19.3), one arrives by use of (VII.19.4) and

$$dr = \mathcal{R}\cos\eta d\eta, \quad dA = \frac{r}{A}\mathcal{R}\cos\eta d\eta, \quad a_R = \frac{\Lambda}{A} \qquad \text{(VII.19.8)}$$

at the metric

$$ds^2 = \left[\tan^2\eta + a_R^2\right]\mathcal{R}^2\cos^2\eta d\eta^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 \ . \qquad \text{(VII.19.9)}$$

The radial part of the metric can also be written as

$$\sqrt{1+\alpha_R^2\tan^2\eta}\, a_R dr \ , \qquad \text{(VII.19.10)}$$

whereby $a_R dr$ is the arc element of the horizontal ellipses. The ascent of the radial curves

$$\tan\chi = \alpha_R(r,\vartheta)\tan\eta$$

(VII.19.11)

depends on ϑ. This means that the ascent of the surface changes, if one orbits on the horizontals around the symmetry axis. Finally, by using (VII.19.11) one obtains the 3-dimensional line element in the form

$$ds^2 = \frac{1}{\cos^2\chi} a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 \ . \qquad \text{(VII.19.12)}$$

The points of the surface are computed with

$$dx^{0'}_{hol} = -\tan\eta dr = -\tan\chi a_R dr, \quad \tan\eta = \mp\frac{r}{\sqrt{\mathcal{R}^2 - r^2}}, \quad x^{0'}_{hol} = \pm\sqrt{\mathcal{R}^2 - r^2} \ . \text{(VII.19.13)}$$

These are relations which one gets from (VII.19.3) and (VII.19.8). They contain only one parameter, i.e. the aperture angle η of the surface at the minor axis or alternatively the parameter $\mathcal{R}$. The metric (VII.19.12) can be matched to the holonomic auxiliary surface of the exterior Kerr metric, however, not to the physical surface. Thus, the same problem is present which we have learned to know from the embedding of the exterior metric. The interior auxiliary surface must be equipped with nonholonomicity. Therefore we define the nonintegrable function

$$dx^{0'}_{anh} = -\tan\eta\, a_R(r,\vartheta) dr \ . \qquad \text{(VII.19.14)}$$

Suppressing again the other dimensions, one has with the BL arc element $dx^{1'} = a_R dr$ of the base plane

$$dx^{0'^2} + dx^{1'^2} = \frac{1}{\cos^2\eta} a_R^2 dr^2 = \frac{1}{1 - \frac{r^2}{\mathcal{R}^2}} a_R^2 dr^2 \ . \qquad \text{(VII.19.15)}$$

The radial arc elements, both the holonomic and the anholonomic ones, have the same projections $dx^{1'} = a_R dr$ onto the base plane. The anholonomic arc elements lie in the anholonomic hyper planes which generally are not tangent to the auxiliary surface. The rigging vector of these hyper panes is not normal to the auxiliary surface as well. Orbiting around the symmetry axis the rigging vector preserves its angle of ascent. The family of the anholonomic hyper planes constitutes the physical surface. Thus, all presuppositions for an adjustment of the interior surface to the exterior physical surface are established.

We return again to the holonomic surface and we perform the matching. We select the lower part $x^{0'}_{hol} = -\sqrt{\mathcal{R}^2 - r^2}$ of the surface. The parameter $\mathcal{R}$ and the center of the surface are adjusted on the symmetry axis in such a way that the surface covers the ergoregion of the exterior surface. Thus, we avoid all oddities of the exterior Kerr metric beneath the ergosphere which is the outer surface of the ergoregion.

If η_g is the aperture angle of the cap which is put into position, η_g must agree up to the sign with the radial angle of ascent of the exterior surface at the junction of the geometries. Both solutions the interior and the exterior have their own co-ordinate systems with different orientations of the local 0-axes. Thus, one has

$$\eta_g = -\varepsilon_g \ . \qquad \text{(VII.19.16)}$$

This condition connects the relations

$$\sin\eta = \frac{r}{\mathcal{R}}, \quad \sin\varepsilon = v_S = -\frac{r}{A}\sqrt{\frac{2M}{r}} \qquad \text{(VII.19.17)}$$

on the boundary surface. One immediately finds the *junction condition*

$$\mathcal{R}_g = A_g\sqrt{\frac{r_g}{2M}} \ . \qquad \text{(VII.19.18)}$$

If one has properly adjusted, one obtains a surface for the *complete Kerr geometry* as one can see in Fig. VII.10.

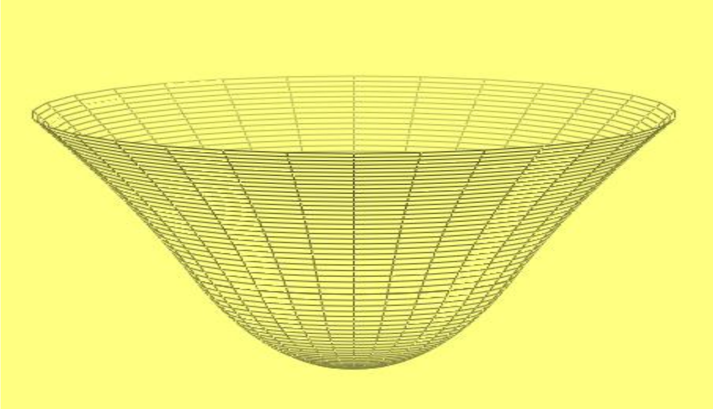

Fig. VII.10

In order to determine the time-like part of the interior Kerr metric, the simple embedding procedure which has been successful for the space-like part is not sufficient. One has again to return to the double-surface theory.

VII.20. Kerr interior solution, the time-like part

In order to find a suitable ansatz for the time-like part of the interior solution, we closely follow the procedure which has enabled us to explain the interior Schwarzschild solution and to find an interior solution for Reissner-Nordström model, respectively. Thereby the curvature vector of the radial curves on the exterior surface is substantial. Most information is read from it and the junction conditions on the boundary surface of the geometries are specified. The only component of the curvature vector at the minor axes of the ellipses reads as

$$\rho = -\frac{2r}{\sin\varepsilon}\frac{r^2+a^2}{r^2-a^2} \qquad \text{(VII.20.1)}$$

in accordance with previous Sections. Considering (VII.19.16) on the boundary surface[132] one has

$$\rho_g = \frac{2r_g}{\sin\eta_g}\Phi_g^2, \quad \Phi_g^2 = \frac{r_g^2+a^2}{r_g^2-a^2}\ . \qquad \text{(VII.20.2)}$$

With (VII.19.18) arises

$$\rho_g = 2\mathcal{R}_g\Phi_g^2\ . \qquad \text{(VII.20.3)}$$

On the boundary surface we put a straight line through the curvature vector up to the symmetry axis. The distance from the base point of the curvature vector up to the intersection of the straight lines with the symmetry axis has the length

$$\bar{\mathcal{R}} = \mathcal{R}_g + \rho_g\ . \qquad \text{(VII.20.4)}$$

If the tip of the curvature vector proceeds on the cap in the radial direction while one holds the base point and if one still calls the new vector X and its angle of ascent ε, then one has in the Cartesian co-ordinates of the embedding space for the components of X

$$\begin{aligned} X^{0'} &= X\cos\varepsilon = \bar{\mathcal{R}}\cos\eta_g - \mathcal{R}\cos\eta \\ X^{1'} &= X\sin\varepsilon = \bar{\mathcal{R}}\sin\eta_g - \mathcal{R}\sin\eta \end{aligned}\ . \qquad \text{(VII.20.5)}$$

The projection of the vector X onto the direction of the extra dimension of the embedding space is $X^{0'} = X\cos\varepsilon$. If one rotates $X^{0'}$ in the $[x^{0'},x^{4'}]$-plane through the imaginary angle $di\psi$ and if one defines the area of the surface swept thereby as proportional to the elapsed time, one has with (VII.20.5)

$$dx^4 = X\cos\varepsilon\, di\psi = \left[\bar{\mathcal{R}}\cos\eta_g - \mathcal{R}\cos\eta\right]di\psi\ . \qquad \text{(VII.20.6)}$$

If one inserts[133] in it (VII.20.4), for ρ_g the relation (VII.20.3), and if one separates $\mathcal{R}$, one first obtains

$$dx^4 = \left[\left(1+2\Phi_g^2\right)\cos\eta_g - \cos\eta\right]\mathcal{R}\, di\psi\ ,$$

a relation which is only valid at the minor axes of the ellipses. Only at this location has a perpendicular slice of the surface the form of a piece a of circular arc with the radius $\mathcal{R}$. In

[132] Suppressing dimensions reduces the boundary surface to an elliptical curve.

[133] Having selected one of the surfaces of the family, the marker g of $\mathcal{R}_g$ is omitted in the 4-dimensional formulae.

order to make appropriate statements in all other positions of the surface, the elliptical factor a_R is to be placed:

$$dx^4 = \frac{1}{2}\left[\left(1+2\Phi_g^2\right)\cos\eta_g - \cos\eta\right]2\mathfrak{R}a_R di\psi \quad .$$

If one has again replaced $2\mathfrak{R}$ in this relation by means of (VII.20.3), one finally obtains

$$dx^4 = \frac{1}{2}\left[\left(1+2\Phi_g^2\right)\cos\eta_g - \cos\eta\right]\Phi_g^{-2} idt, \quad dx^4 = a_T idt, \quad dt = \rho_S^g d\psi \quad , \qquad \text{(VII.20.7)}$$

whereby

$$\rho_S^g = \rho_g a_R^g$$

is the radius of curvature of the exterior geometry on the boundary surface. From (VII.20.7) one reads

$$a_T = \frac{1}{2}\left[\left(1+2\Phi_g^2\right)\cos\eta_g - \cos\eta\right]\Phi_g^{-2}, \quad \alpha_T = a_T^{-1} \qquad \text{(VII.20.8)}$$

and with the time-like arc element one has also supplemented the line element of the auxiliary metric. On the boundary surface one has $\eta = \eta_g$. Thus, one obtains with

$$\cos\eta_g = \cos\varepsilon_g, \quad a_T^g = a_S^g, \quad dx_g^4 = i a_S^g dt \qquad \text{(VII.20.9)}$$

the conformity of the gravitational factor of the interior and of the exterior solutions. If one further inserts the junction condition (VII.19.18) into (VII.19.2) one obtains with

$$a_{Ig}^2 = \frac{r_g^2 + a^2 - 2Mr_g}{r_g^2 + a^2}, \quad a_T^g = a_I^g \qquad \text{(VII.20.10)}$$

the accordance of the radial factor with the time-like gravitational factor on the boundary surface. If the rotation parameter a is put zero, the interior surface will become the cap of a sphere and the time-like arc element

$$dx^4 = \frac{1}{2}\left[3\cos\eta_g - \cos\eta\right]idt = \left[3\mathfrak{R}\cos\eta_g - \mathfrak{R}\cos\eta\right]di\psi \qquad \text{(VII.20.11)}$$

will become the Schwarzschild time-like arc element. Then one has $\rho_g = 2\mathfrak{R}$ and $\bar{\mathfrak{R}} = \rho_g + \mathfrak{R} = 3\mathfrak{R}$. These relations describe the fundamental properties of the Schwarzschild parabola.

So far only the components of the vector X in the [$x^{0'}$, $x^{1'}$]-plane have been considered. However, the ansatz can immediately be extended to the omitted dimensions

$$\begin{aligned} X^{3'} &= X\sin\varepsilon\sin\theta\sin\varphi \\ X^{2'} &= X\sin\varepsilon\sin\theta\cos\varphi \\ X^{1'} &= X\sin\varepsilon\cos\theta \qquad . \\ X^{0'} &= X\cos\varepsilon\cos i\psi \\ X^{4'} &= X\cos\varepsilon\sin i\psi \end{aligned} \qquad \text{(VII.20.12)}$$

Therein the X are the components of the radius vector to a point of the sphere. The value of this vector is computed with

$$X_{a'}X^{a'} = X^2 \quad . \qquad \text{(VII.20.13)}$$

Deforming the pseudo-hyper sphere to a double-surface which permits an interpretation of the interior Kerr solution, the vector X proceeds from the evolute of the radial curve of the surface and points to the elliptical cap of the surface already-discussed. From the rotation of $X^{a'}$ results the before-mentioned double-surface which we also describe with

$$\begin{aligned}
X^{3'} &= \bar{\mathcal{R}}^{3'} - \mathcal{R}^{3'} = \bar{\mathcal{R}} \sin\eta_g \sin\theta \sin\varphi - \mathcal{R} \sin\eta \sin\theta \sin\varphi \\
X^{2'} &= \bar{\mathcal{R}}^{2'} - \mathcal{R}^{2'} = \bar{\mathcal{R}} \sin\eta_g \sin\theta \cos\varphi - \mathcal{R} \sin\eta \sin\theta \cos\varphi \\
X^{1'} &= \bar{\mathcal{R}}^{1'} - \mathcal{R}^{1'} = \bar{\mathcal{R}} \sin\eta_g \cos\theta \qquad - \mathcal{R} \sin\eta \cos\theta \qquad . \\
X^{0'} &= \bar{\mathcal{R}}^{0'} - \mathcal{R}^{0'} = \bar{\mathcal{R}} \cos\eta_g \cos i\psi \qquad - \mathcal{R} \cos\eta \cos i\psi \\
X^{4'} &= \bar{\mathcal{R}}^{4'} - \mathcal{R}^{4'} = \bar{\mathcal{R}} \cos\eta_g \sin i\psi \qquad - \mathcal{R} \cos\eta \sin i\psi
\end{aligned} \tag{VII.20.14}$$

The line element of the 5-dimensional embedding space in pseudo-polar co-ordinates has the form

$$ds^2 = dX^2 + X^2 d\eta^2 + X^2 \sin^2\eta d\theta^2 + X^2 \sin^2\eta \sin^2\theta d\varphi^2 + X^2 \cos^2\eta di\psi^2 \ , \tag{VII.20.15}$$

which is reduced for X = const. to the line element of a pseudo-hyper sphere. Then it is likewise the line element of the double-surface. A dimensional reduction cuts off everything which is not needed for the 4-dimensional representation. The well-known formulae

$$\sin\theta = \frac{r}{\Lambda}\sin\vartheta, \quad \cos\theta = \frac{A}{\Lambda}\cos\vartheta \tag{VII.20.16}$$

ensure again the usual picture of the metric.

In a similar way as we have done with the interior Schwarzschild solution we first refer to the right column of (VII.20.14) to which the line element

$$ds^2 = d\mathcal{R}^2 + \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta d\theta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\theta d\varphi^2 + \mathcal{R}^2 \cos^2\eta di\psi^2 \tag{VII.20.17}$$

is assigned. For $\mathcal{R} = \text{const.}$ that is the metric of a part of the double-surface. From this relation we read the differentials

$$d\mathcal{R}^a = \left\{ d\mathcal{R}, \mathcal{R}d\eta, \mathcal{R}\sin\eta d\theta, \mathcal{R}\sin\eta\sin\theta d\varphi \, , \mathcal{R}\cos\eta di\psi \right\} \tag{VII.20.18}$$

in pseudo-polar co-ordinates.

The connexion coefficients can be determined from the assigned 5-bein which we now designate with $\mathcal{R}_{db}{}^{c}$, in order to refer to the part of the surface in consideration. Here some of the quantities of the pseudo-hyper sphere are to be used, known from some previous Sections, whereby X is to be replaced by $\mathcal{R}$.

For the projectors we write

$$\begin{gathered}
\mathfrak{p}_0^0 = \alpha_R, \quad \mathfrak{p}_1^1 = \alpha_R \\
\mathfrak{p}_0^2 = -\frac{\mathcal{R}}{\rho_H}\sin^2\eta, \quad \mathfrak{p}_1^2 = -\frac{\mathcal{R}}{\rho_H}\sin\eta\cos\eta, \quad \mathfrak{p}_2^2 = \frac{\mathcal{R}}{\rho_E}\sin\eta \ . \\
\mathfrak{p}_3^3 = \frac{\mathcal{R}}{\rho_E} a_R^2 \sin\eta, \quad \mathfrak{p}_4^4 = \frac{\cos\eta}{\left(1 + 2\Phi_g^2\right)\cos\eta_g - \cos\eta} \frac{1}{a_R}
\end{gathered} \tag{VII.20.19}$$

The last projector is somewhat voluminous, but the expression can be improved. If one places a_T by using (VII.20.8), one has

$$\mathfrak{p}^4_4 = \frac{\cos\eta}{2a_T\Phi_g^2}\frac{1}{a_R} .$$

If one expands the fraction with $\mathcal{R}$ and if one also substitutes (VII.20.3), one obtains

$$\mathfrak{p}^4_4 = \frac{\mathcal{R}\cos\eta}{\rho_S^g a_T} . \qquad \text{(VII.20.20)}$$

If one refers to the boundary surface one recognizes with (VII.20.10) and with

$$\left(\mathfrak{p}^4_4\right)_g = \frac{\mathcal{R}}{\rho_S^g}$$

the formal conformity with the correspondent projector of the exterior geometry.

With

$$\partial_a = \mathfrak{p}^b_a \frac{\partial}{\partial\mathcal{R}^b}, \quad d\mathcal{R}^b = \mathfrak{p}^b_a dx^a, \quad Y_{ab}{}^c = \mathfrak{p}^d_a \mathcal{R}_{db}{}^c \qquad \text{(VII.20.21)}$$

the fundamental quantities of the two geometries can be converted into one another. Coevally the physical content is extracted from (VII.20.14) and the dimensional reduction is arranged. From the line element (VII.19.1) we get hold of the expressions for the anholonomic differentials. From the first relation of (VII.20.21) one puts

$$d\mathcal{R}^0 = \mathfrak{p}^0_0 dx^0 = \alpha_R \cdot a_R d\mathcal{R} = d\mathcal{R}$$

and thereby one has deduced the differential of the local extra dimension $dx^0 = a_R d\mathcal{R}$. Moreover,

$$\partial_0 = \frac{\partial}{a_R \partial\mathcal{R}} \qquad \text{(VII.20.22)}$$

is in agreement with the first formula (VII.20.21).

From the second relation of (VII.20.21) one obtains with (45.8)

$$d\mathcal{R}^1 = \mathfrak{p}^1_1 dx^1 = \alpha_R \cdot \alpha_I a_R dr = \frac{1}{\cos\eta} dr = \mathcal{R} d\eta .$$

Further, we transfer a well-known relation of the exterior solution to the interior

$$d\theta = -\frac{1}{\rho_H}\left(\sin\eta\, dx^0 + \cos\eta\, dx^1\right) + \frac{1}{\rho_E} dx^2 \qquad \text{(VII.20.23)}$$

or we deduce it again from (VII.20.16). Thus,

$$d\mathcal{R}^2 = \mathfrak{p}^2_0 dx^0 + \mathfrak{p}^2_1 dx^1 + \mathfrak{p}^2_2 dx^2 = -\frac{\mathcal{R}}{\rho_H}\sin^2\eta\, dx^0 - \frac{\mathcal{R}}{\rho_H}\sin\eta\cos\eta\, dx^1 + \frac{\mathcal{R}}{\rho_E}\sin\eta\, dx^2 = \mathcal{R}\sin\eta\, d\theta$$

is verified. Likewise

$$d\mathcal{R}^3 = \mathfrak{p}^3_3 dx^3 = \frac{\mathcal{R}}{\rho_E} a_R^2 \sin\eta \cdot A\sin\vartheta\, d\varphi = \mathcal{R}\frac{Ar}{\Lambda^3}\cdot\frac{\Lambda^2}{A^2}\cdot\frac{A}{r}\sin\theta\, d\varphi = \mathcal{R}\sin\eta\sin\theta\, d\varphi .$$

With (VII.20.20) one gets

$$d\mathcal{R}^4 = \mathfrak{p}^4_4 dx^4 = \frac{\mathcal{R}\cos\eta}{\rho_S^g a_T} a_T dit = \mathcal{R}\cos\eta\, di\psi ,$$

whereby lastly (VII.20.7) was used. Thus, not only the ansatz (VII.19.1) for the auxiliary metric has been justified, but also all tools have been made available, in order to examine more exactly the geometry which is based on the metric (VII.19.1).

VII.21. Kerr interior solution, field quantities and field equations

In the preceding Section we have described how one unfolds the pseudo spherical geometry into an elliptical geometry by means of projectors. For the computation of the field quantities one harks back to the pseudo-hyper sphere and one replaces the X by the $\mathbb{R}$, and applies it to the projectors by means of (VII.20.21). The projectors (VII.20.19) attract attention in the way that some of them contain the elliptical factors a_R or α_R, respectively. On the one hand they provide the elliptical form of the surface, on the other hand they guarantee the nonholonomicity of the embedding. Facing the interior solution the tangent planes of the surface are rotated in such a way that the rigging vector of the hyper planes obtained in this way always enclose the same angle with those cutting planes on their way around the symmetry axis by moving along a horizontal ellipse.

From the last relation of (VII.20.21) we deduce the 5-dimensional connexion coefficients Y, from which we will obtain the 4-dimensional connection coefficients A in the last step of the dimensional reduction. Just as we have done with the exterior geometry we decompose the connexion coefficients into

$$Y_{ab}{}^{c} = M_{ab}{}^{c} + \tilde{N}_{ab}{}^{c} + B_{ab}{}^{c} + C_{ab}{}^{c} + E_{ab}{}^{c} \qquad \text{(VII.21.1)}$$

with the components

$$\begin{aligned} M_{ab}{}^{c} &= m_a M_b m^c - m_a m_b M^c \\ \tilde{N}_{ab}{}^{c} &= \tilde{m}_a N_b \tilde{m}^c - \tilde{m}_a \tilde{m}_b N^c \\ B_{ab}{}^{c} &= b_a B_b b^c - b_a b_b B^c \qquad , \\ C_{ab}{}^{c} &= c_a C_b c^c - c_a c_b C^c \\ E_{ab}{}^{c} &= -\left[u_a E_b u^c - u_a u_b E^c\right] \end{aligned} \qquad \text{(VII.21.2)}$$

which have the form

$$\begin{gathered} M_b = \left\{\frac{1}{a_R \mathbb{R}},0,0,0,0\right\}, \quad N_b = \left\{0,0,\frac{1}{\rho_H},0,0\right\}, \quad B_b = \left\{\frac{v_I}{\rho_E},\frac{a_I}{\rho_E},0,0,0\right\} \\ C_b = \left\{v_I\frac{r}{\Lambda A},a_I\frac{r}{\Lambda A},\frac{1}{\Lambda}\cot\vartheta,0,0\right\}, \quad E_b = \left\{-\frac{1}{\rho_S^g}\frac{a_I}{a_T},\frac{1}{\rho_S^g}\frac{v_I}{a_T},0,0,0\right\} \end{gathered} . \qquad \text{(VII.21.3)}$$

In (VII.21.2) the

$$\tilde{m}_a = \{\sin\eta,\cos\eta,0,0,0,\} = \{v_I,a_I,0,0,0,\} \qquad \text{(VII.21.4)}$$

are the unit vectors of the radial direction in the elliptical horizontals, while the

$$m_a = \{0,1,0,0,0,\} \qquad \text{(VII.21.5)}$$

are the unit vectors in the anholonomic hyper planes. With (VII.20.9), (VII.20.10), and the analogous relations $v_I^g = v_S^g$ one also recognizes that the field strengths of the interior and the exterior solutions coincide on the boundary surface, except the quantities M_0. These quantities describe quite different curvatures of the two surfaces.

The 5-dimensional Riemann tensor of the flat embedding space

$$R_{abc}{}^{d}(\mathbb{R}) = 2\left[\mathbb{R}_{[b\cdot c\cdot}{}^{d}{}_{,a]} + \mathbb{R}_{[b\cdot c\cdot}{}^{f}\,\mathbb{R}_{a]f}{}^{d} + \mathbb{R}_{[ba]}{}^{f}\mathbb{R}_{fc}{}^{d}\right] \equiv 0, \quad \Phi_{,a} = \frac{\partial\Phi}{\partial X^a} \qquad \text{(VII.21.6)}$$

is identically zero, likewise the Riemann tensor

$$\mathcal{P}_a^g \mathcal{P}_b^h R_{ghc}{}^d(\mathcal{R}) = R_{abc}{}^d(Y)$$
$$R_{abc}{}^d(Y) = 2\left[Y_{[b\cdot c\cdot}{}^d{}_{|a]} + Y_{[b\cdot c\cdot}{}^f Y_{a]f}{}^d + Y_{[ba]}{}^f Y_{fc}{}^d + Y_{gc}{}^d \left(\mathcal{P}^{-1}\right)_f^g \mathcal{P}^f_{[a|||b]} \right] \equiv 0 \qquad \text{(VII.21.7)}$$

of the unfolded geometry. If one establishes the Ricci relation by contraction of this formula and if one decomposes this relation in accordance with (VII.21.1) and (VII.21.2), one obtains the 5-dimensional field equations for the quantities (VII.21.3). This somewhat complex procedure is carried out entirely similarly to that of the exterior geometry. There we have described this problem in detail and we will not repeat it here. We perform the dimensional reduction at once and we deal with

$$A_{mn}{}^s = B_{mn}{}^s + N_{mn}{}^s + C_{mn}{}^s + E_{mn}{}^s \ . \qquad \text{(VII.21.8)}$$

We insert the 4-dimensional connexion coefficient and the 4-dimensional field strengths

$$B_{mn}{}^s = b_m B_n b^s - b_m b_n B^s, \quad B_n = \left\{ \frac{a_I}{\rho_E}, 0, 0, 0 \right\}$$
$$N_{mn}{}^s = m_m N_n m^s - m_m m_n N^s, \quad N_n = \left\{ 0, \frac{1}{\rho_H}, 0, 0 \right\}$$
$$C_{mn}{}^s = c_m C_n c^s - c_m c_n C^s, \quad C_n = \left\{ a_I \frac{r}{A\Lambda}, \frac{1}{\Lambda} \cot\vartheta, 0, 0 \right\} \qquad \text{(VII.21.9)}$$
$$E_{mn}{}^s = -\left[u_m E_n u^s - u_m u_n E^s \right], \quad E_n = \left\{ \frac{1}{\rho_S^g} \frac{v_I}{a_T}, 0, 0, 0 \right\}$$

into the 4-dimensional Ricci

$$R_{mn}(A) = A_{mn}{}^s{}_{|s} - A_{n|m} - A_{rm}{}^s A_{sn}{}^r + A_{mn}{}^s A_s \qquad \text{(VII.21.10)}$$

and we obtain

$$\begin{aligned}
&-\left[B_{n\underset{2}{||}m} - B_{n\underset{2}{||}s} b^s b_m + B_n B_m \right] - b_n b_m \left[B^s{}_{\underset{2}{||}s} + B^s B_s \right] \\
&-\left[N_{n\underset{2}{||}m} - N_{n\underset{2}{||}s} m^s m_m + N_n N_m \right] - m_n m_m \left[N^s{}_{\underset{2}{||}s} + N^s N_s \right] \\
&-\left[C_{n\underset{3}{||}m} + C_n C_m \right] - c_n c_m \left[C^s{}_{\underset{3}{||}s} + C^s C_s \right] \\
&+\left[E_{n\underset{4}{||}m} - E_n E_m \right] + u_n u_m \left[E^s{}_{\underset{4}{||}s} - E^s E_s \right] = -\kappa \left(T_{mn} - \frac{1}{2} g_{mn} T \right)
\end{aligned} \ . \qquad \text{(VII.21.11)}$$

As expected, the field equations of the auxiliary metric are not vacuum field equations. The right side of (VII.21.11) is obtained by relocation of the surplus terms which one gets by the conversion of R(Y) into R(A). It mostly contains the generalized second fundamental forms which can be inferred from (VII.21.3)

$$\begin{aligned}\kappa\left(T_{mn}-\frac{1}{2}g_{mn}T\right) = & -m_m m_n\left(M_0B_0+M_0C_0-M_0E_0\right)\\ & -b_m b_n\left(B_0M_0+B_0C_0-B_0E_0\right)\\ & -c_m c_n\left(C_0M_0+C_0B_0-C_0E_0\right)\\ & +u_m u_n\left(E_0M_0+E_0B_0+E_0C_0\right)\\ & -\left(m_m m_n+b_m b_n\right)\tilde{\Omega}^{s3}\tilde{\Omega}_{3s}\sin^2\eta\\ & +\left(m_m m_n+c_m c_n\right)N_sC^s\sin^2\eta+2N_{(m}E_{n)}\\ & +\left(m_m m_n+u_m u_n\right)E_sF^s\end{aligned} \quad . \qquad \text{(VII.21.12)}$$

The quantity

$$\tilde{\Omega}^{s3}\tilde{\Omega}_{3s}=\tilde{H}_{13}\tilde{H}_{13}-\tilde{H}_{23}\tilde{H}_{23},\quad \tilde{H}_{s3}=i\alpha_R^2\omega\left\{\frac{r}{\Lambda}\sin\vartheta,\frac{A}{\Lambda}\cos\vartheta,0,0\right\} \qquad \text{(VII.21.13)}$$

is made up similarly to the corresponding quantity of the exterior geometry. By contracting the Ricci one finally obtains component by component the stress-energy tensor of the model by applying the Einstein tensor

$$\begin{aligned}\kappa T_{11} &= B_0C_0-B_0E_0-C_0E_0\\ \kappa T_{12} &= E_1N_2\\ \kappa T_{22} &= M_0C_0-M_0E_0-C_0E_0-F_1E_1-N_2C_2\sin^2\eta\\ \kappa T_{33} &= M_0B_0-M_0E_0-C_0E_0+\tilde{\Omega}^{s3}\tilde{\Omega}_{3s}\sin^2\eta\\ \kappa T_{44} &= M_0B_0+M_0C_0+B_0E_0+\tilde{\Omega}^{s3}\tilde{\Omega}_{3s}\sin^2\eta-N_2C_2\sin^2\eta\end{aligned} \quad . \qquad \text{(VII.21.14)}$$

This tensor contains the pressures which are different in all three directions, and the energy density, and in addition the stresses T_{12} which result from the elliptical structure of the source. The stress-energy tensor is covariantly conserved. Although some authors have tried to assume a static object as a source of the Kerr field we underline that in this case the dragging effects caused by the rotation can hardly be explained.

Therefore we perform the intrinsic transformation

$$\begin{aligned}&\Lambda_3^{3'}=\alpha_R,\quad \Lambda_4^{3'}=i\alpha_R\omega,\quad \Lambda_3^{4'}=-i\alpha_R\omega\sigma^2,\quad \Lambda_4^{4'}=\alpha_R\\ &\Lambda_{3'}^{3}=\alpha_R,\quad \Lambda_{4'}^{3}=-i\alpha_R\omega,\quad \Lambda_{3'}^{4}=i\alpha_R\omega\sigma^2,\quad \Lambda_{4'}^{4}=\alpha_R\end{aligned} \quad , \qquad \text{(VII.21.15)}$$

which transforms the static auxiliary metric into the stationary

$$ds^2={dx^1}^2+{dx^2}^2+\left[\alpha_R dx^3+i\alpha_R\omega\sigma dx^4\right]^2+a_T^2\left[-i\alpha_R\omega\sigma dx^3+\alpha_R dx^4\right]^2 \quad . \qquad \text{(VII.21.16)}$$

The above transformation applied to the 4-Beinfeld

$$\underset{m}{e}_i=\Lambda_i^{i'}\underset{m}{e}_{i'},\quad \underset{m}{e}^i=\Lambda_{i'}^{i}\underset{m}{e}^{i'},\quad g_{ik}=\Lambda_{ik}^{i'k'}g_{i'k'}\ , \qquad \text{(VII.21.17)}$$

produces an additional term applied to the connexion coefficients

$$A_{mn}{}^s\to A_{mn}{}^s+G_{mn}{}^s\ . \qquad \text{(VII.21.18)}$$

This additional term can be computed with

$$G_{mns}=\left[g_{sr}\underset{m}{e}^{k'}\underset{n}{e}^{l'}\overset{r}{e}_{j'}+g_{nr}\underset{s}{e}^{k'}\underset{m}{e}^{l'}\overset{r}{e}_{j'}+g_{mr}\underset{s}{e}^{k'}\underset{n}{e}^{l'}\overset{r}{e}_{j'}\right]\Lambda_j^{j'}\Lambda_{[l'|k']}^{j} \qquad \text{(VII.21.19)}$$

which leads to

$$G_{mn}{}^{s} = c_m\left[F_n c^s - c_n F^s\right] - u_m\left[F_n u^s - u_n F^s\right] + H_{mn}{}^{s}$$
$$H_{mns} = -\Omega^T_{nm}u_s + \Omega^T_{sm}u_n + \Omega^T_{sn}u_m + \alpha_T D_{ns}u_m \quad . \qquad \text{(VII.21.20)}$$

The quantity F is already well-known as centrifugal force, the rotational item H has a structure similar to the appropriate expression of the exterior solution

$$\Omega^T_{mn} = -\left[H^T_{nm} + D^T_{nm}\right],\quad H^T_{nm} = a_T\left[H_{nm} + D_{[nm]}\right],\quad D^T_{nm} = \alpha_T D_{(nm)}$$
$$H_{nm} = 2i\alpha_R^2\omega\,\sigma_{|[n}c_{m]},\quad D_{nm} = i\alpha_R^2\omega_{\,|n}\sigma c_m \quad . \qquad \text{(VII.21.21)}$$

The terms with the centrifugal forces in (VII.21.20) are allocated to the corresponding quantities

$$C_m \to C^T_m = C_m + F_m,\quad E_m \to E^T_m = E_m + F_m \ . \qquad \text{(VII.21.22)}$$

Thus, one obtains the Ricci of the rotating system

$$\begin{aligned} R_{mn} = &-\left[N_{n\underset{2}{||}m} - N_{n\underset{2}{||}s}m^s m_m + N_n N_m\right] - m_n m_m\left[N^s{}_{\underset{2}{||}s} + N^s N_s\right] \\ &-\left[B_{n\underset{2}{||}m} - B_{n\underset{2}{||}s}b^s b_m + B_n B_m\right] - b_n b_m\left[B^s{}_{\underset{2}{||}s} + B^s B_s\right] \\ &-\left[C^T_{n\underset{3}{||}m} + C^T_n C^T_m\right] - c_n c_m\left[C^s_{T\underset{3}{||}s} + C^s_T C^T_s - \Omega^{rs}_T\Omega^T_{sr}\right] \\ &+\left[E^T_{n\underset{4}{||}m} - E^T_n E^T_m\right] + u_n u_m\left[E^s_{T\underset{4}{||}s} - E^s_T E^T_s - \Omega^{rs}_T\Omega^T_{sr}\right] \\ &+ u_m\left[\Omega^s_{Tn\underset{4}{||}s} + 2\Omega^T_{[ns]}E^s_T\right] + u_n\left[\Omega^s_{Tm\underset{4}{||}s} + 2\Omega^T_{[ms]}E^s_T\right] - 2\Omega^T_{m3}\Omega^T_{n3} \end{aligned} \qquad \text{(VII.21.23)}$$

and by means of the Einstein tensor the stress-energy tensor

$$\begin{aligned} \kappa T_{11} &= -M_0B_0 + 2B_0C_0 - 2B_0E_0 - \left[\Omega^{\alpha 3}_C\Omega^C_{3\alpha} - \Omega^{\alpha 3}_T\Omega^T_{3\alpha}\right] \\ \kappa T_{22} &= -M_0E_0 + M_0B_0 + 2M_0C_0 - 2B_0C_0 - C_0E_0 + 2F_0E_0 + \left[\Omega^{\alpha 3}_C\Omega^C_{3\alpha} - \Omega^{\alpha 3}_T\Omega^T_{3\alpha}\right] \\ \kappa T_{33} &= -M_0E_0 + M_0B_0 + 2M_0C_0 - 2B_0C_0 - C_0E_0 + 2F_0E_0 \\ &\quad + \left[\Omega^{\alpha 3}_C\Omega^C_{\alpha 3} - \Omega^{\alpha 3}_T\Omega^T_{\alpha 3}\right] + \left[\Omega^{\alpha\beta}_C\Omega^C_{\beta\alpha} - \Omega^{\alpha\beta}_T\Omega^T_{\beta\alpha}\right] \\ \kappa T_{44} &= 2B_0C_0 + M_0B_0 + \left[\Omega^{\alpha 3}_C\Omega^C_{\alpha 3} - \Omega^{\alpha 3}_T\Omega^T_{\alpha 3}\right] - \left[\Omega^{\alpha\beta}_C\Omega^C_{\beta\alpha} - \Omega^{\alpha\beta}_T\Omega^T_{\beta\alpha}\right] \\ \kappa T_{34} &= \Omega^T_{03}M_0 + \Omega^T_{03}E_0 - \left[1 - \frac{a_T^2}{a_I^2}\right]\Omega^T_{13}B_1 - \left[1 - \frac{a_I^2}{a_T^2}\right]\Omega^T_{23}N_2 \end{aligned} \quad , \qquad \text{(VII.21.24)}$$

in which the auxiliary variables

$$F_0 = \alpha_R^2\omega^2\sigma\sigma_{|0},\quad \Omega^T_{03} = i\alpha_T\alpha_R^2\omega\sigma_{|0},\quad \sigma_{|0} = \frac{r}{\Lambda}\sin\vartheta\sin\eta \qquad \text{(VII.21.25)}$$

and the quantities similar to the exterior solution

$$\Omega^C_{nm} = -\left[H^C_{mn} + D^C_{mn}\right],\quad H^C_{mn} = a_I\left[H_{mn} + D_{[mn]}\right],\quad D^C_{mn} = \alpha_I D_{(mn)} \qquad \text{(VII.21.26)}$$

are used. In addition, the Maxwellian relations

$$F_{[m||n]} - 2\Omega^s_{T[m}\Omega^T_{n]s} = 0,\quad E_{[m||n]} = 0,\quad \Omega^T_{[mn\,||s]} + \Omega^T_{[mn}E^T_{s]} = 0 \qquad \text{(VII.21.27)}$$

are satisfied.

The pressure has different values in the three different directions inside the source of the Kerr model. The pressure is anisotropic and the source cannot consist of a perfect fluid. The matter flows with different speeds into the φ-direction in dependence of the distance from the symmetry axis. A computer analysis of the rather complex stress-energy tensor could explain more of its properties. On the boundary surface all brackets vanish in (VII.21.24), likewise the hydrostatic pressure T_{11}. The vanishing of the latter is a basic condition for the stability of the object.

Current and energy have jumps on the boundary surface

$$\kappa T^{\,g}_{34} = \Omega^{Tg}_{03}\left(M^{g}_{0}+E^{\,g}_{0}\right), \quad \kappa T^{\,g}_{44} = B^{g}_{0}\left(M^{g}_{0}+E^{\,g}_{0}\right) , \qquad \text{(VII.21.28)}$$

which are attributed to the different curvatures of the exterior and the interior surfaces. Since $M^{g}_{0}+E^{g}_{0}$ and B^{g}_{0} are positive the energy on the boundary surface is positive as well. The direction of the current T_{34} depends on the direction of the rotation, i.e. on the sign of the rotational parameter a. The progression of the field quantities is well-behaved, except in the positions r = 0, A = a, where the BL ellipses degenerate to a straight line which is limited by the common foci of the BL ellipses. The curvature vectors of the ellipses are infinitely large at these locations and the assigned field strengths are zero. Only for $\vartheta = \pi/2$ at the vertices of the ellipses ρ_E vanishes and the field strengths become infinitely large. For the complete surface this straight line is to be rotated through φ. One obtains a disk with a singular rim, the well-known Kerr singularity. If one puts the rotation parameter a = 0, one obtains the Schwarzschild geometry as a special case of the Kerr geometry. Then the Kerr ring singularity is reduced to the Schwarzschild point singularity at r = 0. It is a property of field theories that the field strengths obey the law $1/r^n$ and possess a singularity at r = 0. The occurrence of singularities inside a source diminishes substantially their physical reliability. Unless gravitation physics does not solve the singularity problem, no improvement of the interior solutions is in sight.

Finally, we add some quotations concerning the Kerr geometry.

Abdelqader [A] and Lake; Alfaro [A], Ali [A] and Ahsan; Anderson [A] and Lemos; Barrabes [B], Bergamini [B] and Viaggiu; Beig [B] and Simon; Bicak [B], Bini [B] , Geralico and Jantzen; Bini [B], de Felice and Gerallico; Bini [B], Geralico and Jantzen; Blandford [B] and Znajek; Böhmer [B] and Hogan; Boyer [B] and Lindquist; Burinskii [B], Burinskii [B] and Kerr; Calvani [C], and Nobili; Carter [C], Chandrasekhar [C], Chellathurai [C] and Dadhich; Chicone [C] and Mashhoon; Clement [C], Cohen [C], Collas [C], Collas [C] and Klein; Comins [C] and Schutz; Cox [C], Cox [C] and Flaherty; Curir [C], Dale [D], De Felice [D], De Felice [D] and Bradley; De Felice [D] and Usseglio-Tomasset; De Felice [D] and Calvani; Debney [D], Kerr and Schild; De La Cruz [D] and Israel; Deser [D], Doran [D], Esteban [E], Peréz, and Díaz; Fackerell [F] and Ipser; Fishbone [F] and Moncrief; Fishbone [F], Frolov [F] and Frolov; Ford [F], Fuchs [F], García-Compeán [G] and Manko; Gergely [L] and Perjes; Glass [G] and Krisch; Glatzmaier [G], Evonuk and Rogers; Godfrey [G], Gürses [G] and Gürsey; Güven [G], Gurtug [G] and Halilsoy;Hansen [H] and Winicour; Herrera [H], Carot and Di Prisco; Herrera [H] and Jiménez; Hayword [H], Heinicke [H] and Hehl; Henry [H], Herdeiro [H] and Rebelo; Hogan [H], Howes [H], Hughston [H], Hugh [H], Choptuik and Matzner; Iyer [I] and Kumar; Iyer [I] and Prasanna; Kerr [K], Kerr [K] and Wilson;

Kihara [K] and Tomimatsu have found closed time-like curves on the symmetry axis of a metric which represents two superposed Kerr lines.

Kim [K], Klotz [K], Kovar [K] and Stuchlik; Kolowski [K], Jaroszynski and Abramowicz; Kramer [K] and Neugebauer; Kramer [K], Krasiński [K], Krasiński [K], Kerr, and Verdaguer; Krori [K], Bhattacharjee and Chaudhury; Kulkarni [K], Chellathurai and Dadhich; Kyrianopoulos [K], Leaute [L], Leaute [L] and Linet; Manko [M], Manko [M] and García-Compeán; Manko [M] and Ruiz; Mark [M], Marsh [M], Martellini [M] and Treves; Martinez [M], Mazur [M], McKellar [M], Stephenson and Thomson; Meinel [M], Miranda [M] et al.; Mishra [M] and Mishra; Mishra [M] and Chakraborty; Moss [M] and Davis; Murenbeld [M] and Trollope; Newman [N] and Winicour; Newman [N] and Janis; Patel [P], Patel [P] and Trivedi; Perjes [P], Prasanna [P] and Chakraborty; Prasanna [P] and

Chakrabarti; Pretorius [P] and Israel; Punsly [P], Ramaswamy [R], Rowan [R], Schiffer [S], Adler, Mark and Scheffield; Scholtz [S], Flandera, and Gürlebeck; Schutz [S] and Comins; Semenov [S] and Dyadechkin; Semerák [S], Semerák [S] and Bičák; Simón [S].

Singh [S] and Upadhyay have examined Newton's analogue of a combined Kerr-NUT-metric.

Stoica [S], Stuchlik [S], Slany and Abramowicz; Stuchlik [S] and Hledik; Taub [T], Teukolsky [T], Thakurta [T], Tsoubelis [T], Economou and Stoghianidis; Quevedo [Q] and Mashhoon; Unruh [U], Vaidya [V] and Patel; Vaidya [V], Venter [V] and Bishop; Visser [V], Wilson [W], Wiltshire [W], Yamazaki [Y], Zakharov [Z], Znajek [Z].

VIII. The Kerr family

VIII.1. Kerr - Newman metric, the seed metric

Newman [N] et al. have generalized the charged Reissner-Nordström metric to a rotating system. The new solution is closely related to the Kerr model. The latter can be obtained from the KN-model by putting zero the new parameter e (the electrical charge of the central mass). Some work has been invested by several authors in order to explain the KN-metric as metric of a charged black hole. However, little work has been done in order to disclose the structures of the model. Here the properties of the field of a charged rotating mass will be examined. Subsequently, the embedding will be accomplished and also an interior solution will be prepared by us [B].

The KN-metric has the same structure as the Kerr metric

$$ds^2 = {dx^1}^2 + {dx^2}^2 + \left[\alpha_R dx^3 + i\alpha_R \omega\sigma dx^4\right]^2 + a_S^2\left[-i\alpha_R\omega\sigma dx^3 + \alpha_R dx^4\right]^2 , \quad \text{(VIII.1.1)}$$

however, the metrical factors are defined in a somewhat different way

$$A^2 = r^2 + a^2, \quad \Lambda^2 = r^2 + a^2\cos^2\vartheta, \quad \delta^2 = r^2 + a^2 - 2Mr + e^2$$
$$a_S = 1/\alpha_S = \frac{\delta}{A}, \quad a_R = 1/\alpha_R = \frac{\Lambda}{A} \quad . \qquad \text{(VIII.1.2)}$$

With an intrinsic transformation the KN-metric is reduced to the seed metric

$$ds^2 = \alpha_S^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 + a_S^2 {dx^4}^2, \quad dx^4 = idt\,. \qquad \text{(VIII.1.3)}$$

It is the metric of a static system to which a double surface is assigned by means of an embedding into a 5-dimensional flat space.

M, a, and e are the mass parameter, the rotational parameter, and the electrical charge. For a = 0 one obtains the Reissner-Nordström model. If one defines

$$a_S = \cos\varepsilon\ , \qquad \text{(VIII.1.4)}$$

one obtains the angle of ascent ε of the radial curves of the surface. From the formula

$$v_S^2 + a_S^2 = \sin^2\varepsilon + \cos^2\varepsilon = 1 \qquad \text{(VIII.1.5)}$$

one gets hold of the velocity

$$v_S = \sin\varepsilon = -\sqrt{\frac{2M}{r} - \frac{e^2}{r^2}} \qquad \text{(VIII.1.6)}$$

of a freely falling observer (without dragging effects) and thus, the ascent

$$\tan\varepsilon = -\frac{1}{\sqrt{\dfrac{r^2 + a^2}{2Mr - e^2} - 1}} \qquad \text{(VIII.1.7)}$$

of the radial curves of a surface which can be deduced by embedding. For the extra dimension $x^{0'}$ of the embedding space we have chosen

$$dx^{0'}_{hol} = -\tan\varepsilon dr\ . \qquad \text{(VIII.1.8)}$$

The points on the surface above the base plane which is parameterized by the elliptical BL-system can be obtained by numerical integration. The solution of the integral

$$x^{0'}(r_1) = -\int_{r_0}^{r_1} \tan\varepsilon\, dr, \quad r_0 = M + \sqrt{M^2 - a^2 - e^2} \tag{VIII.1.9}$$

has no closed form. The boundary of the geometry is situated at r_0 and this surface is called event horizon. For the spatial part of the embedded surface one gets the co-ordinates of the points from

$$x^{a'} = \left\{x^{0'}, r\cos\vartheta, A\sin\vartheta\cos\varphi, A\sin\vartheta\sin\varphi\right\} \ . \tag{VIII.1.10}$$

The metric to which this surface can be assigned is, however, not the KN-metric. Just as in the Kerr solution the normal vector of the surface oscillates cyclically up and down, if one travels on a parallel around the symmetry axis. This can be easily understood because the walls of the elliptically deformed surface are differently scarped. A *rigging vector* is defined as coinciding at the two minor axes of the ellipses with the normal vector and maintains its angle in relation to the base plane throughout a circulation. Anholonomic hyper planes are put up by it.

If the distance of two neighboring ellipses on the base plane are again labeled by

$$dx^{\underline{1}} = a_R dr$$

then the anholonomic differential can be defined by

$$dx^{0'}_{anh} = -\tan\varepsilon\ a_R(r,\vartheta)\, dr = -\tan\varepsilon\, dx^{\underline{1}} \tag{VIII.1.11}$$

in the same manner as we have done previously. The relation

$$ds^2 = dx^{0'2} + dx^{1'2} + dx^{2'2} + dx^{3'2}$$

leads us to the desired metric. The generalized curvature vector of this anholonomic geometry has as only nonvanishing component, i.e. the value

$$\rho_S = a_R A^3 \frac{\sqrt{2Mr - e^2}}{M\left(r^2 - a^2\right) - e^2 r} \ , \tag{VIII.1.12}$$

being the component in the local 0-direction. Thus, it is simpler to interpret

$$a_S dx^4 = a_S dit = \rho_S \cos\varepsilon\, di\psi \tag{VIII.1.13}$$

as the time-like part of the seed metric. One has to hark back to the theory of double-surfaces. We refer to earlier Sections.

VIII.2. Kerr - Newman metric, the rotating metric

Having performed an intrinsic transformation one obtains the KN-metric from the seed metric

$$ds^2 = dx^{1\,2} + dx^{2\,2} + \left[\alpha_R dx^3 + i\alpha_R \omega\sigma dx^4\right]^2 + a_S^2\left[-i\alpha_R\omega\sigma dx^3 + \alpha_R dx^4\right]^2 \quad \text{(VIII.2.1)}$$

with the anholonomic differentials

$$dx^1 = \alpha_S a_R dr, \quad dx^2 = \Lambda d\vartheta, \quad dx^3 = \sigma d\varphi, \quad dx^4 = \rho_S di\psi = idt \ . \quad \text{(VIII.2.2)}$$

From this we read the Carter bein

$$\begin{array}{llllll} \overset{1}{e}_1 = \alpha_S a_R, & \overset{2}{e}_2 = \Lambda, & \overset{3}{e}_3 = \alpha_R\sigma, & \overset{3}{e}_4 = i\alpha_R\omega\sigma, & \overset{4}{e}_3 = -ia_S\alpha_R\omega\sigma^2, & \overset{4}{e}_4 = a_S\alpha_R \\ \underset{1}{e}^1 = a_S\alpha_R, & \underset{2}{e}^2 = \dfrac{1}{\Lambda}, & \underset{3}{e}^3 = \dfrac{\alpha_R}{\sigma}, & \underset{3}{e}^4 = i\alpha_R\omega\sigma, & \underset{4}{e}^3 = -i\alpha_S\alpha_R\omega\,, & \underset{4}{e}^4 = \alpha_S\alpha_R \end{array} \quad \text{(VIII.2.3)}$$

and from it we compute the field quantities

$$B_m = \left\{\frac{a_S}{\rho_E},0,0,0\right\}, \quad N_m = \left\{0,\frac{1}{\rho_H},0,0\right\}, \quad C_m = \left\{\frac{1}{\sigma}\sigma_{|1},\frac{1}{\sigma}\sigma_{|2},0,0\right\} \ . \quad \text{(VIII.2.4)}$$

They differ from the ones of the Kerr geometry only by the gravitational factor. The force of gravity

$$E_m = \left\{\frac{1}{\rho_S}\tan\varepsilon,0,0,0\right\} \quad \text{(VIII.2.5)}$$

has two contributions

$$E_m = E_m^K + \mathcal{E}_m \ , \quad \text{(VIII.2.6)}$$

the first one

$$E_1^K = -\alpha_S\alpha_R \frac{M}{A^4}\left(r^2 - a^2\right) \quad \text{(VIII.2.7)}$$

corresponding to the force of gravity of the Kerr field. To it is added a repulsive force of shorter range which is quadratic in the charge

$$\mathcal{E}_1 = \alpha_S\alpha_R \frac{r}{A}\frac{e^2}{A^3} \ . \quad \text{(VIII.2.8)}$$

Apart from the centrifugal force

$$F_m = \alpha_R^2\omega^2\sigma\sigma_{|m} \quad \text{(VIII.2.9)}$$

the rotational forces come along in the well-known form

$$\begin{array}{c} \Omega_{nm}^C = -\left[H_{mn}^C + D_{mn}^C\right], \quad H_{mn}^C = 2\left[ia_S\alpha_R^2\omega\sigma_{|[m}c_{n]} + ia_S\alpha_R^2\sigma\omega_{|[m}c_{n]}\right] \\ D_{mn}^C = 2i\alpha_S\alpha_R^2\sigma\omega_{|(m}c_{n)} \end{array} \ . \quad \text{(VIII.2.10)}$$

If one processes all quantities mentioned above into the Einstein field equations, one obtains on the right side

$$\kappa T_{mn} = \begin{pmatrix} \mathcal{F}_C^{2} & & & \\ & -\mathcal{F}_C^{2} & & \\ & & -\mathcal{F}_C^{2} & \\ & & & \mathcal{F}_C^{2} \end{pmatrix} \qquad \text{(VIII.2.11)}$$

with the electric field strength[134]

$$\mathcal{F}^{C}_{m} = \left\{ \frac{e}{\Lambda^2}, 0, 0, 0 \right\} . \qquad \text{(VIII.2.12)}$$

Rescaling the charge

$$e^2 \rightarrow \frac{\kappa}{2} e^2$$

the way of writing

$$T_{mn} = \mathcal{F}_m^{\ \ s} \mathcal{F}_{ns} - \frac{1}{4} g_{mn} \mathcal{F}^{rs} \mathcal{F}_{rs}, \quad \mathcal{F}_{14} = \mathcal{F}^{C}_{1} \qquad \text{(VIII.2.13)}$$

is possible.

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[134] Misner, Thorne, and Wheeler write for the electromagnetic field tensor

$$\mathbf{F} = e\Lambda^{-4}\left(r^2 - a^2\cos^2\vartheta\right)\mathbf{d}r \wedge \left[\mathbf{d}t - a^2\sin^2\vartheta\,\mathbf{d}\varphi\right] + 2e\Lambda^{-4} a r \cos\vartheta \sin\vartheta\,\mathbf{d}\vartheta \wedge \left[\left(r^2 + a^2\right)\mathbf{d}\varphi - a\,\mathbf{d}t\right] .$$

Evidently, this expression contains the components of the electromagnetic field strengths with respect to the oblique-angled co-ordinate system of the Kerr geometry:

$$\begin{aligned} \mathcal{F}_{13} &= -\mathcal{F}^{C}_{1}\,\omega\sigma^2 \frac{r^2 - a^2\cos^2\vartheta}{\Lambda^2}, \quad & \mathcal{F}_{14} &= \mathcal{F}^{C}_{1} \frac{r^2 - a^2\cos^2\vartheta}{\Lambda^2} \\ \mathcal{F}_{23} &= \mathcal{F}^{C}_{1} \frac{2ar\cos\vartheta\, A^2 \sin\vartheta}{\Lambda^2}, \quad & \mathcal{F}_{24} &= -\mathcal{F}^{C}_{1} \frac{2ar\sin\vartheta\, a\sin\vartheta}{\Lambda^2} \end{aligned} .$$

Re-writing these quantities in an orthogonal reference system, say the Carter system, one obtains

$$\begin{aligned} \mathcal{F}_{13} &= 0, \quad & \mathcal{F}_{14} &= \mathcal{F}^{C}_{1} \frac{r^2 - a^2\cos^2\vartheta}{\Lambda^2} \\ \mathcal{F}_{23} &= \mathcal{F}^{C}_{1} \frac{2ra\cos\vartheta}{\Lambda^2}, \quad & \mathcal{F}_{24} &= 0 \end{aligned} .$$

The remaining two components, the magnetic field $\mathcal{F}_{23}$ and the electric field $\mathcal{F}_{14}$ are parallel and point into the radial direction. They have a magnetic and an electric monopole as their source. It seems that the Newman-Janis algorithm has not been properly translated into ordinary tensor formalism. Since

$$\Lambda^2 = r^2 + a^2\cos^2\vartheta = \left(r + ia\cos\vartheta\right)\left(r - ia\cos\vartheta\right),$$

we combine the two components $\mathcal{F}_{23}$ and $\mathcal{F}_{14}$ to

$$\left(\mathcal{F}_{14}\mathcal{F}_{14} - \mathcal{F}_{23}\mathcal{F}_{23}\right) = \left(\mathcal{F}_{14} + \mathcal{F}_{23}\right)\left(\mathcal{F}_{14} - \mathcal{F}_{23}\right) = \mathcal{F}^{C}_{1}\mathcal{F}^{C}_{1} .$$

Only one electric component $\mathcal{F}^{C}_{1}$ remains, pointing into the radial direction. Tiomno [T], who has used the components of the electromagnetic field tensor in the co-ordinate way of writing, has stated that the electric and magnetic parts are both radial as well. From the properties of the fields in the exterior he has constructed possible properties of these fields for a still unknown interior solution, and he has assumed a perfect conductor as the source.

The subequations of the Einstein field equations can be computed in such a manner as we have done in previous Sections. We note only the results for the [33-], [34]-, and [44]-components

$$C^{s}{}_{||s} + F^{s}{}_{||s} = \Omega^{rs}_{C}\Omega^{C}_{sr} - \mathcal{F}^{s}_{C}\mathcal{F}^{C}_{s}$$

$$\Omega^{ns}_{C\,\underset{4}{||}n} = 2\Omega^{[sn]}_{C}F_{s} \qquad . \tag{VIII.2.14}$$

$$E^{s}{}_{||s} + F^{s}{}_{||s} = \Omega^{rs}_{C}\Omega^{C}_{sr} - \mathcal{F}^{s}_{C}\mathcal{F}^{C}_{s}$$

On the right side are the field stresses, the field current, and the field energy density. In the Carter system no magnetic field strength is present. Obviously, the observers co-rotate differentially with the electric sources in this system.

From the covariant definition of the electric field tensor

$$\mathcal{F}_{mn} = 2\mathcal{F}^{C}_{[m}u_{n]} \tag{VIII.2.15}$$

and the covariant definition of the time derivative

$$\frac{d}{id\tau}\mathcal{F}^{C}_{m} = \mathcal{F}^{C}_{m||n}u^{n} \tag{VIII.2.16}$$

one obtains the Maxwell equations

$$\mathcal{F}^{\alpha}_{C\underset{4}{||}\alpha} = 0, \quad \frac{d}{id\tau}\mathcal{F}^{C}_{\alpha} = D^{C}_{\alpha\beta}\mathcal{F}^{\beta}_{C}, \quad \alpha,\beta = 1,2,3 \ . \tag{VIII.2.17}$$

The second set can be written as

$$\mathcal{F}_{<mn||r>} = 2\Omega^{C}_{<mn}\mathcal{F}^{C}_{r>}, \quad \mathcal{F}_{<\alpha\beta\underset{4}{||}\gamma>} \equiv 0, \quad \mathcal{F}^{C}_{[\alpha||\beta]} = \mathcal{F}^{C}_{[\alpha}F_{\beta]} \ . \tag{VIII.2.18}$$

While the second relation is trivial, the third relation shows that the electric field is coupled to a gravielectric interaction current. In the symbolic way of writing one has

$$\mathrm{rot}\,\vec{\mathcal{F}} = \vec{\mathcal{F}} \times \vec{F} \ .$$

With these equations one verifies also the conservation of the electric stress-energy tensor

$$T_{m}{}^{n}{}_{||n} = \mathcal{F}_{m}{}^{s}\mathcal{F}^{n}{}_{s||n} - \frac{1}{2}\mathcal{F}^{ns}\mathcal{F}_{<mn||s>} = 0 \ . \tag{VIII.2.19}$$

If one uses other systems which rotate relatively to the Carter system (C), magnetic field components in analogy to the theory of moving media in electrodynamics are expected. This we will show in the next Section.

VIII.3. The Kerr - Newman metric, moving systems

In addition to the Carter system the physics of the Kerr-Newman metric can be described by other preferred systems. Concerning the Kerr metric we have worked out all calculations, also in the system of Iyer and Kumar (A) and of Bardeen (B). The procedures can be extended to the Kerr-Newman metric. In these two systems physics presents itself substantially more complicated. Since magnetic fields arise, we expect further gravimagnetic interaction terms. The calculations for the two systems are similar for a while. One should recall the definitions of the relative velocities of the rotating observers

$$\omega_{AC}=\alpha_S\omega, \quad \omega_{BC}=a_S\omega \tag{VIII.3.1}$$

and the related relativistic expressions

$$\alpha_{AC}\omega_{AC}\sigma, \quad \alpha_{BC}\omega_{BC}\sigma, \quad \alpha_{AC}=1\Big/\sqrt{1-\omega_{AC}^2\sigma^2}, \quad \alpha_{BC}=1\Big/\sqrt{1-\omega_{BC}^2\sigma^2}\ . \tag{VIII.3.2}$$

Between the systems a Lorentz transformation

$$L^{3'}_{3}=\alpha_X, \quad L^{4'}_{3}=i\alpha_X\omega_X\sigma, \quad L^{3'}_{4}=-i\alpha_X\omega_X\sigma, \quad L^{4'}_{4}=\alpha_X, \quad X=AC \text{ or } X=BC \tag{VIII.3.3}$$

mediates which is acting on the field strengths of the system C and provides the expressions

$$\mathcal{H}^{A}_{13}=-i\alpha_{AC}\omega_{AC}\sigma\mathcal{F}^{C}_{1}, \quad \mathcal{F}^{A}_{1}=\alpha_{AC}\mathcal{F}^{C}_{1}, \quad \mathcal{H}^{B}_{13}=-i\alpha_{BC}\omega_{BC}\sigma\mathcal{F}^{C}_{1}, \quad \mathcal{F}^{B}_{1}=\alpha_{BC}\mathcal{F}^{C}_{1}\ . \tag{VIII.3.4}$$

Suppressing the markers A and B

$$\mathcal{F}_{mn}=\mathcal{H}_{mn}+2\,\mathcal{F}_{[m}u_{n]} \tag{VIII.3.5}$$

is valid. The magnetic field can be noted in the symbolic way of writing as

$$\vec{\mathcal{H}}=\alpha\left[\vec{v}\times\vec{\mathcal{F}}\right], \quad \vec{v}=\vec{\omega}\times\vec{\sigma}\ . \tag{VIII.3.6}$$

Rescaling the components of the stress-energy tensor, one has[135]

$$T_{11}=-T_{22}=\frac{1}{2}\left[\mathcal{F}_1\mathcal{F}_1+\mathcal{H}_{13}\mathcal{H}_{13}\right], \quad T_{44}=-T_{33}=\frac{1}{2}\left[\mathcal{F}_1\mathcal{F}_1-\mathcal{H}_{13}\mathcal{H}_{13}\right], \quad T_{34}=\mathcal{F}_1\mathcal{H}_{13}\ , \tag{VIII.3.7}$$

or written covariantly

$$T_{mn}=\mathcal{F}_m{}^s\mathcal{F}_{ns}-\frac{1}{4}g_{mn}\mathcal{F}^{rs}\mathcal{F}_{rs}\ . \tag{VIII.3.8}$$

The Maxwell equations of the system A are

$$\begin{aligned}&\mathcal{F}^{m}{}_{A\underset{4}{\|}m}=\Omega^{mn}_{AC}\mathcal{H}^{A}_{mn}, \quad \mathcal{H}^{m}{}_{An\underset{4}{\|}m}=-\left[\Omega^{AC}_{mn}+\Omega_{mn}\right]\mathcal{F}^{m}_{A}\\ &\mathcal{H}^{A}_{<\alpha\beta\underset{4}{\|}\gamma>}=2\mathcal{F}^{A}_{<\alpha}\Omega^{AC}_{\beta\gamma>}, \quad \mathcal{F}^{A}_{[\alpha\underset{4}{\|}\beta]}=\left[\Omega^{AC}_{[\alpha}+F_{[\alpha}\right]\mathcal{F}^{A}_{\beta]}\end{aligned}\ . \tag{VIII.3.9}$$

The electric and magnetic fields are coupled to gravielectric interaction currents and to sources. The gravitational parts consist of Coriolis-like and centrifugal terms and further ones

[135] The term $-\mathcal{H}_{23}\mathcal{H}_{23}$ is positive in accordance with the above definition. Therefore the electromagnetic energy density T_{44} is positive.

$$\begin{gathered}\Omega^{AC}_{mn} = 2i\alpha^2_{AC}\omega_{AC}\sigma_{|[m}c_{n]} + 2i\alpha^2_{AC}\omega_{AC|[m}c_{n]}\sigma \\ \Omega^{AC}_{m} = \alpha^2_{AC}\omega^2_{AC}\sigma\sigma_{|m} + \alpha^2_{AC}\omega_{AC}\omega_{AC|m}\sigma^2\end{gathered}, \tag{VIII.3.10}$$

which are to be attributed to the differential rotation law. The skew-symmetric quantity

$$\Omega_{3m} = -\Omega_{m3} \stackrel{*}{=} -\Omega^{C}_{3m} \tag{VIII.3.11}$$

is derived from the system C.

For the system B one obtains

$$\begin{gathered}\mathcal{F}^{m}_{B\underset{4}{||}m} = D_m \mathcal{F}^{m}_{B}, \quad \mathcal{H}^{m}_{Bn\underset{4}{||}m} = -2\Omega^{BC}_{mn}\mathcal{F}^{m}_{B} + E^{B}_{m}\mathcal{H}^{m}_{Bn} \\ \mathcal{H}^{B}_{<\alpha\beta\underset{4}{||}\gamma>} = 2\mathcal{F}^{B}_{<\alpha}H^{C}_{\beta\gamma>}, \quad \mathcal{F}^{B}_{[\alpha\underset{4}{||}\beta]} = \mathcal{F}^{B}_{[\alpha}\left[\Omega^{BC}_{\beta]} + F_{\beta]}\right]\end{gathered}. \tag{VIII.3.12}$$

The quantity H^{C}_{mn} is defined by (VIII.2.10). Further, one has

$$\begin{gathered}\Omega^{BC}_{m3} = i\alpha^2_{BC}\omega_{BC}\sigma_{|m} + i\alpha^2_{BC}\omega_{BC|m}\sigma \\ \Omega^{BC}_{m} = \alpha^2_{BC}\omega^2_{BC}\sigma\sigma_{|m} + \alpha^2_{BC}\omega_{BC}\omega_{BC|m}\sigma^2\end{gathered}. \tag{VIII.3.13}$$

The force

$$D_m = \alpha^2_{R}\omega\,\omega_{|m}\sigma^2 \tag{VIII.3.14}$$

has its origin likewise in the differential rotation law and supplies a contribution to the radial total force

$$E^{B}_{m} = E_m - \Omega^{BC}_{m} + F_m + D_m\,. \tag{VIII.3.15}$$

The quantities Ω^{BC}_{mn} and Ω^{BC}_{m} have the same structure as (VIII.3.10). In both systems the Maxwell equations can be symbolically written as

$$\operatorname{div}\vec{\mathcal{F}} = \rho_{eg}, \quad \operatorname{rot}\vec{\mathcal{F}} = \vec{j}_{mg}, \quad \operatorname{div}\vec{\mathcal{H}} = \rho_{mg}, \quad \operatorname{rot}\vec{\mathcal{H}} = \vec{j}_{eg}\,. \tag{VIII.3.16}$$

The charges and currents are gravielectric interaction terms. For the system A one obtains a relation analogous to (VIII.3.15)

$$E^{A}_{m} = E_m + \Omega^{AC}_{m} - \left(F_m + D_m\right), \tag{VIII.3.17}$$

whereby it has to be remembered that in accordance with (VIII.2.6) the quantity E has an electric contribution and that the compilation

$$\Omega^{A}_{mn} = \Omega^{AC}_{mn} - \Omega_{mn} \tag{VIII.3.18}$$

makes the representation clearer. From the Einstein field equations one obtains

$$\begin{gathered}\Omega^{s}_{An\underset{4}{||}s} = j_n, \quad j_n = 2\left[\Omega^{s}_{An}E^{A}_{s} - \mathcal{H}^{s}_{An}\mathcal{F}^{A}_{s}\right] \\ E^{m}_{A\underset{4}{||}m} = j, \quad j = \left[E^{m}_{A}E^{A}_{m} - \Omega^{A}_{mn}\Omega^{mn}_{A}\right] - \left[\mathcal{F}^{A}_{m}\mathcal{F}^{m}_{A} - \mathcal{H}^{A}_{m3}\mathcal{H}^{m3}_{A}\right]\end{gathered}. \tag{VIII.3.19}$$

The total current and the total field energy density both have gravitational and electromagnetic contributions. They are conserved

$$j^{n}_{\underset{4}{||}n} = 0, \quad j^{\cdot} = 0\,. \tag{VIII.3.20}$$

For the system B one gets

$$D^{s}_{\underset{4}{Bn||s}} = j_n, \quad j_n = -2\mathcal{H}^{s}_{Bn}\mathcal{F}^{B}_{s}$$
$$E^{m}_{\underset{4}{B||m}} = j, \quad j = \left[E^{m}_{B}E^{B}_{m} + D^{B}_{mn}D^{mn}_{B}\right] - \left[\mathcal{F}^{B}_{m}\mathcal{F}^{m}_{B} - \mathcal{H}^{B}_{m3}\mathcal{H}^{m3}_{B}\right], \tag{VIII.3.21}$$

whereby the field quantities

$$D^{sn}_{B} = \Omega^{sn}_{B} + \Omega^{ns}_{B} \tag{VIII.3.22}$$

occurring therein are symmetrical. Finally, one has in the same manner as with Kerr geometry

$$u_{[\alpha||\beta]} = 0, \quad u_{(\alpha||\beta)} = D^{B}_{\alpha\beta}, \quad \alpha, \beta = 1,2,3. \tag{VIII.3.23}$$

The locally nonrotating observer does not experience Coriolis-like forces, but shears $D^{B}_{\alpha\beta}$. The current in (VIII.3.21) does not contain gravitational selfinteraction terms.

The authors Aguirregabiria [A], Chamorro and Virbhadra; Berman [B], Bhat [B], Dhurandhar and Dadhich; Calvani [C], de Felice and Nobili; Calvani [C] and Turolla; Curir [C] and Francaviglia; Dadhich [D], Dadhich [D] and Kale; Gonzales [G], De Felice [D], Herrera and Jimenez; Henry [H], Hiscock [H], Krori [K] and Barua; Linet [L], Marck [M], Nashed [N], Nayak [N] and Vishveshwara; Page [P], Pankovic [P], Ciganivic and Glavatovic; Prodanov [P] and Ivanov; Stuchlik [S], Hledik and Juran; Pekeris [P], Pekeris [P] and Frankowski; Semerák [S] and de Felice; Virbhadra [V], Winter [W], Zhytnikov [Z]; and Wald [W] have worked on the KN-solution.

VIII.4. The interior Kerr - Newman solution

Since a new interior solution for the Kerr solution has been suggested by us, it is obvious to extend this to a charged solution and to match their metric to the exterior Kerr-Newman metric [B]. We can rely on the already developed formula apparatus and we will indicate only formulae that deviate from the ones of the Kerr relations. We proceed again from a static seed metric which we subsequently equip with rotational effects.

The seed metric

$$ds^2 = \alpha_I^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 - a_T^2 dt^2 \tag{VIII.4.1}$$

is based on the elliptical-hyperbolic BL co-ordinate system. We read the anholonomic differentials from the line element

$$dx^1 = \alpha_I a_R dr, \quad dx^2 = \Lambda d\vartheta, \quad dx^3 = \sigma d\varphi, \quad dx^4 = a_T i dt. \tag{VIII.4.2}$$

The elliptical factor a_R is maintained unmodified. The gravitational factors

$$\begin{gathered} \alpha_I = \frac{1}{a_I}, \quad a_I^2 = 1 - \frac{r^2}{\mathcal{R}^2}, \quad a_T = \frac{1}{2}\left[\left(1 + 2\Phi_g^2\right)\cos\eta_g - \cos\eta\right]\Phi_g^{-2}, \quad \alpha_T = \frac{1}{a_T} \\ 2\Phi_g^2 = \frac{A_g^2}{r_g}\frac{2Mr_g - e^2}{M\left(r_g^2 - a^2\right) - e^2 r_g}, \quad \cos\eta_g = \sqrt{1 - \frac{r_g^2}{\mathcal{R}_g^2}}, \quad \cos\eta = \sqrt{1 - \frac{r^2}{\mathcal{R}^2}} = a_I \end{gathered} \tag{VIII.4.3}$$

are up to the quantity Φ_g identical with these of the interior Kerr solution and are closely related to the curvature of the surface. As surface we expect an object with elliptical horizontals which can be matched from downside to the exterior surface. $\mathcal{R}$ is the radius of the circular arc which is taken up at the minor axes of the elliptical horizontals and thus, is also the parameter of a family of surfaces. With the *embedding condition*

$$\mathcal{R} = \mathcal{R}_g = \text{const.} \tag{VIII.4.4}$$

a surface of this family is selected and is matched to the exterior surface with the *linking condition*

$$\mathcal{R}_g = \frac{A_g}{\sqrt{\frac{2M}{r_g} - \frac{e^2}{r_g^2}}}. \tag{VIII.4.5}$$

Quantities which have the marker g indicate their values on the boundary surface between interior and exterior solutions. For the curvature radius of the exterior physical surface we have had

$$\rho_S = a_R A^3 \frac{\sqrt{2Mr - e^2}}{M\left(r^2 - a^2\right) - e^2 r}. \tag{VIII.4.6}$$

On the boundary surface at the minor axes it takes the value

$$\rho_g = A_g^3 \frac{\sqrt{2Mr_g - e^2}}{M\left(r_g^2 - a^2\right) - e^2 r_g}. \tag{VIII.4.7}$$

We again point out that the interior surface, just as the exterior one is equipped with anholonomic hyper planes. This requires also the linking condition on the boundary surface. If one inserts (VIII.4.6) into (VIII.4.7) one obtains

$$\rho_g = 2\mathcal{R}_g \Phi_g^2 . \tag{VIII.4.8}$$

If one puts for the time differential

$$idt = \rho_g di\psi , \tag{VIII.4.9}$$

then one has with (VIII.4.3) and (VIII.4.4) for the 4-dimensional theory the expression

$$dx^4 = \left[\left(\mathcal{R}_g + \rho_g\right)\cos\eta_g - \mathcal{R}_g \cos\eta\right] di\psi , \tag{VIII.4.10}$$

which can immediately be explained in the framework of the double-surface theory. For a = 0 one gets the interior solution for the Reissner-Nordström model, for e = 0 the Kerr interior, and for a = 0 and e = 0 the Schwarzschild interior.

VIII.5. The interior Kerr - Newman solution, the field equations

We read from (VIII.4.2) the 4-bein and we compute the field strengths in the usual way

$$\begin{aligned} B_{mn}{}^{s} &= b_m B_n b^s - b_m b_n B^s, \quad B_n = \left\{\frac{a_I}{\rho_E}, 0,0,0\right\} \\ N_{mn}{}^{s} &= m_m N_n m^s - m_m m_n N^s, \quad N_n = \left\{0, \frac{1}{\rho_H}, 0,0\right\} \\ C_{mn}{}^{s} &= c_m C_n c^s - c_m c_n C^s, \quad C_n = \left\{a_I \frac{r}{A\Lambda}, \frac{1}{\Lambda}\cot\vartheta, 0,0\right\} \\ E_{mn}{}^{s} &= -\left[u_m E_n u^s - u_m u_n E^s\right], \quad E_n = \left\{-\frac{1}{\rho_g^S}\frac{a_I}{a_T}, 0,0,0\right\} \end{aligned} \qquad (VIII.5.1)$$

Differences with the Kerr interior exist only due to the special definition of the gravitational factor a_T. After a short calculation one gets on the boundary surface

$$a_T^g = \cos\eta_g = \cos\varepsilon_g, \quad \sin\eta_g = -\sin\varepsilon_g . \qquad (VIII.5.2)$$

The two angles η and ε have opposite orientations for the sake of convenient use. Thus, it can easily be proved that the gravitational forces of the interior and exterior solutions

$$E_1^g = -\frac{1}{\rho_g^S}\tan\eta_g = \frac{1}{\rho_g^S}\tan\varepsilon_g$$

match on the boundary surface. Since also the radial gravitational factors

$$a_I^g = \frac{\delta_g}{A_g} = a_S^g, \quad \delta_g^2 = r_g^2 - 2Mr_g + a^2 + e^2 \qquad (VIII.5.3)$$

match on the boundary surface this applies also to the other field strengths in (VIII.5.1). Thus, a widely accepted requirement is fulfilled, i.e. that by adjusting the solutions the metric and its first derivatives on the boundary surface must coincide. The Ricci supplies

$$\begin{aligned} &-\left[B_{n\,\underset{2}{||}s} - B_{n\underset{2}{||}s} b^s b_m + B_n B_m\right] - b_n b_m\left[B^s{}_{\underset{2}{||}s} + B^s B_s\right] \\ &-\left[N_{n\,\underset{2}{||}s} - N_{n\underset{2}{||}s} m^s m_m + N_n N_m\right] - m_n m_m\left[N^s{}_{\underset{2}{||}s} + N^s N_s\right] \\ &-\left[C_{n\,\underset{3}{||}s} + C_n C_m\right] - c_n c_m\left[C^s{}_{\underset{3}{||}s} + C^s C_s\right] \\ &+\left[E_{n\,\underset{4}{||}s} - E_n E_m\right] + u_n u_m\left[E^s{}_{\underset{4}{||}s} - E^s E_s\right] = -\kappa\left(T_{mn} - \frac{1}{2}g_{mn}T\right) \end{aligned} \qquad (VIII.5.4)$$

Defining the 0-components for the quantities (VIII.5.1) by

$$M_0 = \frac{1}{a_R \mathcal{R}}, \quad B_0 = \frac{v_I}{\rho_E}, \quad C_0 = v_I \frac{r}{A\Lambda}, \quad E_0 = \frac{1}{\rho_g^s}\frac{a_I}{a_T}, \quad v_I = \sin\eta , \qquad (VIII.5.5)$$

one gets with the help of

$$
\begin{aligned}
\kappa\left(T_{mn}-\frac{1}{2}g_{mn}T\right)=&-m_m m_n\left(M_0B_0+M_0C_0-M_0E_0\right)\\
&-b_m b_n\left(B_0M_0+B_0C_0-B_0E_0\right)\\
&-c_m c_n\left(C_0M_0+C_0B_0-C_0E_0\right)\\
&+u_m u_n\left(E_0M_0+E_0B_0+E_0C_0\right)\qquad , \qquad\qquad \text{(VIII.5.6)}\\
&-\left(m_m m_n+b_m b_n\right)\tilde{\Omega}^{s3}\tilde{\Omega}_{3s}\sin^2\eta\\
&+\left(m_m m_n+c_m c_n\right)N_sC^s\sin^2\eta+2N_{(m}E_{n)}\\
&+\left(m_m m_n+u_m u_n\right)E_sF^s
\end{aligned}
$$

a quantity from which the stress-energy tensor of the system can be computed. This quantity is made up by the second fundamental forms of the surface theory and some well-known quantities of the BL system. The latter already indicate the extension to the rotating system. This extension can be established with an intrinsic transformation

$$
\begin{aligned}
&\Lambda_3^{3'}=\alpha_R,\quad \Lambda_4^{3'}=i\alpha_R\omega,\quad \Lambda_3^{4'}=-i\alpha_R\omega\sigma^2,\quad \Lambda_4^{4'}=\alpha_R\\
&\Lambda_{3'}^{3}=\alpha_R,\quad \Lambda_{4'}^{3}=-i\alpha_R\omega,\quad \Lambda_{3'}^{4}=i\alpha_R\omega\sigma^2,\quad \Lambda_{4'}^{4}=\alpha_R
\end{aligned}. \qquad \text{(VIII.5.7)}
$$

The transformation acts on the co-ordinate indices of the 4-bein

$$
\overset{m}{e}_i=\Lambda_i^{i'}\overset{m}{e}_{i'},\quad \underset{m}{e}^i=\Lambda_{i'}^{i}\underset{m}{e}^{i'},\quad g_{ik}=\Lambda_{ik}^{i'k'}g_{i'k'}\,. \qquad \text{(VIII.5.8)}
$$

As we have done with the familiar Kerr metric we implement an oblique-angled co-ordinate system which can be envisaged from the rectangular systems (A,B,C) in various manners. The rotational effects of the Kerr-Newman model are invoked by an operation on the surface. The surface itself remains unaltered. The line element of the interior KN-metric

$$
ds^2={dx^1}^2+{dx^2}^2+\left[\alpha_R dx^3+i\alpha_R\omega\sigma dx^4\right]^2+a_T^2\left[-i\alpha_R\omega\sigma dx^3+\alpha_R dx^4\right]^2,\quad dx^1=\alpha_I a_R dr \qquad \text{(VIII.5.9)}
$$

differs from the line element of the interior Kerr metric only by the definition of the gravitational factors. As a consequence of the transformation (VIII.5.7), new quantities emerge in the Ricci-rotation coefficients. These are the Coriolis-like field strength H^T_{mn} and the shears D^T_{mn} in the equations

$$
\begin{aligned}
&H_{mns}=-\Omega^T_{nm}u_S+\Omega^T_{sm}u_n+\Omega^T_{sn}u_m+\alpha_T D_{ns}u_m\\
&\Omega^T_{nm}=-\left[H^T_{mn}+D^T_{mn}\right],\quad H^T_{mn}=a_T\left[H_{mn}+D_{[mn]}\right],\quad D^T_{mn}=\alpha_T D_{(mn)}\;. \qquad \text{(VIII.5.10)}\\
&H_{m3}=i\alpha_R^2\omega\sigma_{|m},\quad D_{m3}=i\alpha_R^2\omega_{|m}\sigma,\quad D_{3m}=0
\end{aligned}
$$

Further, the centrifugal force is added by

$$
C_m\rightarrow C^T_m=C_m+F_m,\quad E_m\rightarrow E^T_m=E_m+F_m\;. \qquad \text{(VIII.5.11)}
$$

With the new field quantities one gets field equations similar to the one of the Kerr interior. The components of the stress-energy tensor

$$\begin{aligned}
\kappa T_{11} &= -M_0B_0 + 2B_0C_0 - 2B_0E_0 - \left[\Omega_C^{\alpha 3}\Omega_{3\alpha}^C - \Omega_T^{\alpha 3}\Omega_{3\alpha}^T\right] \\
\kappa T_{22} &= -M_0E_0 + M_0B_0 + 2M_0C_0 - 2B_0C_0 - C_0E_0 + 2F_0E_0 + \left[\Omega_C^{\alpha 3}\Omega_{3\alpha}^C - \Omega_T^{\alpha 3}\Omega_{3\alpha}^T\right] \\
\kappa T_{33} &= -M_0E_0 + M_0B_0 + 2M_0C_0 - 2B_0C_0 - C_0E_0 + 2F_0E_0 \\
&\quad + \left[\Omega_C^{\alpha 3}\Omega_{\alpha 3}^C - \Omega_T^{\alpha 3}\Omega_{\alpha 3}^T\right] + \left[\Omega_C^{\alpha\beta}\Omega_{\beta\alpha}^C - \Omega_T^{\alpha\beta}\Omega_{\beta\alpha}^T\right] \\
\kappa T_{44} &= 2B_0C_0 + M_0B_0 + \left[\Omega_C^{\alpha 3}\Omega_{\alpha 3}^C - \Omega_T^{\alpha 3}\Omega_{\alpha 3}^T\right] - \left[\Omega_C^{\alpha\beta}\Omega_{\beta\alpha}^C - \Omega_T^{\alpha\beta}\Omega_{\beta\alpha}^T\right] \\
\kappa T_{34} &= \Omega_{03}^T M_0 + \Omega_{03}^T E_0 - \left[1 - \frac{a_T^2}{a_I^2}\right]\Omega_{13}^T B_1 - \left[1 - \frac{a_I^2}{a_T^2}\right]\Omega_{23}^T N_2
\end{aligned} \tag{VIII.5.12}$$

are likewise equivalent to this solution. Following the 5-dimensional theory the auxiliary variables

$$F_0 = \alpha_R^2\omega^2\sigma\sigma_{|0}\,, \quad \Omega_{03}^T = i\alpha_T\alpha_R^2\omega\,\sigma_{|0}\,, \quad \sigma_{|0} = \frac{r}{\Lambda}\sin\vartheta\sin\eta \tag{VIII.5.13}$$

are used. The Ω_{mn}^C are the rotational field strengths

$$\Omega_{nm}^C = -\left[H_{mn}^C + D_{mn}^C\right], \quad H_{mn}^C = a_I\left[H_{mn} + D_{[mn]}\right], \quad D_{mn}^C = \alpha_I D_{(mn)}\,, \tag{VIII.5.14}$$

which have been discussed in earlier Sections.

If one inserts the values of the boundary surface and also (VIII.5.3) into the stress-energy tensor then the stress-energy tensor decomposes into

$$T_{mn}^g = T_{mn}^e + T_{mn}^r\,, \tag{VIII.5.15}$$

whereby also the well-known electrical stress-energy tensor of the KN exterior

$$\kappa T_{mn}^e = \begin{pmatrix} \mathcal{F}^2 & & & \\ & -\mathcal{F}^2 & & \\ & & -\mathcal{F}^2 & \\ & & & \mathcal{F}^2 \end{pmatrix}, \quad \mathcal{F} = \mathcal{F}_{14} = \frac{e}{\Lambda^2} \tag{VIII.5.16}$$

is retrieved. The hydrostatic pressure vanishes on the boundary surface, but not the stresses $T_{22}^r = T_{33}^r \neq 0$. The jumps in

$$\kappa T_{34}^r = \Omega_{03}^{Tg}\left(M_0^g + E_0^g\right), \quad \kappa T_{44}^r = B_0^g\left(M_0^g + E_0^g\right) \tag{VIII.5.17}$$

can easily be understood. M_0^g and E_0^g are the generalized radial curvatures of the interior and exterior surfaces which both exist on the boundary surface. Since

$$M_0^g + E_0^g = \frac{1}{a_R^g}\left(\frac{1}{\mathcal{R}_g} + \frac{1}{\rho_g}\right) \tag{VIII.5.18}$$

is positive the energy density on the boundary surface is positive as well. For the exterior solution this term vanishes, because with this case one has $M_0^{ext} = -E_0^{ext}$. Mathematically, the KN interior fits into the interior solutions of the whole Kerr family which have been found to date. We will discuss this aspect later on.

VIII.6. The NUT metric, preparation

Newman [N], Unti, and Tamburino have suggested a model which is deduced from the Schwarzschild model and is a vacuum solution as well. However, the solution contains an additional parameter l which is interpreted as monopole parameter. The field equations contain a rotational term which has a only one component whose dual vector points into the radial direction and for this reason has similar properties as the magnetic monopoles, is, however, interpreted as a gravitational effect. The model does not have a counterpart in nature, but is interesting in so far as it can be allocated to the Kerr family. It is also related to the Reissner-Nordström solution and can be combined with the Kerr solution (Singh [S] and Yadav).

The NUT metric has the form

$$ds^2 = \frac{A^2}{\delta^2}dr^2 + A^2 d\vartheta^2 + A^2\sin^2\vartheta d\varphi^2 - \frac{\delta^2}{A^2}\left[dt + 2l(1-\cos\vartheta)d\varphi\right]^2$$
$$A^2 = r^2 + l^2 \quad \delta^2 = r^2 - 2Mr - l^2 \tag{VIII.6.1}$$

and is reduced for l = 0 to the Schwarzschild metric. It is simple to remove the second term in the brackets of (VIII.6.1) by an intrinsic transformation and to re-establish it by this transformation later on. First we deal with the static seed metric

$$ds^2 = \frac{A^2}{\delta^2}dr^2 + A^2 d\vartheta^2 + A^2\sin^2\vartheta d\varphi^2 + \frac{\delta^2}{A^2}{dx^4}^2, \quad dx^4 = idt\,. \tag{VIII.6.2}$$

However, it should be noticed that the radial variable r does not agree with the assumed radius A of the greater circles of this spherically symmetrical system. Therefore we decompose the metrical factors into

$$\alpha_G = \frac{r}{\delta}, \quad a_G = \frac{\delta}{r}, \quad \alpha_D = \frac{A}{r}, \quad a_D = \frac{r}{A} \tag{VIII.6.3}$$

and we write the metric in a more reasonable form

$$ds^2 = \alpha_D^2\left[\alpha_G^2 dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2\right] + a_D^2 a_G^2 dit^2\,. \tag{VIII.6.4}$$

The gravitational factor α_G is related to the Schwarzschild model, but α_D deforms the metric in dependence on the distance from the gravitation center. The model is in some relation to the Weyl geometry, if one converts this model into the tetrad formalism. In the Weyl geometry the quantities are measured with rods which are position-dependent, whereby this dependence goes beyond the dependence of the Riemannian geometry and can be specified by an additional set of equations. These equations have Maxwellian structure. Weyl has faced a unified theory of gravitation and electromagnetism which finally did not work.

The *reduced metric*

$$ds^2 = \alpha_G^2 dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 + a_G^2 dit^2 \tag{VIII.6.5}$$

differs from the Reissner-Nordström metric only by the gravitational factor α_G. With the replacement $l \to ie$ this metric can be obtained immediately from (VIII.6.5). l = 0 leads to the Schwarzschild metric. Accordingly, the reduced model can be treated in a similar way as the Schwarzschild geometry or the Reissner-Nordström geometry. The 4-beins one reads from (VIII.6.5)

$$\overset{1}{e}_1 = \alpha_G, \quad \overset{2}{e}_2 = r, \quad \overset{3}{e}_3 = r\sin\vartheta, \quad \overset{4}{e}_4 = a_G, \quad \partial_1 = a_G \frac{\partial}{\partial r} \ . \tag{VIII.6.6}$$

From this one deduces the field quantities

$$B_1 = \frac{a_G}{r}, \quad C_1 = \frac{a_G}{r}, \quad C_2 = \frac{1}{r}\cot\vartheta, \quad E_1 = -\alpha_G \frac{M}{r^2} - \alpha_G \frac{l^2}{r^3} \ . \tag{VIII.6.7}$$

To the Schwarzschild-like force of gravity enters another second attractive term which decreases in r with the 3rd power. It has a shorter range. Identifying

$$a_G = \cos\varepsilon \tag{VIII.6.8}$$

one gets the angle of ascent ε of the radial curves of the surface which is very similar to the ‘Schwarzschild funnel’ and is embedded into a 5-dimensional flat space.

Herewith and also with

$$v_G^2 = 1 - a_G^2$$

one has deduced the velocity of a freely falling body

$$v_G = \sin\varepsilon = -\sqrt{\frac{2M}{r} + \frac{l^2}{r^2}} \tag{VIII.6.9}$$

and with

$$\tan\varepsilon = \frac{v_G}{a_G} = -\sqrt{\frac{2Mr + l^2}{r^2 - 2Mr - l^2}} \tag{VIII.6.10}$$

one has deduced the curvature radius

$$\rho = \frac{\sqrt{2Mr + l^2}}{Mr + l^2} r^2 = -v_G \frac{r^3}{Mr + l^2} \ . \tag{VIII.6.11}$$

Thus, one also gets the useful formulae

$$v_{G|1} = \frac{a_G}{\rho}, \quad a_{G|1} = -\frac{v_G}{\rho} \ . \tag{VIII.6.12}$$

With (VIII.6.10) and (VIII.6.11) one can convert the force of gravity into the reasonable form

$$E_1 = \frac{1}{\rho}\tan\varepsilon \ . \tag{VIII.6.13}$$

If one defines, with regard to a 5-dimensional formulation, the quantities

$$M_0 = \frac{1}{\rho}, \quad B_0 = \frac{v_G}{r}, \quad C_0 = \frac{v_G}{r}, \quad E_0 = -\frac{1}{\rho} \tag{VIII.6.14}$$

one obtains the well-known structure for the Ricci

$$\begin{aligned}
R_{mn} = &-\left[B_{n\|m \atop 2} + B_n B_m\right] - b_n b_m\left[B^s{}_{\|s \atop 2} + B^s B_s\right] \\
&-\left[C_{n\|m \atop 3} + C_n C_m\right] - c_n c_m\left[C^s{}_{\|s \atop 3} + C^s C_s\right] \\
&+\left[E_{n\|m \atop 4} - E_n E_m\right] + u_n u_m\left[E^s{}_{\|s \atop 4} - E^s E_s\right] \\
= &\, m_m m_n\left(M_0 B_0 + M_0 C_0 - M_0 E_0\right) \\
&+ b_m b_n\left(B_0 M_0 + B_0 C_0 - B_0 E_0\right) \\
&+ c_m c_n\left(C_0 M_0 + C_0 B_0 - C_0 E_0\right) \\
&- u_m u_n\left(E_0 M_0 + E_0 B_0 + E_0 C_0\right) \\
&- \left(m_m m_n + u_m u_n\right) E_1 \frac{1}{\rho}\rho_{|1}
\end{aligned} \quad . \tag{VIII.6.15}$$

The second block in this relation consists of the second fundamental forms of the surface theory. The last line is a consequence of the projection of the Ricci from the pseudo-hyper sphere, as has already been shown. If one computes the auxiliary relations

$$B_0 E_0 = C_0 E_0 = \frac{Mr + l^2}{r^4}, \quad B_0 C_0 = \frac{2Mr + l^2}{r^4}, \quad M_0 = -E_0 ,$$

with (VIII.6.11) and (VIII.6.14) one has

$$B_0 M_0 + B_0 C_0 - B_0 E_0 = C_0 M_0 + C_0 B_0 - C_0 E_0 = -\tilde{\omega}^2 , \tag{VIII.6.16}$$

the monopole field strength being

$$\tilde{\omega} = \frac{l}{r^2} \quad . \tag{VIII.6.17}$$

The computation of the last of all terms in (VIII.6.15) is more complex. One finally arrives at

$$\begin{aligned}
E_1 \frac{1}{\rho}\rho_{|1} = &\quad M_0 B_0 + M_0 C_0 - M_0 E_0 - \tilde{\omega}^2 \\
= &-\left[E_0 M_0 + E_0 B_0 + E_0 C_0\right] - \tilde{\omega}^2
\end{aligned} \quad . \tag{VIII.6.18}$$

Thus, the Ricci takes the form

$$R_{mn} = \begin{pmatrix} \tilde{\omega}^2 & & & \\ & -\tilde{\omega}^2 & & \\ & & -\tilde{\omega}^2 & \\ & & & \tilde{\omega}^2 \end{pmatrix} \tag{VIII.6.19}$$

from which immediately follows R = 0. Rescaling with

$$l^2 \rightarrow \frac{\kappa}{2} l^2 \tag{VIII.6.20}$$

and also defining the gravimagnetic field strength with

$$\mathfrak{H}_{23} = -i\tilde{\omega} , \tag{VIII.6.21}$$

one obtains the dual field strength

$$\mathcal{H}_\alpha = i\varepsilon_\alpha{}^{\beta\gamma}\mathcal{H}_{\beta\gamma}, \quad \mathcal{H}_\alpha = \left\{\tilde{\omega},0,0\right\}, \quad \alpha = 1,2,3 \ . \tag{VIII.6.22}$$

One can write down the stress-energy tensor in a form in analogy to the electrodynamics

$$T_{mn} = -\left[\mathcal{H}_m{}^s\mathcal{H}_{ns} - \frac{1}{4}g_{mn}\mathcal{H}^{rs}\mathcal{H}_{rs}\right]. \tag{VIII.6.23}$$

It supplies (VIII.6.19) with the Einstein field equations. The quantity $\mathcal{H}$ defined in such a way satisfies the Maxwell equations

$$\mathcal{H}^s{}_{n||s} = 0. \quad \mathcal{H}_{<mn||s>} = 0 \ . \tag{VIII.6.24}$$

The reduced NUT metric can be combined with the Reissner-Nordström metric. It turns out to be quite simple to calculate this model because it differs just a little from the preceding one. For the Ricci one gets the components

$$R_{11} = -R_{22} = -R_{33} = R_{44} = \frac{l^2}{r^4} - \frac{e^2}{r^4} \ . \tag{VIII.6.25}$$

Due to the signs which are an outcome of $l \to ie$, it is not possible to set up a common stress-energy tensor for both field strengths which corresponds to the stress-energy tensor of the electrodynamics and whose components could be interpreted as electric and magnetic ones. The l- and e-terms belong to two different worlds.

As to the force of gravity (VIII.6.13) one recognizes with the relations (VIII.6.10) and (VIII.6.11) that it consists of two constituents

$$E_1 = -\alpha_G \frac{M}{r^2} - \alpha_G \frac{l^2}{r^3} \tag{VIII.6.26}$$

which both have an attractive effect. The first part corresponds to the Schwarzschild force of gravity and coincides with it for l = 0. The second part obeys the law $1/r^3$ and thus has a shorter range. The monopole parameter is responsible for this.

If we define

$$\mathcal{E}_1 = -\alpha_G \frac{l^2}{r^3} \ , \tag{VIII.6.27}$$

then a short calculation[136] shows

$$\mathcal{E}^n{}_{||n} = \tilde{\omega}^2 \ , \tag{VIII.6.28}$$

that the quantity $\mathcal{E}$ is coupled to a source with the monopole field strength. For the remaining Schwarzschild-like part

$$\bar{E}_1 = -\alpha_G \frac{M}{r^2}, \quad \bar{E}^n{}_{||n} = 0 \tag{VIII.6.29}$$

is valid in agreement with the 4th line of (VIII.6.19).

[136] $B_1\mathcal{E}_1 = \tilde{\omega}^2$ has been used.

VIII.7. The NUT metric, 5-dimensional form of the reduced metric

Having treated the 4-dimensional field equations of the reduced NUT metric the 5-dimensional field equations can be written down by simply rearranging and by supplementing them with the 0-components. The latter we have already introduced with (VIII.6.14) for the sake of convenience. It remains to be mentioned that for the differentiation in the 0-direction the operators

$$\partial_0 = \frac{\partial}{\partial \rho}, \quad \partial_0 = v_G \frac{\partial}{\partial r} \tag{VIII.7.1}$$

are to be used depending upon the type of the quantity. For the 5-dimensional Ricci one finds the familiar form

$$\begin{aligned} {}^5R_{ab} = &-\left[\underset{1}{M_{b|||a}} - \underset{1}{M_{b|||c}} m^c m_a + M_b M_a\right] - m_a m_b \left[\underset{1}{M^c{}_{|||c}} + M^c M_c\right] \\ &-\left[\underset{2}{B_{b|||a}} + B_b B_a\right] - b_a b_b \left[\underset{2}{B^c{}_{|||c}} + B^c B_c\right] \\ &-\left[\underset{3}{C_{b|||a}} + C_b C_a\right] - c_a c_b \left[\underset{3}{C^c{}_{|||c}} + C^c C_c\right] \\ &+\left[\underset{4}{E_{b|||a}} - E_b E_a\right] + u_a u_b \left[\underset{4}{E^c{}_{|||c}} - E^c E_c\right] \end{aligned} \quad . \tag{VIII.7.2}$$

The relations

$${}^5R_{ab} + 2X_{fb}{}^d \mathfrak{p}^f_{[d|||a]} = 0 \tag{VIII.7.3}$$

must be satisfied with (VIII.7.2). The X are the well-known connexion coefficients of the pseudo-hyper sphere and the $\mathfrak{p}$ the projectors

$$\mathfrak{p}^0_0 = \mathfrak{p}^1_1 = \mathfrak{p}^4_4 = \frac{X}{\rho}, \quad \mathfrak{p}^2_2 = \mathfrak{p}^3_3 = \frac{Xv_G}{r} \ . \tag{VIII.7.4}$$

Since all brackets except the latter vanish in (VIII.7.2)

$$\underset{4}{E_{b|||a}} - E_b E_a = -E_b \frac{1}{\rho} \rho_{|\underline{a}}, \quad \underset{4}{E^c{}_{|||c}} - E^c E_c = -E^c \frac{1}{\rho} \rho_{|\underline{c}} \tag{VIII.7.5}$$

only remains[137]. The two terms on the right side are deduced from the projector term of (VIII.7.3). From (VIII.7.2) one obtains the 4-dimensional components

$$\begin{aligned} &{}^5R_{11} + 2X_{f1}{}^d \mathfrak{p}^f_{[d|||1]} = {}^4R_{11} - \left[M_0 B_0 + M_0 C_0 - M_0 E_0\right] + E_1 \frac{1}{\rho} \rho_{|1} = {}^4R_{11} - \tilde{\omega}^2 = 0 \\ &{}^5R_{22} + 2X_{f2}{}^d \mathfrak{p}^f_{[d|||2]} = {}^4R_{22} - \left[B_0 M_0 + B_0 C_0 - B_0 E_0\right] = {}^4R_{22} + \tilde{\omega}^2 = 0 \\ &{}^5R_{33} + 2X_{f3}{}^d \mathfrak{p}^f_{[d|||3]} = {}^4R_{33} - \left[C_0 M_0 + C_0 B_0 - C_0 E_0\right] = {}^4R_{33} + \tilde{\omega}^2 = 0 \\ &{}^5R_{44} + 2X_{f4}{}^d \mathfrak{p}^f_{[d|||4]} = {}^4R_{44} + \left[E_0 M_0 + E_0 B_0 + E_0 C_0\right] + E_1 \frac{1}{\rho} \rho_{|1} = {}^4R_{44} - \tilde{\omega}^2 = 0 \end{aligned} \tag{VIII.7.6}$$

[137] An underlined index is specified to the values 1, …,4.

in agreement with (VIII.6.19). It turns out that the projector term emerges only in two equations.

VIII.8. The NUT metric, the auxiliary model

After the detailed treatment of the reduced model whose physical interpretation is doubtful, we turn to the auxiliary metric. With an intrinsic transformation

$$\Delta_1^{1'} = \Delta_2^{2'} = \Delta_3^{3'} = \alpha_D, \quad \Delta_4^{4'} = a_D, \quad \alpha_D = \frac{A}{r}, \quad a_D = \frac{r}{A} \tag{VIII.8.1}$$

of the 4-beins[138]

$$\overset{m}{e}_i = \Delta_i^{i'} \overset{m}{e}_{i'}, \quad \underset{m}{e}^i = \Delta_{i'}^i \underset{m}{e}^{i'}, \qquad \partial_1 = a_G a_D \frac{\partial}{\partial r} \tag{VIII.8.2}$$

one has deformed the metric and has prepared the way to the actual NUT metric. Thus, the new 4-beins read as

$$\overset{1}{e}_1 = \alpha_D \alpha_G, \quad \overset{2}{e}_2 = \alpha_D r, \quad \overset{3}{e}_3 = \alpha_D r \sin\vartheta, \quad \overset{4}{e}_4 = a_D a_G \,. \tag{VIII.8.3}$$

From these and the reciprocal values one computes the connexion coefficients or alternatively one can use the intrinsic transformation

$$^*A_{[mn]}{}^s = A_{[mn]}{}^s + \Delta_{mn}{}^s, \quad \Delta_{mn}{}^s = \underset{m}{e}^{k'} \underset{n}{e}^{j'} \overset{s}{e}_{i'} \Delta_i^{i'} \Delta^i_{[j'|k']} \;.$$

One finally has

$$^*A_{mn}{}^s = A_{mn}{}^s + \Delta_{mn}{}^s + \Delta^s{}_{nm} + \Delta^s{}_{mn} \,. \tag{VIII.8.4}$$

A new quantity D_m which has only one component

$$D_1 = \frac{1}{\alpha_D} \alpha_{D|1} = -a_G \frac{l^2}{A^3} \tag{VIII.8.5}$$

can be deduced from the deformation process. It enhances the field quantities of the reduced metric

$$^*B_1 = B_1 + D_1, \quad ^*C_1 = C_1 + D_1, \quad ^*C_2 = C_2, \quad ^*E_1 = E_1 + D_1 \,. \tag{VIII.8.6}$$

In contrast to the reduced metric the 1-components of the quantities

$$B_1 = a_D a_G \frac{1}{r}, \quad C_1 = a_D a_G \frac{1}{r}, \quad E_1 = a_D \frac{1}{\rho} \tan\varepsilon = -a_D \alpha_G \left(\frac{M}{r^2} + \frac{l^2}{r^3} \right) \tag{VIII.8.7}$$

have the additional deformation factor a_D. Likewise, the compact representation

$$^*B_1 = {}^*C_1 = a_D^2 a_G \frac{1}{A} \tag{VIII.8.8}$$

is possible.

With respect to a compact representation and further calculations we introduce the following quantities

$$M_0 = \frac{a_D}{\rho}, \quad B_0 = a_D v_G \frac{1}{r}, \quad B_0 = a_D v_G \frac{1}{r}, \quad E_0 = -\frac{a_D}{\rho}, \quad D_0 = -v_G \frac{l^2}{A^3} \;, \tag{VIII.8.9}$$

[138] The primed indices refer to the co-ordinate indices of the reduced metric.

for a later use of the 5-dimensional representation, whereby v_G has been defined with (VIII.6.9). B, C, and D are horizontal quantities, E is a vertical one. From this follows

$$B_c E^c = C_c E^c = D_c E^c = 0 \tag{VIII.8.10}$$

which is to be considered in some calculations. The asterisked 0-components of the quantities are to be shaped according to the method (VIII.8.6). In order to make the evaluation of the field equations more transparent, numerous auxiliary formulae are listed in the mathematical appendix. For the graded derivatives the asterisked quantities (VIII.8.4) and (VIII.8.6) respectively, are to be used. In the well-known way one obtains the Ricci

$$\begin{aligned} R_{mn} = &-\left[{}^*B_{\underset{2}{n\|m}} + {}^*B_n {}^*B_m \right] - b_n b_m \left[{}^*B^s{}_{\underset{2}{\|s}} + {}^*B^s {}^*B_s \right] \\ &-\left[{}^*C_{\underset{3}{n\|m}} + {}^*C_n {}^*C_m \right] - c_n c_m \left[{}^*C^s{}_{\underset{3}{\|s}} + {}^*C^s {}^*C_s \right] . \\ &+\left[{}^*E_{\underset{4}{n\|m}} - {}^*E_n {}^*E_m \right] + u_n u_m \left[{}^*E^s{}_{\underset{4}{\|s}} - {}^*E^s {}^*E_s \right] \end{aligned} \tag{VIII.8.11}$$

Its calculation again leads to a complex with the second fundamental forms. For the processing of the last line one needs the relation

$$E_1 \frac{1}{\rho} \rho_{|1} = -\left[E_0 M_0 + E_0 B_0 + E_0 C_0 \right] - \alpha_D^2 \omega^2 \ . \tag{VIII.8.12}$$

Using the monopole field strength one finally gets by following (VIII.6.21) with

$$\mathcal{H}_{23} = -i a_D a_G \omega, \quad \mathcal{H}^2 = \mathcal{H}_{mn} \mathcal{H}^{mn} = 2 \mathcal{H}_{23} \mathcal{H}^{23} \tag{VIII.8.13}$$

the Ricci

$$R_{mn} = \begin{pmatrix} 0 & & & \\ & \mathcal{H}^2 & & \\ & & \mathcal{H}^2 & \\ & & & -\mathcal{H}^2 \end{pmatrix} \tag{VIII.8.14}$$

which substantially differs from (VIII.6.19) as a consequence of the vanishing of the [11]-component. Since the Ricci is not trace-free one has

$$R = \mathcal{H}^2 \ . \tag{VIII.8.15}$$

Thus, the stress-energy tensor

$$\kappa T_{mn} = -2 \left[\mathcal{H}_m{}^s \mathcal{H}_{ns} - \frac{1}{4} g_{mn} \mathcal{H}^2 \right] + u_m u_n \mathcal{H}^2 \tag{VIII.8.16}$$

corresponds only very little to that of the electrodynamics and is not trace-free, either. However, it is necessarily covariantly conserved

$$T^{mn}{}_{\|n} = 0 \ . \tag{VIII.8.17}$$

Moreover, the Maxwell-like equations

$$\mathcal{H}^{mn}{}_{\underset{4}{\|n}} = 0, \quad \mathcal{H}_{\underset{4}{<mn\|s>}} + {}^*E_{<m} \mathcal{H}_{ns>} = 0 \tag{VIII.8.18}$$

are valid, they also can be written as

$$\mathrm{rot}\vec{\mathcal{H}} = 0, \quad \mathrm{div}\vec{\mathcal{H}} + {}^*\vec{E}\vec{\mathcal{H}} = 0 \ .$$

The second term in the second equation refers to the selfinteraction of the fields. From (VIII.8.11) one can read for the [44]-component the interesting relation

$$\mathrm{div}\,{}^*\vec{E} = {}^*E^2 - \mathcal{H}^2 \ . \tag{VIII.8.19}$$

On the right side is the energy density of the field. It is positive in accordance with the definition (VIII.8.13). Lynden-Bell [L] has found similar gravimagnetic field equations. Since he has used the co-ordinate method, the field quantities differ from ours by factors. From the relation (VIII.8.19) which can also be written as

$${}^*E^s{}_{\|s} = -\mathcal{H}^2 \tag{VIII.8.20}$$

an equation can be extracted for the Schwarzschild-like force of gravity

$$\bar{E}^s{}_{\|s} = 0, \quad \bar{E}_1 = -a_D \alpha_G \frac{M}{r^2} \ . \tag{VIII.8.21}$$

It corresponds to the second equation (VIII.6.29) of the reduced model. For the second term in the expression of the force of gravity in (VIII.8.7) one gets in analogy to (VIII.6.28)

$$\mathcal{E}^s{}_{\|s} = a_D^2 \tilde{\omega}^2, \quad \mathcal{E}_1 = -a_D \alpha_G \frac{l^2}{r^3} \tag{VIII.8.22}$$

and finally

$$D^s{}_{\|s} = -\mathcal{H}^2 - a_D^2 \tilde{\omega}^2, \quad \mathcal{E}^s{}_{\|s} + D^s{}_{\|s} = -\mathcal{H}^2 \ . \tag{VIII.8.23}$$

One recognizes that the monopole energy density is the source of those terms which deviate from the Schwarzschild-like force of gravity. As the last step towards the actual NUT metric we perform an intrinsic transformation. From the time-like arc element of the NUT metric

$$dx^4 = a_D a_G \left[2il\left(1 - \cos\vartheta\right) d\varphi + dit \right] \tag{VIII.8.24}$$

one reads the coefficients of this transformation

$$\begin{array}{lll} \Lambda^3_{3'} = 1, & \Lambda^4_{3'} = -2il\left(1-\cos\vartheta\right), & \Lambda^4_{4'} = 1 \\ \Lambda^{3'}_3 = 1, & \Lambda^{4'}_3 = \ \ 2il\left(1-\cos\vartheta\right), & \Lambda^{4'}_4 = 1 \end{array} \ . \tag{VIII.8.25}$$

They transform the 4-bein in accordance with

$$\overset{m}{e}_i = \Lambda^{i'}_i \overset{m}{e}_{i'}, \quad \underset{m}{e}^i = \Lambda^i_{i'} \underset{m}{e}^{i'} \ , \tag{VIII.8.26}$$

whereby the co-ordinate indices of the seed metric discussed before now are primed. Similarly to (VIII.8.4), one obtains

$$A_{mn}{}^s = {}^*A_{mn}{}^s + \Lambda_{mn}{}^s + \Lambda^s{}_{nm} + \Lambda^s{}_{mn}, \quad \Lambda_{mn}{}^s = \underset{m}{e}^{k'} \underset{n}{e}^{j'} \overset{s}{e}_{i'} \Lambda^{i'}_i \Lambda^i_{[j'|k']} \ . \tag{VIII.8.27}$$

If one calculates this relation one obtains for the NUT metric

$$A_{mn}{}^s = {}^*A_{mn}{}^s + H_{mn}{}^s, \quad H_{mn}{}^s = \mathcal{H}_{mn} u^s + \mathcal{H}^s{}_m u_n + \mathcal{H}^s{}_n u_m \ . \tag{VIII.8.28}$$

To the connexion coefficients of the seed metric are added expressions with the monopole field strengths. If one processes these into the Ricci one obtains the vacuum field equations

$$R_{mn} = {}^{*}R_{mn} + 2u_{(m}\mathcal{H}^{s}{}_{n)\underset{4}{||}s} - 2\mathcal{H}_{m}{}^{s}\mathcal{H}_{ns} + u_{m}u_{m}\mathcal{H}^{2} = 0 . \qquad \text{(VIII.8.29)}$$

The stress-energy tensor of the seed metric is nullified by the intrinsic transformation. From the Ricci decouples the equation

$$\mathcal{H}^{s}{}_{n\underset{4}{||}s} = 0 \ . \qquad \text{(VIII.8.30)}$$

Thus,

$${}^{*}R_{mn} = 2\mathcal{H}_{m}{}^{s}\mathcal{H}_{ns} - u_{m}u_{n}\mathcal{H}^{2} \qquad \text{(VIII.8.31)}$$

remains, which agrees with (VIII.8.14).

We note that Lynden-Bell [L] and Nouri-Zonoc have tried to attribute a physical meaning to gravitational monopoles. Also Sackfield [S] has made attempts at an explanation. Bonnor [B] has worked out the basic properties of the NUT metric. In contrast to the Schwarzschild metric it does not have a singularity at r = 0. However, the radial and time-like arc elements of the metric change the sign because of the factors $a_D a_G$. The meaning of space and time are exchanged in this case. The double factor mentioned above vanishes at the event horizon $r_H = M \pm \sqrt{M^2 + l^2}$. The force of gravity is infinitely high at this position. Bonnor has assumed a singularity at $\vartheta = 0$ and $\vartheta = \pi$. Although this singularity can be removed by a co-ordinate transformation, he has assumed the singularity at $\vartheta = \pi$ to be physically real and he has explained the source of the field in this manner. The source could be a massless rotating rod. Manko [M] and Ruiz have contradicted Bonnor and have interpreted the source of the NUT metric as two semi-infinite counter-rotating rods which are separated by a massless region. It seems to us that in the literature too much importance has been attached to the ϑ-problem. The field quantities derived from the NUT metric do not differ with respect to the formal structure from the Schwarzschild field strengths or from analogous expressions of other models. The arising singularities at $\vartheta = 0$ and $\vartheta = \pi$ are the well-known co-ordinate singularities of the polar system and can be arbitrarily shifted by a simple co-ordinate transformation. Therefore they represent only a lack of description and they do not have a physical meaning. Only the singularity at A = 0 remains, the singularity in the origin of the system which most models are afflicted with. The monopole field strength is completely independent of the angle ϑ. Since the form of the metric changes for the two special ϑ-values, it has been accepted that the model is not flat in the infinite. However, a glance at the field strengths shows that these vanish in the infinite.

A 5-dimensional formulation of the NUT equations is not necessary. It has been anticipated by the 5-dimensional representation of the field equations of the reduced metric. The intrinsic transformation to the seed metric leads to a local gauge of the rods with which one measures the quantities on this surface. The surface remains unchanged under this transformation. Nor does the second intrinsic transformation which implements the rotation content of geometry change the surface.

Several other authors such as Ashtekar [A] and Sen; Bossard [B], Nicolai, and Stelle; Dowker [D] and Roche; Gautreau [G] and Hoffman; Mena [M] and Natario; Miller [M], Kruskal and Godfrey; Moncrieff [M], Neugebauer [N] and Kramer; Nouri-Zonoz [N], Reina [R] and Treves have been occupied with the NUT metric.

Miller [M] has examined the singularities of the NUT metric and has regarded the maximal analytic extensions. Tomimatsu [T] and Kihara have examined the properties on the symmetry axis of a metric which describes the superposition of two combined Kerr-NUT solutions. Wei [W] and Yamazaki [Y] have extended the combined Kerr-NUT solution to a charged one. Kinnersley [K] has compiled some models in the Newman-Penrose formalism which are related to the NUT metric. Misner [M] has sounded out the singularities of the NUT metric and has computed explicitly the Ricci. Nouri-Zonoz [N], Reina [R], and Treves have shown that the combined Kerr-Newman-NUT solution can be derived with the help of the complex potential formalism which has been introduced by Ernst.

Further: Bradley [B], Fodor, Gergely, Lapedes [L], Markl and Perjés; Dechant [D] and Lasenby; Herlt [H], Hiscock [H] and Konkowski; Magnon [M], Nouri-Zonoz [N], Samuel [S] and Iyer

We will add another Section which is concerned with the possibility of an interior solution.

VIII.9. The interior NUT solution, the construction

Although Nature does not provide evidence that a physical meaning can be assigned to the monopole term of the NUT metric, it is nevertheless a mathematically interesting metric, because it represents a generalization of the Schwarzschild solution and is also in close relation to the Reissner-Nordström solution. It is mathematically quite delightful to supplement the NUT solution with an interior solution although its physical meaning is doubtful. The model which we present can easily be integrated into the Kerr family and is reduced to the Schwarzschild interior solution, if one puts the NUT parameter zero.

The model is developed in three steps just like the exterior one. First the cap of a sphere is matched to the surface of the exterior model at a suitable position. Then rods and clocks are locally gauged on this cap. This leads from the reduced metric to the static seed metric. By an intrinsic transformation the metric finally obtains a cross term which is responsible for the monopole fields. Then the metric is evaluated, the field strengths and field equations are computed. The stress-energy tensor contains a Schwarzschild-like portion and additionally a monopole part which reminds us of the electrodynamics.

The structure of the cap of the sphere takes place similarly to the interior Schwarzschild solution. The $\mathfrak{R}$ again are the radii of a family of spheres, from which a sphere is selected by means of the *embedding condition*

$$\mathfrak{R} = \mathfrak{R}_g = \text{const.}, \tag{VIII.9.1}$$

a sphere which is suitable for the adjustment to the exterior solution. The polar angle is η and is in relation to the Schwarzschild standard co-ordinate by

$$r = \mathfrak{R} \sin\eta \ . \tag{VIII.9.2}$$

Thus, one has

$$\sin\eta = \frac{r}{\mathfrak{R}}, \quad \cos\eta = \sqrt{1 - \frac{r^2}{\mathfrak{R}^2}} \ . \tag{VIII.9.3}$$

The reduced interior metric has the form

$$ds^2 = \mathfrak{R}^2 d\eta^2 + \mathfrak{R}^2 \sin^2\eta d\vartheta^2 + \mathfrak{R}^2 \sin^2\eta \sin^2\vartheta d\varphi^2 + a_T^2 dit^2 \tag{VIII.9.4}$$

which can, with (VIII.9.2), also be written as

$$ds^2 = \frac{1}{1 - \frac{r^2}{\mathfrak{R}^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 + a_T^2 dit^2 . \tag{VIII.9.5}$$

The time factor of the metric is

$$a_T = a_T(\mathfrak{R}, \eta) = \left[\left(\mathfrak{R}_g + \rho_g \right) \cos\eta_g - \mathfrak{R} \cos\eta \right] \frac{1}{\rho_g} , \tag{VIII.9.6}$$

where η_g is the aperture angle of the cap which provides the linking to the exterior solution. ρ_g is the curvature radius of the radial curves of the exterior solution on the boundary surface. It differs from the value of the Schwarzschild solution by the NUT parameter

$$\rho_g = \frac{\sqrt{2Mr_g + l^2}}{Mr_g + l^2} r_g^2. \tag{VIII.9.7}$$

If one puts this factor zero one obtains the appropriate Schwarzschild value. With the interpretation

$$dit = \rho_g di\psi \tag{VIII.9.8}$$

one has for the time-like arc element

$$dx^4 = \left[\left(\mathcal{R}_g + \rho_g\right)\cos\eta_g - \mathcal{R}\cos\eta\right] di\psi \tag{VIII.9.9}$$

which can geometrically be interpreted in the context of a double-surface theory as the ring sector of a pseudo circle. Defining the constant auxiliary variable

$$2\Phi_g^2 = \frac{2Mr_g + l^2}{Mr_g + l^2}\ , \tag{VIII.9.10}$$

a closer relationship can be established between the quantities ρ_g and $\mathcal{R}$

$$\rho_g = 2\mathcal{R}_g\Phi_g^2 = \mathcal{R}_g\frac{2Mr_g + l^2}{Mr_g + l^2} \tag{VIII.9.11}$$

and the time factor can be written in a Schwarzschild-like form

$$a_T = \frac{1}{2}\left[\left(1 + 2\Phi_g^2\right)\cos\eta_g - \cos\eta\right]\Phi_g^{-2}. \tag{VIII.9.12}$$

(VIII.9.11) describes the relation between ρ_g and $\mathcal{R}_g$. From (VIII.9.10) it is evident that for the Schwarzschild case one has $\Phi_g^2 = 1$. Thus, one obtains the well-known Schwarzschild factor if one takes advantage of the embedding condition (VIII.9.1)

$$a_T = \frac{1}{2}\left[3\cos\eta_g - \cos\eta\right]. \tag{VIII.9.13}$$

We also note that one can get the interior Reissner-Nordström metric from the reduced metric with l = ie. The further structure of the reduced model is very similar to the Reissner-Nordström model. From (VIII.9.7) and (VIII.9.11) one obtains the *linking condition*

$$\mathcal{R}_g = \frac{r_g^2}{\sqrt{2Mr_g + l^2}}\ . \tag{VIII.9.14}$$

Having decided for a linking position at r_g, one can compute the assigned radius of the cap.

From (VIII.9.4) one can deduce the operators

$$\partial_0 = \frac{\partial}{\partial\mathcal{R}},\quad \partial_1 = \frac{\partial}{\mathcal{R}\partial\eta},\quad \partial_2 = \frac{\partial}{\mathcal{R}\sin\eta\partial\vartheta},\quad \partial_3 = \frac{\partial}{\mathcal{R}\sin\eta\sin\vartheta\partial\varphi},\quad \partial_4 = \frac{\partial}{a_T\partial it} \tag{VIII.9.15}$$

which are needed for the computation of the field strengths. We emphasize that the time factor a_T is a function of $\mathcal{R}$ and η and is only a function of η after having selecting a cap from the family with the embedding condition (VIII.9.1). As long as one examines the complete geometry viz the extrinsic and the intrinsic, one has to regard $\mathcal{R}$ as variable. Consequently,

$$a_{T|0} = -\frac{1}{\rho_g}\cos\eta, \quad a_{T|1} = \frac{1}{\rho_g}\sin\eta\,. \tag{VIII.9.16}$$

is valid.

Now we are able to compute the 5-dimensional components of the field strengths from the metric (VIII.9.4).

$$\begin{gathered} M_a = \left\{\frac{1}{\mathcal{R}},0,0,0,0\right\}, \quad B_a = \left\{\frac{1}{\mathcal{R}},\frac{1}{\mathcal{R}}\cot\eta,0,0,0\right\} \\ C_a = \left\{\frac{1}{\mathcal{R}},\frac{1}{\mathcal{R}}\cot\eta,\frac{1}{\mathcal{R}\sin\eta}\cot\vartheta,0,0\right\}, \quad E_a = \left\{\frac{1}{\rho_g a_T}\cos\eta,-\frac{1}{\rho_g a_T}\sin\eta,0,0,0\right\} \end{gathered}. \tag{VIII.9.17}$$

Thus, one obtains the 5-dimensional curvature equations

$$\begin{aligned} M_{a\underset{1}{|||}b} + M_a M_b &= 0, \quad M^c{}_{\underset{1}{|||}c} + M^c M_c = 0 \\ B_{a\underset{2}{|||}b} + B_a B_b &= 0, \quad B^c{}_{\underset{2}{|||}c} + B^c B_c = 0 \\ C_{a\underset{3}{|||}b} + C_a C_b &= 0, \quad C^c{}_{\underset{3}{|||}c} + C^c C_c = 0 \\ E_{a\underset{4}{|||}b} - E_a E_b &= 0, \quad E^c{}_{\underset{4}{|||}c} - E^c E_c = 0 \end{aligned}. \tag{VIII.9.18}$$

The 5-dimensional Ricci is composed of these equations. Thus, the geometrical basic structure of the interior NUT solution is entirely described.

VIII.10. The interior NUT solution, the reduced metric

In the preceding Section a metric has been analyzed which is very similar to the interior Schwarzschild metric. The differences of the two metrics come along essentially with the time factor a_T. The difference of the time-like parts is the cause why a single surface is not sufficient for a complete geometrical examination of the model. This is already demanded by the similarity to the Schwarzschild problem. Therefore we will deal with the time-like part of the reduced metric in greater detail. The reduced model can be deduced just like the Schwarzschild one from a pseudo-hyper sphere. We will not repeat this procedure. We will carry out more exactly only the considerations concerning the time-like part.

The time-like arc element dX^4 of the pseudo-hyper sphere is in agreement with (VIII.9.9)

$$dX^4 = -\mathcal{R}\cos\eta \, di\psi \qquad \text{(VIII.10.1)}$$

and is linked to the arc element of the physical surface dx^4 with

$$dX^4 = \mathfrak{p}^4_4 dx^4 \ . \qquad \text{(VIII.10.2)}$$

The latter is well-known from the 4-dimensional ansatz (VIII.9.9). Thus, the projector $\mathfrak{p} = \mathfrak{p}^4_4$ can be made accessible. With (VIII.9.8) and

$$-\mathcal{R}\cos\eta \, di\psi = \mathfrak{p} a_T \rho_g di\psi$$

one can deduce the negative quantity[139]

$$\mathfrak{p} = -\frac{\mathcal{R}\cos\eta}{\rho_g a_T} = -\frac{1}{\left(\frac{\mathcal{R}_g}{\mathcal{R}} + \frac{\rho_g}{\mathcal{R}}\right)\frac{\cos\eta_g}{\cos\eta} - 1} \ . \qquad \text{(VIII.10.3)}$$

With this projector the components of the gravitational field strength can be derived from the appropriate quantities of spherical geometry

$$E_a = \mathfrak{p}\left\{-\frac{1}{\mathcal{R}}, \frac{1}{\mathcal{R}}\tan\eta, 0, 0, 0\right\} . \qquad \text{(VIII.10.4)}$$

This projector will make it possible to represent more simply the stress-energy tensor of this model and compare it with the one of the Schwarzschild model. In order to obtain the stress-energy tensor the 0-components are to be extracted from the 5-dimensional Ricci and are to be shifted to the right side of the equation. One gets the well-known expression

$$\begin{aligned} R_{mn} = & m_m m_n \left(M_0 B_0 + M_0 C_0 - M_0 E_0\right) \\ & + b_m b_n \left(B_0 M_0 + B_0 C_0 - B_0 E_0\right) \\ & + c_m c_n \left(C_0 M_0 + C_0 B_0 - C_0 E_0\right) \\ & - u_m u_n \left(E_0 M_0 + E_0 B_0 + E_0 C_0\right) \end{aligned} \ . \qquad \text{(VIII.10.5)}$$

[139] It should be noted that a_T has a pole. The parameters in a_T and thus the linking to the exterior solution is performed in such a way that a_T always remains finite and positive.

If one contracts this relation[140], and if one evaluates the Einstein tensor, and inserts for the 0-components of the quantities the before-listed values, one has on the right side

$$T_{mn}=\begin{pmatrix} -p & & & \\ & -p & & \\ & & -p & \\ & & & \mu_0 \end{pmatrix}. \qquad \text{(VIII.10.6)}$$

The hydrostatic pressure

$$\kappa p = -(1+2\mathfrak{p})\frac{1}{\mathcal{R}^2} \qquad \text{(VIII.10.7)}$$

is formally identical with that the Schwarzschild and the Reissner-Nordström solution. However, it contains the NUT parameter as a consequence of the projector $\mathfrak{p}$ and the quantity ρ_g being included in $\mathfrak{p}$. The energy density

$$\kappa\mu_0 = \frac{3}{\mathcal{R}^2} \qquad \text{(VIII.10.8)}$$

is formally the same for all three models. Due to the adjustment (VIII.9.14) it depends likewise on the NUT parameter.

In the last paragraphs the connection with other solutions has been discussed. Before we turn to the actual interior NUT solution, the behavior of the reduced metric on the boundary surface of the exterior reduced model is still to be examined. The fastest way to this goal is to compute the quantities a_T and $\mathfrak{p}$ on the boundary surface. One gets

$$a_T^g = \cos\eta_g, \quad \mathfrak{p}_g = -\frac{\mathcal{R}_g}{\rho_g}. \qquad \text{(VIII.10.9)}$$

For the quantities B and C in (VIII.9.17) one immediately obtains agreement. From (VIII.10.4) one gets the expression

$$E_a^g = \left\{-\frac{1}{\rho_g}, \frac{1}{\rho_g}\tan\varepsilon_g, 0, 0, 0\right\}, \qquad \text{(VIII.10.10)}$$

by considering the question of the sign formerly discussed. This expression is likewise known from the exterior solution. If the 4-dimensional components of the field strengths agree on the boundary surface, then this is also valid for the metric and their first derivatives. This is generally demanded as a condition for the linking of solutions. From (VIII.10.7) firstly one has

$$\kappa p_g = -\left(1-2\frac{\mathcal{R}_g}{\rho_g}\right)\frac{1}{\mathcal{R}_g^2}.$$

If one considers the value of ρ_g from (VIII.9.11) and the linking condition (VIII.9.14) one gets with the definition (VIII.6.17)

$$\kappa p_g = \tilde{\omega}_g^2 \qquad \text{(VIII.10.11)}$$

[140] In contrast to the exterior solution one has $R \neq 0$.

in agreement with the first component of (VIII.6.19). Since M_0^{ext} and M_0^{int} differ due to the different curvatures of the exterior and interior surfaces, this is not valid for the surface tensions of the object. For the same reason the energy density on the boundary surface makes a jump. One has

$$\kappa\mu_0^g = \kappa\left(\mu_0^g\right)^{SS} + \tilde{\omega}_g^2\,, \qquad \text{(VIII.10.12)}$$

whereby the first term up to the different definition of ρ_g corresponds to the Schwarzschild expression for the energy density.

VIII.11. The interior NUT solution, the seed metric

In the preceding Section the geometrical basic structure of the interior NUT solution has been represented. The interior and exterior reduced metrics constitute together a complete gravitational model, whereby the exterior solution is endowed with a monopole field and thus formally belongs to the electrovac solutions. In order to approach the actual interior NUT solution, we will again perform an intrinsic transformation. It sets up an additional rule, how local distances and time intervals are to be measured. The transformed 4-bein has the form

$$\overset{1}{e}_1 = \alpha_0 \mathcal{R}, \quad \overset{2}{e}_2 = \alpha_0 \mathcal{R} \sin\eta = \alpha_D r = A, \quad \overset{3}{e}_3 = \alpha_D \mathcal{R} \sin\eta \sin\vartheta = A \sin\vartheta, \quad \overset{4}{e}_4 = a_D a_T$$
$$\alpha_D = \frac{A}{r}, \quad a_D = \frac{r}{A} \tag{VIII.11.1}$$

and thus the metric

$$ds^2 = \alpha_D^2 \left[\mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta\, d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta\, d\varphi^2 \right] + a_D^2 a_T^2 dit^2 . \tag{VIII.11.2}$$

On the surface the radial differential and the radial partial derivatives result in

$$dx^1 = \alpha_D \mathcal{R} d\eta = \frac{1}{\cos\eta} \alpha_D dr = \frac{1}{\sqrt{1 - \frac{r^2}{\mathcal{R}^2}}} \alpha_D dr, \quad \partial_1 = a_D \cos\eta \frac{\partial}{\partial r} \tag{VIII.11.3}$$

and perpendicularly to it

$$dx^0 = \alpha_D d\mathcal{R}, \quad \partial_0 = a_D \frac{\partial}{\partial \mathcal{R}} = a_D \sin\eta \frac{\partial}{\partial r} . \tag{VIII.11.4}$$

In the beginning the deformation factor is a function of $\mathcal{R}$ and η

$$\alpha_D = \alpha_D(\mathcal{R}, \eta) . \tag{VIII.11.5}$$

Only if one has selected a sphere from the family and has applied the embedding condition (VIII.9.1), α_D becomes only a function of η or r, respectively. The deformation factor α_D leads to a new field strength

$$D_a = \frac{1}{\alpha_D} \alpha_{D|a} = \left\{ -\frac{l^2}{A^3} \sin\eta, -\frac{l^2}{A^3} \cos\eta, 0, 0, 0 \right\} . \tag{VIII.11.6}$$

Further all quantities of the reduced metric are to be to revised and listed

$$M_0 = B_0 = C_0 = \frac{a_D}{\mathcal{R}} = \frac{1}{A} \sin\eta , \quad E_0 = -\frac{a_D}{\rho_g a_T} \cos\eta$$
$$B_1 = C_1 = \frac{a_D}{r} \cos\eta = \frac{1}{A} \cos\eta, \quad C_2 = \frac{1}{A} \cot\vartheta, \quad E_1 = -\frac{a_D}{\rho_g a_T} \sin\eta \quad . \tag{VIII.11.7}$$

These are the quantities of the reduced metric, extended by the deformation factor. The listing is still incomplete. With (VIII.11.6) the quantity D joins these expressions

$$
\begin{aligned}
&{}^*B_0 = B_0 + D_0 = \frac{a_D^2}{A}\sin\eta, \quad {}^*C_0 = C_0 + D_0, \quad {}^*E_0 = E_0 + D_0 \\
&{}^*B_1 = B_1 + D_1 = \frac{a_D^2}{A}\cos\eta, \quad {}^*C_1 = C_1 + D_1, \quad {}^*C_2 = C_2, \quad {}^*E_1 = E_1 + D_1
\end{aligned} \quad . \qquad \text{(VIII.11.8)}
$$

We recall the fact that these new quantities do not describe a new surface, but that they are still defined on the surface of the reduced metric, whereby new measuring prescriptions are settled on this surface. If one inserts these quantities into the Ricci one obtains a new set of subequations which have the structure of those equations which we have already found for previous models. However, they are richer. In order to facilitate the way to these relations we indicate some formulae in the mathematical appendix. It should be mentioned that already in these relations arises a quantity which is known as monopole field of the exterior solution

$$
\mathcal{H}_{23} = -i a_D \omega \cos\eta, \quad \mathcal{H}^2 = 2\mathcal{H}_{23}\mathcal{H}^{23} \ , \qquad \text{(VIII.11.9)}
$$

although the seed metric is a static metric. The rotational properties of the model are closely linked with geometry, in a similar way as it is known from the Kerr metric. The quantity $\mathcal{H}$ differs only in the trigonometric function from the corresponding quantity (VIII.8.13) of the exterior solution and coincides with this quantity on the boundary surface of the two solutions. For the subequations of the Ricci we note

$$
\begin{aligned}
{}^*B_{\underset{2}{1\|1}} + {}^*B_1{}^*B_1 &= -M_0{}^*B_0 - \mathcal{H}^2 \\
{}^*C_{\underset{3}{2\|2}} + {}^*C_2{}^*C_2 &= -B_0C_0 - D_1D_1 + \mathcal{H}^2 \\
{}^*C^s{}_{\underset{3}{\|s}} + {}^*C^s{}^*C_s &= -M_0{}^*C_0 - B_0C_0 - D_1D_1 \\
{}^*E_{\underset{4}{1\|1}} - {}^*E_1{}^*E_1 &= -M_0{}^*E_0 + 3D_0E_0 + B_1D_1 - 2\mathcal{H}^2 \\
{}^*E^s{}_{\underset{4}{\|s}} - {}^*E^s\,{}^*E_s &= -M_0{}^*E_0 - 2B_0E_0 + D_0E_0 + B_1D_1 - \mathcal{H}^2
\end{aligned} \quad . \qquad \text{(VIII.11.10)}
$$

If one processes this into the Ricci

$$
\begin{aligned}
R_{mn} = &-\left[{}^*B_{\underset{2}{n\|m}} + {}^*B_n{}^*B_m\right] - b_n b_m\left[{}^*B^s{}_{\underset{2}{\|s}} + {}^*B^s{}^*B_s\right] \\
&-\left[{}^*C_{\underset{3}{n\|m}} + {}^*C_n{}^*C_m\right] - c_n c_m\left[{}^*C^s{}_{\underset{3}{\|s}} + {}^*C^s{}^*C_s\right] \ , \\
&+\left[{}^*E_{\underset{4}{n\|m}} - {}^*E_n{}^*E_m\right] + u_n u_m\left[{}^*E^s{}_{\underset{4}{\|s}} - {}^*E^s{}^*E_s\right]
\end{aligned} \qquad \text{(VIII.11.11)}
$$

one obtains for its nonvanishing components

$$
\begin{aligned}
R_{11} &= \left[M_0{}^*B_0 + M_0{}^*C_0 - M_0{}^*E_0\right] + 3D_0E_0 + B_1D_1 \\
R_{22} &= \left[{}^*B_0M_0 + {}^*B_0C_0 - {}^*B_0E_0\right] + \mathcal{H}^2 + \omega^2 \\
R_{33} &= \left[{}^*C_0M_0 + {}^*C_0C_0 - {}^*C_0E_0\right] + \mathcal{H}^2 + \omega^2 \\
R_{44} &= -\left[{}^*E_0M_0 + {}^*E_0B_0 + {}^*E_0C_0\right] - \mathcal{H}^2 - \omega^2 + E_0D_0 + B_0D_0
\end{aligned} \qquad \text{(VIII.11.12)}
$$

and therefore also these of the stress-energy tensor

$$\begin{aligned}
\kappa T_{11} &= -2\,{}^*B_0E_0 + B_0C_0 + \text{Ħ}_{23}\text{Ħ}_{23} + \omega^2\cos^2\eta \\
\kappa T_{22} &= -2M_0E_0 + M_0B_0 + 2D_0E_0 - \text{Ħ}_{23}\text{Ħ}_{23} - \omega^2\cos^2\eta \\
\kappa T_{33} &= -2M_0E_0 + M_0C_0 + 2D_0E_0 - \text{Ħ}_{23}\text{Ħ}_{23} - \omega^2\cos^2\eta \;. \\
\kappa T_{44} &= -3M_0{}^*B_0 - \left(\text{Ħ}^2 + \omega^2\right) - \text{Ħ}_{23}\text{Ħ}_{23}
\end{aligned} \tag{VIII.11.13}$$

It has no off-diagonal item and is covariantly conserved.

VIII.12. The actual interior NUT solution

Since a position-dependent measuring procedure has been defined on the surface described by the reduced metric, an additional structure is implemented on this very surface by an intrinsic transformation. Thereby only the time-like part of the metric changes to

$$dx^4 = a_D a_T \left[2il(1-\cos\vartheta d\varphi) + idt \right]. \qquad \text{(VIII.12.1)}$$

It differs from that of the exterior solution by the gravitational factor. The 4-bein takes the form

$$\begin{array}{lll} \overset{3}{e}_3 = A\sin\vartheta, & \overset{4}{e}_3 = 2ila_D a_T\,(1-\cos\vartheta), & \overset{4}{e}_4 = a_D a_T \\ \underset{3}{e}^3 = \dfrac{1}{A\sin\vartheta}, & \underset{3}{e}^4 = -2il(1-\cos\vartheta), & \underset{4}{e}^4 = \alpha_D \alpha_T \end{array}. \qquad \text{(VIII.12.2)}$$

To the Ricci-rotation coefficients is attached an aggregate of the well-known structure

$$H_{mn}{}^s = H_{mn}u^s + H^s{}_m u_n + H^s{}_n u_m \qquad \text{(VIII.12.3)}$$

with

$$H_{23} = -ia_D a_T \omega\ . \qquad \text{(VIII.12.4)}$$

First, the field equations are enriched with the relation

$$R_{34} = H^s{}_{\underset{4}{3\|s}}\ . \qquad \text{(VIII.12.5)}$$

However, the Maxwell relations

$$H^s{}_{\underset{4}{n\|s}} = 0, \quad \mathrm{rot}\vec{H} = 0\ , \qquad \text{(VIII.12.6)}$$

are valid. Thus, no energy current arises inside the source. The complete Ricci has the form

$$R_{mn} = {}^*R_{mn} + 2u_{(m}H^s{}_{\underset{4}{n)\|s}} + \begin{pmatrix} 0 & & & \\ & -H^2 & & \\ & & -H^2 & \\ & & & +H^2 \end{pmatrix}, \ R = {}^*R - H^2\ , \qquad \text{(VIII.12.7)}$$

where ${}^*R_{mn}$ and ${}^*R$ are the Ricci and the invariant curvature scalar of the auxiliary model. The Einstein tensor reads as

$$G_{mn} = -\kappa\, {}^*T_{mn} - 2\left[H_m{}^s H_{ns} - \frac{1}{4} g_{mn} H^2 \right] + u_m u_n H^2\ , \qquad \text{(VIII.12.8)}$$

where *T has the form described in (VIII.10.6). The NUT term arising again is far away to be Maxwell-like, but is covariantly conserved. If we prove the *T to be free from divergence, the whole stress-energy tensor of the interior NUT model is divergence-free as it has to be. The divergence of the NUT term in (VIII.12.8) vanishes separately. This can be examined simply by considering the Maxwell-like relations (VIII.12.6) and

$$H_{\underset{4}{<mn\|s>}} + {}^*E_{<m}H_{ns>} = 0, \quad \mathrm{div}\vec{H} + {}^*\vec{E}\vec{H} = 0\ . \qquad \text{(VIII.12.9)}$$

The NUT field has only one component which points into the radial direction. One recognizes this by calculating the dual vector

$$H^{\gamma}=\frac{i}{2}\varepsilon_{\alpha\beta}{}^{\gamma}H^{\alpha\beta}=\left\{a_D a_T\omega,0,0\right\},\quad H^1\stackrel{*}{=}iH_{23},\quad \alpha=1,2,3\,. \tag{VIII.12.11}$$

Lastly, the values on the boundary surfaces of the geometries are to be computed. The somewhat complicated quantity

$$a_T^g=\left[\left(\mathcal{R}_g+\rho_g\right)\cos\eta_g-\mathcal{R}_g\cos\eta_g\right]\frac{1}{\rho_g}=\cos\eta_g=\cos\varepsilon_g \tag{VIII.12.12}$$

is substantially simplified on the boundary surface. Thus, the 0-components of the field quantities take the form

$$M_0^g=B_0^g=C_0^g=\frac{1}{A_g}\sin\eta_g=-\frac{1}{A_g}\sin\varepsilon_g,\quad E_0^g=\frac{a_D}{\rho_g}\,. \tag{VIII.12.13}$$

It is to be noted that first of all M_0^g cannot agree with the appropriate quantity of the exterior geometry, because the exterior and interior surfaces are completely different and have different radial curvatures. Further, one must consider that the local 0-directions of both systems have opposite orientations and therefore the 0-quantities arise with different signs. The 1-components take the form

$$B_1^g=C_1^g=\frac{1}{A_g}\cos\eta_g=\frac{1}{A_g}\cos\varepsilon_g,\quad E_1^g=-\frac{a_D}{\rho_g}\tan\eta_g=\frac{a_D}{\rho_g}\tan\varepsilon_g\,. \tag{VIII.12.14}$$

If one uses the fact that the NUT field quantities of the interior and the exterior solutions have, as a consequence of (VIII.12.12), the same value on the boundary surface

$$H_{23}^g=\mathcal{H}_{23}^g\,, \tag{VIII.12.15}$$

and thus go off continuously from the exterior to the interior, one obtains after some calculations with (VIII.11.13), (VIII.12.7), and with the auxiliary formulae of (VIII.12.13)

$$\kappa T_{mn}^g=\begin{pmatrix}0&&&\\&-2a_{Dg}^2\tilde{\omega}_g^2&&\\&&-2a_{Dg}^2\tilde{\omega}_g^2&\\&&&a_{Dg}^4\left(\dfrac{3}{\mathcal{R}_g^2}+\tilde{\omega}_g^2\right)\end{pmatrix}. \tag{VIII.12.16}$$

The radial hydrostatic pressure in the radial direction vanishes on the boundary surface. This ensures the stability of the construction. If one puts the NUT parameter zero, one obtains the analogous Schwarzschild expression on the boundary surface.

It is of some interest to transform the stress-energy tensor in such a way that it is comparable with the form of the stress-energy tensor of the Schwarzschild model previously written down. If one infers the first component from the relation for the seed metric (VIII.11.13) and if one supplements it with (VIII.12.8) to get to the actual NUT metric, one obtains

$$\kappa T_{11}=-2\,{}^*B_0E_0+B_0C_0+\mathcal{H}_{23}\mathcal{H}_{23}-H_{23}H_{23}+\omega^2\cos^2\eta\,.$$

From (VIII.10.4) one reads for the reduced metric $E_0=-\mathcal{P}/\mathcal{R}$ and one recognizes with (VIII.11.7) that one has to write for the actual metric

$$E_0 = -a_D \frac{\mathfrak{p}}{\mathcal{R}} \; . \tag{VIII.12.17}$$

The first term in the above expression becomes with (VIII.11.7)

$$-2{}^*B_0E_0 = -2B_0E_0 - 2D_0E_0 = 2\left[a_D^2 \frac{\mathfrak{p}}{\mathcal{R}^2} - i\frac{l^2}{A^3}\sin\eta\, a_D \frac{\mathfrak{p}}{\mathcal{R}}\right] = 2\mathfrak{p}\frac{a_D^2}{\mathcal{R}^2} - 2\mathfrak{p}\omega^2\sin^2\eta \; .$$

Altogether this results in

$$\kappa T_{11} = -\kappa p^{SS} - \left(1+2\mathfrak{p}\right)\omega^2\sin^2\eta + \omega^2 + \mathcal{H}_{23}\mathcal{H}_{23} - H_{23}H_{23} \; . \tag{VIII.12.18}$$

In it

$$\kappa p^{SS} = -\left(1+2\mathfrak{p}\right)\frac{a_D^2}{\mathcal{R}^2} \tag{VIII.12.19}$$

is the Schwarzschild-like portion of the hydrostatic pressure. For the other components of the stress-energy tensor one obtains in a similar way

$$\begin{gathered} \kappa T_{22} = \kappa T_{33} = -\kappa p^{SS} + \left(1+2\mathfrak{p}\right)\omega^2\sin^2\eta - \omega^2 - \mathcal{H}_{23}\mathcal{H}_{23} + H_{23}H_{23} \\ \kappa T_{44} = \kappa\mu_0 + \omega^2 + 3\mathcal{H}_{23}\mathcal{H}_{23} - 3H_{23}H_{23} \end{gathered} \; . \tag{VIII.12.20}$$

Therein

$$\kappa\mu_0 = a_D^4 \frac{3}{\mathcal{R}^2} \tag{VIII.12.21}$$

is the Schwarzschild-like energy density. The last conversion has made clear the relationship to and the difference from the Schwarzschild model. To relations with the Reissner-Nordström model we have referred further up. Similarities to other models will be discussed in the next Section. Hardly anybody has been concerned with interior solutions for the NUT metric. We have found a paper of Lukacs [L], Newman, Spaling, and Winicour. They have described an interior solution with a rigidly rotating fluid.

VIII.13. The members of the Kerr family

In the preceding Sections numerous models have been treated which are all of them in a close relationship. Thus, they constitute the *Kerr family*. The primary members of the family are the solutions for a static spherically symmetrical gravitational field found by Karl Schwarzschild in 1915 which are called after him. The charged version has been found by Reissner and Nordström. Both the gravitational fields and the electrical fields are attributed to the curvature of the space. Some decades later Kerr has found a solution for the field of a rotating source which surprisingly is reduced to the Schwarzschild solution, if one puts the rotation parameter zero. This solution has been extended by Newman for electrically charged sources. Thus, it is the generalization of the Reissner-Nordström solution. A little offside of the Kerr models is the NUT solution. It is directly related to the Schwarzschild solution. The rotational content of the metric is interpreted as a monopole field. Quevedo [Q] has extended the theory to a mass with multipole moments. The members of the Kerr family can be mixed at will. Demianski [D] and Newman and also Bini [B] have combined the NUT metric with the Kerr metric and have additionally introduced magnetic monopoles. Demianski and Newman have used complex transformations to deduce the Kerr metric and the NUT metric from the Schwarzschild metric. Krori [K] and Paul have supplemented the NUT metric with an electrical charge term and have considered motions. McGuire [M] and Ruffini have examined all possible combinations of the models. Most members of the Kerr family have been supplemented by interior solutions. For the pure Kerr metric there exist numerous trial solutions. We have suggested new interior solutions for some members of the family which are all based on the same geometrical principles. Interior and exterior solutions can be represented apart from the somewhat remote NUT solution by a common formula which then can be specialized in dependence on the model. All these solutions reduce to the two Schwarzschild solutions with a suitable choice of the parameters. Although the interior solutions are little convincing due to their model-like character and due to their adhering lack, it is nevertheless legitimate to sound out the mathematical possibilities and to get to know their limits. It is possible that new ansätze result from them.

Beyond the Kerr family there still exists an immense amount of axially symmetrical and stationary solutions. The solution of Weyl [W] has been a first ansatz. Some of these papers are mentioned below:

Abramowicz [A], Abramowicz [A] and Wagoner; Abramowicz [A], Miller and Stuchlik; Abramowicz [A] and Muchotrzeb; Abramowicz [A] and Lasota; Abramowicz [A], Lasota and Muchotrzeb; Abramowicz [A] and Szuszkiedwicz; Abramowicz [A], Nurowski and Wex; Abramowicz [A] and Bizak; Akeley [A], Andress [A], Bronnikov [B], Herlt [H], Kramer [K] and Neugebauer; Macdonald [M] and Thorne; Ray [R], Ray and Tiwari; Semerák [S], Teixeira [T], Wolk and Som; Vlachynsky [V], Tresguerres, Obukhov and Hehl; Addy [A] and Data; Aguirregabiria [A] et al; Ahmedov [A] and Ermamatov; Ahmedov [A] and Rakhmatov; Allen [A], Alonso [A], Anastasovski [A], Ardavan [A], Arzelies [A], Ashby [A], Ashworth [A] and Davies; Ashworth [A], Davies and Jennison; Ashworth [A] and Jennison; Aspden [A], Atwater [A], Bach [B], Balasin [B], Böhmer and Grumiller; Balasz [B] and Bertotti; Bampi [B] and Cianci; Bampi [B], Banados [B], Teitelboim and Zanelli; Banerjee [B] and Choudhury; Bardeen [B], Bass [B] and Pirani; Ben-Menahem [B], Berenda [B], Bergamini [B], Reina and Treves; Berger [B], Eardley and Olson; Bergh [B], Bezerra [B], Bicak [B] and Schmidt; Bini [B], Ruffini and Spoliti; Bini [B], Jantzen and Merloni; Bini [B], Cherubini and Mashhoon; Bini [B], Lusanna and Mashhoon; Bini [B], Carini and Jantzen; Bini [B], Mashhoon and Matravers; Bini [B], Geralico and Jantzen; Boachie [B], Bonnor [B], Boyer [B], Boyer [B] and Lindquist; Bratek [B], Brill [B], Brill [B] and Cohen, Brotas [B], Burton [B], Burzlaff [B] and Maison; Butterworth [B], Calvani [C], Salmistraro and Catenacci;

Calvani [C], de Felice and Nobili; Calvani [C] and Turolla; Calvani [C] and Francaviglia; Caporali [C], Carlson [C] and Safko; Carmeli [C] and Malin; Carmeli [C] and Kaye; Carmeli [C], Carmiati [C] and Cooperstock; Catenacci [C], Cavalleri [C], Chakraborty [C] and Sarkar; Chakravarty [C], Champeney [C], Isaak and Khan; Champeney [C] and Moon; Chandrasekhar [C], Culetu [C], Cheng-Deng Kuo [C], Cheo [C] and Heer; Chinea [C], Chitre [C], Hsu and Sherry; Chrobok [C], Obukhov and Scherfner; Chugreev [C], Chugreev [C] and Logunov; Ciufolini [C], Kopeikin and Mashhoon; Clark [C], Clement [C], Gal'tsov, and Guenouche; Clement [C], Cohen [C], Cohen [C] and Brill; Cohen [C] and Sarill; Cohen [C] and Toton; Cohen [C] and Kegeles; Cohen [C]; Collas [C] and Klein; Cooperstock [C] and de la Cruz; Cooperstock [C] and Lim; Corum [C], Cosgrove [C], Cox [D] and Kinnersley; Curir [C], Curir [C] and Francaviglia; Dadhich [D] and Turakulov; Dale [D], Damour [D] and Soffel; Damour [D], Das [D] and Banerji; Davies [D] and Ashworth; Davies [D] and Jennison; De Felice [D], De Felice [D] and Usseglio-Tomasset; De Smet [D], Dehnen [D], Delice [D], Demianski [D], Demianski [D] and Grishchuk; De Oliveira [D], Dietz [D] and Hoenselaers; Drake [D] and Szekeres; Dryuma [D], Duncan [D], Esposito and Lee; Eby [E], Eguchi [E] and Hanson; Embacher [E], Erez [E] and Rosen; Eris [E] and Nutku; Eris [E] and Gürses; Erlichson [E], Ernst [E], Esposito [E] and Witten; Evans [E], Farhoosh [F], and Zimmermann; Fennelly [F], Ferrari [F], Fischer [F], Florides [F], Frank [F], Frolov [F] and Stojkovic; Gariel [G], Garcia [G], Gauthier [G] and Gravel; Geroch [G], Ghosh [G] and Sengupta; Glass [G], Glass [G] and Wilkinson; Goenner [G] and Westphal; Gonzales [G] and Letelier; Gonzales-Martin [G], Grøn [G], Gupta [G] and Mitskiewicz; Gürses [G], Haggag [H] and Mahmoud; Hamilton [H] and Lisle; Hamity [H] and Gleiser; Han [H] Kim and Son; Harrison [H], Hartle [H] and Sharp; Hartle [H], Hawking [H] and Hartle; Hay [H], Schiffer, Cranshaw and Engelstaff; Hayward [H], Heer [H], Herlt [H], Herrera [H], Herrera [H] and MacCallum; Herrera [H] and Jimenez; Hoenselaers [H] and Vishveshvara; Hughes [H], Ibohal [I], Iftime [I], Ingraham [I], Irvine [I], Ise [I] and Uretsky; Islam [I], Israel [I], Israel [I] and Wilson; Israelit [I] and Rosen; Ives [I], Iyer [I] and Kumar; Iyer [I], Iyer [I] and Vishveshwara; Jantzen [J], [J], Jensen [J] and Kucera; Karas [K] and Hure; Kasuya [K], Kegeles [K], King [K], King [K] and Lasota; Kita [K], Kleihaus [K], Kunz and Navorro-Lerida; Klemm [K], Kodama [K], Komatsu [K], Eriguchi and Hachisu; Korotkii [K], Kramer [K] and Neugebauer; Kramer [K], Krasiński [K], Krori [K] and Chakravarty; Krori [K] and Paul; Krori [K] and Chaudhury; Krori [K], Bhattacharjee and Chaudhury; Kandt [K], Kyriakopoulos [K], Lal [L] and Singh; Lanczos [L], Leibowitz [L] and Meinhardt; Lense [L] and Thirring; Letelier [L], Levi-Civita [L], Lindblom [L] and Brill; Lindblom [L], Lobo [L] and Crawford; Longhi [L], Lorentz [L], Lynden-Bell [L] and Pineault; Lynden-Bell [L], Maeda [M] and Cohara; Maeda [M], Sasaki and Nakamura; Maison [M] , Majumdar [M], Maluf [M] and Ulhoa; Marcilhacy [M], Marek [M], Marinov [M], Markland [M], Markland [M] and Perjes; Mars [M] and Senovilla; Mashhoon [M], Mashhoon [M] and Theiss; Mashhoon [M], Hehl and Theiss; Mashkour [M], Matzner [M] and Misner; McFarlane [M] and McGill; McNamara [M], Meinel [M], Meinhardt [M] and Leibowitz; Mena [M], Natario and Tod; Misra [M], Misra [M] and Narain; Morgan [M] and Morgan; Nakamura [N], Maeda and Miyama; Nakamura [N] , Nakamura [N] and Sato; Nashed [N], Nayak [N] and Vishveshvara; Nelson [N], Neugebauer [N], Neugebauer [N] and Kramer; Neugebauer [N] and Meinel; Novello [N] and Rebouças; Obukhov [O], Chrobok and Scherfner; Olum [O] and Everett; Orio [O], Ozsvath [O], Ozsvath [O] and Schücking; Page [P], Papadopoulos [P] and Witten; Papadopoulos [P] and Stewart; Papapetrou [P] and Treder; Papapetrou [P], Papapetrou [P], Macedo and Som; Papini [P], Patel [P] and Misra; Patel [P], Paul [P], Pechlaner [P], Peng [P] and Qin; Perjes [P], Peters [P], Petterson [P], Pfister [P] and Schedel; Pfister [P] and Staudt; Phipps [P], Pietronero [P], Piran [P], Plebanski [P] and Schild; Plebanski [P] and Garcia; Plyatslo [P] and Pucas; Post [P], Prasanna [P], Prasanna [P] and Chakraborty; Prasanna [P] and Iyer; Prechtl [P], Quevedo [Q] and Ryan; Quevedo [Q], Quevedo [Q] and Mashhoon; Racz [R], Rao [R] and Panda; Reina [R] and Treves; Reina [R], Retzloff [R], Defacio and Dennis; Rindler [R] and Perlick; Rindler [R], Rodrigues [R] and Sharif; Rogava [R], Rosen [R] and Shamir; Rosen [R], Rosenthal [R], Rosquist [R], Ross [R], Ruggiero [R] and Tartaglia; Runge [R], Ryan [R], Sachs [S], Sackfield [S], Safko [S] and Witten; Safko [S], Sagnac [S], Sasaki [S], Maeda and Miyama; Schendel [S] and Winicour; Schiff [S], Schwinger [S], Schäfer [S], Semerák [S], Semerák [S] and Felice; Senovilla [S], Silberstein

[S], Singh [S] and Yadav; Singh [S], Rai and Yadav; Singh [S], Sloane [S], Stachel [S], Staudt [S] and Pfister; Stead [S], Donaldson and Donaldson; Stefano [S], Stewart [S], Stodolsky [S], Strauss [S], Strel'tsov [S], Synge [S], Takeno [T], Talbot [T], Tanabe [T], Tartaglia [T] and Ruggiero; Taubner [T] and Weinberg; Teixeira [T], Teyssandier [T], Thirring [T], Thorne [T] and Gürsel; Thorne [T] and Hartle; Tiwari [T], Rao and Kanakamedala; Tombrello [T] and Young. Tomimatsu [T] has interpreted the superposition of two Kerr-NUT metrics as a system of two black holes which are in equilibrium.

Further: Tomimatsu [T] and Sato; Trencevski [T] and Celakoska; Trümper [T], Tsalakou [T], Tsamparlis [T] and Mason; Urani [U] and Carlson; Urani [U] and Kemp; Van den Bergh [V] and Wils; Vishveshvara [V], Wahlquist [W] and Estabrook; Wahlquist [W], Walker [W] and Penrose; Wang [W], Waylen [W], Weinstein [W], Werner [W], Whitman [W], Whitmire [W], Wilkins [W], Will [W], Winicour [W], Whitman [W], Witte [W], Wright [W], Yamazaki [Y], Zhang [Z] and Beesham; Znajek [Z], Zsigrai [Z].

IX. Collapsing systems

IX.1. General remarks on collapsing systems

Since astrophysical observations suggest that stars collapse into dense objects after their nuclear fuel is exhausted, we turn to mathematical models that could describe such a collapse and thus we turn to the most difficult chapters of gravitational physics. So far it was only possible to establish exact solutions of Einstein's field equations for a gravitational collapse under very simplified conditions. Such solutions are those of Oppenheimer and Snyder, McVittie, and Weinberg. Modestly, they describe stellar objects of pressure-free dust which collapse in free fall. Therefore they may represent only a rough approximation of Nature. We will show that the approach to these solutions leads to inconsistencies.

The difficulties of such models consist in the fact that the Einstein's field equations initially are underdetermining. They contain more variables than equations which the system has. Most of the numerous approaches which can be found in the literature do not lead to the determination of the unknown quantities. The few exact solutions either cannot be interpreted physically or are unrealistic. Thus, many authors are of opinion that a determination of these variables is not necessary because the system of equations with these undetermined variables provides sufficiently useful statements. May [M] and White have fitted the unknown quantities with computer techniques in such a way that a physical interpretation of the field equations and the stress-energy-momentum tensor is available.

Tolman [T] has explained that it is even difficult to solve Einstein's field equations for static models. In further research he has dealt with expanding, contracting, and oscillating universes, but generally one or more parameters have been undetermined.

As the interest for the collapsing stars came up, many authors have used the preliminary work of Tolman and have tried to adapt the cosmological solutions for their purposes. This approach is problematic insofar as the cosmological models can deal with the infinite, but collapsing objects should be finite. Furthermore, the geometry which describes these objects has to be adjusted on its boundary to the Schwarzschild geometry which describes its external field. Both suppositions offer considerable difficulties.

Before we turn to *time-dependent* collapsing solutions, we examine how a static solution of Einstein's field equations which is a nonlinear partial differential equations of second order can be derived. In the course of this we use elements of the co-ordinate method, on the one hand, because this is very common, on the other hand, because the integration of the field equations requires a holonomic co-ordinate system. We also mention the Oppenheimer-Volkhoff equation, because it can support the solution process. If we have found a solution, nothing prevents us from formulating the problem covariantly. If a graphical geometry is apparent in the solution one can get a deeper insight by using the covariant method.

We consider the simple case that the stellar object which is to be described by the field equations is spherically symmetric and has the properties of a fluid. The stress-energy- momentum tensor has the form

$$T_{mn} = -p\,g_{mn} + \left(\mu_0 + p\right)u_m u_n \ , \qquad \text{(IX.1.1)}$$

where p is the hydrostatic pressure with equal values according to the three space-like directions, μ_0 the mass density of the object, and

$$u_m = \{0,0,0,1\} \tag{IX.1.2}$$

the observer field in a static system in which (IX.1.1) can be reduced to the simpler form

$$T_{mn} = -p'g_{mn} + \mu_0 u_m u_n \ . \tag{IX.1.3}$$

The line element for a spherically symmetric model is often put in the form

$$ds^2 = e^{\lambda}dr^2 + r^2 d\Omega^2 - e^{\nu}dt^2 \ , \tag{IX.1.4}$$

where $d\Omega^2$ stands for the lateral differentials. For spherically symmetric static systems we have repeatedly calculated the Ricci

$$\begin{aligned} R_{mn} = &-\left[B_{m\underset{2}{\|}n} + B_m B_n\right] - b_m b_n \left[B^s{}_{\underset{2}{\|}s} + B^s B_s\right] \\ &-\left[C_{m\underset{3}{\|}n} + C_m C_n\right] - c_m c_n \left[C^s{}_{\underset{3}{\|}s} + C^s C_s\right] . \\ &+\left[E_{m\underset{4}{\|}n} - E_m E_n\right] + u_m u_n \left[E^s{}_{\underset{4}{\|}s} - E^s E_s\right] \end{aligned} \tag{IX.1.5}$$

With its contraction

$$-\frac{1}{2}R = \left[B^s{}_{\underset{2}{\|}s} + B^s B_s\right] + \left[C^s{}_{\underset{3}{\|}s} + C^s C_s\right] - \left[E^s{}_{\underset{4}{\|}s} - E^s E_s\right] \tag{IX.1.6}$$

we are able to evaluate the Einstein tensor. The field quantities therein are still undetermined. The quantities in (IX.1.5) have the form with respect to (IX.1.4)

$$B_m = \left\{\frac{1}{r}e^{-\lambda/2},0,0,0\right\}, \quad C_m = \left\{\frac{1}{r}e^{-\lambda/2},\frac{1}{r}\cot\vartheta,0,0\right\}, \quad E_m = \left\{-\frac{1}{2}e^{-\lambda/2}\nu',0,0,0\right\} \ . \tag{IX.1.7}$$

Inserting these relations into (IX.1.5) and (IX.1.6), results for the Einstein tensor

$$\begin{aligned} G_{11} &= e^{-\lambda}\left(\frac{1}{r^2} + \frac{1}{r}\nu'\right) - \frac{1}{r^2} = \kappa p \\ G_{22} = G_{33} &= e^{-\lambda}\left[\frac{1}{2}\nu'' + \frac{1}{4}\left(\nu'\nu' - \lambda'\nu'\right) - \frac{1}{2r}\left(\lambda' - \nu'\right)\right] = \kappa p \ . \\ G_{44} &= e^{-\lambda}\left(\frac{1}{r^2} - \frac{1}{r}\lambda'\right) - \frac{1}{r^2} = -\kappa\mu_0 \end{aligned} \tag{IX.1.8}$$

The equations relate the unknown variables ν, λ to the pressure and the energy density at any co-ordinate position r within the fluid.

The Einstein field equations provide three differential equations for the four unknown quantities ν, λ, p, μ_0 which are generally all functions of r. The system of equations is underdetermining. To come to a solution, another independent equation is required which may be derived from the physical properties of the fluid. This may be the equation of state of the fluid, which establishes a relation between the pressure and the energy density.

Due to the high nonlinearity of the equations (IX.1.8) it is generally not expected that there exist explicit analytical solutions of the system, even if we have found an additional equation which associates p with μ_0 or both variables with r. If one sets up such a relation in such a manner that the system can be solved, the solution obtained may be physically useless.

Nevertheless, as there is hope to come to a physically reasonable result, we do up the system (IX.1.8) in such a manner that a solution is more accessible. We equate the first two relations through κp. It is easier to integrate the system in the form

$$\left(\frac{e^{-\lambda}-1}{r^2}+\frac{e^{-\lambda}\nu'}{2r}\right)'+e^{-\lambda-\nu}\left(\frac{e^{\nu}\nu'}{2r}\right)'=0 \ .$$

After a suitable choice for μ and λ Tolman has derived three already known solutions: the Einstein universe, the combined Schwarzschild-de Sitter solution, and the interior Schwarzschild solution. He has added five other solutions which are little if any of physical use.

Differentiating p with respect to r in the first equation (IX.1.8), equating the first two relations in (IX.1.8) through κp, and reordering one obtains

$$p'+\left(\mu_0+p\right)\frac{\nu'}{2}=0 \ . \qquad \text{(IX.1.9)}$$

Oppenheimer [O] and Volkhoff have calculated neutron stars and specified Eq. (IX.1.9) . It is called in this form Oppenheimer-Volkhoff equation. In covariant notation Eq. (IX.1.9) reads as

$$p_{|m}-\left(\mu_0+p\right)E_m=0 \qquad \text{(IX.1.10)}$$

and results without any great computational effort from the conservation law of the stress-energy-momentum tensor with respect to (IX.1.8) in tetrad notation. We have repeatedly used the equation in this form.

Although we have no intention to concern ourselves with neutron stars, we will add further development of the Oppenheimer-Volkhoff equation in the form (IX.1.9) because some authors do refer to this equation. A non-rotating star is to be surrounded by a Schwarzschild field. Thus, the problem of adjusting the quantities on the boundary surface arises. Therefore, OV have introduced the auxiliary variable

$$u\left(r\right)=\frac{r}{2}\left(1-a^2\right), \quad a^2=e^{-\lambda}=1-\frac{2u}{r} \ . \qquad \text{(IX.1.11)}$$

We see that u is a placeholder for the Schwarzschild mass to adjust the interior field to the exterior field. At the boundary of the stellar object we specifically expect

$$a_g^2=1-\frac{2M}{r_g}, \quad u_g=M \ . \qquad \text{(IX.1.12)}$$

Differentiating u with respect to r in (IX.1.11) one obtains

$$u'=-\frac{1}{2}\left[2raa'+a^2-1\right]=-\frac{r^2}{2}\left[e^{-\lambda}\left(\frac{1}{r^2}-\frac{1}{r}\lambda'\right)-\frac{1}{r^2}\right] \ .$$

Thus

$$u'=\frac{1}{2}\kappa\mu_0 r^2 \ , \qquad \text{(IX.1.13)}$$

as it is evident from (IX.1.8). Putting $e^{\nu}=C^2$ and using now the first equation of (IX.1.8) one has

$$\kappa p = \frac{a^2}{r^2} + \frac{2}{r}\frac{1}{C}C' - \frac{1}{r^2} \ .$$

With $2u = r\left(1-a^2\right)$ one arrives, after some reshaping, at

$$\left(\ln C\right)' = \frac{1}{2}\frac{\kappa p r^3 + 2u}{r\left(r-2u\right)} \ . \tag{IX.1.14}$$

Otherwise, one obtains from (IX.1.9)

$$p' = -\left(\mu_0 + p\right)\left(\ln C\right)' \tag{IX.1.15}$$

and one can write the previous equation as

$$p' = -\frac{1}{2}\left(\mu_0 + p\right)\frac{\kappa p r^3 + 2u}{r\left(r-2u\right)} \tag{IX.1.16}$$

Thus, one has with (IX.1.13) and (IX.1.16) two differential equations of 1st order in u and p, which may be integrated with the initial values $p = p_0, u = u_0 = 0$ at $r = 0$ and $p_g = 0$ at the surface of the star $r = r_g$ if in addition the equation of state of the matter distribution, i.e. the relation between p and μ_0 is known. Moreover, from (IX.1.15) one obtains

$$\left(\ln C\right)' = -\frac{p'}{\mu_0 + p}$$

and after integration the metric coefficient

$$C = C_g e^{-\int_0^{p(r)} \frac{dp}{\mu_0 + p}} \ ,$$

wherein the constant of integration C_g is chosen in such a manner that the function $C(r)$ at the boundary surface is continuous.

Considering the pressure gradient on the boundary surface of the geometries and for reasons of stability, also assuming that the pressure on the surface of the star vanishes, it is clear from (IX.1.16) and for $u = M$ that

$$p' = -\mu_0 \frac{M}{r^2\left(1 - \frac{2M}{r}\right)} \ .$$

With $\alpha = 1\big/\sqrt{1-2M/r}$ and $p_{|1} = \frac{1}{\alpha}\frac{\partial p}{\partial r}$ one has a simple relation for the pressure gradient

$$p_{|1} = \mu_0 E_1 \tag{IX.1.17}$$

induced by the Schwarzschild gravity

$$E_1 = -\alpha \frac{M}{r^2}$$

on the surface of the stellar object.

We have seen that the integration of Einstein's field equations under very simplified conditions, such as spherical symmetry and time independence, may be difficult. In case of a collapse, the metric quantities, the pressure, and the energy density are time-dependent

$$\nu = \nu(r,t), \quad \lambda = \lambda(r,t), \quad p = p(r,t), \quad \mu_0 = (t) \ .$$

For simple models, a further relation results from the conservation law in addition to (IX.1.10)

$$T^{4n}{}_{||n} = \mu_{0|4} + 3(\mu_0 + p)U_4 \ . \qquad \text{(IX.1.18)}$$

Therein U_4 is a time-like quantity which takes specific values in dependence on the model.

For a more realistic approach to a stellar object it is to assume that the mass density is not homogeneous, thus $\mu_0 = \mu_0(r,t)$. Not only the density, but also the mass itself can change. Radiating energy the star loses mass: $m = m(r,t)$. Similarly, the structure of a fluid sphere (IX.1.3) is not sufficient. An enhanced approach that takes into account the heating during the contraction, the shears, and the just mentioned energy dissipation would be

$$T_{mn} = -(p+\pi)g_{mn} + (\mu_0 + p + \pi)u_m u_n + 2q_{(m}u_{n)} + \varepsilon k_m k_n + \pi_{mn} \ . \qquad \text{(IX.1.19)}$$

Therein π_{mn} is proportional to the shear tensor

$$D_{mn} = u_{(m||n)} - \frac{1}{3}\Theta' g_{mn} + G_{(m}u_{n)} , \qquad \text{(IX.1.20)}$$

and π proportional to the expansion

$$\Theta = u^m{}_{||m} . \qquad \text{(IX.1.21)}$$

The acceleration of the particles is

$$G_m = u_{m||n}u^n \ , \qquad \text{(IX.1.22)}$$

q_m the heat transfer, ε the density of radiation and k_m a radial null vector. It applies

$$\begin{gathered} u^m q_m = 0, \quad u^m k_m = 1, \quad k^m k_m = 0 \\ \pi_{mn}u^n = 0, \quad \pi_{[mn]} = 0, \quad \pi^m{}_{||m} = 0 \end{gathered} \ . \qquad \text{(IX.1.23)}$$

Herrera [H], Di Prisco, Fuenmayor, and Troconis have investigated such an approach.

If one wants to work in a non-comoving system one has to take for the 4-velocity in the stress-energy-momentum tensor instead of (IX.1.2) the relation

$$u_m = \{-i\alpha v, 0, 0, \alpha\} \ .$$

Thus, another unknown quantity occurs, the speed of contraction $v(r,t)$. Knowing this quantity would be very stimulating. In any case, at the beginning of the collapse must be $v(r,0) = 0$, in the center of the stellar object $v(0,t) = 0$ at any time. The contraction velocity must be less than the velocity of free fall. The pressure inside the star is working against too high contraction speeds.

After having explained the general problems of collapsing systems we turn to some attempts which describe a gravitational collapse, known from the literature.

IX.2. The approach of Narlikar

The most interesting matter would be to find a collapsing solution which joins in a simple manner the exterior Schwarzschild solution. An early attempt stems from Narlikar [N]. He has started with the metric with $\mathcal{R} = \mathcal{R}(t)$

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta\, d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta\, d\varphi^2 - \cos^2\eta\, dt^2 \qquad \text{(VIII.2.1)}$$

which is reduced for $\mathcal{R} = \text{const.}$ to the de Sitter cosmological model. We want to represent the model in the usual way with covariant methods. From (VIII.2.1) we read the 4-bein system

$$\overset{1}{e}_1 = \mathcal{R}, \quad \overset{2}{e}_2 = \mathcal{R}\sin\eta, \quad \overset{3}{e}_3 = \mathcal{R}\sin\eta\sin\vartheta, \quad \overset{4}{e}_4 = \cos\eta \ . \qquad \text{(VIII.2.2)}$$

From this the field strengths are calculated in the same manner as for the de Sitter cosmos

$$\begin{aligned} B_m &= \left\{ \frac{1}{\mathcal{R}} \cot\eta, 0, 0, 0 \right\} \\ C_m &= \left\{ \frac{1}{\mathcal{R}} \cot\eta, \frac{1}{\mathcal{R}\sin\eta} \cot\vartheta, 0, 0 \right\} \ . \\ E_m &= \left\{ \frac{1}{\mathcal{R}} \tan\eta, 0, 0, 0 \right\} \end{aligned} \qquad \text{(VIII.2.3)}$$

The variables B, C, and E are time-dependent. Furthermore, the Ricci-rotation coefficients contain the quantities

$$A_{14}{}^1 = A_{24}{}^2 = A_{34}{}^3 = U_4 \qquad \text{(VIII.2.4)}$$

which can be calculated with

$$A_{14}{}^1 = -\overset{1}{e}_1 \underset{1|4}{e}{}^1 = \frac{1}{\mathcal{R}} \mathcal{R}_{|4}, \quad \partial_4 = \frac{\partial}{\cos\eta\, i \partial t} \ .$$

Defining

$$U_m = \left\{ 0, 0, 0, \frac{1}{\mathcal{R}} \mathcal{R}_{|4} \right\} \qquad \text{(VIII.2.5)}$$

one has for the Ricci-rotation coefficients

$$A_{mn}{}^s = {}^*A_{mn}{}^s + U_{mn}{}^s, \quad U_{mn}{}^s = U_n {}'\delta^s_m - {}'g_{mn} U^s, \quad U_{sn}{}^s = 3U_n \ . \qquad \text{(VIII.2.6)}$$

The prime at the kernel marks the 3-dimensional spatial quantities and A* contains the quantities (VIII.2.3).

Resolving the field equations for A* results in

$${}^*R_{mn} = \frac{3}{\mathcal{R}^2} g_{mn}, \quad {}^*R = \frac{12}{\mathcal{R}^2}, \quad {}^*G_{mn} = -\frac{3}{\mathcal{R}^2} g_{mn} \ . \qquad \text{(VIII.2.7)}$$

Taking into account in the field equations the relations

$$B_{n|4} + B_n U_4 = 0, \quad C_{n|4} + C_n U_4 = 0, \quad E_{n|4} = -E_n U_4, \quad U_{4|1} = U_4 E_1 \qquad \text{(VIII.2.8)}$$

one obtains for the Ricci

$$R_{mn} = {}^*R_{mn} - {}'g_{mn}\left[U_{4|4} + 3U_4U_4\right] - 3u_m u_n\left[U_{4|4} + U_4U_4\right] - 4U_{(m}E_{n)} \ . \qquad \text{(VIII.2.9)}$$

For $\mathcal{R} = \text{const.}$ only the first term remains in the above equation. The term is known from the de Sitter universe. The second and third terms correspond to the structure of the Friedman cosmos, the last term is a current.

With

$$R = {}^*R - 6U_{4|4} - 12U_4U_4, \quad U^n E_n = 0 \qquad \text{(VIII.2.10)}$$

one obtains the Einstein tensor

$$G_{mn} = -g_{mn}\frac{3}{\mathcal{R}^2} + {}'g_{mn}\left[2U_{4|4} + 3U_4U_4\right] + u_m u_n\left[3U_4U_4\right] - 4U_{(m}E_{n)} \ , \qquad \text{(VIII.2.11)}$$

whose divergence must vanish. To check this, we use the commutation relations for the anholonomic differentials

$$\Phi_{|[mn]} = A_{[nm]}{}^s\Phi_{|s} \ .$$

With $A_{41}{}^4 = -E_1$ one obtains

$$U_{4|41} - U_{4|14} = U_4U_{4|1} + E_1U_{4|4} \ .$$

Substituting the last relation of (VIII.2.8) and using the last but one thereof, one finally has

$$U_{4|41} = 2U_{4|4}E_1 \ . \qquad \text{(VIII.2.12)}$$

If we calculate the first component of the divergence of the Einstein tensor using (VIII.2.12) we obtain

$$G^{1n}{}_{||n} = 2U_{4|4}E_1 \ . \qquad \text{(VIII.2.13)}$$

Since this term should vanish it has to be

$$U_{4|4} = 0 \ . \qquad \text{(VIII.2.14)}$$

Hence the suspicion arises that

$$U_4 = 0, \quad \mathcal{R}^{\bullet} = 0 \qquad \text{(VIII.2.15)}$$

and that the system (VIII.2.1) is static even though the ansatz is time-dependent. We see this as a relationship to the Birkhoff theorem, which applies to the exterior Schwarzschild solution. We want to deepen the problem and thereby refer to general difficulties which arise from collapsing approaches.

A collapsing stellar object is described by an interior solution. Thus, the postulation of a non-vanishing stress-energy-momentum tensor is necessary. A simple approach with $p = p(r,t)$ as hydrostatic pressure and $\mu_0 = \mu_0(t)$ as the mass density would be

$$T_{mn} = -p\,g_{mn} + (\mu_0 + p)u_m u_n \ . \qquad \text{(VIII.2.16)}$$

Since we choose a non-comoving reference system the 4-velocity has the form

$$u_m = \{-i\alpha_c v_c, 0, 0, \alpha_c\} \qquad \text{(VIII.2.17)}$$

as seen by an observer in rest. Therein v_c is the radial velocity of the particles following the collapse. The first difficulty is that v_c is an unknown quantity which cannot be determined with Einstein's field equations. The second is that (VIII.2.16) prescribes a specific structure of the stress-energy-momentum tensor.

Inserting (VIII.2.17) into (VIII.2.16) one has

$$\begin{aligned} T_{11} &= -p - \alpha_c^2 v_c^2 \left(\mu_0 + p\right) \\ T_{22} &= T_{33} = -p \\ T_{44} &= \mu_0 + \alpha_c^2 v_c^2 \left(\mu_0 + p\right) \\ T_{14} &= -i\,\alpha_c^2 v_c \left(\mu_0 + p\right) \end{aligned} . \tag{VIII.2.18}$$

The last equation indicates a non-comoving reference system. With (VIII.2.11) one would have

$$\kappa T_{14} = 2E_1 U_4 \ ,$$

which can only with difficulty brought into accordance with (VIII.2.18).

However, Eq. (VIII.2.11) shows that one has $G_{11} = G_{22} = G_{33}$ which is in obvious contradiction to (VIII.2.18). The cause of the three components of the Einstein tensor of the same kind is already contained in the ansatz (VIII.2.1) or (VIII.2.2), respectively. The spatial metric coefficients contain a single time-dependent quantity $\mathcal{R}$ which enters equally into the relation (VIII.2.6) and defines the structure of the stress-energy-momentum tensor.

In addition, the metric (VIII.2.1) reminds one of the metric of the static de Sitter universe, which is the metric on a pseudo-hyper sphere. If the metric of Narlikar describes the interior of a collapsing stellar object, the spatial part of the metric is to be interpreted as the cap of a pseudo-hyper sphere with the radius $\mathcal{R}$. With $\mathcal{R} = \mathcal{R}(t)$ it is admitted that the spherical cap can shrink. Since the approach of Narlikar evidently refers to a non-comoving observer, $r = \mathcal{R}\sin\eta$ is considered where r indicates the position of an observer in this system. Since for a static observer the position does not change with time, $r_{|4} = 0$ applies, from which it would immediately follow that $\mathcal{R}_{|4} = 0$ and consequently $U_4 = 0$. Thus, we have immediately deduced the result (VIII.2.15).

On the other hand, one can define the polar angle η as a function of time. From $r_{|4} = 0$ and $\sin\eta = r/\mathcal{R}$ one recognizes that one obtains a quantity $U_4 \neq 0$. Thus, one has discarded the concept of Narlikar.

We note that Narlikar has not attempted to present his model as an interior solution of the well-known exterior Schwarzschild solution. In this case he would have needed to specify the *linking conditions*. Then the space-like part of the model would have made less trouble. For the time-like part, however, considerable problems were encountered, as other models show as well.

In the next Sections we treat pressure-free solutions, which, however, have deficiencies regarding their physical interpretation.

IX.3. The ansatz of Oppenheimer and Snyder

In 1939, Oppenheimer [O] and Snyder presented a paper now known as that paper which has given rise to the theory of black holes, although the term 'black hole' has been introduced much later. Moreover, the OS approach differs from current methods to implement black holes. The OS model is composed of a collapsing interior and an exterior solution, where the exterior part is the static Schwarzschild solution, which remains static due to the Birkhoff theorem even if the field generating stellar object collapses. Most approaches to a black hole do not use an interior solution. In this case the exterior Schwarzschild solution or the exterior Kerr solution is extended beneath the event horizon. The inner regions of these solutions are to describe black holes.

The OS model is based on an existing cosmological solution by Tolman [T]. The stellar object is made of pressure-free dust with homogeneous density. Since in this case, the internal resistance against a contraction is missing, the object cannot be static. It collapses as a consequence of its own gravitational attraction[141].

A completely pressure-free star is not physically realistic, because the particles of a star finally come so close during the collapse that pressure can be expected at a sufficiently high density. A pressure-free stellar object may approximately describe a dying star. If the thermonuclear processes are exhausted inside a star, they give way to the star's own gravitational attraction, and the star collapses. The just-discussed simplification to $p = 0$ is primarily on practical grounds. The integration of Einstein's field equations without this condition leads to considerable difficulties, and it is hard to find an analytical solution.

In addition to these above-mentioned limitations the OS model permits further criticism. The star collapses with the velocity of observers coming in free fall from *infinity*. The stellar object at the time $t = 0$ would have been infinitely large. We will work out this in the following. Mitra [M] has shown that inconsistencies in the OS model can only be resolved if the mass of a hypothetical OS black hole is $M = 0$. However, a massless star is contrary to the widespread opinion that black holes are supermassive objects. For these and other reasons which we will point out the OS model does not provide an appropriate basis for a black hole.

The linking conditions on the boundary surface between the interior and exterior solutions, which are not treated extensively in the literature, are also of special interest. One has to deal with the question as to why the OS metric though time-dependent appears to be flat.

[141] What could have prevented a pressureless star to contract before its collapse has started remains an open question. A realistic stellar object is kept in balance by inner nuclear and thermodynamic processes. However, this excludes the absence of pressure.

IX.4. The OS interior solution, basic relations

The first considerations are closely related to the original paper by OS, but we have to introduce auxiliary variables which are in close connection with geometrical quantities. Both, the interior and the exterior solutions are treated in two different co-ordinate systems: the one is comoving with the collapsing matter and the other one is not comoving. It is the very transition between the two systems which brings insight into how the collapse proceeds, and thus, sheds light onto the inconsistencies of the model. Our first effort will be to examine the quantities and the relations of the OS-model, to regroup them, and to get contiguity with familiar matters, and to prepare a geometric interpretation.

For the comoving and for the non-comoving co-ordinate system we use the notations

$$\{r',\vartheta,\varphi,t'\}, \quad \{r,\vartheta,\varphi,t\} \ . \tag{VIII.4.1}$$

In the comoving system we write the line element according to OS as

$$ds^2 = e^{\bar{\omega}}dr'^2 + e^{\omega}\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) - dt'^2, \quad \bar{\omega} = \bar{\omega}(r',t'), \quad \omega = \omega(r',t') \ . \tag{VIII.4.2}$$

For the two metric factors OS put

$$e^{\omega} = \left(G + Ft'\right)^{4/3}, \quad e^{\bar{\omega}} = \frac{1}{4}e^{\omega}\left(\frac{\partial\omega}{\partial r'}\right)^2 , \tag{VIII.4.3}$$

wherein one has for the interior solution

$$G = \sqrt{r'^3}, \quad F = -\frac{3}{2}\sqrt{2M}\sqrt{\frac{r'^3}{r_g'^3}} \tag{VIII.4.4}$$

with r_g' the value of r' on the surface of the stellar object, i.e. at the boundary of the interior and exterior solutions. We note the auxiliary variables

$$\mathcal{R}_g' = \sqrt{\frac{r_g'^3}{2M}}, \quad \rho_g' = 2\mathcal{R}_g' , \quad \Lambda = 1 - \frac{3}{\rho_g'}t' \ . \tag{VIII.4.5}$$

As in the non-comoving system the lateral part of the metric has the form

$$r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) ,$$

and this form is conserved under a co-ordinate transformation between these two systems a comparison with (VIII.4.2) gives

$$r^2 = e^{\omega} \ . \tag{VIII.4.6}$$

With (VIII.4.3) and (VIII.4.5) we obtain the relation

$$r = \Lambda^{2/3} r' \ . \tag{VIII.4.7}$$

From the point of view of the comoving observer the radial co-ordinate of the surface does not change

$$\frac{\partial r_g'}{\partial r'} = 0, \quad \frac{\partial r_g'}{\partial t'} = 0 \ . \tag{VIII.4.8}$$

For the quantity Λ one gets the relations

$$\frac{\partial \Lambda}{\partial r'} = 0, \quad \frac{\partial \Lambda}{\partial t'} = -\frac{3}{\rho'_g} \quad , \tag{VIII.4.9}$$

which we need for some calculations. For the radial co-ordinate of the non-comoving observer system one gains, taking advantage of (VIII.4.7) and the above formulae,

$$\frac{\partial r_g}{\partial r'} = \frac{\partial}{\partial r'}\left(\Lambda^{2/3} r'_g\right) = 0, \quad \frac{\partial r_g}{\partial t'} = \frac{\partial}{\partial t'}\left(\Lambda^{2/3} r'_g\right) = -\frac{r_g}{\mathcal{R}_g} = -\sqrt{\frac{2M}{r_g}} \quad . \tag{VIII.4.10}$$

In this calculation the relations

$$\mathcal{R}_g = \Lambda \mathcal{R}'_g \,, \quad \rho_g = \Lambda \rho'_g \tag{VIII.4.11}$$

have been used which can be verified with (VIII.4.5) and (VIII.4.7). Thus, we have also motivated the introduction of these auxiliary variables.

$$\rho = \sqrt{\frac{2r^3}{M}} \tag{VIII.4.12}$$

is the curvature radius of the Schwarzschild parabola. $\mathcal{R}_g$ has half the length of ρ_g, the curvature radius at the boundary surface. If one extends the curvature vector of the Schwarzschild parabola to the directrix of the Schwarzschild parabola, the resulting distance between the Schwarzschild parabola and the directrix has the length

$$\mathcal{R} = \sqrt{\frac{r^3}{2M}} \quad . \tag{VIII.4.13}$$

(VIII.4.11) refers to the values on the boundary surface. The quantities occurring in the OS model allow a geometric interpretation which facilitates the understanding of the theory. We leave the elaboration of the geometrical foundations to a later Section. Since the proper time T' coincides with the co-ordinate time t' in the comoving system, the second relation (VIII.4.10) can be written as

$$\frac{\partial r_g}{\partial t'} = v_g, \quad v_g = -\sqrt{\frac{2M}{r_g}} \,, \tag{VIII.4.14}$$

whereby v_g is the velocity of an observer who is in free fall in the Schwarzschild field, coming from the infinite and reaching the surface of the stellar object. However, this means that the surface itself has this speed and must come from infinity. After a brief calculation we found out an inconsistency of the OS model. This problem will be represented in detail later on.

Using (VIII.4.10) the changes of further variables which relate to the surface can be calculated. We summarize the results with

$$\frac{\partial r_g}{\partial r'} = 0, \quad \frac{\partial r_g}{\partial t'} = v_g, \quad \frac{\partial \mathcal{R}_g}{\partial r'} = 0, \quad \frac{\partial \mathcal{R}_g}{\partial t'} = -\frac{3}{2}, \quad \frac{\partial \rho_g}{\partial r'} = 0, \quad \frac{\partial \rho_g}{\partial t'} = -3 \quad . \tag{VIII.4.15}$$

In the next step we calculate the changes of the relevant variables in the interior of the stellar object. From (VIII.4.7) and (VIII.4.9) one gets

$$\frac{\partial r}{\partial r'} = \Lambda^{2/3} = \frac{r}{r'}, \quad \frac{\partial r}{\partial t'} = -\frac{r}{\mathcal{R}_g} = v_I \quad . \tag{VIII.4.16}$$

v_I is the speed in the interior, i.e. the speed with which the particles draw near in the interior during the collapse. The velocity decreases linearly inwards, and in the center of the object one has

$$v_I(0) = 0 \ . \tag{VIII.4.17}$$

Finally, the second relation (VIII.4.3) should be resolved. According to (VIII.4.16) one has

$$\frac{\partial e^{\omega/2}}{\partial r'} = \frac{r}{r'}$$

and at last

$$e^{\bar{\omega}} = \Lambda^{4/3} = \frac{r^2}{r'^2} \ . \tag{VIII.4.18}$$

Hence all the metric coefficients of the inner OS solution can be written in a form that simplifies the further considerations

$$\overset{1'}{e}_{1'} = \frac{r}{r'}, \quad \overset{2'}{e}_{2'} = r, \quad \overset{3'}{e}_{3'} = r\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1 \ . \tag{VIII.4.19}$$

The metric can be written either as

$$ds^2 = \frac{r^2}{r'^2}dr'^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 - dt'^2 \tag{VIII.4.20}$$

or as

$$ds^2 = \Lambda^{4/3}\left[dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta d\varphi^2\right] - dt'^2 \ . \tag{VIII.4.21}$$

The latter form is entirely written in comoving co-ordinates and the factor Λ contains the time dependence of the otherwise flat 3-dimensional line element.

In order to adapt the notation of authors who deal with collapsing models we introduce the *scale factor*

$$\mathcal{K}(t') = \Lambda^{2/3} \tag{VIII.4.22}$$

with which we can basically describe the relation between comoving and non-comoving co-ordinates

$$r = \mathcal{K}r' \ . \tag{VIII.4.23}$$

With comoving co-ordinates we obtain instead of (VIII.4.19)

$$\overset{1'}{e}_{1'} = \mathcal{K}, \quad \overset{2'}{e}_{2'} = \mathcal{K}r', \quad \overset{3'}{e}_{3'} = \mathcal{K}r'\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1 \tag{VIII.4.24}$$

and instead of (VIII.4.9)

$$\frac{\partial\mathcal{K}}{\partial r'} = 0, \quad \frac{1}{\mathcal{K}}\frac{\partial\mathcal{K}}{\partial t'} = -\frac{1}{\mathcal{R}_g}; \quad \mathcal{K}_{|1'} = 0, \quad \frac{1}{\mathcal{K}}\mathcal{K}_{|4'} = \frac{i}{\mathcal{R}_g} \ . \tag{VIII.4.25}$$

The last expression we shall meet again in the field equations and it is also known for other time-dependent models. From (VIII.4.16) and (VIII.4.24) we read

$$\frac{\partial r}{\partial r'} = \mathcal{K} \ . \tag{VIII.4.26}$$

With this and with (VIII.4.24) one has for $t' = \text{const.}$

$$dx^{1'} = K dr' = dr$$

from which one recognizes that the radial arc element corresponds to a segment on the r-axis. Thus, the OS interior geometry appears to be flat. In a later Section we will critically return to this subject.

IX.5. The OS interior solution, co-ordinate and reference systems

In the previous Section we have already made use of the variable r which designates the radial co-ordinate in the non-comoving system. OS have specified the relation of t' and t. Thus, a matrix can be assembled for the transformation between the two co-ordinate systems. The finding of such a transformation is obviously quite tedious. It is probably for this reason that other authors did not provide such a co-ordinate transformation for their models, or the putting up of a co-ordinate transformation was not possible because the model has no analytical solution. The use of different co-ordinate systems is apparently the only purpose of providing calculations on the simplest possible basis. The great advantage lies in the fact that such a co-ordinate transformation is accompanied by a Lorentz transformation which contains the velocity parameters. If one has found such a Lorentz transformation, and if one refers to the velocity of the surface of the stellar object, one has the *physical* velocity of the collapse at hand.

In the last Section we have already prepared the way for a Lorentz transformation. From (VIII.4.16) and (VIII.4.19) we obtain

$$dr = \frac{r}{r'}dr' - \frac{r}{\mathcal{R}_g}dt' = dx^{1'} - iv_I dx^{4'}, \quad v_I = -\frac{r}{\mathcal{R}_g} \ , \qquad \text{(VIII.5.1)}$$

wherein the $\{dx^{1'}, dx^{4'}\}$ are the anholonomic tetrad differentials. By means of an auxiliary variable

$$y = \frac{1}{2}\left[\left(\frac{r'}{r'_g}\right)^2 - 1\right] + \frac{r'_g}{2M}\frac{r}{r'}, \quad dy = \frac{r'}{r'^2_g}dr' - \frac{1}{2M}\sqrt{\frac{2M}{r_g}}dt' \qquad \text{(VIII.5.2)}$$

and from

$$t = \frac{2}{3}\frac{1}{\sqrt{2M}}\left[r'^{3/2}_g - (2My)^{3/2}\right] - 4M\sqrt{y} + 2M\ln\frac{\sqrt{y}+1}{\sqrt{y}-1} \qquad \text{(VIII.5.3)}$$

one gets, by differentiating,

$$dt = -2M\frac{y^{3/2}}{y-1}\frac{r'}{r'_g}dr' + \sqrt{\frac{2M}{r_g}}\frac{y^{3/2}}{y-1}dt' \ . \qquad \text{(VIII.5.4)}$$

From this one can read the transformation coefficients of the co-ordinate transformation

$$\Lambda^i_{i'} = x^i_{|i'} \ . \qquad \text{(VIII.5.5)}$$

Since these coefficients are orthogonal, one gains also the reciprocal values

$$\Lambda^1_{1'} = \frac{r}{r'}, \quad \Lambda^1_{4'} = i\frac{r}{\mathcal{R}_g} = -iv_I, \quad \Lambda^4_{1'} = -2Mi\frac{y^{3/2}}{y-1}\frac{r'}{r'^2_g}, \quad \Lambda^4_{4'} = \frac{y^{3/2}}{y-1}\sqrt{\frac{2M}{r_g}}$$

$$\Lambda^{1'}_1 = \alpha_I^2\frac{r'}{r}, \quad \Lambda^{4'}_1 = -i\alpha_I^2 v_I, \quad \Lambda^{1'}_4 = -i\alpha_I^2\frac{y-1}{y^{3/2}}\frac{r'}{r_g}, \quad \Lambda^{4'}_4 = \alpha_I^2\frac{y-1}{y^{3/2}}\sqrt{\frac{r_g}{2M}} \ . \tag{VIII.5.6}$$

$$\alpha_I = \frac{1}{\sqrt{1-\frac{2M}{r}\frac{r'^3}{r'^3_g}}} = \frac{1}{\sqrt{1-\frac{r^2}{\mathcal{R}_g^2}}}, \quad v_I = -\frac{r}{\mathcal{R}_g}$$

In these expressions all indices are co-ordinate indices. From

$$g^{ik} = \Lambda^{i\ k}_{i'k'} g^{i'k'}$$

and (VIII.4.19) can be calculated the 4-bein $\underset{m}{e}{}^i$ and from this the reciprocal one

$$\overset{1}{e}_1 = \alpha_I, \quad \overset{2}{e}_2 = r, \quad \overset{3}{e}_3 = r\sin\vartheta, \quad \overset{4}{e}_4 = \alpha_I\sqrt{\frac{r_g}{2M}\frac{y-1}{y^{3/2}}} \ . \tag{VIII.5.7}$$

The indices m number the 4-bein vectors of the reference system which is associated with the observers at rest. Now it is easy to calculate the corresponding Lorentz transformation connecting the comoving observers and the observers at rest

$$L^m_{m'} = \overset{m}{e}_i \Lambda^i_{i'} \underset{m'}{e}{}^{i'} \ . \tag{VIII.5.8}$$

$$L^1_{1'} = \alpha_I, \quad L^1_{4'} = -i\alpha_I v_I, \quad L^4_{1'} = i\alpha_I v_I, \quad L^4_{4'} = \alpha_I$$

Thus, one has the physical quantities describing the collapse at hand. On the surface one has with

$$\overset{1}{e}_1 = \alpha_I^g = \frac{1}{\sqrt{1-\frac{r_g^2}{\mathcal{R}_g^2}}} = \frac{1}{\sqrt{1-\frac{2M}{r_g}}}, \quad \overset{4}{e}_4 = \alpha_I^g\sqrt{\frac{r_g}{2M}}\frac{\frac{r_g}{2M}-1}{\sqrt{\left(\frac{r_g}{2M}\right)^2}} = \sqrt{1-\frac{2M}{r_g}}$$

the Schwarzschild values of the exterior field on the surface of the stellar object. Thus, the two solutions are matched on the boundary surface.

For the velocity on the surface one obtains

$$v_I^g = -\frac{r_g}{\mathcal{R}_g} = -\sqrt{\frac{2M}{r_g}} \ , \tag{VIII.5.9}$$

the velocity of an observer who is in free fall from infinity. Thus, we have once again proved by using physically relevant equations that the surface of the stellar object collapses in free fall from infinity. From (VIII.5.9) we also recognize that for $r_g = \infty$ the initial velocity is $v_I^g\left(r_g = \infty\right) = 0$.

In (VIII.4.9) we had developed

$$\frac{\partial\Lambda}{\partial t'} = -\frac{3}{\rho'_g} \ .$$

Since Λ does not depend on r one has

$$dt' = -\frac{\rho_g'}{3} d\Lambda, \quad t' = -\frac{\rho_g'}{3}\Lambda + C = -\frac{\rho_g}{3} + C \ ,$$

also relying on (VIII.4.11). ρ_g is the radius of curvature of the Schwarzschild parabola at the boundary surface, which has its minimum value $\rho_g\left(2M\right) = 4M$ at the vertex of the Schwarzschild parabola. From this we determine the constant of integration and finally we have

$$t' = \frac{4M}{3} - \frac{\rho_g}{3} \ . \qquad \text{(VIII.5.10)}$$

That is up to sign the formula for the rise time in the Schwarzschild field. At $r_g = 2M$ applies $t' = 0$ and at $r_g = \infty$ applies $t' = \infty$. Because of the invariance under time reversal it can be concluded that the surface of the collapsing object begins to collapse at the time $t' = 0$ at infinity and reaches the event horizon after an infinite proper time. Since for a freely falling observer in the Schwarzschild field the co-ordinate time t' matches the proper time T', the collapsing star would take an infinitely long time to contract to the event horizon.

The OS stellar object in its initial state would have an infinitely large extension, collapses in free fall and leaves empty space behind it in which a Schwarzschild field spreads. However, the collapse velocity would reach the speed of light at $r_g = 2M$ which has to be ruled out by the principle of relativity. At this location the gravity and tidal forces would be infinitely large. Under these conditions a star cannot exist. The OS model is afflicted with all those problems which are known from the Schwarzschild theory. Consequently, the OS model cannot be used as a base model for a black hole.

IX.6. The OS exterior solution

For the exterior solution OS start from the same metric (VIII.4.2) with the ansatz (VIII.4.3) where now we have[142]

$$G=\sqrt{r'^3},\quad F=-\frac{3}{2}\sqrt{2M},\quad \Lambda=1-\frac{3}{\rho'}t',\quad \rho'=\sqrt{\frac{2r'^3}{M}} \qquad \text{(VIII.6.1)}$$

and the following auxiliary formulae

$$\frac{\partial\Lambda}{\partial r'}=\frac{9}{4}\sqrt{\frac{2M}{r'^5}}t',\quad \frac{\partial\Lambda}{\partial t'}=-\frac{3}{\rho'},\quad \frac{\partial\omega}{\partial r'}=\frac{2}{\Lambda r'} \qquad \text{(VIII.6.2)}$$

apply. After similar calculations such as we have performed for the interior solution we obtain the tetrads in the comoving system

$$\overset{1'}{e}_{1'}=\sqrt{\frac{r'}{r}}=\Lambda^{-1/3},\quad \overset{2'}{e}_{2'}=r,\quad \overset{3'}{e}_{3'}=r\sin\vartheta,\quad \overset{4'}{e}_{4'}=1 \qquad \text{(VIII.6.3)}$$

and from

$$r=\Lambda^{2/3}r' \qquad \text{(VIII.6.4)}$$

with (VIII.4.11) and (VIII.4.13) the auxiliary formulae

$$\frac{\partial r}{\partial r'}=\sqrt{\frac{r'}{r}},\quad \frac{\partial r}{\partial t'}=-\frac{r}{\mathfrak{R}}=-\sqrt{\frac{2M}{r}}=v_E\ , \qquad \text{(VIII.6.5)}$$

in which we recognize the Schwarzschild velocity of free fall from infinity. OS put for the time of the system at rest

$$t=\frac{2}{3}\frac{1}{\sqrt{2M}}\left(r'^{3/2}-r^{3/2}\right)-2\sqrt{2Mr}-2M\ln\frac{\sqrt{r}-\sqrt{2M}}{\sqrt{r}+\sqrt{2M}}\ . \qquad \text{(VIII.6.6)}$$

With (VIII.6.4) and (VIII.6.1), last equation, one has

$$r=\left(1-3\sqrt{\frac{M}{2r'^3}}t'\right)^{2/3}r'\ .$$

Isoliating t'

$$t'=\frac{2}{3}\frac{1}{\sqrt{2M}}\left(r'^{3/2}-r^{3/2}\right) \qquad \text{(VIII.6.7)}$$

we finally obtain

$$t'=t+2\sqrt{2Mr}+2M\ln\frac{1-\sqrt{\frac{2M}{r}}}{1+\sqrt{\frac{2M}{r}}}\ , \qquad \text{(VIII.6.8)}$$

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[142] Note that the quantity $\rho'=\rho'(r',t')$ is used while the analogous quantity for the interior solution is the constant quantity ρ'_g, the value of ρ' on the boundary surface between the interior and exterior solutions.

the formula for the transition from the static Schwarzschild time co-ordinate to a time co-ordinate which refers to an observer who is in free fall from infinity. The relation originally was found by Lemaître. By a gauge transformation of the radial co-ordinate

$$r'' = \frac{2}{3}\frac{1}{\sqrt{2M}}r'^{3/2}, \quad t'' = t' \tag{VIII.6.9}$$

one obtains with

$$\overset{1'}{e}_{1'}\,dr' = \sqrt{\frac{r'}{r}}\sqrt{\frac{2M}{r'}}dr'' = \sqrt{\frac{2M}{r}}dr''$$

the relation

$$dx^{1''} = -v_E\,dr'', \quad v_E = -\sqrt{\frac{2M}{r}} \ , \tag{VIII.6.10}$$

well-known from the Lemaître transformation as well. Thus, it is ensured that in spite of different co-ordinate systems the radial arc elements are the same: $dx^{1'} = dx^{1''}$.

After recasting one can specify r as a function of the Lemaître co-ordinates

$$r = \sqrt[3]{2M}\left[\frac{3}{2}\left(r'' - t''\right)\right]^{2/3} \tag{VIII.6.11}$$

and can write the curvature radius of the Schwarzschild parabola and the quantity Λ as

$$\rho = 3\left(r'' - t''\right), \quad \Lambda = 1 - \frac{t''}{r''} \ . \tag{VIII.6.12}$$

Thus, we have explained the OS co-ordinate transformation as a Lemaître transformation. With the help of (VIII.6.5) and by differentiating (VIII.6.8) we evaluate the coefficients of this transformation

$$\begin{aligned} &\Lambda^1_{1''} = -v_E, \quad \Lambda^1_{4''} = -i v_E, \quad \Lambda^4_{1''} = -i\alpha_E^2\, v_E^2, \quad \Lambda^4_{4''} = \alpha_E^2 \\ &\Lambda^{1''}_1 = -\frac{\alpha_E^2}{v_E}, \quad \Lambda^{1''}_4 = -i, \qquad \Lambda^{4''}_1 = -i\alpha_E^2\, v_E, \quad \Lambda^{4''}_4 = 1 \end{aligned} \ . \tag{VIII.6.13}$$

By transvecting with the 4-bein of the comoving system

$$\overset{1''}{e}_{1''} = -v_E, \quad \overset{4''}{e}_{4''} = 1 \tag{VIII.6.14}$$

and the 4-bein of the non-comoving Schwarzschild system

$$\overset{1}{e}_1 = \alpha_E, \quad \overset{4}{e}_4 = \frac{1}{\alpha_E}, \quad \alpha_E = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \tag{VIII.6.15}$$

we obtain the coefficients of a Lorentz transformation

$$L^1_{1''} = \alpha_E, \quad L^1_{4''} = -i\alpha_E v_E, \quad L^4_{1''} = i\alpha_E v_E, \quad L^4_{4''} = \alpha_E \ . \tag{VIII.6.16}$$

In these coefficients are included the *physical* components of the relative velocity v_E of the two systems which we refer again to the surface of the stellar object. One has

$$v_E^g = -\sqrt{\frac{2M}{r_g}} \ . \tag{VIII.6.17}$$

In accordance with previous results the surface has always the speed of an object coming in free fall from infinity. At $r_g = 2M$ arise the well-known Schwarzschild problems.

We want to modify slightly the calculus. If we introduce the *scale factor*

$$\mathcal{K} = \Lambda^{2/3} \qquad \text{(VIII.6.18)}$$

we arrive at

$$r = \mathcal{K} r' \qquad \text{(VIII.6.19)}$$

instead of (VIII.6.4) and at

$$\overset{1'}{e}_{1'} = \frac{1}{\sqrt{\mathcal{K}}}, \quad \overset{2'}{e}_{2'} = \mathcal{K} r', \quad \overset{3'}{e}_{3'} = \mathcal{K} r' \sin\vartheta, \quad \overset{4'}{e}_{4'} = 1 \ . \qquad \text{(VIII.6.20)}$$

By differentiating (VIII.6.19) we obtain, in addition, the relation

$$dr = \frac{1}{\sqrt{\mathcal{K}}} dr' = dx^{1'} \ . \qquad \text{(VIII.6.21)}$$

It should also be noted that according to (VIII.6.1) the scale factor $\mathcal{K}$ is a function $\mathcal{K} = \mathcal{K}(r', t')$ in contrast to the analogous quantity of the interior OS solution. Thus, one has

$$\frac{1}{r} dr = \frac{1}{r'} dr' + \frac{1}{\mathcal{K}} d\mathcal{K} ,$$

wherein

$$\frac{1}{\mathcal{K}} \mathcal{K}_{|1'} = (1 - \Lambda) \frac{1}{r}, \quad \frac{1}{\mathcal{K}} \mathcal{K}_{|4'} = \frac{i}{\mathcal{R}} \ . \qquad \text{(VIII.6.22)}$$

The result of the second expression is known from earlier Sections. Just compare it with the corresponding formulae of the interior OS solution (VIII.4.25). A transformation matrix for the OS co-ordinate system akin to the Lemaître-matrix (VIII.6.13) can be set up

$$\begin{matrix} \Lambda^{1}_{1'} = \dfrac{1}{\sqrt{\mathcal{K}}}, & \Lambda^{1}_{4'} = -i v_E , & \Lambda^{4}_{1'} = i \dfrac{1}{\sqrt{\mathcal{K}}} \alpha_E^2 \, v_E, & \Lambda^{4}_{4'} = \alpha_E^2 \\ \Lambda^{1'}_{1} = \sqrt{\mathcal{K}} \alpha_E^2, & \Lambda^{1'}_{4} = -i \sqrt{\mathcal{K}} v_E, & \Lambda^{4'}_{1} = -i \alpha_E^2 \, v_E, & \Lambda^{4'}_{4} = 1 \end{matrix} \ . \qquad \text{(VIII.6.23)}$$

It leads with (VIII.6.20) and (VIII.6.15) to the same Lorentz transformation (VIII.6.16). To calculate these expressions the relations (VIII.6.4), (VIII.6.19), and (VIII.6.8) can be used. A further extension of the procedure can be omitted because the free fall has already been discussed in detail.

IX.7. The field equations of the interior OS solution

The OS solution describes the collapse of an infinitely large stellar object and is therefore not physically expedient. However, we examine the field equations because the model is mathematically quite interesting and can bring some insights which are supportive for other models.

First, we calculate the stress-energy tensor contained in

$$G_{mn} = R_{mn} - \frac{1}{2} g_{mn} = -\kappa T_{mn} \ .$$

The pressure-free model comprises only the time-dependent energy density

$$T_{44} = \mu_0 \left(t' \right) \ ,$$

whereby OS have specified

$$\kappa\mu_0 = \frac{4}{3} \frac{1}{\left(t' + \frac{G}{F} \right) \left(t' + \frac{\partial G / \partial r'}{\partial F / \partial r'} \right)} \ .$$

With (VIII.4.4) we compute

$$\frac{G}{F} = \frac{\partial G / \partial r'}{\partial F / \partial r'}$$

and

$$\kappa\mu_0 = \frac{4}{3} \frac{1}{\left(t' + \frac{G}{F} \right)^2}, \quad \frac{G}{F} = -\frac{2}{3} \sqrt{\frac{r_g'^3}{2M}} = -\frac{\rho_g'}{3} \ .$$

After some further reshaping one finally obtains with (VIII.4.5) and (VIII.4.11)

$$\kappa\mu_0 = \frac{3}{\mathcal{R}_g^2} \ . \qquad \text{(VIII.7.1)}$$

This is the expression for the energy density and it is identic with the one of the interior Schwarzschild solution. At the beginning of the collapse ($t' = 0$)

$$\Lambda\left(0 \right) = 1, \quad r_g = r_g'$$

is valid, whereby the surface of the stellar object is situated at infinity ($r_g = \infty$). For (VIII.7.1)one can also write

$$\kappa\mu_0 = 3 \frac{2M}{r_g^3} \ .$$

Thus, the infinitely extended object has vanishing mass density

$$\mu_0 \left(0 \right) = 0 \ .$$

Otherwise holds the relation

$$m=\frac{4\pi}{3}r_g^3\mu_0 \qquad \text{(VIII.7.2)}$$

by means of

$$\kappa=\frac{8\pi k}{c^4},\quad M=m\frac{k}{c^4}\ .$$

Therein m is the mass of the object which is enclosed by the sphere with the radius r_g.

For further calculations we need several auxiliary formulae. We use the metric of the comoving system in the form with the scale factor

$$\text{(A)}\quad ds^2=\mathcal{K}^2\left[dr'^2+r'^2\,d\vartheta^2+r'^2\sin^2\vartheta\,d\varphi^2\right]+dit'^2 \qquad \text{(VIII.7.3)}$$

and we read off the 4-bein system

$$\text{(A)}\quad \overset{1'}{e}_{1'}=\mathcal{K},\quad \overset{2'}{e}_{2'}=\mathcal{K}r',\quad \overset{3'}{e}_{3'}=\mathcal{K}r'\sin\vartheta,\quad \overset{4'}{e}_{4'}=1\ . \qquad \text{(VIII.7.4)}$$

The scale factor is a function of the time t', and illustrates the relation between the comoving and non-comoving co-ordinates

$$r=\mathcal{K}r'\ . \qquad \text{(VIII.7.5)}$$

With (VIII.4.16)[143] we have already prepared the quantities needed for further calculations. With the help of tetrads (VIII.7.4) we write

$$r_{|m'}=\{1,0,0,-iv\}=\{\alpha,0,0,-i\alpha v\}a,\quad a=1/\alpha\ . \qquad \text{(VIII.7.6)}$$

With the Lorentz transformation (VIII.5.8) one calculates the change of the non-comoving radial co-ordinate

$$r_{|m}=L_m^{m'}r_{|m'}=\{a,0,0,0\}\ . \qquad \text{(VIII.7.7)}$$

Likewise, one determines the values on the surface, whereby use is made of the compilation (VIII.4.15)

$$r_{g|m}=\left\{-\alpha v v_g,0,0,-i\alpha v_g\right\}\ . \qquad \text{(VIII.7.8)}$$

The 4-velocity of a mass point in the interior of the object relative to the comoving and non-comoving systems is

$$'u_{m'}=\{0,0,0,1\},\quad 'u_m=\{-i\alpha v,0,0,\alpha\}. \qquad \text{(VIII.7.9)}$$

With this and (VIII.4.15) one has

$$\mathcal{R}_{g|m'}=\frac{3}{2}i\,'u_{m'},\quad \rho_{g|m'}=3i\,'u_{m'},\quad \mathcal{R}_{g|m}=\frac{3}{2}i\,'u_m,\quad \rho_{g|m}=3i\,'u_m\ . \qquad \text{(VIII.7.10)}$$

The values of α and v are taken from (VIII.5.6) last line. Thus, we can calculate

143 The marker I of v and α is omitted because in this Section we only deal with the interior OS solution.

$$v_{|m'} = \left\{-\frac{1}{\mathcal{R}_g},0,0,iv\frac{1}{\rho_g}\right\} = \left\{\frac{v_g}{r_g},0,0,iv\frac{1}{\rho_g}\right\} . \tag{VIII.7.11}$$

The change of v consists of a circular part and a parabolic one. The spatial change of v occurs if one moves on a radial line in the interior of the object at a given moment. The time change takes place, however, because the boundary of the interior geometry is sliding down on the Schwarzschild parabola with the radius of curvature ρ_g. If we write the two components separately and if we calculate the components of the system at rest with the Lorentz transformation, we have

$$\begin{aligned} v_{|m'} &= \{\alpha,0,0,-i\alpha v\}\left(-\frac{a}{\mathcal{R}_g}\right) + \{0,0,0,1\}\left(-3iv\frac{1}{\rho_g}\right) \\ v_{|m} &= \{1,0,0,0\}\left(-\frac{a}{\mathcal{R}_g}\right) + \{-i\alpha v,0,0,\alpha\}\left(-3iv\frac{1}{\rho_g}\right) \end{aligned} . \tag{VIII.7.12}$$

It can be seen that the circular quantity is defined in the system at rest. It appears in the comoving system Lorentz transformed while the parabolic component is derived in the comoving system and must be determined for the non-comoving system with a Lorentz transformation. With

$$\frac{1}{\alpha}\alpha_{|m} = \alpha^2 v v_{|m} \tag{VIII.7.13}$$

one has the values for the change of the Lorentz factor

$$\alpha_{|m'} = \left\{-\alpha^3 v\frac{1}{\mathcal{R}_g},0,0,-i\alpha^3 v^2\frac{1}{\rho_g}\right\}, \quad \alpha_{|m} = \left\{-\alpha^4 v\frac{1}{\mathcal{R}_g} - \alpha^4 v^3\frac{1}{\rho_g},0,0,-3i\alpha^4 v^2\frac{1}{\rho_g}\right\} . \tag{VIII.7.14}$$

For the conservation law we note

$$\mu_{0|m'} = \left\{0,0,0,-3i\frac{\mu_o}{\mathcal{R}_g}\right\}, \quad \mu_{0|m} = -3i\frac{\mu_o}{\mathcal{R}_g}{}'u_m . \tag{VIII.7.15}$$

Thus, we are prepared to set up the field equations in both systems. In the comoving system results a similar simple structure of the field equations. With the tetrads (VIII.4.19) we calculate the Ricci-rotation coefficients[144]

$${}'U_{4'} = {}'A_{1'4'}{}^{1'} = -\overset{1'}{e}_{1'}e^{1'}_{1'|4'}, \quad B_{m'} = {}'A_{2'm'}{}^{2'} = -\overset{2'}{e}_{2'}e^{2'}_{2'|m'}, \quad C_{m'} = {}'A_{3'm'}{}^{3'} = -\overset{3'}{e}_{3'}e^{3'}_{3'|m'} . \tag{VIII.7.16}$$

With (VIII.7.6) this results in

$${}'U_{m'} = \left\{0,0,0,\frac{i}{\mathcal{R}_g}\right\}, \quad B_{m'} = \left\{\frac{1}{r},0,0,-\frac{iv}{r}\right\}, \quad C_{m'} = \left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0,-\frac{iv}{r}\right\} . \tag{VIII.7.17}$$

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[144] We use the following notation: A prime on a kernel of a quantity indicates a quantity of the comoving system. A prime on the index of a quantity indicates that it is measured in the comoving system. Thus, for example $\Phi_{m'}$ is a quantity of the non-comoving system which is measured in the comoving, ${}'\Phi_m$ a quantity of the co-moving system which is measured in the non-comoving system.

Thus, the quantity $'U_{4'}$ describes the change in time of the scale factor. With (VIII.7.5), (VIII.7.6), and $v=-r/\mathcal{R}_g$ we obtain

$$'U_{4'}=\frac{i}{\mathcal{R}_g},\quad \frac{1}{\mathcal{K}}\mathcal{K}^{\bullet}=-\frac{1}{\mathcal{R}_g},\quad \mathcal{K}^{\bullet}=\frac{\partial\mathcal{K}}{\partial t'} \tag{VIII.7.18}$$

and thus also

$$'U_{4'}\stackrel{*}{=}B_{4'}\stackrel{*}{=}C_{4'}=\frac{i}{\mathcal{R}_g}\ . \tag{VIII.7.19}$$

The common expression for the 'expansion' results with $'u_{n'}=\{0,0,0,1\}$ in

$$'u^{n'}{}_{||n'}='A_{n'}\,'u^{n'}='U_{4'}+B_{4'}+C_{4'}=3\frac{i}{\mathcal{R}_g} \tag{VIII.7.20}$$

and is here to be interpreted as a spatially uniform contraction which has the same value in all points in space. Nothing changes if one arbitrarily rotates or shifts the local coordinate system. The model has spherical symmetry.

According to (VIII.7.4) the metrical factor of the time-like part of the line element (A) is equal to 1, indicating the free fall from infinity. Thus, no other forces can be derived from the metric (A), in particular, no effect of gravity. Therefore one has

$$'U_{1'}='A_{4'1'}{}^{4'}=-\overset{4'}{e}_{4'}\,e^{4'}_{4'|1'}=0,\quad 'U_{4'}='A_{1'4'}{}^{1'}=-\overset{1'}{e}_{1'}\,e^{1'}_{1'|4'}=\frac{1}{\mathcal{K}}\mathcal{K}_{|4'}\ . \tag{VIII.7.21}$$

Thus, the 4-vector

$$'U_{m'}=\left\{0,0,0,\frac{i}{\mathcal{R}_g}\right\} \tag{VIII.7.22}$$

has only a single component, the time-like one.

With the unit vectors

$$'m_{m'}=\{1,0,0,0\},\quad 'b_{m'}=\{0,1,0,0\},\quad 'c_{m'}=\{0,0,1,0\} \tag{VIII.7.23}$$

and the Ricci-rotation coefficients

$$\begin{aligned} 'A_{m'n'}{}^{s'}&='U_{m'n'}{}^{s'}+B_{m'n'}{}^{s'}+C_{m'n'}{}^{s'}\\ 'U_{m'n'}{}^{s'}&='m_{m'}\,'U_{n'}\,'m^{s'}-'m_{m'}\,'m_{n'}\,'U^{s'}\\ B_{m'n'}{}^{s'}&='b_{m'}B_{n'}\,'b^{s'}-'b_{m'}\,'b_{n'}B^{s'}\\ C_{m'n'}{}^{s'}&='c_{m'}C_{n'}\,'c^{s'}-'c_{m'}\,'c_{n'}C^{s'} \end{aligned} \tag{VIII.7.24}$$

we form the graded derivatives

$$'U_{m'\underset{1}{||}n'}='U_{m'|n'},\quad B_{m'\underset{2}{||}n'}=B_{m'|n'}-'U_{n'm'}{}^{s'}B_{s'},\quad C_{m'\underset{3}{||}n'}=C_{m'|n'}-'U_{n'm'}{}^{s'}C_{s'}-B_{n'm'}{}^{s'}C_{s'}\ . \tag{VIII.7.25}$$

With the submatrices of the metric

$$h_{m'n'} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \quad {}^3g_{m'n'} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \qquad \text{(VIII.7.26)}$$

one finally obtains the Ricci

$$\begin{aligned} R_{m'n'} = &-\left['U^{s'}{}_{\underset{1}{\|}s'} + 'U^{s'}\,'U_{s'} \right] h_{m'n'} \\ &-\left[B_{n'\underset{2}{\|}m'} + B_{n'}B_{m'} \right] - 'b_{n'}\,'b_{m'}\left[B^{s'}{}_{\underset{2}{\|}s'} + B^{s'}B_{s'} \right] \\ &-\left[C_{n'\underset{3}{\|}m'} + C_{n'}C_{m'} \right] - 'c_{n'}\,'c_{m'}\left[C^{s'}{}_{\underset{3}{\|}s'} + C^{s'}C_{s'} \right] \\ = &\frac{3}{2}\frac{1}{\mathcal{R}_g^2}\left[{}^3g_{m'n'} - 'u_{m'}\,'u_{n'} \right] \end{aligned} \qquad \text{(VIII.7.27)}$$

and

$$R = \frac{3}{\mathcal{R}_g^2} \; . \qquad \text{(VIII.7.28)}$$

This results in

$$G_{\alpha'\beta'} = 0, \quad G_{\alpha'4'} = 0, \quad G_{4'4'} = -\kappa\mu_0, \quad \alpha' = 1', 2', 3' \; . \qquad \text{(VIII.7.29)}$$

The conservation law leads to

$$T^{4'n'}{}_{\|n'} = \mu_{0|4'} + 'A_{4'}\mu_0 = 0, \quad 'A_{4'} = 'U_{4'} + B_{4'} + C_{4'} = 3\frac{i}{\mathcal{R}_g}, \quad \partial_{4'} = \frac{\partial}{i\partial t'} \; . \qquad \text{(VIII.7.30)}$$

Thus one obtains

$$\dot{\mu}_0 = \mu_0 \frac{3}{\mathcal{R}_g} \; .$$

(VIII.7.1) is also verified with (VIII.4.15). It is the expression for the mass density which we have taken from the corresponding relation of OS.

The calculation of the field equation in the non-comoving system is considerably more difficult. We start with the metric

$$\text{(B)} \quad ds^2 = \alpha^2 dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 + a_T^2 dit^2$$

$$\alpha = \frac{1}{\sqrt{1 - \frac{r^2}{\mathcal{R}_g^2}}}, \quad a_T = \alpha\sqrt{\frac{r_g}{2M}\frac{y-1}{y^{3/2}}}, \quad y = \frac{1}{2}\left[\left(\frac{r'}{r'_g}\right)^2 - 1\right] + \frac{r'_g}{2M}\frac{r}{r'}, \quad \alpha_T = 1/a_T \qquad \text{(VIII.7.31)}$$

and with the 4-bein system (VIII.5.7)

$$(\mathrm{B})\quad \overset{1}{e}_1=\alpha,\quad \overset{2}{e}_2=r,\quad \overset{3}{e}_3=r\sin\vartheta,\quad \overset{4}{e}_4=a_T\ , \tag{VIII.7.32}$$

and we calculate the field strengths similar to (VIII.7.16). Taking from (VIII.5.2), second equation, the values for

$$y_{|1'}=\underset{1'}{e}^{1'}\frac{\partial y}{\partial r'},\quad y_{|4'}=\underset{4'}{e}^{4'}\frac{\partial y}{i\partial t'}$$

and performing a Lorentz transformation using (VIII.5.8) one obtains

$$y_{|1}=0 \tag{VIII.7.33}$$

which makes further calculations much easier.

The lateral field quantities

$$B_m=\left\{\frac{1}{\alpha r},0,0,0\right\},\quad C_m=\left\{\frac{1}{\alpha r},\frac{1}{r}\cot\vartheta,0,0\right\} \tag{VIII.7.34}$$

are easy to calculate with the 4-bein system (VIII.7.32). Alternatively, one obtains the quantities from (VIII.7.17) with the Lorentz transformation (VIII.5.8)

$$B_m=L_m^{m'}B_{m'},\quad C_m=L_m^{m'}C_{m'}\ .$$

The radial metric factor of the metric (B) contains the curvature quantity α which is time-dependent in accordance with (VIII.4.24) and (VIII.4.22). With (VIII.7.14) one has

$$U_4=A_{14}{}^{1}=-\overset{1}{e}_1\underset{1|4}{e}^{1}=\frac{1}{\alpha}\alpha_{|4}=-3i\alpha^3v^2\frac{1}{\rho_g}\ . \tag{VIII.7.35}$$

Taking into account (VIII.7.33) one gets from (VIII.7.31)

$$U_1=A_{41}{}^{4}=-\overset{4}{e}_4\underset{4|1}{e}^{4}=\frac{1}{a_T}a_{T|1}=\frac{1}{\alpha}\alpha_{|1}+\frac{1}{\sqrt{\frac{r_g}{2M}}}\left(\sqrt{\frac{r_g}{2M}}\right)_{|1}\ . \tag{VIII.7.36}$$

We calculate the first term with (VIII.7.14). For the second term the relation $r_{g|1}=-\alpha v v_g$ derived in (VIII.7.8) is used. Thus, one has in sum

$$\begin{aligned}&U_m=U_m^C+U_m^P,\quad U_m^C=\{1,0,0,0\}\,\alpha v\frac{1}{\mathcal{R}_g},\qquad U_m^P=\{\alpha,0,0,i\alpha v\}\left(-3\alpha^2v\frac{1}{\rho_g}\right)\\ &U_{m'}=U_{m'}^C+U_{m'}^P,\quad U_{m'}^C=\{\alpha,0,0,-i\alpha v\}\,\alpha v\frac{1}{\mathcal{R}_g},\quad U_{m'}^P=\{1,0,0,0\}\left(-3\alpha^2v\frac{1}{\rho_g}\right)\end{aligned} \tag{VIII.7.37}$$

and we have already performed a separation into the circular and into the parabolic part. With the Lorentz transformation (VIII.5.8) we have calculated the second line. The circular part has its origin in the non-comoving system. It is a negative quantity which lies in the radial direction and corresponds to the gravitational force. The parabolic part is positive, has its origin in the comoving system and is a result of the collapse. Both parts are regular for $r > 2M$. Beneath the Schwarzschild event horizon U^C is imaginary and the transformation (VIII.5.8) unphysical.

The field equations take a form such as (VIII.7.27) in the non-comoving system. They are form invariant

$$\begin{aligned} R_{mn} = &-\left[U^s{}_{\underset{1}{||}s} + U^s U_s\right] h_{mn} \\ &-\left[B_{n\underset{2}{||}m} + B_n B_m\right] - b_n b_m \left[B^s{}_{\underset{2}{||}s} + B^s B_s\right] , \\ &-\left[C_{n\underset{3}{||}m} + C_n C_m\right] - c_n c_m \left[C^s{}_{\underset{3}{||}s} + C^s C_s\right] \end{aligned} \qquad \text{(VIII.7.38)}$$

wherein the graded derivatives are constructed similarly to (VIII.7.25)

$$U_{m\underset{1}{||}n} = U_{m|n}, \quad B_{m\underset{2}{||}n} = B_{m|n} - U_{nm}{}^s B_s, \quad C_{m\underset{3}{||}n} = C_{m|n} - U_{nm}{}^s C_s - B_{nm}{}^s C_s \ . \qquad \text{(VIII.7.39)}$$

With

$$-\frac{R}{2} = \left[U^s{}_{\underset{1}{||}s} + U^s U_s\right] + \left[B^s{}_{\underset{2}{||}s} + B^s B_s\right] + \left[C^s{}_{\underset{3}{||}s} + C^s C_s\right] \qquad \text{(VIII.7.40)}$$

one can calculate the Einstein tensor and with the Lorentz transformation

$$T_{mn} = L^{m'n'}_{m\,n} T_{m'n'} \qquad \text{(VIII.7.41)}$$

we obtain the components of the stress-energy-momentum tensor for the system at rest

$$T_{11} = -\alpha^2 v^2 \mu_0, \quad T_{14} = -i\alpha^2 v \mu_0, \quad T_{44} = \alpha^2 \mu_0 \ . \qquad \text{(VIII.7.42)}$$

To verify the field equations one needs a set of formulae. From the last two equations (VIII.4.15) one gains after the transition to the non-comoving system

$$\rho_{g|m} = \left\{3\alpha v, 0, 0, 3i\alpha\right\}. \qquad \text{(VIII.7.43)}$$

For further calculations we need the relations (VIII.7.12) and (VIII.7.14). If we analyze the field equations it is of particular interest how the components of the stress-energy-momentum tensor can be extracted from the Einstein tensor. For this purpose, the second term (VIII.7.14) is decomposed into a circular and a parabolic part.

$$\begin{gathered} \frac{1}{\alpha}\alpha_{|m} = E^C_m + E^P_m \\ E^C_m = \left\{-\alpha v \frac{1}{\mathcal{R}_g}, 0, 0, 0\right\}, \quad E^P_m = \left\{-3\alpha^3 v^3 \frac{1}{\rho_g}, 0, 0 - 3i\alpha^3 v^2 \frac{1}{\rho_g}\right\} . \end{gathered} \qquad \text{(VIII.7.44)}$$

From (VIII.7.37) one needs U^P_1. With (VIII.7.1) we finally have

$$\left(B_1+C_1\right)E_1^P = 2\frac{a}{r}\left(-3\alpha^3 v^3\,\frac{1}{\rho_g}\right) = \alpha^2 v^2 \kappa\mu_0 = -\kappa T_{11}$$

$$\left(B_1+C_1\right)E_4^P = 2\frac{a}{r}\left(-3\alpha^3 v^2\,\frac{1}{\rho_g}\right) = i\alpha^2 v\kappa\mu_0 = -\kappa T_{14}\ . \qquad \text{(VIII.7.45)}$$

$$-\left(B_1+C_1\right)U_1^P = 2\frac{a}{r}\left(3\alpha^3 v\,\frac{1}{\rho_g}\right) = -\alpha^2\kappa\mu_0 = -\kappa T_{44}$$

The left-hand side terms of the above expression are included in the Einstein tensor. It can be seen that the parabolic parts of the field quantities have a significant role in establishing the stress-energy-momentum tensor. The remaining terms in the Einstein tensor cancel each other.

With a little algebra one also finds

$$B^s{}_{\underset{2}{||s}} + B^s B_s = -\frac{2}{\rho_g^2},\quad C^s{}_{\underset{3}{||s}} + C^s C_s = -\frac{6}{\rho_g^2},\quad U^s{}_{|s} + U^s U_s = \frac{2}{\rho_g^2} \qquad \text{(VIII.7.46)}$$

and thereby verifies Einstein's field equations

$$R_{mn} - \frac{1}{2} R\, g_{mn} = -\kappa T_{mn}\,.$$

In the next Section we will use all the relations which we have found here.

IX.8. Discussion of the OS-field equations

In the previous Sections, the OS model was put into a covariant form, although the question of the geometric interpretation of the model has not been presented in sufficient detail. This will be made up now.

In the last Section we have discovered that the OS-expression for the mass density can be converted into the familiar form $\kappa\mu_0 = 3/\mathcal{R}_g^2$. We have encountered such a relation in the cosmological models and in the Schwarzschild interior solution. That implies that the space portion of the interior OS solution can be represented as a spherical cap. We want to pursue this.

We consider a snapshot of the spatial 3-dimensional part of the model during the collapse at an arbitrary time in the non-comoving system. The metric

$$'ds^2 = \alpha^2 dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 \tag{VIII.8.1}$$

applies. By this we separate the time-dependent variables which describe the collapse from those describing the basic geometry. Thus, for the Ricci remains only

$$\begin{aligned} {}^*R_{\alpha\beta} = &-\left[B_{\beta\underset{2}{||}\alpha} + B_\beta B_\alpha\right] - b_\alpha b_\beta\left[B^\gamma{}_{\underset{2}{||}\gamma} + B^\gamma B_\gamma\right] \\ &-\left[C_{\beta\underset{3}{||}\alpha} + C_\beta C_\alpha\right] - c_\alpha c_\beta\left[C^\gamma{}_{\underset{3}{||}\gamma} + C^\gamma C_\gamma\right] \end{aligned} \tag{VIII.8.2}$$

with $\alpha = 1,2,3$ and

$$B_{\beta\underset{2}{||}\alpha} = B_{\beta|\alpha}, \quad C_{\beta\underset{3}{||}\alpha} = C_{\beta|\alpha} - B_{\alpha\beta}{}^\gamma C_\gamma \;. \tag{VIII.8.3}$$

With the lateral field quantities defined in (VIII.7.34) we obtain after a simple calculation

$${}^*R_{\alpha\beta} = \frac{2}{\mathcal{R}_g^2} g_{\alpha\beta} \;. \tag{VIII.8.4}$$

This corresponds to a geometry with constant positive curvature.

According to these considerations we are able to interpret the geometry of the OS model. The metric (B), whose structure is known from other models, suggests to interpret the interior spatial geometry as a cap of a sphere which is placed at a suitable position from the bottom of Flamm's paraboloid up to the Schwarzschild geometry. $\mathcal{R}_g$ is the radius the cap, and we use the relation

$$r = \mathcal{R}_g \sin\eta , \tag{VIII.8.5}$$

where η is the polar angle of the cap. According to the metric (B) is $\alpha = 1/\cos\eta$ and with (VIII.8.5) is $dr = \mathcal{R}_g \cos\eta d\eta$. We have for the metric (B)

$$ds^2 = \mathcal{R}_g^2 d\eta^2 + \mathcal{R}_g^2 \sin^2\eta d\vartheta^2 + \mathcal{R}_g^2 \sin^2\eta \sin^2\vartheta d\varphi^2 + a_T^2 dit^2 \;. \tag{VIII.8.6}$$

Thus, we have found a simple geometric interpretation of the OS-model. During the collapse the cap of a sphere slides down on Flamm's paraboloid of the Schwarzschild environment. While the cap of a sphere shrinks during this process Flamm's paraboloid remains unchanged. This is also a very insightful presentation of the Birkhoff theorem. A collapse has no effect on an observer located in the exterior field. It can also be seen that the cap of a sphere cannot shrink beneath the event horizon. It would uncouple from Flamm's paraboloid. Thus, the OS model does not allow the formation of black holes. Mitra [M] has brought forward another argument for this subject.

All the years since the publication of the paper by Oppenheimer and Snyder have obviously been little investigated the linking conditions of the geometries. On the boundary surface two conditions must be satisfied to ensure the matching of two geometries.

i. The metric or the tetrads, respectively, must match, so that the geometries contact.
ii. The first derivatives of the metric or the field strengths, respectively, must match. This means that the surfaces which are described by the geometries have common tangents, the surfaces merge smoothly into each other and do not buckle.

Nariai [N] and Tomita are of opinion that the interior OS solution does not match the exterior Schwarzschild solution. They have replaced the Schwarzschild solution by a complicated one maintaining the interior OS solution. Nariai [N] has discussed linking conditions of models which consist of interior and exterior solutions in general terms. Israel [I], O'Brien [O], Synge S, Robson [R], Bonnor [B] and Vickers and Lichnerowicz [L] have written extensively about the linking condition of two regions. Nakao [N] has extended the OS model with a cosmological constant.

We want to examine the linking condition for the OS model more closely. In Section IX.5 Eq. (VIII.5.8) - (VIII.5.9) we have calculated the values of the metric coefficients of the metric (B) on the boundary surface of the two geometries.

$$\alpha_g = \frac{1}{\sqrt{1-\dfrac{2M}{r_g}}}, \quad a_T^g = \sqrt{1-\frac{2M}{r_g}}\,.$$

They coincide with the values of the exterior Schwarzschild geometry, if r_g is not less than 2M. Condition (i) is satisfied, the spaces are connected to each other as long one does not under-run the event horizon.

The first derivatives of the metric coefficients correspond to the field strengths presented by us. From (VIII.7.34) one immediately recognizes that the quantities B and C have Schwarzschild values on the boundary. This is certainly not the case with the quantities U_m of (VIII.7.37). This can be understood quite well. The quantity U_m contains the portion U_m^C which corresponds to the body geometry and takes the Schwarzschild value on the boundary surface. The second part U_m^P is allocated to the collapse. U_m^P does not arise immediately from the geometry, but from its change. In accordance with (VIII.4.15) this change comprises the relation

$$\dot{\mathbb{R}}_g = -\frac{3}{2}\,.$$

If one has separated the two quantities, it becomes clear that only the surface of the body geometry can have the same tangents with the surface of the Schwarzschild geometry. Thus, the condition (ii) is satisfied.

The particle velocity in the interior is closely related to the angle of ascent of the radial curves on the cap of the sphere

$$v = -\sin\eta = -\frac{r}{\mathcal{R}_g}.$$

In (VIII.5.9) we have calculated the value of v for the boundary surface. It is

$$\sin\eta_g = \sqrt{\frac{2M}{r_g}} .$$

But this is exactly the value of the angle of ascent of Flamm's paraboloid of the Schwarzschild geometry on the boundary surface to the interior geometry. Thus the condition (ii) is satisfied.

A look at the field strengths of the comoving system immediately shows that the field strengths of the interior and exterior solutions match on the boundary surface. The exterior field is described by a freely falling observer. In several papers [B] we have written down the relation between the OS co-ordinates and the comoving Lemaître co-ordinates and have discussed the free fall in detail in former Sections.

It is now to be clarified why the metric (A) appears to be flat. A reference-system transformation, ie, a transition from an observer being in rest to a comoving observer is not expected to change the geometric structure of the space. The flat appearance of the metric arises because the motion of the freely falling system is tightly coupled to the geometric structure of the model. This will now be examined in more detail. We consider the flat 5-dimensional embedding space, and at first we suppress all dimensions except the first two ones. The global Cartesian co-ordinates in this space are $\left\{x^{0''}, x^{1''}\right\}$ with $x^{0''}$ as the extra dimension and the local co-ordinates are $\left\{x^{0}, x^{1}\right\}$. The relations between the reference systems provide local rotations

$$\begin{aligned} dx^0 &= dx^{0''}\cos\eta - dx^{1''}\sin\eta \\ dx^1 &= dx^{0''}\sin\eta + dx^{1''}\cos\eta \end{aligned} .$$

Since the radial arc element of the line element (A) lies entirely in the local 1-direction

$$dx^0 = 0, \quad dx^{0''} = \tan\eta\, dx^{1''}, \quad dx^{1''} = dr$$

$$dx^{1^2} = dx^{0''^2} + dx^{1''^2} = \tan^2\eta\, dr^2 + dr^2 = \frac{1}{\cos^2\eta}dr^2$$

applies. According to (VIII.8.5) on has

$$\sin\eta = \frac{r}{\mathcal{R}_g}, \quad \cos\eta = \sqrt{1 - \frac{r^2}{\mathcal{R}_g}} = a = \frac{1}{\alpha} , \qquad \text{(VIII.8.7)}$$

whereby the radial part of the metric (A) is geometrically deepened. The parallels of the cap of a sphere have the radius r, and its curvature is defined by

$$B_{a''} = \left\{0, \frac{1}{r}, 0, 0, 0\right\}, \quad a'' = 0'', 1'', \ldots, 4'' . \qquad \text{(VIII.8.8)}$$

In the local system it has the form

$$B_a = \left\{ \frac{1}{r} \sin\eta, \frac{1}{r} \cos\eta, 0, 0, 0 \right\}, \quad a = 0, 1, \ldots, 4 \ . \tag{VIII.8.9}$$

Using the Lorentz angle the Lorentz transformation (VIII.5.8) can be written as

$$L^1_{1'} = \cos i\chi, \quad L^1_{4'} = \sin i\chi, \quad L^4_{1'} = -\sin i\chi, \quad L^4_{4'} = \cos i\chi \tag{VIII.8.10}$$

and is a pseudo-rotation through the imaginary angle $i\chi$ in the local [1,4]-plane. If one operates herewith on (VIII.8.9), one obtains with

$$B_{a'} = \left\{ \frac{1}{r} \sin\eta, \frac{1}{r} \cos\eta \cos i\chi, 0, 0, \frac{1}{r} \cos\eta \sin i\chi \right\} \tag{VIII.8.11}$$

an expression whose 1-component will appear flat because the radial metric coefficient in (B) corresponds to the Lorentz factor. One has

$$\cos\eta \cos i\chi = 1 \ . \tag{VIII.8.12}$$

From (VIII.6.16) and (VIII.8.7) one gathers

$$v = -\sin\eta, \quad a = \cos\eta$$

and one obtains

$$B_{a'} = \left\{ \frac{1}{\mathcal{R}_g}, \frac{1}{r}, 0, 0, \frac{i}{\mathcal{R}_g} \right\} \ . \tag{VIII.8.13}$$

The quantity $B_{a'}$ is generated from (VIII.8.8) by two transformations. A rotation in the [0,1]-plane and a pseudo-rotation in the [1',4']-plane. It should also be noted that this deceptive appearance also occurs in the exterior Schwarzschild field concerning the free fall. The Schwarzschild metric can be brought with a Lemaître transformation into a form which appears flat. This is only valid for the free fall from infinity. If the body starts at a finite position, the metric takes on a much more complicated form.

The metric (A) of the comoving OS model is obtained by means of a Lemaître-like transformation which we have discussed in detail in Section IX.5

$$g_{i'k'} = \Lambda^{i\,k}_{i'k'} g_{ik}, \quad \Lambda^i_{i'} = \overset{m'}{e}_{i'} L^m_{m'} \underset{m}{e}^i \ .$$

A co-ordinate transformation can never change the geometrical or physical background of the model, so that one cannot make any direct conclusions about the geometric structure of the model by means of the apparently flat representation (A). The field quantities, however, show that the basic structure of the geometrical model is invariant under a transformation of the co-ordinates and under a Lorentz transformation as well.

Let us show now how the field equations of the comoving system and the non-comoving system can be converted into one another. Therefore one must recall that the Ricci-rotation coefficients transform inhomogeneously

$$\begin{aligned} {}'A_{m'n'}{}^{s'} &= L^{m\,n\,s'}_{m'n's} A_{mn}{}^{s} + {}'L_{m'n'}{}^{s'} \\ A_{mn}{}^{s} &= L^{m'n's}_{m\,n\,s'} {}'A_{m'n'}{}^{s'} + L_{mn}{}^{s} \end{aligned} \ . \tag{VIII.8.14}$$

We call the respective second terms Lorentz terms. They are defined by

$$'L_{m'n'}{}^{s'} = L^{s'}_{s} L^{s}_{n'|m'}, \quad L_{mn}{}^{s} = L^{s}_{s'} L^{s'}_{n|m}, \quad 'L_{m'n'}{}^{s'} = -L^{m\,n\,s'}_{m'n's} L_{mn}{}^{s} \ . \tag{VIII.8.15}$$

Using

$$L_{1'}^{1}=\alpha, \quad L_{4'}^{1}=-i\alpha v, \quad L_{1'}^{4}=i\alpha v, \quad L_{4'}^{4}=\alpha \tag{VIII.8.16}$$

one can express the components of the Lorentz terms in the following way

$$\begin{gathered} 'L_{4'1'}{}^{4'}='L_{1'}=i\alpha^2 v_{|4'}, \quad 'L_{1'4'}{}^{1'}='L_{4'}=-i\alpha^2 v_{|1'} \\ L_{41}{}^{4}=L_1=-i\alpha^2 v_{|4}, \quad L_{14}{}^{1}=L_4=i\alpha^2 v_{|1} \end{gathered}, \tag{VIII.8.17}$$

and one can write the Lorentz terms in the form

$$'L_{m'n'}{}^{s'}=h_{m'}{}^{s'}\,'L_{n'}-h_{m'n'}\,'L^{s'}, \quad L_{mn}{}^{s}=h_m{}^{s}L_n-h_{mn}L^{s}. \tag{VIII.8.18}$$

With this arrangement results from

$$\begin{gathered} A_{mn}{}^{s}=U_{mn}{}^{s}+B_{mn}{}^{s}+C_{mn}{}^{s} \\ U_{mn}{}^{s}=h_m{}^{s}U_n-h_{mn}U^{s}, \quad B_{mn}{}^{s}=b_mB_nb^{s}-b_mb_nB^{s}, \quad C_{mn}{}^{s}=c_mC_nc^{s}-c_mc_nC^{s} \end{gathered} \tag{VIII.8.19}$$

the transformation law for the U-quantities

$$'U_{m'}=U_{m'}+'L_{m'}, \quad U_{m'}=L_{m'}^{m}U_m, \quad U_m='U_m+L_m, \quad 'U_m=L_m^{m'}\,'U_{m'}\ , \tag{VIII.8.20}$$

if one uses (VIII.8.17) with (VIII.7.12). According to (VIII.8.15)

$$L_m=-L_m^{m'}\,'L_{m'} \tag{VIII.8.21}$$

is to be considered. The relation (VIII.8.20) simplifies the conversion of the field equations of the system (A) into the ones of the system (B) and vice versa. In addition we want to decompose the Lorentz terms into circular and parabolic parts. First we define in each case two circular and parabolic quantities

$$\begin{gathered} L_m^{C}=\{1,0,0,0\}\left(\alpha v\frac{1}{\mathcal{R}_g}\right), \quad L_m^{P}=\{\alpha,0,0,i\alpha v\}\left(-3\alpha^2 v\frac{1}{\rho_g}\right) \\ 'L_{m'}^{C}=\{\alpha,0,0,-i\alpha v\}\left(-\alpha v\frac{1}{\mathcal{R}_g}\right), \quad 'L_{m'}^{P}=\{1,0,0,0\}\left(3\alpha^2 v\frac{1}{\rho_g}\right) \end{gathered} \tag{VIII.8.22}$$

and we again note that the quantity $'U$ has no parabolic part. With this we can deepen the transformation (VIII.8.20). It turns out that the circular and parabolic parts of the U-quantities transform separately. Taking into account (VIII.8.22) one can rearrange (VIII.8.20)

$$\begin{aligned} 'U_{m'} &=\left[U_{m'}^{C}+'L_{m'}^{C}\right]+\left[U_{m'}^{P}+'L_{m'}^{P}\right]+'U_{m'} \\ 'U_m &=\left[U_m^{C}-L_m^{C}\right]+\left[U_m^{P}-L_m^{P}\right]+'U_m \end{aligned}. \tag{VIII.8.23}$$

The lateral field quantities B and C transform as vectors, as we have already noted above. On the basis of

$$L^{n'm'}_{n\ m} B_{n'|m'} = B_{n|m} - L_{mn}{}^{s} B_{s}, \quad U_{mn}{}^{s} = L^{'m'ns}_{m\,n\,s'}\, {}'U_{m'n'}{}^{s'} + L_{mn}{}^{s} \tag{VIII.8.24}$$

results

$$L^{n'm'}_{n\ m}\left[B_{n'|m'} - {}'U_{m'n'}{}^{s'} B_{s'} + B_{m'} B_{n'}\right] = B_{n|m} - U_{mn}{}^{s} B_{s} + B_{m} B_{n} \tag{VIII.8.25}$$

and a corresponding relation for the quantity for C. With (VIII.8.20) and (VIII.8.21) we obtain

$$ {}'U^{s'}{}_{\underset{1}{||}s'} + {}'U^{s'}\,{}'U_{s'} = \frac{1}{2}\frac{1}{\mathcal{R}_g^2}\ . \tag{VIII.8.26}$$

After these preliminaries we are able to investigate the transformation properties of the field equations. We start with the Ricci

$$R_{mn} = A_{mn}{}^{s}{}_{|s} - A_{n|m} - A_{rm}{}^{s} A_{sn}{}^{r} + A_{mn}{}^{s} A_{s} \tag{VIII.8.27}$$

and perform the decomposition according to (VIII.8.19). From this one gets the relation (VIII.7.38) for the non-comoving system and for the comoving system the Ricci (VIII.7.27). As seen from (VIII.8.24) - (VIII.8.26) all the subequations of the Ricci are invariant. Thus, the Ricci is invariant as well

$$L^{m'n'}_{m\ n} R_{m'n'} = R_{mn}\,, \tag{VIII.8.28}$$

which was to be expected.

The stress-energy-momentum tensor of the unprimed system can be calculated from the unprimed field equations or with

$$T_{mn} = L^{m'n'}_{m\ n} T_{m'n'} = \mu_0\, {}'u_m\, {}'u_n, \quad {}'u_m = \{-i\alpha v, 0, 0, \alpha\}\ . \tag{VIII.8.29}$$

Thus, the conservation law results in

$$T_{m}{}^{n}{}_{||n} = {}'u_m\left[\mu_{0|n}\, {}'u^n + \mu_0\, {}'u^n{}_{||n}\right] + \mu_0\, {}'u_{m||n}\, {}'u^n\ . \tag{VIII.8.30}$$

With the results of previous Sections it can be shown that the motion of the particles in the collapsing object is geodesic

$$ {}'u_{m||n}\, {}'u^n = 0\ . \tag{VIII.8.31}$$

Therefore remains

$$\mu_{0|n}\, {}'u^n + \mu_0\, {}'u^n{}_{||n} = 0\ . \tag{VIII.8.32}$$

One has $\mu_{0||n'}\, {}'u^{n'} = \mu_{0|4'}$ and ${}'u^n{}_{||n} = {}'u^{n'}{}_{||n'}$. If one reads from (VIII.7.15) one has verified the conservation law. With (VIII.7.20) one has

$$\dot{\mu}_0 = \frac{3}{\mathcal{R}_g}\mu_0 \tag{VIII.8.33}$$

in accordance with (VIII.4.15). Let's take a look at the equation of motion (VIII.8.31). After reshaping a little we get

$$\left(-i\alpha v\right)_{|4'} = -\alpha\left(\alpha U_1 + i\alpha v U_4\right)$$

and with $m = m_0\alpha$

$$\frac{dmv}{dt'} = -mU_{1'} \ . \tag{VIII.8.34}$$

On the other hand one has with (VIII.8.17)

$$\left(-i\alpha v\right)_{|4'} = -i\alpha^3 v_{|4'} = -\alpha\,'L_{1'}$$

and with (VIII.8.20)

$$'L_{1'} + U_{1'} = 'U_{1'} \equiv 0 \ .$$

It can be seen that the force of gravity acting on the particles in the interior of the stellar object is canceled by the counter force 'L in the comoving system. A comoving observer is not exposed to gravity.

Due to the absence of the parabolic field strengths the possibility to inspect the geometric mechanism of the collapse is limited for the comoving observer. This is reserved for the non-comoving observer. If we isolate the purely spatial parts of the field quantities by means of

$$*B_{\alpha'} = \left\{\frac{1}{r},0,0\right\}, \quad *C_{\alpha'} = \left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0\right\}, \quad \alpha' = 1',2',3' \ , \tag{VIII.8.35}$$

we obtain for the Ricci and the Einstein tensor of these quantities

$$*R_{\alpha'\beta'} = 0, \quad *G_{\alpha'\beta'} = 0. \tag{VIII.8.36}$$

For the comoving observer the spatial part of the geometry appears to be flat. But that this is not the case as we have shown above. We process the remaining components of the 4-dimensional variables to

$$\begin{gathered} 'A_{m'n'}{}^{s'} = *A_{m'n'}{}^{s'} + Q_{m'}{}^{s'}\,'u_{n'} - Q_{m'n'}\,'u^{s'}, \quad 'A_{s'n'}{}^{s'} = 'A_{n'} = *A_{n'} + Q_{s'}{}^{s'}\,'u_{n'} \\ Q_{[m'n']} = 0, \quad Q_{m'n'}\,'u^{n'} = 0 \end{gathered} \ . \tag{VIII.8.37}$$

Included in $*A$ are the quantities (VIII.8.35) and the new symmetric quantity is defined by

$$Q_{1'1'} = 'U_{4'}, \quad Q_{2'2'} = B_{4'}, \quad Q_{3'3'} = C_{4'}. \tag{VIII.8.38}$$

The unit vector $'u^{m'} = \{0,0,0,1\}$ is orthogonal to the surfaces $t' = \text{const}$ of the interior geometry, it is the rigging vector of this shrinking surface. On the basis of

$$'u_{m'||n'} = Q_{m'n'} \tag{VIII.8.39}$$

the Qs are the second fundamental forms of the surface theory. For the Ricci then only remains

$$\begin{aligned} R_{m'n'} &= -\left[Q_{m'n'\wedge s'}{}'u^{s'} + Q_{m'n'}Q_{s'}{}^{s'}\right] \\ &\quad - {}'u_{n'}\left[Q_{s'}{}^{s'}{}_{\wedge \underline{m}'} - Q_{m'}{}^{s'}{}_{\wedge s'}\right] \\ &\quad - {}'u_{m'}\left[{}^*A_{n'|s'}{}'u^{s'} + Q_{r'}{}^{s'}\,{}^*A_{s'n'}{}^{r'}\right] \\ &\quad - {}'u_{m'}{}'u_{n'}\left[Q_{s'}{}^{s'}{}_{\wedge r'}{}'u^{r'} + Q_{r's'}Q^{r's'}\right] \end{aligned} . \qquad \text{(VIII.8.40)}$$

In it is ${}'u_{m'} = \{0,0,0,1\}$ and

$$\Phi_{\underline{m}'\wedge \underline{n}'} = \Phi_{\underline{m}'|\underline{n}'} - {}^*A_{n'm'}{}^{s'}\Phi_{\underline{s}'}, \quad \underline{m}' = 1',2',3' \qquad \text{(VIII.8.41)}$$

the 3-dimensional covariant derivative of a 3-dimensional quantity.

The third brackets in the above block vanish identically and so do the contracted Codazzi equation

$$Q_{s'}{}^{s'}{}_{\wedge \underline{m}'} - Q_{m'}{}^{s'}{}_{\wedge s'} = 0 \qquad \text{(VIII.8.42)}$$

in the second row of the block. Both relations express that in the comoving system must be $R_{\underline{m}'4'} = 0$.

From this the Einstein tensor

$$G_{\alpha'\beta'} = 0, \quad \kappa\mu_0 = -Q_{\alpha'\beta'}Q^{\alpha'\beta'}, \quad \alpha' = 1',2',3' \qquad \text{(VIII.8.43)}$$

can be calculated. The mass density is made up of the field energy density. From (VIII.8.38) and (VIII.7.19) one gets for it the more familiar term

$$\kappa\mu_0 = \frac{3}{\mathcal{R}_g^2} = \frac{6M}{r_g^3} \qquad \text{(VIII.8.44)}$$

with r_g the value associated with the surface of the collapsing star. Since the star collapses in free fall from infinity, the mass density is zero at the beginning of the collapse due to $r_g \to \infty$. On the other hand one has

$$Q_{1'1'} = -i\frac{1}{\mathcal{R}}\dot{\mathcal{R}}, \quad Q_{2'2'} = Q_{3'3'} = -i\frac{1}{r}\dot{r} . \qquad \text{(VIII.8.45)}$$

If one pulls the master switch and if one stops the collapse, the energy density vanishes and so does the mass. The star disappears and the space is empty. From the perspective of a fictional observer the space also appears to be flat. More seriously, one can say that the star of the OS model is gaining its mass only from the collapse.

The solution of Oppenheimer and Snyder is an analytical solution of Einstein's field equations which describes in the same way the interior and the exterior of a stellar object. It has the deficiency to be limited only to pressure-free matter. The model is physically unrealistic because the stellar object is infinitely large at the time $t' = 0$ and its matter is infinitely thinned. The collapse takes place in free fall. OS claim that the stellar object can contract below the event horizon during the collapse. However, our results of previous Sections suggest that the surface of the object can only asymptotically approach the event horizon. The formation of a black hole is not possible. We again refer to the papers of

Mitra [M] and his theory of ECOs and MECOs. Mitra has shown that OS have made a mistake by the less important factor 1/4 at the critical part of the respective calculation. In addition, the crucial quantity has the wrong sign for penetrating the event horizon.

Thus, the physical usability of the model has been questioned. A pressure-free star which fills the vastness of the universe and has vanishing density at the beginning of the collapse and leaves an empty space with a Schwarzschild field during the collapse behind it, does not exist.

Literature: Israel [I], Jhingan [J] and Kaushik.

IX.9. The model of Weinberg

In his textbook Weinberg [W] presents a model for a collapsing star. Demanding spherical symmetry and choosing suitable initial conditions Weinberg obtains a collapsing interior solution which he adapts somewhat intricately to the exterior Schwarzschild solution. Since the solution is already known to us, we can write the metric in a form that allows us to represent the model clearly arranged:

$$\text{(A)}\quad ds^2 = K^2\left[\frac{1}{1-\frac{r'^2}{\mathcal{R}_0^2}}dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta\, d\varphi^2\right] - dt'^2 \ . \qquad \text{(VIII.9.1)}$$

Therein

$$K = K(t') \qquad \text{(VIII.9.2)}$$

is the time-dependent dimensionless *scale factor* describing the collapse. The expression in brackets is the metric of a cap of a sphere, the radius of the beginning of the collapse has the value $\mathcal{R}_0$ and determines the geometry similarly to the Schwarzschild interior solution. During the collapse the cap of the sphere shrinks. $\{r',t'\}$ denotes is the comoving co-ordinate system.

$$u_m = \{0,0,0,1\} \ . \qquad \text{(VIII.9.3)}$$

applies to the 4-velocity.

From (VIII.9.1) one reads the tetrads

$$\overset{1}{e}_1 = K\alpha_I, \quad \overset{2}{e}_2 = Kr', \quad \overset{3}{e}_3 = Kr'\sin\vartheta, \quad \overset{4}{e}_4 = 1, \quad a_I = \sqrt{1-\frac{r'^2}{\mathcal{R}_0^2}}, \quad \alpha_I = \frac{1}{a_I} \qquad \text{(VIII.9.4)}$$

and calculates the Ricci-rotation coefficients

$$\begin{gathered} A_{mn}{}^s = U_{mn}{}^s + B_{mn}{}^s + C_{mn}{}^s \\ U_{mn}{}^s = m_m U_n m^s - m_m m_n U^s, \quad B_{mn}{}^s = b_m B_n b^s - b_m b_n B^s, \quad C_{mn}{}^s = c_m C_n c^s - c_m c_n C^s \end{gathered} \ . \qquad \text{(VIII.9.5)}$$

We again use the unity vectors

$$m_m = \{1,0,0,0\}, \quad b_m = \{0,1,0,0\}, \quad c_m = \{0,0,1,0\}, \quad u_m = \{0,0,0,1\} \ . \qquad \text{(VIII.9.6)}$$

The field strengths are

$$\begin{gathered} U_m = \{0,0,0,U_4\}, \quad U_4 = \frac{1}{K}K_{|4} = -\frac{i}{K}\dot{K}, \quad \partial_4 = \frac{\partial}{i\partial t'} \\ B_m = \left\{\frac{a_I}{Kr'},0,0,\frac{1}{K}K_{|4}\right\}, \quad C_m = \left\{\frac{a_I}{Kr'},\frac{1}{Kr'}\cot\vartheta,0,\frac{1}{K}K_{|4}\right\} \end{gathered} , \qquad \text{(VIII.9.7)}$$

wherein $U_4 \triangleq B_4 \triangleq C_4$ is a relation between the time-like quantities which we will use to simplify some calculations. With the expressions mentioned above, the Ricci has the form

$$R_{mn} = -\left[U^s{}_{\underset{1}{\|}s} + U^s U_s\right] h_{mn}$$
$$-\left[B_{n\underset{2}{\|}m} + B_n B_m\right] - b_n b_m \left[B^s{}_{\underset{2}{\|}s} + B^s B_s\right] , \qquad \text{(VIII.9.8)}$$
$$-\left[C_{n\underset{3}{\|}m} + C_n C_m\right] - c_n c_m \left[C^s{}_{\underset{3}{\|}s} + C^s C_s\right]$$

with

$$h_{mn} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} . \qquad \text{(VIII.9.9)}$$

The graded covariant derivatives are defined by

$$U_{n\underset{1}{\|}m} = U_{n|m}, \quad B_{n\underset{2}{\|}m} = B_{n|m} - U_{mn}{}^s B_s, \quad C_{n\underset{3}{\|}m} = C_{n|m} - B_{mn}{}^s C_s - U_{mn}{}^s C_s \ . \qquad \text{(VIII.9.10)}$$

The stress-energy-momentum tensor of the pressure-free Weinberg model with time-dependent rest mass density μ_0 has the simple form

$$T_{mn} = \mu_0 u_m u_n, \quad \mu_0 = \mu_0(t'), \quad u_m = \{0,0,0,1\} . \qquad \text{(VIII.9.11)}$$

Having calculated (VIII.9.8) with the help of (VIII.9.10) and using the definitions (VIII.9.7) the Einstein tensor is reduced to

$$G_{mn} = \left[2U_{4|4} + 3U_4U_4 - \frac{1}{\mathcal{K}^2\mathcal{R}_0^2}\right]{}'g_{mn} + 3\left[U_4U_4 - \frac{1}{\mathcal{K}^2\mathcal{R}_0^2}\right]u_m u_n = -\kappa\mu_0 u_m u_n \ , \text{(VIII.9.12)}$$

if one uses

$$'g_{mn} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} . \qquad \text{(VIII.9.13)}$$

This relation decomposes into the two equations

$$2U_{4|4} + 3U_4U_4 - \frac{1}{\mathcal{K}^2\mathcal{R}_0^2} = 0, \quad 3\left[U_4U_4 - \frac{1}{\mathcal{K}^2\mathcal{R}_0^2}\right] = -\kappa\mu_0 \qquad \text{(VIII.9.14)}$$

which have to be solved. The latter can be written as

$$3\left[\frac{1}{\mathcal{K}^2}\mathcal{K}^{\cdot 2} + \frac{1}{\mathcal{K}^2\mathcal{R}_0^2}\right] = \kappa\mu_0 \ .$$

At the beginning of the collapse at the time $t' = 0$ one has

$$\mathcal{K}^{\cdot}(0) = 0, \quad \mathcal{K}(0) = 1 \ .$$

With this follows from the above relation

$$\kappa\mu_0(0) = \frac{3}{\mathcal{R}_0^2} \ , \qquad \text{(VIII.9.15)}$$

a construction which is known from previously treated models. Let us take another look at the conservation law. Due to the simplicity of (VIII.9.11) the conservation law is reduced to

$$T^{m}{}_{4\|m} = T^{4}{}_{4|4} + A_m T^{m}{}_{4} = \mu_{0|4} + 3U_4\mu_0 = 0 \ .$$

This becomes

$$\frac{1}{\mu_0}\dot{\mu}_0 + 3\frac{1}{\mathcal{K}}\mathcal{K}^{\bullet} = 0, \quad \left(\ln \mu_0 \mathcal{K}^3\right)^{\bullet} = 0$$

and from this one gets the important relation

$$\mu_0 \mathcal{K}^3 = \text{const.} \ . \tag{VIII.9.16}$$

In addition, multiplying by $\mathcal{K}^3$ one gains from the equation further above

$$3\left[\mathcal{K}\mathcal{K}^{\bullet 2} + \frac{\mathcal{K}}{\mathcal{R}_0^2}\right] = \kappa \mu_0 \mathcal{K}^3 \ . \tag{VIII.9.17}$$

This result can be processed with

$$\mu_0(t')\mathcal{K}^3(t') = \mu_0(0)\mathcal{K}^3(0) = \mu_0(0) = \text{const.} \ .$$

Inserting into (VIII.9.17) and using (VIII.9.15) we obtain the differential equation

$$\mathcal{K}^{\bullet 2} = \frac{1}{\mathcal{R}_0^2}\left(\frac{1}{\mathcal{K}} - 1\right), \quad \mathcal{K}^{\bullet} = -\frac{1}{\mathcal{R}_0}\sqrt{\frac{1}{\mathcal{K}} - 1} \tag{VIII.9.18}$$

after some reshaping. Now the first relation of (VIII.9.14) is to be treated. Since one has

$$U_{4|4} = -\frac{1}{\mathcal{K}}\mathcal{K}^{\bullet\bullet} + \frac{1}{\mathcal{K}^2}\mathcal{K}^{\bullet}\mathcal{K}^{\bullet}$$

and if, in addition, one multiplies with $\mathcal{K}^2$ then

$$2\mathcal{K}\mathcal{K}^{\bullet\bullet} + \mathcal{K}^{\bullet 2} + \frac{1}{\mathcal{R}_0^2} = 0$$

results from (VIII.9.14). If one uses (VIII.9.18) one arrives at the second-order differential equation

$$\mathcal{K}^{\bullet\bullet} = -\frac{1}{2\mathcal{K}^2\mathcal{R}_0^2} \ . \tag{VIII.9.19}$$

It has a solution given by the parameter equations

$$t' = \frac{\mathcal{R}_0}{2}(\psi + \sin\psi), \quad \mathcal{K} = \frac{1}{2}(1 + \cos\psi) \ . \tag{VIII.9.20}$$

Eliminating ψ leads to

$$t' = \mathcal{R}_0\left[\frac{1}{2}\arccos(2\mathcal{K} - 1) + \sqrt{\mathcal{K}(1 - \mathcal{K})}\right] . \tag{VIII.9.21}$$

We supplement Weinberg's calculus with the quantity

$$\mathcal{R} = \sqrt{\mathcal{K}^3}\,\mathcal{R}_0 \ , \tag{VIII.9.22}$$

We will discuss this definition later in more detail. $\mathcal{R}_0$ is the initial value for the radius of a spherical geometry which geometrically describes the collapsing object, and $\mathcal{R}$ is the radius of the sphere at any given time. For $\mathcal{K}=1$ one obtains $\psi=0$ and thus $t'=0$, the initial state of collapse with the initial value $\mathcal{R}=\mathcal{R}_0$. For $\mathcal{K}=0$ and $\psi=\pi$ one has $t'=\frac{\pi}{2}\mathcal{R}_0$ and $\mathcal{R}=0$. The radius of the spherical cap and thus the whole cap of the sphere has shrunk to a point. This can be seen in Fig. IX.1.

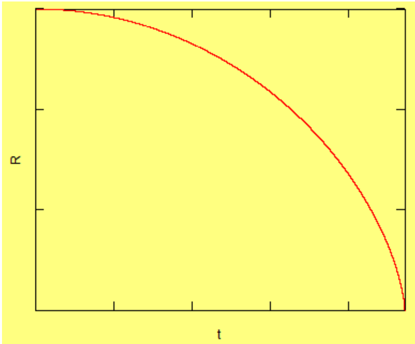

Fig. IX.1

The stellar object collapses after a finite proper time $T'=(\pi/2)\mathcal{R}_0$ from rest to a state of infinitely high energy density, as equation

$$\mu_0=\frac{\mu_0(0)}{\mathcal{K}^3}$$

with $\mathcal{K}(T')=0$ shows. Other models also have this property, which is physically less interpretable. We are taking account to this problem later on. With (VIII.9.15) and (VIII.9.22) one finally has

$$\kappa\mu_0=\frac{3}{\mathcal{R}^2} \tag{VIII.9.23}$$

for an arbitrary time during the collapse.

Redifferentiation of the first equation (VIII.9.20) leads to

$$dt'=\frac{\mathcal{R}_0}{2}(1+\cos\psi)d\psi=\mathcal{K}\mathcal{R}_0 d\psi\ , \tag{VIII.9.24}$$

the approach commonly used by us for the time-like arc element. In addition, we note

$$r'=\mathcal{R}_0\sin\eta'\ , \tag{VIII.9.25}$$

wherein the radial co-ordinate r' and the polar angle η' are time-independent quantities in the comoving co-ordinate system. With this ansatz the metric (VIII.9.1) can be brought into the form

$$ds^2=\mathcal{K}^2\left[\mathcal{R}_0^2 d\eta'^2+\mathcal{R}_0^2\sin^2\eta' d\vartheta^2+\mathcal{R}_0^2\sin^2\eta'\sin^2\vartheta d\varphi^2\right]-dt'^2 \tag{VIII.9.26}$$

Using the relation (VIII.9.24) and inserting into (VIII.9.26) one finds

$$ds=\mathcal{K}(t')ds_0\ , \tag{VIII.9.27}$$

wherein ds_0 is the line element of the spherical space at the time $t'=0$. According to (VIII.9.24) elapses no more time at the end of the collapse.

A comparison with the Friedman cosmological solution of the Section II.7 shows the close formal relationship between the two models. If one puts in formula (II.7.20) the cosmological constant $\lambda=0$ and performs some renaming, then the differential equation of Friedman is reduced to (VIII.9.18). If we had written the metric in the form (VIII.9.26) right away, we would have been able to rely on the Friedman solution and we would have been able to convert the expansion into a contraction by an appropriate choice of sign.

It may be advantageous to combine the time-like quantities and to split them off from the field equations. One obtains the Ricci-rotation coefficients in the form

$$A_{mn}{}^{s} = {}^{*}A_{mn}{}^{s} + {}'\delta^{s}_{m}U_{n} - {}'g_{mn}U^{s}, \quad A_{m} = {}^{*}A_{m} + 3U_{m} \;, \tag{VIII.9.28}$$

wherein $^{*}A$ includes the space-like but time-dependent field strengths B and C. Since the metric factor in the time-like arc element of the Weinberg metric is 1, no gravity occurs. Observers following the collapse are weightless.

For the Ricci with respect to $^{*}A$ only remains

$$^{*}R_{mn} = \frac{2}{\mathcal{K}^{2}\mathcal{R}_{0}^{2}}\,{}'g_{mn}\,.$$

In his book Weinberg has also noted the metric of the collapsing object in non-comoving co-ordinates. This results in the 4-bein

$$\overset{1}{e}_{1} = \alpha_{S} = \frac{1}{\sqrt{1-\frac{r'^{2}}{\mathcal{R}_{0}^{2}}\frac{1}{\mathcal{K}}}}, \qquad \overset{4}{e}_{4} = \sqrt{\frac{\mathcal{K}}{S}}\sqrt[4]{\frac{1-r'^{2}/\mathcal{R}_{0}^{2}}{1-r_{g}'^{2}/\mathcal{R}_{0}^{2}}}\,\frac{1-\frac{r'^{2}}{\mathcal{R}_{0}^{2}}\frac{1}{S}}{\sqrt{1-\frac{r'^{2}}{\mathcal{R}_{0}^{2}}\frac{1}{\mathcal{K}}}}, \,. \tag{VIII.9.29}$$

$$S = 1 - \sqrt{\frac{1-r'^{2}/\mathcal{R}_{0}^{2}}{1-r_{g}'^{2}/\mathcal{R}_{0}^{2}}}\left(1-\mathcal{K}\right)$$

r_{g}' denotes the value of r' on the surface of the stellar object. Since it is assumed that the collapsing star produces a Schwarzschild field around itself which is static due to the Birkhoff theorem, the values (VIII.9.29) have to match on the surface of the stellar object with the Schwarzschild values.

$\mathcal{R}_{0}$ is not only the radius of curvature of the spherical cap at the beginning of the collapse, which defines the geometry of the model of Weinberg. However, it has a meaning at the boundary surface of the interior and exterior geometries concerning the Schwarzschild parabola. The Schwarzschild geometry is usually treated in non-comoving co-ordinates, which we denote by {r, t}.

If one extends the radius of curvature of the Schwarzschild parabola

$$\rho = \sqrt{\frac{2r^{3}}{M}}$$

to the directrix of the Schwarzschild parabola, then the distance $\mathcal{R}$ is cut out. It has half the length of ρ. Thus, one has

$$\mathcal{R}_{g} = \sqrt{\frac{r_{g}^{3}}{2M}}\;. \tag{VIII.9.30}$$

on the boundary surface of the geometries. At the beginning of the collapse at $t' = 0$ one has $r_{g}' = r_{g}$, thus

$$\mathcal{R}_g(0) = \mathcal{R}_0 = \sqrt{\frac{r_g'^3}{2M}} \qquad \text{(VIII.9.31)}$$

applies. Inserting this into (VIII.9.29) we indeed obtain the Schwarzschild values of the metric coefficients

$$\overset{1}{e}_1 = \frac{1}{\sqrt{1-\frac{2M}{r_g}}} = \alpha_g^S, \quad \overset{4}{e}_4 = \sqrt{1-\frac{2M}{r_g}} \; . \qquad \text{(VIII.9.32)}$$

If one still keeps in mind that the lateral part of the metric must be invariant, it follows from (VIII.9.26)

$$\mathcal{K}^2 r'^2 d\vartheta^2 + \mathcal{K}^2 r'^2 \sin^2\vartheta d\varphi^2 = r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2$$

and from this

$$r = \mathcal{K} r' \; , \qquad \text{(VIII.9.33)}$$

the relation between the non-comoving and comoving radial co-ordinate.

At this point we want to substantiate the relation (VIII.9.22). From (VIII.9.30) one obtains

$$\mathcal{R}_g = \sqrt{\frac{\mathcal{K}^3 r_g'^3}{2M}}, \quad \mathcal{R}_g = \sqrt{\mathcal{K}^3}\,\mathcal{R}_0 \qquad \text{(VIII.9.34)}$$

with the above relation. Since this relation is valid not only on the boundary of the geometry but also in the interior of the stellar object, (VIII.9.22) is justified. However, the relation (VIII.9.31) is not transferable to the interior.

The invariance of the lateral arc elements is also applicable to the polar representation of the spherical cap. From

$$r d\vartheta = \mathcal{K}\mathcal{R}_0 \sin\eta' d\vartheta = \mathcal{R}\sin\eta d\vartheta$$

with η as a non co-moving polar angle one obtains

$$\sin\eta' = \sqrt{\mathcal{K}}\sin\eta \qquad \text{(VIII.9.35)}$$

with (VIII.9.22). From this one gets

$$\cos^2\eta = 1 - \frac{1}{\mathcal{K}}\sin^2\eta' = 1 - \frac{r'^2}{\mathcal{R}_0^2}\frac{1}{\mathcal{K}} \; .$$

For the non-comoving bein vector

$$\overset{1}{e}_1 = \frac{1}{\cos\eta} = \frac{1}{\sqrt{1-\frac{r'^2}{\mathcal{R}_0^2}\frac{1}{\mathcal{K}}}} = \frac{1}{\sqrt{1-\frac{r^2}{\mathcal{R}^2}}}$$

a value results which coincides with the one of Weinberg (VIII.9.29).

After these considerations we are able to determine the velocity of the collapse of the stellar object by calculating the velocity on its surface. From (VIII.9.33) and

$$dr = \mathcal{K} dr' + \mathcal{K}^{\bullet} r' dt', \quad \mathcal{K}^{\bullet} = \frac{\partial \mathcal{K}}{\partial t'}$$

the coefficients of the co-ordinate transformation can be calculated as

$$\Lambda^{i}_{i'} = x^{i}_{|i'} \; . \tag{VIII.9.36}$$

With (VIII.9.18) and $\partial_{4'} = -i\dfrac{\partial}{\partial t'}$ one has

$$\Lambda^{1}_{1'} = \mathcal{K}, \quad \Lambda^{1}_{4'} = i\frac{r'}{\mathcal{R}_0}\sqrt{\frac{1}{\mathcal{K}} - 1} \; , \tag{VIII.9.37}$$

if the indices of the co-moving system are primed from now on.

The co-ordinate transformation is associated with a Lorentz transformation which mediates between the reference systems. To get the *physical* value of velocity of the particles in the interior one must determine this Lorentz transformation by transvecting the Λ with the 4-bein

$$L^{m}_{m'} = \overset{m}{e}_{i}\,\Lambda^{i}_{i'}\,\underset{m'}{e}^{i'} \; . \tag{VIII.9.38}$$

With the 4-bein (VIII.9.4) which now is thought to be primed and (VIII.9.29) one gets

$$L^{1}_{1'} = \alpha_c = \frac{\alpha_S}{\alpha_I}, \quad L^{1}_{4'} = -i\alpha_c v_c = \alpha_S i\frac{r'}{\mathcal{R}_0}\sqrt{\frac{1}{\mathcal{K}} - 1} \; . \tag{VIII.9.39}$$

From this one reads the velocity

$$v_c = -\alpha_I\sqrt{v_R^2 - v_I^2}, \quad v_R = -\frac{r}{\mathcal{R}}, \quad v_I = -\frac{r'}{\mathcal{R}_0} \tag{VIII.9.40}$$

whereby (VIII.9.31) and (VIII.9.33) have been used.

As collapse velocity we define the velocity of the surface of the stellar object. Therefore we read the physical value of the collapse velocity from the Lorentz transformation for the boundary surface ($r' = r'_g$) of the geometries:

$$\begin{gathered} L^{1}_{1'} = \alpha_{col}, \quad L^{1}_{4'} = -i\alpha_{col}v_{col} \\ \alpha_{col} = \frac{\sqrt{1 - \dfrac{2M}{r'_g}}}{\sqrt{1 - \dfrac{2M}{r_g}}}, \quad v_{col} = -\frac{\sqrt{\dfrac{2M}{r_g} - \dfrac{2M}{r'_g}}}{\sqrt{1 - \dfrac{2M}{r'_g}}} \end{gathered} \; . \tag{VIII.9.41}$$

Therein $2M/r'_g$ is a constant quantity. The initial velocity is $v_{col} = 0$ at the time $t' = 0$ as a consequence of $r_g = r'_g$ and the Lorentz factor is $\alpha_{col} = 1$. If the surface of the stellar object has reached the event horizon of the Schwarzschild geometry at $r_g = 2M$, then we get $v_{col}(r_g = 2M) = -1$ and $\alpha_{col} = \infty$. The collapse has achieved the speed of light in free fall and the surface having crossed the event horizon would move faster than light. The force of gravity first gets infinite and then would get imaginary.

The relation (VIII.9.41) can be written as

$$v_{col} = \alpha_I^g v_I^g \sqrt{\frac{1}{\mathcal{K}} - 1}.$$

Since the two factors are constant quantities, one immediately recognizes that the collapse velocity takes an infinite value for $\mathcal{K} \to 0$. The star shrinks to a point ($r_g \to 0$), where the mass density $\kappa\mu_0 = 6M/r_g^3$ is infinitely high. The progress of the collapse velocity is shown in Fig. IX.2.

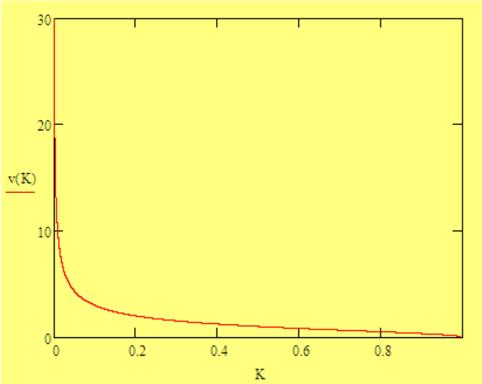

Fig. IX.2

Weinberg also relates the co-ordinate times of the two systems:

$$t = \mathcal{R}_0 \sqrt{1 - \frac{r'^2_g}{\mathcal{R}_0^2}} \int_{S(r',t')}^{1} \frac{1}{1 - \frac{r'^2_g}{\mathcal{R}_0^2}\frac{1}{\mathcal{K}}} \sqrt{\frac{\mathcal{K}}{1-\mathcal{K}}} d\mathcal{K}, \quad d\mathcal{K} = \dot{\mathcal{K}} dt' = -\frac{1}{\mathcal{R}_0}\sqrt{\frac{1}{\mathcal{K}} - 1} dt' . \qquad \text{(VIII.9.42)}$$

The integral has the solution

$$t = -\mathcal{R}_0 \sqrt{1-A} \left[\frac{2A+1}{2} \arcsin(1-2\mathcal{K}) + \sqrt{\mathcal{K}(1-\mathcal{K})} + 2\sqrt{\frac{A^3}{1-A}} \operatorname{arth} \sqrt{\frac{\mathcal{K}}{A}\frac{1-A}{1-\mathcal{K}}} \right]_{S(r',t')}^{1}, \qquad \text{(VIII.9.43)}$$

$$A = \frac{r'^2_g}{\mathcal{R}_0^2}$$

whereby the quantity S given by (VIII.9.29) simplifies to $S_g = \mathcal{K}_g(t')$ on the surface of the star. On this Weinberg takes up further consideration. A light signal which is radially emitted from the surface of the collapsing star needs an infinitely long co-ordinate time in order to reach an observer when the surface coincides with the event horizon. This observer sees the star collapsing eternally. But below the event horizon the star cannot be observed by him. Due to the increasing redshift, he sees the star fade slowly when the surface of the star is approaching the event horizon.

However, we do not find this scenario described by Weinberg realistically. On the one hand, the star collapses in a relatively short proper time to a point with an infinite mass density. On the other hand, for an outside observer this star eternally has a finite extension. In fact, two observers, who are in different states of motion, have different views of the chronological sequences according to the principle of relativity. The principle of relativity does not go so far that an object can have two different, mutually exclusive states. Therefore, we suggest an inconsistency in the model of Weinberg.

Once again we look at the formulae (VIII.9.41) and we interpret them. Then

$$v = -\sqrt{\frac{2M}{r_g}} \qquad \text{(VIII.9.44)}$$

is the velocity of an observer who is in free fall coming from infinity and is located at the very position $r_g \le r'_g$.

$$r_0 = r'_g = r_g\left(t' = 0\right) \tag{VIII.9.45}$$

is the position of an observer who is at the time $t' = 0$, i.e. at the start of the collapse at the position r_0. Thus,

$$v_0 = -\sqrt{\frac{2M}{r'_g}} \tag{VIII.9.46}$$

is the velocity of an observer coming from the infinite who is just passing the position r_0. As we have stated in previous Sections, the fall velocity v of an object that comes from infinity can be directly calculated. But the speed v' of an observer who falls away from r_0 can only be calculated circuitously by using the relativistic difference of v and v_0. From Einstein's law of the addition of velocities results

$$v' = \frac{v - v_0}{1 - vv_0} \ . \tag{9.47}$$

This should also apply to the surface of a star. A glance at (VIII.9.41) shows that

$$v_{col} = -\frac{\sqrt{v^2 - v_0^2}}{\sqrt{1 - v_0^2}} \tag{VIII.9.48}$$

has a different structure and contradicts the relativistic laws. For the associated Lorentz factor, we would expect

$$\alpha' = \alpha\alpha_0\left(1 - vv_0\right) \tag{VIII.9.49}$$

However, in (VIII.9.41) we have

$$\alpha_{col} = \frac{\alpha}{\alpha_0} \ , \tag{VIII.9.50}$$

a term which has, according to the Lorentz equations, the ratio

$$\frac{dT}{dT'} = \frac{\alpha}{\alpha_0} \tag{VIII.9.51}$$

of the proper times of the observer in rest and the observer falling away from r_0, but related to a system that is in free fall coming from *infinity*, but not from r_0, which would be correct.

The formulae (VIII.9.41) are those that correspond to the formulae of free fall in the textbook by Misner, Thorne, and Wheeler and are incorrect, as we have stated in previous Sections. MTW have achieved these formulae by using a not appropriate expression in the line element. But how did the wrong formulae in the model of Weinberg come about? The cause is to be found in the ansatz (VIII.9.1) and (VIII.9.26) for the line element. The first three terms of the line element are written in co-ordinates that are comoving with the particles inside the collapsing object. In particular, the movement of the surface starts at the time $t' = 0$ at the position r_0 according to (VIII.9.45). However, the time-like part of the line element with dt' refers to a co-ordinate system that is comoving with an observer who is in free fall from *infinity*. It should be written as dt'' to mark the meaning of the term correctly.

We will discuss this last statement in more detail. If one has three observers, one of them rests in the Schwarzschild field, the second is released from r_0, and the third comes from infinity, each observer has his specific proper time and Lorentz factor, namely

$$dT = \alpha a_S dt, \quad dT' = \alpha' a_S dt', \quad dT'' = \alpha'' a_S dt'', \quad a_S = \sqrt{1 - \frac{2M}{r}} \ .$$

For the observer at rest is $\alpha = 1$ because $v = 0$. However, for the observer who comes from infinity one has $\alpha'' = 1/a_S$. Finally one has

$$dT = a_S dt, \quad dT' = \alpha' a_S dt', \quad dT'' = dt'', \quad a_S = \sqrt{1 - \frac{2M}{r}} \ . \qquad \text{(VIII.9.52)}$$

In Section IV.12 we have justified these relations with Lemaître-transformations. Only in the system which comes in free fall from infinity, the proper time T" coincides with the co-ordinate time t'. For an observer who falls away from an arbitrary position the redshift factor is different from 1. For this reason we could have been able at the very beginning to state that the Weinberg model is inconsistent. We also want to point out that other models also have this deficiency.

We want to present the facts in abbreviated form once again. The line element is invariant with respect to transformations of the reference system. Thus, it has the same value in all three systems which we have considered

$$ds^2 = dl^2 - dT^2 = dl'^2 - dT'^2 = dl''^2 - dT''^2 \ . \qquad \text{(VIII.9.53)}$$

First, it is clear that is $ds \neq 0$ for a collapsing star and due to the invariance of the line element the value $ds = 0$ is excluded. The collapse velocity of a pressure-free star cannot reach or can only asymptotically reach the velocity of light, ie, only after an infinitely long time. This means that the surface of the star cannot reach let alone cross the event horizon. Comparing Weinberg's approach to the line element (VIII.9.53), the discrepancy is striking. Weinberg's ansatz corresponds to

$$ds^2 = dl'^2 - dT''^2 .$$

Thus, elements which belong to two different states of motion are joined to one another. This ultimately leads to the violation of the Lorentz relations.

If $dx^{4''} = iadt''$ then the force of gravity has to vanish in the comoving system for the free fall from infinity

$$A_{4''\alpha''}{}^{4''} = \frac{1}{a} a_{|\alpha''} = 0 \ .$$

Therefore, $a = a(t'')$ must be a function of the time. With $\bar{t}'' = \int a(t'')dt''$ one can gauge the time in such a way that $dx^{4''} = id\bar{t}''$. Thus one has removed a superfluous factor.

The fact that many authors start with a line element for a collapsing model with the redshift factor 1 is probably due to the fact that they rely on existing spherically symmetric cosmological solutions, most of which originate from Tolman. In these solutions the infinity is of less importance, and there is no need to match the Schwarzschild solution.

The second equation (VIII.9.37) can also be written as

$$r^{\bullet} = -\frac{r'}{r'_g}\sqrt{\frac{2M}{r_g} - \frac{2M}{r'_g}} \; . \tag{VIII.9.54}$$

With (VIII.9.40) one has

$$v_c = \alpha_I r^{\bullet} = \alpha_I \frac{dr}{dT'} \; , \tag{VIII.9.55}$$

where dT' is the proper time of the particles participating in the collapse, which coincides with the co-ordinate time dt' according to the ansatz (VIII.9.1). If one puts

$$\frac{dT}{dT'} = \frac{\alpha_S}{\alpha_I} \; , \tag{VIII.9.56}$$

in analogy to (VIII.9.50) one has a quite obvious relation, even though the ansatz cannot be obtained from a suitable Lorentz transformation

$$v_c = \frac{\alpha_S dr}{dT} = \frac{dx^1}{dT} \; . \tag{VIII.9.57}$$

Florides [F] has followed up the model of Weinberg. Treating a collapse in Newtonian theory, he has shown that the results are identical to those of the relativistic theory, if one restricts oneself to a homogeneous mass distribution and vanishing pressure.

Some models for collapsing stars have been proposed by McVittie [M]. One of them, the pressure-free model, is largely in accord with the model of Weinberg. The paper of Weinberg appeared later. Since the representation of his article is very different from the one by McVittie, one can assume that he has rediscovered the solution.

IX.10. The geometric interpretation of the model of Weinberg

In the following we will deepen the geometric background of a collapsing star and we will investigate its behavior at the event horizon in more detail. Special attention is given to the construction of the stress-energy-momentum tensor in the non-comoving system.

The model of Weinberg describes a non-rotating star consisting of pressure-free incoherent matter collapsing in free fall. For the understanding of the model it is useful to apply geometrical methods. Since the star is to be surrounded by the static Schwarzschild field, we interpret the exterior space-like part of the model as Flamm's paraboloid and the interior geometry as a cap of a sphere which is attached to a suitable position on Flamm's paraboloid. During the collapse of the star the cap of the sphere slides down Flamm's paraboloid, while the exterior Schwarzschild field remains unchanged according to Birkhoff''s theorem. Based on this geometric model some relations which can be derived from the collapsing metric become obvious. They can be decomposed into components which can be attributed to the cap of the sphere or to Flamm's paraboloid, respectively. In addition, it will be explained by this view why the surface of the stellar object cannot go below the Schwarzschild event horizon. Certain limitations are associated to the scale factor, commonly used in literature, which we will discuss later on.

We have written the line element of Weinberg in the form (VIII.9.1). $\{r',t'\}$ are the comoving co-ordinates. In this Section the indices referring to this system are primed, the indices referring to the non-comoving system remain unprimed. However, the connection with the non-comoving co-ordinate time is problematic. From the metric we read the 4-bein

$$\text{(A)}\quad \overset{1'}{e}_{1'} = \mathfrak{K}\alpha_l, \quad \overset{2'}{e}_{2'} = \mathfrak{K}r', \quad \overset{3'}{e}_{3'} = \mathfrak{K}r'\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1, \quad \alpha_l = \frac{1}{\sqrt{1-\frac{r'^2}{\mathfrak{R}_0^2}}}, \quad a_l = \frac{1}{\alpha_l}.$$

(VIII.10.1)

The reciprocal 4-bein is easy to calculate due to the diagonality of the metric. Usually the force of gravity of a gravitational system is derived from the time-like metric factor

$$'A_{4'1'}{}^{4'} = -\overset{4'}{e}_{4'}\underset{4'|1'}{e^{4'}} = -'E_{1'}.$$

Since the metric factor of the Weinberg-line element is $\overset{4'}{e}_{4'}=1$, there are no perceptible acceleration forces in the comoving system. It is $'E_{1'}=0$. The collapse takes place in free fall.

If we temporarily omit the primes which mark the comoving system the stress-energy-momentum tensor has the simple form

$$T_{mn} = -p\,g_{mn} + (p+\mu_0)u_m u_n\,, \qquad \text{(VIII.10.2)}$$

where is p the pressure, μ_0 the mass density, and u_m the velocity of the particles. From the conservation law

$$T^{\alpha m}{}_{||m} = T^{\alpha m}{}_{|m} + A_{mn}{}^{\alpha}T^{nm} + A_m T^{\alpha m} = -g^{\alpha\beta}p_{|\beta} - A_{\beta\gamma}{}^{\alpha}\,p\,g^{\beta\gamma} + A_{44}{}^{\alpha}\mu_0 - A_\beta g^{\alpha\beta}\,p = 0, \quad \alpha = 1,2,3$$

results with $-A_{\beta\gamma}{}^{\alpha}g^{\beta\gamma} = A^{\alpha}$ and $A_{44}{}^{\alpha} = 0$ for the pressure the condition $p_{|\alpha} = 0$. Since the pressure in a collapsing object cannot be constant, the model must be pressure-less.

Again, this is only possible for non-coherent dust, but only as long as the particles do not move too close during the collapse. The pressure-free state is a direct consequence of the ansatz $\overset{4'}{e}_{4'}=1$. The fact that the co-ordinate time t' coincides with the proper time T' of the observer is only valid for an observer which comes in free fall from *infinity*. This is evidently not the case for the surface of the collapsing star. The use of the proper time of an observer who does not participate in the actual collapse ultimately leads to the violation of the addition theorem of velocities and to the destruction of the Lorentz relations, as we discussed in the previous Section.

Although the model of Weinberg already contains inconsistencies concerning its ansatz, we want to further explore the model, because it gives rise to several interesting mechanisms which can be stimulating in the construction of other models. Furthermore, when the surface of the object has reached the event horizon, the very problems known from the Schwarzschild theory occur. The collapse velocity reaches the speed of light at this location.

The metric of the non-comoving system gives more insight into this problem

$$(B) \quad ds^2 = \alpha^2 dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 + a_T^2 dit^2 \ . \qquad (VIII.10.3)$$

From this we read the 4-bein

$$\overset{1}{e}_1 = \alpha_R\,, \quad \overset{2}{e}_2 = r, \quad \overset{3}{e}_3 = r\sin\vartheta, \quad \overset{4}{e}_4 = a_T, \quad \alpha_R = \frac{1}{a_R} = \frac{1}{\sqrt{1-\dfrac{r^2}{\mathcal{R}_g^2}}} \ . \qquad (VIII.10.4)$$

We have little reliance on the time-like metric factor a_T which Weinberg has noted in his textbook. $\mathcal{R}_g$ is the radius of the cap of a sphere and is at the time $t' = \text{const.}$ at all points of the cap by definition equally large. It is closely linked with the radius of curvature of the Schwarzschild parabola on the boundary surface to Flamm's paraboloid

$$\rho_g = \sqrt{\frac{2r_g^3}{M}} \ . \qquad (VIII.10.5)$$

If ρ_g is extended to the directrix of the Schwarzschild parabola, then $\mathcal{R}_g$ is cut out on this straight line. In addition, we get from the properties of the Schwarzschild parabola the simple relation

$$\rho_g = 2\mathcal{R}_g, \quad \mathcal{R}_g = \sqrt{\frac{r_g^3}{2M}} \ , \qquad (VIII.10.6)$$

wherewith we have determined the radius of curvature of that cap which is matched to the Schwarzschild parabola. r_g is the radial non-comoving co-ordinate on the boundary surface. It changes its value during the collapse.

With the polar angle η and

$$r = \mathcal{R}_g \sin\eta \qquad (VIII.10.7)$$

it is possible to bring the metric (B) into the form

$$(B') \quad ds^2 = \mathcal{R}_g^2 d\eta^2 + \mathcal{R}_g^2 \sin^2\eta d\vartheta^2 + \mathcal{R}_g^2 \sin^2\eta \sin^2\vartheta d\varphi^2 - a_T^2 dt^2 . \qquad (VIII.10.8)$$

In the previous Section we have derived the Lorentz transformation from the coefficients of the co-ordinate transformation,

$$L^{1}_{1'} = \alpha_C, \quad L^{1}_{4'} = -i\alpha_C v_C, \quad L^{4}_{1'} = i\alpha_C v_C, \quad L^{4}_{4'} = \alpha_C, \tag{VIII.10.9}$$

wherein is

$$\alpha_C = \frac{\alpha_R}{\alpha_I}, \quad v_C = \alpha_I v_I \sqrt{\frac{1}{\mathcal{K}} - 1}, \quad v_I = -\frac{r'}{\mathcal{R}_0}. \tag{VIII.10.10}$$

If one puts v_I under the root and if one uses the relations $r = \mathcal{K} r'$ and $\mathcal{R}_g = \sqrt{\mathcal{K}^3}\mathcal{R}_0$ one has recognized with

$$\alpha_I = \frac{1}{\sqrt{1 - v_I^2}}, \quad v_C = \alpha_I \sqrt{v_R^2 - v_I^2}, \quad v_R = -\frac{r}{\mathcal{R}_g} \tag{VIII.10.11}$$

that v_C is composed of two velocities, but is violating Einstein's addition theorem of velocities. The latter would read as

$$\alpha_C = \alpha_R \alpha_I (1 - v_R v_I), \quad v_C = \frac{v_R - v_I}{1 - v_R v_I}. \tag{VIII.10.12}$$

If the surface of the collapsing star has reached the Schwarzschild event horizon $r = 2M$, it follows from (VIII.10.6) that $\mathcal{R}_g = 2M$. The cap of the sphere is now a hemisphere and joins with its edge the circle at the waist of Flamm's paraboloid. If one allowed a further contraction, the cap of the sphere would unsolder from Flamm's paraboloid, the linking condition would not be satisfied any longer, the geometric picture would be destroyed.

On the boundary surface the Lorentz factor α_R is equal to

$$\alpha_R^g = \frac{1}{\sqrt{1 - \frac{2M}{r_g}}}$$

and is imaginary for $r_g < 2M$ and also α_C of (VIII.10.10). In addition, it can be seen from (VIII.10.11) with (VIII.10.6) that the velocity of the boundary surface at $2M$ reaches the velocity of light. Below the event horizon the star would collapse faster than light, gravity would be imaginary.

We will avoid such considerations which are often made in the context of the Schwarzschild theory, and we will limit the range of validity of the model in such a way that the model remains in the causal region. However, such a forced restriction reduces the plausibility of the model.

However, this limitation prevents the star from shrinking to a point singularity with an infinitely high mass density and infinitely high spatial curvature. Although this is a concept which many physicists admit, we are not prepared to join it.

From (VIII.10.11) one gets for the surface of the star

$$v_C^g = -\frac{1}{\sqrt{1-\frac{2M}{r'_g}}}\sqrt{\frac{2M}{r'_g}}\sqrt{\frac{1}{\mathfrak{K}}-1} \qquad \text{(VIII.10.13)}$$

and for $v_C^g = -1$ the lowest value for $\mathfrak{K}$. At the beginning of the collapse is $r_g = r'_g$ and therefore is $v_C = 0$. Thus, at $t' = 0$ the scale factor is $\mathfrak{K} = 1$ and one has for it the range of validity

$$\left\{\frac{2M}{r'_g} < \mathfrak{K} \le 1\right\}. \qquad \text{(VIII.10.14)}$$

Below we have plotted in Fig. IX.3 the velocity as function of r on the boundary surface for $r'_g = 10M$. The slung behavior does not agree with a physical progression.

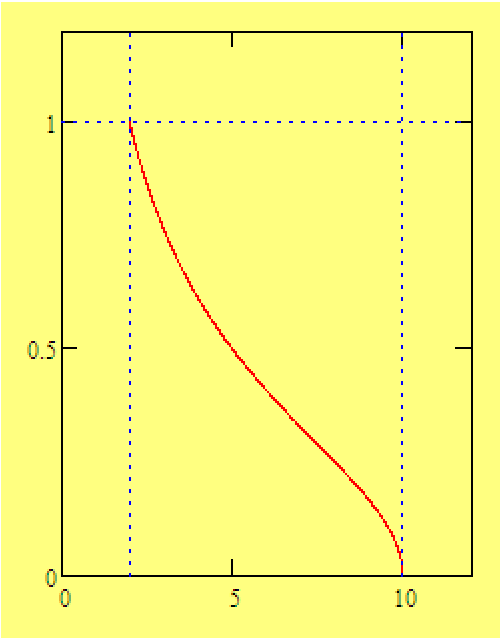

Fig. IX.3

It should also be noted that the collapse velocity which is composed according to Einstein's rules due to (VIII.10.12), provides a convincing progression. The use of the Lorentz relations and their integration with respect to the proper time of the freely falling surface would lead to a model whose surface needs an infinitely long time to reach the event horizon. It would correspond to a model called ECO (eternally collapsing object), proposed by Mitra [M]. However, an analytical solution with the ansatz (VIII.10.12) for the collapse velocity is not known.

IX.11. The field quantities of the model of Weinberg

In Section IX.9 we have shown that in the comoving system the Ricci tensor takes on the form

$$
\begin{aligned}
R_{m'n'} = &-\left['U^{s'}{}_{\underset{1}{||}s'} + 'U^{s'}\,'U_{s'} \right] h_{m'n'} \\
&-\left[B_{n'\underset{2}{||}m'} + B_{n'}B_{m'} \right] - b_{n'}b_{m'}\left[B^{s'}{}_{\underset{2}{||}s'} + B^{s'}B_{s'} \right] . \qquad \text{(VIII.11.1)} \\
&-\left[C_{n'\underset{3}{||}m'} + C_{n'}C_{m'} \right] - c_{n'}c_{m'}\left[C^{s'}{}_{\underset{3}{||}s'} + C^{s'}C_{s'} \right]
\end{aligned}
$$

Therein

$$
h_{m'n'} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \qquad \text{(VIII.11.2)}
$$

is a submatrix of the metric.

$$
'U_{m'} = 'A_{1'm'}{}^{1'} = \left\{ 0,0,0,'U_{4'} \right\}, \quad 'U_{4'} = \frac{1}{K} K_{|4'} = -\frac{i}{K} \dot{K}, \quad \partial_{4'} = -i\frac{\partial}{\partial t'} \qquad \text{(VIII.11.3)}
$$

is the field strength, which describes the collapse and

$$
B_{m'} = 'A_{2'm'}{}^{2'} = \left\{ \frac{a_I}{r}, 0, 0, \frac{1}{K} K_{|4'} \right\}, \quad C_{m'} = 'A_{3'm'}{}^{3'} = \left\{ \frac{a_I}{r}, \frac{1}{r}\cot\vartheta, 0, \frac{1}{K} K_{|4'} \right\} \qquad \text{(VIII.11.4)}
$$

are the lateral field strengths, the

$$
'm_{m'} = \{1,0,0,0\}, \quad b_{m'} = \{0,1,0,0\}, \quad c_{m'} = \{0,0,1,0\}, \quad 'u_{m'} = \{0,0,0,1\} \qquad \text{(VIII.11.5)}
$$

are the unit vectors. The meaning of the graded derivatives

$$
U_{n\underset{1}{||}m} = U_{n|m}, \quad B_{n'\underset{2}{||}m'} = B_{n'|m'} - 'U_{m'n'}{}^{s'}B_{s'}, \quad C_{n'\underset{3}{||}m'} = C_{n'|m'} - B_{m'n'}{}^{s'}C_{s'} - 'U_{m'n'}{}^{s'}C_{s'} \qquad \text{(VIII.11.6)}
$$

was described in former sections in detail.

It can be shown that the field equations in the comoving system have the same form as (VIII.11.1), if the graded derivatives are properly defined. Deriving the field equations we cannot rely on the 4-bein system of the non-comoving system, because we have left open the value for a_T.

But we have a powerful method of calculating the field quantities in the non-comoving system, avoiding the metric factors: the Lorentz transformation. Regarding (VIII.10.10) and (VIII.10.11) we are familiar with the velocity of the particles in the interior of the star, and with the associated Lorentz factor.

The Ricci-rotation coefficients containing the field strengths transform inhomogeneously

$$\begin{array}{c} {'A_{m'n'}}^{s'} = L^{m\,n\,s'}_{m'n's} {A_{mn}}^{s} + {'L_{m'n'}}^{s'}, \quad {'L_{m'n'}}^{s'} = L^{s'}_{s} L^{s}_{s'|m'} \\ {A_{mn}}^{s} = L^{m'n'\,s}_{m\,n\,s'} {'A_{m'n'}}^{s'} + {L_{mn}}^{s}, \quad {L_{mn}}^{s} = L^{s}_{s'} L^{s'}_{s|m} \end{array} . \tag{VIII.11.7}$$

The last terms in each case we call Lorentz terms. With (VIII.10.9) we first compute

$$\begin{array}{c} {'L_{4'1'}}^{4'} = {'L_{1'}} = i\alpha_C^2 v_{C|4'}, \quad {'L_{1'4'}}^{1'} = {'L_{4'}} = -i\alpha_C^2 v_{C|1'} \\ {L_{41}}^{4} = L_1 = -i\alpha_C^2 v_{C|4}, \quad {L_{14}}^{1} = L_4 = i\alpha_C^2 v_{C|1} \end{array} . \tag{VIII.11.8}$$

The expressions can be clearly arranged:

$${'L_{m'n'}}^{s'} = {h_{m'}}^{s'}\,{'L_{n'}} - h_{m'n'}\,{'L^{s'}}, \quad {L_{mn}}^{s} = {h_m}^{s} L_n - h_{mn} L^{s} . \tag{VIII.11.9}$$

Intermediate steps are necessary for the calculation of the changes of the velocity. We remember that the collapse of the radius of the spherical cap is a time-dependent variable that enters into this calculation. Since this change of $\mathcal{R}_g$ is determined by the property of the Schwarzschild parabola, we separate the calculated results into a circular part as would be expected with a non-collapsing object and a parabolic part that stems from the collapse. First, one has for the two velocities, which make up the collapse velocity

$$v^{I}_{|m'} = \{1,0,0,0\}\, a_I v_I \frac{1}{r}, \quad v^{I}_{|m} = \{\alpha_C, 0, 0, -i\alpha_C v_C\}\, a_I v_I \frac{1}{r} , \tag{VIII.11.10}$$

$$\begin{array}{l} v^{R}_{|m'} = \{\alpha_C, 0, 0, -i\alpha_C v_C\}\left(-\dfrac{a_R}{\mathcal{R}_g}\right) + \{0,0,0,1\}\left(-3i\alpha_C v_C \dfrac{a_R}{\rho_g}\right) \\ v^{R}_{|m} = \{1,0,0,0\}\left(-\dfrac{a_R}{\mathcal{R}_g}\right) + \{-i\alpha_C v_C, 0, 0, \alpha_C,\}\left(-3i\alpha_C v_C \dfrac{a_R}{\rho_g}\right) \end{array} . \tag{VIII.11.11}$$

With some computational effort it follows from (VIII.10.11)

$$\begin{array}{l} v^{C}_{|m'} = \dfrac{1}{\alpha_C^2}\{i\alpha_C v_C, 0, 0, \alpha_C\}\, i\alpha_R v_R \dfrac{1}{\mathcal{R}_g} + \dfrac{1}{\alpha_C}\{0,0,0,1\}\left(-3i\alpha_R v_R \dfrac{1}{\rho_g}\right) \\ \qquad + \dfrac{1}{\alpha_C}\{1,0,0,0\}\, v_C \dfrac{a_R}{r} \\ v^{C}_{|m} = \dfrac{1}{\alpha_C^2}\{0,0,0,1\}\, i\alpha_R v_R \dfrac{1}{\mathcal{R}_g} + \dfrac{1}{\alpha_C}\{-i\alpha_C v_C, 0, 0, \alpha_C\}\left(-3i\alpha_R v_R \dfrac{1}{\rho_g}\right) \\ \qquad + \dfrac{1}{\alpha_C}\{\alpha_C, 0, 0, i\alpha_C v_C\}\, v_C \dfrac{a_R}{r} \end{array} \tag{VIII.11.12}$$

and finally with (VIII.11.8)

$$\begin{array}{l} L^{C}_{m} = \{1,0,0,0\}\left(\alpha_R v_R \dfrac{1}{\mathcal{R}_g}\right), \quad L^{P}_{m} = \{\alpha_C, 0, 0, i\alpha_C v_C\}\left(-3\alpha_C \alpha_R v_R \dfrac{1}{\rho_g}\right) \\ {'L^{C}_{m'}} = \{\alpha_c, 0, 0, -i\alpha_C v_C\}\left(-\alpha_R v_R \dfrac{1}{\mathcal{R}_g}\right), \quad {'L^{P}_{m'}} = \{1,0,0,0\}\left(3\alpha_C \alpha_R v_R \dfrac{1}{\rho_g}\right) \end{array} . \tag{VIII.11.13}$$

For the whole Lorentz term one obtains

$$L_m = L^{C}_{m'} + L^{P}_{m} - {'U_m}, \quad {'L_{m'}} = {'L^{C}_{m'}} + {'L^{P}_{m'}} + {'U_{m'}}, \quad {'U_m} = L^{m'}_{m}\,{'U_{m'}} . \tag{VIII.11.14}$$

With this we have the tools in hand to calculate the missing quantities of the non-comoving system. From the inhomogeneous transformation law (VIII.11.7) and (VIII.11.9) one gets the simple relations

$$'U_{m'} = U_{m'} + 'L_{m'}, \quad U_m = 'U_m + L_m . \tag{VIII.11.15}$$

Now we are able to assort all the components of the U-quantities

$$\begin{aligned} U^C_m &= \{1,0,0,0\}\,\alpha_R v_R \frac{1}{\mathcal{R}_g}, \qquad U^P_m = \{\alpha_C,0,0,i\alpha_C v_C\}\left(-3\alpha_C\alpha_R v_R \frac{1}{\rho_g}\right) \\ U^C_{m'} &= \{\alpha_C,0,0,-i\alpha_C v_C\}\,\alpha_R v_R \frac{1}{\mathcal{R}_g}, \quad U^P_{m'} = \{1,0,0,0\}\left(-3\alpha_C\alpha_R v_R \frac{1}{\rho_g}\right) \end{aligned}, \tag{VIII.11.16}$$

whereby we have again made the decomposition into a circular and a parabolic part. We also recognize that the U-variables are already included in the Lorentz terms

$$'L^C_{m'} = -U^C_{m'}, \quad 'L^P_{m'} = -U^P_{m'}, \quad L^C_m = U^C_m, \quad L^P_m = U^P_m . \tag{VIII.11.17}$$

The lateral field quantities transform as vectors

$$B_m = L^{m'}_m B_{m'}, \quad C_m = L^{m'}_m C_{m'} . \tag{VIII.11.18}$$

Thus, we have shown that it is possible to calculate all field quantities of the non-comoving system without complete knowledge of the metric in the non-comoving co-ordinate system. Now the question arises whether the metric coefficients of the non-comoving system can be deduced from the previous results. In the above calculations we have repeatedly relied on the cap of a sphere as basic geometric structure and have written down a corresponding metric (VIII.10.3), (VIII.10.4), and (VIII.10.8). The space-like part of the metric is well known from other models, also the assumption that the radial metric factor (VIII.10.4) corresponds to the Lorentz factor of a motion $v_R = -r/\mathcal{R}_g$.

With (VIII.10.10) and using (VIII.9.18) the components of B in (VIII.11.4) may be brought into the form

$$B_{m'} = \{\alpha_C,0,0,-i\alpha_C v_C\}\frac{a_R}{r} , \tag{VIII.11.19}$$

whereby the familiar structures of the spherical geometry

$$B_m = \left\{\frac{a_R}{r},0,0,0\right\}, \quad C_m = \left\{\frac{a_R}{r},\frac{1}{r}\cot\vartheta,0,0\right\} \tag{VIII.11.20}$$

can be obtained for (VIII.11.18). Since

$$B_1 = \frac{1}{r} r_{|1} = \frac{1}{r} \underset{1}{e}{}^{1} \frac{\partial}{\partial r} r = \frac{a_R}{r}$$

$\underset{1}{e}{}^{1} = a_R,\ \overset{1}{e}{}_1 = \alpha_R = 1/a_R$ must be valid. Therefore the cap of a sphere is a suitable object to give the model a geometric basis. To approach the outstanding time-like metric factor, we calculate with $\frac{1}{\alpha}\alpha_{|1} = \alpha^2 v v_{|1}$ and (VIII.11.11)

$$\frac{1}{\alpha}\alpha_{|1} = -\alpha_R v_R \frac{1}{\rho_g} - 3\alpha_C^2 v_C^2 \alpha_R v_R \frac{1}{\rho_g} . \tag{VIII.11.21}$$

Using the relation $\rho_g = 2\mathcal{R}_g$ one can write the quantity U_1 in the form

$$U_1 = -\alpha_R v_R \frac{1}{\mathcal{R}_g} - 3\alpha_C^2 v_C^2 \alpha_R v_R \frac{1}{\rho_g} + \alpha_R v_R \frac{1}{\rho_g} \ . \qquad \text{(VIII.11.22)}$$

It follows

$$U_1 = \frac{1}{a_T} a_{T|1} = \frac{1}{\alpha} \alpha_{|1} + \alpha_R v_R \frac{1}{\rho_g} \ . \qquad \text{(VIII.11.23)}$$

If one cannot express the last term as a gradient the non-comoving co-ordinate system is anholonomic. Then no relation can be specified between the co-ordinate times t and t'. From this example one can see the difficulty in finding suitable co-ordinate systems for collapsing models. By no means should one imagine that a model can be represented as a 4-dimensional surface in a flat higher dimensional space, whereby the surface is covered by a Gaussian co-ordinate system, and one of these co-ordinates is the time co-ordinate.

As regards the exterior Schwarzschild solution the space-like part of Flamm's paraboloid still fulfills our traditional concepts of the embedding of surfaces into a higher dimensional flat space. The time-like part of the metric needs a sixth variable, whereby two of these variables lie in one and the same dimension, so that the embedding into the 5-dimensional flat space can be sustained.

The complexities of the time-like part of the metric are to be taken into account if the interior solution of a collapsing star is to be linked to the exterior Schwarzschild field. This is a challenge for anyone who deals with this field of problems.

IX.12. The stress-energy momentum tensor

The Ricci in the non-comoving system has the same shape as in the comoving system

$$\begin{aligned} R_{mn} = &-\left[U^s{}_{\underset{1}{||}s} + U^s U_s\right] h_{mn} \\ &-\left[B_{n\underset{2}{||}m} + B_n B_m\right] - b_n b_m \left[B^s{}_{\underset{2}{||}s} + B^s B_s\right] , \\ &-\left[C_{n\underset{3}{||}m} + C_n C_m\right] - c_n c_m \left[C^s{}_{\underset{3}{||}s} + C^s C_s\right] \end{aligned} \qquad \text{(VIII.12.1)}$$

if one appropriately defines the graded derivatives

$$U_{m\underset{1}{||}n} = U_{m|n}, \quad B_{m\underset{2}{||}n} = B_{m|n} - U_{nm}{}^s B_s, \quad C_{m\underset{3}{||}n} = C_{m|n} - U_{nm}{}^s C_s - B_{nm}{}^s C_s \ . \qquad \text{(VIII.12.2)}$$

Therein is

$$U_{mn}{}^s = h_m{}^s U_n - h_{mn} U^s . \qquad \text{(VIII.12.3)}$$

In the previous Section we have calculated the field quantities by means of a Lorentz transformation. The stress-energy-momentum tensor of the non-comoving system can also be calculated with a Lorentz transformation

$$T_{mn} = L_{m\ n}^{m'n'} T_{m'n'} \ . \qquad \text{(VIII.12.4)}$$

Since the stress-energy momentum tensor in the comoving system has the only component

$$T_{4'4'} = \mu_0, \quad \kappa\mu_0 = \frac{3}{\mathcal{R}_g^2} \qquad \text{(VIII.12.5)}$$

one obtains for the non-comoving system

$$T_{11} = -\alpha_C^2 v_C^2 \mu_0, \quad T_{14} = -i\alpha_C^2 v_C \mu_0, \quad T_{44} = \alpha_C^2 \mu_0 \ . \qquad \text{(VIII.12.6)}$$

In particular, we are here interested in how the components of the stress-energy-momentum tensor arise from the geometric components of the Einstein tensor. First, we calculate the quantities

$$\begin{gathered} \frac{1}{\alpha_R}\alpha_{R|m} = E_m^C + E_m^P \\ E_m^C = \{1,0,0,0\}\left(-\alpha_R v_R \frac{1}{\mathcal{R}_g}\right), \quad E_m^P = \{-i\alpha_C v_C, 0, 0, \alpha_C\}\left(-3i\alpha_C v_C \alpha_R v_R \frac{1}{\rho_g}\right) \end{gathered} \ . \qquad \text{(VIII.12.7)}$$

With the parabolic part of the above relation and with U_m^P from (VIII.11.6) we finally obtain the desired relations

$$
\begin{aligned}
\left(B_1 + C_1\right)E_1^P &= \alpha_C^2 v_C^2 \frac{3}{\mathcal{R}_g^2} = \alpha_C^2 v_C^2 \kappa\mu_0 = -\kappa T_{11} \\
\left(B_1 + C_1\right)E_4^P &= i\alpha_C^2 v_C \frac{3}{\mathcal{R}_g^2} = i\alpha_C^2 v_C \kappa\mu_0 = -\kappa T_{14} \ , \qquad \text{(VIII.12.8)} \\
\left(B_1 + C_1\right)U_1^P &= -\alpha_C^2 \frac{3}{\mathcal{R}_g^2} = -\alpha_C^2 \kappa\mu_0 = -\kappa T_{44}
\end{aligned}
$$

whereby the remaining terms of the Einstein tensor are canceled. Thus one has worked out an interesting structure of the field equations of collapsing stars.

We have geometrically deepened an understanding of the model for a collapsing star of Weinberg. We have found general structures which may be helpful for the construction of other models.

IX.13. The solution of McVittie

McVittie [M] has treated a class of collapsing solutions which includes the two discussed by us. He has also treated an ansatz with $p \neq 0$. By this approach pressure and energy density and also the inwards directed speed of the fluid elements become infinite after an infinitely long time. The examples show the difficulties to set up a collapsing model with non-vanishing pressure. Therefore, we only treat the case with $p = 0$.

McVittie starts with the metric

$$ds^2 = \mathcal{K}^2 \left[\frac{1}{1 - k\dfrac{r'^2}{\mathcal{R}_0^2}} dr'^2 + r'^2 d\Omega^2 \right] - dt'^2 \quad . \tag{VIII.13.1}$$

The space *curvature constant* k takes the values

$$k = \{1, 0, -1\} \quad . \tag{VIII.13.2}$$

The time change of the *scale factor* is to be calculated separately for all cases of k.

A quick calculation of the field strengths with the 4-bein to be derived from (I.9.1) results in

$$\begin{aligned} B_m &= \left\{ \frac{a_l}{r}, 0, 0, B_4 \right\} \\ C_m &= \left\{ \frac{a_l}{r}, \frac{1}{r}\cot\vartheta, 0, C_4 \right\} \quad . \\ U_m &= \{0, 0, 0, U_4\} \end{aligned} \tag{VIII.13.3}$$

Therein

$$B_4 \stackrel{*}{=} C_4 \stackrel{*}{=} U_4 = -\frac{i}{\mathcal{K}} \dot{\mathcal{K}} \quad , \tag{VIII.13.4}$$

is the time-independent auxiliary variable

$$a_l = \sqrt{1 - k\frac{r'^2}{\mathcal{R}_0}} \quad , \tag{VIII.13.5}$$

and the non-comoving radial co-ordinate

$$r = \mathcal{K} r' . \tag{VIII.13.6}$$

The derivative of the auxiliary variable a_l provides an expression with the factor k. After a short calculation we get

$$B_{1|1} + B_1 B_1 = -\frac{k}{\mathcal{K}^2 \mathcal{R}_0^2}, \quad C_{1|1} + C_1 C_1 = -\frac{k}{\mathcal{K}^2 \mathcal{R}_0^2}, \quad C_{2|2} - B_{22}{}^1 C_1 + C_2 C_2 = -\frac{k}{\mathcal{K}^2 \mathcal{R}_0^2} \tag{VIII.13.7}$$

With this we obtain for the entirely spatial Ricci

$${}^*R_{\alpha\beta} = 2\frac{k}{\mathcal{K}^2 \mathcal{R}_0^2} g_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3 \quad , \tag{VIII.13.8}$$

which takes a positive, or a negative curvature, or appears flat in dependence of the value of k. We have only to calculate the time-dependent contributions to the field equations for which the relation

$$U_{4|4} + U_4 U_4 = -\frac{1}{\mathcal{K}}\mathcal{K}^{\bullet\bullet} \qquad \text{(VIII.13.9)}$$

is useful. Otherwise, we can refer directly to the relations (VIII.9.12) ff, if we replace $1/\mathcal{K}^2\mathcal{R}_0^2$ by $k/\mathcal{K}^2\mathcal{R}_0^2$.

For $k = 1$ we immediately find the model of Weinberg discussed in Section IX.1. The geometry is that of a cap of a sphere having the radius $\mathcal{R}_0$ at the time $t' = 0$. The cap shrinks to a point in finite time. Corresponding criticism was placed in Section IX.1. McVittie had problems with the transition to a non-comoving co-ordinate system and the adjustment to the exterior Schwarzschild solution. He stopped halfway at a non-orthogonal metric.

The case $k = -1$ is called hyperbolic. In this case the geometry has a constant curvature $\mathcal{R}_0$, but it is not described by a hyperboloid. It can therefore only be a hyperboloid of *constant curvature*. The geometry is based on the cap of a pseudo-sphere. We refer to a situation similar to the Gödel universe.

From the equation

$$G_{44} = -3\left[\frac{1}{\mathcal{K}^2}\mathcal{K}^{\bullet 2} - \frac{1}{\mathcal{K}^2\mathcal{R}_0^2}\right] = -\kappa\mu_0 \qquad \text{(VIII.13.10)}$$

we first get

$$\mathcal{K}\mathcal{K}^{\bullet 2} - \frac{\mathcal{K}^3}{\mathcal{K}^2\mathcal{R}_0^2} = \frac{\kappa\mu_0\mathcal{K}^3}{3}. \qquad \text{(VIII.13.11)}$$

Since the conservation law results in

$$\mu_0(0) = \mu_0\mathcal{K}^3 = \text{const.} \qquad \text{(VIII.13.12)}$$

and in $\mathcal{K}(0) = 1, \mathcal{K}^{\bullet}(0) = 0$ analogous to (VIII.9.16) one finally has

$$\kappa\mu_0(0) = -\frac{3}{\mathcal{R}_0^2}, \qquad \text{(VIII.13.13)}$$

a negative energy density, which leads to the repulsion of mass, if one does not correct the ansatz with

$$\mathcal{R}_0 \to i\mathcal{R}_0. \qquad \text{(VIII.13.14)}$$

As result of this the extradimension is imaginary, which leads to another form of embedding. Since the Schwarzschild geometry, which has to describe the exterior field, has a real extradimension, the adjustment of the interior and the exterior geometry at the boundary surface will cause difficulties.

Inserting the value (VIII.13.13) again into (VIII.13.11) one obtains with $\mathcal{R} = \sqrt{\mathcal{K}^3}\,\mathcal{R}_0$

$$\mathcal{K}^{\bullet} = \pm\frac{1}{i\mathcal{R}_0}\sqrt{\frac{1}{\mathcal{K}} - 1}$$

in which we can now utilize (VIII.13.14).

The case $k = 0$ also leads to a known solution, namely to the one of Oppenheimer and Snyder, which we have dealt with in the Section IX.3 et seq. We only need to show how to convert the quantities of the two versions into one another.

For $k = 0$ and with $a_I = 1$ the metric (VIII.13.1) is reduced to

$$ds^2 = \mathcal{K}^2\left[dr'^2 + r'^2\, d\Omega^2\right] - dt'^2 \ . \qquad \text{(VIII.13.15)}$$

Comparison with (VIII.4.21) leads to

$$\mathcal{K} = \Lambda^{2/3}, \quad \frac{1}{\mathcal{K}}\mathcal{K}^{\bullet} = \frac{2}{3}\frac{1}{\Lambda}\Lambda^{\bullet}, \quad \mathcal{K}^{\bullet} = \frac{\partial \mathcal{K}}{\partial t'} . \qquad \text{(VIII.13.16)}$$

With (VIII.4.9) and (VIII.4.11) one obtains

$$\frac{1}{\mathcal{K}}\mathcal{K}^{\bullet} = -\frac{2}{\Lambda \rho'_g} = -\frac{2}{\rho_g} = -\frac{1}{\mathcal{R}_g} . \qquad \text{(VIII.13.17)}$$

In this case ρ_g is the time-dependent curvature radius of the Schwarzschild parabola on the boundary surface. Its value determines the change of $\mathcal{K}$ when it is sliding down the Schwarzschild parabola during the collapse.

In this geometry $r = \mathcal{K} r'$ applies as well . Here we omit the primes on the indices that indicate the comoving co-ordinate system. Since the geometry appears to be spatially flat,

$$B_{\underset{2}{\alpha\|\beta}} + B_\alpha B_\beta = 0, \quad C_{\underset{3}{\alpha\|\beta}} + C_\alpha C_\beta = 0, \quad {}^*G_{\alpha\beta} = 0 \qquad \text{(VIII.13.18)}$$

also applies, whereby $^*G$ is the purely spatial Einstein tensor.

For collapsing solutions concepts have been taken from expanding cosmological models, since the same mathematical methods are applied to both problems. In contrast to the cosmological models additional difficulties are found for the collapsing models. One has to match the surface of the exterior of the stellar object to the Schwarzschild solution. This requires a considerable mathematical effort. The cosmological line element was written down by Tolman and McVittie in various forms. Since these were also used occasionally for collapsing systems, we want to go into it.

The line element of a collapsing object in comoving co-ordinates is

$$ds^2 = \mathcal{K}^2\left[\frac{dr'^2}{1 - k\dfrac{r'^2}{\mathcal{R}_0^2}} + r'^2\, d\Omega^2\right] - dt'^2 \ . \qquad \text{(VIII.13.19)}$$

For $k = 1$ and with

$$r' = \mathcal{R}_0 \sin\eta', \quad dr' = \mathcal{R}_0 \cos\eta' d\eta'$$

the metric turns into the form

$$ds^2 = \mathcal{K}^2\left[\mathcal{R}_0^2 d\eta'^2 + \mathcal{R}_0^2 \sin^2\eta' d\Omega^2\right] - dt'^2 \ . \qquad \text{(VIII.13.20)}$$

The space-like part of the metric describes a cap of a sphere with time-varying curvature.

For $k = 0$

$$ds^2 = \mathcal{K}^2\left[dr'^2 + r'^2\, d\Omega^2\right] - dt'^2 \qquad \text{(VIII.13.21)}$$

the metric appears flat, but is a function of time.

For $k = -1$ one has

$$r' = \mathcal{R}_0\, sh\eta', \quad dr' = \mathcal{R}_0 ch\eta' d\eta' \ ,$$

$$ds^2 = \mathcal{K}^2\left[\mathcal{R}_0^2 d\eta'^2 + \mathcal{R}_0^2\, sh^2\eta' d\Omega^2\right] - dt'^2 \ . \qquad \text{(VIII.13.22)}$$

The space-like part of the metric describes a time-dependent surface of constant curvature, however, this surface is not to be interpreted as a hyperboloid, as we already have noticed.

With the substitution[145]

$$u = 2\mathcal{R}_0 \tan\frac{\eta}{2}, \quad du = \left(1 + \tan^2\frac{\eta}{2}\right)\mathcal{R}_0 d\eta$$

one obtains

$$\mathcal{R}_0 d\eta = \frac{1}{1 + \frac{u^2}{4\mathcal{R}_0^2}} du, \quad \mathcal{R}_0 \sin\eta = 2\mathcal{R}_0 \sin\frac{\eta}{2}\cos\frac{\eta}{2} = \mathcal{R}_0 \frac{2\tan\frac{\eta}{2}}{1 + \tan^2\frac{\eta}{2}} = \frac{u}{1 + \frac{u^2}{4\mathcal{R}_0^2}} \ .$$

If one renames u into r' one obtains with a similar calculation for $u = 2\mathcal{R}_0\, th\frac{\eta}{2}$

$$ds^2 = \mathcal{K}^2 \frac{1}{\left(1 + k\frac{r'^2}{4\mathcal{R}_0^2}\right)^2}\left[dr'^2 + r'^2 d\Omega^2\right] - dt'^2 \qquad \text{(VIII.13.23)}$$

an isotropic form of the metric which also applies for $k = 0$. If one places $\mathcal{R}_0$ anew, one can also write

$$ds^2 = \frac{\mathcal{K}^2 \mathcal{R}_0^2}{\left(1 + k\frac{r'^2}{4}\right)^2}\left[dr'^2 + r'^2 d\Omega^2\right] - dt'^2 \ . \qquad \text{(VIII.13.24)}$$

With (VIII.13.21) and with

$$r' = \mathcal{R}_0 \frac{u}{1 + k\frac{u^2}{4}} \ ,$$

and with a further renaming one directly arrives at (VIII.13.24).

We finally investigate the physical relevance of the models. With $T_{14} = 0$ one obtains from the conservation law in the comoving co-ordinate system the two relations

$$-p_{|1} + (p + \mu_0)E_1 = 0, \quad \mu_{0|4} + (p + \mu_0)A_4 = 0 \ . \qquad \text{(VIII.13.25)}$$

Therein

[145] From now on will be omitted the prime on η.

$$E_1 = \overset{4}{e}_4 \, e^4_{4|1}$$

is the force of gravity caused by the mass of the stellar object. However, in the models discussed one has $\overset{4}{e}_4 = 1$ and therefore is $E_1 = 0$. This has as a consequence that in the interior of the stellar object the pressure must be constant or vanish. The former is not physically realistic and thus no solutions of Einstein's field equations are known for this case. The case $p = 0$ can only be realized by a non-interacting dust. Such a model can only be a rough approximation of a physically realistic object and only as long as the dust is thinned. If its particles draw nearer interaction forces and pressure immediately occur. For the pressure-free case two models have been discussed, the solution of McVittie with $k = 0$ and the solution of OS, respectively. They describe a star which was initially infinitely large with vanishing matter density and then collapses in free fall. The $k = 1$ solution, or the solution of Weinberg, respectively, collapses from a finite volume in free fall to a zero volume in a finite time. In this model, however, the basic relations of the theory of relativity are violated. Therefore, the Einstein field equations are not a correct solution for a collapsing pressure-free matter and in Nature there do not exist pressure-free collapsing stars. The absence of such a solution is in favor of Einstein's theory.

A paper of Hoyle [H], Fowler, Burbridge, and Burbridge has found general approval under the abbreviation HFBB. The authors have based their ideas on the paper of McVittie (the version $k = 1$) without quoting it. They have discussed the consequences for the collapse of a star with respect to a pressure gradient in the interior of the star, a possible loss of mass of the star, the emission of neutrinos, and the influence of the rotation of the star. However, they insist on believing that a star can collapse beneath the event horizon and can contract to a singularity. Faulkner [F], Hoyle, and Narlikar have relied on it and have investigated the radiation of massive objects. Nariai [N] has presented a model with pressure which undergoes a gravitational collapse. The model is free from singularities and joins the exterior Schwarzschild solution. However, two constants remain undetermined. Taub [T] has calculated models with energy density and pressure. He has given conditions under which the field equations are satisfied. He has adjoined the exterior Schwarzschild solution and discussed the linking conditions. However, the metric coefficients of the model cannot be completely determined by the field equations.

In the last decades the collapse of a star was of topic interest. There have been some exotic solutions and a lot of proposals for solutions with undetermined variables.

Ali [A], Abrahams [A] and Evans; Ames [A] and Thorne; Arnett [A], Banerjee [B], Banerjee [B] and Banerji; Barreto [B], Barreto [B], Herrera, and Santos; Barve [B] , Singh, Vaz, and Witten; Bayin [B], Bekenstein [B], Berezin [B], Berezin and Eroshenko; Bergmann [B], Bini [B] et al., Bochicchio [B] and Laserra; Bochicchio [B], Francaviglia and Laserra; Bondi [B], Bonnor [B], Bonnor [B] and Faulkes; Bonnor [B], Oliveira and Santos; Bonnor [B] and de Oliveira; Brito [B] et al.; Cahill [C] and McVittie; Cahill [C] and Taub; Chakraborty [C], Chakravarty [C], Carr [C] and Coley; Carr [C] et al; Chakraborty [C] and Chakraborty; Chatterjee [C] and Banerjee; Chakraborty [C], Chakraborty, and Debnath; Chan [C], Chan [C], Kichenassamy, Le Denmat and Santos; Chandrasekhar [C] and Tooper; Chandrasekhar [C], Chase [C], Chirenti [C] and Saa; Choudhury and Banerjee; Christodoulou [C], Cissoko [C] et al., Cocke [C], Coley [C] and Tupper; Collins [C] and White; Collins [C], Cooperstock [C], Jhingan, Joshi, and Singh; Cosenza [C], Herrera, Esculpi, and Witten; Cook [C], Corda [C] and Mosquera Cuesta; Culetu [C], Das [D] and Tariq; Das [D] and Kloster; Das [D] Tariq, and Aruliah; Dai [D] and Stojkovic; Datt [D], De [D], De la Cruz [D] and Israel; De la Cruz [D], Chase and Israel; de Oliveira [D] and Santos; Demianski [D] and Lasota; de Oliveira [D], Santos, and Kolassis; de Oliveira [D] , Kolassis, and Santos; Deshingkar [D], Joshi and Dwivedi; Deshingkar [D], Jhingan, Chamorro and Joshi; Di

Criscienzo [D], Nadalini, Vanzo, Zerbini and Zoccatelli; Datta [D], Debnath [D], Chakraborty, and Barrow; Di Prisco [D], Di Prisco [D] et al., Le Denmat, MacCallum and Santos; Doroshkevich [D], Zel'dovich and Novikov; Ehlers [E], Einstein [E] and Strauss; Eisenstaedt [E], Ellis [E], Esposito [E] and Witten; Fackerell [F], Faraoni [F], Gao, Chen, and Shen; Giambo [G], Giannoni, and Magli, Gonçalves [G] and Jhingan; Goswami [G], Gundlach [G] and Martín-García; Faulkes [F], Fayos [F] and Torres; Ferraris [F], Fimin [F] and Chechetkin; Francaviglia, and Spallicci; Foglizzo [F] and Henriksen; Fowler [F], Fujimoro [F], Gao [G] and Chen; Gariel [G], Marcilhacy, and Santos; Glass [G], Glass [G] and Mashhoon; Glass [G], Glazer [G], Gonçalves [G] and Magli; Govender [G], Bogadi and Sharma; Govender [G], Bogadi, and Sharma; Govender [G] et al.; Govinder [G] and Govender; Grundlach [G], Harada [H], Harada [H] and Maeda; Harada [H], Harada [H], Iguchi, and Nakao; Harada [H], Nakao, and Hideo; Hawking [H] and Sciama; Henriksen [H] and Wesson; Hernandez Jr. [H] and Misner; Hernández [H], Núñez and Percoco; Herrera [H] and Santos; Herrera [H], Herrera [H], Di Prisco and Fuenmayor; Herrera [H], Le Denmat, Santos and Wang; Herrera [H], Martin, and Ospino; Herrera [H] et al, Herrera [H] , Le Denmat and Santos; Herrera [H] and Barreto; Herrera [H] and Ruggieri; Herrera [H], Jiménez, Esculpi, and Núñez; Herrera [H], Barreto, Di Prisco and Santos; Herrera [H] and Santos; Herrera [H] and Martinez; Herrera [H], Di Prisco, and Ospino; Di Prisco [D] et al; Herrera [H] and Di Prisco; Herrera [H], Di Prisco, Fuenmayor and Troconis; Israel [I], Ivanov [I], Jhingan [J] and Magli; Kasner [K],Lemos [L], Joshi [J], Joshi [J] and Malafarina; Joshi [J], Malafarina and Narayan; Joshi [J] and Dwivedi; Joshi [J] and Goswami; Jhingan [J], Joshi, and Singh; Jhingan J, Dwivedi, and Barve; Joshi [J] and Singh; Joshi [J] and Królak; Joshi [J], Dadhich, and Maartens; Kanai [K], Siino and Hosoya; Karmakar [K] et al., Kumar [K] and Singh; Kuroda [K], Lake [L], Lake [L] and Hellaby; Lake [L] and Roeder; Mukherjee, Sharma and Maharaj; Knutsen [K], Krishna Rao [K], Kustaanheimo [K], Landau [L], Lapiedra [L], Laserra [L], Leibovitz [L] and Israel; Lin [L], Liu [L], Magli [M], Mestel, and Shu; Malafarina [M], and Joshi; Madhav [M], Goswami, and Joshi; Maeda [M], Mansouri [M], Magli [M], Maartens [M], Maharaj and Tupper; Maharaj [M] and Govender; Markovic [M] and Shapiro; Martínez [M], Pavón and Núñez; Mashhoon [M], Mena [M] and Nolan; Mena [M] and Oliveira; Mena [M], Tavacol, and Joshi; Michel [M], Miller [M], Mitra [M], Moradi [M], Firouzjace, and Mansouri; Müller zum Hagen [M], Müller [M] and Schäfer; Yodzis, and Seifert; Miyamoto [M], Jhingan, and Harada; Naidu [N] and Govender; Nakao [N], Kurita, Morishawa, and Harada; Nariai [N] and Tomita; Narlikar [N] and Vaidya; Narlikar [N] and Moghe; Narlikar [N], Nolan [N], Nolan [N] and Mena; Novikov [N], Ohashi [O], Shiromizu, and Jhingan; Omer [O], Oppenheimer [O] and Volkhoff; Ori [O] and Piran; Pant [P], Pant [P] and Tewari; Penna [P], Penrose [P], Perez [P], Pinheiro [P] and Chan; Pinheiro [P] and Chan; Podurets [P], Rahman [R], Rosales [R], Barreto, Peralta, and Rodrígues-Mueller; Santos [S], Sarwe [S] and Tikekar; Schild [S], Leiter and Robertson; Scheel [S] and Thorne; Shapiro [S] and Teukolsky; Sharma [S] et al., Sharif [S] and Zaeem Ul Hag Bhatti; Singh [S], Singh [S] and Witten; Singh [S] and Joshi; Singh [S] and Pandey; Stephani [S], Som [S] and Santos; Sussman [S], Szekeres [S], Tewari [T] and Charan., Thirukkanesh [T], Rajah and Maharaj; Thirukkanesh [T] and Govender; Rocha [R], Thomas [T], Thompson [T], Thorne [T], Tooper [T], Torres [T] and Fayos; Treciokas [T] and Ellis; Unnikrishnan [U], Vaidya [V], Vickers [V], Wahlquist [W] and Estabrook; Wagh [W] et al, Wang [W] and Wu; Waugh [W] and Lake; Wilson [W], Wyman [W], Seifert, and Müller zum Hagen, Unnikrishnan [U], Unruh [U], Yodzis [Y], Zhang [Z] and Lake.

X. Collapsing interior Schwarzschild solution

We extend the static interior Schwarzschild solution to a collapsing model by applying geometrical methods. We examine the field quantities and field equations in the comoving and non-comoving observer systems. The collapsing stellar object contracts asymptotically to its minimum extent and needs an infinitely long time to arrive at the final state. The event horizon of the the exterior Schwarzschild solution is not reached or even crossed. A geometric model of ECOs (eternally collapsing objects) is presented.

X.1. Introduction

Since Oppenheimer [O] and Snyder, inspired by an expanding cosmological model of Tolman, in 1939 first proposed a model for a collapsing star, many authors have adopted this problem. Among the many suggestions are only a few exact solutions of Einstein's field equations. The reason is that the Einstein field equations are underdetermining and although the conservation laws have been consulted there are not enough equations available to determine the metric coefficients and the physical quantities of the matter configuration.

Therefore we do not try to solve the Einstein field equations, but we have numerous conjectures and make some assumptions. Then we try to assemble them all hoping to obtain a suitable model for the collapse of a star:

i. We rely on geometric ideas which have proven for the exterior and interior Schwarzschild solutions. We interpret the space-like part of the interior solution as the cap of a sphere, the space-like part of the exterior solution as Flamm's paraboloid.

ii. We consider the time-like part of the metric to be an element of a double-surface. This we have discussed in former sections in detail, but here we will indicate this only briefly.

iii. The two solutions, the interior and the exterior, are embeddable into a 5-dimensional flat space, but six variables are required, two of them lie in one and the same dimension. Thus, the theorems of Kasner and Eisenhart are not violated.

iv. The methods of embedding provide strong support for the design of the model. The fact that this is not an academic exercise has been confirmed by us in the discovery of new interior solutions for the models by Kerr, Kerr-Newman, NUT, and Reissner-Nordström.

v. The complete geometry consists of a spherical cap which slides down on Flamm's paraboloid during the collapse. The exterior Schwarzschild geometry remains unchanged according to Birkhoff's theorem. The collapse causes no change in the exterior gravitational field. Above all it produces no gravitational waves.

vi. We perform all calculations by using the method of tetrads. It will prove to be very advantageous for constructing a collapsing model. The Ricci-rotation coefficients guarantee that we will have direct access to the physical and geometrical quantities.

vii. The question of the linking condition on the boundary surface of the interior and exterior solutions should be clarified. Israel [I], O'Brien [O] and Synge, Robson [R], Bonnor [B] and Vickers, Nariai [N], and Lichnerowicz [L] extensively have written on this subject. We want to emphasize that we have to approach to this problem carefully. One has to match a time-dependent model to a static one. At any given moment $t = \text{const.}$ the metric of the interior solution at the boundary should match the exterior one, ie, the two geometries should be connected. Furthermore, the first derivatives of the metric should match, ie the two geometries should have common tangents at the boundary surface. In the present case this can only mean that the spherical cap and Flamm's paraboloid have a common cutting tangent at the boundary surface. Expressions which do not describe the basic geometry, but the change of geometry, ie the collapse, cannot be taken into consideration with respect to the linking, by any means. That these expressions can be problematic shows us a look at historical studies. Although the metrics of the model of Oppenheimer and Snyder match, their derivatives do not. Nariai [N] and Tomita are of opinion that the interior OS solution does not match the exterior Schwarzschild solution and they replace the exterior Schwarzschild solution by a more complicated one while they maintain the interior OS solution. They have attempted to incorporate into the matching also the dynamic quantities which have their origin in the collapse. This presupposes that there is a surface which supports the dynamic properties of the model. We believe that the theory of surfaces does not provide in general such properties.

viii. Accordingly, there is no 'collapsing metric', ie there is no line whose element can be written down. There is in general no surface Σ_c that geometrically describes the collapse, ie a graphic surface or an abstract Riemannian manifold on which one could draw such a line. A look at Fig. X.1 shows that the model can be completely described by the inner surface and the outer surface of the whole Schwarzschild model. There is no global co-ordinate system that could cover such a surface Σ_c. The physical and geometrical quantities are presented in a *reference system* using tetrads. We will use two preferred reference systems. One that is linked to an observer who comoves with the collapse and one that does not comove. Both systems are connected by a Lorentz transformation with non-constant velocity parameters.

ix. For the reasons mentioned above, the allocation of a common co-ordinate system, or the transition from the comoving to the non-comoving co-ordinate system respectively, is problematic. Oppenheimer and Snyder were there partly successful. McVittie [M] gets stuck half way. For the transformation of the time Weinberg [W] writes down an integral whose solution, however, he does not specify. Many authors do not lay a hand on this problem.

x. The collapsing solution is based on the static interior Schwarzschild solution taking into account the pressure inside the matter. The pressure and the mass density are time-dependent. The stellar object cannot be interpreted as an incompressible homogeneous fluid sphere any longer.

xi. Pressure and density of matter should never be infinitely high. The stellar object should not shrink to a point and the curvature of space cannot be infinitely high. As final state no singularity should emerge.

xii. The static model has a horizon. At a relatively small radius of the object the pressure at its center would be infinite. Further, after drilling a hole through the center of the star we make a body oscillate through it. If the object took a certain minimal radius, the body would reach the velocity of light falling through the center of the star. We demand that this *inner horizon* specifies the minimum extension of the collapsing star.

All of that we put together into a toolbox, from which we take out elements as required.

X.2. The interior Schwarzschild solution

Since we intend to create a collapsing model based on the static interior Schwarzschild solution, we are arranging it formally in such a way that an extension is possible. We work out essential parts for the collapse in greater detail and we focus on the geometrical background of the model. This paves the way for us to modify the model so that it can describe a collapse.

The interested reader is supported with numerous detailed calculations in http://members.wavenet.at/arg/PendingPapers/HelpFile.pdf (refered by #). A more detailed description of the interior solution can be found in former sections. The interior Schwarzschild solution is based on the seed metric

$$ds^2 = \alpha_R^2 dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - a_R^2 dt^2, \quad a_R = \sqrt{1 - r^2 / \mathcal{R}_g^2}, \quad \alpha_R = 1/a_R . \qquad \text{(IX.2.1)}$$

Herein $\mathcal{R}_g = \text{const.}$ is the radius of the spherical cap which is used to describe the spatial part of the model and

$$r = \mathcal{R}_g \sin\eta \qquad \text{(IX.2.2)}$$

the radial variable in the 5-dimensional flat embedding space, η the polar angle which is simultaneously the angle of ascent of the spherical cap.

Of the quantity

$$v_R = -\frac{r}{\mathcal{R}_g} = -\sin\eta \qquad \text{(IX.2.3)}$$

we frequently make use later on. From $a_R^2 + v_R^2 = 1$ immediately results $a_R = \cos\eta$. Thus, the seed metric takes the form

$$ds^2 = \frac{1}{\cos^2\eta} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \cos^2\eta dt^2 \qquad \text{(IX.2.4)}$$

and formally corresponds to the de Sitter metric. However, it does not describe a spherical space, but a spherical cap. The connection between the tetrad differentials and the coordinate differentials is established with $dx^m = \overset{m}{e}_i dx^i$, whereby is $dx^i = \{dr, d\vartheta, d\varphi, idt\}$. From (IX.2.1) we read the 4-bein system

$$\overset{1}{e}_1 = \alpha_R, \quad \overset{2}{e}_2 = r, \quad \overset{3}{e}_3 = r\sin\vartheta, \quad \overset{4}{e}_4 = a_R \qquad \text{(IX.2.5)}$$

and we calculate the Ricci-rotation coefficients

$$A_{mn}{}^{s} = \overset{s}{e}_i \underset{[n|m]}{e^i} + g^{sr} g_{mt} \overset{t}{e}_i \underset{[n|r]}{e^i} + g^{sr} g_{nt} \overset{t}{e}_i \underset{[m|r]}{e^i} . \qquad \text{(IX.2.6)}$$

The indices (m, n, ...) number the 4-bein, the (i) are coordinates indices. We prefer to present the variables of the model in the tetrad system ($\Phi_m = \underset{m}{e^i} \Phi_i$).

We decompose the Ricci-rotation coefficients into

$$A_{mn}{}^{s} = B_{mn}{}^{s} + C_{mn}{}^{s} + \hat{U}_{mn}{}^{s} . \qquad \text{(IX.2.7)}$$

With the following unit vectors

$$b_m = \{0,1,0,0\}, \quad c_m = \{0,0,1,0\}, \quad u_m = \{0,0,0,1\} \qquad \text{(IX.2.8)}$$

we write

$$B_{mn}{}^s = b_m B_n b^s - b_m b_n B^s, \quad C_{mn}{}^s = c_m C_n c^s - c_m c_n C^s, \quad \hat{U}_{mn}{}^s = u_m \hat{U}_n u^s - u_m u_n \hat{U}^s. \qquad \text{(IX.2.9)}$$

We calculate the two lateral field quantities B and C and the acceleration $\hat{U}$ from (IX.2.6) with

$$B_1 = -\overset{2}{e}{}_2 e^2_{2|1}, \quad C_1 = -\overset{3}{e}{}_3 e^3_{3|1}, \quad C_2 = -\overset{3}{e}{}_3 e^3_{3|2}, \quad \hat{U}_1 = -\overset{4}{e}{}_4 e^4_{4|1}$$

using

$$\partial_1 = a_R \frac{\partial}{\partial r}, \quad \partial_2 = \frac{1}{r}\frac{\partial}{\partial \vartheta}, \quad \partial_3 = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi}, \quad \partial_4 = \frac{1}{a_R}\frac{\partial}{i\partial t}.$$

Finally we obtain [#1 - #3]

$$B_m = \left\{\frac{a_R}{r},0,0,0\right\}, \quad C_m = \left\{\frac{a_R}{r},\frac{1}{r}\cot\vartheta,0,0\right\}, \quad \hat{U}_m = \left\{\alpha_R v_R \frac{1}{\mathcal{R}_g},0,0,0\right\}. \qquad \text{(IX.2.10)}$$

Thus, we have derived the field quantities of the seed metric and also explained the notation.

We realize that the seed metric is too simple. The acceleration is directed outwards and does not match the corresponding exterior Schwarzschild value. Nevertheless, we calculate the Ricci and the curvature scalar, because the gained structures are retained in the transition to both the genuine interior Schwarzschild model and the collapsing model as well. From

$$R_{mn} = A_{mn}{}^s{}_{|s} - A_{n|m} - A_{rm}{}^s A_{sn}{}^r + A_{mn}{}^s A_s, \quad A_n = A_{rn}{}^r \qquad \text{(IX.2.11)}$$

and (IX.2.7), (IX.2.9) we obtain with [#7]

$$\begin{aligned} R_{mn} = & -\left[\hat{U}^s{}_{\underset{1}{||s}} + \hat{U}^s\hat{U}_s\right]h_{mn} \\ & -\left[B_{n\underset{2}{||m}} + B_nB_m\right] - b_nb_m\left[B^s{}_{\underset{2}{||s}} + B^sB_s\right] \\ & -\left[C_{n\underset{3}{||m}} + C_nC_m\right] - c_nc_m\left[C^s{}_{\underset{3}{||s}} + C^sC_s\right] \\ -\frac{1}{2}R = & \left[\hat{U}^s{}_{\underset{1}{||s}} + \hat{U}^s\hat{U}_s\right] + \left[B^s{}_{\underset{2}{||s}} + B^sB_s\right] + \left[C^s{}_{\underset{3}{||s}} + C^sC_s\right] \end{aligned}, \qquad \text{(IX.2.12)}$$

whereby the *graded derivatives* [2]

$$\hat{U}_{n\underset{1}{||m}} = \hat{U}_{n|m}, \quad B_{n\underset{2}{||m}} = B_{n|m} - \hat{U}_{mn}{}^sB_s, \quad C_{n\underset{3}{||m}} = C_{n|m} - \hat{U}_{mn}{}^sC_s - B_{mn}{}^sC_s \qquad \text{(IX.2.13)}$$

prove to be highly advantageous and are almost mandatory for understanding the structures of the model. Therein is

$$h_{mn} = \mathrm{diag}(1,0,0,1) \qquad \text{(IX.2.14)}$$

a submatrix of the tetrad metric $g_{mn} = \mathrm{diag}(1,1,1,1)$. For later processing we note the subequation of the Ricci [# 6]

$$\hat{U}^s{}_{\underset{1}{||s}} + \hat{U}^s \hat{U}_s = -\frac{1}{\mathcal{R}_g^2} \ . \tag{IX.2.15}$$

The entire Einstein tensor is calculated by using [# 1 - # 5].

We manage the transition to the genuine interior Schwarzschild metric with the projector $\mathcal{P}^s_m$ which has only the few components

$$\mathcal{P}^1_1 = \mathcal{P}^2_2 = \mathcal{P}^3_3 = 1, \quad \mathcal{P}^4_4 = \mathcal{P}, \quad \mathcal{P} = -\frac{1}{2}\frac{a_R}{a_T}, \quad a_T = \frac{1}{2}\left(3a_R^g - a_R\right), \tag{IX.2.16}$$

different from 0. Therein is a_R^g the value of a_R on the surface of the star. If one matches the cap of the sphere to Flamm's paraboloid the relation

$$\rho_g = 2\mathcal{R}_g, \quad \rho = \sqrt{\frac{2r^3}{M}} \tag{IX.2.17}$$

has to apply.

ρ is the radius of curvature of the Schwarzschild parabola, ρ_g its value at the surface of the star. Thus, with

$$\mathcal{R}_g = \sqrt{\frac{r_g^3}{2M}} \tag{IX.2.18}$$

one has made accessible the relation between the radius of the spherical cap and the position r_g of the star's surface in the embedding space. The geometric relations (IX.2.17) are shown graphically in Fig. IV.18. Obviously, the projector (IX.2.16) operates effectively only on the fourth components of the quantities. If we from now on consistently use hats for the quantities of the seed metric, we have

$$dx^m = \left(\mathcal{P}^{-1}\right)^m_s d\hat{x}^s, \quad \partial_m = \mathcal{P}^s_m \hat{\partial}_s, \quad A_{mn}{}^r = \mathcal{P}^s_m \hat{A}_{sn}{}^r . \tag{IX.2.19}$$

To understand the first operation in (IX.2.19) we replace the time differential in the line element (IX.2.1) by

$$i d\hat{t} = -\mathcal{R}_g d i\psi . \tag{IX.2.20}$$

We interpret the time interval as arc element on a pseudo-circle and we let operate the projector on it

$$dx^4 = \left(\mathcal{P}^{-1}\right)^4_4 d\hat{x}^4 = -2\frac{a_T}{a_R}\left(-a_R \mathcal{R}_g d i\psi\right) = a_T 2\mathcal{R}_g d i\psi, \quad i dt = 2\mathcal{R}_g d i\psi, \quad dx^4 = a_T\, i dt$$

and we get the interior Schwarzschild line element

$$ds^2 = \alpha_R^2 dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - a_T^2 dt^2 \ . \tag{IX.2.21}$$

If one takes into account (IX.2.18) and $a_T^g = a_R^g = \sqrt{1 - 2M/r_g}$, then this line element related to the surface of the star coincides with the one of the exterior field. If we write the time-like part as

$$dx^4 = a_T 2\mathcal{R}_g d i\psi = \left(3\mathcal{R}_g \cos\eta_g - \mathcal{R}_g \cos\eta\right) d i\psi$$

we realize that the time which we read from our clock is the real accompanying number of a rotation process through an imaginary angle $i\psi$.

The projector technology gets its actual entitlement only within a 5-dimensional embedding theory [2]. However, here we have shown it in outline, because it plays a useful role in the remodeling of the static model to a collapsing model.

The third operation (IX.2.19) affects only the quantity $\hat{U}$

$$U_{mn}{}^{s} = \mathfrak{p}^{r}_{m}\hat{U}_{rn}{}^{s}, \quad U_1 = \mathfrak{p}\hat{U}_1 . \tag{IX.2.22}$$

A simple calculation [# 9] shows that now holds

$$B_{n\|_{2}m} + B_nB_m = -\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \mathfrak{p} \end{pmatrix}\frac{1}{\mathcal{R}_g^2}, \quad C_{n\|_{3}m} + C_nC_m = -\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & \mathfrak{p} \end{pmatrix}\frac{1}{\mathcal{R}_g^2}, \tag{IX.2.23}$$

because now the quantity $U_{mn}{}^{s}$ containing $\mathfrak{p}$ occurs in the graded derivative. With [# 8] one calculates for U^s

$$U^{s}{}_{\|_{1}s} + U^sU_s = -\frac{\mathfrak{p}}{\mathcal{R}_g^2} . \tag{IX.2.24}$$

The field quantity E_1 in

$$U_1 = A_{41}{}^{4} = -\overset{4}{e}_{4}\, e^{4}{}_{4|1} = \frac{1}{a_T}a_{T|1} = -E_1, \quad E_1 = -\mathfrak{p}\hat{U}_1 = \frac{v_R}{a_T}\frac{1}{\rho_g}$$

is the gravitational force inside the star, is directed inwards, and matches the Schwarzschild term $E_1 = \alpha_R v_R \frac{1}{\rho}$ on the surface.

If one removes the hats from (IX.2.12), one has the structure of the Einstein tensor of the Schwarzschild model and with

$$T_{mn} = -p g_{mn} + (p + \mu_0)u_m u_n \tag{IX.2.25}$$

gains the expressions for the matter configuration

$$\kappa p = -\frac{1}{\mathcal{R}_g^2}(1 + 2\mathfrak{p}), \quad \kappa\mu_0 = \frac{3}{\mathcal{R}_g^2}, \quad 1 - \mathfrak{p} = \frac{3}{2}\frac{a_T^g}{a_T}, \quad p + \mu_0 = (1 - \mathfrak{p})\frac{2}{\mathcal{R}_g^2} = \frac{a_T^g}{a_T}\mu_0 \tag{IX.2.26}$$

and finally the equation of state

$$p = \left(\frac{a_T^g}{a_T} - 1\right)\mu_0 . \tag{IX.2.27}$$

The relations (IX.2.26) are applied to the collapsing model. The pressure can also be written as

$$\kappa p = -\frac{3}{\mathcal{R}_g^2}\frac{\cos\eta_g - \cos\eta}{3\cos\eta_g - \cos\eta} .$$

From (IX.2.2) one obtains $\eta = 0$ for $r = 0$. Thus, in the center of the star $(\eta = 0)$ is

$$\kappa p_c = \frac{3}{\mathcal{R}_g^2}\frac{1-\cos\eta_g}{3\cos\eta_g - 1}.$$

For $\cos\eta_g = 1/3$ the pressure in the center of the star is infinite. If one makes use of this in (IX.2.3) and (IX.2.18), one can see that a star cannot be arbitrarily small. In dependence on its mass its radial coordinate in the embedding space must be

$$r_h > \frac{9}{4}M. \tag{IX.2.28}$$

This is of considerable importance for the collapse. We expect that a star can shrink only close to this minimum value. This is particularly interesting because r_h is above the event horizon of the exterior solution.

For later use, we calculate from (IX.2.16)

$$\frac{1}{\mathfrak{p}}\mathfrak{p}_{|1} = \frac{1}{a_R}a_{R|1} - \frac{1}{a_T}a_{T|1} = \hat{U}_1 - U_1$$

and with (IX.2.22)

$$\mathfrak{p}_{|1} = (1-\mathfrak{p})U_1. \tag{IX.2.29}$$

Thus, we have worked out all the basic relations and we have brought the Schwarzschild model into a form which allows us to extend the interior Schwarzschild solution in such a way that a collapse of a simple matter configuration can be described.

X.3. Collapse, comoving observer system

After the arrangements in the last Section, we are able to reflect, how the inner part of the complete Schwarzschild solution can be related to a collapse. We let the spherical cap, which represents geometrically the region of the interior solution, slide down the Schwarzschild parabola.

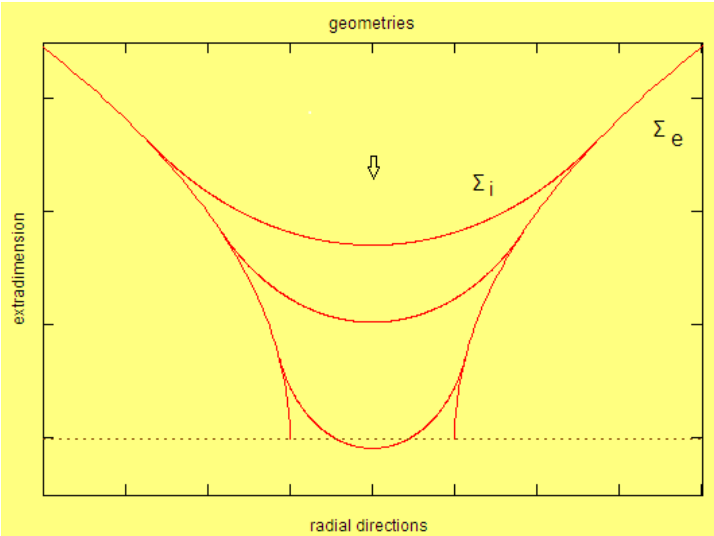

Fig. X.1

Evidently, thereby the radius of the spherical cap changes. Thus, $\mathcal{R}_g$ will be a time-dependent variable but will not change on the spherical cap. Likewise, the Schwarzschild parabola remains unchanged according to Birkhoff's theorem. It governs the course of the collapse by its shape. Therefore the quantity

$$\mathcal{F}_{1'} = 0, \quad \mathcal{F}_{4'} = \frac{1}{\mathcal{R}_g}\mathcal{R}_{g|4'} \qquad \text{(IX.3.1)}$$

enters into the Einstein field equations. Primed indices indicate the comoving frame of reference. If we further make an ansatz for the collapse velocity, we are able to set up a Lorentz transformation which establishes the connection between comoving and non-comoving systems.

Since at any time of the collapse the state of the star is a snapshot of the interior Schwarzschild solution, which can be described by the structures explained in Sec. 2, we can always have access to the relations of this solution.

We note a relation known from the static model for our toolbox and complement it by an analogous one for the comoving system:

$$\frac{1}{r}r_{|1} = \frac{a_R}{r}, \quad r_{|4} = 0, \quad \frac{1}{r'}r'_{|1'} = \frac{a_I}{r}, \quad r'_{|4'} = 0 \qquad \text{(IX.3.2)}$$

The auxiliary variable r' has the range of values $[0,...,r'_g]$. r'_g is the value of r' at the surface of the star. r' is referred to in the literature as comoving radial coordinate. But we do not make use of this interpretation, because we do not use or we are not able to use a coordinate system for the collapsing model. For this we have given reasons in detail above.

In our toolbox we place the following relations:

$$\begin{aligned} a_R^2 &= 1-\frac{r^2}{\mathcal{R}_g^2}, \quad v_R = -\frac{r}{\mathcal{R}_g}, \quad \mathcal{R}_g = \sqrt{\frac{r_g^3}{2M}} \\ a_I^2 &= 1-\frac{r'^2}{\mathcal{R}_0^2}, \quad v_I = -\frac{r'}{\mathcal{R}_0}, \quad \mathcal{R}_0 = \sqrt{\frac{r'^3_g}{2M}} \end{aligned} . \qquad \text{(IX.3.3)}$$

At the beginning of the collapse we have $r_g = r'_g$ and $\mathcal{R}_g = \mathcal{R}_0$. We also demand that the two velocities v_R, v_I defined in (IX.3.3) are composed to the collapse velocity according to Einstein's addition law of velocities

$$v_C = \frac{v_R - v_I}{1-v_R v_I} . \qquad \text{(IX.3.4)}$$

The fact that the collapse velocity consists of two components is basically not new in the case of collapsing models. Besides, the models of McVittie and Weinberg include a similar ansatz. This approach is not immediately obvious, but can be worked out by a few operations. From the classic models only the Oppenheimer-Snyder model can do with a single velocity. However, as a the consequence the OS star collapses in free fall from the infinite.

In the McVittie-Weinberg model, the two velocities are combined in a way which contradicts Einstein's velocity addition theorem and thus the special relativity theory. In our considerations, we strictly adhere to the laws of special and general relativity and use the Einstein addition theorem for the two velocities.

The importance of the two velocities is best seen by looking at their values on the surface of the collapsing star. Due to the linking condition for the two Schwarzschild solutions the velocities must accept values at the surface which are known from the exterior solution. Thus, one obtains for v_R at the surface

$$v_R^g = -\frac{r_g}{\mathcal{R}_g} = -r_g\sqrt{\frac{2M}{r_g^3}} = -\sqrt{\frac{2M}{r_g}} ,$$

the value of the velocity for the free fall from the infinite, which is known from the Schwarzschild theory. For v_I one obtains a thoroughly analogous expression, but with r'_g the value of r' at the surface of the stellar object:

$$v_I^g = -\frac{r'_g}{\mathcal{R}_0} = -r'_g\sqrt{\frac{2M}{r'^3_g}} = -\sqrt{\frac{2M}{r'_g}} = \text{const} .$$

r' is the coordinate comoving with the collapse, r is the non-comoving coordinate.

At the beginning of the collapse, at $T'=0$ the two coordinates coincide

$$r_g = r'_g , \quad \mathcal{R}_g = \mathcal{R}_0$$

and the two speeds are equal

$$v_R^g = v_I^g .$$

Therefore, their difference, the collapse velocity, ie the speed with which the points at the surface move towards the gravitational center, has the value

$$v_C^g = 0.$$

It depends only on the mass M of the star, its respective position r, and the constant initial value v_I. This one is calculated from the fictitious motion from the infinite to the initial position of the surface, and also from the mass of the star. Fig. IX.2 is intended to illustrate this.

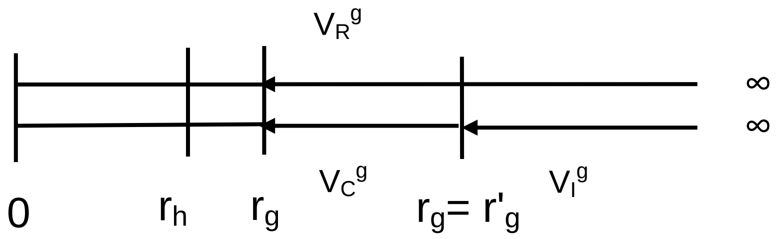

Fig. X.2

If the surface of the star is as close as possible to the inner horizon $r_h = 2.25M$, the velocity constituent v_R^{gh} can be very high at this location, but in any case still smaller than the velocity of light. From this v_I^{gh} is to be subtracted relativistically. However, this quantity is identical with the constant quantity v_I^g.

With this we have set up the collapsing Schwarzschild model. The rest is tedious handwork.

The ansatz (IX.3.3), (IX.3.4) should be compared with historical papers. For $v_I = 0$ one obtains the model of Oppenheimer and Snyder. For the surface of the OS-star one has $v_R^g = -r_g / \mathcal{R}_g = -\sqrt{2M/r_g}$. The pressure-free OS-star collapses in free fall from infinity. A combination of the two velocities other than (IX.3.4), which violates the Einstein addition law of velocities, leads to the models of McVittie and Weinberg.

With (IX.3.4) we are able to set up a Lorentz transformation and the accompanying Lorentz relations. We need a Lorentz transformation which connects a system which participates in the collapse and another which remains at rest relative to an observer system in the outer region.

$$\begin{gathered}
L_{1'}^{1} = \alpha_C, \quad L_{1'}^{4} = i\alpha_C v_C, \quad L_{4'}^{1} = -i\alpha_C v_C, \quad L_{4'}^{4} = \alpha_C \\
v_C = \frac{v_R - v_I}{1 - v_R v_I}, \quad v_R = \frac{v_C + v_I}{1 + v_C v_I}, \quad v_I = \frac{v_R - v_C}{1 - v_R v_C} \\
\alpha_C = \alpha_R \alpha_I (1 - v_R v_I), \quad \alpha_R = \alpha_C \alpha_I (1 + v_C v_I), \quad \alpha_I = \alpha_C \alpha_R (1 - v_C v_R) \\
\alpha_C v_C = \alpha_R \alpha_I (v_R - v_I), \quad \alpha_R v_R = \alpha_C \alpha_I (v_C + v_I), \quad \alpha_I v_I = \alpha_R \alpha_C (v_R - v_C)
\end{gathered} \quad . \qquad \text{(IX.3.5)}$$

With the help of such a Lorentz transformation we can calculate the field quantities in the comoving system. Those field quantities, as set forth above, are components of the Ricci-rotation coefficients. Therefore we have to start from the inhomogeneous transformation law of the Ricci-rotation coefficients

$$'A_{m'n'}{}^{s'} = L_{m'n's}^{m\,n\,s'} A_{mn}{}^{s} + L_{s}^{s'} L_{n'|m'}^{s}. \qquad \text{(IX.3.6)}$$

The Ricci-rotation coefficients themselves are tensors. They describe the curvatures of the normal and oblique cuts through that surface, which we take as a basis of the theory. By the transformation

$$A_{m'n'}{}^{s'} = L^{m\,n\,s'}_{m'n's} A_{mn}{}^{s}$$

the curvatures of the surface are not altered, but only adjusted to the point of view of a new observer. The above inhomogeneous transformation law has the meaning that a new object 'A is assigned to the geometric object A. We call the second term in (IX.3.6) Lorentz term and write it in short as

$$'L_{m'n'}{}^{s'} = L^{s'}_{s} L^{s}_{n'|m'} . \qquad \text{(IX.3.7)}$$

Thus, we end up in

$$'A_{m'n'}{}^{s'} = A_{m'n'}{}^{s'} + 'L_{m'n'}{}^{s'} . \qquad \text{(IX.3.8)}$$

However, concerning the collapse, the process will be somewhat more complex: we go back to the seed metric, we calculate the Lorentz term, we transform to a comoving system, we project to Schwarzschild, and we switch on the collapse by giving up the constraint $\mathcal{R}_g = \text{const.}$. Taken together in a formula this gives

$$'A_{m'n'}{}^{s'} = \mathcal{P}^{r'}_{m'}\left[L^{m\,n\,s'}_{r'n's}\hat{A}_{mn}{}^{s} + '\hat{L}_{r'n'}{}^{s'}\right] = \mathcal{P}^{r'}_{m'}\hat{A}_{r'n'}{}^{s} + 'L_{m'n'}{}^{s'}, \quad 'L_{m'n'}{}^{s'} = \mathcal{P}^{r'}_{m'}{}'\hat{L}_{r'n'}{}^{s'} . \qquad \text{(IX.3.9)}$$

We now expect the field quantities to have a fourth, time-like component in this system. It is easy to see that the lateral field quantities contained in the Ricci-rotation coefficients transform homogeneously

$$\begin{aligned} B_{m'} &= L^{m}_{m'} B_m = \left\{\alpha_C \frac{a_R}{r}, 0, 0, -i\alpha_C v_C \frac{a_R}{r}\right\} \\ C_{m'} &= L^{m}_{m'} C_m = \left\{\alpha_C \frac{a_R}{r}, \frac{1}{r}\cot\vartheta, 0, -i\alpha_C v_C \frac{a_R}{r}\right\} \end{aligned}, \qquad \text{(IX.3.10)}$$

but the quantities U transform inhomogeneously. We put the U-parts of the system into the form

$$\hat{U}_{m'n'}{}^{s'} = h_{m'}{}^{s'}\hat{U}_{n'} - h_{m'n'}\hat{U}^{s'}, \quad 'U_{m'n'}{}^{s'} = h_{m'}{}^{s'}\,'U_{n'} - h_{m'n'}\,'U^{s'} . \qquad \text{(IX.3.11)}$$

First we get with [# 13]

$$\hat{U}_{m'} = \left\{\alpha_C, 0, 0, -i\alpha_C v_C\right\}\alpha_R v_R \frac{1}{\mathcal{R}_g} \qquad \text{(IX.3.12)}$$

and after the $\mathcal{P}$-operation

$$\begin{aligned} U_{1'} &= \mathcal{P}^{4'}_{4'}\hat{A}_{4'1'}{}^{4'}, \quad U_{4'} = \mathcal{P}^{1'}_{1'}\hat{A}_{1'4'}{}^{1'}, \\ U_{m'} &= \left\{\mathcal{P}\hat{U}_{1'}, 0, 0, \hat{U}_{4'}\right\} . \end{aligned} \qquad \text{(IX.3.13)}$$

At this stage of calculations the primed system is a moving system from which the static Schwarzschild metric is observed and is not connected to the collapse. Now the Lorentz term is to be calculated. From [# 13] we take

$$'\hat{L}_{1'} = '\hat{L}_{4'1'}{}^{4'} = L_s^{4'} L^s_{1'|4'} = i\alpha_C^2 v_{C|4'} \quad '\hat{L}_{4'} = '\hat{L}_{1'4'}{}^{1'} = -i\alpha_C^2 v_{C|1'}$$

and we write

$$'\hat{L}_{m'n'}{}^{s'} = h_{m'}{}^{s'}\, '\hat{L}_{n'} - h_{m'n'}\, '\hat{L}^{s'} . \tag{IX.3.14}$$

The terms $'\hat{L}$ are calculated [# 14] and the $\wp$-operation is executed. With the auxiliary variables

$$\begin{aligned} &G_{m'} = \{i\alpha_C v_C, 0, 0, \alpha_C\}\left(-i\alpha_R \frac{1}{\mathcal{R}_g}\right), \quad I_{m'} = \{0,0,0,1\} i\alpha_I v_I \frac{1}{r} \\ &f_{m'} = \{1,0,0,0\}(1-\wp)\alpha_C v_C \alpha_R \frac{1}{\mathcal{R}_g}, \quad g_{m'} = \{-i\alpha_I v_I, 0, 0, \alpha_I\}\frac{i}{\mathcal{R}_g} = \hat{U}_{m'} - G_{m'} \end{aligned} \tag{IX.3.15}$$

we finally obtain

$$'L_{m'n'}{}^{s'} = h_{m'}{}^{s'}\, 'L_{n'} - h_{m'n'}\, 'L^{s'}, \quad 'L_{m'} = -\left[G_{m'} - I_{m'} - f_{m'}\right], \tag{IX.3.16}$$

wherein the quantity f is created by the $\wp$-operation. This reduces the inhomogeneous transformation law of the U-quantities to a vector equation [# 15]

$$'U_{m'} = U_{m'} + 'L_{m'}, \quad 'U_{m'} = \left\{\wp\alpha_I v_I \frac{1}{\mathcal{R}_g}, 0, 0, -i\alpha_C v_C a_R \frac{1}{r}\right\}, \tag{IX.3.17}$$

in which the primes ahead of the kernel indicate a variable of the collapsing system. For the derivation of (IX.3.17) we have used the Lorentz relations (IX.3.5). We have implemented the collapse by considering the field quantities to be a function of time and by demanding that the primed reference system is connected to the collapse.

This is where the problem outlined in item (viii) and (ix) of the Introduction should be illuminated. If the quantity $U_{m'}$ can be derived from the metric coefficients of a line element it can be represented as the gradient of a potential. Thus, $'U_{m'}$ can only be a gradient, if $'L_{m'}$ can also be brought into gradient form. However, this is in general not the case and is not the case in the present model either. The quantities cannot be based on a 4-bein system in such a way that its vectors are tangent to a comoving coordinate system. This in turn, makes the existence of a comoving coordinate system unlikely.

After looking at (IX.3.10) and (IX.3.17) reveals that one has

$$'U_{4'} \overset{*}{=} B_{4'} \overset{*}{=} C_{4'} .$$

From

$$'u^{m'}{}_{||m'} = 'A_{m'}\, 'u^{m'} = 'U_{4'} + B_{4'} + C_{4'}, \quad 'u^{m'} = \{0,0,0,1\}$$

we conclude that the contraction of a volume element is equal in all three directions. It still lacks the quantity mentioned in (IX.3.1). It can be deduced from the field equations. However, we use the conservation law. The stress-energy tensor and its components can be taken over unchanged from the static model (IX.2.25), provided that the indices will be primed. With [# 16] one gains

$$\mu_{0|4'} = -\left(p+\mu_0\right)\left('U_{4'} + B_{4'} + C_{4'}\right) \stackrel{*}{=} -3\left(p+\mu_0\right)'U_{4'} .$$

On the other hand (IX.2.26), second equation, gives

$$\mu_{0|4'} = -2\mu_0 \mathcal{F}_{4'}$$

and with the last two relations (IX.2.26)

$$\mathcal{F}_{4'} = \left(1-\mathfrak{p}\right)'U_{4'} = -i\alpha_C v_C\left(1-\mathfrak{p}\right)\frac{a_R}{r} . \tag{IX.3.18}$$

The conservation law of the comoving system [# 17] is completely treated with

$$T^{m'n'}{}_{||n'} = 0, \quad p_{|1'} = -\left(p+\mu_0\right)'U_{1'}, \quad \mu_{0|4'} \stackrel{*}{=} -3\left(p+\mu_0\right)'U_{4'} . \tag{IX.3.19}$$

The quantity $\mathcal{F}_{4'}$ just derived enters into numerous calculations. The interested reader should follow the calculations [# 18] which allow a more direct derivation of the quantities 'L. Since we did not obtain the field quantities by integration of the field equations, but by extending the static solution with the help of our toolbox, we now have to examine whether the field equations are satisfied with these field quantities. If we derive some auxiliary relations [# 18] we finally obtain by use of the graded derivatives [# 23, # 24]

$$B_{m'\underset{2}{||}n'} = B_{m'|n'} - 'U_{n'm'}{}^{s'}B_{s'} , \quad C_{m'\underset{3}{||}n'} = C_{m'|n'} - 'U_{n'm'}{}^{s'}C_{s'} - B_{n'm'}{}^{s'}C_{s'}$$

$$B_{m'\underset{2}{||}n'} + B_{m'}B_{n'} = -\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \mathfrak{p} \end{pmatrix}\frac{1}{\mathcal{R}_g^2}, \quad B^{s'}{}_{\underset{2}{||}s'} + B^{s'}B_{s'} = -\left(1+\mathfrak{p}\right)\frac{1}{\mathcal{R}_g^2} . \tag{IX.3.20}$$

$$C_{m'\underset{3}{||}n'} + C_{m'}C_{n'} = -\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & \mathfrak{p} \end{pmatrix}\frac{1}{\mathcal{R}_g^2}, \quad C^{s'}{}_{\underset{3}{||}s'} + C^{s'}C_{s'} = -\left(2+\mathfrak{p}\right)\frac{1}{\mathcal{R}_g^2}$$

The verification of the 'U-equation is much more complex. We take from (IX.3.17) the first relation at hand and we differentiate [# 20 - # 22]. It turns out that the U-part of the model is form-invariant

$$'U^{s'}{}_{|s'} + 'U^{s'}\,'U_{s'} = -\frac{\mathfrak{p}}{\mathcal{R}_g^2} . \tag{IX.3.21}$$

For the calculation of this expression we have made use of the relation $\mathfrak{p}_{|1'} = \left(1-\mathfrak{p}\right)'U_{1'}$. Since there is no primed coordinate system, the relation cannot be obtained by direct differentiation, but must be recalculated via the static system. This gives the directly verifiable relation (IX.2.29).

We can take over the structure of the field equations from (IX.2.12), if we remove the hats and equip the kernels and indices with primes. If we substitute the calculated expressions of the subequations [# 25, # 26], the field equations are satisfied.

X.4. Non-comoving observer system

The model is only reasonable, if one succeeds in representing the field quantities in a non-comoving system and in resolving the field equations herewith. Having recalculated some auxiliary variables [# 18], we also find the Lorentz term

$$L_{mn}{}^{s} = L^{s}_{s'} L^{s'}_{n|m}, \quad L_m = G_m - l_m - f_m, \quad L_m = -L^{m'}_{m}{}' L_{m'}$$
$$G_m = \{0,0,0,1\}\left(-i\alpha_R \frac{1}{\mathcal{R}_g}\right), \quad l_m = \{-i\alpha_C v_C, 0, 0, \alpha_C\} i\alpha_l v_l \frac{1}{r} \quad . \qquad \text{(IX.4.1)}$$
$$f_m = \{\alpha_C, 0, 0, i\alpha_C v_C\}(1-\mathcal{p})\alpha_C v_C \alpha_R \frac{1}{\mathcal{R}_g}, \quad g_m = \{-i\alpha_R v_R, 0, 0, \alpha_R\}\frac{i}{\mathcal{R}_g} = \hat{U}_m - G_m$$

With this we can calculate the inhomogeneously transforming U-quantities [# 27]. We realize that the non-comoving system is not identical with the static system, because we have not dismissed the condition $\mathcal{R}_g \neq \text{const.}$. We perform the following operations

$$B_m = L^{m'}_{m} B_{m'}, \quad C_m = L^{m'}_{m} C_{m'}, \quad U_m = L^{m'}_{m}{}' U_{m'} + L_m \ . \qquad \text{(IX.4.2)}$$

The lateral quantities take the static form. However, for the quantity U

$$U_m = \mathcal{p}\hat{U}_m - \alpha_R^2 v_R^2 \mathcal{F}_m = -E_m - \alpha_R^2 v_R^2 \mathcal{F}_m, \quad \mathcal{F}_m = L^{m'}_{m} \mathcal{F}_{m'} \qquad \text{(IX.4.3)}$$

applies.

The first term is the Schwarzschild gravity inside the star, the second the acceleration of the particles in the interior which is caused by the collapse. It can be seen that the expression only goes over into the static one, if we switch off the collapse, thus if we put $\mathcal{R}_g = \text{const.}$, $(\mathcal{F}_m = 0)$. The U-equation again is form-invariant [# 28, # 29]

$$U^{s}{}_{\underset{1}{||}s} + U^{s} U_{s} = -\frac{\mathcal{p}}{\mathcal{R}_g^2} \ . \qquad \text{(IX.4.4)}$$

In contrast, for the B and C equations we get with [# 30]

$$B_{m\underset{2}{||}n} = B_{m|n} - U_{nm}{}^{s}B_{s}, \quad C_{m\underset{3}{||}n} = C_{m|n} - U_{nm'}{}^{s}C_{s} - B_{nm}{}^{s}C_{s}$$

$$\begin{aligned} B_{m\underset{2}{||}n} + B_m B_n &= -\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \mathfrak{p} \end{pmatrix} \frac{1}{\mathfrak{R}_g^2} + \begin{pmatrix} -\hat{U}_1\mathcal{F}_1 & & & -\hat{U}_1\mathcal{F}_4 \\ & 0 & & \\ & & 0 & \\ -\mathcal{F}_4\hat{U}_1 & & & \hat{U}_1\mathcal{F}_1 \end{pmatrix} \\ C_{m\underset{3}{||}n} + C_m C_n &= -\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & \mathfrak{p} \end{pmatrix} \frac{1}{\mathfrak{R}_g^2} + \begin{pmatrix} -\hat{U}_1\mathcal{F}_1 & & & -\hat{U}_1\mathcal{F}_4 \\ & 0 & & \\ & & 0 & \\ -\mathcal{F}_4\hat{U}_1 & & & \hat{U}_1\mathcal{F}_1 \end{pmatrix} \end{aligned} \quad . \qquad \text{(IX.4.5)}$$

$$B^{s}{}_{\underset{2}{||}s} + B^{s}B_{s} = -\left(1+\mathfrak{p}\right)\frac{1}{\mathfrak{R}_g^2}, \quad C^{s}{}_{\underset{3}{||}s} + C^{s}C_{s} = -\left(2+\mathfrak{p}\right)\frac{1}{\mathfrak{R}_g^2}$$

This allows us to represent the Einstein tensor completely. We only have to prepare the right side of the field equations. The stress-energy tensor in the non-comoving system

$$T_{mn} = -p\,g_{mn} + \left(p+\mu_0\right){}'u_m{}'u_n, \quad {}'u_m = \left\{-i\alpha_C v_C, 0, 0, \alpha_C\right\} \qquad \text{(IX.4.6)}$$

we write component by component

$$\begin{aligned} &T_{11} = -p - \alpha_C^2 v_C^2\left(p+\mu_0\right), \quad T_{22} = -p, \quad T_{33} = -p, \\ &T_{41} = -i\alpha_C^2 v_C\left(p+\mu_0\right), \quad T_{44} = \mu_0 + \alpha_C^2 v_C^2\left(p+\mu_0\right) \ . \end{aligned} \qquad \text{(IX.4.7)}$$

The question arises of whether the stress-energy tensor can be geometrized, ie whether the quantities of the right side of the field equations can be brought into connection with the very different field quantities of the left side. If we calculate the quantities in the second brackets of (IX.4.5) [# 31]

$$2\hat{U}_1\mathcal{F}_1 = \alpha_C^2 v_C^2 \kappa\left(p+\mu_0\right), \quad 2\hat{U}_1\mathcal{F}_4 = i\alpha_C^2 v_C \kappa\left(p+\mu_0\right) \qquad \text{(IX.4.8)}$$

it is evident that the lateral subequations establish the necessary connection. If one also takes into consideration the U-equation, one has geometrized the stress-energy tensor.

X.5. Discussion of the model

In the preceding sections it has been suggested that a star which is subjected to the Schwarzschild collapse can contract only up to a minimum radius r_h. We will show that this radius can be reached only asymptotically. First, we derive from

$$r_{|4'} = L^1_{4'} r_{|1} = -i\alpha_C v_C a_R, \quad \frac{\alpha_R dr}{i dT'} = -i\alpha_C v_C$$

the collapse velocity. In the system in rest one has $dx^1 = \alpha_R dr$, in the comoving $dx^{1'} = 0$. For the connection of the proper times the Lorentz relation $dT/dT' = \alpha_C$ applies. With this one obtains from the above expression the collapse velocity

$$v_C = \frac{dx^1}{dT}, \tag{IX.5.1}$$

which we refer to the surface of the star. At this location is $r = r_g, r'_g = r_0 = \text{const.}$ From

$$\frac{\alpha_R dr}{dT'} = \alpha_C v_C = \alpha_R \alpha_I (v_R - v_I)$$

we obtain

$$dr = \alpha_I (v_R - v_I) dT'$$

or taking into account (IX.3.3)

$$dT' = \frac{1}{\alpha_I v_I} \frac{\sqrt{r}}{\sqrt{r_0} - \sqrt{r}} dr.$$

In it are α_I, v_I, and r_0 constants. Integration results in a function

$$f(r) = -\frac{1}{\alpha_I v_I}\left[r + 2\sqrt{r_0}\sqrt{r} + 2r_0 \ln\left(\sqrt{r_0} - \sqrt{r}\right)\right],$$

which is to be regarded in the range $[r_h, r_0]$. Since r is an outgoing coordinate the collapse, however, is directed inwards, we shift the origin of the coordinate system to the position of the surface, and that at the beginning of the collapse. We let run inwards the new radial coordinate: $r \to r_0 - r$, $r_h \to r_0 - r_h$. Then we have

$$g(r) = r_0 - r + 2\sqrt{r_0 - r_h}\sqrt{r_0 - r} + 2(r_0 - r_h)\ln\left(\sqrt{r_0 - r_h} - \sqrt{r_0 - r}\right).$$

If we choose the constant of integration as $g(r_0) = 2(r_0 - r_h)\ln\sqrt{r_0 - r_h}$, then the proper time at the beginning of the collapse is $T' = 0$. Finally, one has in the the range under consideration

$$T'(r) = -\frac{1}{\alpha_I v_I}\left[g(r) - g(r_0)\right] . \qquad \text{(IX.5.2)}$$

The function is depicted in Fig. X.3.

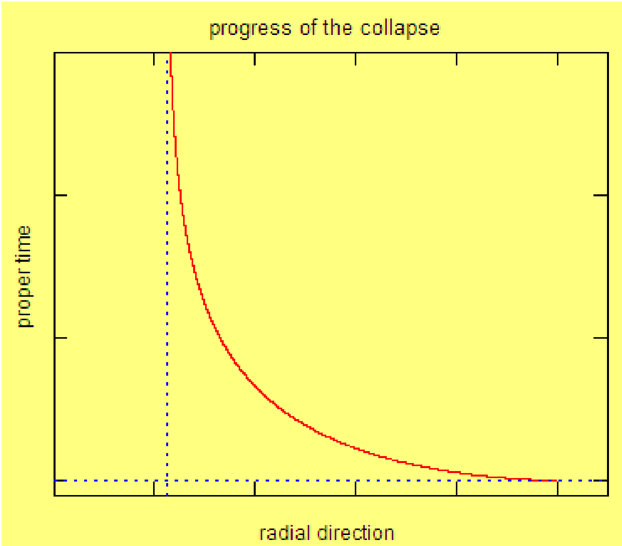

Fig. X.3

From this figure one can gather how much time has passed after the surface of the star has moved a certain distance $r_0 - r$. From $\lim_{r\to r_h} T'(r) = \infty$ is apparent that the star needs an infinitely long time to reach the minimum radius. The collapsing interior Schwarzschild solution has an *inner horizon*. It is identical to the above-mentioned pressure horizon and the velocity horizon. The star can never shrink to a point. The matter density, the pressure, and the curvature of space are never infinite. The inner horizon is above the event horizon of the exterior Schwarzschild solution. Thus, the formation of a black hole in this model is not possible. It describes an ECO (eternally collapsing object), as was predicted by Mitra [M] on the basis of astrophysical considerations. Since the exterior Schwarzschild solution has been proven and describes Nature well, one can assume that the interior solution can describe the interior of a star at least in a rough approximation. Although the two parameters, pressure and mass density are not sufficient to record the properties of a star, there is still hope that at least some basic properties of the model have general validity and that also more pretentious models do not exhibit unusual behaviors.

XI. Epilogue

To our investigations we have prefixed completely general considerations about the space curvature. Following the historical way, one could understand the metric as the metric of a 4-dimensional curved surface which is embedded into a higher-dimensional, if possible, into a 5-dimensional flat space. This vision originally presented by Einstein has soon proved to be untenable. Generally it is not possible to interpret a metric with a cross term as the metric of a surface, whereby the co-ordinate system in use is to be regarded as a Gaussian co-ordinate system of this surface. Thus, Whittaker has dismissed the conception of an actual curvature of the space and has restricted the term curvature to a property of the metric. However, by this somewhat radical aspect one parts with the methods of the surface theory. But these very methods make it possible to get insights which go beyond the understanding of the metric. It has turned out that a surface and its embedding into a higher-dimension flat space are often not sufficient to understand a metric. It is necessary to accept on the surface additional structures like anholonomic hyper planes, oblique-angled bein vectors or locally gauged measuring vectors. With some models it is not sufficient to consider a single surface. A double-surface must be consulted, in order to explain all properties of the metric. Finally, the question arises whether the embedding procedure is a mere academic exercise. We have answered this question. The 5-dimensional formulation leads to sets of compact equations whose geometrical background is simply understood. However, the major advantage is that interior solutions can be added to well-known exterior solutions. We need not make great efforts to find interior solutions, but they can simply be written down. They fit well into the concept of the solution variety of the models.

XII. Mathematical appendix

1. Kerr geometry, seed metric, conversions

$$E_1 \frac{1}{\rho_S} \rho_{S|1} = 2C_0C_0 - M_0E_0 - B_0E_0 - 5C_0E_0 \tag{A1.1}$$

$$2C_0E_0 = C_0C_0 - v_S^2 \tilde{C}_\alpha \tilde{C}^\alpha \tag{A1.2}$$

$$2F_0E_0 = C_0F_0 - v_S^2 \tilde{C}_\alpha \tilde{F}^\alpha \tag{A1.3}$$

$$2B_0E_0 = C_0B_0 + v_S^2 \tilde{H}_{\alpha 3} \tilde{H}^{\alpha 3} \ . \tag{A1.4}$$

(A1.4)

2. Kerr geometry, System C, mixed quantities

$$\begin{aligned}
A_{\alpha 34} &= \ \ a_S\left[H_{\alpha 3} + D_{[\alpha 3]}\right] + \frac{1}{2}\alpha_S D_{\alpha 3} \\
A_{3\beta 4} &= -a_S\left[H_{\beta 3} + D_{[\beta 3]}\right] + \frac{1}{2}\alpha_S D_{\beta 3} \\
A_{4\alpha 3} &= -a_S\left[H_{\alpha 3} + D_{[\alpha 3]}\right] - \frac{1}{2}\alpha_S D_{\alpha 3} \\
A_{43\beta} &= \ \ a_S\left[H_{\beta 3} + D_{[\beta 3]}\right] - \frac{1}{2}\alpha_S D_{\beta 3}
\end{aligned} \tag{A2.1}$$

$$H_{\alpha\beta} = 2i\alpha_R^2 \omega \sigma_{[\alpha} c_{\beta]}, \quad D_{\alpha\beta} = i\alpha_R^2 \omega_{|\alpha} \sigma c_\beta \ . \tag{A2.2}$$

From the first two lines of (A2.1) one gets with

$$H^C_{\alpha\beta} = a_S\left[H_{\alpha\beta} + D_{[\alpha\beta]}\right], \quad D^C_{\alpha\beta} = \alpha_S D_{(\alpha\beta)}, \quad \Omega^C_{\beta\alpha} = -\left[H^C_{\alpha\beta} + D^C_{\alpha\beta}\right] \ . \tag{A2.3}$$

by considering the antisymmetry of the two last indices of the Ricci-rotation coefficients

$$A_{\alpha\beta 4} = -\Omega^C_{\beta\alpha}, \quad A_{\alpha 4\beta} = \Omega^C_{\beta\alpha} \ . \tag{A2.4}$$

From the last two lines of (A2.1) one first gets with

$$A_{4\alpha\beta} = -H^C_{\alpha\beta} + \alpha_S D_{[\alpha\beta]} \ . \tag{A2.5}$$

If one wants therein to place $D^C_{\alpha\beta}$, one at last has with $D_{\alpha\beta} = D_{(\alpha\beta)} + D_{[\alpha\beta]}$

$$A_{4\alpha\beta} = -\Omega^C_{\beta\alpha} + \alpha_S D_{\alpha\beta} \ . \tag{A2.6}$$

The rotational part of the Kerr metric leads to the following expression

$$H_{mns} = -\Omega^C_{nm} u_S + \Omega^C_{sm} u_n + \Omega^C_{sn} u_m + \alpha_S D_{ns} u_m \ . \tag{A2.7}$$

3. Kerr geometry. Centrifugal and rotational forces

$$\alpha_R^2 C_m = \left(1+\alpha_R^2\omega^2\sigma^2\right)C_m = C_m + F_m, \quad C_m = \frac{1}{\sigma}\sigma_m$$

$$\alpha_R^2 C_m = C_m + F_m \tag{A3.1}$$

$$H_{m3}H_{n3} = i\alpha_R^2\omega\sigma_m \cdot i\alpha_R^2\omega\sigma_n = -\alpha_R^4\omega^2\sigma^2 C_m C_n = -\alpha_R^2 C_m F_n = -\left(C_m + F_m\right)F_n$$

$$H_{m3}H_{n3} = -\left(C_m + F_m\right)F_n \ . \tag{A3.2}$$

The components of the quantity $\Omega^C_{\alpha\beta}$ can be represented in such a way that the geometrical and physical meanings are blurred. However, the calculations are simplified by this way of writing. With

$$\Omega^C_{\alpha\beta} = -\left[H^C_{\beta\alpha} + D^C_{\beta\alpha}\right], \quad H^C_{\beta\alpha} = a_S\left[H_{\beta\alpha} + D_{[\beta\alpha]}\right], \quad D^C_{\beta\alpha} = \alpha_S D_{(\beta\alpha)} \tag{A3.3}$$

one firstly has

$$\Omega^C_{13} = a_S\left[H_{13} + \frac{1}{2}D_{13}\right] - \frac{1}{2}\alpha_S D_{13} \ .$$

But since one has $\omega_{|1}\sigma = -2\omega\sigma_1$ one can utilize the expression

$$D_{13} \overset{*}{=} -2\,H_{13} \tag{A3.4}$$

and one discovers that

$$H^C_{13} = 0 \ . \tag{A3.5}$$

One is left with

$$\Omega^C_{13} = -\frac{1}{2}\alpha_S D_{13} \overset{*}{=} \alpha_S H_{13} \ .$$

With $H_{13} = a_S\tilde{H}_{13}$ one lastly gets $\Omega^C_{13} \overset{*}{=} \tilde{H}_{13}$. With $D_{23} = 0$ remains only $\Omega^C_{23} = a_S H_{23} = a_S\tilde{H}_{23}$ and thus

$$\Omega^C_{\alpha 3} \overset{*}{=} \left\{\tilde{H}_{13}, a_S\tilde{H}_{23}\right\}, \quad \Omega^C_{3\alpha} \overset{*}{=} \left\{\tilde{H}_{13}, -a_S\tilde{H}_{23}\right\} \ .$$

Some similar quantities are useful for conversions. The flat $\tilde{\Omega}$, the curved Ω, and the ones curved with respect to the systems A, B, C.

$$\tilde{\Omega}_{\alpha 3} \stackrel{*}{=} \left\{\tilde{H}_{13}, \tilde{H}_{23}\right\}, \quad \tilde{\Omega}_{3\alpha} \stackrel{*}{=} \left\{\tilde{H}_{13}, -\tilde{H}_{23}\right\}$$

$$\Omega_{\alpha 3} \stackrel{*}{=} \left\{H_{13}, H_{23}\right\}, \quad \Omega_{3\alpha} \stackrel{*}{=} \left\{H_{13}, -H_{23}\right\}$$

$$\Omega^{A}_{3\alpha} = H^{AC}_{\alpha 3} + D^{AC}_{\alpha 3} - \Omega_{\alpha 3}, \quad \Omega_{\alpha 3} - \Omega_{3\alpha} \stackrel{*}{=} \Omega^{C}_{3\alpha} \quad . \tag{A3.6}$$

$$\Omega^{B}_{3\alpha} = \left[H^{BC}_{\alpha 3} - H^{C}_{\alpha 3}\right] + \left[D^{BC}_{\alpha 3} - D^{C}_{\alpha 3}\right], \quad \Omega^{B}_{\alpha 3} = 0$$

$$\Omega^{C}_{\alpha 3} \stackrel{*}{=} \left\{\tilde{H}_{13}, a_S \tilde{H}_{23}\right\}, \quad \Omega^{C}_{3\alpha} \stackrel{*}{=} \left\{\tilde{H}_{13}, -a_S \tilde{H}_{23}\right\}$$

4. Kerr geometry. In Ω quadratic relations

$$\tilde{\Omega}^{\alpha\beta}\tilde{\Omega}_{\alpha\beta} = \tilde{H}^{\alpha\beta}\tilde{H}_{\alpha\beta} \tag{A4.1}$$

$$\tilde{\Omega}^{\alpha\beta}\tilde{\Omega}_{\beta\alpha} = 2\left[\tilde{H}^{13}\tilde{H}_{13} - \tilde{H}^{23}\tilde{H}_{23}\right] \tag{A4.2}$$

$$\Omega^{\alpha\beta}\Omega_{\alpha\beta} = H^{\alpha\beta}H_{\alpha\beta} \tag{A4.3}$$

$$\Omega^{\alpha\beta}\Omega_{\beta\alpha} = 2\left[H^{13}H_{13} - H^{23}H_{23}\right] \tag{A4.4}$$

$$v_S^2\tilde{\Omega}^{\alpha\beta}\tilde{\Omega}_{\alpha\beta} = \Omega_C^{\alpha\beta}\Omega^{C}_{\beta\alpha} - \Omega^{\alpha\beta}\Omega_{\beta\alpha} \tag{A4.5}$$

$$v_S^2\tilde{\Omega}^{\alpha 3}\tilde{\Omega}_{3\alpha} = \Omega_C^{\alpha 3}\Omega^{C}_{\alpha 3} - \Omega^{\alpha 3}\Omega_{\alpha 3} \tag{A4.6}$$

$$v_S^2\tilde{\Omega}^{\alpha 3}\tilde{\Omega}_{3\alpha} + 2H^{13}H_{13} - 2\Omega_C^{13}\Omega^{C}_{13} = -v_S^2\tilde{H}^{\alpha 3}\tilde{H}_{\alpha 3} \tag{A4.7}$$

$$v_S^2\tilde{\Omega}^{\alpha 3}\tilde{\Omega}_{3\alpha} + 2H^{23}H_{23} - 2\Omega_C^{23}\Omega^{C}_{23} = v_S^2\tilde{H}^{\alpha 3}\tilde{H}_{\alpha 3} \tag{A4.8}$$

5. Kerr geometry. Further conversions with 0-components

$$\frac{v_S^2}{\Lambda^2} = 2C_0E_0 + 2v_S^2\tilde{C}_\alpha\tilde{F}^\alpha = B_0C_0 - v_S^2N_2C_2 = 2C_0C_0 - 2C_0E_0 \tag{A5.1}$$

$$\begin{aligned} v_S^2N_2C_2 &= -C_0B_0 + 2C_0F_0 + 2C_0E_0 \\ &= -C_0C_0 + C_0F_0 + 2C_0E_0 \\ &= -C_0B_0 - 2C_0C_0 + 2C_0E_0 \end{aligned} \tag{A5.2}$$

$$v_S^2\tilde{\Omega}^{\alpha 3}\tilde{\Omega}_{3\alpha} = C_0B_0 - 2F_0B_0 - 2B_0E_0 \quad . \tag{A5.3}$$

6. Exterior NUT-metric

$$B_1D_1 = -a_G^2\omega^2, \quad *B_1D_1 = -a_D^2 a_G^2\omega^2 = H_{23}H_{23}, \quad H_{23} = -ia_D a_G\omega$$
$$B_0D_0 = -v_G^2\omega^2, \quad *B_0D_0 = -a_D^2 v_G^2\omega^2, \quad B_cD^c = -\omega^2, \quad *B_cD^c = -a_D^2\omega^2$$
$$B_0B_0 = B_0C_0 = \frac{v_G^2}{A^2}, \quad *B_0B_0 = *B_0C_0 = a_D^2\frac{v_G^2}{A^2}$$
$$D_1D_1 = a_G^2\omega^2 - a_D^2 a_G^2\omega^2$$
$$2B_0E_0 = -2M_0B_0 = \frac{v_G^2}{A^2} + \alpha_D^2\omega^2 \quad . \qquad \text{(A6.1)}$$
$$2E_1D_1 = 2M_0D_0 = \alpha_D^2\omega^2 - \alpha_G^2\omega^2$$
$$2M_0{*B_0} + B_0C_0 - B_1D_1 = 0$$
$$*B_0M_0 + *B_0C_0 - *B_0E_0 = \omega^2$$

7. Interior NUT-metric

$$M_0B_0 = \frac{1}{A^2}\sin^2\eta, \quad M_0{*B_0} = \frac{a_D^2}{A^2}\sin^2\eta = \frac{a_D^4}{\mathcal{R}^2}, \quad M_0D_0 = -\omega^2\sin^2\eta$$
$$D_1D_1 = \frac{l^2}{A^2}\omega^2\cos^2\eta = \cos^2\eta - a_D^2\cos^2\eta = *B_1D_1 - B_1D_1 \qquad \text{(A7.1)}$$
$$B_1D_1 = -\omega^2\cos^2\eta, \quad *B_1D_1 = -a_D^2\omega^2\cos^2\eta = \mathcal{H}_{23}\mathcal{H}_{23}$$
$$D_{1|1} = -3\,{*B_1D_1} - M_0D_0 = -\mathcal{H}^2 + a_D^2\omega^2\cos^2\eta + \omega^2\sin^2\eta \ . \qquad \text{(A7.2)}$$

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