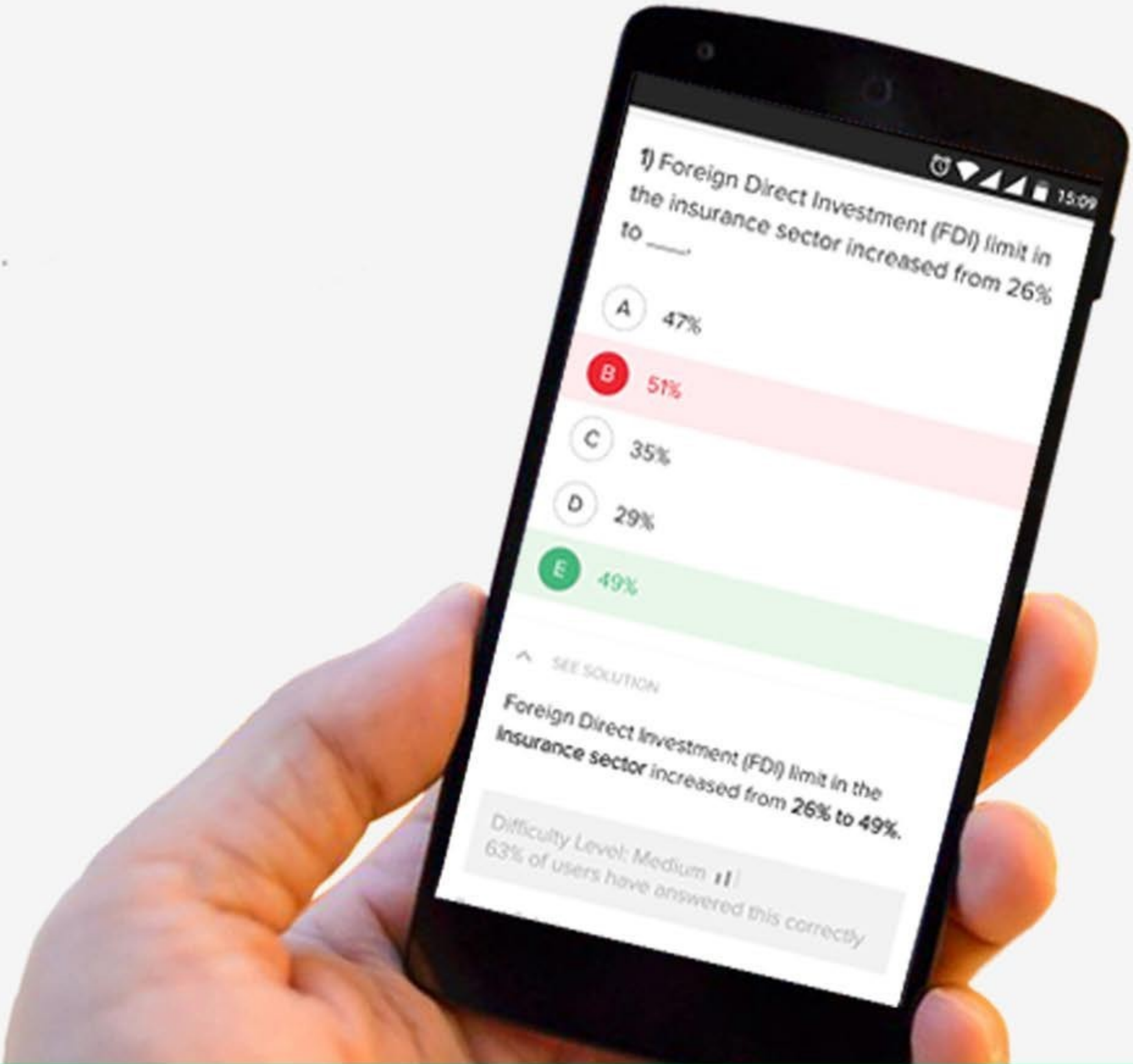
gradeup

Formulas on STRENGTH OF MATERIALS

for GATE ME Exam

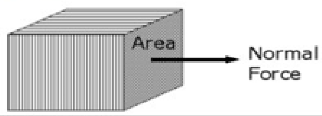
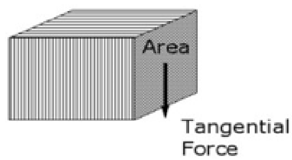
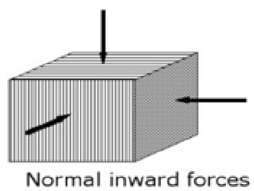


Strength of Material

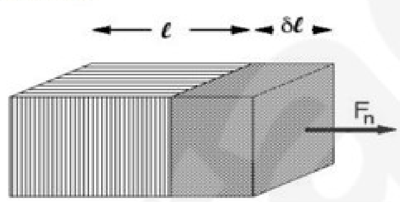
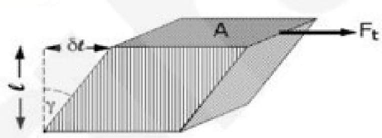
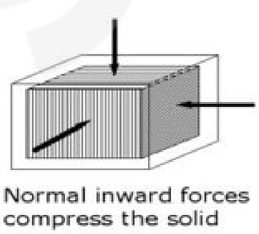
(Formula & Short Notes)

Stress and strain

Stress = Force / Area

Normal stress 	Normal stress = $\frac{\text{Normal force}}{\text{area}}$ $\sigma_n = \frac{F_n}{A}$
Shear stress 	Shear stress = $\frac{\text{tangential force}}{\text{area}}$ $\sigma_t = \frac{F_t}{A}$
Bulk stress 	Bulk Stress = $\frac{\text{normal inward force}}{\text{area}}$ $\sigma_B = P$

$$Tension\ strain(e) = \frac{\Delta L}{L} = \frac{Change\ in\ length}{Initial\ length}$$

Normal strain 	Normal Strain = $\frac{\text{change in normal length}}{\text{original normal length}}$ $\varepsilon_n = \frac{\delta\ell}{\ell}$ Since strain is m/m it is dimensionless.
Shear strain  Note 1: the volume of the solid is not changed by shear strain.	Shear Strain = $\frac{\text{tangential dislacement}}{\text{original normal length}}$ $\varepsilon_t = \frac{\delta t}{\ell} = \gamma(rad)$ Note 2: the angle is is radians, not degrees.
Bulk strain 	Bulk Strain = $\frac{-(\text{change in volume})}{\text{original volume}}$ $\varepsilon_B = \frac{-\delta V}{V}$

Brinell Hardness Number (BHN)

$$HB = \frac{\text{Load (kgf)}}{\text{Surface Area of Indentation (mm}^2\text{)}} = \frac{P}{\frac{\pi D}{2}\left(D - \sqrt{D^2 - d^2}\right)}$$

where D: Diameter of the ball indenter,

d: Diameter at the rim of the permanent impression,

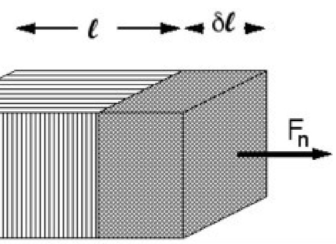
P: Load.

Elastic constants:

$$\frac{\text{Stress}}{\text{Strain}} = \text{Elastic Modulus} = \frac{\sigma}{\varepsilon}$$

where, P = Standard load, D = Diameter of steel ball, and d = Diameter of the indent.

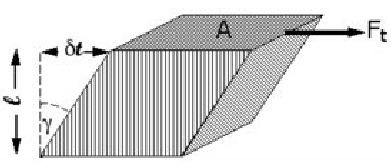
Young's modulus



$$\frac{\text{normal stress}}{\text{normal strain}} = \text{Young's Modulus}$$

$$E = \frac{\sigma_n}{\varepsilon_n} = \frac{F_n/A}{\delta\ell/\ell} = \frac{F_n \cdot \ell}{\delta\ell \cdot A}$$

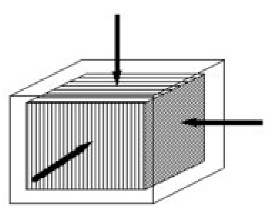
Rigidity modulus



$$\frac{\text{shear stress}}{\text{shear strain}} = \text{Shear Modulus}$$

$$G = \frac{\sigma_s}{\varepsilon_s} = \frac{F_t/A}{\delta t/\ell} = \frac{F_t}{A \cdot \gamma}$$

Bulk modulus



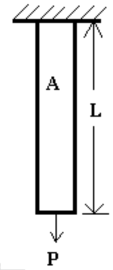
Normal inward forces compress the solid

$$\frac{\text{bulk stress}}{\text{bulk strain}} = \text{Bulk Modulus}$$

$$K = \frac{\sigma_b}{\varepsilon_b} = \frac{P}{-\delta V/V} = -\frac{PV}{\delta V}$$

Axial Elongation of Bar Prismatic Bar Due to External Load

$$\Delta = \frac{PL}{AE}$$



Elongation of Prismatic Bar Due to Self Weight

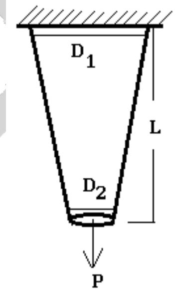
$$\Delta = \frac{PL}{2AE} = \frac{\gamma L^2}{2E}$$

Where γ is specific weight

Elongation of Tapered Bar

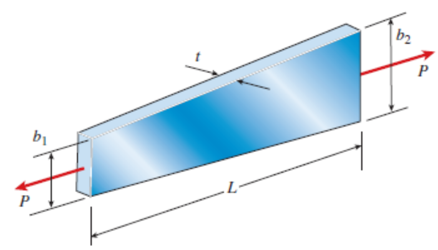
- **Circular Tapered**

$$\Delta = \frac{4PL}{\pi D_1 D_2 E}$$



- **Rectangular Tapered**

$$\Delta = \frac{PL\log_e\left(\frac{B_2}{B_1}\right)}{E.t(B_2 - B_1)}$$



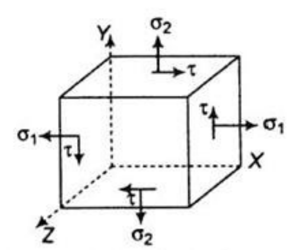
Stress Induced by Axial Stress and Simple Shear

- **Normal stress**

$$\sigma_n = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta + \tau \sin 2\theta$$

- **Tangential stress**

$$\sigma_1 = -\left(\frac{\sigma_1 + \sigma_2}{2}\right)\sin 2\theta + \tau \cos 2\theta$$



Induced stress body diagram

Principal Stresses and Principal Planes

- Major principal stress

$$\sigma_1' = \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\frac{(\sigma_1 - \sigma_2)^2}{2} + \tau^2}$$

- Major principal stress

$$\sigma_2' = \frac{\sigma_1 + \sigma_2}{2} - \sqrt{\frac{(\sigma_1 - \sigma_2)^2}{2} + \tau^2}$$

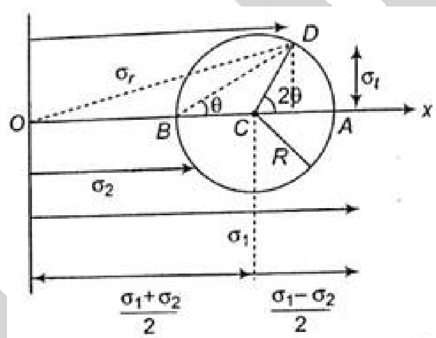
$$\tan 2\theta_p = \frac{2\tau}{\sigma_1 - \sigma_2}$$
$$\sigma_1' + \sigma_2' = \sigma_1 + \sigma_2$$
$$when\ 2\theta_p = 0$$
$$\Rightarrow \sigma_1' = \sigma_1\ and\ \sigma_2' = \sigma_2$$

Principal Strain

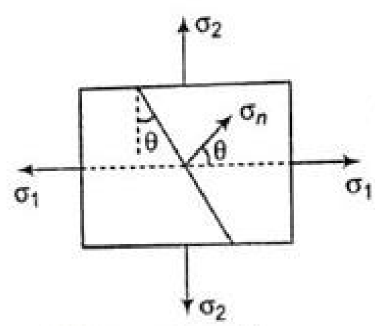
$$\varepsilon_I = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2}$$

$$\varepsilon_{II} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2}$$

Mohr's Circle-



Mohr's circle for plane stress and strain



Different stress diagram

STRAIN ENERGY
Energy Methods:

(i) Formula to calculate the strain energy due to axial loads (tension):

U = ∫ P^2 / (2AE)dx limit 0 toL

Where, P = Applied tensile load, L = Length of the member , A = Area of the member, and E = Young'smodulus.

(ii) Formula to calculate the strain energy due tobending:

U = ∫ M^2 / (2EI) dx limit 0 toL

Where, M = Bending moment due to applied loads, E = Young's modulus, and I = Moment of inertia.

(iii) Formula to calculate the strain energy due totorsion:

U = ∫ T^2 / (2GJ) dx limit 0 toL

Where, T = Applied Torsion , G = Shear modulus or Modulus of rigidity, and J = Polar moment ofinertia

(iv) Formula to calculate the strain energy due to pureshear:

$$U = K \int V^2 / (2GA)\, dx \quad \text{limit 0 to L}$$

Where, V= Shearload

G = Shear modulus or Modulus of rigidity

A = Area of cross section.

K = Constant depends upon shape of cross section.

(v) Formula to calculate the strain energy due to pure shear, if shear stress isgiven:

$$U = \tau^2 V / (2G)$$

Where, τ = ShearStress

G = Shear modulus or Modulus of rigidity

V = Volume of the material.

(vi) Formula to calculate the strain energy , if the moment value isgiven:

$$U = M^2 L / (2EI)$$

Where, M = Bending moment

L = Length of the beam

E = Young'smodulus

I = Moment ofinertia

(vii) Formula to calculate the strain energy , if the torsion moment value isgiven:

$$U = T^2 L / (2GJ)$$

Where, T = AppliedTorsion

L = Length of the beam

G = Shear modulus or Modulus of rigidity

J = Polar moment of inertia

(viii) Formula to calculate the strain energy, if the applied tension load isgiven:

$$U = P^2L / (2AE)$$

Where,

P = Applied tensile load.

L = Length of the member

A = Area of the member

E = Young's modulus.

(ix) Castigliano's first theorem:

$$\delta = \partial U / \partial P$$

Where, δ = Deflection, U= Strain Energy stored, and P = Load

(x) Formula for deflection of a fixed beam with point load at centre:

$$\delta = - wl^3 / 192EI$$

This defection is ¼ times the deflection of a simply supported beam.

(xi) Formula for deflection of a fixed beam with uniformly distributed load:

$$\delta = - wl^4 / 384EI$$

This defection is 5 times the deflection of a simply supported beam.

(xii) Formula for deflection of a fixed beam with eccentric point load:

$$\delta = - wa^3b^3 / 3\,EIl^3$$

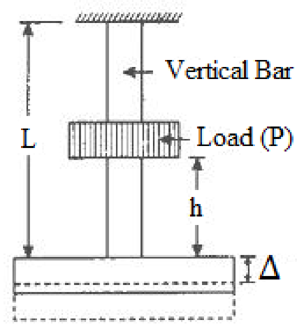
Stresses due to

- **Gradual Loading:-**

$$\sigma = \frac{F}{A}$$

- **Sudden Loading:-**

$$\sigma = {}^{2F}/_{A}$$



- **Impact Loading:-**

$$\sigma = \frac{P}{A}\left(1 + \sqrt{1 + \frac{2AEh}{PL}}\right)$$

Deflection, $If\ \Delta_{st} = \frac{PL}{AE}$

$$\Delta = \Delta_{st} + \sqrt{(\Delta_{st})^2 + 2h\,\Delta_{st}}$$

$$if\ h\ is\ very\ small\ then\ \Delta \approx \sqrt{2h\,\Delta_{st}}$$

Thermal Stresses:-

$$\Delta L = \alpha L \Delta T$$
$$\sigma = \alpha E \Delta T$$

When bar is not totally free to expand and can be expand free by "a"

$$\sigma = E\alpha\Delta T - \frac{aE}{L}$$

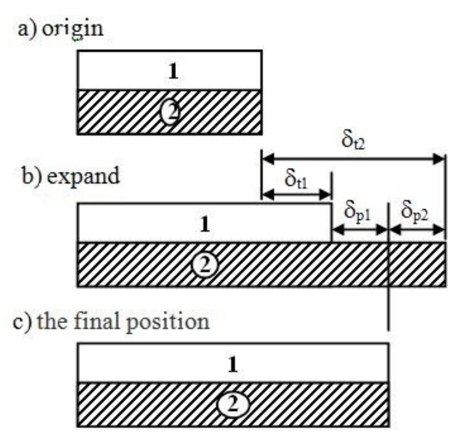
Temperature Stresses in Taper Bars:-

$$\text{Stress} = \alpha L \Delta T = \frac{4PL}{\pi d_1 d_2\, E}$$

Tempertaure Stresses in Composite Bars

$$\delta t_2 = \delta_{t1} + \delta_{p1} + \delta_{p2}$$
$$\delta t_2 - \delta_{t1} = \delta_{p1} + \delta_{p2}$$
$$\Delta t\,(\alpha_1 L_1 - \alpha_2 L_2) = P\left(\frac{L_1}{A_1E_1} + \frac{L_2}{A_2E_2}\right)$$

$$\boxed{P = \frac{\Delta t\,(\alpha_2 - \alpha_1)}{\left(\frac{1}{A_1E_1} + \frac{1}{A_2E_2}\right)}}$$



Hooke's Law (Linear elasticity):

Hooke's Law stated that within elastic limit, the linear relationship between simple stress and strain for a bar is expressed by equations.

$$\sigma \propto \varepsilon,$$
$$\sigma = E\varepsilon$$
$$\frac{P}{A} = E\frac{\Delta L}{L}$$

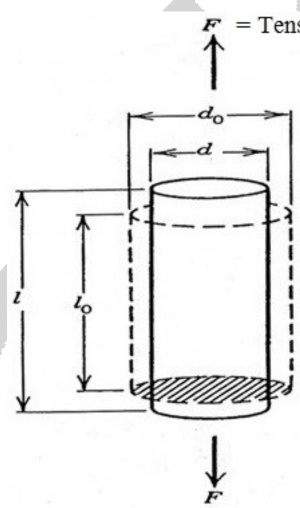
Where, E = Young's modulus of elasticity

P = Applied load across a cross-sectional area

Δl = Change in length

l = Original length

Poisson's Ratio:



$$\text{Poisson Ratio} = \frac{\text{Lateral Strain}}{\text{Linear Strain}}$$

$$\text{Linear strain} = \frac{\Delta l}{l} = \frac{l - l_o}{l_o}$$

$$\text{Lateral Strain} = \frac{\Delta d}{d_o} = \frac{d - d_o}{d_o}$$

Volumetric Strain:

$$\text{Volumetric Strain} = \frac{\text{Change in Volume}(\delta V)}{\text{Original Volume}(V)}$$

$$\epsilon_1 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E}$$
$$\epsilon_2 = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_3}{E}$$
$$\epsilon_3 = \frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$

Futher Volumetric strain $= \epsilon_1 + \epsilon_2 + \epsilon_3$

$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{E} - \frac{2\mu(\sigma_1 + \sigma_2 + \sigma_3)}{E}$$
$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}$$

hence the

$$\boxed{\text{Volumetric strain} = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}}$$

Relationship between E, G, K and μ:

- **Modulus of rigidity:-**

$$Modulus\ of\ rigidity, G = \frac{shear\ stress}{shear\ stress} = \frac{\tau}{\gamma}$$

- **Bulk modulus:-**

$$K = \frac{\text{Volumetric stress}}{\text{Volumetric strain}}$$

$$K = -\frac{dP}{\frac{dV}{V}} = -V\frac{dP}{dV}$$

Negative sign shows decrease in volume.

$$E = 2G(1+\mu) = 3K(1-2\mu)$$

$$E = \frac{9KG}{G+3K}$$

$$\mu = \frac{3K-2G}{G+3K}$$

- **Shear Stress in Rectangular Beam**

Compound Stresses

- **Equation of Pure Bending**

$$\frac{\sigma_a}{y} = \frac{M}{I} = \frac{E}{R}$$

- **Section Modulus**

$$z = \frac{I}{y_{max}} \Rightarrow \frac{M}{I} = \frac{\sigma}{y}$$

$$M = \sigma_{max}\frac{I}{y_{max}} \Rightarrow M = \sigma_{max} x z$$

- **Shearing Stress**

$$\tau = \frac{VA\bar{y}}{Ib}$$

Where,

V = Shearing force

$A\bar{y}$=First moment of area

$$\tau_{max} = \frac{3}{2}\frac{V}{A}$$

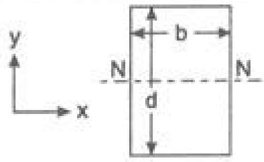
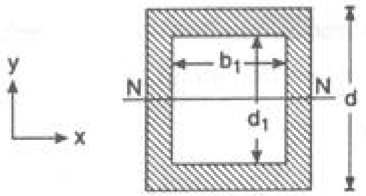
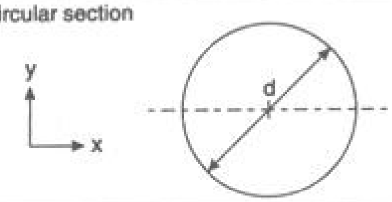
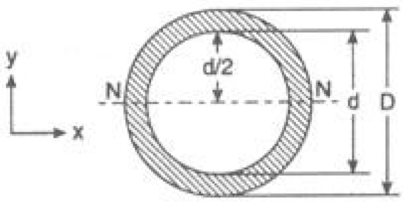
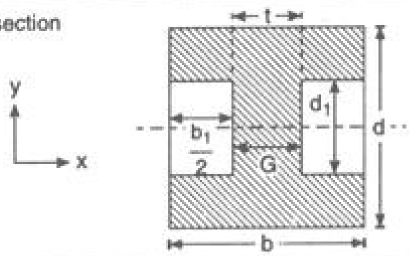
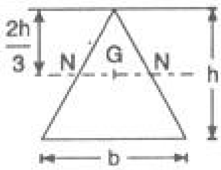
$$\tau_{max} = 1.5\tau_{avg}$$

- **Shear Stress Circular Beam**

$$\tau_{max} = \frac{4V}{3A} = \frac{4}{3}\tau_{av}$$

Moment of Inertia and Section Modulus

Table 11.2.1

Type of section	Moment of Inertia	y_{max}	Section modulus (Z)
Rectangle or paralleogram 	$I_{xx} = \frac{bd^3}{12}$ $I_{yy} = \frac{db^3}{12}$	$\frac{d}{2}$ $\frac{b}{2}$	$Z_{xx} = \frac{bd^2}{6}$ $Z_{yy} = \frac{db^2}{6}$
Hollow rectangular section 	$I_{xx} = \frac{bd^3}{12} - \frac{b_1 d_1^3}{12}$ $I_{yy} = \frac{db^3}{12} - \frac{d_1 b_1^3}{12}$	$\frac{d}{2}$ $\frac{b}{2}$	$Z_{xx} = \frac{1}{6d}(bd^3 - b_1 d_1^3)$ $Z_{yy} = \frac{1}{6b}(db^3 - d_1 b_1^3)$
Circular section 	$I_{xx} = \frac{p}{64} d^4$ $I_{yy} = \frac{p}{64} d^4$	$\frac{d}{2}$ $\frac{d}{2}$	$Z_{xx} = \frac{p}{32} d^3$ $Z_{yy} = \frac{p}{32} d^3$
Hollow curcular section 	$I_{xx} = I_{yy} = I$ $I_{yy} = \frac{p}{64}(D^4 - d^4)$	$\frac{D}{2}$	$Z_{xx} = Z_{yy} = Z$ $Z = \frac{p}{32D}(D^4 - d^4)$
I-section 	$I_{xx} = \frac{bd^3}{12} - \frac{b_1 d_1^3}{12}$ $I_{yy} = \frac{db^3}{12} - \frac{d_1 b_1^3}{12}$ or $I_{xx} = \frac{1}{12}(bd^3 - (b - t)\, d_1^3)$	$\frac{d}{2}$ $\frac{b}{2}$	$Z_{xx} = \frac{1}{6d}(bd^3 - b_1 d_1^3)$ $Z_{yy} = \frac{1}{6b}(db^3 - d_1 b_1^3)$
Triangle 	$I_G = \frac{bh^3}{36}$	$\frac{2}{3}h$	$Z_G = \frac{bh^2}{24}$

- **Direct Stress**

$$\sigma = \frac{P}{A}$$

where P = axial thrust, A = area of cross-section

- **Bending Stress** $$\sigma_b = \frac{My}{I}$$

where M = bending moment, y- distance of fibre from neutral axis, I = moment of inertia.

- **Torsional Shear Stress** $$\tau = \frac{Tr}{J}$$

where T = torque, r = radius of shaft, J = polar moment of inertia.

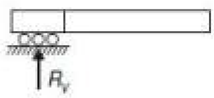
Equivalent Torsional Moment $\sqrt{M^2 + T^2}$

Equivalent Bending Moment $M + \sqrt{M^2 + T^2}$

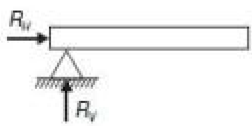
Support: Supports are used to provide suitable reactions (Resisting force) to beams or any body. Following types of supports are used

1. Simple support

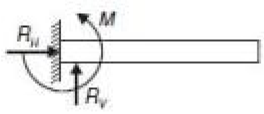
2. Roller support



3. Hinged (Pin) support



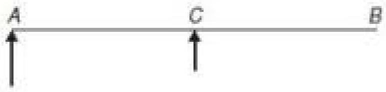
4. Fixed support



Types of Beams

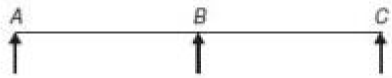
1. Simply supported beams

2. Over hanging beam



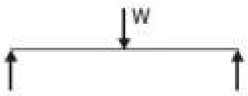
3. Cantilever beams

4. Continuous beams



Types of Loads

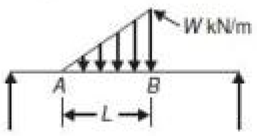
1. Point load



2. Uniformly distributed load (UDL)

Value of UDL = w × L KN point of application → mid point of AB

3. Uniformly varying load (UVL)



Value of UVL = ½ × W × L KN point of application = CG of triangle formed

⇒ $\frac{2}{3}$ L from A, $\frac{L}{3}$ from B

Shear force and Bending Moment Relation $\frac{dV}{dx} = -M$

Load	0	0	Constant
Shear	Constant	Constant	Linear
Moment	Linear	Linear	Parabolic
Load	0	Constant	Linear
Shear	Constant	Linear	Parabolic
Moment	Linear	Parabolic	Cubic

Euler's Buckling Load

$$P_{Critical} = \frac{\pi^2 EI}{l_{equi}^2}$$

For both end hinged $\boldsymbol{l_{equi}} = \boldsymbol{l}$

For one end fixed and other free $\boldsymbol{l_{equi}} = \boldsymbol{2l}$

For both end fixed $\boldsymbol{l_{equi}} = \boldsymbol{l/2}$

For one end fixed and other hinged $\boldsymbol{l_{equi}} = \boldsymbol{l/\sqrt{2}}$

Slenderness Ratio (λ)

$$\lambda = \frac{L_\theta}{r_{min}}$$

$L_\theta = Effective\ length$

$r_{min} = \sqrt{(I_{min}/A)}$

r_{min} = Least radius of gyration

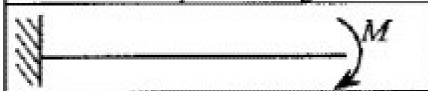
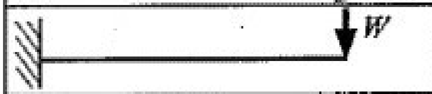
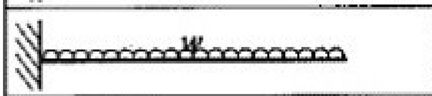
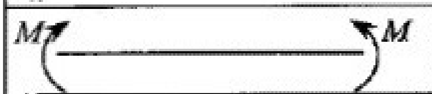
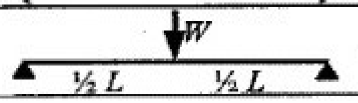
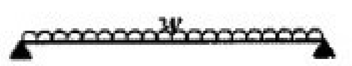
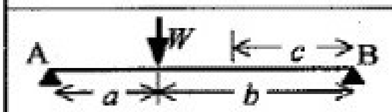
Rankine's Formula for Columns

$$\frac{1}{P_R}=\frac{1}{P_{cs}}+\frac{1}{P_E}$$

- P_R = Crippling load by Rankine's formula
- P_{cs} = σ_{cs} A = Ultimate crushing load for column
- $P_E=\frac{\pi^2 EI}{l_{eff}^2}=$ Crippling load obtained by Euler's formula

Deflection in different Beams

BEAM BENDING

L = overall length W = point load, M = moment w = load per unit length	End Slope	Max Deflection	Max bending moment
	$\frac{ML}{EI}$	$\frac{ML^2}{2EI}$	M
	$\frac{WL^2}{2EI}$	$\frac{WL^3}{3EI}$	WL
	$\frac{wL^3}{6EI}$	$\frac{wL^4}{8EI}$	$\frac{wL^2}{2}$
	$\frac{ML}{2EI}$	$\frac{ML^2}{8EI}$	M
	$\frac{WL^2}{16EI}$	$\frac{WL^3}{48EI}$	$\frac{WL}{4}$
	$\frac{wL^3}{24EI}$	$\frac{5wL^4}{384EI}$	$\frac{wL^2}{8}$
 $a \le b,\ \ c=\sqrt{\frac{1}{3}b(L+a)}$	$\theta_B=\frac{Wac^2}{2LEI}$ $\theta_A=\frac{L+b}{L+a}\theta_B$	$\frac{Wac^3}{3LEI}$ (at position c)	$\frac{Wab}{L}$ (under load)

Torsion

$$\frac{\tau_l}{r}=\frac{T}{J}=\frac{G\theta}{l}$$

Where, T = Torque,

- J = Polar moment of inertia
- G = Modulus of rigidity,
- θ = Angle of twist
- L = Length of shaft,

Total angle of twist

$$\theta = \frac{Tl}{GJ}$$

- GJ = Torsional rigidity
- $\frac{GJ}{l}$ = Torsional stiffness
- $\frac{l}{GJ}$ = Torsional flexibility
- $\frac{EA}{l}$ = Axial stiffness
- $\frac{l}{EA}$ = Axial flexibility

Moment of Inertia About polar Axis

- Moment of Inertia About polar Axis

$$J = \frac{\pi d^4}{32}, \tau_{max} = \frac{16T}{\pi d^3}$$

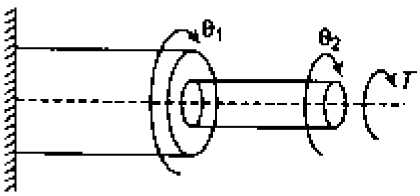
- For hollow circular shaft

$$J = \frac{\pi}{32}(d_0^4 - d_i^4)$$

Compound Shaft

- **Series connection**

$$\theta = \theta_1 + \theta_2$$
$$T = T_1 = T_2$$



Series connection

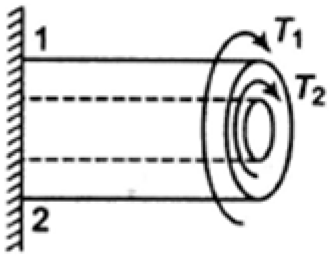
$$\theta = \frac{TL_1}{G_1J_1} + \frac{TL_2}{G_2J_2}$$

Where,

θ_1 = Angular deformation of 1st shaft

θ_2 = Angular deformation of 2nd shaft

- **Parallel Connection**



Parallel connection

$$\theta_1 = \theta_2$$
$$T = T_1 + T_2$$

$$\frac{T_1 L}{G_1 J_1} = \frac{T_2 L}{G_2 J_2}$$

Strain Energy in Torsion

$$U = \frac{1}{2} T\theta = \frac{1}{4} \frac{T^2 L}{GJ}$$

For solid shaft,

$$U = \frac{\tau^2}{4G} \times \text{Volume of shaft}$$

For hollow shaft,

$$U = \frac{\tau^2}{4G}\left(\frac{D^2 + d^2}{D^2}\right) \times \text{Volume of shaft}$$

Thin Cylinder

- **Circumferential Stress /Hoop Stress**

$$\sigma_h = \frac{pd}{2t} \Rightarrow \sigma_h = \frac{pd}{2t\eta}$$

η = Efficiency of joint

- **Longitudinal Stress**

$$\sigma_l = \frac{pd}{4t} \Rightarrow \sigma_l = \frac{pd}{4t\eta}$$

- **Hoop Strain**

$$\varepsilon_h = \frac{pd}{4tE}(2-\mu)$$

- **Longitudinal Strain**

$$\varepsilon_L = \frac{pd}{4tE}(1-2\mu)$$

- **Ratio of Hoop Strain to Longitudinal Strain**

$$\varepsilon_v = \frac{pd}{4tE}(5-4\mu)$$

Stresses in Thin Spherical Shell

- **Hoop stress/longitudinal stress**

$$\sigma_L = \sigma_h = \frac{pd}{4t}$$

- **Hoop stress/longitudinal strain**

$$\varepsilon_L = \varepsilon_h = \frac{pd}{4tE}(1-\mu)$$

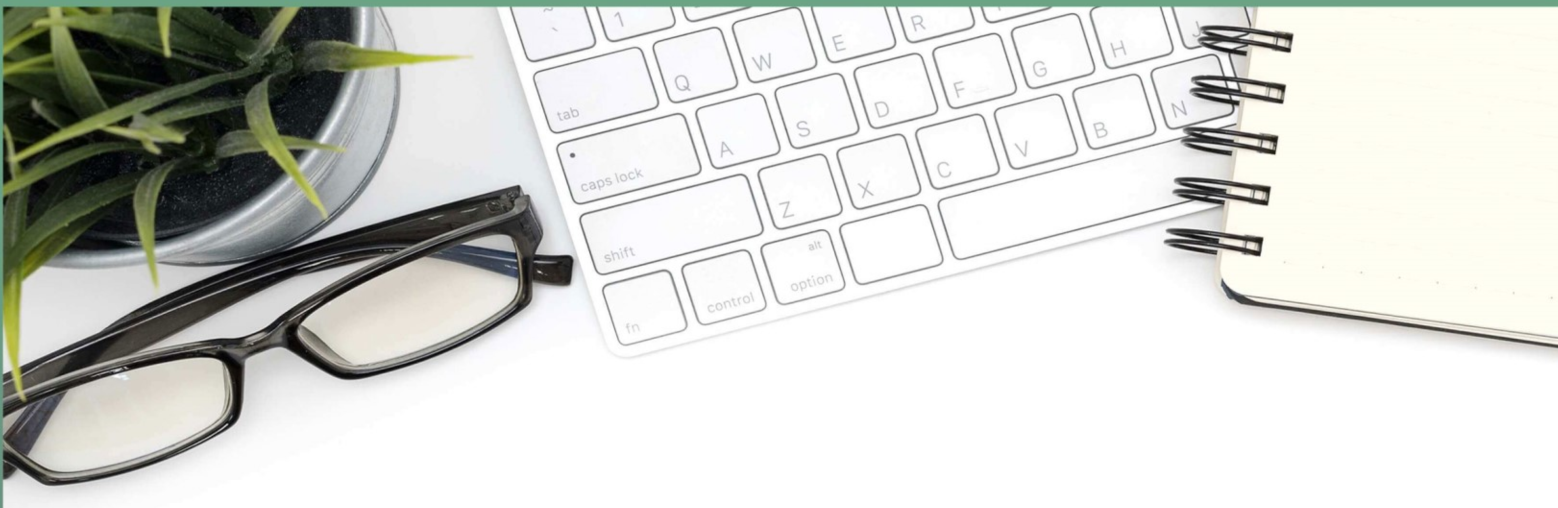
- **Volumetric strain of sphere**

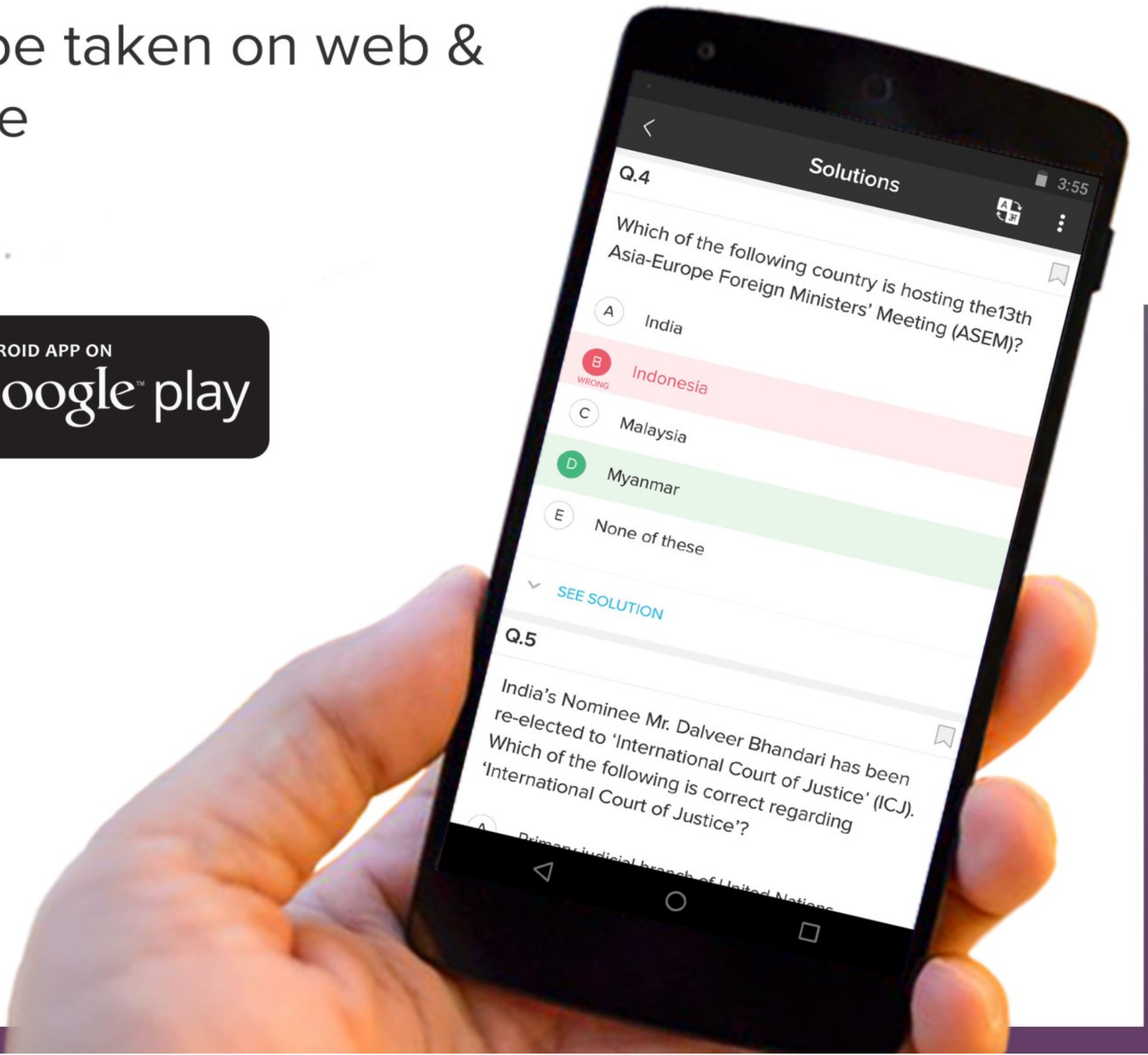
$$\varepsilon_V = \frac{3pd}{4tE}(1-\mu)$$

Thickness ratio of Cylindrical Shell with Hemisphere Ends

$$\frac{t_2}{t_1} = \frac{1-\nu}{2-\nu}$$

Where ν=Poisson Ratio

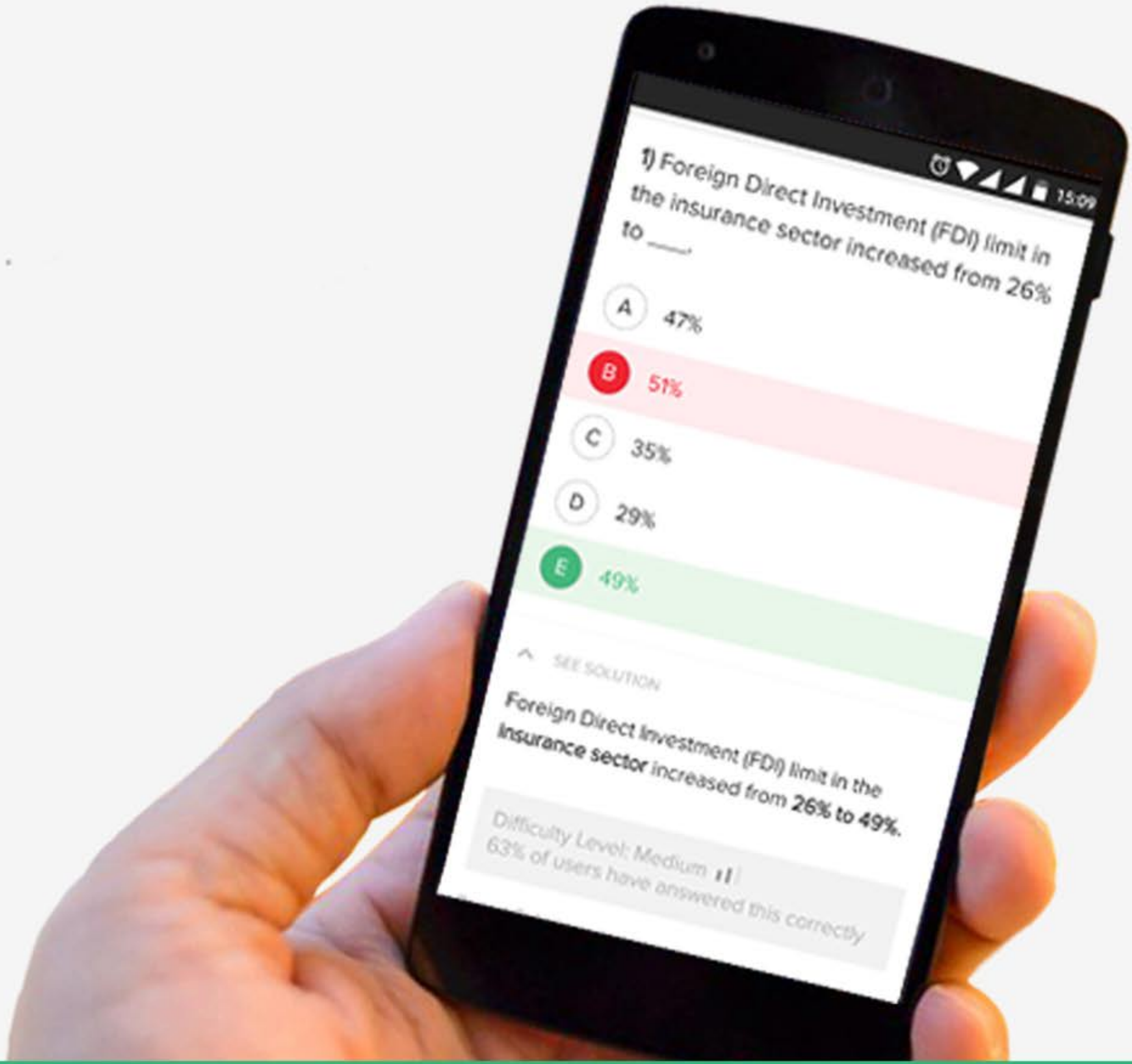




Solutions
3:55
Q.4
Which of the following country is hosting the13th Asia-Europe Foreign Ministers' Meeting (ASEM)?
A India
B Indonesia
WRONG
C Malaysia
D Myanmar
E None of these
SEE SOLUTION
Q.5
India's Nominee Mr. Dalveer Bhandari has been re-elected to 'International Court of Justice' (ICJ). Which of the following is correct regarding 'International Court of Justice'?

gradeup

Formulas on
HEAT TRANSFER
for GATE ME Exam



Short notes for Heat transfer

Fourier's Law of Heat Conduction

$$Q = -kA\frac{dT}{dx}$$

- Q = Heat transfer in given direction.
- A = Cross-sectional area perpendicular to heat flow direction.
- dT = Temperature difference between two ends of a block of thickness dx
- dx = Thickness of solid body
- $\frac{dT}{dx}$ = Temperature gradient in direction of heat flow.

General Heat Conduction Equation

- **Carterisan Coordinates (side parallel to *x, y* and *z*-directions)**

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial t}{\partial z}\right)+q_g=\rho c\frac{\partial t}{\partial \tau}$$

q_g = Internal heat generation per unit volume per unit time

t = Temperature at left face of differential control volume

k_x, k_y, k_z = Thermal conductivities of the material in *x, y* and *z*-directions respectively

c = Specific heat of the material

ρ = Density of the material

α = Thermal diffusivity $\left(\frac{k}{\rho c}\right)$

dτ = Instantaneous time.

- **For homogeneous and isotropic material**

$$k_x = k_y = k_z = k,\ \alpha = \frac{k}{\rho c}$$

$$\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}+\frac{q_g}{k}=\frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

- **For steady state condition (Poisson's equation)**

$$\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}+\frac{q_g}{k}=0$$

- **For steady state and absence of internal heat generation (Laplace equation)**

$$\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}=0$$

- **For unsteady heat flow with no internal heat generation**

$$\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}=\frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

- **Cylindrical Coordinates**
  - **For homogeneous and isotropic material,**

$$\left[\frac{\partial^2 t}{\partial r^2}+\frac{1}{r}\frac{\partial t}{\partial r}+\frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2}\frac{\partial^2 t}{\partial \tau^2}\right]+\frac{q_g}{k}=\frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

  - **For steady state unidirectional heat flow in radial direction with no internal heat generation,**

$$\frac{\partial^2 t}{\partial x^2}+\frac{1}{r}\frac{\partial t}{\partial r}=0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dt}{dr}\right)=0 \Rightarrow r\frac{dt}{dr}=$$

- **Spherical Coordinates**
  - **For homogeneous and isotropic material**

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right)+\frac{q_g}{k}=\frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

  - **For steady state uni-direction heat flow in radial direction with no internal heat generation,**

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dt}{dr}\right)=0$$

- **Thermal resistance of hollow cylinders**

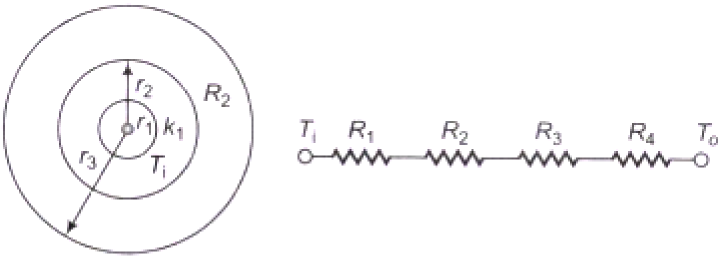
$$R=\frac{\log_e\frac{r_2}{r_1}}{2\pi k L}$$

$$Q=\frac{T_i=T_0}{R}=\frac{\Delta T}{R}$$

- **Thermal Resistance of a Hollow Sphere**

$$R=\frac{r_2-r_1}{4\pi k r_1 r_2}$$

- **Heat Transfer through a Composite Cylinder**



Heat transfer through a composite cylinder

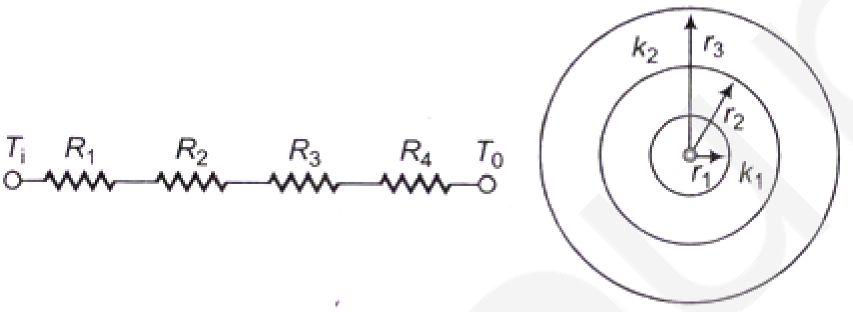
$$Q = \frac{T_i - T_0}{R_{eq}}$$

$$R_{eq} = R_1 + R_2 + R_3 + R_4$$

$$R_1 = \frac{1}{h_i 2\pi r_1 l},\ R_2 = \frac{In\ r_2 / r_1}{2\pi k_1 l}$$

$$R_3 = \frac{In\ r_3 / r_2}{2\pi k_2 l},\ R_4 = \frac{1}{h_0 2\pi r_2 l}$$

- **Heat Transfer through a Composite Sphere**



Heat transfer through a composite sphere

$$R_1 = \frac{1}{h_j A_i} = \frac{1}{h_j 4\pi r_1^2}$$

$$R_3 = \frac{r_2 - r_1}{4\pi k r_1 r_2},\ R_3 = \frac{r_3 - r_2}{4\pi k r_2 r_3}$$

$$R_4 = \frac{1}{h_0 4\pi r_2^2},\ Q\ \frac{T_1 - T_0}{R_1 + R_2 + R_3 + R_4}$$

- **Critical Thickness of Insulation:**
    - In case of cylinder,

$$r_c = \frac{k_0}{h}$$

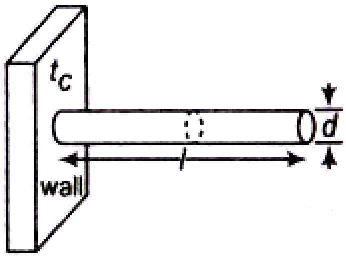
where, k_0 = Thermal conductivity, and h = Heat transfer coefficient

- The drop in temperature across the wall and the air film will be proportional to their resistances, = **hL/k.**

- **Steady Flow of Heat along a Rod Circular fin**

$$\rho = \pi d$$

$$A_c = \frac{\pi}{4} d^2$$



Circular fin diagram

- **Generalized Equation for Fin Rectangular fin**

$$\frac{d^2\theta}{dx^2}+\frac{1}{A_c}\frac{dA_c}{dx}\frac{d\theta}{dx}-\frac{h}{kA_c}\frac{dA_s}{dx}0-0$$

- **Heat balance equation if A_c constant and $A_s \propto P(x)$ linear**

$$\frac{d^2t}{dx^2}-\frac{hp}{k\,A_c}(t-t_a)=0$$

$$\frac{d^2\theta}{dx^2}-m^2\theta=0$$

$$m=\sqrt{\frac{hp}{k\,A_c}}$$

- **General equation of 2nd order**

$$\vartheta = c_1e^{mx} + c_2e^{-mx}$$

  - **Heat Dissipation from an Infinitely Long Fin ($l \to \infty$)**

$$\frac{t-t_{a}}{t_0-t_a}=e^{-mx}$$

    - **Heat transfer by conduction at base**

$$Q_{fin}=\sqrt{kA_cPh}(t_0-t_a)$$

  - **Heat Dissipation from a Fin Insulated at the End Tip**

$$\frac{\theta}{\theta_0}=\frac{t-t_0}{t_0-t_a}=\frac{\cosh\, m(l-x)}{\cosh\, m\, l}$$

$$Q_{fin}=\sqrt{Phk\,A_c}\,(t_0-t_a)\tanh\, ml$$

  - **Heat Dissipation from a Fin loosing Heat at the End Tip**

$$-kA\left(\frac{dt}{dx}\right)_{x=l} = h\, A_S(t - t_a)$$

$$\theta = \theta_0 \left(\frac{\cosh m(l-x) + \frac{h}{mk}\sinh m(l-x)}{\cosh ml + \frac{h}{mk}\sinh ml}\right)$$

$$Q_{fin} = \sqrt{hPkA_c}\ \theta_0 \frac{\tanh ml + \frac{h}{mk}}{1 + \frac{h}{mk}\tanh ml}$$

- **Fin Efficiency**
- Fin efficiency is given by

$$\eta = \frac{\text{Actual heat rate from fin Q}}{\text{Maximum heat transfer rate } Q_{max}}$$

- If $l \rightarrow \infty$ (infinite length of fin),

$$\eta = \frac{\sqrt{hPkA_c}\ \theta_0}{h(Pl + b\delta)\theta_0} = \frac{1}{l}\sqrt{\frac{kA_c}{hP}}$$

$$\eta = \frac{\theta_0\sqrt{hPkA_c}\ \tan h\ ml}{hPl\theta_0}$$

- If finite length of fin,

$$\eta = \frac{\theta_0\sqrt{hPkA_c}\left[\frac{\tan h\ ml + \frac{h}{mk}}{1 + \frac{h}{mk}\tan h\ ml}\right]}{h(Pl + b\delta)\theta_0}$$

- **Fin Effectiveness**

$$\varepsilon = \frac{\text{Actual heat transfer from fin surface (Q)}}{\text{Rate of heat transfer without fin}}$$

$$\varepsilon = \frac{Q}{hA\theta_0} = \frac{\theta_0\sqrt{hPkA_c}\ \tanh ml}{hA\theta_0} = \frac{\tan hml}{\sqrt{\frac{hA_c}{Pk}}}\ (\text{if } l \rightarrow \infty)$$

- **Lumped Parameter System**

$$Q = -\rho VCp\frac{dT}{dt} = hA(T - Ta)$$

$$\int\frac{dT}{(T - Ta)} = -\frac{hA}{\rho VCp}\int dt$$

$$\ln(T - Ta) = -\frac{hA}{\rho VCp}t + C_1$$

$$\ln(T - Ta) = -\frac{hA}{\rho VCp}t + \ln(Ti - Ta)$$

$$\frac{T - Ta}{Ti - Ta} = \exp\left[-\frac{hA}{\rho VCp}t\right]$$

- **Nusselt Number (Nu)**
    - It is a dimensionless quantity defined as= hL/ k,

h = convective heat transfer coefficient,

L is the characteristic length

k is the thermal conductivity of the fluid.

- The Nusselt number could be interpreted physically as the ratio of the temperature gradient in the fluid immediately in contact with the surface to a reference temperature gradient (T_s – T_∞) /L.
- Newton's Law of Cooling says that the rate of heat transfer per unit area by convection is given by

$$\dot{Q}/A = h(T_w - T_\infty)$$

$$\frac{\dot{Q}}{A} = h(T_w - T_\infty) = -k\frac{T_w - T_\infty}{\delta_t}$$

$$h = \frac{k}{\delta_t} \quad Nu = \frac{hL}{k} = \frac{L}{\delta_t}$$

$$Nu = \frac{\text{Rate of heat transfer by convection}}{\text{Rate of heat transfer by conduction}} = \frac{h.A.\Delta T}{kA\frac{\Delta T}{1}}$$

- **Temperature distribution in a boundary layer: Nusselt modulus**
- The heat transfer by convection involves conduction and mixing motion of fluid particles. At the solid fluid interface (y = 0), the heat flows by conduction only, and is given by

$$\frac{\dot{Q}}{A} = -k\left(\frac{dT}{dy}\right)_{Y=0}$$

$$h = \frac{\left(-k\,{}^{dT}\!/_{dy}\right)_{y=0}}{(T_w - T_\infty)}$$

$$\frac{hL}{k} = \frac{\left(\frac{dT}{dy}\right)_{y=0}}{(T_w - T_\infty)/L}, \text{and}$$

In dimensionless form,

$$= \left(\frac{d(T_w - T)/(T_w - T_\infty)}{d(y/L)}\right)_{y=0}$$

- **Reynold Number (Re):**

$$Re = \frac{\text{Inertia force}}{\text{Viscous force}}$$

$$Re = \frac{\rho v l}{\mu}$$

$$Re = \frac{vl}{\nu}$$

- **Critical Reynold Number:** It represents the number where the boundary layer changes from laminar to turbine flow.
    - **For flat plate,**
        - $Re < 5 \times 10^5$ (laminar)
        - $Re > 5 \times 10^5$ (turbulent)
    - **For circular pipes,**
        - Re < 2300 (laminar flow)
        - 2300 < Re < 4000 (transition to turbulent flow)
        - Re > 4000 (turbulent flow)
- **Stanton Number (St)**

$$St = \frac{\text{Heat transfer coefficient}}{\text{Heat flow per unit temperature rise}}$$

$$St = \frac{Nu}{Re \times Pr}$$

- **Grashof Number (Gr)**

If a body with a constant wall temperature T_w is exposed to a qui scent ambient fluid at T_∞, the force per unit volume can be written as:

$$\rho g \beta (t_w - t_\infty)$$

where ρ = mass density of the fluid, β = volume coefficient of expansion and g is the acceleration due to gravity.

$$Gr = \frac{\text{Inertia force} \times \text{buoyancy foce}}{\text{viscous force}}$$

β = Coefficient of volumetric expansion = 1/*T*

$$Gr = \frac{(\rho V^2 L^2) \times \rho g \beta (T_w - T_\infty) L^3}{(\mu V L)^2}$$

$$= \frac{\rho^2 g \beta (T_w - T_\infty) L^3}{\mu^2} = g \beta L^3 (T_w - T_\infty) / \nu^2$$

- The magnitude of Grashof number indicates whether the flow is **laminar or turbulent**.
- If the Grashof number is greater than 10^9, the flow is turbulent and
- For Grashof number less than 10^8, the flow is laminar.
- For $10^8 < Gr < 10^9$, It is the transition range.

- **Prandtl Number (Pr):**

$$Pr = \frac{\text{Momentum diffusivity through the fluid}}{\text{Thermal diffusivity through the fluid}}$$

$$Pr = \mu C_p / k = \nu / \alpha$$

$$Pr = \frac{\nu}{\alpha} = \frac{\mu / \rho}{k / \rho c_p} \Rightarrow Pr = \frac{\mu c_p}{k}$$

- For liquid metal, $Pr < 0.01$
- For air and gases, $Pr \approx 1$
- For water, $Pr \approx 10$
- For heavy oil and grease, $Pr > 10^5$
- For $Pr << 1$ (in case of liquid metals), the thickness of the thermal boundary layer will be much more than the thickness of the momentum boundary layer and vice versa.
- The product of Grashof and Prandtl number is called Rayleigh number. Or,

Ra = Gr × Pr

- **Rayleigh Number (Ra)**

$$Ra = Gr.Pr; Ra = \frac{g \, \beta \, l^3 \, \Delta t}{\nu . \alpha}$$

- Free or natural convection
  - $10^4 < Ra < 10^9$ (laminar flow)
  - $Ra > 10^9$ (turbulent flow)
- **Turbulent flow over flat plate**

$$Nu_x = 0.0292\ (Re)^{0.8}\ (Pr)^{0.33}$$

$$\overline{Nu} = 0.036\ (Re)^{0.8}\ (Pr)^{0.33}$$

$$\delta = \frac{0.37\ x}{(Re_x)^{\frac{1}{5}}}$$

- **Turbulent flow in tubes**

$$Nu = 0.023\ (Re)^{0.8}\ (Pr)^{n}$$

where, n = 0.4 if fluid is being heated,

= 0.3 if fluid is being cooled.

$$\overline{Nu} = 0.0036\ (Re)^{0.8}\ (Pr)^{n}$$

Empirical Correlation for Free Convection

- **Heated surface up or cooled surface down**
- **Laminar flow**

$$2 \times 10^5 < Gr.Pr < 2 \times 10^7$$

$$Nu = 0.54\ (Gr\ Pr)^{0.25}$$

- **Turbulent flow**

$$2 \times 10^7 < Gr.Pr < 3 \times 10^{10}$$

$$Nu = 0.14\ (Gr\ Pr)^{0.33}$$

- **Heated surface down or cooled surface up**
- **Laminar flow**

$$3 \times 10^5 < Gr.Pr < 7 \times 10^8$$

$$Nu = 0.27\ (Gr\ Pr)^{0.25}$$

- **Turbulent flow**

$$7 \times 10^8 < Gr.Pr < 11 \times 10^{10}$$

$$Nu = 0.107\ (Gr\ Pr)^{0.33}$$

- **Vertical plates and Large cylinder**
- **Laminar flow**

$$10^4 < GrPr < 10^9$$

$$Nu = 0.59\ (GrPr)^{0.25}$$

- **Turbulent flow**

$$10^9 < GrPr < 10^{12}$$

$$Nu = 0.13\ (GrPr)$$

Empirical Correlation for Forced Convection

- **Laminar Flow over Flat Plate**

$$Nu_x = \frac{hx}{k} = 0.332(\mathrm{Re}_x)^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}$$

$$Nu_x = \frac{\bar{h}l}{k} = 0.64(\mathrm{Re}_x)^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}$$

- **Hydrodynamic boundary layer thickness**

$$\rho = \frac{5x}{\sqrt{\mathrm{Re}_x}}$$

- **Laminar Flow over Inside Tube**

$$Nu = \frac{hd}{k}, T_{\infty} = \frac{T_o + T_i}{2}$$

  - Constant heat flux, Nu = 4.36

- **Fouling Factor (R_f)**

$$\mathrm{R_f} = \frac{1}{\mathrm{v_{dirty}}} - \frac{1}{\mathrm{v_{clear}}}$$

Fin Efficiency and Fin Effectiveness

- **η_{fin} = (actual heat transferred) / (heat which would be transferred if the entire fin area were at the root temperature)**

- **For a very long fin, effectiveness**:

$$E = \dot{Q}_{\text{with fin}} / \dot{Q}_{\text{without fin}}$$
$$= (hpkA)^{1/2}\theta_0 / hA\theta_0 = (kp/hA)^{1/2}$$

And

$$\eta_{\text{fin}} = (hpkA)^{1/2}\theta_0 (hpL\theta_0) = (hpkA)^{1/2} / (hpL)$$
$$\frac{E}{\eta_{\text{fin}}} = \frac{(kp/hA)^{1/2}}{(hpkA)^{1/2}} \times hpL = \frac{pL}{A} = \frac{\text{Surface area of fin}}{\text{Cross-sectional area of the fin}}$$

i.e., effectiveness increases by increasing the length of the fin but it will decrease the fin efficiency.

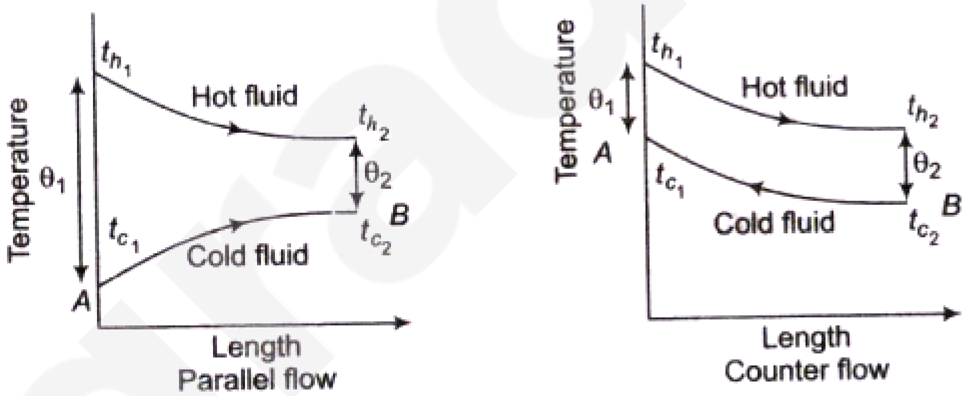
- **Expressions for Fin Efficiency for Fins of Uniform Cross-section:**
  - Very long fins:

$$(hpkA)^{1/2}(T_0 - T_\infty)/[hpL(T_0 - T_\infty)] = 1/mL$$

- For fins having insulated tips

$$\frac{(hpkA)^{1/2}(T_0 - T_\infty)\tanh(mL)}{hpL(T_0 - T_\infty)} = \frac{\tanh(mL)}{mL}$$

Logarithmic Mean Temperature Difference (LMTD)



Temperature distribution for parallel and counter flow heat exchanger

$$Q = UA\frac{\theta_1 - \theta_2}{\log_e\left(\frac{\theta_1}{\theta_2}\right)} = UA\theta_m$$

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

LMTD

- **Capacity Ratio**

- **Capacity ratio *c* = *mc*, where *c* = Specific heat**

$$If\ m_h c_h > m_c c_c,\ R = \frac{m_c c_c}{m_h c_h}$$

$$If\ m_h c_h < m_c c_c,\ R = \frac{m_h c_h}{m_c c_c}$$

- **Effectiveness of Heat Exchanger:**

$$\varepsilon = \frac{(Q_{act})\text{actual heat transfer}}{(Q_{max})\text{maximum possible heat transfer}}$$

$$Q_{act} = m_h c_h (t_{h_1} - t_{h_2})$$

$$= m_c c_c (t_{c_2} - t_{c_1})$$

$$Q_{max} = c_{min} (t_{h_1} - t_{c_1})$$

  - If $m_c c_n < m_h c_h \Rightarrow c_{min} = m_c c_c$

$$\Rightarrow Q_{max} = m_c c_c (t_{h_1} - t_{c_1})$$

$$\varepsilon = \frac{m_c c_c (t_{c_2} - t_{c_1})}{m_c c_c (t_{h_1} - t_{c_1})} = \frac{t_{c_2} - t_{c_1}}{t_{h_1} - t_{c_1}}$$

  - If $m_c c_n < m_h c_h \Rightarrow c_{min} = m_h c_h$

$$\Rightarrow Q_{max} = m_h c_h (t_{h_1} - t_{c_1})$$

$$\varepsilon = \frac{m_h c_h (t_{h_1} - t_{h_2})}{m_h c_h (t_{h_1} - t_{c_1})} = \frac{t_{h_1} - t_{h_2}}{t_{h_1} - t_{c_1}}$$

- **Number of Transfer Units (NTU):**

$$\text{NTU} = \frac{\text{UA}}{\text{c}_{\text{min}}}$$

U = Overall heat transfer coefficient

A = Surface area

C min = Minimum capacity rate

If $m_h c_h < m_c c_c \Rightarrow c_{min} = m_c c_c$

$$\Rightarrow NTU = \frac{UA}{m_h c_c}$$

If $m_h c_h < m_c c_c \Rightarrow c_{min} = m_h c_h$

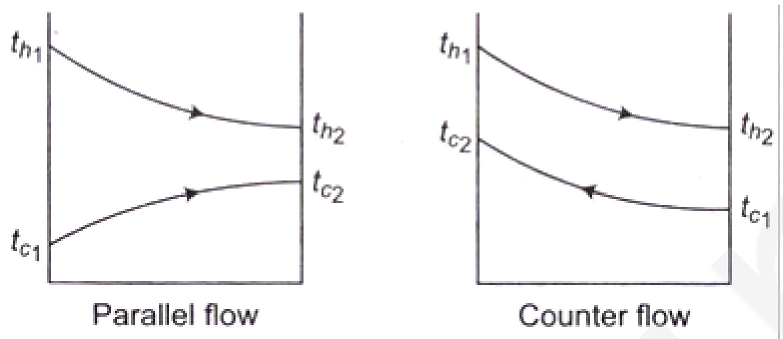
$$\Rightarrow NTU = \frac{UA}{m_h c_h}$$

Effectiveness for Parallel Flow Heat Exchanger

$$\varepsilon = \frac{1 - \exp[-NTU(1+R)]}{1+R}$$

$$R = \frac{c_{\min}}{c_{\max}}$$

$$NTU = \frac{UA}{c_{\min}}$$



Heat exchanger for parallel and counter flow

- **Effectiveness for the Counter Flow Heat Capacity:**

$$\varepsilon = \frac{1 - \exp[-NTU(1-R)]}{1 - R\exp[-NTU\ (1-R)]}$$

- **Heat Exchanger Effectiveness Relation:**
- **Concentric tube:**
  - Parallel flow:

$$\in = \frac{1 - \exp[-N(1+R)]}{(1+R)}; R = C_{\min}/C_{\max}$$

  - Counter flow:

$$\in = \frac{1 - \exp[-N(1-R)]}{1 - R\ \exp[-N(1-R)]}; R < 1$$

$$\in = N/(1+N)\ \text{for}R = 1$$

- **Cross flow (single pass):**
  - Both fluids unmixed:

$$\in = 1 - \exp\left[(1/R)(N)^{0.22}\left\{\exp\left(-R(N)^{0.78}\right) - 1\right\}\right]$$

  - C_{max} mixed , C_{min} unmixed:

$$\in = (1/R)\left[1 - \exp\left\{-R\left(1 - \exp(-N)\right)\right\}\right]$$

  - C_{min} mixed, C_{max} unmixed:

$$\in = 1 - \exp\left[-R^{-1}\left\{1 - \exp(-RN)\right\}\right]$$

Total Emissive Power (E)

- It is defined as the total amount of radiation emitted by a body per unit time and area.

$$E = \sigma T^4 \ W/m^2$$

σ = Stefan Boltzmann constant

$\sigma = 5.67 \times 10^{-8}\ W/m^2\ K^4$

Monochromatic (Spectral) Emissive Power (E_λ)

- It is defined as the rate of energy radiated per unit area of the surface per unit wavelength.

$$E = \int_0^\infty E_\lambda d\lambda \, w/m^2$$

Emission from Real Surface

- The emissive power from a real surface is given by

$$E = \varepsilon \sigma A T^4 \ W$$

ε = Emissivity of the surface,

T = Surface temperature

Emissivity (ε)

- It is defined as the ratio of the emissive power of any body to the emissive power of a black body of same temperature.

$$\varepsilon = \frac{E}{E_b}$$

  - For black body, ε = 1
  - For white body, ε = 0
  - For gray body, 0< ε<1

Reflectivity (ρ)

- It is defined as the fraction of total incident radiation that are reflected by material.

$$\text{Reflectivity}(\rho) = \frac{\text{Energy reflected } (Q_r)}{\text{Total incident radiation } (Q)}$$

Absorptivity

- It is defined as the fraction of total incident radiation that are absorbed by material.

$$\text{Absorptivity}(\alpha) = \frac{\text{Energy absorbed } (Q_a)}{\text{Total incident radiation } (Q)}$$

Transmissivity

- It is defined as the fraction of total incident radiation that are transmitted through the material.

$$\text{Transmissivty}(\tau) = \frac{\text{Energy transmitted } (Q_t)}{\text{Total incident radiation } (Q_w)}$$

$$Q_0 = Q_a + Q_r + Q_t$$

$$\frac{Q_0}{Q_0} = \frac{Q_a}{Q_0} + \frac{Q_r}{Q_0} + \frac{Q_t}{Q_0}$$

$$\alpha + \rho + \tau = 1$$

- **For black body** $\alpha = 1, \rho = 0, \tau = 0$
- **For opaque body** $\tau = 0, \alpha + \rho = 1$
- **For white body** $\rho = 1, \alpha = 1$ and $\tau = 0$

Kirchoff's Law

- The emissivity ε and absorptivity α of a real surface are equal for radiation with identical temperature and wavelength.

$$\alpha = \varepsilon = \frac{E}{E_b}$$

- Emissive power of a black body is directly proportional to the fourth power of its absolute temperature.

$$E_b = \sigma T^4$$

E_b = Emissive power of a black body,

σ = Stefan Boltzmann constant ($5.67 \cdot 10^{-8} W/m^2 \cdot K^4$),

T = Absolute temperature of the emitting surface, K.

Wien's Displacement Law

- Wien's displacement law state that the product of λ_{max} and T is constant.

$$\lambda_{max} T = \text{constant}$$

λ_{max} = Wavelength at which the maximum value of monochromatic emissive power occurs.

Gray Surfaces

The gray surface is a medium whose monochromatic emissivity (ε_λ) does not vary with wavelength.

$$\varepsilon_\lambda = E_\lambda / E_{\lambda, b}$$

But, we know the following.

$$E = \int_0^\infty \varepsilon_\lambda E_{\lambda,b} d\lambda$$

$$E_b = \int_0^\infty E_{\lambda,b} d\lambda = \sigma T^4$$

Therefore,

$$\varepsilon = \frac{\int_0^\infty \varepsilon_\lambda E_{\lambda,b} d\lambda}{\sigma T^4}$$

View Factors:

- Define the view factor, F_{1-2}, as the fraction of energy emitted from surface 1, which directly strikes surface 2.

$$F_{1\to 2} = \frac{q_{1\to 2}}{q_{emitted}} = \frac{\int_{A_2}\int_{A_1} \frac{I \cdot \cos\theta_1 \cdot}{R^2} dA_1 \cdot \cos\theta_2 dA_2}{\pi \cdot I \cdot A_1}$$

$$F_{1\to 2} - \frac{1}{A_1} = \int_{A_2}\int_{A_1} \frac{\cos\theta_1 \cdot \cos\theta_2 \cdot dA_1 \cdot dA_2}{\pi \cdot R^2}$$

Reciprocity:

$$A_i \cdot F_{1\to j} = \int_{Aj}\int_{A_i} \frac{\cos\theta_i \cdot \cos\theta_j \cdot dA_i \cdot dA_j}{\pi \cdot R^2}$$

$$A_j \cdot F_{j\to i} = \int_{Aj}\int_{A_i} \frac{\cos\theta_j \cdot \cos\theta_i \cdot dA_j \cdot dA_i}{\pi \cdot R^2}$$

$$A_i \cdot F_{i\to j} = A_j \cdot F_{j\to i}$$

Planck's Law:

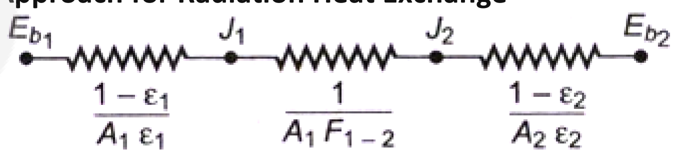
- Planck suggested following formula, monochromatic emissive power of a black body.

$$(E_\lambda)_b = \frac{2\pi c^2 h \lambda^{-5}}{\exp\left(\frac{ch}{\lambda kT}\right) - 1}$$

- Total emissive power

$$E_b = \int_0^\infty (E_\lambda)_b \, d\lambda$$

Electrical Network Approach for Radiation Heat Exchange



An electrical network between two non black surfaces

$$(Q_{1-2})_{net} = \frac{E_{b_1} - E_{b_2}}{\frac{1-\varepsilon_1}{A_1 \varepsilon_1} + \frac{1}{A_1 F_{1-2}} + \frac{1-\varepsilon_2}{A_2 \varepsilon_2}}$$

$$\text{or } (Q_{1-2})_{net} = (F_g)_{1-2} A_1 \sigma_b \left(T_1^4 - T_2^4\right)$$

New Gray Body Factor

$$(G_g)_{1-2} = \frac{1}{\frac{1-\varepsilon_1}{A_1\varepsilon_1} + \frac{1}{A_1 F_{12}} + \frac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

E_1 = Emissivity for body 1

$E2$ = Emissivity for body 2

- **In case of black surfaces, $\varepsilon_1 = \varepsilon_2 = 1$, $(F_g)_{12} = F_{1\text{-}2}$**

$$Q_{net} = F_{1-2} A_1 \sigma \left(T_1^4 - T_2^4 \right)$$

- **In case of parallel planes, $A_1 = A_2$ and $F_{1\text{-}2} = 1$**

$$(F_g)_{1-2} = \frac{1}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

- **In case of concentric cylinder or sphere, $F_{1\text{-}2} = 1$**

$$(F_g)_{1-2} = \frac{1}{\frac{1-\varepsilon_1}{\varepsilon_1} + 1 + \frac{1+\varepsilon_2}{\varepsilon_2}\frac{A_1}{A_2}}$$

Where, $\frac{A_1}{A_2} = \frac{r_1}{r_2}$ (for concentric cylinder)

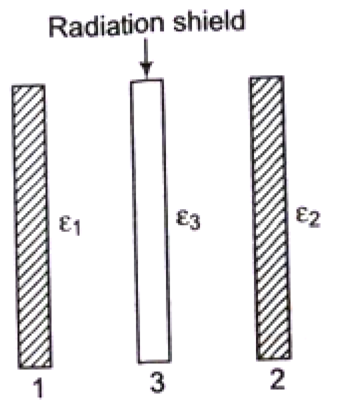
$\frac{A_1}{A_2} = \frac{r_1^2}{r_2^2}$ (for concentric sphere)

- **When a small body lies inside a large enclosure**

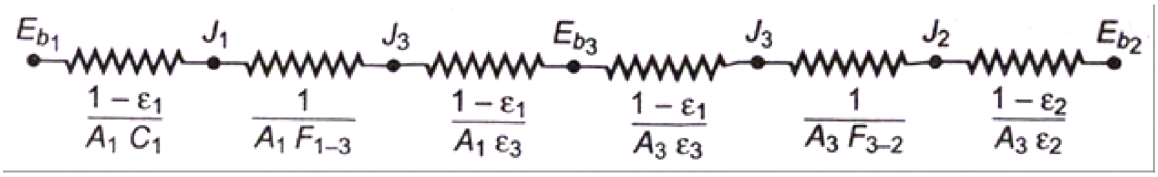
$$F_{1-2} = 1,\ A_1 << A_2 \Rightarrow \frac{A_1}{A_2} = 0$$

$$(F_g)_{1-2} = \frac{1}{\frac{1-\varepsilon_1}{\varepsilon_1} + 1}$$

Radiation Shield



Radiation shield diagram



- **Radiation network for 2 parallel infinite places separated by one shield**

$$(Q_{1-3})_{net} = (Q_{3-2})_{net} \quad (A_1 = A_3 = A_2)$$

$$\frac{A\sigma(T_1^4 - T_3^4)}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_3 - 1}} = \frac{A\sigma(T_3^4 - T_2^4)}{\frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_2} - 1}$$

$$(Q_{1-2})_{net} = \frac{A\sigma(T_1^4 - T_2^4)}{\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_3} - 1\right) + \left(\frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_2} - 1\right)}$$

$$\frac{[(Q_{1-2})_{net}] \text{ with shield}}{[(Q_{1-2})_{net}] \text{ without shield}}$$

$$= \frac{\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1\right)}{\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_3} - 1\right) + \left(\frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_2} - 1\right)}$$

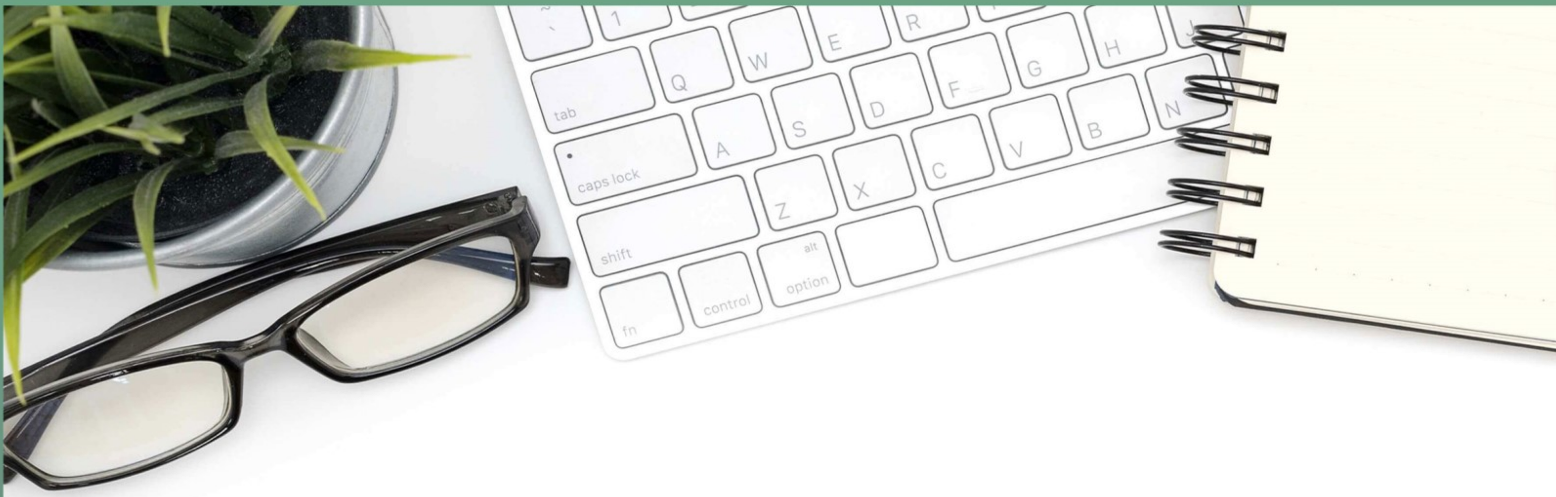
If $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$

Then,

$$(Q_{1-3})_{net} = (Q_{3-2})_{net} = \frac{1}{2}(Q_{1-2})_{net}$$

and

$$T_3^4 = \frac{1}{2}(T_1^4 - T_2^4)$$



Formulae on Manufacturing
for Mechanical Engineering

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Short Notes for Engineering Materials

Crystal Structure of Materials

- When metals solidify from molten state, the atoms arrange themselves into various crderly configurations called crystal.
- There are seven basic crystal structures, they are

Crystal system	Relation between primitives	Interface Angles	Examples
Cubic	$a=b=c$	$\alpha=\beta=\gamma=90^o$	Fe, Al, Cu
Tetragonal	..	$\alpha=\beta=\gamma=90^o$	Sn, SO_2
Orthogonal	$a\neq b\neq c$	$\alpha=\beta=\gamma=90^o$	$KNO_3, BaSO_4$
Hexagonal	$a=b\neq c$	$\alpha=\beta=90^o\ \gamma=120^o$	$SiO_2, AgCI, Zn$
Rhombohedral	$a=b=c$	$\alpha=\beta=\gamma\neq 90^o$	$CaSO_4, CaCO_3$
Monoclinic	$a\neq b\neq c$	$\alpha=\beta=90^o\neq\gamma$	$FeSO_4, NaSO_4$
Friclinic	$a\neq b\neq c$	$\alpha\neq\beta\neq\gamma\neq 90^o$	$CuSO_4, K_2Cr_2O_7$

Simple Cubric Cell (SCC)

- The total number of atoms present in crystal structure,

$$n=\frac{1}{8}\times 8=1$$

- Atomic Packing Factor (APF)

$$=\frac{\text{Volume of atoms in a cell}}{\text{Volume of unit cell}}$$

$$=\frac{4\pi a^3}{8\times 3\times a^3}$$

$$=\frac{\pi}{6}=\frac{3.14}{6}$$

$$=0.52$$

- Percentage APF = **52%**
- Percentage of voids = 100 – 52 = **48%**

Body Centered Cubic (BCC) Structure

- Total effective number of atoms present in the crystal

$$= 1 + 8 \times \frac{1}{8}$$
$$= 1 + 1 = 2$$

- Atomic Packing Factor (APF)

$$= \frac{\pi\sqrt{3}}{8} = \frac{3.14 \times 1.732}{8} = 0.68 \quad \left(\because r = \frac{\sqrt{3}}{4}a\right)$$

- Percentage APF **= 68%**
- Percentage of voids = 100 – 68 = **32%**

Face Centred Crystal (FCC)

In this arrangement, each face has an atom and corners are also occupied by atoms.

Total effective number of atoms in cell**.**

- Atomic Packing Factor (APF)

$$= \frac{n \times \frac{4}{3}\pi r^3}{a^3}$$
$$= \frac{16\pi}{16 \times 3\sqrt{2}}$$
$$= \frac{3.14}{3 \times 1.414}$$
$$= 0.74$$

- Percentage APF = **74%**
- Percentage of voids = 100 – 74 = **26%**

Gibbs phase rule :

- F = C – P + 2
- Number of external factors = 2 (pressure and temperature)**.**
- For metallurgical system, pressure has no appreciable effect on phase equilibrium and hence,
- F = C – P + 1

Engineering and True Stress-Strain Diagrams:

- When we calculate the stress on the basis of the original area, it is called the engineering or nominal stress.
- If we calculate the stress based upon the instantaneous area at any instant of load it is then termed as true stress.
- If we use the original length to calculate the strain, then it is called the engineering strain.

$$\text{True stress } (\sigma_T) = \frac{\text{Instantaneous load}}{\text{Instantaneous cross-sectional area}}$$

$$= \frac{P}{A_i}$$

Brittleness:

- It may be defined as the property of a metal by virtue of which it will fracture without any appreciable deformation.
- This property is just opposite to the ductility of a metal.

Toughness:

- It may be defined as the property of a metal by virtue of which it can absorb maximum energy before fracture takes place.
- Toughness is also calculated in terms of area under stress-strain curve.
- Toughness is the property of materials which enables a material to be twisted, bent or stretched under a high stress before rupture.

Resilience:

- This may be defined as the property of a metal by virtue of which it stores energy and resists shocks or impacts.
- It is measured by the amount of energy absorbed per unit volume, in stressing a material up to elastic limit.

Endurance:

- This is defined as the property of a metal by virtue of which it can withstand varying stresses (same or opposite nature).
- The maximum value of stress, which can be applied for indefinite times without causing its failure, is termed as its endurance limit.

Anelastic Behaviour:

- Recoverable deformation that takes place as a function of time is termed an-elastic deformation.
- Due to some relaxation process within the material, the elastic deformation of the material continues even after the application of the load

Isomorphous system.

- **There are 5 invariant reactions occurring in binary phase system:**
- **Eutectic reaction**: When a liquid phase changes into two different solid phases during cooling or two solid phases change into a single liquid phase during heating, this point is known as eutectic point
- **Eutectoid reaction:** When a solid phase changes into two solid phases during cooling and vice-versa that point is known as eutectoid point
- **Peritectic reaction:** A binary system when solid and liquid phases changes solid phase on cooling and vice-versa on heating, then state of system is known as peritectic point
- **Peritectoid reaction**: If a binary phase diagram when two solid phases changes to one solid phase, then state of system is known as peritectoid point**.**

Normalising

- For this process, the metal is placed in the furnace and heated to just above its 'Upper Critical Temperature'.
- When the new grain structure is formed it is then removed from the furnace and allowed to cool in air as it cools new grains will be formed.
- These grains, although similar to the original ones, will in fact be smaller and more evenly spaced.
- Normalising is used to relieve stresses and to restore the grain structure to normal.

Quenching

- It is a heat treatment when metal at a high temperature is rapidly cooled by immersion in water or oil.
- Quenching makes steel harder and more brittle, with small grains structure

Annealing (Softening)

- Annealing is a heat treatment procedure involving heating the alloy and holding it at a certain temperature (annealing temperature), followed by controlled cooling.
- Annealing results in relief of internal stresses, softening, chemical homogenising and transformation of the grain structure into more stable state.
- The annealing process is carried out in the same way as normalising, except that the component is cooled very slowly. This is usually done by leaving the component to cool down in the furnace for up to 48 hours

Hardening

- Hardening also requires the steel to be heated to its upper critical temperature (plus 50°C) and then quenched.
- The quenching is to hold the grains in their solid solution state called Austenite; cooling at such a rate (called the critical cooling rate) is to prevent the grains forming into ferrite and pearlite.
- Hardening is a process of increasing the metal hardness, strength, toughness, fatigue resistance.

Tempering

- As there are very few applications for very hard and brittle steel, the hardness and brittleness needs to be reduced. The process for reducing hardness and brittleness is called tempering.
- Tempering consists of reheating the previously hardened steel.
- During this heating, small flakes of carbon begin to appear in the needle like structure. (See below) This has the effect of reducing the hardness and brittleness.

Stress Relieving

- When a metal is heated, expansion oc-curs which is more or less proportional to the temperature rise. Upon cooling a metal, the reverse reaction takes place. That is, a contraction is observed.
- When a steel bar or plate is heated at one point more than at another, as in welding or during forging, internal stresses are set up.
- During heating, expansion of the heated area cannot take place unhindered, and it tends to deform.
- On cooling, contraction is prevented from taking place by the unyield-ing cold metal surrounding the heated area.

Casting

- **Solidification Time -** $T_s = k\left(\frac{V}{S.A}\right)^2$
  V=Volume of casting, S.A= Surface area of casting, k=Solidification time (sec/m^2)

- **Caine's Formula-** $X = \frac{a}{Y-b} + c$

  $\text{Freezing Ratio(F.R) } X = \frac{\left(\frac{V}{S.A}\right)_R}{\left(\frac{V}{S.A}\right)_C}$ and $Y = \frac{V_R}{V_C}$ a, b and c are constants

- **Shape Factor-** $\text{S.F} = \frac{L+W}{T}$

  L- Length of casting, W- Width of the casting and T- Avg. thickness of section

- **Modulus Method -** $M_R = 1.2M_C$
  M=$\left(\frac{V}{S.A}\right)$

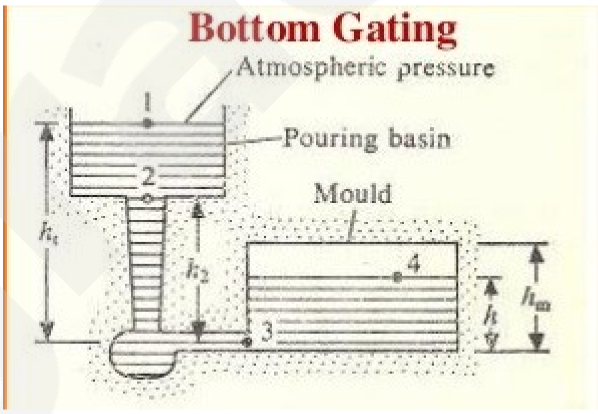
- **Gating Design-**
  1. **Vertical Gating-**

  $$t = \frac{V}{A_g v_g}$$

  where V= Volume of mould, A_g=Cross-section area

  2. **Bottom Gating-**

  $$\boldsymbol{t = \frac{A_m}{A_g}\frac{1}{\sqrt{2g}}2(\sqrt{h_t} - \sqrt{h_t - h_m})}$$



- **Aspiration Effect-**

$$R = \frac{A_2}{A_3} = \sqrt{\frac{h_c}{h_t}}$$

Forging

- **Flow Stress-**

$$\sigma = k\varepsilon^n$$

Rolling

- **Bite Angle-**

$$\mathrm{Cos}\theta = 1 - \frac{\Delta h}{2R}$$

- **Reduction (Draft)-**

$$\Delta h \le \mu^2 R$$

Max DraftΔh **=**$\mu^2 R$

- **Roll Separating Force-**

$F_{sep.roll} = LW\left(1 + \frac{\mu L}{2h}\right)$ $L = \sqrt{R\Delta h}$, W=Width

Extrusion and Wire Drawing

$$\sigma_d = K \ln \frac{A_i}{A_f}$$

Where K=Constant, A_i & A_f = Initial and final Area

Bending

Bend Allowance=**θ(R+Kt)** t= Sheet thickness

Θ=angle in radians, K=Stretch factor (0.33 when R<2t and 0.5 when R>2t) R=bend radius

Punching and Blanking

$$\text{Punch load} = \pi Dt \times K' \frac{\text{Penetration}}{\text{Shear on punch}}$$

K′=Yield strength in shear

Deep Drawing

D=Blank Diameter, d=Cup diameter

$$\frac{D}{d} = Draw\ ratio$$

$$\frac{\pi}{4}D^2 = \frac{\pi}{4}d^2 + \pi dh$$

Metal Cutting

- **Cutting Speed in Turning**

$$\mathbf{V = \pi DN}$$

Where D is the diameter of the work piece, m; N is the rotational speed of the work piece, rev/s

- **Economics of Machining**

1) **Total Minimum Cost**

$$v_{opt} = \frac{C}{\left[\left(\frac{C_e}{C_m} + T_c\right)\left(\frac{1}{n} - 1\right)\right]^n} \Rightarrow t_{opt} = \left[\left(\frac{C_e}{C_n} + T_c\right)\left(\frac{1}{n} - 1\right)\right]$$

where, C_m = Machine cost in /time, T_c = Tool changing time

C = Total cost, C_e = Cost of tool/grind.

2) Maximum Production Rate

$$V_{opt} = \frac{C}{\left[\left(\frac{1}{n}-1\right)T_c\right]^n} \qquad T_{opt} = \left(\frac{1}{n}-1\right)T_c$$

3) Maximum Profit Rate

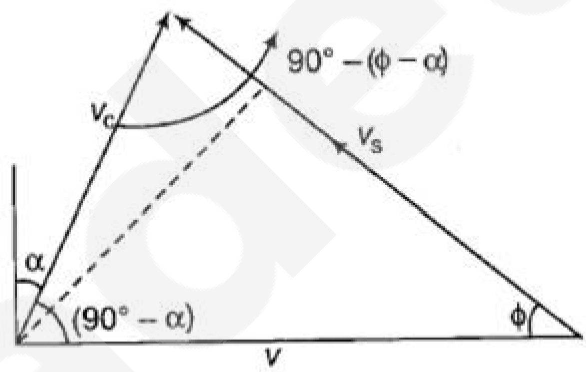
$$\text{Profit rate} = \frac{R - C_0}{T_n + T_n + \frac{T_m}{T} \times T_c} \qquad T_m = \frac{1}{fN}$$

- **Shear angle (β_o)-**

  $\tan \beta_o = r \cos\gamma_o / 1 - r \sin\gamma_o$

Rake angle, γo(in orthogonal plane), r-Chip thickness before and after cut

- **Velocity Ratio**



Velocity ratio diagram

$$\frac{V_s}{\cos\alpha} = \frac{V}{\cos(\phi-\alpha)} = \frac{V_c}{\sin\phi} \qquad \phi = \frac{\pi}{2} + \frac{\alpha}{2} - \frac{\beta}{2}$$

Φ= Shear angle, α=Rake angle, β= Friction angle

- **Shear Strain**

  $\gamma = \tan\phi + \cot(\phi - \alpha)$

- **Tool Life-**

  $VT^n = C$

V = Cutting speed, T = Tool life in minutes, n = A exponent which depends on cutting condition
C = Constant.

$VT^n d^x f^y$ = Constant Where d = Depth of cut, f = Feed rate (in mm/rev) in turning.

- **Yield Shear Strength (τ_s)**

$$\tau_s = \frac{P}{A_s}$$
$$A_s = \frac{tS_o}{Sin\beta_o}$$

t-Thickness, S_0-feed β_o=Shear angle

- **Specific Power Consumption-**

$$U_s = \frac{P_z V_c}{MRR} = \frac{P_z}{tS_o}$$

P_Z=Cutting force, V_c= Cutting Velocity

- **Cutting Force Expression-**

$$F = F_c \sin\alpha + F_t \cos\alpha$$
$$N = F_c \cos\alpha - F_t \sin\alpha$$

The coefficient of friction will be then given as :

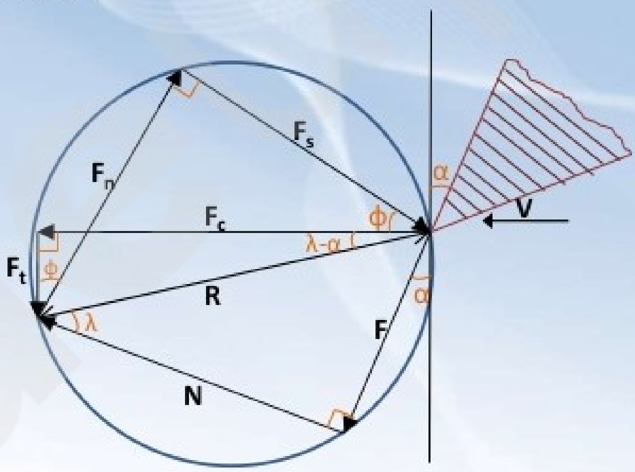
$$\mu = \frac{F}{N} = \frac{F_c \tan\alpha + F_t}{F_c - F_t \tan\alpha}$$
$$\lambda = \tan^{-1}\mu$$

On Shear plane,

$$F_s = F_c \cos\phi - F_t \sin\phi$$
$$F_n = F_c \sin\phi + F_t \cos\phi$$

Now,

$$F_t = F_n \cos\phi - F_s \sin\phi$$
$$F_c = F_n \sin\phi + F_s \cos\phi$$



- **Peak to Valley Height (H_{Max})-**

$$H_{Max} = \frac{f^2}{8R}$$

f- feed, R-Nose radius

$$H_{Max} = \frac{f}{tan\aleph + Cot\aleph_1}$$

$\aleph$=Side Cutting Edge Angle, $\aleph_1$=End Cutting Edge Angle

Non Conventional Machining

- **Electrochemical Machining-**

$$Mass\ removal\ rate\ \dot{m} = \frac{AI}{ZF}$$
$$\text{Volumetric Removal Rate Q} = \frac{AI}{\rho ZF}$$

A=Gram atomic weight of the metallic ions
I=Current (Amps)

P= density of the anode (g/cm^3)
Z= Valency of the cation,
F=mFaraday (96,500 coulombs)

- **Electro Discharge Machining-**

$$V = V_o(1 - e^{-t/RC})$$

V_o=Max. Voltage, t=time, R=Resistance, C= Capacitance

Welding

- **Dilution**

$$= \frac{A_p}{A_P + A_R}$$

A_p=Area of Penetration
A_R= Area of reinforcement

- **Relation in Voltage and Arc Length**

$$V = A + BI$$

l=Arc length, V=Voltage, A&B= Constant

- **Duty Cycle-**

$$I^2 D = Constant$$

D=Duty cycle time, I=current

- **Open Circuit Voltage (OCV) and Short-Circuit Current(SCC) relation-**

$$V_{arc} = OCV - \left[\frac{OCV}{SCC}\right] I$$

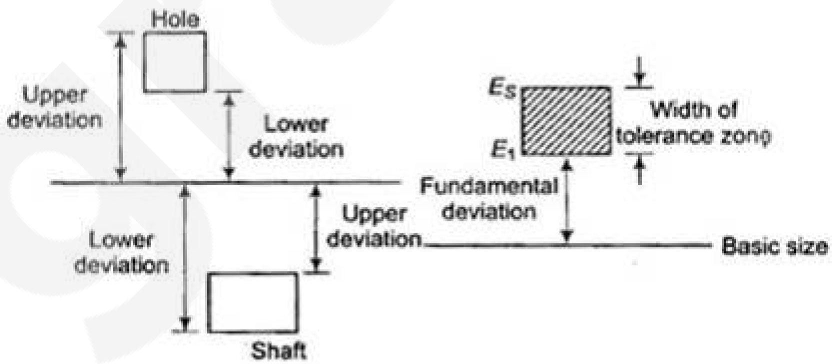
- **Resistance Welding-**

H=I²Rt

For spot welding, diameter of nugget $d_n = 6\sqrt{t}$
Height of the nugget $H_n = 2(t - indentation)/sheet$

Metrology

- **Assembly Representation**



- **Tolerances-**

$$i = 0.45D^{1/3} + 0.001D$$
$$D = \sqrt{D_{max} \times D_{min}}$$

Tolerance Table-

IT01 $\boxed{0.3 + 0.008\,D}$	IT0 $\boxed{0.5 + 0.012\,D}$	IT1 $\boxed{0.8 + 0.02D}$ a	IT2 ar
IT3 ar^2	IT4 ar^3	IT5 $\approx 7i = ar^4$	IT6 10i
IT7 $10(10)^{1/5}i$ = 16i	IT8 $10(10)^{2/5}i$ = 25i	IT9 $10(10)^{3/5}i$ = 40i	IT10 $10(10)^{4/5}i$ = 64i
IT11 100i	IT12 160i	IT13 250i	IT14 400i
IT15 640i	IT16 1000i		

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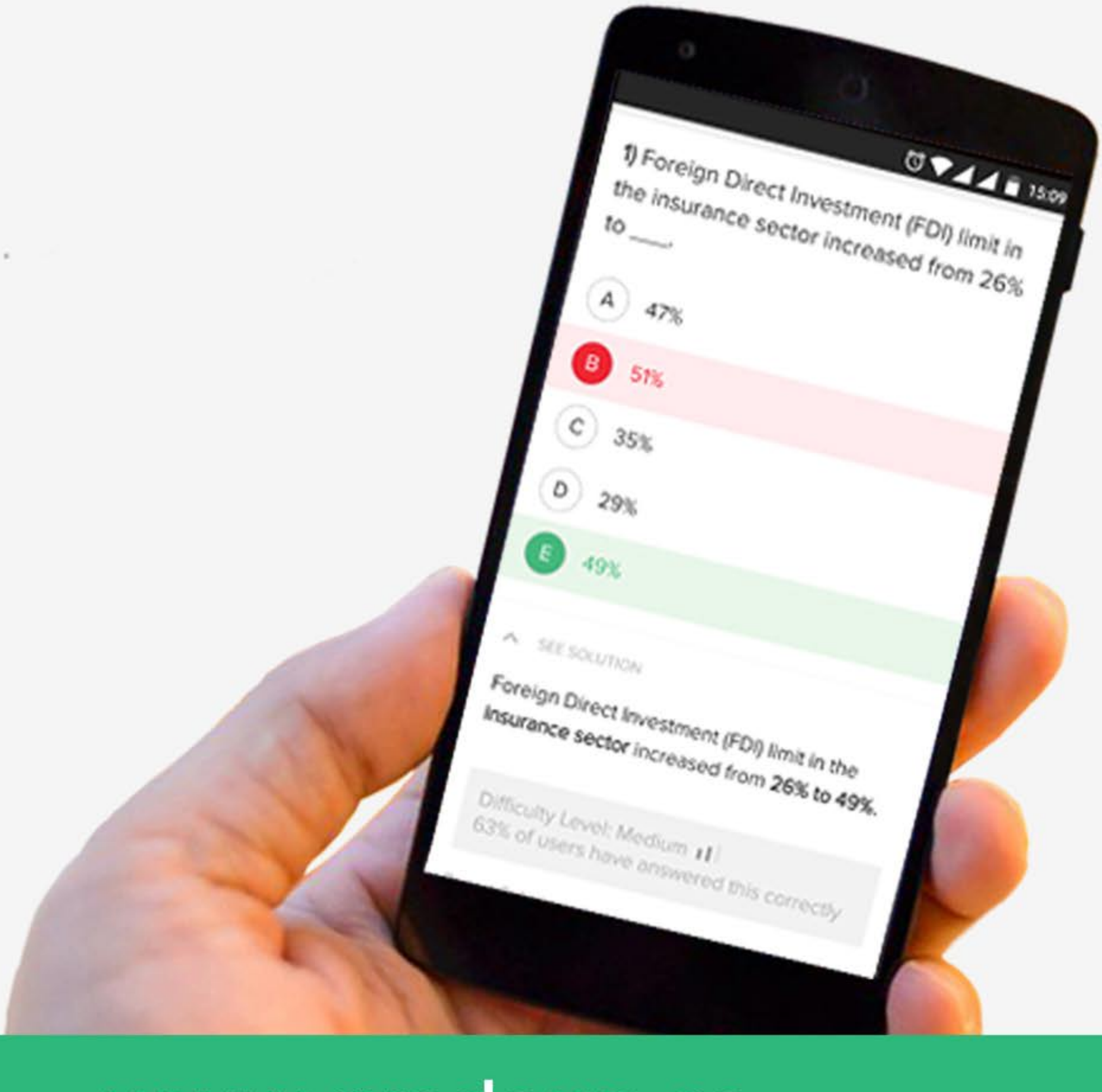
Formulas on
FLUID MECHANICS
for GATE ME Exam



Pressure (P):

- If F be the normal force acting on a surface of area A in contact with liquid, then pressure exerted by liquid on this surface is: $P = F/A$
- **Units :** N/m^2 or Pascal (S.I.) and Dyne/cm^2 (C.G.S.)
- **Dimension :** $[P] = \frac{[F]}{[A]} = \frac{[MLT^{-2}]}{[L^2]} = [ML^{-1}T^{-2}]$
- **Atmospheric pressure:** Its value on the surface of the earth at sea level is nearly $1.013 \times 10^5\ N/m^2$ or Pascal in S.I. other practical units of pressure are atmosphere, bar and torr (*mm* of *Hg*)
- $1atm = 1.01 \times 10^5\ Pa = 1.01\,\text{bar} = 760\ \text{torr}$
- **Fluid Pressure at a Point:** $\rho = \frac{dF}{dA}$

Density (ρ):

- In a fluid, at a point, density $\boldsymbol{\rho}$ is defined as: $\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$
- In case of homogenous isotropic substance, it has no directional properties, so is a scalar.
- It has dimensions $[ML^{-3}]$ and S.I. unit kg/m^3 while C.G.S. unit g/cc with $1g/cc = 10^3 kg/m^3$
- Density of body = Density of substance
- Relative density or specific gravity which is defined as : $RD = \frac{\text{Density of body}}{\text{Density of water}}$
- If m_1 mass of liquid of density ρ_1 and m_2 mass of density ρ_2 are mixed, then as

  $m = m_1 + m_2$ and $V = (m_1/\rho_1) + (m_2/\rho_2)$ [As $V = m/\rho$]

  $$\rho = \frac{m}{V} = \frac{m_1 + m_2}{(m_1/\rho_1) + (m_2/\rho_2)} = \frac{\Sigma m_i}{\Sigma(m_i/p_i)}$$

  If $m_1 = m_2$, $\rho = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2}$ = Harmonic mean
- If V_1 volume of liquid of density ρ_1 and V_2 volume of liquid of density ρ_2 are mixed, then as: $m = \rho_1 V_1 + \rho_2 V_2$ and $V = V_1 + V_2$ [As $\rho = m/V$]

  If $V_1 = V_2 = V$ $\rho = (\rho_1 + \rho_2)/2$ = Arithmetic Mean

- With rise in temperature due to thermal expansion of a given body, volume will increase while mass will remain unchanged, so density will decrease, *i.e.*,

$$\frac{\rho}{\rho_0} = \frac{(m/V)}{(m/V_0)} = \frac{V_0}{V} = \frac{V_0}{V_0(1+\gamma\Delta\theta)} \qquad [\text{As } V = V_0(1+\gamma\Delta\theta)]$$

or

$$\rho = \frac{\rho_0}{(1+\gamma\Delta\theta)} \simeq \rho_0(1-\gamma\Delta\theta)$$

- With increase in pressure due to decrease in volume, density will increase, *i.e.*,

$$\frac{\rho}{\rho_0} = \frac{(m/V)}{(m/V_0)} = \frac{V_0}{V} \qquad [\text{As } \rho = \frac{m}{V}]$$

- By definition of **bulk-modulus**: $B = -V_0 \frac{\Delta p}{\Delta V}$ *i.e.*, $V = V_0\left[1 - \frac{\Delta p}{B}\right]$

$$\rho = \rho_0\left(1 - \frac{\Delta p}{B}\right)^{-1} \simeq \rho_0\left(1 + \frac{\Delta p}{B}\right)$$

Specific Weight (*w*):

- It is defined as the weight per unit volume.
- Specific weight $= \frac{Weight}{Volume} = \frac{m.g}{Volume} = \rho.g$

Specific Gravity or Relative Density (s):

- It is the ratio of specific weight of fluid to the specific weight of a standard fluid. Standard fluid is water in case of liquid and H_2 or air in case of gas.

$$s = \frac{\gamma}{\gamma_w} = \frac{\rho.g}{\rho_{w.}g} = \frac{\rho}{\rho_w}$$

Where, γ_w = Specific weight of water, and ρ_w = Density of water specific.

Specific Volume (*v*):

- Specific volume of liquid is defined as volume per unit mass. It is also defined as the reciprocal of specific density.
- Specific volume $= \frac{V}{m} = \frac{1}{\rho}$

Inertial force per unit area $= \frac{dp/dt}{A} = \frac{v(dm/dt)}{A} = \frac{v\,Av\,\rho}{A} = v^2\rho$

Viscous force per unit area: $F/A = \frac{\eta v}{r}$

Reynold's number: $N_R = \frac{\text{Inertial force per unit area}}{\text{Viscous force per unit area}} = \frac{v^2 \rho}{\eta v / r} = \frac{v \rho r}{\eta}$

Pascal's Law: $p_x = p_y = p_z$; where, p_x, p_y and p_z are the pressure at point x,y,z respectively.

Hydrostatic Law:

- $\frac{\partial p}{\partial z} = pg$ or $dp = pg\ dz$
- $\int_o^p dp = pg \int_o^h dz$
- $p = pgh$ and $h = \frac{p}{pg}$; where, h is known as **pressure head.**

Pressure Energy	Potential energy	Kinetic energy
It is the energy possessed by a liquid by virtue of its pressure. It is the measure of work done in pushing the liquid against pressure without imparting any velocity to it.	It is the energy possessed by liquid by virtue of its height or position above the surface of earth or any reference level taken as zero level.	It is the energy possessed by a liquid by virtue of its motion or velocity.
Pressure energy of the liquid PV	Potential energy of the liquid mgh	Kinetic energy of the liquid $mv^2/2$
Pressure energy per unit mass of the liquid P/ρ	Potential energy per unit mass of the liquid gh	Kinetic energy per unit mass of the liquid $v^2/2$
Pressure energy per unit volume of the liquid P	Potential energy per unit volume of the liquid ρgh	Kinetic energy per unit volume of the liquid $\rho v^2/2$

Quantities that Satisfy a Balance Equation

Quantity	mass	x momentum	y momentum	z momentum	Energy	Species
Φ	m	mu	mv	mw	$E + m\mathbf{V}^2/2$	$m^{(K)}$
ϕ	1	u	v	w	$e + \mathbf{V}^2/2$	$W^{(K)}$

In this table, u, v, and w are the x, y and z velocity components, E is the total thermodynamic internal energy, e is the thermodynamic internal energy per unit mass, and $m^{(K)}$ is the mass of a chemical species, K, $W^{(K)}$ is the mass fraction of species K.

The other energy term, m**V**^{2}/2, is the kinetic energy.

- $Storage = \frac{\partial \Phi}{\partial t} = \frac{\partial (m\varphi)}{\partial t} = \frac{\partial (\rho \Delta x \Delta y \Delta z \varphi)}{\partial t} = \frac{\partial (\rho\varphi)}{\partial t} \Delta x \Delta y \Delta z$
- $Inflow = \rho\, u\varphi\big|_x \Delta y\ \Delta z \;\; + \rho\, v\varphi\big|_y \Delta x\ \Delta z \;\; + \rho\, w\varphi\big|_z \Delta y\ \Delta x$
- $Outflow = \rho\, u\varphi\big|_{x+\Delta x} \Delta y\ \Delta z \;\; + \rho\, v\varphi\big|_{y+\Delta y} \Delta x\ \Delta z \;\; + \rho\, w\varphi\big|_{z+\Delta z}\ \Delta y\ \Delta x$
- $Source = S_\varphi\ \Delta x\ \Delta y\ \Delta z$
- $\frac{\partial \rho\varphi}{\partial t} + \frac{\rho\, u\varphi\big|_{x+\Delta x} - \rho\, u\varphi\big|_x}{\Delta x} + \frac{\rho\, v\varphi\big|_{y+\Delta y} - \rho\, v\varphi\big|_y}{\Delta y} + \frac{\rho\, w\varphi\big|_{z+\Delta z} - \rho\, w\varphi\big|_z}{\Delta z} = S_\varphi$
- $\frac{\partial \rho\varphi}{\partial t} + \frac{\partial \rho u\varphi}{\partial x} + \frac{\partial \rho v\varphi}{\partial y} + \frac{\partial \rho w\varphi}{\partial z} = S_\varphi^*$
- $S_\varphi^* = \lim_{\Delta x \Delta y \Delta z \to 0} S_\varphi$

The Mass Balance Equations:

- $\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$
- $\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$
- $\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0$
- $\frac{\partial \rho}{\partial t} + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0$
- $\frac{D\Psi}{Dt} = \frac{\partial \Psi}{\partial t} + u\frac{\partial \Psi}{\partial x} + v\frac{\partial \Psi}{\partial y} + w\frac{\partial \Psi}{\partial z}$ *or* $\frac{D\Psi}{Dt} = \frac{\partial \Psi}{\partial t} + u_i\frac{\partial \Psi}{\partial x_i}$
- $\frac{D\rho}{Dt} + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0 \qquad \frac{D\rho}{Dt} + \rho\frac{\partial u_i}{\partial x_i} = 0 \qquad \frac{D\rho}{Dt} + \rho\Delta = 0$
- $\Delta \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ *or* $\Delta \equiv \frac{\partial u_i}{\partial x_i} = 0$

- $\rho \dfrac{\partial \varphi}{\partial t} + \varphi \left[\dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u_i}{\partial x_i} \right] + \rho u_i \dfrac{\partial \varphi}{\partial x_i} = S_\varphi$
- $\rho \dfrac{\partial \varphi}{\partial t} + \rho u_i \dfrac{\partial \varphi}{\partial x_i} = S_\varphi$

Momentum Balance Equation:

- $Net\ j-direction\ source\,term = \dfrac{\partial \sigma_{1j}}{\partial x_1} + \dfrac{\partial \sigma_{2j}}{\partial x_2} + \dfrac{\partial \sigma_{3j}}{\partial x_3} + \rho B_j = \dfrac{\partial \sigma_{ij}}{\partial x_i} + \rho B_j$
- $\dfrac{\partial \rho u_j}{\partial t} + \dfrac{\partial \rho u_i u_j}{\partial x_i} = \dfrac{\partial \sigma_{ij}}{\partial x_i} + \rho B_j \qquad j = 1, \ldots 3$
- For a Newtonian fluid, the stress, σ_{ij}, is given by the following equation:

$$\sigma_{ij} = -P\delta_{ij} + \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] + (\kappa - \frac{2}{3}\mu)\Delta \delta_{ij}$$

- $\dfrac{\partial \rho u_j}{\partial t} + \dfrac{\partial \rho u_i u_j}{\partial x_i} = \dfrac{\partial}{\partial x_i} \left[-P\delta_{ij} + \mu \left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} \right) + (\kappa - \dfrac{2}{3}\mu)\Delta \delta_{ij} \right] + \rho B_j \qquad j = 1, \ldots 3$
- $\dfrac{\partial \rho u_j}{\partial t} + \dfrac{\partial \rho u_i u_j}{\partial x_i} = -\dfrac{\partial P}{\partial x_j} + \dfrac{\partial}{\partial x_i} \left[\mu \left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} \right) \right] + \dfrac{\partial}{\partial x_j} \left[(\kappa - \dfrac{2}{3}\mu)\Delta \right] + \rho B_j \qquad j = 1, \ldots 3$
- $\dfrac{\partial \rho u}{\partial t} + \dfrac{\partial \rho uu}{\partial x} + \dfrac{\partial \rho vu}{\partial y} + \dfrac{\partial wu}{\partial z} = \rho B_x$
- $-\dfrac{\partial P}{\partial x} + 2\dfrac{\partial}{\partial x}\mu \left(\dfrac{\partial u}{\partial x} \right) + \dfrac{\partial}{\partial y} \left[\mu \left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \right) \right] + \dfrac{\partial}{\partial z} \left[\mu \left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} \right) \right] + \dfrac{\partial}{\partial x} \left[(\kappa - \dfrac{2}{3}\mu)\Delta \right]$

Energy Balance Equation:

- This directional heat flux is given the symbol q_i: $q_i = -k \dfrac{\partial T}{\partial x_i}$
- $\dfrac{Net\ \ xDirection\,heat}{Unit\,Volume} = -\dfrac{q_x|_{x+\Delta x} - q_x|_x}{\Delta x \Delta y \Delta z} \Delta y \Delta z \quad = -\dfrac{q_x|_{x+\Delta x} - q_x|_x}{\Delta x}$
- $\underset{\Delta x \to 0}{Limit} \dfrac{Net\ \ xDirection\,heat\,source}{Unit\,Volume} = -\dfrac{\partial q_x}{\partial x}$

- $Heat\ Rate = -\dfrac{\partial q_x}{\partial x} - \dfrac{\partial q_y}{\partial y} - \dfrac{\partial q_x}{\partial x} = -\dfrac{\partial q_i}{\partial x_i}$
- *Body-force work rate* = $\rho(uB_x + vB_y + wB_z) = \rho u_i B_i$

- The work term on each face is given by the following equation:

$$y\text{-face surface force work} = (u\sigma_{yx} + v\sigma_{yy} + w\sigma_{yz})\Delta x\, \Delta z = u_i\sigma_{iy}\, \Delta x\, \Delta z$$

- $Net\ yFace\ Surface\ Force\ Work = \dfrac{\partial(u\sigma_{yx} + v\sigma_{yy} + w\sigma_{yz})}{\partial y} = \dfrac{\partial u_i \sigma_{yi}}{\partial y}$
- $Net\ Surface\ Force\ Work = \dfrac{\partial u_i\sigma_{xi}}{\partial x} + \dfrac{\partial u_i\sigma_{yi}}{\partial y} + \dfrac{\partial u_i\sigma_{zi}}{\partial z} = \dfrac{\partial u_i\sigma_{ji}}{\partial x_j}$
- Energy balance equation:

$$\frac{\partial\rho(e+\mathbf{V}^2/2)}{\partial t} + \frac{\partial\rho u_i(e+\mathbf{V}^2/2)}{\partial x_i} = -\frac{\partial q_i}{\partial x_i} + \frac{\partial u_i\sigma_{ji}}{\partial x_j} + \rho u_i B_i$$

Substitutions for Stresses and Heat Flux:

Using only the Fourier Law heat transfer, the source term involving the heat flux in the energy balance equation:

- $-\dfrac{\partial q_i}{\partial x_i} = -\dfrac{\partial}{\partial x_i}\left(-k\dfrac{\partial T}{\partial x_i}\right) = \dfrac{\partial}{\partial x_i}k\dfrac{\partial T}{\partial x_i} = \dfrac{\partial}{\partial x}k\dfrac{\partial T}{\partial x} + \dfrac{\partial}{\partial y}k\dfrac{\partial T}{\partial y} + \dfrac{\partial}{\partial z}k\dfrac{\partial T}{\partial z}$
- $\dfrac{\partial\rho e}{\partial t} + \dfrac{\partial\rho u_i e}{\partial x_i} = \dfrac{\partial}{\partial x_i}k\dfrac{\partial T}{\partial x_i} + \left[-P\delta_{ij} + \mu\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) + (\kappa - \dfrac{2}{3}\mu)\Delta\delta_{ij}\right]\dfrac{\partial u_i}{\partial x_j}$
- $\delta_{ij}\dfrac{\partial u_i}{\partial x_j} = \dfrac{\partial u_j}{\partial x_j} = \Delta$
- $\dfrac{\partial\rho e}{\partial t} + \dfrac{\partial\rho u_i e}{\partial x_i} = \dfrac{\partial}{\partial x_i}k\dfrac{\partial T}{\partial x_i} - P\Delta + \mu\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)\dfrac{\partial u_i}{\partial x_j} + (\kappa - \dfrac{2}{3}\mu)\Delta^2$

Dissipation to avoid confusion with the general quantity in a balance equation:

- $\boldsymbol{\Phi_D} = \mu\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)\dfrac{\partial u_i}{\partial x_j} + (\kappa - \dfrac{2}{3}\mu)\Delta^2$

- $$\frac{\partial \rho e}{\partial t}+\frac{\partial \rho u_i e}{\partial x_i}=\frac{\partial}{\partial x_i}k\frac{\partial T}{\partial x_i}-P\Delta+\boldsymbol{\Phi_D}$$

The temperature gradient in the Fourier law conduction term may also be written as a gradient of enthalpy or internal energy:

- $$\frac{\partial T}{\partial x_i}=\frac{1}{c_v}\frac{\partial e}{\partial x_i}+\frac{1}{c_v}\left[\frac{T\beta_P}{\kappa_T}-P\right]\frac{1}{\rho^2}\frac{\partial \rho}{\partial x_i}$$
- $$\frac{\partial T}{\partial x_i}=\frac{1}{c_p}\frac{\partial h}{\partial x_i}-\frac{1-T\beta_P}{\rho c_p}\frac{\partial P}{\partial x_i}$$

- $$\frac{\partial \rho e}{\partial t}+\frac{\partial \rho u_i e}{\partial x_i}=\frac{\partial}{\partial x_i}\frac{k}{c_v}\frac{\partial e}{\partial x_i}-P\Delta+\boldsymbol{\Phi_D}+\frac{\partial}{\partial x_i}\frac{1}{c_v}\left[\frac{T\beta_P}{\kappa_T}-P\right]\frac{1}{\rho^2}\frac{\partial \rho}{\partial x_i}$$
- $$\frac{\partial \rho h}{\partial t}+\frac{\partial \rho u_i h}{\partial x_i}=\frac{\partial}{\partial x_i}\frac{k}{c_p}\frac{\partial h}{\partial x_i}+\boldsymbol{\Phi_D}+\frac{\partial}{\partial x_i}\left[\frac{1-T\beta_P}{\rho c_p}\right]\frac{\partial P}{\partial x_i}+\frac{DP}{Dt}$$
- $$c_p\left[\frac{\partial \rho T}{\partial t}+\frac{\partial \rho u_i T}{\partial x_i}\right]=\frac{\partial}{\partial x_i}k\frac{\partial T}{\partial x_i}+\boldsymbol{\Phi_D}+\beta_P T\frac{DP}{Dt}$$

General Balance Equations			
φ	**c**	$\Gamma^{(\varphi)}$	$S^{(\varphi)}$
1	1	0	0
$u = u_x = u_1$	1	μ	$-\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\mu\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}\mu\frac{\partial v}{\partial x}+\frac{\partial}{\partial z}\mu\frac{\partial w}{\partial x}+\frac{\partial}{\partial x}\left[(\kappa-\frac{2}{3}\mu)\Delta\right]+\rho B_x$
$v = u_y = u_2$	1	μ	$-\frac{\partial P}{\partial y}+\frac{\partial}{\partial x}\mu\frac{\partial u}{\partial y}+\frac{\partial}{\partial y}\mu\frac{\partial v}{\partial y}+\frac{\partial}{\partial z}\mu\frac{\partial w}{\partial y}+\frac{\partial}{\partial y}\left[(\kappa-\frac{2}{3}\mu)\Delta\right]+\rho B_y$
$w = u_z = u_3$	1	μ	$-\frac{\partial P}{\partial z}+\frac{\partial}{\partial x}\mu\frac{\partial u}{\partial z}+\frac{\partial}{\partial y}\mu\frac{\partial v}{\partial z}+\frac{\partial}{\partial z}\mu\frac{\partial w}{\partial z}+\frac{\partial}{\partial z}\left[(\kappa-\frac{2}{3}\mu)\Delta\right]+\rho B_z$
e	1	k/c_v	$-P\Delta+\boldsymbol{\Phi_D}+\frac{\partial}{\partial x_i}\frac{1}{c_v}\left[\frac{T\beta_P}{\kappa_T}-P\right]\frac{1}{\rho^2}\frac{\partial\rho}{\partial x_i}$
h	1	k/c_P	$\boldsymbol{\Phi_D}+\frac{\partial}{\partial x_i}\left[\frac{1-T\beta_P}{\rho c_p}\right]\frac{\partial P}{\partial x_i}+\frac{DP}{Dt}$
T	c_P	k	$\boldsymbol{\Phi_D}+\beta_P T\frac{DP}{Dt}$
T	c_v	k	$\boldsymbol{\Phi_D}+\frac{T\beta_P}{\kappa_T}\Delta$
$W^{(K)}$	1	$\boldsymbol{\rho D}_{K,Mix}$	$r^{(K)}$

Momentum equation:

- $$\frac{\partial\rho u_j}{\partial t}+\frac{\partial\rho u_i u_j}{\partial x_i}=\frac{\partial}{\partial x_i}\mu\frac{\partial u_j}{\partial x_i}+S^{(j)}=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_i}\mu\frac{\partial u_j}{\partial x_i}+S^{*(j)}$$

General Momentum Equations			
φ	**c**	$\Gamma^{(\varphi)}$	$S^{*(\varphi)}$
1	1	0	0
$u = u_x = u_1$	1	μ	$\frac{\partial}{\partial x}\mu\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}\mu\frac{\partial v}{\partial x}+\frac{\partial}{\partial z}\mu\frac{\partial w}{\partial x}+\frac{\partial}{\partial x}\left[(\kappa-\frac{2}{3}\mu)\Delta\right]+\rho B_x$
$v = u_y = u_2$	1	μ	$+\frac{\partial}{\partial x}\mu\frac{\partial u}{\partial y}+\frac{\partial}{\partial y}\mu\frac{\partial v}{\partial y}+\frac{\partial}{\partial z}\mu\frac{\partial w}{\partial y}+\frac{\partial}{\partial y}\left[(\kappa-\frac{2}{3}\mu)\Delta\right]+\rho B_y$
$w = u_z = u_3$	1	μ	$\frac{\partial}{\partial x}\mu\frac{\partial u}{\partial z}+\frac{\partial}{\partial y}\mu\frac{\partial v}{\partial z}+\frac{\partial}{\partial z}\mu\frac{\partial w}{\partial z}+\frac{\partial}{\partial z}\left[(\kappa-\frac{2}{3}\mu)\Delta\right]+\rho B_z$

Bernoulli's Equation:

This equation has four variables: velocity (v), elevation (z), pressure (p), and density (ρ). It also has a constant (g), which is the acceleration due to gravity. Here is Bernoulli's equation:

- $\frac{v^2}{2} + gz + \frac{p}{\rho} = \text{constant}$
- $P + \rho gh + \frac{1}{2}\rho v^2 = \text{constant}$
- $\frac{P}{\rho g} + h + \frac{v^2}{2g} = \text{constant}$; $\frac{P}{\rho g}$ is called pressure head, h is called gravitational head and $\frac{v^2}{2g}$ is called velocity head.

Factors that influence head loss due to friction are:

- Length of the pipe (l)
- Effective diameter of the pipe (D_h)
- Velocity of the water in the pipe (v)
- Acceleration of gravity (g)
- Friction from the surface roughness of the pipe (λ)
- *The head loss due to the pipe is estimated by the following equation:*

$$h_{f,major} = \lambda \frac{lv^2}{2D_h g}$$

- To estimate the total head loss in a piping system, one adds the head loss from the fittings and the pipe:

$$h_{f,total} = \sum h_{f,minor} + \sum h_{f,major}$$

- Note that the summation symbol ($\sum$) means to add up the losses from all the different sources. A less compact-way to write this equation is:

$$h_{f,total} = h_{f,minor\,1} + h_{f,minor\,2} + h_{f,minor\,3} + \cdots$$
$$h_{f,major\,1} + h_{f,major\,2} + h_{f,major\,3} + \cdots$$

Combining Bernoulli's Equation With Head Loss:

$$\frac{p_1}{\gamma} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{v_2^2}{2g} + z_2 + h_{f,total}$$

Relation between coefficient of viscosity and temperature:

$$\text{Andrade formula } \eta = \frac{A e^{C\rho/T}}{\rho^{-1/3}}$$

Stoke's Law: $F = 6\pi\eta rv$

Terminal Velocity:

- Weight of the body (W) = mg = ($volume \times density$) $\times g = \frac{4}{3}\pi r^3 \rho g$
- Upward thrust (T) = weight of the fluid displaced

$$= (volume \times density) \text{ of the fluid} \times g = \frac{4}{3}\pi r^3 \sigma g$$

- Viscous force (F) = $6\pi\eta rv$
- When the body attains terminal velocity the net force acting on the body is zero.
- $W–T–F = 0$ or $F = W – T$
- $6\pi\eta rv = \frac{4}{3}\pi r^3 \rho g - \frac{4}{3}\pi r^3 \sigma g = \frac{4}{3}\pi r^3 (\rho - \sigma) g$
- Terminal velocity $v = \frac{2}{9}\frac{r^2(\rho - \sigma) g}{\eta}$
- Terminal velocity depend on the radius of the sphere so if radius is made n - fold, terminal velocity will become n^2 times.
- Greater the density of solid greater will be the terminal velocity
- Greater the density and viscosity of the fluid lesser will be the terminal velocity.
- If $\rho > \sigma$ then terminal velocity will be positive and hence the spherical body will attain constant velocity in downward direction.
- If $\rho < \sigma$ then terminal velocity will be negative and hence the spherical body will attain constant velocity in upward direction.

Poiseuille's Formula:

- $V \propto \frac{P r^4}{\eta l}$ or $V = \frac{KP r^4}{\eta l}$
- $V = \frac{\pi P r^4}{8\eta l}$; where $K = \frac{\pi}{8}$ is the constant of proportionality.

Buoyant Force:

- Buoyant force = Weight of fluid displaced by body

- Buoyant force on cylinder =Weight of fluid displaced by cylinder
- $V_{s_{in}}$ = Value of immersed part of solid
- $F_B = p_{water} \times g \times$ Volume of fluid displaced
- $F_B = p_{water} \times g \times$ Volume of cylinder immersed inside the water
- $F_B = mg$
- $F_B = p_w g \frac{\pi}{4} d^2$ $(\because w = mg = pVg)$
- $V_{s_{in}} plg = V_s p_s g$
- $p_w g \frac{\pi}{4} d^2 x = p_{cylinder} g \frac{\pi}{4} d^2 h$
- $p_w x = p_{cylinder} h$

Relation between B, G and M:

- GM $= \frac{l}{V} - BG$; where l = Least moment of inertia of plane of body at water surface, G = Centre of gravity, B = Centre of buoyancy, and M = Metacentre.
- $l = \min(l_{xx}, l_{yy}), \quad l_{xx} = \frac{bd^3}{12}, \quad l_{yy} = \frac{bd^3}{12}$
- $V = bdx$

Energy Equations:

- $F_{net} = F_g + F_p + F_v + F_c + F_t$; where Gravity force F_g, Pressure force F_p, Viscous force F_v, Compressibility force F_c, and Turbulent force F_t.
- If fluid is incompressible, then $F_c = 0$

  $\therefore F_{net} = F_g + F_p + F_v + F_t$; This is known as **Reynolds equation** of motion.
- If fluid is incompressible and turbulence is negligible, then

$F_c = 0, F_t = 0 \quad \therefore F_{net} = F_g + F_p + F_v$; This equation is called as **Navier-Stokes equation**.

- If fluid flow is considered ideal then, viscous effect will also be negligible. Then

  $F_{net} = F_g + F_p$; This equation is known as **Euler's equation**.
- Euler's equation can be written as: $\frac{dp}{\rho} + gdz + vdv = 0$

Dimensional analysis:

Quantity	Symbol	Dimensions
Mass	m	M
Length	l	L
Time	t	T
Temperature	T	θ
Velocity	u	LT^{-1}
Acceleration	a	LT^{-2}
Momentum/Impulse	mv	MLT^{-1}
Force	F	MLT^{-2}
Energy - Work	W	$ML^{2}T^{-2}$
Power	P	$ML^{2}T^{-3}$
Moment of Force	M	$ML^{2}T^{-2}$
Angular momentum	-	$ML^{2}T^{-1}$
Angle	η	$M^{0}L^{0}T^{0}$
Angular Velocity	ω	T^{-1}
Angular acceleration	α	T^{-2}
Area	A	L^{2}
Volume	V	L^{3}
First Moment of Area	Ar	L^{3}
Second Moment of Area	I	L^{4}
Density	ρ	ML^{-3}
Specific heat-Constant Pressure	C_p	$L^{2}T^{-2}\theta^{-1}$
Elastic Modulus	**E**	$\mathbf{ML^{-1}T^{-2}}$
Flexural Rigidity	EI	$ML^{3}T^{-2}$
Shear Modulus	G	$ML^{-1}T^{-2}$
Torsional rigidity	GJ	$ML^{3}T^{-2}$
Stiffness	k	MT^{-2}

Angular stiffness	T/η	$ML^{2}T^{-2}$
Flexibiity	1/k	$M^{-1}T^{2}$
Vorticity	-	T^{-1}
Circulation	-	$L^{2}T^{-1}$
Viscosity	μ	$ML^{-1}T^{-1}$
Kinematic Viscosity	τ	$L^{2}T^{-1}$
Diffusivity	-	$L^{2}T^{-1}$
Friction coefficient	f /μ	$M^{0}L^{0}T^{0}$
Restitution coefficient		$M^{0}L^{0}T^{0}$
Specific heat- Constant volume	C_v	$L^{2}T^{-2}\theta^{-1}$

Boundary layer:

- **Reynolds number** $=\dfrac{\rho v\cdot x}{\mu}(\mathrm{Re})_x=\dfrac{v\cdot x}{v}$
- **Displacement Thickness (δ*):** $\delta^*=\int_0^\delta\left(1-\dfrac{u}{U}\right)dy$
- **Momentum Thickness (θ):** $\theta=\int_0^\delta\dfrac{u}{U}\left(1-\dfrac{u}{U}\right)dy$
- **Energy Thickness (δ**):** $\delta^{**}=\int_0^\delta\dfrac{u}{U}\left(1-\dfrac{u^2}{U^2}\right)dy$
- **Boundary Conditions for the Velocity Profile:** Boundary conditions are as
  - (a) $At\ y=0, u=0, \dfrac{du}{dy}\neq 0$; (b) $At\ y=\delta, u=U, \dfrac{du}{dy}=0$

Turbulent flow:

- **Shear stress in turbulent flow:** $\tau=\tau_v+\tau_t=\mu\dfrac{d\bar{u}}{dy}+\eta\dfrac{d\bar{u}}{dy}$
- **Turbulent shear stress by Reynold:** $\tau=\rho u'v'$
- **Shear stress in turbulent flow due to Prndtle :** $\tau=\rho l^2\left(\dfrac{du}{dy}\right)^2$

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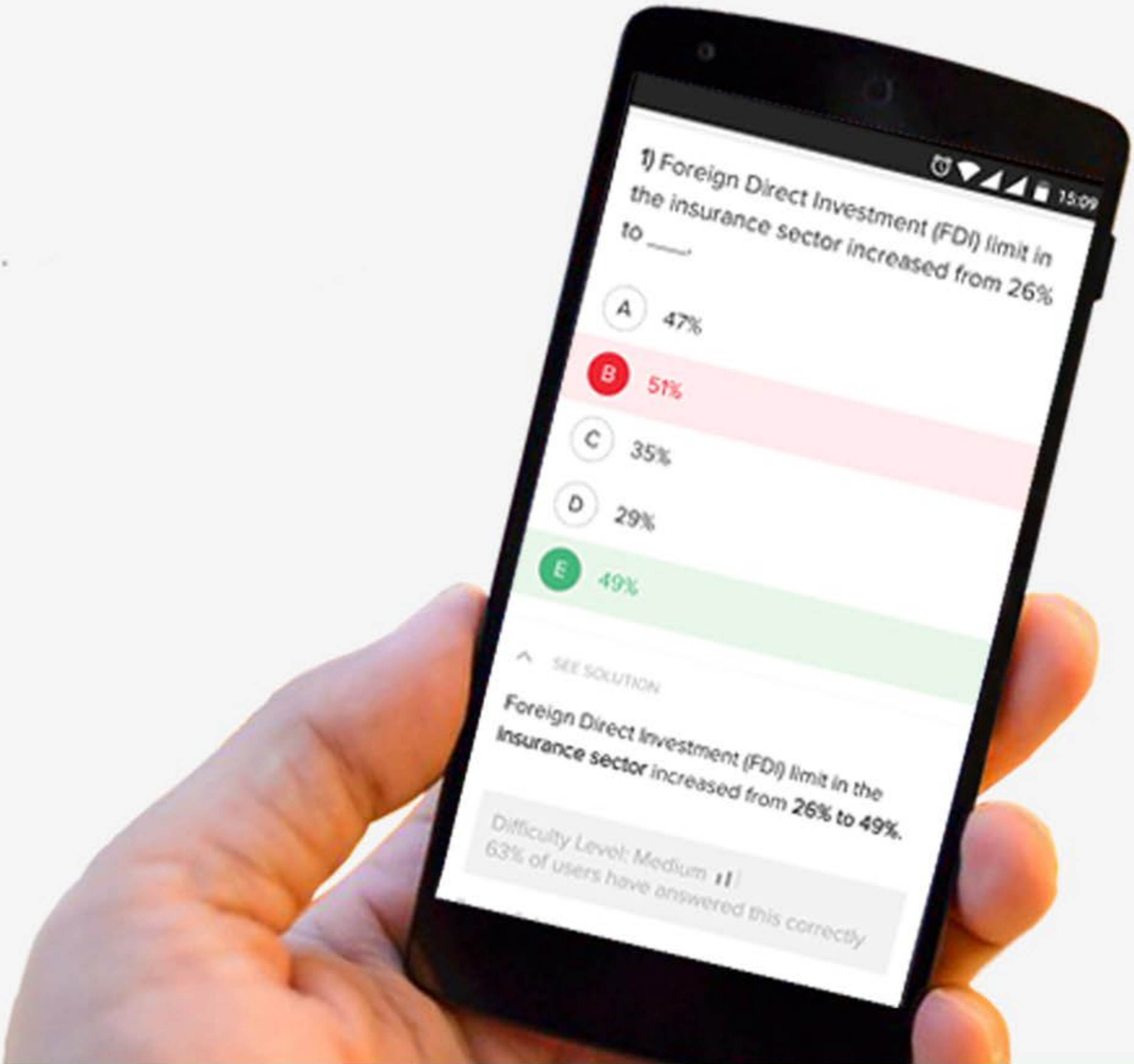
Formulas on THERMODYNAMICS for GATE ME Exam



Thermodynamics

Symbol/Formula	Parameter
M	Molar mass (M/μ)
m	Mass (M)
$n = \frac{m}{M}$	Number of moles (μ)
E	Energy or general extensive property
$e = \frac{E}{m}$	Specific molar energy (energy per unit mass) or general extensive property per unit mass
$\bar{e} = \frac{E}{n} = eM$	Specific energy (energy per unit mole) or general extensive property per unit mole
P	Pressure ($ML^{-1}T^{-2}$)
V	Volume (L^3); Specific volume or volume per unit mass, v (L^3M^{-1}) and the volume per unit mole $\bar{v}$ ($L^3\mu^{-1}$)
T	Temperature (Θ)
ρ	Density (ML^{-3}); ρ = 1/v.
x	Quality
U	Thermodynamic internal energy (ML^2T^{-2}); Internal energy per unit mass, u (L^2T^{-2}), and the internal energy per unit mole, $\bar{u}$ ($ML^2T^{-2}\mu^{-1}$)
H = U + PV	Thermodynamic enthalpy (ML^2T^{-2}); Enthalpy per unit mass, h = u + Pv (dimensions: L^2T^{-2}) and the internal energy per unit mole $\bar{h}$ ($ML^2T^{-2}\mu^{-1}$)
S	Entropy ($ML^2T^{-2}\Theta^{-1}$); Entropy per unit mass, s($L^2T^{-2}\Theta^{-1}$) and the internal energy per unit mole $\bar{s}$ ($ML^2T^{-2}\Theta^{-1}\mu^{-1}$)
W	Work (ML^2T^{-2})
Q	Heat transfer (ML^2T^{-2})
$\dot{W}_u$:	The useful work rate or mechanical power (ML^2T^{-3})
$\dot{m}$:	The mass flow rate (MT^{-1})

$\frac{\vec{V}^2}{2}$:	The kinetic energy per unit mass (L^2T^{-2})
gz:	The potential energy per unit mass (L^2T^{-2})
E_{tot}:	The total energy = m(u + $\frac{\vec{V}^2}{2}$ + gz) (ML^2T^{-2})
$\dot{Q}$:	The heat transfer rate (ML^2T^{-3})
$\frac{dE_{CV}}{dt}$:	The rate of change of energy for the control volume.(ml^2t^{-3})
M	Molar mass (M/μ)
m	Mass (M)
$n = \frac{m}{M}$	Number of moles (μ)
E	Energy or general extensive property
$e = \frac{E}{m}$	Specific molar energy (energy per unit mass) or general extensive property per unit mass
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U	Thermodynamic internal energy (ML^2T^{-2}); Internal energy per unit mass, u (L^2T^{-2}), and the internal energy per unit mole, $\bar{u}$ ($ML^2T^{-2}\mu^{-1}$)
H = U + PV	Thermodynamic enthalpy (ML^2T^{-2}); we also have the enthalpy per unit mass, h = u + Pv (dimensions: L^2T^{-2}) and the internal energy per unit mole $\bar{h}$ ($ML^2T^{-2}\mu^{-1}$)
S	Entropy ($ML^2T^{-2}\Theta^{-1}$); Entropy per unit mass, s($L^2T^{-2}\Theta^{-1}$) and the internal energy per unit mole $\bar{s}$ ($ML^2T^{-2}\Theta^{-1}\mu^{-1}$)

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$n = \frac{m}{M}$	Number of moles (μ)
E	Energy or general extensive property
$e = \frac{E}{m}$	Specific molar energy (energy per unit mass) or general extensive property per unit mass
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T	Temperature (Θ)
ρ	Density (ML^{-3}); $\rho = 1/v$.
x	Quality
U	Thermodynamic internal energy (ML^2T^{-2}); we also have the internal energy per unit mass, u (L^2T^{-2}), and the internal energy per unit mole, $\bar{u}$ ($ML^2T^{-2}\mu^{-1}$)
H = U + PV	Thermodynamic enthalpy (ML^2T^{-2}); we also have the enthalpy per unit mass, h = u + Pv (dimensions: L^2T^{-2}) and the internal energy per unit mole $\bar{h}$ ($ML^2T^{-2}\mu^{-1}$)

S	Entropy ($ML^2T^{-2}\Theta^{-1}$); we also have the entropy per unit mass, s($L^2T^{-2}\Theta^{-1}$) and the internal energy per unit mole $\bar{s}$ ($ML^2T^{-2}\Theta^{-1}\mu^{-1}$)
W	Work (ML^2T^{-2})
Q	Heat transfer (ML^2T^{-2})
$\dot{W}_u$:	The useful work rate or mechanical power (ML^2T^{-3})
$\dot{m}$:	The mass flow rate (MT^{-1})
$\frac{\vec{V}^2}{2}$:	The kinetic energy per unit mass (L^2T^{-2})
gz:	The potential energy per unit mass (L^2T^{-2})
E_{tot}:	The total energy = $m(u + \frac{\vec{V}^2}{2} + gz)$ (ML^2T^{-2})
$\dot{Q}$:	The heat transfer rate (ML^2T^{-3})
$\frac{dE_{CV}}{dt}$:	The rate of change of energy for the control volume. (ml^2t^{-3})

Unit conversion factors

For metric units

- **Basic:**
  - 1 N = 1 kg·m/s^2;
  - 1 J = 1 N·m;
  - 1 W = 1 J/s;
  - 1 Pa = 1 N/m^2.
- **Others:**
  - 1 kPa·m^3 = 1 kJ;
  - T(K) = T(^{o}C) + 273.15;
  - 1 L (liter) = 0.001 m^3;
  - 1 m^2/s^2 = 1 J/kg.
- **Prefixes (and abbreviations):**
  - nano(n) – 10^{-9};
  - micro(μ) – 10^{-6};
  - milli(m) – 10^{-3};
  - kilo(k) – 10^3;
  - mega(M) – 10^6;
  - giga(G) – 10^9.
  - A metric ton (European word: tonne) is 1000 kg.

For engineering units

- **Energy:**
  - 1 Btu = 5.40395 psia·ft^3 = 778.169 ft·lb$_f$ = (1 kWh)/3412.14 = (1 hp·h)/2544.5 = 25,037 lb$_m$·ft^2/s^2.
- **Pressure:**
  - 1 psia = 1 lb$_f$/in^2 = 144 psfa = 144 lb$_f$/ft^2.
- **Others:**
  - T(R) = T(^{o}F) + 459.67;
  - 1 lb$_f$ = 32.174 lb$_m$·ft/s^2;
  - 1 ton of refrigeration = 200 Btu/min.

Concepts & Definitions

	Formula	Units
Pressure	$P = \frac{F}{A}$	Pa
• **Units**	$1\ Pa = 1\ N/m^2$ $1\ bar = 10^5\ Pa = 0.1\ Mpa$ $1\ atm = 101325\ Pa$	
Specific Volume	$v = \frac{V}{m}$	m^3/kg
Density	$\rho = \frac{m}{V} \rightarrow \rho = \frac{1}{v}$	kg/m^3
Static Pressure Variation	$\Delta P = \rho g h$ $\uparrow = -\ ,\ \downarrow = +$	Pa
Absolute Temperature	$T(K) = T(°C) + 273.15$	

Properties of a Pure Substance

	Formula	Units
Quality	$x = \frac{m_{vapor}}{m_{tot}}$ (vapour mass fraction) $1 - x = \frac{m_{liquid}}{m_{tot}}$ (Liquid mass fraction)	
Specific Volume	$v = v_f + x v_{fg}$	m^3/kg
Average Specific Volume	$v = (1-x)v_f + x v_g$ (only two phase mixture)	m^3/kg
Ideal –gas law	$P << P_c \quad T << T_c \quad Z = 1$	
• **Equations**	$Pv = RT \qquad PV = mRT = n\bar{R}T$	

• Universal Gas Constant	$\bar{R} = 8.3145$	$kJ / kmol\ K$
• Gas Constant	$R = \frac{\bar{R}}{M}$ M = molecular mass	$kJ / kg\ K$
Compressibility Factor Z	$Pv = ZRT$	
Reduced Properties	$P_r = \frac{P}{P_c}$, $T_r = \frac{T}{T_c}$	

Work & Heat

	Formula	Units
Displacement Work	$W = \int_1^2 Fdx = \int_1^2 PdV$	J
Integration	$W = \int_1^2 PdV = P(V_2 - V_1)$	J
Specific Work	$w = \frac{W}{m}$ (work per unit mass)	J / kg
Power (rate of work)	$\dot{W} = F\bar{V} = P\dot{V} = T\omega$	W
• Velocity	$\bar{V} = r\omega$	rad / s
• Torque	$T = Fr$	Nm
Polytropic Process $(n \neq 1)$	$PV^n = Const = P_1V_1^n = P_2V_2^n$ $Pv^n = C$	
• Polytropic Exponent	$n = \frac{\ln\left(P_2/P_1\right)}{\ln\left(V_1/V_2\right)}$	
• n=1	$PV = Const = P_1V_1 = P_2V_2$	
Polytropic Process Work	${}_1W_2 = \frac{1}{1-n}(P_2V_2 - P_1V_1)$ $n \neq 1$	J
• n=1	${}_1W_2 = P_2V_2 \ln\left(V_2/V_1\right)$	J
Adiabatic Process	$Q = 0$	
Conduction Heat Transfer	$\dot{Q} = -kA\frac{dT}{dx}$, k =conductivity	W
Convection Heat Transfer	$\dot{Q} = hA\Delta T$, h =convection coefficient	W
Radiation Heat Transfer	$\dot{Q} = \varepsilon\sigma A(T_s^4 - T_{amb}^4)$	W

Terminology:

Q₁ = heat

Q₂ = heat transferred during the process between state 1 and state 2

$\dot{Q}$ = rate of heat transfer
W = work
${}_1W_2$ = work done during the change from state 1 to state 2
$\dot{W}$ = rate of work = Power. 1 W=1 J/s

The First Law of Thermodynamics

	Formula	Units
Total Energy	$E = U + KE + PE \rightarrow dE = dU + d(KE) + d(PE)$	J
Energy	$dE = \delta Q - \delta W \rightarrow E_2 - E_1 = {}_1Q_2 - {}_1W_2$	J
Kinetic Energy	$KE = 0.5m\bar{V}^2$	J
Potential Energy	$PE = mgZ \rightarrow PE_2 - PE_1 = mg(Z_2 - Z_1)$	J
Internal Energy	$U = U_{liq} + U_{vap} \rightarrow mu = m_{liq}u_f + m_{vap}u_g$	
Specific Internal Energy of Saturated Steam (two-phase mass average)	$u = (1-x)u_f + xu_g$ $u = u_f + xu_{fg}$	kJ / kg
Total Energy	$U_2 - U_1 + \dfrac{m(\bar{V}_2^2 - \bar{V}_1^2)}{2} + mg(Z_2 - Z_1) = {}_1Q_2 - {}_1W_2$	J
Specific Energy	$e = u + 0.5\bar{V}^2 + gZ$	
Enthalpy	$H = U + PV$	
Specific Enthalpy	$h = u + Pv$	kJ / kg
For Ideal Gasses	$Pv = RT \text{ and } u = f(T)$	
• **Enthalpy**	$h = u + Pv = u + RT$	
• **R Constant**	$u = f(t) \rightarrow h = f(T)$	
Specific Enthalpy for Saturation State (two-phase mass average)	$h = (1-x)h_f + xh_g$ $h = h_f + xh_{fg}$	kJ / kg
Specific Heat at Constant Volume	$C_v = \frac{1}{m}\left(\frac{\delta Q}{\delta T}\right)_v = \frac{1}{m}\left(\frac{\delta U}{\delta T}\right)_v = \left(\frac{\delta u}{\delta T}\right)_v$ $\rightarrow (u_e - u_i) = C_v(T_e - T_i)$	
Specific Heat at Constant Pressure	$C_p = \frac{1}{m}\left(\frac{\delta Q}{\delta T}\right)_p = \frac{1}{m}\left(\frac{\delta H}{\delta T}\right)_p = \left(\frac{\delta h}{\delta T}\right)_p$ $\rightarrow (h_e - h_i) = C_p(T_e - T_i)$	
Solids & Liquids	Incompressible, so v=constant $C = C_c = C_p$ (Tables A.3 & A.4) $u_2 - u_1 = C(T_2 - T_1)$ $h_2 - h_1 = u_2 - u_1 + v(P_2 - P_1)$	

Ideal Gas	$h = u + Pv = u + RT$ $u_2 - u_1 \cong C_v(T_2 - T_1)$ $h_2 - h_1 \cong C_p(T_2 - T_1)$
Energy Rate	$\dot{E} = \dot{Q} - \dot{W} \quad (rate = +in - out)$ $\rightarrow E_2 - E_1 = {}_1Q_2 - {}_1W_2 \quad (change = +in - out)$

First-Law Analysis for Control Volume

	Formula	Units
Volume Flow Rate	$\dot{V} = \int \bar{V} dA = A\bar{V}$ (using average velocity)	
Mass Flow Rate	$\dot{m} = \int \rho \bar{V} dA = \rho A \bar{V} = A\frac{\bar{V}}{v}$ (using average values)	kg/s
Power	$\dot{W} = \dot{m} C_p \Delta T \qquad \dot{W} = \dot{m} C_v \Delta T \qquad \dot{m} = \dot{V}/v$	W
Flow Work Rate	$\dot{W}_{flow} = P\dot{V} = \dot{m}Pv$	
Flow Direction	From higher P to lower P unless significant KE or PE	
• **Total Enthalpy**	$h_{tot} = h + \frac{1}{2}\bar{V}^2 + gZ$	
Instantaneous Process		
• **Continuity Equation**	$\dot{m}_{C.V.} = \sum \dot{m}_i - \sum \dot{m}_e$	
• **Energy Equation**	$\dot{E}_{C.V.} = \dot{Q}_{C.V.} - \dot{W}_{C.V.} + \sum \dot{m}_i h_{tot\,i} - \sum \dot{m}_e h_{tot\,e}$ → First Law $\rightarrow \dot{Q} + \sum \dot{m}_i (h_i + \frac{1}{2}\bar{V}^2 + gZ_i) = \frac{dE}{dt} + \sum \dot{m}_e \left(h_e + \frac{1}{2}\bar{V}^2 + gZ_e\right) - \dot{W}$	
Steady State Process	A steady-state has no storage effects, with all properties constant with time	
• **No Storage**	$\dot{m}_{C.V.} = 0, \quad \dot{E}_{C.V.} = 0$	
• **Continuity Equation**	$\sum \dot{m}_i = \sum \dot{m}_e$ (in = out)	
• **Energy Equation**	$\dot{Q}_{C.V.} + \sum \dot{m}_i h_{tot\,i} = \dot{W}_{C.V.} + \sum \dot{m}_e h_{tot\,e}$ (in = out) → First Law $\rightarrow \dot{Q} + \sum \dot{m}_i (h_i + \frac{1}{2}\bar{V}^2 + gZ_i) = \dot{W} + \sum \dot{m}_e \left(h_e + \frac{1}{2}\bar{V}^2 + gZ_e\right)$	
• **Specific Heat Transfer**	$q = \frac{\dot{Q}_{C.V.}}{\dot{m}}$	kJ/kg
• **Specific Work**	$w = \frac{\dot{W}_{C.V.}}{\dot{m}}$	kJ/kg
• **SS Single Flow Eq.**	$q + h_{tot\,i} = w + h_{tot\,e}$ (in = out)	
Transient Process	Change in mass (storage) such as filling or emptying of a container.	

• **Continuity Equation**	$m_2 - m_1 = \sum m_i - \sum m_e$
• **Energy Equation**	$E_2 - E_1 = Q_{C.V} - W_{C.V.} + \sum m_i h_{tot\,i} - \sum m_e h_{tot\,e}$ $E_2 - E_1 = m_2\left(u_2 + \frac{1}{2}\bar{V}_2^2 + gZ_2\right) - m_1\left(u_1 + \frac{1}{2}\bar{V}_1^2 + gZ_1\right)$

$$\rightarrow \quad Q_{C.V} + \sum m_i h_{tot\,i} = \sum m_e h_{tot\,e} + \left[m_2\left(u_2 + \frac{1}{2}\bar{V}_2^2 + gZ_2\right) - m_1\left(u_1 + \frac{1}{2}\bar{V}_1^2 + gZ_1\right)\right]_{C.V.} - W_{C.V.}$$

The Second Law of Thermodynamics

	Formula	Units
	All W, Q can also be rates $\dot{W}$, $\dot{Q}$	
Heat Engine	$W_{HE} = Q_H - Q_L$	
• **Thermal efficiency**	$\eta_{HE} = \frac{W_{HE}}{Q_H} = 1 - \frac{Q_L}{Q_H}$	
• **Carnot Cycle**	$\eta_{Thermal} = 1 - \frac{Q_L}{Q_H} = 1 - \frac{T_L}{T_H}$	
• **Real Heat Engine**	$\eta_{HE} = \frac{W_{HE}}{Q_H} \le \eta_{Carnot\ HE} = 1 - \frac{T_L}{T_H}$	
Heat Pump	$W_{HP} = Q_H - Q_L$	
• **Coefficient of Performance**	$\beta'_{HP} = \frac{Q_H}{W_{HP}} = \frac{Q_H}{Q_H - Q_L}$	
• **Carnot Cycle**	$\beta'_{HP} = \frac{Q_H}{Q_H - Q_L} = \frac{T_H}{T_H - T_L}$	
• **Real Heat Pump**	$\beta_{HP} = \frac{Q_H}{W_{HP}} \le \beta_{Carnot\ HP} = \frac{T_H}{T_H - T_L}$	
Refrigerator	$W_{REF} = Q_H - Q_L$	
• **Coefficient of Performance**	$\beta_{REF} = \frac{Q_L}{W_{REF}} = \frac{Q_L}{Q_H - Q_L}$	
• **Carnot Cycle**	$\beta = \frac{Q_L}{Q_H - Q_L} = \frac{T_L}{T_H - T_L}$	
• **Real Refrigerator**	$\beta_{REF} = \frac{Q_L}{W_{REF}} \le \beta_{Carnot\ REF} = \frac{T_L}{T_H - T_L}$	
Absolute Temperature	$\frac{T_L}{T_H} = \frac{Q_L}{Q_H}$	

Entropy

	Formula	Units
Inequality of Clausis	$\oint \frac{\delta Q}{T} \leq 0$	
Entropy	$dS \equiv \left(\frac{\delta Q}{T}\right)_{rev}$	kJ/kgK
Change of Entropy	$S_2 - S_1 = \int_1^2 \left(\frac{\delta Q}{T}\right)_{rev}$	kJ/kgK
Specific Entropy	$s = (1-x)s_f + xs_g$ $s = s_f + xs_{fg}$	kJ/kgK
Entropy Change		
• **Carnot Cycle**	Isothermal Heat Transfer: $S_2 - S_1 = \frac{1}{T_H}\int_1^2 \delta Q = \frac{{}_1Q_2}{T_H}$	
	Reversible Adiabatic (Isentropic Process): $dS = \left(\frac{\delta Q}{T}\right)_{rev}$	
	Reversible Isothermal Process: $S_4 - S_3 = \int_3^4 \left(\frac{\delta Q}{T}\right)_{rev} = \frac{{}_3Q_4}{T_L}$	
	Reversible Adiabatic (Isentropic Process): Entropy decrease in process 3-4 = the entropy increase in process 1-2.	
• **Reversible Heat-Transfer Process**	$s_2 - s_1 = s_{fg} = \frac{1}{m}\int_1^2 \left(\frac{\delta Q}{T}\right)_{rev} = \frac{1}{mT}\int_1^2 \delta Q = \frac{{}_1q_2}{T} = \frac{h_{fg}}{T}$	
Gibbs Equations	$Tds = du + Pdv$ $Tds = dh - vdP$	
Entropy Generation	$dS = \frac{\delta Q}{T} + \delta S_{gen}$ $\delta W_{irr} = PdV - T\delta S_{gen}$ $S_2 - S_1 = \int_1^2 dS = \int_1^2 \frac{\delta Q}{T} + {}_1S_{2\,gen}$	
Entropy Balance Equation	$\vartriangle Entropy = +in - out + gen$	
Principle of the Increase of Entropy	$dS_{net} = dS_{c.m.} + dS_{surr} = \sum \delta S_{gen} \geq 0$	
Entropy Change		
• **Solids & Liquids**	$s_2 - s_1 = c\ln\frac{T_2}{T_1}$ Reversible Process: $ds_{gen} = 0$ Adiabatic Process: $dq = 0$	

• **Ideal Gas**	Constant Volume: $s_2 - s_1 = \int_1^2 Cv_0 \frac{dT}{T} + R\ln {v_2}/{v_1}$ Constant Pressure: $s_2 - s_1 = \int_1^2 Cp_0 \frac{dT}{T} - R\ln {P_2}/{P_1}$ Constant Specific Heat: $s_2 - s_1 = Cv_0 \ln {T_2}/{T_1} + R\ln {v_2}/{v_1}$ $s_2 - s_1 = Cp_0 \ln {T_2}/{T_1} - R\ln {P_2}/{P_1}$
Standard Entropy	$s_T^0 = \int_{T_0}^{T} \frac{C_{p0}}{T} dT$ kJ / kgK
Change in Standard Entropy	$s_2 - s_1 = \left(s_{T2}^0 - s_{T1}^0\right) - R\ln {P_2}/{P_1}$ kJ / kgK
Ideal Gas Undergoing an Isentropic Process	$s_2 - s_1 = 0 = Cp_0 \ln {T_2}/{T_1} - R\ln {P_2}/{P_1}$ $\rightarrow \frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{R/Cp0}$ but $\frac{R}{C_{p0}} = \frac{C_{p0} - C_{v0}}{C_{p0}} = \frac{k-1}{k}$, $k = \frac{C_{p0}}{C_{v0}}$ = ratio of specific heats $\Rightarrow \frac{T_2}{T_1} = \left(\frac{v_1}{v_2}\right)^{k-1}$, $\frac{P_2}{P_1} = \left(\frac{v_1}{v_2}\right)^{k}$ Special case of polytropic process where k = n: $Pv^k = const$
Reversible Polytropic Process for Ideal Gas	$PV^n = const = P_1V_1^n = P_2V_2^n$ $\rightarrow \frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^n$, $\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{n-1/n} = \left(\frac{V_1}{V_2}\right)^{n-1}$
• **Work**	${}_1W_2 = \int_1^2 PdV = const\int_1^2 \frac{dV}{V^n} = \frac{P_2V_2 - P_1V_1}{1-n} = \frac{mR(T_2 - T_1)}{1-n}$
• **Values for n**	Isobaric process: $n = 0,\quad P = const$ Isothermal Process: $n = 1,\quad T = const$ Isentropic Process: $n = k,\quad s = const$ Isochronic Process: $n = \infty,\quad v = const$

Second-Law Analysis for Control Volume

	Formula	Units
2nd Law Expressed as a Change of Entropy	$\frac{dS_{c.m.}}{dt} = \sum \frac{\dot{Q}}{T} + \dot{S}_{gen}$	
Entropy Balance Equation	$rate\,of\;change = +in - out + generation$ $\rightarrow \frac{dS_{C.V.}}{dt} = \sum \dot{m}_i s_i - \sum \dot{m}_e s_e + \sum \frac{\dot{Q}_{C.V.}}{T} + \dot{S}_{gen}$ where $S_{C.V.} = \int \rho s dV = m_{c.v.} s = m_A s_A + m_B s_B + \ldots$ and $\dot{S}_{gen} = \int \rho \dot{s}_{gen} dV = \dot{S}_{gen.A} + \dot{S}_{gen.B} + \ldots$	
Steady State Process	$\frac{dS_{C.V.}}{dt} = 0$ $\rightarrow \sum \dot{m}_e s_e - \sum \dot{m}_i s_i = \sum_{C.V.} \frac{\dot{Q}_{C.V.}}{T} + \dot{S}_{gen}$	
• **Continuity equation**	$\dot{m}_i = \dot{m}_e = \dot{m} \qquad \Rightarrow \dot{m}(s_e - s_i) = \sum_{C.V.} \frac{\dot{Q}_{C.V.}}{T} + \dot{S}_{gen}$	
• **Adiabatic process**	$s_e = s_i + s_{gen} \geq s_i$	
Transient Process	$\frac{d}{dt}(ms)_{C.V.} = \sum \dot{m}_i s_i - \sum \dot{m}_e s_e + \sum \frac{\dot{Q}_{C.V.}}{T} + \dot{S}_{gen}$ $\rightarrow (m_2 s_2 - m_1 s_1)_{C.V.} = \sum m_i s_i - \sum m_e s_e + \int_0^t \frac{\dot{Q}_{C.V.}}{T} dt + {}_1\dot{S}_{2gen}$	
	Reversible Steady State Process	
If Process Reversible & Adiabatic	$s_e = s_i$ $h_e - h_i = \int_i^e vdP$ $w = (h_i - h_e) + \frac{\bar{V}_i^2 - \bar{V}_e^2}{2} + g(Z_i - Z_e)$ $= -\int_i^e vdP + \frac{\bar{V}_i^2 - \bar{V}_e^2}{2} + g(Z_i - Z_e)$	
If Process is Reversible and Isothermal	$\dot{m}(s_e - s_i) = \frac{1}{T}\sum_{C.V.} \dot{Q}_{C.V.} = \frac{Q_{C.V.}}{T}$ or $T(s_e - s_i) = \frac{\dot{Q}_{C.V.}}{\dot{m}} = q$ $\rightarrow T(s_e - s_i) = (h_e - h_i) - \int_i^e vdP$	

Incompressible Fluid	$v\left(P_e - P_i\right) + \frac{\bar{V}_e^2 - \bar{V}_i^2}{2} + g\left(Z_e - Z_i\right) = 0$ → Bernoulli Eq.
Reversible Polytrophic Process for Ideal Gas	$w = -\int_i^e vdP \quad and \quad Pv^n = const = C^n$ $w = -\int_i^e vdP = -C\int_i^e \frac{dP}{P^{1/n}}$ $= -\frac{n}{n-1}\left(P_e v_e - P_i v_i\right) = -\frac{nR}{n-1}\left(T_e - T_i\right)$
Isothermal Process (n=1)	$w = -\int_i^e vdP = -C\int_i^e \frac{dP}{P} = -P_i v_i \ln\frac{P_e}{P_i}$
Principle of the Increase of Entropy	$\frac{dS_{net}}{dt} = \frac{dS_{C.V.}}{dt} + \frac{dS_{surr}}{dt} = \sum \dot{S}_{gen} \geq 0$
Efficiency	
Turbine	$\eta = \frac{w_a}{w_s} = \frac{h_i - h_e}{h_i - h_{es}}$ Turbine work is out
Compressor (Pump)	$\eta = \frac{w_s}{w_a} = \frac{h_i - h_{es}}{h_i - h_e}$ Compressor work is in
Cooled Compressor	$\eta = \frac{w_T}{w}$
Nozzle	$\eta = \frac{1/2\,\bar{V}_e^2}{1/2\,\bar{V}_{es}^2}$ Kinetic energy is out

Note:

°F = (°C x 9/5) + 32

°C = (°F - 32) x (5/9)

°K = °C + 273

Q = mCΔT thermal energy = mass x specific heat x change in T

Q = mHf thermal energy = mass x heat of fusion

Q = mHv thermal energy = mass x heat of vaporization

ΔL = αLiΔT change in length = coefficient of expansion x initial length x change in T

ΔV = βViΔT change in volume = coefficient of expansion x initial volume x change in T

ΔU = Q – W internal energy = heat energy - work

Plausible Physical Situations		Name	State Variables	P	V	T	ΔE_{th}	Q	W_s	W
Insulated sleeve or rapid process	Add weight to or push down on piston	Adiabatic compression	PV^{γ} = Const; $TV^{\gamma-1}$ = Const	Up	Down	Up	$nC_v\Delta T > 0$	0	$-nC_v\Delta T < 0$	$nC_v\Delta T > 0$
Insulated sleeve or rapid process	Remove weight from or pull up on piston	Adiabatic expansion	PV^{γ} = Const; $TV^{\gamma-1}$ = Const	Down	Up	Down	$nC_v\Delta T < 0$	0	$-nC_v\Delta T > 0$	$nC_v\Delta T < 0$
Heat gas	Locked piston or rigid container	Isochoric	V fixed; $P \propto T$	Up	Fixed	Up	$nC_v\Delta T > 0$	$nC_v\Delta T > 0$	0	0
Cool gas	Locked piston or rigid container	Isochoric	V fixed; $P \propto T$	Down	Fixed	Down	$nC_v\Delta T < 0$	$nC_v\Delta T < 0$	0	0
Heat gas	Piston free to move, load unchanged	Isobaric expansion	P fixed; $V \propto T$	Fixed	Up	Up	$nC_v\Delta T > 0$	$nC_p\Delta T > 0$	$P\Delta V > 0$	$-P\Delta V < 0$
Cool gas	Piston free to move, load unchanged	Isobaric compression	P fixed; $V \propto T$	Fixed	Down	Down	$nC_v\Delta T < 0$	$nC_p\Delta T < 0$	$P\Delta V < 0$	$-P\Delta V > 0$
Immerse gas in large bath	Add weight to piston	Isothermal compression	T fixed at temperature of bath, PV = Const	Up	Down	Fixed	$nC_v\Delta T = 0$	$nRT*\ln(V_f/V_i) < 0$	$nRT*\ln(V_f/V_i) < 0$	$-nRT*\ln(V_f/V_i) > 0$
Immerse gas in large bath	Remove weight from piston	Isothermal expansion	T fixed at temperature of bath, PV = Const	Down	Up	Fixed	$nC_v\Delta T = 0$	$nRT*\ln(V_f/V_i) > 0$	$nRT*\ln(V_f/V_i) > 0$	$-nRT*\ln(V_f/V_i) < 0$
Unknown	Unknown	No Name	PV/T = Const	?	?	?	$nC_v\Delta T$	$\Delta E_{th} + W_s$	$\int PdV = \pm$ area under curve in PV diagram	$-\int PdV = \pm$ area under curve in PV diagram

I.C Engine

- **Mean Effective Pressure MEP:-** $W_{net}=P_{MEP}\Delta V$

  $=MEP \times Piston\ area \times Stroke$

  $= MEP \times Displacement$

  $MEP= W_{net}/Displacement= W_{net}/V_{max}-V_{min}$

- **Otto Cycle Efficiency**:- $\eta_{th,Otto} = 1 - (1/r^{k-1})$

  where r = compression ratio k = cp/cv for air at room T

- **Diesel Efficiency:-** $\eta th,_{Diesel} = 1 -(1/r^{k}-1)[(r_c)^k - 1/k(r_c-1)]$

  r = compression ratio, r_c = cutoff ratio (r_c = V3/V2 = v3/v2) k = c_p/c_v for air at room T.

- **Indicated Thermal Efficiency:-** $\acute{\eta}_{indi} = \frac{Indicated\ Power}{Energy\ in\ fuel\ per\ second} = \frac{I_p}{m_f \times C_v}$

  Here, I_p = Indicated power, C_v = Calorific value of fuel, and M_f = Mass of fuel/second

- **Brake Thermal Efficiency:-** $\acute{\eta}_{\text{Brake}} = \frac{\text{Brake Power}}{\text{Energy in fuel per second}} = \frac{b_p}{m_f \times C_v}$

Psychrometry

- **Partial Pressure of Water Vapour:-**

$$p_v = (p_{vs})_{wb} \frac{-[p-(p_v)_{wb}](t_{db}-t_{wb})}{1527.4-1.3\,t_{wb}}$$

$$or\ p_v = (p_{vs})_{wb} \frac{1.8 \times p(t_{db}-t_{wb})}{2700}$$

p_v = partial pressure of water vapour

p_s = partial pressure of dry air

where, $(p_{vs})_{wb}$ = saturation pressure of water vapour corresponding to wet bulb temperature

p = atmospheric pressure of moist air

T_{wb} = wet bulb temperature

T_{db} = dry bulb temperature

- **Absolute Humidity:-**

$$= \frac{\text{Weight of Water Vapour}}{\text{Volume of Air(Mixture)}}$$

- **Specific Humidity(w):-**

$$= \frac{\text{Mass of water vapour in air}}{\text{Mass of Dry Air}}$$

$$\omega = 0.622 \frac{P_V}{P-P_V} \text{Kg water vapour/kg of dry air}$$

- **Relative Humidity :-**

$$\emptyset = \frac{m_v}{m_s} = \frac{P_v}{P_{vs}}$$

- **Degree of Saturation (μ):-**

$$\mu = \frac{\text{Specific humidity of air}}{\text{Specific humidity of Saturated air}} = \frac{\omega}{\omega_s}$$

- **Dew Point Depression:-**

DPD = DBT – DPT

- **Enthalpy of Air (*h*):-**

$$\mathbf{h = C_{pm}\, t_{ab} + w(h_g - C_{pv}\, t_{db})}$$

- **Sensible Heat Factor-**

$$= \frac{S_H}{S_H + S_L}$$

- **By Pass Factor:-**

$$m = \frac{t_2 - t_{Coil}}{t_1 - t_{Coil}}$$

Air temperature at entry and exit is t_1 and t_2 respectively

- **Volumetric Efficiency :-**

$$\eta_V = 1 + c - c(r_p)^{1/\gamma}$$

$$\text{C = Compression ratio}, r_p = p_2/p_1$$

- **Vapour Absorption System C.O.P**

$$C.O.P = \left(\frac{T_e}{T_c - T_e}\right) \times \left(\frac{T_g - T_c}{T_g}\right)$$

T_e=Evaporator Temperature
T_g=Generator Temperature
T_C= Condenser Temperature

- **Designation of Refrigerant:-**

For a hydrocarbon Chemical formula is $C_r H_s F_1 CI_y$

If $S+t+y=2r+2$, designation of refrigerant is $R(r-1)(S+1)t$

It $S+t+y=2r$, designation of refrigerant is $R1(r-1)(S+1)t$

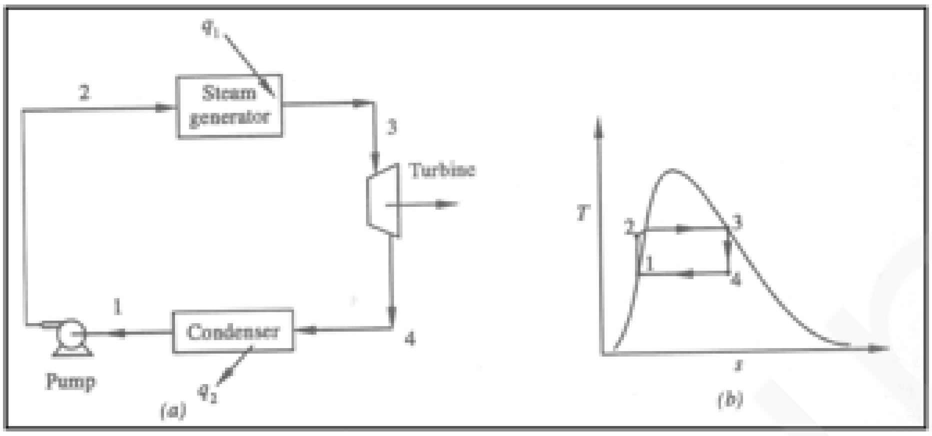
For an Inorganic Refrigerant

Designation is R(700 + molecular weight)

Power Plant Engineering

- **Ranking Cycle efficiency-**

$$\eta = \frac{q_1 - q_2}{q_1} = \frac{(h_3 - h_2) - (h_4 - h_1)}{h_3 - h_2} = \frac{(h_3 - h_4) - (h_2 - h_1)}{h_3 - h_2}$$



- **Overall efficiency:-**

  =Eff. Boiler x Eff. of Cycle x Eff. of turbine x Eff. of generator x Eff. of auxiliary

- **Efficiency of Brayton Cycle:-**

$$\eta_{th,Brayton} = 1 - \frac{1}{r_p^{(\gamma-1)/\gamma}}$$

Where r_p is pressure ratio

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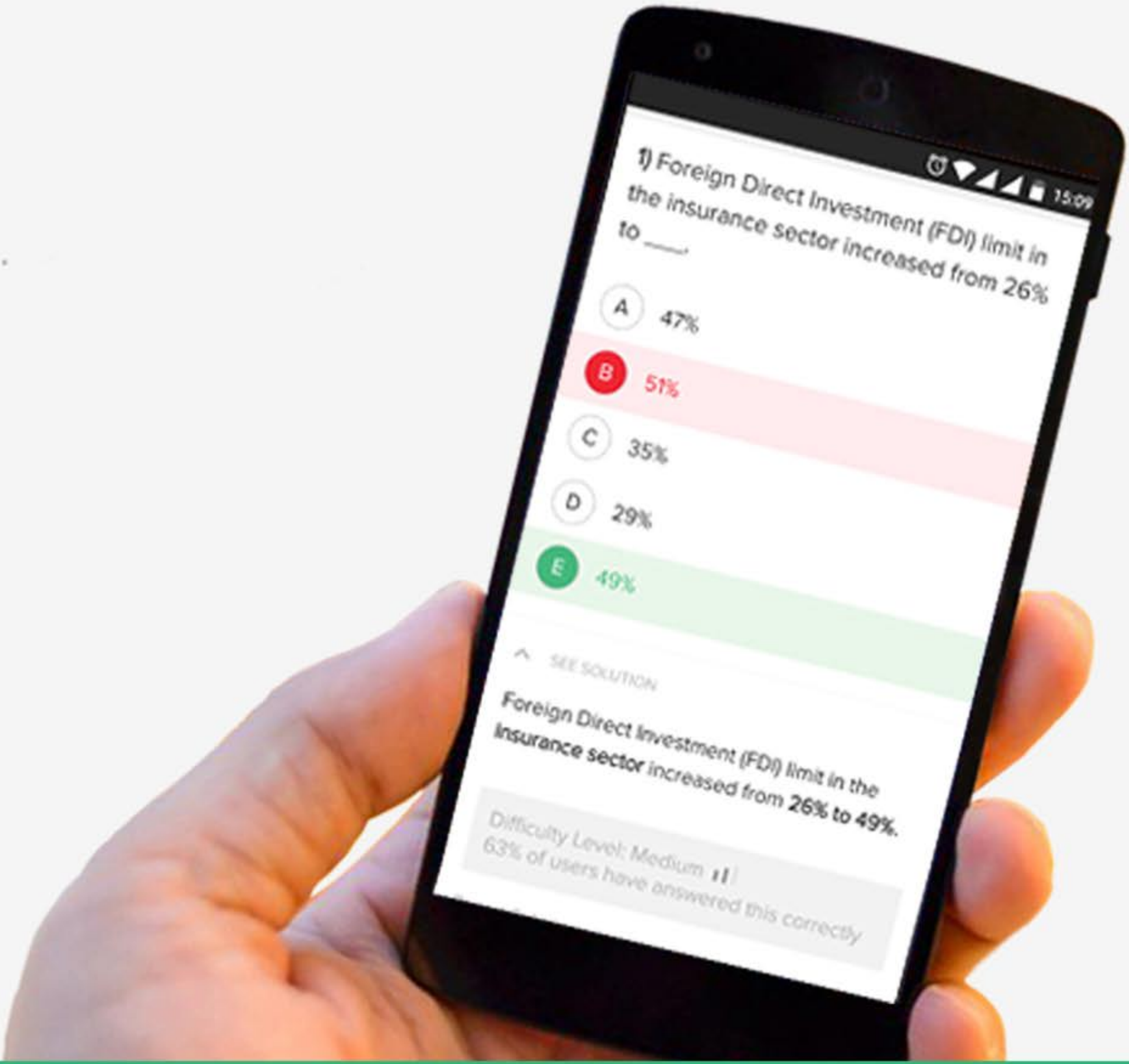
Formulas on
MACHINE DESIGN
for GATE ME Exam



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Short Notes on Machine Design

Static Load

- A static load is a mechanical force applied slowly to an assembly or object.Load does not change in magnitude and direction and normally increases gradually to a steady value
- This force is often applied to engineering structures on which peoples' safety depends on because engineers need to know the maximum force a structure can support before it will collapse.

Dynamic load

- A dynamic load, results when loading conditions change with time. Load may change in magnitude for example, traffic of varying weight passing a bridge.
- Load may change in direction, for example, load on piston rod of a double acting cylinder. Vibration and shock are types of dynamic loading.

Factor of safety (F.O.S):

- The ratio of ultimate to allowable load or stress is known as factor of safety i.e. The factor of safety can be defined as the ratio of the material strength or failure stress to the allowable or working stress.
- The factor of safety must be always greater than unity. It is easier to refer to the ratio of stresses since this applies to material properties.

F.O.S = failure stress / working or allowable stress

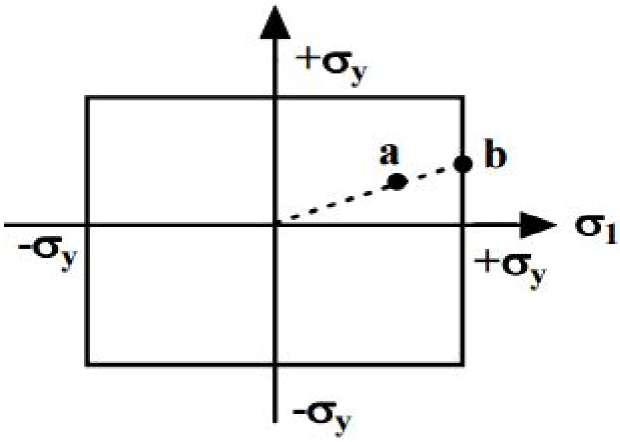
Static Failure Theories

Maximum Principal Stress Theory (Rankine Theory):

- The principal stresses ***σ1 (maximum principal stress)***, ***σ2 (minimum principal stress)*** or **σ3** exceeds the yield stress, yielding would occur.
- For two dimensional loading situation for a ductile material where tensile and compressive yield stress are nearly of same magnitude:

$$\sigma_1 = \pm \sigma_y$$

$$\sigma_2 = \pm \sigma_y$$



- Yielding occurs when the state of stress is at the boundary of the rectangle.

Maximum Principal Strain Theory (St. Venant's theory):

- If ε1 and ε2 are maximum and minimum principal strains corresponding to σ1 and σ2, in the limiting case:

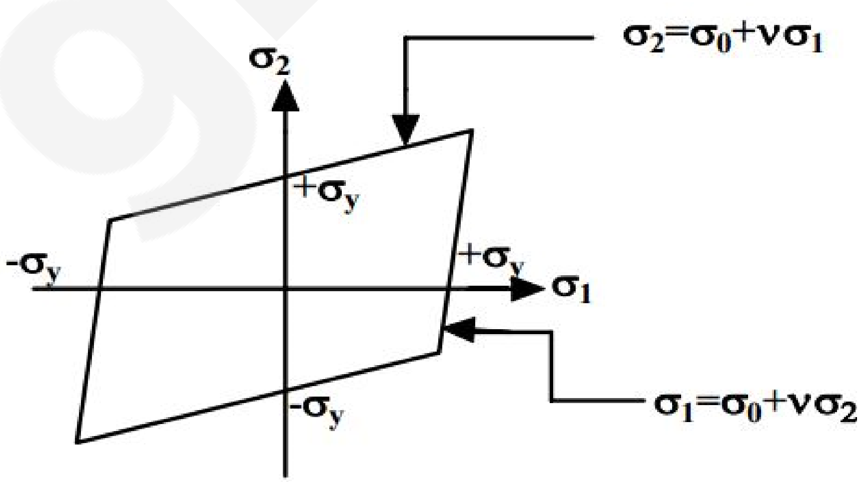
$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) \qquad |\sigma_1| \geq |\sigma_2|$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) \qquad |\sigma_2| \geq |\sigma_1|$$

$$E\varepsilon_1 = \sigma_1 - \nu\sigma_2 = \pm\sigma_0$$

$$E\varepsilon_2 = \sigma_2 - \nu\sigma_1 = \pm\sigma_0$$

- Boundary of a yield surface in Maximum Strain Energy Theory is given below



Maximum Shear Stress Theory (Tresca Theory):

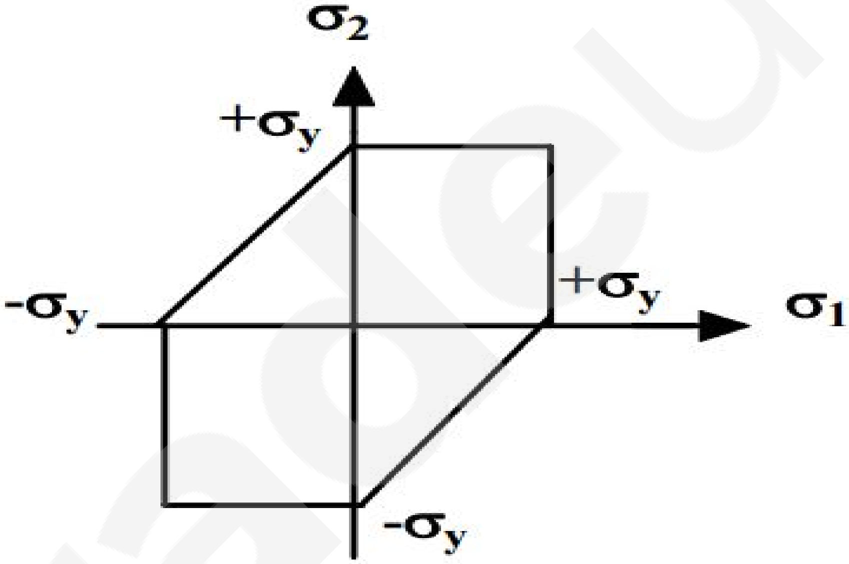
- At the tensile yield point σ2= σ3 = 0 and thus maximum shear stress is σy/2.

$$\sigma_1 - \sigma_2 = \pm\sigma_y$$

$$\sigma_2 - \sigma_3 = \pm\sigma_y$$

$$\sigma_3 - \sigma_1 = \pm\sigma_y$$

- Yield surface corresponding to maximum shear stress theory in biaxial stress situation is given below :



Maximum strain energy theory (Beltrami's theory):

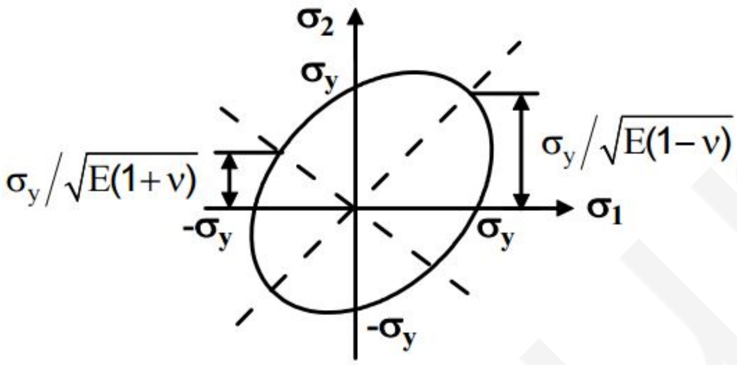
- Failure would occur when the total strain energy absorbed at a point per unit volume exceeds the strain energy absorbed per unit volume at the tensile yield point.

$$\frac{1}{2}\left(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3\right) = \frac{1}{2}\sigma_y\varepsilon_y$$

$$\sigma_1^{\,2} + \sigma_2^{\,2} + \sigma_3^{\,2} - 2\upsilon\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) = \sigma_y^{\,2}$$

$$\left(\frac{\sigma_1}{\sigma_y}\right)^2 + \left(\frac{\sigma_2}{\sigma_y}\right)^2 - 2\nu\left(\frac{\sigma_1\sigma_2}{{\sigma_y}^2}\right) = 1$$

- Above equation results in Elliptical yield surface which can be viewed as:



Distortion energy theory (Von Mises yield criterion):

- Yielding would occur when total distortion energy absorbed per unit volume due to applied loads exceeds the distortion energy absorbed per unit volume at the tensile yield point. Total strain energy E_T and strain energy for volume change E_V can be given as:

$$E_T = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3) \quad \text{and } E_V = \frac{3}{2}\sigma_{av}\varepsilon_{av}$$

$$E_d = E_T - E_V = \frac{2(1+\nu)}{6E}\left({\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right)$$

At the tensile yield point, σ1 = σy , σ2 = σ3 = 0 which gives,

$$E_{dy} = \frac{2(1+\nu)}{6E}{\sigma_y}^2$$

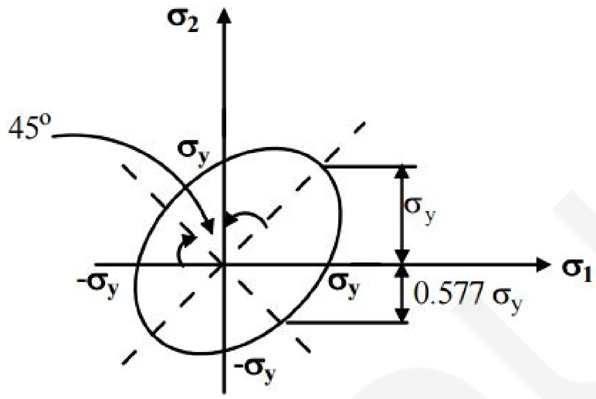
The failure criterion is thus obtained by equating Ed and Edy , which gives

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2$$

In a 2-D situation if σ3 = 0, so the equation reduces to,

$$\left(\frac{\sigma_1}{\sigma_y}\right)^2+\left(\frac{\sigma_2}{\sigma_y}\right)^2-\left(\frac{\sigma_1}{\sigma_y}\right)\left(\frac{\sigma_2}{\sigma_y}\right)=1$$

- This is an equation of ellipse and yield equation is an ellipse.
- This theory is widely accepted for ductile materials



Cotter and Knuckle Joints

A cotter joint is a temporary fastening and is used to connect rigidly two co-axial road or bars which are subjected to axial tensile or compressive forces.

Socket and Spigot Cotter Joints

In a socket and spigot cotter joint, one end of the rods is provided with a socket type of end as shown in figure and the other end of the rod is inserted into a socket. The end of the rod which goes into a socket is also called spigot.

Failures in Socket and Spigot Cotter Joints

Failure Cases	Tensile Force
Failure of the rod in tension	$P = \frac{\pi}{4} \times d^2 \times \sigma_t$
Failure of spigot in tension across the weakest section	$P = \left[\frac{\pi}{4}(d_2)^2 - d_2 \times t\right]\sigma_t$
Failure of the rod or cotter in crushing	$P = d_2 \times t \times \sigma_t$
Failure of the socket in tension across the slot	$P = \left\{\frac{\pi}{4}\left[(d_1)^2 - (d_2)^2\right] - (d_1 - d_2)\,t\right\}\sigma_t$
Failure of cotter in shear	$P = 2b \times t \times \tau$
Failure of the socket collar in crushing	$P = (d_4 - d_2)\,t \times \sigma_c$
Failure of rod end in shear	$P = 2(d_4 - d_2)\,c \times \tau$
Failure of spigot collar in crushing	$P = 2a \times d_2 \times \tau$
Failure of the spigot collar in shearing	$P = \frac{\pi}{4}\left[(d_3)^2 - (d_2)^2\right]\sigma_c$
	$P = \pi d_2 \times t_1 \times \tau$

Failures in Sleeve and Cotter Joints

Failure Cases	Tensile Force
Failure of the rod in tension	$P = \frac{\pi}{4} \times d^2 \times \sigma_t$
Failure of rod in tension across the weakest section	$P = \left[\frac{\pi}{4}(d_2)^2 - d_2 \times t\right]\sigma_t$
Failure of the rod or cotter in crushing	$P = d_2 \times t \times \sigma_t$
Failure of sleeve in tension across the slot	$P = \left\{\frac{\pi}{4}\left[(d_1)^2 - (d_2)^2\right] - (d_1 - d_2)\,t\right\}\sigma_t$
Failure of cotter in shear	$P = 2b \times t \times \tau$
Failure of rod end in shear	$P = 2a \times d_2 \times \tau$
Failure of sleeve end in shearing	$P = 2(d_1 - d_2)\,c \times \tau$

Knuckle Joint

- It is used to connect two rods whose axis either coincide or intersect and lie in one plane.
- This joint generally found in the link of a cycle chain tie rod joint for roof truss, valve rod joint with eccentric rod tension link in bridge structure, lever and rod connection of various types.
- It is sometimes also called forked pin joint.

Failures in Knuckle Joint

Failure Cases	Tensile Force
Failure of the solid rod in tension	$P = \frac{\pi}{4} \times d^2 \times \sigma_t$
Failure of the knuckle pin in shear	$P = 2\frac{\pi}{4}(d_2)^2 \tau$
Failure of the single eye or rod end in tension	$P = (d_2 - d_1)\, t \times \sigma_t$
Failure of the single eye or rod end in shearing	$P = (d_2 - d_1)\, t \times \tau$
Failure of forked end in tension	$P = (d_2 - d_1)\, 2t_1 \times \sigma_t$
Failure of the forked end in crushing	$P = d_1 \times 2t_1 \times \sigma_c$

- To connect the transmission shaft to rotating machine elements like pulley, gear, sprocket or flywheel.
- Cotter and knuckle joints are not used for connect

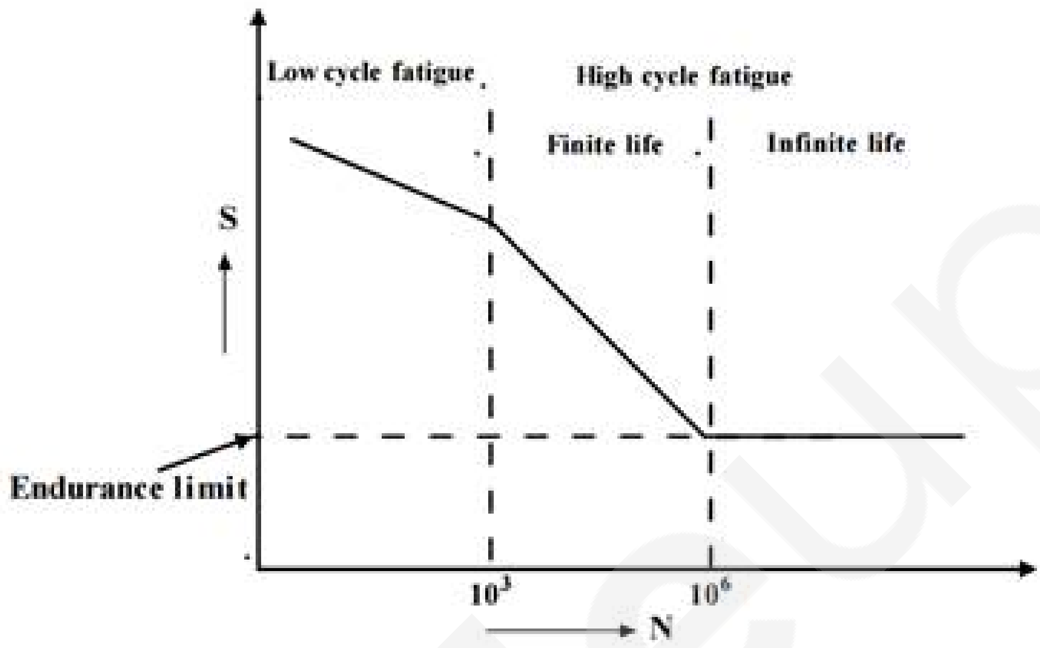
Fatigue

- Fatigue loading is primarily the type of loading which causes cyclic variations in the applied stress or strain on a component.
- Variable loading due to: -Change in the magnitude of applied load Example: punching or shearing operations- -Change in direction of load application Example: a connecting rod -Change in point of load application Example: a rotating shaft.

Fatigue Failure:

- Machine elements subjected to fluctuating stresses usually fail at stress levels much below their ultimate strength and in many cases below the yield point of the material too.
- These failures occur due to very large number of stress cycle and are known as fatigue failure.
- Fatigue failures are influenced by
  - Nature and magnitude of the stress cycle

- Endurance limit
- Stress concentration
- Surface characteristics



Riveted joints:

- There are two basic components of riveted joints:
  - Rivets
  - Two or more plates.
- The popular materials for the rivets are: *Steel, Brass, Aluminium* & *Copper* as per the requirement of the application for fluid tight joints the steel rivets are used

Welded Joints

- It is a permanent joint.
- When the two parts are joined by heating to a suitable temperature with or without application of pressure.

Welding Processes

- **Fusion Welding**
- **Thermit Welding**
- **Gas Welding**
- **Electric Arc Welding**
- **Forge Welding**

Types of Welded Joints:

Lap Joint or Fillet Joint

- In lap joint, overlapping the plate and welding the edge of the plates takes place in welding process.
- The strength of different types of fillet joint can be given according to their welding process as

- **Shear strength in parallel fillet weld**,

$$\tau = \frac{P}{0.707\,hl} \text{ or } P = 0.707\,h/\tau$$

where, *P* = Tensile force on the plates

h = Leg of the weld

l = Length of the weld

τ = Permissible shear stress

- For double parallel fillet weld,

$$P = 1.414\,h/\tau$$

- **Strength of Transverse Fillet Weld**
    - *p* = Throat area Allowable tensile stress
    - = 0.707 $s \times l \times \sigma_t$
- For double transverse fillet joint

$$P = 1.414\,hl\sigma_t$$

Special Cases of Fillet Welded Joint

- **Circular Fillet Weld Subjected to Torsion**
    - Shear stress $\tau = \dfrac{2T}{\pi t d^2}$

or $\tau_{max} = \dfrac{2.83T}{\pi h d^2}$

where, *T* = Torque acting or rod

h = size of weld

t = Throat thickness

Circular Fillet Weld Subjected to Bending Moment

- **Bending stress:**

$$\sigma_b = \frac{4M}{\pi t d^2}$$

$$\sigma_{b\,(\max)} = \frac{5.66\,M}{\pi h d^2}$$

- **Long Fillet Weld Subjected to Torsion**
  - Shear stress: $\tau = \frac{3T}{tl^2}$

$$\tau_{\max} = \frac{4.242\,T}{hl^2}$$

Butt Joint

- **Strength of Butt Joint**
  - For single V – butt joint,

$$\mathbf{P = t \times l \times \sigma_t}$$

  - For double V-butt joint,

$$\mathbf{P = (t_1 + t_2) l \times \sigma_t}$$

Eccentric Loaded Welded Joints

When the shear and bending stresses are simultaneously present in a joint.

- Maximum normal stress

$$\sigma_{t_{\max}} = \frac{\sigma_b}{2} + \frac{1}{2}\sqrt{(\sigma_b)^2 + 4\tau^2}$$

- Maximum shear stress

$$\tau_{\max} = \frac{1}{2}\sqrt{(\sigma_b)^2 + 4\tau^2}$$

- Direct or primary shear stress

$$\tau_1 = \frac{\text{Load}}{\text{Throat area}} = \frac{P}{2tl}$$

$$\tau_1 = \frac{P}{1.414\,hl}$$

Strength of Bolted Joint

- Maximum tensile stress in the bolt

$$\sigma_t = \frac{P}{\left(\frac{\pi}{4} d_c^2\right)}$$

where, d_c = Core diameter

$$\frac{\sigma_{yt}}{f_s} = \frac{P}{\frac{\pi}{4} d_c^2}$$

Torque Requirement for Bolt Tightening

$$M_t = \frac{P_i\, d_m}{2}\left(\frac{\mu \sec\theta + \tan\alpha}{1 - \mu \sec\theta \tan\alpha}\right)$$

where,

Pi = Pretension in bolt, d_m = 0.9 *d*

d = Nominal diameter

For ISO metric screw thread ϑ = 30°

For ISO metric α = 25°

Eccentric Load on Bracket with Circular Base

- If there are n number of bolts, then load in a bolt

$$w_{b_1} = \frac{2wL\,(R - r\cos\alpha)}{n\,(2R^2 + r^2)}$$

- In above case when *n* = 4

$$w_b = \frac{w.L\,(R - \alpha\cos\alpha)}{2\,(2R^2 + r^2)}$$

Maximum load in bolt

$$\left(w_b\right)_{max} = \frac{2wL}{n}\left(\frac{R+r}{2R^2+r^2}\right)$$

where, cos α = -1

Factor of Safety (FOS) in Bolted Joints

- It is defined as the ratio of failure stress to allowable stress.

$$\text{FOS} = \frac{\text{Failure stress}}{\text{Allowable stress}}$$

- For ductile material,

$$\text{FOS} = \frac{S_{yt}}{\sigma}$$

- For brittle material,

$$\text{FOS} = \frac{S_{ut}}{\sigma}$$

where, Syt = Yield strength of component material

Sut = Ultimate tensile stress of components material

σ = Allowable stress.

Stress concentration Factor

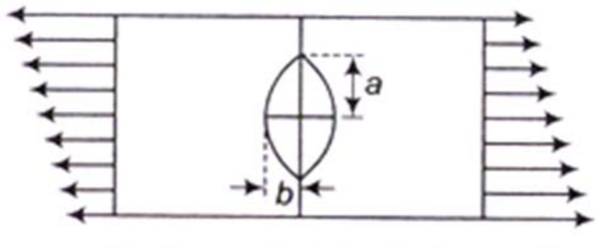
- It is defined as the ratio of highest value of actual stress near discontinuity to nominal stress obtained by elementary equations for minimum cross-section. It is denoted by k_t.

$$k_t = \frac{\text{Highest value of actual stress near discontinuity}}{\text{Nominal stress obtained by elementary equations}}$$

$$or\ k_t = \frac{\sigma_{max}}{\sigma_0} = \frac{\tau_{max}}{\tau_0}$$

where, σ_0, τ_0 = Nominal stresses

- The magnitude of stress concentration factor depends upon the geometry of the component.
- In this case, k_t = 1+2(a/b)



Design against fluctuating load

where,

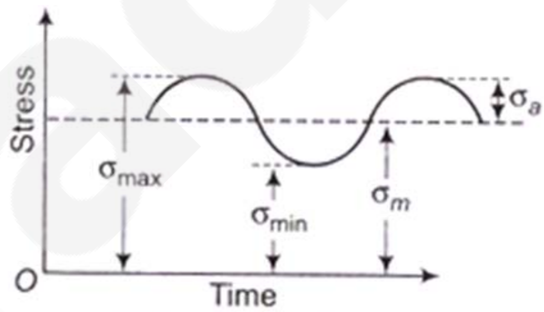
a = Semi-axis of ellipse perpendicular to the direction of load

b = Semi-axis of ellipse parallel to the direction of load

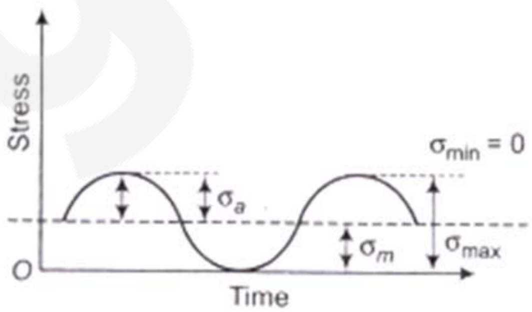
- If b = 0 then, hole is like as very sharp crack then, $k_t = \infty$
- If a = b then, hole becomes a circular hole then, k_t = 1+2=3

Fluctuating Load

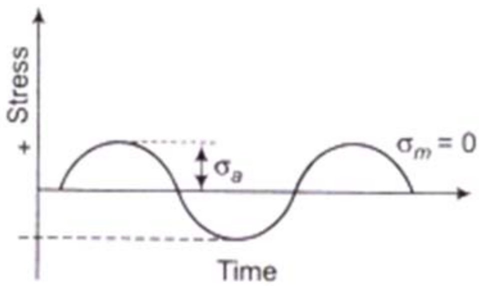
- It is defined as the load, of which magnitude and direction both changes with respect to time.



(a) Fluctuating load



(b) Repeated load



σ_m = Mass stress, σ_a = Stress amplitude

- Mass stress and stress amplitude

$$\sigma_m = \frac{1}{2}(\sigma_{max} + \sigma_{min})$$

$$\sigma_a = \frac{1}{2}(\sigma_{max} - \sigma_{min})$$

- For repeated stress, $\sigma_{min} = 0, \sigma_m = \frac{\sigma_{max}}{2}$ and $\sigma_a = \frac{\sigma_{max}}{2}$
- For reversed stress, $\sigma_m = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = 0$

$$\sigma_a = \frac{1}{2}(\sigma_{max} + \sigma_{max}) = \sigma_{max}$$

Gears

Gear can be defined as the mechanical element used for transmitted power and rotary motion from one shaft to another by means of progressive engagement of projections called teeth.

Classification of Gears

- Spur Gear
- Helical Gear
- Bevel Gear
- Worm Gear

Spur Gear

In spur gears, teeth are cut parallel to axis of the gear.

- Circular pitch

$$P = \frac{\pi d}{z}$$

- Diametrical pitch

$$P = \frac{z}{d}$$

- Module

$$m = \frac{d}{z}$$

- Torque transmitted by gear

$$M_t = \frac{60 \times 10^6 (kW)}{2\pi n}$$

- **Dynamic load or incremental dynamic load**

$$P_d = \frac{21v(ceb + P_t)}{21v + \sqrt{ceb + P_t}}$$

Where, v = Pitch line velocity

c = Deformation factor

b = Face width of tooth

P_t = Tangential force due to rated torque. e = Sum of errors between two meshing teeth

- **Estimation of module based on beam strength**

$$m = \left[\frac{60 \times 10^6}{\pi} \left\{ \frac{(kw) c_s (f_s)}{z n c_v \left(\frac{b}{m} \right) \left(\frac{S_w}{3} \right) Y} \right\} \right]^{-1/3}$$

Where, c_s = Service factor,

c_v = Velocity factor

f_s = Factor of safety,

n = Speed (rpm)

- **Estimation of module based on wear strength**

$$m = \left[\frac{60 \times 10^6}{\pi} \left\{ \frac{(kw) c_s (f_s)}{z_p^2 n_p c_v \left(\frac{b}{m} \right) pK} \right\} \right]^{-1/3}$$

Helical Gear

- The teeth of helical gear cut in the form of helix or an angle on the pitch cylinder.

$$P_n = \frac{P}{\cos \psi}$$

Where, P_n = Normal diametrical pitch

P = Transverse diametrical pitch

Ψ = Helix angle

m_n = m cos ψ

m_n = Normal module

m = transverse module

- Axial pitch

$$P_a = \frac{P}{\tan \psi}$$

- Pitch circular diameter

$$d = \frac{zm_n}{\cos \psi}$$

- Tooth proportions
    - Addendum $h_a = m_n$
    - Dedendum hf = 1.25 m_n
    - Clearance c = 0.25 m_n

- Addendum circle diameter $d_a = d + 2h_a$ or

$$d_a = \frac{z\, m_n}{\cos\psi} + 2m_n$$

- Dedendum circle diameter

$$d_f = \frac{z\, m_n}{\cos\psi} + 2.5m_n$$

- Component of tooth forces

$$P_t = \frac{2M_t}{d_p}$$

$$P_r = P_t \left[\frac{\tan \alpha_n}{\cos \psi} \right]$$

$$P_a = P_t \tan \psi$$

- Beam strength of helical gear

$$\mathbf{S_b = m_n\ b\ \sigma_b\ Y}$$

Where, m = Module,

σ_b = Permissible bending stress

y = Lawis form factor

- Dynamic load or incremental dynamic load P_d

$$P_d = \frac{21v\,(ceb \cos^2 \psi + P_t) \cos \psi}{21v + \sqrt{(ceb \cos^2 \psi + P_t)}}$$

Where, e = Sum of errors,

C = Deformation factor

- Wear strength of helical gear

$$S_w = \frac{bQ\, d_p K}{\cos^2 \psi}$$

Herringbone Gear

- In order to avoid an axial thrust on the shaft and the bearings, the double helical gears or Herringbone gears are used.

Bevel Gears

- Use to transmit power between two intersecting shafts.
- High speed high power transmission.

Classification of Bevel Gear

- Mitre Gear: When two bevel gears are mounted on shafts that are intersecting at right angle.
- Crown Gear: In pair of bevel gear, when one of the gear has a pitch angle of 90°.
- Internal Bevel Gear: When the teeth of bevel gear are cut on the inside of the pitch.
- Skew Bevel Gear: Mounted on non-parallel and non-intersecting shafts. It constant of straight teeth.

- Hypoid Gear: Similar to skew bevel gear, non-parallel and non-intersecting shafts. It consists of curved teeth.
- Zerol Gear: Sprial bevel gear with zero spiral angle.
- Force Gear: Consists of a spur or helical pinion meshing with a conjugate gear or disk form.

- **Beam strength of bevel gear**

$$S_b = mb\sigma_b Y\left[1 - \frac{b}{A_o}\right]$$

Where, $\left[1 - \frac{b}{A_o}\right] =$ bevel factor.

- **Wear strength of bevel gears**

$$S_w = \frac{0.75\, b\, Q\, D_p K}{\cos\gamma}$$

Where, K = Material constant, $K = 0.16\left(\frac{BHN}{100}\right)^2$

Bearing

- A bearing is a mechanical element that permits relative motion between two components or parts, such as the shaft and housing, with minimum friction.

Plain Bearings (Sliding Contact Bearings)

- A plain bearing is any bearing that works by sliding action, with or without lubricant. This group encompasses essentially all types other than rolling-element bearings.[i]

Journal or Sleeve Bearings

- These are cylindrical or ring-shaped bearings designed to carry radial loads.
- The simplest and most widely used types of sleeve bearings are cast-bronze and porous-bronze (powdered-metal) cylindrical bearings.

Thrust Bearings

- This type of bearing differs from a sleeve bearing in that loads are supported axially rather than radially which is shown in the following figure. Thin, disk like thrust bearings are called *thrust washers.*

Bearing Materials

- **Babbitts**

- **Bronzes and Copper Alloys**
- **Aluminium**
- **Porous Metals**
- **Plastics**

Anti Friction Bearings

- *Ball, roller, and needle bearings* are classified as antifriction bearings since friction has been reduced to a minimum.

Bearing Loads

- **Radial Load**
    - Loads acting perpendicular to the axis of the bearing are called *radial loads.* Although radial bearings are designed primarily for straight radial service, they will withstand considerable thrust loads when deep ball tracks in the raceway are used.
- **Thrust Load**
    - Loads applied parallel to the axis of the bearing are called *thrust loads.* Thrust bearings are not designed to carry radial loads.

Ball Bearings

- Angular-contact bearings are used for combined radial and thrust loads and where precise shaft location is needed. Uses of the other two types are described by their names: radial bearings for radial loads and thrust bearings for thrust loads (See the following figure).

Radial Bearings

- *Deep-groove* bearings are the most widely used ball bearings. In addition to radial loads, they can carry substantial thrust loads at high speeds, in either direction.
- *Self-aligning* bearings come in two types: internal and external. In internal bearings, the outer-ring ball groove is ground as a spherical surface.Externally self-aligning bearings have a spherical surface on the outside of the outer ring, which matches a concave spherical housing.
- *Double-row, deep-groove* bearings embody the same principle of design as single-row bearings. Double-row bearings can be used where high radial and thrust rigidity is needed and space is limited.
- *Angular-contact thrust* bearings can support a heavy thrust load in one direction combined with a moderate radial load.

Thrust Bearings

- *Flat-race* bearings consist of a pair of flat washers separated by the ball complement and a shaft-piloted retainer, so load capacity is limited. Contact stresses are high, and torque resistance is low.
- *One-directional, grooved-race* bearings have grooved races very similar to those found in radial bearings.

- *Two-directional, groove-race* bearings consist of two stationary races, one rotating race, and two ball complements.

Roller Bearing (Rolling Contact Bearings)

- The principal types of roller bearings are *cylindrical, needle, tapered,* and *spherical.*
- They have higher load capacities than ball bearings of the same size and are widely used in heavy-duty, moderate-speed applications..

Cylindrical Bearings

- Cylindrical roller bearings have high radial capacity and provide accurate guidance to the rollers. Their low friction permits operation at high speed, and thrust loads of some magnitude can be carried through the flange-roller end contacts.

Needle Bearings

- Needle bearings are roller bearings with rollers that have high length-to-diameter ratios. Compared with other roller bearings, needle bearings have much smaller rollers for a given bore size.
- *Loose-needle* bearings are simply a full complement of needles in the annular space between two hardened machine components, which form the bearing raceways. They provide an effective and inexpensive bearing assembly with moderate speed capability, but they are sensitive to misalignment.
- *Caged assemblies* are simply a roller complement with a retainer, placed between two hardened machine elements that act as raceways. Their speed capability is about 3 times higher than that of loose-needle bearings, but the smaller complement of needles reduces load capacity for the caged assemblies.
- *Thrust bearings* are caged bearings with rollers assembled like the spokes of a wheel in a wafer like retainer.

Tapered Bearings

- Tapered roller bearings are widely used in roll-neck applications in rolling mills, transmissions, gear reducers, geared shafting, steering mechanisms, and machine-tool spindles. Where speeds are low, grease lubrication suffices, but high speeds demand oil lubrication, and very high speeds demand special lubricating arrangements.

Spherical Bearings

- Spherical roller bearings offer an unequaled combination of high load capacity, high tolerance to shock loads, and self-aligning ability, but they are speed-limited.
- *Single-row* bearings are the most widely used tapered roller bearings. They have a high radial capacity and a thrust capacity about 60 percent of radial capacity.
- *Two-row* bearings can replace two single-row bearings mounted back-to-back or face-to-face when the required capacity exceeds that of a single-row bearing.

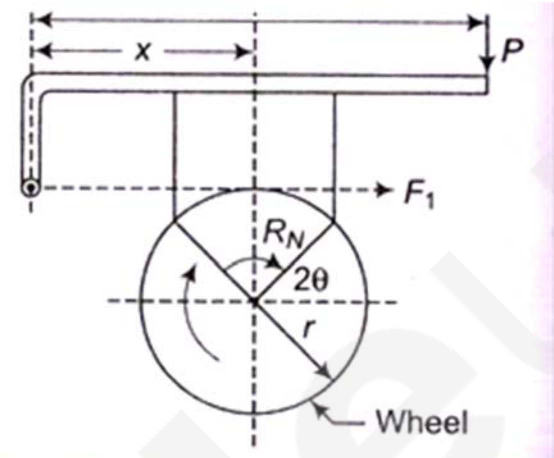
Brake

- A brake is a device by means of which artificial frictional resistance is applied to a moving machine member, in order to retard or shop the motion of a machine.

- The most commonly brakes use friction to convert kinetic energy into beat, though other methods of energy conversion may be employed.

Single Block or Shoe Brake

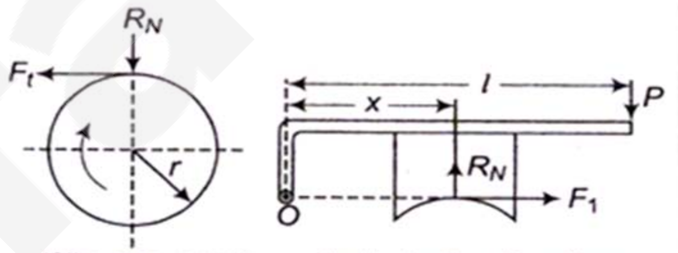
- It consists of a block or shoe which is passed against the rim of revolving brake wheel drum. The block is made of a softer material than the rim of the wheel.
- If the angle of contact is less than 60° then, it may be assumed that normal pressure or force between the block and the wheel is uniform.



A schematic diagram of shoe brake

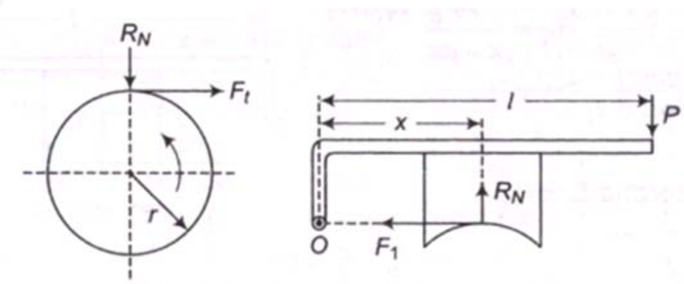
Case I: When the line of action of tangential braking force passes through the fulcrum O of the lever.

- If wheel is rotating in clockwise direction then, Free Body Diagram (FBD) of wheel and block is



Wheel is rotating anti-clockwise direction

- If wheel is rotating in anticlockwise direction then, FBD of wheel and block is



Wheel is rotating anti-clockwise direction

- Braking force

$$\mu R_N = \frac{\mu P l}{x}$$

- Braking torque

$$T_B = \frac{\mu P l r}{x}$$

- When wheel is rotating in anticlockwise direction then, the braking torque is same as above

$$T_B = \frac{\mu P l r}{x}$$

Case II: When the line of acting of the tangential braking force (F_t) passes through a distance a below the fulcrum *O*. Then, there are two cases:

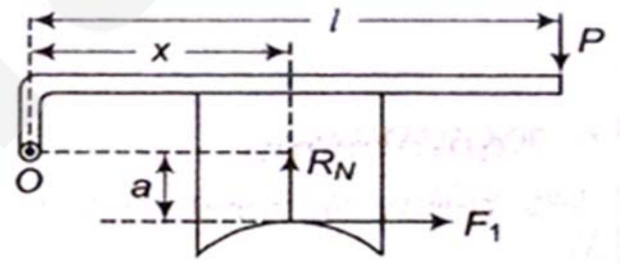
For Clockwise:

- Braking force

$$F_t = \mu R_N = \frac{\mu P l}{x + \mu a}$$

- Braking torque

$$T_B = \frac{\mu P l r}{x + \mu a}$$



FBD of block

For Anti Clockwise:

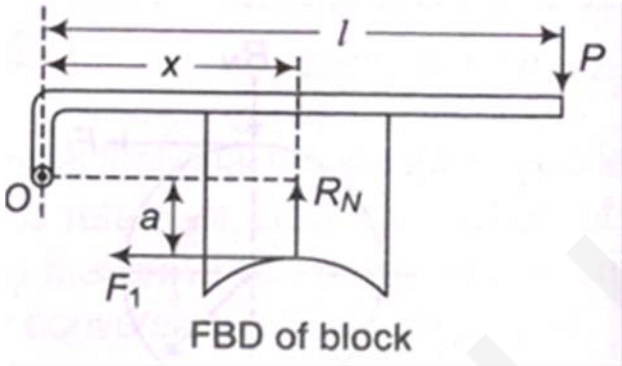
- Braking force

$$F_t = \frac{\mu P l}{x - \mu a}$$

- Braking torque

$$T_B = \frac{\mu P l r}{x - \mu a}$$

(as $T_B = F_t \times r$)



Case III: When the line of action of tangential braking force (F_t) passes through a distance 'a' above the fulcrum O.

For clockwise,

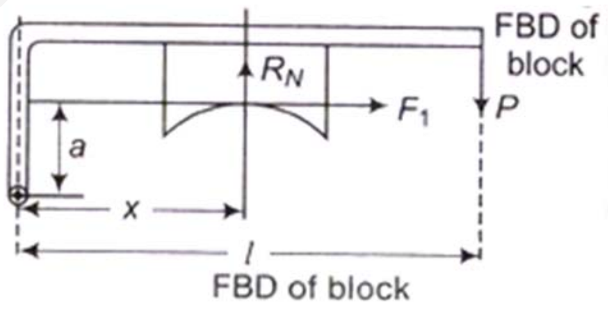
- Braking force

$$F_t = \frac{\mu P l}{x - \mu a}$$

- Braking torque

$$T_B = \frac{\mu P l r}{x - \mu a}$$

($T_B = F_t \times r$)



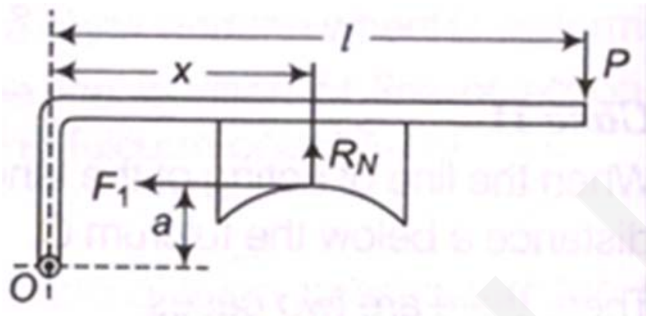
For Anti- Clockwise

- Braking force

$$F_t = \frac{\mu P l}{x + \mu a}$$

- Braking torque

$$T_B = \frac{\mu P l r}{x - \mu a}$$



FBD of block

- When the frictional force helps to apply the brakes then, such type of brakes are said to self-energizing brakes.
- When P is negative or equal to zero then, these are known as self-locking brakes.

Simple Band Brake

- A band brake consists of a flexible band of leather, one or more ropes, or steel lined with friction material, which embraces a part of the circumference of the drum is called simple band brake.

We know,

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

$$(\theta = 360^\circ - \theta')$$

$$or\ 2.3\log\left(\frac{T_1}{T_2}\right) = \mu\,(360^\circ - \theta') = \mu\theta$$

- Braking force on the drum = ($T_1 - T_2$)
- Braking torque on the drum (T_B) = (T_1 - T_2) × r
- When wheel rotates in the clockwise direction and taking moment about fulcrum O

$$Pl = T_1 \times b$$

$$T_1 = \frac{Pl}{b}$$

- For anticlockwise rotation of the drum $Pl = T_2b$

$$T_2 = \frac{Pl}{b}$$

where, b = Perpendicular distance from O to the line of action T_1 or T_2

l = Length of the lever from the fulcrum

T_1 = σt wt

w = Width of the band

t = Thickness of the band

σ_t = Permissible stress in the band.

Clutch

- A clutch is a mechanical device that provides for the transmission of power (and therefore usually motion) from one component (the driving member) to another (the driven member) when engaged, but can be disengaged.

Friction Clutch

- The friction clutch is used to transmit power of shafts and machines which must be started and stopped frequently.
- Friction surfaces of a clutch remain in contact to each other by applying an axial thrust or load w.

Considering Uniform Pressure

- The uniform pressure p can be evaluated as.

$$p = \frac{w}{\pi(r_1^2 - r_2^2)}$$

- **Total frictional torque** given in this case,

$$T = \frac{2}{3}\mu w\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right) = mw\, R_m$$

where, Rm = Mean radius of friction surfaces

$$R_m = \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$$

Considering Uniform Wear

- Total frictional torque acting on clutch

$$T = \frac{1}{2}\mu w\,(r_1 + r_2) = \mu w R_m$$

where R_m= Mean radius of friction surfaces $= \frac{r_1 - r_2}{2}$

- In uniform wear theory, Maximum pressure acts at the inner radius and minimum pressure acts at the outer radius.

$$p_{\max} \times r_2 = c,\ p_{\min} \times r_1 = c$$
$$\frac{p_{\max}}{p_{\min}} = \frac{r_1}{r_2}$$

- Average pressure on the friction surfaces

$$p_{av} = \frac{w}{\pi(r_1^2 - r_2^2)}$$

Multiple Disc Clutch

- Number of pairs of contact surfaces

$$\boldsymbol{n = n_1 + n_2 - 1}$$

where, n_1 = Number of discs on the driving shaft

n_2 = Number of discs on the driven shaft

- **Total frictional torque** acting on the frictional surface

$$\boldsymbol{T = \eta \mu w R_m}$$

where,

$$R_m = \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$$ [in case of uniform pressure]

$$= \frac{1}{2}(r_1 + r_2)$$ [in case of uniform wear]

where, r_1 and r_2 are outer and inner radii of the friction plates.

Cone Clutch

- In cone clutch, driver is keyed to the driving shaft by a sunk key and has an inside conical surface or face which exactly fits into the outside conical surface of the driven.

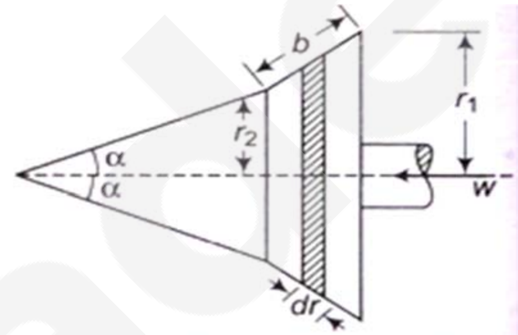
- Total torque on the clutch,

$$T = \frac{2}{3}\mu w \cos ec\left[\frac{r_1^3 - r_2^3}{r_1^2 - r_1^2}\right]$$ (for uniform pressure)

$$= \frac{1}{2}\mu w \cos ec\ \alpha\left(r_1 + r_2\right)$$ (for uniform wear)

α = Semi angle of cone or face angle of the cone

$$w_n = \frac{w}{\sin\alpha}$$



Cone clutch

where, *w* = Axial load or thrust

- Axial force required for engaging the clutch,

$$= w_n(\sin\alpha + \mu\cos\alpha)$$

- Axial force required to disengaged the clutch

$$w_d = w_n(\mu\cos\alpha - \sin\alpha)$$

- If face width b and mean radius of cone clutch is R_m.

Then,

$$R_m = \left(\frac{r_1 - r_2}{2}\right)$$

$$R_n = \frac{w}{2\pi R_m \, b \sin\alpha}, T = 2\pi\mu . P_n R_m b$$

- Power transmitted by clutch $= \frac{2\pi NT}{60}$

Centrifugal Clutch

- Centrifugal force acting on each shoe at running speed

$$p_c = m\,\omega^2 r$$

Where,

$$\omega = \left(\frac{2\pi N}{60}\right)$$

- Friction force acting on each shoe = $\mu(P_c - P_s)$

The direction of force is perpendicular to the radius of the rim pulley.

- Frictional torque on each shoe = $\mu(P_c - P_s) \times R$
- Total torque transmitted = Number of shoes $\mu(P_c - P_s)R$

$$= n\mu(P_c - P_s)R$$

Arc = Angle (in radian) × Radiua l = θR

Where, area of contact = *lb*

- Force exerted on each shoe = **plb**

$$P_c - P_s = lbp$$

Where, *l* = Contact length of the shoe

b = Width of the shoe

p = Pressure intensity on shoe

q = Angle made by shoe at the centre of spider in radian

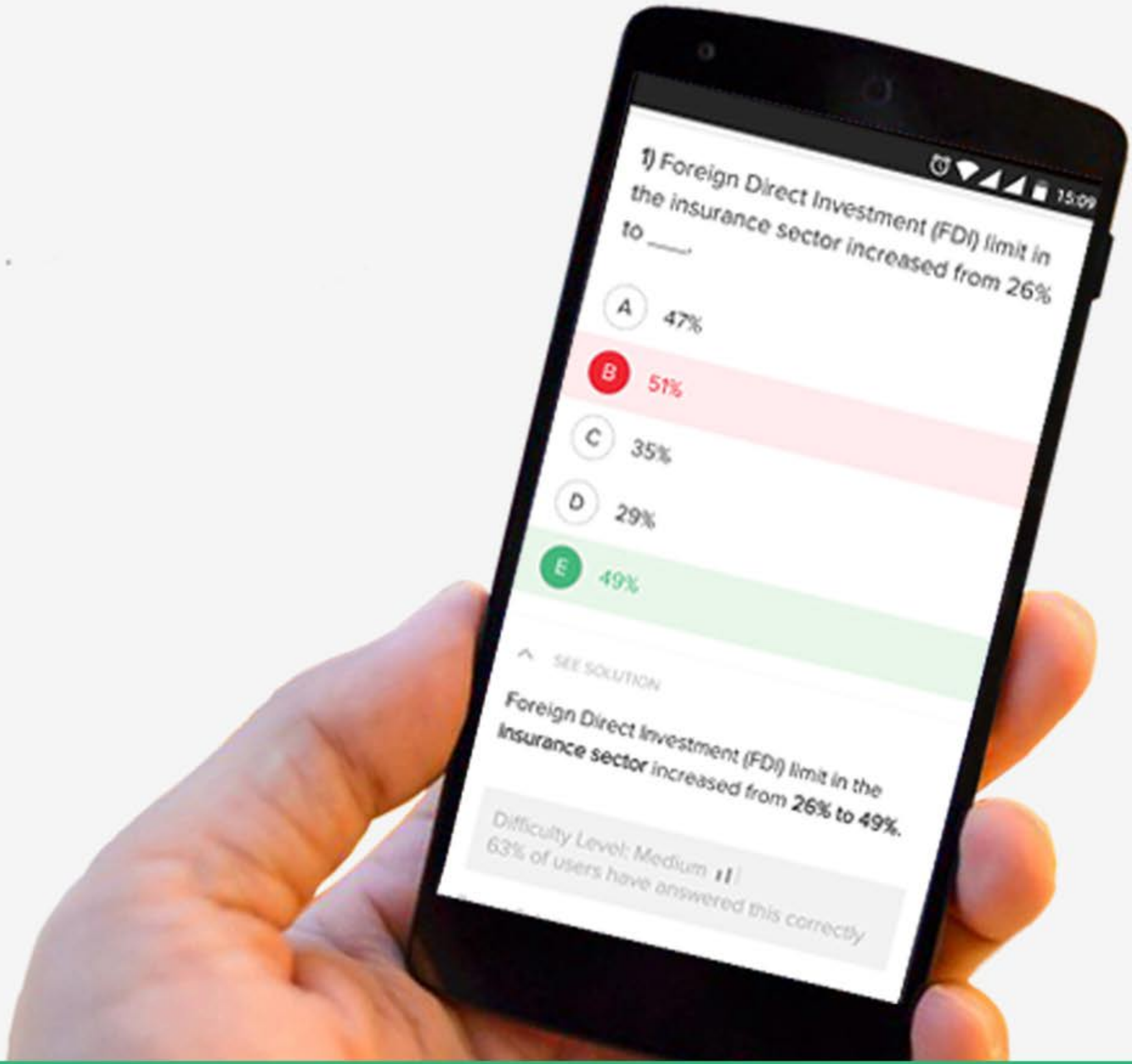
R = Contact radius of shoe = inside radius of the rim of the pulley

grade**up**

Formulas on

THEORY OF MACHINES

for GATE ME Exam



Theory of Machines and Vibrations – Short Notes

Instantaneous Centre of Velocity (I-centre)

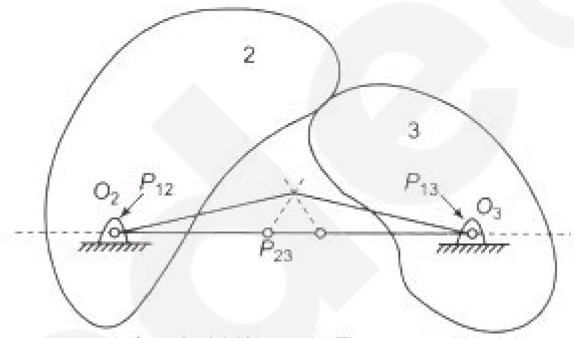
- The instantaneous centre of velocity can be defined as a point which has no velocity with respect to the fixed link.

Centro

- Instantaneous centre is also called centro
- Primary Centro One which can be easily located by a mere observation of the mechanism.
- Secondary Centro Centros that cannot be easily located

Aronhold-Kennedy Theorem of Three Centre

- It state that if three bodies are in relative motion with respect to one another, the three relative instantaneous centers of velocity ar collinear.



Aronhold-Kennedy Theorem of three centre 1, 2 and 3

Number of Centros in a Mechanism

- For a mechanism of n links, the number of centros (Instantaneous centre) N is

$$N = \frac{1}{2}n(n-1)$$

Linkages are the basic building blocks of all mechanisms

- Links: rigid member having nodes.
- Node: attachment points.
- Binary link: 2 nodes
- Ternary link: 3 nodes
- Quaternary link: 4 nodes
- Joint: connection between two or more links (at their nodes) which allows motion; (Joints also called kinematic pairs)

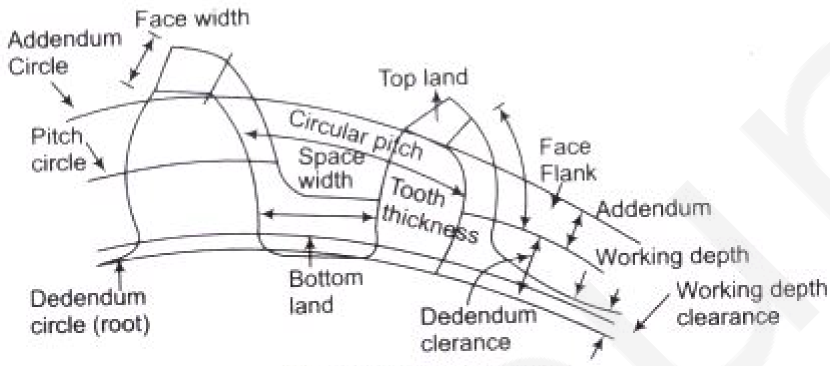
D'Alembert's Principle and Inertia Forces

- D'Alembert's principle states that the reverse-effective forces and torques and the external forces and torques on a body together give statical equilibrium

$$F + (-ma_G) = 0$$

$$T_{eG} + (-I_G\alpha) = 0$$

Gear Terminology



A schematic gear diagram

Circular Pitch (p):

- It is a distance measured along the circumference of the pitch circle from a point on one tooth to the corresponding point on the adjacent tooth.

$$p = \frac{\pi d}{T}$$

Diametrical Pitch (P)

- It is the number of teeth per unit length of the pitch circle diameter in inches.

$$P = \frac{T}{d}$$

Module (m)

- It is the ratio of pitch diameter in mm to the number of teeth. The term is used SI units in place of diametrical pitch.

$$m = \frac{d}{T}$$

$$\Rightarrow p = \pi m$$

Gear Ratio (G)

- It is the ratio of the number of teeth on the gear to that on the pinion.

$$G = \frac{T}{t}$$

where, T = number of teeth on the gear

t = number of teeth on the pinion

Velocity Ratio

- The velocity ratio is defined as the ratio of the angular velocity of the follower to the angular velocity of the driver gear

$$VR = \frac{\omega_2}{\omega_1} = \frac{N_2}{N_1} = \frac{d_1}{d_2} = \frac{T_1}{T_2}$$

Gear Train

- A gear train is a combination of gears used to transmit motion from one shaft to another. Gear trains are used to speed up or stepped down the speed of driven shaft. The following are main types of gear trains.

Simple Gear Train

- Series of gears, capable of receiving and transmitting motion from one gear to another is called a simple gear train.

Train value

$$= \frac{N_3}{N_1} = \frac{T_1}{T_3}$$

$$= \frac{\text{Number of teeth on driving gears}}{\text{Number of teeth on driver gear}}$$

Speed ratio

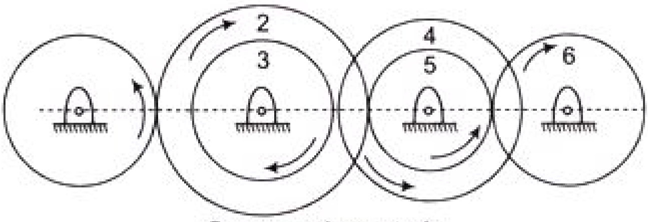
$$= \frac{1}{\text{Train value}}$$

Gears-and-gear-trains

- The intermediate gears have no effect on the speed ratio and therefore they are known as idlers.

Compound Gear Train

- When a series of gears are connected in such a way that two or more gears rotate about an axis with the same angular velocity.



Compound gear train

Train value

$$= \frac{\text{Product of number of teeth on driving gears}}{\text{Product of number of teeth on driven gears}}$$

Planetary or Epicyclic Gear Train

- A gear train having a relative motion of axes is called a planetary or an epicyclic gear train. In an epicyclic train, the axis of at least one of the gears also moves relative to the frame.
- If the arm a is fixed the wheels S and P constitute a simple train. However if the wheel S is fixed so that arm a can rotate about the axis of S. The P would be moved around S therefore it is an epicyclic train

Flywheel

- A flywheel is used to control the variations in speed during each cycle of an operation. A flywheel acts as a reservoir of energy which stores energy during the period when the supply of energy is more than the requirement and releases the energy during the period when the supply energy is less than the requirement.

Maximum fluctuation of energy (e),

$$e = (\triangle KE) = \frac{1}{2}/(\omega_{max}^2 - \omega_{min}^2)$$

$$e = I\omega(\omega_1 - \omega_2) \left(\omega = \frac{\omega_1 + \omega_2}{2}\right)$$

$$e = I\omega^2 k_s \qquad \left[k_s = \frac{\omega_1 - \omega_2}{2}\right]$$

$$k_s = \frac{e}{\frac{1}{2} I\omega^2 \times 2}, \; k_s = \frac{e}{2E}$$

where,

ωmax and ωmin are the maximum and minimum angular speed respectively.

E = kinematic energy of the flywheel at mean speed.

Flywheel in Punching Press

- Generally, flywheel is used to reduce fluctuation of speed where the load on the crank shaft constant while the applied torque varies.
- However, the flywheel can also be used to reduce fluctuation of speed when the torque is constant but load varies during the cycle *e.g.*, in punching press in riveting machine.
- Let *E* be energy required for one punch energy supplied to crank shaft from the motor during punching

$$= E\left(\frac{\theta_2 - \theta_1}{2\pi}\right)$$

Governors

The function of a governor is to maintain or regulate the speed of an engine within specified limits whenever there is variation of load.

Types of Governors

The broadly classification of the governors are given below.

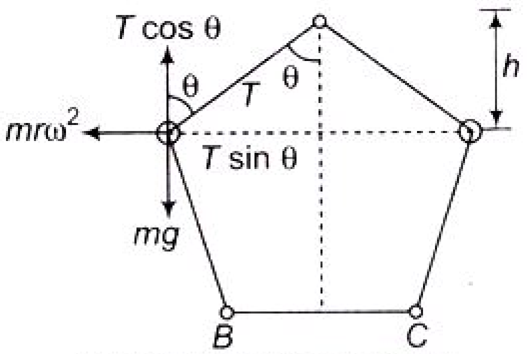
Centrifugal Governor

- In this type of governor, the action of governor depends upon the centrifugal effects produced by the masses of two balls.

Inertia Governor

- In this type of governor, positions of the balls are effected by the forces set up by an angular acceleration or deceleration of the given spindle in addition to centrifugal forces on the balls.

Pendulum Type Watt Governor



Simple conical governor

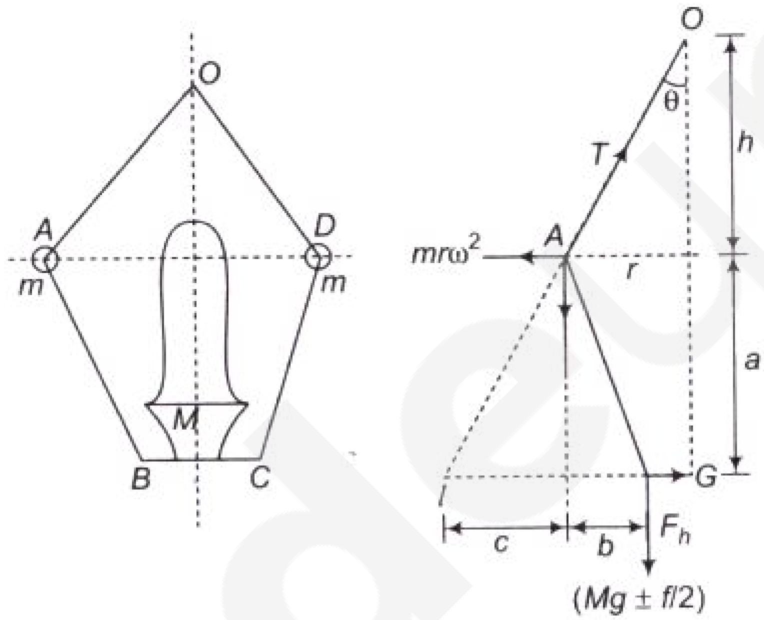
height of each bal

$$h = \frac{g}{w^2}$$

Porter Type Governor

Porter governor can be shown as

$$h = \frac{895}{N^2}\left(\frac{2mg + (mg \pm f)(1+k)}{2mg}\right)$$



Wilson Hartness Governor

- **Main spring force**

$$F'_{s_2} - F'_{s_1} = 4s(r_2 - r_1)$$

- **Net auxiliary spring force**

$$F'_{s_2} - F'_{s_1} = h_2\ s_a = \left(h_1 \frac{y}{x}\right)\frac{s}{a}$$

$$= (r_2 - r_1)\frac{b}{a}\frac{y}{x}\ s_a$$

Pickering Governor

$$f = \frac{m(e + f)\omega^2}{192\ EI}$$

Where, E = modulus of elasticity of the spring material

I = moment of inertia of the cross-section of the spring about neutral axis

Sensitiveness of a Governor

- The governor is said to be sensitive when it readily responds to a small change of speed.
  - Sensitiveness of a governor is defined as the ratio of difference between the maximum and minimum speeds to the mean equilibrium speed.

$$\text{Sensitiveness} = \frac{\text{range of speed}}{\text{mean speed}}$$

$$= 2\frac{(N_2 - N_1)}{N_1 + N_2}$$

where, N = mean speed

$N1$ = minimum speed corresponding to full load conditions

$N2$ = maximum speed corresponding to no load conditions.

Huning

- Sensitiveness of a governor is a desirable quality. However, if a governor is too sensitive, it may fluctuate continuously. This phenomenon of fluctuation is known as hunting.

Isochronism

- If a governor is at equilibrium only for a particular speed, it is called isochronous governor, for which we can say that an isochronous governor is infinitely sensitive.

$$\frac{dF}{dr} = m\omega^2$$

Stability

- A governor is said to be stable if it brings the speed of the engine to the required value and there is not much hunting. Obviously, the stability and sensitivity are two opposite characteristics.

Cam:

- A cam is a mechanical member used to impart desired motion (displacement) to a follower by direct contact (either point or line contact).
- Cam mechanisms belong to higher pair mechanism.
- A driver member known as cam.
- A driven member called the follower.
- A frame is one which supports the cam and guides the follower.

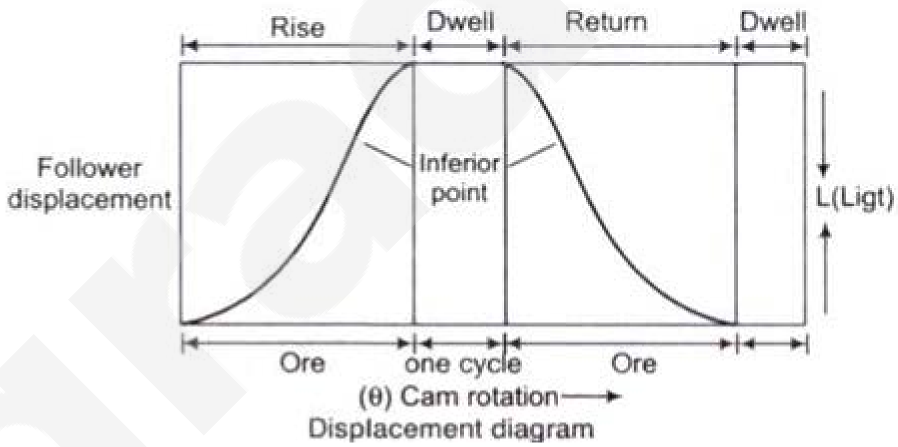
Key Points for Cams

- For a roller follower, the trace point is at the roller centre.
- For a flat-face follower, it is at the point of contact between the follower and cam surface when the contact is along the base circle of the cam.
- During a complete rotation, the pressure angle varies from its maximum to its minimum value.
- The greater the pressure angle, the higher will be side thrust and consequently the changes of the translating follower jamming in its guide will increase.
- It is not desirable to increase the pressure angle

Follower Displacement Diagram

The following terms are used with reference to the angular motion of the cam

- **Angle of Ascent (φa):** It is the angle through which the cam turns during the time the follower rises.
- **Angle of Dwell (*f*)** Angle of dwell is the angle through which the cam turns while the follower remains stationery at the highest or the lowest position.



Balancing

- Balancing is defined as the process of designing a machine in which unbalance force is minimum.

- If the centre of mass of rotating machines does not lie on the axis of rotation, the inertia force is given by $F_1 = m\omega^2 e$

m = mass of the machine

ω = angular speed of the machine

e = eccentricity i.e., the distance from the centre of mass to the axis of rotation

Internal and external Balancing

- The shaft can be completely balancing by adding a mass m_1 at a distance e_1 from the axis of rotation diametrically opposite to m so that,

$$m\omega 2e = m_1\omega^2 e_1$$

Static Balancing

- If a shaft carries a number of unbalanced masses such that the centre of mass of the system is said to be statically balanced

Dynamic Balancing

- A system of rotating masses in dynamic balance when there does not exist any resultant centrifugal force as well as resultant couple.

Vibrations

Time Period for Simple Pendulum

Vibrations: Vibration refers to mechanical oscillations about an equilibrium point. In its simplest form, vibration can be considered to be the oscillation or repetitive motion of an object around an equilibrium position.

Vibrations or mechanical oscillations are of many types as given below

- **Free Vibration** (Natural vibration) Vibration over an interval of time during which the system is free from excitation is known as free vibration.
- **Damped and Undamped Vibration:** Energy of a vibrating system is gradually dissipated by friction and other resistance.
- **Forced Vibration** When a repeated force continuously acts on a system, the vibrations are said to be forced.
- **Linear Vibration**: If all the basic components of a vibratory system-the spring, the mass, and the damper, behave linearly, the resulting vibration is known as linear vibration. The differential equations that govern the behaviour of vibratory linear systems are linear. Therefore, the principle of superposition holds.
- Nonlinear vibration: vibration: If however, however, any of the basic components behave nonlinearly, the

- **Harmonic Vibration:** Vibration in which the motion is a sinusoidal function of time.
- **Fundamental Vibration:** Harmonic component of a vibration with the lowest frequency.
- **Steady State Vibration:** When the particles of the body move in steady state condition or continuing period vibration is called steady state vibration.
- **Transient Vibration:** Vibratory motion of a system other than steady state.
- **Longitudinal Vibration:** Vibration parallel to the longitudinal axis of a member
- **Transverse Vibration:** Vibration in a direction perpendicular to the longitudinal axis or central plane of a member.
- **Torsional Vibration:** Vibration that involves torsion of a member.

Mode of Vibration: Configuration of points of a SHM is called the mode of vibration.

Natural Frequency: Frequency of free simple harmonic vibration of an undamped linear system.

Time Period: Time taken for one oscillation is called time period.

Simple Pendulum: If time period of the pendulum is 1s, then pendulum is called simple pendulum.

$$\left|\frac{\theta}{\alpha}\right| = \frac{l}{g}\left(where,\ \alpha = \frac{d^2\theta}{dt^2}\right)$$

Time period is given by

$$T = 2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|}$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

(for small amplitude $\sin\theta \approx \theta$)

$$T = 2\pi\sqrt{\frac{l}{g}\left(1 + \frac{\theta_0^3}{16}\right)}$$

[large amplitude (θ_0)]

Free Vibration of Damped One Degree-of-Freedom Systems

- Damping factor:

$$\zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}}$$

- **The system response when under-damped: $\xi < 1$**

$$x(t) = e^{-\xi\omega_n t}(x_0 \cos\omega_d t + \frac{\dot{x}_0 + \xi\omega_n x_0}{\omega_d}\sin\omega_d t)$$

- **The system response when critically damped: ξ = 1**

$$x(t) = e^{-\omega_n t}(x_0 + (\dot{x}_0 + \omega_n x_0)t)$$

- **The system response when over-damped: ξ > 1**

$$x(t) = C_1 e^{(-\xi+\sqrt{\xi^2-1})\omega_n t} + C_2 e^{(-\xi-\sqrt{\xi^2-1})\omega_n t}$$

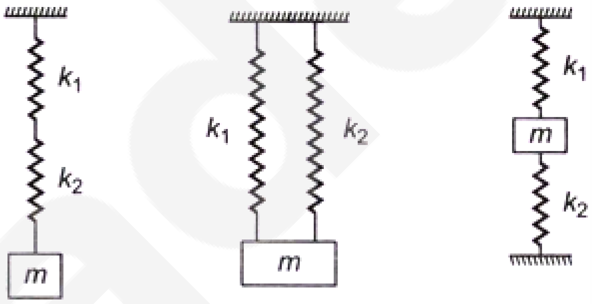
C_1 and C_2 are the constants that are lengthy in closed-form. They can be found numerically by the initial conditions.

- **Spring Block System**

$$ma = -kx$$

$$\left|\frac{x}{a}\right| = \frac{m}{k} \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$$

- **Equivalent force constant** (*k*) is given by



Spring block system

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$k = \frac{k_1 k_2}{k_1 + k_2}$$

$$k = k_1 + k_2$$

$$k = k_1 + k_2$$

- If spring has a mass m_s and a mass m is suspended from it,

$$T = 2\pi\sqrt{\frac{m + \frac{m_s}{3}}{k}}$$

Mass spring system

$$m\ddot{x} + kx = 0$$

$\omega_n = \sqrt{k/m}$: Natural frequency

$T = \dfrac{2\pi}{\omega_n}$: Period of motion

Damping

Any influence which tends to dissipate the energy of a system.

Damping Factor or Damping Ratio: It is the ratio of actual to critical damping coefficient.

$$\xi = \sqrt{\frac{(c/2m)^2}{k/m}} = \frac{c}{2\sqrt{mk}}$$

- ξ = 1, the damping is known as critical under critical damping condition Critical damping coefficient

$$C_c = 2\xi\sqrt{mk}$$
$$\Rightarrow C_c = 2\,m\omega_n$$

- ξ < 1 *i.e.*, system is underdamped.

$$\alpha_{1,2} = \left(-\xi \pm i\sqrt{1-\xi^2}\right)\omega_n$$

$$x = Ae^{\left(-\xi + i\sqrt{1-\xi^2}\right)\omega_n t} + Be^{\left(-\xi - i\sqrt{1-\xi^2}\right)\omega_n t}$$

- Damped frequency

$$\omega_d = \sqrt{1-\xi^2}\;\omega_n$$

Logarithmic Decrement

- In an underdamped system, arithmetic ratio of two successive oscillations is called logarithmic decrement (constant).

Since,

$$\frac{X_n}{X_{n+1}} = e^{\xi\omega_n T_d} = \frac{X_0}{X_1} = \frac{X_1}{X_2} = \frac{X_2}{X_3}$$

Logarithmic decrement,

$$\delta = In\left(\frac{X_n}{X_{n+1}}\right) = In\, e^{\xi\omega_n T_d}\ \delta = \xi\omega_n T_d$$

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi}}$$

$$\delta = \frac{1}{n}\ \text{ln}\left(\frac{X_0}{X_n}\right)$$

Forced Vibration

- Amplitude of the steady state response is given by

case of steady state response first term zero ($e^{-\infty} = 0$).

$$A = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$$

$$A = \frac{F_0/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\,\xi\,\frac{\omega}{\omega_n}\right)^2}}\ \tan\phi = \left(\frac{c\omega}{k-m\omega^2}\right)$$

Magnification Factor

- Ratio of the amplitude of the steady state response to the static deflection under the action force F0 is known as magnification factor

$$MF = \frac{\frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}}{F_0/k} = \frac{k}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}$$

$$= \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\,\xi\,\frac{\omega}{\omega_n}\right)^2}}$$

Resonance

- When the frequency of external force(driving frequency) is equal to the natural frequency of a vibrating body, the amplitude of vibration becomes excessively large. This phenomenon is known as resonance

Critical Speed

- Critical or whirling or whipping speed is the speed at which the shaft tends to vibrate violently in transverse direction

Critical speed essentially depends on

- The eccentricity of the C.G of the rotating masses from the axis of rotation of the shaft.
- Diameter of the disc
- Span (length) of the shaft, and
- Type of supports connections at its ends

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Forecasting

- **Simple Moving Average-**

$$F_n = \frac{1}{N}(D_n + D_{n-1} + D_{n-2} + D_{n-3} + \cdots)$$

- **Moving Weight Average-**

$$\text{Weight Moving Average} = \frac{\sum_{t=1}^{n} W_i D_i}{\sum_{i=1}^{n} W_i}$$

- **Single (Simple) Exponential Smoothing-**

$$F_t = F_{t-1} + \alpha(D_{t-1} - F_{t-1})$$
$$or\ F_t = (1-\alpha)F_{t-1} + \alpha D_{t-1}$$

if previous forecasting is not given

$$F_t = \alpha D_t + \alpha(1-\alpha)D_{t-1} + \alpha(1-\alpha)^2 D_{t-2} \ldots \ldots$$

Where F_t= Smoothed average forecast for period t
F_{t-1}=Previous period forecast
α=Smoothing constant

- **Linear Regression-**

$$Y = a + bX$$
$$\sum y = na + b\sum x$$
$$\sum xy = n\sum x + b\sum x^2$$

- **Forecasting Error-**

$$e_t = (D_t - F_t)$$

- **Bias-**

$$Bias = \frac{1}{N}\sum_{t=1}^{n}(D_t - F_t)$$

- **Mean Absolute Deviation-**

$$MAD = \frac{1}{N}\sum_{t=1}^{n}|D_t - F_t|$$

- **Mean Square Error-**

$$MSE = \frac{1}{N}\sum_{t=1}^{n}(D_t - F_t)^2$$

- **Mean Absolute Percentage Error-**

$$MAPE = \frac{1}{N}\sum_{t=1}^{n}\frac{|D_t - F_t|}{D_t} \times 100$$

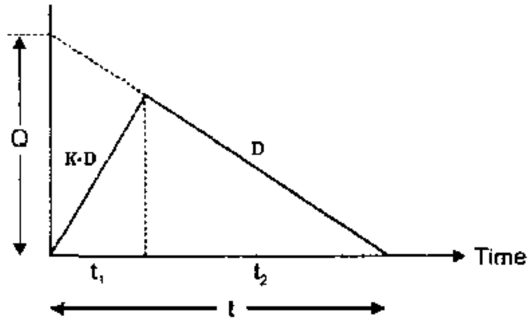
Inventory

If D= demand/year, C_o=Order cost, C_c= Carrying cost, P= Purchase price/unit
Q^*=Economic Order Quantity, K= Production Rate and C_s =Shortage Cost/unit/period

- **Case-1** Purchase Model With Instantaneous Replenishment and Without Shortage-
  1. $EOQ\ Q^* = \sqrt{\frac{2DC_o}{C_c}}$ at EOQ $Invertory\ cost = Order\ cost$
  2. $No. of\ order = \frac{D}{Q^*}$
  3. $Time\ Taken\ Per\ Order = \frac{Q^*}{D}$
  4. $Total\ Cost = Unit\ cost + Invertory\ cost + Order\ cost$

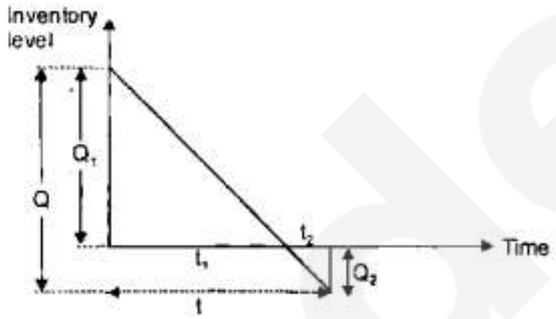
$$= (D \times P) + \left(\frac{Q}{2} \times C_c\right) + \left(\frac{D}{Q} \times C_o\right) = (D \times P) + \sqrt{2DC_cC_o}$$

- **Case-2** Manufacturing Model Without Shortage-



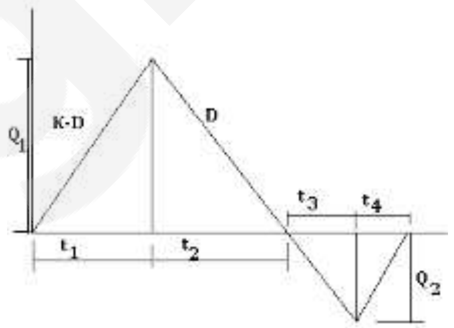
1. $EOQ\ Q^* = \sqrt{\frac{2C_oD}{C_C\left(1-\frac{D}{K}\right)}}$
2. $t_1 = \frac{Q^*}{K}$
3. $t_2 = \frac{Q^*\left(1-\frac{D}{K}\right)}{D}$
4. Total optimum cost $= \sqrt{2DC_cC_o\left(1-\frac{D}{K}\right)}$

- **Case 3** Purchase Model With Shortage-



1. $Q = EOQ = \sqrt{\frac{2DC_o}{C_C}\left(\frac{C_S+C_C}{C_S}\right)}$
2. $Q_1 = \sqrt{\frac{2DC_o}{C_C}\left(\frac{C_S}{C_S+C_C}\right)}$, $Q_2 = Q - Q_1$
3. $t = \frac{Q}{D}$, $t_1 = \frac{Q_1}{D}$ *and* $t_2 = \frac{Q_2}{D}$
4. Total optimum cost $= \sqrt{2DC_cC_o\left(\frac{C_S}{C_S+C_C}\right)}$

- **Case 4** Manufacturing Model With Shortfall



1. $Q = EOQ = \sqrt{\frac{2DC_o}{C_C\times\left(1-\frac{D}{K}\right)}\left(\frac{C_S+C_C}{C_S}\right)}$
2. $Q_1 = \sqrt{\frac{2DC_o}{C_C}\left(\frac{C_S}{C_S+C_C}\right)\times\left(1-\frac{D}{K}\right)}$

3. $Q_1 = \left(1 - \frac{D}{K}\right) Q - Q_2$
4. $t = \frac{Q}{D},\ t_1 = \frac{Q_1}{K-D}$ *and* $t_2 = \frac{Q_1}{D}$
1. $t_3 = \frac{Q_3}{D}$ *and* $t_4 = \frac{Q_2}{K-D}$

- Lead Time Demand + Safety Stock = Reorder Point

PERT and CPM

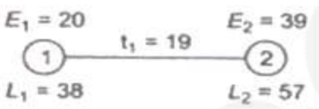
- EFT = EST + activity time
- LFT = LST + Duration of activity

- **Total Float-** $(TF_0) = L_j - (E_i + t_{ij}) = LF_{ij} - EF_{ij}$
  $= (L_j - t_{ij}) = E_i$
  $= LS_{ij} - ES_{ij}$

- **Free Float-** $FF_0 =$ **$(E_j - E_i) - T_{ij}$**

- **Independent Float** $IF_0 = (E_j - L_j) - t_{ij}$ -
  $= FF_j - (\text{Slack of event } i)$

Example-



1. Total float = $L_2 - (E_1 + t_{12}) = 57 - (20 + 19) = 18$
2. Free float = $E_2 - E_1 - t_{12} = 0$
3. Independent float = $E_2 - (L_1 + t_{12}) = -18$

- **PERT Expected time-** $t_e = \frac{t_0 + 4t_m + t_p}{6}$
  1. t_0 = Optimistic time i.e., shortest possible time to complete the activity if all goes well.
  2. t_p = Pessimistic time i.e., longest time that an activity could take if everything goes wrong.
  3. t_m = Most likely time i.e., normal time of an activity would take.

- **Standard deviation-** $(\sigma) = \frac{t_p - t_0}{6}$

- **Variance -** $(\sigma^2) = \left(\frac{t_p - t_0}{6}\right)^2$

- **Crashing-** $\text{Cost Slope (CS)} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}$

- **Standard Normal Variation** $Z = \frac{T_s - T_e}{\sigma_e}$ **(SNV)-**

Linear Programming

Simplex Method Case 1. Maximization Problem

$$Max\ Z = 3x1 + 5x2$$

$$s/t \quad 3x1 + 2x2 \le 18 \quad -(I)$$

$$x1 \le 4 \quad -(II)$$

$$x2 \le 6 \quad -(III)$$

$$x1, x2 \ge 0$$

Standard Form:

Max Z = $3x_1 + 5x_2 + 0w_1 + 0w_2 + 0w_3$

$3x_1 + 2x_2 + w_1 + 0w_2 + 0w_3 = 18$

$x_1 + 0x_2 + 0w_1 + w_2 + 0w_3 = 4$

$0x_1 + x_2 + 0w_1 + 0w_2 + w_3 = 6$

To prepare initial Table:

Table - I

	c_j	3	5	0	0	0	
c_i	x_i	x_1	x_2	w_1	w_2	w_3	b_i
0	w_1	3	2	1	0	0	18
0	w_2	1	0	0	1	0	4
0	w_3	0	1	0	0	1	6
	I_j	-3	-5	0	0	0	Z=0

- $I_j = (Z_j - c_j) = (\Sigma a_{ij}.c_i) - c_j$

Interpretation of Simplex Table

Table - I

	c_j	3	5	0	0	0		
c_i	x_i	x_1	x_2	w_1	w_2	w_3	b_i	
0	w_1	3	2	1	0	0	18	(18/2=9)
0	w_2	1	0	0	1	0	4	(4/0=infinity)
0	w_3	0	1	0	0	1	6	(6/1=6)
	I_j	-3	-5	0	0	0	Z=0	

- Key Column $\rightarrow$ Min I_j [Most Negative]
- Key Row $\rightarrow$ Min positive ratio.

How to get next table ?

- Leaving variable : w3
- Entering variable : x2
- Key no. = 1
- For old key row : New No.= Old No./key No.
- For other rows:

$$New\ No. = Old\ No. - \frac{(Corresponding\ Key\ Row\ No.).(Corresponding\ Key\ Column\ No.)}{Key\ No.}$$

- 18 $\rightarrow$ 18 - (6*2)/1 = 6
- I(w3) = 0 $\rightarrow$ 0 - [1*(-5)]/1 = 5

Table - II

c_i	x_i \ c_j	3	5	0	0	0	
c_i	x_i	x_1	x_2	w_1	w_2	w_3	b_i Ratio
0	w_1	3	0	1	0	-2	6 (6/3=2)
0	w_2	1	0	0	1	0	4 (4/1=4)
5	x_2	0	1	0	0	1	6 (6/0=infinity)
	I_j	-3	0	0	0	5	Z=30

- Key Column → Min I_j
- Key Row → Min positive ratio

Table - III

	c_j	3	5	0	0	0	
c_i	x_i	x_1	x_2	w_1	w_2	w_3	b_i Ratio
3	x_1	1	0	1/3	0	-2/3	2
0	w_2	0	0	-1/3	1	2/3	2
5	x_2	0	1	0	0	1	6
	I_j	0	0	0	0	3	Z=36

- **This is the final Table**

The Optimal Solution is x1 = 2, x2 = 6

giving Z = 36

Type of Solutions : Basic, Feasible/Infeasible, Optimal/ Non-Optimal, Unique/Alternative Optimal, Bounded/Unbounded, Degenerate/Non-Degenerate

- **Analysis of Solution**
  1. This is a Basic solution, as values of basic variables are Positive
  2. This is a feasible solution, as values of basic variables, not containing Artificial Variable, are Positive and all constraints are satisfied
  3. This Feasible solution is an Optimal, as all values in Index Row are positive.
  4. If there is an Artificial Variable, as Basic variable in final table, it is called as Infeasible solution
  5. This solution is unique Optimal, as the number of zeroes are equal to number of basic variables in Index Row in final Table.
  6. If the number of zeroes are more than number of basic variables in Index Row in final Table, it is a case of more than one optimal solutions.
  7. This is a Bounded Solution, as the values of all Basic variables in final table, are finite positive.
  8. This is a Non-degenerate Solution, as value of none of the basic variables is Zero , in final table.
  9. If value of at least one of the basic variables is Zero in Index Row in final Table, it is a Degenerate Solution.

- **Duality With Example**

1. Case-1

Max.Z = $x_1 - x_2 + 3x_3$

s/t $x_1 + x_2 + x_3 \le 10$

$2x_1 - x_2 - x_3 \le 2$

$2x_1 - 2x_2 - 3x_3 \le 6$

$x_1, x_2, x_3 \ge 0$

Dual of this would be

Min Z = $10y_1 + 2y_2 + 6y_3$

s/t $y_1 + 2y_2 + 2y_3 \ge 1$

$y_1 - y_2 - 2y_3 \ge -1$

$y_1 - y_2 - 3y_3 \ge 3$

$y_1, y_3, y_3 \ge 0$,

2. Case -2

Min Z = $20x_1 + 23x_2$

s/t $-4x_1 - x_2 \leq -8$

$5x_1 - 3x_2 \leq -4$

$x_1, x_2 \geq 0$

Dual of this would be

Max Z = $8y_1 + 4y_2$

S/t $4y_1 - 5y_2 \leq 20$

$y_1 + 3y_2 \leq 23$

$y_1, y_2 \geq 0$,

3. Case-3

Max.Zp = $x_1+2x_2-3x_3$

s/t $2x_1 + x_2 + x_3 \leq 10$

$3x_1 - x_2 + 2x_3 \geq 110$(this need to be converted in less than form)

$x_1 + 2x_2 - x_3 = 4$

$x_1, x_3 \geq 0$, x_2 unrestricted

Dual of this would be

Min. $Z_d = 10y_1 - 110y_2 + 4y_3$

s/t $2y_1 - 3y_2 + y_3 \geq 1$

$y_1 + y_2 + 2y_3 = 2$

$y_1 - 2y_2 - y_3 \geq -3$

$y_1, y_2 \geq 0$, y_3 unrestricted

Assignment

- **Steps of Solving Assignment Problem-**

  1. Identify the minimum element in each row and **subtract** it from every element of that row.

  2. Identify the minimum element in each column and subtract it from every element of that column

  3. Make the assignments for the reduced matrix obtained from **steps 1 and 2** in the following way:

  I. For each row or column with a single zero value cell that has not be assigned or eliminated, box ☐ that zero value as an assigned cell.

  II. For every zero that becomes assigned, cross out (X) all other zeros in the same row and the same column.

  III. If for a row and a column, there are two or more zeros and one cannot be chosen by inspection, choose the cell arbitrarily for assignment.

  IV. The above process may be continued until every zero cell is either assigned ☐ or crossed (X)

  4. An optimal assignment is found, if the number of assigned cells equals the number of rows (and columns). In case you have chosen a zero cell arbitrarily, there may be alternate optimal solutions. If no optimal solution is found, go to step 5.

  5. Draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix obtained from step 3 by adopting the following procedure:

  I. Mark all the rows that do not have assignments.

  II. Mark all the columns (not already marked) which have zeros in the marked rows.

  III. Mark all the rows (not already marked) that have assignments in marked columns.

  IV. Repeat steps 5 (ii) and (iii) until no more rows or columns can be marked.

  V. Draw straight lines through all unmarked rows and marked columns.

1. Select the smallest element (i.e., 1) from all the uncovered elements. Subtract this smallest element from all the uncovered elements and add it to the elements, which lie at the intersection of two lines. Thus, we obtain another reduced matrix for fresh assignment.

Note- For maximization problem the matrix is converted into minimization problem by **subtracting** each element by the element having maximum value in the matrix.

Queuing Theory

Arrival rate is λ and mean service rate is denoted by μ.

- **Kendall's notation:-** (a/b/c) : (d/e)
  - a = Probability law for the arrival time
  - b = Probability law according to which the customers are being served.
  - c = number of channels
  - d = capacity of the system
  - e = queue discipline.
- **Traffic Intensity(ρ)** = Mean arrival rate/ Mean service rate = λ / μ
- **Formulas of Queuing Theory**
  1. Expected number of customers in the system (Ls) = $\lambda / (\mu - \lambda)$
  2. Expected number of customers in the queue (Lq) = $\lambda^2 / \mu (\mu - \lambda)$
  3. Expected waiting time for a customer in the queue(Wq) = $\lambda / \mu (\mu - \lambda)$
  4. Expected waiting time for a customer in the system (Ws) = $1 / (\mu - \lambda)$
  5. Probability that the queue is non-empty $P(n>1) = (\lambda / \mu)^2$
  6. Probability that the number of customers ,n in the system exceeds a given number, $P(n>=k) = (\lambda / \mu)k$
  7. Expected length of non-empty queue = $\mu / (\mu - \lambda)$.

Sequencing and Scheduling

Sequencing

1. Johnson's Problem

1. 2 machines and n jobs

Step 1: Find the minimum among various t_{i1} and t_{i2}.

Step 2a : If the minimum processing time requires machine 1, place the associated job in the first available position in sequence. Go to Step 3.

Step 2b : If the minimum processing time requires machine 2, place the associated job in the last available position in sequence. Go to Step 3.

Step 3: Remove the assigned job from consideration and return to Step 1 until all positions in sequence are filled.

2. 3 machines and n jobs

two conditions of this approach

- The smallest processing time on machine A is greater than or equal to the greatest processing time on machine B, i.e.,
  Min. $(A_i) \geq$ Max. (B_i)
- The smallest processing time on machine C is greater than or equal to the greatest processing time on machine B, i.e.,
  Max. $(B_i) \leq$ Min. (C_i)

If **either or both** of the above conditions are satisfied, then we replace the three machines by two fictitious machines G & H with corresponding processing times given by

$G_i = A_i + B_i$

$H_i = B_i + C_i$

Where G_i & H_i are the processing times for ith job on machine G and H respectively

- **Time Study-**
  Normal time = Observed time x Rating factor
  Standard time = Normal time + allowances

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Vector Algebra

If $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are orthonormal vectors and $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ then $|\boldsymbol{A}|^2 = A_x^2 + A_y^2 + A_z^2$. [Orthonormal vectors $\equiv$ orthogonal unit vectors.]

Scalar product

$$\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}|\cos\theta \qquad \text{where } \theta \text{ is the angle between the vectors}$$

$$= A_xB_x + A_yB_y + A_zB_z = [\,A_xA_yA_z\,]\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Scalar multiplication is commutative: $\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{A}$.

Equation of a line

A point $\boldsymbol{r} \equiv (x, y, z)$ lies on a line passing through a point $\boldsymbol{a}$ and parallel to vector $\boldsymbol{b}$ if

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$$

with λ a real number.

Equation of a plane

A point $\boldsymbol{r} \equiv (x, y, z)$ is on a plane if either

(a) $\boldsymbol{r} \cdot \widehat{\boldsymbol{d}} = |\boldsymbol{d}|$, where $\boldsymbol{d}$ is the normal from the origin to the plane, or

(b) $\dfrac{x}{X} + \dfrac{y}{Y} + \dfrac{z}{Z} = 1$ where X, Y, Z are the intercepts on the axes.

Vector product

$\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{n}\,|\boldsymbol{A}|\,|\boldsymbol{B}|\sin\theta$, where θ is the angle between the vectors and $\boldsymbol{n}$ is a unit vector normal to the plane containing $\boldsymbol{A}$ and $\boldsymbol{B}$ in the direction for which $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{n}$ form a right-handed set of axes.

$\boldsymbol{A} \times \boldsymbol{B}$ in determinant form

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$\boldsymbol{A} \times \boldsymbol{B}$ in matrix form

$$\begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Vector multiplication is not commutative: $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$.

Scalar triple product

$$\boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B} \times \boldsymbol{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = -\boldsymbol{A} \times \boldsymbol{C} \cdot \boldsymbol{B}, \quad \text{etc.}$$

Vector triple product

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}, \qquad (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{B} \cdot \boldsymbol{C})\boldsymbol{A}$$

Non-orthogonal basis

$$\begin{aligned} \boldsymbol{A} &= A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3 \\ A_1 &= \boldsymbol{\epsilon}' \cdot \boldsymbol{A} \quad \text{where} \quad \boldsymbol{\epsilon}' = \frac{\boldsymbol{e}_2 \times \boldsymbol{e}_3}{\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)} \end{aligned}$$

Similarly for A_2 and A_3.

Summation convention

$$\begin{aligned} \boldsymbol{a} &= a_i\boldsymbol{e}_i && \text{implies summation over } i = 1\ldots3 \\ \boldsymbol{a} \cdot \boldsymbol{b} &= a_i b_i \\ (\boldsymbol{a} \times \boldsymbol{b})_i &= \varepsilon_{ijk} a_j b_k && \text{where } \varepsilon_{123} = 1; \quad \varepsilon_{ijk} = -\varepsilon_{ikj} \\ \varepsilon_{ijk}\varepsilon_{klm} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \end{aligned}$$

Matrix Algebra

Unit matrices

The unit matrix I of order n is a square matrix with all diagonal elements equal to one and all off-diagonal elements zero, i.e., $(I)_{ij} = \delta_{ij}$. If A is a square matrix of order n, then $AI = IA = A$. Also $I = I^{-1}$.
I is sometimes written as I_n if the order needs to be stated explicitly.

Products

If A is a $(n \times l)$ matrix and B is a $(l \times m)$ then the product AB is defined by

$$(AB)_{ij} = \sum_{k=1}^{l} A_{ik} B_{kj}$$

In general $AB \neq BA$.

Transpose matrices

If A is a matrix, then transpose matrix A^T is such that $(A^T)_{ij} = (A)_{ji}$.

Inverse matrices

If A is a square matrix with non-zero determinant, then its inverse A^{-1} is such that $AA^{-1} = A^{-1}A = I$.

$$(A^{-1})_{ij} = \frac{\text{transpose of cofactor of } A_{ij}}{|A|}$$

where the cofactor of A_{ij} is $(-1)^{i+j}$ times the determinant of the matrix A with the j-th row and i-th column deleted.

Determinants

If A is a square matrix then the determinant of A, $|A|$ ($\equiv \det A$) is defined by

$$|A| = \sum_{i,j,k,\ldots} \epsilon_{ijk\ldots} A_{1i} A_{2j} A_{3k} \ldots$$

where the number of the suffixes is equal to the order of the matrix.

2×2 matrices

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then,

$$|A| = ad - bc \qquad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Product rules

$$(AB \ldots N)^T = N^T \ldots B^T A^T$$

$$(AB \ldots N)^{-1} = N^{-1} \ldots B^{-1} A^{-1} \qquad \text{(if individual inverses exist)}$$

$$|AB \ldots N| = |A|\,|B| \ldots |N| \qquad \text{(if individual matrices are square)}$$

Orthogonal matrices

An orthogonal matrix Q is a square matrix whose columns q_i form a set of orthonormal vectors. For any orthogonal matrix Q,

$$Q^{-1} = Q^T, \quad |Q| = \pm 1, \quad Q^T \text{ is also orthogonal.}$$

Solving sets of linear simultaneous equations

If A is square then $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ if A^{-1} exists, i.e., if $|A| \neq 0$.
If A is square then $A\boldsymbol{x} = 0$ has a non-trivial solution if and only if $|A| = 0$.
An over-constrained set of equations $A\boldsymbol{x} = \boldsymbol{b}$ is one in which A has m rows and n columns, where m (the number of equations) is greater than n (the number of variables). The best solution x (in the sense that it minimizes the error $|A\boldsymbol{x} - \boldsymbol{b}|$) is the solution of the n equations $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$. If the columns of A are orthonormal vectors then $\boldsymbol{x} = A^T\boldsymbol{b}$.

Hermitian matrices

The Hermitian conjugate of A is $A^\dagger = (A^*)^T$, where A^* is a matrix each of whose components is the complex conjugate of the corresponding components of A. If $A = A^\dagger$ then A is called a Hermitian matrix.

Eigenvalues and eigenvectors

The n eigenvalues λ_i and eigenvectors $\boldsymbol{u}_i$ of an $n \times n$ matrix A are the solutions of the equation $A\boldsymbol{u} = \lambda\boldsymbol{u}$. The eigenvalues are the zeros of the polynomial of degree n, $P_n(\lambda) = |A - \lambda I|$. If A is Hermitian then the eigenvalues λ_i are real and the eigenvectors u_i are mutually orthogonal. $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A.

$$\mathrm{Tr}\, A = \sum_i \lambda_i, \quad \text{also } |A| = \prod_i \lambda_i.$$

If S is a symmetric matrix, Λ is the diagonal matrix whose diagonal elements are the eigenvalues of S, and U is the matrix whose columns are the normalized eigenvectors of A, then

$$U^T S U = \Lambda \qquad \text{and} \qquad S = U \Lambda U^T.$$

If x is an approximation to an eigenvector of A then $\boldsymbol{x}^T A\boldsymbol{x}/(\boldsymbol{x}^T\boldsymbol{x})$ (Rayleigh's quotient) is an approximation to the corresponding eigenvalue.

Commutators

$$\begin{aligned}
[A, B] &\equiv AB - BA \\
[A, B] &= -[B, A] \\
[A, B]^\dagger &= [B^\dagger, A^\dagger] \\
[A + B, C] &= [A, C] + [B, C] \\
[AB, C] &= A[B, C] + [A, C]B \\
[A, [B, C]] &+ [B, [C, A]] + [C, [A, B]] = 0
\end{aligned}$$

Hermitian algebra

$$\boldsymbol{b}^\dagger = (b_1^*, b_2^*, \ldots)$$

	Matrix form	Operator form	Bra-ket form
Hermiticity	$\boldsymbol{b}^* \cdot A \cdot \boldsymbol{c} = (A \cdot \boldsymbol{b})^* \cdot \boldsymbol{c}$	$\int \psi^* O \phi = \int (O\psi)^* \phi$	$\langle \psi \vert O \vert \phi \rangle$
Eigenvalues, λ real	$A\boldsymbol{u}_i = \lambda_{(i)} \boldsymbol{u}_i$	$O\psi_i = \lambda_{(i)} \psi_i$	$O \vert i\rangle = \lambda_i \vert i\rangle$
Orthogonality	$\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0$	$\int \psi_i^* \psi_j = 0$	$\langle i \vert j \rangle = 0 \qquad (i \neq j)$
Completeness	$\boldsymbol{b} = \sum_i \boldsymbol{u}_i (\boldsymbol{u}_i \cdot \boldsymbol{b})$	$\phi = \sum_i \psi_i \left(\int \psi_i^* \phi \right)$	$\phi = \sum_i \vert i\rangle \langle i \vert \phi \rangle$
Rayleigh–Ritz			
Lowest eigenvalue	$\lambda_0 \leq \dfrac{\boldsymbol{b}^* \cdot A \cdot \boldsymbol{b}}{\boldsymbol{b}^* \cdot \boldsymbol{b}}$	$\lambda_0 \leq \dfrac{\int \psi^* O \psi}{\int \psi^* \psi}$	$\dfrac{\langle \psi \vert O \vert \psi \rangle}{\langle \psi \vert \psi \rangle}$

Pauli spin matrices

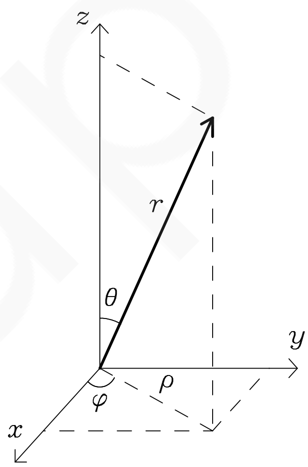
$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_x\sigma_y = \mathrm{i}\sigma_z, \qquad \sigma_y\sigma_z = \mathrm{i}\sigma_x, \qquad \sigma_z\sigma_x = \mathrm{i}\sigma_y, \qquad \sigma_x\sigma_x = \sigma_y\sigma_y = \sigma_z\sigma_z = I$$

Vector Calculus

Notation

ϕ is a scalar function of a set of position coordinates. In Cartesian coordinates $\phi = \phi(x, y, z)$; in cylindrical polar coordinates $\phi = \phi(\rho, \varphi, z)$; in spherical polar coordinates $\phi = \phi(r, \theta, \varphi)$; in cases with radial symmetry $\phi = \phi(r)$. $\boldsymbol{A}$ is a vector function whose components are scalar functions of the position coordinates: in Cartesian coordinates $\boldsymbol{A} = \boldsymbol{i}A_x + \boldsymbol{j}A_y + \boldsymbol{k}A_z$, where A_x, A_y, A_z are independent functions of x, y, z.



In Cartesian coordinates ∇ ('del') $\equiv \boldsymbol{i}\dfrac{\partial}{\partial x} + \boldsymbol{j}\dfrac{\partial}{\partial y} + \boldsymbol{k}\dfrac{\partial}{\partial z} \equiv \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix}$

$$\operatorname{grad}\phi = \nabla\phi, \qquad \operatorname{div}\boldsymbol{A} = \nabla\cdot\boldsymbol{A}, \qquad \operatorname{curl}A = \nabla\times\boldsymbol{A}$$

Identities

$$\operatorname{grad}(\phi_1 + \phi_2) \equiv \operatorname{grad}\phi_1 + \operatorname{grad}\phi_2 \qquad \operatorname{div}(\boldsymbol{A}_1 + \boldsymbol{A}_2) \equiv \operatorname{div}\boldsymbol{A}_1 + \operatorname{div}\boldsymbol{A}_2$$

$$\operatorname{grad}(\phi_1\phi_2) \equiv \phi_1\operatorname{grad}\phi_2 + \phi_2\operatorname{grad}\phi_1$$

$$\operatorname{curl}(\boldsymbol{A_1} + \boldsymbol{A_2}) \equiv \operatorname{curl}\boldsymbol{A}_1 + \operatorname{curl}\boldsymbol{A}_2$$

$$\operatorname{div}(\phi\boldsymbol{A}) \equiv \phi\operatorname{div}\boldsymbol{A} + (\operatorname{grad}\phi)\cdot\boldsymbol{A}, \qquad \operatorname{curl}(\phi\boldsymbol{A}) \equiv \phi\operatorname{curl}\boldsymbol{A} + (\operatorname{grad}\phi)\times\boldsymbol{A}$$

$$\operatorname{div}(\boldsymbol{A}_1\times\boldsymbol{A}_2) \equiv \boldsymbol{A}_2\cdot\operatorname{curl}\boldsymbol{A}_1 - \boldsymbol{A}_1\cdot\operatorname{curl}\boldsymbol{A}_2$$

$$\operatorname{curl}(\boldsymbol{A}_1\times\boldsymbol{A}_2) \equiv \boldsymbol{A}_1\operatorname{div}\boldsymbol{A}_2 - \boldsymbol{A}_2\operatorname{div}\boldsymbol{A}_1 + (\boldsymbol{A}_2\cdot\operatorname{grad})\boldsymbol{A}_1 - (\boldsymbol{A}_1\cdot\operatorname{grad})\boldsymbol{A}_2$$

$$\operatorname{div}(\operatorname{curl}\boldsymbol{A}) \equiv 0, \qquad \operatorname{curl}(\operatorname{grad}\phi) \equiv 0$$

$$\operatorname{curl}(\operatorname{curl}\boldsymbol{A}) \equiv \operatorname{grad}(\operatorname{div}\boldsymbol{A}) - \operatorname{div}(\operatorname{grad}\boldsymbol{A}) \equiv \operatorname{grad}(\operatorname{div}\boldsymbol{A}) - \nabla^2\boldsymbol{A}$$

$$\operatorname{grad}(\boldsymbol{A}_1\cdot\boldsymbol{A}_2) \equiv \boldsymbol{A}_1\times(\operatorname{curl}\boldsymbol{A}_2) + (\boldsymbol{A}_1\cdot\operatorname{grad})\boldsymbol{A}_2 + \boldsymbol{A}_2\times(\operatorname{curl}\boldsymbol{A}_1) + (\boldsymbol{A}_2\cdot\operatorname{grad})\boldsymbol{A}_1$$

Grad, Div, Curl and the Laplacian

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Cartesian Coordinates		$x = \rho\cos\varphi \quad y = \rho\sin\varphi \quad z = z$	$x = r\cos\varphi\sin\theta \quad y = r\sin\varphi\sin\theta$ $z = r\cos\theta$
Vector $\boldsymbol{A}$	$A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$	$A_\rho\widehat{\boldsymbol{\rho}} + A_\varphi\widehat{\boldsymbol{\varphi}} + A_z\widehat{\boldsymbol{z}}$	$A_r\widehat{\boldsymbol{r}} + A_\theta\widehat{\boldsymbol{\theta}} + A_\varphi\widehat{\boldsymbol{\varphi}}$
Gradient $\nabla\phi$	$\frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$	$\frac{\partial\phi}{\partial\rho}\widehat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\widehat{\boldsymbol{\varphi}} + \frac{\partial\phi}{\partial z}\widehat{\boldsymbol{z}}$	$\frac{\partial\phi}{\partial r}\widehat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\widehat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\widehat{\boldsymbol{\varphi}}$
Divergence $\nabla\cdot\boldsymbol{A}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times\boldsymbol{A}$	$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{\rho}\widehat{\boldsymbol{\rho}} & \widehat{\boldsymbol{\varphi}} & \frac{1}{\rho}\widehat{\boldsymbol{z}} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{r^2\sin\theta}\widehat{\boldsymbol{r}} & \frac{1}{r\sin\theta}\widehat{\boldsymbol{\theta}} & \frac{1}{r}\widehat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian $\nabla^2\phi$	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

Transformation of integrals

L = the distance along some curve 'C' in space and is measured from some fixed point.

S = a surface area

τ = a volume contained by a specified surface

$\widehat{\boldsymbol{t}}$ = the unit tangent to C at the point P

$\widehat{\boldsymbol{n}}$ = the unit outward pointing normal

$\boldsymbol{A}$ = some vector function

$\mathrm{d}\boldsymbol{L}$ = the vector element of curve ($= \widehat{\boldsymbol{t}}\,\mathrm{d}L$)

$\mathrm{d}\boldsymbol{S}$ = the vector element of surface ($= \widehat{\boldsymbol{n}}\,\mathrm{d}S$)

Then $\int_C \boldsymbol{A}\cdot\widehat{\boldsymbol{t}}\,\mathrm{d}L = \int_C \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{L}$

and when $\boldsymbol{A} = \nabla\phi$

$$\int_C (\nabla\phi)\cdot\mathrm{d}\boldsymbol{L} = \int_C \mathrm{d}\phi$$

Gauss's Theorem (Divergence Theorem)

When S defines a closed region having a volume τ

$$\int_\tau (\nabla\cdot\boldsymbol{A})\,\mathrm{d}\tau = \int_S (\boldsymbol{A}\cdot\widehat{\boldsymbol{n}})\,\mathrm{d}S = \int_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}$$

also

$$\int_\tau (\nabla\phi)\,\mathrm{d}\tau = \int_S \phi\,\mathrm{d}\boldsymbol{S} \qquad \int_\tau (\nabla\times\boldsymbol{A})\,\mathrm{d}\tau = \int_S (\widehat{\boldsymbol{n}}\times\boldsymbol{A})\,\mathrm{d}S$$

Stokes's Theorem

When C is closed and bounds the open surface S,

$$\int_S (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} = \int_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{L}$$

also

$$\int_S (\widehat{\boldsymbol{n}} \times \nabla\phi)\ \mathrm{d}S = \int_C \phi\ \mathrm{d}\boldsymbol{L}$$

Green's Theorem

$$\begin{aligned}\int_S \psi\nabla\phi \cdot \mathrm{d}\boldsymbol{S} &= \int_\tau \nabla \cdot (\psi\nabla\phi)\ \mathrm{d}\tau \\ &= \int_\tau \left[\psi\nabla^2\phi + (\nabla\psi)\cdot(\nabla\phi)\right]\ \mathrm{d}\tau\end{aligned}$$

Green's Second Theorem

$$\int_\tau (\psi\nabla^2\phi - \phi\nabla^2\psi)\ \mathrm{d}\tau = \int_S [\psi(\nabla\phi) - \phi(\nabla\psi)] \cdot \mathrm{d}\boldsymbol{S}$$

Complex Variables

Complex numbers

The complex number $z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{e}^{\mathrm{i}(\theta+2n\pi)}$, where $\mathrm{i}^2 = -1$ and n is an arbitrary integer. The real quantity r is the modulus of z and the angle θ is the argument of z. The complex conjugate of z is $z^* = x - \mathrm{i}y = r(\cos\theta - \mathrm{i}\sin\theta) = r\,\mathrm{e}^{-\mathrm{i}\theta}$; $zz^* = |z|^2 = x^2 + y^2$

De Moivre's theorem

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \mathrm{e}^{\mathrm{i}n\theta} = \cos n\theta + \mathrm{i}\sin n\theta$$

Power series for complex variables.

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \qquad \text{convergent for all finite } z$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad \text{convergent for all finite } z$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \qquad \text{convergent for all finite } z$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad \text{principal value of } \ln(1+z)$$

This last series converges both on and within the circle $|z| = 1$ except at the point $z = -1$.

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

This last series converges both on and within the circle $|z| = 1$ except at the points $z = \pm\mathrm{i}$.

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + \cdots$$

This last series converges both on and within the circle $|z| = 1$ except at the point $z = -1$.

Trigonometric Formulae

$$\cos^2 A + \sin^2 A = 1 \qquad \sec^2 A - \tan^2 A = 1 \qquad \operatorname{cosec}^2 A - \cot^2 A = 1$$

$$\sin 2A = 2\sin A\cos A \qquad \cos 2A = \cos^2 A - \sin^2 A \qquad \tan 2A = \frac{2\tan A}{1-\tan^2 A}.$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B \qquad \cos A\cos B = \frac{\cos(A+B)+\cos(A-B)}{2}$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B \qquad \sin A\sin B = \frac{\cos(A-B)-\cos(A+B)}{2}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B} \qquad \sin A\cos B = \frac{\sin(A+B)+\sin(A-B)}{2}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \qquad \cos^2 A = \frac{1+\cos 2A}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \qquad \sin^2 A = \frac{1-\cos 2A}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \qquad \cos^3 A = \frac{3\cos A + \cos 3A}{4}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2} \qquad \sin^3 A = \frac{3\sin A - \sin 3A}{4}$$

Relations between sides and angles of any plane triangle

In a plane triangle with angles A, B, and C and sides opposite a, b, and c respectively,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter of circumscribed circle.}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a = b\cos C + c\cos B$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

$$\text{area} = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \sqrt{s(s-a)(s-b)(s-c)}, \qquad \text{where } s = \frac{1}{2}(a+b+c)$$

Relations between sides and angles of any spherical triangle

In a spherical triangle with angles A, B, and C and sides opposite a, b, and c respectively,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

$$\cos a = \cos b\cos c + \sin b\sin c\cos A$$

$$\cos A = -\cos B\cos C + \sin B\sin C\cos a$$

Hyperbolic Functions

$$\cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \text{valid for all } x$$

$$\sinh x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{valid for all } x$$

$\cosh \mathrm{i}x = \cos x$

$\cos \mathrm{i}x = \cosh x$

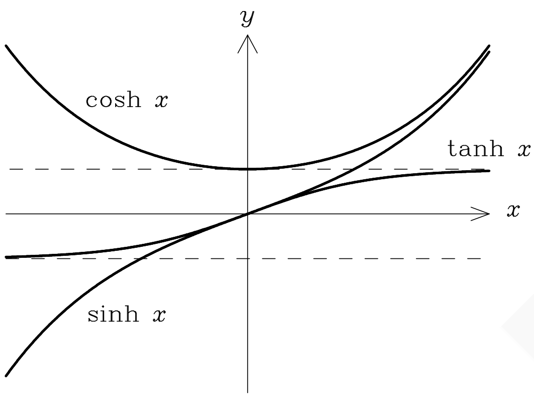
$\sinh \mathrm{i}x = \mathrm{i} \sin x$

$\sin \mathrm{i}x = \mathrm{i} \sinh x$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x} \qquad \operatorname{cosech} x = \frac{1}{\sinh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$



For large positive x:

$$\cosh x \approx \sinh x \to \frac{\mathrm{e}^x}{2}$$

$$\tanh x \to 1$$

For large negative x:

$$\cosh x \approx -\sinh x \to \frac{\mathrm{e}^{-x}}{2}$$

$$\tanh x \to -1$$

Relations of the functions

$\sinh x = -\sinh(-x)$

$\operatorname{sech} x = \operatorname{sech}(-x)$

$\cosh x = \cosh(-x)$

$\operatorname{cosech} x = -\operatorname{cosech}(-x)$

$\tanh x = -\tanh(-x)$

$\coth x = -\coth(-x)$

$$\sinh x = \frac{2\tanh(x/2)}{1 - \tanh^2(x/2)} = \frac{\tanh x}{\sqrt{1 - \tanh^2 x}}$$

$$\cosh x = \frac{1 + \tanh^2(x/2)}{1 - \tanh^2(x/2)} = \frac{1}{\sqrt{1 - \tanh^2 x}}$$

$$\tanh x = \sqrt{1 - \operatorname{sech}^2 x} \qquad \operatorname{sech} x = \sqrt{1 - \tanh^2 x}$$

$$\coth x = \sqrt{\operatorname{cosech}^2 x + 1} \qquad \operatorname{cosech} x = \sqrt{\coth^2 x - 1}$$

$$\sinh(x/2) = \sqrt{\frac{\cosh x - 1}{2}} \qquad \cosh(x/2) = \sqrt{\frac{\cosh x + 1}{2}}$$

$$\tanh(x/2) = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$$\sinh(2x) = 2\sinh x \cosh x \qquad \tanh(2x) = \frac{2\tanh x}{1 + \tanh^2 x}$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

$$\sinh(3x) = 3\sinh x + 4\sinh^3 x \qquad \cosh 3x = 4\cosh^3 x - 3\cosh x$$

$$\tanh(3x) = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\sinh x + \sinh y = 2 \sinh \frac{1}{2}(x+y) \cosh \frac{1}{2}(x-y) \qquad \cosh x + \cosh y = 2 \cosh \frac{1}{2}(x+y) \cosh \frac{1}{2}(x-y)$$

$$\sinh x - \sinh y = 2 \cosh \frac{1}{2}(x+y) \sinh \frac{1}{2}(x-y) \qquad \cosh x - \cosh y = 2 \sinh \frac{1}{2}(x+y) \sinh \frac{1}{2}(x-y)$$

$$\sinh x \pm \cosh x = \frac{1 \pm \tanh(x/2)}{1 \mp \tanh(x/2)} = e^{\pm x}$$

$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}$$

$$\coth x \pm \coth y = \pm \frac{\sinh(x \pm y)}{\sinh x \sinh y}$$

Inverse functions

$$\sinh^{-1} \frac{x}{a} = \ln\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right) \qquad \text{for } -\infty < x < \infty$$

$$\cosh^{-1} \frac{x}{a} = \ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) \qquad \text{for } x \geq a$$

$$\tanh^{-1} \frac{x}{a} = \frac{1}{2} \ln\left(\frac{a+x}{a-x}\right) \qquad \text{for } x^2 < a^2$$

$$\coth^{-1} \frac{x}{a} = \frac{1}{2} \ln\left(\frac{x+a}{x-a}\right) \qquad \text{for } x^2 > a^2$$

$$\operatorname{sech}^{-1} \frac{x}{a} = \ln\left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2} - 1}\right) \qquad \text{for } 0 < x \leq a$$

$$\operatorname{cosech}^{-1} \frac{x}{a} = \ln\left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2} + 1}\right) \qquad \text{for } x \neq 0$$

Limits

$n^c x^n \to 0$ as $n \to \infty$ if $|x| < 1$ (any fixed c)

$x^n/n! \to 0$ as $n \to \infty$ (any fixed x)

$(1 + x/n)^n \to e^x$ as $n \to \infty$, $x \ln x \to 0$ as $x \to 0$

If $f(a) = g(a) = 0$ then $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$ (l'Hôpital's rule)

Differentiation

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \cdots + {}^nC_r u^{(n-r)}v^{(r)} + \cdots + uv^{(n)}$$ Leibniz Theorem

$$\text{where } {}^nC_r \equiv \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \cos x & \frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) &= \cosh x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) &= -\sin x & \frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) &= \sinh x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) &= \sec^2 x & \frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) &= \mathrm{sech}^2\, x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) &= \sec x \tan x & \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{sech}\, x) &= -\mathrm{sech}\, x \tanh x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) &= -\mathrm{cosec}^2\, x & \frac{\mathrm{d}}{\mathrm{d}x}(\coth x) &= -\mathrm{cosech}^2\, x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosec}\, x) &= -\mathrm{cosec}\, x \cot x & \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosech}\, x) &= -\mathrm{cosech}\, x \coth x
\end{aligned}$$

Integration

Standard forms

$$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$$ for $n \neq -1$

$$\int \frac{1}{x}\,\mathrm{d}x = \ln x + c \qquad \int \ln x \,\mathrm{d}x = x(\ln x - 1) + c$$

$$\int e^{ax}\,\mathrm{d}x = \frac{1}{a}e^{ax} + c \qquad \int x e^{ax}\,\mathrm{d}x = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right) + c$$

$$\int x \ln x \,\mathrm{d}x = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$$

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c$$ for $x^2 < a^2$

$$\int \frac{1}{x^2-a^2}\,\mathrm{d}x = -\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + c = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c$$ for $x^2 > a^2$

$$\int \frac{x}{(x^2 \pm a^2)^n}\,\mathrm{d}x = \frac{-1}{2(n-1)}\frac{1}{(x^2 \pm a^2)^{n-1}} + c$$ for $n \neq 1$

$$\int \frac{x}{x^2 \pm a^2}\,\mathrm{d}x = \frac{1}{2}\ln(x^2 \pm a^2) + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}}\,\mathrm{d}x = \ln\left(x + \sqrt{x^2 \pm a^2}\right) + c$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}}\,\mathrm{d}x = \sqrt{x^2 \pm a^2} + c$$

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right] + c$$

$$\int_0^\infty \frac{1}{(1+x)x^p}\,\mathrm{d}x = \pi \operatorname{cosec} p\pi \qquad \text{for } p<1$$

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\int_{-\infty}^\infty \exp(-x^2/2\sigma^2)\,\mathrm{d}x = \sigma\sqrt{2\pi}$$

$$\int_{-\infty}^\infty x^n \exp(-x^2/2\sigma^2)\,\mathrm{d}x = \begin{cases} 1\times 3\times 5\times\cdots(n-1)\sigma^{n+1}\sqrt{2\pi} & \text{for } n\geq 2 \text{ and even} \\ 0 & \text{for } n\geq 1 \text{ and odd} \end{cases}$$

$$\int \sin x\,\mathrm{d}x = -\cos x + c \qquad \int \sinh x\,\mathrm{d}x = \cosh x + c$$

$$\int \cos x\,\mathrm{d}x = \sin x + c \qquad \int \cosh x\,\mathrm{d}x = \sinh x + c$$

$$\int \tan x\,\mathrm{d}x = -\ln(\cos x) + c \qquad \int \tanh x\,\mathrm{d}x = \ln(\cosh x) + c$$

$$\int \operatorname{cosec} x\,\mathrm{d}x = \ln(\operatorname{cosec} x - \cot x) + c \qquad \int \operatorname{cosech} x\,\mathrm{d}x = \ln\left[\tanh(x/2)\right] + c$$

$$\int \sec x\,\mathrm{d}x = \ln(\sec x + \tan x) + c \qquad \int \operatorname{sech} x\,\mathrm{d}x = 2\tan^{-1}(\mathrm{e}^x) + c$$

$$\int \cot x\,\mathrm{d}x = \ln(\sin x) + c \qquad \int \coth x\,\mathrm{d}x = \ln(\sinh x) + c$$

$$\int \sin mx \sin nx\,\mathrm{d}x = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c \qquad \text{if } m^2 \neq n^2$$

$$\int \cos mx \cos nx\,\mathrm{d}x = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + c \qquad \text{if } m^2 \neq n^2$$

Standard substitutions

If the integrand is a function of:	substitute:
(a^2-x^2) or $\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
(x^2+a^2) or $\sqrt{x^2+a^2}$	$x = a\tan\theta$ or $x = a\sinh\theta$
(x^2-a^2) or $\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $x = a\cosh\theta$

If the integrand is a rational function of $\sin x$ or $\cos x$ or both, substitute $t = \tan(x/2)$ and use the results:

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2}.$$

If the integrand is of the form:	substitute:
$\displaystyle\int \frac{\mathrm{d}x}{(ax+b)\sqrt{px+q}}$	$px+q = u^2$
$\displaystyle\int \frac{\mathrm{d}x}{(ax+b)\sqrt{px^2+qx+r}}$	$ax+b = \dfrac{1}{u}$.

gradeup

Integration by parts

$$\int_a^b u\ \mathrm{d}v = uv\Big|_a^b - \int_a^b v\ \mathrm{d}u$$

Differentiation of an integral

If $f(x,\alpha)$ is a function of x containing a parameter α and the limits of integration a and b are functions of α then

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\ \mathrm{d}x = f(b,\alpha)\frac{\mathrm{d}b}{\mathrm{d}\alpha} - f(a,\alpha)\frac{\mathrm{d}a}{\mathrm{d}\alpha} + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x,\alpha)\ \mathrm{d}x.$$

Special case,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(y)\ \mathrm{d}y = f(x).$$

Dirac δ-'function'

$$\delta(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[\mathrm{i}\omega(t-\tau)]\ \mathrm{d}\omega.$$

If $f(t)$ is an arbitrary function of t then $\int_{-\infty}^{\infty} \delta(t-\tau) f(t)\ \mathrm{d}t = f(\tau)$.

$\delta(t) = 0$ if $t \neq 0$, also $\int_{-\infty}^{\infty} \delta(t)\ \mathrm{d}t = 1$

Reduction formulae

Factorials

$n! = n(n-1)(n-2)\ldots 1, \qquad 0! = 1.$

Stirling's formula for large n: $\quad \ln(n!) \approx n\ln n - n.$

For any $p > -1$, $\int_0^{\infty} x^p\, \mathrm{e}^{-x}\ \mathrm{d}x = p\int_0^{\infty} x^{p-1}\, \mathrm{e}^{-x}\ \mathrm{d}x = p!. \qquad (-1/2)! = \sqrt{\pi}, \qquad (1/2)! = \sqrt{\pi}/2$, etc.

For any $p, q > -1$, $\int_0^1 x^p(1-x)^q\ \mathrm{d}x = \frac{p!q!}{(p+q+1)!}$.

Trigonometrical

If m, n are integers,

$$\int_0^{\pi/2} \sin^m\theta\ \cos^n\theta\ \mathrm{d}\theta = \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2}\theta\ \cos^n\theta\ \mathrm{d}\theta = \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m\theta\ \cos^{n-2}\theta\ \mathrm{d}\theta$$

and can therefore be reduced eventually to one of the following integrals

$$\int_0^{\pi/2} \sin\theta\ \cos\theta\ \mathrm{d}\theta = \frac{1}{2}, \qquad \int_0^{\pi/2} \sin\theta\ \mathrm{d}\theta = 1, \qquad \int_0^{\pi/2} \cos\theta\ \mathrm{d}\theta = 1, \qquad \int_0^{\pi/2} \mathrm{d}\theta = \frac{\pi}{2}.$$

Other

If $I_n = \int_0^{\infty} x^n \exp(-\alpha x^2)\ \mathrm{d}x \quad$ then $\quad I_n = \frac{(n-1)}{2\alpha} I_{n-2}, \quad I_0 = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}, \quad I_1 = \frac{1}{2\alpha}.$

Differential Equations

Diffusion (conduction) equation

$$\frac{\partial \psi}{\partial t} = \kappa \nabla^2 \psi$$

Wave equation

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

Legendre's equation

$$(1 - x^2) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x \frac{\mathrm{d}y}{\mathrm{d}x} + l(l+1)y = 0,$$

solutions of which are Legendre polynomials $P_l(x)$, where $P_l(x) = \frac{1}{2^l l!} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^l (x^2 - 1)^l$, *Rodrigues' formula* so $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$ etc.

Recursion relation

$$P_l(x) = \frac{1}{l} \left[(2l - 1) x P_{l-1}(x) - (l - 1) P_{l-2}(x) \right]$$

Orthogonality

$$\int_{-1}^{1} P_l(x) P_{l'}(x) \, \mathrm{d}x = \frac{2}{2l + 1} \delta_{ll'}$$

Bessel's equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - m^2) y = 0,$$

solutions of which are Bessel functions $J_m(x)$ of order m.

Series form of Bessel functions of the first kind

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{m+2k}}{k!(m+k)!} \qquad \text{(integer } m\text{)}.$$

The same general form holds for non-integer $m > 0$.

Laplace's equation

$$\nabla^2 u = 0$$

If expressed in two-dimensional polar coordinates (see section 4), a solution is

$$u(\rho, \varphi) = \left[A\rho^n + B\rho^{-n}\right]\left[C\exp(in\varphi) + D\exp(-in\varphi)\right]$$

where A, B, C, D are constants and n is a real integer.

If expressed in three-dimensional polar coordinates (see section 4) a solution is

$$u(r, \theta, \varphi) = \left[Ar^l + Br^{-(l+1)}\right] P_l^m \left[C\sin m\varphi + D\cos m\varphi\right]$$

where l and m are integers with $l \geq |m| \geq 0$; A, B, C, D are constants;

$$P_l^m(\cos\theta) = \sin^{|m|}\theta \left[\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}\right]^{|m|} P_l(\cos\theta)$$

is the associated Legendre polynomial.

$$P_l^0(1) = 1.$$

If expressed in cylindrical polar coordinates (see section 4), a solution is

$$u(\rho, \varphi, z) = J_m(n\rho)\left[A\cos m\varphi + B\sin m\varphi\right]\left[C\exp(nz) + D\exp(-nz)\right]$$

where m and n are integers; A, B, C, D are constants.

Spherical harmonics

The normalized solutions $Y_l^m(\theta, \varphi)$ of the equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] Y_l^m + l(l+1)Y_l^m = 0$$

are called spherical harmonics, and have values given by

$$Y_l^m(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}} P_l^m(\cos\theta)\ \mathrm{e}^{\mathrm{i}m\varphi} \times \begin{cases} (-1)^m & \text{for } m \geq 0 \\ 1 & \text{for } m < 0 \end{cases}$$

i.e., $Y_0^0 = \sqrt{\frac{1}{4\pi}}$, $\quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, $\quad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\ \mathrm{e}^{\pm\mathrm{i}\varphi}$, etc.

Orthogonality

$$\int_{4\pi} Y_l^{*m} Y_{l'}^{m'}\ \mathrm{d}\Omega = \delta_{ll'}\delta_{mm'}$$

Calculus of Variations

The condition for $I = \int_a^b F(y, y', x)\ \mathrm{d}x$ to have a stationary value is $\frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)$, where $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. This is the Euler–Lagrange equation.

Functions of Several Variables

If $\phi = f(x, y, z, \ldots)$ then $\dfrac{\partial \phi}{\partial x}$ implies differentiation with respect to x keeping $y, z, \ldots$ constant.

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{d}z + \cdots \quad \text{and} \quad \delta\phi \approx \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z + \cdots$$

where $x, y, z, \ldots$ are independent variables. $\dfrac{\partial \phi}{\partial x}$ is also written as $\left(\dfrac{\partial \phi}{\partial x}\right)_{y,\ldots}$ or $\left.\dfrac{\partial \phi}{\partial x}\right|_{y,\ldots}$ when the variables kept constant need to be stated explicitly.

If ϕ is a well-behaved function then $\dfrac{\partial^2 \phi}{\partial x\,\partial y} = \dfrac{\partial^2 \phi}{\partial y\,\partial x}$ etc.

If $\phi = f(x, y)$,

$$\left(\frac{\partial \phi}{\partial x}\right)_y = \frac{1}{\left(\dfrac{\partial x}{\partial \phi}\right)_y}, \qquad \left(\frac{\partial \phi}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_\phi \left(\frac{\partial y}{\partial \phi}\right)_x = -1.$$

Taylor series for two variables

If $\phi(x, y)$ is well-behaved in the vicinity of $x = a, y = b$ then it has a Taylor series

$$\phi(x, y) = \phi(a + u, b + v) = \phi(a, b) + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + \frac{1}{2!}\left(u^2\frac{\partial^2 \phi}{\partial x^2} + 2uv\frac{\partial^2 \phi}{\partial x\,\partial y} + v^2\frac{\partial^2 \phi}{\partial y^2}\right) + \cdots$$

where $x = a + u, y = b + v$ and the differential coefficients are evaluated at $x = a, \quad y = b$

Stationary points

A function $\phi = f(x, y)$ has a stationary point when $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \phi}{\partial y} = 0$. Unless $\dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{\partial^2 \phi}{\partial x\,\partial y} = 0$, the following conditions determine whether it is a minimum, a maximum or a saddle point.

$$\left.\begin{array}{ll} \text{Minimum:} & \dfrac{\partial^2 \phi}{\partial x^2} > 0, \text{ or } \dfrac{\partial^2 \phi}{\partial y^2} > 0, \\ \text{Maximum:} & \dfrac{\partial^2 \phi}{\partial x^2} < 0, \text{ or } \dfrac{\partial^2 \phi}{\partial y^2} < 0, \end{array}\right\} \text{ and } \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} > \left(\frac{\partial^2 \phi}{\partial x\,\partial y}\right)^2$$

$$\text{Saddle point: } \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} < \left(\frac{\partial^2 \phi}{\partial x\,\partial y}\right)^2$$

If $\dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{\partial^2 \phi}{\partial x\,\partial y} = 0$ the character of the turning point is determined by the next higher derivative.

Changing variables: the chain rule

If $\phi = f(x, y, \ldots)$ and the variables $x, y, \ldots$ are functions of independent variables $u, v, \ldots$ then

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + \cdots$$

$$\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial v} + \cdots$$

etc.

Changing variables in surface and volume integrals – Jacobians

If an area A in the x, y plane maps into an area A' in the u, v plane then

$$\int_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{A'} f(u,v) J\,\mathrm{d}u\,\mathrm{d}v \quad \text{where} \quad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian J is also written as $\dfrac{\partial(x,y)}{\partial(u,v)}$. The corresponding formula for volume integrals is

$$\int_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{V'} f(u,v,w) J\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \qquad \text{where now} \qquad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Fourier Series and Transforms

Fourier series

If $y(x)$ is a function defined in the range $-\pi \le x \le \pi$ then

$$y(x) \approx c_0 + \sum_{m=1}^{M} c_m \cos mx + \sum_{m=1}^{M'} s_m \sin mx$$

where the coefficients are

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(x)\,\mathrm{d}x$$

$$c_m = \frac{1}{\pi}\int_{-\pi}^{\pi} y(x)\cos mx\,\mathrm{d}x \qquad (m = 1, \ldots, M)$$

$$s_m = \frac{1}{\pi}\int_{-\pi}^{\pi} y(x)\sin mx\,\mathrm{d}x \qquad (m = 1, \ldots, M')$$

with convergence to $y(x)$ as $M, M' \to \infty$ for all points where $y(x)$ is continuous.

Fourier series for other ranges

Variable t, range $0 \le t \le T$, (i.e., a periodic function of time with period T, frequency $\omega = 2\pi/T$).

$$y(t) \approx c_0 + \sum c_m \cos m\omega t + \sum s_m \sin m\omega t$$

where

$$c_0 = \frac{\omega}{2\pi}\int_0^T y(t)\,\mathrm{d}t, \quad c_m = \frac{\omega}{\pi}\int_0^T y(t)\cos m\omega t\,\mathrm{d}t, \quad s_m = \frac{\omega}{\pi}\int_0^T y(t)\sin m\omega t\,\mathrm{d}t.$$

Variable x, range $0 \le x \le L$,

$$y(x) \approx c_0 + \sum c_m \cos\frac{2m\pi x}{L} + \sum s_m \sin\frac{2m\pi x}{L}$$

where

$$c_0 = \frac{1}{L}\int_0^L y(x)\,\mathrm{d}x, \quad c_m = \frac{2}{L}\int_0^L y(x)\cos\frac{2m\pi x}{L}\,\mathrm{d}x, \quad s_m = \frac{2}{L}\int_0^L y(x)\sin\frac{2m\pi x}{L}\,\mathrm{d}x.$$

Fourier series for odd and even functions

If $y(x)$ is an *odd* (anti-symmetric) function [i.e., $y(-x) = -y(x)$] defined in the range $-\pi \le x \le \pi$, then only sines are required in the Fourier series and $s_m = \frac{2}{\pi}\int_0^{\pi} y(x)\sin mx \, dx$. If, in addition, $y(x)$ is symmetric about $x = \pi/2$, then the coefficients s_m are given by $s_m = 0$ (for m even), $s_m = \frac{4}{\pi}\int_0^{\pi/2} y(x)\sin mx \, dx$ (for m odd). If $y(x)$ is an *even* (symmetric) function [i.e., $y(-x) = y(x)$] defined in the range $-\pi \le x \le \pi$, then only constant and cosine terms are required in the Fourier series and $c_0 = \frac{1}{\pi}\int_0^{\pi} y(x)\, dx$, $c_m = \frac{2}{\pi}\int_0^{\pi} y(x)\cos mx \, dx$. If, in addition, $y(x)$ is anti-symmetric about $x = \frac{\pi}{2}$, then $c_0 = 0$ and the coefficients c_m are given by $c_m = 0$ (for m even), $c_m = \frac{4}{\pi}\int_0^{\pi/2} y(x)\cos mx \, dx$ (for m odd).

[These results also apply to Fourier series with more general ranges provided appropriate changes are made to the limits of integration.]

Complex form of Fourier series

If $y(x)$ is a function defined in the range $-\pi \le x \le \pi$ then

$$y(x) \approx \sum_{-M}^{M} C_m \, e^{imx}, \quad C_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(x)\, e^{-imx}\, dx$$

with m taking all integer values in the range $\pm M$. This approximation converges to $y(x)$ as $M \to \infty$ under the same conditions as the real form.

For other ranges the formulae are:

Variable t, range $0 \le t \le T$, frequency $\omega = 2\pi/T$,

$$y(t) = \sum_{-\infty}^{\infty} C_m \, e^{im\omega t}, \quad C_m = \frac{\omega}{2\pi}\int_0^{T} y(t)\, e^{-im\omega t}\, dt.$$

Variable x', range $0 \le x' \le L$,

$$y(x') = \sum_{-\infty}^{\infty} C_m \, e^{i2m\pi x'/L}, \quad C_m = \frac{1}{L}\int_0^{L} y(x')\, e^{-i2m\pi x'/L}\, dx'.$$

Discrete Fourier series

If $y(x)$ is a function defined in the range $-\pi \le x \le \pi$ which is sampled in the $2N$ equally spaced points $x_n = nx/N$ $[n = -(N-1)\dots N]$, then

$$\begin{aligned} y(x_n) = {} & c_0 + c_1\cos x_n + c_2\cos 2x_n + \cdots + c_{N-1}\cos(N-1)x_n + c_N\cos Nx_n \\ & + s_1\sin x_n + s_2\sin 2x_n + \cdots + s_{N-1}\sin(N-1)x_n + s_N\sin Nx_n \end{aligned}$$

where the coefficients are

$$\begin{aligned} c_0 &= \frac{1}{2N}\sum y(x_n) \\ c_m &= \frac{1}{N}\sum y(x_n)\cos mx_n && (m = 1, \dots, N-1) \\ c_N &= \frac{1}{2N}\sum y(x_n)\cos Nx_n \\ s_m &= \frac{1}{N}\sum y(x_n)\sin mx_n && (m = 1, \dots, N-1) \\ s_N &= \frac{1}{2N}\sum y(x_n)\sin Nx_n \end{aligned}$$

each summation being over the $2N$ sampling points x_n.

Fourier transforms

If $y(x)$ is a function defined in the range $-\infty \le x \le \infty$ then the Fourier transform $\widehat{y}(\omega)$ is defined by the equations

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{y}(\omega)\, e^{i\omega t}\, d\omega, \qquad \widehat{y}(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-i\omega t}\, dt.$$

If ω is replaced by $2\pi f$, where f is the frequency, this relationship becomes

$$y(t) = \int_{-\infty}^{\infty} \widehat{y}(f)\, e^{i2\pi f t}\, df, \qquad \widehat{y}(f) = \int_{-\infty}^{\infty} y(t)\, e^{-i2\pi f t}\, dt.$$

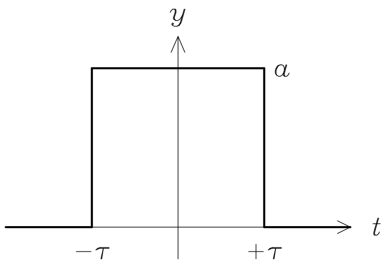
If $y(t)$ is symmetric about $t = 0$ then

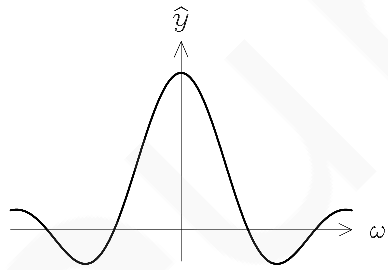
$$y(t) = \frac{1}{\pi}\int_{0}^{\infty} \widehat{y}(\omega) \cos\omega t\, d\omega, \qquad \widehat{y}(\omega) = 2\int_{0}^{\infty} y(t) \cos\omega t\, dt.$$

If $y(t)$ is anti-symmetric about $t = 0$ then

$$y(t) = \frac{1}{\pi}\int_{0}^{\infty} \widehat{y}(\omega) \sin\omega t\, d\omega, \qquad \widehat{y}(\omega) = 2\int_{0}^{\infty} y(t) \sin\omega t\, dt.$$

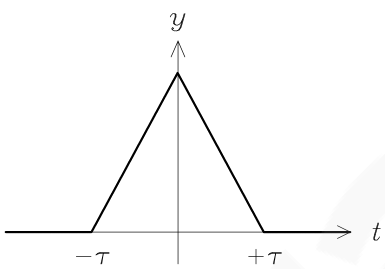
Specific cases

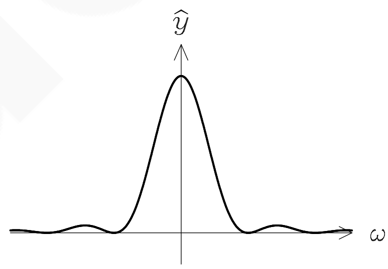




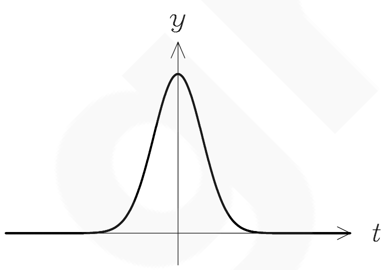
$$y(t) = a, \quad |t| \le \tau \\ \quad = 0, \quad |t| > \tau \Big\} \quad \text{('Top Hat')}, \qquad \widehat{y}(\omega) = 2a\frac{\sin\omega\tau}{\omega} \equiv 2a\tau\,\mathrm{sinc}(\omega\tau)$$

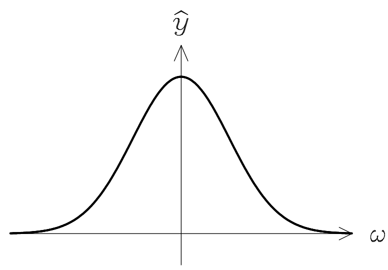
where $\quad \mathrm{sinc}(x) = \dfrac{\sin(x)}{x}$





$$\left.\begin{array}{ll} y(t) = a(1 - |t|/\tau), & |t| \le \tau \\ \quad = 0, & |t| > \tau \end{array}\right\} \quad \text{('Saw-tooth')}, \qquad \widehat{y}(\omega) = \frac{2a}{\omega^2\tau}(1 - \cos\omega\tau) = a\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2}\right)$$





$$y(t) = \exp(-t^2/t_0^2) \quad \text{(Gaussian)}, \qquad \widehat{y}(\omega) = t_0\sqrt{\pi}\exp\left(-\omega^2 t_0^2/4\right)$$

$$y(t) = f(t)\, e^{i\omega_0 t} \quad \text{(modulated function)}, \qquad \widehat{y}(\omega) = \widehat{f}(\omega - \omega_0)$$

$$y(t) = \sum_{m=-\infty}^{\infty} \delta(t - m\tau) \quad \text{(sampling function)} \qquad \widehat{y}(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n/\tau)$$

Convolution theorem

If $z(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)\,\mathrm{d}\tau \quad \equiv x(t) * y(t)$ then $\quad \widehat{z}(\omega) = \widehat{x}(\omega)\,\widehat{y}(\omega)$.

Conversely, $\widehat{xy} = \widehat{x} * \widehat{y}$.

Parseval's theorem

$$\int_{-\infty}^{\infty} y^*(t)\; y(t)\;\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{y}^*(\omega)\;\widehat{y}(\omega)\;\mathrm{d}\omega$$

(if $\widehat{y}$ is normalised as on page 21)

Fourier transforms in two dimensions

$$\begin{aligned}\widehat{V}(\boldsymbol{k}) &= \int V(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\;\mathrm{d}^2\boldsymbol{r} \\ &= \int_0^{\infty} 2\pi r V(r) J_0(kr)\;\mathrm{d}r \qquad \text{if azimuthally symmetric}\end{aligned}$$

Fourier transforms in three dimensions

$$\begin{aligned}\widehat{V}(\boldsymbol{k}) &= \int V(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\;\mathrm{d}^3\boldsymbol{r} \\ &= \frac{4\pi}{k}\int_0^{\infty} V(r)\;r\sin kr\;\mathrm{d}r \qquad \text{if spherically symmetric} \\ V(\boldsymbol{r}) &= \frac{1}{(2\pi)^3}\int \widehat{V}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\;\mathrm{d}^3\boldsymbol{k}\end{aligned}$$

Examples

$V(\boldsymbol{r})$	$\widehat{V}(\boldsymbol{k})$
$\dfrac{1}{4\pi r}$	$\dfrac{1}{k^2}$
$\dfrac{\mathrm{e}^{-\lambda r}}{4\pi r}$	$\dfrac{1}{k^2+\lambda^2}$
$\nabla V(\boldsymbol{r})$	$\mathrm{i}\boldsymbol{k}\widehat{V}(\boldsymbol{k})$
$\nabla^2 V(\boldsymbol{r})$	$-k^2\widehat{V}(\boldsymbol{k})$

Laplace Transforms

If $y(t)$ is a function defined for $t \geq 0$, the Laplace transform $\overline{y}(s)$ is defined by the equation

$$\overline{y}(s) = \mathcal{L}\{y(t)\} = \int_0^\infty \mathrm{e}^{-st} y(t)\ \mathrm{d}t$$

Function $y(t)$ $\quad(t > 0)$	Transform $\overline{y}(s)$	
$\delta(t)$	1	Delta function
$\theta(t)$	$\dfrac{1}{s}$	Unit step function
t^n	$\dfrac{n!}{s^{n+1}}$	
$t^{1/2}$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{s^3}}$	
$t^{-1/2}$	$\sqrt{\dfrac{\pi}{s}}$	
e^{-at}	$\dfrac{1}{(s+a)}$	
$\sin \omega t$	$\dfrac{\omega}{(s^2+\omega^2)}$	
$\cos \omega t$	$\dfrac{s}{(s^2+\omega^2)}$	
$\sinh \omega t$	$\dfrac{\omega}{(s^2-\omega^2)}$	
$\cosh \omega t$	$\dfrac{s}{(s^2-\omega^2)}$	
$\mathrm{e}^{-at} y(t)$	$\overline{y}(s+a)$	
$y(t-\tau)\ \theta(t-\tau)$	$\mathrm{e}^{-s\tau}\ \overline{y}(s)$	
$t y(t)$	$-\dfrac{\mathrm{d}\overline{y}}{\mathrm{d}s}$	
$\dfrac{\mathrm{d}y}{\mathrm{d}t}$	$s\overline{y}(s) - y(0)$	
$\dfrac{\mathrm{d}^n y}{\mathrm{d}t^n}$	$s^n\overline{y}(s) - s^{n-1}y(0) - s^{n-2}\left[\dfrac{\mathrm{d}y}{\mathrm{d}t}\right]_0 \cdots - \left[\dfrac{\mathrm{d}^{n-1}y}{\mathrm{d}t^{n-1}}\right]_0$	
$\int_0^t y(\tau)\ \mathrm{d}\tau$	$\dfrac{\overline{y}(s)}{s}$	
$\left.\begin{array}{l} \int_0^t x(\tau)\ y(t-\tau)\ \mathrm{d}\tau \\ \int_0^t x(t-\tau)\ y(\tau)\ \mathrm{d}\tau \end{array}\right\}$	$\overline{x}(s)\ \overline{y}(s)$	Convolution theorem

[Note that if $y(t) = 0$ for $t < 0$ then the Fourier transform of $y(t)$ is $\widehat{y}(\omega) = \overline{y}(\mathrm{i}\omega)$.]

Numerical Analysis

Finding the zeros of equations

If the equation is $y = f(x)$ and x_n is an approximation to the root then either

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \text{(Newton)}$$

$$\text{or, } x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \qquad \text{(Linear interpolation)}$$

are, in general, better approximations.

Numerical integration of differential equations

If $\dfrac{dy}{dx} = f(x, y)$ then

$$y_{n+1} = y_n + hf(x_n, y_n) \quad \text{where } h = x_{n+1} - x_n \qquad \text{(Euler method)}$$

$$\text{Putting} \quad y^*_{n+1} = y_n + hf(x_n, y_n) \qquad \text{(improved Euler method)}$$

$$\text{then} \quad y_{n+1} = y_n + \frac{h[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]}{2}$$

Central difference notation

If $y(x)$ is tabulated at equal intervals of x, where h is the interval, then $\delta y_{n+1/2} = y_{n+1} - y_n$ and $\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2}$

Approximating to derivatives

$$\left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_n}{h} \approx \frac{y_n - y_{n-1}}{h} \approx \frac{\delta y_{n+\frac{1}{2}} + \delta y_{n-\frac{1}{2}}}{2h} \quad \text{where} \quad h = x_{n+1} - x_n$$

$$\left(\frac{d^2y}{dx^2}\right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = \frac{\delta^2 y_n}{h^2}$$

Interpolation: Everett's formula

$$y(x) = y(x_0 + \theta h) \approx \overline{\theta} y_0 + \theta y_1 + \frac{1}{3!}\overline{\theta}(\overline{\theta}^2 - 1)\delta^2 y_0 + \frac{1}{3!}\theta(\theta^2 - 1)\delta^2 y_1 + \cdots$$

where θ is the fraction of the interval h $(= x_{n+1} - x_n)$ between the sampling points and $\overline{\theta} = 1 - \theta$. The first two terms represent linear interpolation.

Numerical evaluation of definite integrals

Trapezoidal rule

The interval of integration is divided into n equal sub-intervals, each of width h; then

$$\int_a^b f(x)\, dx \approx h\left[c\frac{1}{2}f(a) + f(x_1) + \cdots + f(x_j) + \cdots + \frac{1}{2}f(b)\right]$$

where $h = (b - a)/n$ and $x_j = a + jh$.

Simpson's rule

The interval of integration is divided into an even number (say $2n$) of equal sub-intervals, each of width $h = (b - a)/2n$; then

$$\int_a^b f(x)\, dx \approx \frac{h}{3}[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(b)]$$

Gauss's integration formulae

These have the general form $\int_{-1}^{1} y(x)\,\mathrm{d}x \approx \sum_{1}^{n} c_i y(x_i)$

For $n = 2:\quad x_i = \pm 0{\cdot}5773;\quad c_i = 1, 1$ (exact for any cubic).

For $n = 3:\quad x_i = -0{\cdot}7746, 0{\cdot}0, 0{\cdot}7746;\quad c_i = 0{\cdot}555, 0{\cdot}888, 0{\cdot}555$ (exact for any quintic).

Treatment of Random Errors

Sample mean $\quad \overline{x} = \frac{1}{n}(x_1 + x_2 + \cdots x_n)$

Residual: $\quad d = x - \overline{x}$

Standard deviation of sample: $\quad s = \frac{1}{\sqrt{n}}(d_1^2 + d_2^2 + \cdots d_n^2)^{1/2}$

Standard deviation of distribution: $\quad \sigma \approx \frac{1}{\sqrt{n-1}}(d_1^2 + d_2^2 + \cdots d_n^2)^{1/2}$

Standard deviation of mean:

$$\sigma_m = \frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{n(n-1)}}(d_1^2 + d_2^2 + \cdots d_n^2)^{1/2}$$

$$= \frac{1}{\sqrt{n(n-1)}}\left[\sum x_i^2 - \frac{1}{n}\left(\sum x_i\right)^2\right]^{1/2}$$

Result of n measurements is quoted as $\overline{x} \pm \sigma_m$.

Range method

A quick but crude method of estimating σ is to find the range r of a set of n readings, i.e., the difference between the largest and smallest values, then

$$\sigma \approx \frac{r}{\sqrt{n}}.$$

This is usually adequate for n less than about 12.

Combination of errors

If $Z = Z(A, B, \ldots)$ (with A, B, etc. independent) then

$$(\sigma_Z)^2 = \left(\frac{\partial Z}{\partial A}\sigma_A\right)^2 + \left(\frac{\partial Z}{\partial B}\sigma_B\right)^2 + \cdots$$

So if

(i) $Z = A \pm B \pm C,\qquad (\sigma_Z)^2 = (\sigma_A)^2 + (\sigma_B)^2 + (\sigma_C)^2$

(ii) $Z = AB$ or $A/B,\qquad \left(\frac{\sigma_Z}{Z}\right)^2 = \left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2$

(iii) $Z = A^m,\qquad \frac{\sigma_Z}{Z} = m\frac{\sigma_A}{A}$

(iv) $Z = \ln A,\qquad \sigma_Z = \frac{\sigma_A}{A}$

(v) $Z = \exp A,\qquad \frac{\sigma_Z}{Z} = \sigma_A$

Statistics

Mean and Variance

A random variable X has a distribution over some subset x of the real numbers. When the distribution of X is discrete, the probability that $X = x_i$ is P_i. When the distribution is continuous, the probability that X lies in an interval δx is $f(x)\delta x$, where $f(x)$ is the probability density function.

$$\text{Mean } \mu = E(X) = \sum P_i x_i \text{ or } \int x\, f(x)\, \mathrm{d}x.$$

$$\text{Variance } \sigma^2 = V(X) = E[(X-\mu)^2] = \sum P_i (x_i - \mu)^2 \text{ or } \int (x-\mu)^2 f(x)\, \mathrm{d}x.$$

Probability distributions

Error function: $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x \mathrm{e}^{-y^2}\, \mathrm{d}y$

Binomial: $f(x) = \dbinom{n}{x} p^x q^{n-x}$ where $q = (1-p)$, $\quad \mu = np$, $\sigma^2 = npq$, $p < 1$.

Poisson: $f(x) = \dfrac{\mu^x}{x!} \mathrm{e}^{-\mu}$, and $\sigma^2 = \mu$

Normal: $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[-\dfrac{(x-\mu)^2}{2\sigma^2}\right]$

Weighted sums of random variables

If $W = aX + bY$ then $E(W) = aE(X) + bE(Y)$. If X and Y are independent then $V(W) = a^2V(X) + b^2V(Y)$.

Statistics of a data sample $x_1, \ldots, x_n$

$$\text{Sample mean } \overline{x} = \frac{1}{n}\sum x_i$$

$$\text{Sample variance } s^2 = \frac{1}{n}\sum (x_i - \overline{x})^2 = \left(\frac{1}{n}\sum x_i^2\right) - \overline{x}^2 = E(x^2) - [E(x)]^2$$

Regression (least squares fitting)

To fit a straight line by least squares to n pairs of points (x_i, y_i), model the observations by $y_i = \alpha + \beta(x_i - \overline{x}) + \epsilon_i$, where the ϵ_i are independent samples of a random variable with zero mean and variance σ^2.

Sample statistics: $s_x^2 = \dfrac{1}{n}\sum (x_i - \overline{x})^2$, $\quad s_y^2 = \dfrac{1}{n}\sum (y_i - \overline{y})^2$, $\quad s_{xy}^2 = \dfrac{1}{n}\sum (x_i - \overline{x})(y_i - \overline{y})$.

Estimators: $\widehat{\alpha} = \overline{y}$, $\widehat{\beta} = \dfrac{s_{xy}^2}{s_x^2}$; $E(Y \text{ at } x) = \widehat{\alpha} + \widehat{\beta}(x - \overline{x})$; $\widehat{\sigma}^2 = \dfrac{n}{n-2}$(residual variance),

where residual variance $= \dfrac{1}{n}\sum \{y_i - \widehat{\alpha} - \widehat{\beta}(x_i - \overline{x})\}^2 = s_y^2 - \dfrac{s_{xy}^4}{s_x^2}$.

Estimates for the variances of $\widehat{\alpha}$ and $\widehat{\beta}$ are $\dfrac{\widehat{\sigma}^2}{n}$ and $\dfrac{\widehat{\sigma}^2}{ns_x^2}$.

Correlation coefficient: $\widehat{\rho} = r = \dfrac{s_{xy}^2}{s_x s_y}$.