TRANSMISSION LINE FORMULAS

A COLLECTION OF METHODS OF CALCULATION FOR THE ELECTRICAL DESIGN OF TRANSMISSION LINES

BY

HERBERT BRISTOL DWIGHT, D.Sc.

Member of the American Institute of Electrical Engineers
Associate Member of the Engineering Institute of Canada
Author of "Constant-Voltage Transmission"

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PREFACE TO THE SECOND EDITION

The object of this book is to compile a set of instructions for engineers, which will enable them to make electrical calculations for transmission lines with the least possible amount of work. For this purpose, the chart, tables of formulas and tables of line constants have been arranged so as to make a convenient working tool.

The first five tables of formulas do not involve complex quantities or higher mathematics. However, the use of complex quantities, distinguished by the letter "*j*," is extremely useful in transmission line calculations, and a special effort has been made in Chapter VI to make it easy for anyone who is accustomed to simple algebra to use the complex quantities, even without a previous acquaintance with them.

Calculations for constant-voltage transmission lines and for the application of synchronous condensers have been included, as they are now necessary for the most important transmission systems.

A number of chapters have been included as a reference section, in which the derivation of the principal formulas can be found.

H. B. Dwight.

Hamilton, Canada
March, 1925

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TRANSMISSION LINE FORMULAS

PART I

WORKING FORMULAS

CHAPTER I

INTRODUCTION

THE determination of the electrical characteristics of transmission lines is an engineering problem of considerable practical importance. It occurs frequently in electrical engineering, being required not only for the design of new lines, but for determining the operation, loading and stability of existing transmission lines and networks.

Working methods and formulas are given in the tables in Part III of this book, a sufficient number of different formulas having been selected to cover the range required for power transmission lines at commercial frequencies, and for the problems occurring most frequently in practice. The formulas for short lines are very concise, and those for long lines have been planned with a view to simplicity and the saving of time and effort in making calculations. Groups of problems with answers, some of them worked out in detail, are given to illustrate in a practical manner the use of the formulas.

The first method, which is described in Chapter III, is in the form of a chart, shown in the frontispiece, which gives the regulation or voltage drop of a line, and which also shows directly the required size of conductor for given conditions.

In Chapter IV are described formulas for distribution lines and transmission lines only a few miles long. These are extended by means of the constant K, as described in Chapter V, to apply to lines up to approximately 200 miles in length at commercial frequencies.

For the calculation of very long lines, for unusual problems, and for checking different formulas, the fundamental relations of transmission lines are expressed by rapidly converging series, as described in Chapter VI. While these series require more work than the K formulas, they give exact results to any degree of accuracy desired. The method of convergent series involves the use of complex numbers, that is, numbers in which "j" terms appear. They are easier to handle, however, than sines and cosines of angles or hyperbolic functions of complex quantities, and therefore, the use of these other mathematical functions has been avoided where possible.

Each of the above groups of working formulas is printed in a table, ready for practical use, given in Part III at the end of the book.

When any formula is given which uses approximations, the limits of its accuracy should be clearly stated so that one can tell at a glance whether the method is sufficiently accurate for the purpose in hand, or whether a longer method giving greater accuracy is desirable. This is especially necessary in the calculation of transmission lines, because approximate formulas are quite permissible for lines only a few miles long, but become very untrustworthy when the length is increased to one hundred miles or more. For this reason, each table of formulas has its percentage and range of accuracy printed in a prominent position, so that the most suitable method for any case may be quickly chosen. For estimating accuracy, the convergent series, which give the exact results of the standard hyperbolic theory, are the criterion by which other methods are judged.

CHAPTER II

ELEMENTS OF A TRANSMISSION LINE

THE essential elements of a transmission line have been described many times, but a short discussion of them, with an explanation of some of the terms used in connection with the subject, may be useful before proceeding with the actual calculations.

A transmission line consists of two or more conductors insulated from each other so that they can carry energy by electric currents to some more or less distant point.

The conductors may be solid copper wires, copper cables, or aluminum cables. The diameters and resistances of various standard conductors are given in Tables 14 and 15, in Part III. It will be noted that the exact value of the resistance of a conductor differs slightly when a direct current, and an alternating current of 25 or 60 cycles, is flowing. This is due to the "skin effect," by which an alternating current tends to flow near the surface of a conductor, as explained in Chapter XVII. The drop in voltage due to resistance is proportional to the current and is in phase with it when the current is alternating.

Only overhead lines, carrying alternating currents, will be considered in this book. Such lines are supported by poles or steel towers at a considerable height above the ground. The conductors are separated from each other by a distance which may be several inches or several feet. The distance is called the "spacing" of the conductors and it has an important bearing on the electrical characteristics of the line.

An alternating magnetic field is formed around, and inside of, conductors carrying alternating currents. This field generates a voltage along the conductors which is proportional to the current, like the voltage drop due to resistance, but which is 90° out of phase with the current. This voltage is called the reactance drop. Tables of reactance of transmission lines will be found in Part III.

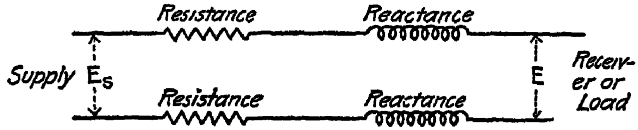


FIG 1.

Since the voltage drop in a transmission line is due to resistance and reactance, a simple line may be considered to be made up of the elements shown in Fig. 1. If R is the total resistance, the voltage drop in phase with the current I will be IR, and if X is the total reactance, the voltage drop in quadrature with the current will be IX.

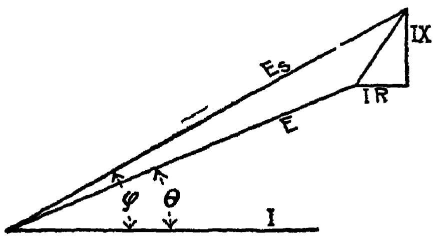


FIG. 2.

The vector diagram of the above quantities will be as in Fig. 2. The current is in general not in phase with the voltage E, but lags behind it by an angle θ, according to the power factor, $\cos \theta$, of the load. The resistance drop IR will therefore not be added directly to E, but must be added vectorially, along with the reactance drop IX, as in Fig. 2. It is evident that the voltages E and E_s, and the power factors $\cos \theta$ and $\cos \phi$, at the two ends of the line, are not the same in value.

A long transmission line acts as a condenser and this fact also must be taken into account. A condenser consists of two electrical conductors placed close together but insulated from each other so that a direct current cannot pass between them. However, if an alternating voltage be applied between them, a charge of electricity proportional to the capacitance, or electrostatic capacity, of the condenser will flow into and out of the conductors. The result is that an alternating current will appear to flow between them, proportional to the capacity susceptance of the condenser. This current, called the charging current, will be 90° out of phase with the voltage, and, unlike most currents in ordinary practice, it will lead the voltage in phase, instead of lagging behind it. The amount of the charging current may be determined by means of the tables of capacity susceptance of transmission lines, in Part III.

A current in phase with the voltage will flow between the conductors, but it is only noticeable at very high voltages. Part of it is a leakage current flowing over the insulators, and part is a discharge through the air, and produces the glow called corona, on high-voltage conductors.

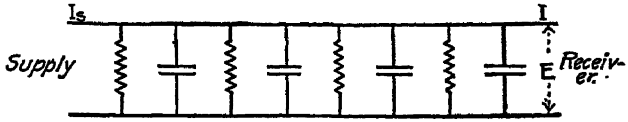


Fig. 3.

The elements of a transmission line accounting for the leakage current and charging current are shown in Fig. 3, in which resistances and condensers are shunted across the line all along its length.

Considering for the present that the voltage of the line is the same at all parts and is equal to E, the current in phase with E flowing across from one conductor to the other will be EG, where G is the total conductance between the wires. So also, if B is the capacity susceptance of the line considered as a condenser, EB will be the value of the shunted current in quadrature with E.

The vector diagram for the line indicated in Fig. 3 (neglecting the voltage drop in the conductors) is shown in Fig. 4. It is seen that the current I_s at the supply end is different in magnitude and phase from the current I at the receiver.

In order to calculate the combined effect of the above phenomena, formulas must be used which will take into account the fact that the resistance, capacitance, etc., are uniformly distributed along the line, and that the line current and voltage are different at all parts of the line.

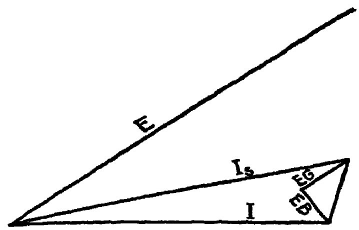


FIG. 4.

While the values of reactance and capacity susceptance are tabulated in Part III for 25 and 60 cycles only, the 50-cycle values can be very easily obtained by multiplying the 25-cycle values by 2.

CHAPTER III

TRANSMISSION LINE CHART

THE characteristic of a transmission line, as ordinarily operated, which limits the load it may carry, is its regulation, or the variation in voltage which occurs when the load is thrown on and off. This is especially true when the load has a low power factor, which is the case in most instances at the present time.

For estimating the regulation of a line, or the size of conductor required, the regulation chart which forms the frontispiece of the book may be used, and it will give the required result much quicker than any method of calculation.

An article by the author describing the calculation and construction of this type of chart has been published in the "Electric Journal," July, 1915, page 306.

The chart is accurate within approximately ½ of 1% of full line voltage, for the range of conductors indicated, when the line is not more than 100 miles long, and the load is such as to produce not more than 15% resistance or reactance volts.

Wherever a voltage is mentioned in this chapter, it refers to line voltage, that is, voltage between conductors, and not voltage to neutral.

In using the chart,* one places a straightedge across it from the point on the left corresponding to the spacing of the transmission line, to the point on the right, corresponding to the resistance of the conductor per mile. The regulation factor, V, is then read directly from the chart for the

* The process of using the chart is similar to that used with the transformer regulation and efficiency charts published by J. F Peters, *Electric Journal*, December, 1911.

power factor of load considered. The regulation is taken as the change in load voltage when the load is thrown on or off, assuming constant supply voltage.

The total regulation is quickly figured on the slide rule from the following formula for two-phase (four wire) or three-phase lines:

$$\text{Regulation Volts} = \frac{1000 \text{ Kv-a.} \times lV}{E},$$

where Kv-a. = kilovolt-amperes of load, at the receiver end;
E = line voltage at the load, or receiver end;
l = length of line in miles.

For single-phase lines use $2V$ instead of V, making the formula as follows:

$$\text{Regulation Volts} = \frac{1000 \text{ Kv-a.} \times l \times 2V}{E}.$$

The regulation volts may be expressed as a percentage of E to give the per cent regulation, and a formula is given on the chart for obtaining this result directly.

The line drop, or difference in voltage between the supply end and the receiver end of the line, is the same as the regulation for lines less than about 20 miles long, but for longer lines the effect of the charging current must be taken into account by the formula

$$\text{Line Drop} = \text{Regulation Volts} - EK,$$

where

$$K = 6\left(\frac{fl}{100{,}000}\right)^2,$$

and

f = frequency in cycles per second.

It is seen that the voltage due to the charging current is proportional to the line voltage E, and to the square of the number of miles, but is independent of the size or spacing of the conductors, within the assigned limit of accuracy. The constant K does not need to be used in the formula for regulation, since the charging current is present at both no load and full load.

In selecting the spacing point on the chart, one notes whether the frequency is 25, 50 or 60 cycles, and whether the conductor is of copper or aluminum. The spacing points are the same for both solid wire and cable. When the wires of a three-phase line are not spaced at the corners of an equilateral triangle, but are at irregular distances a, b, and c from each other, as in Figs. 5 and 6, the equivalent spacing

$$s = \sqrt[3]{abc}$$

should be used.

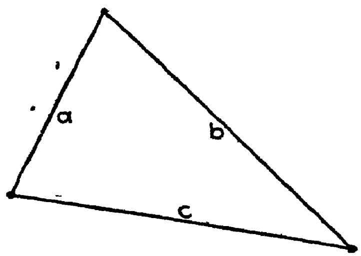


FIG. 5.—Irregular Triangular Spacing

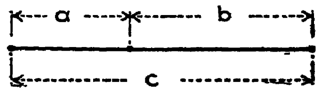


FIG. 6.—Irregular Flat Spacing.

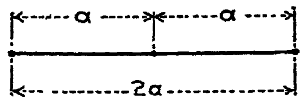


FIG. 7.—Regular Flat Spacing.

With regular flat spacing, as in Figs. 7 and 8, the equation for the equivalent spacing becomes simply

$$s = 1.26a.$$

It makes no difference whether the plane of the wires with flat spacing is horizontal, vertical or inclined.

The spacing of a two-phase line is the average distance between wires of the same phase. The distance between wires of different phases is not considered.

The points marked on the resistance scale at the right of the chart are for cables at 20° C., assuming hard-drawn copper of a conductivity equal to 97.3% of the Annealed Copper Standard, and hard-drawn aluminum of 61% conductivity, and allowing an increase of 2% in resistance for the effect of spiraling of the wires in the cable. However, these resistance points are placed on the chart for convenience only, and are not essential. If other assumptions are made, or if other sizes of conductor are used, all that is needed is to find the resistance of the conductor per mile, and use the corresponding resistance point on the chart to find "V." For instance, the resistance of English standard conductors could be marked on the right-hand scale in red ink.

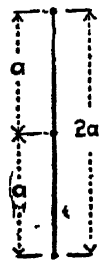


FIG. 8.—Regular Flat Spacing

One of the most common problems in estimating new projects is to determine the size of wire needed for any given value of regulation, and the chart will be found especially applicable to this work. "V" is first found from the equation,

$$V = \frac{\text{Per cent Regulation} \times E^2}{100{,}000\ \text{Kv-a.} \times l}.$$

Then lay a straightedge through "V" and the point for the spacing to be used, and the nearest size of conductor can be seen, at a glance, on the resistance scale at the right.

The chart is quite as useful for finding the voltage drop, or required size of conductor, for distribution lines a few hundred feet long as it is for transmission lines many miles long.

PROBLEM A

Find, by means of the chart, the regulation and line drop for the following set of conditions:

Length of line	100 miles.
Spacing	8 feet.
Conductor	No. 3 copper cable.
Load (measured at receiver end), 3000 kv-a., 66,000 volts between conductors, 90% P.F., three-phase, 60 cycles.	

Lay a straightedge from the 8-foot spacing point (60 cycles, copper conductor) to the point on the resistance scale for No. 3 copper cable.

It is found to cross the 90% P.F. line at the reading 1.350. Then, by the formula on the chart,

$$\text{Per Cent Regulation} = \frac{100{,}000\times3000\times100\times1.350}{66{,}000\times66{,}000}$$
$$=9.30\%.$$

The calculated value of the regulation of this line is 9.46% (Chap. VI, Prob. 2), so that the error involved in using the chart is less than $\frac{1}{5}$ of 1% of line voltage.

The per cent line drop, according to the chart, is

$$9.30-2.16=7.14\%.$$

As the calculated value is 7.15% (Chap. VI, Prob. 2), the error from the chart is less than $\frac{1}{10}$ of 1% of the line voltage.

PROBLEM B

To find the size of copper required to give approximately 10% voltage drop in the following case:

Length of line.................................. 3 miles.
Flat spacing as in Fig. 7. Wires 2 feet apart.
Load (measured at receiver end), 250 kv-a., 2200 volts between conductors, 85% P.F., three-phase, 60 cycles.

First, find V from the formula on the chart.

$$V=\frac{10\times2200\times2200}{100{,}000\times250\times3}=0.64.$$

The equivalent spacing is 1.26×2, or 2.52 feet. The proper spacing point will therefore be just below the spacing point for $2\frac{1}{2}$ feet, copper conductor, 60 cycles. Lay a straightedge from this point to the reading 0.64 on the line for 85% P.F. and it cuts the resistance scale at 0.36 ohm per mile. The nearest size of copper is seen to be No. 000.

PROBLEM C

Find the voltage drop of the following two-phase line:

Length of line.................. 80 miles.
Spacing........................ 10 feet.
Conductor...................... No. 00 aluminum cable, steel reinforced.
Load (measured at receiver end), 15,000 kv-a., 100,000 volts between conductors, 95% P.F., two-phase, 25 cycles.

Laying a straightedge across the chart from the 10-foot spacing point for 25 cycles and aluminum conductor, to the resistance point for No. 00 aluminum, the value of V for 95% P.F. is found to be 0 755. Then the line drop, in volts, is equal to

$$\frac{1000\times 15{,}000\times 80\times 0\ 755}{100{,}000}-100{,}000\times 6\times 0\ 02\times 0\ 02$$
$$=9060-240$$
$$=8820 \text{ volts.}$$

The calculated value is 8820 volts (Chap. VI, Prob 1).

PROBLEM D

Find the regulation of the following single-phase line

Length of line..................	15 miles
Spacing.......................	3 feet.
Conductor..........................	No. 0 copper wire.
Load (at receiver end), 300 kv-a., 50% P.F., 11,000 volts between conductors, single-phase, 60 cycles.	

From the chart, $V=0\ 845$.

$$\text{Therefore Regulation}=\frac{100{,}000\times 300\times 15\times 2\times 0\ 845}{11{,}000\times 11{,}000}=6\ 29\%.$$

[Calculated value, 6 40% (Chap. IV, Prob. 6). Error from chart, 0 11% of line voltage]

PROBLEM E

Find the kv-a., which can be delivered at the end of the following line, with 8% regulation:

Length of line............ ..	75 miles.
Spacing.................. ..	8 feet, regular flat spacing.
Conductor......	No. 00 aluminum cable.
Character of load (at receiver end), 88,000 volts between conductors, 85% P F., three-phase, 25 cycles.	

Equivalent spacing $s=8\times 1.26=10\ 08$ feet.

$$V=0\ 755.$$

$$\text{Kv-a.}=\frac{8\times 88{,}000\times 88{,}000}{100{,}000\times 0.755\times 75}$$
$$=10{,}900.$$

PROBLEMS, CHAPTER III

(Transmission Line Chart)

1. Find the size of copper cable which is needed to deliver 200 kv-a. at a distance of 3 miles with 10% drop or less.

Spacing of line .... 2 feet.
Character of load (at receiver end), 2200 volts between conductors, 80% P F., three-phase, 60 cycles.

[*Ans.* No. 0.]

2 Assuming No 0 copper cable for the previous problem, find the volts drop in the line.

[*Ans.* 219 volts. Calculated, 222 volts (Chap. IV., Prob. 1). Error 0 14% of line voltage.]

3. Find the size of copper required for a drop of 6% or less in the following case:

Length of line............ . .. . 5000 feet.
Spacing .... 18 inches.
Load (at receiver end), 75 kw. (79 kv-a.), 95% P.F., 2000 volts between conductors, single-phase, 60 cycles.

[*Ans.* No. 4 copper.]

4 Assuming No. 4 copper wire of 1 312 ohms per mile, find the per cent line drop and the supply voltage.

[*Ans.* 5.43%, 2109 volts between conductors, calculated 5 43%, 2109 volts (Chap. IV., Prob. 3).]

5. Find the required size of copper for 9% regulation in the following case:

Length of line...................... .... 25 miles.
Spacing.......... .... 3 feet.
Load (at receiver end), 2500 kv-a , 20,000 volts between conductors, 60% P.F , three-phase, 25 cycles.

[*Ans.* No 0 copper]

6. Assuming No. 0 copper wire of 0.520 ohm per mile, find the per cent regulation.

[*Ans.* 8 33%, calculated 8 51% (Chap. IV., Prob. 5). Error 0 18% of line voltage.]

7. Find the per cent regulation and the voltage drop of the following line:

Length of line	75 miles.
Spacing, 8 feet, regular flat spacing	
Conductor .	No 00 aluminum cable
Load (at receiver end), 10,000 kv-a, 88,000 volts between conductors, 85% P F, three-phase, 25 cycles.	

[*Ans* 7 29% Reg'n, 7 08% line drop, calc 7 37% Reg'n., 7 15% line drop (Chap V, Prob 3)]

8. Find the kv-a which can be delivered at the end of the following line, with 10% regulation, at 75% and at 90% P F..

Length of line .	100 miles
Spacing..	10 feet
Conductor . . .	No 0000 aluminum cable.
Receiver voltage.. .	110,000 volts between conductors, three-phase, 60 cycles

[*Ans* 14,500 kv-a., 16,400 kv-a]

CHAPTER IV

FORMULAS FOR SHORT LINES

THE effect of capacitance, or electrostatic capacity, is inappreciable with short lines as it amounts to only $\frac{1}{10}$ of 1% for a line about 20 miles long. Thus distribution lines and many short transmission lines can be quite accurately calculated without considering the line capacitance at all. The formulas in Tables 1, 2 and 3 enable one to solve many problems connected with such lines.

The formulas are divided into three groups, those in Table 1 being used when all the particulars describing the load, such as kv-a., voltage and power factor, are specified at the receiver end. Table 2 is used when these particulars are specified at the supply end, and Table 3 when part are specified at each end.

One first finds the quantities P and Q or P_s and Q_s. These are the values of in-phase current, and reactive or quadrature current, per conductor, at the point where the conditions are specified.

The next step is to find the quantities A and B, or F and H. One is then ready to find the value of any of the quantities, whose formulas are given in the tables. It should be remembered that each of these quantities may be determined independently of all the others. Thus it is not necessary to work out the first six equations in order to obtain the value of the seventh, since the seventh, like any of the others, may be calculated directly.

In Fig. 9, the same vector diagram is shown as in Fig. 2, and, in addition, the components of the various quantities in phase with E and in quadrature with it are shown. The quantities represented by the letters are defined in Table 1.

From Fig. 9 is obtained directly the equation for the supply voltage, E_s, as follows:

$$E_s^2 = (E + PR - QX)^2 + (PX + QR)^2,$$

$$E_s = \sqrt{A^2 + B^2} = A + \frac{B^2}{2A}, \text{ approximately,}$$

when B is small compared with A. It is to be remembered that Q is a negative quantity when the current is lagging.

It is better not to multiply E by $\cos\theta$, but to do all the calculating on the drop itself since it is a relatively small

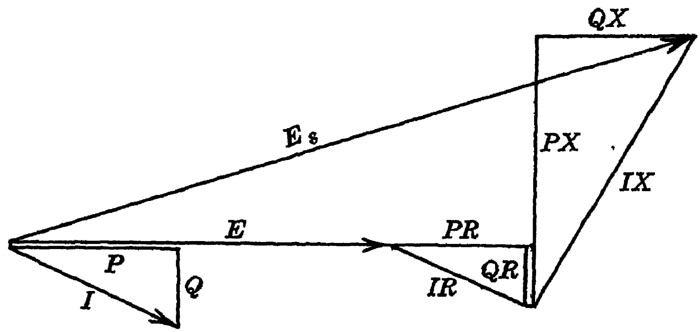


FIG. 9—Vector Diagram of Short Transmission Line with Lagging Current, Conditions Given at Receiver End.

quantity. Thus, one is handling large quantities with liability of large errors, if one uses the equation:

$$E_s^2 = (E \cos\theta + IR)^2 + (E \sin\theta + IX)^2.$$

On the other hand, one is calculating only small quantities, with correspondingly small errors, if one uses the following form of equation:

$$E_s = E + PR - QX + \frac{(PX + QR)^2}{2(E + PR - QX)}.$$

It is also an advantage to engineers who are accustomed to slide rule work, to use rectangular co-ordinates, and to avoid as much as possible the use of angles and trigonometrical tables.

The derivation of the formula for kw. at the supply end is given in Chapter XV.

When all the conditions are given at the supply end, the vector diagram is as shown in Fig. 10. From this diagram is obtained the equation:

$$E^2 = (E_s - P_sR + Q_sX)^2 + (P_sX + Q_sR)^2,$$

$$E = \sqrt{F^2 + H^2} = F + \frac{H^2}{2F}, \text{ approximately,}$$

when H is small compared with F. Q_s is a negative quantity when the current is lagging.

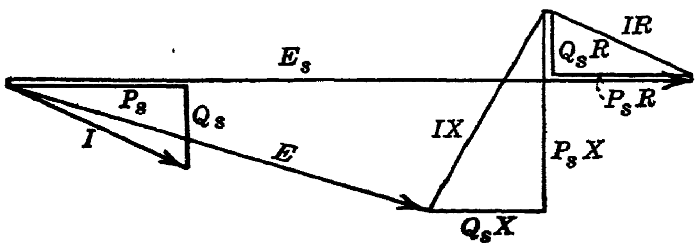


FIG. 10.—Vector Diagram of Short Line with Lagging Current, Conditions Given at Supply End.

If some of the conditions of the transmission line problem are given at the receiver end, and some at the supply end, it is possible, but not very desirable, to obtain a solution by a "trial and error" method. That is, one can assume values for the unknown conditions at the receiver end, and solve the problem by the formulas of Table 1. The results will probably not agree with the conditions which were specified at the supply end, and so the assumptions must be altered and the calculation gone over again. This must be repeated until a satisfactory result is obtained.

It is worth while having a direct formula to give the correct solution the first time the calculation is performed. Such a formula is given in Table 3 for the case where the voltage, E_s, is given at the supply end, and the load kv-a. and power factor are given at the receiver end.

From the vector diagram, Fig. 9, we have, as in deriving the formulas of Table 1,

$$E_s^2=(E+PR-QX)^2+(PX+QR)^2.$$

Multiplying by E^2,

$$\begin{aligned}E_s^2E^2&=(E^2+PER-QEX)^2+(PEX+QER)^2\\&=(E^2+L^2)^2+M^4,\end{aligned}$$

where PE and QE, and therefore L and M, are known from the specified load kv-a. and power factor, but where E itself is unknown.

Then

$$E^4+E^2(2L^2-E_s^2)+L^4+M^4=0.$$

The solution of this quadratic equation is:

$$E^2=\tfrac{1}{2}E_s^2-L^2+\tfrac{1}{2}\sqrt{E_s^4-4E_s^2L^2-4M^4},$$

as in Table 3.

The convergent series given in the same table is obtained by expanding the expression for E by means of the binomial or multinomial theorem.

PROBLEM A (TABLE 1)

Find the regulation in the following case:

Length of line	15 miles
Spacing	3 feet
Conductor	No. 0 copper wire
Load (at receiver end), 500 kv-a, 11,000 volts between conductors, 75% P.F. lagging, three-phase, 60 cycles.	

From the tables in Part III,

Voltage to neutral (star voltage) = 6351 volts.

$$R=0.5338\times 15=8.01$$

$$X=0.686\times 15=10.29$$

$$P=\frac{1000\times 500\times 0.75}{3\times 6351}=19.70$$

$$Q = -\frac{1000 \times 500 \times 0.6614}{3 \times 6351} = -17.35$$

$$A = 6351 + 19.70 \times 8.01 + 17.35 \times 10.29$$

$$= 6351 + 158 + 179 = 6688$$

$$B = 19.70 \times 10.29 - 17.35 \times 8.01$$

$$= 203 - 139 = 64$$

Voltage at supply end

$$= 6688 + \frac{64 \times 64}{2 \times 6688}$$

$= 6688$ volts to neutral

Regulation $= 6688 - 6351$

$= 337$ volts to neutral.

Regulation in volts between conductors

$$= 337 \times 1.732 = 584$$

$$\text{Per cent regulation} = \frac{100 \times 337}{6351}$$

$= 5.31$ per cent of receiver voltage.

PROBLEM B (TABLE 1)

Find the regulation for the same conditions as in Problem A, except including at each end of the line a bank of transformers rated at 500 kv-a., 11,000 to 2200 volts, with 1% resistance drop and 5% reactance drop.

Refer all transformer characteristics to the 11,000 volt side.

Voltage to neutral = 6351 volts,

Full load current of transformer bank, assumed star connected,

$$= \frac{1000 \times 500}{3 \times 6351} = 26.25 \text{ amperes.}$$

Resistance drop to neutral, $\frac{1}{100} \times 6351 = 63.5$ volts at each end.

Reactance drop to neutral, $\frac{5}{100} \times 6351 = 317.5$ volts at each end.

Resistance ohms to neutral, $\frac{63.5}{26.25} = 2.42$ ohms at each end.

Reactance ohms to neutral, $\frac{317.5}{26.25} = 12.10$ ohms at each end.

The above calculation applies whether the transformers are star or delta connected, but the values are virtual and not actual in the case of delta connected transformers.

$$R = 8\ 01 + 2\ 42 + 2\ 42 = 12\ 85$$

$$X = 10\ 29 + 12\ 10 + 12\ 10 = 34.49$$

$$A = 6351 + 19\ 70 \times 12\ 85 + 17\ 35 \times 34\ 49$$

$$= 6351 + 253 + 598 = 7202$$

$$B = 19.70 \times 34.49 - 17\ 35 \times 12\ 85$$

$$= 680 - 223 = 457$$

$$\text{Voltage at supply end} \quad = 7202 + \frac{457 \times 457}{2 \times 7202}$$

$$= 7216 \text{ volts to neutral.}$$

Regulation $= 7216 - 6351 =$ 865 volts to neutral.

$$\text{Per Cent Regulation} \quad = \frac{100 \times 865}{6351}$$

$$= 13\ 6 \text{ per cent of receiver voltage.}$$

PROBLEM C (TABLE 2)

Calculate the volts drop for the following case, where all the conditions are specified at the supply, or generator, end of the line.

Length of line............ .... 10 miles.
Spacing.................. .. .... 3 feet.
Conductor................. .. . .. No. 2 copper wire.
Quantities measured at supply end: 600 kv-a , 6600 volts between conductors, 80% P.F., lagging, three-phase, 60 cycles.

From the tables,

Voltage to neutral $= 3811$

$$R = 0.8483 \times 10 = 8\ 483$$

$$X = 0.714 \times 10 = 7.14$$

$$P_s = \frac{1000 \times 600 \times 0\ 80}{3 \times 3811} = 42.0$$

$$Q_s = -\frac{1000\times600\times0\ 60}{3\times3811} = -31\ 5$$

$$F = 3811 - 42\ 0\times8.483 - 31\ 5\times7\ 14$$
$$= 3811 - 356 - 225 = 3230$$

$$H = -42.0\times7\ 14 + 31.5\times8\ 48$$
$$= -\ 300 + 267 = -33$$

Voltage at receiver end

$$= 3230 + \frac{33\times33}{2\times3230}$$

$= 3230$ volts to neutral.

Voltage drop $= 3811 - 3230$

$= 581$ volts to neutral.

Voltage drop $= 581\times\sqrt{3}$

$= 1006$ volts between conductors

Per cent drop $= \dfrac{100\times581}{3230}$

$= 18\ 0$ per cent of receiver voltage.

PROBLEM D (TABLE 3)

Find the voltage at the receiver end in the following case:

Length of line . . .	20 miles.
Conductor	No. 0000 copper cable.
Spacing	5 feet.

Voltage at supply, or generator end, 33,000 volts between conductors.

Load at receiver end, 8000 kv-a. at 75% P F., lagging, 3-phase, 60 cycle.

$E_s = 33{,}000 \div 1\ 732 = 19{,}050$ volts to neutral, from table 27.

$R = 0\ 2724\times20 = 5\ 45$ ohms, from Table 14.

$X = 0\ 698\times20 = 13\ 96$ ohms, from Table 19.

$$PE = \frac{1000\times8000\times0\ 75}{3} = 2{,}000{,}000 \text{ watts.}$$

$$QE = -\frac{1000\times8000\times0\ 6614}{3} = -1{,}764{,}000 \text{ v-a.}$$

$$L^2 = 10{,}900{,}000 + 24{,}620{,}000 = 35{,}520{,}000$$

$$M^2 = 27{,}920{,}000 - 9{,}620{,}000 = 18{,}300{,}000$$

$$E^2 = 1\ 815 \times 10^8 - 0\ 355 \times 10^8 + 1\ 405 \times 10^8$$
$$= 2\ 865 \times 10^8$$

$E = 16{,}930$ volts to neutral.

$16{,}930 \times 1\ 732 = 29{,}300$ volts between conductors.

By series,

$$E = 19{,}050\ [1 - 0\ 0973 - 0\ 0095 - 0\ 0013 - 0\ 0004$$
$$- 0\ 0004 - 0\ 0001 - . \quad]$$
$$= 19{,}050 - 2080 = 16{,}970 \text{ volts to neutral.}$$

$16{,}970 \times 1\ 732 = 29{,}400$ volts between conductors at receiver end.

PROBLEMS, CHAPTER IV

(Formulas for Short Lines, Tables 1, 2 and 3)

1. Find the voltage drop in the following case

Length of line. . .	3 miles
Spacing.. . .	2 feet
Conductor	No. 0 copper cable.
Load (at receiver end), 200 kv-a., 2200 volts between conductors, 80% P F , three-phase, 60 cycles (Prob. 2, Chap. III)	

[*Ans.* 222 volts]

2. Find (*a*) the P F. at the supply end;
(*b*) the per cent efficiency of the line, for the case in Prob. 1.
[*Ans.* (*a*) 78 8% P.F (*b*) 92 2% efficiency.]

3 Find the supply voltage in the following case·

Length of line	5000 feet
Spacing	18 inches.
Conductor, No. 4 copper wire of 1 312 ohms per mile.	
Load (at receiver end), 75 kw , 2000 volts between conductors, 95% P.F., single-phase, 60 cycles. (Prob 4, Chap III.)	

[*Ans.* 2109 volts.]

4. Find the volts drop and the watts loss, in the following line:

Length of line	20 miles
Spacing .	5 feet.
Conductor .	No 1 aluminum cable.
Two-phase, 25 cycles	
Kv-a at supply end .	10,000
Line volts at supply end .	35,000.
P.F at supply end.	80%

[*Ans.* 5990 volts between conductors, 1790 kw.

5 Find the per cent regulation of the following line.

Length of line	25 miles.
Spacing	3 feet.

Conductor, No 0 copper wire of 0 520 ohm per mile

Load (at receiver end), 2500 kv-a, 20,000 volts between conductors, 60% P F., three-phase, 25 cycles (Prob. 6, Chap III.)

[*Ans.* 8 51%.]

6. Find the regulation of the following single-phase line:

Length of line..	15 miles.
Spacing	3 feet.
Conductor. ..	No. 0 copper wire.

Load, at receiver end, 300 kv-a, 11,000 volts, 50% P F., lagging, single-phase, 60 cycles

(Same as Prob. D, Chap. III.)

[*Ans* 6 40%.]

CHAPTER V

K FORMULAS

WHEN a transmission line is more than 20 miles long, the formulas for short lines given in Chapter IV are no longer accurate, and other formulas must be used, which will take into account the capacitance of the line. Such formulas, called *K* formulas, will be found in Tables 4 and 5, Part III.

The *K* formulas will be found very similar to those of the last chapter, and while they require more arithmetical work, they should not be found any more difficult to understand. No more values of line constants need to be looked up for the *K* formulas than for the "Short Line" formulas. The capacitance of the line does not enter into the calculations, since its effect is allowed for by means of the constant *K* which is the same, at any one frequency, for all values of line capacitance.

The formulas of this chapter assume that the leakage current is zero; that is, that no power is lost from leakage over the insulators or from corona. This is a correct assumption to make for all voltages except the very highest in use. If it is desired to make allowance for corona loss, the formulas of Chapter VI should be used.

The accuracy of the *K* formulas is given as approximately $\frac{1}{10}$ of 1% of line voltage for lines up to 100 miles long and with regulation up to 20%, and as $\frac{1}{2}$ of 1% for lines up to 200 miles long, and with the same regulation. These limits are close enough for commercial work, so that the *K* formulas can be recommended for all ordinary engineering calculations of the performance of long power transmission lines under steady conditions, where the corona loss is small. The accuracy of the electrical calculations will be better than the accuracy with which the resistance and the physical dimensions of the line are generally known.

The *K* formulas are well adapted to the solution of long transmission lines which have substations at intermediate points between the ends. In such cases each section of the line between substations must be calculated separately, beginning with the end where conditions are known. The first step is to find the voltage, in-phase current and quadrature current at the first substation. The load taken by the substation, expressed as in-phase current and quadrature current, must be added to, or subtracted from, the above values of current. When conditions are given at the receiver end and one is proceeding toward the supply end, the substation load must be added to the line load. When conditions are given at the supply end, the substation load must be subtracted from the line load, since one is proceeding away from the supply. Having thus found complete conditions at one end of the second section of the line, the calculation of this section may be taken up in the same way as for the first section. In this manner the entire line may be calculated and the voltage and current at the unknown end may be determined.

Examples are worked out, which will give a clear idea of the manner in which the *K* formulas are used. Many other such examples have been calculated and carefully compared with the fundamental formulas. As these examples have covered the range of practicable transmission lines, a sound basis is afforded for the estimate of the accuracy of the *K* formulas and for the statement that they are sufficiently reliable for all ordinary engineering purposes in the calculation of electric power transmission lines, up to 200 miles in length.

If it is desired to take the step-up and step-down transformers into account, they should be treated as separate short sections of the transmission line.

PROBLEM A (TABLE 4)

Find by the K formulas the line drop in the following case:

Length of line . . .	100 miles.
Spacing.	8 feet
Conductor . . .	No. 3 copper cable.
Load (at receiver end), 3000 kv-a., 66,000 volts between conductors, 90% P.F., lagging, three-phase, 60 cycles.	

(Prob. A, Chap. III).

E = star voltage

$$=66{,}000 \div 1.732 = 38{,}110.$$

$$R = 1.091 \times 100 = 109.1.$$

$$X = 0.840 \times 100 = 84.0.$$

$$K = 6 \times 0.06 \times 0.06 = 0.0216.$$

$$P = \frac{1000 \times 3000 \times 0.90}{3 \times 38{,}110} = 23.63$$

$$Q = -\frac{1000 \times 3000 \times 0.4359}{3 \times 38{,}110} = -11.44$$

$$A = 38{,}110 - 38{,}110 \times 0.0216$$

$$+23.63 \times 109.1\,(1-0.0144) + 11.44 \times 84.0\,(1-0.0036)$$

$$= 40{,}790 \text{ volts.}$$

$$B = \frac{38{,}110 \times 109.1 \times 0.0216}{84.0}$$

$$+23.63 \times 84.0\,(1-0.0036) - 11.44 \times 109.1\,(1-0.0144)$$

$$= 1820 \text{ volts.}$$

Supply voltage $= 40{,}790 + \dfrac{1820 \times 1820}{2 \times 40{,}790} = 40{,}830$

Line drop $= 2720$ volts to neutral

$= 7.13\%$ of receiver voltage.

By the fundamental formulas, using the same line constants, the line drop is 7.15% (Prob. 2, Chap. VI). The discrepancy is 0.02% of receiver voltage.

PROBLEM B (TABLE 4)

Find, by the K formulas, the voltage at the supply end of the following line:

Total length of line	250 miles.
Spacing	15 feet, effective.
Conductor	300,000 c.m. copper cable.

Load at receiver end of line, 10,000 kv-a., 90% P.F., lagging, 110,000 volts between conductors, three-phase, 60 cycles.

Load taken by a substation at the middle of the line, 125 miles from either end, 5000 kv-a. at the line voltage and at 85% P.F. lagging. (Prob. B, Chap. VI.)

Solution of first section of line, 125 miles long.

E = star voltage at receiver end

$=110{,}000 \div \sqrt{3}=63{,}510.$

$R=0.1929 \times 125=24.1$ ohms.

$X=0.804 \times 125=100.5$ ohms.

$K=6 \times 0.075 \times 0.075=0.03375.$

$$P=\frac{1000 \times 10{,}000 \times 0.9}{3 \times 63{,}510}=+47.3 \text{ amperes.}$$

$$Q=-\frac{1000 \times 10{,}000 \times 0.4359}{3 \times 63{,}510}=-22.9 \text{ amperes.}$$

$A=63{,}510-2140+1110+2290=64770$ volts.

$B=510+4730-540=4700$ volts.

$C=47.3-1.6+0.2-0.1=+45.8$ amperes.

$D=0.4-22.9+0.8+42.2=+20.5$ amperes.

$$A+\frac{B^2}{2A}=E_1=64{,}940 \text{ volts to neutral.}$$

$64{,}940 \times \sqrt{3}=112{,}480$ volts between conductors.

$$\text{In-phase current}=\frac{AC+BD}{E_1}=+47.2 \text{ amperes.}$$

Reactive current $=\dfrac{AD-BC}{E_1}=+17\ 0$ amperes (leading).

Solution of second section of line.

Conditions at middle of line:

$$E_1=64{,}940 \text{ volts.}$$

In-phase current of substation load

$$=\frac{1000\times5000\times0\ 85}{3\times64{,}940}=+21\ 8 \text{ amperes.}$$

Reactive current of substation load

$$=-\frac{1000\times5000\times0\ 5268}{3\times64{,}940}=-13\ 5 \text{ amperes.}$$

$$P_1=+47\ 2+21\ 8=+69\ 0 \text{ amperes.}$$

$$Q_1=+17\ 0-13\ 5=+\ 3\ 5 \text{ amperes.}$$

Then, by the K formulas,

$$A_1=64{,}940-2190+1630-350=64{,}030 \text{ volts.}$$

$$B_1=+520+6900-80=7340 \text{ volts.}$$

$$A_1+\frac{B_1^2}{2A_1}=64{,}450 \text{ volts to neutral.}$$

$64{,}450\times\sqrt{3}=111{,}630$ volts between conductors at the supply end of the line

(By the fundamental formulas, supply voltage $=111{,}800$ line volts (Prob B, Chap. VI.)

PROBLEMS, CHAPTER V

(*K* FORMULAS, TABLES 4 AND 5)

1. Find, by means of the K formulas, the voltage drop from the supply end to the receiver end (the line drop) of the following line:

Length of line......	200 miles
Spacing....................	9 feet.
Conductor	No. 000 aluminum cable.
Load (at receiver end), 4500 kv-a., 66,000 volts between conductors, 80% P.F., three-phase, 60 cycles.	

Ans. 6650 volts between conductors.

[By the fundamental formulas, 6700 volts.]

2 Find the regulation of the line in Prob A, Chap. V. [See Prob A, Chap. III]

Ans 9.45%.

[By the fundamental formulas 9 46% (Chap VI, Prob. 2) Error 0 01% of line voltage]

3 Find the per cent regulation and voltage drop of the following line

Length of line	75 miles.
Spacing, 8 feet, regular flat spacing.	
Conductor	No. 00 aluminum cable
Load (at receiver end), 10,000 kv-a , 88,000 line volts, 85% P.F., three-phase, 25 cycles. (Prob. 7, Chap III.)	

Ans. 7 37% reg'n, 7 15% drop.

4 Find, by the K formulas, the per cent voltage drop, the per cent loss, and the power factor at the supply end of the following line:

Length of line.	100 miles.
Spacing.	6 feet.
Conductor	No. 0000 copper wire.
Take $r=0\ 267$, $x=0\ 727$, $b=6\ 03\times 10^{-6}$.	
Load (at receiver end), 100 amperes per wire, 60,000 line volts, 95% P F , three-phase, 60 cycles. [Problem of Pender and Thomson, *Proc A. I. E. E* , July, 1911.]	

Ans. 13 09% drop, 7 61% loss, 96 58% P.F.

[Calc. by series, 13 03% drop, 7 60% loss, 96 66% P.F. (Prob. 4, Chap VI)]

5. Find, by the K formulas, the kv-a. and voltage at the supply end, and the efficiency of the following line·

Length of line 250 km = 155.34 miles.	
Spacing.	6 feet.
Conductor.	No 000 copper wire
Total resistance of one conductor. .	51 5 ohms.
Total reactance of one conductor. .	48 0 ohms.
Load (at receiver end), 15,000 kv-a , 86,600 line volts, 80% P.F., three-phase, 25 cycles.	

[See page 91, "Application of Hyperbolic Functions," by A. E. Kennelly, University of London Press, 1912]

Ans. 15,130 kv-a., 97,920 line volts, 89.68%.

[By series, 15,153 kv-a , 97,934 volts, 89 67%.]

6. Find (*a*) star voltage at supply end at full load,
   (*b*) star voltage at supply end at no load,
   (*c*) regulation volts (star) at the supply end,
   (*d*) amperes per wire at supply end at full load,
   (*e*) power factor at supply end at full load,
   (*f*) loss in line at full load,
   (*g*) efficiency of the tranmission line,
   (*h*) amperes per wire at supply end at no load (*i.e.*, the "charging current"),
   (*i*) power factor at supply end at no load,
   (*j*) loss in line at no load,

for the following line:

Length of line .... 300 miles.
Spacing........ .. .. 10 feet.
Conductor, No. 000 copper cable of 0 330 ohm per mile.
Load (at receiver end), 18,000 kv-a , 104,000 line volts, 90% P F., three-phase, 60 cycles.

[Prob. A, page 2, *G. E. Review Supplement*, May, 1910.]

Answers·

By *K* Formulas	By Fundamental Formulas	Error in Per Cent of Full Voltage or Current
(*a*) 69,820	69,670 volts	0 3%
(*b*) 48,610	48,950 volts	0 6%
(*c*) 21,210	20,720 volts	0 9%
(*d*) 97.0	96.59 amperes	0.5%
(*e*) 92.2	92.35%	0 2%
(*f*) 2530	2440 kw.	0.6%
(*g*) 86.5	86.90%	0 5%
(*h*) 91.3	90.97 amperes	0 5%
(*i*) 7.1	6 47%	0 7%
(*j*) 940	860 kw.	0.5%

7. Find, by the K formulas, the voltage at the supply end of the following line:

Total length of line	400 miles.
Spacing	15 feet.
Conductor	No 0000 copper cable of 0.2693 ohm per mile.
Load at receiver end of line, 5000 kv-a , 85% P.F. (lagging), 110,000 line volts, three-phase, 60 cycles.	

Load taken by a substation at the middle of the line, 200 miles from either end, 2500 kv-a., at the line voltage and at 90% P.F. (lagging).

Ans. 89,720 line volts.

[By the convergent series, 90,190 volts (Prob. 6, Chap. VI). Error 0 5%.]

CHAPTER VI

CONVERGENT SERIES

THE complete mathematical solution of the problem of calculating the electrical behavior of a transmission line in the steady state involves hyperbolic sines and cosines of complex quantities. Complex quantities are those which consist of a real and an imaginary term. The hyperbolic formulas are given, with their derivation, in Chapter XIV. By means of them can be calculated the voltage and current at any point, and hence the operating characteristics, of a transmission line in the steady state. The hyperbolic formulas take exact account of the fact that all the electrical properties of the line, such as resistance, capacitance, etc., are uniformly distributed along it.

The results of the hyperbolic theory constitute the criterion by which all approximate transmission line calculations must be judged. In order to estimate the accuracy and range of an approximate formula, problems should be selected referring to lines of different lengths. These problems should be solved by both the approximate and the hyperbolic methods and so the percentage error of the approximate method can be determined for different classes of problems.

The hyperbolic sines and cosines of complex quantities, and the transmission line formulas involving them, can be expanded in the form of convergent series,* and these series

* Prof. T. R. Rosebrugh, *Applied Science Magazine*, University of Toronto, Mar., 1909, Prof T. R. Rosebrugh, *Trans. A. I. E E*, 1909, p. 687; J F. H. Douglas, *Electrical World*, April 28, 1910; Dr. C. P. Steinmetz, *Electrical World*, June 23, 1910, Dr. C. P. Steinmetz, "Engineering Mathematics," Chap. V, 1911.

are very convergent and convenient for commercial frequencies such as 60 cycles or less, and are adaptable to slide rule work. They do not involve hyperbolic or trigonometrical functions, and so do not require any mathematical tables, the only operations being multiplication and addition. The series can be carried to any accuracy desired by merely using enough of the terms, which diminish very rapidly when commercial frequencies are involved.

The fundamental formulas as expressed by convergent series have been rearranged, and some new convergent series have been added, to make the formulas in tables 6, 7 and 8 directly applicable to the exact solution of all the problems treated by the K formulas. Exactly the same final formulas in A, B, C, D, etc., are used with the convergent series as with the K formulas.

Unlike the K formulas, which are expressed in the simplest algebraical form, the convergent series involve the use of complex numbers, that is, numbers containing the well-known "j" terms. No difficulty should be experienced on this account, however, as the rules for using complex quantities are quite straightforward, and even one who has never worked with them should be able to make use of the formulas described in this chapter by closely following the instructions.

Each of the complex quantities, $(A+jB)$, $(P+jQ)$, $Z=(r+jx)l$,* $Y=(g+jb)l$, etc., is composed of two parts, the first, a so-called "real" term, and the second, a j term. In adding complex numbers, the j terms must be kept separate from the others. Thus

$$4+j\,5 \text{ added to } 7+j\,3=11+j\,8.$$

In multiplying two complex quantities, the simple rules

* The notation $Z=(r+jx)l$, etc., is used in accordance with the resolution adopted by the International Electrotechnical Commission in Turin, Sept., 1911, and the Standardization Rules of the American Institute of Electrical Engineers.

of ordinary algebra are followed, and it must be remembered that

$$j \times j = j^2$$
$$= -1,$$

and, therefore,

$$-j \times j = +1$$
$$j^3 = -j$$
$$j^4 = +1$$
$$j^5 = +j, \text{ etc.}$$

Thus $(4+j\,5) \times (7+j\,3)$ is worked out as follows:

$$\begin{array}{r} 4+j\,5 \\ 7+j\,3 \\ \hline +28+j\,35 \\ -15+j\,12 \\ \hline +13+j\,47. \end{array}$$

In some kinds of problems fractions will be encountered whose denominators are complex quantities, or, what amounts to the same thing, division by complex quantities must be performed. A fraction with a complex denominator can be changed directly into a plain complex quantity by multiplying both numerator and denominator by the conjugate of the denominator, that is, a quantity which is the same except that j is replaced by $-j$. This process is called "rationalizing the denominator" and it obviously does not change the value of the fraction.

As an example, consider the fraction

$$\frac{8+j\,5}{3+j\,7}.$$

Multiplying above and below by the conjugate of the denominator, $3-j\,7$,

$$
\begin{array}{lll}
8+j\,5 & \qquad & 3+j\,7 \\
3-j\,7 & & 3-j\,7 \\
\hline
+24+j\,15 & & +\ \,9+j\,21 \\
+35-j\,56 & & +49-j\,21 \\
\hline
+59-j\,41 & & +58
\end{array}
$$

the fraction becomes

$$\frac{59-j\,41}{58}=1.02-j\,0.71.$$

This is in a workable form and can be added to other quantities.

In using the convergent series, E, P, and Q are the same as used with the K formulas, E being expressed as a real number without any j term. Z is equal to $(r+jx)l$, where r and x are taken from Tables 14 to 19, Part III, for resistance and reactance per mile. Y is equal to $(g+jb)l$. The leakage conductance, g, per mile, should be estimated from the most suitable data available, giving insulator leakage and corona loss under conditions similar to those of the line considered. The capacity susceptance, b, per mile, will be found in Tables 20 to 23, Part III.

After Y and Z have been written down in the form of complex numbers, the product YZ should be found, as described above for the multiplication of complex quantities. From this are obtained

$$\frac{YZ}{2},\quad \frac{YZ}{4},\quad \text{and}\quad \frac{YZ}{6},$$

each expressed as a complex number of a single real term and a single j term. Multiplying the last two together gives $\frac{Y^2Z^2}{2\cdot3\cdot4}$, from which $\frac{Y^2Z^2}{2\cdot3\cdot4\cdot5}$ may be written down.

In most cases no more terms need to be calculated, even for very accurate work, but this is to be determined while doing the work, as one usually figures out the terms of these series until they become too small to be considered when added to $\frac{YZ}{2}$.

By addition of terms obtained above, the values of

$$\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\text{etc.} \quad \text{and} \quad \frac{YZ}{2\ 3}+\frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5}+\text{etc.}$$

are obtained, each as a complex number of two terms.

Multiply E by the value found for $\left(\frac{YZ}{2}+\text{etc.}\right)$ and add it to E. Multiply $(P+jQ)$ by Z, or $(r+jx)l$, and by the value of $\left(\frac{YZ}{2\cdot 3}+\text{etc.}\right)$ and add it to $(P+jQ)Z$. The above quantities are added together to give $A+jB$, the sum of all the real parts being equal to A, and the sum of all the j terms being equal to B.

Similarly, $C+jD$ is found by adding

$$(P+jQ),\ (P+jQ)\left(\frac{YZ}{2}+\text{etc.}\right),\ EY, \text{ and } EY\left(\frac{YZ}{2\ 3}+\text{etc.}\right).$$

These values of A, B, etc., are inserted in equations 1 to 23 given with the K formulas, in exactly the same way as the values of A, B, etc., found according to the second page of Table 4. Each step of the above procedure is shown in the examples in this chapter.

The use of Table 7 is the same as that of Table 6 described above.

PROBLEM A (TABLE 6)

Find the voltage drop, by means of the convergent series, for the following line:

Length of line....	200 miles.
Conductor....	250,000 c. m. copper cable.
Spacing	13 feet, effective.

Load, at receiver end, 14,000 kv-a , 110,000 line volts, 80% P F. lagging, three-phase, 60 cycles.

$$rl=0\ 2309\times200=46\ 2 \text{ ohms.}$$

$$xl=0\ 798\times200=159\ 6 \text{ ohms}$$

$$bl=5\ 35\times10^{-6}\times200=0\ 001070 \text{ mho.}$$

$$Z=46\ 2+j\,159\ 6$$

$$Y=\qquad +j\,0\ 001070$$

$$YZ=-0\ 1708+j\,0\ 0494$$

$$\frac{YZ}{2}=-0\ 0854+j\,0\ 0247$$

$$\frac{YZ}{4}=-0\ 0427+j\,0.0123$$

$$\frac{YZ}{6}=-0\ 0285+j\,0\ 0082$$

$$+0\ 0012-j\,0\ 0004$$

$$-0\ 0001-j\,0.0004$$

$$\frac{Y^2Z^2}{2\cdot3\ 4}=+0\ 0011-j\,0\ 0008$$

$$\frac{YZ}{2}=-0\ 0854+j\,0\ 0247$$

$$\left(\frac{YZ}{2}+\text{etc}\right)=-0\ 0843+j\,0\ 0239$$

$$\frac{Y^2Z^2}{2\ 3\ 4\ 5}=+0\ 0002-j\,0\ 0002$$

$$\frac{YZ}{2\ 3}=-0\ 0285+j\,0\ 0082$$

$$\left(\frac{YZ}{2\cdot3}+\text{etc.}\right)=-0\ 0283+j\,0\ 0080$$

$$\left(1+\frac{YZ}{2\ 3}+\text{etc.}\right)=+0\ 9717+j\,0.0080$$

$E=63{,}510=$star voltage at receiver end.

$$\left(\frac{YZ}{2}+\text{etc.}\right)=-0.0843+j\,0\ 0239.$$

$$E\left(\frac{YZ}{2}+\text{etc.}\right)=-5350+j\,1520$$

$$P=\frac{1000\times 14{,}000\times 0\ 8}{3\times 63{,}510}=58\ 8\text{ amperes.}$$

$$Q=-\frac{1000\times 14{,}000\times 0\ 6}{3\times 63{,}510}=-44\ 1\text{ amperes}$$

$$Z=46\ 2+j\,159\ 6$$

$$\left(1+\frac{YZ}{2\ 3}+\text{etc.}\right)=0\ 9717+j\,0\ 0080$$

$$+44\ 9+j\,155\ 1$$

$$-1\ 3+j\,0\ 4$$

$$Z\left(1+\frac{YZ}{2\ 3}+\text{etc.}\right)=+43\ 6+j\,155\ 5$$

$$P+jQ=+58.8-j\,44\ 1$$

$$+6850+j\,9140$$

$$+2560-j\,1920$$

$$(P+jQ)Z\left(1+\frac{YZ}{2\ 3}+\text{etc.}\right)=+9410+j\,7220$$

$$E=63{,}510$$

$$E\left(\frac{YZ}{2}+\text{etc.}\right)=-5350+j\,1520$$

$$A+jB=67{,}570+j\,8740$$

$$A+\frac{B^2}{2A}=68{,}140\text{ volts to neutral.}$$

$$E=63{,}510\text{ volts to neutral.}$$

Voltage Drop = 4630 volts to neutral.

$4630\times 1.732=8020$ volts between conductors.

$$\frac{8020}{110{,}000}\times 100=7.29\%\text{ of receiver voltage.}$$

PROBLEM B (TABLE 6)

Find, by the convergent series, the voltage at the supply end of the following line:

Total length of line	250 miles.
Spacing	15 feet, effective.
Conductor	300,000 c. m. copper cable.

Load at receiver end of line, 10,000 kv-a., 90% P.F. lagging, 110,000 volts between conductors, three-phase, 60 cycles.

Load taken by a substation at the middle of the line, 125 miles from either end, 5000 kv-a, at the line voltage and at 85% P.F. lagging. (Prob. B, Chap. V.)

Solution of first section of line.

$$r=0.1929,\ x=0.804,\ b=5.31\times10^{-6},\ l=125$$

$$Z=24.1+j\,100.5$$

$$Y=+j\,0.000664$$

$$YZ=-0.0667+j\,0.0160$$

$$\left(\frac{YZ}{2}+\frac{Y^2Z^2}{2.3.4}+\text{etc.}\right)=-0.0332+j\,0.0079$$

$$\left(\frac{YZ}{2.3}+\frac{Y^2Z^2}{2.3.4.5}+\text{etc.}\right)=-0.0111+j\,0.0027$$

$$E=+63{,}510$$

$$E\left(\frac{YZ}{2}+\text{etc.}\right)=-2110+j\,500$$

$$(P+jQ)Z=+3440+j\,4200$$

$$(P+jQ)Z\left(\frac{YZ}{2.3}+\text{etc.}\right)=-50\quad -j\,60$$

$$A+jB=64{,}790+j\,4640$$

$$E_1=A+\frac{B^2}{2A}=64{,}960 \text{ volts to neutral.}$$

$$64{,}960\times\sqrt{3}=112{,}500 \text{ volts between conductors.}$$

In a similar manner, it is found that

$$C+jD=+45.8+j19.9$$

In-phase current, $\dfrac{AC+BD}{A+\dfrac{B^2}{2A}}=+47.0$ amperes.

Reactive current, $\dfrac{AD-BC}{A+\dfrac{B^2}{2A}}=+16.6$ amperes (leading).

Solution of second section of line:

Conditions at middle of line:—

$$E_1=64{,}960 \text{ volts to neutral.}$$

In-phase current of substation load

$$=\frac{1000\times 5000\times 0.85}{3\times 64{,}960}=+21.8 \text{ amperes.}$$

Reactive current of substation load

$$=-\frac{1000\times 5000\times 0.5268}{3\times 64{,}960}=-13.5 \text{ amperes.}$$

Current of substation load $=21.8-j\,13.5$

Current of first section $=47.0+j\,16.6$

$$P_1+jQ_1=68.8+j\,3.1$$

$$E_1=+64{,}960$$

$$E_1\left(\frac{YZ}{2}+\text{etc.}\right)=-2160+j\,510$$

$$(P_1+jQ_1)Z=+1350+j\,6980$$

$$(P_1+jQ_1)Z\left(\frac{YZ}{2\cdot 3}+\text{etc}\right)=-30\quad -j\,70$$

$$A_1+jB_1=64{,}120+j\,7420$$

$$A_1+\frac{B_1^2}{2A_1}=64{,}550 \text{ volts to neutral.}$$

$64{,}550\times\sqrt{3}=111{,}800$ volts between conductors at supply end.

PROBLEM C (TABLE 6)

Find the equation for the supply voltage of two lines in parallel, where conditions are given at the receiver end, one of the lines being that of Problem A, and the other being a 125-mile section as in Problem B. For this problem, it is assumed that the longer line takes a circuitous route and that the two lines are paralleled at both ends and have the same supply and receiver voltages.

For the line in Problem A,

$$E_s = E\alpha_1 + I\beta_1$$

where
$$\alpha_1 = 1 - 0.0843 + j\,0.0239$$
$$= +0.9157 + j\,0.0239$$

and
$$\beta_1 = +43.6 + j\,155.5$$

Then
$$\epsilon_1 = \frac{1}{\beta_1} = +0.00167 - j\,0.00596$$

and
$$\alpha_1\epsilon_1 = +0.00168 - j\,0.00542$$

From Problem B.

$$\alpha_2 = 1 - 0.0332 + j\,0.0079$$
$$= +0.9668 + j\,0.0079$$
$$\beta_2 = +23.6 + j\,99.5$$
$$\epsilon_2 = \frac{1}{\beta_2} = +0.00226 - j\,0.00951$$
$$\epsilon_1 = +0.00167 - j\,0.00596$$
$$\epsilon_1 + \epsilon_2 = +0.00393 - j\,0.01547$$
$$\beta_0 = \frac{1}{\epsilon_1 + \epsilon_2} = +15.4 + j\,60.7$$
$$\alpha_1\epsilon_1 = +0.00168 - j\,0.00542$$
$$\alpha_2\epsilon_2 = +0.00227 - j\,0.00918$$
$$\alpha_1\epsilon_1 + \alpha_2\epsilon_2 = +0.00395 - j\,0.01460$$
$$\alpha_0 = +0.947 + j\,0.014$$
$$E_s = E(0.947 + j\,0.014) + I(15.4 + j\,60.7)$$

for the two lines in parallel.

PROBLEMS, CHAPTER VI

(CONVERGENT SERIES, TABLES 6, 7 AND 8)

1. Find, by the convergent series, the voltage drop of the following line:

Length of line	80 miles.
Spacing	10 feet.
Conductor	No. 00 aluminum cable, steel reinforced.
Load (at receiver end), 15,000 kv-a., 100,000 line volts, 95% P F., two-phase, 25 cycles. (Prob. C, Chap. III.)	

Ans. 8820 volts between conductors.

2. Find, by the convergent series, the per cent line drop and the per cent regulation of the following line:

Length of line	100 miles.
Spacing	8 feet.
Conductor	No. 3 copper cable.
Load (at receiver end), 3000 kv-a, 66,000 line volts, 90% P.F., three-phase, 60 cycles. (See Prob. A., Chap. III, and Prob. A, Chap V)	

Ans. 7 15% drop, 9 46% reg'n.

3. Find, by the convergent series, the kv-a. and voltage at the supply end, and the efficiency of the following line:

Length of line	250 km. = 155.34 miles.
Spacing	6 feet.
Conductor	No. 000 copper wire.

Total resistance of one conductor, 51.5 ohms.
Total reactance of one conductor, 48 0 ohms.
Total susceptance of one conductor, 3.724×10^{-4} mhos.
Load (at receiver end), 15,000 kv-a , 86,600 line volts, 80% P. F., three-phase, 25 cycles. (Prob. 5, Chap. V.)

Ans. 15,153 kv.-a., 97,934 volts, line; 56,542 volts, star; 89 67% efficiency.

4. Find, by the convergent series, the per cent voltage drop, the per cent loss, and the power factor at the supply end of the following line:

Length of line . . . . . . 100 miles.
Spacing.. .. . . 6 feet.
Conductor . . . . No 0000 copper wire.
Take $r=0\ 267$, $x=0\ 727$, $b=6\ 03\times10^{-6}$.
Load (at receiver end), 100 amperes per wire, 60,000 line volts, 95% P.F., three-phase, 60 cycles. [Prob. 4, Chap. V.]

Ans. 13 03% drop, 7.60% loss, 96 66% P.F.

5. Find, by the convergent series,

(*a*) star voltage at supply end at full load,
(*b*) star voltage at supply end at no load,
(*c*) regulation volts (star), at the supply end,
(*d*) amperes per wire at supply end at full load,
(*e*) power factor at supply end at full load,
(*f*) loss in line at full load,
(*g*) efficiency of the transmission line,
(*h*) amperes per wire at supply end at no load (*i.e.*, the "charging current"),
(*i*) power factor at supply end at no load,
(*j*) loss in line at no load, for the following line:

Length of line .. . . 300 miles.
Spacing...... . . .. 10 feet.
Conductor, No. 000 copper cable 0 330 ohm per mile.
Load (at receiver end), 18,000 kv-a , 104,000 line volts, 90% P.F., three-phase, 60 cycles. (Prob. 6, Chap. V.)

Ans. (*a*) 69,670 volts, (*b*) 48,950 volts, (*c*) 20,720 volts, (*d*) 96 59 amperes, (*e*) 92 35%, (*f*) 2440 kw , (*g*) 86 90%, (*h*) 90 97 amperes, (*i*) 6 47%, (*j*) 860 kw.

6. Find, by the convergent series, the voltage at the supply end of the following line:

Total length of line . .. . 400 miles.
Spacing.... ... . 15 feet.
Conductor.............. .. .... No. 0000 copper cable of 0 2693 ohm per mile.
Load at receiver end of line, 5000 kv-a , 85% P.F. (lagging), 110,000 line volts, three-phase, 60 cycles.

Load taken by a substation at the middle of the line, 200 miles from either end, 2500 kv-a., at the line voltage and at 90% P.F. (lagging). (Prob. 7, Chap. V)

Ans. 90,190 volts.

CHAPTER VII

CONSTANT-VOLTAGE LINES

IN the problems dealt with in the chapters so far, a line has been calculated for one load at a definite power factor, and in calculating the performance at no load, it has been assumed that when the real power is zero, the reactive power is also zero.

In a great many transmission lines, however, and especially in all of the very longest and most important transmission lines, artificial control of the power factor by synchronous condensers is used, so as to hold the voltage at both ends of the line constant. At no load, as well as at other loads, a large amount of reactive power is taken over the line, of sufficient quantity to hold the voltages at their normal full load values.

It is evident that in constant-voltage lines a different set of given conditions is assumed from those considered in the preceding chapters. The voltage is given at both ends of the line; the kilowatts are known at one end, but the reactive power and the power factor are unknown and must be calculated. Special formulas for this set of given conditions are tabulated in Tables 9 to 13.

It is found that the relation of kw. to reactive kv-a. required to maintain constant voltage in a transmission line is represented by a true circle. The graphical method of drawing this circle on accurate cross section paper is very convenient and useful, since it shows the solution of the problem for all possible values of load. Further, a number of concentric circles can easily be calculated and drawn, which will show the effect of changes in the voltage at one end of the line.

The circle diagram shows the operating characteristics of a constant-voltage line so clearly and completely, and it is such a good method of checking calculations, that it is recommended that the circle diagram be drawn in connection with the solution of any constant-voltage line. The tables for constant-voltage lines show, first, the instructions for making the circle diagram, and then give formulas for calculating the various problems connected with constant-voltage lines.

Formulas for the circle diagram of a transmission line, including the effect of the distributed capacitance of the line, were published by the writer for the case in which conditions are given at the receiver end.* Later, formulas were presented for the case of conditions given at the supply end,† and for transmission systems including transformer characteristics along with the line characteristics.‡ A method of solving complicated networks of transmission lines has been given by Prof. Rosebrugh.§

In the case of transmission lines not more than 20 miles long, the capacitance of the line may be neglected. The calculation of the line when conditions are given at the receiver end is based directly on the vector diagram shown in Fig. 9, Chapter IV. The relations shown by the vector diagram are given by the following equation, as in Chapter IV:

$$E_s^2 = (E + PR - QX)^2 + (PX + QR)^2,$$

* "The Use of Synchronous Condensers with Transmission Lines," by H. B. Dwight, Transactions Engineering Institute of Canada, Nov., 1913, p 251, and "The Calculation of Constant-Voltage Transmission Lines," by H. B. Dwight, *The Electric Journal*, Sept., 1914, p. 487.

† "Constant-Voltage Transmission," by H B. Dwight, Tables II and IV, published by John Wiley & Sons, 1915

‡ "Circle Diagrams for Transmission Systems," by R. D. Evans and H K. Sels, *The Electric Journal*, Dec., 1921, p 530.

"Electrical Characteristics of Transmission Systems," by H. B. Dwight, *Trans. A. I. E. E*, 1922, p. 781.

§ "The Calculation of Transmission Line Networks," by Prof. T. R. Rosebrugh, Bulletin No. 1, 1919, of the School of Engineering Research, University of Toronto.

where Q is a positive quantity for leading current, and a negative quantity for lagging current. Now, the voltage is constant at both ends of a constant-voltage line, and so E_s and E are constant, as are also R and X. The equation can therefore be rearranged as an equation between Q and P and it will give the reactive power in the line, $\frac{3EQ}{1000}$, required to maintain constant voltage for a true power load, $\frac{3EP}{1000}$. The equation between P and Q is:

$$P^2+Q^2+\frac{2ER}{R^2+X^2}P-\frac{2EX}{R^2+X^2}Q-\frac{E_s^2-E^2}{R^2+X^2}=0.$$

This is the equation of a circle. The formulas for calculating the position of the center and the length of the radius are given in Table 9.

The circle diagram may be drawn in terms of either current or power, that is, Q may be plotted against P or $\frac{3EQ}{1000}$ against $\frac{3EP}{1000}$. The latter is recommended, because loads are usually given in kw. and kv-a. If the diagram is in terms of current, the Standardization Rules of the American Institute of Electrical Engineers (see Rule 3230, 1922 Edition), as well as those of the International Electrotechnical Commission, expressly state that when an in-phase current is drawn horizontally to the right, a leading current shall be drawn upward, in the positive direction. It seems almost obvious that when the above currents are all multiplied by the constant transmission voltage, their relationship should still be shown by the same shape of diagram. It would be inconsistent to have two different shapes of diagrams to show the same relationship. The power diagram is to all intents and purposes a vector diagram, since it conveys exactly the same information as the vector diagram of currents referred to above. Standardization Rule No. 3230 of the A. I. E. E. explicitly refers to *any* vector diagram. However, kw. and kv-a. are in a strict technical sense not vector quantities, and circle diagrams are sometimes drawn with leading reactive kv-a. downward, in the negative direction, although it is inconsistent with the current diagram, and with the general meaning of Standardization Rule 3230.

The use of conjugate quantities, which are very advantageous in many calculations, does not require that leading kv-a. be plotted downward, but merely that the conjugate of the proper quantities be taken. Where the conjugate of an impedance is required, no difficulty is encountered.

The "conjugate" of a complex quantity is obtained by changing the sign of the imaginary part, that is, by changing j to $-j$.

The use of conjugates is both easy and natural. It will be remembered that a complex fraction is multiplied above and below by the conjugate of the denominator, in order to rationalize the denominator. In a somewhat similar way, when multiplying a complex voltage by a complex current, if the conjugate of the voltage is used, the resulting expression can be used to represent the volt-amperes, with the additional advantage that the real part is equal to the watts. It is necessary to use the conjugate of the voltage in order not to conflict with the precedent set by Standardization Rule No. 3230 of the A. I. E. E., but this can easily be done.

While the most usual and general problem is the multiplication of a voltage by a current, the matter is very clearly set forth by an example of a complex value of current,

$$I(\cos\theta + j\sin\theta) = Ie^{j\theta},$$

flowing through an impedance, $R+jX$. The voltage across the impedance is $I_e^{j\theta}(R+jX)$, and the conjugate of the voltage is $I_e^{-j\theta}(R-jX)$. If now, in accordance with the previous paragraph, the current be multiplied by the conjugate of the voltage, the resulting expression for v-a. is:

$$I_e^{j\theta}I_e^{-j\theta}(R-jX) = I^2R - jI^2X.$$

It is seen that the real part is equal to the watts and the unreal part is equal to the reactive volt-amperes, which are negative when lagging, thus agreeing with the method of drawing vector diagrams given by Standardization Rule 3230.

If, on the other hand, the conjugate of the current be multiplied by the voltage, as is sometimes done, the resulting expression for v-a. is:

$$I_e^{-j\theta} I_e^{j\theta}(R+jX) = I^2R + jI^2X.$$

In this, the lagging quantity is positive, and would be plotted in a counter-clockwise direction from the in-phase quantity, which is in disagreement with the method of Standardization Rule 3230.

Therefore, the procedure of using the conjugate of the current should not be followed.*

The circles in the circle diagram show the reactive kv-a. *in the line* required to maintain constant voltage. It is of practical importance to know also the reactive kv-a. of synchronous condensers required, and for this the values of reactive kv-a. of the load must be added to the values given by the circle, because the synchronous condensers must correct the lagging kv-a. of the load, as well as provide the reactive kv-a. required in the line. In the tables it is assumed that the power factor of the load is constant from no load to full load, and this gives a straight line for the curve of load kv-a. Values read from this straight line, when added to those of the circle, give an ellipse. If the power factor of the load from light load to full load is known for a certain case, by experience, the curve of load kv-a. will not be a straight line, though it will usually not depart far from it. This would produce corresponding changes in the curve of synchronous condenser capacity.

The straight line of load kv-a. is most easily drawn by

* See discussions by J. R. Dunbar and H B Dwight, *Trans. A. I. E E.*, 1922, p. 789.

plotting an abscissa, cos θ, and an ordinate, $-\sin\theta$, to give one point on the line. Values of sin θ corresponding to values of the power factor, cos θ, are tabulated in Table 28.

The theoretical limit of the load of the line for the given constant supply and receiver voltages may be read directly from the circle diagram or it may be calculated by the formulas given in the Tables. It is shown by the farthest distance to the right reached by the circle. This theoretical limit is so much larger than the regular load of an ordinary line with rather small conductors, such as No. 000, that it is of little importance in such cases. However, with heavy conductors such as 500,000 circular mil copper, as used in the largest transmission systems, the theoretical limit of load becomes very important, for the efficiency is high right up to this limit of load.

If the voltage of a station supplying a lighting load be lowered, the kilowatts and current are both reduced. This is evident by considering the lighting load as a pure resistance through which current is flowing.

If the load, however, is entirely a motor load, the kilowatts will remain practically constant when the voltage changes, and so the in-phase current will increase when the voltage decreases. It has been shown that for most cases where the kilowatts are constant, and in fact for most loads encountered in actual practice, the operation of synchronous condensers with a transmission line will become impracticable and the condensers will drop out of step at a load somewhat less than the maximum load indicated by the circle diagram. This point may be called the "stability limit" of load of the transmission system, since the system becomes unstable when the load is increased to this amount.

For such a heavy load the synchronous condensers are essential for holding up the receiver voltage, and when the condensers drop out of step the receiver voltage will immediately fall very low, and the transmission line will behave almost as though it were short-circuited at the receiver end.

The determination of the stability limit of load of a transmission system is carried out by a process of drawing curves. See Fig. 10a, which is reproduced by permission of the author from Fig. 3a of a paper by E. B. Shand.*

The first step is to draw the characteristic curves of the condensers used for maintaining constant voltage on the transmission system. Such curves, in amperes plotted on

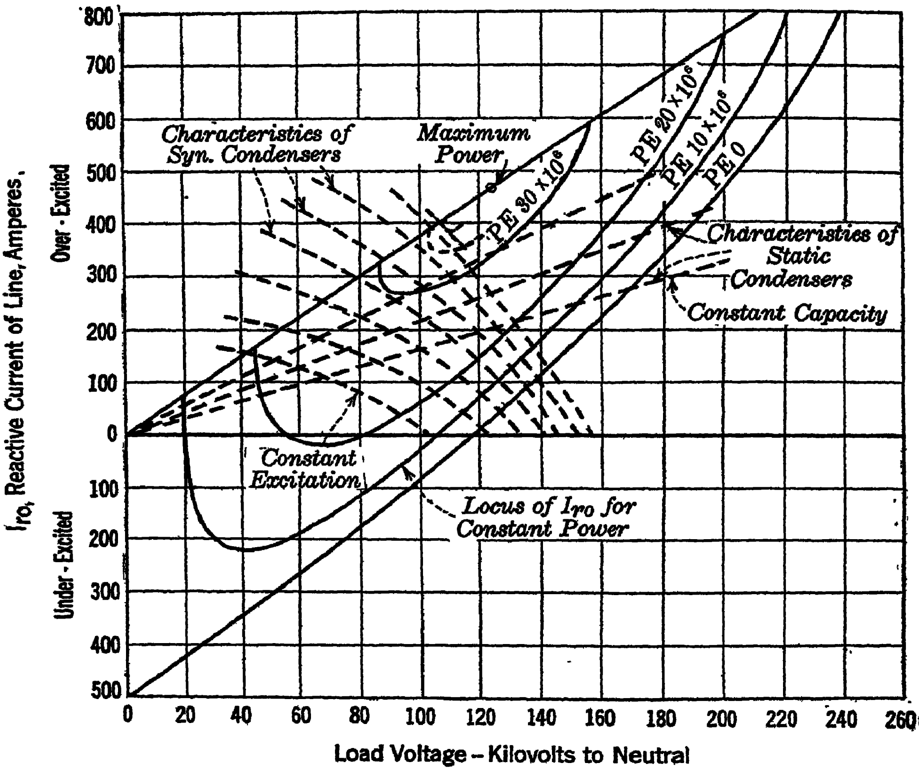


FIG. 10a.—Curves Used in Finding the Stability Limit.

Constants, $a_1 = 0.85$, $a_2 = 0.075$, $R' = 73.1$, $X' = 168$ $E_S = 100$ Kilovolts to Neutral.

receiver voltage, are represented by the dotted lines in Fig. 10a, for various values of excitation. A group of synchronous condensers with constant excitation will carry less leading current when the voltage is increased, and so

* "The Limitations of Output of a Power System Involving Long Transmission Lines," by E. B. Shand, *Journal, A. I. E. E.*, March, 1924, page 219.

their curves slope downward to the right as shown in Fig. 10*a*. The drawing of these curves is a problem in the design or testing of synchronous machines. The curves are easily drawn by taking values from the well-known load saturation curves of the machines.

The leading current of static condensers, on the other hand, increases in proportion as the voltage increases and so their characteristics are straight lines pointing upward toward the right as shown.

The next step is to draw characteristic curves of the transmission line, as shown by the full line curves of Fig. 10*a*. In a manner similar to that used on the first page of Table 11, let

$$E(a_1+ja_2)=E\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot3\cdot4}+\ldots\right)$$

and

$$R'+jX'=(R+jX)\left(1+\frac{YZ}{2\cdot3}+\frac{Y^2Z^2}{2\cdot3\cdot4\cdot5}+\ldots\right).$$

Then

$$E_s=E(a_1+ja_2)+(P+jQ)(R'+jX').$$

Multiplying by E,

$$E_sE=E^2a_1+PER'-QEX'+j(E^2a_2+PEX'+QER'),$$

or, using absolute values,

$$E_s^2E^2=(E^2a_1+PER'-QEX')^2+(E^2a_2+PEX'+QER')^2.$$

Each curve is drawn for a specified load PE, and E_s is assumed constant. Then, by specifying different values of E, corresponding values of Q can be calculated, and the curves can be plotted. For the most accurate work, the effect of the transformer impedances should be included.

When a numerical value is given to E, the equation becomes

$$aQ^2+bQ+c=0,$$

where a, b and c have numerical values.

Then

$$Q = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

In this way the curves of Fig. 10*a* can be plotted.

The point of instability is where the transmission line curve is tangent to one of the dotted synchronous condenser curves. Thus, in Fig. 10*a*, the curve for $PE = 30 \times 10^6$ watts per phase becomes tangent to one of the synchronous condenser regulation curves at 90 kilovolts. The full line curve goes on to 86 kilovolts where it reaches the straight line corresponding to the circle diagram limit. According to the circle diagram limit, therefore, a larger load than 30×10^6 watts per phase can be carried when the receiver voltage is 90 kilovolts. This would, however, be beyond the stability limit.

Fig. 10*a* therefore shows how to find the stability limit of a transmission system supplying a load of constant power, and having condensers of a definite total rating, and it shows that the stability limit is in some usual cases slightly lower than the circle diagram limit.

If economic and engineering conditions justify a load for a transmission line approaching or exceeding the stability limit or the circle diagram limit of load, synchronous condenser stations may be placed intermediate between the supply and the receiver. The stability limit of such a line with intermediate condenser stations is found by drawing curves similar to those described above, of reactive amperes on a voltage base for each point where there are synchronous condensers. For the calculation of the stability limit, it is not possible to consider each section of line separately, but the entire transmission line must be taken into consideration.

For further discussion of the stability limit, reference can be made to the paper by E. B. Shand, previously mentioned, and to papers published in the Journal of the A. I. E. E., 1924, by R. C. Bergvall, R. D. Evans, C. L. Fortescue, H. K. Sels and C. F. Wagner.

The formulas used for constructing the circle diagram when the effect of line capacitance is included, as given in Table 11, are derived as follows:

From Table 6, or equation (14) of Chap. XIV, we have the fundamental equation for conditions given at the receiver end·

$$E_s = E\left(1+\frac{YZ}{2}+\ .\ .\ .\right)+(P+jQ)(R+jX)\left(1+\frac{YZ}{2\ 3}+\ .\ .\ .\right)$$

$$= E' + jE'' + (P+jQ)(R'+jX'),$$

using the notation of Table 11. Then

$$E_s^2 = (E'+PR'-QX')^2+(E''+PX'+QR')^2.$$

Rearranging this as an equation between P and Q,

$$P^2+Q^2+\frac{2(E'R'+E''X')}{R'^2+X'^2}P-\frac{2(E'X'-E''R')}{R'^2+X'^2}Q$$

$$-\frac{E_s^2-E'^2-E''^2}{R'^2+X'^2}=0.$$

or

$$\left(P+\frac{E'R'+E''X'}{R'^2+X'^2}\right)^2+\left(Q-\frac{E'X'-E''R'}{R'^2+X'^2}\right)^2-\frac{E_s^2}{R'^2+X'^2}=0.$$

This is the equation of a circle, the units being in amperes per conductor. Multiply by $\frac{3E}{1000}$ to change the units to kw. and kv-a. and the formulas for a', b' and c' of Table 11 are obtained.

Constant-voltage Lines with Transformers.—In using synchronous condensers for voltage control, it is often desired to hold the voltage constant at the low-tension side of the transformers. Formulas are given in Table 13, whereby the transformers and synchronous condensers are included as an integral part of the transmission system. The diagram of the constant-voltage line is still a true circle, even when transformer impedance and average values of transformer core loss and magnetizing current and of synchronous condenser loss, are included.

The formulas as given apply to long lines with distributed capacitance and impedance, but they are applicable to short lines by merely changing the series $\left(1+\frac{YZ}{2}+\ldots\right)$ and $\left(1+\frac{YZ}{2\cdot 3}+\ldots\right)$ to unity.

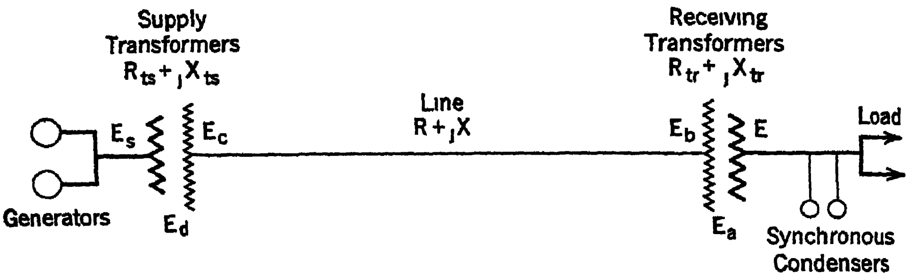


FIG. 11—SCHEME OF CONNECTIONS OF CONSTANT-VOLTAGE TRANSMISSION LINE

Load Current $= P + jQ_l$
Load Current + Reactive Current from Synchronous Condensers $= P + jQ$
Current for Average Loss in Synchronous Condensers $= P_c$
Admittance for Core Loss and Magnetizing Current of Receiving Transformers at Average Voltage $= G_{tr} + jB_{tr}$
Admittance for Core Loss and Magnetizing Current of Supply Transformers at Average Voltage $= G_{ts} + jB_{ts}$

It may be noted that

$$1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\ 3\ 4}+\ldots=\cosh\sqrt{YZ},$$

and

$$1+\frac{YZ}{2\cdot 3}+\frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5}+\ldots=\frac{\sinh\sqrt{YZ}}{\sqrt{YZ}}.$$

The scheme of connections of the transmission line, with transformers and synchronous condensers, is shown in Fig. 11.

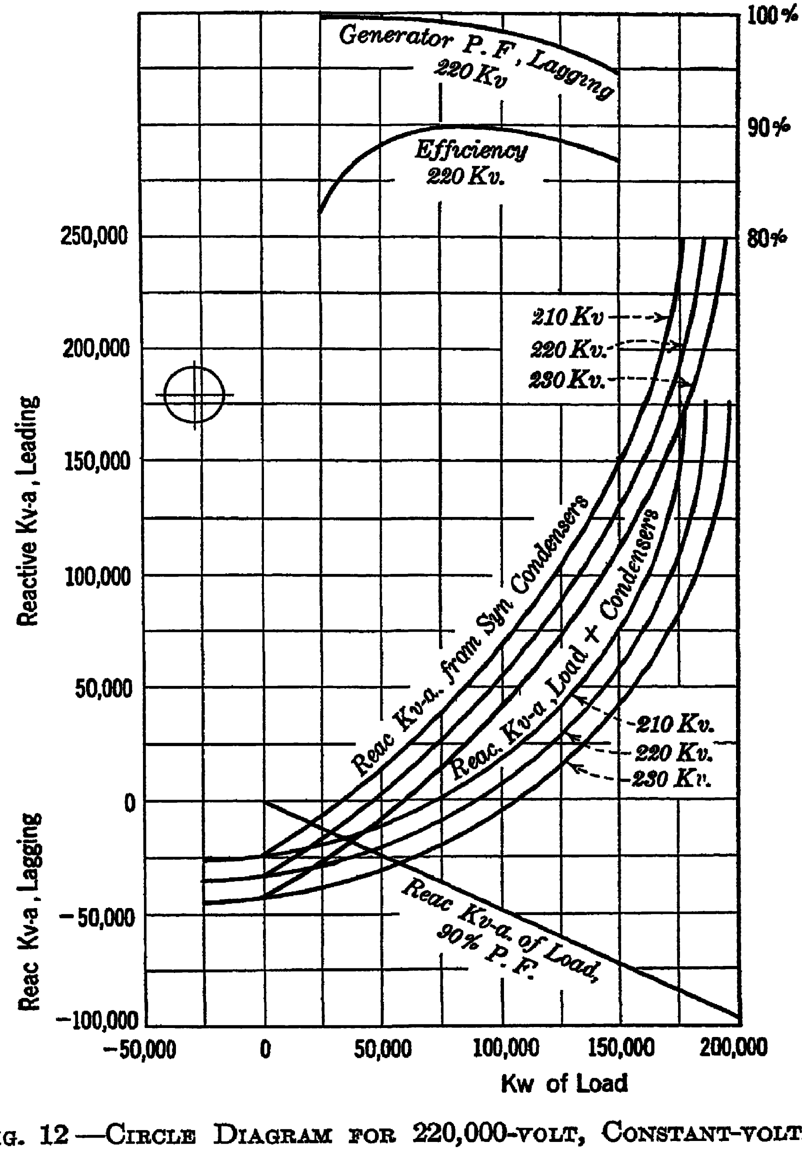


FIG. 12—CIRCLE DIAGRAM FOR 220,000-VOLT, CONSTANT-VOLTAGE TRANSMISSION LINE.

(See Example 1)

Example 1

Length of line = 200 miles

Frequency = 60 cycles

$R+jX$ = 23 2 $+j$ 160 ohms

$Y=+j$ 0 00106 mho

$$1+\frac{YZ}{2}+\ldots=0.91637+j\,0\ 01195$$

$$1+\frac{YZ}{2\cdot 3}+\ldots=0.97197+j\,0\ 00403$$

$P_c=8.66$ amperes

$R_{tr}+jX_{tr}=1.33+j\,24\ 0$ ohms

$G_{tr}+jB_{tr}=0.000{,}022{,}5-j\,0\ 000{,}187{,}5$ mho

$R_{ts}+jX_{ts}=1\ 61+j\,29.0$ ohms

$G_{ts}+jB_{ts}=0\ 000{,}018{,}6-j\,0\ 000{,}155{,}0$ mho

$E=115{,}470$ volts to neutral (line voltage 200,000)

$I_a=P+jQ+8.7$

$E_a=115{,}480+j\,100+(P+jQ)(0\ 7+j\,12.0)$

$I_b=(P+jQ)(1.002+j\,0.0002)+11.3-j\,21\ 6$

$E_b=115{,}740+j\,220+(P+jQ)(1\ 3+j\,24.0)$

$E_c=109{,}680+j\,2870+(P+j\,Q)(22.9+j\,178.0)$

$I_c=(P+jQ)(0.894+j\,0\ 013)+9.9+j\,99.5$

$E_d=108{,}240+j\,3090+(P+jQ)(23.4+j\,191.0)$

$I_d=(P+jQ)(0.924+j\,0\ 013)+12.4+j\,82.8$

$E_s=107{,}050+j\,3340+(P+jQ)(23\ 9+j\,204\ 4)$

$\quad=E'+jE''+(P+jQ)(R'+jX')$

$a'=-26{,}600$ kw.

$b'=178{,}300$ kv-a.

$c'=213{,}800$ kv-a.

when $E_s=127{,}020$ volts to neutral (line voltage = 220,000). See Fig. 12, which shows the desired characteristics of the system.

CHAPTER VIII

POWER FACTOR CORRECTION

The question of power factor correction, while not inherently a transmission line problem, occurs so frequently in the calculation of the loads on transmission lines that a short discussion of it will be given.

Electric power loads, in themselves, have usually a lagging power factor, due to the lagging magnetizing current taken by induction motors and transformers, which are generally more numerous than other classes of apparatus having higher power factor characteristics. For instance, incandescent lamps, synchronous converters and many synchronous motors have practically unity power factor and when they are added to a lagging power factor load they raise its power factor, though they cannot make it leading. Synchronous motors of generous design can operate at leading power factor, and synchronous condensers and static condensers can furnish current at practically zero power factor, leading.

The fundamental problem suggested by the term "power factor correction," is the determination of the percentage power factor which is obtained by adding a certain load of high or leading power factor to a given lagging load. This problem is of importance in city distribution where the total current of the load is of importance, even if the voltage drop in the lines is not. For long transmission lines, where the effect of voltage drop is predominant, the effect of leading current may be calculated in the manner described in Chapter VII on constant-voltage lines.

The change in load power factor caused by the addition of a certain load, can be conveniently calculated by using rectangular co-ordinates and slide rule calculations. Thus, if the original load be represented by an in-phase current, P_1, and a reactive current, Q_1, which is negative when lagging, as in Table 1, and if the corresponding currents for the added load are P_2 and Q_2, then the final load currents are P_3, in-phase, and Q_3, reactive, where

$$P_3 = P_1 + P_2,$$

and

$$Q_3 = Q_1 + Q_2.$$

The power factor of the final load is

$$\frac{P_3}{\sqrt{P_3^2 + Q_3^2}}.$$

The final load is leading or lagging according as Q^3 is positive or negative.

This method of calculating problems in power factor correction is simple and well adapted to slide rule work. It does not require the use of trigonometrical tables or the drawing of a vector diagram.

A vector diagram for a power factor correction problem can easily be drawn and can sometimes help give a physical idea of the changes in the load characteristics. Such a diagram is given in Fig. 13 as an illustration for Problem B. It is not necessary, however, in the calculation of the problem.

It is obvious that if each value of current is multiplied by the star voltage, E, the calculation can be carried out in terms of kilowatts, p, and reactive kv-a., q, instead of amperes. This is usually more convenient.

Where loads commonly described as zero power factor loads are involved, an in-phase component should be used, equal to the losses in the zero power factor apparatus such as synchronous condensers or static condensers.

PROBLEM A

Find the power factor resulting by adding a 1000 kv-a. synchronous condenser, having 5% losses, to a 5000 kv-a load of 80% lagging power factor

$$p_1 = 5000 \times 0\ 8 = 4000$$

$$p_1^2 = 16{,}000{,}000$$

$$p_1^2 + q_1^2 = 25{,}000{,}000$$

$$q_1^2 = 9{,}000{,}000$$

$$q_1 = -3000 \text{ (lagging)}$$

$$p_2 = 1000 \times 0\ 05 = 50$$

$$p_2^2 = 2500$$

$$p_2^2 + q_2^2 = 1{,}000{,}000$$

$$q_2^2 = 997{,}500$$

$$q_2 = +999 \text{ (leading)}$$

$$p_3 = p_1 + p_2 = 4050$$

$$q_3 = q_1 + q_2 = -3000 + 999 = -2001 \text{ (lagging).}$$

$$p_3^2 = 16{,}402{,}500$$

$$q_3^2 = \underline{4{,}004{,}001}$$

$$p_3^2 + q_3^2 = 20{,}406{,}501$$

$$\sqrt{p_3^2 + q_3^2} = 4517$$

$$\text{Power factor} = \frac{p_3}{\sqrt{p_3^2 + q_3^2}} = \frac{4050}{4517}$$

$$= 0.897$$

$$= 89\ 7\% \text{ lagging.}$$

The power factor is lagging because q_3 is negative.

PROBLEM B

Find the power factor of the load made up of the following loads: 10,000 kv-a. at 85% P.F , lagging, a 1000 kw. synchronous converter adjusted for unity P.F. and having 4% losses, and a 2000 kv-a synchronous condenser having 5% losses.

$$
\begin{array}{ll}
p_1 = 8500 & q_1 = -5268 \quad \text{(See Table 28)} \\
p_2 = 1040 & q_2 = \quad 0 \\
p_3 = \ \ 100 & q_3 = +1998 \\
\hline
p_4 = 9640 & q_4 = -3270
\end{array}
$$

$$\text{Power factor} = \frac{p_4}{\sqrt{p_4^2+q_4^2}} = \frac{9640}{10{,}180} = 0\ 947$$

$$= 94\ 7\%, \text{ lagging.}$$

The power factor is lagging since q_4 is negative.

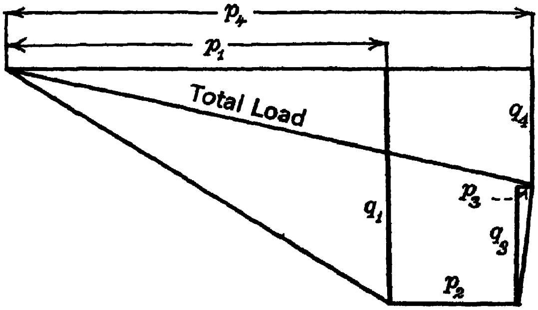


FIG 13.—Vector Diagram Illustrating Power Factor Correction Problem B.

PROBLEM C

Find the increase in power factor due to adding a 1500 kw. synchronous motor-generator set of 91% overall efficiency operating at 80% P.F. leading, to an 8000 kv-a. load of 70% lagging power factor.

$$p_1 = 8000 \times 0.7 = 5600$$

$$p_2 = \frac{1500}{0\ 91} = 1649$$

$$p_3 \qquad\qquad = 7249$$

$$q_1 = -8000 \times 0.7141 = -5713 \quad \text{(lagging)}$$

$$q_2 = +1649 \times \frac{0.6}{0.8} = +1237 \quad \text{(leading)}$$

$$q_3 = -4476 \quad \text{(lagging)}$$

$$\text{Final power factor} = \frac{p_3}{\sqrt{p_3{}^2 + q_3{}^2}} = \frac{7249}{8520}$$

$$= 0.851$$

$$= 85.1\% \text{ P.F., lagging.}$$

The power factor is lagging because q_3 is negative.

CHAPTER IX

EFFECT OF A TIE-LINE BETWEEN TWO SUBSTATIONS

When the transmission line to a substation becomes overloaded, it is sometimes easier to put in a tie-line from another near-by substation (see Fig. 14), than to duplicate or enlarge the main line. The construction of such a tie-line would give two circuits for supplying each substation, where there had been only one before. Reserve transmission circuits are thereby provided, which are valuable in case of line trouble.

In order to determine whether a tie-line is worth installing, and in order to design it, it is useful to have methods

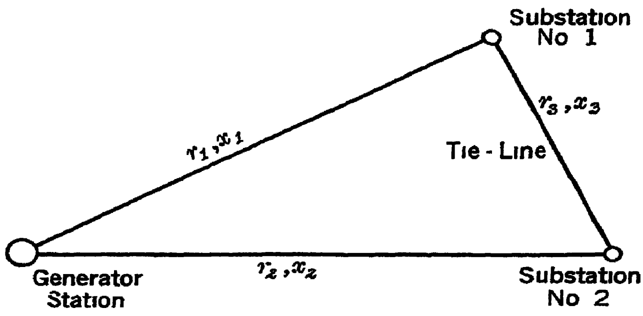


Fig 14.—Diagram showing Substations and Tie-Line.

of calculation of the effect of the tie-line on the currents and voltages of the system, as given in this chapter.*

The method of calculation depends on where the various factors such as voltage, power factor, etc., are measured or specified. Three cases will be taken up.

* "The Effect of a Tie-Line between Two Substations," by H. B. Dwight, *The Electrical Review*, Chicago, Dec 21, 1918.

Case I.—The simplest case of the tie-line problem is where the current, power factor and voltage of each line are measured at the generator station. Any of the lines shown in Fig. 14 may of course consist of several cables or overhead lines in parallel.

Let the combined resistance per phase of the lines to substation No. 1 be r_1, of the lines to substation No. 2, be r_2, and of the tie-line be r_3. Let the corresponding reactances be x_1, x_2 and x_3. Let

$$R = r_1 + r_2 + r_3, \quad \ldots \ldots \quad (1)$$

and let

$$X = x_1 + x_2 + x_3. \quad \ldots \ldots \quad (2)$$

Let the current and power factor for the lines to the two substations be measured at the generator station, before the tie-line is connected. Let P_1 be the current per conductor in phase with the generator voltage, flowing to substation No. 1, and let Q_1 be the reactive current. The quantity Q_1 will be positive for leading current and negative for lagging. Let P_2 and Q_2 be the corresponding currents for substation No. 2.

Find the quantities,

$$S = P_2 r_2 - Q_2 x_2 - P_1 r_1 + Q_1 x_1, \quad \ldots \quad (3)$$

and

$$T = P_2 x_2 + Q_2 r_2 - P_1 x_1 - Q_1 r_1. \quad \ldots \quad (4)$$

Then, after the tie-line is connected, the in-phase and reactive currents flowing in the tie-line from the first substation to the second, are

$$P_c = \frac{RS + XT}{R^2 + X^2}, \quad \ldots \ldots \quad (5)$$

and

$$Q_c = \frac{RT - XS}{R^2 + X^2}. \quad \ldots \ldots \quad (6)$$

The current per conductor in the tie-line is $\sqrt{P_c^2 + Q_c^2}$. The current in the line to Substation No. 1, in phase with the generator voltage, is $P_a = P_1 + P_c$ and the corresponding reactive current is $Q_a = Q_1 + Q_c$, so that the current per conductor in this line is $\sqrt{P_a^2 + Q_a^2}$. Similarly, the current in the line to Substation No. 2, in phase with the generator voltage, is $P_b = P_2 - P_c$ and the reactive current is $Q_b = Q_2 - Q_c$, so that the current in this line is $\sqrt{P_b^2 + Q_b^2}$. From these currents the voltages at the two substations may be calculated by the formulas for short transmission lines, when conditions are given at the supply end. See Table 2.

Case II.—If the currents, voltages and power factors are measured at the substations instead of at the generator station, the quantities P_1, Q_1, etc., must be calculated in the way used for short transmission lines, as follows:

Let P and Q be the in-phase current and the reactive current per conductor at Substation No. 1, and let E_1 be the voltage to neutral at the substation.

Find the quantities:

$$A = E_1 + Pr_1 - Qx_1, \qquad (7)$$

and

$$B = Px_1 + Qr_1. \qquad (8)$$

The voltage to neutral at the generators is

$$E_s = \sqrt{A^2 + B^2}. \qquad (9)$$

If the drop in the line is not more than 20%, the simpler formula may be used:

$$E_s = A + \frac{B^2}{2A}. \qquad (10)$$

The current in phase with E_s is

$$P_1 = \frac{AP + BQ}{E_s}, \qquad (11)$$

and the current in quadrature with E_s is

$$Q_1 = \frac{AQ - BP}{E_s}. \qquad (12)$$

In the same way P_2 and Q_2 may be calculated. Then the effect of the tie-line may be calculated as in *Case I.*

To a certain extent, the above calculation for *Case II* is a trial and error method, for the values of E_s found from the two substations should be equal. If they are not equal, due to the readings of the voltage or other quantities being wrongly taken, or taken at different times, then the given values of E should be slightly altered until they give equal values of E_s.

An approximate solution of *Case II* may be obtained, and the calculation of P_1, P_2, Q_1 and Q_2 may be avoided, by assuming that the power factor at each substation is equal to the power factor of the corresponding line at the generator station, and therefore, $P_1 = P$ and $Q_1 = Q$. The calculation required will then be only the calculation of *Case I.*

Case III.—If the voltage is given at the generator station and the load in kilowatts and the power factor are given at each substation, then the voltage at each substation before the tie-line is connected may be directly calculated as in Table 3.

Now that E is known, the currents P and Q can be immediately found. Then P_1 and Q_1 can be calculated from formulas (11) and (12), and finally the tie-line problem can be solved as in *Case I.*

In all the methods of calculation given above, the assumption is made that the load currents of the substations are unchanged when the tie-line is connected, even though the voltages at the substations may be slightly changed by that operation.

Example

Case I.—Let the line to Substation No. 1 be 10 miles long and composed of No. 0000 copper cables at 3 ft. spacing. Let the line to Substation No. 2 be 10 miles long and composed of No. 0 copper cables at 3 ft. spacing. Let the tie-line be $2\frac{1}{2}$ miles long and composed of No. 0 copper cables at 3 ft. spacing. Let the system be 3 phase, 60 cycles.

Let $P_1=35$, $Q_1=23.3$, $P_2=58.4$ and $Q_2=43.8$.

Then $r_1=2.69$, $r_2=5.38$, $r_3=1.35$, $R=9.42$;

$$x_1=6.36,\ x_2=6.78,\ x_3=1.69, \text{ and } X=14.83;$$

$$S=314.2+297.0-94.2-148.2=368.8;$$

$$T=396.0-235.6-222.6+62.7=0.5;$$

$$P_c=\frac{3475+7}{309}=11.3;$$

$$Q_c=\frac{5475-5}{309}=17.7;$$

$\sqrt{P_c^2+Q_c^2}=21.0$ amperes per phase in the tie-line;

$$P_b=47.1, \text{ and } Q_b=26.1.$$

Therefore, the current in this line is 53.8 amperes per phase. The current in this line before the tie-line was connected was 73.0 amperes, and so the tie-line reduces the current in the overloaded line to 74 per cent of its original value.

Example

Case III.—Let the lines be the same as in the preceding example and let the generator station voltage be 6600 volts between wires. Then $E_s=3810$ volts to neutral. Let the load on Substation No. 1 be $PE_1=120{,}000$ watts and $QE_1=80{,}000$ volt-amperes lagging, per phase. Let the load on Substation No. 2 be $P'E_2=200{,}000$ watts and $Q'E_2=150{,}000$ volt-amperes lagging, per phase.

For Substation No. 1, by Table 3,

$$L_1^2=831{,}600;$$

$$M_1^2=548{,}000,$$

and

$$E_1=3574 \text{ volts.}$$

Now that E_1 is known, we can obtain $P=33\ 6$ and $Q=22.4$ amperes per phase.

By formulas (7) and (8),

$$A=3574+90+143=3807;$$

$$B=213-60=153;$$

$$E_s=A+\frac{B^2}{2A}=3810 \text{ volts,}$$

which checks the calculation.

By formulas (11) and (12),

$$P_1=32\ 6,$$

and

$$Q_1=23.7 \text{ amperes per phase.}$$

It is seen that these quantities are not much different from P and Q.

For Substation (2), by Table 3,

$$L_2^2=2{,}093{,}000;$$

$$M_2^2=549{,}000;$$

and

$$E=3140 \text{ volts.}$$

We can now obtain

$$P'=63\ 7;$$

and

$$Q'=47\ 8 \text{ amperes per phase.}$$

Then, by formulas (11) and (12),

$$P_2=61\ 4,$$

and

$$Q_2=50.6 \text{ amperes per phase.}$$

All the data for *Case I* are now definitely known, and we obtain, by formulas (3) to (6), $P_c=13.3$ and $Q_c=20\ 9$ amperes per phase. The current in the tie-line is, therefore, 24 7 amperes per phase. The current in the line to Substation No. 2 is 56.6 amperes. Before the tie-line was connected it was 79.6 amperes, and the current in the overloaded line has therefore been reduced to 71 per cent of its original value.

Approximate Solution

Case III.—In order to find the load currents, assume that the voltage at both substations is 10% lower than at the generator station, that is, $E=3430$ volts. This is inaccurate by several per cent, but a fair estimate of the effect of the tie-line can be obtained in this way. We thus obtain $P=35$ amperes, and, as in the approximate method for *Case II*, P_1 may be assumed to be the same. Similarly, values of Q_1, P_2 and Q_2 are obtained which are the same as those used for the example of *Case I*. It is seen that the current in the overloaded line is reduced to 74% of its original value, according to the approximate method, and to 71%, by the more accurate method.

Accordingly, the approximate method, which uses the calculations of *Case I* only, and in which the substation voltages are assumed, and the change in power factor from one end of the line to the other is neglected, may be recommended for estimating work.

Derivation of Formulas (5) and (6)

Case I.—Use the same notation as in *Case I*. Let the currents after the tie-line is connected be P_a+jQ_a amperes per wire in the first line, P_b+jQ_b in the second line, and P_c+jQ_c in the tie-line, P_a being in phase with the generator voltage, etc. Then, assuming that P_c+jQ_c flows from the first substation to the second,

$$P_a=P_1+P_c;$$

$$Q_a=Q_1+Q_c;$$

$$P_b=P_2-P_c;$$

and

$$Q_b=Q_2-Q_c.$$

Let E_a and E_b be the voltages to neutral at the two substations after the tie-line is connected.

$$E_a = E_s - (P_a + jQ_a)(r_1 + jx_1)$$
$$= E_s - (P_1 + jQ_1)(r_1 + jx_1) - (P_c + jQ_c)(r_1 + jx_1). \quad . \quad . \quad (13)$$
$$E_b = E_s - (P_b + jQ_b)(r_2 + jx_2)$$
$$= E_s - (P_2 + jQ_2)(r_2 + jx_2) + (P_c + jQ_c)(r_2 + jx_2). \quad . \quad . \quad (14)$$

and

$$E_b = E_a - (P_c + jQ_c)(r_3 + jx_3). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

From the above three equations,

$$E_s - (P_2 + jQ_2)(r_2 + jx_2) + (P_c + jQ_c)(r_2 + jx_2)$$
$$= E_s - (P_1 + jQ_1)(r_1 + jx_1) - (P_c + jQ_c)(r_1 + r_3 + jx_1 + jx_3).$$

Therefore, $(P_c + jQ_c)(r_1 + r_2 + r_3 + jx_1 + jx_2 + jx_3)$

$$= (P_2 + jQ_2)(r_2 + jx_2) - (P_1 + jQ_1)(r_1 + jx_1)$$

That is,

$$P_c + jQ_c = \frac{1}{(R + jX)}[P_2r_2 - Q_2x_2 - P_1r_1 + Q_1x_1$$
$$+ j(P_2x_2 + Q_2r_2 - P_1x_1 - Q_1r_1)]$$
$$= \frac{(R - jX)(S + jT)}{R^2 + X^2}. \quad \text{(See equations 3 and 4).}$$

Therefore,

$$P_c = \frac{RS + XT}{R^2 + X^2}, \text{ as in equation (5)}$$

and

$$Q_c = \frac{RT - XS}{R^2 + X^2}, \text{ as in equation (6).}$$

One way to look at the above calculation is that the difference in voltage between the two substations before the tie-line is connected, drives a circulating current through all three lines after the tie-line is connected.

CHAPTER X

EFFECT OF MUTUAL INDUCTANCE ON ADJACENT CIRCUITS

THE choice of the phases to which each conductor should be connected when two transmission circuits are close together is of considerable importance.* Such a problem arises in the case of the usual arrangement of two circuits on one tower, shown in Fig. 15. It is found that by choosing the proper arrangement of conductors, and without changing the mechanical design of the towers, or changing the amount of the transposition or rotation of the circuits, the inductance of each circuit can be reduced by as much as 7% when the two circuits are carrying equal loads. With the same arrangement, the greater part of the unbalance may be eliminated.

The arrangements suggested in this chapter do not involve any expense, since they refer only to the phases to which the various conductors shall be connected, after the line is built. The design of the towers can be settled by mechanical considerations only.

When two circuits are carried on the same line of towers, they often have flat spacing, that is, the three conductors of a circuit lie in a plane, which usually is vertical or nearly so. When the conductors are loaded with ice, one conductor may lose its ice load on one span before the others, and when suspension insulators are used, they can allow a large vertical rise in that conductor, so that it is safer not to have two adjacent conductors placed one vertically above the other.

* "Reducing Inductance on Adjacent Transmission Circuits," by H. B. Dwight, *The Electrical World*, Jan. 12, 1924, p. 89.

Accordingly, one conductor is frequently offset somewhat, as shown in Fig. 15 or Fig. 16.

The reactance drop in each conductor can be calculated when the loads on the two circuits are equal. This is an average condition and includes the case when the circuits

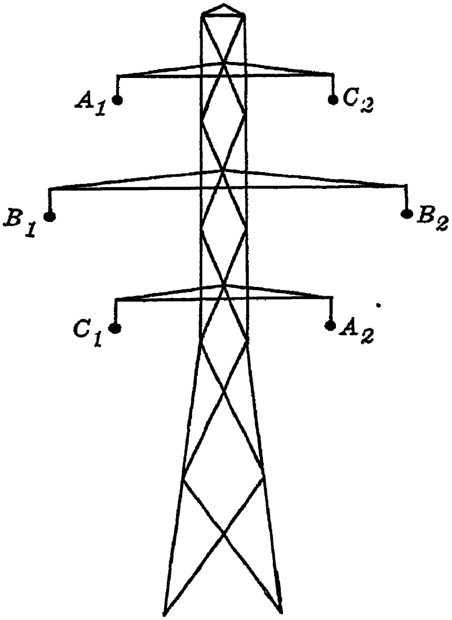


FIG. 15—Adjacent Transmission Circuits, Arranged Oppositely.

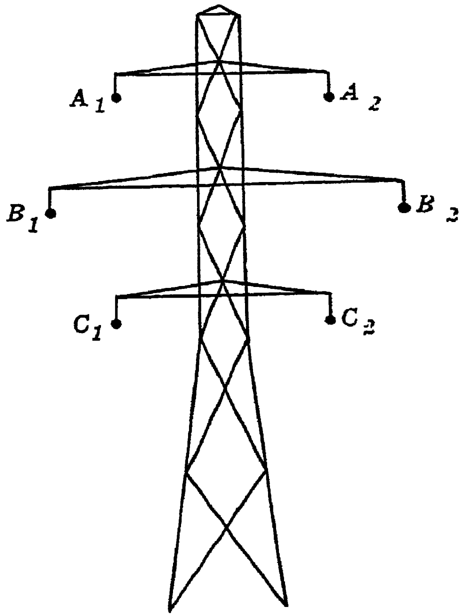


FIG 16—Adjacent Transmission Circuits, Arranged Alike.

are in parallel. Formulas will first be derived for the general case for any shape for the spacing arrangement of the conductors.

Let

$$a = \cos 120° + j \sin 120°$$

$$= -0.5 + j\,0.866. \qquad (1)$$

$$a^2 = (-0.5 + j\,0.866)^2$$

$$= -0.5 - j\,0.866$$

$$= \cos 240° + j \sin 240°. \qquad (2)$$

Then

$$1 + a + a^2 = 0. \qquad (3)$$

Find the reactance drop in conductor A_1, taking into account magnetic flux up to a certain large distance p. This quantity p vanishes later, by applying equation (3).

Let the current in conductors A_1 and A_2, due to the load, which is the same in both circuits, be I. Then the current in B_1 and B_2 is Ia and the current in C_1 and C_2 is Ia^2.

The reactance drop in conductor A_1 due to its own current is

$$j\frac{\omega 2I}{10^9}\left(\frac{1}{4}+\text{logh}\,\frac{p}{r}\right) \text{ volts per cm.}$$

The reactance drop in conductor A_1, due to current I in another conductor at a distance m, is

$$j\frac{\omega 2I}{10^9}\left(\text{logh}\,\frac{p}{m}\right) \text{ volts per cm.}$$

The reactance drop in A_1 due to the currents in the six conductors is

$$\begin{aligned} j\frac{\omega 2I}{10^9}\Big[\frac{1}{4}&+\text{logh}\,\frac{p}{r}+a\,\text{logh}\,\frac{p}{A_1B_1} \\ &+a^2\,\text{logh}\,\frac{p}{A_1C_1}+\text{logh}\,\frac{p}{A_1A_2} \\ &+a\,\text{logh}\,\frac{p}{A_1B_2}+a^2\,\text{logh}\,\frac{p}{A_1C_2}\Big] \\ =j\,\frac{\omega 2I}{10^9}\,[\tfrac{1}{4}&-\text{logh}\,(A_1A_2\times r) \\ &-a\,\text{logh}\,(A_1B_1\times A_1B_2) \\ &-a^2\,\text{logh}\,(A_1C_1\times A_1C_2)] \quad . \quad . \quad . \quad . \quad (4) \end{aligned}$$

using equation (3).

The reactance drop in B_1 is

$$\begin{aligned} j\frac{\omega 2Ia}{10^9}[\tfrac{1}{4}&-\text{logh}\,(B_1B_2\times r) \\ &-a\,\text{logh}\,(B_1C_1\times B_1C_2) \\ &-a^2\,\text{logh}\,(B_1A_1\times B_1A_2)]. \quad . \quad . \quad . \quad (5) \end{aligned}$$

The reactance drop in C_1 is

$$j\frac{\omega 2Ia^2}{10^9}[\tfrac{1}{4} - \text{logh } (C_1C_2 \times r)$$
$$-a \text{ logh } (C_1A_1 \times C_1A_2)$$
$$-a^2 \text{ logh } (C_1B_1 \times C_1B_2)]. \quad . \quad . \quad . \quad (6)$$

$$\omega = 2\pi f$$
$$= 2\pi \times \text{cycles per second.}$$

I is the absolute, or measured, value of current in each conductor in amperes.

r = radius of conductor, measured in the same units as the spacings.

These general formulas can now be applied to the particular cases of Figs. 15 and 16. For Fig. 15, putting

$$A_1B_1 = B_1C_1 = m$$
$$A_1C_2 = 1\ 5m$$

and

$$B_1B_2 = 2\ 5m$$

we obtain

$$A_1C_1 = 1.732m$$
$$A_1A_2 = 2.29m$$

and

$$A_1B_2 = 2.18m.$$

Then E_a = reactance drop in A_1

$$= j\frac{\omega 2I}{10^9}[\tfrac{1}{4} - \text{logh } 2.29mr$$
$$+(0.5 - j\,0.866) \text{ logh } (2\ 18m^2)$$
$$+(0.5 + j\,0.866) \text{ logh } (1.5 \times 1.732m^2)]$$
$$= j\frac{\omega 2I}{10^9}\left(\frac{1}{4} + \text{logh } \frac{m}{r} + 0\ 038 + j\,0\ 152\right) \text{ volts per cm.}$$

E_b = Reactance drop in B_1

$$= j\frac{\omega 2Ia}{10^9}\left(\frac{1}{4} + \text{logh}\,\frac{m}{r} - 0\ 137\right)$$

E_c = Reactance drop in C_1

$$= j\frac{\omega 2Ia^2}{10^9}\left(\frac{1}{4} + \text{logh}\,\frac{m}{r} + 0.038 - j\,0.152\right).$$

It is now useful to apply three equations given by Chas. F. Fortescue.*

First, the average reactive drop is

$$E_{a2} = \tfrac{1}{3}(E_a + a^2E_b + aE_c)$$

$$= j\frac{\omega 2I}{10^9}\left(\frac{1}{4} + \text{logh}\,\frac{m}{r} - 0.020\right).$$

Also

$$E_{a1} = \tfrac{1}{3}(E_a + aE_b + a^2E_c)$$

$$= j\frac{\omega 2I}{10^9}(+0\ 073 + j\,0.127).$$

This is a measure of the unbalance, and is known as the "reversed phase sequence voltage."

Finally

$$E_{a0} = \tfrac{1}{3}(E_a + E_b + E_c)$$

$$= j\frac{\omega 2I}{10^9}(-0.015 + j\,0.025).$$

This is an equal voltage drop in all three conductors of the circuit. It is small in the cases here considered.

If the same quantities are calculated for Fig. 16, we obtain

$$E_a = j\frac{\omega 2I}{10^9}\left(\frac{1}{4} + \text{logh}\,\frac{m}{r} + 0\ 673 + j\,0\ 519\right)$$

$$E_b = j\frac{\omega 2Ia}{10^9}\left(\frac{1}{4} + \text{logh}\,\frac{m}{r} - 0\ 137\right)$$

* "Method of Symmetrical Co-ordinates Applied to the Solution of Polyphase Networks," by C. L. Fortescue, *Trans.*, *A I. E E*, 1918, equations (5) and (6), p 1034.

$$E_c = j\frac{\omega 2Ia^2}{10^9}\left(\frac{1}{4}+\text{logh}\,\frac{m}{r}+0.673-j\,0.519\right)$$

$$E_{a2} = j\frac{\omega 2I}{10^9}\left(\frac{1}{4}+\text{logh}\,\frac{m}{r}+0.403\right)$$

$$E_{a1} = j\frac{\omega 2I}{10^9}(+0\ 285+j\,0\ 493)$$

$$E_{a0} = j\frac{\omega 2I}{10^9}(-0\ 015+j\,0.025).$$

The quantity $\left(\frac{1}{4}+\text{logh}\,\frac{m}{r}\right)$ may be taken as 5 5 for an average transmission circuit. Then the value of E_{a2} is 5.480 for Fig. 15 and 5.903 for Fig. 16.

It is seen that the average voltage drop, E_{a2}, is 7% less for Fig. 15 than for Fig. 16. The reactance drop for Fig. 15 is less than if there were no load in the second circuit on the tower, and it is even less than for one isolated circuit with equilateral triangular spacing of side m.

The unbalance, represented by E_{a1}, is less than one third as great in Fig. 15 as it is in Fig. 16.

While transposition of the conductors every few miles gives theoretically equal reactance drops in the conductors, and removes the unbalance, the correction cannot be perfect, and it is better if the unbalance in each section of the line is small.

It is evident that the arrangement of Fig. 15 is preferable in every way to that of Fig. 16, and no expense is involved in adopting it.

If B_1 and B_2, in Figs. 15 and 16, are brought in line with the upper and lower conductors, so that each circuit has flat spacing, the method of connection of Fig. 15 gives 9% less reactance drop than the method of Fig. 16, but the unbalance is slightly greater with the method of Fig. 15.

An interesting case, although not a practical one, is when the six conductors are spaced at the corners of a regular hexagon, having all its sides and angles equal. Here, the method of connection of Fig. 15 gives 11% less reactance drop than the method of Fig. 16. The unbalance is zero with the method of connection of Fig. 15 and with equal loads on the two circuits, but there is considerable unbalance with the method of Fig. 16.

This general method of calculation can be used for determining the reactance drop and the effect of mutual inductance and of different methods of transposing conductors, etc., with various combinations of alternating-current circuits.

CHAPTER XI

RESISTANCE AND REACTANCE OF STEEL CONDUCTORS

IRON and steel wires and cables are used to a considerable extent as conductors for branch power lines where the currents to be carried are not large. This is partly due to the necessity in sparsely settled districts of taking advantage of every possible economy in design, so that the use of a copper or aluminum conductor is prohibited when a cheaper steel conductor can do the work. The utilization of steel conductors is also partly due to the increased knowledge of their electrical properties, so that they may be applied to a projected line with more confidence that the results desired will be obtained.

The resistance and reactance of iron and steel wires and cables vary considerably with the grade of the metal and with the number of amperes of current which is being carried. Accordingly, these properties are shown by a set of curves (see Tables 24 and 25), which have been drawn up so as to apply as much as possible to sizes and grades of steel conductors commercially used in America.* These curves show the average results of the tests which have been published. As the tests published so far are not very complete and do not agree very closely with each other, the probable characteristics of a steel conductor can best be obtained for estimating purposes from a set of average curves, as in Tables 24 and 25.

The tests of a given type of steel conductor show two main kinds of variations, when different samples are tested. First, there are variations in the conductivity for direct

* "Resistance and Reactance of Commercial Steel Conductors," by H. B. Dwight, *The Electric Journal*, Jan., 1919, p. 25.

current, which is the same as the conductivity for a very small alternating current, and second, there are variations in the percentage increase in resistance when alternating current is carried.

In order to make the set of curves consistent as regards direct-current conductivity, the following values, expressed as percentages of the annealed copper standard, have been assumed.

Grade E. B. B. (Extra Best Best)	16%
Grade B. B. (Best Best)	14%
Ordinary Steel Grade	12%
Siemens–Martin	9%
High Strength	8%

The percentage increase of resistance of a given size of wire or cable at a given strength of alternating current is greatest for Grade E. B. B. and least for High Strength Steel. The other grades have intermediate values in the order given in the above list. The curves show that a low-priced, medium grade of steel may have a lower resistance than Grade E. B. B. or B. B., for certain alternating-current loads. The percentage increase in resistance is practically proportional to the frequency at commercial frequencies, and this fact enables curves for 25 and 60 cycles to be compared.

The reactance plotted in Tables 24 and 25 is the internal reactance, that is, the reactance due to magnetic flux inside the conductors. In order to obtain the total reactance of the electric power line, the external reactance should be added. This may be taken from tables prepared for use with circuits using copper conductors, but in general, it will be sufficient to add 0.8 ohm per mile for 60 cycles and 0 3 ohm per mile for 25 cycles. The reactance of steel conductors is practically proportional to the frequency at commercial frequencies.

The conductors referred to in Tables 24 and 25 are wires of the Birmingham Wire Gauge (B. W. G.) and cables made up of such wires, except where the sizes are given in circular mils. There is some discrepancy between the nominal diameters of the cables and their actual diameters, as is shown by the following table:

CABLES COMPOSED OF 7 STANDARD B. W. G WIRES

Nominal Diameter	Size of Wires, B W G.	Diameter of Wires	Actual Diameter of Cable	Sectional Area in Circular Mils
$\frac{1}{4}$	No 14	0 083	0 249	48,200
$\frac{9}{32}$	No. 13	0 095	0 285	63,200
$\frac{5}{16}$	No. 12	0 109	0 327	83,200
$\frac{3}{8}$	No. 11	0 120	0 360	100,800
$\frac{1}{2}$	No. 8	0 165	0 495	190,600

The Birmingham wire gauge is used largely for telegraph wires of iron and steel. The electrical properties of steel conductors of other gauges can be estimated by changing the values given by Tables 24 and 25 in proportion to the change in sectional area of the conductor.

The tests on which the curves of Tables 24 and 25 are based, were described in the following articles:

"Effective Resistance and Inductance of Iron and Bimetallic Wires," by John M. Miller, Scientific Paper No. 252 of the Bureau of Standards, Aug., 1915.

"Iron Wire for Distribution and Transmission Lines," The Electrical World, April 8, 1916.

"Iron and Steel Wire for Transmission Conductors," by T. A. Worcester, General Electric Review, June, 1916, page 488.

"Steel Conductors for Transmission Lines," by H. B. Dwight, Transactions A. I. E. E., Sept. 18, 1916, page 1237.

"Characteristics of Iron and Steel Conductors," by C. E. Oakes and W. Eckley, The Electrical World, Oct. 14, 1916.

"Characteristics of Iron Wire for Transmission Purposes," by L. W. Morrow, The Electrical World, July 14, 1917.

"Iron Wire for Short High-voltage Lines," by W. T. Ryan, The Electrical Review, Sept. 22, 1917.

"Iron and Steel Conductors," by R. C. Powell, Journal of Electricity, April 1, 1918.

"Characteristics of Iron and Steel Conductors," by C. E. Oakes and P. A. Sahm, The Electrical World, July 27, 1918.

The capacitance of steel conductors is the same as that of copper conductors of the same size, and therefore, the usual tables and formulas may be used for determining the capacitance of steel conductors.

Example

Length of line	7 miles
Voltage between conductors at receiver	11,000 volts
Voltage to neutral at receiver	6350 volts
Frequency	60 cycles
Power factor of load	85%
Phases	3
Conductor, $\frac{5}{16}''$ ordinary steel cable.	
Full load	200 kv-a.
Full load amperes per cable	10.5 amperes
Resistance of cable per mile at full load	6.0 ohms, from Table 25
Reactance of cable per mile at full load	1 9 ohms, from Table 25
In-phase current per cable	8.9 amperes
Reactive current per cable	5 5 amperes

Voltage drop per cable

$$=8\ 9\times 6\ 0\times 7+5\ 5\times 1.9\times 7$$

$$+\frac{(8\ 9\times 1\ 9\times 7-5.5\times 6\ 0\times 7)^2}{2\times 6350}=450 \text{ volts}$$

Voltage drop in per cent of receiver voltage

$$=\frac{450\times 100}{6350}=7.1\%$$

CHAPTER XII

CORONA LOSS AND VOLTAGE LIMITS

In designing a transmission line, the limitations due to corona should be kept in mind. If the voltage of a line is higher than a certain critical amount, corresponding to a certain voltage gradient in the air at the conductor surface, the air, which at lower voltages is a good insulator, breaks down and allows an electric discharge to pass between the conductors. This is accompanied by a glow on the conductors which may be seen on a dark night. The energy of this discharge is the "corona loss" and the phenomenon could be represented by a conductance g per mile, between conductors. This conductance, however, is subject to large variations with changing conditions.

It is accepted as a general approximate rule that a transmission line should not be operated at a voltage higher than its fair weather disruptive critical voltage, e_0. Accordingly, values of e_0 are of considerable importance and are given in Table 26. The data in that table and in this chapter are taken, by permission of the author, from the book "Dielectric Phenomena in High Voltage Engineering," by F. W. Peek, Jr.

If a line is operated at the fair weather value of e_0, then in stormy weather there will be considerable corona loss per mile, which, however, may not take place along the entire line in the case of long lines. This occasional loss in unfavorable weather is considered reasonable and allowable in most cases, and so the general rule is put forward that the fair weather value of e_0 is the corona limit of operating voltage. It is questionable if it is ever desirable to operate above this voltage.

Two values of corona loss can be calculated by the formula for p in Table 26, one for fair weather and one for stormy weather. The first uses the fair weather value of e_0, as given directly by the equation for e_0 in Table 26. The stormy weather value of corona loss is calculated by the same formula, but a value of e_0 is used which is 80% as large as that used for the fair weather calculation. See the example in this chapter.

It is evident from the values in Table 26 that a change in the diameter of the conductor produces a large change in the corona limit of voltage. Thus the steel core usually used with aluminum cables increases their outside diameter and reduces the possibility of trouble from corona. A core of low-priced, medium strength steel might also be worth while using in copper cables at high voltages, in order to avoid excessive corona loss.

Example

Conditions given:

Number of phases	3
Frequency	60 cycles per sec.
Length	100 miles
Spacing, triangular	120 in.
Conductor	No. 0 cable, dia. 0.374 in.
Maximum temperature	100° F.
Elevation	1000 ft.
Barometer	28 85 in.

Then

$$\log_{10}\frac{s}{r}=\log_{10} 642=2.81;$$

$$\sqrt{\frac{r}{s}}=0.0394;$$

$$\delta=\frac{17\ 9\times 28.85}{459+100}=0\ 925;$$

$m_0 = 0.87;$

$e_0 = 123 \times 0.87 \times 0.187 \times 0.925 \times 2.81$

$= 52.0$ kv. to neutral, in fair weather;

$p = 0.014(e - 52.0)^2 \times 3$ kw. per mile of three-phase line, in fair weather.

In stormy weather,

$e_0 = 0.8 \times 52.0 = 41.6$ kv. to neutral;

and

$p = 0.14(e - 41.6)^2 \times 3$ kw. per mile of three-phase line.

At 10,000 ft. elevation, 20.4 in. barometer,

$e_0 = 36.8$ kv. to neutral in fair weather,

and

$p = 0.0599(e - 36.8)^2$ kw. per mile of three-phase line.

In stormy weather at 10,000 ft. elevation,

$e_0 = 0.8 \times 36.8 = 29.4$ kv. to neutral,

and

$p = 0.0599(e - 29.4)^2$ kw. per mile of three-phase line.

For the above, the temperature has been assumed to be the same under all conditions, but the storm loss will generally be less due to lower temperature, as will also the losses during winter.

The corona loss calculated from the above formulas, is listed below for different voltages.

EXAMPLE

LINE WITH NO. 0 CONDUCTORS

Kv. between Conductors	e=Kv. to Neutral	Corona Loss in Kw. for 100 Miles of 3-phase Line			
		1000 Ft. Elevation		10,000 Ft Elevation	
		Fair Weather	Storms	Fair Weather	Storms
50	28.9	0	0	0	0
60	34.7	0	0	0	170
70	40.5	0	0	80	800
80	46.3	0	90	540	1,700
90	52.0	0	460	1,400	3,000
100	57.8	140	1100	2,600	4,800
110	63.6	570	2000	4,300	7,000
120	69.4	1300	3300	6,300	9,600
130	75.1	2200	4700	8,800	12,500
140	80.9	3500	6500	11,600	15,800
150	86.7	5100	8600	14,900	19,600

PART II

THEORY

CHAPTER XIII

CONDUCTORS

THREE main classes of conductors are used for overhead lines for the transmission of electric power; namely, copper wires, copper cables and aluminum cables. Steel conductors are occasionally used, for which see Chapter XI. The cables used are generally strands of seven wires; that is, they consist of a central straight wire with six wires wound spirally around it, as indicated by the cross section in Fig. 17.

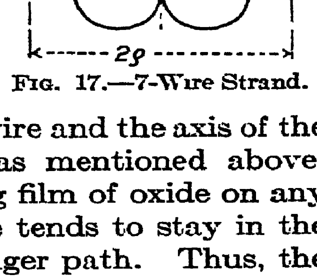


FIG. 17.—7-Wire Strand.

From this figure it is seen that the maximum diameter of a 7-wire strand is equal to 3 times the diameter of one of the wires

The outside wires do not follow a straight path parallel to the central wire and the axis of the cable, but lie in a spiral around it, as mentioned above. As there is always a slight insulating film of oxide on any wire, the current flowing in the cable tends to stay in the individual wires, and so follows the longer path. Thus, the resistance of a cable is greater than that of a solid wire of the same area of cross section. The amount of the difference depends on the number of wires in the cable and the pitch of the spiraling, but an average value of 2% is assumed in making up the tables in Part III. The cross section of the cable is assumed to be equal to the sum of the cross sections of the individual wires. The weight per unit length of the cable calculated from this cross section must be increased by the same percentage as the above increase in resistance, due to the extra length of the outside wires. Since the cross section in Fig. 17 does not cut the outside wires exactly at right angles, their sections as shown in the figure are really ellipses, and the diameter of the cable is slightly greater than $6\rho_1$. However, this difference is small and has been neglected in the figures for diameter of cable tabulated in Part III.

The number of wires in a strand varies in practice according to the degree of flexibility and mechanical strength desired by the user. The number of wires per strand in the tables represents average practice for overhead lines. The larger cables have 19, 37 or 61 wires. See Table 904, A. I. E. E. Standards, Edition of 1922.

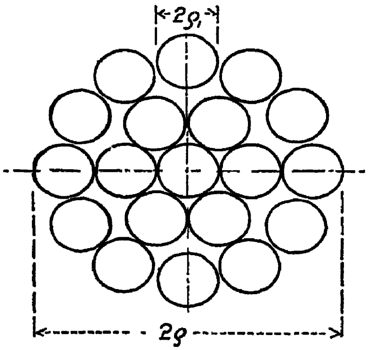


FIG. 18.—19-Wire Strand.

The section of a 19-wire strand is shown in Fig. 18, and it is seen that the maximum diameter is 5 times the diameter of one of the individual wires. The same increase of 2% in resistance is allowed as with a 7-wire strand.

There is only a very slight difference in the reactance and capacity of a 7-wire and a 19-wire strand of the same sectional area, so that values listed for 7 wires may be used for 19 wires, and vice versa, without very much error.

The resistances for direct current tabulated in Part III have been calculated in accordance with the recommendations of the Bureau of Standards for the preparation of wire tables.* The Standardization Rules of the American Insti-

* "Copper Wire Tables," Circular No. 31 of the Bureau of Standards, 1914, and para. 9050, Standards of the A. I E. E., 1922.

tute of Electrical Engineers are in agreement with these recommendations. According to the Bureau of Standards and the A. I. E. E., the "Annealed Copper Standard," which is of 100% conductivity, is represented by a resistivity of 0 15328 ohm (meter, gram) at 20° C. This is equivalent to 1 7241 microhm-centimeter at 20° C., assuming a density of 8 89. The conductivity of hard drawn copper recommended for wire tables by the Bureau of Standards is 97.3%, this value representing an average for good commercial copper. The average conductivity given by the Bureau of Standards for hard drawn aluminum on the centimeter cube basis, assuming a density of 2.70, is 61%. The above values have been used in preparing the tables in Part III, 2% being added to the resistance for the effect of spiraling, as already noted.

If it is desired to calculate the resistance of copper conductors for other temperatures than 20° C., the temperature coefficient, α_{20}, for hard drawn copper of 97.3% conductivity should be used in connection with the formula

$$R_t = R_{20}\{1 + \alpha_{20}(t - 20)\}$$

where t is the temperature in degrees Centigrade for which the resistance R_t is desired and where

$$\alpha_{20} = 0\ 00382.$$

For other initial temperatures and other conductivities, temperature coefficients should be used as given in the table of temperature coefficients in Part III, which is taken from Table II, Circular No. 31 of the Bureau of Standards.

For the temperature coefficient of hard drawn aluminum, a value of

$$\alpha_{20} = 0.0039,$$

which is recommended by the Bureau of Standards, may be used.

CHAPTER XIV

THEORY OF CONVERGENT SERIES

The well-known fundamental formulas for a transmission line without branches, in which the load is delivered only at the end of the line, are as follows:

$$E_s = E \cosh \sqrt{YZ} + \frac{1}{\sqrt{YZ}} IZ \sinh \sqrt{YZ},$$

and

$$I_s = I \cosh \sqrt{YZ} + \frac{1}{\sqrt{YZ}} EY \sinh \sqrt{YZ},$$

where

E_s and I_s are the voltage and current at the supply end,

E and I are the voltage and current at the receiver end,

Y is the line admittance

and

Z is the line impedance.

The above equations have been published at various times. They are obtained as follows:

Let

r = resistance of conductor per unit length,

and

x = reactance of conductor per unit length.

Then

$z = r + jx$

= impedance of conductor per unit length.

Let g = leakage conductance from conductor per unit length,

and

b = capacity susceptance of conductor per unit length.

Then

$$y = g + jb$$

$= $ admittance of conductor per unit length.

Let

$E_l = $ voltage of line at a distance l from the receiver end,

and let

$$I_l = P_l + jQ_l$$

$= $ current in the line at a distance l from the receiver end.

(Since I_l is usually not in phase with the voltage, it must be expressed as a complex quantity.)

Now in an element of length, dl, of the line, the voltage consumed by impedance is

$$dE_l = zI_l dl.$$

The current consumed by admittance is

$$dI_l = yE_l dl.$$

Thus

$$\frac{dE_l}{dl} = zI_l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{dI_l}{dl} = yE_l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Differentiating (1)

$$\frac{d^2E_l}{dl^2} = z\frac{dI_l}{dl}.$$

Substituting (2) in this gives

$$\frac{d^2E_l}{dl^2} = zyE_l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is a differential equation of the second order and may be expressed in the form

$$(D^2 - yz)E_l = 0,$$

and we have

$$E_l = A_1\epsilon^{l\sqrt{yz}} + A_2\epsilon^{-l\sqrt{yz}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and from (1),

$$I_l = \frac{1}{z}\frac{dE_l}{dl}$$

$$= \sqrt{\frac{y}{z}}(A_1\epsilon^{l\sqrt{yz}} - A_2\epsilon^{-l\sqrt{yz}}). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now at the supply end,

$$yl = Y,$$

and

$$zl = Z.$$

Therefore,

$$E_s = A_1\epsilon^{\sqrt{YZ}} + A_2\epsilon^{-\sqrt{YZ}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$I_s = \sqrt{\frac{Y}{Z}}(A_1\epsilon^{\sqrt{YZ}} - A_2\epsilon^{-\sqrt{YZ}}). \quad . \quad . \quad . \quad . \quad (7)$$

At the receiver end,

$$l = 0$$

and

$$E = A_1 + A_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$I = \sqrt{\frac{Y}{Z}}(A_1 - A_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From (6)

$$E_s = \tfrac{1}{2}(A_1 + A_2)(\epsilon^{\sqrt{YZ}} + \epsilon^{-\sqrt{YZ}})$$

$$+ \tfrac{1}{2}(A_1 - A_2)(\epsilon^{\sqrt{YZ}} - \epsilon^{-\sqrt{YZ}}), \quad . \quad . \quad (10)$$

which, by the definition of cosh θ and sinh θ, is

$$(A_1 + A_2)\cosh\sqrt{YZ} + (A_1 - A_2)\sinh\sqrt{YZ}.$$

Therefore, from (8) and (9),

$$E_s = E\cosh\sqrt{YZ} + \frac{1}{\sqrt{YZ}}IZ\sinh\sqrt{YZ}. \quad . \quad . \quad (11)$$

Similarly, from (7)

$$I_s = \sqrt{\frac{Y}{Z}}\,\frac{1}{2}(A_1 - A_2)(\epsilon^{\sqrt{YZ}} + \epsilon^{-\sqrt{YZ}})$$

$$+\sqrt{\frac{Y}{Z}}\frac{1}{2}(A_1+A_2)(\epsilon^{\sqrt{YZ}}-\epsilon^{-\sqrt{YZ}}), \quad . \quad . \quad . \quad (12)$$

$$=I \cosh \sqrt{YZ}+\frac{1}{\sqrt{YZ}}EY \sinh \sqrt{YZ}. . \quad . \quad (13)$$

Equations (11) and (13) are the fundamental formulas of transmission lines, as generally written.

Now

$$\epsilon^{\sqrt{YZ}}+\epsilon^{-\sqrt{YZ}}$$

$$=1+\sqrt{YZ}+\frac{(\sqrt{YZ})^2}{2}+\frac{(\sqrt{YZ})^3}{2\cdot 3}+\frac{(\sqrt{YZ})^4}{2\cdot 3\cdot 4}+\text{etc.}$$

$$+1-\sqrt{YZ}+\frac{(\sqrt{YZ})^2}{2}-\frac{(\sqrt{YZ})^3}{2\cdot 3}+\frac{(\sqrt{YZ})^4}{2\cdot 3\cdot 4}-\text{etc.}$$

$$=2\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\text{etc.}\right).$$

Similarly

$$\epsilon^{\sqrt{YZ}}-\epsilon^{-\sqrt{YZ}}$$

$$=2\sqrt{YZ}\left(1+\frac{YZ}{2\cdot 3}+\frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5}+\text{etc.}\right).$$

Substituting these results in (10) and (12), we can express the fundamental equations as follows:

$$E_s=E\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\frac{Y^3Z^3}{2\cdot 3\cdot 4\cdot 5\cdot 6}+\text{etc.}\right)$$

$$+IZ\left(1+\frac{YZ}{2\cdot 3}+\frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5}+\frac{Y^3Z^3}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}+\text{etc.}\right) \qquad (14)$$

and

$$I_s=I\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\frac{Y^3Z^3}{2\cdot 3\cdot 4\cdot 5\cdot 6}+\text{etc.}\right)$$

$$+EY\left(1+\frac{YZ}{2\cdot 3}+\frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5}+\frac{Y^3Z^3}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}+\text{etc.}\right). \qquad (15)$$

* See references to T. R. Rosebrugh, J. F. H. Douglas, and C. P. Steinmetz, Chapter VI.

Equations (14) and (15) are the same as those tabulated in Table 6 for obtaining E_s or $A+jB$, and I_s or $C+jD$. For no-load values, all that is necessary is to put the load current, I, equal to zero.

When conditions are given at the supply end, the same equations for full-load conditions are obtained, except that the second half of the expressions for voltage and current is negative, since power is now flowing away from the point where the voltage is specified, instead of toward it. Thus we have the series for $F+jH$ and $M+jN$.

At no load, the conditions are really not all specified at the supply end, but the current is specified to be zero at the receiver end, and this necessitates the use of special series. From Table 6 we have the ratio of the voltage at the two ends of the line,

$$\frac{E_{0s}}{E}=\frac{A_0+jB_0}{E}$$

$$=\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot3\cdot4}+\text{etc.}\right).$$

This ratio is independent of the voltage E, and depends only on the constants of the line. Thus, if E_s, the voltage at the supply end at no load, is given, we can obtain the no-load voltage at the receiver end from the equation,

$$\frac{E_s}{E_0}=\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot3\cdot4}+\text{etc.}\right),$$

or

$$E_0=E_s\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot3\cdot4}+\text{etc.}\right)^{-1},$$

which, when expanded by the binomial theorem, gives

$$E_0=F_0+jH_0$$

$$=E_s(1-\tfrac{1}{2}YZ+\tfrac{5}{24}Y^2Z^2-\tfrac{61}{720}Y^3Z^3+\tfrac{277}{8064}Y^4Z^4-\text{etc.}). \quad (16)$$

as in Table 7.

The no-load current at the supply end is

$$I_{0s} = E_0 Y\left(1 + \frac{YZ}{2\cdot 3} + \frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5} + \text{etc.}\right),$$

as in the equation for $C_0 + jD_0$.

Substituting the value of E_0 from equation (16), we have

$$I_{0s} = E_s Y(1 - \tfrac{1}{2}YZ + \tfrac{5}{24}Y^2Z^2 - \tfrac{61}{720}Y^3Z^3 + \tfrac{277}{8064}Y^4Z^4 - \text{etc.})$$

$$\times\left(1 + \frac{YZ}{2\cdot 3} + \frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5} + \frac{Y^3Z^3}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\right.$$

$$\left. + \frac{Y^4Z^4}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9} + \text{etc.}\right).$$

Multiplying the two series together by the ordinary algebraical method, we obtain

$$I_{0s} = M_0 + jN_0$$

$$= E_s Y(1 - \tfrac{1}{3}YZ + \tfrac{2}{15}Y^2Z^2 - \tfrac{17}{315}Y^3Z^3 + \tfrac{62}{2835}Y^4Z^4 - \text{etc.})$$

as given in Table 7 of convergent series.

When the supply voltage and the receiver load are given for a long transmission line, as in Table 8, we have the following equation, as in Table 6:

$$\text{Supply voltage} = E\left(1 + \frac{YZ}{2} + \frac{Y^2Z^2}{2\cdot 3\cdot 4} + \dots\right)$$

$$+ (P + jQ)(R + jX)\left(1 + \frac{YZ}{2\cdot 3} + \frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5} + \dots\right),$$

$$\frac{\text{Supply voltage}}{\left(1 + \frac{YZ}{2} + \dots\right)} = E + \frac{(P + jQ)(R + jX)\left(1 + \frac{YZ}{2\cdot 3} + \dots\right)}{\left(1 + \frac{YZ}{2} + \dots\right)}$$

$$= E + (P + jQ)(R_1 + jX_1),$$

as indicated in Table 8.

The absolute value of the supply voltage is the known quantity E_s. Find the absolute value of

$$\frac{E_s}{\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\ 3\cdot 4}+\ldots\right)}$$

and let it be E_c. Then

$$E_c^2=(E+PR_1-QX_1)^2+(PX_1+QR_1)^2.$$

This is the same form of equation as in the derivation of Table 3 in Chapter IV, and so the remaining formulas of Table 8 are derived, as shown in Chapter IV.

Lines Connected in Parallel.—When two or more long transmission lines are connected in parallel, the constants of the combined circuit, based on the exact hyperbolic theory, can be written in a form similar to that for a single transmission line. This was described by Prof. T. R. Rosebrugh in the Engineering Research Bulletin, University of Toronto, 1919. He also gave methods of calculating the combined constants of groups of transmission lines in series, series-parallel and in networks of different kinds.

From the first part of Table 6, or from equations (11) to (15) of this chapter, we have

$$E_s=E\alpha+I\beta, \qquad (17)$$

and

$$I_s=E\gamma+I\delta, \qquad (18)$$

where

$$\alpha=\delta=1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\ldots$$

$$=\cosh\sqrt{YZ}$$

$$\beta=Z\left(1+\frac{YZ}{2\cdot 3}+\frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5}+\ldots\right)$$

$$=\sqrt{\frac{Z}{Y}}\sinh\sqrt{YZ},$$

and

$$\gamma = Y\left(1+\frac{YZ}{2\cdot 3}+\frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5}+\dots\right)$$

$$=\sqrt{\frac{Y}{Z}}\sinh\sqrt{YZ}$$

$$=\frac{Y\beta}{Z}.$$

The above constants are for a uniform transmission line, without transformers. If the line is not uniform, or if transformers are included in the circuit, α will in general not be equal to δ. For such cases, which represent circuits in series, use the general method of Table 13, or of Section III of the paper by Prof. Roseburgh referred to above.

For a check on numerical computations, it is useful to note that

$$\alpha\delta-\beta\gamma=\cosh^2\sqrt{YZ}-\sinh^2\sqrt{YZ}=1. \quad . \quad . \quad (19)$$

Let there be two long lines connected in parallel, which may have different routes and lengths and different sizes of conductors and spacings. The receiver voltage E will be the same for both lines, as will also the supply voltage E_s. The total current at the receiver end is

$$I=I_1+I_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where I_1 is the current in the first transmission line at the receiver end and I_2 that in the second line. Similarly,

$$I_s=I_{s1}+I_{s2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The above quantities are in general complex quantities.

From 17,

$$I_1=\frac{E_s}{\beta_1}-\frac{E\alpha_1}{\beta_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and

$$I_2=\frac{E_s}{\beta_2}-\frac{E\alpha_2}{\beta_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

ut

$$\frac{1}{\beta_1}=\epsilon_1 \quad \text{and} \quad \frac{1}{\beta_2}=\epsilon_2.$$

From equation (20), by adding (22) and (23),

$$I = E_s(\epsilon_1 + \epsilon_2) - E(\alpha_1\epsilon_1 + \alpha_2\epsilon_2).$$

Therefore,

$$E_s = \frac{E(\alpha_1\epsilon_1 + \alpha_2\epsilon_2)}{(\epsilon_1 + \epsilon_2)} + \frac{I}{(\epsilon_1 + \epsilon_2)}. \quad . \quad . \quad . \quad (24)$$

Or,

$$E_s = E\alpha_0 + I\beta_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where

$$\beta_0 = \frac{1}{\epsilon_1 + \epsilon_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and

$$\alpha_0 = (\alpha_1\epsilon_1 + \alpha_2\epsilon_2)\beta_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

From (18) and (22),

$$I_{s1} = I_1\delta_1 + E\gamma_1$$
$$= E_s\delta_1\epsilon_1 - E\alpha_1\delta_1\epsilon_1 + E\gamma_1.$$

Similarly,

$$I_{s2} = E_s\delta_2\epsilon_2 - E\alpha_2\delta_2\epsilon_2 + E\gamma_2.$$

Adding together,

$$I_s = I_{s1} + I_{s2}$$
$$= E_s(\delta_1\epsilon_1 + \delta_2\epsilon_2) - E(\alpha_1\delta_1\epsilon_1 + \alpha_2\delta_2\epsilon_2) + E(\gamma_1 + \gamma_2),$$

or, from (24),

$$I_s = \frac{E(\alpha_1\epsilon_1 + \alpha_2\epsilon_2)(\delta_1\epsilon_1 + \delta_2\epsilon_2)}{(\epsilon_1 + \epsilon_2)} + \frac{I(\delta_1\epsilon_1 + \delta_2\epsilon_2)}{(\epsilon_1 + \epsilon_2)}$$
$$- E(\alpha_1\delta_1\epsilon_1 + \alpha_2\delta_2\epsilon_2) + E(\gamma_1 + \gamma_2)$$
$$= E\gamma_0 + I\delta_0,$$

where

$$\gamma_0 = (\gamma_1 + \gamma_2) - (\alpha_1\delta_1\epsilon_1 + \alpha_2\delta_2\epsilon_2) + (\delta_1\epsilon_1 + \delta_2\epsilon_2)\alpha_0, \quad . \quad (28)$$

and

$$\delta_0 = (\delta_1\epsilon_1 + \delta_2\epsilon_2)\beta_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

It is now desirable to show that

$$\alpha_0\delta_0 - \beta_0\gamma_0 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

We have, from (29),

$$\alpha_0\delta_0=(\delta_1\epsilon_1+\delta_2\epsilon_2)\alpha_0\beta_0,$$

and from (28),

$$-\beta_0\gamma_0=(\alpha_1\delta_1\epsilon_1+\alpha_2\delta_2\epsilon_2)\beta_0-(\gamma_1+\gamma_2)\beta_0-(\delta_1\epsilon_1+\delta_2\epsilon_2)\alpha_0\beta_0.$$

Therefore,

$$\alpha_0\delta_0-\beta_0\gamma_0=(\alpha_1\delta_1\epsilon_1+\alpha_2\delta_2\epsilon_2)\beta_0-(\gamma_1+\gamma_2)\beta_0. \quad . \quad (31)$$

If each side of this equation is equal to 1, then

$$(\alpha_1\delta_1\epsilon_1+\alpha_2\delta_2\epsilon_2)-(\gamma_1+\gamma_2)=\frac{1}{\beta_0}$$
$$=\epsilon_1+\epsilon_2. \quad . \quad . \quad . \quad . \quad (32)$$

Now

$$\alpha_1\delta_1\epsilon_1-\gamma_1=\epsilon_1(\alpha_1\delta_1-\beta_1\gamma_1)$$
$$=\epsilon_1$$

by equation (19) for uniform transmission lines. Similarly,

$$\alpha_2\delta_2\epsilon_2-\gamma_2=\epsilon_2,$$

and so equations (32), (31) and (30) are shown to be true for uniform transmission lines in parallel. This theorem is capable of extension to more complicated cases.

Equation (30) gives a useful check on the correctness of numerical values of the constants, or it may be used to give a shorter formula for γ_0, namely,

$$\gamma_0=\frac{\alpha_0\delta_0-1}{\beta_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

as given in Table 6.

If there are n lines in parallel, the constants α_0, β_0, γ_0 and δ_0 for the combined circuit are derived in the same way, and are

$$\beta_0=\frac{1}{\epsilon_1+\epsilon_2+\ .\ .\ .\ +\epsilon_n}. \quad . \quad . \quad . \quad . \quad . \quad (34)$$

$$\alpha_0=(\alpha_1\epsilon_1+\alpha_2\epsilon_2+\ .\ .\ .\ +\alpha_n\epsilon_n)\beta_0. \quad . \quad . \quad . \quad . \quad (35)$$

$$\gamma_0=(\gamma_1+\gamma_2+\ .\ .\ .\ +\gamma_n)-(\alpha_1\delta_1\epsilon_1+\alpha_2\delta_2\epsilon_2+\ .\ .\ .\ \alpha_n\delta_n\epsilon_n)$$
$$+(\delta_1\epsilon_1+\delta_2\epsilon_2+\ .\ .\ .\ +\delta_n\epsilon_n)\alpha_0. \quad . \quad . \quad . \quad . \quad (36)$$

$$\delta_0=(\delta_1\epsilon_1+\delta_2\epsilon_2+\ .\ .\ .\ +\delta_n\epsilon_n)\beta_0. \quad . \quad . \quad . \quad . \quad (37)$$

For a check on numerical values, we have the equation

$$\alpha_0 \delta_0 - \beta_0 \gamma_0 = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

or this may be used to give the following short formula for γ_0:

$$\gamma_0 = \frac{\alpha_0 \delta_0 - 1}{\beta_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

as in Table 6.

The voltages and currents for the combined circuit are now given by

$$E_s = E\alpha_0 + I\beta_0 = A + jB, \quad . \quad . \quad . \quad . \quad . \quad (40)$$

and

$$I_s = E\gamma_0 + I\delta_0 = C + jD. \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Thus, the characteristics of the group of parallel lines can be calculated exactly as if there were only one equivalent transmission line, using the full load formulas of Table 4, or using Table 11 if the problem relates to a constant-voltage system.

CHAPTER XV

TRANSMISSION LINE PROBLEMS

WHEN conditions are given at the receiver, or load, end of a transmission line, the convergent series of Table 6 give at once the voltage, $A+jB$, and the current, $C+jD$, at the other end of the line. By putting the load current equal to zero, we obtain the following expression for the no-load voltage at the supply end:

$$A_0+jB_0=E\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\text{etc.}\right).$$

Thus the ratio of the voltages at the two ends of the line at no load is

$$\frac{A_0+jB_0}{E}=\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\text{etc.}\right), \quad . \quad . \quad . \quad (1)$$

which is independent of the voltage E, and depends only on the constants of the line.

The absolute value of a complex quantity like the voltage A_0+jB_0, is its total numerical value independent of its phase relation. This is the same, in the case of the voltage A_0+jB_0, as its measured value, and is equal to

$$\sqrt{A_0^2+B_0^2}, \quad \text{or} \quad A_0+\frac{B_0^2}{2A_0},$$

to a very close approximation when B_0 is smaller than A_0. Since the two complex quantities making up equation (1) are equal in all respects, their absolute values are equal, and hence

$$\frac{A_0+\dfrac{B_0^2}{2A_0}}{E}=\text{absol. value of }\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\text{etc.}\right). \quad (2)$$

When the line is carrying full load, the measured value of the receiver voltage is E, and of the supply voltage, $A+\frac{B^2}{2A}$. If the load be thrown off and the supply voltage be kept constant at $A+\frac{B^2}{2A}$, then the receiver voltage will rise to a value E_0. The line is now at no load, and the ratio of the voltages at the two ends is, by equation (2),

$$\frac{A+\frac{B^2}{2A}}{E_0} = \text{absolute value of } \left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\ 4}+\text{etc.}\right)$$

$$=\frac{A_0+\frac{B_0{}^2}{2A_0}}{E}.$$

Thus

$$E_0=\frac{A+\frac{B^2}{2A}}{A_0+\frac{B_0{}^2}{2A_0}}E$$

(equation 2, Table 4).

We are now in a position to obtain the regulation of the line, since by the definition in the A. I. E. E. Standardization Rules,

$$\text{Per cent regulation} = \frac{E_0-E}{E}\times 100.$$

Thus, the regulation volts at the receiver end, which are to be expressed as a percentage of E, are

$$E_0-E=\frac{A+\frac{B^2}{2A}}{A_0+\frac{B_0{}^2}{2A_0}}E-E$$

as in equation (3), Table 4.

It is often desirable to find the regulation of a line at the supply end, that is, the per cent change in supply voltage from full-load conditions to no-load conditions, when the receiver voltage is kept constant. If the receiver voltage is E, we have seen in the preceding paragraph that the full-load supply voltage is equal to

$$E_s = A + \frac{B^2}{2A},$$

and the no-load supply voltage is

$$E_{0_s} = A_0 + \frac{B_0^2}{2A_0}.$$

The per cent regulation at the supply end is

$$\frac{E_s - E_{0s}}{E_s} \times 100,$$

and the regulation volts at the supply end are, thus

$$E_s - E_{0s} = A + \frac{B^2}{2A} - A_0 - \frac{B_0^2}{2A_0},$$

as in equation (6), Table 4.

In the expression $C + jD$ for current at the supply end, the quantity C denotes the component of current which is

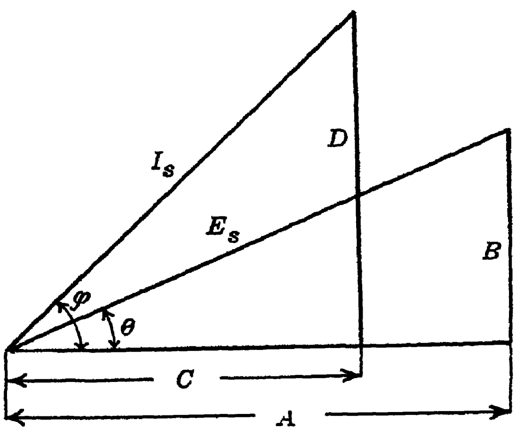


FIG 19.

in phase with the voltage E at the other end of the line. We can, however, find the component of supply current which is in phase with the supply voltage, by first finding the watts at the supply end.

Let the supply voltage be

$$E_s = A + jB,$$

and let its phase be denoted by the angle θ, Fig. 19, where

$$\tan\theta = \frac{B}{A},$$

and, therefore,

$$\sin\theta = \frac{B}{\sqrt{A^2+B^2}},$$

and

$$\cos\theta = \frac{A}{\sqrt{A^2+B^2}}.$$

Similarly, let the current at the supply end be $C+jD$, at a phase angle ϕ, where

$$\tan\phi = \frac{D}{C}$$

and, therefore

$$\sin\phi = \frac{D}{\sqrt{C^2+D^2}}$$

and

$$\cos\phi = \frac{C}{\sqrt{C^2+D^2}}.$$

The watts at the supply end are equal to the current, multiplied by the voltage, multiplied by the power factor; that is,

$$\begin{aligned}
\text{Watts} &= \text{absolute value of } I_s \times \text{absolute value of } E_s \\
&\qquad \times \cos(\phi-\theta) \\
&= \sqrt{C^2+D^2} \times \sqrt{A^2+B^2} \times (\cos\phi\cos\theta + \sin\phi\sin\theta) \\
&= \sqrt{C^2+D^2} \times \sqrt{A^2+B^2} \\
&\qquad \left\{ \frac{C}{\sqrt{C^2+D^2}} \times \frac{A}{\sqrt{A^2+B^2}} + \frac{D}{\sqrt{C^2+D^2}} \times \frac{B}{\sqrt{A^2+B^2}} \right\} \\
&= AC + BD,
\end{aligned}$$

see equation (11), Table 4.

The quadrature volt-amperes, or reactive power, are given by the following equation:

Reactive volt-amperes

$$=\text{absolute value of } I_s \times \text{absolute value of } E_s \times \sin(\phi-\theta)$$

$$=\sqrt{C^2+D^2}\times\sqrt{A^2+B^2}\ (\sin\phi\cos\theta-\cos\phi\sin\theta)$$

$$=\sqrt{C^2+D^2}\times\sqrt{A^2+B^2}$$

$$\left\{\frac{D}{\sqrt{C^2+D^2}}\times\frac{A}{\sqrt{A^2+B^2}}-\frac{C}{\sqrt{C^2+D^2}}\times\frac{B}{\sqrt{A^2+B^2}}\right\}$$

$$=AD-BC.$$

When the expression $AD-BC$ has a negative value, the current at the supply end is lagging behind the supply voltage, and when the expression has a positive value, the current leads the voltage in phase.

We can now obtain the in-phase component of current, which is equal to watts divided by voltage (equations 15 and 16), and in the same way the quadrature component of current, which is equal to reactive volt-amperes divided by voltage (equations 22 and 23). The power factor at the supply end is equal to watts divided by volt-amperes (equations 13 and 14). Since the power supplied is known, being $AC+BD$, and the power delivered at the receiver is also known, being equal to EP, their difference represents the loss of power in the line due to resistance of the conductors, leakage over the insulators and corona loss.

The equations in F, H, M and N are quite similar to the above equations in their derivation, and they give the solutions of similar problems when conditions are given at the supply end of the line.

CHAPTER XVI

REACTANCE OF WIRE, SINGLE-PHASE

Effect of Flux in Air.—Let there be an alternating current, I, in the transmission line wire, A, indicated in Fig. 20.

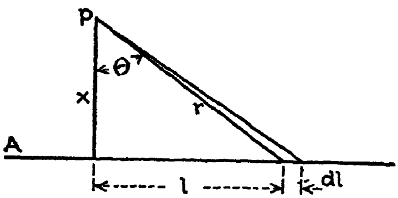


FIG. 20.

The magnetic field set up by the current at P, a distance x away from the wire, will be at right angles to the wire. The intensity of the field will be equal to the force on a unit magnetic pole at P due to the current in the wire. The force due to the current in a short length, dl, of the wire will be

$$\frac{Idl}{r^2} \cos \theta = \frac{I}{x} \cos \theta d\theta,$$

since

$$dl \cos \theta = r d\theta$$

and

$$r = \frac{x}{\cos \theta}.$$

The total force at P due to the current in the wire A is equal to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{I \cos \theta d\theta}{x} \text{ (where } x \text{ is a constant)}$$

$$= \left[\frac{I \sin \theta}{x}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2I}{x},$$

where I is measured in absolute electromagnetic units. When I is in amperes, the field at distance x is

$$\frac{2I}{10x} \text{ lines per sq. cm.} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In Fig. 21 is shown the cross section of a single-phase transmission line. The lines of force in the path of thickness dx surrounding the wire A are

$$\frac{2I}{10x}dx$$

per centimeter of the transmission line. These lines cut

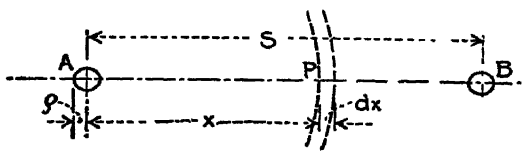


FIG. 21.

the wire A and produce an alternating voltage in it which is 90° out of phase with the current and is equal to

$$j\omega\frac{2I}{x}dx \times 10^{-9} \text{ volts,}$$

where $\omega = 2\pi \times$number of cycles per second, and where I is in amperes.

The voltage drop between the wires A and B, due to flux in the air produced by the current in A, is obtained by integrating the above expression from $x = \rho$ to $x = s$. The integration is not carried beyond $x = s$, since flux which cuts both A and the return wire B does not produce any voltage between them. The voltage drop is equal to

$$\int_{\rho}^{s} j\omega\frac{2I}{x}dx \times 10^{-9} = j\omega 2I \operatorname{logh}\frac{s}{\rho} \times 10^{-9}$$

where "logh" denotes the hyperbolic or natural logarithm. Note that

$$\operatorname{logh} a = 2\ 3026 \log_{10} a.$$

There will be an equal drop due to the flux produced by the current in the wire B, so that the total drop due to flux in air is

$$j\omega 4I \text{ logh} \frac{s}{\rho} \times 10^{-9}$$

volts per centimeter of line,

$$= 2j\omega \times 741.1 \log_{10} \frac{s}{\rho} \times 10^{-6}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

volts per ampere per mile of single-phase line.

In order to show that it is mathematically correct to use the distance s between the centers of the wires, in formula (2), it is necessary to find the average value of logh v, Fig. 22, where v is the distance from the center of wire A

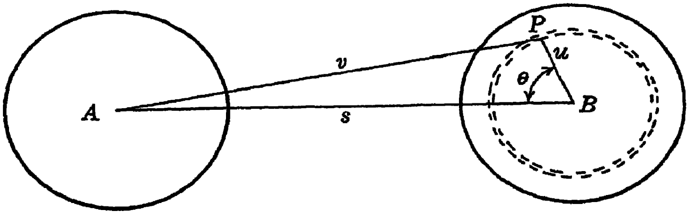


FIG. 22—Single-Phase Circuit.

to any point P on the section of wire B. First, let P be a point on the circle of radius u, that is let u be constant. We have

$$v^2 = s^2 + u^2 - 2us \cos \theta$$

$$= s^2 + u^2 - use^{j\theta} - use^{-j\theta}$$

since

$$\cos \theta = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}).$$

Then

$$\frac{v^2}{s^2} = 1 - \frac{u}{s}e^{j\theta} - \frac{u}{s}e^{-j\theta} + \frac{u^2}{s^2}$$

$$= \left(1 - \frac{u}{s}e^{j\theta}\right)\left(1 - \frac{u}{s}e^{-j\theta}\right).$$

Now

$$\text{logh } (1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^n}{n} + \ldots\right).$$

Therefore,

$$2 \operatorname{logh} v = 2 \operatorname{logh} s - \left(\frac{u}{s}e^{j\theta} + \frac{1}{2}\frac{u^2}{s^2}e^{j2\theta} + \dots + \frac{1}{n}\frac{u^n}{s^n}e^{jn\theta} + \dots\right)$$
$$- \left(\frac{u}{s}e^{-j\theta} + \frac{1}{2}\frac{u^2}{s^2}e^{-j\theta} + \dots + \frac{1}{n}\frac{u^n}{s^n}e^{-jn\theta} + \dots\right)$$

$$\operatorname{logh} v = \operatorname{logh} s$$
$$- \left(\frac{u}{s}\cos\theta + \frac{1}{2}\frac{u^2}{s_2}\cos 2\theta + \dots + \frac{1}{n}\frac{u^n}{s^n}\cos n\theta + \dots\right).$$

Multiply this expression by the element of area $udud\theta$ at the point P, and integrate from $\theta = 0$ to $\theta = 2\pi$. The cosine terms vanish, and the result is $2\pi u$ (logh s)du. Now let u vary, and integrate from $u = 0$ to $u = \rho$ and the result is $\pi\rho^2$ logh s. Divide by the area of the circle, $\pi\rho^2$, and the average value of logh v is found to be exactly logh s. This is in agreement with the calculation of "proximity effect," in which the distortion of the current in one wire by the proximity of the current in the neighboring wire, is calculated.* The proximity effect is negligible in overhead transmission line conductors.

Effect of Flux in the Conductor.—Let i be the current per unit area of section at any point in the wire shown in Fig. 23. (For the present assume that i is the same at all points of the section.)

The total area of section of the wire is $\pi\rho^2$ and therefore the total current in the wire is

$$I = \pi i \rho^2.$$

The total current inside the circle of radius x is

$$I_1 = \pi i x^2.$$

This is the only current forcing flux around the circular path of width dx, since currents flowing nearer the surface of the wire do not tend to produce magnetic lines in a path

* "Proximity Effect in Wires and Thin Tubes," by H. B Dwight, *Transactions, A. I. E E.*, 1923, p. 850.

which does not surround them. Thus the flux density at the radius x is

$$\frac{2I_1}{10x}=\frac{2\pi i x^2}{10x}$$

$$=\frac{2\pi i x}{10}.$$

The total flux in the outer ring of the section is

$$\int_x^\rho \frac{2\pi i x dx}{10}=\frac{\pi i(\rho^2-x^2)}{10}.$$

This cuts the element dx of the wire and produces a voltage along it equal to

$$j\omega\pi i(\rho^2-x^2)10^{-9} \text{ volts per cm.} \quad . \quad . \quad . \quad (3)$$

This voltage leads the current by 90° in phase at all sections. It is greatest at the center and zero at the surface and so is an unbalanced voltage; it therefore causes a local quadrature current to flow along the center of the wire and return near the surface.

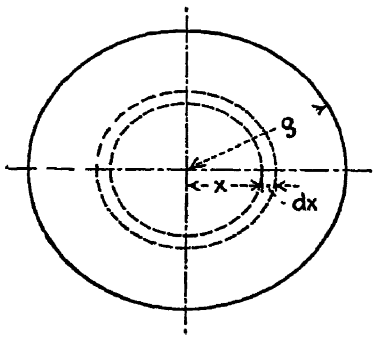


FIG. 23—Section of Wire.

Let the local current at the element dx be $(i_{(x)})$ per unit area of section. Then the average voltage drop along the wire due to the flux inside it and the resulting local balancing current, is equal to

$$j\omega I L_1=j\omega\pi i(\rho^2-x^2)10^{-9}+i_{(x)}r,$$

where L_1 is the self-inductance of the wire due to the abovementioned flux inside it, and where r is the specific resistance of the metal in centimeter units, that is, the resistance in ohms of a centimeter cube of the metal. The current $i_{(x)}$

adjusts itself so that the drop is the same at all parts of the section. From the last equation, we have

$$i_{(x)} = \frac{j\omega\pi i\rho^2 L_1}{r} - \frac{j\omega\pi i(\rho^2 - x^2)}{r} \times 10^{-9}, \quad . \quad . \quad . \quad (4)$$

since

$$I = \pi i \rho^2.$$

As $i_{(x)}$ is a local current in the wire, and does not increase or decrease the main current I, its sum when added up all over the section must be zero, and thus

$$\int_0^\rho 2\pi x i_{(x)} dx = 0,$$

that is,

$$j\frac{2\omega\pi^2 i}{r}\int_0^\rho (\rho^2 L_1 x - \rho^2 x 10^{-9} + x^3 10^{-9}) dx = 0.$$

Now L_1 is a constant, independent of x, and so

$$\rho^4 L_1 - \rho^4 10^{-9} + \frac{\rho^4}{2} 10^{-9} = 0.$$

Therefore

$$L_1 = \tfrac{1}{2} \times 10^{-9}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The voltage drop between the wires A and B due to the flux inside both wires is

$$2j\omega I L_1 = 2j\omega I \times \tfrac{1}{2} \times 10^{-9} \text{ volts per cm.}$$
$$= 2j\omega \times 80\ 47 \times 10^{-6}$$

volts per ampere per mile. The total reactive drop between the wires is thus

$$2j\omega\left(80.47 + 741\ 1 \log_{10}\frac{s}{\rho}\right) \times 10^{-6}, \quad . \quad . \quad . \quad (6)$$

volts per ampere per mile of single-phase line.

This may be written in the following form which is more convenient for computation by means of logarithm tables:

$$\text{Reactance drop} = 2j\omega \times 741.1 \log_{10}\frac{s}{0.779\rho} \times 10^{-6}. \quad (7)$$

volts per ampere per mile of single-phase line.

The above is the usual formula for reactance of a single-phase line. The proof is longer than that generally given, but it has the advantage of giving a correct idea of the distribution of current and magnetic flux inside the wire. As the irregular distribution of current produces the "skin effect" described in the next chapter, and necessitates slight corrections in the above formula for reactance and in the resistance, the importance of calculating the correct current distribution is evident. The above formula is sufficiently accurate, however, for calculating the tables of reactance of wire in Part III.

CHAPTER XVII

SKIN EFFECT

In the last chapter a local quadrature current $i_{(x)}$ was assumed, whose resistance drop balances up the unequal voltages produced at the center and near the surface by the flux inside the wire. This local current, $i_{(x)}$, when added up over all parts of the section of the wire, amounts to zero, and so cannot produce magnetic lines in the air outside the wire. But it can produce lines inside the wire, and the effect of these will now be calculated.

The reactive drop in one wire due to the flux inside it produced by the main current i is

$$j\omega\pi i\rho^2 L_1 = j\omega\pi i\rho^2 \times \tfrac{1}{2} \times 10^{-9} \text{ volts,}$$

where i is in amperes.

Then at a distance x from the center we have, from equation (4), Chapter XVI,

$$i_{(x)} = \frac{j\omega\pi i\rho^2}{r} \times \frac{1}{2} \times 10^{-9} - \frac{j\omega\pi i}{r}(\rho^2 - x^2) \times 10^{-9}$$

$$= \frac{j\omega\pi i}{r}\left(-\frac{\rho^2}{2} + x^2\right) \times 10^{-9}.$$

This is a lagging current at the center and a leading current at the surface, and it equals zero when integrated over the entire section.

The current $i_{(x)}$, integrated over the circle of radius x, is

$$I_{(x)} = \int_0^x \frac{2j\omega\pi^2 i}{r} \times 10^{-9}\left(-\frac{\rho^2 x}{2} + x^3\right)dx$$

$$= \frac{2j\omega\pi^2 i}{r} \times 10^{-9}\left(-\frac{\rho^2 x^2}{4} + \frac{x^4}{4}\right).$$

The flux density at the element dx, due to the above current, is

$$\frac{2I_{(x)}}{10x}=\frac{j\omega\pi^2 i}{10r}\times 10^{-9}(-\rho^2 x+x^3).$$

The flux in the ring outside of the circle of radius x, due to $I_{(x)}$, is

$$\phi_{(x)}=\frac{j\omega\pi^2 i}{10r}\times 10^{-9}\int_x^\rho(-\rho^2 x+x^3)dx$$

$$=\frac{j\omega\pi^2 i 10^{-9}}{10r}\left(-\frac{\rho^4}{2}+\frac{\rho^2 x^2}{2}+\frac{\rho^4}{4}-\frac{x^4}{4}\right)$$

$$=\frac{j\omega\pi^2 i 10^{-9}}{40r}(-\rho^4+2\rho^2 x^2-x^4).$$

This flux produces a voltage at the element dx, equal to

$$-\frac{\omega^2\pi^2 i 10^{-18}}{4r}(-\rho^4+2\rho^2 x^2-x^4). \quad . \quad . \quad . \quad . \quad (1)$$

A local current, $i_{(2x)}$, will flow in order to keep the voltage drop uniform over the section. Let the average drop due to $\phi_{(x)}$ be

$$j\omega IL_2=j\omega\pi\rho^2 iL_2;$$

then

$$j\omega\pi\rho^2 iL_2=\frac{\omega^2\pi^2 i 10^{-18}}{4r}(\rho^4-2\rho^2 x^2+x^4)+i_{(2x)}r. \quad . \quad . \quad (2)$$

Integrate $i_{(2x)}$ over the entire surface and as it is a local current

$$\int_0^\rho 2\pi x i_{(2x)}dx=0$$

$$=\frac{2j\pi^2\omega i}{r}\int_0^\rho \rho^2 L_2 x dx$$

$$-\frac{\omega^2\pi^3 i 10^{-18}}{2r^2}\int_0^\rho(\rho^4 x-2\rho^2 x^3+x^5)dx.$$

Therefore,

$$jL_2\rho^4=\frac{\omega\pi 10^{-18}}{2r}\left(\frac{\rho^6}{2}-\frac{\rho^6}{2}+\frac{\rho^6}{6}\right),$$

and

$$L_2 = -j\frac{1}{12}\,\frac{\omega\pi\rho^2 10^{-18}}{r}. \qquad (3)$$

Thus

$$i_{(2x)} = -\frac{\omega^2\pi^2 i 10^{-18}}{12r^2}(2\rho^4 - 6\rho^2x^2 + 3x^4). \qquad (4)$$

This current is in phase with the main current and, as it is negative at the center and positive at the surface, it produces a stronger resultant current near the surface of the wire. This is the well-known "skin effect." The effect of the quadrature current $i_{(x)}$ is to increase the resultant current both at the center and near the surface, but its effect is not as large as that of the in-phase current $i_{(2x)}$ and so the net result is a crowding of current toward the surface.

The above process may be continued indefinitely, each step adding a smaller correction than the one before to the current at radius x and to the average drop in the wire.

Thus the expression for $i_{(2x)}$, equation (4), may be integrated over the circle of radius x, and will give the value of $I_{(2x)}$. This current produces a flux density at the radius x, and by integrating this over the outer ring of the section, the value of $\phi_{(2x)}$ is obtained. The flux $\phi_{(2x)}$ produces an unbalanced voltage which must be corrected by a local current $i_{(3x)}$, so as to give a uniform drop over the section, due to the inductance L_3. Equating the total local current to zero, as before, gives

$$L_3 = -\frac{1}{48}\,\frac{\omega^2\pi^2\rho^4 10^{-27}}{r^2}.$$

In the same way it is found that

$$L_4 = j\frac{1}{180}\,\frac{\omega^3\pi^3\rho^6 10^{-36}}{r^3},$$

and

$$L_5 = \frac{13}{8640}\,\frac{\omega^4\pi^4\rho^8 10^{-45}}{r^4}.$$

Let the resistance of the wire per centimeter be R, where

$$R = \frac{r}{\pi\rho^2} \text{ ohms per cm.,}$$

and let

$$m = \frac{\omega\pi\rho^2 10^{-9}}{r}$$

$$= \frac{\omega 10^{-9}}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Then the total drop in the wire is

$$IR + j\omega I(L + L_1 + L_2 + \ldots)$$

$$= IR + j\omega I 10^{-9}\Big(2 \operatorname{logh} \frac{s}{\rho} + \frac{1}{2} - j\frac{1}{12}m$$

$$- \frac{1}{48}m^2 + j\frac{1}{180}m^3 + \frac{13}{8640}m^4 - \ldots\Big), \quad . \quad . \quad . \quad . \quad (6)$$

volts per centimeter.

The total drop in phase with the current is

$$IR\left(1 + \frac{1}{12}m^2 - \frac{1}{180}m^4 + \ldots\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The total copper loss due to all the currents in the wire is therefore equal to

$$I^2R\left(1 + \frac{1}{12}m^2 - \frac{1}{180}m^4 + \ldots\right).$$

This can be checked by integrating the losses due to the total in-phase and quadrature currents in all parts of the section of the wire, the above result being obtained by this method also. Thus, in every respect, both as to voltage drop and watts loss, the resistance of the wire to the alternating current is

$$R' = R\left(1 + \frac{1}{12}m^2 - \frac{1}{180}m^4 + \frac{11}{10080}m^6 - \ldots\right).$$

Values of R' for both 25 and 60 cycles are tabulated in Part III. When taking the resistance of a conductor from the tables, R' should always be used for alternating current, and R should be used only when the conductor carries direct current.

The total drop in quadrature with the current is

$$j\omega I10^{-9}\left(2 \text{ logh } \frac{s}{\rho}+\frac{1}{2}-\frac{1}{48}m^2+\frac{13}{8640}m^4-\ldots\right)$$

$$=j\omega I10^{-9}\left\{2 \text{ logh } \frac{s}{\rho}+\frac{1}{2}\left(1-\frac{1}{24}m^2+\frac{13}{4320}m^4-\ldots\right)\right\}. \quad (8)$$

The series

$$1-\frac{1}{24}m^2+\frac{13}{4320}m^4-\ldots$$

is thus a correction factor for the term $\frac{1}{2}$ or 80.47 in the ordinary formula for reactance. Its effect is too small, however, to make any appreciable change in the tabulated values of reactance.

Proof by Infinite Series.—The above formulas for the resistance and inductance of a wire carrying alternating current are sufficiently accurate for transmission line calculations with ordinary frequencies. They may also be extended to include more terms without undue labor. However, as skin effect formulas are generally obtained and expressed by means of infinite series which can be carried out to any degree of accuracy for high-frequency work, a short outline of the derivation of the infinite series will be given. It will prove a check upon the correctness of the formulas given above, but it will probably not give as clear an idea as they do of the actual distribution of current in the wire.

Let an alternating current, I, of sine wave form and of steady value, flow in a round wire of radius ρ. (See Fig. 23, Chap. XVI.) Let it take up such a distribution that the drop at all parts of the section of the wire, due to resistance and to magnetic flux, is the same. Then if i' be the current density at radius x, we may assume

$$i'=a_0+a_1x^2+a_2x^4+\ldots+a_nx^{2n}+\ldots$$

where a_0, a_1, . . . a_n, etc., are constants, independent of x. (As the same value of i' would be obtained for both $+x$ and $-x$, only even powers of x need be assumed for the series.)

The total current in the part of the section inside a circle of radius x will be

$$I'=\int_0^x 2\pi x i' dx$$

$$=2\pi\left(\frac{a_0x^2}{2}+\frac{a_1x^4}{4}+\ldots+\frac{a_{n-1}x^{2n}}{2n}+\ldots\right). \quad . \quad . \quad . \quad . \quad (9)$$

The flux density at the radius x is

$$\frac{2I'}{10x}=\frac{2\pi}{10}\left(a_0x+\frac{a_1x^3}{2}+\ldots+\frac{a_{n-1}x^{2n-1}}{n}+\ldots\right),$$

and the total flux in the outer ring of the section, outside the circle of radius x, is

$$\phi'=\int_x^\rho \frac{2I'}{10x}dx$$

$$=\frac{\pi}{10}\left(a_0\rho^2+\frac{a_1\rho^4}{2^2}+\frac{a_2\rho^6}{3^2}+\ldots+\frac{a_{n-1}\rho^{2n}}{n^2}+\ldots\right)$$

$$-\frac{\pi}{10}\left(a_0x^2+\frac{a_1x^4}{2^2}+\frac{a_2x^6}{3^2}+\ldots+\frac{a_{n-1}x^{2n}}{n^2}+\ldots\right). \quad . \quad . \quad . \quad (10)$$

The drop at radius x due to the flux ϕ' is

$$j\omega\phi'\times 10^{-8}$$

and the resistance drop due to the current at the same part is $i'r$. Thus the total drop per centimeter of wire, which is the same at all parts of the section, is

$$V=j\omega\phi' 10^{-8}+i'r$$

$$=j\omega\pi 10^{-9}\left(a_0\rho^2+a_1\frac{\rho^4}{2^2}+\frac{a_2\rho^6}{3^2}+\ldots+\frac{a_{n-1}\rho^{2n}}{n^2}+\ldots\right)$$

$$-j\omega\pi 10^{-9}\left(a_0x^2+\frac{a_1x^4}{2^2}+\frac{a_2x^6}{3^2}+\ldots+\frac{a_{n-1}x^{2n}}{n^2}+\ldots\right)$$

$$+r(a_0+a_1x^2+a_2x^4+\ldots+a_nx^{2n}+\ldots). \quad . \quad . \quad . \quad . \quad (11)$$

The above expression for V is the same for all values of x, and we may therefore equate each coefficient of x^2, x^4, etc., to zero. Thus, putting

$$\frac{\omega\pi\rho^2 10^{-9}}{r} = m,$$

we have

$$a_1 = \frac{jma_0}{\rho^2},$$

$$a_2 = \frac{jma_1}{2^2\rho^2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_n = \frac{jma_{n-1}}{n^2\rho^2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{etc.}$$

and

$$V = a_0 r + j\omega\pi 10^{-9}\left(a_0\rho^2 + \frac{a_1\rho^4}{2^2} + \ldots + \frac{a_{n-1}\rho^{2n}}{n^2} + \ldots\right).$$

Substituting the values of a_1, a_2, etc., we obtain

$$V = a_0 r + j\pi\omega 10^{-9}$$

$$\left\{a_0\rho^2 + \frac{jma_0\rho^2}{(\lfloor 2)^2} + \frac{(jm)^2 a_0\rho^2}{(\lfloor 3)^2} + \ldots + \frac{(jm)^{n-1}a_0\rho^2}{(\lfloor n)^2} + \ldots\right\}$$

$$= a_0 r\left\{1 + jm + \frac{(jm)^2}{(\lfloor 2)^2} + \frac{(jm)^3}{(\lfloor 3)^2} + \ldots + \frac{(jm)^n}{(\lfloor n)^2} + \ldots\right\}. \quad (12)$$

Now by putting $x = \rho$ in the expression for I', equation (9), we obtain the value of the total current in the wire,

$$I = \pi\left(a_0\rho^2 + \frac{a_1\rho^4}{2} + \frac{a_2\rho^6}{3} + \ldots + \frac{a_{n-1}\rho^{2n}}{n} + \ldots\right)$$

$$= \pi a_0\rho^2\left\{1 + \frac{2jm}{(\lfloor 2)^2} + \frac{3(jm)^2}{(\lfloor 3)^2} + \ldots + \frac{n(jm)^{n-1}}{(\lfloor n)^2} + \ldots\right\}.$$

Therefore,

$$a_0 = \frac{I}{\pi\rho^2\left\{1 + \frac{2jm}{(\lfloor 2)^2} + \frac{3(jm)^2}{(\lfloor 3)^2} + \ldots + \frac{n(jm)^{n-1}}{(\lfloor n)^2} + \ldots\right\}}.$$

Substituting this value of a_0 in equation (12), and putting

$$\frac{r}{\pi \rho^2} = R,$$

the resistance per centimeter of the wire, we obtain

$$V = IR \frac{1 + jm + \frac{(jm)^2}{(\underline{|2})^2} + \ldots + \frac{(jm)^n}{(\underline{|n})^2} + \ldots}{1 + \frac{2jm}{(\underline{|2})^2} + \frac{3(jm)^2}{(\underline{|3})^2} + \ldots + \frac{n(jm)^{n-1}}{(\underline{|n})^2} + \ldots} \qquad (13)$$

This expression can evidently be carried to any accuracy desired. It will give the same results as were previously obtained in equation (6), by expanding the denominator as a binomial of the form $(1+x)^{-1}$ and multiplying by the numerator. This gives

$$V = IR\{1 + \tfrac{1}{2}jm - \tfrac{1}{12}(jm)^2 + \tfrac{1}{48}(jm)^3 - \tfrac{1}{180}(jm)^4 + \tfrac{13}{8640}(jm)^5 - \ldots\},$$

or,

$$V = IR(1 + \tfrac{1}{12}m^2 - \tfrac{1}{180}m^4 + \ldots) + \tfrac{1}{2}j\omega I 10^{-9}(1 - \tfrac{1}{24}m^2 + \tfrac{13}{4320}m^4 - \ldots).$$

This is the voltage drop, omitting the effect of the flux outside the wire, and is the same as the value previously obtained. (See equations 7 and 8.)

The series of (13) can be expressed as Bessel functions.

References

Maxwell, *Elec and Magn*, Vol. II, Para 689–690

Rayleigh, *Phil Mag*, 1886, Vol. 21, page 381.

Kelvin, *Math. Papers*, 1889, Vol. 3, page 491.

Rosa and Grover, *Bulletin of Bureau of Standards*, Washington, 1911, Vol. 8, No. 1, pages 173–185 and pages 226–230

Skin effect formulas for non-magnetic wires are very closely applicable to round cables of non-magnetic material. It is evident that the formulas apply to solid wires of either copper or aluminum, by using the proper value of r, the resistivity. A copper cable is really a mixture of copper and air, and if a higher value for the resistivity be used, corresponding to this mixture, then the skin effect can be quite closely calculated, for the current distribution in the round cable will be very similar to that in a round wire. This has been closely checked by test. See Fig. 9, "Experimental Researches on Skin Effect in Conductors," by A. E. Kennelly, F. A. Laws and P. H. Pierce, Transactions A. I. E. E., 1915, page 1971. The correction for spirality is inappreciable for cables used in overhead transmission lines, at commercial power frequencies.

The choice of values for resistivity and cross section of the conductors may be avoided by expressing the formulas in terms of the resistance of the conductor per unit length. Thus, if in the formulas in this chapter, one uses equation (5)

$$m = \frac{\omega 10^{-9}}{R},$$

the value of the resistivity is not required, but only the total resistance of the conductor per unit length, which is usually known accurately. This gives the most convenient way of calculating the skin effect of round conductors, and it also provides a means of allowing for the effect of changes in resistance due to temperature and other causes.

Compact engineering formulas for skin effect in round non-magnetic conductors at commercial frequencies can therefore be given in terms of the resistance per mile of the conductors:

$$\frac{R_{ac}}{R_{dc}} = 1 + \frac{3\ 07}{(100R_{dc})^2} - \frac{7\ 5}{(100R_{dc})^4} + \frac{54}{(100R_{dc})^6} - \cdots$$

for 60 cycles. . . . . (14)

and

$$\frac{R_{ac}}{R_{dc}} = 1 + \frac{0.53}{(100R_{dc})^2} - \frac{1.3}{(100R_{dc})^4} + \frac{9}{(100R_{dc})^6} - \ldots$$

for 25 cycles. . . . . (15)

where R_{dc} is the resistance of the conductor to direct current in ohms per mile. See Tables 14 and 15. These formulas may be used for any round, non-magnetic, coreless wire or cable, of copper or aluminum, at high or low temperature. The formulas should not be used unless the last term is small, but this limitation will apply only for larger conductors than are used on overhead transmission lines, or higher frequencies than 60 cycles. Similar formulas were published by the author in the Transactions A. I. E. E., 1915, page 2019.

CHAPTER XVIII

REACTANCE OF CABLE, SINGLE-PHASE

OWING chiefly to its greater flexibility, a cable made up of small wires is much better than a solid wire, where a comparatively large conductor is required. Accordingly, cables are used very frequently as transmission line conductors.

The reactance of a stranded conductor is appreciably different from that of a solid wire. For example, the reactance of No. 0000 cable of seven wires at 60 cycles and 18-inch spacing is 0.552 ohm per mile, while the reactance of a solid wire of the same sectional area is 0.560 ohm, and of a solid wire of the same diameter as the cable, 0.544 ohm. Therefore, the error in the rather frequent practice of using a solid wire formula for a cable is in either case about 1.5 per cent, which is too large an error to be neglected. The formulas of this chapter are therefore given, from which can be obtained very closely the reactance of most common types of stranded non-magnetic conductors.*

It may be assumed that a cable is equivalent to a bundle of straight parallel wires, without spiraling. Although the spiraling of the wires has a marked effect on the resistance of cables, amounting to one or two per cent, the spiraling produces very little effect on the reactance of non-magnetic cables in overhead circuits.

This may be shown by estimating the increase in inductance due to spiraling in a seven-wire cable. The chief effect of the spiraling is to produce an alternating magnetic

* "The Reactance of Stranded Conductors," by H. B. Dwight, *The Electrical World,* April 19, 1913, and "Constant-Voltage Transmission," by H. B Dwight, John Wiley & Sons, 1915, p. 109

field along the central part of the cable parallel to the axis, the same as is produced in a long solenoid. The density of this field is $4\pi n$ lines per sq. cm for 1 absampere, that is, one absolute unit of current, where n is the number of turns per centimeter Multiplying by the area of the path for these lines, which is $\frac{\pi d^2}{4}$, where d is the mean diameter of the coil in centimeters, we obtain the total number of lines passing through the coil parallel to its axis,

$$\pi^2 d^2 n \text{ lines.} \qquad (1)$$

In a length of the solenoid containing N turns, the above alternating field cuts all the turns, and the self-inductance of the coil due to this field is

$$\pi^2 d^2 n N \text{ abhenrys.} \qquad (2)$$

This agrees with Havelock's formula for the self-inductance of a long solenoid.*

Considering the outside wires of one mile of a seven-wire No. 0000 cable, and assuming that the wires make one complete turn about the cable every 5 2 inches,

$$n=\frac{1}{5\ 2\times 2.54}, \quad N=\frac{5280\times 12}{5.2}=12{,}200,$$

and $d=0.347\times 2.54$ centimeters, measured to the centers of the outside wires. Therefore, the self-inductance of the outside wires due to spiraling is

$$L=\pi^2\times 0\ 347^2\times 6\ 45\times\frac{1}{5\ 2}\times\frac{1}{2\ 54}\times 12{,}200$$

$$=7100 \text{ abhenrys per mile.}$$

Now the outside wires constitute only $\frac{6}{7}$ of the cable,

* Scientific Paper No 169 of the Bureau of Standards, Washington, D. C, equation (79), p 121.

and therefore, the reactance of the entire cable at 60 cycles is

$$2\pi \times 60 \times 7100 \times 10^{-9} \times \frac{6^2}{7^2} = 0.002 \text{ ohm per mile,}$$

due to the flux produced by spiraling.

The reactance of this cable for 18-inch spacing is 0.552 ohm per mile, as mentioned above. Therefore, the effect of spiraling in this case is about $\frac{1}{3}$ of 1 per cent. With larger spacings or with smaller 7-wire cables, the percentage effect of spiraling will be less than in the above case. With larger cables than No. 0000, there will be more than one layer of wires. Alternate layers are arranged with opposite rotations about the axis, and so will tend to neutralize each other in producing a field parallel to the axis. Accordingly, the effect of spiraling has been neglected in calculating the reactance of overhead non-magnetic cables in this chapter.

It may be mentioned that the effect of spiraling is not negligible in iron or steel cables, but is very important in that case. The reason is that the path for the flux parallel to the axis is through solid iron, with practically no air gaps to cross, and so the relatively small magnetizing force produces several hundred times as much flux as it would if the metal were copper or aluminum, of permeability equal to unity.*

Reactance of 7-wire Cable.—As in deriving the formula for reactance of round wire, the current will be assumed to be uniformly distributed over the section of the cable, and the average reactive voltage drop for all parts of the section will be calculated.

If the current in cable A, Fig. 24, is 1 absampere, then the current in each small wire of the 7-wire cable will be $\frac{1}{7}$ absampere. The flux density due to the current in one small wire, at a distance x centimeters from its center, is

$$\frac{2}{7x}, \text{ lines per sq. cm.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* "Steel Conductors for Transmission Lines," by H. B. Dwight, *Trans. A. I. E. E.*, Sept., 1916, p. 1244, Fig. 8.

If the distance between any two wires of cable A is t centimeters, the flux caused by current in the first wire, which cuts the second wire but does not cut cable B, is

$$\frac{1}{7}M=\frac{1}{7}\int_t^s \frac{2}{x}dx$$

$$=\frac{2}{7}\operatorname{logh}\frac{s}{t}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

lines per centimeter of wire, where M is the mutual inductance of the two small wires, neglecting flux cutting cable B.

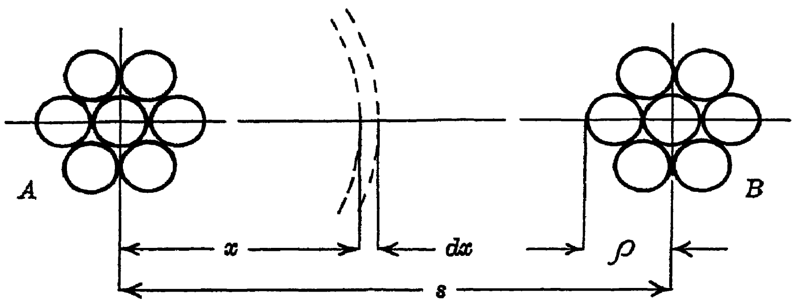


FIG. 24.—Single-Phase Circuit of Cables.

The required voltage induced in the second wire by the first is

$$\frac{2\pi fM}{7} \text{ abvolts per cm.,} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

where f is the frequency in cycles per second.

The inductance of a wire due to its own current is given by the usual formula for self-inductance,

$$L_1=2\operatorname{logh}\frac{s}{q}+\frac{1}{2}, \text{ abhenrys per cm.,} \qquad . \quad . \qquad (6)$$

in which q is the radius of the wire, and in which $\frac{1}{2}$ expresses the effect of the flux inside the wire. The voltage drop due to self-inductance in each wire, which carries $\frac{1}{7}$ absampere, is

$$\frac{2\pi fL_1}{7}, \text{ abvolts per cm.} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Adding the voltage drops in all the wires in cable A, and dividing by 7, the average voltage drop in cable A, which carries 1 absampere, is

$$2\pi fL = \frac{1}{7}\left[\frac{7\times 2\pi fL_1}{7} + \frac{12\times 2\pi fM_1}{7} + \frac{30\times 2\pi fM_2}{7}\right] \text{ abvolts per cm.,} \quad (8)$$

where L_1 is the inductance of a wire due to its own current. M_1 is the inductance for a pair consisting of the center wire and an outer wire, and M_2 is the average inductance for pairs consisting of two outer wires. It is to be noted that each pair of wires is counted twice in forming expression (8), since the first wire of a pair produces a voltage drop in the second, and the second also produces a voltage drop in the first. Therefore,

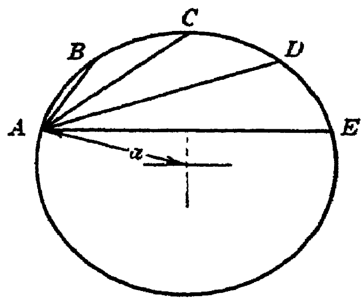


FIG. 25.—Geometrical Division of Circle.

$$L = \tfrac{1}{49}(7L_1 + 12M_1 + 30M_2) \text{ abhenrys per cm.} \quad (9)$$

L_1 is given by formula (6) and

$$M_1 = 2 \text{ logh } \frac{s}{2q} \quad (10)$$

according to (4).

For calculation of M_2 it is convenient to use the following theorem based on Cotes' theorem in trigonometry:

If a circle of radius a (Fig. 25) be divided into m equal parts at A, B, C, D, etc., then

$$AB, AC, AD, \text{ etc. (to } m-1 \text{ factors)} = ma^{m-1}\,*. \quad (11)$$

Therefore, the average value of logh t is

$$\text{logh } (am^{\frac{1}{m-1}}). \quad (12)$$

* C. E. Guye, "Comptes Rendus," Vol CXVIII, p 1329, 1894. C. E. Guye, "L'Éclairage Électrique," Vol III, p 20, 1895 E. B Rosa, "Bulletin of the Bureau of Standards," Vol. IV, No. 2, p. 335, 1907.

Thus, the average value of M_2 is

$$2 \text{ logh } \frac{s}{2q6^{\frac{1}{5}}} \text{ abhenrys per cm.} \quad . \quad . \quad . \quad (13)$$

Therefore,

$$L = \frac{1}{49}\Big[98 \text{ logh } \frac{s}{q} + \frac{7}{2} - 24 \text{ logh } 2$$
$$-60 \text{ logh } 2 - 12 \text{ logh } 6\Big] \text{ abhenrys per cm.} \quad (14)$$

Changing to practical units, and referring to the maximum radius of the cable

$$\rho = 3q,$$

we have

$$L = \left(103\ 3 + 741.13 \log_{10} \frac{s}{\rho}\right) 10^{-6}$$
$$\text{henrys per mile of cable.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Applying the above method to a cable of n wires composed of p layers around a central wire (Fig. 26, b and c),

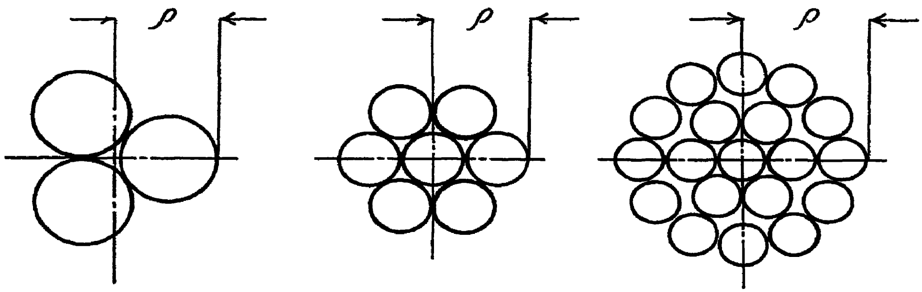


FIG 26

(a) 3-Wire Cable (b) 7-Wire Cable (c) 19-Wire Cable

$$n = 1 + 6 + 2 \times 6 + 3 \times 6 + \ldots + p \times 6$$
$$= 3p^2 + 3p + 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$L = 2 \text{ logh } \frac{s}{\rho} + \frac{1}{2n} + 2 \text{ logh } (2p+1) - \frac{2(n-1)}{n^2} \text{ logh } 3$$
$$-\frac{24}{n^2}\{4 \text{ logh } 2 + 2 \times 13 \text{ logh } 4 + 3 \times 28 \text{ logh } 6 + \ldots$$
$$+ p(n-3p) \text{ logh } (2p)\}$$
$$\text{rys per cm.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This gives,

for $n = 7$ and $p = 1$,

$$L = \left(103.3 + 741.13 \log_{10} \frac{s}{\rho}\right) 10^{-6} \text{ henrys per mile}, \quad (18)$$

for $n = 19$ and $p = 2$,

$$L = \left(89.3 + 741.13 \log_{10} \frac{s}{\rho}\right) 10^{-6} \text{ henrys per mile}; \quad (19)$$

for $n = 37$ and $p = 3$,

$$L = \left(85.1 + 741.13 \log_{10} \frac{s}{\rho}\right) 10^{-6} \text{ henrys per mile}; \quad (20)$$

for $n = 61$ and $p = 4$,

$$L = \left(83.3 + 741.13 \log_{10} \frac{s}{\rho}\right) 10^{-6} \text{ henrys per mile}; \quad (21)$$

and for solid wire,

$n = 1$ and $p = 0$,

$$L = \left(80.5 + 741.13 \log_{10} \frac{s}{\rho}\right) 10^{-6} \text{ henrys per mile}. \quad (22)$$

A three-wire cable (Fig. 26a) is a special case, since there is no central wire. The inductance is

$$L = 2 \text{ logh} \frac{s}{\rho} + \frac{1}{6} + 2 \text{ logh} \left(\frac{2\sqrt{3}+3}{3}\right) - \frac{4}{3} \text{ logh } 2$$

abhenrys per cm. . . . . (23)

Therefore,

$$L = \left(125.2 + 741.13 \log_{10} \frac{s}{\rho}\right) 10^{-6}, \quad (24)$$

henrys per mile of cable, where ρ is the radius of the circumscribing circle of the cable.

The above equations may be put in the following simplified form:

Number of Wires in Cable	Inductance per Mile, in Henrys	
3	$L=741.13\left(\log_{10}\frac{2.951s}{d}\right)\times10^{-6}.$	(25)
7	$L=741.13\left(\log_{10}\frac{2.756s}{d}\right)\times10^{-6}.$	(26)
19	$L=741.13\left(\log_{10}\frac{2.640s}{d}\right)\times10^{-6}.$	(27)
37	$L=741.13\left(\log_{10}\frac{2.605s}{d}\right)\times10^{-6}.$	(28)
61	$L=741.13\left(\log_{10}\frac{2.590s}{d}\right)\times10^{-6}.$	(29)
1 (single wire)	$L=741.13\left(\log_{10}\frac{2.568s}{d}\right)\times10^{-6}.$	(30)

In the above, d is the diameter of the circumscribing circle of the cables, measured in the same units as s, the spacing between the centers of the cables.

The significance of the equations derived above is indicated by the following examples of calculated reactance. The values tabulated are in ohms per mile at 60 cycles, where

$$\text{Reactance in ohms} = 2\pi \times 60L.$$

REACTANCE PER MILE AT 60 CYCLES OF NO. 0000 WIRE AND CABLE, 211,600 CIRCULAR MILS

Feet Spacing	Solid Wire	3-Wire Cable	7-Wire Cable	19-Wire Cable	37-Wire Cable	61-Wire Cable
1½	0.560	0.550	0.552	0.546	0.544	0.543
6	0.728	0.718	0.721	0.714	0.712	0.711

The reactance of the cables is less than that of the wire, because the diameter is larger, though the sectional area is the same. The formulas, therefore, show that stranding reduces the reactance of a conductor of a certain cross-section by one or two per cent. As is well known, an increase of about the same percentage takes place in the resistance, due to stranding.

CHAPTER XIX

REACTANCE OF TWO-PHASE AND THREE-PHASE LINES

Reactance, Two-phase.—The reactance drop in a single-phase line, in which round wire is used, is given in Chapter XVI, and is

$$2jI \times 2\pi f 741.1 \log_{10} \frac{s}{0.779\rho} \times 10^{-6}$$

volts per mile of transmission line. This is equal to $2jIx$ volts per mile of line, where x is the reactance per mile, as given in Tables 16 to 19, and where I is the current per conductor. The formulas in Tables 1, 2, and 3 for short transmission lines, give the voltage drop to neutral, which is

$$jIx.$$

Since a two-phase, four-wire line is made up of two-single-phase circuits, the reactance drop to neutral is jIx, and the line drop due to reactance is $2jIx$, volts per mile of line, I being the current per conductor.

Reactance, Three-phase, Irregular Spacing.—When the conductors of a three-phase line are spaced so that they are not exactly equidistant, the voltage drop due to reactance is not the same in the different phases. It is the practice with such lines to interchange, or transpose, the conductors at intervals along the line, so that the different reactive voltages are applied to an equal extent to all three conductors. Such a line, when carrying a balanced load (equal currents in each conductor, at 120° in phase from each other), will have the voltages of the three phases equal at the end of the line.

The average reactance of an irregularly spaced three-phase line, in which the conductors are transposed at regular intervals, may be calculated as follows.

Let Fig. 27 represent the spacing of the conductors A, B, and C of a transmission line. Let the currents in the conductors be represented by the vectors OP, OQ, and OR (Fig. 28). If the power factor is 100%, these vectors may also represent the star voltages, and the line voltages will be represented by the vectors PQ, QR and RP.

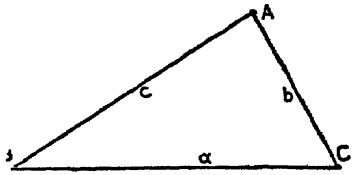


FIG 27.

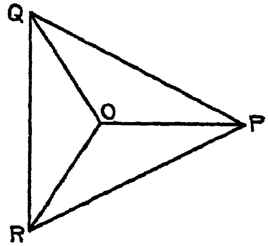


FIG 28.

Let the current in conductor A be

$$I_A = OP = 1.00I \text{ amperes,}$$

and let

$$I_B = OQ = (-0.50 + 0.866j)I \text{ amperes,}$$

and

$$I_C = OR = (-0.50 - 0.866j)I \text{ amperes.}$$

Let also

Voltage from neutral to $A = 1.00V$.

Voltage from neutral to $B = (-0.50 + 0.866j)V$.

And voltage from neutral to $C = (-0.50 - 0.866j)V$.

Then the measured, or absolute, value of voltage between B and C is $1.732V$, where V is the star voltage of the line.

The reactive voltage on conductor A per mile is

$$1.00\left(80.5 + 741.1 \log \frac{u}{\rho}\right) Ij2\pi f \times 10^{-6}$$

(due to flux from I_A)

$$+(-0\ 50+0\ 866j)\left(741.1\log\frac{u}{c}\right)Ij2\pi f\times 10^{-6}$$

(due to flux from I_B)

$$+(-0.50-0.866j)\left(741.1\log\frac{u}{b}\right)Ij2\pi f\times 10^{-6}$$

(due to flux from I_C),

where ρ is the radius of the wire and where u is a certain large distance to which flux is taken.

The reactive voltage on conductor B per mile is

$$1.00\left(741.1\log\frac{u}{c}\right)Ij2\pi f\times 10^{-6}$$

(due to flux from I_A)

$$+(-0.50+0.866j)\left(80.5+741.1\log\frac{u}{\rho}\right)Ij2\pi f\times 10^{-6}$$

(due to flux from I_B)

$$+(-0.50-0.866j)\left(741.1\log\frac{u}{a}\right)Ij2\pi f\times 10^{-6}$$

(due to flux from I_C).

The reactive voltage between A and B is therefore

$$(1.50-0.866j)\left(80.5+741.1\log\frac{c}{\rho}\right)Ij2\pi f\times 10^{-6}$$

$$+(0.50+0.866j)741.1\left(\log\frac{b}{\rho}-\log\frac{a}{\rho}\right)Ij2\pi f\times 10^{-6}.$$

Suppose at the end of one mile the line is transposed so that the above two conductors occupy the positions B and C. Then the reactive voltage between these conductors for the next mile is

$$(1.50-0.866j)\left(80.5+741.1\log\frac{a}{\rho}\right)Ij2\pi f\times 10^{-6}$$

$$+(0.50+0.866j)741.1\left(\log\frac{c}{\rho}-\log\frac{b}{\rho}\right)Ij2\pi f\times 10^{-6}.$$

Let the line be transposed again. Then for the third mile the reactance voltage between these same two conductors is

$$(1.50-0.866j)\left(80.5+741.1\log\frac{b}{\rho}\right)Ij2\pi f\times 10^{-6}$$

$$+(0.50+0.866j)741.1\left(\log\frac{a}{\rho}-\log\frac{c}{\rho}\right)Ij2\pi f\times 10^{-6}.$$

The total reactive voltage for the three miles is

$$(1.50-0.866j)\left\{80.5\times 3+741.1\left(\log\frac{a}{\rho}+\log\frac{b}{\rho}+\log\frac{c}{\rho}\right)\right\}$$
$$\times Ij2\pi f\times 10^{-6}.$$

Thus the average reactive voltage between conductors, per mile, which is the same for all phases, is, in absolute value,

$$1.732\left(80.5+741.1\log\frac{\sqrt[3]{abc}}{\rho}\right)I2\pi f\times 10^{-6}$$
$$=1.732Ix,$$

where x is the tabulated value of reactance.

Reactance, Three-phase, Regular Triangular Spacing.—When the conductors are spaced at the corners of an equilateral triangle of side s, then the expression for reactance is the same as the usual formula:

$$x=\left(80.5+741.1\log_{10}\frac{s}{\rho}\right)2\pi f\times 10^{-6}.$$

Reactance, Three-phase, Regular Flat Spacing.—When the three conductors lie in one plane (either horizontal or vertical), the center one being equidistant from the other two, as in Fig. 29, the reactance per mile is

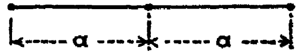


FIG. 29.

$$80.5+741.1\log\frac{a\sqrt[3]{1\times 1\times 2}}{\rho}$$
$$=80.5+741.1\log\frac{1.26a}{\rho}.$$

This is approximately 4% higher in an ordinary case than the reactance for spacing on an equilateral triangle of side *a*.

The formulas of this chapter will apply to cable as well as to wire, if the term 80 5 is changed as per Chapter XVIII.

CHAPTER XX

RESISTANCE AND REACTANCE OF LARGE CABLES WITH STEEL CORES

STEEL cores are frequently used in transmission line conductors. Practically all aluminum conductors used for overhead transmission lines are cables with steel cores, since these are necessary for the requisite mechanical strength. Cores of medium grade steel may also be used in copper cables on very high voltage lines in order to increase the diameter and so reduce the corona loss.

The steel core has a distinct effect on the electrical characteristics of a cable, and the amount of this effect for large cables can be estimated in the manner described in this chapter.*

The effect of the addition of a steel core to a transmission line cable is, first, to increase the diameter, second, to decrease the resistance by an amount which may be two per cent, more or less, and third, to decrease the reactance, usually by a smaller percentage. It will be shown later in this chapter that a useful approximate rule for transmission calculations is to take the conductivity of the steel cored cable as being equal to the sum of the conductivities for alternating current of the core and the copper or aluminum, and to take the reactance of the complete cable as if the core were made of non-magnetic material, the same as the rest of the cable.

This procedure does not apply very closely to the smaller cables which have only 7 wires, since the spiraling of the outer wires is all in one direction and so produces a certain

* "The Electrical Characteristics of Transmission Conductors with Steel Cores," by H. B. Dwight, *The Electric Journal*, Jan., 1921, p. 9.

amount of longitudinal magnetization in the central steel wire. This results in relatively more iron loss and reactance than in the larger cables where some layers are spiraled in one direction and some in the other. The effect of longitudinal magnetization is very small in cables of more than 7 wires.

The reactance of a transmission line is due to magnetic flux in the air surrounding the conductors, and also to flux inside of the conductors. The flux in the air produces the greater part of the reactance, but it is not of interest in the present calculation, for it cuts both the core and the outer wires of the cable equally. The effect of the flux inside of the cable should be calculated, since it alters the distribution of current between the core and the remainder of the cable.

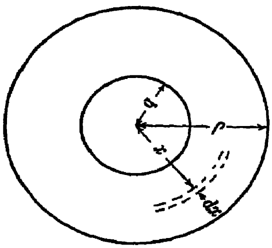


FIG 30.—Section of Cable and Core.

The non-magnetic wires of a cored cable form a tube of outer radius ρ and inner radius q, Fig. 30. Neglecting the current in the core, which is small, the total current inside the circle of radius x is

$$I_{(x)} = \pi i(x^2 - q^2) \text{ absamperes, } \quad . \quad . \quad . \quad . \quad (1)$$

where i is the current density in absamperes per sq. cm., considered constant for the present, and where the dimensions are in centimeters. The flux density at radius x is

$$\frac{2I_{(x)}}{x} = 2\pi i\left(x - \frac{q^2}{x}\right) \text{ lines per sq. cm. } \quad . \quad . \quad . \quad (2)$$

The total flux in the ring outside of the circle of radius x is obtained by integrating from x to ρ and is equal to

$$\phi_{(x)} = \pi i\left(\rho^2 - x^2 - 2q^2 \operatorname{logh} \frac{\rho}{x}\right) \text{ lines per cm. } \quad . \quad . \quad . \quad . \quad (3)$$

The reactance drop at radius x due to the above flux is

$$j\omega\phi_{x}=j\omega\pi i\left(\rho^2-x^2-2q^2\ \mathrm{logh}\ \frac{\rho}{x}\right)\ \text{abvolts per cm.,} \quad . \quad . \quad (4)$$

where

$$\omega=2\pi f,$$

and where f is the frequency in cycles per second.

To find the average reactance drop due to the above flux, multiply the element of area, $2\pi x dx$, by the drop in that element, given by (4), then integrate over the section of the tube and divide by the area of section of the tube. This gives

$$\frac{j\omega\pi i}{2}\left(\rho^2-3q^2+\frac{4q^4}{\rho^2-q^2}\ \mathrm{logh}\ \frac{\rho}{q}\right)\ \text{abvolts per cm.} \quad . \quad (5)$$

Assume that the diameter of the core is one-third that of the complete cable, which is usually very nearly the case, then

$$q=\tfrac{1}{3}\rho,$$

and the average reactance drop in the tube is

$$\frac{j\omega\pi i\rho^2}{2}\times\frac{13.10}{18}\ \text{abvolts per cm.} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Let the total current in the tube, equal to $\pi i(\rho^2-q^2)$, be represented by $(a+jb)$ amperes. Then the reactance drop in the tube is

$$\frac{j\omega}{2}\times\frac{13.10}{18}\times\frac{9}{8}\ \frac{(a+jb)}{10}\ \text{abvolts per cm.} \quad . \quad . \quad (7)$$

This is equivalent at 60 cycles to

$$0.0248j(a+jb)\ \text{volts per mile.} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Let R be the alternating-current resistance of the tube in ohms per mile.

* "The Inductance of Tubular Conductors," by H. B. Dwight, *The Electrical Review*, Feb. 9, 1918, p. 224.

Since the current in the steel core is a very small percentage of the total current, the calculation for skin effect in a tube will apply very closely.* The increase in resistance at 60 cycles for tubes whose inside diameter is one-third the outside diameter, is as follows:

Resistance of Cable, Ohms per Mile	Per Cent Increase in Resistance of the Aluminum or Copper at 60 Cycles	Resistance of Cable, Ohms per Mile	Per Cent Increase in Resistance of the Aluminum or Copper at 60 Cycles
0.08	2.6	0.12	1.2
0.09	2.1	0.13	1.0
0.10	1.8	0.14	0.9
0.11	1.5	0.15	0.8

If the inside diameter is larger than one-third the outside diameter, the per cent increase will be less than is given above. It is to be noted that the increases in resistance given above are less than those given in Tables 14 and 15 for coreless cables.

The impedance drop, taking into account only the flux considered above, is

$$(a+jb)(R+j0.0248) \text{ volts per mile.} \quad . \quad . \quad (9)$$

Now the alternating magnetic flux considered above cuts the core as well as the tube. The total flux is given by (3), putting $x=q$, and is

$$\pi i\rho^2(1-\tfrac{1}{9}-\tfrac{8}{9}\times 1.10) \text{ lines per cm.} \quad . \quad . \quad . \quad (10)$$

The reactance drop in the core due to this flux is

$$j\omega\pi i\rho^2\times\frac{5.80}{9} \text{ abvolts per cm.} \quad . \quad . \quad . \quad . \quad (11)$$

which is equivalent to

$$j(a+jb)0.0440 \text{ ohm per mile.} \quad . \quad . \quad . \quad (12)$$

* See the curves of Fig. 1, "Skin Effect and Proximity Effect in Tubular Conductors," by H. B. Dwight, *Trans.*, *A. I. E. E.*, 1922, p. 189, and "A Precise Method of Calculation of Skin Effect in Isolated Tubes," by H. B. Dwight, *Journal, A. I. E. E.*, August, 1923, p. 827.

Let the current in the steel core be $c+jd$ amperes. The impedance of the core due to its effective resistance and the flux inside the steel, may be taken with reasonable accuracy from the curves in Tables 24 and 25.

Let the impedance given by the curves be R_1+jX_1 for a certain assumed current in the core, which does not require to be estimated very closely for this purpose. Then the drop in the steel core is

$$(c+jd)(R_1+jX_1)+j(a+jb)(0.0440) \text{ volts per mile.} \quad . \quad (13)$$

This may be equated to (9), since the core and the tube are in electrical contact and take up a distribution of current such as to give the same voltage drop in each. Therefore,

$$\begin{aligned}(c+jd)(R_1+jX_1) &= (a+jb)(R+j0.0248-j0.0440)\\ &= (a+jb)(R-j0.0192)\end{aligned}$$

$$\text{volts per mile at 60 cycles.} \quad . \quad . \quad . \quad . \quad (14)$$

The term 0.0192 becomes 0.0080 at 25 cycles.

Equation (14) is the same as the usual equation for two impedances in parallel. Thus, let each side be equal to E. Then

$$a+jb=\frac{E}{R-j0.0192},$$

and

$$c+jd=\frac{E}{R_1+jX_1}.$$

The current in the complete cable

$$=E\times\text{total admittance}$$

$$=E\left(\frac{1}{R-j0.0192}+\frac{1}{R_1+jX_1}\right).$$

In this way, the effective resistance R' and reactance X' of the complete cable including the core, may be calculated, since the total admittance is

$$\frac{1}{R'+jX'}=\frac{1}{R-j0.0192}+\frac{1}{R_1+jX_1}. \quad . \quad . \quad (15)$$

EXAMPLES OF TRANSMISSION LINE CABLES WITH STEEL CORES

Outer Wires of Cables	Core	Current Assumed for Core	R_1	R	R'	% Decrease in Resistance Due to Core	% Decrease Calculated from R_1 and R Only	X_1	X	X'	% Decrease in Reactance Due to Core
600,000 c.m. Aluminum	$\frac{5}{16}''$ Cable, ordinary steel	30 amps., 60 cy.	7.6	0.153	0.150	1.8	2.0	2.0	0.760	0.754	0.8
600,000 c.m. Aluminum	$\frac{5}{16}''$ Cable, ordinary steel	15 amps., 60 cy.	6.4	0.153	0.149	2.2	2.3	1.3	0.760	0.754	0.8
600,000 c.m. Aluminum	$\frac{5}{16}''$ Cable, ordinary steel	30 amps., 25 cy.	6.5	0.152	0.148	2.2	2.3	1.1	0.317	0.315	0.7
400,000 c.m. Aluminum	$\frac{9}{32}''$ Cable, ordinary steel	20 amps., 60 cy.	8.6	0.228	0.223	2.3	2.5	2.0	0.760	0.751	1.2
250,000 c.m. Aluminum	No. 6 B.W.G. Wire, ordinary steel	12.5 amps., 60 cy.	20.8	0.364	0.359	1.3	1.8	9.8	0.767	0.761	0.8
150,000 c.m. Aluminum	No. 8 B.W.G. Wire, ordinary steel	7.5 amps., 60 cy.	23.3	0.605	0.592	2.1	2.5	9.0	0.777	0.773	0.6

Resistance and Reactance are in Ohms per Mile. X is the reactance, at an average high voltage spacing, of a 600,000 c.m. or 400,000 c.m., etc., aluminum cable having no core.

It is found that the current in the core is usually so small that its effect in producing magnetic flux in the outer part of the cable may be neglected, as was done in the above calculation. The current in the outer part of the cable is to the current in the core in the ratio of their admittances, and that is very closely in the inverse ratio of their resistances. An additional loss in the core, due to the nearness of the current in the outside wires, may also take place, especially with the larger sizes of individual wires.

A few examples are tabulated herewith. The reactances are based on spacings which would be usual for transmission line work. It may be observed from the table that a cable composed of 600,000 circular mils of aluminum and an 83,000 circular mil steel core, has a larger diameter and therefore nearly one per cent less reactance than a 600,000 circular mil cable without a core. The same cable, composed of 600,000 circular mils of aluminum and 83,000 of steel, has only 0.2 per cent more reactance than a 683,000 circular mil, all-aluminum cable. Thus, although the core offers a magnetic path to the flux of self-inductance, the amount of effective flux in the core is very small, since a steel core of $\frac{1}{3}$ the diameter of the cable carries only about $\frac{1}{50}$ of the total current.

In conclusion, the examples show that a close approximation to the electrical characteristics of a steel cored cable as used on transmission lines may be obtained by taking the resistance as equal to that of the core and the outer conductors connected in parallel, and by taking the reactance as equal to that of a non-magnetic cable of the same outside diameter. The direct-current resistance of the steel core should not be used, but only the value of the resistance to alternating current.

A practical feature not included in the calculations is that with comparatively heavy currents the increase in diameter caused by the core makes a decrease in the temperature and resistance of the cable, because of the larger cooling surface.

CHAPTER XXI

CAPACITANCE OF SINGLE-PHASE LINE

Capacitance of Two Round Wires.—The conductors of a transmission line form a condenser, the electrostatic capacity of which can be calculated from the dimensions of the line. The simplest line to calculate is a single-phase line consisting of two round wires, and this case will be investigated first.

Suppose that A and B (Fig. 31) are two long parallel wires of infinitesimal section and that they are spaced a

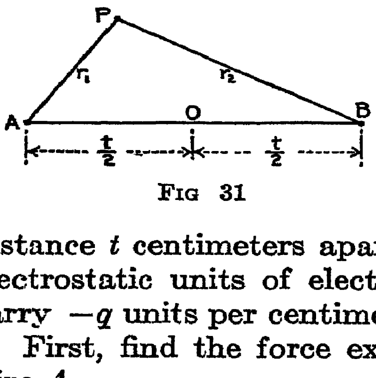


FIG 31

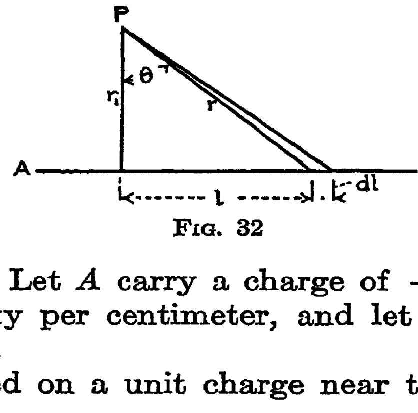


FIG. 32

distance t centimeters apart. Let A carry a charge of $+q$ electrostatic units of electricity per centimeter, and let B carry $-q$ units per centimeter.

First, find the force exerted on a unit charge near the wire A.

From the symmetry of the arrangement it is evident that the resultant force on a unit charge at P (Fig. 32) will be a repulsion away from the wire at right angles to it, since the total effect of the right half of the wire must be equal to the total effect of the left half. The force at right angles to the wire exerted by the charge on the element dl will be

$$q\frac{dl}{r^2}\cos\theta = \frac{q}{r_1}\cos\theta d\theta,$$

since

$$dl \cos \theta = r d\theta$$

and

$$r = \frac{r_1}{\cos \theta}.$$

The total force exerted by the wire will be

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{q}{r_1} \cos \theta d\theta = \frac{2q}{r_1}.$$

The potential of the point O (Fig. 31) midway between the wires, will be zero, since the effect of the positive charge on A will be equal to the effect of the negative charge on B. The potential difference between P and O is the work done in moving a unit charge from one point to the other. The force due to the wire A on a unit charge at any point is $\frac{2q}{r}$, acting directly away from A. Therefore the work done in moving a distance dr toward A is

$$\frac{2q}{r} dr$$

and thus the total work in moving from P to O against the force due to A is

$$\int_{r_1}^{\frac{t}{2}} \frac{2q}{r} dr = 2q \operatorname{logh} \frac{t}{2r_1}.$$

Similarly, the work against the force due to B is equal to

$$-2q \operatorname{logh} \frac{t}{2r_2}.$$

Therefore the potential difference between P and O is equal to the total work, and is

$$2q \operatorname{logh} \frac{r_2}{r_1}.$$

At P (Fig. 33) make the angle APD equal to the angle PBD. Then the triangle PBD is similar to the triangle APD, and therefore

$$\frac{PB}{AP}=\frac{r_2}{r_1}$$

$$=\frac{DB}{\rho}.$$

If we draw a circle of radius ρ about the fixed point D, then at any point on this circle similar triangles are formed by ρ, r_1, and r_2, as in Fig. 33, and therefore

$$\frac{r_2}{r_1}=\frac{DB}{\rho}=\text{constant},$$

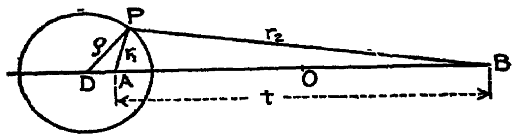


FIG. 33.

where r_1 and r_2 are the distances of the point on the circle from A and B respectively. Therefore, the potential

$$2q \operatorname{logh} \frac{r_2}{r_1}$$

will be the same at all points on this circle.

Now let a solid cylindrical conductor fill all the space inside the circle of radius ρ. All points on its surface will be at the same potential. The distribution of potential outside of the cylinder will not be altered from the previous condition when all points on the circle of radius ρ were also at the same potential. The potential of the cylinder will be

$$2q \operatorname{logh} \frac{r_2}{r_1}=2q \operatorname{logh} \frac{DB}{\rho}.$$

In the same way, let the wire B be replaced by a solid cylinder of radius ρ and center E, as in Fig. 34.

The potential of this cylinder, which carries $-q$ units per centimeter, will be

$$-2q \operatorname{logh} \frac{AE}{\rho} = -2q \operatorname{logh} \frac{DB}{\rho},$$

since the second cylinder is symmetrical with the first.

Thus the potential difference between the two cylinders is

$$4q \operatorname{logh} \frac{DB}{\rho} = \frac{q}{C},$$

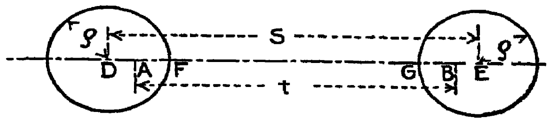


FIG 34.

where C is the capacitance per centimeter of line.

Thus

$$C = \frac{1}{4 \operatorname{logh} \frac{DB}{\rho}}.$$

Now

$$\frac{DB}{\rho} = \frac{FB}{FA},$$

since F is a point on the circle.

Therefore

$$\frac{DB}{\rho} = \frac{DB - \rho}{DB + \rho - s},$$

where s is the interaxial distance between the two solid cylinders, and is equal to DE, Fig. 34.

Therefore,

$$DB^2 + DB \cdot \rho - DB \cdot s = DB \cdot \rho - \rho^2$$

or

$$DB^2 - DB \cdot s + \rho^2 = 0.$$

Solving this quadratic equation in DB, we have

$$DB = \frac{s + \sqrt{s^2 - 4\rho^2}}{2}.$$

[The negative value of the radical must not be used, since it would give the value of DA instead of DB.]

Therefore,

$$C=\frac{1}{4\ \mathrm{logh}\left(\frac{s}{2\rho}+\sqrt{\frac{s^2}{4\rho^2}-1}\right)},$$

which may be expressed as

$$C=\frac{1}{4\cosh^{-1}\left(\frac{s}{2\rho}\right)},$$

or it may be expanded by the Binomial Theorem to give the approximate value

$$C=\frac{1}{4\ \mathrm{logh}\left(\frac{s}{\rho}-\frac{\rho}{s}\right)}\text{ per centimeter,}$$

or, very nearly,

$$C=\frac{1}{4\ \mathrm{logh}\ \frac{s}{\rho}}.*$$

Transferring to other units,

$$C=\frac{1}{\log_{10}\left(\frac{s}{\rho}-\frac{\rho}{s}\right)}\times\frac{4343\times2\ 540\times12\times5280}{4\times9\times10^{11}}$$

$$=\frac{1}{2}\times\frac{38.83\times10^{-9}}{\log_{10}\left(\frac{s}{\rho}-\frac{\rho}{s}\right)}$$

farads per mile of single-phase line.

* It is evident that the expression

$$C=\frac{1}{4\ \mathrm{logh}\ \frac{s-\rho}{\rho}},$$

which is sometimes published, is less accurate than the simpler expression

$$C=\frac{1}{4\ \mathrm{logh}\ \frac{s}{\rho}}.$$

The capacity susceptance is

$$2\pi fC = 2\pi f \times \frac{1}{2} \times \frac{38.83 \times 10^{-9}}{\log_{10}\left(\frac{s}{\rho} - \frac{\rho}{s}\right)}$$

mhos per mile of single-phase line. The charging current in this line will be

$$\frac{1}{2}E \times 2\pi f \times \frac{38.83 \times 10^{-9}}{\log_{10}\left(\frac{s}{\rho} - \frac{\rho}{s}\right)} \text{ amperes.}$$
$$= \tfrac{1}{2}Eb,$$

where b is the tabulated value of capacity susceptance.

REFERENCE.—"A Treatise on the Theory of Alternating Currents," by Alexander Russell, 1904, Vol I, page 99.

Capacitance of Cable.—The formula for capacitance of a line using stranded cables will be the same as the above formula for solid wires, ρ being taken as the maximum radius of the cable. All the electrostatic charge on the cable does not lie at the maximum radius from the center, but as actual cables are generally slightly larger than the calculated diameters in the tables, it will be sufficiently close to take ρ from the tables and use it in the regular formula for capacitance.

Effect of the Earth on Capacitance of Line.—The effect of bringing a conducting plane, such as the earth, near to two charged wires is to change their electrostatic field and increase their capacitance.

Consider two long parallel wires, A and A_1 (Fig. 35), of infinitesimal section and carrying $+q$ and $-q$ units of electricity per centimeter respectively. As in Fig. 31, the point O midway between the two wires will be at zero potential. All points having the same potential must have $\frac{r_1}{r_2}$ equal to a constant. It is evident that all points at the same potential as O lie in the plane MON, perpendicular to AA_1, since for all such points

$$r_1 = r_2.$$

Therefore, the wire A_1 may be replaced by a solid conducting plane MN, which will be at zero potential. Thus, when the conducting plane is the earth, its effect is the same as that of a charged wire at a depth below the surface equal to the height of the original wire.* The assumed wire is called an image wire, since it occupies the same position as the image of the real wire, considering the surface of the ground as a mirror.

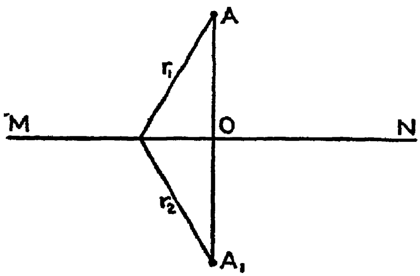


FIG. 35.

In the case of a single-phase transmission line, image wires A' and B' must be assumed for both wires A and B (Fig. 36) and the capacitance of the entire system of four wires is then calculated as follows:

Let h be the distance of the wires from the ground and s their distance apart from center to center. Let A carry a charge of $+q$ units per centimeter; A', $-q$ units; B, $-q$ units; and B', $+q$ units. Let a unit charge be carried from the surface of A to that of B. Assuming that the charges are concentrated at the centers of the wires, the total work done is equal to

$$\int_{\rho}^{s-\rho}\frac{2q\,dx}{x}+\int_{\rho}^{s-\rho}\frac{2q}{s-x}dx-\int^{s-\rho}\frac{2qx}{4h^2+x^2}dx$$
$$-\int_{\rho}^{s-\rho}\frac{2q(s-x)}{4h^2+(s-x)^2}dx.$$

* See "Elements of Electricity and Magnetism," by J. J. Thomson, page 138

The sum of the first two integrals has been shown to be approximately

$$4q \operatorname{logh} \frac{s}{\rho}.$$

The sum of the last two integrals is approximately

$$-q \operatorname{logh} \frac{4h^2+s^2}{4h^2}+q \operatorname{logh} \frac{4h^2}{4h^2+s^2}$$

$$=2q \operatorname{logh} \frac{4h^2}{4h^2+s^2}.$$

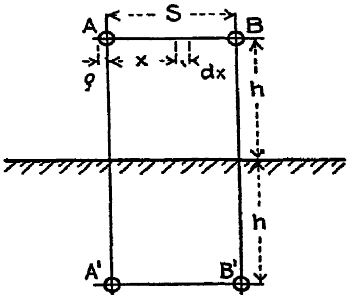


FIG. 36.

Therefore, the total work is equal to

$$4q \operatorname{logh} \frac{s}{\rho}+4q \operatorname{logh} \frac{2h}{\sqrt{4h^2+s^2}}=\frac{q}{C}.$$

Therefore, C is approximately

$$\frac{1}{4 \operatorname{logh} \frac{s}{\rho}+4 \operatorname{logh} \frac{2h}{\sqrt{4h^2+s^2}}}.$$

Taking as an average case,

$h=360$ inches (30 feet),
$s=120$ inches (10 feet),
$\rho=0.25$ inch,

we have

$$\frac{s}{\rho}=480,$$

and

$$\frac{2h}{s}=6.$$

Therefore,

$$\frac{2h}{\sqrt{4h^2+s^2}}=\frac{6}{\sqrt{37}}.$$

Now,

$$\log_{10} 480 = 2\ 681,$$

while

$$\log_{10} \frac{\sqrt{37}}{6} = 0\ 006.$$

Thus the capacitance is changed by the nearness of the ground by less than $\frac{1}{4}$ of 1%, even with the comparatively wide spacing of 10 feet.

Tests have shown that the effect of the ground in increasing the capacitance is even less than the above amount, due partly to the fact that the ground is a poor conductor. As the effect of the ground is so slight, it has been neglected entirely in the calculations in this book.

REFERENCE.—For an alternative proof, see "A Treatise on the Theory of Alternating Currents," by Alexander Russell, 1904, Vol. I.

See also "The Calculation of Capacity Coefficients for Parallel Suspended Wires," by Frank F. Fowle, *Elec. World*, Aug. 19, 1911.

CHAPTER XXII

CAPACITANCE OF TWO-PHASE AND THREE-PHASE LINES

Capacitance, Two-phase.—The charging current of a single-phase line was shown in Chapter XXI to be

$$\frac{1}{2}E\times\frac{38.83\times10^{-9}}{\log_{10}\left(\frac{s}{\rho}-\frac{\rho}{s}\right)}\times2\pi f$$

amperes per mile of line

$$=\tfrac{1}{2}Eb,$$

where b is the tabulated value of capacity susceptance per mile.

In a two-phase, four-wire line, each phase is quite similar to a single-phase line, and so the charging current per wire is

$$\tfrac{1}{2}Eb.$$

Capacitance, Three-phase, Irregular Spacing.—When the wires of a three-phase transmission line are not spaced at the corners of an equilateral triangle, but the transposition of the conductors is carried out at regular intervals, the charging current in the wires will be a balanced, three-phase current, since each wire will have passed through the same average conditions. This is shown in an approximate manner as follows:

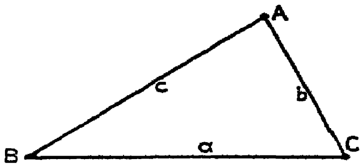


Fig. 37.

As when calculating the self-induction of an irregularly spaced line, consider a line three miles long which is transposed at the end of each mile.

The work in carrying a unit charge from C to B (Fig. 37) assuming the charges concentrated at the centers of the wires, is approximately

$$E_a = q_B 2 \text{ logh } \frac{a}{\rho} - q_C 2 \text{ logh } \frac{a}{\rho} + q_A 2 \text{ logh } \frac{c}{b}.$$

Now q_B is a periodic quantity, which alternates in value at the same frequency as the voltage or current.

We have

$$q_B = C_1 E$$

$$= -j \frac{I'_B}{2\pi f},$$

where I'_B is the charging current flowing into the capacitance C_1 of the wire B.

Thus

$$E_a = \frac{-j}{2\pi f}\left(I'_B 2 \text{ logh } \frac{a}{\rho} - I'_C 2 \text{ logh } \frac{a}{\rho} + I'_A 2 \text{ logh } \frac{c}{b}\right).$$

In the second mile the conductors are transposed into new positions. Let the currents in them remain the same. Therefore,

$$E_a = \frac{-j}{2\pi f}\left(I'_B 2 \text{ logh } \frac{b}{\rho} - I'_C 2 \text{ logh } \frac{b}{\rho} + I'_A 2 \text{ logh } \frac{a}{c}\right),$$

and in the third mile

$$E_a = \frac{-j}{2\pi f}\left(I'_B 2 \text{ logh } \frac{c}{\rho} - I'_C 2 \text{ logh } \frac{c}{\rho} + I'_A 2 \text{ logh } \frac{b}{a}\right).$$

Adding together and dividing by 3, we obtain the approximate average value per mile,

$$E_a = \frac{-j}{2\pi f}(I'_B - I'_C) 2 \text{ logh } \frac{\sqrt[3]{abc}}{\rho};$$

that is,

$$E_a = \frac{-j}{2\pi f}(I'_B - I'_C) 2 \text{ logh } \frac{s}{\rho},$$

where

$$s = \sqrt[3]{abc}.$$

Similarly,

$$E_b = \frac{-j}{2\pi f}(I'_C - I'_A)2 \text{ logh } \frac{s}{\rho},$$

and

$$E_c = \frac{-j}{2\pi f}(I'_A - I'_B)2 \text{ logh } \frac{s}{\rho}.$$

Let

$$E_a = -1.00E, \text{ as in Fig. 38,}$$

and

$$E_b = (0.50 - 0.866j)E,$$

and

$$E_c = (0.50 + 0.866j)E.$$

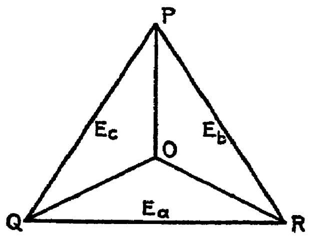


FIG. 38.

Then

$$I'_A - I'_B = \frac{j2\pi fE(0.50 + 0.866j)}{2 \text{ logh } \frac{s}{\rho}}, \quad . \quad . \quad . \quad (1)$$

$$I'_B - I'_C = \frac{j2\pi fE}{2 \text{ logh } \frac{s}{\rho}} \times (-1.00), \quad . \quad . \quad . \quad . \quad (2)$$

$$I'_C - I'_A = \frac{j2\pi fE}{2 \text{ logh } \frac{s}{\rho}}(0.50 - 0.866j), \quad . \quad . \quad . \quad (3)$$

also

$$I'_A + I'_B + I'_C = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

since they are currents flowing in a three-phase line.

From equations (1) and (3)

$$2I'_A - I'_B - I'_C = -\frac{j2\pi fE(-1.732j)}{2 \text{ logh } \frac{s}{\rho}}. \quad . \quad . \quad . \quad (5)$$

Adding (4) and (5), we have

$$I'_A = -\frac{1}{\sqrt{3}} \times \frac{2\pi f E \times 1\ 00}{2 \operatorname{logh} \frac{s}{\rho}}.$$

From this,

$$I'_B = -\frac{1}{\sqrt{3}} \times \frac{2\pi f E(-0\ 50 + 0\ 866j)}{2 \operatorname{logh} \frac{s}{\rho}}.$$

and

$$I'_C = -\frac{1}{\sqrt{3}} \times \frac{2\pi f E(-0\ 50 - 0\ 866j)}{2 \operatorname{logh} \frac{s}{\rho}}.$$

The vectors for I'_A, I'_B and I'_C may now be plotted as in Fig. 39, and it is seen that the vectors are the same length and are at 120° to each other. Thus the charging current is a balanced three-phase current.

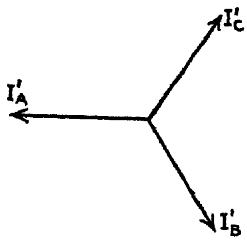


FIG. 39.

The power factor of the charging current is zero, since the current in any wire I_A' (Fig. 39) is at right angles to the direction OP (Fig. 38) of the corresponding star voltage or in-phase current.

Capacitance, Three-phase, Regular Triangular Spacing.—When the conductors are placed an equal distance, s, from each other, the formula for b is

$$b = \frac{38.83 \times 2\pi f}{\log_{10} \frac{s}{\rho}} \times 10^{-9} \text{ mhos per mile.}$$

Capacitance, Three-phase Regular Flat Spacing.—When the wires lie in one plane (either horizontal or vertical), the center one being at a distance a from the other two (see

Fig. 29), and the wires being transposed at regular intervals, the formula for susceptance is

$$b = \frac{38.83 \times 2\pi f}{\log_{10} \dfrac{a\sqrt[3]{1 \times 1 \times 2}}{\rho}} \times 10^{-9}$$

$$= \frac{38.83 \times 2\pi f}{\log_{10} \dfrac{1.26a}{\rho}} \times 10^{-9}$$

mhos per mile.

Part III

TABLES

N B.—The Transmission Line Chart, which forms the frontispiece of this book, can be used for rapid estimation of some of the results of the following tables.

TABLE 1.—FORMULAS FOR SHORT LINES

Conditions Given at Receiver End

These formulas are exact when the line is short. When the line is 20 miles long, they are correct within approximately $\frac{1}{10}$ of 1% of line voltage, for frequencies up to 60 cycles.

Conditions given, for 3-phase circuit:

Kv-a. = Kv-a. at receiver end.

E = Full load voltage to neutral at receiver end,

$= \frac{1}{\sqrt{3}} \times$ voltage between conductors in 3-phase circuit.

$\cos\theta$ = Power factor at receiver end.

Kw. = Kv.-a. $\cos\theta$ = Power at receiver end.

r = Resistance of conductor per mile.

x = Reactance of conductor per mile.

l = Length of line in miles.

$R = rl$.

$X = xl$.

Then $P = \frac{1000 \text{ Kv-a.} \cos\theta}{3E}$ = In-phase current per conductor at receiver end.

$Q = \frac{1000 \text{ Kv-a.} \sin\theta}{3E}$ = Reactive current per conductor at receiver end, when current is leading.

$Q = -\frac{1000 \text{ Kv-a.} \sin\theta}{3E}$ when current is lagging.

TABLE 1. (*Continued*)

Find the following

$$A = E + PR - QX.$$

$$B = PX + QR.$$

Formulas (capacity neglected):

(1) Voltage to neutral at supply end $= A + \dfrac{B^2}{2A}$.

(2) Regulation of line $= A + \dfrac{B^2}{2A} - E$ volts to neutral. (Same as line drop.)

(3) Per cent regulation of line at receiver end $= \dfrac{100\left(A + \dfrac{B^2}{2A} - E\right)}{E}$ per cent. (Same as per cent line drop.)

(4) Kv-a. at supply end $= \dfrac{A + \dfrac{B^2}{2A}}{E} \times$ Kv-a.

(5) Kw. at supply end $= \dfrac{3}{1000}(AP + BQ)$.

(6) Power factor at supply end $= \dfrac{3}{1000} \dfrac{(AP + BQ)E}{\left(A + \dfrac{B^2}{2A}\right) \times \text{Kv-a.}}$ (in decimals).

(7) In-phase current at supply end $= \dfrac{AP + BQ}{A + \dfrac{B^2}{2A}}$ amperes per conductor.

(8) Kw. loss in line $= \dfrac{3}{1000}(AP + BQ - EP)$.

(9) Per cent efficiency of line $= \dfrac{100EP}{AP + BQ}$ per cent.

(10) Reactive Kv-a. at supply end $= \frac{3}{1000}(AQ - BP)$.
When this quantity is positive, the current is leading.
When this quantity is negative, the current is lagging.

(11) Reactive or quadrature current at supply end $= \dfrac{AQ - BP}{A + \dfrac{B^2}{2A}}$ amperes per conductor.

TABLE 1. (*Continued*)

In a single-phase circuit, the voltage to neutral is one-half the voltage between conductors, and in a two-phase circuit, the voltage to neutral is one-half the voltage between conductors of the same phase In the formulas for P and Q, and in formulas 5, 6, 8 and 10, the figure 3 represents the number of conductors For single-phase lines, change this figure to 2, and for two-phase lines, change it to 4, thus making the table applicable to single-phase or two-phase (4-conductor) lines.

In the case of short transmission lines where capacitance is not considered, the resistance and reactance to neutral of star-connected step-up and step-down transformers may be added to the resistance and reactance of the line, without producing any error or approximation. If the transformers are delta-connected, the percentage resistance and reactance should be found, and then the resistance and reactance to neutral of star-connected transformers having the same percentages should be calculated and added to the line, as described above.

TABLE 2.—FORMULAS FOR SHORT LINES

Conditions Given at Supply End

These formulas are exact when the line is short. When the line is 20 miles long, they are correct within approximately $\frac{1}{10}$ of 1% of line voltage, for frequencies up to 60 cycles

Conditions given, for 3-phase circuit:

Kv-a = Kv-a. at supply end.

E_s = Full load voltage to neutral at supply end

$= \frac{1}{\sqrt{3}} \times$ voltage between conductors in 3-phase circuit.

$\cos\theta$ = Power factor at supply end.

Kw = Kv-a. $\cos\theta$ = Power at supply end.

r = Resistance of conductor per mile.

x = Reactance of conductor per mile.

l = Length of line in miles.

$R = rl$

$X = xl.$

Then $P_s = \frac{1000 \text{ Kv-a. } \cos\theta}{3E_s}$ = In-phase current per conductor at supply end.

$Q_s = \frac{1000 \text{ Kv-a. } \sin\theta}{3E_s}$ = Reactive current per conductor at supply end, when current is leading

$Q_s = -\frac{1000 \text{ Kv-a. } \sin\theta}{3E_s}$ when current is lagging.

Find the following quantities:

$$F = E_s - P_sR + Q_sX.$$

$$H = -P_sX - Q_sR.$$

Formulas (capacity neglected):

(1) Voltage to neutral at receiver end $= F + \frac{H^2}{2F}$.

(2) Regulation of line $= E_s - F - \frac{H^2}{2F}$ volts to neutral. (Same as line drop.)

TABLE 2.—*Continued*

(3) Per cent regulation of line at receiver end $= \dfrac{100\left(E_s - F - \dfrac{H^2}{2F}\right)}{F + \dfrac{H^2}{2F}}$ per cent.

(Same as per cent line drop.)

(4) Kv-a. at receiver end $= \dfrac{F + \dfrac{H^2}{2F}}{E_s} \times$ Kv-a.

(5) Kw. at receiver end $= \frac{3}{1000}(FP_s + HQ_s)$.

(6) Power factor at receiver end $= \dfrac{3}{1000} \dfrac{(FP_s + HQ_s)E_s}{\left(F + \dfrac{H^2}{2F}\right) \times \text{Kv-a.}}$

(in decimals).

(7) In-phase current at receiver end $= \dfrac{FP_s + HQ_s}{F + \dfrac{H^2}{2F}}$ amperes per conductor.

(8) Kw. loss in line $= \frac{3}{1000}(E_sP_s - FP_s - HQ_s)$.

(9) Per cent efficiency of line $= \dfrac{100(FP_s + HQ_s)}{E_sP_s}$ per cent.

(10) Reactive Kv-a. at receiver end $= \frac{3}{1000}(HP_s - FQ_s)$.

When this quantity is positive, the current is leading.
When this quantity is negative, the current is lagging.

(11) Reactive or quadrature current at receiver end $= \dfrac{HP_s - FQ_s}{F + \dfrac{H^2}{2F}}$

amperes per conductor.

For single-phase and 2-phase circuits, and for the effect of transformers, see the paragraphs at the end of Table 1.

TABLE 3.—FORMULAS FOR SHORT LINES

SUPPLY VOLTAGE AND RECEIVER LOAD GIVEN

These formulas are exact when the line is short. When the line is 20 miles long, they are correct within approximately $\frac{1}{10}$ of 1% of line voltage, for frequencies up to 60 cycles.

Conditions given, for 3-phase circuit:

Kv-a. = Kv-a. at *receiver* end.

E_s = Full load voltage to neutral at *supply* end

$= \frac{1}{\sqrt{3}} \times$ voltage between conductors in 3-phase circuit

$\cos\theta$ = Power factor at *receiver* end.

Kw = Kv-a. $\cos\theta$

r = Resistance of conductor per mile.

x = Reactance of conductor per mile.

l = Length of line in miles.

$R = rl.$

$X = xl.$

Then $PE = \dfrac{1000 \text{ Kv-a. } \cos\theta}{3}$ = Watts per conductor at receiver end, where E is the unknown voltage to neutral at the receiver end.

$QE = \dfrac{1000 \text{ Kv-a } \sin\theta}{3}$ = Reactive volt-amperes per conductor at receiver end, when current is leading.

$QE = -\dfrac{1000 \text{ Kv-a } \sin\theta}{3}$ when current is lagging.

PE and QE are known, but the separate quantities P, Q and E are not known.

Find the quantities:

$$L^2 = PER - QEX,$$

and

$$M^2 = PEX + QER.$$

Then,

$$E^2 = \tfrac{1}{2}E_s^2 - L^2 + \tfrac{1}{2}\sqrt{E_s^4 - 4E_s^2L^2 - 4M^4}.$$

If the drop in the line is not more than 20%, the following series may be used:

$$E = E_s\left[1 - \frac{L^2}{E_s^2} - \frac{L^4}{E_s^4} - \frac{M^4}{2E_s^4} - \frac{2L^6}{E_s^6} - \frac{3}{2}\frac{L^2M^4}{E_s^6} - \frac{5L^8}{E_s^8} - \frac{5L^4M^4}{E_s^8} - \frac{5}{8}\frac{M^8}{E_s^8} - \cdots\right].$$

Now that E is known, P and Q can be found, and the quantities A and B and the various characteristics of the line can be calculated as in Table 1.

TABLE 4—K FORMULAS FOR TRANSMISSION LINES

Conditions Given at Receiver End

Accurate within approximately $\frac{1}{10}$ of 1% of line voltage up to 100 miles, and $\frac{1}{2}$ of 1% up to 200 miles, for lines with regulation up to 20%.

Conditions given, for 3-phase circuit:

Kv-a. = Kv-a at receiver end.
E = Full load voltage to neutral at receiver end
$\doteq \frac{1}{\sqrt{3}} \times$ voltage between conductors in 3-phase circuit.
$\cos\theta$ = Power factor at receiver end.
Kw. = Kv-a $\cos\theta$ = Power at receiver end.
r = Resistance of conductor per mile.
x = Reactance of conductor per mile.
l = Length of transmission line in miles.
$R = rl.$
$X = xl.$
f = Frequency in cycles per second.

Then $K = 6\left(\frac{fl}{100{,}000}\right)^2.$

$P = \frac{1000 \text{ Kv-a.} \cos\theta}{3E}$ = In-phase current per conductor at receiver end.

$Q = \frac{1000 \text{ Kv-a.} \sin\theta}{3E}$ = Reactive current per conductor at receiver end, when current is leading.

$Q = -\frac{1000 \text{ Kv-a.} \sin\theta}{3E}$ when current is lagging.

Find the following quantities:

Full Load

$$A = E(1-K) + PR(1-\tfrac{2}{3}K) - QX(1-\tfrac{1}{6}K).$$

$$B = \frac{ERK}{X} + PX(1-\tfrac{1}{6}K) + QR(1-\tfrac{2}{3}K).$$

$$C = P(1-K) - \frac{QRK}{X} - \frac{2}{3}\frac{ERK^2}{X^2}.$$

$$D = \frac{PRK}{X} + Q(1-K) + \frac{2EK}{X}(1-\tfrac{1}{3}K).$$

TABLE 4—*Continued*

CONDITIONS GIVEN AT RECEIVER END

No Load

$$A_0=E(1-K).$$

$$B_0=\frac{ERK}{X}.$$

$$C_0=-\frac{2}{3}\,\frac{ERK^2}{X^2}.$$

$$D_0=\frac{2EK}{X}(1-\tfrac{1}{3}K).$$

Formulas for 3-phase circuits:

Full Load | **No Load (Constant Supply Voltage)**

Voltage to neutral at receiver end.

(1) $E.$

(2) $E_0=\dfrac{A+\dfrac{B^2}{2A}}{A_0+\dfrac{B_0^2}{2A_0}}E.$

Regulation at receiver end in volts to neutral, for constant supply voltage.

(3) $\dfrac{A+\dfrac{B^2}{2A}}{A_0+\dfrac{B_0^2}{2A_0}}E-E.$

N.B. The regulation at receiver end may be expressed as a percentage of E.

Full Load | **No Load (Constant Receiver Voltage)**

Voltage to neutral at supply end.

(4) $E_s=A+\dfrac{B^2}{2A}.$

(5) $E_{0s}=A_0+\dfrac{B_0^2}{2A_0}.$

Regulation at supply end in volts to neutral, for constant receiver voltage.

(6) $A+\dfrac{B^2}{2A}-A_0-\dfrac{B_0^2}{2A_0}.$

N.B.—The regulation at supply end may be expressed as a percentage of E_s.
Current at supply end, in amperes per conductor.

(7) $\sqrt{C^2+D^2}.$

(8) $\sqrt{C_0^2+D_0^2}.$

TABLE 4—*Continued*

Full Load	No Load (Constant Receiver Voltage)
Kv-a at supply end.	
(9) $\frac{3}{1000}\left(A+\frac{B^2}{2A}\right)\sqrt{C^2+D^2}$.	(10) $\frac{3}{1000}\left(A_0+\frac{B_0^2}{2A_0}\right)\sqrt{C_0^2+D_0^2}$.
Kw. at supply end.	
(11) $\frac{3}{1000}(AC+BD)$.	(12) $\frac{3}{1000}(A_0C_0+B_0D_0)$.
Power factor at supply end, in per cent.	
(13) $\frac{100(AC+BD)}{\left(A+\frac{B^2}{2A}\right)\sqrt{C^2+D^2}}$.	(14) $\frac{100(A_0C_0+B_0D_0)}{\left(A_0+\frac{B_0^2}{2A_0}\right)\sqrt{C_0^2+D_0^2}}$.
In-phase current at supply end, in amperes per conductor.	
(15) $\frac{AC+BD}{A+\frac{B^2}{2A}}$.	(16) $\frac{A_0C_0+B_0D_0}{A_0+\frac{B_0^2}{2A_0}}$.
Kw. loss in line.	
(17) $\frac{3}{1000}(AC+BD-EP)$.	(18) $\frac{3}{1000}(A_0C_0+B_0D_0)$. [Same as No. 12.]
Per cent efficiency of line.	
(19) $\frac{100EP}{AC+BD}$ per cent.	
Reactive Kv-a. at supply end.	
(20) $\frac{3}{1000}(AD-BC)$.	(21) $\frac{3}{1000}(A_0D_0-B_0C_0)$.

When this quantity is positive, the current is leading.
When this quantity is negative, the current is lagging.

Reactive or quadrature current at supply end, in amperes per conductor.

(22) $\frac{AD-BC}{A+\frac{B^2}{2A}}$.	(23) $\frac{A_0D_0-B_0C_0}{A_0+\frac{B_0^2}{2A_0}}$.

Transformer banks are to be treated as separate sections of the line.

TABLE 5—*K* FORMULAS FOR TRANSMISSION LINES

CONDITIONS GIVEN AT SUPPLY END

Accurate within approximately $\frac{1}{10}$ of 1% of line voltage up to 100 miles and $\frac{1}{2}$ of 1% up to 200 miles, for lines with regulation up to 20%.

Conditions given, for 3-phase circuit:

Kv-a. = Kv-a. at supply end

E_s = Full load voltage to neutral at supply end

$= \frac{1}{\sqrt{3}} \times$ voltage between conductors in 3-phase circuit

$\cos\theta$ = Power factor at supply end

Kw = Kv-a $\cos\theta$ = Power at supply end

r = Resistance of conductor per mile

x = Reactance of conductor per mile.

l = Length of transmission line in miles.

$R = rl$.

$X = xl$.

f = Frequency in cycles per second.

Then $$K = 6\left(\frac{fl}{100{,}000}\right)^2.$$

$P_s = \dfrac{1000 \text{ Kv-a. } \cos\theta}{3E_s}$ = In-phase current per conductor at supply end.

$Q_s = \dfrac{1000 \text{ Kv-a. } \sin\theta}{3E_s}$ = Reactive current per conductor at supply end, when current is leading.

$Q_s = -\dfrac{1000 \text{ Kv-a } \sin\theta}{3E_s}$ when current is lagging.

Find the following quantities:

Full Load

$$F = E_s(1-K) - P_sR(1-\tfrac{2}{3}K) + Q_sX(1-\tfrac{1}{6}K).$$

$$H = \frac{E_sRK}{X} - P_sX(1-\tfrac{1}{6}K) - Q_sR(1-\tfrac{2}{3}K).$$

$$M = P_s(1-K) - \frac{Q_sRK}{X} + \frac{2}{3}\frac{E_sRK^2}{X^2}.$$

$$N = \frac{P_sRK}{X} + Q_s(1-K) - \frac{2E_sK}{X}(1-\tfrac{1}{3}K).$$

TABLE 5—*Continued*

CONDITIONS GIVEN AT SUPPLY END

No Load

$F_0 = E_s(1+K).$

$H_0 = -\dfrac{E_sRK}{X}.$

$M_0 = \dfrac{4}{3}\dfrac{E_sRK^2}{X^2}.$

$N_0 = \dfrac{2E_sK}{X}(1+\tfrac{2}{3}K).$

Formulas for 3-phase Circuits:

Full Load | **No Load**

Voltage to neutral at receiver end.

(1) $E = F + \dfrac{H^2}{2F}.$

(2) $E_0 = F_0 + \dfrac{H_0^2}{2F_0}$ (for constant supply voltage).

Regulation at receiver end in volts to neutral, for constant supply voltage.

(3) $F_0 + \dfrac{H_0^2}{2F_0} - F - \dfrac{H^2}{2F}.$

N.B.—The regulation at receiver end may be expressed as a percentage of E.

Voltage to neutral at supply end.

(4) E_s.

(5) $E_{0s} = \dfrac{F+\dfrac{H^2}{2F}}{F_0+\dfrac{H_0^2}{2F_0}}E_s.$ (for constant receiver voltage.)

Regulation at supply end in volts to neutral, for constant receiver voltage.

(6) $E_s - \dfrac{F+\dfrac{H^2}{2F}}{F_0+\dfrac{H_0^2}{2F_0}}E_s.$

N.B.—The regulation at supply end may be expressed as a percentage of E_s

TABLE 5—*Continued*

	Full Load	**No Load (Constant Supply Voltage $= E_s$**
Kv-a.	(7) $\sqrt{M^2+N^2}$ at receiver end.	(8) $\sqrt{M_0^2+N_0^2}$ at supply end
Kw.	(9) $\frac{3}{1000}\left(F+\frac{H^2}{2F}\right)\sqrt{M^2+N^2}$ at receiver end.	(10) $\frac{3}{1000}E_s\sqrt{M_0^2+N_0^2}$ at supply end.
	(11) $\frac{3}{1000}(FM+HN)$ at receiver end	(12) $\frac{3}{1000}E_sM_0$ at supply end
Power factor, in per cent.	(13) $\frac{100(FM+HN)}{\left(F+\frac{H^2}{2F}\right)\sqrt{M^2+N^2}}$ at receiver end.	(14) $\frac{M_0}{\sqrt{M_0^2+N_0^2}}$ at supply end.
In-phase current, in amperes per conductor.	(15) $\frac{FM+HN}{F+\frac{H^2}{2F}}$ at receiver end.	(16) M_0 at supply end.
Kw. loss in line.	(17) $\frac{3}{1000}(E_sP_s-FM-HN)$.	(18) $\frac{3}{1000}E_sM_0$ [Same as No. 12]
Per cent efficiency of line.	(19) $\frac{100(FM+HN)}{E_sP_s}$ per cent.	
Reactive Kv-a.	(20) $\frac{3}{1000}(FN-HM)$ at receiver end.	(21) $\frac{3}{1000}E_sN_0$ at supply end.

When this quantity is positive the current is leading
When this quantity is negative, the current is lagging.

Reactive or quadrature current, in amperes per conductor.	(22) $\frac{FN-HM}{F+\frac{H^2}{2F}}$ at receiver end.	(23) N_0 at supply end.

Transformer banks are to be treated as separate sections of the line.

TABLE 6.—CONVERGENT SERIES FOR TRANSMISSION LINES

Conditions Given at Receiver End

The convergent series give the results of the fundamental formulas as accurately as desired, if a sufficient number of terms is used.

When conditions are given at the receiver end, the same as with the K formulas, find the quantities:

Full Load

$$A+jB=E\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\frac{Y^3Z^3}{2\cdot 3\cdot 4\cdot 5\cdot 6}+\text{etc.}\right)$$

$$+(P+jQ)Z\left(1+\frac{YZ}{2\cdot 3}+\frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5}+\frac{Y^3Z^3}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}+\text{etc.}\right).$$

$$C+jD=(P+jQ)\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\frac{Y^3Z^3}{2\cdot 3\cdot 4\cdot 5\cdot 6}+\text{etc}\right)$$

$$+EY\left(1+\frac{YZ}{2\cdot 3}+\frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5}+\frac{Y^3Z^3}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}+\text{etc.}\right).$$

No Load

$$A_0+jB_0=E\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\frac{Y^3Z^3}{2\cdot 3\cdot 4\cdot 5\cdot 6}+\text{etc.}\right),$$

$$C_0+jD_0=EY\left(1+\frac{YZ}{2\cdot 3}+\frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5}+\frac{Y^3Z^3}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}+\text{etc.}\right),$$

where $Z=(r+jx)l$ = impedance.
r = Resistance of conductor per mile.
x = Reactance of conductor per mile.
l = Length of transmission line in miles.
$Y=(g+jb)l$ = admittance.
g = Leakage conductance of conductor per mile.
b = Capacity susceptance of conductor per mile.

Use A, B, C, D, etc., with the equations for 3-phase circuits in formulas 1–23 of Table 4 to solve transmission line problems.

In the formulas, $A+\frac{B^2}{2A}$ is used instead of $\sqrt{A^2+B^2}$. This approximation may be used for very accurate work, as it is correct within approximately $\frac{2}{100}$ of 1% when the regulation is not more than 20%.

TABLE 6—*Continued*

LINES CONNECTED IN PARALLEL

For n lines in parallel, find the constants α, β, γ and δ for each line, such that

$$E_s = E\alpha + I\beta,$$

and

$$I_s = E\gamma + I\delta.$$

For a uniform transmission line, without transformers,

$$\alpha = \delta = 1 + \frac{YZ}{2} + \frac{Y^2Z^2}{2\cdot 3\cdot 4} + \ldots$$

$$\beta = Z\left(1 + \frac{YZ}{2\cdot 3} + \frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5} + \ldots\right)$$

$$\gamma = \frac{Y\beta}{Z}.$$

If the line is not uniform, or if transformers are included in the circuit, α will in general not be equal to δ. For such cases, which represent circuits in series, use the method of Table 13.

Check on accuracy of numerical values.

$$\alpha\delta - \beta\gamma = 1.$$

Put

$$\epsilon = \frac{1}{\beta}.$$

Find the constants α_1, β_1, γ_1, δ_1, and ϵ_1 for the first line, and α_2, β_2, etc , for the second line, and so on. The combined circuit consisting of n lines in parallel is equivalent to a single transmission line having constants α_0, β_0, γ_0 and δ_0, where

$$\beta_0 = \frac{1}{\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n}.$$

$$\alpha_0 = (\alpha_1\epsilon_1 + \alpha_2\epsilon_2 + \ldots + \alpha_n\epsilon_n)\beta_0.$$

$$\gamma_0 = (\gamma_1 + \gamma_2 + \ldots + \gamma_n) - (\alpha_1\delta_1\epsilon_1 + \alpha_2\delta_2\epsilon_2 + \ldots + \alpha_n\delta_n\epsilon_n)$$
$$+ (\delta_1\epsilon_1 + \delta_2\epsilon_2 + \ldots + \delta_n\epsilon_n)\alpha_0.$$

$$\delta_0 = (\delta_1\epsilon_1 + \delta_2\epsilon_2 + \ldots + \delta_n\epsilon_n)\beta_0.$$

For a check on numerical values, note that

$$\alpha_0\delta_0 - \beta_0\gamma_0 = 1,$$

TABLE 6—*Continued*

or this equation may be used to give a short formula for γ_0, namely,

$$\gamma_0 = \frac{\alpha_0 \delta_0 - 1.}{\beta_0}$$

The voltages and currents for the combined circuit are given by

$$E_s = E\alpha_0 + I\beta_0 = A + jB,$$

and

$$I_s = E\gamma_0 + I\delta_0 = C + jD.$$

Problems relating to the combined circuit may now be solved by the full load formulas of Table 4.

TABLE 7.—CONVERGENT SERIES FOR TRANSMISSION LINES

Conditions Given at Supply End

The convergent series give the results of the fundamental formulas as accurately as desired, if a sufficient number of terms is used.

When conditions are given at the supply end, the same as with the K formulas, find the quantities:

Full Load

$$F+jH=E_s\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot3\cdot4}+\frac{Y^3Z^3}{2\cdot3\cdot4\cdot5\cdot6}+\text{etc.}\right)$$

$$-(P_s+jQ_s)Z\left(1+\frac{YZ}{2\cdot3}+\frac{Y^2Z^2}{2\cdot3\cdot4\cdot5}+\frac{Y^3Z^3}{2\cdot3\cdot4\cdot5\cdot6\cdot7}+\text{etc.}\right).$$

$$M+jN=(P_s+jQ_s)\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot3\cdot4}+\frac{Y^3Z^3}{2\cdot3\cdot4\cdot5\cdot6}+\text{etc.}\right)$$

$$-E_sY\left(1+\frac{YZ}{2\cdot3}+\frac{Y^2Z^2}{2\cdot3\cdot4\cdot5}+\frac{Y^3Z^3}{2\cdot3\cdot4\cdot5\cdot6\cdot7}+\text{etc.}\right).$$

No Load

$$F_0+jH_0=E_s(1-\tfrac{1}{2}YZ+\tfrac{5}{24}Y^2Z^2-\tfrac{61}{720}Y^3Z^3+\tfrac{277}{8064}Y^4Z^4-\text{etc.}),$$

$$M_0+jN_0=E_sY(1-\tfrac{1}{3}YZ+\tfrac{2}{15}Y^2Z^2-\tfrac{17}{315}Y^3Z^3+\tfrac{62}{2835}Y^4Z^4-\text{etc.}),$$

where $Z=(r+jx)l=$ impedance.
- $r=$ Resistance of conductor per mile.
- $x=$ Reactance of conductor per mile
- $l=$ Length of transmission line in miles.
- $Y=(g+jb)l=$ admittance.
- $g=$ Leakage conductance of conductor per mile.
- $b=$ Capacity susceptance of conductor per mile.

Use F, H, M, N, etc., with the equations for 3-phase circuits in formulas 1–23 of Table 5 to solve transmission line problems.

Note.—In the formulas, $F+\frac{H^2}{2F}$ is used instead of $\sqrt{F^2+H^2}$. This approximation may be used for very accurate work, as it is correct within approximately $\frac{2}{100}$ of 1% when the regulation is not more than 20%.

TABLE 8.—CONVERGENT SERIES FOR TRANSMISSION LINES

SUPPLY VOLTAGE AND RECEIVER LOAD GIVEN

Conditions given, for 3-phase circuit:

Kv-a. = Kv-a. at *receiver* end.

E_s = Full load voltage to neutral at *supply* end

$= \frac{1}{\sqrt{3}} \times$ voltage between conductors in 3-phase circuit.

$\cos\theta$ = Power factor at *receiver* end.

Kw. = Kv-a. $\cos\theta$.

$Z = R + jX = (r + jx)l$ = impedance.

r = resistance of conductor per mile.

x = reactance of conductor per mile.

l = length of transmission line in miles.

$Y = (g + jb)l$ = admittance.

g = leakage conductance of conductor per mile.

b = susceptance of conductor per mile, due to capacitance.

Then $PE = \frac{1000 \text{ Kv-a. } \cos\theta}{3}$ = Watts per phase at receiver end, where E is the unknown voltage to neutral at the receiver end.

$QE = \frac{1000 \text{ Kv-a } \sin\theta}{3}$ = reactive volt-amperes per phase at receiver end, when current is leading.

$QE = -\frac{1000 \text{ Kv-a. } \sin\theta}{3}$ when current is lagging.

PE and QE are known, but the separate quantities P, Q and E are not known.

Find the quantities:

E_c = the absolute value of $\dfrac{E_s}{\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot3\cdot4}+\ldots\right)}$

$$R_1 + jX_1 = \frac{(R+jX)\left(1+\frac{YZ}{2\cdot3}+\frac{Y^2Z^2}{2\cdot3\cdot4\cdot5}+\ldots\right)}{\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot3\cdot4}+\ldots\right)}.$$

$$L^2 = PER_1 - QEX_1,$$

and

$$M^2 = PEX_1 + QER_1.$$

Then $E^2 = \frac{1}{2}E_c^2 - L^2 + \frac{1}{2}\sqrt{E_c^4 - 4E_c^2L^2 - 4M^4}.$

TABLE 8—*Continued*

If the drop in the line is not more than 20%, the following series may be used:

$$E = E_c\left[1 - \frac{L^2}{E_c^2} - \frac{L^4}{E_c^4} - \frac{M^4}{2E_c^4} - \frac{2L^6}{E_c^6} - \frac{3}{2}\frac{L^2M^4}{E_c^6} - \frac{5L^8}{E_c^8} - \frac{5L^4M^4}{E_c^8} - \frac{5}{8}\frac{M^8}{E_c^8} - \cdots\right].$$

Now that E is known, P and Q can be found, and the quantities A, B, C and D can be calculated as in Table 6. Then the formulas given in the last part of Table 4 can be used to solve the various transmission line problems.

TABLE 9.—SHORT CONSTANT-VOLTAGE LINES*

Conditions Given at Receiver End

These formulas are exact when the line is short. When the line is 20 miles long, they are correct within approximately $\frac{1}{10}$ of 1 per cent of line voltage.

Conditions given, for 3-phase circuit:

E = Constant voltage to neutral, at receiver end

$= \frac{1}{\sqrt{3}} \times$ voltage between conductors in 3-phase circuit.

E_s = Constant voltage to neutral, at supply end.

R = Resistance of one conductor.

X = Reactance of one conductor.

P = In-phase current of the load, in amperes per conductor.

$\cos\theta$ = Power factor, lagging, of the load.

(1) Circle Diagram.

Describe a circle with center (a, b) and radius c, where

$$a = -\frac{3}{1000}\frac{E^2R}{R^2+X^2} \quad \ldots\ldots \quad \text{Kw.},$$

$$b = +\frac{3}{1000}\frac{E^2X}{R^2+X^2} \quad \ldots\ldots \quad \text{Kv-a.},$$

and

$$c = +\frac{3}{1000}\frac{EE_s}{\sqrt{R^2+X^2}} \quad \ldots\ldots \quad \text{Kv-a.}$$

The ordinates to the circle give the reactive Kv-a., $\frac{3EQ}{1000}$, in the line at the receiver end for a given Kw. load, $\frac{3EP}{1000}$. Positive values of $\frac{3EQ}{1000}$, plotted upward, represent leading Kv-a., and negative values, plotted downward, represent lagging Kv-a. See Fig 12, Chap. VII.

Draw a straight line at an angle θ below the base line, where $\cos\theta$ is the power factor, lagging, of the load By means of a pair of dividers, add the ordinates of the straight line to those of the circle, thus plotting the ellipse giving the Kv-a. of synchronous condensers required.

(2) Theoretical Limit of Load, in Kilowatts.

$$\text{Maximum load} = c + a \quad \ldots\ldots \quad \text{Kw.}$$

This is numerically less than c, since a is a negative quantity. It may be read from the circle diagram, as it is the farthest distance to the right reached by the circle.

* For Tables 9 to 13, see Tables I to IV, Constant-Voltage Transmission, by H B. Dwight, published by John Wiley & Sons, New York.

TABLE 9.—*Continued*

(3) Reactive Kv-a. in the Line at Receiver End.

For a more precise value than that obtained from the circle diagram, calculate $\frac{3EQ}{1000}$ by the equation

$$\left(b-\frac{3EQ}{1000}\right)^2=c^2-\left(\frac{3EP}{1000}-a\right)^2.$$

The quantity a is negative, so that $-a$ is positive

(4) Reactive Kv-a. of Synchronous Condensers.

This may be calculated by means of the formula

$$\frac{3EQ}{1000}+\frac{3EP}{1000}\frac{\sin\theta}{\cos\theta} \quad \ldots\ldots\ldots \quad \text{Kv-a.},$$

where the power load is $\frac{3EP}{1000}$ Kw. at a lagging power factor $\cos\theta$.

It is worth while checking the results of (3) and (4) by drawing the circle diagram and obtaining the same results graphically.

(5) Concentric Circles.

Since a and b, which give the center, are independent of the supply voltage E_s, and since the radius c is directly proportional to E_s, it is evident that a number of circles corresponding to different values of E_s may be drawn about the same center. See Fig 12, Chap. VII.

Now that Q is known for a given value of P, A and B and the various problems connected with the line can be calculated by the formulas of Table 1.

For single-phase and two-phase circuits and for the effect of transformers, see the paragraphs at the end of Table 1.

TABLE 10.—SHORT CONSTANT-VOLTAGE LINES

CONDITIONS GIVEN AT SUPPLY END

These formulas are exact when the line is short. When the line is 20 miles long, they are correct within approximately $\frac{1}{10}$ of 1 per cent of line voltage.

Conditions given, for 3-phase circuit:

E = Constant voltage to neutral at receiver end
$= \frac{1}{\sqrt{3}} \times$ voltage between conductors in 3-phase circuit.
E_s = Constant voltage to neutral at supply end.
R = Resistance of one conductor.
X = Reactance of one conductor.
P_s = Current at the supply end in phase with E_s, in amperes per conductor.

(1) Circle Diagram.

Describe a circle with center (a, b) and radius c, where

$$a = +\frac{3}{1000}\frac{E_s^2 R}{R^2+X^2} \quad \ldots\ldots\ldots \quad \text{Kw.},$$

$$b = -\frac{3}{1000}\frac{E_s^2 X}{R^2+X^2} \quad \ldots\ldots\ldots \quad \text{Kv-a},$$

and

$$c = +\frac{3}{1000}\frac{E E_s}{\sqrt{R^2+X^2}} \quad \ldots\ldots\ldots \quad \text{Kv-a.}$$

The ordinates to this circle represent the reactive Kv-a., $\frac{3E_sQ_s}{1000}$, in the line at the supply end.

(2) Concentric Circles.

Since a and b, which give the center, are independent of the receiver voltage E, and since the radius c is directly proportional to E, it is evident that a number of circles corresponding to different values of E may be drawn about the same center.

(3) Theoretical Limit of the Load in Kilowatts at the Supply End of the Line.

The maximum load is:

$$c+a \quad \ldots\ldots\ldots\ldots \quad \text{Kw.}$$

This is numerically greater than c since a is positive. It may be read from the circle diagram, as it is the farthest distance to the right reached by the circle.

TABLE 10.—*Continued*

(4) Calculated Value of Reactive Kv-a. in the Line at Supply End.

The value of the line reactive Kv-a., $\frac{3E_sQ_s}{1000}$, for a given power load at the supply end, $\frac{3E_sP_s}{1000}$, may be found from the circle diagram described in 1, or it may be calculated precisely by means of the equation:

$$\left(b-\frac{3E_sQ_s}{1000}\right)^2=c^2-\left(\frac{3E_sP_s}{1000}-a\right)^2.$$

When Q_s, as found above, is positive, it represents a leading load on the generators, and when it is negative, a lagging load

Now that Q_s is known for a given value of P_s, the various problems such as power factor at receiver end, efficiency, etc., can be solved by means of the formulas in Table 2.

For single-phase and two-phase circuits and for the effect of transformers, see the paragraphs at the end of Table 1.

TABLE 11.—LONG CONSTANT-VOLTAGE LINES

Conditions Given at Receiver End

These formulas give the results of the fundamental hyperbolic formulas as accurately as desired, if a sufficient number of terms of the convergent series is used.

Conditions given, for 3-phase circuit:

E = Constant voltage to neutral, at receiver end.
E_s = Constant voltage to neutral, at supply end.
$Z = R + jX$ = Impedance of one conductor.
R = Resistance of one conductor
X = Reactance of one conductor.
$Y = (g + jb)l$ = Admittance of one conductor.
g = Leakage conductance of one conductor, per mile
b = Capacity susceptance of one conductor, per mile.
l = Length of line in miles.
P = In-phase current of the load, in amperes per conductor.
$\cos\theta$ = Power factor, lagging, of the load.

Find

$$E' + jE'' = E\left(1 + \frac{YZ}{2} + \frac{Y^2Z^2}{2\ 3\ 4} + \ldots\right),$$

and

$$R' + jX' = (R + jX)\left(1 + \frac{YZ}{2\cdot 3} + \frac{Y^2Z^2}{2\cdot 3\ 4\cdot 5} + \ldots\right).$$

(1) Circle Diagram.

Describe a circle with center (a', b') and radius c', where

$$a' = -\frac{3E}{1000}\frac{E'R' + E''X'}{R'^2 + X'^2} \quad \ldots\ldots \quad \text{Kw},$$

$$b' = +\frac{3E}{1000}\frac{E'X' - E''R'}{R'^2 + X'^2} \quad \ldots\ldots \quad \text{Kv-a.}$$

$$c' = +\frac{3E}{1000}\frac{E_s}{\sqrt{R'^2 + X'^2}} \quad \ldots\ldots \quad \text{Kv-a.}$$

The ordinates to the circle give the reactive Kv-a., $\frac{3EQ}{1000}$, in the line at the receiver end, for a given Kw. load, $\frac{3EP}{1000}$. Positive values of $\frac{3EQ}{1000}$, plotted upward, represent leading Kv-a., and negative values, plotted downward, represent lagging Kv-a. See Fig. 12, Chap. VII.

TABLE 11.—*Continued*

Draw a straight line at an angle θ below the base line, where cos θ is the power factor, lagging, of the load. By means of a pair of dividers, add the ordinates of the straight line to those of the circle, thus plotting the ellipse giving the Kv-a. of synchronous condensers required.

If the power factor is not the same at all loads, the line representing the load will not be straight but will be a curve showing the reactive Kv-a. of the load from no load to full load, and the curve of Kv-a. of synchronous condensers will not be an exact ellipse.

(2) Theoretical Limit of Load, in Kilowatts.

$$\text{Maximum Load} = c' + a' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{Kw.}$$

This is numerically less than c' since a' is a negative quantity. It may be read from the circle diagram as it is the farthest distance to the right reached by the circle.

(3) Reactive Kv-a. in the Line at Receiver End.

For a more precise value than that obtained from the circle diagram, calculate $\frac{3EQ}{1000}$ by the equation

$$\left(b' - \frac{3EQ}{1000}\right)^2 = c'^2 - \left(\frac{3EP}{1000} - a'\right)^2.$$

The quantity a' is negative, so that $-a'$ is positive.

(4) Reactive Kv-a. of Synchronous Condensers.

This may be calculated by means of the formula:

$$\frac{3EQ}{1000} + \frac{3EP}{1000}\frac{\sin\theta}{\cos\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{Kv-a.;}$$

where the power load is $\frac{3EP}{1000}$ Kw. at a lagging power factor cos θ.

It is worth while checking the results of (3) and (4) by drawing the circle diagram and obtaining the same results graphically.

(5) Concentric Circles.

Since a' and b', which give the center, are independent of the supply voltage E_s, and since the radius c' is directly proportional to E_s, it is evident that a number of circles corresponding to different values of E_s may be drawn about the same center. See Fig. 12, Chap. VII.

TABLE 11.—*Continued*

Now that Q is known for a given value of P, one can find A, B, C and D by the equations of Table 6, and the various transmission line problems can be calculated by the formulas in the latter part of Table 4. If it is desired to include the transformers with the line, use Table 13.

For drawing the circle diagram of two or more lines or circuits in parallel, find the constants α_0 and β_0 of the combined circuit, as described in Table 6. Then, since

$$E_s = E\alpha_0 + I\beta_0$$

we have

$$E' + jE'' = E\alpha_0,$$

and

$$R' + jX' = \beta_0.$$

The circle diagram for the combined circuit can now be drawn, or the values calculated, as described in this table.

TABLE 12 —LONG CONSTANT-VOLTAGE LINES

Conditions Given at Supply End

These formulas give the results of the fundamental hyperbolic formulas as accurately as desired, if a sufficient number of terms of the convergent series is used.

Conditions given, for 3-phase circuit:

E = Constant voltage to neutral, at receiver end.
E_s = Constant voltage to neutral, at supply end.
$Z = R + jX$ = Impedance of one conductor.
R = Resistance of one conductor.
X = Reactance of one conductor.
$Y = (g + jb)l$ = Admittance of one conductor.
g = Leakage conductance of one conductor, per mile.
b = Capacity susceptance of one conductor, per mile.
l = Length of line in miles.
P_s = Current in the line, in amperes per conductor, at the supply end, in phase with E_s.

Find

$$E'_s + jE''_s = E_s\left(1 + \frac{YZ}{2} + \frac{Y^2Z^2}{2\cdot 3\cdot 4} + \ldots\right),$$

and

$$R' + jX' = (R + jX)\left(1 + \frac{YZ}{2\cdot 3} + \frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5} + \ldots\right).$$

(1) Circle Diagram.

Describe a circle with center (a', b') and radius c', where

$$a' = +\frac{3E_s}{1000}\,\frac{E'_sR' + E''_sX'}{R'^2 + X'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{Kw.,}$$

$$b' = -\frac{3E_s}{1000}\,\frac{E'_sX' - E''_sR'}{R'^2 + X'^2} \quad . \quad . \quad . \quad . \quad . \quad , \quad \text{Kv-a.,}$$

and

$$c' = +\frac{3E_s}{1000}\,\frac{E}{\sqrt{R'^2 + X'^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{Kv-a.}$$

The ordinates to the circle give the reactive Kv-a., $\frac{3E_sQ_s}{1000}$, in the line at the supply end, for a given value of kilowatts, $\frac{3E_sP_s}{1000}$ in the line at the supply end.

(2) Theoretical Limit of the Load in Kilowatts at the Supply End of the Line.

$$\text{Maximum Load} = c' + a' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{Kw.}$$

TABLE 12.—*Continued*

This is numerically greater than c' since a' is positive It may be read from the circle diagram as it is the farthest distance to the right reached by the circle.

(3) Reactive Kv-a. in the Line at Supply End.

For a more precise value than that obtained from the circle diagram, calculate $\frac{3E_sQ_s}{1000}$ by the equation:

$$\left(b' - \frac{3E_sQ_s}{1000}\right)^2 = c'^2 - \left(\frac{3E_sP_s}{1000} - a'\right)^2.$$

The quantity a' is positive, so that $-a'$ is negative.

(4) Concentric Circles.

Since a' and b', which give the center, are independent of the receiver voltage E_s, and since the radius c' is directly proportional to E, it is evident that a number of circles corresponding to different values of E may be drawn about the same center.

Now that Q_s is known for a given value of P_s, one can find F, H, M and N by the equations of Table 7, and the various transmission line problems can be calculated by the formulas in the latter part of Table 5.

TABLE 13.—CONSTANT-VOLTAGE LINES WITH TRANSFORMERS

Conditions given, for 3-phase circuit:

All quantities referred to high tension.

E = equivalent high-tension voltage to neutral, held constant at the low-tension side of the receiving transformers. See Fig. 11, p. 71.

$3EP_c$ = average value of loss in the synchronous condensers.

$G_{tr}+jB_{tr}$ = Admittance corresponding to the core loss and magnetizing current of the receiving transformers at the average operating high-tension voltage, for equivalent star-connected transformers.

$G_{ts}+jB_{ts}$ = Similar quantity for the supply transformers.

$R_{tr}+jX_{tr}$ = Impedance of receiving transformers, referred to high tension.

$R_{ts}+jX_{ts}$ = Similar quantity for the supply transformers.

$Y=(g+jb)l$ = Admittance of the transmission line.

$Z=(r+jx)l$ = Impedance of the transmission line.

Find numerical values for the following quantities, except that the letters P and Q will appear in all expressions:

Current in secondary of receiving transformers:

(1) $I_a=P+jQ+P_c$ amperes per conductor.

Voltage induced in receiving transformers:

(2) $E_a=E+\frac{1}{2}I_a(R_{tr}+jX_{tr})$ volts to neutral.

Current in primary of receiving transformers:

(3) $I_b=I_a+E_a(G_{tr}+jB_{tr})$.

Voltage at receiving end of transmission line.

(4) $E_b=E_a+\frac{1}{2}I_b(R_{tr}+jX_{tr})$.

Voltage at supply end of transmission line:

$$(5)\quad E_c=E_b\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\dots\right)+I_bZ\left(1+\frac{YZ}{2\cdot 3}+\frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5}+\dots\right).$$

Current at supply end of transmission line:

$$(6)\quad I_c=I_b\left(1+\frac{YZ}{2}+\frac{Y^2Z^2}{2\cdot 3\cdot 4}+\dots\right)+E_bY\left(1+\frac{YZ}{2\cdot 3}+\frac{Y^2Z^2}{2\cdot 3\cdot 4\cdot 5}+\dots\right).$$

Equations for voltage E_{b1} and current I_{b1}, etc., at an intermediate point or points where the line characteristics change, and where there is no load nor any synchronous condenser, are of the same form as the two preceding equations.

TABLE 13—*Continued*

(7) Current in secondary of supply transformers $= I_c$.

Voltage induced in supply transformers:

(8) $E_d = E_c + \frac{1}{2} I_c (R_{ts} + jX_{ts})$.

Current in primary of supply transformers:

(9) $I_d = I_c + E_d (G_{ts} + jB_{ts})$.

Voltage at generator terminals:

(10) $E_s = E_d + \frac{1}{2} I_d (R_{ts} + jX_{ts}) = E' + jE'' + (P + jQ)(R' + jX')$.

Current at generator terminals:

(11) $C + jD = I_d$.

The instructions for drawing the circle diagram, and the remaining formulas, are the same as in Table 11.

TABLE 14.—RESISTANCE OF COPPER WIRE AND CABLE

Data assumed:

Temperature, 20° C. (68° F.)

Conductivity of hard drawn copper, 97 3% of the annealed copper standard.

Increase of resistance and weight of cables due to spiraling, 2%.

COPPER WIRE

B & S Gauge	Circular Mils	Diameter (2ρ) Inch	Resistance, Ohms per Mile				
			Direct Current	25 cycles	Increase	60 cycles	Increase
					Per cent		Per cent
0000	211,600	4600	2600	2662	08	2671	43
000	167,800	4096	3354	3355	05	.3363	.27
00	133,100	3648	4229	4230	03	.4236	.17
0	105,500	3249	5333	5334	02	.5338	.11
1	83,690	2893	.6725	6725	.01	.6729	.07
2	66,370	2576	8480	8480	01	.8483	04
3	52,630	2294	1 069	1 069	.	1 070	.03
4	41,740	.2043	1 348	1 348	.	1.348	.02
5	33,100	1819	1 700	1 700	.....	1.700	01
6	26,250	1620	2 144	2 144	......	2.144	01
7	20,820	.1443	2 703	2 703	....	2.703	
8	16,510	1285	3 409	3 409	..	3.409	

WEIGHT OF CONDUCTORS

Copper Wire		Copper Cable		
B. & S. gauge	Weight, Pounds per mile	B & S. gauge	Circular Mils	Weight, pounds per mile
0000	3380	..	700,000	11,400
000	2680	..	600,000	9,780
00	2130	...	500,000	8,150
0	1690	.....	450,000	7,340
1	1340	......	400,000	6,520
2	1060	......	350,000	5,710
3	841	.....	300,000	4,890
4	667	......	250,000	4,080
5	529	0000	....	3,450
6	420	000	....	2,740
7	333	00	......	2,170
8	264	0	......	1,720
		1	....	1,360
		2	....	1,080
		3	......	858
		4		681

TABLE 14—*Continued*

COPPER CABLE

B & S Gauge	Circular Mils	Diameter (2ρ) Inches	No of Wires Assumed	Resistance, Ohms per Mile				
				Direct current	25 cycles	Increase	60 cycles	Increase
						Per cent		Per cent
	700,000	9641	61	0820	0826	76	0856	4 42
	600,000	8914	37	0957	0962	56	0988	3 27
	500,000	8137	37	1148	1153	39	1174	2 29
	450,000	7720	37	1276	1280	.32	.1299	1 86
	400,000	7255	19	1435	1439	26	1456	1 47
	350,000	6786	19	1640	1643	20	1659	1 13
	300,000	.6283	19	1913	1916	15	1929	83
	250,000	5735	19	2296	2298	10	.2309	58
0000	211,600	5216	7	2713	2715	07	2724	.42
000	167,800	.4645	7	3421	3422	05	.3430	26
00	133,100	4137	7	4313	4315	03	.4321	17
0	105,500	3683	7	5439	5440	02	.5445	11
1	83,690	3280	7	6859	6860	01	6864	07
2	66,370	.2921	7	8649	8650	.01	8653	04
3	52,630	2601	7	1 091	1 091	. . .	1 091	03
4	41,740	.2317	7	1 375	1 375	. ..	1 375	.02

TEMPERATURE COEFFICIENTS OF COPPER

For different initial temperatures (centigrade) and different conductivities

Ohms per Meter-Gram at 20 Deg. Cent.	Per Cent Conductivity	α_0	α_{15}	α_{20}	α_{25}	α_{30}	α_{50}
.16134	95	00403	.00380	.00373	00367	00360	00336
.15966	96	.00408	.00385	00377	00370	.00364	00339
.15802	97	00413	.00389	00381	00374	00367	.00342
.15753	97.3	.00414	.00390	00382	00375	00368	00343
.15640	98	00417	.00393	00385	00378	00371	00345
.15482	99	00422	.00397	.00389	00382	00374	.00348
15328	100	00427	.00401	.00393	00385	00378	.00352
.15176	101	00431	00405	.00397	.00389	00382	.00355

$$R_t = R_{t1}(1+\alpha_{t1}[t-t_1]),$$

where R_t is the resistance at any temperature t deg. cent and R_{t1} is the resistance at any "initial temperature" t_1 deg. cent.

From Table II, Circular No. 31, of the Bureau of Standards, 1914.

TABLE 15.—RESISTANCE OF ALUMINUM CABLE, STEEL REINFORCED

Data assumed

Temperature, 20° C (68° F)
Conductivity of hard drawn aluminum, 61% of the annealed copper standard
Values are from the table of the Aluminum Company of America.

B & S Gauge	Circular Mils	Diameter (2ρ), Inches	Number of Wires		Resistance, Ohms per Mile	Weight, Pounds per Mile		
			Aluminum	Steel		Aluminum	Steel	Total
	900,000	1 162	54	7	1019	4462	1658	6120
	795,000	1 093	54	7	1146	3944	1463	5407
	715,500	1 036	54	7	.1272	3542	1315	4857
	605,000	953	54	7	.1510	2999	1114	4113
	500,000	.904	30	7	.1832	2477	1658	4135
	397,500	806	30	7	2297	1969	1315	3284
	336,400	.741	30	7	.2719	1669	1114	2783
	266,800	.633	6	7	.3422	1319	492	1811
0000	211,600	564	6	1	4309	1052	504	1556
000	167,800	501	6	1	.5417	830	397	1227
00	133,100	447	6	1	.6832	660	317	977
0	105,500	398	6	1	8653	525	251	776
1	83,690	355	6	1	1 093	417	200	617
2	66,370	316	6	1	1 378	330	158	488
3	52,630	.281	6	1	1 738	262	125	387
4	41,740	.250	6	1	2 191	207	99	306

Temperature coefficient of aluminum at 20° C, $\alpha_{20} =$ 0039

TABLE 16.—REACTANCE OF WIRE, 25 CYCLES

$x = 2\pi f \times 741.1 \log_{10} \frac{2\ 568s}{d} \times 10^{-6}$ ohms per mile

d = diameter of wire. s = spacing, measured in the same units as d.

For three-phase irregular spacing use $s = \sqrt[3]{abc}$. For three-phase regular flat spacing use $s = 1\ 26\ a$.

For a two-phase line the spacing is the mean distance between centers of conductors of the same phase.

B & S Gauge	No 0000	No 000	No 00	No 0	No 1	No. 2	No. 3	No. 4	No 5	No 6	No 7
Circular Mils	211,600	167,800	133,100	105,500	83,690	66,370	52,630	41,740	33,100	26,250	20,820
Diameter, Inches	.4600	.4096	.3648	.3249	.2893	.2576	2294	2043	1819	1620	1443
Spacing, Feet											
1	.213	.219	224	230	.236	242	248	.254	.259	265	271
1.5	.233	.239	245	.251	256	262	268	274	.280	286	.292
2	.248	.254	.259	265	271	277	283	289	295	300	306
2.5	.259	.265	271	276	282	288	294	300	306	312	.318
3	.268	.274	.280	.286	.292	297	303	309	.315	321	327
3.5	.276	.282	.288	.293	.299	.305	311	317	.323	329	335
4	.283	289	294	300	306	312	318	324	330	335	341
4.5	.289	.295	.300	.306	.312	318	324	330	335	341	347
5	.294	.300	306	.311	.317	.323	329	335	341	.347	.353
6	.303	.309	.315	321	.327	333	338	344	350	356	362
7	.311	.317	.323	.329	.334	340	.346	352	358	364	370
8	.318	.324	.330	.336	.341	347	353	359	365	370	.376
9	.324	.330	.335	341	.347	353	359	365	370	376	382
10	.329	.335	.341	347	352	358	364	.370	376	382	388
11	334	340	.346	351	357	.363	369	.375	381	387	.392
12	.338	.344	.350	356	362	368	374	379	385	.391	397
13	.342	.348	.354	360	366	372	378	383	389	395	.401
14	346	.352	358	364	.369	375	381	387	393	399	405
15	350	.356	361	.367	.373	379	385	391	396	402	408
16	.353	.359	365	.370	.376	.382	388	394	400	.406	411
18	.359	365	371	.376	382	.388	.394	.400	406	411	417
20	364	.370	.376	382	388	.394	.399	.405	411	.417	.422

For 50 cycles, multiply the above values by 2.

TABLE 17A.—REACTANCE OF LARGE COPPER CABLES, 25 CYCLES

Values in table are in ohms per mile

For formulas for reactance of cables see Chapter XVIII.

Circular mils.........	700,000	600,000	500,000	450,000	400,000	350,000	300,000	250,000
Number of wires.....	61	37	37	37	19	19	19	19
Diameter, inches ..	.9641	8914	8137	.7720	.7255	6786	.6283	.5735
Spacing, Feet								
1	.176	.180	.184	.187	.191	.194	.198	203
1.5	.196	.200	.205	.208	.211	.215	219	223
2	.211	.215	219	.222	.226	.229	233	238
2 5	.222	.226	.231	.233	.237	.241	.245	249
3	.231	.235	.240	.243	.246	.250	.254	258
3 5	239	243	248	.251	.254	.258	.262	266
4	.246	.250	.254	.257	.261	.264	.268	273
4 5	.252	.256	.260	.263	.267	.270	274	279
5	.257	.261	.266	.268	.272	276	.279	284
6	.266	.270	.275	.278	.281	.285	.289	293
7	.274	.278	.283	.285	.289	.293	.297	301
8	.281	.283	.290	.292	.296	.209	.303	.308
9	.287	.291	.296	298	302	305	309	.314
10	.292	.296	.301	.304	307	311	.315	.319
11	.297	.301	.306	.308	.312	.316	.319	.321
12	.301	.305	.310	.313	.317	.320	.324	.328
13	.305	.309	.314	.317	.321	.324	.328	.332
14	.309	.313	318	.320	.325	.328	332	.336
15	.313	.317	321	.324	.328	331	335	340
16	.316	.320	325	.327	.331	.335	.338	.343
18	.322	.326	.331	.333	.337	.341	.344	.349
20	.327	.331	.336	.339	.342	346	.350	.354
25	.338	.343	.348	.350	.354	.357	.361	.366
30	.318	.352	.356	.359	.363	.366	370	.375

For 50 cycles, multiply the above values by 2.

TABLE 17*B*.—REACTANCE OF LARGE ALUMINUM CABLES, STEEL REINFORCED, 25 CYCLES

Values in table are in ohms per mile.

Circular mils . .	900,000	795,000	715,500	605,000	500,000	397,500	336,400	266,800	Spacing, Feet
Number of wires .	61	61	61	61	37	37	37	13	
Diameter, inches . .	1 162	1 093	1 036	953	904	806	741	633	
Spacing, Feet									
1	166	.169	.172	.176	.179	185	189	200	1
1 5	.187	.190	.193	.197	.200	206	210	.220	1 5
2	201	204	.207	211	214	220	.224	235	2
2 5	.213	.216	.219	.223	.226	231	235	246	2 5
3	.222	225	.228	.232	235	241	.245	.255	3
3 5	.230	233	.236	240	243	248	.252	.263	3 5
4	.237	.240	242	246	249	255	259	270	4
4 5	243	246	.248	.252	.255	261	265	.276	4 5
5	248	251	.254	.258	.261	266	271	281	5
6	.257	260	263	.267	.270	276	280	291	6
7	.265	268	271	.275	278	284	288	298	7
8	272	275	.277	.281	.284	290	294	305	8
9	.278	.281	283	.287	290	296	.300	311	9
10	283	286	289	293	296	302	.306	316	10
11	.288	291	.294	298	301	306	311	321	11
12	292	.295	.298	.302	305	311	315	326	12
13	296	299	.302	.306	309	315	319	330	13
14	300	303	.306	.310	.313	.319	323	334	14
15	.303	306	309	313	316	322	326	337	15
16	306	.309	.312	.316	319	325	329	340	16
18	312	.315	.318	.322	.325	331	335	346	18
20	318	.321	324	328	331	337	341	351	20
25	329	332	.335	339	342	348	352	.363	25
30	338	341	.344	.348	351	357	361	372	30

For 50 cycles, multiply the above values by 2

TABLE 17C.—REACTANCE OF CABLE, 25 CYCLES

Sizes for Both Aluminum Cables, Steel Reinforced, and Copper Cables

For diameters see Resistance Tables 14 and 15.

Values in table are in ohms per mile.

B. & S Gauge	No. 0000	No 000	No 00	No 0	No 1	No 2	No 3	No 4	Spacing, Feet
Circular mils	211,600	167,800	133,100	105,500	83,690	66,370	52,630	41,740	
Number of wires	7	7	7	7	7	7	7	7	
Spacing, Feet									
1	.210	215	221	227	233	239	245	.251	1
1.5	230	.236	242	248	254	259	265	271	1.5
2	.245	.251	256	.262	268	274	280	286	2
2.5	.256	.262	268	274	279	285	291	297	2.5
3	265	.271	277	.283	289	294	300	306	3
3.5	273	279	.285	.290	296	302	308	314	3.5
4	280	286	291	297	303	.309	.315	321	4
4.5	286	292	297	303	309	315	321	.327	4.5
5	291	.297	303	309	315	320	326	332	5
6	300	.306	312	318	324	330	335	341	6
7	308	.314	.320	.326	331	337	343	349	7
8	315	.321	.326	332	338	344	350	356	8
9	321	.326	332	338	344	350	.356	362	9
10	326	.332	.338	344	350	355	.361	367	10
11	331	337	343	348	354	360	366	372	11
12	335	.341	347	353	359	365	371	376	12
13	339	.345	.351	357	363	369	375	380	13
14	343	.349	355	.361	367	372	378	381	14
15	347	352	358	364	370	376	.382	388	15
16	350	356	361	367	.373	.379	.385	391	16
18	356	.362	367	.373	379	385	391	397	18
20	361	.367	373	379	385	390	396	.402	20

For 50 cycles, multiply the above values by 2

TABLE 18.—REACTANCE OF WIRE, 60 CYCLES

$x = 2\pi f \times 741.1 \log_{10} \frac{2\ 568s}{d} \times 10^{-6}$ ohms per mile.

d = diameter of wire. s = spacing, measured in the same units as d.

For three-phase irregular spacing use $s = \sqrt[3]{abc}$. For three-phase regular flat spacing use $s = 1.26\ a$.

For a two-phase line the spacing is the mean distance between centers of conductors of the same phase

B & S Gauge	No 0000	No 000	No 00	No 0	No 1	No 2	No 3	No 4	No 5	No 6	No 7	No 8	
Circular mils	211,600	167,800	133,100	105,500	83,690	66,370	52,630	41,740	33,100	26,250	20,820	16,510	Spacing, Feet
Diameter, inches....	.4600	4096	.3648	.3249	2893	.2576	2294	.2043	1819	1620	1443	.1285	
Spacing, Feet													
1	.510	.524	.539	.553	567	581	595	609	623	637	651	665	1
1 5	560	.574	.588	.602	616	630	644	658	672	686	700	714	1 5
2	.594	.608	.623	.637	651	665	679	693	707	721	735	749	2
2.5	.621	636	.650	.664	678	692	.706	720	734	748	762	776	2 5
3	.644	658	.672	686	700	714	.728	.742	756	770	784	798	3
3 5	.662	.676	.690	.705	719	733	747	761	.775	789	803	817	3 5
4	.678	693	.707	.721	735	749	763	777	.791	805	819	833	4
4 5	.693	.707	.721	.735	749	763	777	791	805	819	.833	847	4 5
5	.706	720	.734	.748	762	776	.790	804	818	832	.846	860	5
6	.728	742	.756	.770	784	.798	812	826	840	854	868	882	6
7	.747	761	775	.789	803	817	831	845	859	873	887	901	7
8	.763	.777	.791	.805	819	.833	847	.861	875	889	903	917	8
9	.777	.791	805	.819	833	847	861	.875	.889	903	918	932	9
10	.790	.804	.818	.832	.846	860	.874	.888	.902	916	.930	944	10
11	.801	816	.830	844	858	872	886	900	914	928	.942	956	11
12	812	826	.840	.854	868	882	896	910	924	938	.952	967	12
13	822	836	.850	864	.878	892	906	920	934	948	962	976	13
14	.831	845	859	873	887	901	915	929	943	957	971	985	14
15	839	853	867	881	895	909	923	937	951	965	980	994	15
16	.847	861	875	.889	903	917	931	945	959	973	987	1 001	16
18	.861	.875	889	903	917	931	946	960	973	987	1 002	1.016	18
20	874	888	902	916	930	944	958	972	986	1 000	1 014	1 029	20

TABLE 19A.—REACTANCE OF LARGE COPPER CABLES, 60 CYCLES

Values in table are in ohms per mile.

For formulas for reactance of cables, see Chapter XVIII

Circular mils....	700,000	600,000	500,000	450,000	400,000	350,000	300,000	250,000	Spacing, Feet
Number of wires....	61	37	37	37	19	19	19	19	
Diameter, inches.....	.9641	8914	8137	7720	7255	6786	6283	5735	
Spacing, Feet									
1	.421	.432	.443	.449	.458	466	476	187	1
1 5	.471	.481	.492	.498	507	.515	.525	536	1 5
2	506	516	.527	533	.542	550	560	571	2
2.5	.533	.543	.554	560	569	577	.587	598	2 5
3	.555	.565	576	582	.591	599	609	620	3
3 5	.573	.584	.595	601	610	618	628	639	3 5
4	.590	.600	.611	617	626	634	644	655	4
4.5	.604	.614	.625	.632	.641	.649	.658	.669	4 5
5	.617	.627	.638	644	653	.662	.671	682	5
6	.639	.649	.660	666	676	684	693	704	6
7	.658	668	679	685	694	702	712	723	7
8	.674	.684	695	.702	.710	719	728	739	8
9	.688	.698	.709	.716	.725	733	712	753	9
10	.701	711	.722	728	.738	746	.755	766	10
11	712	723	.734	740	749	757	767	778	11
12	.723	.733	.744	750	.760	768	777	788	12
13	.733	.743	.754	.760	769	778	787	798	13
14	.742	.752	763	.769	778	786	796	807	14
15	.750	.760	.771	.778	.787	795	801	815	15
16	.758	.768	.779	.786	.795	803	.812	823	16
18	.772	.782	793	.800	809	817	826	838	18
20	.785	.795	.808	812	822	830	839	850	20
25	.812	.822	.833	840	849	857	866	877	25
30	.834	.844	.855	862	871	879	.888	.900	30

TABLE 19*B*.—REACTANCE OF LARGE ALUMINUM CABLES, STEEL REINFORCED, 60 CYCLES

Values in table are in ohms per mile.

Circular mils	900,000	795,000	715,500	605,000	500,000	397,500	336,400	266,800	Spacing, Feet
Number of wires.....	61	61	61	61	37	37	37	13	
Diameter, inches....	1.162	1.093	1.036	.953	.904	.806	741	633	
Spacing, Feet									
1	.400	.407	.413	.423	.430	444	.454	.480	1
1 5	.449	.456	.462	.472	.479	493	.503	.529	1 5
2	.484	.491	497	.507	.514	.528	.538	564	2
2 5	.511	.518	.524	.534	.541	.555	565	591	2 5
3	.533	.540	546	.556	.563	.577	.588	614	3
3 5	.552	.559	.565	.575	.582	.596	606	632	3 5
4	.568	.575	582	.592	.598	.612	622	648	4
4 5	.582	589	595	.605	.612	.626	.636	663	4 5
5	.595	.602	.608	618	625	.639	.650	676	5
6	.617	.624	630	640	648	662	671	697	6
7	636	.643	.649	659	666	680	690	716	7
8	653	.660	.666	.676	683	696	706	732	8
9	667	674	680	.690	.697	.710	720	.746	9
10	679	686	693	703	.710	.724	733	.760	10
11	691	698	704	714	722	735	.745	771	11
12	.701	.708	715	725	732	745	.756	.782	12
13	711	.718	.724	.734	742	755	765	791	13
14	720	727	733	743	750	764	774	800	14
15	729	.736	.742	.752	759	773	782	809	15
16	737	744	750	.760	.767	780	.790	816	16
18	.751	758	764	774	781	.795	804	831	18
20	.763	.770	777	787	794	.808	817	844	20
25	.790	.797	.804	813	821	835	.844	.870	25
30	.812	.820	.826	.836	843	856	.867	.892	30

TABLE 19C.—REACTANCE OF CABLE, 60 CYCLES

SIZES FOR BOTH ALUMINUM CABLES, STEEL REINFORCED, AND COPPER CABLES

For diameters see Resistance Tables 14 and 15

Values in table are in ohms per mile

B & S Gauge.....	No 0000	No 000	No 00	No 0	No 1	No 2	No 3	No 4	Spacing, Feet
Circular mils ...	211,600	167,800	133,100	105,500	83,690	66,370	52,630	41,740	
Number of wires	7	7	7	7	7	7	7	7	
Spacing, Feet									
1	.503	.517	.531	.545	.560	.574	588	.602	1
1 5	.552	.566	580	595	609	623	637	651	1 5
2	.587	601	.615	.629	644	658	672	686	2
2 5	.614	.628	.642	656	671	685	699	713	2 5
3	636	651	664	678	693	707	.721	735	3
3 5	.655	669	683	697	711	725	739	754	3 5
4	.671	.685	.699	.714	.728	.742	756	770	4
4 5	.686	.700	.714	.728	742	756	770	784	4 5
5	.698	712	727	741	755	769	783	797	5
6	.721	734	.749	763	777	791	805	819	6
7	739	.753	767	781	796	810	824	838	7
8	755	770	.784	798	812	826	840	851	8
9	770	.784	.798	812	826	840	854	868	9
10	.782	.797	811	825	839	853	867	881	10
11	.794	.808	.822	836	850	865	878	893	11
12	.805	819	.833	847	861	875	889	903	12
13	814	828	.842	857	871	885	899	913	13
14	.823	837	.851	866	880	891	908	922	14
15	832	.846	860	.874	.888	.902	916	930	15
16	.839	.854	868	.882	898	910	924	938	16
18	.854	.868	882	896	910	924	938	952	18
20	.867	881	895	.909	923	937	951	965	20

TABLE 20.—CAPACITY SUSCEPTANCE OF WIRE, 25 CYCLES

$$b\ (\text{approx}) = \frac{2\pi f \times 38\ 83 \times 10^{-9}}{\log_{10}\left(\frac{s}{\rho} - \frac{\rho}{s}\right)} \text{ mhos per mile.}$$

ρ = radius of wire.
s = spacing
For three-phase irregular spacing use $s = \sqrt[3]{abc}$
For three-phase regular flat spacing use $s = 1\ 26\ a$

For a two-phase line the spacing is the average distance between centers of conductors of the same phase

B. & S Gauge	No 0000	No. 000	No. 00	No 0	No 1	No 2	No 3	No 4	No 5	No 6	No 7	No 8	
Circular mils	211,600	167,800	133,100	105,500	83,690	66,370	52,630	41,740	33,100	26,250	20,820	16,510	Spacing, Feet
Diameter, (2ρ) inches....	.4600	.4096	.3648	3249	2893	.2576	2294	2043	1819	1620.	1443	.1285	
Spacing, Feet	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	
1	3 55	3 45	3 35	3 26	3 18	3 10	3 02	2 95	2 88	2 81	2 75	2 69	1
1 5	3 22	3 14	3 06	2 98	2 91	2 84	2 78	2 72	2 66	2 60	2 54	2 49	1 5
2	3 02	2 95	2 88	2 81	2 75	2 69	2 63	2 57	2 52	2 47	2 42	2 37	2
2 5	2 88	2 82	2 75	2 69	2 63	2 58	2 52	2 47	2 42	2 38	2 33	2 29	2 5
3	2 78	2 72	2 66	2 60	2 55	2 49	2 44	2 40	2 35	2 30	2 26	2 22	3
3 5	2 70	2.64	2 58	2 53	2 48	2 43	2 38	2 33	2 29	2 25	2 21	2 17	3 5
4	2 63	2 57	2 52	2 47	2 42	2 37	2 33	2 28	2 24	2 20	2 16	2 12	4
4 5	2 57	2 52	2 47	2 42	2 37	2 33	2 28	2.24	2 20	2 16	2 12	2 09	4 5
5	2 52	2 47	2 42	2 38	2 33	2 29	2 24	2 20	2 16	2 13	2 09	2 05	5
6	2 44	2 40	2 35	2 30	2 26	2 22	2 18	2 15	2 10	2 07	2 03	2 00	6
7	2.38	2 34	2 29	2 25	2 21	2 17	2 13	2 09	2 06	2 02	1 99	1 96	7
8	2.33	2 28	2 24	2 20	2 16	2.12	2 09	2 05	2 02	1 98	1 95	1 92	8
9	2 28	2 24	2 20	2 16	2 12	2 09	2 05	2 02	1 98	1 95	1 92	1 89	9
10	2 24	2 20	2.16	2 13	2 09	2 05	2 02	1 99	1 96	1 92	1 89	1 86	10
11	2 21	2 17	2 13	2 10	2 06	2 03	1 99	1 96	1 93	1 90	1 87	1 84	11
12	2.18	2 14	2 11	2 07	2 03	2 00	1 97	1 94	1 91	1 88	1 85	1 82	12
13	2.15	2 12	2 08	2 04	2 01	1 98	1 95	1 92	1 89	1 86	1 83	1 80	13
14	2 13	2 09	2 06	2 02	1 99	1 96	1 93	1 90	1 87	1 84	1 81	1 79	14
15	2 11	2 07	2 04	2 00	1 97	1 94	1 91	1 88	1 85	1 82	1 80	1 77	15
16	2.09	2.05	2 02	1 99	1 95	1 92	1 89	1 86	1 84	1 81	1 78	1 75	16
18	2 05	2 02	1 99	1 95	1 92	1 89	1 86	1 83	1 81	1 78	1 75	1 73	18
20	2 02	1 99	1.96	1 92	1 90	1 87	1 84	1 81	1 78	1 76	1 73	1 71	20

For 50 cycles, multiply the above values by 2

TABLE 21A.—CAPACITY SUSCEPTANCE OF LARGE COPPER CABLES, 25 CYCLES

$$b \text{ (approx.)} = \frac{2\pi f \times 38.83 \times 10^{-9}}{\log_{10}\left(\frac{s}{\rho} - \frac{\rho}{s}\right)} \text{ mhos per mile.}$$

ρ = maximum radius of cable.
s = spacing, measured in the same units as ρ
For three-phase irregular spacing use $s = \sqrt[3]{abc}$
For three-phase regular flat spacing use $s = 1.26\,a$

For a two-phase line the spacing is the mean distance between centers of conductors of the same phase

Circular mils	700,000	600,000	500,000	450,000	400,000	350,000	300,000	250,000	Spacing, Feet
Number of wires	61	37	37	37	19	19	19	19	
Diameter, inches .	.9641	.8914	.8137	.7720	.7255	.6786	.6283	.5735	
Spacing, Feet	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	
1	4.37	4.27	4.15	4.09	4.01	3.94	3.86	3.76	1
1.5	3.88	3.80	3.71	3.66	3.60	3.54	3.47	3.39	1.5
2	3.60	3.52	3.45	3.40	3.35	3.30	3.24	3.17	2
2.5	3.40	3.34	3.27	3.23	3.18	3.13	3.08	3.02	2.5
3	3.26	3.20	3.13	3.10	3.06	3.01	2.96	2.91	3
3.5	3.14	3.09	3.03	3.00	2.95	2.91	2.87	2.82	3.5
4	3.05	3.00	2.94	2.91	2.87	2.84	2.79	2.74	4
4.5	2.98	2.93	2.87	2.84	2.81	2.77	2.73	2.68	4.5
5	2.91	2.86	2.81	2.78	2.75	2.71	2.67	2.63	5
6	2.81	2.76	2.71	2.69	2.65	2.62	2.58	2.54	6
7	2.72	2.68	2.64	2.61	2.58	2.55	2.51	2.47	7
8	2.65	2.61	2.57	2.55	2.52	2.49	2.45	2.41	8
9	2.60	2.56	2.52	2.49	2.47	2.44	2.41	2.37	9
10	2.55	2.51	2.47	2.45	2.42	2.39	2.36	2.33	10
11	2.50	2.47	2.43	2.41	2.38	2.35	2.32	2.29	11
12	2.46	2.43	2.39	2.37	2.35	2.32	2.29	2.26	12
13	2.43	2.40	2.36	2.34	2.32	2.29	2.26	2.23	13
14	2.40	2.37	2.33	2.31	2.29	2.26	2.21	2.20	14
15	2.37	2.34	2.31	2.29	2.26	2.24	2.21	2.18	15
16	2.35	2.32	2.28	2.26	2.24	2.21	2.19	2.16	16
18	2.30	2.27	2.24	2.22	2.20	2.17	2.15	2.12	18
20	2.26	2.23	2.20	2.18	2.16	2.14	2.12	2.09	20
25	2.18	2.16	2.13	2.11	2.09	2.07	2.05	2.02	25
30	2.12	2.10	2.07	2.05	2.04	2.02	1.99	1.97	30

For 50 cycles, multiply the above values by 2.

TABLE 21*B*.—CAPACITY SUSCEPTANCE OF LARGE ALUMINUM CABLES, STEEL REINFORCED, 25 CYCLES

Values in table are in mhos per mile.

Circular mils ..	900,000	795,000	715,500	605,000	500,000	397,500	336,400	266,800	Spacing, Feet
Number of wires..	61	61	61	61	37	37	37	13	
Diameter, inches ..	1 162	1 093	1 036	953	.904	806	741	633	
Spacing, Feet	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	
1	4 64	4 55	4 47	4 36	4 29	4 14	4 04	3 87	1
1 5	4 09	4 02	3 96	3 87	3 81	3 70	3 62	3 48	1 5
2	3 78	3 72	3 66	3 59	3 54	3 44	3 37	3 25	2
2 5	3 56	3 51	3 46	3 39	3 35	3 26	3 20	3 09	2 5
3	3 41	3 36	3 31	3 25	3 21	3 12	3 07	2 97	3
3 5	3 28	3 24	3 20	3 13	3 10	3 02	2 97	2 88	3 5
4	3 18	3 14	3 10	3 04	3 01	2 93	2 89	2 80	4
4.5	3 10	3 06	3 02	2 97	2 94	2 86	2 82	2 74	4 5
5	3 03	2 99	2 95	2 90	2 87	2 81	2 76	2 68	5
6	2 92	2 88	2 83	2 80	2 77	2 71	2 67	2 59	6
7	2 82	2 79	2 76	2 71	2 69	2 63	2 59	2 52	7
8	2 75	2 72	2 69	2 64	2 62	2 56	2 53	2 46	8
9	2 69	2 66	2 63	2 59	2 57	2 51	2 47	2 41	9
10	2 64	2 61	2 58	2 54	2 52	2 47	2 43	2 37	10
11	2 59	2 56	2 53	2 49	2 48	2 42	2 39	2 33	11
12	2 55	2 52	2 50	2 45	2 44	2 39	2 36	2 30	12
13	2 51	2 49	2 46	2 42	2 41	2 36	2 33	2 27	13
14	2 48	2 45	2 43	2 39	2 38	2 33	2 30	2 25	14
15	2 45	2 42	2 40	2 36	2 35	2 31	2 27	2 22	15
16	2 42	2 40	2 37	2 34	2 33	2 28	2 25	2 20	16
18	2 37	2 35	2 33	2 29	2 28	2 24	2 21	2 15	18
20	2 33	2 31	2 29	2 25	2 24	2 20	2 17	2 12	20
25	2 25	2 23	2 21	2 18	2 16	2 12	2 10	2 05	25
30	2 18	2 17	2 15	2 12	2 10	2 07	2 04	2 00	30

For 50 cycles, multiply the above values by 2

TABLE 21C.—CAPACITY SUSCEPTANCE OF CABLE, 25 CYCLES

SIZES FOR BOTH ALUMINUM CABLES, STEEL REINFORCED, AND COPPER CABLES

For diameters see Resistance Tables 14 and 15
Values in table are in mhos per mile
For Aluminum Cable, Steel Reinforced, add 1 5% to the tabulated values

B & S Gauge	No 0000	No 000	No 00	No 0	No 1	No 2	No 3	No 4	Spacing, Feet
Circular mils	211,600	167,800	133,100	105,500	83,690	66,370	52,630	41,740	
Number of wires	7	7	7	7	7	7	7	7	
Spacing, Feet	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	
1	3 67	3 56	3 46	3 36	3 27	3 19	3 10	3 03	1
1 5	3 32	3 23	3 14	3 06	2 99	2 92	2 85	2 78	1 5
2	3 10	3 03	2 95	2 88	2 82	2 75	2 69	2 63	2
2 5	2 96	2 89	2 82	2 76	2 70	2 64	2 58	2 53	2 5
3	2 85	2 79	2 72	2 66	2 61	2 55	2 50	2 45	3
3.5	2 76	2 70	2 64	2 59	2 53	2 48	2 43	2 38	3 5
4	2 68	2 63	2 58	2 52	2 47	2 42	2 38	2 33	4
4 5	2 63	2 58	2 52	2 47	2 42	2 38	2 33	2 29	4 5
5	2 58	2 53	2 48	2 43	2 38	2 33	2 29	2 25	5
6	2 50	2 45	2 40	2 35	2 31	2 26	2 22	2 18	6
7	2 43	2 38	2 34	2 29	2 25	2 21	2 17	2 13	7
8	2 38	2.33	2 29	2.25	2 20	2 16	2 13	2 09	8
9	2 33	2 29	2 24	2 20	2 16	2 13	2 09	2 05	9
10	2 29	2 25	2 21	2 17	2 13	2 09	2 06	2 02	10
11	2 25	2 21	2 17	2 14	2 10	2 06	2 03	2 00	11
12	2 22	2 18	2 14	2 11	2 07	2 04	2 00	1 97	12
13	2 20	2 16	2 12	2 08	2 05	2 01	1 98	1 95	13
14	2 17	2 13	2 10	2 06	2 03	1 99	1 96	1 93	14
15	2 15	2 11	2 07	2 04	2 01	1 97	1 94	1 91	15
16	2 13	2 09	2 05	2 02	1 99	1 96	1 92	1 89	16
18	2 09	2 05	2 02	1 99	1 96	1 92	1 89	1 86	18
20	2 06	2 02	1 99	1 96	1 93	1 90	1 87	1 84	20

For 50 cycles, multiply the above values by 2.

TABLE 22.—CAPACITY SUSCEPTANCE OF WIRE, 60 CYCLES

$$b \text{ (approx.)} = \frac{2\pi f \times 38\ 83 \times 10^{-9}}{\log_{10}\left(\frac{s}{\rho} - \frac{\rho}{s}\right)} \text{ mhos per mile.}$$

ρ = radius of wire.
s = spacing.
For three-phase irregular spacing use $s = \sqrt[3]{abc}$
For three-phase regular flat spacing use $s = 1\ 26\ a$

For a two-phase line the spacing is the average distance between centers of conductors of the same phase

B & S. Gauge.	No 0000	No 000	No 00	No 0	No 1	No 2	No 3	No. 4	No. 5	No 6	No 7	No 8	Spacing, Feet
Circular mils .	211,600	167,800	133,100	105,500	83,690	66,370	52,630	41,740	33,100	26,250	20,820	16,510	
Diameter, (2ρ) inches	.4600	4096	.3648	.3249	.2893	2576	2294	2043	1819	1620	1443	1285	
Spacing, Feet	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	
1	8 52	8 28	8 05	7 83	7 63	7 43	7 25	7 07	6 90	6 74	6 59	6 45	1
1 5	7.73	7.53	7 34	7 16	6 99	6 82	6 67	6 52	6 38	6 24	6 11	5 98	1 5
2	7 25	7 08	6 91	6 75	6 60	6 45	6 31	6 18	6 05	5 92	5 80	5 69	2
2 5	6 92	6 76	6.61	6 46	6 32	6 18	6 06	5 93	5 81	5 70	5 59	5 48	2 5
3	6 67	6 52	6 38	6 24	6 11	5 98	5 86	5 75	5 64	5 53	5 44	5 33	3
3 5	6 47	6 33	6 20	6 07	5 94	5 83	5 71	5 60	5 49	5 39	5 29	5 20	3 5
4	6 31	6 18	6 05	5 93	5 81	5 69	5 58	5 48	5 38	5 28	5 19	5 10	4
4.5	6 18	6 05	5 92	5 81	5 69	5 58	5.48	5 38	5 28	5 18	5 09	5 01	4 5
5	6 06	5 93	5 82	5 70	5 59	5 49	5 39	5 29	5.19	5 10	5 01	4 93	5
6	5 87	5 75	5 64	5 53	5 43	5 33	5 23	5 15	5 05	4 97	4 88	4 80	6
7	5 71	5 60	5 50	5 40	5 30	5 20	5 11	5 02	4 94	4 85	4 77	4 70	7
8	5 59	5 48	5 38	5 28	5 19	5 10	5 01	4 92	4 84	4 76	4 69	4 61	8
9	5 48	5 38	5 28	5 19	5 09	5 01	4 92	4 84	4 76	4 69	4 61	4 54	9
10	5 39	5 29	5 19	5 10	5 02	4 93	4 85	4 77	4 69	4 62	4 55	4 48	10
11	5 31	5 21	5 12	5 03	4 95	4 86	4 78	4 70	4 63	4 56	4 49	4 42	11
12	5 23	5 14	5 05	4 97	4 88	4 80	4.72	4 65	4.58	4 50	4 44	4 37	12
13	5 17	5 08	4 99	4 91	4 83	4 75	4 67	4 60	4 53	4 46	4 39	4 32	13
14	5 11	5 02	4 94	4 86	4 78	4 70	4 62	4 55	4 48	4 41	4 35	4 29	14
15	5 06	4 97	4 89	4 81	4 73	4 65	4 58	4 51	4 44	4 37	4 31	4 25	15
16	5 01	4 93	4 84	4.76	4 69	4 61	4 54	4 47	4 40	4 34	4 27	4 21	16
18	4 92	4 84	4 76	4 69	4 61	4 54	4 47	4 40	4 34	4 27	4 21	4 15	18
20	4 85	4 77	4 69	4 62	4 55	4 48	4 41	4 34	4 28	4 22	4 16	4 10	20

TABLE 23A.—CAPACITY SUSCEPTANCE OF LARGE COPPER CABLES, 60 CYCLES

$$b \text{ (approx)} = \frac{2\pi f \times 38.83 \times 10^{-9}}{\log_{10}\left(\frac{s}{\rho} - \frac{\rho}{s}\right)} \text{ mhos per mile}$$

ρ = maximum radius of cable
s = spacing, measured in the same units as ρ
For three-phase irregular spacing use $s = \sqrt[3]{abc}$
For three-phase regular flat spacing use $s = 1.26\ a$.

For a two-phase line the spacing is the mean distance between centers of conductors of the same phase

Circular mils	700,000	600,000	500,000	450,000	400,000	350,000	300,000	250,000	Spacing, Feet
Number of wires	61	37	37	37	19	19	19	19	
Diameter, inches	.9641	.8914	.8137	.7720	.7255	.6786	.6283	.5735	
Spacing, Feet	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	
1	10.49	10.24	9.96	9.81	9.63	9.45	9.25	9.03	1
1.5	9.31	9.12	8.90	8.77	8.63	8.49	8.33	8.14	1.5
2	8.63	8.46	8.27	8.16	8.04	7.91	7.77	7.61	2
2.5	8.16	8.01	7.84	7.74	7.64	7.52	7.39	7.25	2.5
3	7.82	7.68	7.52	7.43	7.33	7.23	7.11	6.97	3
3.5	7.55	7.42	7.27	7.19	7.09	7.00	6.88	6.76	3.5
4	7.33	7.20	7.07	6.99	6.90	6.81	6.70	6.58	4
4.5	7.14	7.03	6.90	6.82	6.74	6.65	6.55	6.44	4.5
5	6.99	6.88	6.75	6.68	6.60	6.51	6.42	6.31	5
6	6.73	6.63	6.51	6.45	6.37	6.29	6.20	6.10	6
7	6.53	6.44	6.33	6.26	6.19	6.12	6.03	5.94	7
8	6.37	6.27	6.17	6.11	6.04	5.97	5.89	5.80	8
9	6.23	6.14	6.04	5.98	5.92	5.85	5.77	5.68	9
10	6.11	6.02	5.93	5.87	5.81	5.74	5.67	5.58	10
11	6.01	5.92	5.83	5.78	5.72	5.65	5.58	5.50	11
12	5.91	5.84	5.74	5.69	5.63	5.57	5.50	5.42	12
13	5.83	5.76	5.66	5.62	5.56	5.50	5.43	5.35	13
14	5.76	5.68	5.60	5.55	5.49	5.43	5.37	5.29	14
15	5.69	5.62	5.53	5.48	5.43	5.37	5.31	5.23	15
16	5.63	5.56	5.47	5.43	5.37	5.32	5.25	5.18	16
18	5.52	5.45	5.37	5.33	5.28	5.22	5.16	5.09	18
20	5.43	5.36	5.28	5.24	5.19	5.14	5.08	5.01	20
25	5.24	5.18	5.11	5.07	5.02	4.97	4.91	4.85	25
30	5.10	5.04	4.97	4.93	4.89	4.84	4.78	4.73	30

TABLE 23*B*.—CAPACITY SUSCEPTANCE OF LARGE ALUMINUM CABLES, STEEL REINFORCED, 60 CYCLES

Values in table are in mhos per mile

Circular mils. .	900,000	795,000	715,500	605,000	500,000	397,500	336,400	266,800	Spacing, Feet
Number of wires	61	61	61	61	37	37	37	13	
Diameter, inches. . .	1 162	1 093	1 036	953	904	806	.741	633	
Spacing, Feet	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	
1	11 13	10.92	10 73	10 45	10 28	9 93	9 70	9 27	1
1 5	9 82	9 66	9 50	9 28	9 15	8 88	8 68	8 34	1 5
2	9 06	8 92	8 79	8 60	8 49	8 25	8 08	7 79	2
2.5	8 55	8 42	8 30	8 14	8 04	7.82	7 67	7 41	2 5
3	8 17	8 06	7 95	7 80	7 70	7 50	7 37	7 12	3
3.5	7 88	7 77	7 67	7 53	7 44	7 25	7 13	6 90	3.5
4	7.64	7.54	7 45	7 31	7 23	7 06	6 93	6 71	4
4 5	7.44	7 34	7 26	7 13	7 05	6 88	6 76	6 56	4 5
5	7 27	7 18	7 09	6.97	6 90	6 74	6 63	6 43	5
6	7 00	6 91	6 80	6 72	6 65	6 50	6 40	6 21	6
7	6 78	6 70	6 62	6 52	6 45	6 32	6 22	6 04	7
8	6 60	6 52	6 46	6 35	6 29	6 16	6 07	5 90	8
9	6 45	6 38	6 32	6 22	6 16	6 03	5 94	5 78	9
10	6 33	6 26	6 19	6 10	6 04	5 92	5 83	5 68	10
11	6 21	6 15	6 08	5.99	5 94	5 82	5 74	5 59	11
12	6 12	6 05	5 99	5 90	5 85	5 73	5 65	5 51	12
13	6 03	5 96	5 91	5 82	5 77	5 65	5 58	5 44	13
14	5 95	5 89	5 83	5 75	5 70	5 59	5 51	5 38	14
15	5 88	5 82	5 76	5 68	5 64	5 52	5 45	5 32	15
16	5 81	5 75	5 70	5 62	5 58	5 46	5 39	5 26	16
18	5 70	5 64	5 59	5 51	5 47	5 36	5 29	5 17	18
20	5 60	5 54	5 50	5 42	5 37	5 27	5 21	5 08	20
25	5 40	5 35	5 30	5 23	5 19	5 10	5 04	4 92	25
30	5 24	5 20	5 15	5 09	5 05	4 96	4 90	4 79	30

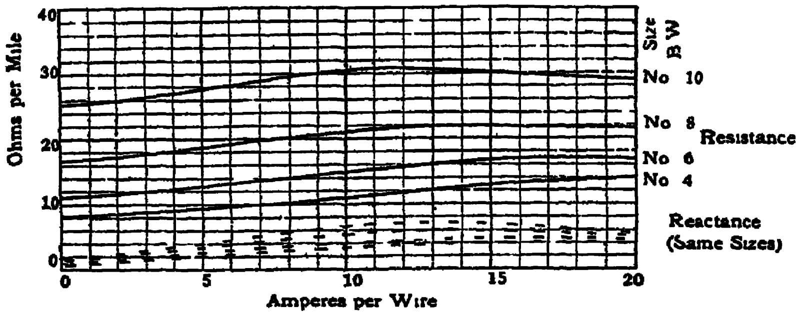
TABLE 23C.—CAPACITY SUSCEPTANCE OF CABLE, 60 CYCLES

SIZES FOR BOTH ALUMINUM CABLES, STEEL REINFORCED, AND COPPER CABLES

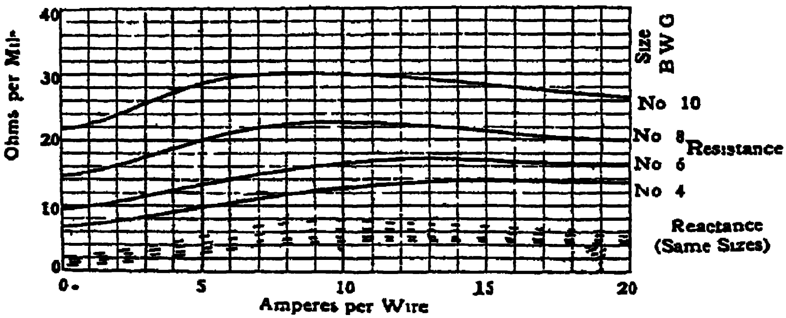
For diameters see Resistance Tables 14 and 15.
Values in table are in mhos per mile.
For Aluminum Cable, Steel Reinforced, add 1 5% to the tabulated values

B & S Gauge	No 0000	No 000	No 00	No 0	No 1	No 2	No 3	No 4	Spacing, Feet
Circular mils.	211,600	167,800	133,100	105,500	83,690	66,370	52,630	41,740	
Number of wires .	7	7	7	7	7	7	7	7	
Spacing, Feet	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	
1	8 80	8 54	8 30	8 07	7 85	7 64	7 45	7 26	1
1 5	7 96	7 75	7 55	7 36	7 17	7 00	6 84	6 68	1 5
2	7 45	7 27	7 09	6 92	6 76	6 61	6 46	6 32	2
2 5	7 10	6 93	6 77	6 62	6 47	6 33	6 20	6 07	2 5
3	6 84	6 68	6 53	6 39	6 25	6 12	5 90	5 87	3
3.5	6 63	6 48	6 34	6 21	6 08	5 95	5 83	5 72	3 5
4	6 46	6 32	6.19	6 06	5 94	5 82	5 70	5 59	4
4 5	6 32	6 19	6 06	5 93	5 81	5 70	5 59	5 49	4 5
5	6 20	6 07	5 94	5 82	5 71	5 60	5 49	5 39	5
6	6 00	5 88	5 76	5 65	5 54	5 44	5 34	5 24	6
7	5 84	5 72	5 61	5 50	5 40	5 30	5 21	5 12	7
8	5 70	5 59	5 49	5 39	5 29	5 19	5 10	5 02	8
9	5 59	5 49	5 39	5 29	5 19	5 10	5 01	4 93	9
10	5 50	5 39	5 30	5 20	5 11	5 02	4 94	4 85	10
11	5 41	5 31	5 22	5 13	5 04	4 95	4 87	4 79	11
12	5 34	5 24	5 15	5 06	4 97	4 89	4 81	4 73	12
13	5 27	5 18	5 09	5 00	4 92	4 83	4 75	4 68	13
14	5 21	5 12	5 03	4 95	4 86	4 78	4 70	4 63	14
15	5 16	5 07	4 98	4 90	4 82	4 74	4 67	4 59	15
16	5 11	5 02	4 93	4 85	4 77	4 69	4 62	4 55	16
18	5 02	4 93	4 85	4 77	4 69	4 62	4 55	4 18	18
20	4 94	4.86	4 78	4.70	4 63	· 4 55	4 18	4 11	20

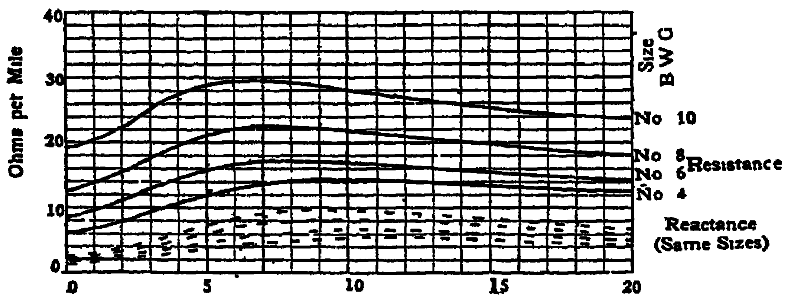
TABLE 24.—STEEL CONDUCTORS, 25 CYCLES



ORDINARY STEEL GRADE WIRES, 25 CYCLES

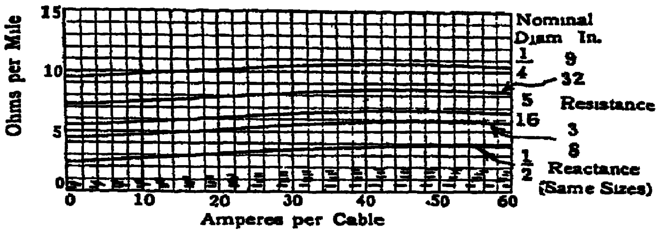


GRADE BB WIRES, 25 CYCLES

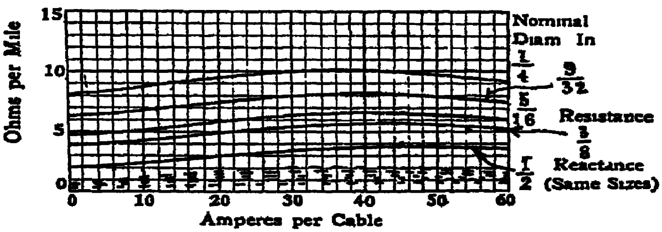


GRADE EBB WIRES, 25 CYCLES

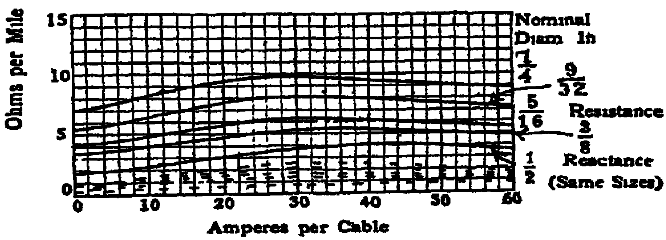
TABLE 24—*Continued*



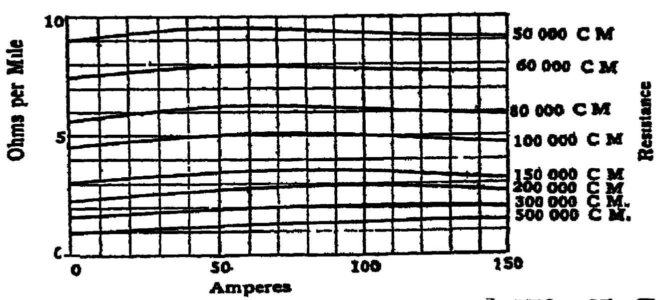
ORDINARY STEEL GRADE SEVEN STRAND CABLES, 25 CYCLES



GRADE BB SEVEN STRAND CABLES, 25 CYCLES

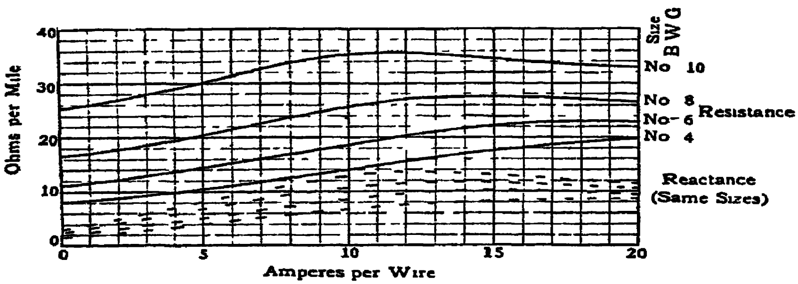


GRADE EBB SEVEN STRAND CABLES, 25 CYCLES

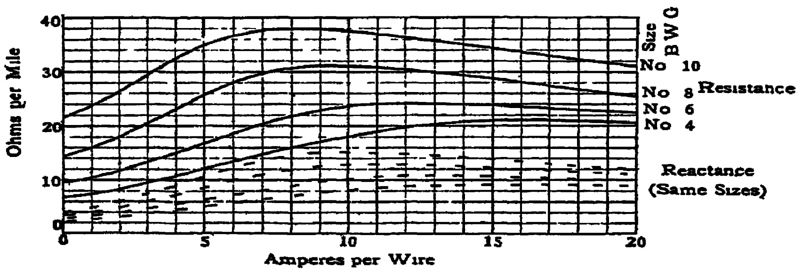


ORDINARY STEEL NINETEEN STRAND CABLES, 25 CYCLES

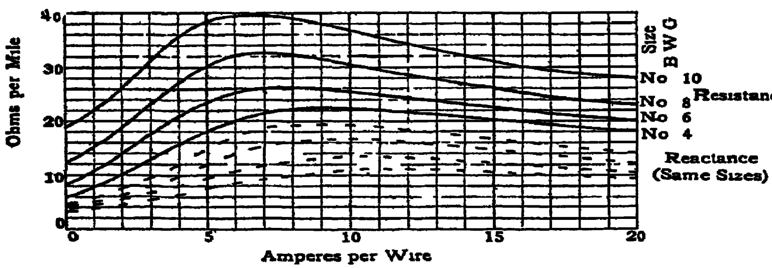
TABLE 25.—STEEL CONDUCTORS, 60 CYCLES



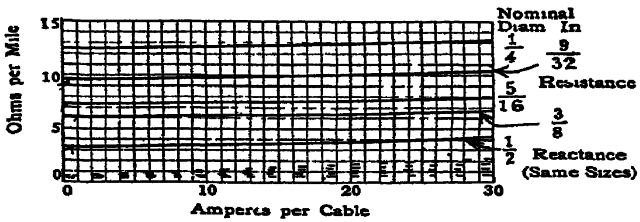
ORDINARY STEEL GRADE WIRES, 60 CYCLES



GRADE BB WIRES, 60 CYCLES

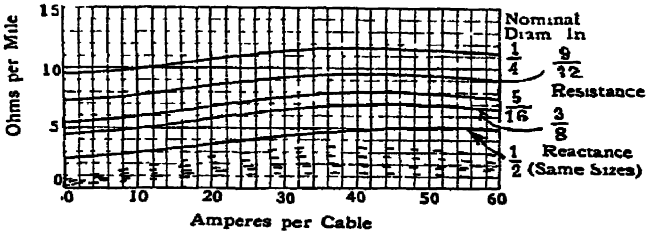


GRADE EBB WIRES, 60 CYCLES

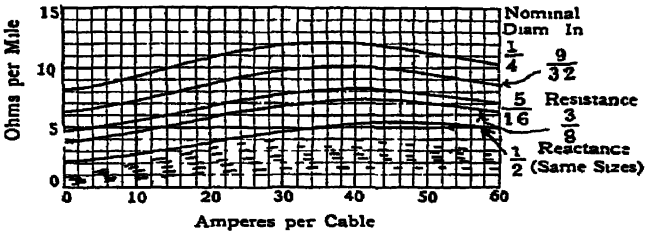


SIEMENS-MARTIN SEVEN STRAND CABLES, 60 CYCLES

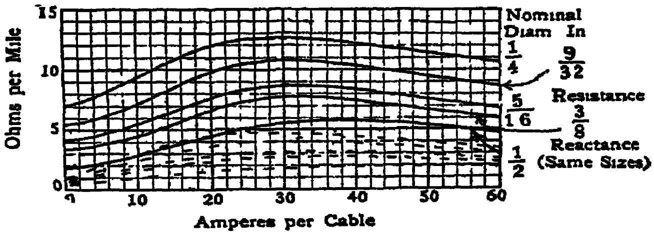
TABLE 25.—*Continued*



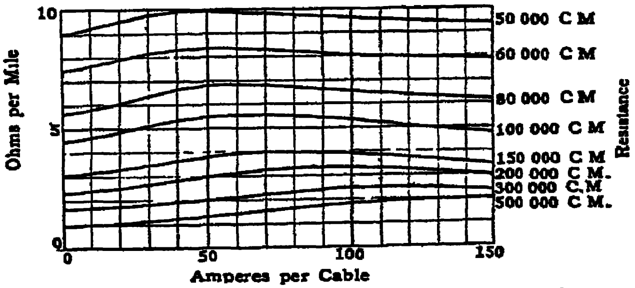
ORDINARY STEEL GRADE SEVEN STRAND CABLES, 60 CYCLES



GRADE BB SEVEN STRAND CABLES, 60 CYCLES



GRADE EBB SEVEN STRAND CABLES, 60 CYCLES



ORDINARY STEEL NINETEEN STRAND CABLES, 60 CYCLES

TABLE 26.—CORONA LOSS AND VOLTAGE LIMITS

Disruptive Critical Voltage in fair weather:

$$e_0 = 123 m_0 r \delta \left(\log_{10} \frac{s}{r} \right) \text{ kilovolts to neutral.}$$

Power Loss:

$$p = \frac{390}{\delta} (f+25) \sqrt{\frac{r}{s}} (e-e_0)^2 10^{-5} \text{ Kw. per mile of single conductor,}$$

where m_0 = Irregularity factor.
- = 1 for polished wires.
- = 0 98 to 0 93 for roughened or weathered wires.
- = 0 87 to 0 83 for stranded cables.†

r = Radius of conductor in inches.
s = Axial spacing of conductors in inches, for triangular spacing
$\delta = \dfrac{17.9b}{459+t}$.
b = Barometric pressure in inches.
t = Temperature in degrees Fahrenheit.
f = Frequency in cycles per second.
e = Effective kilovolts to neutral applied to the line
$= \dfrac{1}{\sqrt{3}}$ times voltage between conductors for three-phase line
$= \frac{1}{2}$ times voltage between conductors for single-phase or two-phase lines.

For stormy weather, consider $e_0 = 80\%$ of fair weather value for e_0. It is impossible to predict the storm loss with great accuracy because of the many variables that enter.

The operating voltage should not be higher than the fair weather value for e_0 at the given altitude

* From "Dielectric Phenomena in High Voltage Engineering," by F. W Peek, Jr, pages 204–211.

† Tests on 1 inch diameter cables showed that the irregularity factor was approximately the same for 19, 37 and 61 strand cables of standard concentric lay The slight difference was in favor of the 37 strand conductor Seven strand cables in large sizes are undesirable since the strands become mutilated in manufacture, which lowers m_0. For special types of conductors m_0 should be determined by measurement.

TABLE 26.—*Continued*

CORONA LIMIT OF VOLTAGE AT SEA LEVEL

Kilovolts between conductors at 76 cm barometric pressure and 25° C.
Three-phase triangular or flat spacing.
For single-phase or two-phase, multiply the tabulated voltages by 1 16.
Multiply the voltages by the Altitude Correction Factor, tabulated below.

CABLES, (Without Steel Cores).

Size, B & S. or Circular Mils	Spacing, Feet								
	4	5	6	8	10	12	14	16	20
4	56	58	60	62	64	66	68	69	71
3	62	65	67	70	72	74	76	77	80
2	....	71	73	76	79	81	83	85	87
1	....	79	81	85	88	91	93	95	97
0	....	....	90	95	98	102	104	108	109
00	....	.	98	104	108	111	114	117	121
000	....	.	....	114	118	121	124	127	132
0000	...	....	..	125	130	135	138	141	146
250,000	...	...	...	138	144	149	152	156	161
300,000	....	...	...	..	151	156	161	165	171
350,000	...	...	..	..	161	166	170	175	180
400,000	...	...	...	....	171	176	180	185	192
450,000	...	..	..	...	178	184	190	194	200
500,000	...	....	...	..	188	194	199	205	210
800,000	.....	...	...	......	....	234	241	244	256
1,000,000	.....	....	...	......	.....	256	264	270	281

TABLE 26 —*Continued*

CORONA LIMIT OF VOLTAGE AT SEA LEVEL

Wires

Size, B. & S	Spacing, Feet									
	3	4	5	6	8	10	12	14	16	20
4	51	54	56	58	60	62	64	65	66	68
3		59	62	64	66	68	70	72	74	76
2	..	.	69	70	74	76	78	80	82	84
1	...	.	75	77	81	83	86	88	90	92
0	.	.		85	89	92	95	97	99	102
00	..			94	98	102	105	107	110	113
000	.	.	.	.	109	113	116	119	121	124
0000	. .	...	.	.	120	125	128	131	134	138

Altitude Correction Factor at 25° C.

Altitude, Feet	δ	Altitude, Feet	δ
0	1 00	5,000	82
500	.98	6,000	79
1,000	.96	7,000	77
1,500	.94	8,000	74
2,000	.92	9,000	71
2,500	.91	10,000	68
3,000	.89	12,000	63
4,000	.86	14,000	.58

TABLE 27.—STAR VOLTAGES

STAR VOLTAGE, OR VOLTAGE TO NEUTRAL = LINE VOLTAGE ÷ $\sqrt{3}$ FOR THREE-PHASE CIRCUITS

Line Voltage	Star Voltage	Line Voltage	Star Voltage
10,000	5,774	33,000	19,050
10,500	6,062	35,000	20,210
11,000	6,351	40,000	23,090
11,500	6,640	44,000	25,400
12,000	6,928	45,000	25,980
12,500	7,217	50,000	28,870
13,000	7,506	55,000	31,750
13,200	7,621	60,000	34,640
13,500	7,794	65,000	37,530
14,000	8,083	66,000	38,110
14,500	8,372	70,000	40,410
15,000	8,660	75,000	43,300
16,000	9,238	77,000	44,460
17,000	9,815	80,000	46,190
18,000	10,390	85,000	49,070
19,000	10,970	88,000	50,810
20,000	11,550	90,000	51,960
22,000	12,700	95,000	54,850
25,000	14,430	99,000	57,160
30,000	17,320	100,000	57,740

TABLE 28 —POWER FACTOR AND REACTIVE FACTOR

Cos θ Power Factor	Sin θ Reactive Factor	Cos θ Power Factor	Sin θ Reactive Factor
.50	.8660	.75	6614
.51	8602	.76	6499
52	.8542	.77	6380
.53	.8480	78	.6258
54	.8417	.79	.6131
.55	.8352	.80	6000
.56	8285	.81	.5864
57	.8216	82	5724
58	8146	83	5578
59	8074	84	5426
.60	8000	.85	5268
61	7924	.86	5103
62	7846	.87	.4931
.63	.7766	88	4750
64	7684	.89	.4560
65	7599	.90	4359
.66	.7513	91	.4146
67	.7424	92	3919
.68	7332	93	3676
.69	.7238	.94	3412
70	.7141	95	.3123
71	.7042	96	2800
.72	6940	97	2431
73	6834	98	.1990
74	.6726	99	.1411
		1 00	0

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