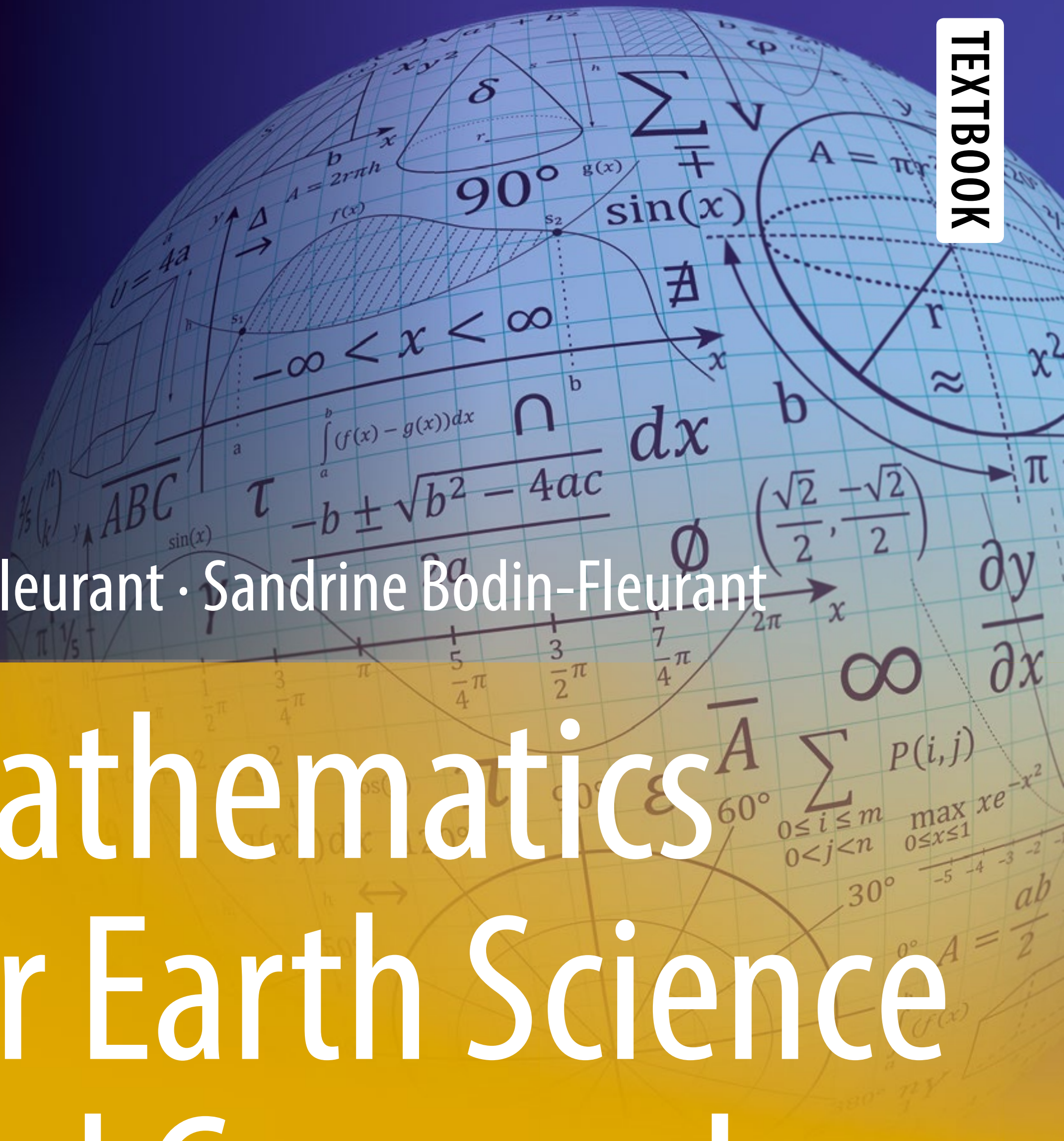

TEXTBOOK

Cyril Fleurant · Sandrine Bodin-Fleurant

Mathematics for Earth Science and Geography

Introductory Course with Practical Exercises and R/Xcas Resources

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Sandrine Bodin-Fleurant and Cyril Fleurant
Angers, August 2018
To Victor, Abel, Martin and Jeanne our beloved children

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Introduction

Geography (whether physical, human or social) and earth science are complex sciences that require elaborate quantification tools to understand their foundations and their processes.

This textbook includes exercises to acquire and consolidate the mathematical basics necessary to practice physical geography and earth science. The aim of this textbook is to be of use in real geographical and earth science situations, and to illustrate how mathematical tools can be used to enhance these disciplines.

The textbook is organized around issues in geography and in earth science that will motivate the need for theoretical contributions and mathematical practice. The text is illustrated with examples developed and derived from them. This textbook is based on the authors' experiences and needs of students in their construction of mathematical tools. Mathematical theoretical contributions will be those that are actually useful for the construction of a solid geographical or earth science culture.

Priority is also given to digital applications and computing. The authors want the links between geography, earth science and mathematics to be as complete as possible, so computer tools (freeware R and Xcas) will be used.

Who Is This Book For?

This textbook is mainly intended for bachelor's degree students in geography and earth science and their teachers—and, of course, also beyond (for instance master's students wishing to upgrade their mathematical knowledge).

These students are divided into two main groups:

- Students with advanced mathematics training who have taken a secondary science course. These students need to consolidate their mathematical knowledge, and especially to learn to use this knowledge in non-mathematical contexts, coming from the disciplines studied.
- Students who have not studied advanced mathematics at school for which it is necessary to start with the basics of mathematics. These basics are given here directly in the context of geography and earth science.

This textbook is also intended for teachers in secondary schools (whether in earth science, geography or mathematics), who will find in it many ideas for interdisciplinary teaching.

Why This Book?

The authors are used to working with pupils, students and their teachers. An observation was made on the difficulties that many students face when using simple mathematical knowledge in different contexts (drawing regression lines, manipulating units, using derivatives). We felt it was necessary to emphasize these needs in order to complement university lectures and existing works.

The textbook voluntarily takes over many notions and gives methodological advice (e.g. about making conversions or plotting functions). Useful mathematical functions are discussed with contextualized examples. The textbook also addresses (always in a very contextualized way) more advanced notions, such as integrals, differential equations and partial derivatives.

Please note that we chose not to deal with statistics, for which there are already many references. For example, the interested reader may refer to Dadson S.J., 2017, *Statistical analysis of geographical data: an introduction*, Wiley.

Another aim of this text is to help readers acquire an understanding of orders of magnitude and units commonly used in university courses, as well as a greater confidence in the resolution method and encrypted data.

Contents

The book is divided into six chapters that progressively lead to ever more advanced notions.

Chapter 1, "Quantities and Measures", treats decimal and sexagesimal numeration systems. It presents the notions of symbol, measurement, unit and approximate value (rounded, significant digits, uncertainty, error), as well as conversion units, percentages and rates. It also discusses scientific notation and the powers of ten.

Chapter 2, "Variables and Functions", discusses the variables and their values and links them together to define the notion of function. We also show the interest of the representation of these functions to illustrate natural phenomena. Some common functions in geography and earth science are developed, as are the notions of increasing, decreasing, minimum and maximum, useful for the study of their evolution and trends.

Chapter 3, "Trigonometry, Geometry of Plane and Space", develops 2D trigonometry, trigonometric functions and 2D and 3D geometry.

Chapter 4, "Cartography", provides mathematical foundations for understanding a mapping system, in particular concerning the identification of items on a globe, projections and spherical trigonometry.

Chapter 5, "Derivation", shows the use of derivatives in the study of common functions. It will shed light on the advantages of derivation for the study of function variations. The elements of computational techniques are provided, and the more intensive calculations can be carried out using formal calculation software.

Chapter 6, "Integration and Differential Equations", presents two fundamental tools related to functions: integration and the concept of differential equations. The concept of partial differential equations is also presented.

Each chapter offers **two types of exercises:**

- Mathematical application exercises, which allow students to take into account the main elements of the text and help them to memorize them. They are not contextualized but are nonetheless focused on the knowledge and know-how that will be useful for students in earth science and geography;
- Exercises in earth science and geography in which problem situations are identified. These exercises aim to contextualize mathematical tools in concrete problems encountered by students.

Half of these exercises are contextualized and all **(over 110 in total)** are fully corrected.

How to Use This Manual

Each series of exercises in each chapter begins with three series of **flash questions.** These questions allow readers to determine whether the content of the chapter is familiar to them. First try to answer the questions without looking at the content and then use it to check or reinforce the methods and calculations. Do not work on all three series at a time.

Depending on their knowledge and goals, students can:

- Start directly with a series of flash questions (as a diagnostic test) and then work on (or not) the content of the text according to the result;
- Begin by reading the key points at the end of the chapter to assess their familiarity with the contents and then decide whether to work first on the text or the exercises;
- First work on the text and then check its level of familiarity based on the various exercises;
- Start directly with the geography or geology exercises and refer to the text and the mathematical exercises as required.

General Advice When performing the calculations, first seek an order of magnitude of the result. Then carry out the calculations without the use of a calculator, and finally finish by checking your answers with a calculator or with R (calculation software).

Good Luck and Enjoy!

The authors hope everyone finds this textbook enjoyable. It requires **active reading** to be effective: with pencil in hand, try to solve the exercises and take time to gather your personal knowledge before turning to the solutions.

Contents

About the Authors

Cyril Fleurant obtained an undergraduate degree in applied geology from the University of Bordeaux and a master's degree and a PhD in quantitative hydrology and hydrogeology from Mines Paris Tech (Paris) and University Pierre and Marie Curie (Paris). He held several teaching and research positions at the University of Nantes, École des mines de Nantes and Agrocampus Ouest. Since 2012 he holds a Full Professor position at the University of Angers (France) – Department of Geography. He has recently been elected director of the Faculty of Literature, Languages and Human Sciences.

Prof. Fleurant's research is pluridisciplinary within the research group LETG – UMR CNRS 6554 and involves projects on geomorphology, hydrology and hydrogeology.

Sandrine Bodin-Fleurant obtained her master's degree in fundamental mathematics from the University of Angers, France. She has taught mathematics for 15 years, mostly at the university level (calculus, algebra, statistics, probability, computer science). Her (former) students include many Earth Science students.

Sandrine Bodin-Fleurant has also been a regional school inspector in mathematics (with a strong emphasis on pedagogy and didactics) for 5 years. She worked regularly with earth science and geography teachers and inspectors and she has an excellent knowledge of the abilities and needs of both high school and undergraduate students. She is now deputy head inspector for the whole “Rhône” area.

She has published various documents at the national level (about Fast Fourier Transform, the new probability curriculum at secondary school level and many others).

1 Quantities and Measures

Abstract

In geography as in Earth science, data are quantified, and numbers need to be manipulated constantly. This chapter deals with a base 10 system and a sexagesimal numeral system (for the use of time and angles). It deals with the concepts of measurement, unit, and approximate value (rounded, significant digits, uncertainties), as well as unit conversions, percentages, and rates. It also discusses scientific notation and the powers of ten. These notions are fundamental in applied science. The exercises show various contexts, for example, flow, planets, temperature, and albedo.

Keywords

Quantities · Measure · Base 10 · Time conversion · Unit conversion · Approximate value · Rates · Scientific notation · Powers of 10

Aims and Objectives

- To know how to manipulate numbers, to know how to move from one scale to another, and to manipulate different orders of magnitude.
- To be comfortable with numbers and their comparison ($0.1 < 0.2$ but $-0.1 > -0.2$) as well as with scientific notation and its meaning (it should be immediately seen that 10^{-1} is greater than 10^{-2}).
- For objects encountered frequently, to know their units and to have in mind orders of magnitude of their value.
- To know how to convert units (for example, to convert without error cubic meters to cubic centimeters and meters per second [m/s] to centimeters per hour [cm/h]).

1.1 Numbers

1.1.1 Numeration and Integers

A **numeral system** (or a system of numeration) is a set of symbols and rules for expressing numbers (it is a way of counting). A number is an idea, and a numeral is how we write it (number is often used instead of numeral, and this book will be no exception). The symbols are grouped, and the group size is called the **base**.

Our common numeral system is in base 10: the usual symbols are 0, 1, 2, 3, …, 9. Notations are given to ten **units** (**ten**, denoted 10^1; read: "10 to the power of 1"), to ten tens (**one hundred**, value $100 = 10 \times 10$, denoted 10^2), to ten hundreds (**one thousand**, value $1000 = 10 \times 10 \times 10$, denoted 10^3), and so forth.

Electronic supplementary material The online version of this article (https://doi.org/10.1007/978-3-319-69242-5_1) contains supplementary material, which is available to authorized users.

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Other bases are used: base 60 (for time and angle measurements) or base 2, in which the numbers are written using only 0s and 1s. This latter numeral system revolutionized the modern world since it is the key to the functioning of computers and their binary nature: 1, charged; 0, uncharged (all information, coded in numbers, can thus be translated by a digit and vice versa).

1.1.2 Decimals

The most often used counting system is the decimal numeral system, called the base 10 system ("decimal" from the Latin *decimalis* meaning "that has base 10," because we have 10 fingers). Two important elements should be noted:

- The **base 10 system of numeration** uses 10 symbols (**digits**): 0, 1, 2,. . ., 9;
- When writing a number, the position of a digit indicates the power of 10, and the digit specifies the number of times that power intervenes. A zero indicates the absence of power at this position.

This is how positive integers are described.

For example, in the number "525,005," the digit 5 takes three different values. Reading from left to right: the first 5 indicates the value "five hundred thousand," the second five "five thousand," and the third 5 "five units"). We have thus:

$$525,005 = 5 \times 100,000 + 2 \times 10,000 + 5 \times 1000 + 0 \times 100 + 0 \times 10 + 5 \times 1.$$

Note 1 Everyday language often mixes the concepts of digit and numerals (numbers). In mathematics, the digits are the symbols used to write numerals, just as letters are the symbols used to write words. In addition, just as some words have one letter ("I" in the sentence "I am happy"), some numbers are composed of one digit ("Victor is 3 years old").

Adding + or − signs to these **natural numbers** describes the **integers** (Fig. 1.1). For example, the number −246 is smaller than the number 37.

1.1.3 Powers of 10 and Decimal Numbers

Definition 1
For n positive integer, we define 10^n by $\underbrace{10 \times 10 \times \ldots \times 10}_{n \ \textit{times}}$, using the convention $10^0 = 1$ (Table 1.1).

Note 2 10^n is written with a 1 followed by n zeros.

Thus the number 525,005 from the preceding example can be written as follows (Fig. 1.2):

$$525,005 = 5 \times 10^5 + 2 \times 10^4 + 5 \times 10^3 + 0 \times 10^2 + 0 \times 10^1 + 5 \times 10^0.$$

Definition 2
We define 10^{-1} by $10^{-1} = \frac{1}{10^1}$. This number reads "one *tenth.*"

For positive integer n, we define 10^{-n} by $\underbrace{10^{-1} \times 10^{-1} \times \ldots \times 10^{-1}}_{n \ \textit{times}}$, with $10^0 = 10^{-0} = 1$ (Table 1.1).

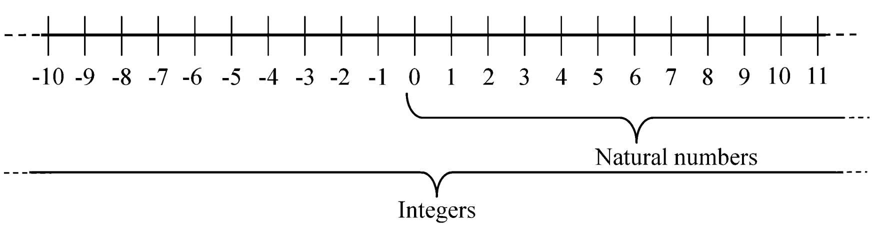

Fig. 1.1 The set of positive integers (natural numbers) is included in the set of integers

Note 3 10^{-n} is written $0.0\ldots1$, with n zeros: a zero before the decimal point (position of units) and $n-1$ zeros thereafter.

A **tenth** is one part of a unit divided into 10 equal parts, and a **hundredth** is one part of a unit divided into 100 equal parts (Fig. 1.3). The number 7×10^{-5} is therefore smaller than the number 2×10^{-4}. Using tenths (10^{-1} also denoted 0.1), hundredths (10^{-2} or 0.01), and so forth, we can describe **decimals**.

For example, in the number "323,003.03", the number 3 takes four different values from left to right: the value "three hundred thousand" for the first 3, the value "three thousand" for the second 3, the value "three units" for the third, and the value "three hundredths" for the last 3, which gives (Fig. 1.4)

$$\begin{aligned}323,003.03 = 3 \times 10^5 + 2 \times 10^4 + 3 \times 10^3 + \\ 0 \times 10^2 + 0 \times 10^1 + 3 \times 10^0 + \\ 0 \times 10^{-1} + 3 \times 10^{-2}.\end{aligned}$$

By associating the signs + or − with these numbers, we can describe the **decimals**. For example, the number −0.03 is a negative decimal number greater than the number −1.2.

Table 1.1 Scientific notation vs. classical notation

Scientific notation	Classical notation
10^n	$\underbrace{10 \times 10 \times \ldots \times 10}_{n \text{ times}}$
...	...
10^3	$10 \times 10 \times 10 = 1000$
10^2	$10 \times 10 = 100$
10^1	10
10^0	1
10^{-1}	0.1
10^{-2}	$0.1 \times 0.1 = 0.01$
10^{-3}	$0.1 \times 0.1 \times 0.1 \times = 0.001$
...	...
10^{-n}	$\underbrace{10^{-1} \times 10^{-1} \times \ldots \times 10^{-1}}_{n \text{ times}}$

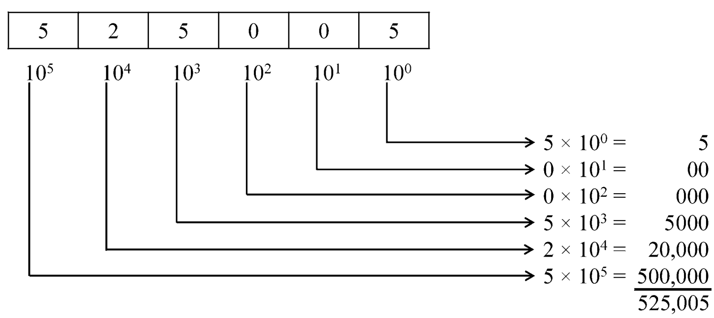

Fig. 1.2 Principle of decomposition of numbers in base 10

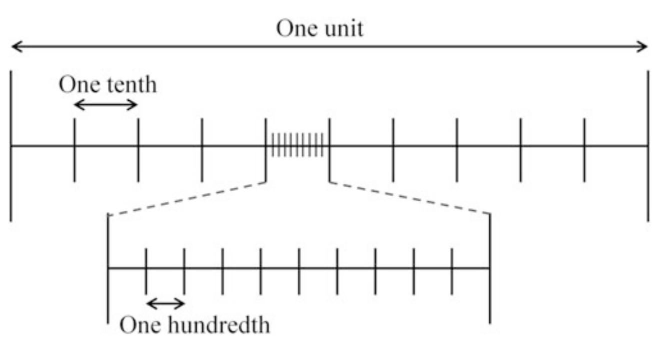

Fig. 1.3 Decimal subdivision: each interval is divided into ten other intervals, and so on

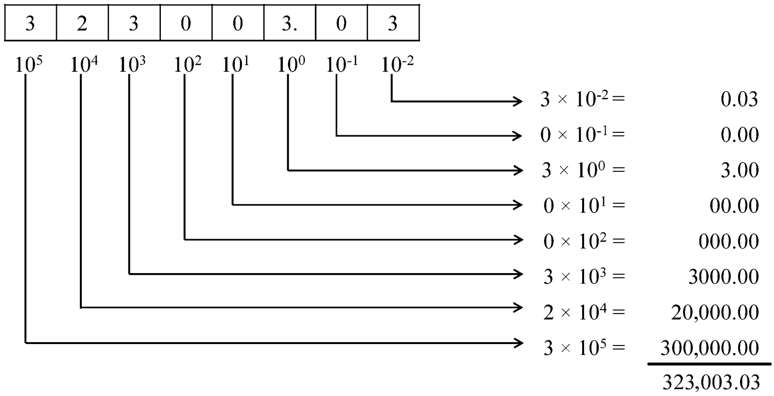

Fig. 1.4 Decomposition of 323,003.03 in base 10

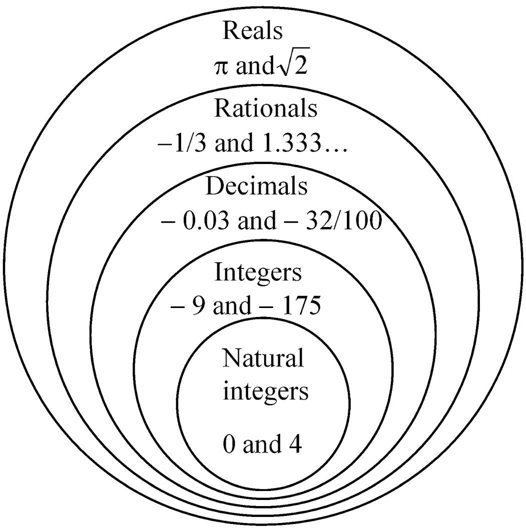

Fig. 1.5 Sets of numbers. Natural integers (or positive integers) are included in the sets of integers, themselves included in decimals, and so on. These inclusions stop in this book at real numbers, but they continue with other numbers such as complex numbers

1.1.4 Set of Numbers

All positive integers are integers (Figs. 1.1 and 1.5). All integers are decimals. Indeed, 4 can be written as 4.0. There are other types of numbers such as **rational numbers**, which can be written as a ratio of two integers, $-\frac{1}{3}$, for example. All decimal numbers are rational numbers: $-3 \times 10^{-1} - 2 \times 10^{-2} = -\frac{32}{100}$. There are also **real numbers**, which can be rational or irrational. For example, π and $\sqrt{2}$ are irrationals.

Table 1.2 Properties of powers of 10. Here n and m are positive integers

General form	Example
$10^n \times 10^m = 10^{n+m}$	$10^2 \times 10^4 = 10^6 = 1{,}000{,}000$
$10^{-n} = \frac{1}{10^n}$	$10^{-4} = \frac{1}{10^4} = 0.0001$
$\frac{10^n}{10^m} = 10^{n-m}$	$\frac{10^2}{10^3} = 10^{-1} = 0.1$
$(10^n)^m = 10^{n \times m}$	$(10^2)^3 = 10^6 = 1{,}000{,}000$

1.1.5 Properties of Powers of 10

These properties are synthesized and illustrated in Table 1.2.

Example 1

$$33 \times 10^{13} / \left(0.3 \times 8 \times 7.132 \times 10^6\right) = \frac{33 \times 10^{13}}{0.3 \times 8 \times 7.132 \times 10^6} = \frac{33 \times 10^{13}}{17.1168 \times 10^6} = \frac{33 \times 10^{13-6}}{17.1168}$$

The result is thus approximately $1.9279 \times 10^7 = 19{,}279{,}000$

$$\left(17.83 \times 10^{-8}\right)^2 = 317.9089 \times 10^{-8\times 2} = 317.9089 \times 10^{-16} = 3.179089 \times 10^{-14}$$

Note 4 There is no specific property for the sum and difference of powers of 10. In this case we use the classic decimal form to perform sums and differences.

Example 2

$$\begin{aligned}&28.53 \times 10^4 - 13.1432 \times 10^2\\&= 285{,}300 - 1314.32\\&= 283{,}985.68\\&= 2.8398568 \times 10^5\end{aligned}$$

1.1.6 Scientific Notation of a Decimal Number

Definition 3 The scientific notation of a decimal number d is written in the form $\pm a \times 10^n$ where:

- a is a decimal number greater than or equal to 1 and strictly less than 10 (which means that there is only one digit, not zero, to the left of the decimal point in the writing of the number a);
- n is a relative integer (if n is positive, the number $a \times 10^n$ is greater than 1; if n is strictly negative, the number $a \times 10^n$ is smaller than 1).

Scientific notation is a compact way of writing very small and very large numbers.

Example 3: Scientific Notation
- 323,003.03 is written 3.2300303×10^5
- 0.000000897 is written 8.97×10^{-7}
- -4.512 can be written -4.512×10^0

1.1.7 Sexagesimal System

The sexagesimal system is a numeral system with 60 at its base. Two main examples illustrate this: hours, minutes, and seconds for the measurement of duration (time), and degrees, minutes, and seconds for the measurement of angles.

For times, the unit of measure is the hour (h), then the minute (60 minutes = 60 min = 60′ = 1 h), and then the second (60 seconds = 60 s = 60″ = 1 min).

Similarly, for angles, the basic unit of measurement is the degree (°), then the minute (60 min = 1 degree), and then the second (60 s = 1 min). A full circle measures 360°. Chapter 4 on mapping will address these angle units in more detail.

Example 4: Duration Calculations
- Going from the sexagesimal system to the decimal system:
  7 min 14 s = 7 min + 14/60 min $\simeq$ 7.23 min.
  Indeed, 1 s corresponds to 1/60 min.

- Going from the decimal system to the sexagesimal system:
  7.92 h = 7 h and 0.92 × 60 min = 7 h 55.2 min = 7 h 55 min and 0.2 × 60 s = 7 h 55 min 12 s.

- Switching from the decimal system to time units:
  0.042 year = 0.042 × 365 days = 15.33 days = 15 days and 0.33 × 24 h = 15 days and 7.92 h = 15 days 7 h and 0.92 × 60 min = 15 days 7 h 55 min and 0.2 × 60 s = 15 days 7 h 55 min 12 s.

- Calculation in the sexagesimal system.
  The difference 48 h 6 min 3 s − 29 h 15 min 15 s is calculated starting from the smallest unit, the second. Since it is not possible to calculate 3 − 15 s, we convert 1 min into 60 s:
  48 h 6 min 3 s − 29 h 15 min 15 s = 48 h 5 min 63 s − 29 h 15 min 15 s
  = 48 h 5 min 48 s − 29 h 15 min.
  The calculation is now on the minutes. Since it is not possible to calculate 5 − 15 min, 1 h is converted into 60 min:
  48 h 5 min 48 s − 29 h 15 min = 47 h 65 min 48 s − 29 h 15 min
  = 47 h 50 min 48 s − 29 h.
  We finish the calculation and finally get
  48 h 6 min 3 s − 29 h 15 min 15 s = 18 h 50 min 48 s.

1.2 Measurement of Quantities

1.2.1 Quantities and Units of Measure

Earth science specialists and geographers deal with **quantities**. The most frequently encountered quantities are length, mass, time, temperature, area, volume, force, energy, pressure, power, and flow.

These amounts are measured. Measurement units of the International System of Units (which is the modern form of the metric system) are often used: the meter (m) as the unit of measurement of length, the kilogram (kg) as the unit of measurement of mass, the second (s) as the unit of time, and the kelvin (K) as the unit of measurement of temperature (Table 1.3). Many other units exist that are multiples of these basic units (Table 1.4) or other units, for example, the degree Celsius (°C) or degree Fahrenheit (°F) for temperature, liter (l and sometimes L) for volume (or rather capacity), or pascal (Pa) for pressure (Table 1.5).

Table 1.3 Some units of the International System of Units

Quantity name	Unit name	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Temperature	kelvin	K

Table 1.5 Examples of units derived from International System of Units and expressed using base units

Quantity name	Unit name	Symbol
Area	square meter	m^2
Volume	cubic meter	m^3
Speed	meter per second	$m\ s^{-1}$
Debit	cubic meter per second	$m^3\ s^{-1}$
Acceleration	meter per square second	$m\ s^{-2}$
Density	kilogram per cubic meter	$kg\ m^{-3}$
Force	newton	N or $m\ kg\ s^{-1}$
Pressure	pascal	Pa or $N\ m^{-2}$ or $m^{-1}\ kg\ s^{-1}$

Table 1.4 Relations between powers of ten, some prefixes of simple units: tera (T), giga (G), mega (M), kilo (k), hecto (h), deca (da), deci (d), centi (c), milli (m), micro (μ), nano (n), and pico (p) and orders of magnitude of distances in meters

Symbol	Power of 10	Value	Example
T	10^{12}	1,000,000,000,000	Sun–Saturn distance (1.4 Tm)
G	10^{9}	1,000,000,000	Diameter of Sun (1.4 Gm)
M	10^{6}	1,000,000	Length of Amazon River (6.4 Mm)
k	10^{3}	1000	International mile (1609 km)
h	10^{2}	100	Height of Eiffel Tower (3.2 hm)
da	10^{1}	10	Size of house (2 to 3 dam)
unit	10^{0}	1	Average height of UK woman (1.61 m)
d	10^{-1}	0.1	Height of this book (3 dm)
c	10^{-2}	0.01	Width of average finger (1.5 cm)
m	10^{-3}	0.001	Size of a grain of coarse sand (1 to 2 mm)
μ	10^{-6}	0.000001	Size of clay particle (2 to 4 μm)
n	10^{-9}	0.000000001	Diameter of DNA molecule (2 nm)
p	10^{-12}	0.000000000001	X-ray wavelength (5 pm)

Insert 1 (Googol, Google)
When handling numbers, one often needs to name them: 1200 reads "one thousand two hundred," 5×10^{-2} is pronounced "five hundredths." But it is sometimes very difficult to give a name to numbers when they are very small or very large. The prefixes help us for the small ones (deci, centi, milli, micro, nano, pico, femto atto, zepto, or yocto for 10^{-24}) as well as for large ones (deca, hecto, kilo, mega, giga, tera, peta, exa, zeta, or yotta for 10^{24}). But how does one name, for example, 10^{100}, that is, the digit 1 followed by 100 zeros? This number is called googol. There is also the googolplex, which has a value of 10^{googol}, or the digit 1 followed by 10^{100} zeros (that is, a 1 followed by a googol of zeros) and the googolplexian which has a value of $10^{\text{googolplex}}$, that is, a 1 followed by 10^{googol} zeros: the world's largest number with a name! All this can make you dizzy!

Let us return to the googol. One sees immediately the similarity of this word to the name of a famous search engine. The story goes that the two founders of Google (Sergey Brin and Larry Page) wanted to give a name to their search engine that would symbolize the large amount of information available on the Internet. After soliciting ideas from their students, one of them (Sean Anderson) suggested googol. Then the stories differ, with some saying that Anderson misspelled "Googol" as "Google" when registering the domain. Others claim that the name "Googol" was already taken, so it was changed intentionally. This naming problem had no adverse effect on the company, which in 2017 had a capitalization of around USD 570 billion!

1.2.2 Operations on Units of Measurement

Certain quantities have no unit of measure; they are called dimensionless or bare numbers. Only quantities of the same unit can be added (or subtracted). For example, to add 1.35 km and 247 m, the two lengths must be converted into the same unit (meters, for example). We cannot add two quantities that are not of the same nature (meters and seconds, for example).

The products (multiplication) and quotients (division) of magnitudes are always possible, whatever their units. The result is expressed in a new unit derived from the calculation.

Example 5 Density is calculated by dividing the mass of a material (e.g., in kg) per unit volume (e.g., in m^3). The density is then expressed in kg/m^3 (read "kilograms per cubic meter") or $kg\ m^{-3}$. For example, the density of Earth is about $5.52\ g/cm^3$.

The flow of a river is calculated by dividing the volume of water (e.g., in m^3) that flows in a given time (e.g., in s). The flow rate is therefore expressed in m^3/s (read "cubic meters per second") or $m^3\ s^{-1}$.

For example, the interannual average flow of the Loire River at its outlet is about $930\ m^3/s$ (measurements carried out between 2004 and 2008 in Saint-Nazaire, Loire-Atlantique, France).

If we know that $-\alpha \times t$ is a dimensionless quantity and that t is expressed in seconds (s), then necessarily α is expressed in s^{-1}, so that $-\alpha \times t$ is without a unit ($s \times s^{-1} = s^0 = 1$).

The previously used density is an example of a quantity with a unit using fractions: it indicates the mass contained in the given unit of volume. Speed (velocity) is another example: $\textbf{\textit{speed}} = \frac{\textbf{\textit{distance}}}{\textbf{\textit{time}}}$. For a moving object it corresponds to the distance traveled (in the chosen unit of length) in a given time (in the chosen

Table 1.6 Examples of converting a number into its multiples and submultiples

		k	h	da		d	c	m	
Example 6	1	5	3	0	0	0	0	0	0
Example 7	0	0	0	0	1	7	6	2	5

unit). Its unit is denoted by either m/s or m s^{-1}, and this unit is expressed as a fraction (similarly for km/h or km h^{-1}, g/cm^3 or g cm^{-3}).

1.2.3 Conversion of Units of Measure

For simple units, Examples 6 and 7 below show how to convert a number into its multiples and submultiples using two different methods. The first method is the one used in the USA/UK, and you need to know the relation between the two units; the second method is the French way of doing conversion using a conversion table (Table 1.6).

Example 6: Convert 15.3 kg into mg

Method 1:

We have 1 kg = 1000 g; 1 g = 1000 mg.
So 1 kg = 1000 × 1000 mg = 10^6 mg.
Thus, 15.3 kg = 15.3 × 10^6 mg = 1.53 10^7 mg.

Method 2:

To convert 15.3 kg to mg, 15.3 is inserted in the conversion table by placing the 5 of the 15.3 in the kg column (because 5 is the unit figure of 15.3). Complete with the number of 0s necessary to reach the column of mg and read the result: 15.3 kg = 15,300,000 mg (Table 1.6).

Example 7: Convert 176 cm into km

Method 1:

Use the fact that 1 m = 10^{-3} km and that 1 cm = 10^{-2} m.

So 1 cm = $10^{-3} \times 10^{-2}$ km = 10^{-5} km.

Therefore, 176.25 cm = 176.25 × 10^{-5} km = 1.7625 × 10^{-3} km.

Method 2:

To convert 176.25 cm to km, position 176.25 in the conversion table by placing the unit figure 6 of 176.25 in the cm column. Complete using the number of zeros necessary to reach the km column and read the result: 176.25 cm = 0.0017625 km (Table 1.6).

To convert quotient units, first use common sense and proportionality.

Example 8 You want to convert a speed measurement from 2 m/s to km/h. If you walk 2 m in a second, you will have traveled 2 × 3600 m = 7200 m in 1 h (since 1 h = 3600 s). We will have traveled 7.2 km in 1 h. We thus have 2 m/s = 7.2 km/h.

Note 5 The calculation to be carried out in a formula makes it possible to know in advance what unit the result will be expressed in, and this makes it possible to verify the consistency of the reasoning. This is **the homogeneity of dimensions.** For example, if one knows the mass of a component expressed in kg and its density ρ expressed in kg/m^3, then one can determine its volume V in m^3. Moreover, we can find how to determine this volume thanks to the units: if one writes (by mistake!) $V = \frac{\rho}{M}$, this indicates that V should be expressed in $\frac{kg/m^3}{kg}$, that is to say, in m^{-3}, which is wrong. This leads to the correct way to write, $V = \frac{M}{\rho}$, whose unit is m^3.

1.2.4 Areas and Volumes

Areas and volumes are expressed in units that are products of units of length (e.g., cm^2 or dm^3), so 1 cm^2 corresponds to the area of a square with an edge length of 1 cm, 1 dm^3 corresponds to the volume of a cube with an edge length of 1 dm.

For conversions of units of area or volume, one first converts the base unit that composes it and then carries out the operation (square or cubic). You can also use a double or triple-column conversion table for each unit.

Table 1.7 Examples of conversion for areas (Examples 9 and 11)

km^2		hm^2		dam^2		m^2		dm^2		cm^2		mm^2	
0	0	0	0	0	0	0	0	0	0	1	2	5	6
0	0	2	5	0	0	0	0	0	0	0	0	0	0

Table 1.8 Volume of conversion examples (Examples 10 and 12)

dam^3			m^3			dm^3			cm^3		
0	0	0	0	0	0	1	3	0	0	0	0
0	0	0	0	0	0	0	0	0	0	2	5

Example 9 We want to express 12.56 cm^2 in m^2.

Method 1: We have 1 cm = 0.01 m,

so 1 $cm^2 = 0.01 \times 0.01$ m^2 = 0.000 1 m^2.

And therefore 12.56 $cm^2 = 12.56 \times 0.000\,1$ m^2 = 0.001256 m^2,

which in scientific notation is written 1.256×10^{-3} m^2.

Method 2: We use a double-column conversion table as in Table 1.7. Position 2, figure of unit of 12.56, in the right column of the double column of cm^2. Then complete using the number of 0s necessary to reach the column on the right of the double column of m^2. This gives 12.56 cm^2 = 0.001256 m^2.

Example 10 We want to express 0.13 m^3 in dm^3.

Method 1: we have 1 m = 10 dm.

Therefore, 1 $m^3 = 10 \times 10 \times 10$ dm^3 = 1000 dm^3.

Hence, 0.13 $m^3 = 0.13 \times 1000$ dm^3 = 130 dm^3.

Method 2: With a three-column conversion table (Table 1.8), place the 0s of the units of 0.13 m^3 in the right-hand column of the triple column of m^3. Then complete using the number of 0s necessary to reach the column on the right of the triple column of dm^3. This gives 0.13 m^3 = 130 dm^3.

Surface units are often expressed in **hectares** (ha). In "hectare" we find the prefix "hecto-," associated with 100 (1 ha is 1 hectometer squared). To visualize the surface obtained, the following means can be used: a square surface of 100 m on each side has an area of 1 ha. An area of 1 ha therefore corresponds to 100×100 m^2, i.e., 10,000 m^2. A soccer field measures approximately 0.7 ha.

Example 11 We wish to convert 25 ha into km^2.

Method 1: We know that 1 ha = 0.01 km^2 (if you don't know it by heart, see subsequent discussion on how to calculate this).

To get from 1 to 0.01, we divide by 1000. Thus, 25 ha = 25/1000 km^2 = 0.025 km^2.

Method 2: We first use the equality 1 ha = 1×10^4 m^2, so 25 ha = 25×10^4 m^2 = 250,000 m^2.

Having placed 250,000 m^2 in Table 1.7, we obtain 25 ha = 0.25 km^2.

The volume units are often expressed in **liters** (L) or its multiples and submultiples (hL and mL, for example). Two conversions should be memorized: 1 m^3 contains 1000 L and 1 dm^3 contains 1 L (abusively written 1 m^3 = 1000 L and 1 dm^3 = 1 L).

Example 12 To convert 25 cm^3 to liters, insert 25 cm^3 in Table 1.8 (with the 5 in the right-hand column of cm^3). This gives 25 cm^3 = 0.025 dm^3, and therefore 25 cm^3 = 0.025 L = 25 mL.

In the case of quotient units involving surfaces or volumes, the rules of proportionality and the conversion tables are also used.

Example 13 We want to convert the density of the Earth, which is 5.52 g/cm^3, into kg/m^3. It is known that 1 $m^3 = 10^6$ cm^3. Thus we have 5.52 g/cm^3 = 5.52×10^6 g/m^3. Because 1000 g = 10^3 g = 1 kg, we therefore have 5.52 g/cm^3 = 5.52×10^3 kg/m^3 (approximately 6 tons/m^3).

1.3 Approximate Values

Can the statements "there were 64,596,800 inhabitants in the United Kingdom in 2014" (which is an estimate) and "the temperature on Mars is on average −65.53°C" (which is a measure) be accurate to the nearest unit? To the nearest hundredth?

If you round up, you must indicate a domain in which the actual value is located.

1.3.1 Rounding

We will see in the following sections that we do not always retain all the numbers of a result. Quite often, one does not need to know the exact value of a quantity (for example, you do not need to know the distance between Brussels and Paris or between Toronto and Quebec to the nearest millimeter). This section, however, is not intended to indicate the precision to which a number should be taken; this notion depends on the context of the geographical and Earth science situations.

Once the level of accuracy is chosen (e.g., one chooses to round to the nearest hundredth, that is to say, to round to two decimal places), the **rounding rule** is as follows:

- The digit in the position to be rounded (e.g., the number of hundredths) is marked;
- The digit to the right of the position to be rounded is considered. If it is 0, 1, 2, 3, or 4, then the digit of the position to be rounded is not changed. If it is 5, 6, 7, 8, or 9, then the digit of the position to be rounded is increased by 1;
- We discard the digits after the position to be rounded.

Example 14
- 12.674 rounded to two decimal places (rounded to the nearest hundredth) is 12.67
- 12.679 rounded to two decimal places is 12.68
- 12.675 rounded to two decimal places is 12.68
- 12.696 rounded to two decimal places is 12.70

Note 6
- When carrying out a calculation, only the end result is rounded; any rounding that takes place before will affect the end result, giving an incorrect answer.
- There are other delicate rounding rules for when the digit to the right of the position to be rounded is 5. We do not elaborate on these methods here.

1.3.2 Significant Figures

Significant figures indicate the degree of precision. Results are sometimes obtained with too many figures, suggesting illusory precision: care should be taken to limit oneself to figures that are reasonably reliable. In a calculation, an estimate, or a measure, not all the obtained figures are significant and so should not be included in the answer. Table 9 gives some rules on the choice of significant figures.

Note 7 In the case of the last line of Table 1.9, writing the number in scientific notation makes it possible to indicate precisely the status of these zeros: we write 2180×10^3 if the zero is significant; otherwise we write 2.18×10^3.

Calculation Rules
- When two numbers are added or subtracted, the result is rounded according to the precision of the least precise number.

For example, $31.66 + 123.4 = 155.0\cancel{6} = 155.1$ (only one decimal is kept, as 123.4 has only one decimal).

On the other hand, we have $31.66 + 123.400 = 155.06$.

Table 1.9 Rules for significant figures

	Examples of numbers	
Nonzero digits (different from zero) are significant.	0.00**1** **4**0**6** **7**00	**218**0
Zeros placed between two nonzero digits are significant.	0.001 4**0**6 700	2180
Zeros placed to the left of the first significant digit are not significant.	$\cancel{0}\cancel{0}\cancel{0}, \cancel{0}\cancel{0}1$ 406 700	$\cancel{0}2180$
Zeros placed to the right and after the decimal point are significant.	0.001 406 7**00**	
Zeros placed just to the left of the decimal point may or may not be significant.		218**0**

- When two numbers are multiplied or divided, the result is expressed with the same number of significant digits as the least accurate number.

For example, $3.14 \times 12.75 \times 10^3 = 40\,0\cancel{3}\cancel{5} \simeq 40{,}000 = 4.00 \times 10^4$. Only three significant digits are shown, as 3.14 has only three.

1.3.3 Absolute Uncertainty

Definition 4
The **absolute uncertainty** of a quantity A is the margin of inaccuracy of measurement of this quantity. It is noted by the symbol ΔA and it has the same unit as the quantity.

Without indication, the last significant figure is considered to be ± 0.5.

- $T = 23.597^\circ C$ *means that* $23.5975^\circ C \geq T \geq 23.5965^\circ C$;
- $T = 23.60^\circ C$ *means that* $23.605^\circ C \geq T \geq 23.595^\circ C$;
- $T = 23.6^\circ C$ *means that* $23.65^\circ C \geq T \geq 23.55^\circ C$.

When writing $T = 23.5970^\circ C \pm 0.0005^\circ C$, $T = 23.5970^\circ C$ is the central value and $\Delta T = 0.005^\circ C$ is the absolute uncertainty of this measure.

The absolute uncertainty must be rounded to a single significant figure, always upward so as not to underestimate the actual uncertainty (e.g., $\Delta x = 0.2$ m and not $\Delta x = 0.13$ m). The measure is then itself rounded (this time with the usual rounding rules) according to the number of decimals of the absolute uncertainty.

Example 15

13.467 ± 0.14 cm $= 13.5 \pm 0.2$ cm.
1.671×10^2 s ± 47 s $= 167 \pm 5$ s $=$
$(1.67 \pm 0.05) \times 10^2$ s.

For a sum or a difference, add the absolute uncertainties (ensuring beforehand the coherence of the units and rounding the sum of the uncertainties).

Example 16 We consider $x_1 = 1.678 \pm 0.007$ m and $x_2 = 12.45 \pm 0.04$ m. Let us write $x = x_1 + x_2$; we then have $\Delta x = \Delta x_1 + \Delta x_2 = 0.007 + 0.04$ m $= 0.047$ m $= 0.05$ m, and thus, $x = 12.128 \pm 0.05$ m $= 12.13 \pm 0.05$ m.

1.3.4 Relative (or Percentage) Uncertainty

Definition 5
The **relative (or percentage) uncertainty** of a quantity A is the ratio of the absolute uncertainty of the value to the measured quantity. It is denoted $\frac{\Delta A}{A}$ and is therefore dimensionless.

Example 17 In $T = 23^\circ C \pm 0.5^\circ C$, the relative uncertainty is $\frac{\Delta T}{T} = \frac{0.5}{23} \simeq 0.022 = 2.2\%$. The central value is accurate to within 2.2%.

Relative uncertainty provides information about the accuracy of a value. Two values with the same absolute uncertainty may not at all be known with the same precision. The smaller the relative uncertainty, the better the approximation.

Example 18 We know the masses M_1 and M_2 of two objects with an absolute uncertainty of 1 kg each: $M_1 = 2.697$ kg $\pm$ 1 kg and $M_2 = 15$ kg $\pm$ 1 kg. M_1 is known with a relative accuracy of $\frac{\Delta M_1}{M_1} = 0.037\%$, whereas M_2 is known to $\frac{\Delta M_2}{M_2} = 6.7\%$: M_1 is known with much greater accuracy than M_2.

For a product (two numbers multiplied together) or a quotient (the division of one number by another) of values, the relative uncertainties are added. If x and y are quantities and $z = x \times y$ $(= xy)$ and $t = \frac{x}{y}$, we therefore have

$$\frac{\Delta z}{z} = \frac{\Delta x}{x} + \frac{\Delta y}{y}$$
and $$\frac{\Delta t}{t} = \frac{\Delta x}{x} + \frac{\Delta y}{y}.$$

The measures of the sides of a rectangular basin are given by $L = 1250\ \text{m} \pm 5\ \text{m}$ and $l = 875\ \text{m} \pm 5\ \text{m}$. We want to calculate its area. We have thus $L = 1250\ \text{m} \pm 0.40\%$ and $l = 875\ \text{m} \pm 0.57\%$. The area A of the basin therefore has the value

$A = 1250 \times 875\ \text{m}^2 \pm (0.40 + 0.57)\% = 1{,}093{,}750\ \text{m}^2 \pm 0.97\% = 1{,}093{,}750\ \text{m}^2 \pm 10{,}609\ \text{m}^2 = 1{,}090{,}000\ \text{m}^2 \pm 10{,}000\ \text{m}^2$.

The result contains as many significant digits (three) as the initial measure with the least significant digits.

1.3.5 Errors

Error gives an indication of the vagueness of a value or measure. To calculate it, we need two values:

- The actual value of a quantity or a value considered as a reference value;
- A measured value for this quantity.

Definition 6
The *absolute error* of a measured variable is the difference between this measure (denoted by x^*) and the reference value of the quantity (x). The absolute error is denoted by $\Delta x = |x^* - x|$ (the absolute value function is used so that the result is always positive). The absolute error therefore has the same unity as the measured quantity.

The *relative error* is defined as $\frac{\Delta x}{x}$ and is thus dimensionless.

Example 19 The speed of light is currently considered to be $c = 299{,}729$ km/s. An experiment allows us to obtain a value of the speed of light of $c_m = 300{,}120$ km/s.

The absolute error is $|c - c_m| = 391$ km/s (the same unit since this is a difference).

The relative error is $\frac{391}{299{,}729} \approx 1.3 \times 10^{-3} = 0.13\%$ (the relative error is dimensionless).

Insert 2 (Standard Meter and Unit)
The meter is the basic unit of the length of the International System of Units, but this has not always been the case. For a long time several basic units of length coexisted: the Roman foot (296 mm), the mile (1609 m), the arpent (71.851 m)... While this heterogeneity was not necessarily a problem in times of reduced world trade, single basic units are needed today. The meter was born on March 26, 1791, in the climate of reforms following the French Revolution as a reference to a universal distance: the ten-millionth part of a quarter of the terrestrial meridian. Thus a quarter of the terrestrial meridian is ten million meters, or 10,000 km, so that the circumference of the Earth is, by definition, 40,000 km. The International Bureau of Weights and Measures (Paris) redefined the meter in 1889 as the length of the standard bar (made of platinum and iridium) that it retains within its walls. Since 1983, the meter has been defined as the distance traveled in a vacuum at the speed of light in 1/299792.458 s.

The coexistence of several measuring systems can have serious consequences. In December 1998, the Mars Climate Orbiter robotic space probe was launched into space and reached Mars 9 months later. During its automatic entry into the orbit of Mars, a mistake in the calculation of the trajectory led to the destruction of the probe. A commission of inquiry pointed to the problem: the propulsion of the probe was calculated in a non-SI unit, whereas the navigation was all in SI units. The cost of this mission is estimated to be hundreds of millions of dollars!

1.4 Percentages

To calculate a percentage, the partial value (that for which the percentage is considered) is divided by the total value.

Example 20 The calculation of the relative uncertainty consists in calculating the percentage of uncertainty relative to the central value: $\frac{\Delta v}{v}$. The result is a number between 0 and 1. You can also write, more classically, $\left(\frac{\Delta v}{v} \times 100\right)\%$.

For example, $0.05/0.12 \approx 0.42 = 42\%$.

To apply a percentage to a value, multiply this value by the percentage (or the number between 0 and 1).

Example 21 According to the Office for National Statistics, the UK's population was around 64,596,800 in 2014, of which 50.8% were female. So in that year there were 64,596,800 × (50.8/100) ≃ 32,815,000 females in the UK.

A *rate of change* is calculated by identifying the starting value (v_d) and final value (v_a) using the formula $\frac{v_a - v_d}{v_d} = \left(\frac{v_a - v_d}{v_d} \times 100\right)\%$. The result may be positive (increase) or negative (decrease).

Example 22 A population decreased from 80,700 to 80,200 people. The rate of change is $\frac{80,200 - 80,700}{80,700} = -0.6\%$.

Key Points
- In the decimal system, the numbers are decomposed according to powers of 10. Scientific writing is based on this decomposition. In the sexagesimal system (hours, minutes, seconds, and angles), it is multiplication or division by 60 that intervenes in changes of unity.
- Multiplication and division can be performed on units of measurement, and a new unit results from the operation (e.g., m divided by s gives m/s). Additions and subtractions can be performed only if the quantities are expressed in the same unit.
- A one-column table can be used for conversions of single units (e.g., mass or length) and double- or triple-column tables for conversions of units of area or volume.
- The rules of proportionality and common sense are important to reason soundly about the magnitudes (to know in advance the unity of a result, with an order of magnitude of what will be obtained).
- The precision of the results obtained obeys rules (rounded, significant figures) to better understand the uncertainty and the error.

Exercises

Mathematical Exercises

Exercise 1.1: Flash Questions. Series 1

(1) Convert 12.67 dm into km.
(2) Convert 1 mm^2/s into m^2/year.
(3) Simplify $\frac{3 \times 10^5}{0.1 \times 10^7}$.
(4) What are the significant digits of 345, of 2.40×10^{-4}, and of 250?
(5) When $x = 12.78 \pm 0.05$ and $y = 1659 \pm 0.001$, express $s = x + y$ and $p = \frac{x}{y}$.

Exercise 1.2: Flash Questions. Series 2

(1) Write using the scientific notation $8 \times 10^{-5} \times 3.71 \times 10^4$.
(2) How many seconds are there in 2 h 48 min 10 s?
(3) Which is the highest speed between 10 m/s and 10 km/h?
(4) Write the results with the correct number of significant figures: 12, 7 − 3, 68; $4, 5 \times 10^3/(2.111 \times 10^2)$.
(5) What is the rate of change for a transition from 12.7 to 13.2?

Exercise 1.3: Flash Questions. Series 3

(1) Write using the scientific notation: $\frac{1.8 \cdot 10^{-3}}{2 \cdot 10^{-5}}$.
(2) Express in decimal hours: 48 h 12 min 46 s.
(3) Convert 17.88 cm^3 into liters.
(4) When $x = (1.59 \pm 0.03)$ and $y = 1\,659 \pm 0.001$, express $p = xy$.
(5) Calculate the relative error and absolute error committed by replacing $x = 15.91$ per $x^* = 16.02$.

Exercise 1.4: Powers of 10 and Scientific Notation

(1) Simplify: $10^4 \times 10^{-3}$; $10^{-12} \times 10^7$; $\frac{10^5}{10^{11}}$; $\frac{10^{-2}}{10^5}$; $(10^{-3})^2$; $\left(\frac{1}{10^4}\right)^3$; $-3.8 \times 10^4 + 252 \times 10^5$.
(2) Write the following numbers using scientific notation: $\frac{21}{1000 \times 9.81 \times 0.000035}$; $\frac{1}{3.85 \times 10^6}$.
(3) Perform the following calculation giving the result in scientific notation:
- 29.83/1000
- $17.32 \times 10^{15}/(0.002\,681 \times (6370 \times 10^3)^2)$
- $\frac{(1 \times 10^{-3})^2 \times 325}{23.65 \times 18 \times 10^{-7}}$
- $\frac{1.374 \times 10^{23} \times 5.981 \times 10^{-4}}{6.590 \times 10^{17} \times 2.367 \times 10^{-3}}$

(4) Repeat the preceding three questions using R. This training is essential to learn how to use the basic elements of R. Special attention should be paid to the notation in R of the powers of 10 and the positioning of brackets.

Exercise 1.5: Time Measures

(1) Convert into hours, minutes, and seconds: 3.5 h; 12.75 h; 8.86 h; 0.026 7 h.
(2) Convert to decimal hours: 287 h 28 min 42 s; 18 h 48 min 10 s.

Exercise 1.6: Conversion of Simple Units

(1) Express the following quantities in meters (the use of scientific notation is permitted): 42,000 km; 15.67 cm; 298.4 μm.
(2) Express the following quantities in square meters: 3.5 mm^2; 1 km^2; 3.7 ha.
(3) Express the following quantities in cubic meters: 1986 L; 29.8 km; 478.6 cm.

Exercise 1.7: Conversion of Quotient Units

(1) Convert the following values to meters per second: 300,000 km/s; 98 km h^{-1}; 2.9 mm/year; 35 m/min.
(2) Perform the required conversions: 5.9 mm^2/min $= \ldots$ m^2/s; 0.0549 kg/cm^3 $= \ldots$ g/m^3.
(3) A light-year is the distance traveled by light in vacuum in 1 year (the speed of light is around 300,000 km/s). Convert a light-year to meters.

Exercise 1.8: Significant Figures

(1) How many significant figures are there in the following numbers: 143; 143.6; 143,670; 0.014 3; 1.43×10^3; $1.430\,0 \times 10^3$; 1043; 1430?
(2) Express the results with the correct number of significant figures: 6.46 + 5.2; 8.59 × 13; 1589.400–1.0067; $\frac{1.207 \times 10^3}{1.4 \times 10^{-4}}$.

Exercise 1.9: Uncertainties

(1) Rewrite the following values with their uncertainty according to the rounding rules: 456.255 ± 3.56; 57.287 ± 0.15.
(2) Rewrite the following values expressing the result in scientific notation: 312,000 m^3 ± 3000 m^3; 0.004560 m ± 0.00006 m; 5.19×10^3s ± 17 s.

(3) Let us write: $x = 47.70 \pm 0.05$; $y = 8.62 \pm 0.01$; $z = 0.081 \pm 0.004$.
Calculate the central value and the absolute and relative uncertainties of the following values:

$$x + y; x \times y; x - 2z; \frac{x}{z}; y + z; \frac{y \times z}{x}.$$

Exercises in Geography and Geology

Note 8 In the following exercises, there are sometimes missing data or formulas (Earth's radius, volume formula of a sphere). It is up to the reader to discover them.

Exercise 1.10: Orders of Magnitude

It is important to have comparison scales to be able to quickly visualize whether a result is absurd or coherent. For example: is a lake surface of 0.03 km^2 possible? Is it possible for a river to have a flow of 1500 m^3/s?

The authors suggest some examples in this exercise. It is up to the readers to find and use personal orders of magnitude in their close proximity and to memorize some important orders of magnitude.

Complete Table 1.10 with the sizes of your choice (to be memorized).

Example given by authors in Table 1.11.

Exercise 1.11: Duration of One Day

(1) Using a calendar showing the astronomical time (rounded to the nearest minute) of sunrise and sunset for a place and day of your choice, calculate the length of that day (length between sunrise and sunset/daylight hours).

Table 1.10 Some orders of magnitude

Measure	Unit	Your choices
Length	m, km	
Area	m^2, km^2, ha	
Volume	m^3, L	
Mass	g, kg, ton	
Density	kg/m^3	
Speed	m/s, km/h	
Flow	m^3/s, L/s	

Table 1.11 Some orders of magnitude

Measure	Unit	Choice of authors
Length	m, km, mile, ft., inch	Length of Loire River: about 1000 km; height of Kilimanjaro: 5900 m. There are 26.2 miles in a marathon. The average UK woman is 5 ft., 4 inches tall.
Area	m^2, km^2, ha,	Area of Paris: 105 km^2; area of a football field: 0.7 ha.
Volume	m^3, L	Capacity of a bathtub: 150 L; volume of seas and oceans on Earth: 1340 million km^3.
Mass	g, kg, ton, lb., oz.	Weight of an African elephant: 7 tons; mass of Earth: 6×10^{24} kg. Average weight of a US woman: 166 lb. There are 6 oz. of sugar in a cake recipe.
Density	kg/m^3	Density of water: 1000 kg/m^3 (1 kg/L); density of limestone: 2700 kg/m^3.
Speed	m/s, km/h	Speed of a walker: 5 km/h (1.4 m/s); speed of light: 300,000 km/s.
Flow	m^3/s, L/s	Flow of Loire River: 840 m^3/s; flow of a tap: 12 L/min.

(2) Are the length between the day and the night at the equinoxes (around March 20 and September 22) equal?

Exercise 1.12: Around the Earth

The questions in this exercise are independent of each another.

(1) The original definition of kilometer is as follows: it corresponds to 1/10,000th of the distance between the equator and the North Pole (along a meridian passing through Paris). Find the radius of the Earth from this definition and compare the result with the real value of the radius of the Earth.
(2) You are sitting in a chair located at the equator. How fast are you moving? Here, we consider only the rotation of the Earth on its own axis.
(3) What error is committed when approaching the number of seconds in a year (nonleap year) using $\pi\ 10^7$?

Exercise 1.13: Wavelength

Visible light is composed of a set of wavelengths ranging from 390 to 780 nm.

(1) The Sun emits radiation at wavelengths ranging from 3×10^{-1} to 3 μm. Does the Sun emit visible light?
(2) The Earth emits radiation at wavelengths ranging from 3×10^{-6} to 3×10^{-5} m. Does the Earth emit visible light?

Exercise 1.14: Albedo

Albedo is the reflective power of a surface. It makes it possible to know the amount of incident sunlight that is reflected by this surface. In terrestrial climatology, this variable expresses the rate of solar radiation that will be reflected back by the atmosphere and the Earth and that will not serve to warm up our planet. It is a unitless quantity (since it is the ratio of two quantities with the same unit).

(1) Knowing that the average incident solar radiation is of the order of 342 W/m^2 and that 102 W/m^2 are reflected by clouds, the atmosphere, and Earth's surface, what is the value of the average albedo of the Earth system? What rounding is it reasonable to do on this value?
(2) Considering that the reflected solar radiation (102 W/m^2) is known with an absolute uncertainty of 2 W/m^2, what are the lower and upper limits of this radiation? What is the value of the relative uncertainty?
(3) If we consider the value of $\frac{102}{342}$ as the reference albedo, what is the relative error due to this uncertainty of 2 W/m^2?

Exercise 1.15: Temperature Gradient

The theoretical value of the temperature gradient in the troposphere (the first 14 km of the atmosphere) is 6.49°C/km, i.e., the temperature decreases from 6.49°C every kilometer.

(1) What is this gradient when expressed in °C/m?
(2) Give the value of the gradient in K/km.

Exercise 1.16: Erosion Rate

The rate of erosion is a variable that varies greatly depending on the environment, geological properties, and the local climate. In the Jamaica Cockpit Country, limestone erosion can be estimated theoretically using several equations (carbonate chemistry, carbon dioxide pressure, precipitation) and has a value of the order of 130 mm/1000 years.

Note 9 We typically write "erosion rate," but it is actually a speed (a rate would be dimensionless).

(1) What is the value of the rate of erosion in mm/year and in m/year (to the decimal and in scientific notation)?
(2) In fact, this erosion rate has been estimated to an average value of 145 mm/1000 years. What is the absolute error of the model compared to the measurements? What is the relative error?

Exercise 1.17: Flow Rate

The average velocity of a river is calculated using reel measurements on a section perpendicular to the flow. One method of calculation makes it possible to obtain an average speed of 1.3 m/s.

(1) Express this speed in km/h.
(2) This average speed makes it possible to estimate the flow of the river. Knowing that the flow can be expressed in m^3/s, what type of quantity should be associated with the average speed to determine the flow?
(3) Since the cross section of the flow is 10 m^2, what is the value of the flow in m^3/s, then in L/min?
(4) Compare this value with the flow of a river near you.

Table 1.12 Earth, Mars, and Mercury planet data

Planets	Radius (km)	Weight (kg)
Earth	6378	5.98×10^{24}
Mercury	2439	3.3×10^{23}
March	3398	6.4×10^{23}

Exercise 1.18: Density Volumetric Mass of Planets
We are interested in the volumetric masses of Earth, Mercury, and Mars.

(1) Using only your own intuition, classify the densities of these planets in increasing order.
(2) Then calculate the densities of Earth, Mars, and Mercury in g.cm^{-3} using the data in Table 1.12.

Note 10 You will need to use the formula giving the volume of a sphere.

(3) Use R to calculate the volumetricmass of the planets in two different ways:
  - Method 1: Create vectors (`numeric()`) `Radius` and `Mass` then calculate the corresponding vector `VolumetricMass`;
  - Method 2: Create a table (`data.frame()`) containing the columns `PlanetName`, `Radius`, `Mass`, and `VolumetricMass`.

Exercise 1.19: Water of the Oceans
The amount of ocean water represents about 97% of all water on Earth, approximately 1.3×10^9 km^3. Evaporation of the oceans is estimated at 5×10^5 km^3/year. It is estimated that three-quarters of the surface of the Earth is occupied by the oceans.

(1) Assuming that all ocean water can evaporate, how long would it take to fully renew the oceans (that is to say, until all the water is evaporated)?
(2) What is the height of water evaporated from the oceans every year?
(3) The last report of the Intergovernmental Panel on Climate Change (IPCC) reports that sea levels rose by 19 cm between 1901 and 2010 (IPCC, 2014). If we assume that this increase is solely due to the melting of polar ice, how much water does this melt correspond to?
(4) It is estimated that the water contained in glaciers at the poles is 25×10^6 km^3 (De Marsily, 2000). What percentage of this represents the quantity of water calculated in the previous question?

Solutions

Solution 1.1
(1) 12.67 dm = 0.001267 km = 1.267×10^{-3} km.
(2) 1 year = 365 × 24 × 60 × 60 s = 31,536,000 s. Therefore, 1 mm^2/s = 31,536,000 mm^2/year = 31.536 m^2/ year because 1 m^2 = 1,000,000 mm^2.
(3) $\frac{3 \times 10^5}{0.1 \times 10^7} = \frac{3}{0.1} \times 10^{5-7} = 30 \times 10^{-2} = 3 \times 10^{-1} = 0.3$.
(4) The significant figures of 345 are 3, 4, and 5; those of 2.40×10^{-4} are 2, 4, and 0; those of 250 are 2, 5, and possibly 0 (250 should be written in scientific notation to be able to conclude).
(5) $x + y = 14.439 \pm 0.051 = 14.44 \pm 0.06$ (the absolute uncertainty is rounded first by upward). $x = 12.78 \pm 0.40\%$ and $y = 1.659 \pm 0.061\%$; therefore $\frac{x}{y} = 7.7034 \pm 0.47\% = 7.7034 \pm 0.0354 = 7.70 \pm 0.04$ (absolute uncertainties were added with rounding upward).

Solution 1.2
(1) $8 \times 10^{-5} \times 3.71 \times 10^4 = 29.68 \times 10^{-5+4} = 29.68 \times 10^{-1} = 2.968 \times 10^0$.
(2) 2 h 48 min 10 s = 2 × 3600 + 48 × 60 + 10 s = 10,090 s.
(3) 10 m/s = 10 × 3 600 m/h = 36 km/h. 10 m/s is greater than 10 km/h. Note: 10 m/s is the speed of a 100 m champion sprinter (about 100 m in 10 s), while 10 km/h is the speed of a moderate running stroke.

(4) 12.7–3.68 = 9.0 (to one decimal because 12.7 contains only one), $4.5 \times 10^3/(2.111 \times 10^2) = 21$ (two significant figures as for 4.5×10^3).

(5) $\dfrac{13.2 - 12.7}{12.7} \simeq 0.039$. The rate of change is 3.9%.

Solution 1.3

(1) $\dfrac{1.8 \times 10^{-3}}{2 \times 10^{-5}} = 0.9 \times 10^{-3-(-5)} = 0.9 \times 10^2 = 9 \times 10^1$.

(2) 48 h 12 min 46 s = 48 + 12/60 + 46/3600 h $\simeq$ 48.21 h.
The following conversions were used: 1 min = 1/60 h and 1 s = 1/3600 h (because 1 h = 3600 s).

(3) $17.88\ \text{cm}^3 = 0.017\,88\ \text{dm}^3 = 1.788\ \text{L} \times 10^{-2}$ (recall that $1\ \text{dm}^3 = 1$ L).

(4) $\dfrac{0.03 \times 10^2}{1.59 \times 10^2} \simeq 1.89\%$ and $\dfrac{0.001}{1.659} \simeq 0.07\%$ (rounded upward).
We have $x = 1.59 \times 10^2 \pm 1.89\%$ and $y = 1.659 \pm 0.07\%$ therefore $p = x \times y = 263.781 \pm 1.96\% = 263.781 \pm 5.17 = 264 \pm 6$ (since 1.96% of 263.81 is $263.81 \times 1.96/100 \simeq 5.17$).

(5) The absolute error is $|15.91 - 16.02| = 0.1$ and the relative error is $\dfrac{0.11}{15.91} = 0.69\%$.

Solution 1.4

Note 11 Perform the calculations mentally as much as you can (to manipulate the powers of 10) to remain critical with respect to the result found.

(1)

- $10^4 \times 10^{-3} = 10^{4-3} = 10^1 = 10$;
- $10^{-12} \times 10^7 = 10^{-12+7} = 10^{-5}$;
- $\dfrac{10^5}{10^{11}} = 10^{5-11} = 10^{-6}$;
- $\dfrac{10^{-2}}{10^5} = 10^{-2-5} = 10^{-7}$;
- $(10^{-3})^2 = 10^{-3 \times 2} = 10^{-6}$;
- $\left(\dfrac{1}{10^4}\right)^3 = \dfrac{1}{10^{4\times3}} = \dfrac{1}{10^{12}} = 10^{-12}$;
- $-3.8 \times 10^4 + 252 \times 10^5 = -3.8 \times 10^4 + 2520 \times 10^4 = 2516.2 \times 10^4 = 2.5162 \times 10^7$.

(2)

- $\dfrac{21}{1000 \times 9.81 \times 0.000\,035} = \dfrac{2.1 \times 10^1}{10^3 \times 9.81 \times 3.5 \times 10^{-5}} \simeq 0.0612 \times 10^3 = 6.12 \times 10^1$;
- $\dfrac{1}{3.85 \times 10^6} \simeq 0.26 \times 10^{-6} = 2.6 \times 10^{-7}$.

(3)

- $29.83/1.000 = 2.983 \times 10^{-2}$;
- $17.32 \times 10^{15}/(0.002\,681 \times (6370 \times 10^3)^2)$
$= \dfrac{1.732 \times 10^{16}}{2.681 \times 10^{-3} \times 40,576,900 \times 10^6} \simeq 1.882 \times 10^{-11} \times 10^{16} = 1.882 \times 10^5$;
- $\dfrac{(1 \cdot 10^{-3})^2 \times 325}{23.65 \times 18 \times 10^{-7}} \simeq 0.763 \times 10^{-6+7} = 7.63 \times 10^0$;
- $\dfrac{1.374 \times 10^{23} \times 5.981 \times 10^{-4}}{6.590 \times 10^{17} \times 2.367 \times 10^{-3}} \simeq 0.5268 \times 10^{-23-4-17+3} = 5.268 \times 10^4$.

(4)

```
# ------------------------------------
# Powers of 10 and scientific notation
# ------------------------------------
MyComputation1 <- 1e4 * 1e-3
MyComputation1
MyComputation2 <- 1e-12 * 1e7
MyComputation2
MyComputation3 <- 1e5/1e11
MyComputation3
MyComputation4 <- 1e-2/1e5
MyComputation4
MyComputation5 <- (1e-3) ^ 2
MyComputation5
MyComputation6 <- (1/1e4) ^ 3
MyComputation6
MyComputation7 <- -3.8 * 1e4 + 252e5
MyComputation7
MyComputation8 <- 21/(1000 *
   9.81 * 0.000035)
```

```
MyComputation8
MyComputation9 <- 1/(3.85e6)
MyComputation10 <- 29.83/1000
MyComputation10
MyComputation11 <- 17.32e15/(0.002681 *
   (6370e3) ^ 2)
MyComputation11
MyComputation12 <- (1e-3) ^ 2 *
   325/(23.65 * 18e-7)
MyComputation12
MyComputation13 <- 1.374e23 * 5.981e-
   4/(6.590e17 * 2.367e-3)
MyComputation13
```

Solution 1.5

(1)

- 3.5 h = 3 h 30 min;
- 12.75 h = 12 h 45 min;
- 8.86 h = 8 h + 0.86 × 60 min = 8 h + 51.6 min = 8 h + 51 min + 0.6 × 60 s = 8 h 51 min 36 s;
- 0.026 7 h = 0 h + 0.026 7 × 60 s = 0 h 1.602 min = 0 h 1 min 36.12 s.

(2)

- 287 h 28 min 42 s $= 287 + \frac{28}{60} + \frac{42}{3600}$h $\simeq 287.48$h;
- 18 h 48 min 10 s $= 18 + \frac{48}{60} + \frac{10}{3600}$ h $\simeq$ 18.80 h.

Solution 1.6

(1)

- 42,000 km = 42,000,000 m = 4.2×10^7 m;
- 15.67 cm = 0.1567 m = 1.567×10^{-1} m;
- 298.4 µm = 298.4 × 10–6 m = 2984 × 10^{-4} m.

(2)

- 3.5 mm^2 = 3.5×10^{-6} m^2;
- 1 km^2 = 10^6 m^2;
- 3.7 ha = 37,000 m^2 .

(3)

- 1.986 l = 1.986 dm^3 = 1986 × 10^{-3} m^3;
- 29.8 km^3 = 29.8 $(10^3)^3$ m^3 = 2.98 × 10^{10} m^3;
- 478.6 cm^3 = 478.6 $(10^{-2})^3$ m^3 = 4.786 × 10^{-4} m^3.

Solution 1.7

(1) Try to have an idea in advance about whether it is a high or low speed and verify that the result is consistent with the prediction. We have:

- 300,000 km/s = 300,000,000 m/s;
- 98 km.h^{-1} = 98/3600 × 1000 m/s $\simeq$ 27 m/s;
- 2.9 mm/year = 2.910^{-3}/ (365 × 24 × 3600) m/s $\simeq$ 9.2 10^{-11} m/s;
- 35 m/min = 35/60 m/s $\simeq$ 0.58 m/s.

(2)

- 5.9 mm^2/ min = 5.9 × 10^{-6}/ 60 m^2/ s $\simeq$ 9.8 10^{-8} m^2/s;
- 0.0549 kg/cm^3 = 0.054 9 × $(10^2)^3$kg/m^3 = 54.9 10^6 g/m^3 = 5.49 10^7 g/m^3.

(3) In a vacuum, light travels 300,000 km in 1 s. How much does it travel in 1 year?

$$1 \text{ year} = \underbrace{365}_{\text{days in a year}} \times \underbrace{24}_{\text{hours in a day}} \times \underbrace{3\,600}_{\text{seconds in an hour}}$$
$$= 31{,}536{,}000\,\text{s}.$$

If in 1 s light travels 300,000 km, then in 31,536,000 s (1 year), it travels 300,000 × 31,536,000 km = 9,460,800,000,000 km, or 9.46 × 10^{12} km, or 9.46 × 10^{15} m. A light-year is therefore 9.46 petam (peta is the prefix of the International System of Units for 10^{15}).

Solution 1.8

(1) There are 3 significant figures for 143; 4 for 143.6; 6 for 143,670; 3 for 0.014 3; 3 for 1.43 × 10^3; 5 for 1.4300 × 10^5; 4 for 1 043; 3 or 4 for 1430.

(2) 6.46 + 5.2 = 11.7 (one decimal digit, as for 5.2); 8.59 × 13 = 110 = 1.1 × 10^2 (two significant digits, as for 13); 1589.400 − 1.006 7 = 1588.393 (with three decimal figures as for 1589.400); $\frac{1.207 \times 10^3}{1.4 \times 10^{-4}} = 8.6 \times 10^6$ (two significant figures as for 1.4 × 10^{-4}).

Solution 1.9

(1) Remember that the error has only one significant digit.

- $456.255 \pm 3.56 = 456 \pm 4$;
- $57.287 \pm 0.15 = 57.3 \pm 0.2$.

(2)

- $312{,}000\ \mathrm{m}^3 \pm 3000\ \mathrm{m}^3 = (3.12 \pm 0.03) \times 10^5\ \mathrm{m}^3$;
- $0.004560\,\mathrm{m} \pm 0.00006\,\mathrm{m} = (4.56 \pm 0.06) \times 10^{-3}\ \mathrm{m}$;
- $5.19\ 10^3\ \mathrm{s} \pm 17\ \mathrm{s} = (5.19 \pm 0.02) \times 10^3 \mathrm{s}$.

(3)

- $x + y = 56.32 \pm 0.06 = 56.32 \pm 0.11\%$;
- $x - 2y = x - y - y = 30.46 \pm 0.07 = 30.46 \pm 0.23\%$;
- $x = 47.70 \pm 0.105\%$ and $z = 0.081 \pm 4.94\%$, therefore $\frac{x}{z} = 3.863\ 7 \pm 5.1\% = 3.9 \pm 0.2$;
- $y + z = 8.7 \pm 0.02 = 8.7 \pm 0.23\%$;
- $y = 8.62 \pm 0.12\%$, therefore $\dfrac{yz}{x} = 0.014\ 63 \pm 5.2\% = 0.014\ 6 \pm 0.000\ 8$.

Solution 1.10

Examples were suggested in the exercises. We reiterate here the importance of having a personal culture of orders of magnitude (and also a culture related to the disciplines studied: geography or Earth science).

Solution 1.11

(1) In San Francisco, on May 5, 2018, the Sun rose at 06: 10 and set at 20:05.

– We deal first with the minutes. Because 05–10 does not work, we subtract 1 h = 60 min from the 20 h we have. 20 h 05 min = 19 h 65 min.

– we then have: 20 h 05 min − 6 h 10 min = 19 h 65 min − 6 h 10 min = 13 h 55 min.

Therefore, the day lasted 13 h 55 min.

(2) March 18, 2018 (day of the equinox), the Sun rose in Paris at 6:59 a.m. and set at 7 p.m. The day therefore had a duration of 12 h 01 min and the night 11 h 59 min.

Solution 1.12

(1) The distance between the equator and the North Pole (along a meridian through Paris) is one-quarter of a circle around the Earth. The definition given for 1 km thus indicates that the circumference of the Earth is of 40,000 km. Let us denote by R the radius; we then have $2 \times \pi R = 40{,}000$ and therefore $R = 40{,}000/(2 \times \pi) = 6366.2$ km ± 0.1 km (the result is given with too much accuracy with regard to the starting approximation).

The polar radius is actually 6352.75 km ± 0.01 km. The approximation gives an absolute error of 13.5 km and a relative error of $\dfrac{13.5}{6352.75} = 0.21\%$.

(2) A point on the Earth returns to its initial position within 24 h. The length of the Earth at the equator is 40,075.02 km. The desired speed is 40,075.02/(24 × 3600) km/s = 0.464 km/s = 464 m/s.

(3) An astronomical year has 365.2422 days. So on the one hand a year lasts $365.242\ 2 \times 24 \times 60 \times 60 = 31{,}556{,}926$ s. On the other hand, $\pi \times 10^7 \approx 31{,}415{,}927$.

The relative error is $\dfrac{31,556,926 - 31,415,927}{31,556,926} = 0.45\%$.

Solution 1.13

(1) The Sun emits radiation at wavelengths between 300 and 3000 nm. This interval contains some values (and even all of the values) of visible light (but also nonvisible waves), so the Sun emits visible light (we knew it!).

Note 12 The Sun (“hot” body) radiates short wavelengths in the ultraviolet, visible, and infrared spectra (between 0.2 and 10 μm).

(2) 3×10^{-6} m = 3000 nm and 3×10^{-5} m = 30,000 nm. This interval contains no visible light value. The Earth therefore does not emit radiation in the visible domain.

Solution 1.14

(1) $\text{Albedo}_{\text{Earth}} = \frac{102}{342} = 0.2982 = 29.82\times 10^{-2}$ $= 29.82\%$, or about 30%. The given values are of course orders of magnitude, which are themselves rounded, so the two digits after the comma would have no meaning.

(2) If it were 102 W/m^2 with an absolute uncertainty of 2 W/m^2, this would mean that the reflected solar radiation was between 100 and 104 W/m^2. Relative uncertainty is equal to $\frac{2}{102} = 1.96\%$.

(3) The relative error is $\frac{\left|\frac{102}{342} - \frac{100}{342}\right|}{\frac{102}{342}} =$ $\frac{2}{342} \simeq 0.0058 = 0.58\%$.

Solution 1.15

(1) 1 km is 1000 m. If the temperature decreases from 6.49°C over 1000 m, it decreases 1000 times less over 1 m, so the gradient has the following value:

$$\frac{-6.49}{1000} = -0.00649 = -6.49 \times 10^{-3°}\text{C/m}.$$

(2) The gradient is still −6.49 K/km: if we consider two temperatures expressed in degrees Celsius, $T_{1,\,C}$ and $T_{2,\,C}$, with $T_{1.\,C} - T_{2.\,C} = -6.49$, then, in kelvin, we will have the difference $T_{1.\,C} + 273.15 - (T_{2.C} + 273.15) = -6.49$ after simplifying.

Solution 1.16

(1) In 1000 years, 130 mm of limestone will have been eroded. In 1 year, it will have been 1000 times less, so the rate of erosion is $\frac{130}{1000} = 0.13 = 13 \times 10^{-2}\text{mm/year}$. 13×10^{-2} mm of eroded rock is 0.13 mm = 0.00013 m. The rate of erosion is thus 0.00013 m/year 1.3×10^{-4} m/year.

(2) The statement indicates that the exact value to be considered is the measured value. The absolute error of the model with respect to the measurements is $|130 - 145| = 15$ mm/1000 years. The relative error of the model with respect to the $\frac{|130 - 145|}{145} = 0.103 = 10.3\% \approx 10\%$.

Solution 1.17

(1) In 1 s, the water runs 1.3 m = 0.13 dam = 0.013 hm = 0.0013 km = 1.3×10^{-3} km. The speed is thus 1.3×10^{-3} m/s. In 1 s, water flows 1.3×10^{-3} km; in 3600 s or 1 h, it will cover 3600 times more distance, i.e., $1.3 \times 10^{-3} \times 3600 = 4.68 \approx 4.7$ km/h.

(2) The flow has a unit in m^3/s, the speed can be expressed in m/s. With these units, we see that if we multiply the mean velocity by a surface in m^2, we fall back on the m^3/s (m/s × m^2 = m^3/s) unit. The quantity we need is a surface.

(3) The flow rate is 1.3 m/s × 10 m^2 = 13 m^3/s. The flow rate means that in 1 s, 13 m^3 of water flows through the section we have considered. 1 m^3 of water is equivalent to 1000 L of water, so 13 m^3 represents 13 × 1000 L = 13,000 L = 13 × 10^3 L. A flow rate of 13 m^3/s is equal to 13 × 10^3 L/s. In 1 s, 13 × 10^3 L crosses the section of river in question; in 1 min, or 60 s, 60 times more does. The flow rate is therefore 13 × 10^3 × 60 L/min = 780 × 10^3 L/min = 78 × 10^4 L/min.

Solution 1.18

(1) The volumetric mass is obtained by dividing the mass by the volume.
The volume of a sphere is calculated by the formula $\frac{4}{3} \times \pi \times \text{radius}^3$. Hence, Table 1.13 (by converting the initial units).

Table 1.13 Calculating the density of Earth, Mercury, and Mars

Planet	Radius (cm)	Volume (cm^3)	Mass (g)	Volumetric mass (g/cm^3)
Earth	6.378×10^8	1.086×10^{27}	5.98×10^{27}	5.50
Mercury	2.439×10^8	6.077×10^{25}	3.3×10^{26}	5.43
Mars	3.398×10^8	1.643×10^{26}	6.4×10^{26}	3.90

(2)

```
# ------------------------------- ----
# Volumetric mass of planets
# ------------------------------- ----
#METHOD 1: Computation with vectors
(numeric)
# Create vectors
# The vector "Radius" of the planet in
cm
# The vector "Mass" of the planet in g
Radius <- c(6.378e8,2.439e8,3.398e8)
Mass <- c(5.98e27,3.3e26,6.4e26)

# Calculate the volumetric mass for
each planet
# VolumetricMass = Mass/Volume
# VolumetricMass = Mass/(4/3 * pi *
Radius ^ 3)
# The vector "VolumetricMass " of the
planet in g/cm3
VolumetricMass <- Mass/(4/3 * pi *
Radius^3)
VolumetricMass

# ------------------------------- ----
# Volumetric mass of planets
# ------------------------------- ----
# METHOD 2: create an array (data.
frame)
# Column 1: Name of the planet
# Column 2: Radius of the planet in cm
# Column 3: Mass of the planet in g
# Column 4: Volumetric mass of the
planet in g/cm3
PlanetName = c("Earth", "Mercury",
"Mars")
Radius <- c(6.378e8,2.439e8,3.398e8)
Mass <- c(5.98e27,3.3e26,6.4e26)
VolumetricMass <-c(NA, NA, NA)
# Create an array (data.frame)
Table <- data.frame(PlanetName, Ra-
dius, Mass, VolumetricMass )
# Assign the name of each column
colnames (Table) <- c( "PlanetName,"
"Radius," "Mass", " VolumetricMass ")

Table
# Computation of the volumetric mass
for each planet
Table$VolumetricMass <-
Table$Mass/(4/3 * pi * Table$Radius^3)
Table
# Round the result to 2 digits after
the decimal dot
Table$VolumetricMass <-
round(Table$Mass/(4/3 * pi * Table$
Radius^3), 2)
Table
```

Solution 1.19

(1) $5 \times 10^5 km^3$ evaporates per year, out of a total of 1.3×10^9 km^3.Therefore, we need $\frac{1.3 \times 10^9}{5 \times 10^5} \approx 3 \times 10^3$ years, or 3000 years, to completely replenish the oceans. We kept only one significant figure, as is the case in 5×10^5.

(2) The total surface of the Earth is $4 \times \pi \times \text{Radius}^2 \approx 4 \times \pi \times (6378)^2$ km^2 $= 5.112 \times \pi \times 10^8$ km^2.
As the ocean surface occupies three-quarters of the total area, this surface has a value of $0.75 \times 5.112 \times 10^5 = 3.834 \times 10^8$ km^2.
The evaporated water height is assimilated to a cylinder of volume 5×10^5 km^3 having a base area of 3.834×10^8 km^2. Using the relation Cylinder volume = Base area $\times$ Height, it is deduced therefrom that the height of the water sought is equal to $\frac{5 \times 10^5}{3.834 \times 10^8} km \approx 1.10^{-3} km$, i.e., about 1 m.

(3) We use once again the formula of the volume of a cylinder: Base area $\times$ Height. The unknown quantity this time is the volume of water sought. We have volume $= 3.834 \times 10^8$ km^2 $\times$ 19×10^{-5} km $\approx$ 73,000 km^3.

(4) 25×10^6 km^3 represents 100% of the water contained in the glaciers of the poles, 73,000 km^3, so it represents $\dfrac{73,000}{25 \times 10^6} \approx 0.3\%$.

2 Variables and Functions

Abstract

Geographers and Earth science specialists are confronted with phenomena that they try to apprehend and formalize by identifying the variables involved and the relations among them. This chapter introduces the notions of variables and functions (as a link between variables). These functions provide fundamental information to geographers and Earth science specialists, enabling them to study and conceptualize these phenomena.

This chapter explains how to represent functions (2D and 3D) using the appropriate scale. Usual functions such as linear, quadratic, square root, exponential, and logarithm functions are introduced and their formulas, properties, and graphic representations given. Indications on how to solve some equations are presented. The notion of increase and decrease are discussed, as is the rate of increase (slope) for a line.

In the exercises, indications on how to draw a line and find a line equation on logarithmic scales are added.

Keywords

Variables · Functions · Graphic representation · Equation of a line · Linear function · Exponentials · Logarithms · Equation solving · Slope

Aims and Objectives

- To know that quantities used are variables linked together by functions.
- To know the difference between variable and parameter.
- To know the common functions, their algebraic expression, their graphical representation, and their properties: linear functions, quadratic functions, root functions, exponential functions, and logarithm functions.
- To understand the basics of the growth and decay of functions, which are useful for the study of their evolution and trends.
- To understand the interest of functions and their representation to illustrate natural phenomena.

The problems raised by geographers and Earth science specialists are numerous; the following are some examples:

- *What is the change in pressure in the atmosphere when altitude increases?*
- *What is the influence of depth on temperature in the Earth's crust?*
- *Does the viscosity of water affect the speed of its flow?*
- *Does the density of a mudflow affect the flow length?*

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We will see that temperature, pressure, altitude, viscosity, velocity, distance, and time are called "variables" and that they are linked to one another by relations. Mathematically speaking, these relationships are functions that provide fundamental information to geographers and Earth science specialists, enabling them to study and conceptualize these phenomena.

2.1 Variables and Values

"The standard temperature measured at the Earth's surface is 15 °C": This sentence indicates that the **variable** (or the **magnitude of**) temperature takes the value 15 °C on the surface of the Earth. We can write: on the surface of the Earth, $T = 15^{\circ}\mathrm{C}$.

The letter T is used both to name the temperature variable and to be replaced by values. Variables are so called because they vary, i.e., they can take a variety of values.

There are three types of variables, discussed in what follows.

Qualitative Variables (Also Known as Categorical Variables) Examples of qualitative variables are the name of a continent, a person's residence, and the canopy color in a satellite image. The variables *continent* and *color* can take a finite number of values that are nominal (i.e., not digital). The value can be "Europe," "Asia," "America," "Oceania," or "Africa" for the continent variable and "blue," "red," or "green" for the color variable. Variables that are not qualitative are known as **quantitative variables**.

Discrete Variables Examples of discrete variables include the number of drains in a river system and the number of agricultural plots within a given territory. The values taken by the variable *number of drains* and the variable *number of plots* are integers. It can be "50" or "200" for the variable *number of drains* and "6" or "1500" for the variable *number of plots*.

Continuous Variables These include air pressure and azimuth. These variables can take all possible values between their extremes. The variable *atmospheric pressure* can take the value "1013.7 hPa," the *azimuth* variable can take all possible values between 0 and 360 degrees.

2.2 Link Between Variables, Functions

Definition 7 *A **function** defines the relationships between one or more variables and one or more other variables: to each value of the first variable (or variables) corresponds a value of the other variable (or variables).*

The following situations define functions:

- Situation 1: Pressure varies with altitude;
- Situation 2: The number of people in a population is expressed by age group;
- Situation 3: At each time t, we measure the temperature T;
- Situation 4: The temperature is measured at the position(x, y).

Translation of the foregoing situations:

- Situation 1: The pressure variable (P) is a function of the altitude variable. If you call f this function, you write $P = f(\text{altitude})$. If this function f is known, it is possible to calculate the values of P from the altitude values;
- Situation 2: The number of people (the variable is the number of people) is known according to the age group variable. The two variables involved take discrete values;
- Situation 3: The temperature variable is a function of the time variable. We can use the notation $T = f(t)$ or call the function directly T, in which case the values then be $T(t)$;
- Situation 4: The temperature variable is a function of the variables x and y: $T = f(x, y)$. Here we have a function of two variables.

Note 13 *Most often in practice we know experimentally two associated values (elevation at different temperatures, for example). Among the usual mathematical functions, the one that best fits these values is chosen. We talk about* ***modeling****. This is why it is necessary to know well the curves and properties of the common functions (see following discussion).*

The functions cited in the preceding examples are taken from a context. Functions can also be defined out of context, directly, using a mathematical formula.

Example 23 *Let us define the function* f *by* $f(t) = t^2 + 1$ *(we write* $t \mapsto t^2 + 1$*) or the function* g *by* $g(x) = x + 1 - \frac{1}{x}$. *One needs to know for what input values of* t *the number* $f(t)$ *is defined and for what values of* x *the number* $g(x)$ *exists. The set of all permitted inputs to a given function is called the* ***domain*** *of the function. Outside of any practical context, one considers the largest possible set of definitions: for* f*, it is the set of all real numbers, that is, the interval* $\mathbb{R} =]-\infty; +\infty[$. *For* g*, it is the set* $\mathbb{R}\backslash\{0\}$ *consisting of all real numbers except 0. In a practical context, it is the situation that allows one to know the extreme values taken by the variable.*

To calculate the values taken, the variable is replaced by a numerical value; then the calculation is carried out. Pay attention to operational priorities! Using the foregoing notations, $f(3) = 3^2 + 1 = 9 + 1 = 10$ and $g(2) = 2 + 1 - \frac{1}{2} = 2.5$.

2.3 Formulas and Parameters

We can express the relation between variables by a formula (which can be obtained by modeling).

Example 24 *The relationship* $T_F = T_C \times 1.8 + 32$ *makes it possible to connect the temperature* T_C *expressed in degrees Celsius to that of* T_F *expressed in degrees Fahrenheit. It expresses directly the variable* T_F *as a function of the variable* T_C *but also allows one to find the expression of* T_C *in terms of* T_F.

In the troposphere, the temperature decreases by 6.5 °C per kilometer. This decrease with altitude can thus be written $T(z) = T_0 - 6.5 \times z$, *with* T_0 *the value of the temperature at altitude 0 km (*e.g., *18 °C), the value of the altitude at which the value of the temperature is to be calculated. Note that in this example, the unit of* z *must be in km and that of* T *in degrees Celsius. It is interesting to note that the units of* z *and of* $T(z)$ *depend on the unit of constant 6.5 (°C/km).*

This relationship between variables can be interpreted in three different ways:

- Function T is a function of two variables: the variable T_0 (surface temperature) and the variable z (altitude);
- If one fixes T_0, that is to say, if one gives a precise value to T_0, then T is a function of the altitude variable z. In that case, T_0 (with its value set at $T_0 = 20$°C, for example) is a **parameter** of function T;
- If one fixes z, then T is a function of the ground temperature variable T_0. Therefore z (with its value set at 1 km, for instance) is a parameter of function T.

2.4 Representation of Functions, First Elements

The graphical representation of a function f of a variable x is obtained by placing the coordinate $(x, f(x))$ points in a frame (usually in a frame where the axes are orthogonal, that is to say, where they form a right angle). The horizontal axis is called the **x-axis** and the vertical axis the **y-axis.**

A curve is obtained if the variable is continuous (Figs. 2.1 and 2.2). If the variable is discrete, we obtain dots separated from each other (Fig. 2.3). A number x has at most one number $f(x)$ associated with it; an x coordinate is therefore associated with at most one point of the curve.

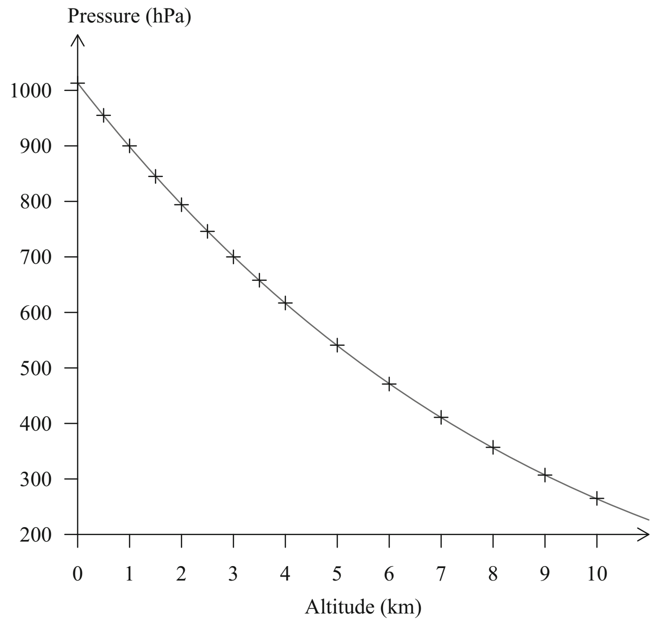

Fig. 2.1 Evolution of pressure in the troposphere

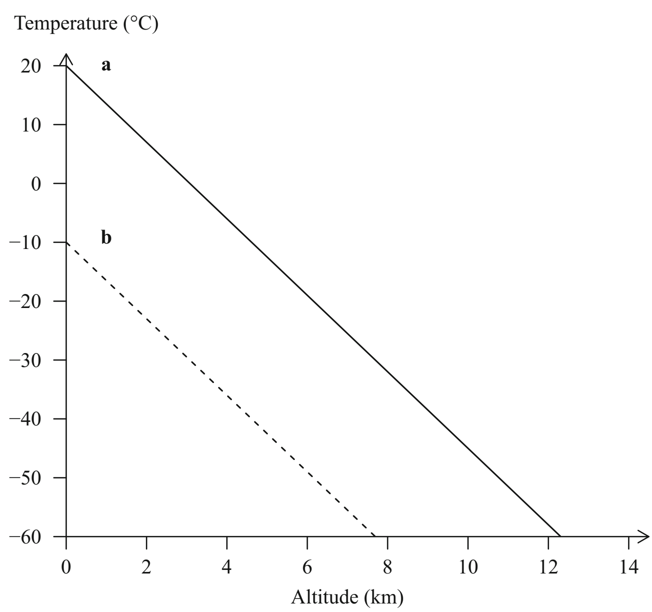

Fig. 2.2 Temperature changes in troposphere for a ground temperature of $T_0 = 20\ °\mathrm{C}$ **(a)** and $T_0 = -10\ °\mathrm{C}$ **(b)**

Example 25 *The graph in Fig. 2.1 shows the evolution of pressure depending on altitude. Pressure decreases as altitude increases: on the ground surface ($z = 0$ km), the pressure is slightly higher than 1000 hPa, whereas it is only 200 hPa for an altitude greater than 10 km. This graph is composed of measurement points (represented by +) from a balloon probe. Each point is located*

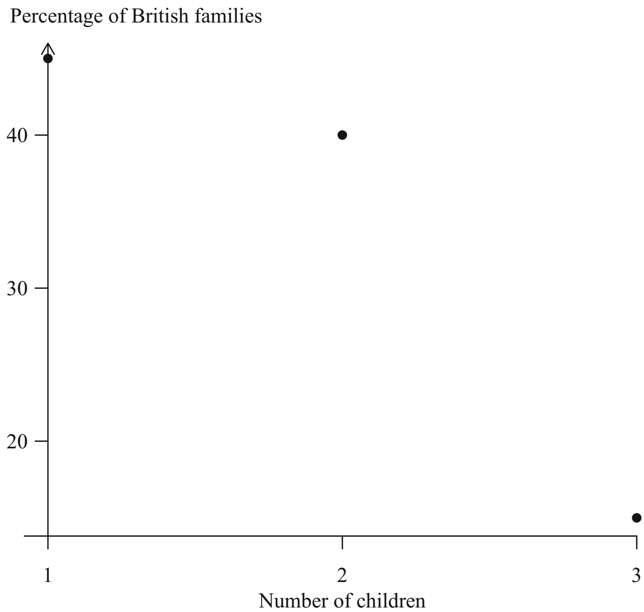

Fig. 2.3 Percentage of British families in 2016 based on the number of children under 18 per household (Source: www.ons.gov.uk). For this representation, connecting the points does not make sense because the variable of the x-axis is not a continuous quantitative variable

on the graph with its two coordinates: the altitude variable (x) and the pressure variable (f(x)). These two variables are continuous, so the measurement points can be connected in a curve. This curve represents a function that is not known a priori *but that, given its shape, can be modeled by*

$$f(x) = 1013.25\left(\frac{288 - 6.5x}{288}\right)^{5.255}.$$

The graph in Fig. 2.2 shows the evolution of temperature as a function of altitude. It was seen previously that the temperature decreases by about 6.5 °C per kilometer in the troposphere and therefore can be expressed mathematically by $T(z) = T_0 - 6.5\,z$, with T_0 the value of the temperature at altitude 0 km and z the value of the altitude.

*If one sets T_0, the latter is then a parameter of the equation. For each value of this parameter, a different line can be drawn. Curve **a** represents the case $T_0 = 20$ °C and curve **b** represents the case $T_0 = -10$ °C. We note that the two lines are parallel: whether the surface temperature is $T_0 = 20$ °C or $T_0 = -10$ °C, the decrease is always 6.5 °C per kilometer.*

The graph in Fig. 2.3 illustrates the percentage of UK families in 2016 that have 1, 2, 3 or more children in their households. Contrary to the two previous examples, the x-axis represents a discrete variable: the number of children. Thus, there would be no sense in linking the points obtained.

For a function f of two variables x and y, the graphical representation is a three-dimensional (3D) space surface, as shown in Fig. 2.4. In this figure, the graph represents the function defined by $f(x, y) = \sqrt{x^2 + y^2}$. The graph of Fig. 2.5 is a representation of a topographic surface whose coordinate point (x, y) is characterized by an elevation z = f(x, y).

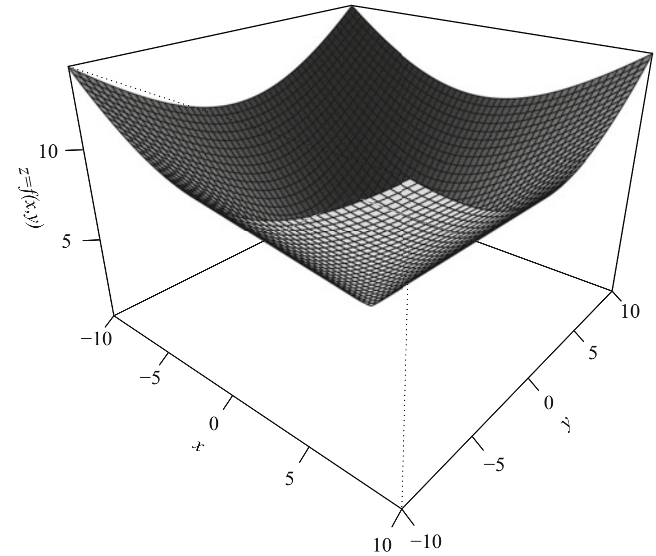

Fig. 2.4 In the case of a function of two variables x and y, the value of $f(x, y)$ reads on the z-axis from the 3D surface and the coordinate point (x,y)

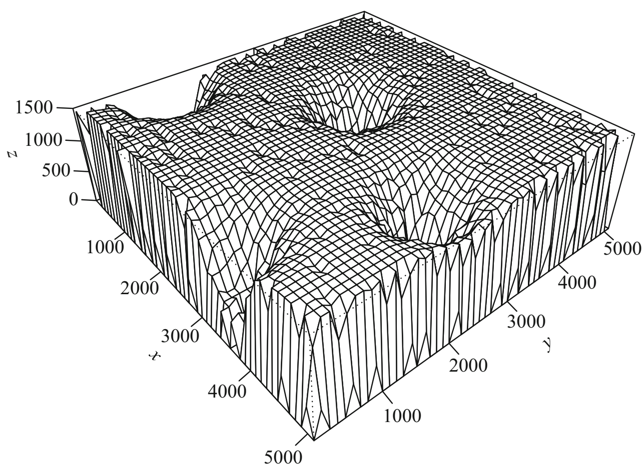

Fig. 2.5 This is not the result of an equation but of points, each corresponding to an altitude (z) and connected to one another. Each pair of coordinates (x, y) corresponds to an altitude z

2.5 Scales and Graphic Representation

2.5.1 Choice of Variables on X- and Y-Axes

Whether it is to position measurement points on a graph or to plot the graph of a function, the choice of the x- and y-axis scale (abscissa and ordinate scale) is very important. The scales chosen (which are generally different for the x- and y-axis) will depend on the data or the function and the size of the medium on which one wishes to represent them.

First, you must choose which axis (x or y) will represent which variable. In the case of a function, this is imposed: the variable is on the abscissa (x-axis) and the values of the function on the ordinate (y-axis) (Fig. 2.6).

In the case of a data set, there is no predetermined rule. In the case of the data (altitude, temperature) in Fig. 2.7, the altitude was

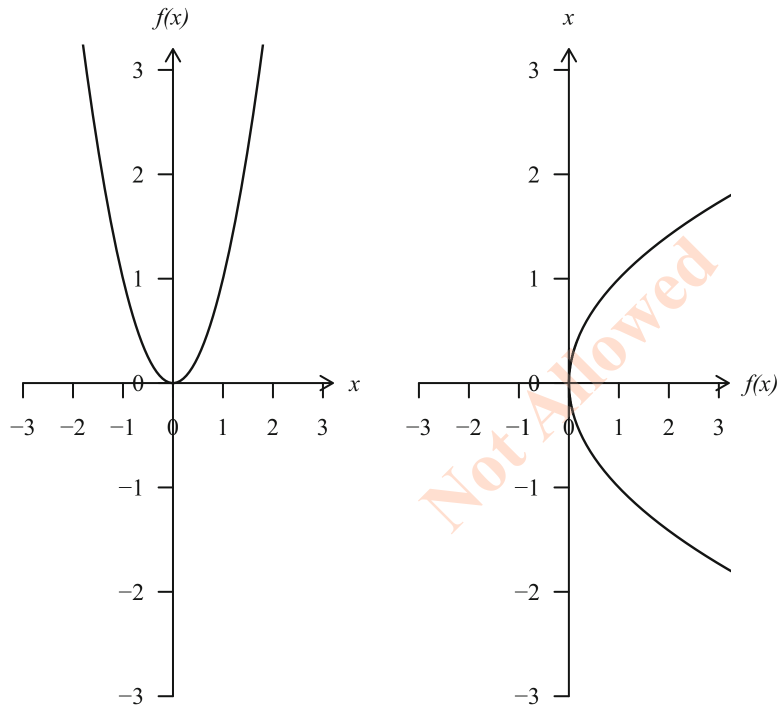

Fig. 2.6 In the case of a function, the variables are on the x-axis and the values of the function on the y-axis (left graph). The converse is not allowed (right graph)

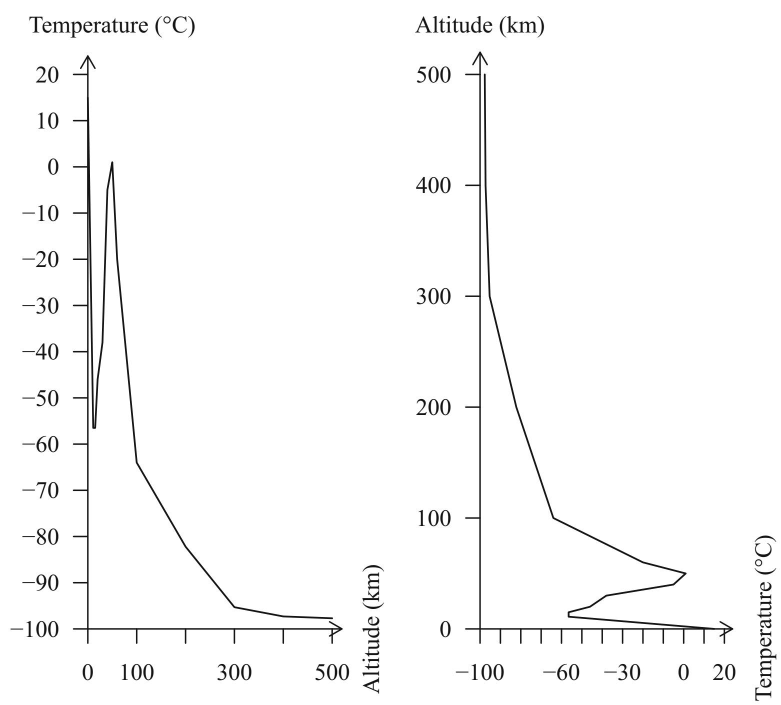

Fig. 2.7 In the case of data, we choose the position variable (x-axis and y-axis) as we want

plotted on the x-axis and the temperature on the y-axis (left) and then the opposite (on the right). It is common to place the altitude on the y-axis (Fig. 2.7 on the right) because it is natural for the reader to read altitude vertically. On the other hand, we see that this representation is a problem: to a single value of the temperature variable (e.g., -40 °C) correspond several values of the altitude variable. This configuration is not compatible with the representation of mathematical functions, and therefore, if we want to model this curve by a mathematical function, we prefer the representation given on the left-hand side in Fig. 2.7.

2.5.2 Graduation of Axes

Then determine the **range** of the data or function, that is to say, the maximum and minimum values that are plotted on the graph. Finally, a proportional relation is established between the scope and the desired media size to determine the axes of **graduations**.

Example 26 *Table 2.1 presents temperature data (°C) and the corresponding distances x (km).*

The data in Table 2.1 indicate that it is reasonable to draw distances over an interval [0; 6] *km and temperatures over a range* $[-10; 20]$ *°C. The range of the temperature data is therefore equal to thirty* $20-(-10) = 30$ *°C. The range of the distance data is* $6 - 0 = 6$ *km.*

If one wishes to represent the evolution of the temperature as a function of the distance on a graph of width 12 cm (x-axis) and height 10 cm (y-axis), the graduations on each of the axes will be determined by means of the following relations (using proportionality):

For temperature	*For distance*
12 cm $\leftrightarrow$ 30°C	10 cm $\leftrightarrow$ 6 km
$1\,\text{cm} \leftrightarrow \dfrac{1\times 30}{12} = 2.5^\circ\text{C}$	$1\,\text{cm} \leftrightarrow \dfrac{1\times 6}{10} = 0.6\,\text{km}$

See Fig. 2.8 to illustrate this example.

Table 2.1 Data table: distance x (km) and temperature (°C)

Distance x (km)	Temperature (°C)
1	17
2	7.5
3	_3
4	_5.5
5	_8

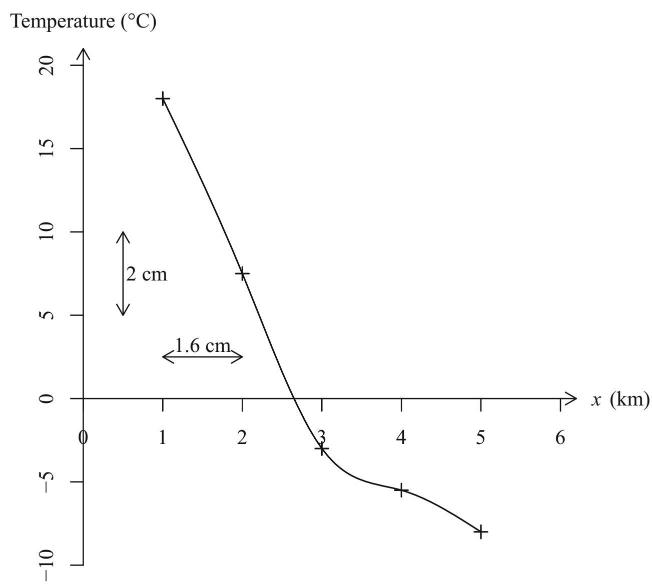

Fig. 2.8 Representation of data in Table 2.1

Note 14

(1) *On each axis (x-axis and y-axis), indicate the name of the quantity represented and its unit.*
(2) *The axes must be graduated.*
(3) *The graph should have a legend indicating, if needed, the source of the data.*
(4) *In the case of a map, do not forget to note the scale and the north arrow.*
(5) *The origin point of the graph does not necessarily have the coordinates* (0; 0).
(6) *We can connect the points between them if and only if the variable studied is continuous. In this case, connect the points continuously and not with segments (see example in Fig. 2.8).*

2.6 The Common Functions

In geography and geology, it is important to know the main functions that are used in modeling: their graphic representation and their main properties.

2.6.1 Linear Functions

Example 27 *In hydrogeology, for a liquid flowing through a porous medium, Darcy's law relates the liquid flow rate through the medium (Q) to the hydraulic gradient. The relationship is given by $Q = A \times k \times i$, where A is the surface perpendicular to the flow and k is a parameter called the permeability of the porous medium (Fig. 2.9).*

The expression Q in terms of i defines a **linear function** of the type $f(x) = a \times x$, where a is a real number (parameter) (here: $a = A \times k$). Its graph is a **straight line through the origin** of the frame (for the situation presented the flow is zero when the hydraulic gradient is zero). This occurs in situations of proportionality between variables.

The function f, which makes it possible to link the temperature in degrees Celsius to degrees Fahrenheit is given by $f(x) = 1.8 \times x + 32$ is also a **linear function.** It is defined by a relation of the type $f(x) = a \times x + b$, where a and b are real

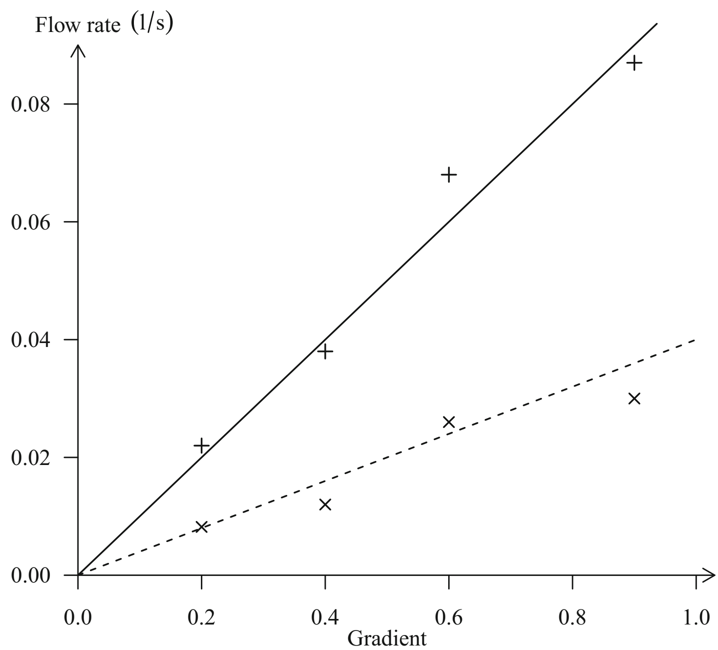

Fig. 2.9 According to Darcy's law, each line represents the proportionality between the flow Q and the hydraulic gradient i, here for two different porous media, one having a high permeability (continuous line) and the other a lower one (dotted line)

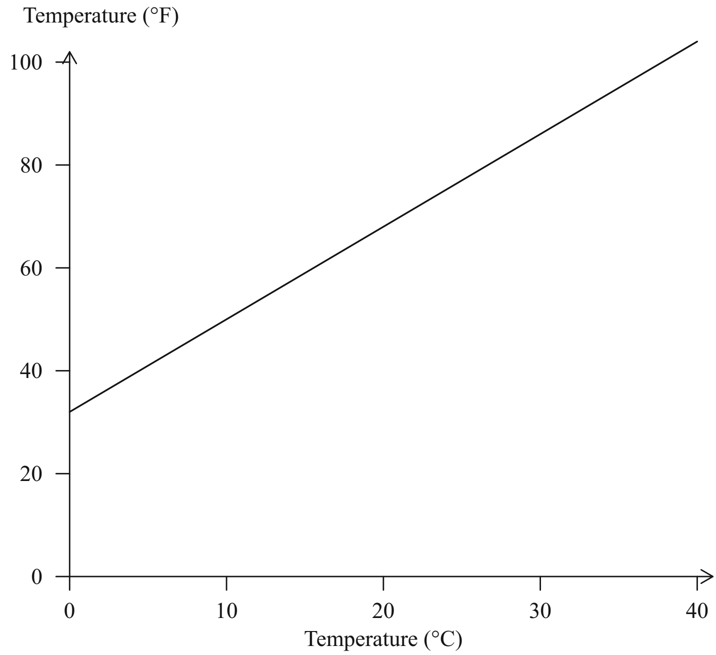

Fig. 2.10 Relation between temperature in degrees Celsius and Fahrenheit

numbers (parameters). Its graphical representation is a (nonvertical) **line** of the equation $y = ax + b$ (Fig. 2.10). With the foregoing notations, the number a is referred to as the **slope of the line** representing the function f. If $a = 0$, then the function is constant; b is the **intercept.**

Insert 3 (Modeling)

A model is a simplified representation – physical or abstract – of a real system that is itself often complicated or complex. The common point of all models is that they are wrong! As Professor Box said in 1976: "All models are wrong but some are useful." The errors produced by the models can be conceptual (and therefore inherent in their construction), numerical (they stem from the resolution methods used), or predictive (they stem from uncertainty).

A physical model often represents, on a small scale, a real system to be studied. For example, a model of the Mont Saint-Michel (France) and its sedimentary environment was designed to study siltation. An abstract model can be constructed by two distinct approaches. The first consists in mathematically formalizing the real system to be studied: its components, its limits, or its interactions. Most of the time this mathematical formalization of complicated equations is accompanied by a computer numerical solution – we speak of in silico *experiments, referring to computer silicon chips. The second approach consists in finding mathematical functions that are closest to the data provided by the actual system to be studied. To do this, we must make assumptions about the behavior of the data in order to find functions that could be suitable: linear functions, exponential functions, and so forth. The adjustment of the chosen model to the data is made on the basis of tests that make it possible to validate or not this choice. Figure 2.11 gives an example of this method of data modeling.*

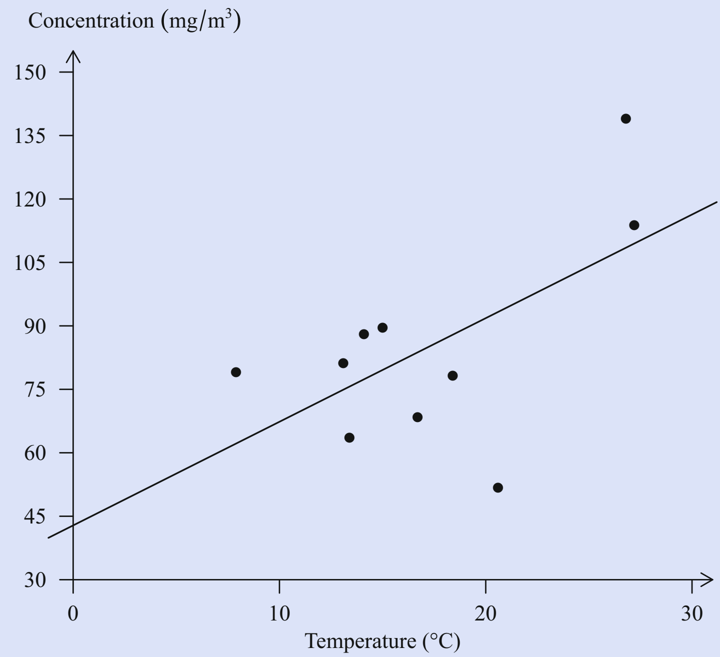

Fig. 2.11 The points represent the concentration of an air pollutant (unit in milligrams per cubic meter of air) measured according to the temperature of the air. The so-called regression line is based on the assumption that the relationship between the concentration of this pollutant and the air temperature is linear: linear modeling was chosen. The line is adjusted to the nearest point. To do this, the parameters of the line are computed automatically so that the distance between the points and the line is as small as possible: the linear least-squares method (the errors of the distances – between the real points and the regression line – squared are minimized)

2.6.2 The Quadratic Functions

A **quadratic** (or polynomial function of degree 2) is a function defined on $]-\infty\,;+\infty[$ (i.e., for all real numbers) by $f(x) = ax^2 + bx + c$, where a, b, c are real parameters, with $a \neq 0$ (otherwise the function is linear and not quadratic). The name quadratic comes from *quad*, meaning "square," because the variable x gets squared (like x^2).

Its graphical representation is a curve called a parabola. It passes through the point $(0, c)$.

Example 28 *An object dropped at time $t = 0$ s at altitude z_0 and initial speed v_0 and forming an angle of +45° to the horizontal operates at a trajectory whose expression of its altitude z at instant t is given by $z(t) = -\frac{1}{2}gt^2 + \frac{\sqrt{2}}{2}v_0t + z_0$ (where $g \approx 9.81$ m/s^2 is the acceleration of terrestrial gravity). z is a quadratic function, with $a = \frac{1}{2}g$, $b = \frac{\sqrt{2}}{2}v_0$, and $c = z_0$ (Fig. 2.12).*

2.6.3 Square Root Function

The **square root function** is defined on $[0; +\infty[$ (that is, for non-negative real numbers) by $f(x) = \sqrt{x}$; we also use the notation $\sqrt{x} = x^{1/2}$. Its graphical representation is given in Fig. 2.13.

The square root function is the **inverse function** of the square function ($x \mapsto x^2$): that is to say, for x non-negative we have $(\sqrt{x})^2 = x$ and $\sqrt{x^2} = x$. This reciprocity makes it possible to solve equations (solving an equation means finding all values that fit an equality).

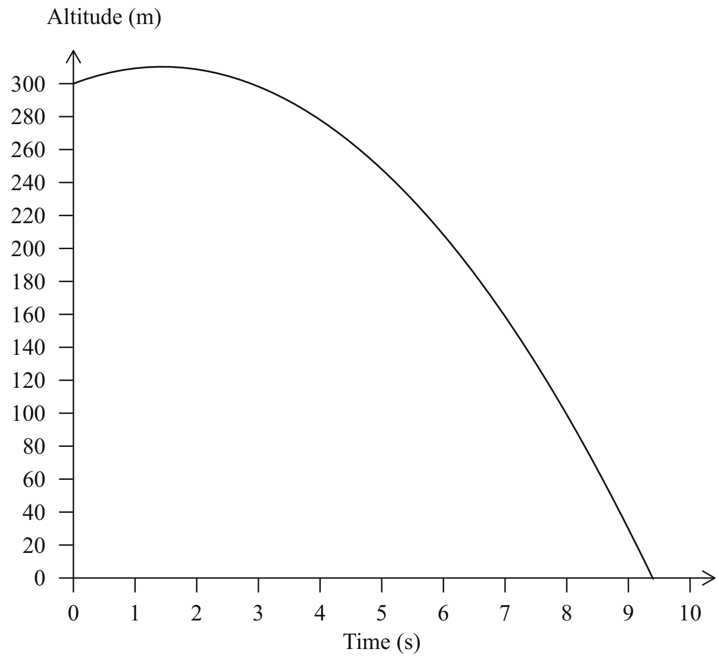

Fig. 2.12 Curve of altitude of an object in free fall as a function of time, dropped at an altitude $z_0 = 300$ m and an initial speed of $v_0 = 20$ m/s at an angle of 45° to the horizontal

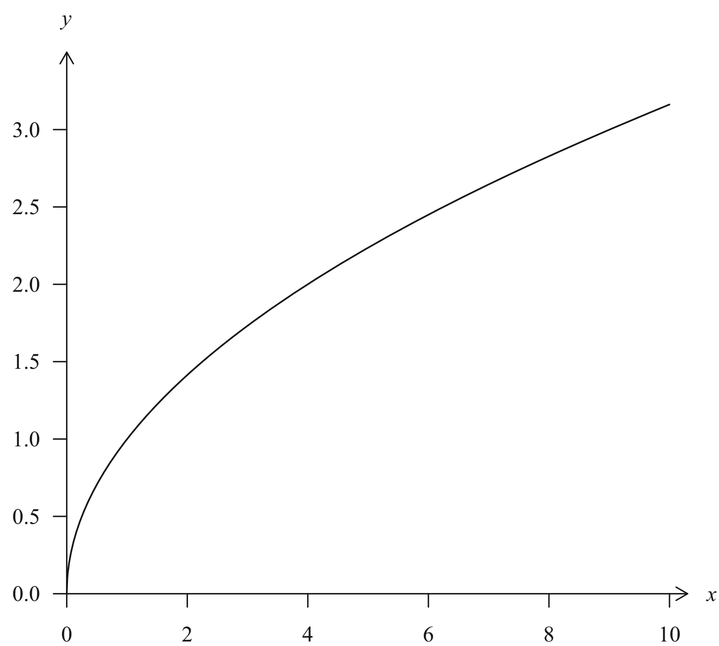

Fig. 2.13 Graphical representation of square root function $y = f(x) = \sqrt{x}$

Example 29 *We drop an object from height* $z_0 = 200$ *m without initial speed and we want to know how much time it will take to reach the ground.*

The altitude is a function of time given by $z(t) = -\frac{9,81}{2}t^2 + 200.$

The value of t *(non-negative), for which* $z(t) = 0 : -\frac{9.81}{2}t^2 + 200 = 0$*, is* $t^2 = \frac{-200}{-4.905}$ *and so* $t = \sqrt{\frac{200}{4.905}} \approx 6.4\,s$*. The object will reach the ground after* 6.4 *s approximately.*

2.6.4 Exponential Functions

Here we do not define exponential functions (or logarithm functions). The interested reader is invited to explore this question in other works.

Definition 8 (Some elements of natural exponential functions) *A* ***natural exponential function****'s domain is* $]-\infty\,;+\infty[$. *This function is denoted* exp. *It is a strictly increasing function. Its values (i.e., the results) are always positive. Two particular values are to be retained:* $\exp(0) = 1$ *and* $\exp(1) = e \approx 2.71$ *(Fig. 2.14).*

The properties of the function exp are similar to those seen on the powers of 10 in Chap. 1 (i.e., there are properties related to product and quotient but not for sum or difference). This is why we often use the notation $\exp(x) = e^x$.

Note 15 (Properties) *For all real numbers a and b and for any integer n we have*

$$e^{a+b} = e^a e^b,$$

$$e^{-a} = \frac{1}{e^a},$$

$$e^{na} = (e^a)^n,$$

$$\frac{e^a}{e^b} = e^{a-b}.$$

Frequently encountered in Earth science and geography are functions called **exponentially**

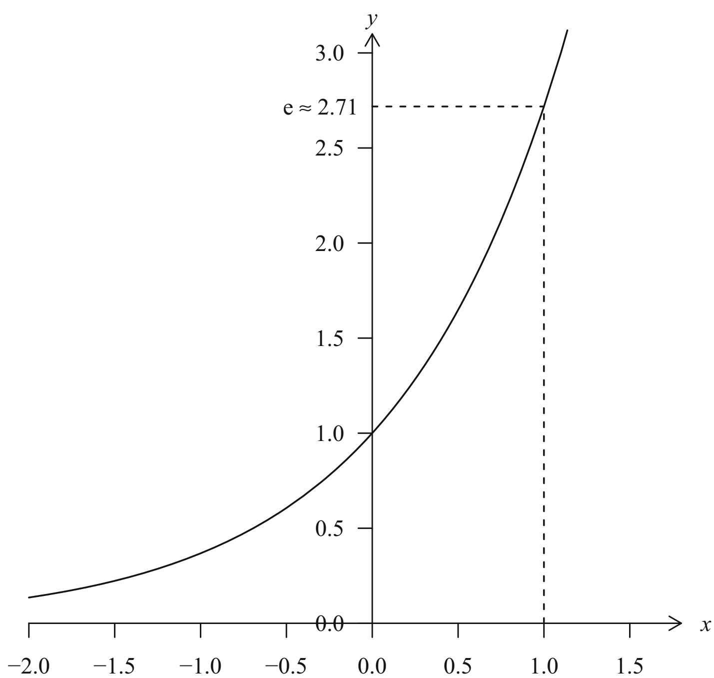

Fig. 2.14 Graphic representation of natural exponential function $y = f(x) = \exp(x)$

decreasing functions. They are found, for example, to translate the radioactive decay. They are of the form $f(x) = Ae^{-x/b}$, where A and b are parameters. It should be noted that the value of A is obtained for $x = 0$: $f(0) = Ae^0 = A$ (Figs. 2.15 and 2.16).

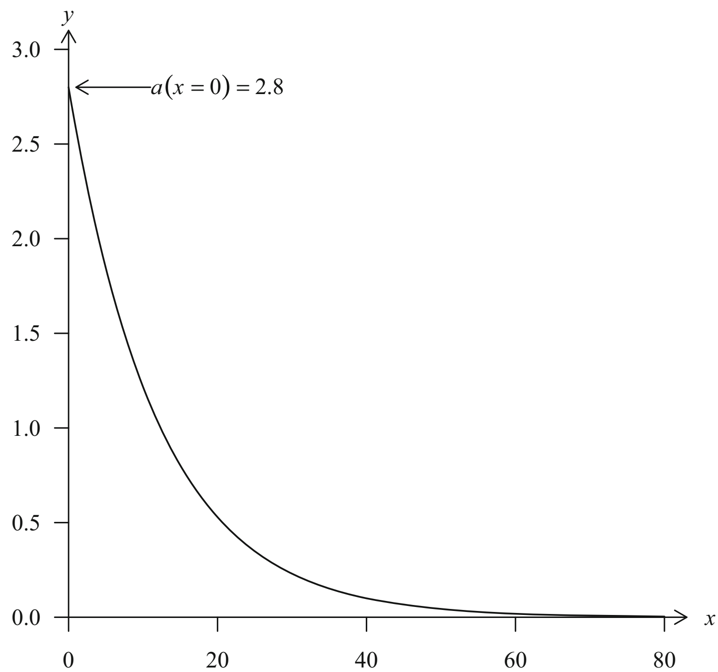

Fig. 2.15 Graphic representation of exponentially decreasing function $f(x) = 2.8\exp\left(-\frac{x}{12}\right)$

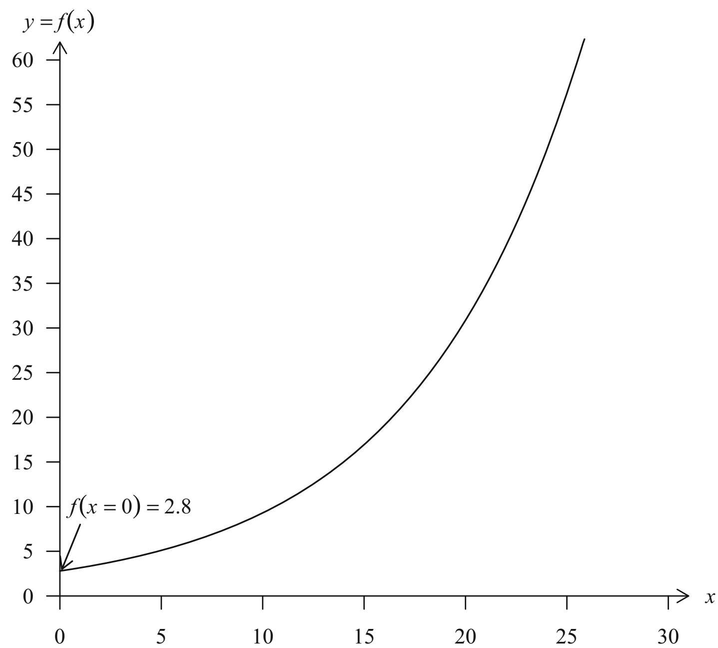

Fig. 2.16 Graphic representation of exponentially increasing function $f(x) = 2.8 \exp(0.12x)$

Example 30

$$a(t) = a_0 \times \exp\left(-\frac{t}{\tau}\right).$$

Similarly, there are functions called **increasing exponential functions**, which are used to describe phenomena of rising exponentially. They are of the form $f(x) = Ae^{x/b}$.The value of A is again obtained for $x = 0$.

Example 31

$$f(x) = 2.8e^{0,12x}.$$

2.6.5 Natural Logarithm Function

Definition 9 (Some features of the natural logarithm function) *The **natural logarithm function** is defined on* $]0; +\infty[$ *(that is, only for non-negative values) and is denoted* ln. *It is a strictly increasing function. Two values should be noted:* $\ln(1) = 0$ *and* $\ln(e) = 1$ *(Fig. 2.17).*

Note 16 It is pronounced *"ell-enn-of-x." (Note: That's "ell-enn," as in the letter, not "one-enn" or "eye-enn"!)*

Historically, the ln function was introduced to make it easier to manipulate (and multiply) large numbers (at a time when there were no calculators). For example, in astronomy, it transforms multiplications into additions. This makes it possible to retain the following properties.

Note 17 (Properties)

- *For all nonnegative numbers a and b, we have*

$$\ln(a \times b) = \ln(a) + \ln(b);$$

- *For all nonnegative numbers a and b and for any number n, we have*

$$\ln(a^n) = n \times \ln(a),$$

$$\ln\left(\frac{1}{a}\right) = -\ln(a),$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b).$$

Example 32 $3.1 \times \ln(10^4) = 3.1 \times 4 \times \ln 10 = 12.4 \times \ln(10)$.

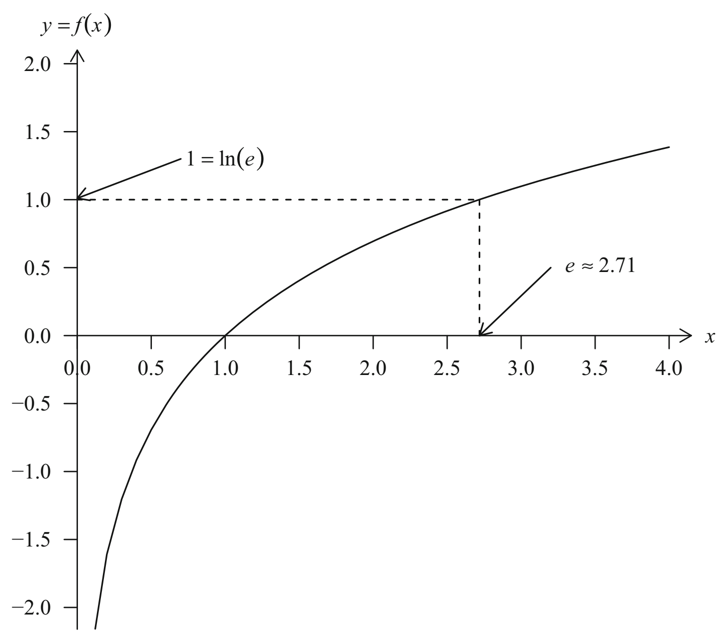

Fig. 2.17 Graphic representation of natural logarithm function $y = f(x) = \ln x$

To solve an **equation** *where the* **variable** *(that is, the letter whose value we are searching to make the equation true) is an exponent, use the* ln *function to "lower the exponent."*

To solve $(10.4) \times (49.2)^n = 15$ *using the variable* n*, we write* $\ln(10.4 \times 49.2^n) = \ln(15)$*; then, using two properties of* ln, $\ln(10.4) + n \times \ln(49.2) = \ln(15)$.

We conclude by writing $n = \dfrac{\ln(15) - \ln(10.4)}{\ln(49.2)} \approx 0.094.$

2.6.6 Link Between Exp and ln, Solving Equations

We observe that $\exp(\ln(1)) = \exp(0) = 1$ and $\ln(\exp(1)) = 1$.In fact, the natural exponential and logarithm functions are inverse of one another.

Note 18 (Properties) *For any positive number* x *we have*

$$\exp(\ln(x)) = x.$$

And for any number x *we have*

$$\ln(\exp(x)) = x.$$

These properties allow us to solve equations involving the natural exponential function or the natural logarithm function.

Example 33 *To solve the equation* $\exp(-3x) = 2$ *(that is to say, to find the values of* x *that verify the equality), we use the* ln *function to "neutralize" the* exp *function:*

We write: $\ln(\exp(-3x)) = \ln(2)$*, so* $-3x = \ln(2)$*, and finally* $x = -\dfrac{1}{3}\ln(2) \approx -0.23$.

To solve the equation $\ln(8, 9y) = -12$*, use the function* exp *to "neutralize" function* ln.

We write $\exp(\ln(8, 9y)) = \exp(-12)$*, then* $8, 9y = \exp(-12)$*,*

and finally $y = \dfrac{1}{8.9}\exp(-12) \approx 6.9 \times 10^{-7}$.

Insert 4 (Exponential and Lily – Tribute to Albert Jacquard)
Albert Jacquard (1925–2013) was a French researcher who specialized in population genetics but was above all a great humanist. He contributed greatly to the popularization of science through books published in French.

In "L'équation du nénuphar" (The Water Lily Equation), Calmann-Lévy, 1996, Albert Jacquard is interested in forging the critical spirit of future generations. Albert Jacquard places himself in a situation where each water lily is able to produce each day a new water lily. In mathematical terms, this evolution is expressed using the equation $x(t) = x(0)2^t$*, with* $x(0)$ *the initial number of water lilies and* $x(t)$ *the number of water lilies at time* t*. It is found that after 30 days the entire lake is covered with water lilies, which choke and die, and the question arises: after how long did the water lilies cover only half of the lake? We would like to answer after 15 days. But in reality, the answer is 29 days, because the process is not linear! Indeed, it is a question of solving the equation of variable* t*,* $x(0)2^t = 0.5$*, knowing that* $x(0)2^{30} = 1$ *and therefore that* $x(0) = \dfrac{1}{2^{30}}$.

This example illustrates the question of the resources of our planet vis-à-vis its population. On the 29th day a water lily familiar with math would be concerned about the future of its peers and would warn them: "Tomorrow we will die." But why would the others listen? Because on the 29th day 50% of the surface of the lake is still not covered.

2.6.7 Power Functions

Power functions are functions defined for x non-negative by x^α, where α is any real number (Fig. 2.18).

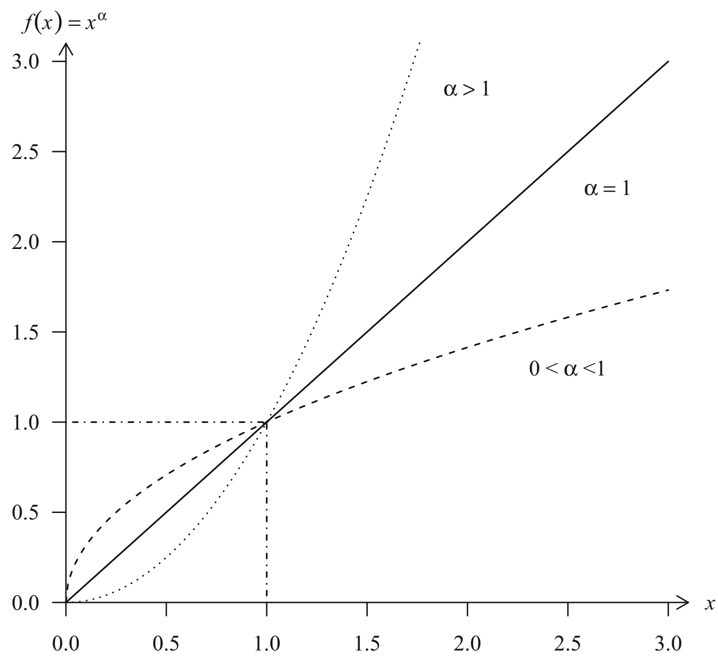

Fig. 2.18 Graphical representation of power function $y = f(x) = x^\alpha$ for $\alpha = 0.5$, $\alpha = 1$, and $\alpha = 2$

Note 19 (Remarks) *Power functions satisfy the same properties as powers of 10.*

For the product:

$$x^\alpha \times x^\beta = x^{\alpha+\beta},$$
$$(x^\alpha)^\beta = x^{\alpha \times \beta}.$$

For the quotient:

$$\frac{x^\alpha}{x^\beta} = x^{\alpha-\beta}.$$

There is no property for addition and subtraction.

The function defined by $x^{1/2}$ is the square root function, as stated previously. Indeed, $(x^{1/2})^2 = x$ and $(x^2)^{1/2} = x$ according to the properties of the powers. In addition, $(x^{1/4})^4 = x$, and so forth.

Example 34 *The temperature equilibrium (in Kelvin) of a planet is given by the formula $T_e = 280\left(\frac{1-A}{D^2}\right)^{1/4}$, where A and D are parameters (A: dimensionless albedo, D: distance to Sun in astronomical unit).* The *function T_e is a power function.*

2.7 Evolution and Trends

2.7.1 Increase and Decrease

The function defined by $a(t) = 2.8e^{-t/0.012}$ for $t > 0$ is a decreasing exponential function: when t increases, the value of $a(t)$ decreases. The following definitions are given (Fig. 2.19).

A **function** f **is increasing** if the value of $f(x)$ increases when x increases. In other words, for x_1 smaller than x_2 we have $f(x_1)$ smaller than $f(x_2)$. Or again, if $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) > 0$.

A **function** f **is decreasing** if the value of $f(x)$ decreases when x increases. In other words, for x_1 smaller than x_2, we have $f(x_1)$ bigger than $f(x_2)$. Or again, if $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) < 0$.

Note 20 *We will see in Chap. 5 how to determine by calculation (using derivatives) when a function is increasing and when it is decreasing (that is to say, its variations).*

Example 35 *Functions* ln *and* exp *and positive exponent power function are increasing. The negative exponent power functions are decreasing.*

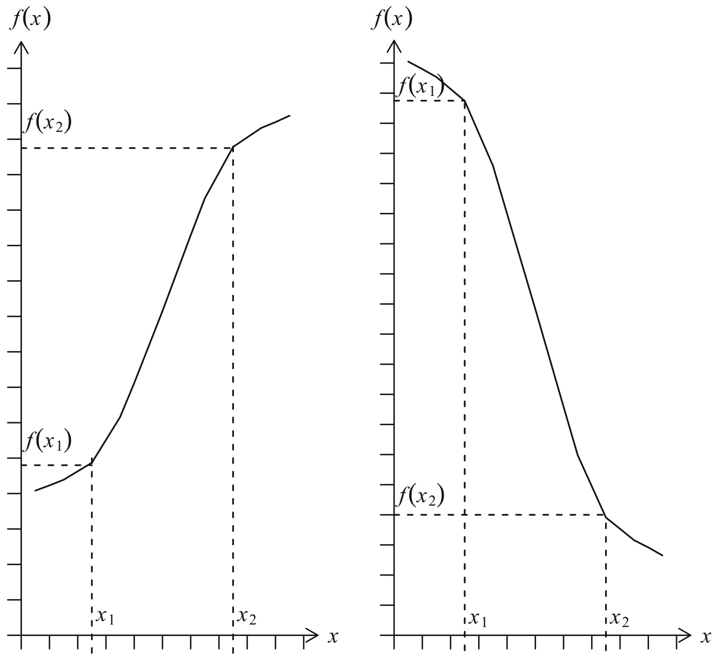

Fig. 2.19 Graphic representations of an increasing function (left) and a decreasing function (right). For all choices $x_1 < x_2$ on the left-hand curve, we have $f(x_1) < f(x_2)$. For all choices of $x_1 < x_2$ on the right-hand curve, we have $f(x_1) > f(x_2)$.

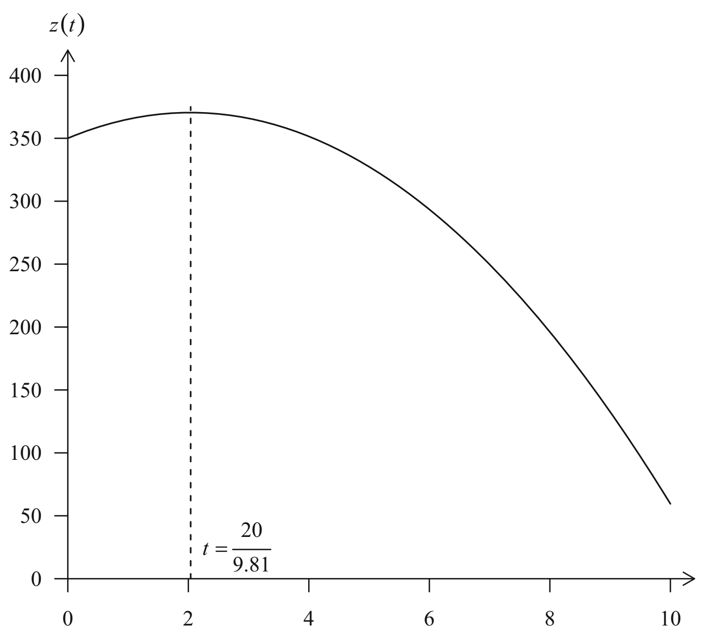

Fig. 2.20 When an object is dropped with an initial velocity at an angle above the horizontal, the altitude first increases, then decreases. The tipping point can be found here, $t = \frac{20}{9.81} \approx 2$

The function $z(t) = -\frac{9.81}{2}t^2 - 2t + 350$ *is decreasing on* $[0; +\infty[$. *The function* $z(t) = -\frac{9.81}{2}t^2 + 20t + 350$ *(modeling the height of an object dropped vertically with an initial velocity upwards) is increasing on* $\left[0; \frac{20}{9.81}\right]$ *and decreasing on* $\left[\frac{20}{9.81}; +\infty\right[$, *with* $\frac{20}{9.81} \approx 2$ *(Fig. 2.20).*

In a market economy, demand is a decreasing function of price, while supply is an increasing

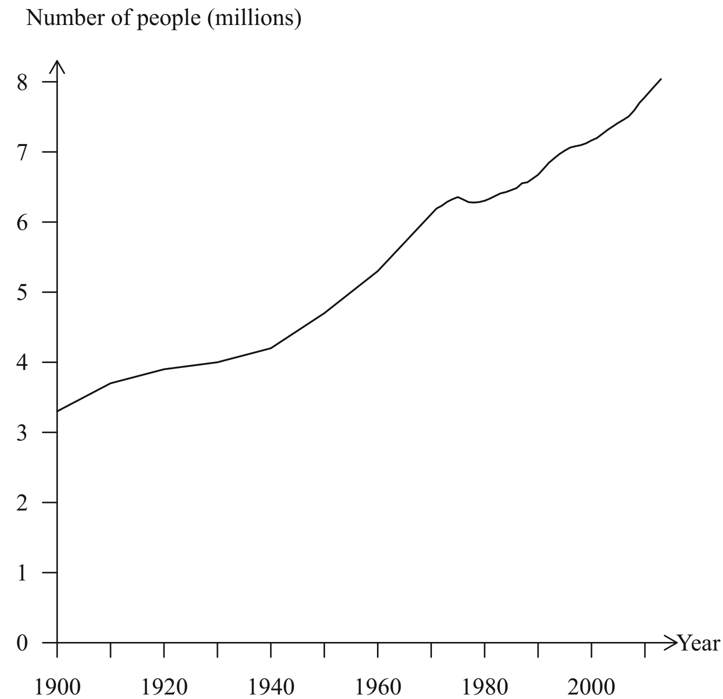

Fig. 2.21 Evolution of Swiss population from 1900 to 2013. The Swiss population has been increasing over the years, except for the period 1976–1981

function of price. When prices rise, demand decreases as supply increases. Conversely, if the price decreases, demand increases while supply decreases.

Example of the evolution of the Swiss population over a little more than 100 years: apart from a slight decrease during the decade 1970–1980, the Swiss population has been increasing. As the years go by, the number of inhabitants increases (Fig. 2.21).

2.7.2 Special Case of Linear Functions

Linear functions have a constant rate of increase (positive or negative). Indeed, for the linear function f expressed by $f(x) = ax + b$ and for $x_1 < x_2$ we have

$$f(x_2) - f(x_1) = ax_2 - ax_1 = a(x_2 - x_1).$$

The increase $f(x_2) - f(x_1)$ is proportional to $x_2 - x_1$, and the rate of increase $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is constant and is a.

This property provides a method for determining a: one chooses two points of the line, with coordinates $(x_1, f(x_1))$ and $(x_2, f(x_2))$, then $a = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Graphically (Fig. 2.22), a displacement of one unit on the abscissa induces a displacement of a units on the y-axis (upwards if a is positive and downwards if a is negative).

2.7.3 Maximum, Minimum

Figure 2.23 shows a decreasing function from $[-3; -1.57]$, increasing from $[-1.57; 1.57]$, then decreasing on $[1.57; 3]$. We say that the function admits a minimum of -1.57 and a maximum of 1.57.

2.7.4 Evolution for Large Values, Limit

Let us use again the function defined by $a(t) = a_0 \times \exp\left(-\frac{t}{\tau}\right)$, with $a_0 = 2.8$ and $\tau = 12$ (Fig. 2.15).

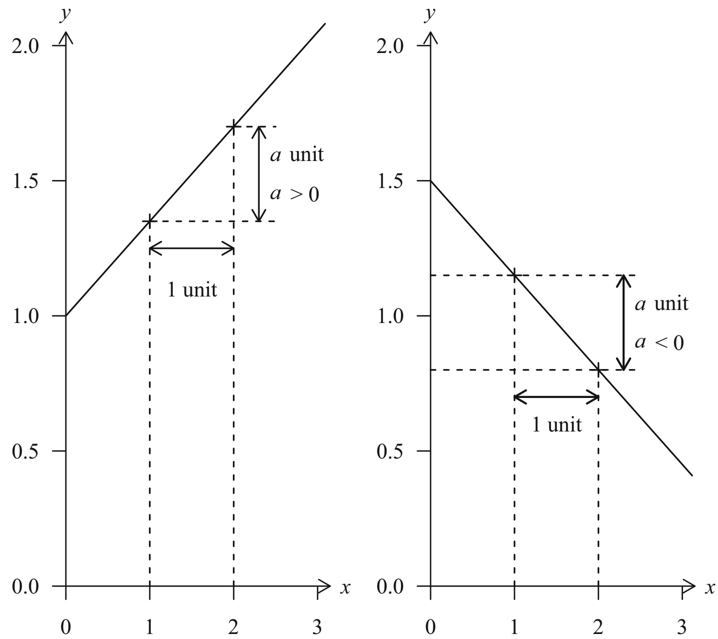

Fig. 2.22 Diagram of growth rate a in the case of linear functions

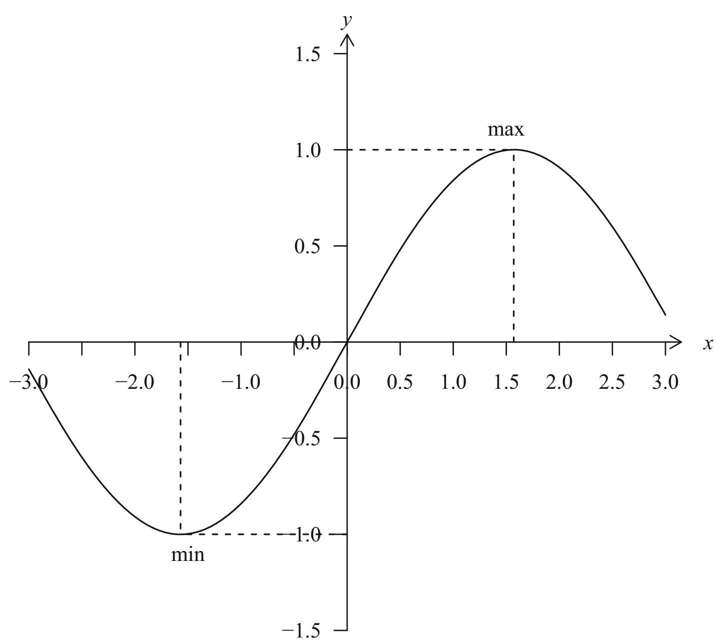

Fig. 2.23 Example of a function with a minimum and a maximum

For very large values of t, $a(t)$ is very small and positive; it is said that when t tends toward infinity, the limit of $a(t)$ is 0.

Key Points

– In a function or a formula, letter(s) (expressing quantities) can have the status of variables (which can take many values) or parameters (the value is fixed for a given reasoning).
– The usual functions must be well known: formulas, properties, and especially graphical representation.
– The link between some functions helps solve equations: square/square root, logarithm/exponential.
– The increase of a function between x_1 and x_2 (with $x_1 < x_2$) is obtained by studying the sign of $f(x_2) - f(x_1)$. For a linear function, the rate of increase $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is constant and equal to the slope of the corresponding line.

Exercises

Mathematical Exercises

Exercise 2.1: Flash Questions. Series 1

(1) Draw the curve of function ln.
(2) Let us give the following relationship: $f(x) = 0.243 \times \alpha - 4.2 \times x^2$. Find the expression of f for the value 1.4 of the parameter α. Is the function f linear?
(3) Can we transform the expression $\exp(a + b)$? What about $\ln(a + b)$?
(4) Solve the equation of variable x: $7.5x - 12 = 19.5$.

Exercise 2.2: Flash Questions. Series 2

(1) Draw the shape of a decaying exponential function.
(2) Recall the definition of a decreasing function.
(3) Can we transform the expression $\ln(a) + \ln(b)$? What about $\exp(a) + \exp(b)$?
(4) Solve the equation of variable a : $7.2 \times 3.4^a = 1$.

Exercise 2.3: Flash Questions. Series 3

(1) Draw the curve of the function exp.
(2) Function f is defined by $f(x) = 0.415 \times \exp(-3.2 \times x) + 252$. What value does $f(0)$ have? To what does $f(x)$ tend when x takes very large values?
(3) Solve the equation of variable t: $9.81t^2 - 150 = 0$ (where $t > 0$).
(4) Solve the equation of variable z: $8.2 - 4.3z = 6.4$.

Exercise 2.4: Drawing Lines

Drawing a line

Method 1

Using two points on the line (which is sufficient to trace it).

Example 36 To draw the line of the equation $y = 1.8 \times 10^{-4}x + 1$, we look for two points on this line. For $x = 0$, we have $y = 1$; for $x = 10{,}000$, $y = 2.8$. Place the coordinate points (0; 1) and (10,000; 2.8) on a graph (scaled to adapt to the situation, of course), and draw the corresponding line (Fig. 2.24).

Note 21 (Remarks)

- *To check the accuracy of the drawing, you can calculate the coordinates of a third point and insert it. No need to take more: this would show that you did not understand, given the equation, that we would obtain a straight line and that two points were enough!*
- *One can also choose a value of* y *and find the corresponding value of* x.

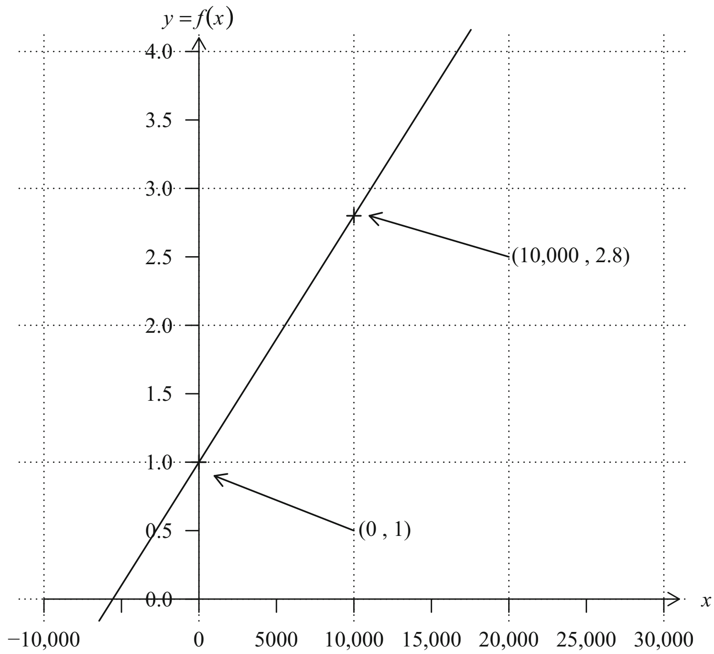

Fig. 2.24 A line can be drawn by means of two points

Using a suitable range on your axes, draw the following equation lines:

(1) $y = -3x + 4$
(2) $5x + 2y = 1$
(3) $y = 2 \times 10^3 x - 235$

Method 2

Using one point and the slope

A point M of coordinates (α, β) is known, and the slope of the line is denoted a. Insert point M. A second point is obtained from the point M by moving it from+1 unit along the x-axis and from a (positive or negative) vertically (Fig. 2.25).

(4) Draw the lines from questions 1, 2, and 3 using this second method.
(5) Plot with R the previous lines: generate the abscissas x using the command `seq()`, and use the function `plot()` to draw the lines. Do not forget to give names to the x- and y-axes (options `xlab` and `ylab`).

Exercise 2.5: Equation of a Line
Finding the equation of a line

- The slope of a line is determined as follows:

Method 1: Using two points of the line (e.g., by reading the graphic representation) $A(x_A, y_A)$ and $B(x_B, y_B)$. The slope is then $\dfrac{y_B - y_A}{x_B - x_A}$.

Method 2: Graphically determining the slope: when moving along the right axis in +1 units, the displacement (positive or negative) is measured vertically and represents the slope (don't forget to consider the scale of this y-axis). See the text in section 2.7 for a graphic illustration.

- We conclude on the equation of the line:

Method 1 (point slope): The line (AB) passing through the points $A(x_A, y_A)$ and $B(x_B, y_B)$ has the following equation: $y = \dfrac{y_B - y_A}{x_B - x_A}(x - x_A) + y_A$.

Method 2 (slope intercept): The intercept b can be determined by graphic representation (we know that $b = f(0)$). If the slope is a, then the line has the equation $y = ax + b$.

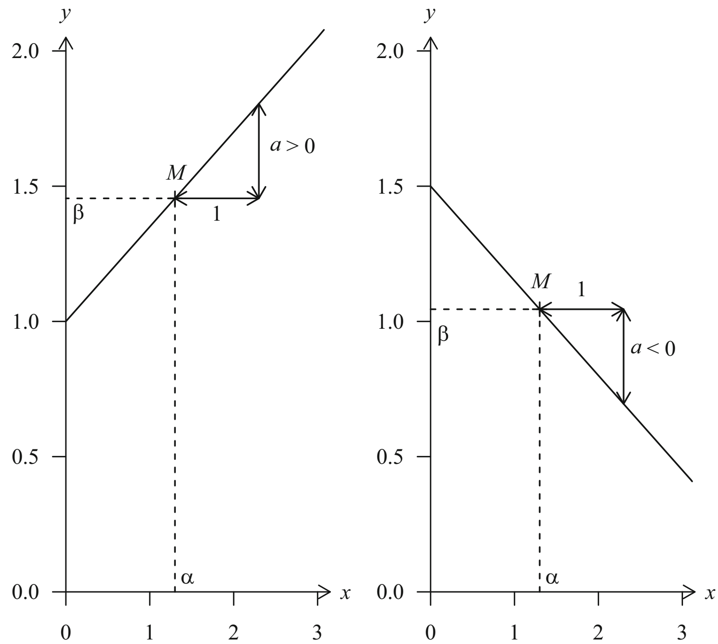

Fig. 2.25 A line can be drawn knowing a point and its slope

Note 22 *Whatever the quantities used on the x- and y-axes (e.g., temperature, pressure), a line equation is written using x and y: avoid writing "the line of equation $T = T_0 - 6.5 \times z$" and write "the line of equation $y = T_0 - 6.5 \times x$."*

(1) Find the equation of the line shown in Fig. 2.26.
(2) Find the equation of the line shown in Fig. 2.27.

Exercise 2.6: Celsius and Fahrenheit
Let us use again the relation $T_F = T_C \times 1.8 + 32$ connecting the temperature T_C expressed in degrees Celsius to T_F expressed in degrees Fahrenheit.

(1) Fill in the following table:

Temperature, °C	0	100		
Temperature, °F			0	100

(2) Express the absolute zero temperature in degrees Celsius and in degrees Fahrenheit.
(3) What is the nature of the function expressing T_F in terms of T_C? Graph this function.
(4) Find the function expressing T_C in terms of T_F and graphically represent it.
(5) Plot with R the function T_C with variable T_F and place on the line the four points of the table in question 1.

Exercise 2.7: Exponential Function
The following questions allow us to perform calculations using the exponential function. They are to be seen as musical scales of music, to be practiced regularly.

Simplify or calculate (use mainly the properties of the exponential function and a minimum number of numerical tools):

(1) $\exp(0)$; $e^3 \times e^{-3}$; $e^{-3.5} \times e^{3.5}$;
(2) $e^{7.3}/e^{-7.3}$; $\dfrac{\exp(-6.3)}{\exp(8.2)}$; $\exp(8.9) \cdot \exp(-6.57)$;
(3) $\dfrac{1.9 \times \exp(-34.5)}{5.7 \times \exp(12.8)}$;

$$\frac{8.34 \times \exp(24) \times 0.54 \times \exp(-0.15)}{4.42 \times \exp(-7)};$$

(4) $8.3 \times 10^{-2} \times \exp(5.7) \times 1.45 \times 10^5 \times \exp(-2.7)$; $\dfrac{3.6 \times 10^4 \times e^{-3.52}}{9.25 \times 10^{-2} \times \exp(4.6)}$.

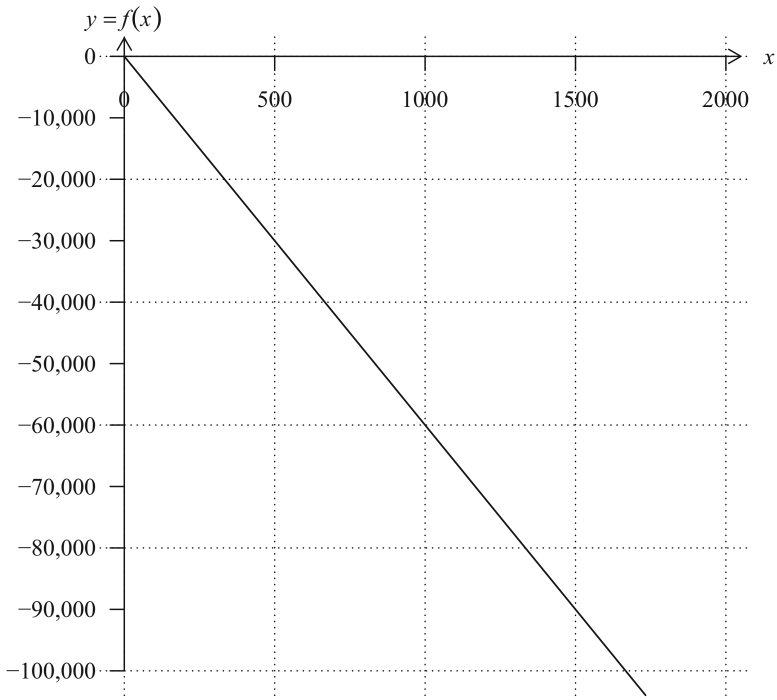

Fig. 2.26 Find the equation of this line (first example)

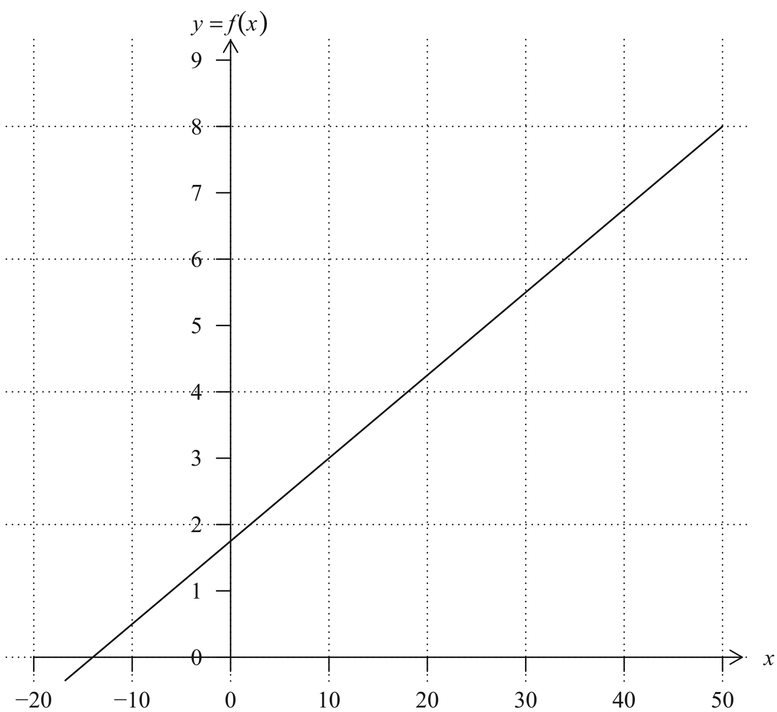

Fig. 2.27 Find the equation of this line (second example)

Exercise 2.8: Exponential Function
Let us consider the function $f(t) = 15.7e^{7.4t}$, where t is non-negative.

(1) What is the value of $f(0)$?
(2) First without a digital tool, draw the shape of the curve representing f, then check with R.

Exercise 2.9: Exponential Function
Let us consider the function $f(t) = 1013.8e^{-1.2t}$ where t is non-negative.

(1) What is the value of $f(0)$?
(2) First without a digital tool, draw the shape of the curve representing f, then check with R.

Exercise 2.10: Exponential Function
Let us consider the function g, whose curve is given in Fig. 2.28.
(1) What is the value of $g(0)$?
(2) Find a and b to express $g(t)$ in the form $g(t) = ae^{-t/b}$.

Note 23 *This is often done when modeling; from a curve obtained one seeks the expression of a function to approach* that curve "*as well as possible*". *There is usually not a single solution.*

(3) Draw with R the curves of $g(t) = ae^{-t/b}$ for $a = 1$ and $b = 0$, 3, 10, and 30 (functions `plot()` and `lines()`). To do this, define a function (`function()`) that calls the constant a and vector b (vector function `c()`). Indicate in the figure the values of the parameter b next to each of the curves (function `text()`).

Exercise 2.11: Natural Logarithm Function
(1) Using a calculator, a graphic tool, or R, draw the graph of the function f defined by $f(x) = \dfrac{\ln(x-40)}{x}$.
(2) Find graphically (and approximately) the maximum of f and the value of x for which this maximum is reached.
(3) With R, answer the previous question using the function `max()`.

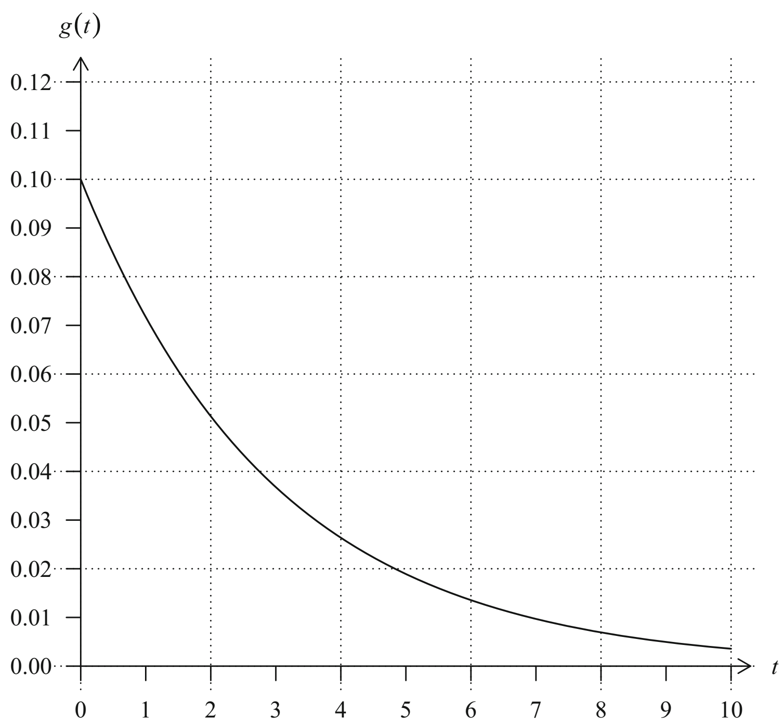

Fig. 2.28 Example of exponential decay

Exercise 2.12: Solving Equations with Natural Logarithm and Exponential
In the following equations, the variable is specified: if there are other letters, then they are parameters.

(1) Find number a such that $20.9 \times 10^a = 41$.
(2) Solve the equation $\ln(8t) = 25$ with variable t, first directly, then using the package `rootSolve` of R.
(3) Solve the equation $\exp(3 \times 10^{-2} t) = 0.2$ with variable t, first directly, then using the package `rootSolve` of R.
(4) Solve the equation $e^{-z/z_0} = K$ with variable z (where z_0 and K are parameters that will appear in the solution).

Exercise 2.13: Logarithm Function in Base 10 and Powers of 10
The function $\log_{10}$ (read "log base 10") is a logarithm function (also called a common logarithm) that takes the value 0 in 1 (as ln) but is 1 in 10 (instead of e for ln). It is frequently used and often denoted log.

It is defined for $x > 0$ by $\log(x) = \dfrac{\ln x}{\ln(10)}$.

(1) Without a calculator, check that it has $\log(1) = 0$ and $\log(10) = 1$.
(2) Always without a calculator, draw approximately the graph of the function log. Check the answer using a calculator or software.
(3) Justify the following sentence: "When you multiply a number by 10, its value increase by 1." Deduct the values of $\log(10^3)$, $\log(10^7)$, and $\log(10^{-2})$.

Note 24 (Remarks) *Like* exp *and* ln *are inverse functions of each other, the foregoing shows that* $\log_{10}$ *and the power function of 10 are reciprocals of each other: we have* $\log(10^x) = x$ *and* $10^{\log x} = x$..

For example, if the variable t is such that $10^{2.5t} = 12.3 \times 10^{-2}$ *(that is, if one seeks to solve the equation* $10^{2.5t} = 12.3 \times 10^{-2}$*), we write* $2.5t = \log(12.3 \times 10^{-2})$*; then we obtain* $t = \dfrac{\log\left(12.3 \times 10^{-2}\right)}{2.5} \simeq -0.36$.

(4) Solve the following equations:

$$10^{1000.5t} = 4.67;$$

$$10^{-2.5x} = 4.56 \times 10^5 \times 8.2\exp(4.2);$$

$$5.3 \times 10^{3,1y} = 7.8 \times 10^4.$$

(5) With R graphically, verify the solutions of the three equations of question 4.

Exercise 2.14: Logarithm Function in Base 2 and Powers of 2
Define the function $\log_2$, and take questions 1–3 of the previous exercise (with adaptation) for the functions $\log_2$ and power of 2.

(1) Solve the equation $5.3 \times 2^{3.1x} = 0.151$.
(2) With R, graphically check the solution of the equation $5.3 \times 2^{3.1x} - 0.151 = 0$.

Exercise 2.15: Logarithmic Scale
The scales presented in the course are called **linear**. The next exercise aims to present another type of scale (nonlinear).

Table 2.2 shows the pressure (hPa) and the corresponding distances x (km).

(1) Represent the data by choosing suitable x-axis (distance x) and y-axis (pressure) scales.

The resulting graph does not allow all pressure data to be displayed correctly. Indeed, the data for $x = 30$ km and $x = 40$ km are "crushed" on the x-axis. The pressure values at $x = 30$ km and $x = 40$ km, with the authors' chosen scale (see solutions), are to be placed respectively at 6.0×10^{-5}cm and 1.2×10^{-5}cm from the origin, which is less than a micrometer!

Table 2.2 Data table: distance x (km) and pressure (hPa)

Distance x (Km)	Pressure (hPa)
5	900
10	250
20	50
30	5
40	5×10^{-3}

One can reverse the problem and choose a scale that makes it possible to visualize the small pressures. For example, 1 cm on the graph represents 10^{-3} hPa in reality. In this case, the 900 hPa of reality must be placed with a linear scale at $\frac{900}{10^{-3}} = 900,000$ cm from the origin on the graph, i.e., we would need a 9 km sheet to represent all of these data!

For this type of data, which have large amplitudes in their values, linear scales are not adopted.

The point of a **logarithmic scale** is to represent data or functions that have a large range of values.

The method of setting a logarithmic scale involves moving from one mark to another by a constant factor (10 in the case of a decimal logarithmic scale). Thus, the number a is positioned at $y = \log(a)$ on a logarithmic scale:

- Numbers $1 < a < 10$ are represented by the points $0 < y < 1$;
- Numbers $10 < a < 100$ are represented by the points $1 < y < 2$;
- and so forth;
- Numbers $10^n < a < 10^{n+1}$ are represented by the points $n < y < n + 1$;
- Numbers $10^{-n} < a < 10^{-n+1}$ are represented by the points $-n < y < -n + 1$.

(2) Apply a logarithmic scale to the pressure data in Table 2.2 (while keeping a linear scale x-axis).

Exercises in Geography and Geology

Exercise 2.16: Evapotranspiration and Carbon Dioxide Pressure

There is an **empirical** relationship (that is to say, based on experience) between the pressure of carbon dioxide P_{CO_2} (Atm) in the atmosphere and evapotranspiration Evt (Mm/year). This relationship is given by Brook and Hanson (1991) in the form $\log(P_{CO_2}) = -3.47 + 2.09(1 - \exp(-0.00172 Evt))$.

(1) We give $Evt = 1$ m/year. Calculate P_{CO_2} in atm.
(2) We give $P_{CO_2} = 0.005$ atm. With R, graphically determine the approximate value of evapotranspiration in m/year.

Exercise 2.17: Age of Oceanic Crust

The relationship between water depth P and the age of the oceanic crust t below it is given by $P = 2500 + 350\sqrt{t}$ (Pomerol and Renard, 1995). P is expressed in meters and t in millions of years (My).

(1) Calculate the depth of water above a crust of 15 My.
(2) What is the age of oceanic crust located at a depth of 4400 m?
(3) The relationship is valid for a lithosphere whose age is less than 60 My. How can these data be mathematically interpreted?
(4) Use R to represent graphically P in terms of t and place points from questions 1 and 2 on the curve.

Exercise 2.18: Evolution of Population of Quebec (Canada)

The evolution of Quebec's population between 1951 and 2011 is shown in Fig. 2.29.

(1) From the data (1951 and 2011), calculate the average annual growth rate.
(2) We choose to assimilate the graphical representation to a line: propose an equation of this line.
(3) From the straight line obtained in the previous question, find the annual growth rate of the population and compare it with the result of the first question.
(4) Use the linear model function of `R (lm())` to determine the average annual growth rate.

Exercise 2.19: Carbon-14 Dating

Carbon-14 (radioactive isotope of carbon) is constantly being renewed in living beings. This renewal ceases upon the death of the creature, and the amount of carbon-14 present then decreases in a decreasing exponential way.

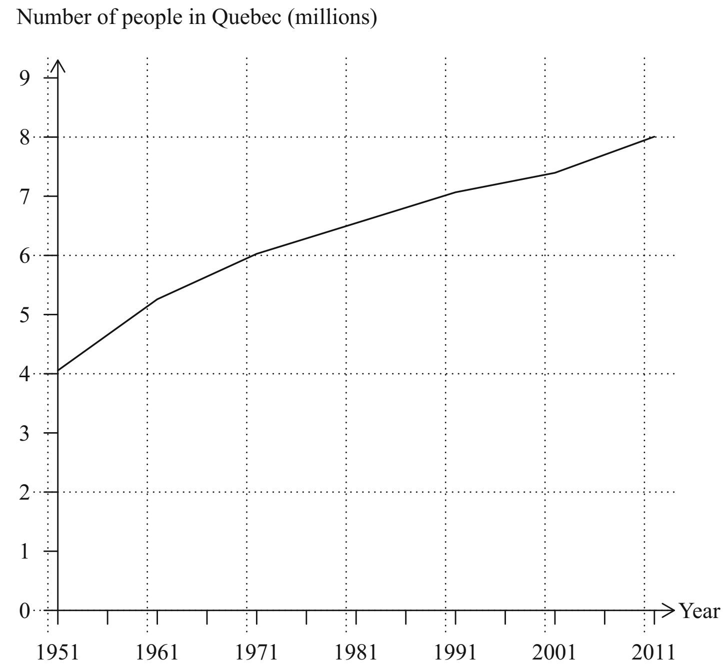

Fig. 2.29 Evolution of Quebec's population

The living being dies at $t = 0$; at that time there is then N_0 carbon-14 atoms present. The number of carbon-14 atoms present decreases according to the law $N(t) = N_0 \times \exp(-1.210 \times 10^{-4} t)$, where t is the number of years elapsed since the death of the individual.

The proportion of carbon-14 remaining determines the time that has elapsed since death (and there is no need to know the value of N_0).

(1) What percentage of carbon-14 atoms is lost after a century? After 10,000 years?
(2) We call the half-life of carbon-14 the time after which half of the atoms have disintegrated. Calculate this half-life.
(3) Fragments found contain 60% of the expected carbon-14 content. How old are these fragments?

Exercise 2.20: Scenario in Hydrogeology

To determine the flow rate of water in an aquifer, Darcy's law can be used. It gives the relation between the flow through this medium (Q), the hydraulic gradient i, the surface perpendicular to the flow A, and a parameter that characterizes the aquifer k: permeability (expressed in m/s).

Darcy's law is given by the relation $Q = A \times k \times i$.

The hydraulic gradient is the ratio between the difference in water depth in two boreholes and the distance between these two boreholes. The hydraulic gradient is $i = \frac{H}{L}$ (Fig. 2.30).

(1) What is the unit of measure of the gradient? Of the flow?
(2) Justify mathematically and physically that $Q = 0$ m^3/s when $i = 0$.
(3) It is assumed that this aquifer consists of limestone whose estimated permeability is 8.64 m/day and that the river is 15 m deep. From the graph in Fig. 2.30, determine the flow (in L/s) that goes from the aquifer to the river.
(4) By imagining that a soluble contaminant escapes from the tank shown in the figure by a cylinder, calculate the time it will take the contaminant to reach the river.

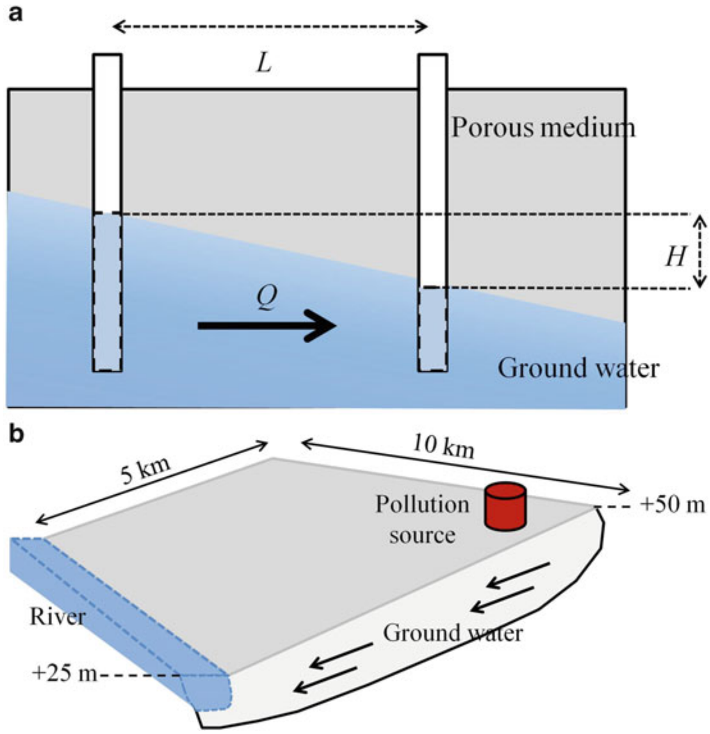

Fig. 2.30 Principle of flow in an aquifer: the flow rate is proportional to the hydraulic gradient

(5) It is considered that the gradient i varies between 0 and 0.2 and that the permeability k is also a variable with values ranging from 7 to 10 m/day. Represent the values taken by the flow Q in L/s. To do this, use the R function `perps()`: an axis for i, an axis for k, and the third axis for Q.

Exercise 2.21: Earth Science Scenario

Depending on its speed of entry into the atmosphere and its cohesion, a meteor arriving on Titan (main moon of Saturn) can fragment during its fall. The fragmentation altitude h_f is given by $h_f \simeq H \ln\left(\frac{\rho_{atm} v^2}{S}\right)$, where v is the speed of the meteor (in m/s), H the thickness of the gas of Titan's atmosphere, ρ_{atm} the density of Titan's atmosphere, and S the shear resistance of the meteor. The goal of the exercise is to find the speed necessary for a meteor entering Titan's atmosphere to reach the ground without fragmenting.

Information: $S = 10^4$ N m^{-2}; $H = 20345$ m; $\rho_{atm} = 5.3$ kg m^{-3}.

(1) In this relationship, what are the parameters? What are the variables? Using numerical data, rewrite this formula as a function f of a variable x.
(2) Mathematically, what are the possible values taken by x for the function f (its domain)? Mathematically, are there any values of x for which $f(x)$ is negative?
(3) Physically, how do you interpret a negative result for h_f?
(4) For a meteor entering Titan's atmosphere what speed does it have to be traveling at to reach the ground without fragmenting?

Solutions

Solution 2.1: Flash Questions. Series 1

(1) See the text.
(2) $\alpha = 1.4$, we have $f(x) = 0.3402 - 4.2 \times x^2$. The function is not linear because of the presence of the square x^2.
(3) We have $\exp(a + b) = \exp(a) \times \exp(b)$. There is no property for $\ln(a + b)$.

(4) The equation $7.5x - 12 = 19.5$ gives $7.5x = 19.5 + 12 = 31.5$, that is, $x = \frac{31.5}{7.5} = 4.2$. The equation has only one solution: 4.2.

Solution 2.2: Flash Questions. Series 2

(1) See the text.
(2) See the text.
(3) We have $\ln(a) + \ln(b) = \ln(a \times b)$. There is no property for $\exp(a) + \exp(b)$.
(4) The equation $7.2 \times 3.4^a = 1$ gives $3.4^a = \frac{1}{7.2}$, and therefore (using the ln function) $\ln(3.4^a) = \ln\left(\frac{1}{7.2}\right)$, then $a \times \ln(3.4) = -\ln(7.2)$, and finally: $a = -\frac{\ln(7.2)}{\ln(3.4)} \approx -1.61$.

Solution 2.3: Flash Questions. Series 3

(1) See the text.
(2) $f(0) = 252.415$. As $\exp(-3.2x)$ tends toward 0 when x takes very large values, $f(x)$ tends toward 252 when x takes very large values.
(3) The equation $9.81t^2 - 150 = 0$ gives $9.81t^2 = 150$, so that $t^2 = \frac{150}{9.81}$. This leads to two solutions: $t = -\sqrt{\frac{150}{9.81}}$ and $t = +\sqrt{\frac{150}{9.81}}$. Since the statement (or often the concrete situation) indicates that t is positive, we keep only the solution $t = +\sqrt{\frac{150}{9.81}} \approx 3.91$.
(4) Equation $8.2 - 4.3 \times z = 6.4$ gives $-4.3z = 6.4 - 8.2 = -1.8$ and so $z = -\frac{1.8}{-4.3} \approx 0.42$.

Solution 2.4: Line Drawing

For straight lines D_1, D_2, and D_3 of questions 1–3, we obtain the following drawings (Fig. 2.31):

(1) Line $D_1 : y = -3x + 4$ passes through points (0, 4) and (3, −5).
(2) Line $D_2 : 5x + 2y = 1$ passes through points (0, 0.5) and (3, −7).
(3) Line $D_3 : y = 2 \times 10^3 x - 235$ passes through points (0, −235) and (3, 5765).

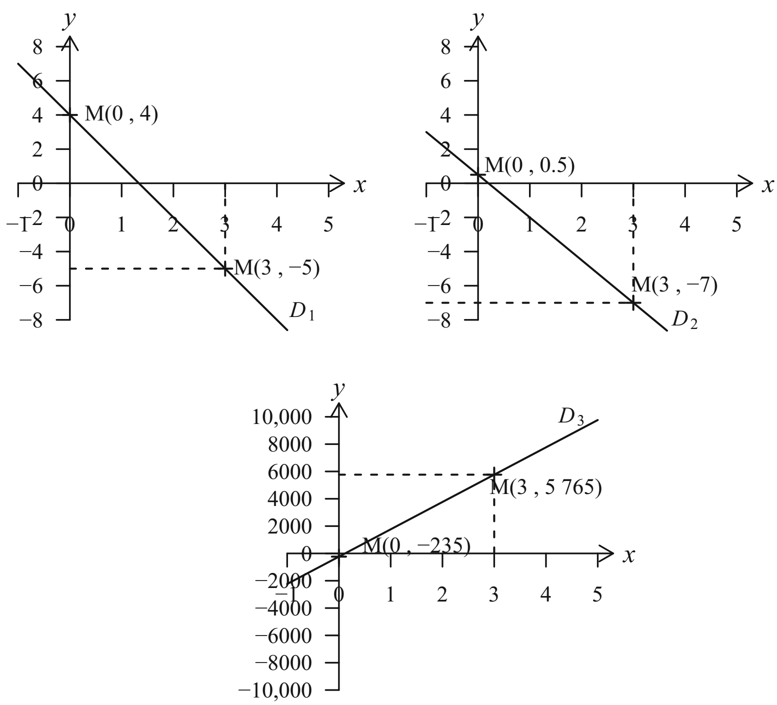

Fig. 2.31 Graphing straight lines $D_1 : y = -3x + 4$ (top left); $D_2 : 5x + 2y = 1$ (top, right); and $D_3 : y = 2 \times 10^3 x - 235$ (bottom left)

(4) Line D_1 passes through (0, 4) and has a slope of -3; line D_2 passes through (0; -0.5) and has a slope of -2.5 (since the line has an equation of $y = 0.5 - 2.5x$); line D_3 passes through (0; -235) and has a slope of 2×10^3.

(5)

```
#-------------------------------------
# Plot lines
#-------------------------------------
#------------D1 : y=-3x+4
x <- seq(-1,5,0.1)
plot(x,-3*x+4,type="l",col=1,lwd=1.2,
     xlim=c(-1,5),ylim=c(-8,8),
     axes=T,xlab="x",ylab="y")
points(0,4,lwd=1.2,pch=3)
points(3,-5,lwd=1.2,pch=3)
text(1,4,"M(0;4)")
text(4,-5,"M(3;-5)")

#------------D2 : 2x+2y=1
x <- seq(-1,5,0.1)
plot(x,-2.5*x+.5,type="l",col=1,
lwd=1.2,
     xlim=c(-1,4),ylim=c(-8,5),
     axes=T,xlab="X",ylab="y")
points(0,0.5,lwd=1.2,pch=3)
points(3,-7,lwd=1.2,pch=3)
text(0.5,1,"M(0;0.5)")
text(3.5,-7,"M(3;-7)")

#------------D3 : y=2.10^3x-235
x <- seq(-1,5,0.1)
plot(x,2e3*x-235,type="l",col=1,
lwd=1.2,
     xlim=c(-1,5),ylim=c(-5000,10000),
     axes=T,xlab="x",ylab="y")
points(0,-235,lwd=1.2,pch=3)
points(3,5765,lwd=1.2,pch=3)
text(1.5,500,"M(0;-235)")
text(4,4765,"M(3;5765)")
```

Solution 2.5: Equation of a Line

(1) For the first line, the coordinate points (0; 0) and (500; $-30{,}000$) belong to this line. Its slope is $\dfrac{-30,000 - 0}{500 - 0} = -60$ (this negative slope is consistent with the shape of the curve). An equation of the line is $y = -60x$.

(2) The second line passes through (10, 3) and (50, 8), so an increase of 40 on the x-axis leads to an increase of 5 on the y-axis, i.e., 0.125 on the y-axis for 1 on the x-axis: the slope is 0.125. An equation of the line is $y = 0.125 \times (x - 10) + 3$ or $y = 0.125x + 1.75$.

Solution 2.6: Celsius and Fahrenheit

The relation $T_F = T_C \times 1.8 + 32$ connects the temperature T_C expressed in degrees Celsius to T_F expressed in degrees Fahrenheit.

(1)

Temperature, °C	0	100	−17.8	37.8
Temperature, °F	32	212	0	100

(2) Absolute zero is −273.15 °C or −459.67 °F.

(3) The function expressing T_F in terms of T_C is a linear function (see the text for its graphical representation).

(4) We have $T_C = (T_F - 32)/1.8 = \dfrac{1}{1.8}T_F - \dfrac{32}{1.8}$, which is written in the classical form of a line equation by $y = \dfrac{1}{1.8}x - \dfrac{32}{1.8} + 32$ (Fig. 2.32).

(5)

```
#-------------------------------------
# Celsius vs Fahrenheit
#-------------------------------------
tf <- seq(0,250,0.1)
plot(tf,(tf-32)/1.8,type="l",col=1,
lwd=1.2,
     xlim=c(0,250),ylim=c(-40,100),
     axes=T, xlab="Temperature (°F)",
     ylab="Temperature (°C)")
points(32,0,lwd=1.2,pch=3)
points(212,100,lwd=1.2,pch=3)
points(0,-17.8,lwd=1.2,pch=3)
points(100,37.8,lwd=1.2,pch=3)
```

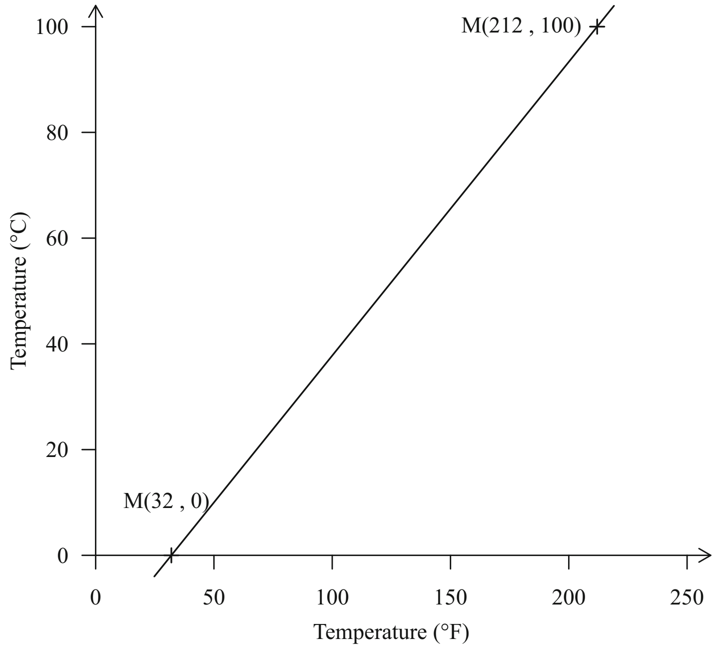

Fig. 2.32 Graphical representation of the line expressing the relation $T_F = 1.8T_C + 32$

Solution 2.7: Exponential Function

Recall that rounding is to be done as late as possible in a calculation.

(1) The answer is 1 for the three calculations.

(2)

- $e^{7.3}/e^{-7.3} = e^{7.3\ -\ (-7.3)} = e^{14.6}$
- $\dfrac{\exp(-6.3)}{\exp(8.2)} = \exp(-6.3 - 8.2)$ $= \exp(-14.5)$
- $\exp(8.9) \cdot \exp(-6.57) = \exp(8.9 - 6.57) = \exp(2.33)$

(3)

- $\dfrac{1.9 \times \exp(-34.5)}{5.7 \times \exp(12.8)} = \dfrac{1.9}{5.7}\exp(-34.5 - 12.8)$ $\simeq 0.33\exp(-47.3)$
- $\dfrac{8.34 \times \exp(24) \times 0.54 \times \exp(-0.15)}{4.42 \times \exp(-7)} = \dfrac{8.34 \times 0.54}{4.42}\exp(24 - 0.15 - (-7)) \simeq 1.02\exp(30.85)$

(4)

- $8.3 \times 10^{-2} \times \exp(5.7) \times 1.45 \times 10^5 \times \exp(-2.7) = 8.3 \times 1.45 \times 10^{-2+5} \exp(5.7 - 2.7) = 12.035 \times 10^3 \exp(3) \simeq 2.4 \times 10^5$
- $\dfrac{3.6 \times 10^4 \times e^{-3.52}}{9.25 \times 10^{-2} \times \exp(4.6)} = \dfrac{3.6}{9.25} \times 10^{4-(-2)}\exp(-3.52 - 4.6) = \dfrac{3.6}{9.25} \times 10^6\exp(-8.12) \simeq 115.8$

Solution 2.8: Exponential Function

(1) $f(0) = 15.7$.

(2) The curve is one of an increasing exponential function (Fig. 2.33).

```
#------------------------------------
# Increasing Exponential
#------------------------------------
# At t = 0
Ft = 15.7 * exp(7.4 * 0)
print(Ft)
cat("at t=0, the value of the function
f(t)=",Ft,"\n")

# Q2 : plot the function
t <- seq(0,1,0.01)
plot(t,15.7*exp(7.4*t),type="l",col=1,
lwd=1.2,
     xlim=c(0,1),ylim=c(0,10000),
     axes=T,,xlab="t",ylab="f(t)")
# One can zoom in to check that f(t=0)
```

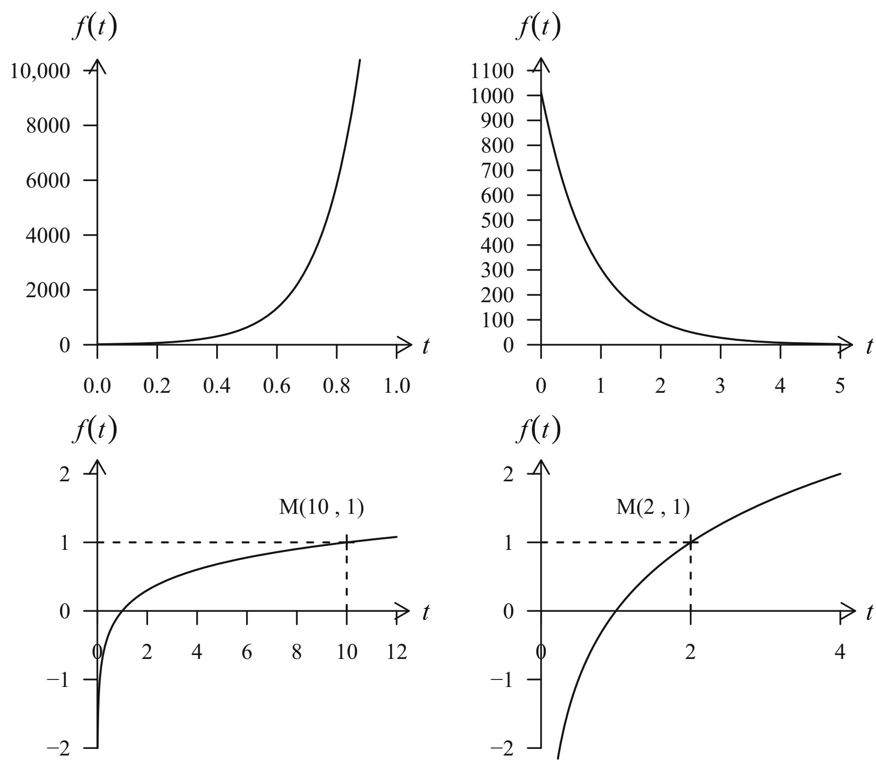

Fig. 2.33 Graphing functions $f(t) = 15.7 \exp{(7.4t)}$ (top left), $f(t) = 1013.8 \exp{(-1.2t)}$ (top right), $\log_{10}$ (bottom left), and $\log_2$ (bottom right)

```
=15.7
# by means of xlim and ylim
plot(t,15.7*exp(7.4*t),type="l",col=1,
lwd=1.2,
     xlim=c(0.002),ylim=c(15.5,16),
          axes=T,xlab="t",ylab="f(t)")
```

Solution 2.9: Exponential Function

(1) $f(0) = 1013.8$.

(2) The curve is one of a decreasing exponential function (Fig. 2.33).

```
#--------------------------------------
# Decreasing Exponential
#--------------------------------------
# At t = 0
Ft = 1013.8 * exp(-1.2 * 0)
print(Ft)
cat("At t=0, the value of the function
f(t)=",Ft,"\n")
# Plot the function
t <- seq(0,10,0.1)
plot(t,1013.8*exp(-1.2*t),type="l",
col=1,lwd=1.2,
     xlim=c(0,10),ylim=c(0,1200),
     axes=T,xlab="t",ylab="f(t)")
# One can zoom in to check that f(t=0)
=1013.8
# by means of xlim and ylim
plot(t,1013.8*exp(-1.2*t),type="l",
col=1,lwd=1.2,
     xlim=c(0,.002),ylim=c(1012,1014),
          axes=T,xlab="t",ylab="f(t)")
```

Solution 2.10: Exponential Function

(1) $g(0) \simeq 0.1$. Note that the values read graphically are approximate values.

(2) As $g(0) \simeq 0.1$, we have $a \simeq 0.1$. We search b taking another value; for example: $g(5) \simeq 0.02$. So $0.1 \times e^{-5/b} = 0.02$, and so

$e^{-5/b} = 0.02/0.1 = 0.2$, from which $-5/b = \ln(0.2)$, and finally $b = -5/\ln(0.2) \simeq 3.1$. The function g defined by $g(t) = 0.1 \times e^{-t/3.1}$ has a graphical representation close to the proposed curve.

(3)

```
#-------------------------------------
# Exponential g(t)=a*exp(-t/b)
#-------------------------------------
Gt <- function(a,b,t) {
  return (a * exp(-t/b))
}
t <- seq(0,100,0.1)
# Parameter "a" is set to 0.1 (con-
stant)
a <- 1.0
# Parameter "b" can have several values
b <- c(3,10,30,60)

plot(t,Gt(a,b[1],t),type="l",col=1,
lwd=1.2,
     xlim=c(0,100),ylim=c(0,1.1),
     axes=T,xlab="t",ylab="g(t)")
lines(t,Gt(a,b[2],t))
lines(t,Gt(a,b[3],t))
lines(t,Gt(a,b[4],t))
text(8,0.3,"b=3")
text(20,0.3,"b=10")
text(45,0.3,"b=30")
text(85,0.3,"b=60")
```

Solution 2.11:

(1) The graph of the function $f(x) = \dfrac{\ln(x-40)}{x}$ is given in Fig. 2.34.

(2) Graphically (and very roughly), the maximum is reached at $x = 60$ and is 0.05.

(3)

```
#-------------------------------------
# Function f(x)=ln(x-40)/x
#-------------------------------------
    x <- seq(41,100,0.1)
    plot(x,log(x-40)/x,type="l",col=1,
    lwd=1.2,
       xlim=c(40,100),ylim=c(0,0.06),
       axes=T,xlab="x",ylab="f(x)")
       # One may create a vector close
       to xmax with a small step
       x <- seq(50,70,0.000001)
       # One may use the function max()
       XMax <- max(log(x-40)/x)
```

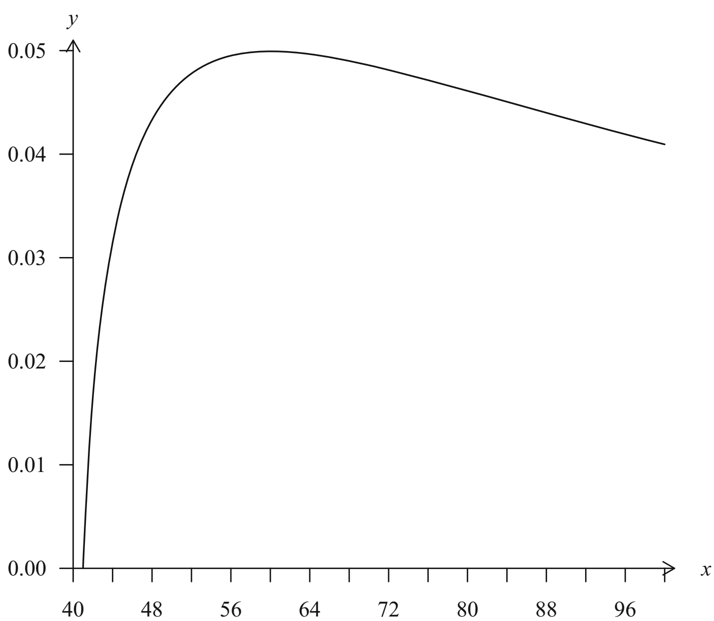

Fig. 2.34 Graphical representation of function $f(x) = \frac{\ln(x-40)}{x}$

Solution 2.12: Solving Equations with ln and exp

(1) The function ln is used so that the exponent appears in a multiplication: $20.9 \times 10^a = 41$ becomes $10^a = 41/20.9$, then $\ln(10^a) = \ln(41/20.9)$, and so $a \times \ln(10) = \ln(41/20.9)$, and finally $a = \frac{\ln(41/20.9)}{\ln(10)} \simeq 0.29$.

(2) The exponential is used: $\ln(8t) = 25$ becomes $\exp(\ln(8t)) = \exp(25)$ (because $\exp(\ln(x)) = x$), from which $8t = e^{25}$, and so $t = \frac{e^{25}}{8} \simeq 9 \times 10^9$.

(3) We use the natural logarithm: $\exp(3 \times 10^{-2}t) = 0.2$ becomes $\ln(\exp(3 \times 10^{-2}t)) = \ln(0.2)$ (because $\ln(\exp(x)) = x$), from which $3 \times 10^{-2}t = \ln(0.2)$, and so $t = \frac{\ln(0.2)}{3 \times 10^{-2}} \simeq -53.6$.

```
#-----------------------------------------
# Solving equations Exp and Ln
#-----------------------------------------
# Load library "rootSolve"
library(rootSolve)

#------------ y(t) = ln(8t) - 25
fun <- function (x) log(8*x) -25
curve(fun(x),0,1e11)
abline(h = 0, lty = 3)
uni <- uniroot(fun, c(0, 1e11))
$root
points(uni, 0, pch = 16, cex = 2)
      cat("The value of t, such as
y(t) = 0 is t =",
uni,"\n")

#------------ y(t) = exp(3.10^-2 * x)
-0.2
fun <- function (x) exp(3e-2 * x) -0.2
curve(fun(x),-100,0)
abline(h = 0, lty = 3)
uni <- uniroot(fun, c(-100, 0))$root
points(uni, 0, pch = 3, lwd=1.2)
cat("The value of t, such as y(t) =
0 is t =",
      uni,"\n")
```

(4) We use the natural logarithm: $e^{-z/z_0} = K$ with variable z becomes $\ln\left(e^{-z/z_0}\right) = \ln(K)$, and so $-z/z_0 = \ln(K)$, from which $z = -z_0 \times \ln(K)$. This is valid if $K > 0$. There is no solution if $K \leq 0$.

Solution 2.13: $\log_{10}$ and Powers of 10

(1) We have $\log(1) = \frac{\ln(1)}{\ln(10)} = 0$ because $\ln(1) = 0$ and $\log(10) = \frac{\ln(10)}{\ln(10)} = 1$.

(2) The function curve log has the same appearance as that of ln but passes through the point (10, 1) (Fig. 2.33).

(3) If we multiply a number a by 10, the value of log goes from $\log(a)$ to $\log(10 \times a)$. The function log has the same property as ln.

$$\log(10 \times a) = \log(10) + \log(a) = 1 + \log(a).$$

It can be deduced from this that

$$\log\left(10^3\right) = 3\left(\text{because } 10^3 = 10 \times 10 \times 10\right),$$
$$\log\left(10^7\right) = 7, \log\left(10^{-2}\right) = -2.$$

(4) The equation $10^{100.5t} = 4.67$ becomes $\log(100.5t) = \log(4.67)$, whereby $100.5\,t = \log(4.67)$, and so $t = \frac{\log(4.67)}{100.5} \simeq 6.7 \times 10^{-3}$.

The equation $10^{-2.5x} = 4.56 \cdot 10^5 \times 8.2 \exp(4{,}2)$ becomes $-2.5x = \log(4.56 \times 10^5 \times 8.2 \exp(4.2))$, and so $x = -\log(4.56 \times 10^5 \times 8.2 \exp(4.2))/2.5 \simeq -3.36$.

The equation $5.3 \cdot 10^{3.1y} = 7.8 \times 10^4$ becomes $10^{3.1y} = 7.8 \times 10^4/5.3$, and so $3.1y = \log(7.8 \times 10^4/5.3)$, from which $y = \log(7.8 \times 10^4/5.3)/3.1 \simeq 1.34$.

(5)

```
#-----------------------------------------
# log10 : graphical solutions
#-----------------------------------------
#------------ y(t) = 10^(100.5 * t) -
4.67
t <- seq(0,7e-3,0.0001)
```

```
plot(t,10^(100.5*t)-4.67,type="l",
col=1,lwd=1.2,
   axes=T,xlab="x",ylab="f(x)")
t0 <- log(4.67,10)/100.5
points(t0,0,pch=3,lwd=1.2)
segments(t0,-4,t0,0,lty=2)
segments(0,t0,t0,0,lty=2)

#--- y(x) = 10^(-2.5 * x) - 4.56 * 10^5
* 8.2 * exp(4.2) x <- seq(-4,-3,0.01)
plot(x,10^(-2.5*x)-4.56e5 * 8.2 * exp
(4.2),type="l",
   col=1,lwd=1.2, axes=T,xlab="x",
   ylab="f(x)",
   ylim=c(-1e8,5e8))
x0 <- -log(4.56e5*8.2*exp(4.2),10)/2.5
points(x0,0,pch=3,lwd=1.2)
segments(x0,-2e8,x0,0,lty=2)
     segments(-4,0,x0,0,lty=2)

#----------- f(y) = 5,3.10^(3,1 * y) -
7,8.10^4
y <- seq(1,1.6,0.01)
plot(y,5.3*10^(3.1*y)-7.8e4,type="l",
col=1,lwd=1.2,
  axes=T,xlab="x",ylab="f(x)",ylim=c
(-1e5,1e5))
     y0 <- log(7.8e4/5.3,10)/3.1
     points(y0,0,pch=3,lwd=1.2)
     segments(y0,-1e6,y0,0,lty=2)
     segments(-1e5,0,y0,0,lty=2)
```

Solution 2.14: $\log_2$ and Powers of 2

The function $\log_2$ is defined for $x > 0$ by $\log_2(x) = \dfrac{\ln(x)}{\ln(2)}$.

(1) We have $\log_2(1) = \dfrac{\ln(1)}{\ln(2)} = 0$ because $\ln(1) = 0$ and $\log_2(2) = \dfrac{\ln(2)}{\ln(2)} = 1$.

(2) The curve of the function $\log_2$ has the same appearance as that of ln but passes through the point (2; 1) (Fig. 2.33).

(3) It has a similar property to that of $\log_{10}$; when multiplying a number by 2, the value of $\log_2$ increases by 1. Indeed, if one multiplies a number a by 2, the value of $\log_2$ goes from $\log_2(a)$ to $\log_2(2 \times a)$. The function $\log_2$ has the same property as ln: $\log_2(2 \times a) = \log_2(2) + \log_2(a) = 1 + \log_2(a)$.

It can be deduced from this that $\log_2(2^3)=3$ (because $2^3 = 2 \times 2 \times 2$), $\log_2(2^7) = 7$, $\log_2(2^{-2}) = -2$.

(4) The equation $5.3 \times 2^{3.1x} = 0.151$ becomes $2^{3.1x} = \dfrac{0.151}{5.3}$, which means $3.1x = \log_2\left(\dfrac{0.151}{5.3}\right)$, and so $x = \log_2\left(\dfrac{0.151}{5.3}\right)/3.1 \simeq -1.66$.

(5)

```
#--------------------------------------
log 2 : graphical solutions
#--------------------------------------
#------------ y(t) = 5,3.2^(10,1 * x) -
0,151
x <- seq(-2,-1,0.0001)
plot(x,5.3*2^(3.1*x)-0.151,type="l",
col=1,lwd=1.2,
axes=T,xlab="x",ylab="f(x)"),ylim=c
(-0.5,0.5)
     x0 <- log(0.151/5.3,2)/3.1
     points(x0,0,pch=3,lwd=1.2)
     segments(x0,-0.5,x0,0,lty=2)
     segments(-2,0,x0,0,lty=2)
```

Solution 2.15: Logarithmic Scale

(1) The data in Table 2.2 indicate that it is reasonable to plot the distances over an interval of [0; 50] km and pressures over an interval of $[10^{-3}; 1000]$ hPa. The extent of the pressure data is thus $1000 - 10^{-3} = 999.999$ hPa. The range of the distance data is $50 - 0 = 50$ km. Let us imagine that we want to represent the evolution of the temperature as a function of the altitude on a plot of width 12 cm (x-axis) and height 10 cm (y-axis). The graduations to be carried out on each of the axes are obtained by the following relations:

For pressure:

$$12 \text{ cm} \leftrightarrow 999,999 \text{ hPa}$$

$$1 \text{ cm} \leftrightarrow \frac{1 \times 999,999}{12} \simeq 83.33 \text{ hPa};$$

For distance:

$$10 \text{ cm} \leftrightarrow 50 \text{ km}$$

$$1 \text{ cm} \leftrightarrow \frac{1 \times 50}{10} = 5 \text{ km}.$$

These rules allow the points to be placed on the graph (Table 2.3).

Figure 2.35 shows the data in Table 2.2 on a linear scale.

(2) With a logarithmic scale, we obtain Fig. 2.36.

Table 2.3 Rule for reporting data on graph

Distance: 5 km ↔ 1 cm	Pressure: 83.33 hPa ↔ 1 cm
0 km ↔ 0 cm	1000 hPa ↔12 cm
5 km ↔ 1 cm	900 hPa ↔ 10.8 cm
10 km ↔ 2 cm	250 hPa ↔ 3.0 cm
20 km ↔ 4 cm	50 hPa ↔ 0.6 cm
30 km ↔ 6 cm	5 hPa ↔ 0.06 cm
50 km ↔ 8 cm	$5 \cdot 10^{-3}$ hPa ↔ $6.0 \cdot 10^{-5}$ cm
30 km 10 cm	10^{-3} hPa ↔ $1.2 \cdot 10^{-5}$ cm

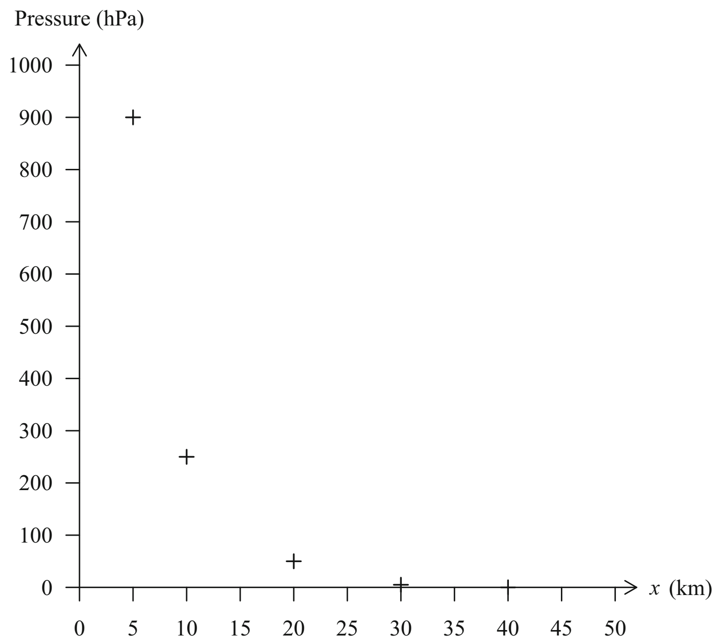

Fig. 2.35 Representation of Table 2.2 data on linear scale

Note 25

- *On a linear scale, two graduations with a difference of 10 are at a constant distance.*
- *On a logarithmic scale, two graduations with a ratio of 10 are at constant distance.*

Solution 2.16: Evapotranspiration and Carbon Dioxide Pressure

(1) Evapotranspiration is 1 m/year, or 1000 mm/year, so

$$\begin{aligned} P_{CO_2} &= 10^{-3.47+2.09(1-\exp(-0.00172\times1000))} \\ &= 0.0176\,\text{atm}. \end{aligned}$$

(2) We find a value of 0.476 m/year.

```
#--------------------------------------
# PCO2 and EVt (Brook et Hanson, 1991)
#--------------------------------------
# Evapotranspiration in mm/year
evt <- seq(0,600,0.1)
plot(evt,10^(-3.47+2.09*(1-exp
               (-0.00172*evt))),
     type="l", col=1,lwd=1.2, axes=T,
     xlab="Evt (mm/an)",ylab="PCO2
           (atm)",
     ylim=c(0,0.005))
```

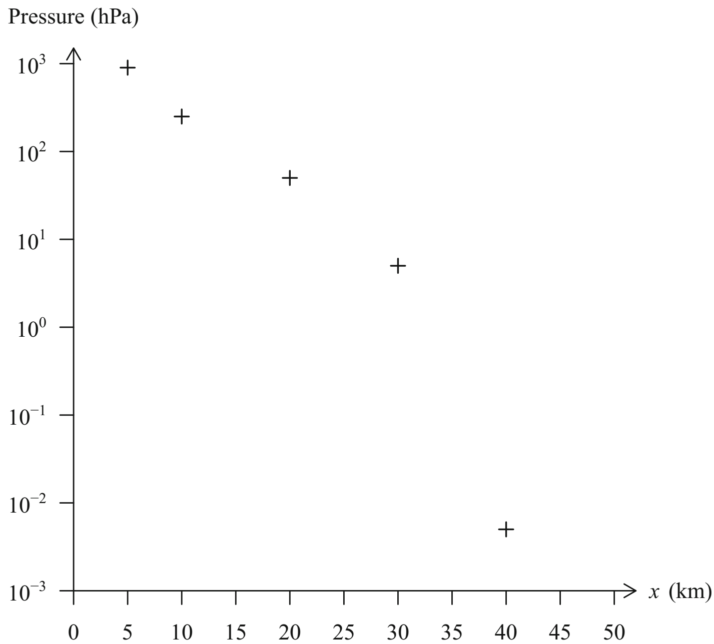

Fig. 2.36 Representation of Table 2.2 data on a logarithmic scale on y-axis

```
# One may search the values of Evt0
that
# correspond to the segment intersec-
tions
Evt0 <- 476
segments(0,0.005,Evt0,0.005,lty=2)
segments(Evt0,0,Evt0,
  10^(-3.47 + 2.09 *
    ( 1 - exp(-0.00172 * Evt0 ))),lty=2)
```

Solution 2.17: Age of Oceanic Crust

We use the relationship $P = 2500 + 350\sqrt{t}$, where P (depth of water) is expressed in meters and t (age of oceanic crust) in millions of years (My).

(1) $t = 15$ My, we have $P = 2500 + 350\sqrt{15} \simeq 3856$. The depth sought is 3860 m (within 10 m).

(2) To find the age of the oceanic crust at a depth of 4400 m, we solve the equation with the variable $t: 2500 + 350\sqrt{t} = 4400$, $350\sqrt{t} = 4400 - 2500 = 1900$, and so $\sqrt{t} = 1900/350$. From there $t = (1900/350)^2 \simeq 29.47$, where the age of such a crust is about 29 My.

(3) If the proposed relationship is valid for a lithosphere whose age is less than 60 My, that means that t is valid between 0 and 60 (in My) or that the function P has a domain of [0; 60].

(4) The graphical representation is given in Fig. 2.37. For the digital solution:

```
#-------------------------------------
# Relationship between age of oceanic
crust
# and depth of water
#-------------------------------------
t <- seq(0,60,0.01)
plot(t,2500+350*sqrt(t),type="l",
col=1,lwd=1.2,
xlim=c(0,60),ylim=c(0,5000),
axes=F,xlab=NA,ylab=NA)
# Axe X
axis(side=1,at=seq(0,60,10),pos=0)
arrows(60,0,62,0,length=0.1)
mtext("Age (million years)",1,at=30,
line=1,las=1)
# Axis Y
axis(side=2,at=seq(0,5000,500),pos=0)
arrows(0,5000,0,5200,length=0.1)
mtext("P(t) (m)",2,at=3000,line=1.5)
```

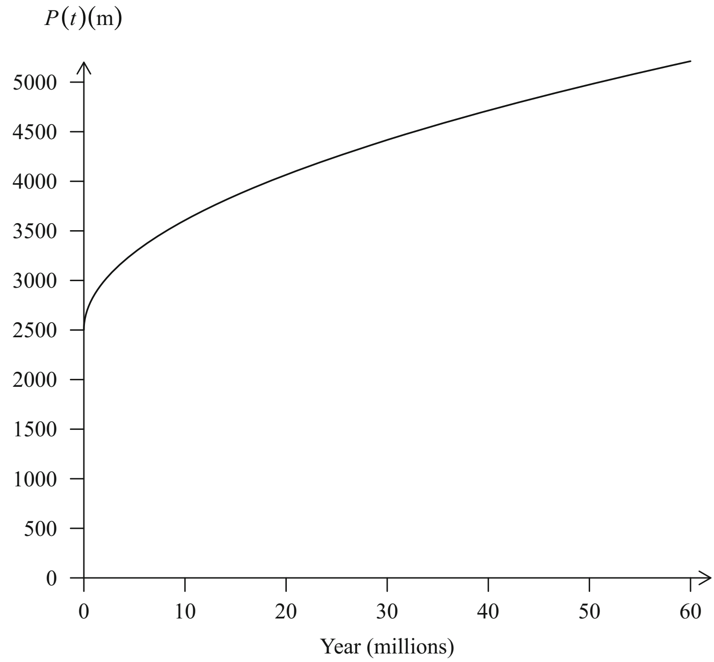

Fig. 2.37 Relationship between water depth $P(t)$ expressed in meters and age of oceanic crust t in millions of years

```
points(15,2500+350*sqrt(15),pch=3,
lwd=1.2)
points(((4400-2500)/350)^2,4400,pch=3,
lwd=1.2)
```

Solution 2.18: Evolution of Population of Quebec

(1) If we take the extreme values (four million in 1951 and eight million in 2011), there is an average increase over the period of

$$\frac{8-4}{2011-1951} \simeq 0.066$$

$$(66,000 \text{ inhabitants per year}).$$

(2) We assimilate the curve to a straight line (called modeling): a straight line is drawn that seems to "stick" the best in the curve and we reason from this line, for example, the line passing through the points (1951, 4) and (1995, 7) (the chosen line does not necessarily pass through the points of the sample).

(3) The line chosen in the previous question has a slope of $\dfrac{7-4}{1995-1951} \simeq 0.07$. The population increase is about 70,000 inhabitants per year over the period [1951; 1995].

If we take up the reasoning on the extreme values of question 1, it amounts to approaching the curve by the straight line passing through the two extreme points (1951, 4) and (2011, 8). In tracing it, we see that this straight line models the real curve less faithfully.

(4)

```
#--------------------------------------
# Population of Quebec
#--------------------------------------
# One chooses two points on the line
year <- c(1951,2011)
pop <- c(4,8)

# Linear model
Model <- lm(pop~year)
```

```
# Correlation coefficient
cor(pop,year,method="pearson")
# Value of gradient °C/km
Model
cat("the rate is 0.066, so 66,000 peo-
ple/year")
```

Solution 2.19: Carbon-14 Dating

(1) After a century $N(100) \simeq N_0 \times 0.988$, so $\dfrac{N(100)}{N_0} \simeq 0.988$: about 1.20% of carbon-14 was lost in a century.

$\dfrac{N(10,000)}{N_0} \simeq 0.298$, therefore 70.2% of the carbon-14 disappeared after 10,000 years.

(2) To calculate the half-life, we need to determine the value of t for which $N(t) = \dfrac{N_0}{2}$ (that is, $\dfrac{N(t)}{N_0} = 0.5 = 50\%$). A first method is to plot the curve of the function f defined by $f(t) = \exp(-1.210 \times 10^{-4} t)$ and see at what value of t we have $f(t) = 0.5$ (Fig. 2.38).

As a second method, we can solve the equation $0.5 = \exp(-1.210 \times 10^{-4} t)$,
so $\ln(0.5) = -1.210 \times 10^{-4} t$, and therefore

$$t = \frac{\ln(0.5)}{-1.210 \cdot 10^{-4}} \simeq 5730 \text{ years.}$$

(3) We proceed as in the previous question: $t = \dfrac{\ln(0.6)}{-1.210 \times 10^{-4}} \simeq 4220$ years.

Solution 2.20: Scenario in Hydrogeology

(1) The gradient i is the ratio of the height of the water H between the two piezometers and the distance L between them. The gradient is the ratio between two lengths, so it is dimensionless. The flow rate Q is the product of A (m^2), k (m/s), and i (without dimension). The flow therefore has the unit $m^2 \times$ m/s, which is m^3/s.

(2) $Q = A \times k \times i$; if the gradient is zero ($i = 0$), then the flow rate Q is also zero. Physically,

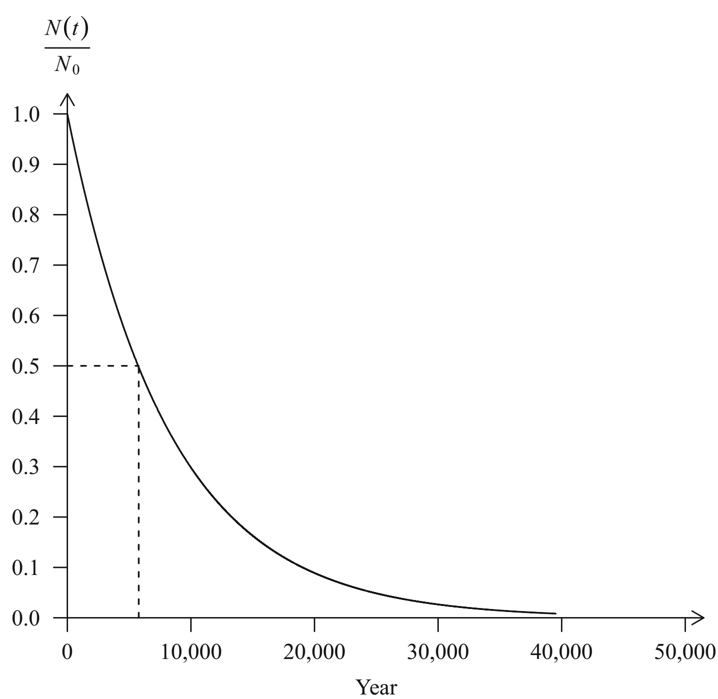

Fig. 2.38 Decreasing exponential function that describes the radioactive decay of carbon-14

this is explained as follows: if the gradient is zero ($i = 0$), this means that H is zero (for $i = \frac{H}{L}$), so that the slope between the two piezometers is zero. In other words, there is no flow between the two piezometers, so the flow rate is zero.

(3) The starting point is the Darcy equation $Q = A \times k \times i$. The value of the permeability is $k = 8.64$ m/day. What are the values of A and i? A is the surface perpendicular to the flow, the river being 15 m deep and 10 km long, so the perpendicular surface is $15 \times 10 \times 10^3$ m^2, or 150×10^3 m^2. The gradient is the ratio of H to L, where H is the difference in height between the top of the aquifer (50 m) and the river (25 m), so $H = 25$ m. The distance between the top of the aquifer and the river is 5 km, or 5×10^3 m. If we apply Darcy's law, we have $Q = 150 \times 10^3 \times 8.64 \times \frac{25}{5 \cdot 10^3} = 8.64 \times \frac{150 \times 25}{5} = 8.64 \times 750 = 6480\,\mathrm{m}^3/\mathrm{day} = 648 \times 10^3\ 1/\mathrm{day} = 75\ 1/\mathrm{s}$.

(4) The Darcy flow is expressed by $Q = A \times k \times i$, so the associated Darcy velocity is equal to $U = \frac{Q}{A} = k \times i$. The velocity value is $\frac{Q}{A} = 8.64 \times \frac{25}{5 \times 10^3}$ m/day or 43.2×10^{-3} m/day. The pollutant travels a distance $d = 5$ km between the tank and the river at speed U, so the travel time will be $t = \frac{d}{U} = \frac{5 \times 10^3}{43.2 \times 10^{-3}} \simeq 115{,}741$ days, a little over 317 years.

(5)

```
#-------------------------------------
# Hydrogeology problem
#-------------------------------------
# Permeability between 7 and 10 m/day
# One divides by 24 * 3600 to transform
into seconds
k <- seq(7,10,0.1)/(24*3600)
# Gradient between 0 and 0.2
i <- seq(0,0.2,0.01)
# Flow rate in L/s
Q <- function(k,i) {
# Surface in m2
A <- 150e3
return (A * k * i)
}
z <- outer(k,i,Q)
z[is.na(z)] <-1
persp(k,i,z/1000, theta = 30, phi =
30, expand = 0.5,
shade=.5, axes=T,
xlab = "Permeability (m/s)",
ylab = "Gradient",
zlab = "Flow rate (L/s)",
ticktype="detailed")
```

Solution 2.21:

(1) Parameters here are H, ρ_{atm}, and S. The given relation makes it possible to write the variable h_f as a function of the variable v. We can write $h_f(v) = H \ln\left(\frac{\rho_{atm} v^2}{S}\right)$ or $h_f = f(v)$, where f is defined by $f(v) = H \ln\left(\frac{\rho_{atm} v^2}{S}\right)$ or $f(x) = H \ln\left(\frac{\rho_{atm} x^2}{S}\right)$.

(2) The mathematical expression "inside" ln must be positive so that the function f is well defined. This is the case for all numbers v except for 0 (notably the function f is well defined mathematically for strictly negative numbers; obviously, this does not make sense physically since v is a speed). When the expression within ln is less than 1, the result is negative.

(3) Physically, this means that the fragmentation height is negative (below ground level), so arrival at height 0 (ground) will be without fragmentation. In practice, a negative height will not be reached: if the meteor arrives intact at ground level, it crashes there!

(4) In view of the preceding answer, we try to solve $h_f \leq 0$ (it is an inequality), and one will

obtain an infinite number of values of v satisfying this inequality. We will simply solve $h_f = 0$, and the speed v found will be the highest speed for which there is no fragmentation upon arrival on the ground.

Resolution: $h_f = 0$ for $20,385 \ln\left(\frac{5.3 \times v^2}{10^4}\right) = 0$ so for $\ln\left(\frac{5.3 \times v^2}{10^4}\right) = 0$.

Therefore, we try to find v so that $\frac{5.3 \times v^2}{10^4} = 1$, so $v^2 = \frac{10^4}{5.3}$, and by using the square root (we only keep the positive solution) $v = \sqrt{\left(\frac{10^4}{5.3}\right)} \simeq 43$ m s^{-1} (it should be noted that the value of H does not intervene).

For speeds v between 0 and 43 m/s, there is no fragmentation before arrival on the ground.

3 Trigonometry, Geometry of Plane and Space

Abstract

In geography and Earth science, the use of geometric considerations is frequent, whether in a plane or in space. The use of angles is particularly important and leads to the introduction of notions of trigonometry as well as the functions cosine, sine, and tangent.

These tools play a fundamental role in mapping, which will be discussed in Chap. 4.

This chapter introduces trigonometry, with some basics on angle measurement (in degrees and radians) and introducing sine, cosine, and tangent.

It also presents some basics on the geometry of planes and space, such as the distance formula between two points, polar and spherical coordinates, and the use of some properties such as Pythagoras' theorem and the intercept theorem.

The exercises give examples of uses in various contexts (e.g., law of refraction, measuring the radius of the Earth or the depth of Moho, urban density)

Keywords

Geometry · Trigonometry · Angle and length calculation · Sine · Cosine · Tangent · Slope · Pythagoras' theorem · Cartesian and polar coordinates.

Electronic supplementary material: The online version of this article (https://doi.org/10.1007/978-3-319-69242-5_3) contains supplementary material, which is available to authorized users.

Aims and Objectives

- To know how to calculate lengths and angles in a plane using trigonometry and the Pythagorean and intercept theorems.
- To know the trigonometric functions sine, cosine, and tangent and their properties.
- In the geometry of space, to calculate a length from Cartesian coordinates and develop notions about the spherical coordinates of a point.

3.1 Measurements of Angles

There are two main units of measurements used for angles (Table 3.1): the degree and the radian (mathematical unit). These angle units are dimensionless in a physical sense.

Definition 10

- *The **radian,** denoted "rad", verifies that a complete turn of a circle is 2π rad (and a half turn π rad).*
- *For the **degree,** denoted "°", a full turn is 360 degrees. As for the hours, decimal degrees or the sexagesimal system are used for*

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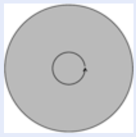
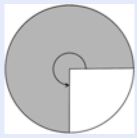
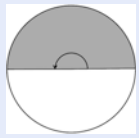
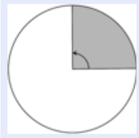
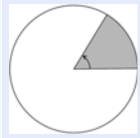
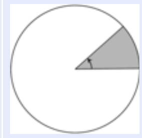
Table 3.1 Summary of angle units and some particular values

Unit						
Degree (°)	360	270	180	90	60	45
Radian (rad)	2π	$\frac{3\pi}{2}$	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$

submultiples: minutes of an angle (denoted by the symbol ′) and seconds of an angle (denoted by the symbol ″). Hence, we have $1^\circ = 60' = 3600''$.

Example 37 *An angle of* 72.53° *is* $72^\circ\ 0.53 \times 60' = 72^\circ\ 31.8' = 72^\circ\ 31'0.8 \times 60'' = 72^\circ 31' 48''$.

Conversely, an angle of $34^\circ 13' 15''$ *is* $(34 + 13/60 + 15/3600)^\circ \simeq 34.22^\circ$.

3.2 Plane Geometry (Part 1): Trigonometry, Slope

In this section, we place ourselves in a plane (that is to say on a 2D flat surface, but not necessarily horizontal) or something considered as such. The notions dealt with are essential not only in cartography but also in many other fields of geography and geology: they enable the identification and description of what is observed in a plane. The quantities used in this paragraph will be lengths and angles.

3.2.1 Calculation of Lengths and Angles in a Right Triangle

Let *ABC* represent a right triangle, with the right angle located at point *C* (small box in corner) (Fig. 3.1). Several properties make it possible to calculate the lengths in this triangle.

If we know two of the three lengths, we can obtain the third by the Pythagorean theorem:

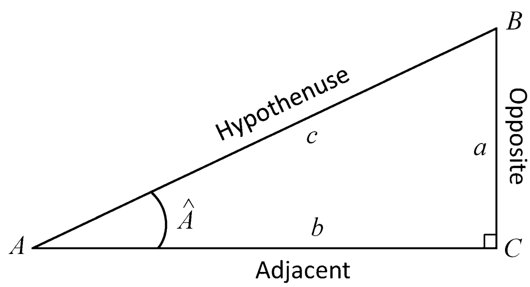


Fig. 3.1 Right triangle

Theorem 1 (Pythagoras' Theorem) *In the triangle ABC with the right angle at point C, there is an equality between the lengths* $AB = c$, $AC = b$, *and* $BC = a$ *given by* $BC^2 + AC^2 = AB^2$, *which is* $a^2 + b^2 = c^2$. *This can also be stated as: among the three sides of a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides.*

Example 38 *In Fig. 3.2, we know that* $DA = 1552$ *m, that* $DH = 1500$ *m, and that DHA is a right triangle at point H. We thus have the relation* $AH^2 + DH^2 = DA^2$, *which leads to* $AH^2 = 1552^2 - 1500^2 = 158{,}704$. *Finally,* $AH = \sqrt{158{,}704} \approx 398$ *m.*

Thanks to trigonometry, if a length and an angle are known, a second length can be obtained. If two lengths are known, an angle can be obtained.

Theorem 2 (Property (trigonometry)) *In a right triangle, the ratios of two lengths are expressed as a function of an angle using the* *cosine (cos), sine (sin), or tangent (tan) functions. As shown in Fig. 3.1, the longest side is the hypotenuse, the adjacent side is next to angle $\widehat{A}$, and the opposite side is opposite angle $\widehat{A}$.*

$$\sin\widehat{A} = \frac{BC}{AB} = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\widehat{A} = \frac{AC}{AB} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\widehat{A} = \frac{BC}{AC} = \frac{\text{opposite}}{\text{adjacent}}$$

Note 26 (Remarks)

- *Since these are length ratios, the values of the trigonometric functions are dimensionless (no units).*

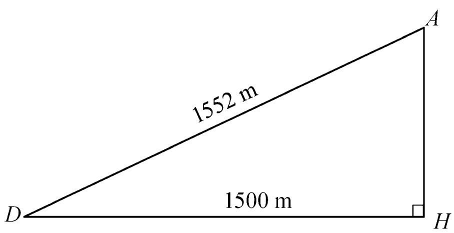


Fig. 3.2 Figure not to scale

- *From the relationship* $\sin\widehat{A} = \frac{BC}{AB}$ *we deduce that* $\widehat{A} = \sin^{-1}\left(\frac{BC}{AB}\right)$, *where* $\sin^{-1}$ *is the inverse sine function, also called* arcsin. *Similarly, we can use the inverse cosine functions (denoted by* $\cos^{-1}$ *or* arccos*) and inverse tangent function (*$\tan^{-1}$ *or* arctan*).*

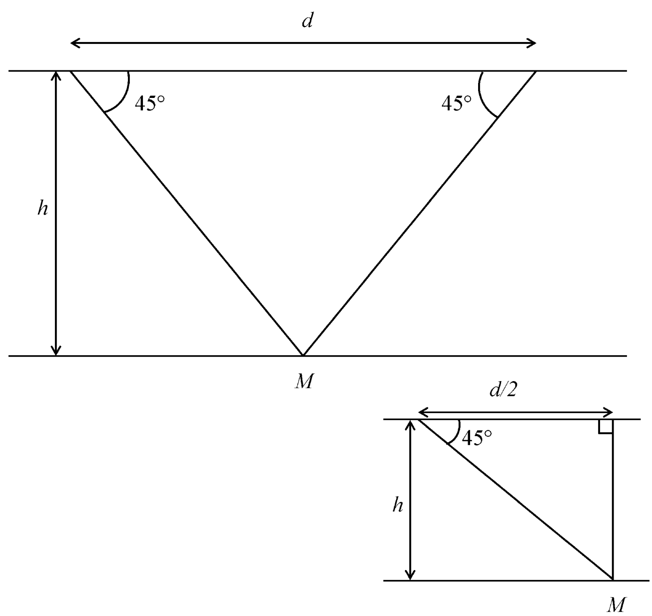
Do not confuse the inverse sine function $\sin^{-1}$ *with* $\frac{1}{\sin}$.

- *Don't forget to set your calculator to the correct angle unit (degree or radian).*

Example 39 *In accretionary wedges, h is known and we seek d (Fig. 3.3). We divide the triangle into two identical right triangles, with one side of length h and the other d/2. Using the 45° angle, we obtain* $\tan(45^\circ) = \frac{h}{d/2}$, *and therefore we have* $d = 2h$ *since* $\tan(45^\circ) = 1$.

Other situations in connection with accretion will be presented in the exercises.

Fig. 3.3 Schematic of an accretionary wedge



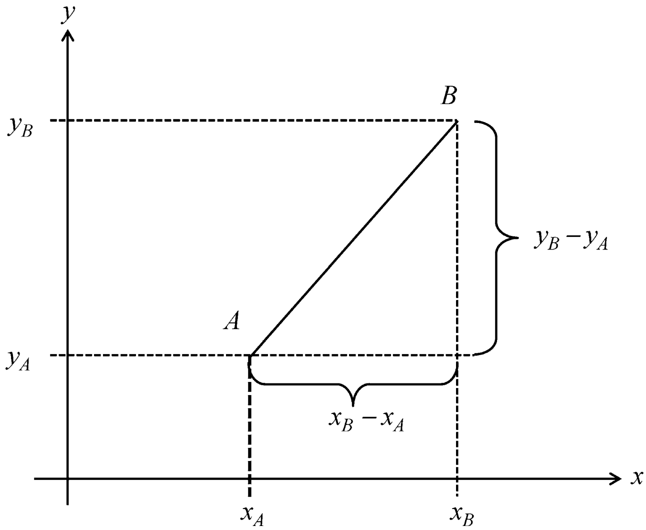


Fig. 3.4 Principle of calculating the distance between two points, A and B, of respective coordinates (x_A, y_A) and (x_B, y_B)

3.2.2 Calculating the Distance Between Two Points Using Coordinates

Theorem 3 (Calculation of Distance Between Two Points) *We are given two points, A and B, of respective coordinates* (x_A, y_A) *and* (x_B, y_B) *using orthonormal axes (that is to say, the x- and y-axes are perpendicular and graduated on the same scale). Then the distance between points A and B is (Fig. 3.4)*

$$d(A,B) = AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}.$$

Example 40 *To set the positioning error of a point P on a map, the distance between the actual point and the point on the map is calculated. We note* δx_P *the error in* $x : \delta x_P = x_{\text{map}} - x_{\text{actual}}$ *and* δy_P *the error in* $y : \delta y_P = y_{\text{map}} - y_{\text{actual}}$. *These errors can be positive or negative. The positioning error of P is then (Fig. 3.5)*

$$\Delta_P = \sqrt{(\delta x_P)^2 + (\delta y_P)^2}.$$

Δ_P is *always positive.*

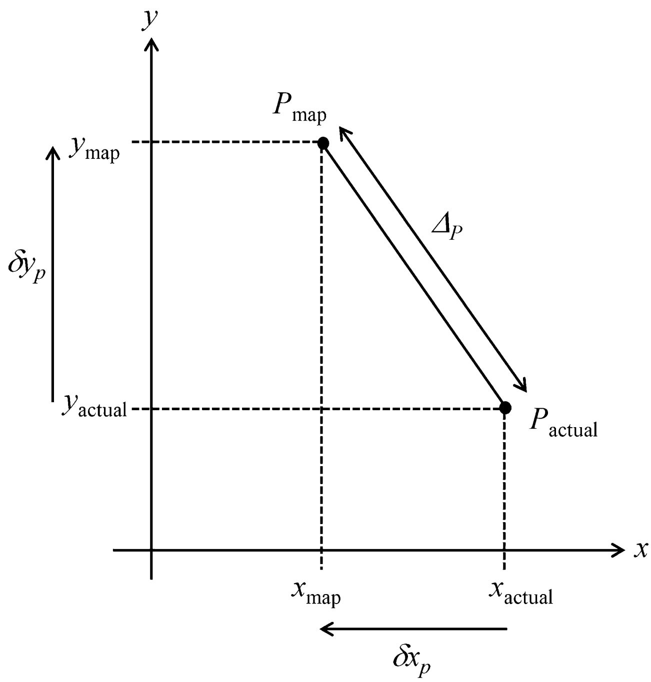
Insert 5 (Distances)
Distance is a concept that is natural to us because we use it in our everyday lives: we walk 250 m (or yards) to go to the coffee shop or drive 170 km (or miles) to go on vacation. Intuitively the distance between two points is the length that separates them. Mathematically, the notion of distance is more complex. It is well understood that to reach a point $B(x_B, y_B)$ *from a point* $A(x_A, y_A)$, *there are often several possible paths and, therefore, several distances.*

The Euclidean distance is the most intuitive and is defined by $d(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$. *It corresponds to the shortest distance between two points, and is shown by the solid line in Fig. 3.6.*

There is also the Manhattan distance (or taxicab geometry) defined by $d(A, B) = |x_B - x_A| + |y_B - y_A|$ *(dashed lines). It is the sum of horizontal and vertical components. This distance is named after one of the boroughs of New York City (United States) because the urban structure is very similar to a grid and the distance is*

(continued)

Fig. 3.5 Principle of calculation of the positional error of a point on a map



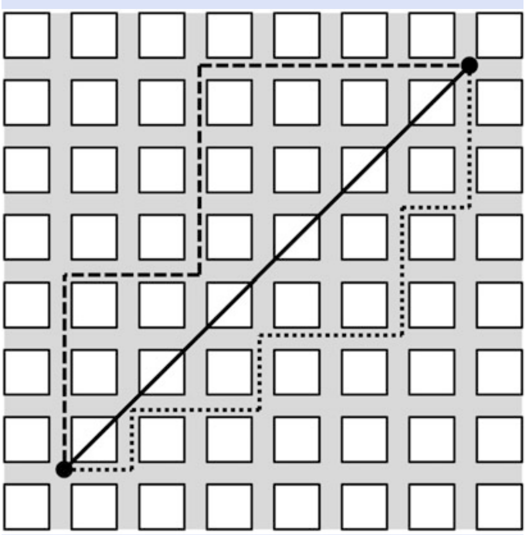


Fig. 3.6 The only Euclidean distance (solid line) and two examples of the Manhattan distance (dashed lines)

precisely defined by the possible paths on this grid.

Note that the notion of distance is not necessarily linked to geographical space. For example, the Levenshtein distance is defined as the minimum number of operations (add/delete/replace) needed to transform one string into another. Thus, dLevenshtein(“mum”, “sum”) = 1 because it is necessary and sufficient to replace the “m” with the “s” so that the two words are the same.

3.2.3 Slope Calculation

Definition 11 *The* ***slope*** *of a land is the ratio* $\frac{\Delta h}{d}$ *of the height difference to the horizontal distance (Fig. 3.7). This ratio is the tangent of the angle* $\widehat{A}$ *between the ground (slanted line) and the horizontal line,* $\tan \widehat{A} = \frac{\Delta h}{d}$.

Example 41 *A 10% slope indicates an increase of 10 m when 100 m is traveled horizontally. The angle* $\widehat{A}$ *corresponds to* $\tan^{-1}\left(\frac{10}{100}\right) \approx 5.7°$.

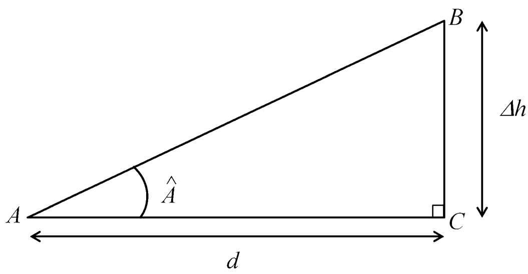


Fig. 3.7 The slope is defined as the ratio of two lengths

Note 27 *The slope (indicated as a percentage) is sometimes the ratio of the difference in altitude and distance (along the hypotenuse). In this case, the angle is calculated by the reverse function* $\sin^{-1}$.

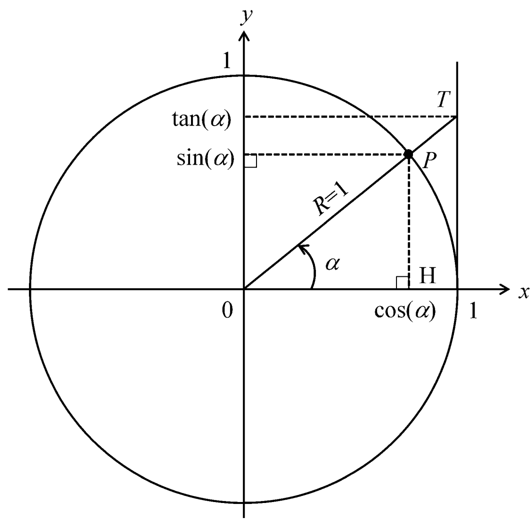


Fig. 3.8 Unit circle with only one angle α

3.3 Trigonometric Functions

3.3.1 Unit Circle

Definition 12 *A* ***unit circle*** *is a circle of radius 1 centered at the origin O (0,0) (Fig. 3.8* and *Fig. 3.9). Let P be a point of coordinates* (x_P, y_P) *on this circle; note* α *the angle between the positive x-axis and line OP.*

The ***cosine of angle*** α *is defined by* $\cos(\alpha) = x_P$ *and the* ***sine of angle*** α *by* $\sin(\alpha) = y_P$.

The ***tangent of angle*** α *is defined as the y-coordinate of point T, which is the intersection of line OP and the tangent to the trigonometric circle at the point* (1, 0).

Note 28 *Using trigonometry in triangle OHP, which has a right angle at point H* of *Fig. 3.10, we have*

$$\cos\alpha = \frac{OH}{OP} \text{ and } \sin\alpha = \frac{HP}{OP}.$$

Since the circle has a radius of 1, we have $OP = 1$, *and so* $\cos\alpha = OH = x_P$ *and* $\sin\alpha = HP = y_P$*: the two definitions are coherent.*

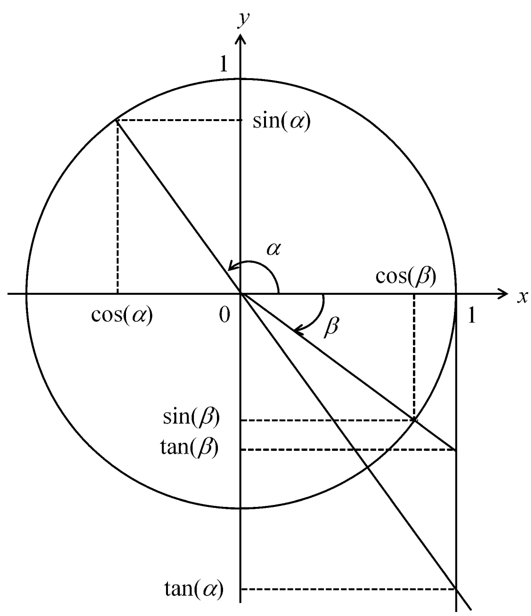


Fig. 3.9 Unit circle with two angles α and β

We also have, thanks to the Pythagorean theorem, $\cos^2\alpha + \sin^2\alpha = 1$ (or (cos(alpha))^2 + (sin (alpha))^2 = 1), and thanks to the intercept theorem, we have tan = sin/cos.

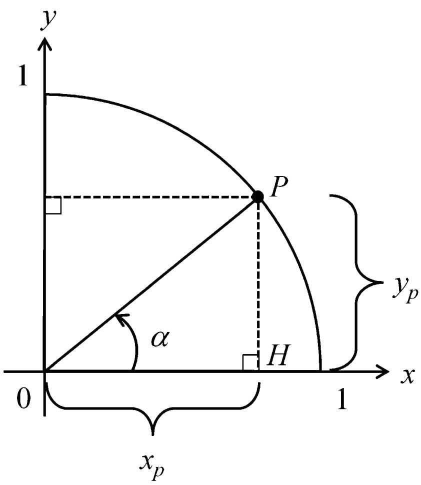


Fig. 3.10 Unit circle and Pythagorean theorem

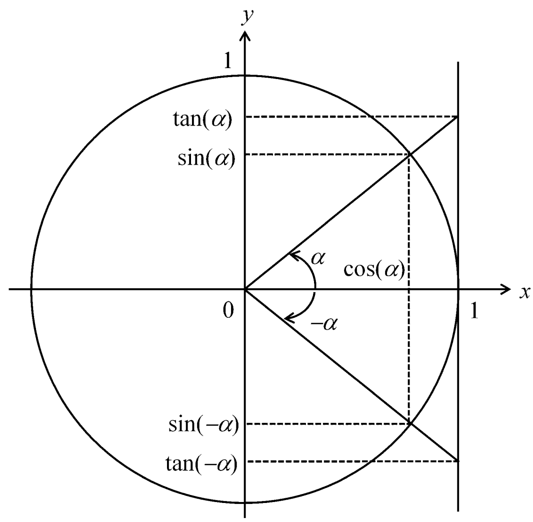


Fig. 3.11 Properties of opposite angles

Theorem 4 (Properties) *Some properties are shown graphically:*

- $\cos\alpha$ *and* $\sin\alpha$ *take values between −1 and 1.*
- *We have relations between opposite angles (Fig. 3.11):*

$$\cos(-\alpha) = \cos\alpha,$$

$$\sin(-\alpha) = -\sin\alpha,$$

$$\text{and} \quad \tan(-\alpha) = -\tan\alpha.$$

Important angles and their cosine, sine, and tangent values are given in Table 3.2.

Table 3.2 Some important cosine, sine, and tangent values

Angle in radians	0	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$
Angle in degrees	0	90	180	−90
Cosine	1	0	−1	0
Sine	0	1	0	−1
Tangent = (sin/cos)	0	$+\infty$	0	$-\infty$

3.3.2 Sine, Cosine, and Tangent Functions

The previous section defined the notions of sine, cosine, and tangent of an angle from the position of the corresponding point on a unit circle. We have only mentioned certain angles (those corresponding to a maximum of one turn around the unit circle). It is possible to perform more than one turn (in the positive or negative direction): this is equivalent to winding a line around the unit circle and corresponds to angles greater than 2π rad (360°) or smaller than -2π rad (−360°). With each angle is associated a point of the unit circle, and the cosine and the sine of this angle are defined as previously.

Figure 3.12 illustrates these three steps: passing a number α from the line of the real numbers to the positioning of this angle by wrapping this line around the unit circle (counterclockwise if $\alpha \geq 0$ and clockwise if $\alpha < 0$ and then obtaining $\cos(\alpha)$, $\sin(\alpha)$, and $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$). Thus, the real numbers α, $\alpha + 2\pi$, and $\alpha - 2\pi$ correspond to

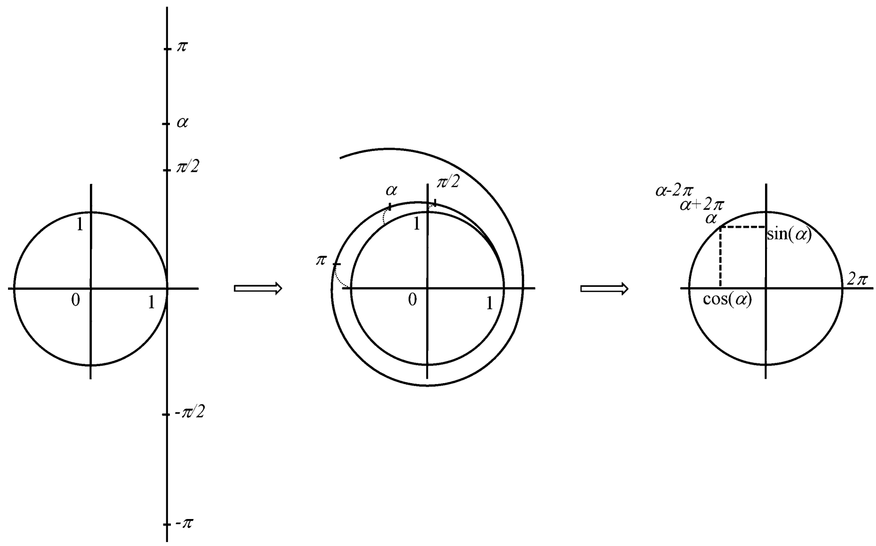
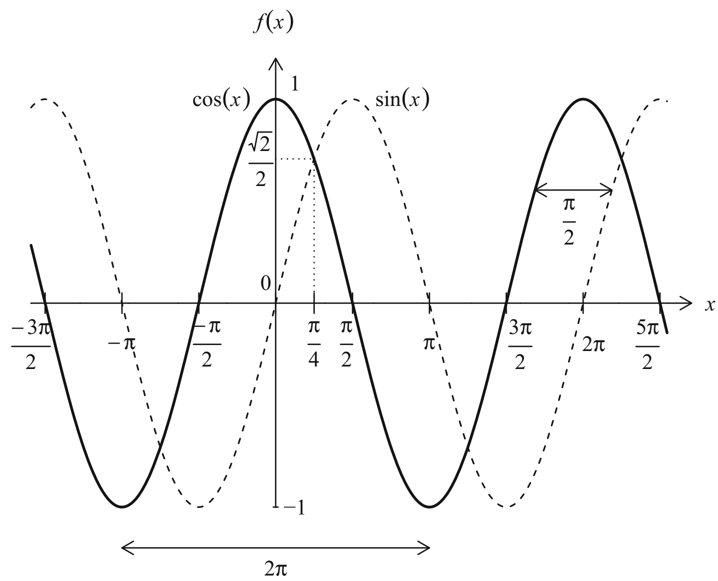


Fig. 3.12 Principle of wrapping of real numbers around unit circle

Fig. 3.13 Graph of cosine and sine functions



the same point on the unit circle and, hence, have the same value of cosine, sine, and tangent because the length of the unit circle is 2π.

We deduce that the **cosine** and **sine** functions are defined for all real numbers, and they are both periodic of period 2π (which means that knowing these functions over an interval of length 2π allows us to know the whole functions).

In addition, the curve of the cosine function is symmetrical about the y-axis: the cosine function is **even.** The curve of the sine function is symmetrical about the origin: the sine function is **odd** (Fig. 3.13).

The **tangent** function is defined for all real numbers, except those for which the cosine is 0 (i.e., numbers of the form $\pi/2 + n\pi$, where n is a positive or negative integer: e.g., $-3\pi/2$, $-\pi/2$, $\pi/2$, $3\pi/2$, $5\pi/2$). It is a periodic function of period π. The tangent function is odd, i.e., its representative curve is symmetrical about the origin (Fig. 5.6).

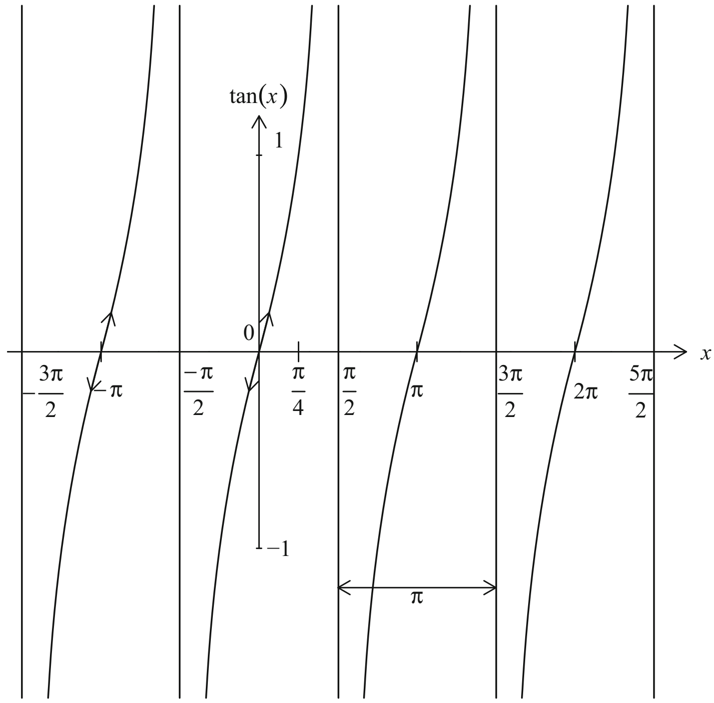


Fig. 3.14 Graph of tangent function

3.4 Plane Geometry (Part 2)

3.4.1 Polar Coordinates: Using Distance and Angles

In some situations with cylindrical symmetry, it is more convenient to use polar coordinates than Cartesian coordinates. This is the case, for example, when studying the properties of a circular ice cap.

It is located in an orthonormal basis, that is to say, in a mark (O, x, y) where the axes are perpendicular and the same scale is on the two axes. Any point M of the plane is identified by its coordinates (x, y). It is also possible to fully describe the position of M by its distance r to point O (so we have $r > 0$) and the angle θ between (Ox) and (OM) (Fig. 3.15). (r, θ) are the **polar coordinates** of point M.

Note 29 (Remarks)

- *To obtain a unique value of* θ, *we restrict ourselves to the interval* $[0, 2\pi[$ *(or* $]-\pi, \pi]$*) if the reasoning is in radians and* $[0; 360[$ *(or* $]-180, 180]$*) if we reason in degrees.*

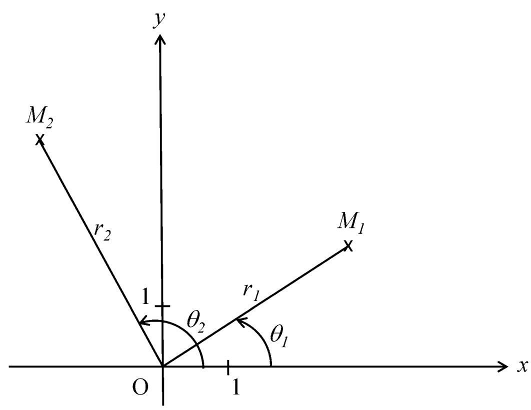


Fig. 3.15 Polar coordinate representation of points $M_1(r_1, \theta_1)$ and $M_2(r_2, \theta_2)$

- *We have the following relation between Cartesian and polar coordinates (Fig. 3.16):*

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

Example 42 *Let a be a positive constant. All the points of the plane for which* $r = a$ *are a circle of center O and radius a.*

The graphical representation of $r(\theta) = \dfrac{2}{1 - 0.25\cos\theta}$ *is obtained by calculating, for each value of* θ*, the corresponding value of* r *and plotting the related point. For example, for* $\theta = 0$ *we have* $r = \dfrac{8}{3}$ *and point* M_1 *(Fig. 3.17); for* $\theta = \dfrac{\pi}{2}$ *we have* $r = 2$ *and point* M_2.

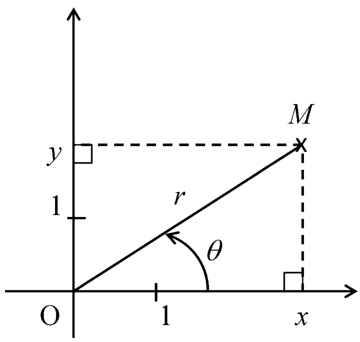


Fig. 3.16 Relation between polar and Cartesian coordinates

3.4.2 Enlarging and Reducing, Intercept Theorem

Theorem 5 (Property (Enlarging, Reducing))
In a situation of ***enlargement*** *(or* ***reduction****) of a figure, there is proportionality between the initial lengths and the lengths after processing* ***(intercept theorem).*** *The corresponding length ratios are therefore all equal.*

Note 30 *The intercept theorem is known in some countries as Thales' theorem.*

In Fig. 3.18, the triangle OCD is an enlargement of the triangle OAB. This is due in particular to

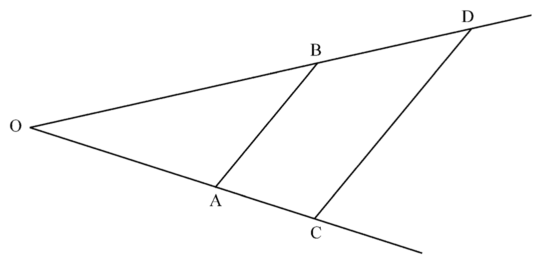


Fig. 3.18 Enlargement of a triangle

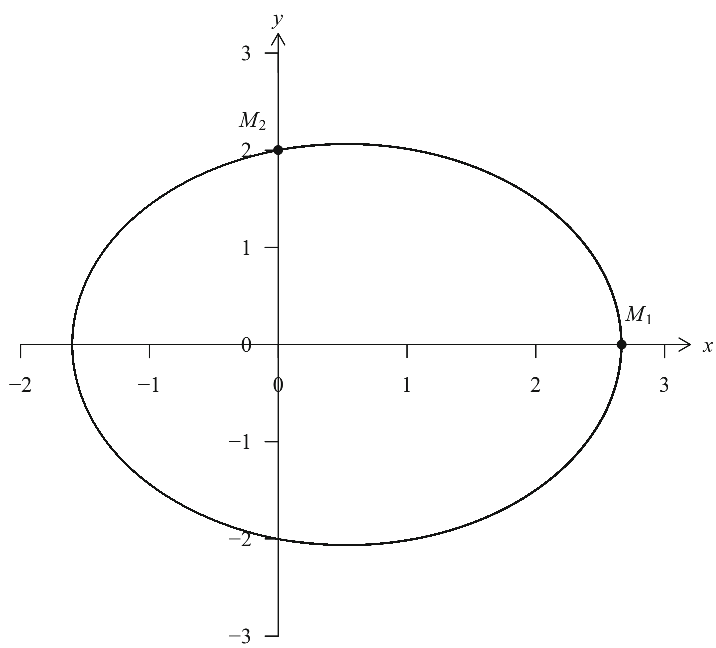


Fig. 3.17 Drawing ellipse $r = \dfrac{2}{1 - 0.25\cos\theta}$

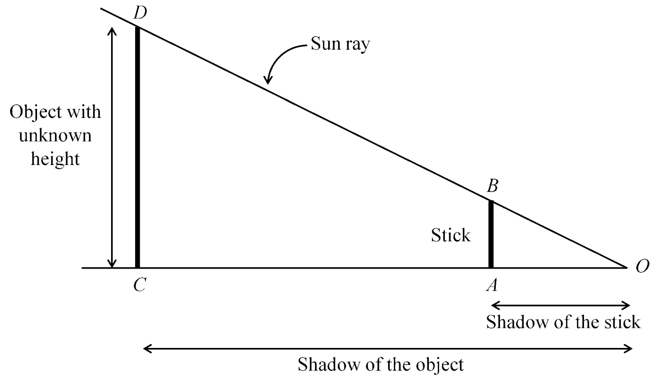


Fig. 3.19 Determining height using a stick

the fact that lines AB and CD are parallel. So we have the equalities $\dfrac{OA}{OC} = \dfrac{OB}{OD} = \dfrac{AB}{CD}$.

Note 31 *If all of the lengths of an object are enlarged by the same coefficient k, then the area is enlarged by the coefficient* k^2 *and the volume by the coefficient* k^3.

Example 43 *A method for determining the height of an object is to use shadows (Fig. 3.19). A stick is planted in front of an object whose height is to be determined. We know the height of the stick and the lengths of the two shadows. One can then calculate the desired height. The legend says that Thales of Miletus used this method to calculate the height of the pyramid of Khufu (in Egypt). If the stick measures 1.5 m, its shadow 2.25 m, and the shadow of the pyramid 219 m, the height h of the pyramid comes to, through the intercept theorem,* $\dfrac{1.5}{h} = \dfrac{2.25}{219}$. *Thus,* $h = \dfrac{1.5 \times 219}{2.25} \approx 146\,m$.

Insert 6 (History: Measure of Earth's Radius by Eratosthenes)

Eratosthenes (276 BC–194 BC), born in Cyrene (now Libya), was a Greek scholar and curator of the Library of Alexandria. He proposed a method to determine the radius of the Earth with precision, which even more than 2000 years later still seems to be extraordinarily accurate. What's more, he did so without leaving Egypt! The city of Syene (currently Aswan in Egypt) is located at a latitude of 23.5° North, that is, exactly on the Tropic of Cancer. Already at that time astronomical knowledge was advanced and it was known that in the tropics, the Sun is directly overhead at least once a year. For the Tropic of Cancer this date is June 21 on the summer solstice.

At Syene, as at any other location with a North latitude of 23.5°, on June 21 at noon, the Sun is at its zenith. This means that at this precise moment one can see the sunlight at the bottom of a well because its rays are perpendicular to the surface of the ground and therefore parallel to the Earth's radius (Fig. 3.20). But on the same date and at the same time, in the city of Alexandria (31°North), the Sun's rays do not reach the bottoms of wells because the Sun is not at its zenith, so its rays form an angle, producing a shadow. Therefore, by measuring only the relation between the size of the obelisk at Alexandria and the shadow it produces on the ground, as well as the distance between Syene and Alexandria, Eratosthenes was able to determine the radius of the Earth! The exercise "Measure Earth's radius by Eratosthenes" proposes to reproduce his method.

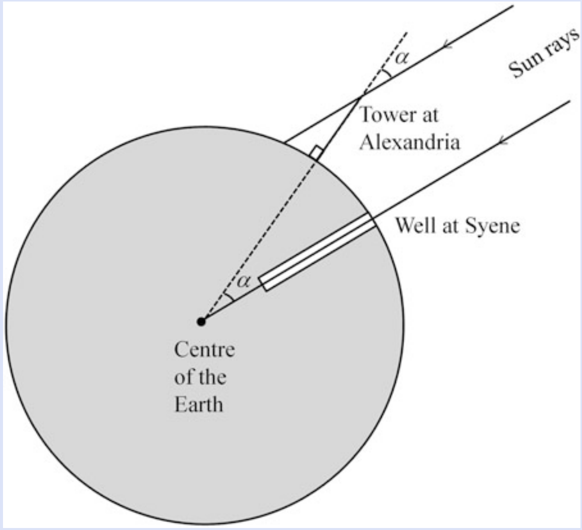


Fig. 3.20 Method used by Eratosthenes to calculate Earth's radius

3.4.3 Calculation of Angles in Any Triangle

Let ABC be a triangle whose angles are denoted $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$ (Fig. 3.21).

Theorem 6 (Property (Sum of Angles)) *The three angles of a triangle add up to 180° (or π rad).*

Note 32 *In a right triangle, the sum of the two other angles is therefore 90°, which is $\frac{\pi}{2}$ rad.*

Theorem 7 (Property (Law of Cosines, Cosine Rule)) *In the notation of Fig. 3.21, we have* $AB^2 = BC^2 + AC^2 - 2ab\cos\widehat{C}$, *which is also* $c^2 = a^2 + b^2 - 2\ ab\ \cos\ \widehat{C}$.

Note 33 *The Pythagorean theorem is a specific case of the foregoing relationship. When the right triangle ABC has its right angle at C, we have* $\cos\widehat{C} = \cos 90° = 0$; *then we find, as expected,* $c^2 = a^2 + b^2$.

Theorem 8 (Property (Law of Sines, Sine Rule))

$$\frac{a}{\sin\widehat{A}} = \frac{b}{\sin\widehat{B}} = \frac{c}{\sin\widehat{C}}$$

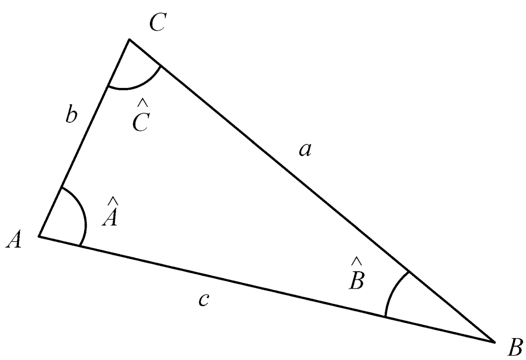


Fig. 3.21 Triangle

With those three properties, you can solve any triangle (that is, find the three missing sides/ angles when you know the other three).

3.5 Geometry of Space

3.5.1 In a Cartesian Basis

Theorem 8 (Calculation of Distance Between Two Points in Space) *The formula has a form similar to that indicated in the plane. Let there be two points A and B of respective Cartesian* *coordinates* (x_A, y_A, z_A) *and* (x_B, y_B, z_B) *in an* ***orthonormal coordinate system*** *of space, that is, a system whose axes are at right angles, and the scales of the three axes are equal. The Euclidean distance between points A and B has a value of*

$$\begin{aligned} d(A,B) &= AB \\ &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}. \end{aligned}$$

3.5.2 Spherical Coordinate System

Spherical coordinates are often used in geography and Earth science because the shape of the Earth is close to that of a sphere.

We use an orthonormal basis. Another way to uniquely characterize a point M from a space is to use its **spherical coordinates** (r, θ, ψ) (Fig. 3.22). Please note that we use here geographical spherical coordinates instead of mathematical ones: the angles used are not the same.

- The radial distance $r = OM$ can vary from 0 to $+\infty$;
- The angle ψ is the angle between the plane xOy and OM and can vary between $-\pi/2$ rad and $\pi/2$ rad;

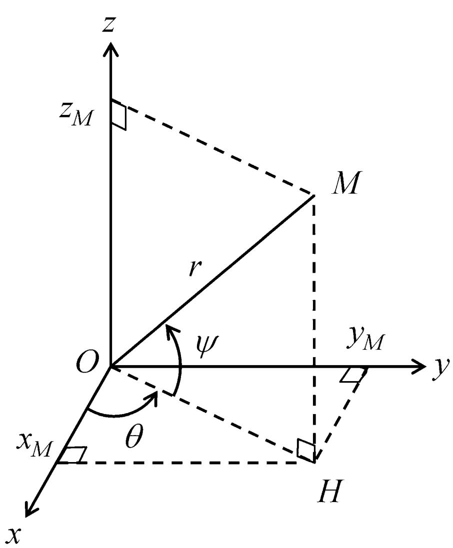


Fig. 3.22 Spherical coordinates

- The angle θ is the angle between Ox and OH, where H is the orthogonal projected point M on the plane xOy, which means H belongs to this plane and MH is orthogonal to the plane. The angle θ can vary between 0 and 2πrad.

Note 34 (Remarks)

- *In Chap. 4, it will be seen that these coordinates correspond to the latitude and longitude of a point on the globe.*
- *We have the following relationships:*

$$\begin{cases} x_M = r\cos\theta\cos\psi, \\ y_M = r\sin\theta\cos\psi, \\ z_M = r\sin\psi. \end{cases}$$

Key Points

- *Radian* is an angle measurement such that a complete circle turn is equal to 2π rad (i.e., 360°).
- To calculate a length in a right triangle we have
  - The Pythagorean theorem if two sides are known;
  - Trigonometric relations if a side and an angle are known.
- Calculation of angles:
  - To calculate the measure of angle in a right triangle, one uses the trigonometric relations and inverse trigonometric functions ($\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$). Be sure to program the calculation tool in the desired angle measurement (degrees or radians);
  - A slope is the ratio of the height of elevation to the horizontal length traveled, and it is the tangent of the inclination angle of the ground;
  - In any triangle the formula connecting angles and side lengths is given by a generalization of the Pythagorean theorem: $AB^2 = BC^2 + AC^2 - 2ab\cos(\widehat{C})$ or $c^2 = a^2 + b^2 - 2ab\cos(\widehat{C})$.
- In an orthonormal basis, the distance between two points A and B of the plane is given by $AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$; for

two points A and B in space, it is obtained using the formula $AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$.

- The sin and cos functions are defined for all numbers; they have a 2π period thanks to the wrapping around the unit circle. Their graphical representations must be known. The tangent function is defined by $\tan = \frac{\sin}{\cos}$.
- The *spherical* coordinates of a point in space make it possible to locate a point knowing its distance to the center of the chosen coordinate system and two angles.

Exercises

Mathematical Exercises

Exercise 3.1: Flash Questions. Series 1

(1) Let us consider M and N points of the space of respective coordinates (18.4, 5.3, −7.9) and (3.7, −12.8, 10.0) in an orthonormal basis. Calculate the length MN.

(2) From memory, recall the relationship between the lengths of the sides and an angle in any triangle.

(3) Let ABC be a right triangle with $\widehat{B}$ being the right angle. The length AB is 13.2 cm and the length BC is 19 cm. Calculate the value of the angle in A (using a calculator or software).

Exercise 3.2: Flash Questions. Series 2

(1) Convert 30° in radians.

(2) Let ABC be a right triangle with $\widehat{A}$ being the right angle. The length AB is 13.2 cm and the length BC is 28.4 cm. Calculate the length AC (using a calculator or software).

(3) Place the angles $\pi/4$, $3\pi/4$, and $-\pi/4$ on a unit circle. It is known that $\cos(\pi/4) = \sin(\pi/4) = \frac{\sqrt{2}}{2}$. Calculate the cosine, sine, and tangent values of $3\pi/4$ and $-\pi/4$.

Exercise 3.3: Flash Questions. Series 3

(1) Let ABC be an isosceles triangle with $\widehat{A}$ being the right angle (that is, the angles in B and in C have the same value). Calculate the angles in B and in C depending on the angle in A.

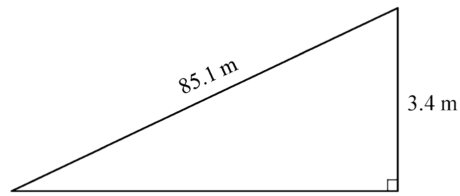


Fig. 3.23 Right triangle with two known lengths

(2) ABC is a right triangle with $\widehat{C}$ being the right angle. The length AB is 13.2 cm and the angle in B is 35°. Calculate the length AC (using a calculator or software).

(3) From memory, recall the diagram showing the spherical coordinates (in space).

Exercise 3.4: Miscellaneous Calculations in a Right Triangle

(1) In the triangle in Fig. 3.23, calculate the missing length and the measures of the two angles that are not right angles.

(2) A 6 m tall ladder is placed against a vertical wall and forms an angle with the ground of 75°. How high up the wall does the ladder go? How far from the wall is the bottom of the ladder positioned?

Exercise 3.5: Unit Circle

(1) Place the angles shown in the table (below) on a unit circle. Find the corresponding angle situated between $-\pi$ and $+\pi$, then the one between 0 and 2π, then the cosine, sine, and tangent values.

Angle (rad)	2π	$-\frac{3\pi}{2}$	7π	$\frac{7\pi}{6}$	$\frac{9\pi}{4}$	$\frac{17\pi}{3}$
Associated angle between $-\pi$ and $+\pi$						
Associated angle between 0 and 2π						
Cosine						
Sine						
Tangent						

(2) We have positioned on the unit circle (Fig. 3.24) an angle α and the angles $\frac{\pi}{2} - \alpha$ and $\pi - \alpha$.

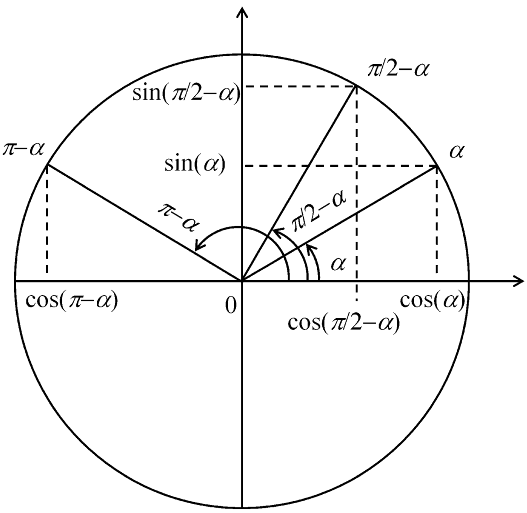


Fig. 3.24 Position of angle α on unit circle

Consideration of this unit circle makes it possible to deduce, for example, that $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$ and $\cos(\pi - \alpha) = -\cos\alpha$.

Express in a similar way $\sin\left(\frac{\pi}{2} - \alpha\right)$, $\sin(\pi - \alpha)$, $\cos\left(\frac{\pi}{2} + \alpha\right)$, and $\cos(\pi + \alpha)$.

Exercise 3.6: Enlargement
Consider Fig. 3.25, where the lines AC and DE are parallel. Calculate the lengths DE and BE.

Exercise 3.7: Spherical Coordinates
We use an orthonormal basis and consider a point M of Cartesian coordinates (x, y, z) and spherical coordinates (r, θ, ψ).

Justify the relations seen in the text:

$$\begin{cases} x_M = r\cos\theta\cos\psi, \\ y_M = r\sin\theta\cos\psi, \\ z_M = r\sin\psi. \end{cases}$$

Exercises in Geography and Geology

Exercise 3.8: Snell-Descartes' Law of Refraction
Snell-Descartes' (or Snell's) law of refraction of a wave between two media is given by $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where n_1 and n_2 are the

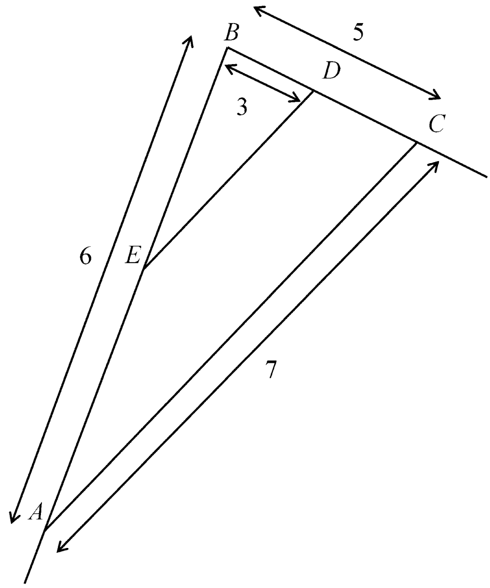


Fig. 3.25 Enlargement (figure not to scale)

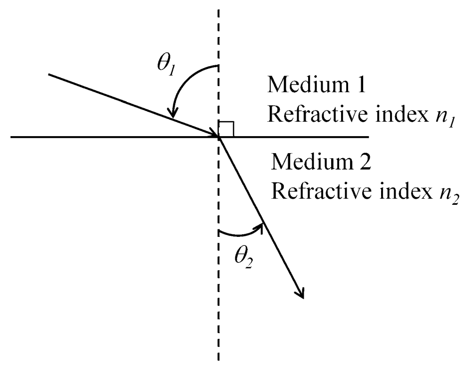


Fig. 3.26 Principle of Snell's law

refractive indices of the two environments, θ_1 is the angle of incidence of the wave between the two media, and θ_2 is the angle of refraction of the wave at this interface (Fig. 3.26).

A light ray passes from the air to the water at an angle of incidence in the air of 50.0°. What is the angle of refraction in the water? (given $n_{air} = 1.00$ and $n_{water} = 1.33$).

Exercise 3.9: Measuring Height like a Woodcutter
We present here another method than that of shadows (see text) to determine heights "in nature."

There are two rods of identical length. The first rod is placed horizontally at eye level and the other vertically at the end of the first rod (the whole forms a T).

An observer is positioned in front of an object (a tree, for example) whose height she wishes to determine (Fig. 3.27). Using the T, she must be able to conceal completely and precisely the target object (the bottom of the second stick coincides with the bottom of the object and the top with its top). To do this, she can slide the second rod vertically or move closer to or farther away from the object.

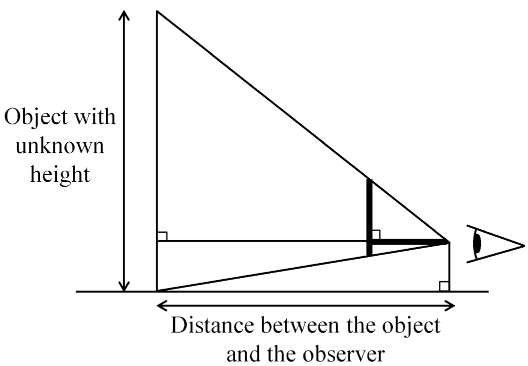


Fig. 3.27 Principle of calculation of height like a woodcutter

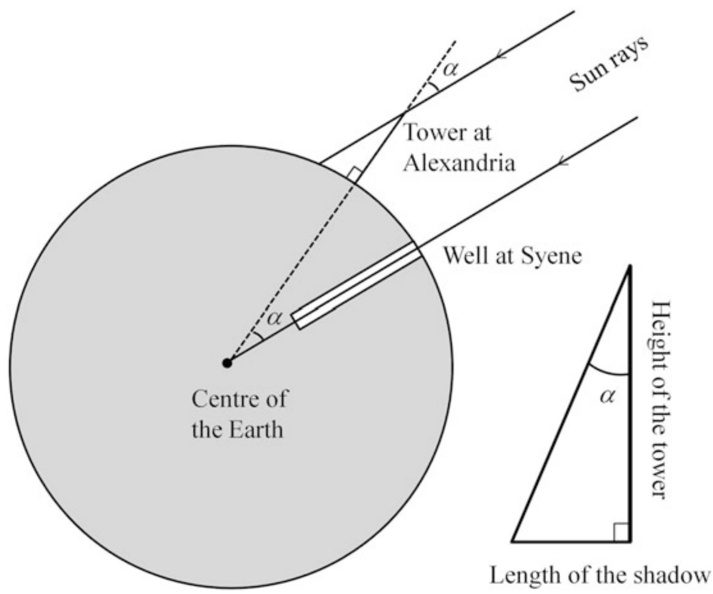
The following data are known: the length of the rods of the T, the height of the observer, and the distance between the observer and the object to be measured.

(1) Explain how to find the height of the object (you can consider using the intercept theorem twice).
(2) Application: an observer 1.65 m tall seeks to evaluate the size of the tree. Once the cross of the lumberjack is positioned correctly, it is located 22 m from the tree. What is the size of this tree?

Exercise 3.10: Measurement of Earth's Radius by Eratosthenes

In the insert on the measurement of the radius of the Earth, the method of Eratosthenes was presented. In Fig. 3.28 we see that the data that were useful to Eratosthenes were the ratio of the size of the obelisk at Alexandria to the shadow it produces. On June 21 the shadow is eight times smaller than the size of the obelisk. Finally, Eratosthenes estimated the distance between the cities of Syene and Alexandria at about 5000 stadia, using the unit of measurement of the time (1 stadium = approximately 157 m).

Fig. 3.28 Method used by Eratosthenes to calculate Earth's radius



(1) Identify the places on a map or using Google Earth and verify that the two cities have the same longitude (see Chap. 4 for the definition of longitude).
(2) Calculate the value of angle α and deduce the circumference of the Earth.
(3) What radius of the Earth can be deduced from this calculation? What is the error in relation to the radius known today (6371 km)?

Exercise 3.11: Accretionary Prism

(1) Level 1: Work again on the example given in the text (in the section on the geometry of the plane).
(2) Level 2: From Fig. 3.29, express the length d in terms of h and angle α.
(3) Level 3: From Fig. 3.29, express the length d in terms of h and angles α and β.

Exercise 3.12: Area of a Watershed

On a map, the area of a small catchment is $A = 2.50\ \text{km}^2$. In reality (our world is in three dimensions), area A on the map is the orthogonal projection of the area of surface A_s, and we have the relation

$$A_s = \frac{A}{\cos(\alpha)},$$

with α the angle that defines the average slope of the watershed.

(1) If the angle is 10°, what is the value of the actual area A_s in km^2? What is the value of this same area in hectares (ha)?
(2) On a small mountain catchment of 1.50 ha (on a map), it is considered that the slope has a constant value. The latter is measured by GPS: walking 20 m on the slope means climbing 3.5 m in altitude.

What is the value of this slope, given as a percentage? What is the value of the actual area of the watershed, given in ha and km^2?

Exercise 3.13: Depth of Mohorovicic Discontinuity

The energy developed by an earthquake at its focus (point F in Fig. 3.30) causes seismic waves that propagate in all directions within the globe. Their propagation velocity depends on the type of material in which they travel: the denser the material, the higher the speed.

These waves are of different types.

The primary, or P, waves (P for pressure) are the fastest (on the order of 6.5 km/s) and are therefore the first to be recorded by seismographs (point S in Fig. 3.30).

PmP waves are P waves that reflect on the surface of the Mohorovicic Discontinuity (Moho) (named after the Croatian meteorologist who highlighted this discontinuity) owing to the difference in density between the Earth's crust and the upper part of the mantle. PmP waves have the same velocity as P waves, but their path is longer.

The difference in the arrival times of the P and PmP waves gives an order of magnitude of the depth of the Moho: the limit between the Earth's crust and the upper part of the mantle.

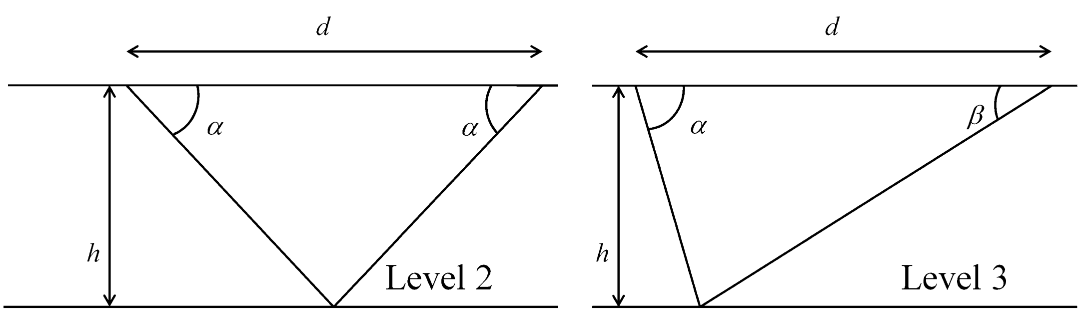


Fig. 3.29 Schematic of accretionary prism in case of two equal angles (left) and an angle α and an angle β with different values (right)

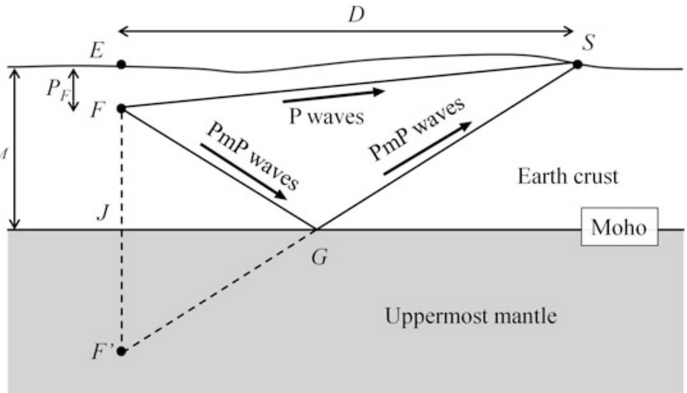


Fig. 3.30 Routes of P and PmP waves in crust. Point E: earthquake epicenter; F: focus; S: seismograph measuring station

Using Fig. 3.30, which models the system under study, we aim to determine the depth of the Moho (P_M). For this, several steps must be followed:

(1) Determine the position of point G of reflection of the PmP waves on the Moho (to do this, calculate the distance JG depending on distances P_M, P_F, and D).
(2) Express the distances FS and $FG + GS$ respectively traversed by the P and PmP waves as a function of the distances P_M, P_F, and D.
(3) For an earthquake with a depth of 11 km whose epicenter is located 70 km from the measuring station, P waves were detected at the measuring station at 9 h 15 min 12.490 s, and a PmP wave at 9 h 15 min 15.579 s. What is the depth of the Moho?
(4) From the expressions obtained in Question 2 and using the reasoning of Question 3 find the expression of the depth of the Moho P_M in terms of P_F and D (this is the same calculation as in question 3 but using variables – that is to say, in literal form – in the computation instead of numerical values).
(5) An explosion occurs in an open sky mine with seismographs placed at different distances D from the explosion source. The seismographs indicate the arrival times of the P waves (t_P) and PmP waves (t_{PmP}). These measurements are given in the following table:

D (km)	t_P (s)	t_{PmP} (s)
1	0.19	12.72
10	1.86	13.07
20	3.61	13.47
30	5.42	14.16
40	7.29	14.70
50	9.01	15.61
60	10.92	16.74
70	12.75	18.19
80	14.51	19.26
90	16.34	21.07
100	18.18	22.19

Using R software, calculate the 11 values of the depth of the Moho and give an average value.

Exercise 3.14: Alluvial Plain Aquifer

In a groundwater aquifer, there is a well with a flow rate q (expressed in m^3/s). The depth $h(x, y)$ of this underground water table at each point $M(x, y)$ can be determined by

$$h_M = \frac{q}{2\pi T} \ln\left(\frac{r_2}{r_1}\right),$$

with r_1 the distance between point M and the well, r_2 the distance between point M and the image of the well with respect to the river (Fig. 3.31), and T the transmissivity of the aquifer (m^2/s).

(1) Give the literal expression of the hydraulic head $h_M(x, y)$ in any location $M(x, y)$ of the aquifer, according to the variables x,y and the

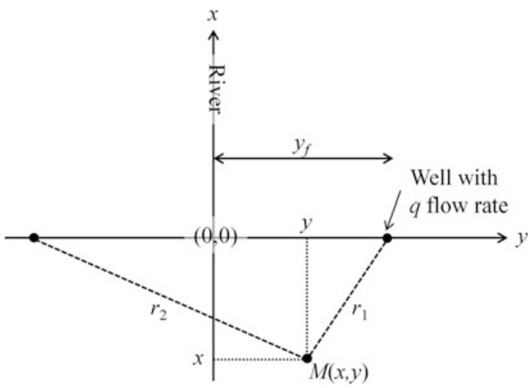


Fig. 3.31 From above the well flow rate q in an underground water table is located at a distance y_f from the river. The depth of the groundwater $h(x, y)$ can be determined at each point $M(x, y)$

parameters q, T, and y_f, where y_f is the distance between the well and the river.

(2) Using R, plot the shape of the hydraulic head $h(x, y)$ in an area of 400×400 m using the functions `image()` and `contour()`. Observe how the hydraulic head changes when the well's flow rate increases. Use the value T=0.001 m^2/s for transmissivity.

Exercise 3.15: Urban Density

On the SPRINGER editor's website (http://extras.springer.com), download the Data49.csv file. This file contains data on about 363 villages of Maine-et-Loire (49, France):

- Column 1: x-coordinates of the village in hm;
- Column 2: y-coordinates of the village in hm;
- Column 3: number of inhabitants of the village in thousands of persons;
- Column 4: area of the village in ha.

(1) Using R, draw a point cloud with the density of the population on the y-axis and on the x-axis the distance between each village and the center of Angers, whose coordinates are $X_A = 4322$ hm and $Y_A = 67{,}140$ hm.

(2) The Clark model (1951) gives the relationship between the density and distance using an exponential such as $d = A \exp{(br)}$, where d is the density, r the distance, A by definition the density in $r = 0$ (here the town of Angers), and b a strictly negative parameter.

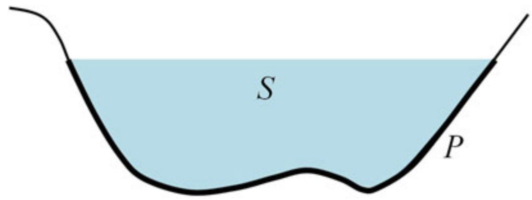


Fig. 3.32 Transverse river profile. The hydraulic radius R is defined as the ratio of the wetted section S to the wet perimeter P

What values of A and b make it possible to superimpose this function on the graph in question 1?

Exercise 3.16: Density of Bus Stops

On the SPRINGER editor's website (http://extras.springer.com), download the file BusAngers.csv. This file contains data on the position of 1600 bus stops in the city of Angers (France):

- Column 1: x-coordinates of stop in m;
- Column 2: y-coordinates of stop in m.

(1) Using R, draw a scatter plot with the bus stop density on the y-axis and on the x-axis the distance between each bus stop and the center of Angers, whose coordinates are $X_A = 432{,}200$ m and $Y_A = 6{,}714{,}000$ m.

(2) How far from the center of Angers is this density at its maximum?

Exercise 3.17: Current Speed

One can estimate the speed V of a river's current using the Manning–Strickler formula:

$$V = R^{\frac{2}{3}} \frac{J^{\frac{1}{2}}}{n},$$

where R is the hydraulic radius (Fig. 3.32), i.e., the ratio $\frac{S}{P}$ of the wetted section S and the wet perimeter P; J the average slope of the river and n

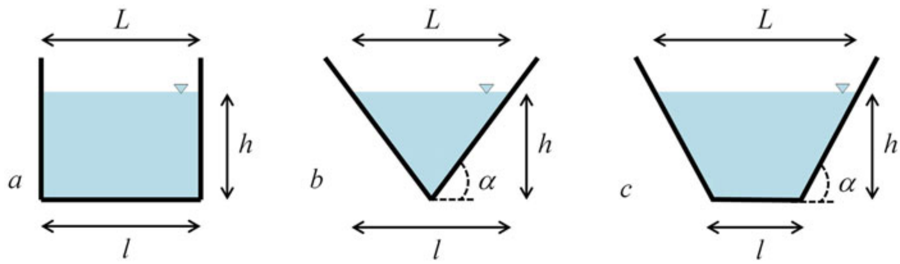


Fig. 3.33 Three rive shapes: rectangular (*a*), triangular (*b*), and V (*c*)

a Manning coefficient that depends on the roughness of the river bed. For this exercise, $n = 0.03$ and $J = 0.5\%$.

(1) It is considered that the transverse profile of the river has a rectangular shape of width l and height h (Fig. 3.33a). What is the expression for the speed V in terms of l, h, n, and J?
(2) It is considered that the orthogonal profile of the river has a triangular shape of width l, height h, and angle α (Fig. 3.33 b). What is the expression for this speed V in terms of h, α, n, and J?
(3) It is considered that the profile of the wide river has a trapezoidal form with width l, height h, and angle α (Fig. 3.33 c). What is the expression for the speed V in terms of h, α, l, n, and J?
(4) Using R, plot the speed values in the river for the triangular and trapezoidal shapes and when the angle ranges from 10° to 45°.

Solutions

Solution 3.1: Flash Questions. Series 1

(1) We have $d(M,N) = \sqrt{(3.7-18.4)^2 + (-12.8-5.3)^2 + (10-(-7.9))^2} \approx 29.4$.
(2) See the text.
(3) We have $\tan\widehat{A} = \dfrac{BC}{AB} = \dfrac{19}{13.2}$, so $\widehat{A} = \tan^{-1}\left(\dfrac{19}{13.2}\right) \approx 55^\circ$.

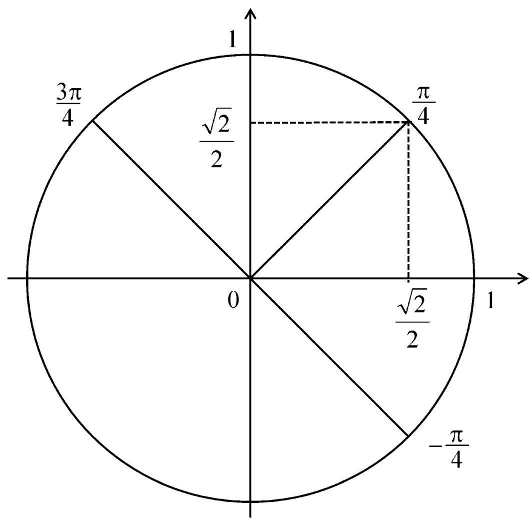


Fig. 3.34 Trigonometric circle with angles $-\frac{\pi}{4}, \frac{\pi}{4}$, and $\frac{3\pi}{4}$

Solution 3.2: Flash Questions. Series 2

(1) We have $30^\circ = \frac{\pi}{6}$ rad.
(2) Pythagoras' theorem gives $AB^2 + AC^2 = BC^2$,

so $AC^2 = BC^2 - AB^2 = 28.4^2 - 13.2^2 = 632.32$.
And we have $AC \approx 25.1$ cm.

(3) Fig. 3.34

The following values can be found:

$$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2},\quad \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2},$$
$$\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

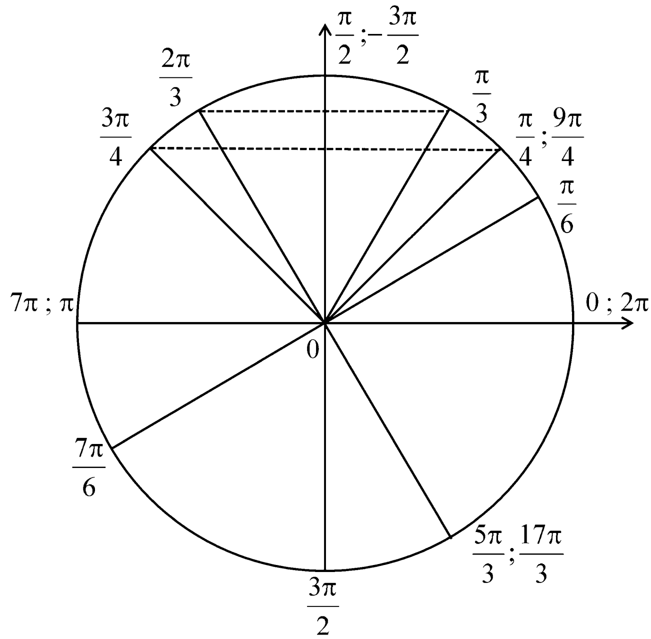


Fig. 3.35 Placement of angles on unit circle

Solution 3.3: Flash Questions. Series 3

(1) The sum of the angles of triangle ABC is 180°; thus, $\widehat{A}+\widehat{B}+\widehat{C}=180°$, hence $2\widehat{B}=180-\widehat{A}$, and finally $\widehat{B}=\widehat{C}=\dfrac{180-\widehat{A}}{2}$.

(2) We have $\sin\widehat{B}=\dfrac{AC}{AB}$, so $AC=AB\times\sin\widehat{B}$ $=13.2\times\sin(35)\approx 7.6\,\text{cm}$.

(3) See the text.

Solution 3.4: Miscellaneous Calculations on the Right Triangle

(1) Using Pythagoras' theorem, the missing side measures $\sqrt{85.1^2-3.4^2}\approx 85.0\,\text{m}$.

The angle between the side measuring 85.1 m and that measuring 3.4 m is equal to $\cos^{-1}(3.4/85.1)\approx 87.7^{\circ}$.

That between the sides of 85.1 m and 85.0 m measures $\sin^{-1}(3.4/85.1)\approx 2.3^{\circ}$.

(2) Let the desired height be called h. We have $\sin(75)=h/6$, so $h=6\times\sin(75)\approx 5.8$ m. The ladder is at a distance from the wall of $6\times\cos(75)\approx 1.6$ m.

Solution 3.5: Unit Circle

(1) The angles of the table are presented in Fig. 3.35.

We can deduce:

Angle (rad)	2π	$-\frac{3\pi}{2}$	7π	$\frac{7\pi}{6}$	$\frac{9\pi}{4}$	$\frac{17\pi}{3}$
Associated angle between $-\pi$ and $+\pi$	0	$\frac{\pi}{2}$	π	$-\frac{5\pi}{6}$	$\frac{\pi}{4}$	$-\frac{\pi}{3}$
Associated angle between 0 and 2π	0	$\frac{\pi}{2}$	π	$\frac{7\pi}{6}$	$\frac{\pi}{4}$	$\frac{5\pi}{3}$
Cosine	1	0	-1	$-\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
Sine	0	1	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$
Tangent	0	Not defined	0	$\sqrt{3}$	1	$-\sqrt{3}$

(2) We have $\sin\left(\frac{\pi}{2}-\alpha\right)=\cos(\alpha)$, $\sin(\pi-\alpha)=\sin(\alpha)$, $\cos\left(\frac{\pi}{2}+\alpha\right)=-\sin(\alpha)$, and $\cos(\pi+\alpha)=-\cos(\alpha)$.

(3) See the unit circle from question 1.

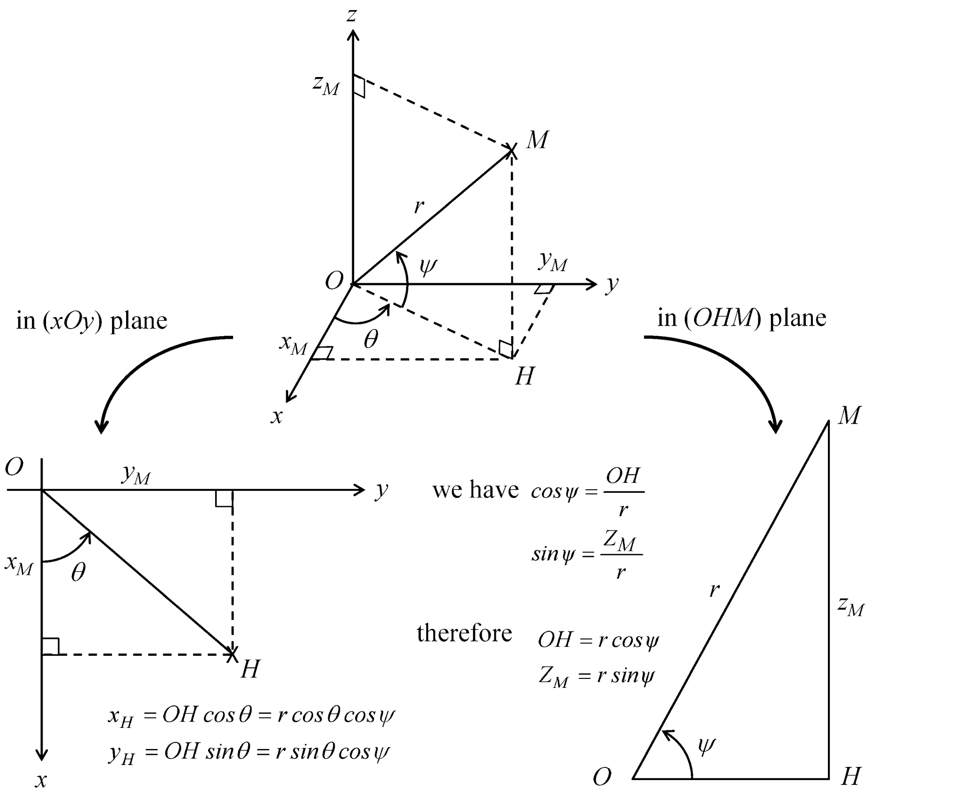


Fig. 3.36 Principles of spherical coordinates

Solution 3.6: Enlargement
Triangle *ABC* is an enlargement of triangle *EBC*. Using the intercept theorem, we therefore have the following equalities:
$\frac{BE}{BA} = \frac{BD}{BC} = \frac{DE}{AC}$, from which $\frac{BE}{6} = \frac{3}{5} = \frac{DE}{7}$, and so $BE = \frac{18}{5} = 3.6$ cm and $DE = \frac{21}{5} = 4.2$ cm.

Solution 3.7: Spherical Coordinates (refer to Fig. 3.36 for solution)

Solution 3.8: Refraction of a Wave; Snell-Descartes Law
The law of refraction gives the equation of a variable (expressed in degrees) $1 \times \sin 50.0 = 1.33 \sin\theta_2$, from which $\sin\theta_2 = \frac{\sin 50}{1.33}$, and so $\theta_2 = \sin^{-1}\left(\frac{\sin 50}{1.33}\right) \simeq 35.2^{\circ}$.

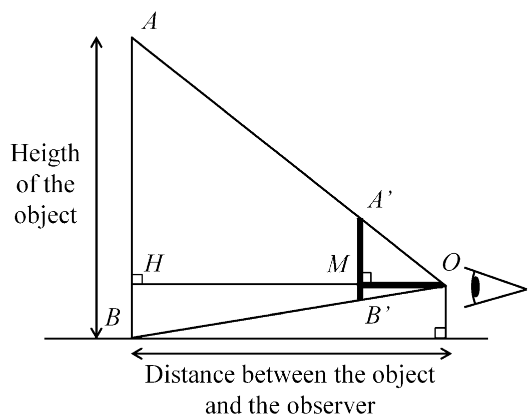


Fig. 3.37 Principle of calculation of height of an object like a woodcutter

Solution 3.9: Measuring Height like a Woodcutter
(1) We use the notation of Fig. 3.37.

First of all, since the two rods are the same length, $OM = A'B'$.

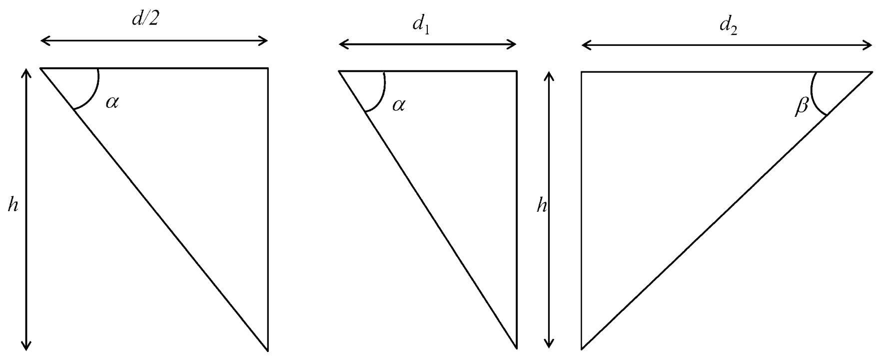


Fig. 3.38 Schematic of an accretionary prism in the case of two equal angles (left) and an angle α and an angle β with different values (right)

We first consider triangle OAB and use the intercept theorem. Triangle OAB is an enlargement of triangle $OA'B'$, so we have the equality

$$\frac{A'B'}{AB} = \frac{OA'}{OA}.$$

Then consider triangle OMA', its enlargement OHA, and once again the intercept theorem: $\frac{OM}{OH} = \frac{OA'}{OA}$.

From the two previous equations we draw $\frac{A'B'}{AB} = \frac{OM}{OH}$, and therefore that $AB = \frac{A'B' \times OH}{OM}$; simplifying (recall that $A'B' = OM$), we obtain $AB = OH$. The desired height is equal to the distance between the observer and the object (this comes from the particular situation of the equal length of the two rods).

(2) The height of the observer is not needed. The tree measures 22 m, which is the distance of the observer to the tree!

Solution 3.10: Historical Earth Radius Measurement

(2) Let us consider the triangle formed by the obelisk, its shadow, and a solar ray. It is then known that

$$\tan(\alpha) = \frac{\text{height of shadow}}{\text{height of the obelisk}} = \frac{1}{8}.$$

We thus deduce the value of angle α as $\tan^{-1}(1/8)$ (which is about 7.1°, but we use the exact value $\tan^{-1}(1/8)$ in the following calculations to avoid using an approximate value too early in the calculations).

Note that if the instrument of calculation gives the answer in radians, multiply the result by $180/\pi$ if the answer is desired in degrees.

An interesting symmetry about angle α is observed: this angle, which Eratosthenes measured thanks to the obelisk and its shadow, is the same as that formed by the vertical of the well (at Syene) and the obelisk (at Alexandria). We can thus approach the problem by using a relation of proportionality: since angle α shows the distance between Syene and Alexandria, which is 5000 stadia, we estimate the associated portion of the circle and by proportionality obtain the circumference. Hence, the circumference C of the Earth is equal to

$$C = \frac{5000 \times 157}{\tan^{-1}(1/8)} \times 360$$
$$\approx 39{,}700{,}000 \text{ m, or } 39{,}700 \text{ km}.$$

(3) We deduce a value of the radius of the Earth of $\frac{39{,}700}{2 \times \pi} \approx 6320\,\text{km}$.

The error of Eratosthenes is thus $\frac{|r_{\text{today}} - r_{\text{Eratosthenes}}|}{r_{\text{today}}} = \frac{|6371 - 6320|}{6371} \approx 0.8\%$.
It is noted that the very low relative error is quite exceptional, which gives the full measure of the genius of Eratosthenes.

Note 35 We have chosen to round to three significant figures (the number 5000 allowing us to choose 1–4 significant digits).

Solution 3.11: Accretionary Prism

(1) We repeat the same method as in the text, replacing tan (45°) with (tan (α)). We obtain the relation $d = 2 \times \dfrac{h}{\tan(\alpha)}$.

(2) The division gives two right triangles, with $d = d_1 + d_2$. As earlier, we have $d_1 = \dfrac{h}{\tan(\alpha)}$ and $d_2 = \dfrac{h}{\tan(\beta)}$, which leads to $d = \dfrac{h}{\tan(\alpha)} + \dfrac{h}{\tan(\beta)}$.

Solution 3.12: Watershed

(1) $A_s = \dfrac{2.50}{\cos(10)} \approx 2.54\ \text{km}^2$ (don't forget to configure the calculation instrument in degrees).

1 ha = 1 hm^2 = 0.01 km^2. 1 km^2 is therefore 100 times larger than a hectare or a square hectometer: 1 km^2 = 100 ha. So 2.54 km^2 = 2.54 × 100 = 254 ha.

(2) The slope is considered constant, moving on the pool side is along the hypotenuse of a hypothetical right triangle. In this configuration, we know that $\sin(\alpha) = \dfrac{h}{d}$ (Fig. 3.39).

When we travel $d = 20$ m on the slope, we rise $h = 3.5$m in altitude. The average slope is therefore $\dfrac{d}{h} = \dfrac{3.5}{20} = 0.175$, or 17.5%.

The angle of this slope is therefore arcsin(0.175) $\simeq 10^\circ$.

If we use the equality $A_s = \dfrac{A}{\cos(\alpha)}$, we have (the angles are expressed in degrees) $A_s = \dfrac{1.50}{\cos(10)} \approx 1.52$ ha, or $\dfrac{1.52}{100} \simeq 0.0152\ \text{km}^2$.

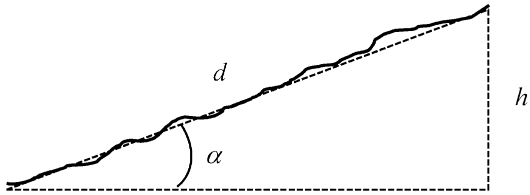


Fig. 3.39 Path on slope of a watershed. When we travel $d = 20$ m, we rise $h =$ 3.5 m

Solution 3.13: Depth of Moho

(1) Consider F' the central symmetric of point F with central point J (Fig. 3.40). Let us consider triangle $F'SE$. The intercept theorem assures that $\dfrac{JG}{D} = \dfrac{F'J}{F'E}$, which gives, noting that $F'E = 2P_M - P_F$ and $F'J = FJ = P_M - P_F$,

$$JG = \frac{P_M - P_F}{2P_M - P_F} D.$$

(2) The distance traveled by the P wave is FS. In right triangle FSE, we have $FS^2 = SE^2 + EF^2$, so $FS = \sqrt{D^2 + P_F^2}$.

The distance traveled by the PmP waves is $FG + GS$. Therefore, $FG = F'G$ because F' is the central symmetric of point F with the Moho being the central point. So $FG + GS = F'G + GS = F'S$ because the three points F', G, and S are aligned. In the right triangle $EF'S$, we have $F'S^2 = SE^2 + EF'^2$, and we deduce that $F'S = \sqrt{(2P_M - P_F)^2 + D^2}$.

(3) In this question, $P_F = 11$ km and $D = 70$ km.

The difference in arrival times between the two wave types is 3.1089 s. These waves travel 6.5 km/s; there is therefore a difference in distance of 3.089 × 6.5 km = 20.0785 km.

However, this difference is also $F'S - FS = \sqrt{(2P_M - 11)^2 + 70^2} - \sqrt{70^2 + 11^2}$.

We solve the equation of the variable $P_M : \sqrt{(2P_M - 11)^2 + 70^2} - \sqrt{5021} =$ 20.0785. This gives $\sqrt{(2P_M - 11)^2 + 70^2} =$ 70.859 + 20.0785 = 90.94.

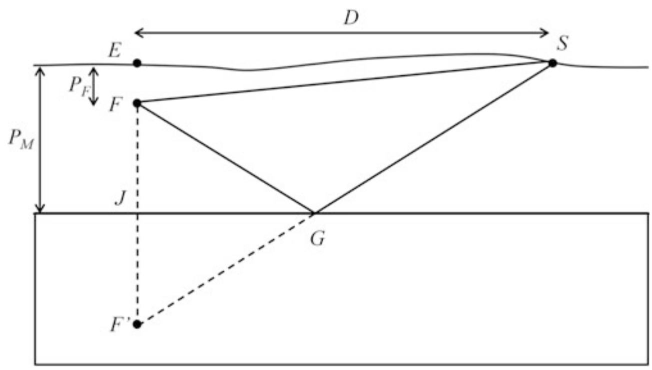


Fig. 3.40 Diagram of scenario

From which $(2P_M - 11)^2 + 70^2 = 8270.1$, and so $(2P_M - 11)^2 = 3370.1$.

From which $2P_M - 11 = 58.1$, and therefore $P_M \approx 35$ km (the result is given with two significant digits as for the distance 11 km).

(4) The recorded time offset is equal to $\Delta t = \dfrac{F'S - FS}{6.5}$; this shift Δt is a known measure. It is now necessary to extract P_M of this expression. We have successively

$$\Delta t = \frac{F'S - FS}{6.5}$$
$$= \frac{\sqrt{(2P_M - P_F)^2 + D^2} - \sqrt{D^2 + P_F^2}}{6.5}$$
$$6.5\Delta t = \sqrt{(2P_M - P_F)^2 + D^2} - \sqrt{D^2 + P_F^2}$$
$$6.5\Delta t + \sqrt{D^2 + P_F^2} = \sqrt{(2P_M - P_F)^2 + D^2}$$
$$(2P_M - P_F)^2 + D^2 = \left(6.5\Delta t + \sqrt{D^2 + P_F^2}\right)^2$$
$$(2P_M - P_F)^2 = \left(6.5\Delta t + \sqrt{D^2 + P_F^2}\right)^2 - D^2$$
$$2P_M - P_F = \sqrt{\left(6.5\Delta t + \sqrt{D^2 + P_F^2}\right)^2 - D^2}$$
$$P_M = \frac{1}{2}\left[P_F + \sqrt{\left(6.5\Delta t + \sqrt{D^2 + P_F^2}\right)^2 - D^2}\right]$$

(5) To calculate the depth of the Moho, we use the preceding relationship posing $P_F = 0$ km. Indeed, the mine fire indicates that here the focal point (point F) and epicenter (point E) are the same.

```
#-----------------------------------------
# Depth of Moho
#-----------------------------------------
# P-waves speed (km/s)
V <- 6.5
# P-wave arrival time (s)
P <- c(0.19,1.86,3.61,5.42,7.29,
9.01,10.92,12.75,14.51,16.34,18.18)
# PmP-wave arrival time (s)
PmP <- c(12.72,13.07,13.47,14.16,
14.70,15.61,16.74,18.19,19.26,21.07,2-
2.19)
# Time difference between P waves and
PmP waves (s)
dt <- PmP-P
# Seismograph distance (km)
D <- c(1,10,20,30,40,50,60,70,80,90,100)
# Depth of Moho (km)
PM <- 0.5 * sqrt((V*dt+D)^2 - D^2)

# Average value
mean(PM)
```

Solution 3.14: Alluvial Plain Aquifer

(1) Figure 3.31 allows us to change from radial coordinates (polar coordinates) to Cartesian coordinates(x, y). For this, we see that r_1 and r_2 are the lengths of the hypotenuses of two right triangles. We then have

$$r_1^2 = \sqrt{(y - y_f)^2 + x^2},$$
$$r_2^2 = \sqrt{(y + y_f)^2 + x^2}.$$

Replacing the values of r_1 and r_2 in the equation we have

$$\begin{aligned}h_M(x,y) &= \frac{q}{2\pi T}\ln\left(\frac{r_2}{r_1}\right)\\ &= \frac{q}{2\pi T}\ln\left(\frac{\sqrt{\left(y+y_f\right)^2+x^2}}{\sqrt{\left(y-y_f\right)^2+x^2}}\right)\\ &= \frac{q}{4\pi T}\ln\left(\frac{\left(y+y_f\right)^2+x^2}{\left(y-y_f\right)^2+x^2}\right).\end{aligned}$$

(2)

```
#------------------------------------
# Images method
#------------------------------------
# Location of well in Y (m)
yf <- 200
# Flow rate of well (m3/s)
q <- 0.05
# Transmissivity (m2/s)
Transm <- 0.001
x <- seq(-200,200,1)
y <- seq(0,400,1)
f <- function(x,y) {
return ( - q *
log(((y+yf)^2+x^2)/((y-yf)^2+x^2)) /
(4* pi *Transm) )
}
z <- outer(x, y, f)
image(x,y,z)
contour(x,y,z,add=TRUE)
```

Solution 3.15: Density

(1) To determine the distance between each village of Maine-et-Loire and the city of Angers, use the equation of the Euclidean distance:

$$d(A,B) = \sqrt{\left(x_B - x_A\right)^2 + \left(y_B - y_A\right)^2}.$$

The density of the population of a village is determined simply by dividing the number of inhabitants of each village by the area of the village.

(2) The Clark model is used in spatial analysis to study many urban processes, for example, residential location, urban forms, and density of shops. To verify that this model is usable in this exercise, we superimpose the function of the model $y = A\ \exp\ (bx)$ on actual data. For this, it is necessary to determine the value of the two parameters A and b. The parameter A is by definition the density of the population in $x = 0$ (here the town of Angers) whose value corresponds to the maximum density (function `max()`). The parameter b is manually adjusted (note that R makes it possible to find automatically the best value, but this is beyond the scope of this exercise). Clark's law illustrates a good trend but does not seem to be the most representative model of the relationship between density and distance to the city of Angers.

```
#------------------------------------
# Density of Maine-et-Loire cities
#------------------------------------
# Read data
setwd("D:/DirectoryOfYourData/")

Data <- read.csv("Data49.csv",sep=";")

# Angers X and Y coordinates (hm)
x1 <- 4322
y1 <- 67140

# Distance to each city from Angers
Data$Distance  <-  sqrt((Data$X-x1)^2
+(Data$Y-y1)^2)

# Density hab / ha
# Area is in ha in the data file
Data$Density <- Data$Pop *1000 / Data
$Surf

# Plot the data
plot(Data$Distance/10,Data$Density,
pch=16,
xlab = "Distance From Angers city cen-
ter (km)",
ylab = "Number of inhabitants/ha")
# Model: Clark, 1951
A <- max(Data$Density)
b <- -0.2
x <- seq(0,700)
y <- A*exp(b*x)
lines(x,y,col=2,lwd=2)
```

Solution 3.16: Density of Bus Stops

(1) To calculate the density of bus stops, count the number of these bus stops between two successive concentric circles of radius iD_R and $(i + 1)D_R$. The increment is $i = 0, 1, \ldots, N$, with N being the number of intervals ($N = \frac{D_{Max}}{D_R}$, where D_{Max} is the distance from the center of Angers to the most distant bus stop) and D_R the space step, fixed by the operator. This method is equivalent to plotting a histogram of constant classes (intervals between two successive concentric circles), as shown in Fig. 3.41.

(2) The largest concentration of bus stops is located at a distance of 2.9 km from the center of Angers. This value can be found using different space step D_R, which gradually decreases.

```
#-----------------------------------
# Density of Angers bus stops
#-----------------------------------
# Read data
setwd("D:/DirectoryOfYourData/")

Bus <- read.csv("BusAngers.csv",
sep=";")

# Bus stop locations
plot(Bus$X,Bus$Y,pch=16,cex=0.5,
xlab="X",ylab="Y")

# Angers X and Y coordinates (hm)
x1 <- 432200
y1 <- 6714000
points(x1,y1,pch=16,cex=0.5,col=2)

# Distance to each bus stop from Angers
city center
Bus$Distance <- sqrt((Bus$X-x1)^2+(Bus
$Y-y1)^2)

# Max distance
Dmax <- max(Bus$Distance)

# The radius varies from 0 to Dmax and
# the number of bus stops is counted
# between two successive distant cir-
cles
# with Dr space step (m)
Dr <- 100
# Number of density values
N <- floor(Dmax/Dr)
Density<- matrix(data=.0,ncol=2,
nrow=N)
```

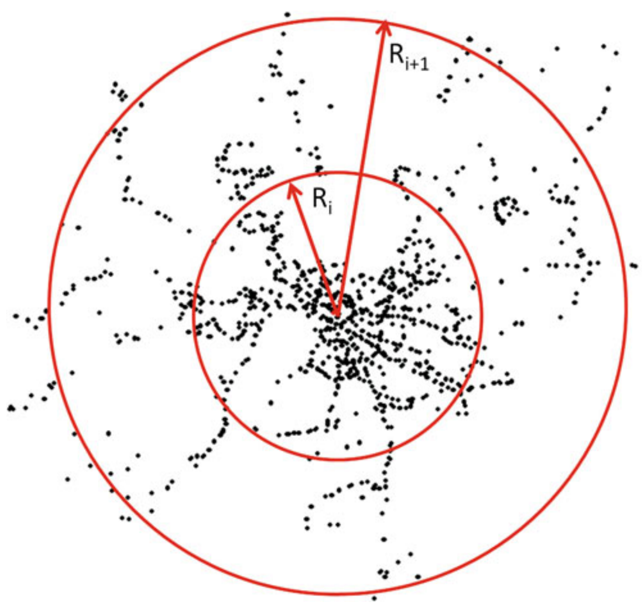


Fig. 3.41 Principle of calculation of density of bus stops: concentric circles

```
for(i in 0:N) {
Rayon1 <- i * Dr
Rayon2 <- (i+1) * Dr
cpt <- 0
for(j in 1:nrow(Bus)) {
if(Bus$Distance[j] <= Rayon2 & Bus
$Distance[j] > Rayon1)
cpt <- cpt + 1
}
Density[i,1] <- 0.5 *(Rayon1 + Rayon2)
Density[i,2] <- cpt
}

plot(Density,type="h",
                  xlab="Distance (m)",
ylab="Number of bus stops")
```

Solution 3.17: Current Speed

(1) When the river has a rectangular shape (Fig. 3.33 a), then $l = L$. The wet perimeter is equal to $P = l + 2h$ and the wetted section $S = lh$. Thus, the hydraulic radius is $R = \frac{S}{P} = \frac{lh}{l+2h}$. From this we deduce the value of the mean velocity of the current:

$$V = \left(\frac{lh}{l+2h}\right)^{\frac{2}{3}} \frac{J^{\frac{1}{2}}}{n}.$$

(2) In the case of a triangular river, $l = L$ and angle α links the width l with the height h.

Let us first determine the wet perimeter P. In a right triangle (Fig. 3.42), we have (using Pythagoras'theorem)

$$\left(\frac{P}{2}\right)^2 = \left(\frac{l}{2}\right)^2 + h^2.$$

It can be deduced from this that

$$P = 2\sqrt{\left(\frac{l}{2}\right)^2 + h^2}.$$

Remembering that in a right-angled triangle the tangent of angle α is the ratio of the lengths of the opposite sides h and adjacent $-\frac{l}{2}$. The relationship between l and h is as follows:

$$\tan(\alpha) = \frac{h}{\frac{l}{2}}.$$

Replacing the value of $\frac{l}{2} = \frac{h}{\tan(\alpha)}$ in the wet perimeter equation, one obtains successively

$$\begin{aligned} P &= 2\sqrt{\left(\frac{l}{2}\right)^2 + h^2} \\ &= 2\sqrt{\left(\frac{h}{\tan(\alpha)}\right)^2 + h^2} \\ &= 2\sqrt{h^2\left(\frac{1}{\tan^2(\alpha)} + 1\right)}. \end{aligned}$$

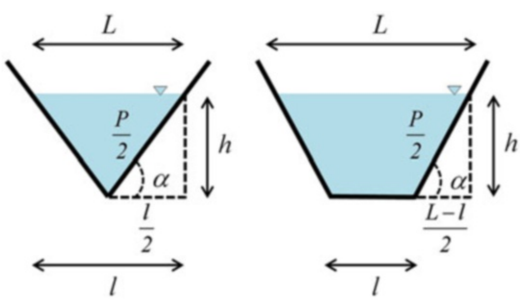


Fig. 3.42 Schematic representation of the shape of a river

Thus,

$$P = 2h\sqrt{1 + \frac{1}{\tan^2(\alpha)}}.$$

The wetted surface S is, by symmetry, equal to twice that of the right triangle, that is to say, that of a rectangle of width $\frac{l}{2}$ and height h. The wetted surface is therefore successively equal to

$$\begin{aligned} S &= h\frac{l}{2} \\ &= h\frac{h}{\tan(\alpha)} \\ &= \frac{h^2}{\tan(\alpha)}. \end{aligned}$$

From this we deduce the expression of the hydraulic radius:

$$\begin{aligned} R &= \frac{S}{P} \\ &= \frac{\frac{h^2}{\tan(\alpha)}}{2h\sqrt{1 + \frac{1}{\tan^2(\alpha)}}} = \frac{h}{2\tan(\alpha)\sqrt{1 + \frac{1}{\tan^2(\alpha)}}} \\ &= \frac{h}{2\sqrt{1 + \tan^2(\alpha)}}. \end{aligned}$$

Finally, it is deduced that the velocity of the flow is of the form

$$V = \left(\frac{h}{2\sqrt{1 + \tan^2(\alpha)}}\right)^{\frac{2}{3}} \frac{J^{\frac{1}{2}}}{n}.$$

(3) The case of a trapezoidal river is relatively similar to the previous case. For wet perimeter P, it is clear (Fig. 3.42) that it is equal to that of the triangular form plus the length of the bottom whose width is l:

$$\begin{aligned} P &= 2h\sqrt{1 + \frac{1}{\tan^2(\alpha)}} + l \\ &= 2h\sqrt{1 + \frac{1}{\tan^2(\alpha)}} + \frac{2h}{\tan(\alpha)} \\ &= 2h\left(\sqrt{1 + \frac{1}{\tan^2(\alpha)}} + \frac{1}{\tan(\alpha)}\right). \end{aligned}$$

The case of the wetted surface is identical since we see that it is equal to that of the triangular form plus the surface of the rectangle of width l and height h:

$$\begin{aligned} S &= \frac{h^2}{\tan(\alpha)} + lh \\ &= \frac{h^2}{\tan(\alpha)} + \frac{2h^2}{\tan(\alpha)} \\ &= \frac{3h^2}{\tan(\alpha)}. \end{aligned}$$

The hydraulic radius is thus

$$\begin{aligned} R &= \frac{S}{P} \\ &= \frac{\frac{3h^2}{\tan(\alpha)}}{2h\left(\sqrt{1 + \frac{1}{\tan^2(\alpha)}} + \frac{1}{\tan(\alpha)}\right)} \\ &= \frac{3h}{2}\frac{1}{1 + \sqrt{1 + \tan^2(\alpha)}}. \end{aligned}$$

The expression for the velocity is thus

$$V = \left(\frac{3h}{2}\frac{1}{1 + \sqrt{1 + \tan^2(\alpha)}}\right)^{\frac{2}{3}} \frac{J^{\frac{1}{2}}}{n}.$$

(4) We use the equations of the preceding speed and varying angle α of 10° to 45° through, for example, instruction `seq()`.

```
#-------------------------------------
# River flow
#-------------------------------------
# River depth (m)
h <- 10
# River bank angle (°)
alpha <- seq(10,45,0.1)
# Transformation into radian
alphaRad <- alpha * pi / 180
# Average slope
J <- 0.005
# Manning rugosity coefficient
n <- 0.03

# Triangular shape
VTriangle <-
```

```
(h/(2 * sqrt(1 + (tan(alphaRad))^2)))^
(2/3) * J^(1/2)/n

# Trapezoidal shape
VTrapeze <-
(2*h/2  *  1/(1+sqrt(1+(tan(alphaRad))
^2)))^(2/3) * J^(1/2)/n
```

```
# Plot velocities
plot(alpha,VTriangle,xlab="Angle (°)",
ylab="Velocity  (m/s)",type="l",lwd=2,
col=1)
lines(alpha,VTrapeze,lwd=2,col=2)
```

4 Cartography

Abstract

How should the Earth be represented on a plane map? This is impossible while preserving all the initial properties (e.g., distances, angles, areas). Hence, choices must be made, leading to the use of properties of geometry of the plane or of space and, in particular, trigonometry.

Various methods can be used to switch from the sphere to the plane: these are called **projections**. The purpose of this chapter is not to present and study the various existing projections but to give all the mathematical tools necessary to understand a presentation on this topic.

The chapter presents basic concepts on map scales, angles on Earth, and geographical coordinates (latitude, longitude, and height). It also introduces spherical geometry and some mapping systems.

Keywords

Cartography · Scales · Geographical coordinates · Latitude · Longitude · Parallel · Meridian · Ellipsoid height · Spherical geometry · Projections · Mapping system

Aims and Objectives

- To acquire a basis for understanding mapping.
- To be able to find yourself on the globe.
- To understand how to use spherical trigonometry results.
- To acquire notions about various projections.

4.1 Scales

Definition 13 *The cartographic* ***scale*** *of a map is the reduction ratio between the reality on the ground and the map. A scale of 1:100,000 indicates that 1 cm on the map represents 100,000 cm = 1 km on the ground.*

Example 44

- *On Ordnance Survey (OS) (UK) Explorer 1:25,000 topographic maps, 1 cm on the map = 250 m on the ground.*
- *On Hema (Australia): 1:850,000 travel maps, 1 cm on the map = 850 m on the ground.*
- *On 1:5,000,000 atlas maps, 1 cm on the map = 50 km in the field.*

Note 36

- *Be careful with the words "small scale" and "large scale": the ratio 1:x is small when x is big. A small scale is used for viewing a larger area (example: 1:1,000,000) and a large scale for a local description, that is, a smaller area (example: 1:25,000).*

C. Fleurant, S. Bodin-Fleurant, *Mathematics for Earth Science and Geography*, Springer Textbooks in Earth Sciences, Geography and Environment, https://doi.org/10.1007/978-3-319-69242-5_4

4.2 Measurement of Angles, Geographical Coordinates

4.2.1 Links Between Time, Angles, and Distances

The Earth completes a rotation on its axis in 24 h (360°). In 1 h, it turns at an angle of 360/24°, or $15^\circ \approx 0.26$ rad.

Similarly, the circumference of the Earth at the equator is about 40,000 km (360°), and therefore 10,000 km per 90° and 5000 km for 45°. It should be noted that the distance corresponding to one degree (at the equator) is about 111 km. With the equator measuring about 40,000 km, 1 h at the equator corresponds to a distance of about 1700 km. A point at the equator rotates around the Earth at a speed of about 1700 km/h.

Insert 7 (Calendars)
The measurement of time came into existence with the birth of civilization. From clepsydres to atomic clocks, its measurement became increasingly precise (to a precision of five pico-seconds, that is $5 \cdot 10^{-12}$ s *currently). On a daily basis, we can identify ourselves in time by means of a calendar. If the latter seems simple to us because it is familiar, in reality it is much more complex. This complexity derives from the fact that our time division does not fall right. For example, our day is 24 h, and the sidereal day (rotation of the Earth on its axis) lasts 23 h 56 min 4 s!*

Many calendars have been developed. Whether by the Hebrews, Babylonians, or Greeks, the approximations were very important. In the era of the Roman Empire, the Julian calendar was created (Julius Caesar, 46 B.C.), which is based on the astronomical sciences and is organized on a solar year, i.e., 365.25 days. The Julian calendar has 12 months plus a quarter of a day every 4 years, whence comes the leap year. Similarly, the Augustan calendar (named after Emperor Augustus, 8 B.C.) has 7 months of 31 days, 4 months of 30 days, and 1 month of 28 or 29 days. Despite these adjustments, the Roman calendar lags behind by 11 min 14 s per year, or 4 days every four centuries. Our current Gregorian calendar (Pope Gregory XIII, sixteenth century) is much more precise but not yet perfect: it runs fast by 3 days every 10,000 years!

The months of July and August were named in honor of Julius and Augustus and both have 31 days: certainly one emperor couldn't have fewer days than the other!

4.2.2 Geographical Coordinates

The Earth is not exactly spherical (there is a flattening at the North and South Poles): it is more accurately modeled as an ellipsoid rotating around the North Pole/South Pole axis (Fig. 4.1).

Definition 14 *The* ***equator*** *is an imaginary line halfway between the North and South Poles in a plane perpendicular to the axis of the poles. The equator is a circle whose center is also the center of the Earth.*

Note 37 (Remarks)

- *The length of the equator is about 40,075 km, and its radius is 6378 km (this is the equatorial radius). The distance (on the sphere/on the surface of the Earth) from the equator to each of the two poles is 10,002 km. The distance (in a straight line) from the center of the Earth to the poles is 6356 km. The Earth's mean radius, which is the average distance between its center and the surface, is 6371 km.*
- *In textbooks, the Earth is represented in different ways* (Fig. 4.2). *From a classical perspective* (Fig. 4.2a), *Earth is slightly inclined so that the North Pole appears closer to the reader and the South Pole further (here in gray). In this case the equator is represented by an ellipse with half of it with dots for the invisible part. If we place our eye in the plane of the equator, we get Fig. 4.2b if it is parallel to the East–West axis and Fig. 4.2c if it is*

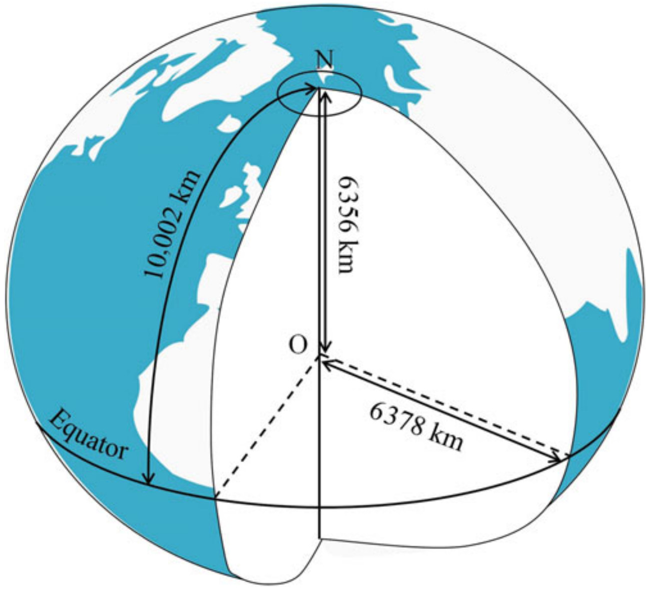


Fig. 4.1 Schematic view of Earth

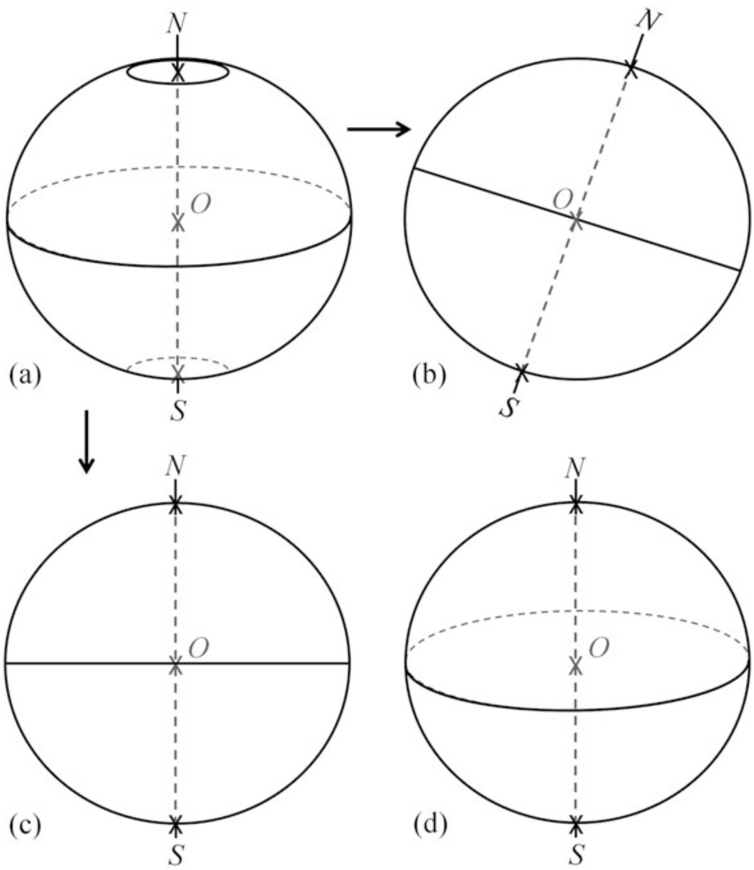


Fig. 4.2 Representations of Earth in (**a**) a classical perspective, (**b**) a perspective parallel to the East–West axis with the eye in the plane of the equator, (**c**) a perspective perpendicular to the East–West axis and with the eye in the plane of the equator, and (**d**) in a false perspective

perpendicular to the East–West axis. In the latter two cases, we see that the equator is represented by a line since the eye is in its plane and since the North and South Poles are exactly on the circumference of the Earth. In many textbooks, the Earth is represented as in Fig. 4.2d, that is to say, that the equator is represented by an ellipse to give a certain perspective but that the North and South Poles are on the circumference of the Earth. This last representation is false, but since it simplifies the Earth's in space, we will adopt it hereafter.

Let us consider a point P located on the terrestrial globe (on the ellipsoid). We call:

- O the center of the ellipsoid;
- NS the polar axis (N: North Pole, S: South Pole).

Definition 15 (Meridian)
Let P be a point on the ellipsoid. The plane that contains the rotation axis (that is to say, the line NS passing through the North Pole, N, and the South Pole, S) is called the ***meridian plane*** *at point P. The intersection between this meridian plane and the terrestrial ellipsoid is an ellipse (Fig. 4.3). In astronomy, this ellipse is called the meridian at point P. In geography the meridian at point P is defined by the half-ellipse determined from the aforementioned ellipse, passing through P and joining one pole to the other. We will use this latter definition in the remainder of this chapter.*

Definition 16 (Longitude) *A particular meridian has been chosen to be the international reference: it is called* ***the prime meridian*** *or* ***international reference meridian*** *and passes close to* ***Greenwich*** *(London, UK). The longitude of a point P is the angle λ between the meridian plane at point P and the prime meridian plane (Fig. 4.4). This measure is between 180°E (East) and 180°W (West). Points with the same longitude are located on the same* ***meridian.*** *In other words, meridians are lines of constant longitude.*

Note 38 (Remark) *The longitude of a point on the prime meridian is 0°. The meridian opposite this prime meridian has a longitude of 180°E = 180°W.*

Example 45 *The city of Beirut in Lebanon is located at a longitude of approximately 35°30′E.*

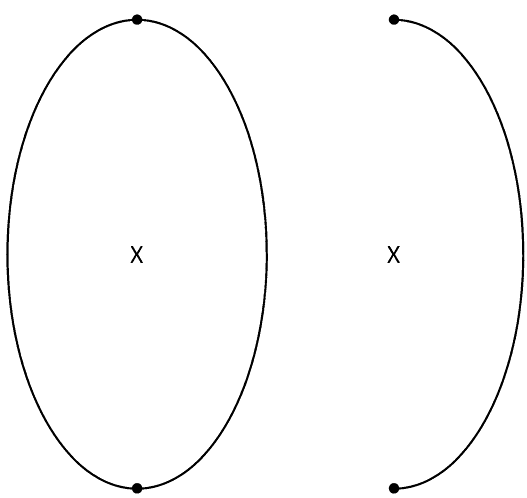


Fig. 4.3 Schematic of an ellipse (left) and a half-ellipse (right)

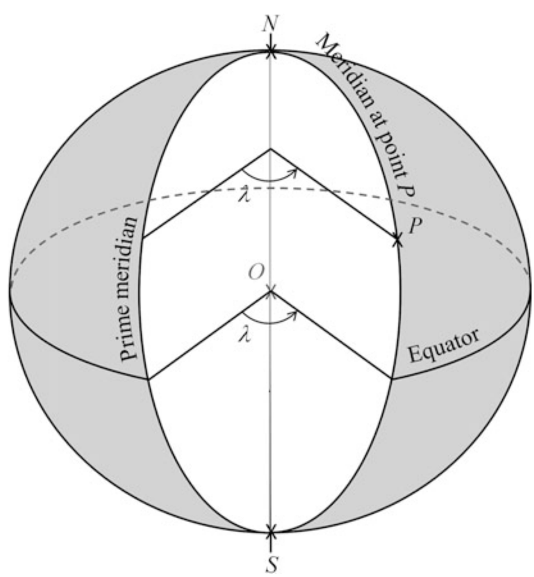


Fig. 4.4 Schematic representation of longitude

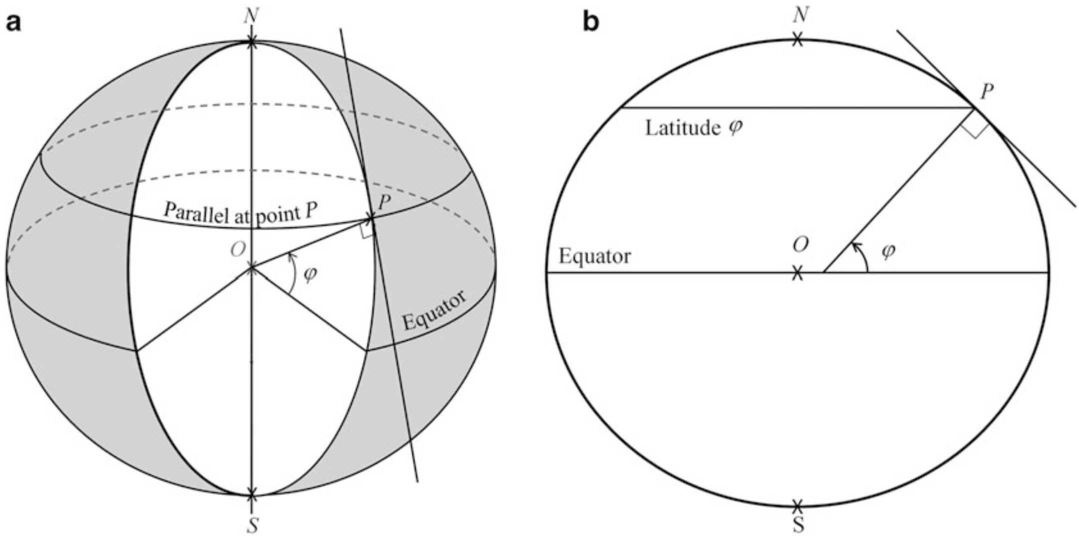


Fig. 4.5 Schematic representation of latitude in false perspective (**a**) and plane view (**b**). The line passing through point *P* represents the plane tangent to the ellipsoid at this point

Definition 17 (Parallel) *A* ***parallel*** *is an imaginary circle parallel to the equator. All points on the same parallel have the same* ***latitude****, expressed in angle units (e.g., degrees). By convention, the equator has a latitude of 0°. The* ***latitude*** *of a point P is defined by measuring the angle φ between the plane of the equator and the line perpendicular to the plane tangent to the ellipsoid at point P (Fig. 4.5). This measure is between 90°S (South) and 90°N (North).*

Note 39 (Remarks)

- *Points with the same latitude are located on the same parallel. In other words, parallels are circles of constant latitude.*
- *Because the Earth is not an exact sphere, the line perpendicular to the ellipsoid does not pass exactly through the center of the Earth.*

Definition 18 (Ellipsoid Height) *The* ***ellipsoid height*** *h of a point Q (also called the elevation above the ellipsoid) is defined by the distance between the measurement point Q and the value 0 at the ellipsoid (the plane tangent to the ellipsoid, along the vertical of the site) (Fig. 4.6).*

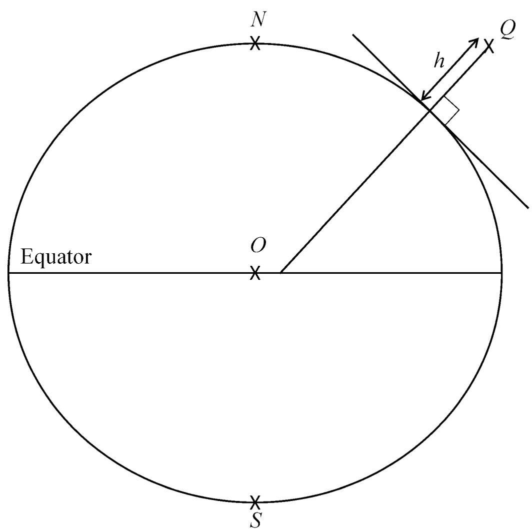


Fig. 4.6 Height *h* at point *Q*

Note 40

- *The Earth is not a perfect ellipsoid because there are large differences in surface relief. We choose a reference surface (usually the mean sea level) whose altitude is 0. The* ***elevation*** *of a point on Earth is the distance between the reference surface and the point under consideration (at the vertical of the site).*

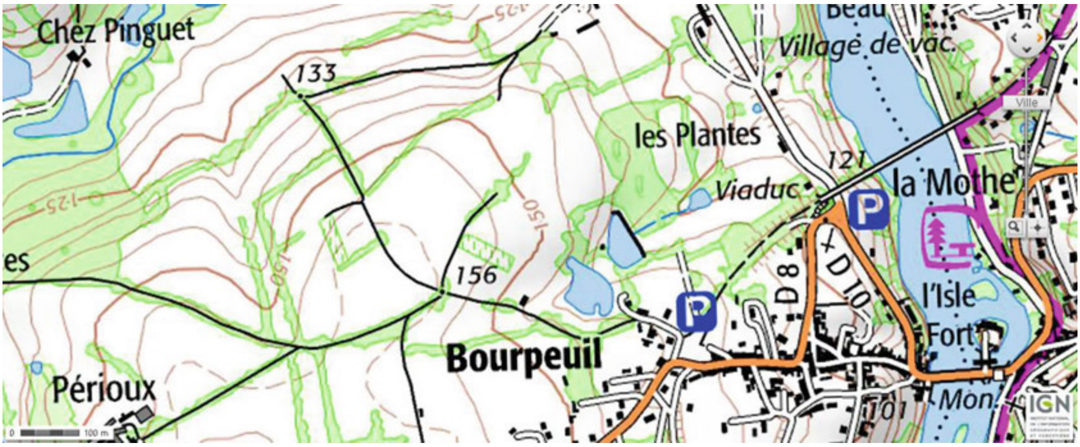


Fig. 4.7 Main contour lines at 125 m and 150 m in Vienne Valley (Isle-Jourdain, France). Extracted from a SCAN25®, © IGN – 2015, Authorization 40-18.29 (Copy is forbidden)

- *Figure 4.7 shows* ***contour lines*** *(or* ***isolines****), which are lines connecting points having the same altitude.*

Insert 8 (Obliquity and Astrology)

Astrology, popular beliefs, and other superstitions hold that human personality traits are allocated according to the position of the Sun on one's day of birth over the north–south axis of the Earth. The various astrological signs were fixed according to the configurations of the second century AD.

But since the signs were established, the constellations have drifted by a whole month and the Earth's axis no longer points in the same direction. For example, if you were born on April 9, 1971, your supposed astrological sign is Aries, and your daily horoscope for Aries might read: "You may have short involvements with attractive strangers. You will be able to convince them easily with your charm and sugar-coated talk." *But in reality, you are Pisces and should read* "Singles, you will tend to put the bar so high that few contenders will find favor in your eyes." *These are obviously not the same! The issue of changing signs comes from the obliquity of Earth. What is it?*

Obliquity somewhat undermines astrological predictions, because if the Sun was on the axis of a certain constellation 1800 years ago, this constellation, and therefore the associated astrological sign, is no longer the same today. How does that work?

In its orbit around the Sun, the Earth is inclined on the plane of the ecliptic of 23.5°, which is the obliquity. That is, the north–south axis is not perpendicular to the plane of rotation of the Earth around the Sun. In addition, the Earth rotates on its axis but not as a perfect sphere. Indeed, the joint attraction of the Moon and the Sun makes this movement equivalent to that of a top. The consequence is that the north–south axis of the Earth does not always point in the same direction but describes a circle whose complete rotation is accomplished in just over 25,700 years. Thus, the polar star, which is by definition the star pointed to by the north–south axis of the Earth, has not always been the same over the course of Earth's history. Currently the polar star of the Northern Hemisphere is Alpha Ursae Minoris *and is located in the constellation of Ursa Minor. However, 12,000 years ago,* Alpha Lyrae *was our polar star.*

4.3 Spherical Geometry

Spherical geometry is the geometry of the two-dimensional surface of a sphere. In this part, the Earth is assimilated to a sphere of radius $R = 6371$ km and of center O.

4.3.1 Geometry of Space

We saw in Chap. 3 that a point in space can be characterized by its coordinates (x, y, z) in a coordinate system or by its spherical coordinates.

It is thus possible to characterize a point M of the Earth using longitude and latitude. The coordinates (ρ, λ, φ) are such that (Fig. 4.8):

- ρ refers to the distance from point M to the center of the Earth O;
- The angle λ is the longitude of this point (expressed in degrees, $-180^\circ \leq \lambda \leq 180^\circ$, or in radians, $-\pi \leq \lambda \leq \pi$);
- The angle φ is the latitude (expressed in degrees, $-90^\circ \leq \varphi \leq 90^\circ$, or in radians, $-\dfrac{\pi}{2} \leq \varphi \leq \dfrac{\pi}{2}$).

These are the spherical coordinates seen in the previous chapter, with different notations. Recall that the link between the Cartesian coordinates and the spherical coordinates of a point $M(x_M, y_M, z_M)$ is given by

$$\begin{cases} x_M &= \rho \cos\lambda \cos\varphi, \\ y_M &= \rho \sin\lambda \cos\varphi, \\ z_M &= \rho \sin\varphi. \end{cases}$$

4.3.2 Elements of Spherical Trigonometry

Consequently, the geometry and trigonometry used are those concerning the sphere. The shortest path between two points along the surface of the Earth (called the great circle distance or **orthodrome** or **geodesic)** is the distance measured along a great circle (that is to say, a circle of the same diameter as the Earth) and not on a straight line. The angles and triangles with which we work must be redefined.

Definition 19 (Property (Calculating the Length of an Arc)) *Let us consider two points A and B of the surface of the Earth, placed on the same great circle with center O (center of the Earth) and radius R. We call α the angle in O in triangle AOB* (Fig. 4.9).

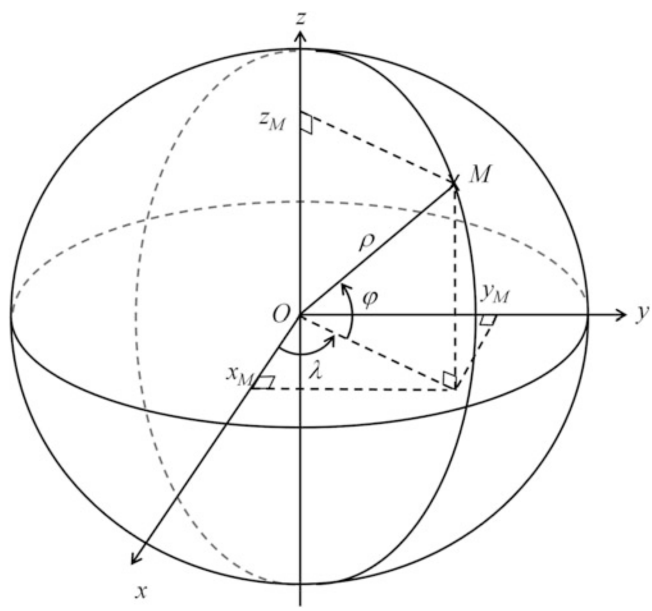


Fig. 4.8 Principle of spherical coordinates. A point M is characterized by its distance to the center O (ρ) and two angles (λ and φ)

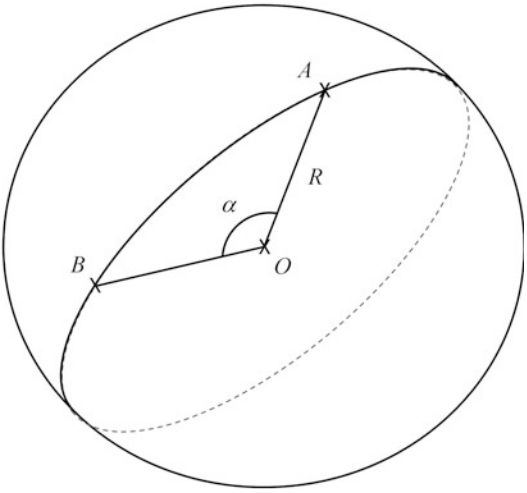


Fig. 4.9 Principles of spherical trigonometry: length of an arc

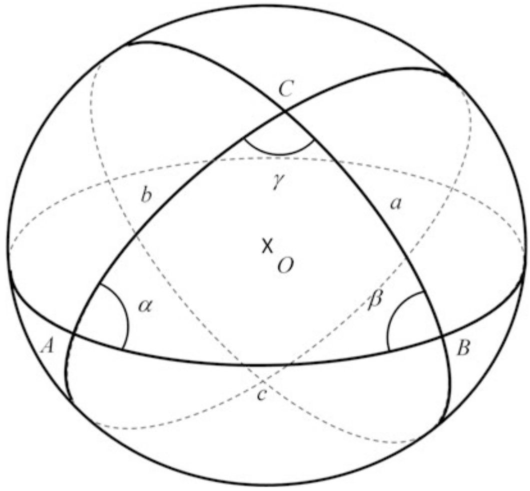


Fig. 4.10 Principles of spherical trigonometry: spherical triangle

If α is expressed in radians, then the length of arc $\widehat{AB}$ is αR.

If α is expressed in degrees, then the length of arc $\widehat{AB}$ is $\alpha R \times \dfrac{\pi}{180}$.

Note 41 (Remarks)

- *The length of an arc α with radius R is proportional to 2πR (with the coefficient of proportionality the same as for switching from α to 2π).*
- *The angle between two points on the globe and the center of the Earth is not easy to define. It uses a different type of angle: the angle between two arcs with center O. This configuration makes it possible to introduce the concept of* ***spherical triangle*** (Fig. 4.10).

Example 46 *The* ***azimuth*** *is the angle between a chosen reference direction* (e.g., *the geographical North) and a direction AB* (Fig. 4.11).

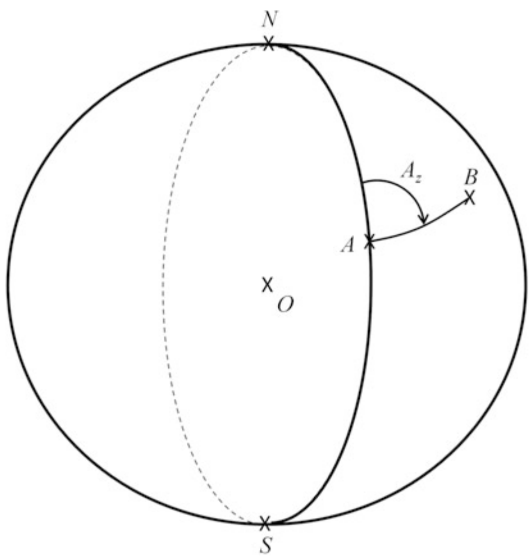


Fig. 4.11 Principles of spherical trigonometry: azimuth

In a spherical triangle (Fig. 4.10), we can use the following **useful results**:

- The sum of the angles of a spherical triangle is always greater than π radians (180°).
- We have the spherical law of cosines: $\cos\gamma = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c$ and, in a similar manner, $\cos c = \cos a\cos b + \sin a\sin b\cos\gamma$.
- The law of sines in spherical trigonometry: $\dfrac{\sin a}{\sin\alpha} = \dfrac{\sin b}{\sin\beta} = \dfrac{\sin c}{\sin\gamma}$.
- The distance $d(X, Y)$ between two points X and Y of the Earth (whose latitudes φ_X and φ_Y are known and with the longitude difference $\Delta\lambda$) is called the **geodesic** or **orthodrome**. Its formula is given by the spherical law of cosines:

$$d(X, Y) = R\cos^{-1}(\sin\varphi_X\sin\varphi_Y + \cos\varphi_X\cos\varphi_Y\cos\Delta\lambda),$$

where the angles are expressed in radians. Be sure that the calculation tool is configured in the same unit as the angles used.

- Same formula for angles expressed in degrees:

$$d(X,Y) = R \times \frac{\pi}{180} \times \cos^{-1}(\sin\varphi_X \sin\varphi_Y + \cos\varphi_X \cos\varphi_Y \cos\Delta\lambda),$$

where the angles are expressed in degrees.

Example 47 *Consider the cities of Detroit (Michigan, USA): point X, 42°20′N, 83°03′W and Toronto (Ontario, Canada): point Y, 43°39′N, 79°23′W. The distance between Detroit and Toronto is*

$$\begin{aligned} d(X,Y) = \quad & 6371 \times \frac{\pi}{180} \times \cos^{-1}(\sin(42.33) \\ & \times \sin(43.65) + \cos(42.33) \\ & \times \cos(43.65) \times \cos(3.67)) \\ \simeq \quad & 333 \text{ km}. \end{aligned}$$

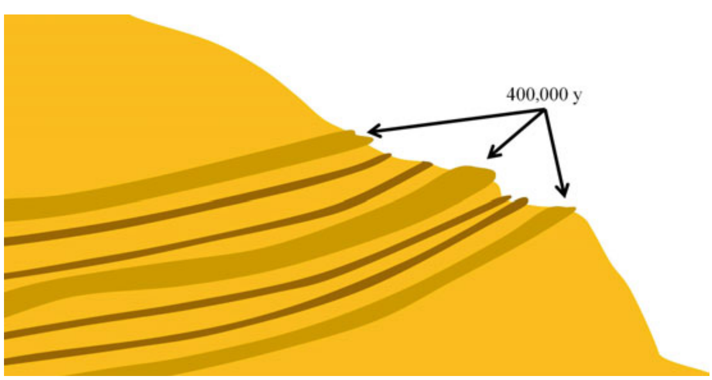
To carry out this calculation, the angles must be converted into decimal degrees. For example, $42^{\circ}20' = 12 + \frac{20^{\circ}}{60} \approx 42.33^{\circ}$, and remember to configure the calculation tool in degrees.

Insert 9 (Why Do Natural Climate Changes Occur in Geological Layers?)
The main cause of natural climate change is the position of the Earth with respect to the Sun. The parameters that characterize this configuration are called Milankovic parameters and it defines cycles of the same name. We present two of these parameters:

– ***Eccentricity*** *is relative to the Earth–Sun distance and varies according to a cycle of about 400,000 years. It has a very important effect on the natural climate: when the Earth is closest to the Sun, it receives 25% more energy than when it is furthest from the Sun.*
– ***Obliquity*** *corresponds to the inclination of the Earth with respect to its plane of rotation around the Sun. This angle is now 23.5° and varies between 22.1° and 24.5° almost every 40,000 years. It influences the seasons very strongly: the more inclined the Earth, the more pronounced the seasons.*

These cycles are imperceptible on the scale of a human life, but they show up in certain sedimentary layers of carbonates. Indeed, the cycle of eccentricity causes the Earth to warm and then cool with a periodicity of about 400,000 years. These variations in the temperature of the atmosphere influence the temperature of the ocean and the solubility of carbon dioxide, the primary element of carbonate rocks that can be found in geological layers. The principle is the same for the obliquity, with a periodicity of 40,000 years (see picture).

(continued)



Thus, there is a direct link between Milankovic cycles and the nature of the carbonates that have settled at the bottom of the oceans and that have ermerged by the tectonic activity. This is why climate change can be read in geological layers.

4.4 Different Mapping Systems

Representing the Earth on a map means connecting a sphere (or an ellipsoid) to a plane: such a mapping is called a **projection.**

As previously stated, it is not possible to represent the Earth on a flat map without losing information. There are many types of projections, some of which retain areas, others retain angles locally.

It is not the purpose of this section or this book to detail the different types of projections. We will only present some of them and emphasize the mathematical tools used. In the cases presented in what follows, the projections of a sphere on a plane are equivalent to projecting a circle on a line.

Definition 20 *In each case described in Fig. 4.12, a point M of the Earth is associated with a point M_i. The relationship $M \mapsto M_i$ defines a projection.*

Four projections are defined:

- $M \mapsto M_1$ *defines an* ***orthogonal projection;***
- $M \mapsto M_2$ *defines an* ***azimuthal projection;***
- $M \mapsto M_3$ *defines a* ***steregraphic projection;***
- $M \mapsto M_4$ *defines a* ***gnomonic projection.***

Note 42 (Remarks)

- *We invite the reader to search for images of the representation of the Earth by each of these processes.*
- *Azimuthal projections retain areas (measurements of surfaces) and give a good representation of a hemisphere.*
- *Stereographical projections, known since Antiquity, preserve angles. Circles of a sphere are represented by circles on the map (except those passing through the North Pole, which are transformed into straight lines).*
- *Gnomonic projections deform figures dramatically and may only represent a hemisphere. But they have the advantage of retaining geodesics – the shortest route on the Earth between two points: geodesics on the sphere become geodesics on the map.*
- *No projection retains the distances of all the points of the map. The notion of scale is therefore not true: there is not a single reduction lengths between the sphere and the map. The*

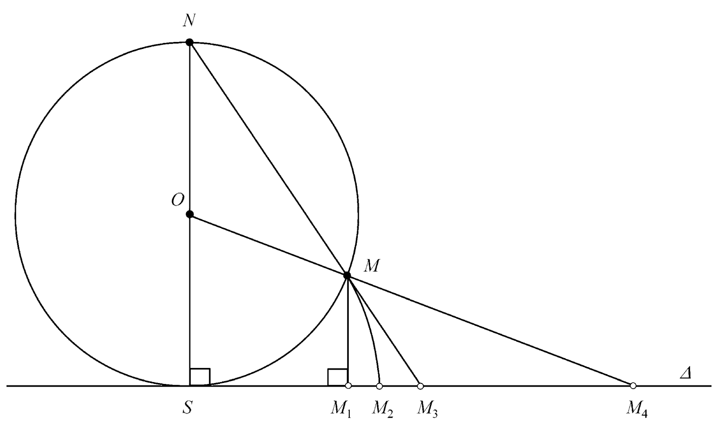


Fig. 4.12 Principle of mapping system (Bourguet 1992)

scale of a map is accurate locally at certain points of the map only.

Key Points

- The scale of a map is the ratio between the length on the map and the length on the ground.
- Imaginary lines around the Earth (equator, meridians, prime meridian, parallels) are used to define the geographical coordinates of a point (latitude and longitude).
- The distance along the globe between two points of the Earth is given by the formula

$$d(X,Y) = R\cos^{-1}(\sin\varphi_X\sin\varphi_Y + \cos\varphi_X\cos\varphi_Y\cos\Delta\lambda),$$

where the angles are expressed in radians, φ_X and φ_Y are the corresponding latitudes of the points, and $\Delta\lambda$ is the difference in longitudes.

The same formula with the angles expressed in degrees is given by

$$d(X,Y) = R \times \frac{\pi}{180} \times \cos^{-1}(\sin\varphi_X\sin\varphi_Y + \cos\varphi_X\cos\varphi_Y\cos\Delta\lambda).$$

- A projection is a mapping between a sphere (or an ellipsoid) and a plane. There are many different projections. None retain all initial properties.

Exercise

Mathematical Exercises

Exercise 4.1: Flash Questions. Series 1

(1) True or false: a large-scale map is used to represent a large territory?
(2) Recall the information defining the longitude of a point.
(3) Calculate the speed of rotation of a point at the equator (use 40,075 km for the circumference of the equator). The answer will be given in km/h and m/s.

Exercise 4.2: Flash Questions. Series 2

(1) Convert 45° into radians.
(2) Calculate the distance between Brussels (50°50′N, 4°21′E) and Quebec (46°48′N, 71°13′W). The formula from the text will be used:

$$d(X,Y) = R \times \frac{\pi}{180} \times \cos^{-1}(\sin\varphi_X\sin\varphi_Y + \cos\varphi_X\cos\varphi_Y\cos\Delta\lambda).$$

Make sure that the calculation tool is configured in the same unit as the angles used.

(3) Recall the formula for the length of an arc α with radius R.

Exercise 4.3: Flash Questions. Series 3

(1) Convert 78°15′ into decimal degrees.
(2) Recall the information defining the latitude of a point.
(3) Give the coordinates of your favorite place. What are the geographical coordinates of its antipode (diametrically opposite place on the globe)?

Exercise 4.4: Equivalence Between Angle, Distance, and Time

In this exercise, we will use a value of 40,000 km for the circumference of the Earth.

(1) A nautical mile is the distance corresponding to 1 min of latitude angle. What is the value of this distance?
(2) Fill in the following table (the location is on the equator):

	°	km	hour	minute	second
1°					
1 km					

Exercise 4.5: Study of Different Projections

We use Fig. 4.12 and assume being in a plane (in two dimensions).

(1) From the observation of the figure, specify how points M_1, M_2, M_3, and M_4 are defined.
(2) For each of the projections, specify:
   - The points of the circle that can be projected;
   - Where the projected points are located;
   - Whether the same projected point can correspond to two different points at the start.
(3) Note α the angle $\widehat{SOM}$ (recall that angle $\widehat{SNM}$ is then $\alpha/2$). Show that points M_1, M_2, M_3, and M_4 are in this order on line Δ.
(4) Calculate the lengths SM_1, SM_2, SM_3, and SM_4 as a function of the variable α (you can use the angle $\beta = \widehat{OSM} = \widehat{OMS}$ and express β in terms of α).

Exercises in Geography and Geology

Exercise 4.6: Scales

(1) A rectangular building measuring 30×50 m is represented on a 1/25,000 map. What will be the dimensions of the rectangle on the map?
(2) On a map of Mali, Africa the straight-line distance between Bamako and Segou is 17.5 cm. In reality, it is 206 km. What is the scale of the map?

Exercise 4.7: Reducing the Scaling

The scale of a map is halved. By how much will the area of a represented object be reduced?

Exercise 4.8: Representing Circles on a Map

On a map, the population of cities is represented by circles (disks) whose area is proportional to the number of inhabitants.

On a map, a town with a population of 250,000 inhabitants is represented by a circle 3 mm in diameter.

(1) What is the diameter of the circle representing a city of 700,000 inhabitants?
(2) What population will be represented by a circle of 2 mm in diameter?

Exercise 4.9: Slope

A hiker climbs along a slope: his starting altitude is 750 m and that of his destination 1275 m. As the crow flies, the distance between departure and arrival points is 3 km.

(1) Represent the situation in a drawing whose scale will be specified.
(2) Calculate the angle of the slope (in degrees and percentage).

Exercise 4.10: Locating a Site on the Globe

(1) Choose a location and place it on the globe (Fig. 4.1) presented in Sect. 4.2.2, "Geographical coordinates," indicating latitudes and longitudes by angles.
(2) Place the geographical coordinates (0°N, 0°E) on the globe.
(3) What are the geographical coordinates of the North Pole

Exercise 4.11: Time Difference

(1) Choose two places in the world and look up their longitude.
(2) Figure out the real time difference between these two locations (independently of the official time zones).

Exercise 4.12: Meridians and Parallels

In this exercise, take the value of 40,075 km as the circumference of the Earth at the equator.

(1) If located on the equator, what is the distance between two meridians separated by an angle of 1°? How does this distance change if one places oneself somewhere else besides the equator?
(2) What is the distance between two parallels separated at an angle of 1°?
(3) How do you measure a meridian arc between latitudes 35°South and 40°South?

Exercise 4.13: Parallels

(1) What is the circumference of the parallel at latitude 60°North?

(2) Recall that the distance between two meridians separated by 1° at the equator is about 111 km. What is this distance at the parallel of latitude 60°North?

Exercise 4.14: Projections and Errors
We want to determine the distance between two major cities, Helsinki (Finland) and Anchorage, Alaska (USA), using three different methods (you will need to research any missing information not provided in the book).

(1) What is the distance between Helsinki and Anchorage as measured on a world map?
(2) What is the distance between Helsinki and Anchorage as measured by Google Earth?
(3) What is the distance between Helsinki and Anchorage according to the spherical trigonometry law of cosines? For this purpose, recall the distance between two points A and B:

$$\begin{aligned} &d(A,B)\\ &= R\cos^{-1}[\sin(L_A)\sin(L_B)\\ &\quad +\cos(L_A)\cos(L_B)\cos(l_A-l_B)] \end{aligned}$$

where L_A, L_B, l_A, and l_B are the latitudes and longitudes of points A and B expressed in radians.
(4) What conclusion can be drawn?

Exercise 4.15: Plate Tectonics
The Eurasian plate is considered to revolve around a point (called the Euler pole) whose geographical coordinates are (50.6°N, 67.6°W) at the rotational angular velocity of 2.3° per million years.

The aim of the exercise is to calculate an approximate value of the linear travel velocity of Paris (France) located on this plate.

(1) Locate the Euler pole on the globe.
(2) Calculate the distance between the pole and Paris (distance along the globe).
(3) Determine the linear velocity Paris displacement in millimeters per year (use the equation speed = distance × angular velocity in degrees$\times\frac{\pi}{180}$).

Solutions

Solution 4.1: Flash Questions. Series 1
(1) It is wrong; a large scale is used to represent a local space.
(2) See the text.
(3) The speed of a point situated at the equator is $\frac{40,075}{24} \approx 1669.8$ km/h, which is $\frac{1670 \times 1000}{3600} \approx 463.83\,\mathrm{m/s}$ (we have used five significant figures, as for 40,075).

Solution 4.2: Flash Questions. Series 2
(1) We have $45^\circ = \frac{\pi}{4}$rad.
(2) We first convert the coordinates to decimal degrees (e.g., 50°50′ = 50 + 50/60° ≈ 50.83°). We have

$$\begin{aligned} &d(Brussels, Quebec) = 6371 \times \frac{\pi}{180} \times \cos^{-1}\\ &[\sin(50.83)\sin(46.8)\\ &\quad +\cos(50.83)\cos(46.8)\cos(75.57)]\\ &\quad \approx 5300\ km. \end{aligned}$$

Note 43 To obtain 75.57, we add $4°21' + 71°13'$ because the cities are located on either side of the prime meridian.

(3) See the text.

Solution 4.3: Flash Questions. Series 3
(1) We have $78^\circ 15' = \left(78 + \frac{15}{60}\right)^\circ = 78,25^\circ$.
(2) See the text.
(3) We choose, for example, Bern (Switzerland, Europe), whose geographical coordinates are 46°57′N and 7°27′E. The point at the antipode has the coordinates 46°57′S and (180 + 7)°27′E = 172°33′W (located east of New Zealand).

Solution 4.4: Angle, Distance, and Time Equivalences
(1) The circumference of the Earth is 40,000 km and $360^\circ = 21,600'$. A minute of an angle has a value of 40,000/21,600 ≈ 1.852 km.

(2) Calculation example: a full rotation around the Earth takes 24 h for 360°. So 1° corresponds to $\frac{24}{360} \approx 0.067\,\mathrm{h}$.

	°	km	hour	minute	second
1°		111	0.067	4	240
1 km	0.009		$6\ 10^{-4}$	0.036	2.16

Solution 4.5: Study of Different Projections

Please refer to Fig. 4.13

(1) Point M_1 is the orthogonal (involving right angles) projected point of M on line Δ (point M_1 is the intersection point of line Δ and the line perpendicular to Δ passing through M).
Point M_2 verifies $SM = SM_2$. In other words, M_2 is the point on the circle of center S and radius SM, which is located in the same half-plane as M with respect to the axis NS.
Point M_3 is the intersection point of lines NM and Δ.
Point M_4 is the intersection point of the half-lines (OM) and of Δ.

(2) For the orthogonal projection (M_1), all the points of the circle have a projection; the projected points are located on a segment of center S and width $2R$ (where R is the radius of the Earth). Each point of this segment (except the two ends) is the projection of two points of the circle.
For the azimuth projection (M_2), all the points of the circle have a projection; the projected points are located on a segment of center S and width $4R$. Each point of this segment is the projection of a single point of the circle.
For the stereographic projection (M_3), all points of the circle have a projection except point N; the projected points describe the whole line Δ. Each point of Δ is the projection from a single point of the circle (except point N, for which there is ambiguity regarding its projection).
For the gnomonic projection (M_4), only the points of the lower semicircle with center O and radius R containing S have a projection; the projected points describe the whole line Δ and each point of Δ is the projection of a single point of this semicircle.

(3) Let r be the length SM. We have $SM_2 = r$. The triangle SMM_1 is a right triangle in M_1, so its longest side is $SM = r$. We have $SM_1 \leq SM_2$.
If we denote by α the angle O in triangle SOM (this is also the angle in M in triangle M_1MM_4), the angle in N in triangle SNM then measures $\alpha/2$ (the angle at the center theorem makes it possible to assess it). α is also the angle in M in the triangle M_1MM_3. In the triangle M_1MM_3 with the right angle at point M_1, the angle at M is smaller than in triangle M_1MM_4, so $M_1M_3 \leq M_1M_4$.
The order between points M_2 and M_3 remains to be seen: points S, M, and N are on the same circle with diameter $[SN]$. Therefore, triangle NSM is a right triangle at M: lines NM and SM are perpendicular. Line NM is therefore tangent to the circle of center S and with radius SM, it is therefore outside this circle and point M_3 is indeed located “after” M_2.

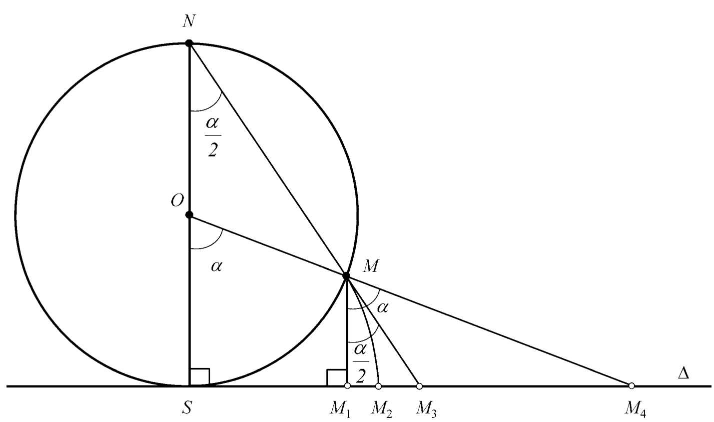


Fig. 4.13 Study of different projections

(4) In triangle OMS_1 with right angle at S_1 (Fig. 4.14), we have $\sin\alpha = \frac{S_1M}{R}$. Thus, $S_1M = SM_1 = R\sin\alpha$.

In triangle OMH with right angle at H (Fig. 4.15), we have $HM = R\sin\left(\frac{\alpha}{2}\right)$. Therefore, $SM = SM_2 = 2R\sin\left(\frac{\alpha}{2}\right)$.

In triangle NSM_3 with right angle at S, we have $SM_3 = 2R\tan\left(\frac{\alpha}{2}\right)$.

In triangle OSM_4 with right angle at S, we have $SM_4 = R\tan\alpha$.

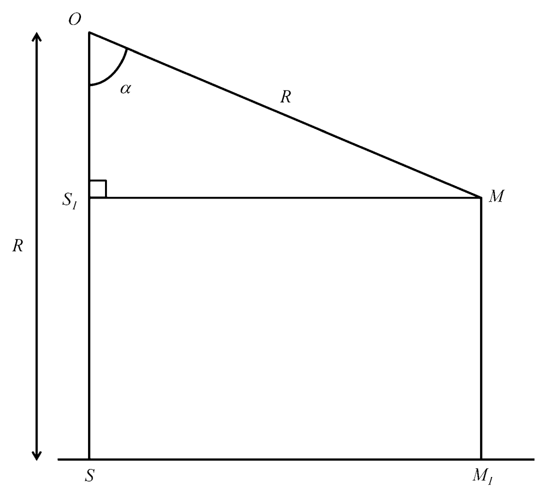


Fig. 4.14 Zoom of orthogonal projection

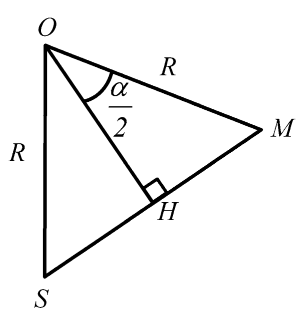


Fig. 4.15 Zoom of projections

Solution 4.6: Scales

(1) In a 1/25,000 scale, 1 cm on the map represents 25,000 cm = 250 m in reality. We have a situation of proportionality:

Distance on map	1 cm	0.12 cm	0.2 cm
Distance in field	250 m	30 m	50 m

The desired dimensions are therefore $\frac{30}{250}$ cm $= 0.12$ cm $= 1.2$ mm and $\frac{50}{250}$ cm $= 0.2$ cm $= 2$ mm. The building is represented by a 1.2 mm × 2 mm rectangle on the map.

Note 44 This calculation does not take into account mapping representation rules that may lead to a representation with dimensions other than those calculated here.

(2) 17.5 cm on the map represents 206 km = 20,600,000 cm. The scale of the map is therefore 17.5/20,600,000 ≈ 1/1,180,000.

Solution 4.7: Reducing Scale

If a distance of 1 cm is x centimeters in reality for the first map, a distance of 1 cm represents $x/2$ cm on the second map whose scale is reduced by half.

A distance of d cm is represented by a distance d/x on the first map and $d/(2x)$ on the second map.

A square of side d cm in reality is represented by a square of area $(d/x)^2$ on the first map and $(d/(2x))^2 = d^2/(4x^2) = (d/x)^2/4$ on the second map.

The area is therefore reduced to a quarter.

Note 45 In Chap. 3 (enlargement and reduction) we saw that when distances are multiplied by k, the areas are multiplied by k^2 and volumes by k^3; here, $k = \frac{1}{2}$ (with $k^2 = ¼$).

Solution 4.8: Representing Circles on a Map

We have the following table (where the first two lines are proportional):

Population	250,000	700,000	
Area (mm)			
Diameter (in mm)	3		5

Using the proportionality of the first two lines and the formula of the area of a disk (area $= \pi \times (\text{diameter}/2)^2$), the following results are obtained:

(1) The area of the disk corresponding to 250,000 inhabitants is $\pi \times (3/2)^2 \approx 7.1\ \text{mm}^2$.
Thus, by proportionality, the area of a disk for 700,000 inhabitants has a value of $\frac{700{,}000 \times 7.1}{250{,}000} \approx 19.9\ \text{mm}^2$ and the corresponding diameter satisfies $(\text{diameter}/2)^2 \approx 19.9/\pi$. Therefore, diameter $= 2 \times \sqrt{19.9/\pi} \approx 5\,\text{mm}$.

(2) The area of a disk 2 mm in diameter is approximately 3.1 mm^2, and this disk represents a population of $250{,}000 \times 3.1/7.1 \approx 110{,}000$ inhabitants.

Solution 4.9: Slope

(1) The situation is represented in Fig. 4.16 by a right triangle, with 1 cm in the drawing corresponding to 500 m (=50,000 cm) in reality. The scale is 1/50,000.
A distance of 3 km is represented by a distance of $3000/500 = 6$ cm and the height difference $1275 - 750 = 525$ m is represented by $525/500 \approx 1.1$ cm (Fig. 4.16).

(2) Calculation of the slope in degrees: let α be the desired angle. We have $\sin \alpha = 525/3000$, so $\alpha = \sin^{-1}(525/3000) \approx 10^{\circ}$.

Calculation of the slope in percentage: we seek the length of the third side using Pythagoras' theorem, which is $\sqrt{3000^2 - 525^2} \approx 2950\,\text{m}$.

The slope is therefore $525/2950 \approx 17.8\%$.

Solution 4.10: Locating Sites on Globe

(1) The geographical coordinates (0°N, 0°E) are off Gabon (Africa).

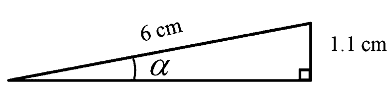


Fig. 4.16 Slope calculation

(2) The North Pole has the geographical coordinates 90°N, 0°E.

Solution 4.11: Time Difference

(1) The cities of Dakar, Senegal, and Ouagadougou in Burkina Faso have the respective longitudes 17°25′ West and 1°31′ West.

(2) The angle difference is $17°25' - 1°31' = 16°85' - 1°31' = 15°54' \approx 15.9°$.
The time difference between Dakar and Ouagadougou is $15.9 \times 24/360 \approx 1.06\,\text{h} \approx 1\,\text{h}$ 04 min.

Note 46 If one of the chosen sites is located in the west and the other in the east, sum the longitudes.

Solution 4.12: Meridians and Parallels

(1) The circumference of the Earth at the equator is 40,075 km and corresponds to 360°. The value of 1° is therefore

$$\begin{aligned} 1^{\circ} &\leftrightarrow \frac{40{,}075}{360}\ \text{km} \\ &\leftrightarrow 111\ \text{km}. \end{aligned}$$

It is clear (in Google Earth, for example, if we show the "grid") that the distance between two meridians is greatest at the equator and decreases when moving toward the poles. At the poles, all meridians converge; the distance between two meridians is zero.

(2) The distance between two parallels separated by an angle of 1° is constant over the entire surface of the Earth. The calculation is the same as in the previous question (if we consider the Earth as a sphere); the required distance is about 111 km.

(3) We proceed as if a meridian were a circular arc and use proportionality, then 360° (circle around the globe) corresponds to a distance of 40,075 km, so 5° corresponds to $\frac{5 \times 40{,}075}{360} \approx 557\,\text{km}$.

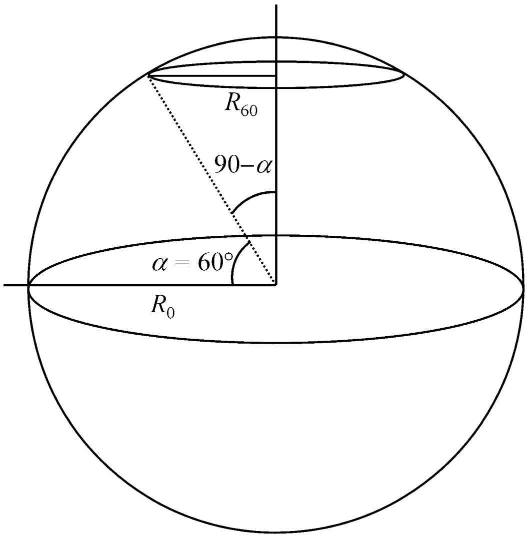


Fig. 4.17 Relationship between parallels at latitudes 60° and 0° (equator)

Solution 4.13: Parallels

(1) In Fig. 4.17, it is seen that the sine of the angle $90 - \alpha$ is the ratio of R_{60} to the radius of the Earth R_0 (dashed). Thus, we have $\sin(90 - \alpha) = \sin(30) = \frac{R_{60}}{R_0}$.
We therefore deduce that $R_{60} = \sin(30)R_0 = \frac{R_0}{2} \approx 3186\,\text{km}$.
The circumference of the parallel at 60° North is therefore $3186 \times 2\pi \approx 20{,}020$ km.

(2) 360° thus corresponds to 20,020 km, so 1° is about 56 km.

Solution 4.14: Projections and Errors

(1) This will depend on the world map and its chosen scale. On Google Maps, we measured 28 cm between the two cities knowing that the scale is 1.9 cm for 1000 km. It therefore follows that the distance given by this world map is $\frac{28 \times 1000}{1.9} \approx 14{,}700\,\text{km}$.

(2) The distance indicated by Google Earth is 6535 km.

(3) To apply a spherical trigonometric relationship, we must first find the longitude and latitude of Anchorage and Helsinki. It is easy to find these data on the Internet:

City	Longitude	Latitude
Anchorage	149°53′57″West	61°13′06″North
Helsinki	24°56′55″East	60°10′24″North

Conversion of sexagesimal into decimal coordinates gives:

City	Longitude (°)	Latitude (°)	Longitude (rad)	Latitude (rad)
Anchorage	149.899	61.218	2.616	1.068
Helsinki	−24.948611	60.17333	−0.435	1.050

We apply the relation (use radian mode on the calculator):

$$\begin{aligned} d(A,B) &= 6\,371 \cos^{-1}[\sin(L_A)\sin(L_B) \\ &\quad + \cos(L_A)\cos(L_B)\cos(l_A - l_B)] \\ &= 6\,371 \cos^{-1}[\sin(1.068)\sin(1.050) \\ &\quad + \cos(1.068)\cos(1.050)\cos(2.616 + 0.435)] \\ &\approx 6\,510\ \text{km}. \end{aligned}$$

(4) The three distances between the same two points are as follows:

	Distance (km)
Calculation on Google Maps	14,737
Calculation on Google Earth	6535
Calculation by spherical trigonometry	6510

The nature of the projection of (Mercator) Google Maps involves errors in calculating distances. These errors are especially important when the locations are far from the equator. We see that for latitudes of about 60° this error is of a factor 2. The error exceeds a factor of 6 beyond a latitude of 80°.

Solution 4.15: Plate Tectonics

(1) The geographical coordinates (50.6°N, 67.6°W) are located in North America (Province of Quebec, Canada). Note that this is not itself on the Eurasian plate.

(2) To calculate the distance between this pole and Paris, we use the formula given in the text (in degrees):

$$d(\text{Euler pole}, \text{Paris}) = 6\,371 \times \frac{\pi}{180} \times \cos^{-1}[\sin(50.6)\sin(48.9) + \cos(50.6)\cos(48.9)\cos(69.9)] \approx 4830\text{ km}.$$

(3) The linear velocity of Paris' displacement on the Eurasian plate is therefore $4830 \times 2.3 \times \frac{\pi}{180}$ km per million years $\approx$ 200 km per million years, about 200 mm/year.

5 Derivation

Abstract

In the phenomena studied in geography and Earth science, we are often interested in variations, in growth. We are familiar with the notion of the slope of a line. How can this notion be adapted for a curve? The slope will not be constant; each point on the curve will be associated with a given slope (valid only locally); we thus define a new function, the derived function. The free software used for the calculations is Xcas.

This chapter introduces the notion of slope (gradient) and derivative for a curve at one point and the derivative function. It shows how to use the basic derivative formulas but also to use free software for the computation of the derivative calculation. It shows how to calculate the equation of a tangent line and to use differentiation to find the variation of a function. Some basics on partial derivative, second derivative, and concavity are also given.

Keywords

Differentiation · Slope and gradient · Tangent of a curve · Derivative formulas · Calculation software · Variation of a function · Partial derivative · Second derivative · Concavity.

Electronic supplementary material The online version of this article (https://doi.org/10.1007/978-3-319-69242-5_5) contains supplementary material, which is available to authorized users.

Aims and Objectives

- To establish a connection between derivation, rate of increase, and tangent to a curve.
- To understand how to manipulate the derivation formulas for common functions in simple cases.
- To understand how to use formal calculation software to calculate derivatives in more complex cases.
- To understand the use of derivations in the study of the variations of a function, in particular in the search for a maximum or a minimum.
- To understand how to calculate partial derivatives.
- To understand how to calculate two-order derivatives and establish a connection with the shape of a curve (concepts of convexity and concavity).

5.1 Slope and Derivative at One Point

In Chap. 2, the concept of the slope of a line was introduced; it was established that if f is a linear function expressed by $f(x) = ax + b$ (with its graphical representation being a line), then the slope a is calculated using the formula

$$a = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

where x_1 and x_2 are two numbers.

C. Fleurant, S. Bodin-Fleurant, *Mathematics for Earth Science and Geography*, Springer Textbooks in Earth Sciences, Geography and Environment, https://doi.org/10.1007/978-3-319-69242-5_5

Consider now a function f reflecting the relationship between the horizontal distance of a landscape x and its altitude $h : h = f(x)$. In what follows, we do not need to know precisely the function f. Let us imagine that its representative curve is that given by Fig. 5.1.

Let us imagine someone walking from the valley and moving from left to right along this curve. As long as the walker has not reached the top of the curve, the walker's height is greater as the distance traveled is great since the function f is increasing (Chap. 2). The change in height depending on the distance is called the **slope** or **gradient.**

Let us denote by Δx the variation in distance and by Δh the variation in height. The gradient is expressed as $gradient = slope = \frac{variation\ in\ height}{variation\ in\ distance} = \frac{\Delta h}{\Delta x}$.

Note 47 (Remarks) *We recognize a formula similar to the "rate of change" formula encountered in Chap. 1.*

- *When the function is increasing, the gradient is positive. When it is decreasing, the gradient negative.*
- *Using the notation of Fig. 5.1, because points A and B are points of the curve representing f when x takes the value x_A, the corresponding height y_A has the values $f(x_A)$. Points A and B have as their coordinates $(x_A, f(x_A))$ and $(x_B, f(x_B))$.*

The slope (or gradient) between points A and B is thus given by $\frac{f(x_B) - f(x_A)}{x_B - x_A}$.

This is consistent with what was seen for lines.

- *The assertions here are, of course, valid for functions of all types of variables. For example, f can express a relationship between a height h and a time t (in this case there is a gradient of time).*

Example 48 *Let f be defined by* $f(x) = 0.03x^2 + 1230$. *The slope between the points of the curve of the corresponding x-coordinate* 0.1 *and* 0.4 *has a value of* $\frac{f(0.4) - f(0.1)}{0.4 - 0.1} = 0.015$.

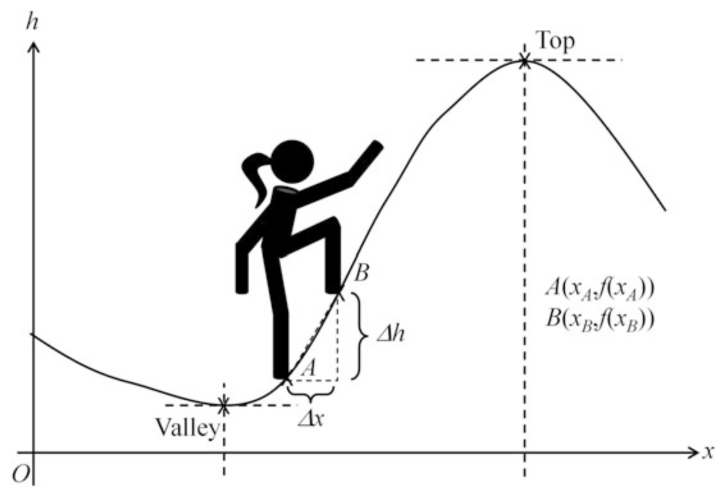


Fig. 5.1 Slope and derivative: the principle

Insert 10 (Fractals – Tribute to Benoit Mandelbrot)

Fractals Everywhere! *This is the title of the book by Barnsley (1993) in which the author shows that our environment is composed of a large number of fractal objects. These objects, whether natural or anthropogenic, are characterized by the complexity of their form. These include, for example, plants, urban structures, and hydrographic networks.*

Science has remained indifferent to shapes for centuries. In the physical sciences, complicated forms have long been reduced to their center of gravity. In mathematics, objects too irregular or too discontinuous were treated as "mathematical monsters." The "noble" domains of the hard sciences such as relativistic physics and quantum physics have always focused primarily on the macrocosm and the microcosm. The forms of geography and geology that belong to intermediate scales have therefore been neglected for centuries.

Faced with this indifference to forms, so-called morphological theories appeared in the 1960s. These are the theories of catastrophes (Rene Thom – 1968), the theory of chaos (initiated by Henri Poincare and formalized by James York and Tie-Yen Li in 1975), the theory of dissipative structures (Ilya Prigogine – 1960), or fractal geometry (Benoit Mandelbrot – 1960).

Benoit Mandelbrot (1924–2010) discovered invariance laws between the short- and long-term price forecast curves while conducting research at IBM on the signal optimization of noisy objects. The genius of Benoit Mandelbrot would be to establish the connection between the forms of very diverse objects, complex by their nature and their scale, and to unite their description in a new geometry: fractal geometry. The word fractal comes from the Latin fractus, *or "broken." This geometry, therefore, has to do with objects whose irregularity makes it impossible to describe them properly using Euclidean geometry (our classical geometry). As Mandelbrot put it, "Clouds are not spheres, mountains are not cones." Fractal objects have very special properties such as self-similarity: one part of an object has the same shape as another on a different scale (Fig. 5.2).*

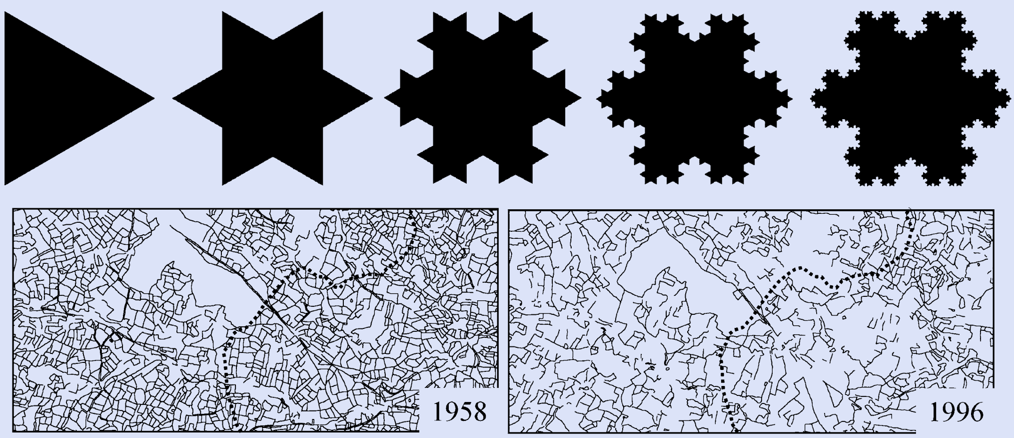


Fig. 5.2 Top: mathematical fractal: the von Koch flake is produced by successive iterations. It is self-similar; its outline has an infinite length, while its surface is finite! Below: natural fractal: aerial view of hedgrow lattice on two different dates (Roland and Fleurant, 2004)

5.2 Derivative Functions and Common Formulas

The preceding section presented a variation in distance and height (possibly small but not zero), as if one were moving along a curve by making jumps. In the case of continuous movement, it would be interesting to know the slope at each point on the curve.

5.2.1 Slope, Derivative Function

To determine the slope (gradient) at a point, we will consider infinitely small variations (Fig. 5.3). Consequently, it will take an infinite number of points to cover the curve to be traveled.

Definition 21 *Let us consider once again the situation* $h = f(x)$*. Infinitely small variations along the x-axis and along the h-axis are denoted respectively by* $\mathrm{d}x$ *and* $\mathrm{d}h$*. The slope of the curve at the point with horizontal coordinate x is defined by slope* $= \frac{\mathrm{d}h}{\mathrm{d}x}$.

For each x a slope is obtained. There is therefore a relationship between x and the slope of the curve at the abscissa point x. This relationship defines a new function from the function f called the ***derivative function*** *of f, denoted by* f' *(read: "f prime").*

Thus, $f'(x) = \frac{\mathrm{d}h}{\mathrm{d}x}$.

We perform ***differentiation*** *to obtain a* ***derivative****.*

Without a practical calculation formula, this definition is of little interest. Let us see using an example how to find f' when we know f.

Example 49 *We choose f expressed in the form* $f(x) = 0.03x^2 + 1230$*, and we will compute* $f'(x)$ *for any value x (we will therefore determine the function* f'*).*

According to Fig. 5.3, when the x-coordinate is equal to $x + \mathrm{d}x$*, the y-coordinate is equal to* $h + \mathrm{d}h$*. Thus, we have by expansion* $h + \mathrm{d}h = f(x + \mathrm{d}x) = 0.03(x + \mathrm{d}x)^2 + 1230 = 0.03(x^2 + 2x\mathrm{d}x + (\mathrm{d}x)^2) + 1230$.

On the one hand, $h = f(x) = 0.03x^2 + 1230$.

On the other hand, $\mathrm{d}h = h + \mathrm{d}h - h = 0.03(x^2 + 2x\mathrm{d}x + (\mathrm{d}x)^2) + 1230 - (0.03x^2 + 1230)$.

By simplifying and grouping, we obtain $\mathrm{d}h = 0.03(2x\mathrm{d}x + (\mathrm{d}x)^2) = 0.06x\mathrm{d}x + 0.03(\mathrm{d}x)^2$.

$\mathrm{d}x$ *expresses a very small quantity,* $(\mathrm{d}x)^2$ *expresses a very small amount of a very small quantity that will be considered negligible (recall*

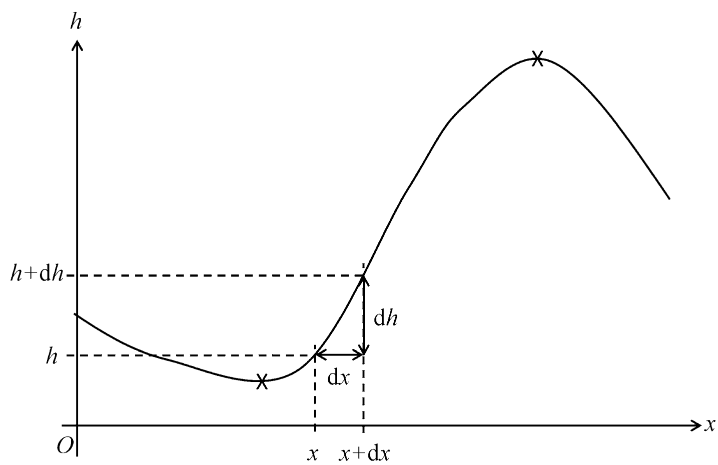


Fig. 5.3 Slope and derivative: calculation

that 0.001 × 0.001 is very small compared to 0.001). So we will do without it. This gives $\mathrm{d}h = 0.06\mathrm{d}x$. *We are almost there, dividing by* $\mathrm{d}x$: $f'(x) = \frac{\mathrm{d}h}{\mathrm{d}x} = 0.06x$.

The derivative function of f is defined by $f'(x) = 0.06x$.

Note 48 (Remarks)

- *When the variable is time, the derived function is sometimes denoted by* $\dot{f}$ *(read: "f overdot") instead of* f' *or* $\frac{\mathrm{d}f}{\mathrm{d}t}$.
- *If f expresses the distance traveled as a function of time t, the derivative* f' *is the speed s (instantaneous speed) and the function* s' *the acceleration. For example, the height z as a function of time t of an object being released without initial speed from a height of 200 m is given by the formula* $z(t) = -\frac{9.81}{2}t^2 + 200$. *The speed function s is therefore expressed by* $s(t) = \frac{\mathrm{d}z}{\mathrm{d}t}(= -9.81t)$ *and the acceleration function a has a value of* $a(t) = s'(t)$ $(= -9.81)$ *(here the acceleration is constant).*

5.2.2 Tangent of a Curve

Let f be a function and M the point of the curve representing f by x-coordinate m (the y-coordinate of this point is therefore $f(m)$). The **tangent** to the curve representing f at point M has for its slope the number $f'(m)$ (Fig. 5.4).

Note 49 (Remarks)

- *Reread all the material on equations of lines from Chap. 2.*
- *The tangent at point M has an equation of the form* $y = f'(m)x + b$, *where b can be calculated using the fact that point M is a point of the tangent (and therefore the coordinates of M verify the equation of the line).*

The tangent to point M has the equation $y = f'(m)(x - m) + f(m)$.

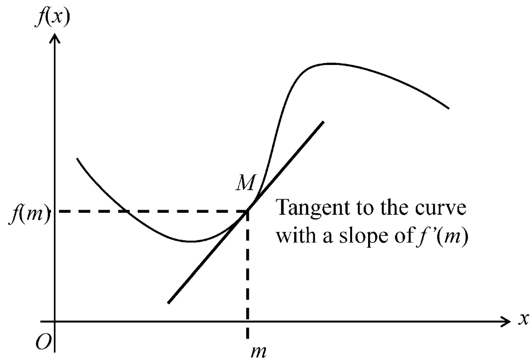


Fig. 5.4 Slope, derivative, and tangent

5.2.3 Derivative Calculations: Use of Derivative Formulas

In practice, hopefully, one does not proceed each time as in the previous example to determine a derivative function! The known formulas of the derivatives of the common functions are used (see Table 5.1).

Note 50 *According to these formulas, the slope of a constant function is zero at any point, and the slope of a linear function written in the form ax + b is a at any point: this is consistent with what was said in Chap. 2.*

Example 50 *Function f from the preceding discussion, verifying* $f(x) = 0.03x^2 + 1230$, *is of the form* $ax^2 + bx + c$, *with* $a = 0.03$, $b = 0$, *and* $c = 1230$. *From the preceding formulas, the derivative* f' *is defined by* $f'(x) = 2 \times a \times x + b = 0.06x$, *which is what we found previously.*

Not all functions are common functions. How does one find the derivative function of f defined by $f(t) = 0.012\exp(-2t) + 1.45\sqrt{t} \times t^2$? The following rules for deriving sums and products of functions are used (see Table 5.2).

Example 51 *For* $f(t) = 0.012\exp(-2t) + 1.45\sqrt{t} \times t^2$, *we use the formula of the derivative of a product and a sum, and we have*

Table 5.1 Table of derivatives of common functions

Function name	Expression	Derivative
Constant	$f(x) = a$	0
Linear	$f(x) = ax + b$	a
	$f(x) = 1.8x + 32$	$f'(x) = 1.8$
Quadratic	$f(x) = ax^2 + bx + c$	$f'(x) = 2ax + b$
	$z(t) = -\frac{1}{2}gt^2 + \frac{\sqrt{2}}{2}v_0 t + z_0$	$z'(t) = -gt + \frac{\sqrt{2}}{2}v_0$
Square root	$f(x) = \sqrt{x}$	$f'(x) = \frac{1}{2\sqrt{x}}$
Exponential	$f(x) = e^x$	$f'(x) = e^x$
	$f(x) = e^{ax}$	$f'(x) = ae^{ax}$
	$a(t) = a_0 \times \exp\left(-\frac{t}{\tau}\right)$	$a'(t) = -\frac{a_0}{\tau} \times \exp\left(-\frac{t}{\tau}\right)$
Logarithm	$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$
Power	$f(x) = x^a$	$f'(x) = ax^{a-1}$
Cosine	$f(x) = \cos x$	$f'(x) = -\sin x$
Sine	$f(x) = \sin x$	$f'(x) = \cos x$
Tangent	$f(x) = \tan x = \frac{\sin x}{\cos x}$	$f'(x) = \frac{1}{\cos^2 x} = 1 + \tan^2 x$

Table 5.2 Derivation formulas

Name	Derivative rule	Example/comment
Multiplication	$(a \times f)' = a \times f'$ where a is a constant.	The derivative undergoes the same operation as the function
Sum	$(f + g)' = f' + g'$	If $f(x) = 0.3x^2 + 2e^{-x}$, then $f'(x) = 0.6x + (-2e^{-x}) = 0.6x - 2e^{-x}$
Product	$(f \times g)' = f' \times g + f \times g'$	If $f(x) = -1.47x \ln x$, then $f'(x) = -1.47 \times 1 \times \ln x + \left(-1.47 \times \underbrace{x \times \frac{1}{x}}_{=1}\right)$ $= -1.47 \ln x - 1.47$ Be careful: the derivative of a product is not the product of the derivatives
Quotient	$\left(\frac{f}{g}\right)' = \frac{f' \times g - f \times g'}{g^2}$	If $f(x) = \frac{\ln x}{x}$, then $f'(x) = \frac{\frac{1}{x} \times x - \ln x \times 1}{x^2} = \frac{1 - \ln x}{x^2}$
Composition (chain rule)	If u is a function, $(f(u))' = u' \times f'(u)$	If $f(v) = H \ln\left(\rho_{atm} \times \frac{v^2}{S}u\right)$, where u is a function and v a variable, f is of the form $H \ln(u)$, then f' is written $H\frac{u'}{u}$ where $u'(v) = \rho_{atm} \times \frac{2v}{S}$, from which $f'(v) = H \times \rho_{atm} \times \frac{2v}{S} \times \frac{1}{\rho_{atm} \times \frac{v^2}{S}} = H \times \frac{2}{v}$

$$
\begin{aligned}
f'(t) &= 0.012 \times (-2\exp(-2t)) \\
&\quad +1.45 \times \left(\frac{1}{2\sqrt{t}} \times t^2 + \sqrt{t} \times 2t\right) \\
&= -0.024\exp(-2t) + 1.45(0.5+2) \\
&\quad \times t \times \sqrt{t} = -0.024\exp(-2t) \\
&\quad +3.625 \times t \times \sqrt{t}.
\end{aligned}
$$

Proof of the derivative formulas for the tangent function: to derive the function $\tan = \dfrac{\sin}{\cos}$, *use the derivation formula* $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$, *with* $f = \sin$, $g = \cos$, $f' = \cos$, $g' = -\sin$. *We thus obtain*

$$
\begin{aligned}
\tan' &= \frac{\cos \times \cos - \sin \times (-\sin)}{\cos^2} \\
&= \frac{\cos^2 + \sin^2}{\cos^2} = \frac{1}{\cos^2}
\end{aligned}
$$

on the one hand; on the other hand, taking up part of the previous calculation yields

$$
\begin{aligned}
\tan' &= \frac{\cos^2 + \sin^2}{\cos^2} \\
&= \frac{\cos^2}{\cos^2} + \frac{\sin^2}{\cos^2} = 1 + \tan^2.
\end{aligned}
$$

5.2.4 Calculation of Derivatives: Use of Calculation Software

The software R makes it possible to perform calculations of derivatives (including with parameters, that is to say, formally).

Example 52 *The command for obtaining the derivative of the function* $f(t) = 0.012\exp(-2t) + 1.45\sqrt{t} \times t^2$ *(see the preceding paragraph) is written*

```
> D (expression (0.012 * exp. (−2 * t) + 1.45 * sqrt (t) * t ^ 2), "t").
```

Note that multiplications are to be written using the symbol "∗", the exponent is preceded by "^", and the command for the root function is sqrt (for square root). The result is

```
1.45 * (0.5 * t ^ -0.5) * t^2 + 1.45 * sqrt(t) * (2 * t) - 0.012 * (exp (−2 * t) * 2).
```

It remains to simplify the result, in particular by using $t^{0.5} = \sqrt{t}$.

Note 51 *There are softwares dedicated to computer algebra systems (CASs) for more intuitive handling and producing more readable results. For example, refer to the Xcas software developed by two French academics. A short presentation, summary of the main commands, and additional links can be found in Appendix B. You can also use the Sage Math software package.*

Example 53 *In the following two examples, Xcas is used to calculate derivatives (Fig. 5.5).*

It is possible to calculate the derivative of a function having one or more parameters; note that the Xcas software treats these parameters as new variables. For example, the derivative of the function defined by $z(t) = -\dfrac{1}{2} \times 9{,}81 \times t^2 + \dfrac{\sqrt{2}}{2} v_0 t + z_0$ *is given in Fig. 5.6.*

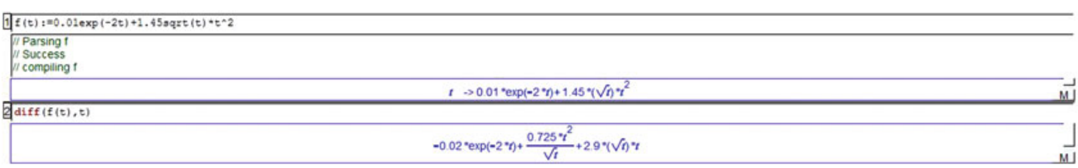


Fig. 5.5 As with R, use *sqrt* for controlling the root function; declare a function using ":="; the command *diff* allows calculation of derivatives

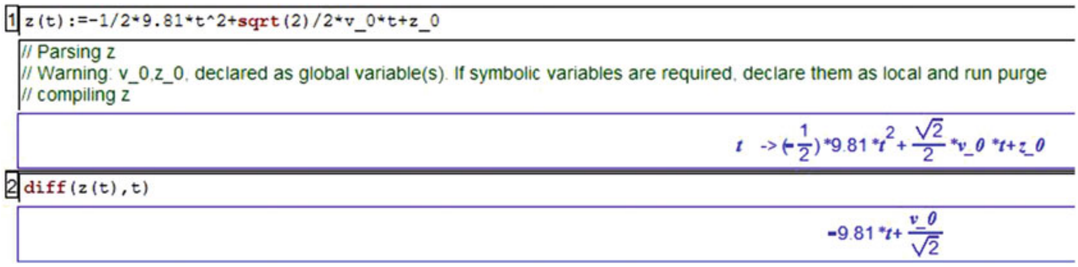


Fig. 5.6 Add the symbol "*" to indicate the multiplication of variables or parameters

5.3 Variations of a Function and Differentiation

For $f(x) = 0.03x^2 + 1230$, recall that $f'(x) = 0.06x$. In Fig. 5.7, it can be seen that the function f is a decreasing function before 0 and increasing after. This illustrates the following result: a positive derivative f' means that the function f is increasing, and a negative derivative f' means that the function f is decreasing.

Property 1 *The minimum and maximum are obtained when f' takes a value of zero: if $f(a)$ is a minimum or maximum for f, then $f'(a) = 0$.*

This leads to the following method for precisely determining a maximum or minimum of a function f when the latter is not at the extreme values of an interval:

- Obtain the derivative of the function f;
- Solve the equation $f'(x) = 0$;
- The solutions of this equation are the sole possible values for which f admits a minimum or a maximum;
- The observation of the graph of the function f lets you know whether it is a maximum or a minimum.

Example 54 *We drop an object at time instant $t = 0$ s, at altitude 300 m, with an initial angle of 45° with the horizontal line, and an initial speed of 20 m/s. The trajectory (altitude as a function of time) of this object is expressed by the function z with $z(t) = -\frac{1}{2}gt^2 + \frac{\sqrt{2}}{2} \times 20t + 300$, where $g \approx 9.81$. Its graphical representation is given in Fig. 5.8.*

The aim is to determine precisely the maximum of this function z.

We have $z'(t) = -gt + \frac{\sqrt{2}}{2} \times 20$. Therefore, the equation $z'(t) = 0$ is written $-9.81t + 14.14 = 0$, which gives $-9.81t = -14.14$, and so $t = \frac{-14.14}{-9.81} \approx 1.44$.

With the graphical representation, we know that we have just obtained the value of z for which the maximum is reached: the maximum height is reached at 1.44 s and is equal to $z(1.44) = -\frac{1}{2}g \times 1.44^2 + \frac{\sqrt{2}}{2} \times 20 \times 1.44 + 300 \approx 310.2$ m (Fig. 5.9).

5.3.1 Partial Derivative

We now consider a function of two variables, for example the altitude z function of the coordinates x and y. An infinitely small variation along the x-axis is no longer dx but ∂x (read: "d round x"): there has been an infinitely small variation along the x-axis, **keeping the value of y constant.**

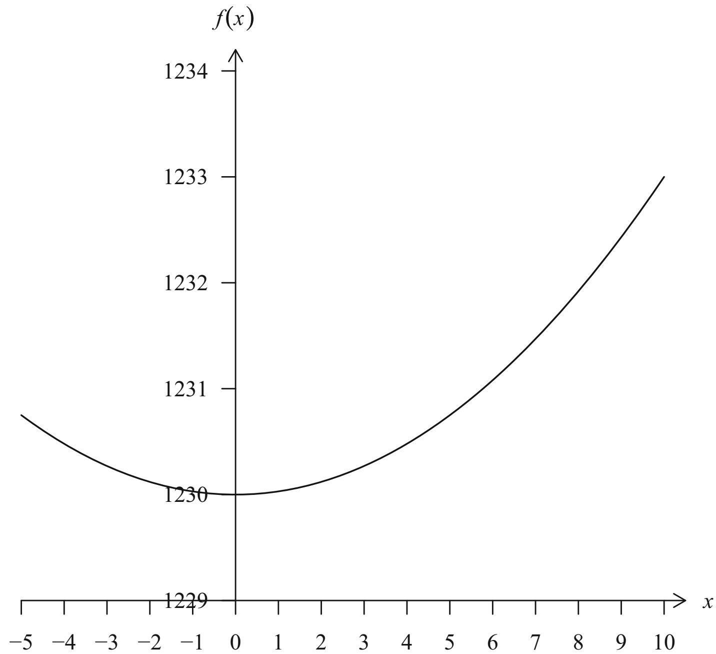


Fig. 5.7 Graphical representation of function f defined by $f(x) = 0.03x^2 + 1230$

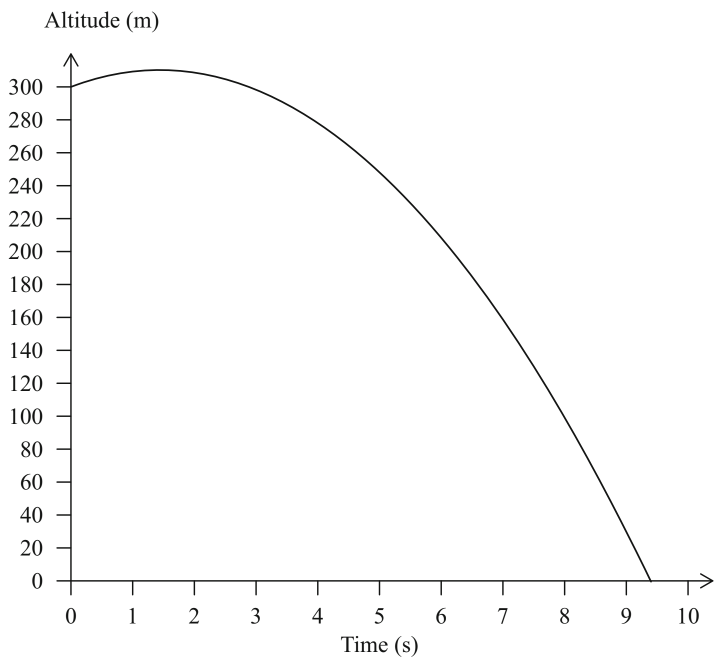


Fig. 5.8 Trajectory of a falling object, dropped at an altitude $z_0 = 300$ m, with an initial speed of $v_0 = 20$ m/s and at an angle of 45° with the horizontal line

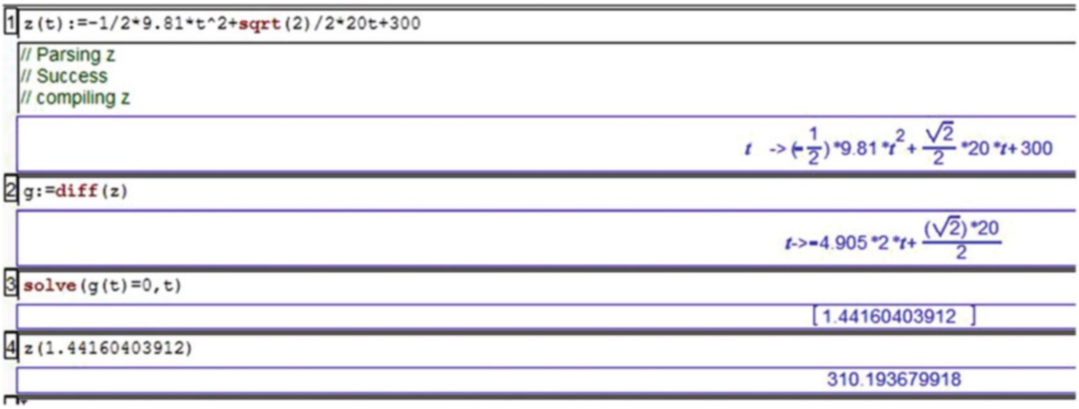


Fig. 5.9 The function *solve* in Xcas allows one to solve equations

Similarly, the slope in the x-direction is denoted by $\frac{\partial z}{\partial x}$ (partial derivative of z with respect to x). The slope in the y-direction is denoted by $\frac{\partial z}{\partial y}$: this is the slope obtained with an infinitely small displacement along the y-axis, keeping the value of x constant.

The partial derivative calculation rules are the same as for the derivatives of a single variable function: to calculate a partial derivative with respect to x of the function $z(x, y)$, we treat y as constant (that is, as a parameter or a number) in the formula and differentiate with respect to x. Similarly, to calculate a partial derivative with respect to y, treat x as a constant in the formula, and differentiate with respect to y.

Example 55 *Let f be the function defined by $f(x, y) = x^2 \exp(-y) + xy$. We have (Fig. 5.10)*

$$\frac{\partial f}{\partial x}(x,y) = 2x\exp(-y) + y$$
$$\text{and}\quad \frac{\partial f}{\partial y}(x,y) = -x^2\exp(-y) + x.$$

The atmospheric pressure P in the vicinity of the surface of the Earth is a function of the two independent variables T, temperature, and z, altitude (these two variables are considered to be independent at small altitudes). This function is expressed by $P(T,z) = P_0\exp\left(-\frac{Mgz}{RT}\right)$, where P_0, M, g, and R are parameters.

The partial derivative $\frac{\partial P}{\partial T}$ expresses the variation of pressure as a function of temperature (at constant altitude), and $\frac{\partial P}{\partial z}$ expresses the variation of pressure as a function of altitude (at constant temperature).

We obtain

$$\frac{\partial P}{\partial T}(T,z) = P_0 \times \left(-\frac{Mgz}{R}\right) \times \left(-\frac{1}{T^2}\right)\exp\left(-\frac{Mgz}{RT}\right)$$
$$= P_0 \times \frac{Mgz}{RT^2}\exp\left(-\frac{Mgz}{RT}\right)$$
$$\text{and}\quad \frac{\partial P}{\partial z}(T,z) = -P_0\frac{Mg}{RT}\exp\left(-\frac{Mgz}{RT}\right).$$

These partial derivatives satisfy the equation $T\frac{\partial P}{\partial T} + z\frac{\partial P}{\partial z} = 0$. We say that P is a solution of ***the partial differential equation*** *$T\frac{\partial P}{\partial T} + z\frac{\partial P}{\partial z} = 0$ (see Chap. 6).*

```
1 f(x,y):=x^2*exp(-y)+x*y
// Parsing f
// Success
// compiling f
                                   (x, y )->x^2*exp(-y)+x*y
2 diff(f(x,y),x)
                                   2*x*exp(-y)+y
3 diff(f(x,y),y)
                                   -x^2*exp(-y)+x
```

Fig. 5.10 This is again the command *diff*, which is used to specify the variable for differentiation

5.3.2 Second Derivative and Concavity

Figure 5.11 shows an increasing function f; the derivative function f' is therefore positive. We see that the slopes are decreasing (increasingly smaller), then increasing; f' is therefore decreasing and increasing. Therefore, the derivative of f' is negative and then positive. We denote by f'' (read "double prime") the derivative function of the derivative **function**, called the **second derivative.** It is also denoted by $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$.

When the derivative is decreasing, the function f is **concave down – or concave –** (we then have $f''(x) \leq 0$). When the derivative is increasing, the function f is **concave up – or convex** (then $f''(x) \geq 0$). The change in concavity (or the inflection change) is called an **inflection point** then one has $f''(x) = 0$.

Example 56 *Acceleration is the second derivative of the distance. For example, if* $z(t) = -\frac{1}{2}gt^2 + \frac{\sqrt{2}}{2}v_0 t + z_0$, *then the acceleration is* $a(t) = z''(t) = -g$.

The cube function has a point of inflection at the origin (Fig. 5.12). Indeed, if $f(x) = x^3$, *then* $f'(x) = 3x^2$ *and* $f''(x) = 6x$, *which has a value of zero and changes sign for* $x = 0$. *The point of coordinates* (0, 0) *is the inflection point of the curve.*

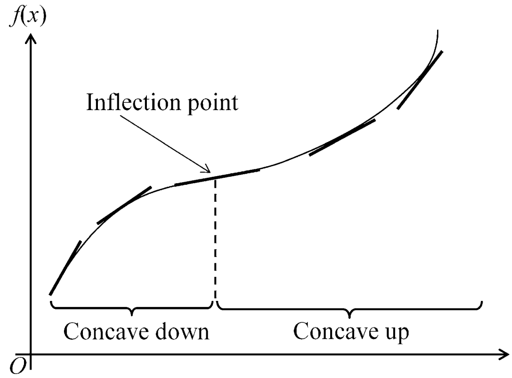


Fig. 5.11 Inflection point, curvature

For a function of two variables, $f(x, y)$, the second-order partial derivatives are $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial y \partial x}$, and $\frac{\partial^2 f}{\partial y \partial x}$.

Note 52 *The second derivative* $\frac{\partial^2 f}{\partial y \partial x}$ *is obtained by first differentiating f with respect to the variable x (with y constant): we get* $\frac{\partial f}{\partial x}$. *Then differentiate* $\frac{\partial f}{\partial x}$ *with respect to y with x constant.*

In all common situations, one obtains in the same way $\frac{\partial^2 f}{\partial y \partial x}$ *by first differentiating f with respect to y, then the result with respect to x. That is to say, we have* $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

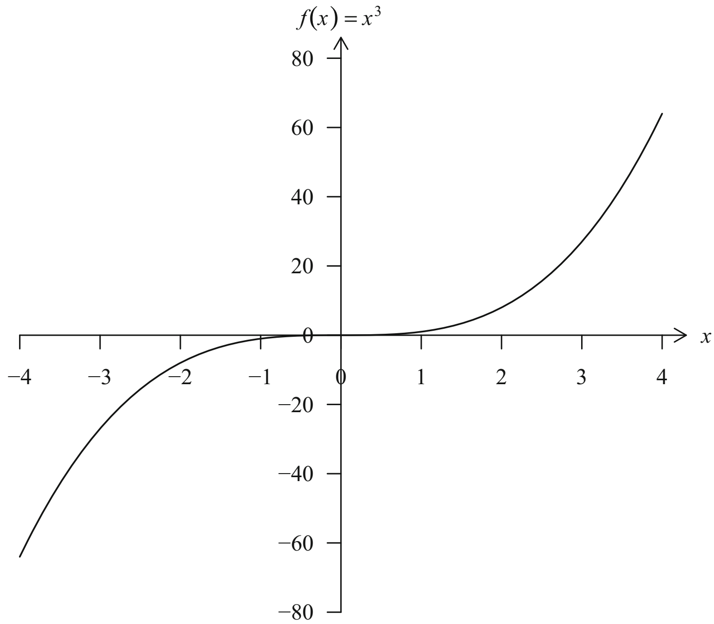


Fig. 5.12 Graphical representation of function f defined by $f(x) = x^3$

Example 57 *Let us take again the function f defined by* $f(x, y) = x^2 \exp(-y) + xy$.

We have seen that $\dfrac{\partial f}{\partial x}(x, y) = 2x\exp(-y) + y$,

and so $\dfrac{\partial^2 f}{\partial y \partial x} = -2x\exp(-y) + 1$.

On the other hand,

$$\frac{\partial f}{\partial y}(x, y) = -x^2\exp(-y) + x,$$

and so $\dfrac{\partial^2 f}{\partial x \partial y} = -2x\exp(-y) + 1$.

We indeed have the equality $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$.

A digital elevation model (DEM) is a numerical representation of topography. To each coordinate point (x, y) *we associate its altitude z; the shape of the relief is thus known. DEM is used very frequently in geography and Earth science to visualize relief in 3D or to analyze the morphology of natural structures.*

Each point (x, y) *can be defined as the center of a mesh of dimensions* $\mathrm{d}x \times \mathrm{d}y$. *The central mesh therefore has eight neighbors, and one can determine the gradient:*

$$g_{DTM} = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}.$$

Key Points

- The slope between two points A and B of the curve of a function f is given by $\dfrac{f(x_B) - f(x_A)}{x_B - x_A}$.
- The line tangent to the curve of f at the point with x-coordinate m has for a slope $f'(m)$ (with f' the derivative function of f).
- The derivative formulas make it possible to calculate derivatives. These calculations can also be performed using the Xcas software (formal calculation).
- For a function f, the study of the sign of the derivative f' allows one to know the variations of f (and in particular the maximum and minimum). When the derivative f' is positive, the function f is increasing; when the derivative f' is negative, the function f is decreasing.

– For a function of several variables x and y, the partial derivative with respect to x is obtained by considering the function as a single variable (x) with y constant.
– When the second derivative of a function is positive, this function is concave up (convex); when the second derivative is negative, the function is concave down (concave).

Exercises

Mathematical Exercises

Exercise 5.1: Flash Questions. Series 1

(1) Why is the derivative of a constant function zero (that is to say, is constantly 0)?
(2) What is the derivative of the function defined by $f(x) = 4.5x - 7.0$? And the derivative of the function g defined by $g(t) = 20.3 \exp(-1.1t)$?
(3) What is the derivative of the function defined by $z(t) = 3.6 \ln(2.4t^2 - 5.3)$?
(4) Find the maximum of the function defined by $f(x) = -x^2 + 4.5x + 12.4$.

Exercise 5.2: Flash Questions. Series 2

(1) How do you calculate the slope of a line?
(2) We have $h(t) = -4.9t^2 + 25.6t + 150.3$. Calculate $\frac{\mathrm{d}h}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2h}{\mathrm{d}t^2}$.
(3) Let f be the function defined by $f(t) = A\exp\left(-\frac{t}{b}\right)$. Calculate the derivative of f (depending on the parameters A and b).
(4) We consider the function of two variables $f(x, y) = \cos(x)\sin(y)$. Calculate the partial derivative of f with respect to the variable x, then with respect to the variable y.

Exercise 5.3: Flash Questions. Series 3

(1) What is the equation of the line tangent to the curve of a function f at a point of the x-coordinate m?
(2) What is the derivative of the function defined by $f(x) = 3.5x^3$? And function g defined by $g(y) = \ln y$?
(3) Let f be a function defined by $f(x) = 2x^3 - 7x^2 + 8x + 5$. Show that the representative curve of f has an inflection point whose coordinates will be determined.
(4) Let T be a function defined by $T(x, y, z) = ax^2 + bxy + cyz + \mathrm{d}x$. Calculate the partial derivatives with respect to the variables x, y, and z.

Exercise 5.4: Derivative Calculations, Level 1

It is essential to be comfortable with the basic calculations of derivatives. There are many questions in this exercise. It is important to work on at least some of them.

(1) We have $f(x) = \sin x$ and $g(x) = 3\cos x$. Calculate $f'(x)$ and $g'(x)$.
(2) We have $g(t) = 3.5t^2 - 4t + 12.1$ and $h(t) = 8\sqrt{t}$. Calculate $g'(t)$ and $h'(t)$.
(3) We have $f(x) = \frac{2}{x}$ and $g(x) = 4x^3 - \ln x$. Calculate $f'(x)$ and $g'(x)$.
(4) We have $h(t) = \exp(-3t)$ and $T(z) = 4.7z - 54$. Calculate $\frac{\mathrm{d}h}{\mathrm{d}t}$ and $\frac{\mathrm{d}T}{\mathrm{d}z}$.
(5) We have $f(x) = a\cos x$ and $g(x) = ax^3 + b\ln x$. Calculate $f'(x)$ and $g'(x)$.
(6) We give $h(x, y, z) = \sin x + 3\cos y + 2\sqrt{z}$. Calculate $\frac{\partial h}{\partial x}$, $\frac{\partial h}{\partial y}$, and $\frac{\partial h}{\partial z}$.
(7) We give $v(u) = u - 2\exp(2u) + 5c$. Calculate $v'(u)$ and $v''(u)$.

Exercise 5.5: Derivative Calculations, Level 2

For those who wish to be able to control the consistency of derivative calculations given by a formal calculation software.

(1) $f(x) = \exp(-x^2)$ and $g(x) = \frac{\cos x}{\sin x}$. Calculate $f'(x)$ and $g'(x)$.
(2) We have $g(t) = 3\sin^2(t)$ and $h(t) = 5t + \frac{8}{3}\ln\left(3t^2 + 1\right)$. Calculate $g'(t)$ and $h'(t)$.
(3) We have $f(x) = \sin x \cos x$ and $g(x) = \sqrt{\sin x}$. Calculate $f'(x)$ and $g'(x)$.

(4) $h(t) = \cos\left(\sqrt{t}\right)$ and $T(z) = 1013.25\left(\dfrac{295 - 7.1z}{295}\right)^{4.34}$. Calculate $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}T}{\mathrm{d}z}$.

(5) We have $f(x) = kg_0 \sin(kx + \phi)$ and $g(x) = \sqrt{1 - a^2 \cos x}$. Calculate $f'(x)$ and $g'(x)$.

(6) $h(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Calculate $\dfrac{\partial h}{\partial x}$, $\dfrac{\partial h}{\partial y}$, and $\dfrac{\partial h}{\partial z}$.

(7) $v(u) = A \exp(-bu^2)$. Calculate $v'(u)$ and $v''(u)$.

Exercise 5.6: Derivative Calculations, Level 3
For those who wish to be able to complete successfully any derivative calculation.

(1) $f(x) = \dfrac{\exp(3x)}{x^2}$ and $g(x) = \dfrac{1 + x^2}{1 - x^2}$. Calculate $f'(x)$ and $g'(x)$.

(2) We have $g(t) = A \ln\left(\dfrac{3 + t^2}{2 + t^2}\right)$ and $h(t) = t\sqrt{1 + 3t^2}$. Calculate $g'(t)$ and $h'(t)$.

(3) We have $f(x) = \sin(ax)\cos(bx)$ and $g(x) = (\sin x + bx^4)^3$. Calculate $f'(x)$ and $g'(x)$.

(4) $h(t) = \cos\left(a\sqrt{t + b}\right)$ and $T(z) = P_0\left(\dfrac{b - az}{b}\right)^k$. Calculate $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}T}{\mathrm{d}z}$.

(5) We have $f(a) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\dfrac{(a - m)^2}{2\sigma^2}\right)$ and $g(x) = \sqrt{1 - a^2 \cos x}$. Calculate $f'(a)$ and $g'(x)$.

(6) $T(z, t) = \dfrac{1 - \exp(-t_0 z)}{1 - \exp(-tz)}$. Calculate $\dfrac{\partial T}{\partial z}$ and $\dfrac{\partial T}{\partial t}$.

(7) $v(u) = A \sin(u + \omega u^2)$. Calculate $v'(u)$ and $v''(u)$.

Exercise 5.7: Tangents Everywhere

(1) Let f be a function defined by $f(x) = x^3$. Draw the representative curve of f, then determine an equation of the tangent line to the latter at the point of the x-coordinate 2 and trace this tangent.

(2) Read in the graph (Fig. 5.13) the derivative numbers at the points of the x-coordinate 0, 1 and 2.

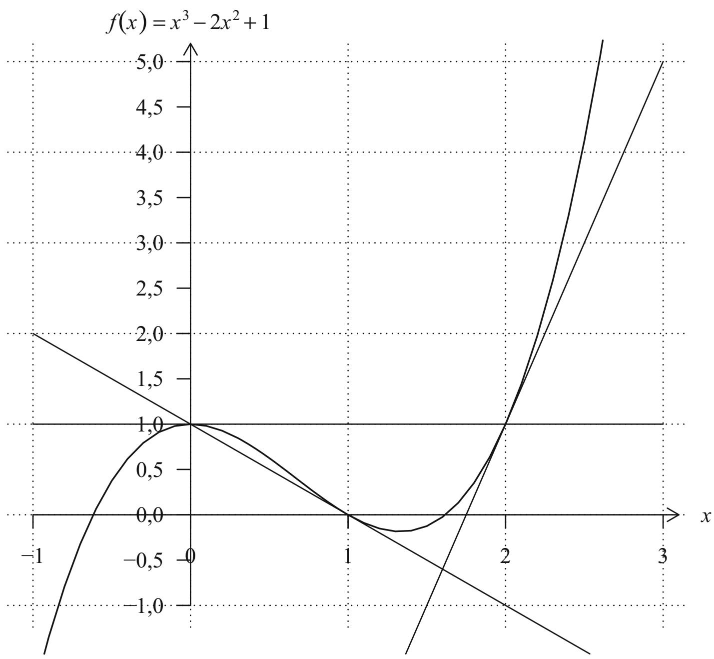


Fig. 5.13 Graphic of a function and some tangents

Exercise 5.8: Parabola
Let z be a function defined by $z(t) = at^2 + bt + c$, where a, b, and c are parameters, with a nonzero. Show that z admits a minimum or a maximum (it is then said that z admits an **extremum**), and provide its coordinates.

Exercise 5.9: Cubic Function
f is a function defined by $f(x) = x^3 + 16x^2 - 195x + 20$ on the domain $[-20; +10]$.

(1) Find the coordinates of the maximum and minimum of f.
(2) Find the coordinates of point(s) of inflection of the curve representing f.

Exercise 5.10: Maximum
We use an exercise already discussed in Chap. 2: find, with the help of formal calculation software, the approximate values of the coordinates of the maximum of the function f defined by $f(x) = \dfrac{\ln(x-40)}{x}$ (use the function *fsolve*(equation, x = *guess*) of Xcas for an approximate solution).

Exercise 5.11: Inflection Points
f is a function defined by $f(x) = \exp(-x^2)$. Determine the coordinates of the inflection points of the curve representative of f.

Exercise 5.12: Partial Derivatives
In each of the following cases, calculate $\dfrac{\partial^2 T}{\partial x^2} - \dfrac{\partial^2 T}{\partial y^2}$.

(1) $T(x, y) = x^2 + \exp(3y)$.
(2) $T(x, y) = \cos(x - y)$.
(3) $T(x, y) = A\dfrac{x}{y}$, where A is a parameter.

Exercises in Geography and Earth Science

Exercise 5.13: Gravity Formulas
As a first approximation, the gravity of Earth has a value of $g = 9.81$ m/s^2. But in fact, gravity varies according to latitude. Several formulas give the expression of gravity g as a function of latitude ϕ. In the following two formulas, give the expression $\dfrac{\mathrm{d}g}{\mathrm{d}\phi}$:

(1) $g = g_0[1 + a\sin^2(\phi) + b\sin^4(\phi) + c\sin^6(\phi) + d\sin^8(\phi)]$, where g_0, a, b, c, d are constants (parameters).
(2) $g = g_e \times \dfrac{1 + k\sin^2(\phi)}{\sqrt{1 - e^2\sin^2(\phi)}}$, where g_e, k, e are parameters (use formal calculation software for this question).

Exercise 5.14: Free Fall
A child leaning on a balcony (situated at altitude z_0) throws a ball up in the air vertically. The ball goes up, then comes down to the ground (altitude 0). The altitude of the ball in meters, depending on the time t in seconds elapsed since the throw, is expressed as $z(t) = -4.9t^2 + 8t + 5$.

(1) What is the original height? What is the maximum height reached?
(2) What is the speed when the object reaches the ground? What is the speed of the ball when it reaches 8 m on the ascent?
(3) What is the acceleration of the ball when it hits the ground? What is the acceleration when it reaches 8 m on the ascent?

Exercise 5.15: Heat Received on Earth
The amount of solar heat received on Earth as a function of the day of the year can be computed by $E(t) = C - C_0\cos\left[\dfrac{2\pi}{365.25}(t - 2.72)\right]$, where E is in Joules/day, t in days, C and C_0 are constants, and the angles are expressed in radians.

(1) On what days of the year is the energy received the most important?
(2) What time of year does the heat increase most quickly?

Exercise 5.16: Temperature Gradient
We take the equation, given in Chap. 2, of the evolution of temperature $T(z)$ (°C) as a function of altitude z (km) in the troposphere:

$$T(z) = T_0 - 6.5z.$$

(1) Calculate the temperature gradient.
(2) What is the significance of this gradient?

Exercise 5.17: Pressure Gradient

We take the equation, given in Chap. 2, of the evolution of atmospheric pressure $P(z)$ (hPa) as a function of altitude z (km) in the troposphere:

$$P(z) = 1013.25\left(\frac{288 - 6.5z}{288}\right)^{5.255}.$$

(1) Calculate the pressure gradient.
(2) With R, trace the evolution of this gradient with altitude z in km ($0 < z < 12$).
(3) What is the significance of this curve?

Exercise 5.18: Ideal Gas Law

The ideal gas law is written $PV = nRT$, where n and R are constants and V, T, and P are variables (volume, temperature, pressure). Show that $\frac{\partial V}{\partial T} \times \frac{\partial T}{\partial P} \times \frac{\partial P}{\partial V} = -1$.

Exercise 5.19: Van Der Waals Equation

The van der Waals state equation for gases (for one mole of gas) can be written $\left(P + \frac{a}{V^2}\right)(V - b) = RT$, where P is the pressure, V the volume, and T the temperature (these are variables), and a and b are constants (parameters, respectively cohesion pressure and covolume).

There is a critical temperature, T_c, such that the liquid state can only be obtained for temperatures below this critical temperature T_c for a critical volume V_c.

(1) Express P in terms of V and T.
(2) Calculate T_c and V_c knowing that one has $\left[\frac{\partial P}{\partial V}\right]_{T=T_c, V=V_c} = 0$ and $\left[\frac{\partial^2 P}{\partial V^2}\right]_{T=T_c, V=V_c} = 0.$
(3) Deduce the value of the critical pressure P_c.

Exercise 5.20: Soil Temperature

The temperature of a soil varies with depth x considered (in meters) and the time t of day (in hours) according to the function $T(x, t) = T_0 \exp(-\lambda x) \sin(24t - \lambda x)$, where T_0 and λ are parameters.

(1) Compute the partial derivatives $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial t}$. How can these derivatives be interpreted?
(2) Show that function T is a solution of a partial differential equation of the form $\frac{\partial T}{\partial t}(x, t) = c\frac{\partial^2 T}{\partial x^2}(x, t)$, where c is a constant. This equation is the heat equation.

Exercise 5.21: Drinking Water Resource

A coastal aquifer (Fig. 5.14) has a surface area of $5 \times 10\ \text{km}^2$ and has an average thickness of $e = 80$ m. This aquifer is supplied by precipitation that flows into the ocean, once it has infiltrated the aquifer, with a slope of about $\beta = 1\%$. It is desired to capture this freshwater through a well with a flow rate q, located at a distance d from the shoreline. This type of well leads to freshwater pollution by salt water from the ocean. Indeed, if the flow rate q of the well is too high, then one will "pump" freshwater from the aquifer but also the salt water from the ocean. It is shown that the condition for collecting only freshwater is $\left.\frac{\partial h}{\partial y}\right|_{\substack{x=0\\y=0}} > 0$, with $h(x, y)$ the height of the water in the aquifer, given by the equation

$$h(x, y) = \beta y - \frac{q}{4\pi T}\ln\left[\frac{(y + d)^2 + x^2}{(y - d)^2 + x^2}\right].$$

(1) What should the threshold value of the flow rate be to capture only freshwater? Useful information: $d = 300$ m, T the transmissivity of the aquifer is $T = k \times e$ (m^2/s), the permeability $k = 1.6 \cdot 10^{-4}$m/s.
(2) Using R, view in two dimensions the values of the hydraulic head $h(x, y)$. Consider an extension zone of 400×400 m (the x-axis

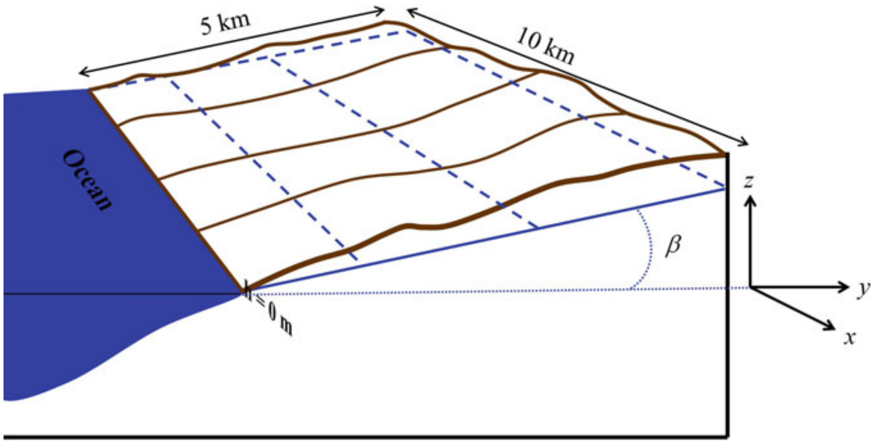


Fig. 5.14 In this coastal aquifer, the freshwater naturally flows to the ocean

ranges from −200 to +200 m and the y-axis from 0 to 400 m). Use, for example, the functions `image()` and `contour()`.

Exercise 5.22: Pollution of an Aquifer

We consider the case of an aquifer that is polluted by an agricultural source (Fig. 5.15). A well of drinking water with flow rate q in this aquifer can be "pumped". It is shown that the velocity vector of the pollutant $\vec{U}\left(u_x, u_y, u_z\right)$ in the aquifer has as its coordinates

$$u_x = -\frac{k}{\omega}\frac{\partial h}{\partial x}(x, y),$$
$$u_y = -\frac{k}{\omega}\frac{\partial h}{\partial y}(x, y),$$
$$u_z = -\frac{k}{\omega}\frac{\partial h}{\partial z}(x, y),$$

with porosity $\omega = 15\%$ and permeability $k = 10^{-4}$ m/s. The height of the water in the aquifer is expressed by

$$h(x, y) = \beta y - \frac{q}{4\pi T}\ln\left[\frac{(y+d)^2 + x^2}{(y-d)^2 + x^2}\right],$$

where q is the flow rate (m^3/s), T the transmissivity of the aquifer (m^2/s), and d the distance from the borehole to the river (m) (in this case, $(0, d)$). The transmissivity of the aquifer is equal to $T = k \times e$, where k is the permeability (m/s) and e the average thickness of the aquifer (m).

(1) At which average speed (give speed in m/s and in cm/day) will the pollutant flow in the aquifer, knowing that:
- this average speed is defined by $u = \sqrt{u_x^2 + u_y^2 + u_z^2}$;
- the well flow rate is $q = 0$ m^3/s?

(2) If the pollution zone is located at a distance of 500 m from the river, how long will it take the pollutant to reach the river (give the result in years)?

(3) We consider the case of a well with a flow rate q not equal to zero and located at a distance d along the y-axis. Deduce the expression of the coordinates u_x, u_y, and u_z of the average speed of the pollutant.

(4) Using R, plot the values of the average speed $u = \sqrt{u_x^2 + u_y^2 + u_z^2}$. Plot this along the y-axis, between the borehole and the river, and for $x = 0$ m. The average thickness of the aquifer is $e = 80$ m. The drilling takes place at a distance $d = 200$ m.

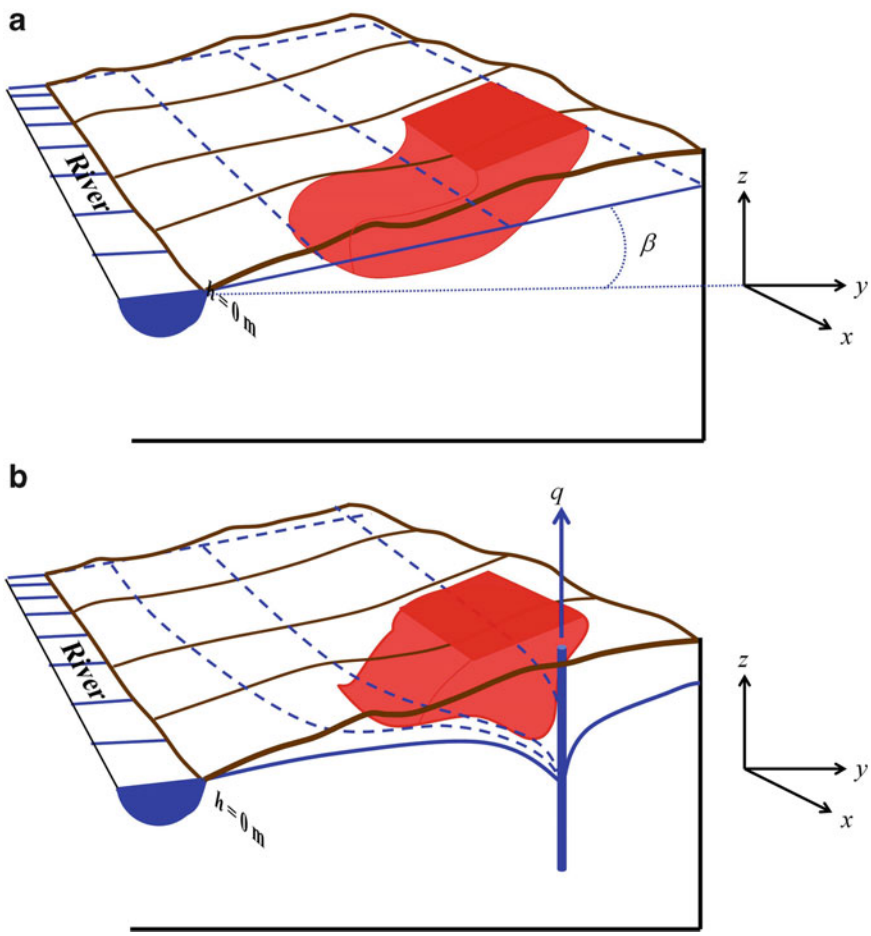


Fig. 5.15 The hydraulic head of an aquifer can be considered a plane in the case of natural flow behavior (top). The values of the hydraulic head are changed following the introduction of a well (below). The hydraulic behavior of a pollutant will therefore be very different from one case to another

Solutions

Solution 5.1: Flash Questions. Series 1

(1) A constant function has a constant slope equal to 0: the derivative function is therefore 0.

(2) We have $f'(x) = 4.5$ (for all x, f is a linear function).

$$g'(t) = -20.3 \times (-1.1)\exp(-1.1t) = -22.33\exp(-1.1t).$$

(3) We have

$$z'(t) = 3.6 \times \frac{2.4 \times 2 \times t}{2.4t^2 - 5.3} = \frac{17.28t}{2.4t^2 - 5.3}.$$

(4) The curve representing f (Fig. 5.16) clearly indicates the presence of a maximum. The derivative of f is defined by $f'(x) = -2x + 4.5$ and has a value of zero at $x = \frac{4.5}{2} = 2.25$. The function f has its maximum at 2.25, and its value is $f(2.25) = -2 \times 2.25^2 + 4.5 \times 2.25 + 12.4 \approx 17.5$.

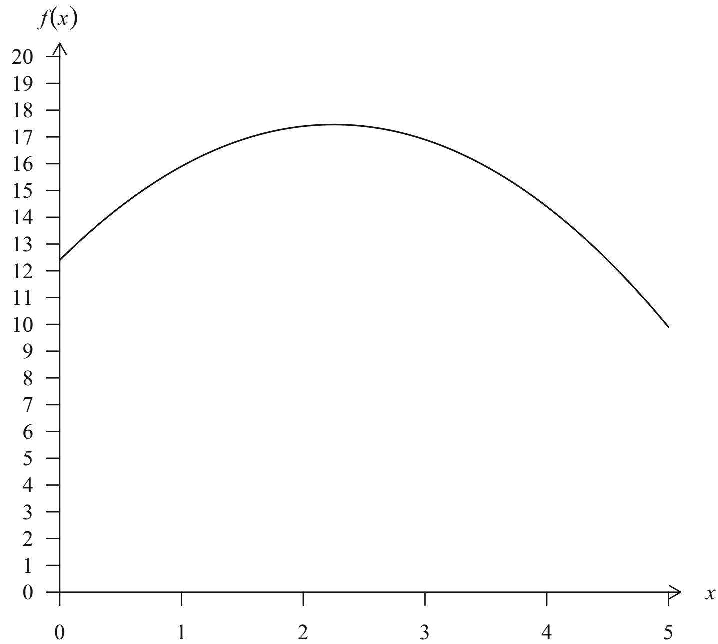


Fig. 5.16 Representative curve of $f(x) = -x^2 + 4.5x + 12.4$

Solution 5.2: Flash Questions. Series 2

(1) To calculate the slope of a line, consider two points of this line and calculate (or determine graphically) their coordinates. If we call these points A and B and their coordinate information (x_A, y_A) and (x_B, y_B), then the slope of the line is $\frac{y_B - y_A}{x_B - x_A}$. Please reread what was said on this matter in Chap. 2.

(2) We have

$\frac{\mathrm{d}h}{\mathrm{d}t} = -4.9 \times 2t + 25.6 = -9.8t + 25.6$, and so $\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = -9.8$. More generally, if the motion of a point is given by a quadratic function, then the speed (derivative) is a linear function and the acceleration (second derivative) is a constant.

(3) A and b are parameters; they are therefore to be treated as constant, and differentiation is with respect to the variable t. Function f is of the form $A \times \exp(u)$, where $u(t) = -\frac{t}{b}$, and so $u'(t) = -\frac{1}{b}$. We have $f'(t) = A \times u'(t)\exp(u(t)) = A \times \left(-\frac{1}{b}\right) \times \exp\left(-\frac{t}{b}\right) = -A/b\exp\left(-\frac{t}{b}\right)$.

(4) To derive with respect to the variable x, consider that y is a constant. We have, thus, $\frac{\partial f}{\partial x}(x,y) = -\sin(x)\sin(y)$. Likewise, $\frac{\partial f}{\partial y}(x,y) = \cos(x)\cos(y)$.

Solution 5.3: Flash Questions. Series 3

(1) You'll find the answer in the text, also remember that the slope of this tangent line is $f'(m)$.

(2) $f'(x) = 3.5 \times 3x^2 = 10.5x^2$ and $g'(y) = \frac{1}{y}$.

(3) We have $f'(x) = 6x^2 - 14x + 8$ and $f''(x) = 12x - 14$. The second derivative has a value of zero and changes sign at $x = \frac{14}{12} = \frac{7}{6}$. Therefore, the graph of f admits a point of inflection at the point of the x-coordinate $\frac{7}{6}$ and y-coordinate $f\left(\frac{7}{6}\right) = \frac{431}{54}$. The inflection point has the approximate coordinates (1.7, 7.9).

(4) We have $\dfrac{\partial T}{\partial x}(x,y,z) = 2ax + by + d$, $\dfrac{\partial T}{\partial y}(x,y,z) = bx + cz$, and $\dfrac{\partial T}{\partial z}(x,y,z) = cy$.

Solution 5.4: Derivation Calculations, Level 1

(1) $f'(x) = \cos x$ and $g'(x) = -3\sin x$.

(2) $g'(t) = 7t - 4$ and $h'(t) = \dfrac{4}{\sqrt{t}}$.

(3) $f'(x) = -\dfrac{2}{x^2}$ and $g'(x) = 12x^2 - \dfrac{1}{x}$.

(4) $\dfrac{\mathrm{d}h}{\mathrm{d}t} = -3\exp(-3t)$ and $\dfrac{\mathrm{d}T}{\mathrm{d}z} = 4.7$.

(5) $f'(x) = -a\sin x$ and $g'(x) = 3ax^2 + \dfrac{b}{x}$.

(6) $\dfrac{\partial h}{\partial x} = \cos x$, $\dfrac{\partial h}{\partial y} = -3\sin y$, and $\dfrac{\partial h}{\partial z} = \dfrac{1}{\sqrt{z}}$.

(7) $v'(u) = 1 - 4\exp(2u)$ and $v''(u) = -8\exp(2u)$.

Solution 5.5: Derivation Calculations, Level 2

(1) $f'(x) = -2x\exp(-x^2)$ and $g'(x) = \dfrac{-\sin x \times \sin x - \cos x \times \cos x}{(\sin x)^2} = -\dfrac{1}{\sin^2(x)}$.

(2) $g'(t) = 3 \times 2\cos t \sin t$ (it is of the form u^2, with $u(t) = \sin t$ and $u'(t) = \cos t$).

$$h'(t) = 5 + \frac{8}{3}\,\frac{3 \times 2t}{3t^2 + 1} = 5 + \frac{16t}{3t^2 + 1}.$$

(3) $f'(x) = \cos x \cos x + \sin x \times (-\sin x) = \cos^2 x - \sin^2 x$ and $g'(x) = \dfrac{\cos x}{2\sqrt{\sin x}}$ (it is of the form $\sqrt{u}$ whose derivative is $\dfrac{u'}{2\sqrt{u}}$ with $u(x) = \sin x$ and $u'(x) = \cos x$).

(4) $\dfrac{\mathrm{d}h}{\mathrm{d}t} = -\dfrac{1}{2\sqrt{t}}\sin\left(\sqrt{t}\right)$ and $\dfrac{\mathrm{d}T}{\mathrm{d}z} = 1013.25 \times 4.34 \times \dfrac{-7.1}{295} \times \left(\dfrac{295 - 7.1z}{295}\right)^{3.34} \approx 105.8 \times \left(\dfrac{295 - 7.1z}{295}\right)^{3.34}$ (the derivative of au^{β} is of the form $au'u^{\beta-1}$, with $a = 1013.25$, $\beta = 4.34$, $u(z) = \dfrac{295 - 7.1z}{295}$, $u'(z) = -\dfrac{7.1}{295}$).

(5) $f'(x) = kg_0 \times k\cos(kx + \phi) = k^2 g_0 \cos(kx + \phi)$ and $g'(x) = \dfrac{a^2 \sin x}{2\sqrt{1 - a^2\cos x}}$.

(6) $\dfrac{\partial h}{\partial x}(x,y,z) = \dfrac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}$.

Similarly, $\dfrac{\partial h}{\partial y}(x,y,z) = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}$ and $\dfrac{\partial h}{\partial z}(x,y,z) = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$.

(7)

$$\begin{aligned} v'(u) &= A \times (-b \times 2u)\exp\left(-bu^2\right) = -2Abu\exp\left(-bu^2\right) \text{ and} \\ v''(u) &= -2Ab\exp\left(-bu^2\right) + (-2bu) \times \left(-2Abu\exp\left(-bu^2\right)\right) \\ &= -2Ab\exp\left(-bu^2\right) + 4Ab^2u^2\exp\left(-bu^2\right) \\ &= \left(-2Ab + 4Ab^2u^2\right)\exp\left(-bu^2\right). \end{aligned}$$

Solution 5.6: Derivation Calculations, Level 3

(1) We use in both cases the formula $(u/v)' = \dfrac{u'v - uv'}{v^2}$.

We have $f'(x) = \dfrac{3\exp(3x) \times x^2 - \exp(3x) \times 2x}{x^4} = \dfrac{(3x - 2)\exp(3x)}{x^3}$ and $g'(x) = \dfrac{2x(1 - x^2) - (1 + x^2)(-2x)}{(1 - x^2)^2} = \dfrac{2x - 2x^3 + 2x + 2x^3}{(1 - x^2)^2} = \dfrac{4x}{(1 - x^2)^2}$.

(2) g is of the form $A\ \ln(u)$ and $g' = A\dfrac{u'}{u}$, with $u(t) = \dfrac{3+t^2}{2+t^2}$ and $u'(t) = \dfrac{2t(2+t^2)-(3+t^2)\times(2t)}{(2+t^2)^2} = \dfrac{4t+2t^3-6t-2t^3}{(2+t^2)^2} = \dfrac{-2t}{(2+t^2)^2}$.

From there $g'(t) = A\dfrac{\frac{-2t}{(2+t^2)^2}}{\frac{3+t^2}{2+t^2}} = A\dfrac{-2t}{(3+t^2)(2+t^2)}$.

h is of the form $u \times v$, with $u(t) = t$, $u'(t) = 1$, and $v(t) = \sqrt{1+3t^2}$; $v'(t) = \dfrac{3\times 2t}{2\sqrt{1+3t^2}} = \dfrac{3t}{\sqrt{1+3t^2}}$.

From there $h'(t) = 1 \times \sqrt{1+3t^2} + t \times \dfrac{3t}{\sqrt{1+3t^2}} = \dfrac{1+3t^2+3t^2}{\sqrt{1+3t^2}} = \dfrac{1+6t^2}{\sqrt{1+3t^2}}$.

(3) $f'(x) = a\cos(ax)\cos(bx) - b\sin(ax)\sin(bx)$ and $g'(x) = 3 \times (\cos x + 4bx^3)(\sin x + bx^4)^2$.

(4) $\dfrac{\mathrm{d}h}{\mathrm{d}t} = -\dfrac{a}{2\sqrt{t+b}}\sin\left(a\sqrt{t+b}\right)$ and $\dfrac{\mathrm{d}T}{\mathrm{d}z} = P_0 \times k \times \left(\dfrac{-a}{b}\right)\left(\dfrac{b-az}{b}\right)^{k-1}$ (see the similar calculation in the previous exercise, without parameter).

(5) $f'(a) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \times \dfrac{(-2(a-m))}{2\sigma^2}\exp\left(-\dfrac{(a-m)^2}{2\sigma^2}\right) = \dfrac{-a+m}{\sigma^2\sqrt{2\pi\sigma^2}}\exp\left(-\dfrac{(a-m)^2}{2\sigma^2}\right)$.

$$g'(x) = \frac{a^2\sin x}{2\sqrt{1-a^2\cos x}}.$$

(6) We have

$$\begin{aligned}\frac{\partial T}{\partial z}(z,t) &= \frac{(t_0\exp(-t_0z))(1-\exp(-tz)) - (1-\exp(-t_0z))(t\exp(-tz))}{(1-\exp(-tz))^2}\\ &= \frac{t_0\exp(-t_0z) - t_0\exp(-t_0z)\exp(-tz) - t\exp(-tz) + t\exp(-t_0z)\exp(-tz)}{(1-\exp(-tz))^2}\\ &= \frac{t_0\exp(-t_0z) - t\exp(-tz) + (t-t_0)\exp(-t_0z)\exp(-tz)}{(1-\exp(-tz))^2} \text{ and}\\ \frac{\partial T}{\partial t}(z,t) &= -(1-\exp(-t_0z))\frac{z\exp(-tz)}{(1-\exp(-tz))^2}.\end{aligned}$$

(7) We have $v'(u) = A(1 + 2\omega u)\cos(u + \omega u^2)$ and $v''(u) = A \times 2\omega\cos(u + \omega u^2) - A(1 + 2\omega u)^2\sin(u + \omega u^2)$.

Solution 5.7: Tangents Everywhere

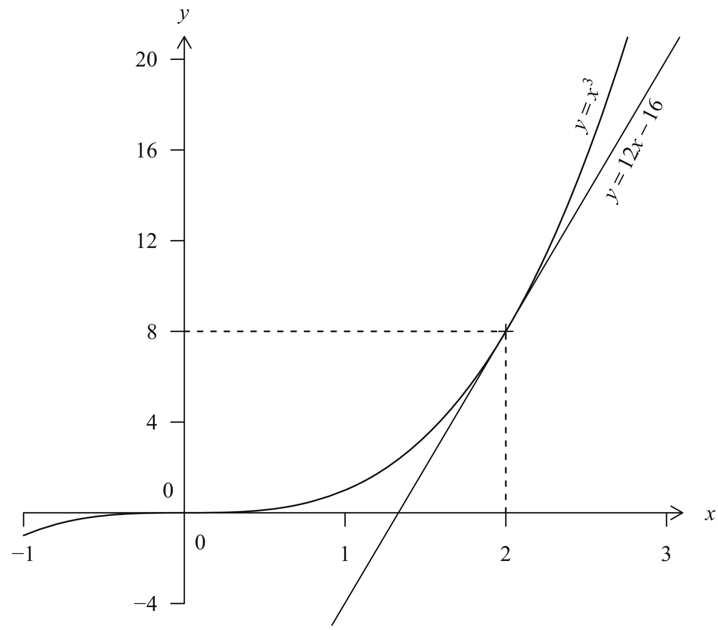
(1) We have $f'(x) = 3x^2$; therefore, an equation of the tangent line to the curve of f at the point of the x-coordinate 2 is $y = f'(2)(x-2) + f(2) = 12(x-2) + 8 = 12x - 16$ (Fig. 5.17).

(2) The **tangent line** at the point of the x-coordinate 0 is horizontal: $f'(0) = 0$. To the tangent at the point of the x-coordinate 1 (respectively 2), when moving a unit on the x-axis on the tangent, there is a displacement of -1 along the y-axis (or 4); thus, $f'(1) = -1$ (respectively $f'(2) = 4$).

Solution 5.8: Parabola

We have $z'(t) = 2at + b$, and $2at + b$ has a value of zero and changes sign at $-\dfrac{b}{2a}$. Thus, the

Fig. 5.17 Graph $y = x^3$



function admits an extreme with coordinates $\left(-\frac{b}{2a}; z\left(-\frac{b}{2a}\right)\right)$, which is $\left(-\frac{b}{2a}; -\frac{b^2}{4a}+c\right)$.

Solution 5.9: Cubic Function

The graphical representation of f is given by (Fig. 5.18)

(1) We have $f'(x) = 3x^2 + 32x - 195$. A resolution with Xcas gives the solutions $x = -15$ and $x = \frac{13}{3}$ for the equation $f'(x) = 0$. The graphical representation of the function indicates that f admits a maximum at -15 and a minimum at $\frac{13}{3}$. As $f(-15) = 3170$ and $f\left(\frac{13}{3}\right) \approx -443$, the maximum has the coordinates $(-15, 3170)$ and the minimum has the approximate coordinates $(4.3, -443)$.

(2) We have $f''(x) = 6x + 32$, which has a value of zero and changes sign at $-\frac{16}{3}$.

As $f\left(-\frac{16}{3}\right) \approx 1363$, the graph of f admits an inflection point with the approximate coordinates $(-5.3, 1363)$.

Solution 5.10: Maximum

The solution is given by Xcas software in Fig. 5.19. f admits a maximum with the approximate coordinates $(60, 0.050)$.

Solution 5.11: Inflection Points

We use the second derivative of f. f has the form $\exp(u)$, where $u(x) = -x^2$, and so $u'(x) = -2x$. Thus, we have $f'(x) = u'(x) \times \exp(u(x)) = -2x \exp(-x^2)$. And therefore $f''(x) = -2\exp(-x^2) - 2x \times (-2x)\exp(-x^2) = (4x^2 - 2)\exp(-x^2)$. As the expression $\exp(-x^2)$ is always positive, $f''(x)$ has a value of zero and changes sign where $4x^2 - 2$ has a value of zero and changes sign. We solve $4x^2 - 2 = 0$, and the solution is $4x^2 = 2$, from which $x^2 = \frac{1}{2}$, and so $x = \pm\sqrt{\frac{1}{2}}$. The corresponding ordinates both have a value of $f\left(\sqrt{\frac{1}{2}}\right) = \exp\left(-\frac{1}{2}\right) \approx 0.61$.

The function f admits two inflection points with the approximate coordinates $(-0.71, 0.61)$ and $(+0.71, 0.61)$.

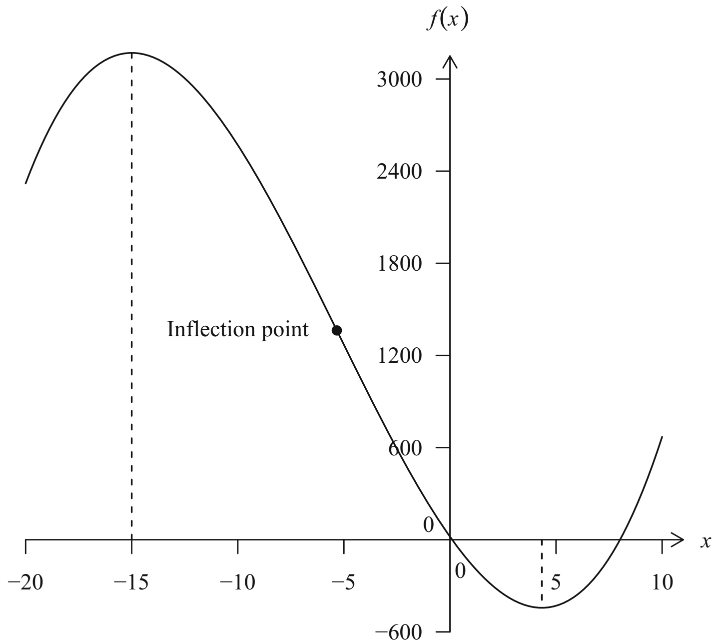


Fig. 5.18 Representation of a cubic function

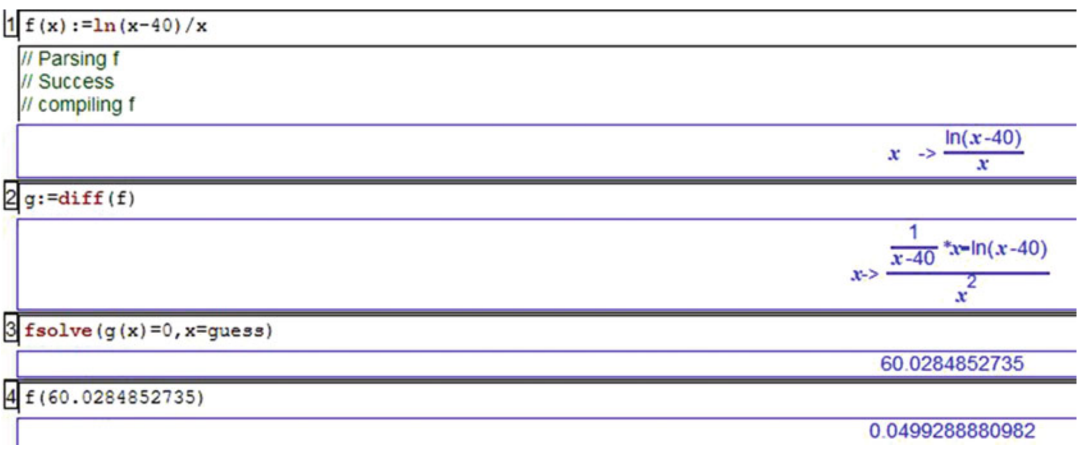


Fig. 5.19 Calculation of coordinates of maximum of function f defined by $f(x) = \dfrac{\ln(x-40)}{x}$ using Xcas

Solution 5.12: Partial Derivatives

(1) We have $\dfrac{\partial T}{\partial x} = 2x$, from where $\dfrac{\partial^2 T}{\partial x^2} = 2$, and $\dfrac{\partial T}{\partial y} = 3\exp(3y)$, so $\dfrac{\partial^2 T}{\partial y^2} = 9\exp(3y)$.

Therefore, we have $\dfrac{\partial^2 T}{\partial x^2} - \dfrac{\partial^2 T}{\partial y^2} = 2 - 9\exp(3y)$.

(2) We have $\dfrac{\partial T}{\partial x} = -\sin(x-y)$, from where $\dfrac{\partial^2 T}{\partial x^2} = -\cos(x-y)$, and $\dfrac{\partial T}{\partial y} = +\sin(x-y)$, from where $\dfrac{\partial^2 T}{\partial y^2} = -\cos(x-y)$.

Thus, we have $\dfrac{\partial^2 T}{\partial x^2} - \dfrac{\partial^2 T}{\partial y^2} = 0$; it is said that this function T is a **solution of the partial differential equation** $\dfrac{\partial^2 T}{\partial x^2} - \dfrac{\partial^2 T}{\partial y^2} = 0$.

(3) We have $\dfrac{\partial T}{\partial x} = \dfrac{A}{y}$, from where $\dfrac{\partial^2 T}{\partial x^2} = 0$, and $\dfrac{\partial T}{\partial y} = -\dfrac{Ax}{y^2}$, from where $\dfrac{\partial^2 T}{\partial y^2} = 2\dfrac{Ax}{y^3}$.

So we have: $\dfrac{\partial^2 T}{\partial x^2} - \dfrac{\partial^2 T}{\partial y^2} = -2\dfrac{Ax}{y^3}$.

Solution 5.13: Gravity Formulas

The differentiation is with respect to the variable ϕ (the other variables being considered constant).

(1) We must differentiate an expression of the form $\sin^n$, that is to say, u^n, with $u = \sin$. The derivative of u^n is $nu'u^{n-1}$, here with $u' = \cos$. So we have

$$g'(\phi) = g_0\big[2a\cos(\phi)\sin(\phi) + 4b\cos(\phi)\sin^3(\phi)$$

$$+6c\cos(\phi)\sin^5(\phi) + 8d\cos(\phi)\sin^7(\phi)\big].$$

(2) We use Xcas (remember to write the multiplication symbol *; do not use the letter e, which means exp(1) in Xcas). We note here $c = e^2$ (Fig. 5.20).

Solution 5.14: Free Fall

(1) We have $z_0 = z(0) = 5$; therefore, the initial height is 5 m.

The maximum is reached when $z'(t) = 0$. We have $z'(t) = -4.9 \times 2t + 8$.

$z'(t) = 0$ for $t = \dfrac{8}{(4.9 \times 2)} \simeq 0.82\,\text{s}$. The corresponding height is $z(0.82) \simeq 8.3$ m.

(2) The speed $v(t)$ of the ball is $z'(t)$. First we solve the equation $z(t) = 0$ for when the ball reaches the ground. Xcas gives $t \simeq 2.12$ s (Fig. 5.21).

Xcas then indicates a speed of about -12.7 m/s (the sign comes from the upward orientation of the axis).

Then we solve the equation $z(t) = 8$. Xcas indicates two solutions: the first corresponds to the rise of the ball and the other to its descent. We have $t \simeq 0.58$ s to reach the height of 8 m on the ascent. The speed is then about 2.28 m/s.

(3) The acceleration a is obtained by differentiating the speed v. We have $v'(t) = -9.8$: it is a constant throughout the trajectory of the ball (the only force acting on the ball is its weight).

Solution 5.15: Heat Received on Earth

We consider a domain of 0 to 2π for the cosine function, so values between 0 and 365.25 days (1 year) for the variable t.

(1) We calculate the derivative of the function E,

$$E'(t) = C_0 \times \frac{2\pi}{365.25}\sin\left[\frac{2\pi}{365.25}(t - 2.72)\right],$$

and we have $E'(t) = 0$ for $\dfrac{2\pi}{365.25}(t - 2.72) = 0$ or $\dfrac{2\pi}{365.25}(t - 2.72) = \pi$ (the values 0 and π are zeroes of the sine function). This corresponds to $t = 2.72$ (early in the year) or $t = \dfrac{365.25}{2} + 2.72$ (midyear).

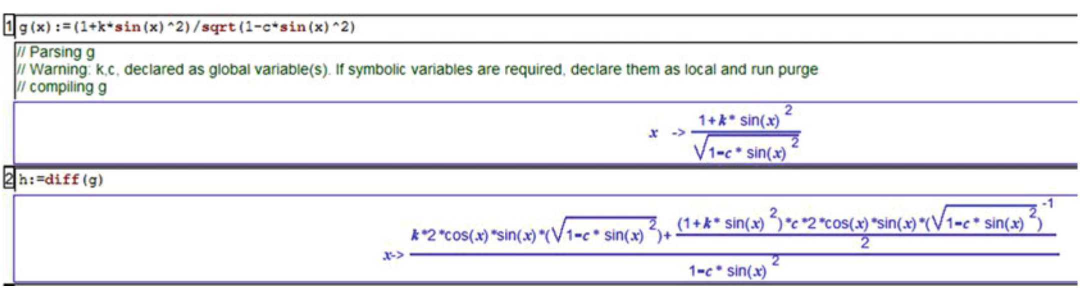


Fig. 5.20 Gravity and latitude

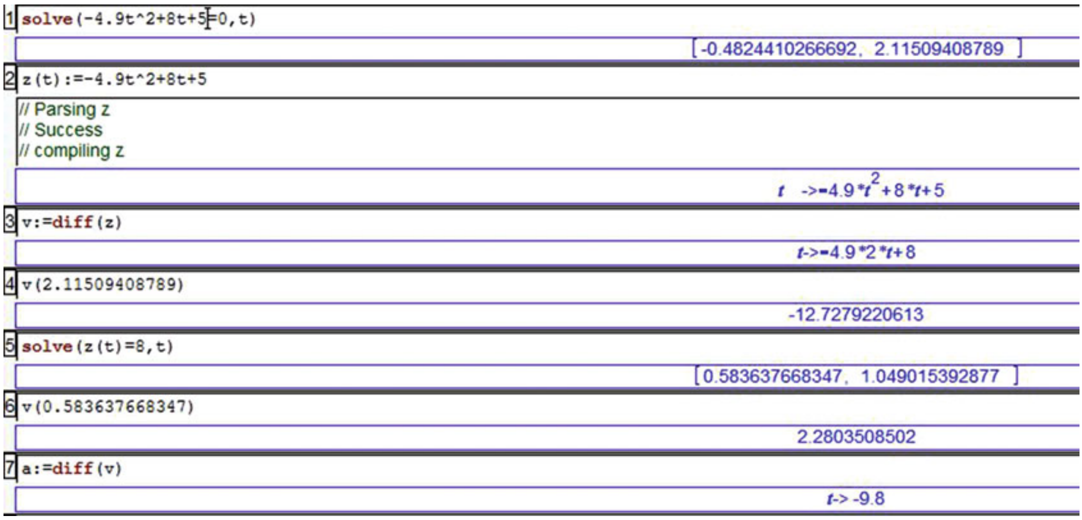
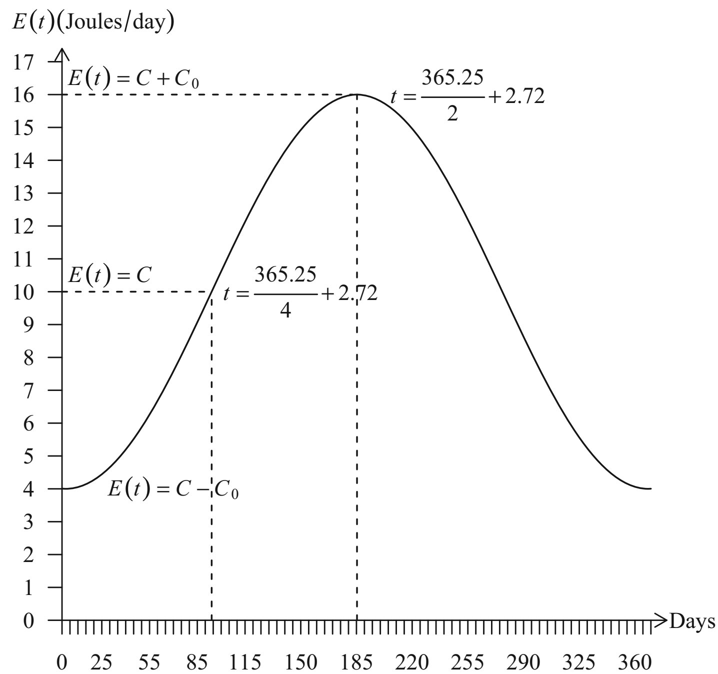


Fig. 5.21 Solving equations and calculations of speed and acceleration with Xcas

Fig. 5.22 Graph of function

$$E(t) = C - C_0 \cos\left[\frac{2\pi}{365.25}(t - 2.72)\right]$$



The graphical representation of the function E (Fig. 5.22) shows that the value $t = 0$ corresponds to a minimum of the energy received and the value $t = \frac{365.25}{2}$ to a maximum: the amount of heat received is most important in the middle of the year (early summer), which is consistent with common sense!

(2) The rate of increase of the function E is given by E'. To find the maximum E', we calculate its derivative E''. We have

$$E''(t) = C_0\left(\frac{2\pi}{365.25}\right)^2 \cos\left[\frac{2\pi}{365.25}(t-2.72)\right],$$

which has a value of zero for $\frac{2\pi}{365.25}(t-2.72) = \frac{\pi}{2}$ or $\frac{2\pi}{365.25}(t-2.72) = \frac{3\pi}{2}$, so has a value of zero for $t = \frac{365.25}{4} + 2.72$ (end of March) or $t = 3 \times \frac{365.25}{4} + 2.72$ (end of September). The curve representing E' has the appearance of a sine: a maximum first then and a minimum. The growth in the amount of heat received is at its maximum at the end of March (and the decrease is at its maximum at the end of September).

Solution 5.16: Temperature Gradient

(1) The temperature gradient is written $\frac{\mathrm{d}T}{\mathrm{d}z}$ and

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}z}(z) &= \mathrm{d}z[T_0 - 6.5z]\\ &= -6.5.\end{aligned}$$

The temperature gradient is constant and equal to -6.5 °C/km.

(2) The gradient is negative and indicates that the temperature decreases continuously with increasing altitude. In addition, the gradient is constant and equal to -6.5 °C/km, and this reduction is always the same regardless of altitude.

Solution 5.17: Pressure Gradient

(1) The pressure gradient is written $\frac{\mathrm{d}P}{\mathrm{d}z}$, and so we have

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}z}(z) &= 1013.25 \times 5.255\mathrm{d}z\left[\frac{288-6.5z}{288}\right] \times \left(\frac{288-6.5z}{288}\right)^{5.255-1}\\ &= 1013.25 \times 5.255 \times \left(\frac{-6.5}{288}\right)\left(\frac{288-6.5z}{288}\right)^{4.255}\\ &= -\frac{1013.25 \times 5.255 \times 6.5}{288}\left(\frac{288-6.5z}{288}\right)^{4.255}\\ &\simeq 120.2\left(\frac{288-6.5z}{288}\right)^{4.255}.\end{aligned}$$

(2)

```
#------------------------------------
# Pressure gradient in the troposphere
#------------------------------------
# Altitude (km)
z <- seq(0,12,0.1)

# Pressure gradient (hPa/m)
Gp <- -120.17*((288-6.5*z)/288)^4.255

# Plot
plot(z,Gp,type="l",
     xlab="Altitude (km)",
     ylab="Pressure gradient (hPa/m)")
```

(3) The gradient takes negative values over the entire range of altitude (between 0 and 12 km). This indicates that the pressure decreases as the altitude increases. In addition, the gradient values range from -120 hPa/km (at 0 km) to -40 hPa/km (12 km); this decrease becomes lower as altitude increases. In other words, the decrease in pressure is greater at low altitudes.

Solution 5.18: Ideal Gas Law

$PV = nRT$, so $V = \frac{nRT}{P}$, and so $\frac{\partial V}{\partial T} = \frac{nR}{P}$.

Similarly, $T = \frac{PV}{nR}$, and so $\frac{\partial T}{\partial P} = \frac{V}{nR}$.

Finally, $P = \frac{nRT}{V}$, and so $\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$.

So we have $\frac{\partial V}{\partial T} \times \frac{\partial T}{\partial P} \times \frac{\partial P}{\partial V} = \frac{nR}{P} \times \frac{V}{nR} \times \left(-\frac{nRT}{V^2}\right) = -\frac{nRT}{PV} = -1$

because $PV = nRT$, so the quotient $\frac{nRT}{PV}$ is 1.

Solution 5.19: van der Waals Equation

(1) We express the variable P depending on variables T and V:

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT, \quad \text{so} \quad P = \frac{RT}{V-b} - \frac{a}{V^2}.$$

(2) We have $\frac{\partial P}{\partial V}(T, V) = -\frac{RT}{(V-b)^2} + 2\frac{a}{V^3}$

and $\frac{\partial^2 P}{\partial V^2}(T, V) = 2\frac{RT}{(V-b)^3} - 6\frac{a}{V^4}$.

We have two equations (both with variables T_c and V_c):

$$-\frac{RT_c}{(V_c-b)^2} + 2\frac{a}{V_c^3} = 0 \quad \text{and} \quad 2\frac{RT_c}{(V_c-b)^3} - 6\frac{a}{V_c^4} = 0\,.$$

Multiplying the first equality by $\frac{2}{(V_c - b)}$ yields $-2\frac{RT_c}{(V_c-b)^3} + 4\frac{a}{(V_c^3)(V_c-b)} = 0$.

By adding the second equality, we obtain

$$4\frac{a}{(V_c^3)(V_c-b)} - 6\frac{a}{V_c^4} = 0,$$

which gives $\frac{4aV_c - 6a(V_c-b)}{V_c^4(V_c-b)} = 0$, and so $4aV_c - 6a(V_c - b) = 0$, from which $-2aV_c + 6ab = 0$, or, finally, $V_c = 3b$.

Replacing V_c with $3b$ in the equality $-\frac{RT}{(V_c-b)^2} + 2\frac{a}{V_c^3} = 0$, we obtain $-\frac{RT_c}{(2b)^2} + 2\frac{a}{(3b)^3} = 0$, and so

$$T_c = -2\frac{\frac{a}{(3b)^3}}{-\frac{R}{(2b)^2}} = \frac{8a}{27bR}.$$

In the end, we have

$V_c = 3b$ and $T_c = \frac{8a}{27bR}$.

(3) P_c satisfies the equality $\left(P_c + \frac{a}{V_c^2}\right)(V_c - b) = RT_c$, so substituting the results obtained in the previous question yields

$$\left(P_c - \frac{a}{9b^2}\right) \times (2b) = \frac{8a}{27b}, \text{ and so}$$

$$P_c = \frac{8a}{2b \times 27b} + \frac{a}{9b^2} = \frac{a}{27b^2}\,.$$

Solution 5.20: Soil Temperature

(1) We have (using the formula for differentiating a product $u \times v$)

$\frac{\partial T}{\partial x}(x,t) = T_0 \times (-\lambda) \times \exp(-\lambda x)\sin(24t - \lambda x)$ $-\lambda T_0 \exp(-\lambda x)\cos(24t - \lambda x)$; this reflects how the temperature increases (or decreases) as a function of the depth (at a fixed time).

In addition, $\frac{\partial T}{\partial t}(x,t) = 24T_0\exp(-\lambda x)\cos(24t - \lambda x)$; this reflects how the temperature increases (or decreases) depending on the time of day (at a fixed depth).

(2) We calculate $\frac{\partial^2 T}{\partial x^2}(x,t)$ again using the differentiation of a product:

$$\frac{\partial^2 T}{\partial x^2}(x,t) = \left[\lambda^2 T_0\exp(-\lambda x)\sin(24t - \lambda x) + \lambda^2 T_0\exp(-\lambda x)\cos(24t - \lambda x)\right]$$
$$- \lambda[-\lambda T_0\exp(-\lambda x)\cos(24t - \lambda x) + \lambda T_0\exp(-\lambda x)\sin(24t - \lambda x)].$$

So after simplification of two terms

$$\frac{\partial^2 T}{\partial x^2}(x,t) = 2\lambda^2 T_0 \exp(-\lambda x)\cos(24t - \lambda x).$$

We indeed recognize $\frac{\partial T}{\partial t}(x,t)$ apart from a multiplicative constant, and we have $\frac{\partial T}{\partial t}(x,t) = \frac{24}{2\lambda^2}\frac{\partial^2 T}{\partial x^2}(x,t)$ (heat equation).

Solution 5.21: Drinking Water Resource

(1) The mathematical prerequisite to capture only a freshwater aquifer and not “to pump” salt water from the ocean is written $\left.\frac{\partial h}{\partial y}\right|_{\substack{x=0\\y=0}} > 0$. We must therefore calculate the partial derivative $\frac{\partial h}{\partial y}$. We have successively

$$\begin{aligned}\frac{\partial h}{\partial y} &= \beta - \frac{q}{4\pi T}\frac{\partial}{\partial y}\left[\ln\left(\frac{(y+d)^2+x^2}{(y-d)^2+x^2}\right)\right]\\ &= \beta - \frac{q}{4\pi T}\frac{(y-d)^2+x^2}{(y+d)^2+x^2}\frac{\partial}{\partial y}\left[\left(\frac{(y+d)^2+x^2}{(y-d)^2+x^2}\right)\right]\\ &= \beta - \frac{qd}{\pi T}\frac{d^2-y^2+x^2}{\left[(y+d)^2+x^2\right]\left[(y-d)^2+x^2\right]}.\end{aligned}$$

In $x = 0$ and $y = 0$, this gives

$$\begin{aligned}\left.\frac{\partial h}{\partial y}\right|_{\substack{x=0\\y=0}} &= \beta - \frac{qd}{\pi T}\times\frac{d^2}{d^4}\\ &= \beta - \frac{q}{\pi T d}.\end{aligned}$$

If we apply the condition $\left.\frac{\partial h}{\partial y}\right|_{\substack{x=0\\y=0}} > 0$, we have

$$q \le \beta\pi T d,$$

and therefore the values given show that q must be lower than 0.122 m^3/s or 122 l/s so as not to capture the salt water of the ocean.

(2) We see that for the velocity threshold $q = \beta\pi T d$, the value of the hydraulic head $h(x, y) = 0$ m is set by the ocean. Once this limit is exceeded, the hydraulic head becomes negative and the direction of the flow is then from the ocean to the aquifer.

```
#-----------------------------------
# Water resource
#-----------------------------------
# Distance well-river (m)
d <- 300
# Permeability (m/s)
k <- 1.6e-4
# Aquifer thickness (m)
e <- 80
# Transmissivity (m2/s)
Transm <- e * k
# Slope of the aquifer
a <- 0.01
# Flow rate’s threshold (m3/s)
q <- a*pi*Transm*d*1.2

# Hydraulic head of the aquifer
h <- function(x,y) {
  return ( a*y - q/(4*pi*Transm)*
    log(((y+d)^2+x^2)/((y-d)^2+x^2)) )
}

# Plots
x <- seq(-200,200,1)
y <- seq(0,400,1)
z <- outer(x,y,h)
image(x,y,z)
contour(x,y,z,add=TRUE)
```

Solution 5.22: Pollution of an Aquifer

(1) If $q = 0$ m^3/s, the height of the water in the aquifer is $h(x, y) = \beta y$, so the coordinates of the velocity vector of the pollutant are

$$\begin{aligned}u_x &= 0,\\ u_y &= -\frac{k\beta}{\omega},\\ u_z &= 0.\end{aligned}$$

With the values $\omega = 15$ % (0.15), $k = 10^{-4}$m/s, and $\beta = 1$ % (=0.01) it gives:

$$\begin{aligned}u_x &= 0;\\ u_y &= -6.6\times 10^{-6};\\ u_z &= 0.\end{aligned}$$

The average velocity of the pollutant in the aquifer is $u = \sqrt{u_x^2 + u_y^2 + u_z^2} = 6.6 \times 10^{-6}$ m/s.

(2) It takes 1 s for the pollutant to travel 6.6×10^{-6} m; how long does it take to travel 500 m? A proportionality rule is used: $\frac{500 \times 1}{6.6 \times 10^{-6}} = 75{,}757{,}575.75$ s. A year is $365 \times 24 \times 3600$ s. How many years are in 75,757,575.75 s? Proportionality indicates that $\frac{75{,}757{,}575.75}{365 \times 24 \times 3600} = 2.4$ years.

(3) When the flow rate q is not zero, the coordinates of the average velocity of the pollutant can be calculated from

$$u_x = -\frac{k}{\omega}\frac{\partial h(x,y)}{\partial x};$$
$$u_y = -\frac{k}{\omega}\frac{\partial h(x,y)}{\partial y};$$
$$u_z = -\frac{k}{\omega}\frac{\partial h(x,y)}{\partial z}.$$

We have

$$\begin{aligned}
u_x &= -\frac{k}{\omega}\frac{\partial h(x,y)}{\partial x}\\
&= -\frac{k}{\omega}\frac{\partial}{\partial x}\left[\beta y - \frac{q}{4\pi T}\ln\left(\frac{x^2+(y+d)^2}{x^2+(y-d)^2}\right)\right]\\
&= \frac{k}{\omega}\frac{q}{4\pi T}\frac{x^2+(y-d)^2}{x^2+(y+d)^2}\frac{\partial}{\partial x}\left[\frac{x^2+(y+d)^2}{x^2+(y-d)^2}\right]\\
&= \frac{kq}{4\pi T\omega}\frac{x^2+(y-d)^2}{x^2+(y+d)^2}\times\frac{2x\left[x^2+(y-d)^2\right]-2x\left[x^2+(y+d)^2\right]}{\left[x^2+(y-d)^2\right]^2}\\
&= -\frac{2kqd}{\pi T\omega}\frac{xy}{\left[x^2+(y+d)^2\right]\left[x^2+(y-d)^2\right]};\\
u_y &= -\frac{k}{\omega}\frac{\partial h(x,y)}{\partial y}\\
&= -\frac{k}{\omega}\frac{\partial}{\partial y}\left[\beta y - \frac{q}{4\pi T}\ln\left(\frac{x^2+(y+d)^2}{x^2+(y-d)^2}\right)\right]\\
&= -\frac{k}{\omega}\left[\beta - \frac{q}{2\pi T}\frac{x^2+(y-d)^2}{x^2+(y+d)^2}\frac{\partial}{\partial y}\left[\frac{x^2+(y+d)^2}{x^2+(y-d)^2}\right]\right]\\
&= -\frac{k}{\omega}\left[\beta - \frac{qd}{\pi T}\frac{x^2-y^2+d^2}{\left[x^2+(y+d)^2\right]\left[x^2+(y-d)^2\right]}\right];\\
u_z &= -\frac{k}{\omega}\frac{\partial h(x,y)}{\partial z}\\
&= 0.
\end{aligned}$$

Therefore, the average velocity of the pollutant $\vec{u}$ has the coordinates

$$u_x = -\frac{2kqd}{\pi T\omega}\frac{xy}{\left[x^2+(y+d)^2\right]\left[x^2+(y-d)^2\right]};$$

$$u_y = -\frac{k}{\omega}\left[\beta - \frac{qd}{\pi T}\frac{x^2-y^2+d^2}{\left[x^2+(y+d)^2\right]\left[x^2+(y-d)^2\right]}\right];$$

$$u_z = 0.$$

```
#-------------------------------------
# Aquifer pollution
#-------------------------------------
# Slope of the aquifer
beta <- 0.01
# Drilling location in Y (m)
d <- 200
# Flow rate of the well (m3/s)
q <- 0.01
# Permeability (m/s)
k <- 1e-4
# Aquifer's thickness (m)
e <- 80
# Transmissivity (m2/s)
Transm <- k * e
# Delay parameter
R <- 1
# Porosity
w <- 0.15
# The study area
x <- 0
dy <- 0.1 # Spatial step
y <- seq(0,d,dy)
# Velocity along the X axis
ux <- function(x,y) {
  return ( - (2*k*q*d)/(pi*Transm*R*w)
*x*y/
              ((x^2+(y+d)^2)*(x^2+(y-d)
^2)) )
}
# Velocity along the Y axis
uy <- function(x,y) {
  return ( - k/(R*w)*(beta -
          (q*d)/(pi*Transm)*(x^2-y^2
+d^2)/
          ((x^2+(y+d)^2)*(x^2+(y-d)^2))
) )
}
# Velocity along the Z axis
uz <- function(x,y) {
  return ( .0 )
}
# Average velocity
# at x=0 and y=d, one have a singular-
ity,
# so we introduce a default value:
# a value close to the well
u <- function(x,y) {
  return (ifelse (y == d,
          u<- u(0,d-dy),
          u <- sqrt(ux(x,y)^2+uy(x,y)^2
+uz(x,y)^2)))
}
plot(y,u(x,y),type="l",
       xlab="Y",ylab="Average velocity
(m/s)")
```

Integration and Differential Equations

6

Abstract

This chapter focuses on two fundamental function-related tools that are frequently used in Earth science and geography: integration and differential equations. Partial differential equations are presented very summarily to indicate to the reader their existence, but their study goes beyond the scope of this chapter and book.

We can calculate the area of common geometric shapes (e.g., area of a rectangle), but how can we compute the area of a domain bounded by a more complex curve? We will use the integration tool, introducing a method of calculating integrals using, or not, calculation software and approximating integrals with the rectangle method.

When one is interested in the variations of a magnitude in time and space (which frequently happens in geography and Earth science), it is often useful to invoke a particular type of function: solutions of differential equations and partial differential equations. This chapter explains what solving a differential equation means and how to do it (with or without the help of calculation software).

Keywords

Integration · Primitive (antiderivative) · Integral · Rectangle method · Differential equation · Ordinary differential equation · Partial differential equation

Aims and Objectives

- To understand the relationship between area under a curve and integration.
- To know the link between the primitive of a function and an integral.
- To know how to calculate integrals and how to use formal calculation software for integrals.
- To understand the rectangle method for the approximate computation of an integral.
- To recognize a differential equation, an unknown function, and a variable.
- To know how to solve simple ordinary differential equations.
- To know how to use Xcas to solve differential equations.
- To recognize a partial differential equation.

6.1 Integrals: Introduction

The volume of a liquid that flows at a known constant flow rate can be calculated using the formula $V = Q \times t$, where V is the volume, Q the flow of liquid, and t the duration of observation.

Electronic supplementary material The online version of this article (https://doi.org/10.1007/978-3-319-69242-5_6) contains supplementary material, which is available to authorized users.

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In the preceding expression, the flow rate was considered to be constant throughout the duration t of the observation. If the flow is Q_1 for duration t_1 and Q_2 for duration t_2, then volume V is obtained by (Fig. 6.1) $V = Q_1 \times t_1 + Q_2 \times t_2$.

In reality, the flow Q can vary continuously and is then expressed by a function of time $Q(t)$. How is it possible to calculate the volume that has flowed during a given time interval?

To do this, let us return to the previous situation with the flows Q_1 and Q_2. The function $Q(t)$ has a value of Q_1 over a period t_1 (e.g., from 0 to t_1) and Q_2 over a period t_2 (e.g., from t_1 to $t_1 + t_2$).

The amount $Q_1 \times t_1$ corresponds to the area of the first rectangle and $Q_2 \times t_2$ to the area of the second rectangle. In the end, the volume V between the moments 0 and $t_1 + t_2$ is represented by the sum of the areas of the two rectangles, which corresponds to **the area under the curve** representing Q (gray parts in Fig. 6.1).

It is this idea that we will try to follow in the case of a function Q defined as a function of t on the interval $[0; T]$. Take, for example, the expression of the flow of a watershed $Q(t) = \frac{3t^2}{80{,}000}\exp\left(-\frac{t}{200}\right)$, where the duration t is expressed in seconds and the flow rate Q in $\mathrm{m^3/s}$ over a period $T = 5$ min $= 300$ s. The graphical representation of the function Q on the interval $[0; 300]$ is given in Fig. 6.2.

What rectangles shall we introduce to calculate the sum of the areas? For the height, it is given by the different values $Q(t)$. We will take an infinitely small width $\mathrm{d}t$, and the area of the corresponding rectangle will be $Q(t) \times \mathrm{d}t$, denoted also by $Q(t)\mathrm{d}t$ (we choose rectangles below the curve representing Q).

It is now necessary to add the areas of all these slices. There is an infinity of them. To calculate such a sum of infinitesimal areas of an infinite number of rectangles, the mathematical tool used is called **integral of the function** Q between the bounds 0 and T (Fig. 6.3):

$$V = \int_0^T Q(t)\mathrm{d}t.$$

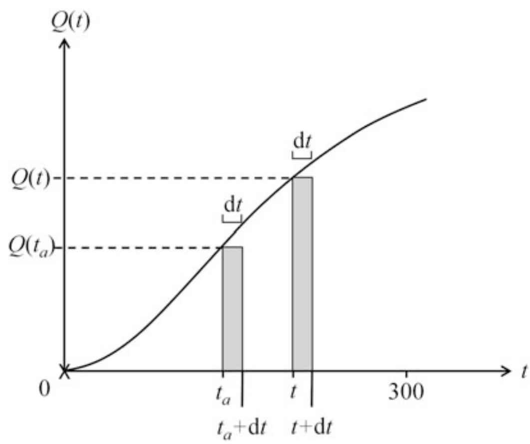

Fig. 6.2 Graph of flow function $[0; 300]$, $Q : t \mapsto \frac{3t^2}{80{,}000}\exp\left(-\frac{t}{200}\right)$

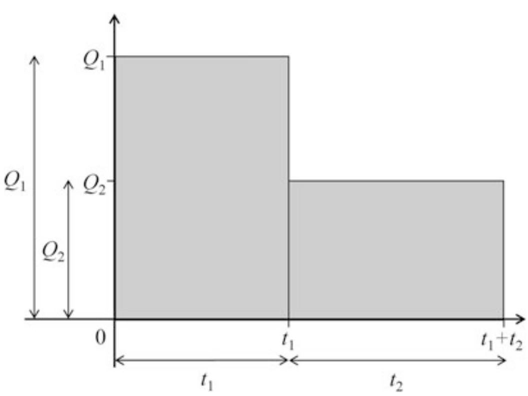

Fig. 6.1 $Q(t) = Q_1$ for $0 \le t \le t_1$ and $Q(t) = Q_2$ for $t_1 < t \le t_1 + t_2$

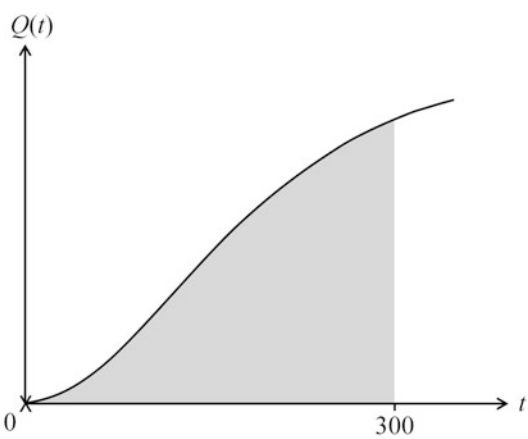

Fig. 6.3 Area under curve representing Q

Note 53 *The symbol* $\int$ *is a stylish S, S as in "sum."*

We will see in the following sections how to calculate an integral. For the example given, we have

$$\int_0^{300} Q(t)\mathrm{d}t = \int_0^{300} \frac{3t^2}{80{,}000}\exp\left(-\frac{t}{200}\right)\mathrm{d}t \approx 115 \;\; \mathrm{m}^3.$$

Note 54 *In the preceding example, the integral was defined on a bounded interval (the upper and lower bounds are finite numbers). In other situations, it may be necessary to consider the area under the curve between 0 and* $+\infty$ *or* $-\infty$ *and* $+\infty$*. The result of some integrations can also be infinite.*

For example, it has been demonstrated that

$$\int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}.$$

Here the result is finite and the upper and lower bounds of the integral are infinite.

6.2 Notion of Primitive (Antiderivative)

6.2.1 Introduction, Definition

We studied in Chap. 5 the notion of derivative: to a function f we associate a new function f'. We are now interested in reversing this process: if one takes on a function f, how does one find a function F whose f is the derivative, that is to say, such that $F' = f$?

Definition 22 *Let f be a function. We call the "**primitive** of f" (or "**antiderivative** of f") any function F such as* $F' = f$*.*

Example 58 *In Chap. 5, we wrote that the derivative v of the function z defined by* $z(t) = -\dfrac{9.81t^2}{2} + 200$ *is expressed by* $v(t) = -9.81t$*. We can therefore also write that the function z is a primitive of the function v.*

By simple observation of derivation formulas Table 5.1, we can find some primitives:

- *If* $f(x) = 0$*, then for* any *constant, F defined by* $F(x) = a$ is *a primitive of f.*
- *If* $f(x) = a$ *(with "a" a* constant*), then for any constant b, F defined by* $F(x) = ax + b$ *is a primitive of f.*
- *If* $f(x) = \exp(x)$*, then* $F(x) = \exp(x)$ *defines a primitive of f.*
- *If* $f(x) = \dfrac{1}{x}$*, then* $F(x) = \ln(|x|)$ *defines a primitive of f.*
- *If* $f(x) = \cos(x)$*, then* $F(x) = \sin(x)$ *defines a primitive of f.*

Note 55 *In the first example, the function defined by* $z(t) = -\dfrac{9.81}{2}t^2 + 200$ *is a primitive of v (where* $v(t) = -9.81t$*) because, by deriving z, we find* $z'(t) = v(t)$*. In the derivation, the constant 200 plays no role (its derivative is 0), and another constant would have given the same derivative.*

The function v therefore has an infinity of primitives, which are all the functions of the form $V(t) = -\dfrac{9.81}{2}t^2 + C$ *(where C is a constant).*

For example, the function V defined by $V(t) = -\dfrac{9.81}{2}t^2 - 153.7$ *is also a primitive of v.*

*Therefore, there is not a unique primitive for a given function (there are even an infinite number of primitives). When we find one, we say that it is **a** primitive function and not **the** primitive function (although such an abuse of language is common).*
• To complete the preceding discussion on uniqueness, if we impose a given value for the primitive function F at a given number (it is desired that F, in addition to verifying $F' = f$*, should confirm* $F(x_0) = y_0$*), then there is only one function F answering the question.*

For example, taking the previous notations, the function z is the unique primitive of v for which $z(0) = 200$*.*

6.2.2 Calculating Primitives

From the formulas of the common derivatives, we can establish primitives of the common functions (Table 6.1).

Example 59 *Let us prove that A defined by* $A(t) = -\tau a_0 \exp\left(-\frac{t}{\tau}\right)$ *is a primitive of a defined by* $a(t) = a_0 \times \exp\left(-\frac{t}{\tau}\right)$.

We derive: A is of the form b exp (*u*) *whose derivative is equal to* $b \times u' \times \exp(u)$, *with* $b = -\tau a_0$, $u(t) = -\frac{t}{\tau}$, *and* $u'(t) = -\frac{1}{\tau}$.

So we have $A'(t) = -\tau a_0 \times \left(-\frac{1}{\tau}\right) \times \exp\left(-\frac{t}{\tau}\right) = a_0 \exp\left(-\frac{t}{\tau}\right) = a(t)$. *The function A is indeed a primitive of a.*

Note 56 *The primitive of some functions cannot be expressed from the functions used in this book, such as:* ϕ *defined by* $\phi(x) = \exp(-x^2)$.

Operations on the derivatives are simple for multiplication by constants and sums; they are more complicated for products and quotients. The properties of primitives are given in Table 6.2.

Example 60 *If* $h(x) = 0.3x^2 + 2e^{-x}$, *then H defined by* $H(x) = 0.1x^3 + (-2e^{-x}) = 0.1x^3 - 2e^{-x}$ *is a primitive of h.*

We considered $f(x) = 0.3x^2$ *and* $g(x) = 2e^{-x}$, *so that* $h = f + g$ *and* $F(x) = 0.1x^3$ *and* $G(x) = -2e^{-x}$.

If f is a function defined by $f(x) = 3.1x^3 - 7.4x + 9.0$, *then the function F defined by* $F(x) = \frac{3.1}{4}x^4 - \frac{7.4}{2}x^2 + 9.0x$ *is the*

Table 6.1 Primitives of common functions

Function name f	Expression	Primitive F (C Constant)
Constant	$f(x) = a$	$F(x) = ax + C$
Linear	$f(x) = ax + b$	$F(x) = \frac{1}{2}ax^2 + bx + C$
Power	$f(x) = x^a$, with $a \neq -1$	$F(x) = \frac{1}{a+1}x^{a+1} + C$
Exponential	$f(x) = e^x$	$F(x) = e^x + C$
	$f(x) = e^{ax}$, with $a \neq 0$	$F(x) = \frac{1}{a}e^{ax} + C$
	$a(t) = a_0 \times \exp\left(-\frac{t}{\tau}\right)$	$A(t) = -a_0 \times \tau \times \exp\left(-\frac{t}{\tau}\right) + C$
Inverse	$f(x) = \frac{1}{x}$	$F(x) = \ln(\lvert x\rvert) + C$
Cosine	$f(x) = \cos x$	$F(x) = \sin x + C$
Sine	$f(x) = \sin x$	$F(x) = -\cos x + C$
	$f(t) = \sin(\omega t + \phi)$, with $\omega \neq 0$	$F(t) = -\frac{1}{\omega}\cos(\omega t + \phi) + C$

Table 6.2 Properties of primitives (Let F and G be the respective primitives of f and g. a is a constant and u a function)

	Function	Primitive
Multiplication by constant	$a \times f$	$a \times F$
Sum	$f + g$	$F + G$
Composition (chain rule)	$u' \times f(u)$	$F(u)$
Example of composition	$g(x) = f(ax)$ with $a \neq 0$	$G(x) = \frac{1}{a}F(ax)$

primitive of f, which has values 0 in 0 (we have considered the primitives term by term).

A primitive of the function g defined by $g(x) = xe^{-x^2}$ *is G defined by* $G(x) = -\frac{1}{2}e^{-x^2}$.

Indeed, the function g is of the form $au' \exp(u)$, *where* $u(x) = -x^2$ *and* $u'(x) = -2x$; *therefore, with* $a = -\frac{1}{2}$, *we have a primitive of the form* $a \exp(u)$.

6.3 Calculation of Integrals

6.3.1 Calculating Integrals: Using Primitives

We have introduced the notion of primitives in order to invoke the link with the computation of integrals (this is what interested us in the first place). We admit the following property.

Theorem 9 (Property (Calculation of an Integral Using a Primitive)) *Let f be a function and F a primitive of f. We have:*

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Note 57 *We have* $\int_a^a f(x)\mathrm{d}x = 0$. *This can be justified in two ways:*

- *The area of the field considered is clearly zero (it has no "thickness");*
- *The integral is* $F(a) - F(a) = 0$.

Example 61 *We want to calculate* $\int_0^{\frac{\pi}{4}} \cos(2x)\mathrm{d}x$.

A primitive of f (with $f(x) = \cos(2x)$*) is F with* $F(x) = \frac{1}{2}\sin(2x)$. *Thus, we have* $\int_0^{\frac{\pi}{4}} \cos(2x)\mathrm{d}x = \frac{1}{2}\sin\left(2 \times \frac{\pi}{4}\right) - \frac{1}{2}\sin(2 \times 0) = \frac{1}{2} \times 1 - 0 = \frac{1}{2}$.

There is no need to introduce the notations f and F to perform the calculation; you can directly write the primitive between brackets:

$$\int_0^{\frac{\pi}{4}} \cos(2x)\mathrm{d}x = \left[\frac{1}{2}\sin(2x)\right]_0^{\pi/4}$$
$$= \frac{1}{2}\sin\left(2 \times \frac{\pi}{4}\right) - \frac{1}{2}\sin(2 \times 0) = \frac{1}{2}.$$

The brackets on $[F(x)]_a^b$ *mean* $F(b) - F(a)$:

$$\int_0^a 3te^{-2t^2}\mathrm{d}t = \left[-\frac{3}{4}e^{-2t^2}\right]_0^a$$
$$= \frac{3}{4}\left(1 - e^{-2a^2}\right).$$

Note 58 *The variable used is no longer displayed after calculation of the integral (which is normal: an area is calculated; we get a constant, not a function), so the choice of the letter used for integration is not relevant. For example, we have*

$$\int_0^{\frac{\pi}{4}} \cos(2x)\mathrm{d}x = \int_0^{\frac{\pi}{4}} \cos(2t)\mathrm{d}t$$
$$= \int_0^{\frac{\pi}{4}} \cos(2y)\mathrm{d}y\left(= \frac{1}{2}\right).$$

Note 59 *If we choose another primitive for f, for example G = F + c instead of F, the result of the integral is the same (fortunately, the area under the curve does not depend on the method used to calculate it!). Indeed,* $G(b) - G(a) = (F(b) + C) - (F(a) + C) = F(b) - F(a)$ *(the constants cancel each other).*

6.3.2 Some Integral Properties

An integral corresponds to the area under the curve representing a function, and if the domain is divided in two, the sum of the two areas corresponds to the total area. This gives the following property for integrals (Fig. 6.4).

Theorem 10 (Property (Breaking Intervals, Additive Property)) *Let f be a function for which an integral is computed over an interval* $[a, b]$, *and let c be an element of this interval* $[a, b]$. *We have*

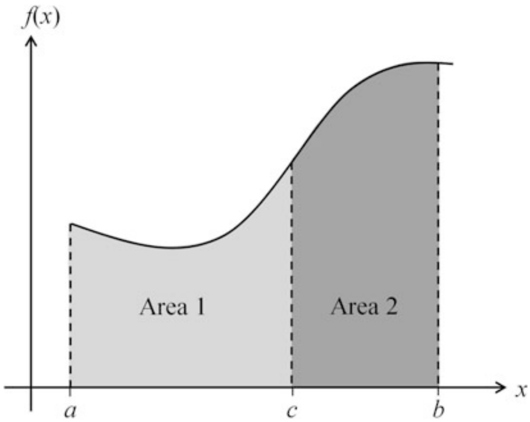

Fig. 6.4 The additive property relation establishes that $\int_a^b f(x)\mathrm{d}x = \text{Area}\,1 + \text{Area}\,2$, with $\text{Area}\,1 = \int_a^c f(x)\mathrm{d}x$ and $\text{Area}\,2 = \int_c^b f(x)\mathrm{d}x$.

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$$

Example 62 *We consider again the situation of a flow Q over a period $[0; T]$ defined from two functions Q_1 and Q_2 as follows:*

$$Q(t) = \begin{cases} Q_1(t) & \text{if } t \in [0; T_a], \\ Q_2(t) & \text{if } t \in]T_a; T]. \end{cases}$$

We then have

$$V = \int_0^T Q(t)\mathrm{d}t = \int_0^{T_a} Q_1(t)\mathrm{d}t + \int_{T_a}^T Q_2(t)\mathrm{d}t.$$

The properties of multiplication by a constant and of sum seen for primitives lead to analogous properties for integrals.

Theorem 11 (Property (Multiplication by a Constant and Sum)) *Let f and g be two functions, $[c; d]$ be an interval, and a be a constant. We have*

$$\int_c^d a \times f(x)\mathrm{d}x = a\int_c^d f(x)\mathrm{d}x,$$

$$\text{or } \int_c^d (af) = a\int_c^d f,$$

$$\text{and } \int_c^d (f(x)+g(x))\mathrm{d}x = \int_c^d f(x)\mathrm{d}x + \int_c^d g(x)\mathrm{d}x,$$

$$\text{or } \int_c^d (f+g) = \int_c^d f + \int_c^d g.$$

Example 63 *It is advisable to use the two foregoing properties (particularly to remove the constants from the integral) in order to make the function that must be primitive easier to recognize.*

For example, to calculate

$$I = \int_0^T \left[\frac{\lambda\phi_0}{k}\cos(\omega t + \phi) - \mu t^2\right]\mathrm{d}t,$$

$$\text{write } I = \frac{\lambda\phi_0}{k}\int_0^T \cos(\omega t + \phi)\mathrm{d}t - \mu\int_0^T t^2\mathrm{d}t,$$

where the constants λ, ϕ_0, k, and μ are removed from the integrals. Thus, we can more easily see the primitives to carry on the calculation.

The integral of a function is a number (quantity) having a unit. We have spoken of an area under the curve for an the integral, but the integral is not necessarily an area quantity; it depends on the unit of x and that of $f(x)$. The integral is obtained by the product of quantities x by quantities $f(x)$ (and then there the results are summed). So we have the following property.

Theorem 12 (Property (Dimension of Integral)) *The integral of a function is a number whose unit is the product of the unit of the function and that of the variable of integration.*

Example 64 *On the previous example concerning the calculation of volume:*

$$V = \int_0^{300} Q(t)\mathrm{d}t = \int_0^{300} \frac{3t^2}{80{,}000}\exp\left(-\frac{t}{200}\right)\mathrm{d}t.$$

The flow function Q is expressed in m^3/s and the variable t in s. Volume V is therefore expressed in $m^3/s \times s$, or in m^3: it is a unit of volume, and it is coherent. Here, the "area under the curve" is a volume!

We have introduced the integral as the area under the curve of a function, and we have presented this for a positive function.

Nevertheless, we can integrate a negative function or a function that does not keep a constant sign. We can no longer speak of an area (or rather: we associate a negative sign with the area obtained when the function is negative), but integral calculus is possible.

Example 65 *Consider the function* sin *and calculate its integral between the bounds $-\pi$ and π (Fig. 6.5). We have*

$$\begin{aligned} I &= \int_{-\pi}^{\pi} \sin(x)\mathrm{d}x \\ &= [-\cos(x)]_{-\pi}^{\pi} \\ &= -\cos(\pi) + \cos(-\pi) \\ &= -(-1) + (-1) \\ &= 0. \end{aligned}$$

It is also possible to calculate I by dividing it into two integrals (thanks to the additive property):

$$\begin{aligned} I &= \int_{-\pi}^{0} \sin(x)\mathrm{d}x + \int_{0}^{\pi} \sin(x)\mathrm{d}x \\ &= [-\cos(x)]_{-\pi}^{0} + [-\cos(x)]_{0}^{\pi} \\ &= (-1-1) + (-(-1)+1) \\ &= -2+2 \\ &= 0. \end{aligned}$$

Definition 23 (Average Value of a Function) *The* ***average value of a function*** *f over an interval* $[a; b]$ *is given by*

$$\frac{1}{b-a}\int_{a}^{b} f(t)\mathrm{d}t.$$

This definition is consistent with the classical definition of an average for which the sum of the results is divided by the total number of values. Here the "total number of values" is equal to $b - a$ *(width of the interval over which the function f is considered), and the sum of the values f(t) is achieved thanks to the integral.*

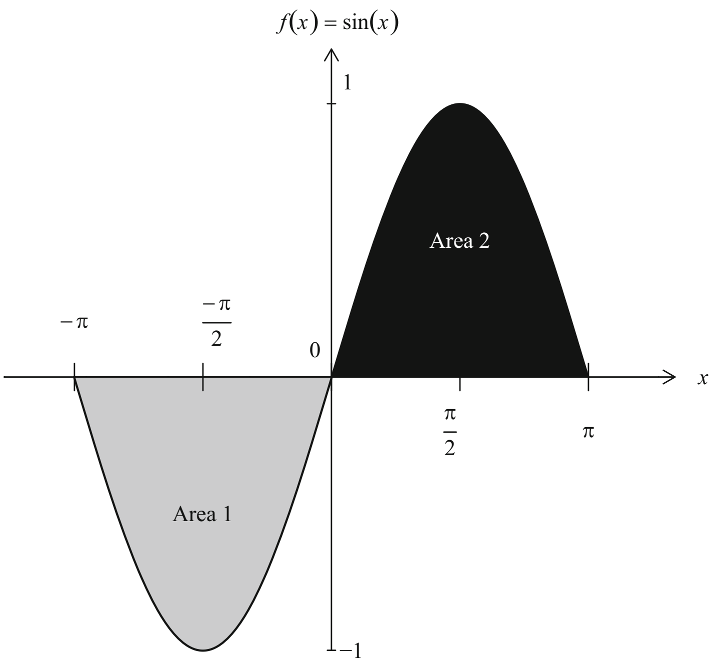

Fig. 6.5 Use of the additive property to calculate $\int_{-\pi}^{\pi} \sin(x)\mathrm{d}x$. It gives us: $\int_{-\pi}^{\pi} \sin(x)\mathrm{d}x = 0 = \text{Area 1} + \text{Area 2}$, with $\text{Area 1} = \int_{-\pi}^{0} \sin(x)\mathrm{d}x = -2$ and $\text{Area 2} = \int_{0}^{\pi} \sin(x)\mathrm{d}x = +2$.

Example 66 *In the previous example concerning the calculation of volume, the average flow rate is given by* $\frac{1}{T}\int_0^T Q(t)\mathrm{d}t$ *in units of* m^3/s.

6.3.3 Calculation of Integrals: Using Calculation Software

With the Xcas software (Appendix B), primitives can be calculated using the command *int* or the symbol $\int$ in the menu (written *integrate* in the script). If the variable is not specified, the variable x is selected.

Example 67 *Figure 6.6 shows the calculation of primitive functions* $x \mapsto x^2$, $x \mapsto x \times y$, $t \mapsto \cos(\omega t + \phi)$, *and* $x \mapsto \exp(-x^2)$.

The last example shows in the primitive a particular function, denoted erf, which confirms that we cannot find a primitive of the function $x \mapsto \exp(-x^2)$ *expressed by means of the usual functions studied in this book.*

By specifying the integration bounds (with the possibility of the bounds $\pm\infty$ using the command *infinity* or the keyboard), we can calculate integrals (Fig. 6.7).

One often needs only the approximate value of an integral: use the command *evalf* before carrying out the calculation (Fig. 6.8).

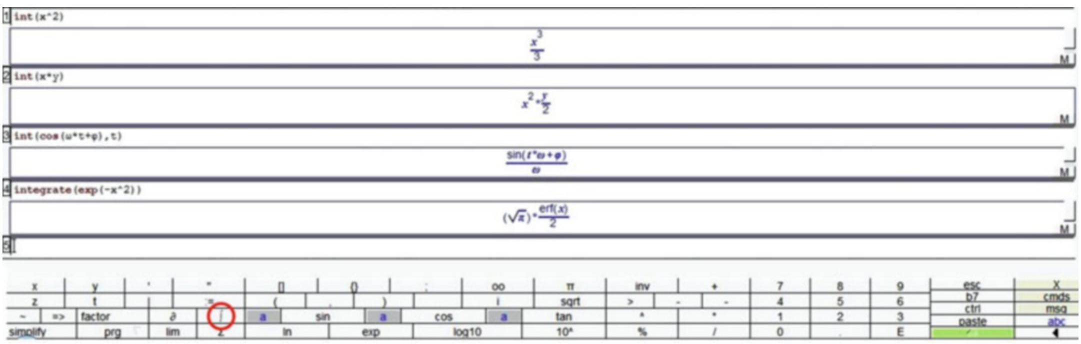

Fig. 6.6 Calculation of primitives in Xcas software. The symbol $\int$ is enclosed in the figure to locate it

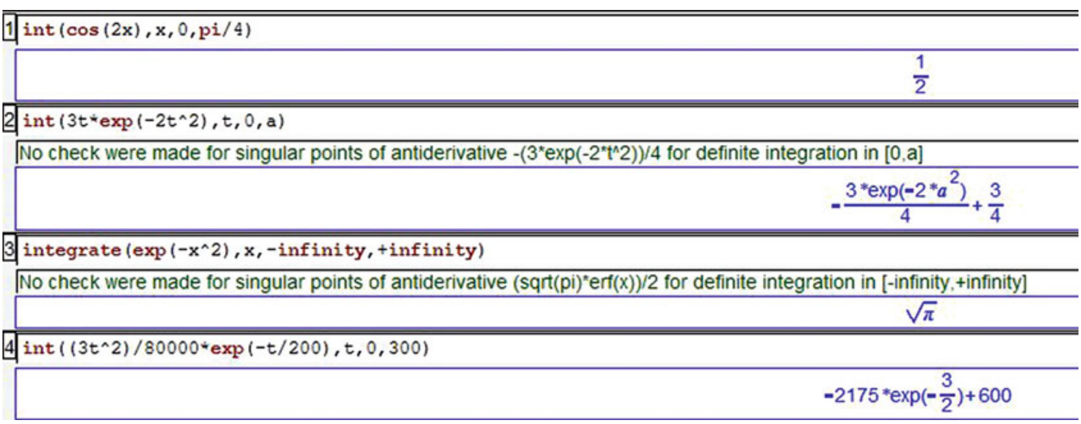

Fig. 6.7 Integral calculation with Xcas software

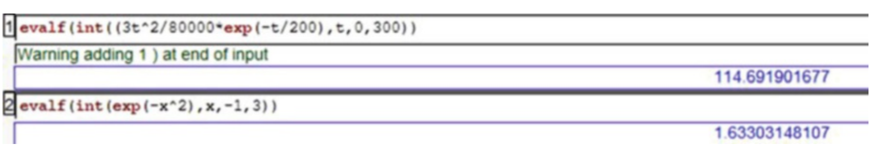

Fig. 6.8 Approximate values of integrals with Xcas software

6.4 Calculating Approximate Values of Integrals by the Rectangle Method

Let us return to the initial situation of the calculation of the global volume from the knowledge of the flow rate (continuous function Q of the variable t). Suppose now that the function Q is not known, but that regular measurements are taken at t_i, giving the associated value $Q(t_i)$. How is it possible to calculate the integral of Q based on these data?

We must first know what values are known. Most often, measurements are made regularly over the entire duration: the measurements are spaced apart with a fixed duration Δt and N measurements are made. The first measurement at $t = \Delta t$ is denoted t_1, the second measurement at $t = 2\Delta t$ is denoted $t_2, \ldots$, the *ith* measurement at $t = i \times \Delta t$ is denoted t_i (Fig. 6.9).

For the values of t_i thus indicated, the values $Q(t_i)$ are measured and denoted $Q_i = Q(t_i)$ (Fig. 6.10).

Next, we create around t_i an interval of width Δt and the corresponding rectangle of height Q_i (Fig. 6.11).

An approximate value of the integral of the function Q between the bounds 0 and T is given by the sum of the areas of the (adjacent) rectangles obtained, and this is written

$$\int_0^T Q(t)\mathrm{d}t \simeq \sum_{i=1}^{i=N} Q_i \times \Delta t$$
$$= \Delta t \sum_{i=1}^{i=N} Q_i.$$

We used the notation $\sum$ for the sums: this is the symbol used in the case of a discrete sum. Exercise 6.7 goes over how to use this symbol.

Note 60 *There are other approximation methods, for example that use trapezoids rather than rectangles or in which rectangles are placed otherwise.*

Example 68 *We consider again the function* $Q(t) = \dfrac{3t^2}{80{,}000}\exp\left(-\dfrac{t}{200}\right)$ *defined over the interval* $[0; 300]$. *A measurement is made every 10 s (that is,* $\Delta t = 10$ *s, and there are 300/10* $= 30$ *measurements in total). The following values are obtained:*

$$Q_i = \frac{3(\Delta t)^2 \times i^2}{80{,}000}\exp\left(-\frac{\Delta t}{200}i\right)$$
$$= \frac{3i^2}{800}\exp\left(-\frac{1}{200}i\right).$$

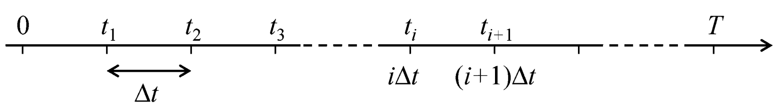

Fig. 6.9 Discretization of time: continuous time is divided into a finite number of time intervals

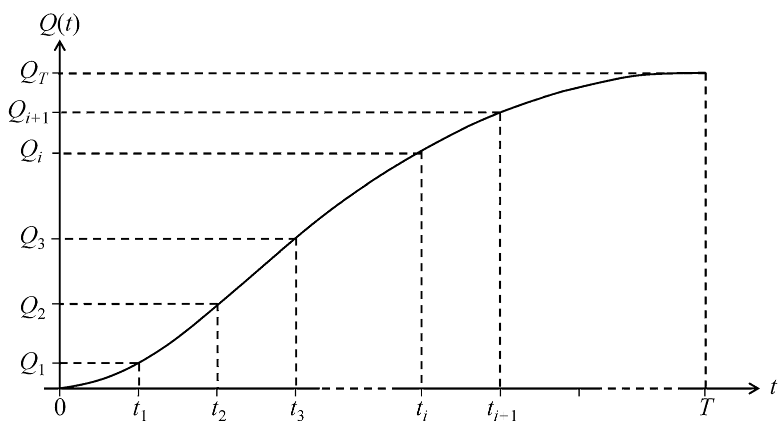

Fig. 6.10 Discretization of curve

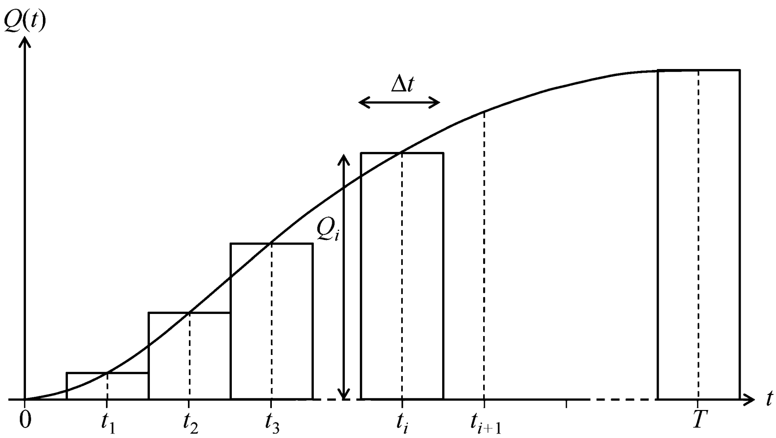

Fig. 6.11 Principle of rectangle method

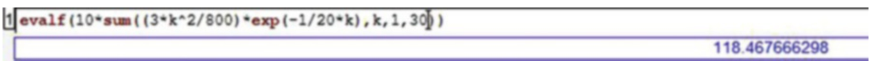

Fig. 6.12 Rectangle method for calculating V

This gives the approximate value of the volume:

$$V \approx 10\sum_{i=1}^{i=30}\frac{3\times i^2}{800}\exp\left(\frac{1}{200}i\right)$$
$$\approx \frac{3}{80}\sum_{i=1}^{i=30} i^2\exp\left(-\frac{1}{200}i\right).$$

With Xcas (using the letter k instead of i, which has another meaning in Xcas), we find V $\approx$ *118 m*^{3} *(instead of about 115 m*^{3}*, which is a good approximation) (Fig. 6.12).*

Insert 11 (Zeno's Dichotomy Paradox)

"That which is in locomotion must arrive at the half-way stage before it arrives at the goal"– Aristotle.

A runner on a racetrack must first reach half the distance to the goal, then half the remaining distance, then half of what is left, and so on. Does that mean that the runner will never reach the finish line? Reality proves this wrong, and therein lies the paradox. This paradox was stated by, among others, Zeno of Elea (a pre-Socratic Greek philosopher).

Let us say that the distance to run is L meters. The runner travels half the distance $\left(\frac{L}{2}\right)$*, then half the remaining distance* $\left(\frac{L}{4}\right)$*, and so on. Therefore, the total distance to run is* $\frac{L}{2}+\frac{L}{4}+\frac{L}{8}+\cdots$*. Those three small dots mean that there are an infinity of distances to reach, these distances becoming smaller and smaller as the runner approaches her target. This total distance to be run can thus also be written*

$$\sum_{k=1}^{\infty}\frac{L}{2^k} = L\sum_{k=1}^{\infty}\frac{1}{2^k}.$$

The paradox arises from the intuition that an infinite sum cannot give a finite result. But this intuition is wrong. Indeed, it can be shown that the sum $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ *is a finite number (here 1 in this case) and therefore that the infinite sum* $\frac{L}{2}+\frac{L}{4}+\frac{L}{8}+\cdots = L\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)$ *gives a finite result, the distance L to the finish line. Good news: this is consistent with observation.*

6.5 Differential Equations

6.5.1 Introduction

Think of a watershed by a tank. Rain falling on the watershed is an entrance signal of the system. This amount of water entering the system accumulates and eventually flows to the outlet of the watershed. The various quantities are quantified in functions of time (Fig. 6.13): at time t, R(t) is the intensity of the rain (m/s), $Q(t)$ the outflow (m^3/s), and $V(t)$ the volume of water stored in the catchment area (m^3).

Changes in the water stock depend on incoming rain and outflow. The change in stock over a short period dt is expressed by $\mathrm{d}V(t) = (A \times R(t) - Q(t))\mathrm{d}t$. So we have

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) = A \times R(t) - Q(t), \qquad (6.1)$$

where A is the area of the watershed (m^2).

If we put forth the hypothesis (it is a model) that there is a linear relation between the stock $V(t)$ and the flow rate $Q(t)$ (that is, $Q(t) = \alpha V(t)$), relation (6.1) becomes

$$\frac{1}{\alpha}\frac{\mathrm{d}Q}{\mathrm{d}t}(t) + Q(t) = A \times R(t). \qquad (6.2)$$

In this relationship, the intensity of the rain $R(t)$ is known and the flow rate $Q(t)$ at the outlet of the watershed is unknown. The function Q satisfies a relationship between itself and its derivative: this relationship is called a **differential equation.**

In geography and Earth science, approaches analogous to that just described are numerous and can be modeled by a system-type approach (Fig. 6.14). The changes in mass or energy in the system are evaluated according to its inputs and outputs. The transfer and storage of this mass or energy are subject to conservation and transport laws, and this balance of mass and energy leads to differential equations (thanks to the link between the variations of a function and its derivative).

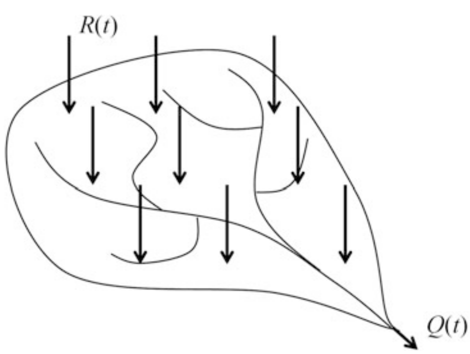

Fig. 6.13 Conceptualization of flows on a watershed

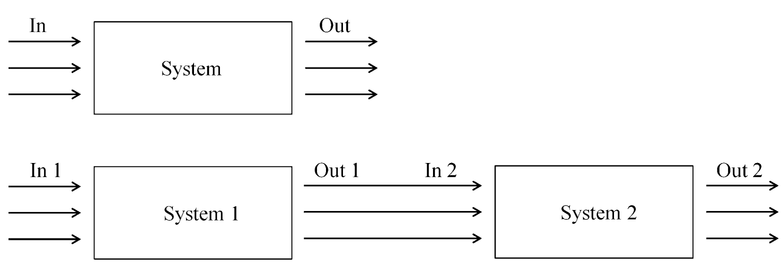

Fig. 6.14 Systemic approach: the variation of mass or energy within the system depends on the outputs and inputs

6.5.2 First Elements on Differential Equations

We have seen that solving an equation amounts to finding all the values of a variable (unknown) satisfying an equality (e.g., the equation $3x^2 = 12$ has two solutions: $x = -2$ and $x = 2$).

Similarly, a differential equation is an equation involving an unknown function (of a variable, for example, x or t) and where the equation relates this function and one or more of its successive derivatives. We call these equations **ordinary differential equations** (**ODEs**). Solving a differential equation means finding all the functions that satisfy the equality.

Example 69 (Examples of ODEs)

- $\frac{1}{\alpha}\frac{\mathrm{d}}{Q\mathrm{d}t}(t) + Q(t) = P(t)$, *where the function P is an unknown function of t;*
- $f' = f$ *(also written $y' = y$);*
- $f'' = -ax$ *(also written* $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -ax$ *or* $\frac{\partial^2 f}{\partial x^2} = -ax$*);*
- $y'' - 3y' + 2y = \cos(x)$.

Example 70 (Example of Solution) *The function f defined by* $f(x) = \exp(-2x) - \frac{1}{10}(\cos x + 3\sin x)$ *is a solution of the differential equation* $y'' + y' - 2y = \sin x$.

Indeed, we have

$$f'(x) = -2\exp(-2x) - \frac{1}{10}(-\sin x + 3\cos x)$$
$$\text{and } f''(x) = 4\exp(-2x) - \frac{1}{10}(-\cos x - 3\sin x).$$

So we have

$$\begin{aligned} f''(x) + f'(x) - 2f(x) &= 4\exp(-2x) - \frac{1}{10}(-\cos x - 3\sin x) \\ &\quad - 2\exp(-2x) - \frac{1}{10}(-\sin x + 3\cos x) - 2\left[\exp(-2x) - \frac{1}{10}(\cos x + 3\sin x)\right] \\ &= 4\exp(-2x) - 2\exp(-2x) - 2\exp(-2x) \\ &\quad - \frac{1}{10}[-\cos x - 3\sin x - \sin x + 3\cos x - 2\cos x - 6\sin x] \\ &= \sin x. \end{aligned}$$

This is what was expected.

6.5.3 Solving a Differential Equation

In this section, only one method of analytically solving differential equations is given; there are others (e.g., Bernoulli, Ricatti), which are not mentioned here.

When trying to **solve** a differential equation, we want to find all the possible functions satisfying the equation.

Example 71 *We want to solve $y' = y$ (where y is a function of the variable x).*

The constantly zero function is obviously a solution.

If y does not take the value zero, then we can write $\frac{y'}{y} = 1$, *and therefore, by considering the antiderivatives on both sides, we have*

$$\begin{aligned} \ln|y| &= x + C, \\ \text{which leads to } |y| &= \exp(x + C) \\ &= e^x \times e^C, \\ \text{which gives } y &= \pm e^C e^x, \end{aligned}$$

so y can be written in the form Ke^x, where K is a constant with respect to the variable x.

It can be easily verified that all functions of the form $y(x) = Ke^x$ (where K is a number) are indeed solutions (including for $K = 0$, which gives the null function).

Let us prove that every solution function is of the form $y(x) = Ke^x$, with K a real number. Suppose that y is a solution (that is, $y' = y$), and consider $u(x) = y(x)e^{-x}$ for all x. By deriving (a product formula), we have $u'(x) = y'(x)e^{-x} - y(x)e^{-x}$. As $y'(x) = y(x)$ by hypothesis, therefore $u'(x) = y(x)e^{-x} - y(x)e^{-x} = 0$, so u is a constant function K. Thus, $u(x) = K = y(x)e^{-x}$, and so $y(x) = Ke^x$. That is what we wanted.

Let us consider the differential equation $f'' = ax$. Thus, we have $f'(x) = \frac{1}{2}ax^2 + C$, and so $f(x) = \frac{1}{6}ax^3 + Cx + D$, where C and D can take any constant values.

We will give only one theoretical result and one solution method and focus mainly on the use of formal calculation software.

Theorem 13 (Property (Linear Homogeneous Differential Equation of the First Order)) *The solutions of the differential equation $y' = ay$ (which can also be written $y' - ay = 0$) are the functions $f(x) = Ke^{ax}$, where K is any real number.*

Example 72 *The differential equation $\frac{1}{\alpha}\frac{\mathrm{d}Q}{\mathrm{d}t}(t) + Q(t) = 0$ (i.e., the homogeneous version of the rain/flow equation indicated earlier) has the solution $Q(t) = Ke^{\alpha t}$.*

Definition 24 (Solution Method in the Case of an ODE of the First Order with Separable Variables) *This is a differential equation of the form $y'g(y) = f(x)$ (we also have $y' = \frac{f(x)}{g(y)}$). It can also be written $\frac{\mathrm{d}y}{\mathrm{d}x}g(y) = f(x)$.*

Method for solution:

- *First separate the variables x and y: $g(y)\,\mathrm{d}y = f(x)\mathrm{d}x$;*
- *Calculate both primitives (they are also equal): $\int g(y)\mathrm{d}y = \int f(x)\mathrm{d}x$;*
- *Use the initial condition $y(0) = y_0$ for $x = 0$;*
- *Integrate between the bounds y_0 and $y(x)$ ($y(x)$ becomes a variable) and from 0 to x. For more clarity, the variables' name is changed inside integrals. It becomes $\int_{y_0}^{y(x)} g(u)\mathrm{d}u = \int_0^x f(v)\mathrm{d}v$;*
- *Finally, calculate these integrals and obtain a relation between y(x) and x, without y'. This is what we wanted.*

Example 73 *Let us consider again the differential equation $y' = y$, which is also written $\frac{y'}{y} = 1$. We are in the situation of separable variables.*

Thus, we have $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, or $\frac{1}{y}\mathrm{d}y = \mathrm{d}x$. From there $\int_{y_0}^{y(x)} \frac{1}{u}\mathrm{d}u = \int_0^x 1\mathrm{d}x$; and so $\ln(y(x)) - \ln(y_0) = x - 0$, which gives $\ln\left(\frac{y(x)}{y_0}\right) = x$. This equation of variable y(x) is solved using the function exp to "neutralize" the logarithm: $\frac{y(x)}{y_0} = \ln x$, and so $y(x) = \ln(y_0)e^x$. This is indeed the form obtained previously. It should be noted that the method can only be carried out where the expressions are defined (for example, to write $\ln(y)$, it is necessary that $y > 0$).

Example 74 *Consider the differential equation $y' = \frac{1}{1+y}$.*

We write $(1+y)y' = 1$, which is indeed the form of the previous method (separable variables). We have successively

$$\begin{aligned}(1+y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 1,\\ (1+y)\mathrm{d}y &= \mathrm{d}x,\\ \int_{y_0}^{y(x)}(1+u)\mathrm{d}u &= \int_0^x \mathrm{d}v,\\ \left[u+\frac{1}{2}u^2\right]_{y_0}^{y(x)} &= [v]_0^x,\\ y(x)+\frac{1}{2}y(x)^2-y_0-\frac{1}{2}y_0^2 &= x.\end{aligned}$$

$y(x)$ is a solution of an equation (which is not a differential equation) that can be solved to find the expression of $y(x)$ in terms of x. For example, for $y_0 = 0$, we solve the equation $\frac{1}{2}y(x)^2 + y(x) - x = 0$ of variable $y(x)$.

Xcas can be used to complete the solution. This leads to two solutions, one of which is negative (this negative solution is most often discarded because the variables used are generally positive). Here, $y(x) = \sqrt{1+2x} - 1$.

6.5.4 Use of Formal Calculation Software

Xcas (Appendix B) solves differential equations very efficiently (formally, with or without initial conditions, or in an approximate way).

Example 75 *Use the command "desolve" for exact resolutions. Caution: use only y (and its derivatives y', $y''\ldots$) for the unknown function and not another letter (Fig. 6.15).*

Example 76 *The rain/flow model: we consider the case where the rain function is given by $P(t) = 1.2e^{-0.2t}$ with t in minutes, 1.2 in mm/min, and 0.2 in min^{-1}.*

The solution is proposed in three cases (Fig. 6.16):

- *Solution of differential equation in the general case: $\frac{1}{\alpha}\frac{\mathrm{d}Q}{\mathrm{d}t}(t) + Q(t) = 1.2e^{-0.2t}$;*

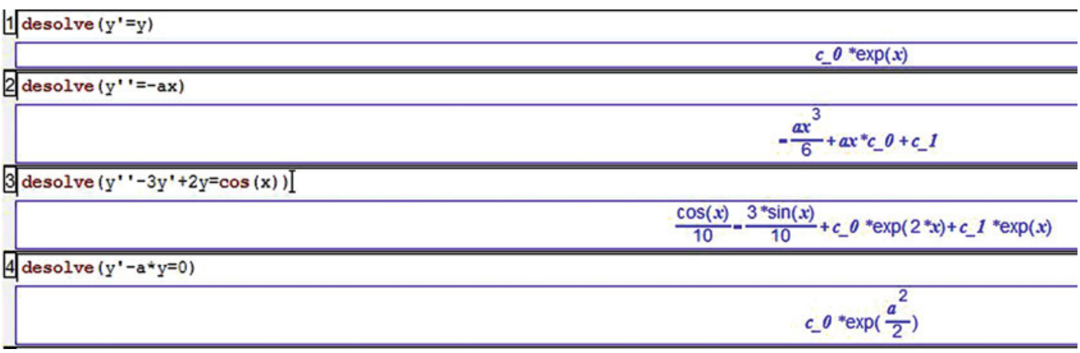

Fig. 6.15 General solution of differential equations with Xcas

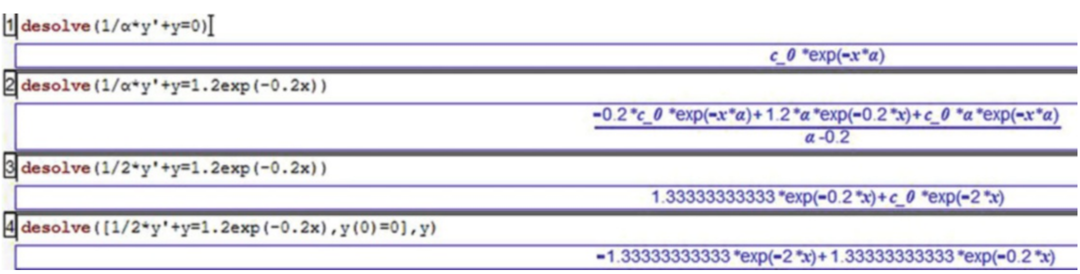

Fig. 6.16 Solving rain/flow differential equation with Xcas

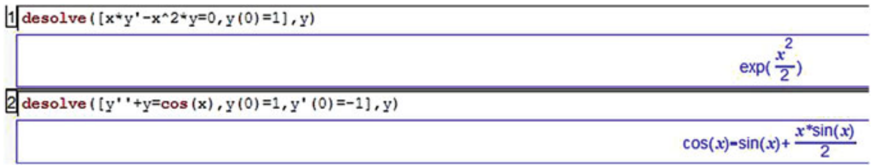

Fig. 6.17 Solution with initial condition

- *Solution of differential equation taking* $\alpha = 2\,\mathrm{min}^{-1}: \frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}t}(t) + Q(t) = 1.2e^{-0.2t}$;
- *Solution of differential equation taking the initial condition of no initial flow* ($Q(0) = 0$).

Example 77 *Solution with one or more initial conditions (Fig. 6.17).*

We can also seek a solution using numerical methods (approximate resolution) for a differential equation of the type $y' = f(x, y)$, *for example. The command*

*odesolve (cos (x * y-1), [x, y], [0.2],1)*

gives a value of y(1), where y is the solution of the differential equation $y' = \cos(x \times y - 1)$ *with* $y(0) = 2$.

Insert 12 (The Finite Difference Method)

In Earth science and geography, modeling a system often involves a series of equations. The equation of continuity, or of conservation of mass, makes it possible to take into account the amount of matter that enters and leaves the system. This equation very often produces one or more differential equations that describe the functioning of the system. Some differential equations are relatively easy to solve: a sheet of paper and a pencil are sufficient to find a solution, which is then called analytical.

But in most cases the differential equations do not have analytical solutions, so another method is needed to solve them. These are usually computer-assisted numerical methods. They are varied and more or less complicated to implement: they are, for example, the finite-element method, the multiagent method, and the finite-difference method.

The latter is one of the easiest to implement. Consider the steady-state 1D transport function

$$D\frac{\partial^2 C}{\partial x^2} - U\frac{\partial C}{\partial x} = 0. \qquad (6.3)$$

This function is used to calculate the concentration C of a pollutant in an aquifer whose Darcy velocity is U and dispersion is D. To solve this equation using the finite-difference method, we discretize the space x with a step Δx *(same as in the rectangle method to approximate the value of an integral). One obtains points of the function C with coordinates* (x_i, C_i). *The derived number at x between*x_i *and* x_{i+1} *is assimilated to the slope between two consecutive points:*

$$\frac{\partial C}{\partial x}(x) = \frac{C_{i+1} - C_i}{\Delta x}. \qquad (6.4)$$

Similarly, for the second derivative, we have

$$\frac{\partial^2 C}{\partial x^2}(x) = \frac{C_{i+1} + C_{i-1} - 2C_i}{\Delta x^2}. \qquad (6.5)$$

(continued)

Replacing Eqs. 6.4 and 6.5 in 6.3 one obtains

$$C_i = \frac{1}{2 - \frac{U\Delta x}{D}}\left[\left(1 - \frac{U\Delta x}{D}\right)C_{i+1} + C_{i-1}\right].$$

One can thus express the concentration C on the interval i (C_i) as a function of its neighboring values (C_{i-1} and C_{i+1}). If the axis of the variable x has intervals, we will obtain a system with n equations and n unknown values that can be solved through a matrix method (e.g., Gauss-Seidel or Jacobi). That would take us beyond the scope of this book.

6.6 Partial Differential Equations

6.6.1 Introduction, Definition

The purpose of this section is simply to explain what a **partial differential equation** (PDE) is, but without considering its solution.

A PDE is an equation with one or more unknown functions **of several variables** and where the equation involves one or more partial derivatives.

Example 78 *For a function f of variables x and y, the PDE $\frac{\partial f}{\partial x} = 0$ has the solution $f(x, y) = g(y)$ (where g is any function: in reality f only depends on y or is a constant).*

This is to be compared to the differential equation $\frac{\mathrm{d}f}{\mathrm{d}x} = 0$ (it is then implied that f is a function of the single variable x), where the solution is $f(x) = c$, with c constant.

Definition 25 (Some Common PDEs)

- *Diffusion equation: $\frac{\partial C}{\partial t} = K\frac{\partial^2 C}{\partial x^2}$, where C is the concentration, a function of time t and the x-coordinate, and K is a constant.*
- *Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.*
- *Equation of heat: $\frac{\partial T}{\partial t} = \frac{\kappa}{\rho c}\frac{\partial^2 T}{\partial x^2}$. Mathematically, this is the same equation as the preceding diffusion equation.*

Example 79 *The function T defined by $T(x,t) = T_0 \exp\left(-\sqrt{\frac{\omega\rho c}{2\kappa}}x\right)\sin\left(\omega t - \sqrt{\frac{\omega\rho c}{2\kappa}}x\right)$* is a solution of the heat equation (with T_0 and ω parameters).

To verify this, it is sufficient to show that $\frac{\partial T}{\partial t} = \frac{\kappa}{\rho c}\frac{\partial^2 T}{\partial x^2}$. Calculations are complicated (but can be done manually if desired). The Xcas software makes it possible to calculate partial derivatives (Fig. 6.18). If E is an expression (a function of x and y), use the command diff(E, $x\$$ n, $y\$$ m) to have the partial derivative of order n in x and m in y.

6.6.2 Solving a PDE

Solving a PDE is often difficult (if not impossible) in a complete way. We can sometimes consider additional conditions.

Definition 26 (Stationary Solutions) ***Stationary solutions*** *are solutions that are functions of only one variable.*

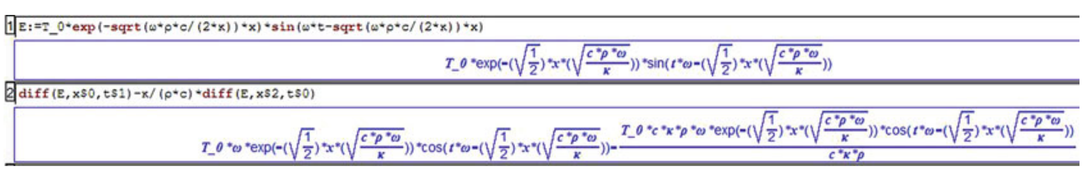

Fig. 6.18 Solution of a PDE

Example 80 *For the heat equation* $\frac{\partial T}{\partial t} = \frac{\kappa}{\rho c}\frac{\partial^2 T}{\partial x^2}$, *if we consider in addition that* $T(x, t) = T(x)$, *then we have* $\frac{\partial T}{\partial t} = 0$, *and so* $\frac{\partial^2 T}{\partial x^2} = 0$. *From this we deduce* $\frac{\partial T}{\partial x} = a$ *and that* $T(x) = ax + b$, *where a and b are constants.*

Most elements on formal (or exact) solutions go beyond the scope of this book. The same holds true for approximate solutions (e.g., by the finite-difference method).

Key Points

- The integral $\int_a^b f(t)\mathrm{d}t$ is the area under the representative curve of f between the abscissa a and b (taking into account the sign of the function).
- If f is a function, then a primitive of f is a function F such that $F' = f$.
- With the previous notation, $\int_a^b f(t)\mathrm{d}t = F(b) - F(a)$. If we know a primitive of the function f, we can easily calculate any integral of f.
- Use the Xcas *int* command to calculate integrals.
- The rectangle method gives an approximate computation of an integral.
- An ODE is an equation involving an unknown function and one or more of its derivatives.
- The linear ODE $y' = ay$ has the solution $y(x) = Ke^{ax}$, where K is any constant.
- The *desolve* Xcas command solves ODEs.
- PDEs involve an unknown function with several variables. It is often impossible to fully solve a PDE.

Exercise

Mathematical Exercises

Exercise 6.1: Flash Questions. Series 1

(1) Is the function f defined by $f(x) = 3x^4 - 5x + 2$ a primitive of the function g defined by $g(x) = 12x - 5$?

(2) Calculate $\int_0^{\pi/2} \sin(x)\mathrm{d}x$.

(3) True or false: the function $t \mapsto \cos t$ is a solution of the differential equation $y' - 2y = t$?

(4) Find at least one common function solution of the differential equation $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -f$.

Exercise 6.2: Flash Questions. Series 2

(1) Find an antiderivative for the function $x \mapsto e^{-x}$.

(2) Calculate $\int_0^1 e^{-t}\mathrm{d}t$.

(3) Show that the function $u(t) = 2e^t - t^2 - 2t - 2$ is a solution of the differential equation $u' = u + t^2$.

(4) Solve the PDE $\frac{\partial f}{\partial x}(x, t) = t$.

Exercise 6.3: Flash Questions. Series 3

(1) Find a primitive for the cosine function and a primitive for the sine function.

(2) Calculate the following integral (depending on y): $\int_2^3 x^2 y^3 \mathrm{d}x$.

(3) Give all the solutions of the differential equation $y' - 3y = 0$.

(4) Show that the function defined by $u(x, t) = \cos(x - \alpha t)$ is a solution of the PDE $\frac{\partial u}{\partial x} + \frac{1}{\alpha}\frac{\partial u}{\partial t} = 0$.

Exercise 6.4: Calculating Primitives

(1) Find the primitive function of f defined by $f(x) = x^2 - 1$ that has a value of zero in 0, then one that has a value of zero in 1.

(2) Find the primitive function of g defined by $g(t) = \frac{2}{t} - t$ that has a zero value in 1, then that which has a value of 3 in e (recall that $\ln e = 1$).

Exercise 6.5: Computation of Integrals, Level 1

Calculate the following integrals (ideally without using calculation software):

(1) $\int_0^{\pi/2} \cos t \mathrm{d}t$

(2) $\int_0^1 \left(2t^3 - 3t + 4\right)\mathrm{d}t$

(3) $\int_0^{\ln(2)} (1 - e^t)\mathrm{d}t$

(4) $\int_0^{\pi} \sin\left(\frac{1}{4}x\right)\mathrm{d}x$

Exercise 6.6: Calculation of Integrals, Level 2

Calculate the following integrals (you may use calculation software):

(1) $\int_2^3 \frac{1}{x-1}\mathrm{d}x$

(2) $\int_0^{\pi/4} \cos(3.7t + 0.31)\mathrm{d}t$

(3) $\int_0^{+\infty} t\exp\left(-t^2\right)\mathrm{d}t$

(4) $\int_{-2}^{2} \left(7x^2 - 3x + 3\right)\mathrm{d}x$

Exercise 6.7: Manipulation of the Symbol $\sum$

The symbol $\sum$ (with the uppercase Greek letter Sigma, for "sum") is used to describe sums concisely.

For example, the sum $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$ is written $\sum_{k=1}^{k=7} k^2$ (read "sum for k varying from 1 to 7 of k^2") and $\sum_{k=3}^{k=100} k^2$ replaces $3^2 + 4^2 + 5^2 + \ldots + 99^2 + 100^2$.

(1) Using the symbol $\sum$ write the sums $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ and $\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{21}+\frac{1}{22}$.

Without using the symbol $\sum$ (and possibly by means of dotted lines) write the sums $\sum_{k=1}^{k=4} 3^k$ and $\sum_{k=1}^{k=1000} \frac{1}{k^2}$.

(2) In the equality

$$\sum_{k=1}^{k=7} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2,$$

we see that the choice of the letter used (called **index of summation**) does not matter. Thus, we have $\sum_{k=1}^{k=7} k^2 = \sum_{i=1}^{i=7} i^2 = \sum_{n=1}^{n=7} n^2$.

Moreover, the same sum can be described by an index with different initial and final values.

Show that $\sum_{k=1}^{k=100} k(k+1) = \sum_{i=2}^{101} i(i-1)$.

(3) Similarly to integrals, the linearity property applies for both the addition of sums and the multiplication in a sum by a constant independant of the index k

$$\sum_k af(k) = a\sum_k f(k) \text{ and}$$

$\sum k\left[f(k) + g(k)\right] = \sum_k f(k) + \sum_k g(k)$. This property makes it possible to simplify calculations to be carried out and to use known relations.

Calculate $\sum_{k=1}^{100} (2k-3)$ (use the formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$).

(4) Reread what was said about the rectangle method for calculating integrals. For each interval chosen, the central value was chosen for t_i. Redo the problem by retaining the left value for each interval (with for first value $t = 0$ for the interval [0; 10], then $t = 10$ for the interval [10; 20], and so forth) and write the formula giving an approximate value of the integral of the function Q studied. Calculate the approximate value thus obtained.

Exercise 6.8: Differential Equation and Initial Condition

Let us consider the differential equation $y' - 3.5y = 0$.

(1) Give the expression of the general solution of this differential equation.
(2) Draw three such solutions.
(3) Find the solution that satisfies the **initial condition** $y(0) = 8.2$.

Exercise 6.9: Solving Partial Differential Equations
In each of the following cases, solve the PDE (i.e., find all the functions satisfying the equation):

(1) $\frac{\partial u}{\partial x} = 1 + x + y^2$, where u is a function of (x, y)
(2) $\frac{\partial f^2}{\partial y^2} = \frac{1}{xy^2}$, where f is a function of (x, y) and $y > 0$
(3) $\frac{\partial u^2}{\partial x \partial y} = 1$, where u is a function of (x, y)

Exercises in Geography and Earth Science

Exercise 6.10: Heat Received on Earth, Continued
We revisit the situation examined in an exercise in Chap. 5 on the heat received on Earth. Recall that the amount of solar heat received on Earth depending on the day of the year can be modeled by $E(t) = C - C_0 \cos\left[\frac{2\pi}{365.25}(t - 2.72)\right]$, where E is in Joules/day, t is in days, and C and C_0 are constants.

(1) What is the amount of heat received during the first quarter of the year?
(2) What is the average amount of heat received per day?

Exercise 6.11: Calculation of the Volume of a Polar Ice Cap
We have seen that an integral is used to calculate an area (we cut the surface into rectangles of width dx, then we sum with an integral). An analogous method can also be used to calculate a volume: the volume is cut into elementary cylinders of height dz, then we sum up using an integral.

We will calculate the volume of ice contained in a Martian polar ice cap. The cap is a circular dome of radius $R_{\max}$ and height $H_{\max}$. The section of this cap confirms the relation $z^2 = 2\tau_0 \frac{R_{\max} - x}{\rho_G \times g}$ (the origin of the axes is at the center and at the base of the cap; x is the width and z the height of the dome), where τ_0 is a constant, $R_{\max}$ the (horizontal) radius of the cap, ρ_G the density of ice, and g the acceleration of gravity on Mars (Fig. 6.19).

Numerical data: $\tau_0 = 25{,}000$ Pa, $R_{\max} = 600$ km, $H_{\max} = 3$ km, $\rho_G = 900$ kg/m^3, and $g = 3.7$ m/s^2

(1) Express the radius r of a cylinder of ice at an altitude z in terms of z and the parameters given earlier (see also Fig. 6.20).
(2) Obtain from this the expression of the volume of a basic cylinder according to its radius $r(z)$ and thickness dz.
(3) By integrating the volume from the substratum/ice interface (altitude 0) to the top of the cap (altitude $H_{\max}$), determine the total volume of this cap, then its mass.

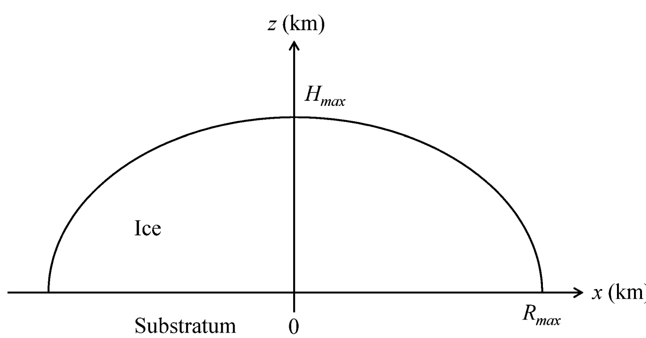

Fig. 6.19 Schematic of a Martian polar ice cap

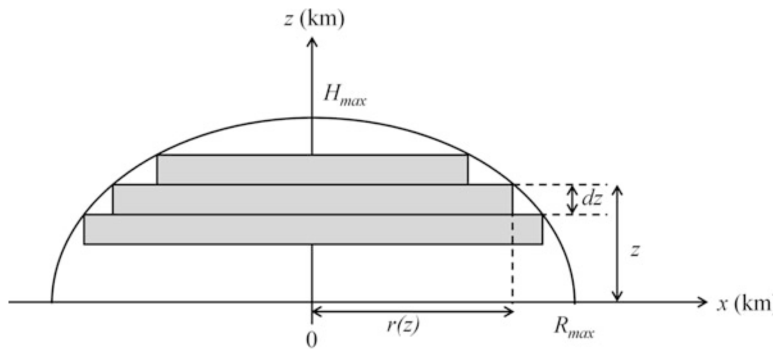

Fig. 6.20 Application of principle of rectangle method for calculating the volume of a Mars polar cap

Exercise 6.12: Moment of Inertia

For a ball of radius R whose density (variable ρ) is given by $\rho(r)$, where r is the distance to the center of the ball, the moment of inertia I is given by $I = \frac{8\pi}{3}\int_0^R \rho(r) r^4 \mathrm{d}r$. The unit of the moment of inertia is kg m^2

(1) Calculate the moment of inertia in the case of a homogeneous ball of constant density ρ.

In this case we are looking at the Earth, where $R = 6370$ km and $\rho = 5480$ kg/m^3.

(2) Calculate the moment of inertia in the case of a ball having two homogeneous layers: a radius of core R_N and density ρ_N and a mantle of density ρ_M.

Compute the numerical solution in the case of Earth, where $R_N = 3490$ km, $\rho_N = 12{,}700$ kg/m^3, and $\rho_M = 4150$ kg/m^3.

Exercise 6.13: Horton Infiltration Model

In the case of a partially saturated soil, Horton's infiltration model assumes that, when it rains, the infiltration capacity of the soil decreases with time, as the soil becomes saturated. This model is expressed by

$$F(t) = F_C + (F_0 - F_C)e^{-kt},$$

where $F(t)$ is the infiltration capacity of the soil at time t ($F(t)$, in mm/min and t in min). F_C (mm/min) is the asymptotic infiltration capacity of the soil, i.e., when $t \to \infty$, F_0 (mm/min) is the initial infiltration capacity of the soil and k is a parameter (min^{-1}).

In this exercise, F_C, F_0, and k are measured on the Bunder watershed in Indonesia (Fleurant et al. 2006).

F_C	F_0	k
0.119	1.254	0.211

(1) With R, plot F as a function of time t and plot the amount of water infiltrated in the first 10 min.
(2) Calculate (with or without the help of Xcas software) the quantity H of water (in mm) that can infiltrate in 10 min.
(3) With R, assess the amount of water that can infiltrate in 10 min using the rectangle method for calculating an approximate value of an integral of F. Use this method with several rectangles and observe the precision obtained (by comparing with the exact value calculated in the previous question).

Exercise 6.14: Topographical Uplift

In geomorphological modeling, one can take into account the uplift of a region by the following law:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = U,$$

where z is the topography (m) and U the uplift velocity (m/s).

We consider an area initially at altitude $z(t = 0) = 10$ m and whose uplift velocity is constant and equal to $U = 10^{-2}$ mm/year.

(1) Give the expression of the altitude $z(t)$ as a function of time.
(2) Plot this function with R and give the altitude of the area after 10 million years.

Exercise 6.15: Terrain Erosion
Ahnert's law (1970) describes the erosion of reliefs in temperate zones by

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -kh,$$

where h is the elevation of the topography (m), t the time (in Myr), and k an erosion parameter. It is a linear diffusion equation.

(1) What is the expression of the evolution over time of a relief given an initial altitude of $h_0 = 2000$ m?
(2) Using R, plot $h(t)$ over a period of 35 million years (35 Myr) with $k = 0.2$ Myr^{-1}.
(3) How long does it take to erode 50% of the surface?

Exercise 6.16: Radiocarbon Dating (or Carbon-14 Dating)
We call $N(t)$ the number of atoms of carbon-14 present at the instant t (in years) in an organism. Once it dies, the carbon-14 disintegrates (without any new additions). See the exercise on the same subject in Chap. 2.

The function N confirms the differential equation $\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\lambda N(t)$, where λ is a positive constant characterizing carbon-14.

(1) Find the expression of the function N of the variable t (in years) depending on the parameter λ and the initial number N_0 of carbon-14 atoms.
(2) The half-life of carbon-14 is 5730 years, meaning that after 5730 years, exactly half the original carbon-14 has disappeared. Deduct the value of λ.
(3) In 1949, in Lascaux (France), a carbon-14 content of about 16% was found in the pigments of frescoes. How old are these frescoes?

Exercise 6.17: Population Dynamics, Verhulst Model
The growth model proposed by Pierre-François Verhulst (1804–1849) concerns the dynamics of populations. Denoting by y the size of a population, $m(y)$ the mortality rate, and $n(y)$ the birth rate, the function y confirms the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = y[n(y) - m(y)]$.

We made the following **modeling assumption**: the more people, the lower the birth rate is (n is a decreasing linear function) and the higher the mortality rate is (m is an increasing linear function: therefore, the opposite, $-m$, is a decreasing linear function). Then, small populations tend to grow. As the function $n - m$ is a decreasing linear function of the population, one can find a and b constants (with $a < 0$) such as $n(y) - m(y) = ay + b$. We assume in addition that $b > 0$.

The so-called "logistic" equation is written $\frac{\mathrm{d}y}{\mathrm{d}t} = y[ay + b]$, with $a < 0$ and $b > 0$.

(1) Find a nonzero constant function K solution of this logistic equation. This constant K is called the carrying capacity.
(2) Use a computer algebra system to show that the solutions are of the form $\frac{K}{1 + \mu e^{-bx}}$, where μ is a constant.
(3) We use the notation $y_0 = y(0)$ (population at the origin of time). Find the unique function y that satisfies this condition. What value does $\lim_{t \to +\infty} y(t)$ have?

Exercise 6.18: Model of Unit Hydrograph
Nash (1959) proposed to assimilate the processes of a watershed to a succession of tanks that spill into each other. It is a conceptual model based on the assumption that the flow Q of a river is

proportional to the volume V stored in its subwatershed:

$$V = \tau Q.$$

However, Maillet's law indicates that when a tank is being emptied, its flow Q is equal to the temporal variations of the volume V, so

$$Q = -\frac{\mathrm{d}V}{\mathrm{d}t}.$$

A minus sign is written to compensate for the fact that variations in volume dV are negative since the tank is emptying.

(1) Combining these two equations, find a differential equation satisfied by the tank drain flow Q and deduce an expression of Q as a function of time.
(2) In Nash's model (1959), a watershed is modeled not by a single tank, but by a succession of several tanks that flow into each other (Fig. 6.21). In this case, it can be shown that the nth tank flow is a function of the preceding tank's flow $(n-1)$th:

$$Q_n(t) = \int_0^t Q_1(x)Q_{n-1}(t-x)\mathrm{d}x.$$

Give the mathematical expression of $Q_2(t)$ and $Q_3(t)$.

(3) The general formula is $Q_n(t) = \frac{V_0^n t^{n-1}}{(n-1)!\tau^n} e^{-\frac{t}{\tau}}$, with $(n-1)! = (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$.

Using R, plot $Q_n(t)$ for a number of tanks ranging from $n = 1$ to $n = 10$.

A small surface watershed of area $S = 1$ ha will be conceptualized as having received a uniformly distributed rainfall of $h = 0.1$ mm and the value $\tau = 1$ h.

Exercise 6.19: Opening Fractures in Karstic Zone

A limestone aquifer is considered with fractures having an average thickness of about $x_0 = 2$ mm. The groundwater flowing in this aquifer can dissolve the limestone under certain physico-chemical conditions, leading to the widening of the fractures. The evolution of the thickness of the fractures is written (Kaufmann, 2003)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{mF}{\rho},$$

where $m = 0.1$ kg/mol is the atomic mass of the calcite, $\rho = 2700$ kg/m^3 is the density of the calcite, and F is the eroded calcium flux:

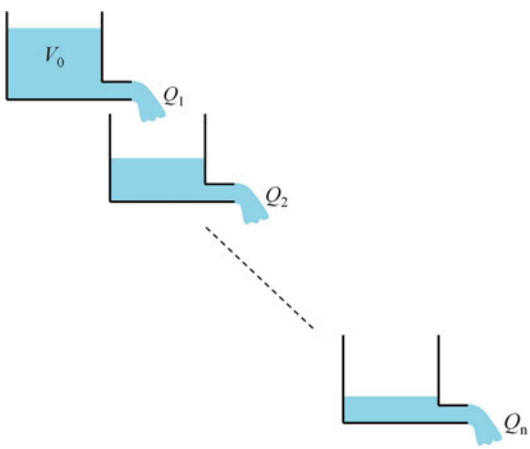

Fig. 6.21 Nash's model (1959) of the unit hydrograph. The watershed is conceptualized as a succession of tanks that flow into each other

$$F = k_0\left(1 + \frac{k_0 x}{6Dc_{eq}}\right)^{-1}\left(1 - \frac{c_s}{c_{eq}}\right)^{-3},$$

with $k_0 = 4 \cdot 10^{-7}$ mol/m^2/s a dissolution coefficient, $D = 10^{-9}\text{m}^2/\text{s}$ a diffusion coefficient, $c_{eq} = 2$ mol/m^3 the concentration of calcium at equilibrium, and $c_s = 0.9c_{eq}$ a threshold of calcium concentration.

(1) Give the expression of the evolution of the thickness of fractures over time.
Solution tips:
- Write F in the form $F = A \times \frac{1}{1 + Bx}$, indicating the expressions of A and B according to the parameters of the exercise.
- Combine the two given relationships to establish a differential equation satisfied by $x(t)$.
- Solve this differential equation by separating the variables.
- $x(t)$ is then a solution of a second-order equation (a classical equation, not a differential equation) of the form $ay^2 + by + c = 0$. To determine the solutions, calculate the quantity $\Delta = b^2 - 4ac$. In the case of a positive result, the equation has two solutions: $y_1 = \frac{-b - \sqrt{\Delta}}{2a}$ and $y_2 = \frac{-b + \sqrt{\Delta}}{2a}$.
- One can also use the Xcas software to solve the differential equation or the second-order equation.

(2) Using R, plot the evolution of the thickness of fractures over time between 0 and 2000 years and comment on this evolution.

Solutions

Solution 6.1: Flash Questions. Series 1

(1) We have $f'(x) = 12x^3 - 5 \neq g(x)$: f is therefore not a primitive of the function g.

(2) $\int_0^{\pi/2} \sin(x)\mathrm{d}x = [-\cos(x)]_0^{\pi/2} = 0 + 1 = 1.$

(3) False. If you use the notation $y(t) = \cos t$, you have $y'(t) = -\sin t$, and so $y'(t) - 2y(t) = -\sin t - 2\cos t \neq t$.

(4) Let us seek among the usual functions a function for which the second derivative is close to the original function. This is the case for exp, sin, and cos. For exp, we have $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = +f$. On the other hand, sin and cos answer the desired question and we have for these two functions $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -f$. Consequently, the functions of the form $\lambda\sin + \mu\cos$ are solutions of the proposed differential equation.

Solution 6.2: Flash Questions. Series 2

(1) $x \mapsto -e^{-x}$ is a primitive of $x \mapsto e^{-x}$.

(2) $\int_0^1 e^{-t}\mathrm{d}t = [-e^{-t}]_0^1 = -e^{-1} + 1 \approx 0.63.$

(3) We have $u'(t) = 2e^t - 2t - 2$ and $u(t) + t^2 = 2e^t - t^2 - 2t - 2 + t^2 = 2e^t - 2t - 2$.

Indeed, $u'(t) = u(t) + t^2$.

(4) $\frac{\partial f}{\partial x}(x, t) = t$; therefore (by taking a primitive with respect to the variable x, at t constant), $f(x, t) = tx + c$, where c is a constant with respect to the variable x (but not necessarily with respect to the variable t). We prefer to write this: $f(x, t) = tx + g(t)$, where g is any function of a single variable (the variable t).

Solution 6.3: Flash Questions. Series 3

(1) sin is a primitive of the function cos and $-$cos is a primitive of the function sin.

(2)
$$\begin{aligned}\int_2^3 x^2y^3\mathrm{d}x &= y^3\int_2^3 x^2\mathrm{d}x \\ &= y^3\left[\frac{1}{3}x^3\right]_2^3 \\ &= \frac{y^3}{3}[27 - 8] \\ &= \frac{19y^3}{3}.\end{aligned}$$

(3) According to the text, the solutions of the differential equation $y' - 3y = 0$ are the functions y defined by $y(x) = ae^{3x}$, where a is any real number.

(4) We have $\frac{\partial u}{\partial x} = -\sin(x - \alpha t)$ and $\frac{\partial u}{\partial t} = +\alpha \sin(x - \alpha t)$.

Thus, we have $\frac{\partial u}{\partial x} + \frac{1}{\alpha}\frac{\partial u}{\partial t} = 0$.

Solution 6.4: Primitive Calculation

(1)

- The primitives of f are in the form $F(x) = \frac{1}{3}x^3 - x + C$, where C is a constant to be determined with the help of the given condition.
- For the condition $F(0) = 0$, this gives $C = 0$.
- For the condition $F(1) = 0$, this gives $\frac{1}{3}1^3 - 1 + C = 0$, which leads to $C = 1 - \frac{1}{3} = \frac{2}{3}$.

(2)

- The primitives of g are of the form $G(t) = 2\ln(|t|) - \frac{1}{2}t^2 + C$, where C is a constant to be determined with the help of the given condition.
- For the condition $G(1) = 0$, this gives $2\ln(1) - \frac{1}{2} \times 1^2 + C = 0$, which is $C = \frac{1}{2}$ (because $\ln(1) = 0$).
- For the condition $G(e) = 3$, this gives $2\ln(e) - \frac{1}{2}e^2 + C = 3$, that is, $C = \frac{e^2}{2} + 1$ (because $\ln(e) = 1$).

Solution 6.5: Computation of Integrals, Level 1

(1) $$\int_0^{\pi/2} \cos t\mathrm{d}t = [\sin t]_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1 - 0 = 1.$$

(2) $$\int_0^1 \left(2t^3 - 3t + 4\right)\mathrm{d}t = \left[\frac{2}{4}t^4 - \frac{3}{2}t^2 + 4t\right]_0^1 = \frac{1}{2} - \frac{3}{2} + 4 = 3.$$

(3) $$\int_0^{\ln(2)} (1 - e^{-t})\mathrm{d}t = [t + e^{-t}]_0^{\ln(2)}$$
$$= \ln(2) + \exp(-\ln(2)) - 1$$
$$= \ln(2) + \exp\left(\ln\left(\frac{1}{2}\right)\right) - 1$$
$$= \ln(2) + \frac{1}{2} - 1$$
$$= \ln(2) - \frac{1}{2}.$$

(4) $$\int_0^{\pi} \sin\left(\frac{1}{4}x\right)\mathrm{d}x = 4\left[-\cos\left(\frac{1}{4}x\right)\right]_0^{\pi} = 4\left[-\frac{\sqrt{2}}{2} + 1\right].$$

Solution 6.6: Calculation of Integrals, Level 2

(1) $$\int_2^3 \frac{1}{x-1}\mathrm{d}x = [\ln(x-1)]_2^3 = \ln(2) - \ln(1) = \ln(2).$$

(2) $$\int_0^{\pi/4} \cos(3.7t + 0.31)\mathrm{d}t$$
$$= \frac{1}{3.7}[\sin(3.7t + 0.31)]_0^{\pi/4}$$
$$= \frac{1}{3.7}(\sin(3.7 \times \pi/4 + 0.31) - \sin(0.31))$$
$$\approx -0.10.$$

(3) $\int_0^{+\infty} t\exp\left(-t^2\right)\mathrm{d}t = \left[-\frac{1}{2}\exp\left(-t^2\right)\right]_0^{+\infty} = 0 + \frac{1}{2} = \frac{1}{2}$ (because the limit of the function $t \mapsto \exp(-t^2)$ in $+\infty$ is 0).

(4) $$\int_{-2}^{2} \left(7x^2 - 3x + 3\right)\mathrm{d}x = \left[\frac{7}{3}x^3 - \frac{3}{2}x^2 + 3x\right]_{-2}^{2}$$

$$= \frac{7}{3} \times 2^3 - \frac{3}{2} \times 2^2 + 3 \times 2$$
$$- \left(\frac{7}{3} \times (-2)^3 - \frac{3}{2} \times (-2)^2 + 3 \times (-2)\right)$$
$$= \frac{56}{3} - 6 + 6 - \left(-\frac{56}{3} - 6 - 6\right) = \frac{148}{3}.$$

Solution 6.7: Manipulating the Symbol $\sum$

(1) We have $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} = \sum_{k=2}^{k=5} \frac{1}{k} = \sum_{k=2}^{5} \frac{1}{k}$ (pay attention to the first value of the index k) and $\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{21}+\frac{1}{22} = \sum_{k=3}^{22} \frac{1}{k}$.

We have $\sum_{k=1}^{k=4} 3^k = 3^1 + 3^2 + 3^3 + 3^4$ and $\sum_{k=1}^{k=1000} \frac{1}{k^2} = 1^2 + 2^2 + 3^2 + \ldots + 999^2 + 1000^2$.

(2) On the one hand, $\sum_{k=1}^{k=100} k(k+1) = 1 \times 2 + 2 \times 3 + \ldots + 100 \times 101$, and on the other hand, $\sum_{i=2}^{101} i(i-1) = 2 \times 1 + 3 \times 2 + \ldots + 101 \times 100$.

So we have indeed

$$\sum_{k=1}^{k=100} k(k+1) = \sum_{i=2}^{101} i(i-1).$$

(3) $$\sum_{k=1}^{100} (2k-3) = 2\sum_{k=1}^{100} k - 3\sum_{k=1}^{100} 1$$
$$= 2 \times \frac{100 \times 101}{2} - 3 \times 100 = 9800.$$

(4) We have, as in the text, $\Delta t = 10$, but with $t_i = 10 \times (i-1)$ for i varying from 1 to 30, always with $Q_i = Q(t_i)$. The approximation of the integral of the function Q is given by the same formula as in the text: $\int_0^{300} Q(t)\mathrm{d}t \simeq 10 \sum_{i=1}^{i=N} Q_i \simeq 125\,\mathrm{m}^3$ (Fig. 6.22).

Solution 6.8: Differential Equation and Initial Condition

We work on the differential equation $y' - 3.5y = 0$.

(1) According to the text, the general solution is of the form $y(x) = Ke^{3.5x}$, where K is any constant.

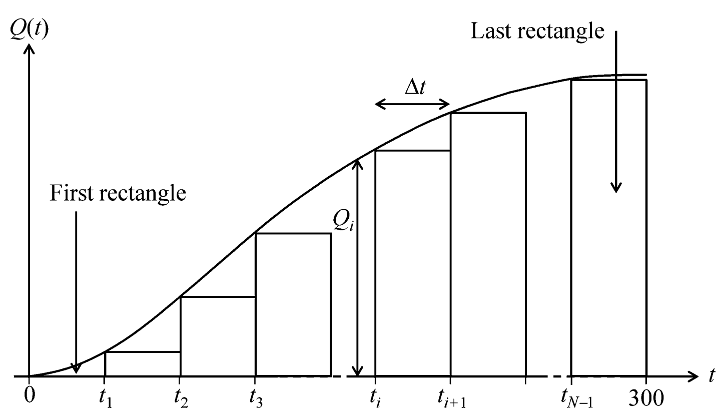

Fig. 6.22 Rectangle method

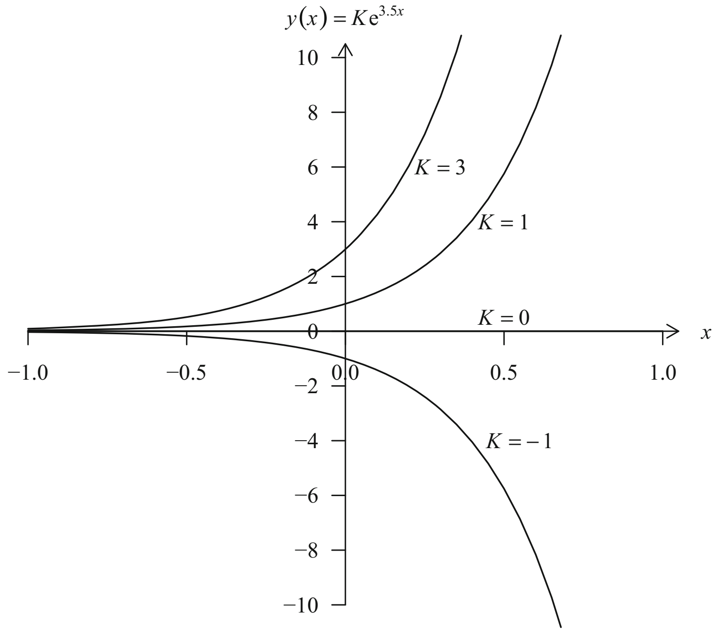

Fig. 6.23 Solutions $y(x) = Ke^{3.5x}$ for parameter values $K = 0$, $K = 1$, $K = 3$, and $K = -1$.

(2) Figure 6.23 shows the graphical representations of the solutions for $K = 0$, $K = 1$, $K = 3$, and $K = -1$.

(3) On the one hand, $y(0) = 8.2$ and $y(0) = Ke^0 = K$ on the other hand; thus, $K = 8.2$, and the solution sought is expressed by $y(x) = 8.2e^{3.5x}$.

Solution 6.9: Solving Partial Differential Equations

(1) We consider the primitives with respect to the variable x (the variable y being considered constant): $u(x, y) = x + \frac{1}{2}x^2 + y^2x + c$, where c is a constant with respect to the variable x, but not necessarily vis-à-vis y: we write instead $c(y)$. The solutions of the PDE $\frac{\partial u}{\partial x} = 1 + x + y^2$ are the functions u defined by $u(x, y) = x + \frac{1}{2}x^2 + y^2x + c(y)$, where c is an arbitrary function.

(2) We consider twice the primitives with respect to the variable y (with x constant): $\frac{\partial f}{\partial y} = -\frac{1}{x} \times \frac{1}{y} + c(x)$ and $f(x, y) = -\frac{1}{x}\ln(y)$ $+c(x)y + d(x)$, where c and d are any functions of the variable x.

(3) We first consider the primitives with respect to x (with y constant), and then with respect to y (with x constant): $\frac{\partial u}{\partial y} = x + c(y)$, then $u(x, y) = xy + C(y) + d(x)$, where C is any function that has a derivative (a primitive of any function c) with respect to the variable y and d any function of the variable x. We could have first considered the primitives with respect to y, then to x.

Solution 6.10: Heat Received on Earth, Continued

(1) We calculate the following integral:

$$\int_0^{365.25/4} E(t)\mathrm{d}t = \int_0^{365.25/4} \left[C - C_0 \cos\left(\frac{2\pi}{365.25}(t-2.72)\right)\right]\mathrm{d}t$$
$$= \left[Ct - C_0 \times \frac{365.25}{2\pi}\sin\left(\frac{2\pi}{365.25}(t-2.72)\right)\right]_0^{365.25/4}$$
$$= \frac{365.25}{4}C - C_0 \times \frac{365.25}{2\pi}\times$$
$$= \left[\sin\left(\frac{2\pi}{365.25}\left(\frac{365.25}{4} - 2.72\right)\right) - \sin\left(\frac{2\pi}{365.25}(0-2.72)\right)\right]$$
$$\simeq 91.31C - 55.35C_0.$$

(2) The amount of heat received on average per day is given by

$$\int_0^{365.25} E(t)\mathrm{d}t = \frac{1}{365.25}\int_0^{365.25} \left[C - C_0 \cos\left(\frac{2\pi}{365.25}(t-2.72)\right)\right]\mathrm{d}t$$
$$= \frac{1}{365.25}\left[Ct - C_0 \times \frac{365.25}{2\pi}\sin\left(\frac{2\pi}{365.25}(t-2.72)\right)\right]_0^{365.25}$$
$$= C.$$

We used the 2π periodicity of the sine function. The formula of $E(t)$ is based on the variations around this mean value.

Solution 6.11: Calculation of Volume of Polar Ice Cap

(1) The radius r of a cylinder of ice at an altitude z corresponds to the horizontal coordinate x. We have the relation $z^2 = 2\tau_0 \dfrac{R_{\max} - x}{\rho_G g}$, so $R_{\max} - x = z^2 \times \rho_G \times g\dfrac{1}{2\tau_0}$, and so $x = r(z) = R_{\max} - z^2 \times \rho_G \times g\dfrac{1}{2\tau_0}$.

(2) The volume of a basic cylinder of radius $r(z)$ and thickness $\mathrm{d}z$ is calculated by the following formula: Area of base $\times$ Height, or $\pi r(z)^2 \times \mathrm{d}z$. The elementary volume $\mathrm{d}V(z)$ is expressed by

$$\mathrm{d}V(z) = \pi\left(R_{\max} - z^2 \times \rho_G g\frac{1}{2\tau_0}\right)^2 \mathrm{d}z.$$

(3) The total volume of the cap is obtained by calculation of the integral $V = \displaystyle\int_0^{H_{\max}} \mathrm{d}V(z)$

$$= \int_0^{H_{\max}} \pi\left(R_{\max} - z^2 \times \rho_G g\frac{1}{2\tau_0}\right)^2 \mathrm{d}z.$$

This gives the following equations (one can also perform the calculation with formal calculation software):

$$V = \int_0^{H_{\max}} \pi \left[R_{\max}^2 - R_{\max} \times z^2 \times \rho_G g \frac{1}{\tau_0} + \left(z^2 \times \rho_G g \frac{1}{2\tau_0} \right)^2 \right] \mathrm{d}z$$
$$= \pi \int_0^{H_{\max}} R_{\max}^2 \mathrm{d}z - \pi R_{\max} \times \frac{\rho_G g}{\tau_0} \int_0^{H_{\max}} z^2 \mathrm{d}z + \pi \frac{\rho_G^2 g^2}{4\tau_0^2} \int_0^{H_{\max}} z^4 \mathrm{d}z.$$

So we have

$$V = \pi R_{\max}^2 [z]_0^{H_{\max}} - \pi R_{\max} \times \frac{\rho_G g}{\tau_0} \left[\frac{z^3}{3} \right]_0^{H_{\max}} + \pi \frac{\rho_G^2 g^2}{4\tau_0^2} \left[\frac{z^5}{5} \right]_0^{H_{\max}}$$
$$= \pi R_{\max}^2 \times H_{\max} - \pi R_{\max} \times \frac{\rho_G g}{\tau_0} \frac{H_{\max}^3}{3} + \pi \frac{\rho_G^2 g^2}{4\tau_0^2} \frac{H_{\max}^5}{5}.$$

The numerical application gives (remember to convert kilometers to meters)

$$V = \pi \times (600{,}000)^2 \times 3000 - \pi \times 600{,}000 \times \frac{900 \times 3.7}{25{,}000} \frac{3000^3}{3} + \pi \frac{900^2 \times 3.7^2}{4 \times 25{,}000^2} \frac{3000^5}{5} \approx 1.8 \times 10^{15}\ \mathrm{m}^3 = 1.8 \times 10^9\ \mathrm{km}^3.$$

This gives a mass of $m = V \times \rho_G = 1.8 \times 10^{15} \times 900 = 1.6 \times 10^{18}$ kg.

Solution 6.12: Moment of Inertia

(1) We have $I = \frac{8\pi}{3} \int_0^R \rho r^4 \mathrm{d}r$, where ρ is a constant; therefore,

$$I = \frac{8\pi}{3} \times \rho \left[\frac{1}{5} r^5 \right]_0^R = \frac{8\pi}{15} \times \rho \times R^5.$$

For the Earth, this gives (using $R = 6370$ km $=$ $6.37\ 10^6$ m) $I = \frac{8\pi}{15} \times 5480 \times \left(6.37 \times 10^6\right)^5 = 9.63 \times 10^{37}$ kg m^2.

(2) Use the additive property of integrals:

$$I = \int_0^{R_N} \frac{8\pi}{3} \rho_N r^4 \mathrm{d}r + \int_{R_N}^R \frac{8\pi}{3} \rho_M r^4 \mathrm{d}r$$
$$= \frac{8\pi}{15} \times \rho_N \times R_N^5 + \frac{8\pi}{15} \times \rho_M \left(R^5 - R_N^5\right).$$

In the case of the Earth, this gives

$$I = \frac{8\pi}{15} \times 12{,}700 \times \left(3.49 \times 10^6\right)^5 + \frac{8\pi}{15} \times 4150 \times \left(\left(6.37 \times 10^6\right)^5 - \left(3.49 \times 10^6\right)^5 \right) =$$

8.03×10^{37} kg m^2. This is close to the value of 8.02×10^{37} kg m^2 obtained by astronomical measurements.

Solution 6.13: Horton Infiltration Model

(1) The height of water that has infiltrated during the first 10 min is the area under the curve (in black in Fig. 6.24).

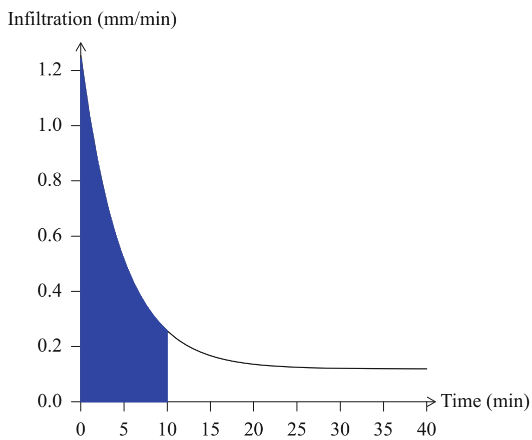

Fig. 6.24 Horton infiltration model: the amount of water that has infiltrated over time corresponds to the area under the curve

(2) The height of water H that infiltrates corresponds to the surface under the curve; it is, therefore, in mathematical terms, the integral of $F(t)$ between the bounds 0 and 10:

$$
\begin{aligned}
H &= \int_0^{10} F(t)\mathrm{d}t \\
&= \int_0^{10} \left(F_C + (F_0 - F_C)e^{-kt}\right)\mathrm{d}t \\
&= \int_0^{10} F_C\mathrm{d}t + (F_0 - F_C)\int_0^{10} e^{-kt}\mathrm{d}t \\
&= F_C[t]_0^{10} + (F_0 - F_C)\left[\frac{e^{-kx}}{-k}\right]_0^{10} \\
&= F_C(10-0) + (F_0 - F_C)\left(\frac{e^{-10k}}{-k} - \frac{e^{-k\times 0}}{-k}\right) \\
&= 10F_C + \frac{(F_0 - F_C)}{k}\left[1 - e^{-10k}\right] \\
&= 10 \times 0.119 + \frac{(1.254 - 0.119)}{0.211}\left[1 - e^{-10\times 0.211}\right] \\
&\approx 5.92\ \text{mm}.
\end{aligned}
$$

(3) Applying the rectangle method, there is a value of H that varies according to the number of rectangles:

Number of rectangles	1	2	4	8	16	32
Value $H(t)$	5.142	5.704	5.862	5.903	5.913	5.916

A fair approximation of the exact value of H.

```
#-------------------------------------
# Horton's model
#-------------------------------------
# Time (min)
t <- seq(0,40,1)
# Asymptotic infiltration capacity
(mm/min)
FC <- 0.119
# Initial infiltration capacity
(mm/min)
F0 <- 1.254
# Parameter (min^-1)
k <- 0.211
# Infiltration capacity at t (mm/min)
Ft <- FC + (F0-FC)*exp(-k*t)
# Plot
plot(t,Ft,type="l",col=1,
          lwd=1.2,xlim=c(0,40),ylim=c
          (0,1.3),
     axes=F,xlab=NA,ylab=NA)
# X axis
axis(side=1,at=seq(0,40,5),pos=0)
arrows(40,0,41,0,length=0.1)
mtext("Time (min)",1,at=20,line=1.5)
# Y axis
axis(side=2,at=seq(0,1.2,0.2),pos=0)
arrows(0,1.2,0,1.3,length=0.1)
mtext("Infiltration (mm/min)",2,
at=0.6,line=1.5)

# Number of rectangles
N <- 256
# When computing from 0 to 10, the time
step is then
dt <- (10-0)/ N
sum <- 0
for(i in 0:(N-1)) {
  print(i)
  # Center of the rectangle
  xr <- dt/2 + i*dt
  # Area of the rectangle
  sr <- dt * (FC + (F0-FC)*exp(-k*xr))
  sum <- sum + sr
  rect(xr-0.5*dt,.0,xr+0.5*dt,
       FC + (F0-FC)*exp(-k*xr),
       border=4,lwd=1.2)
}
cat("With",N,"rectangles,
    The value of the area is",
    round(sum,5),"mm","\n")
```

Solution 6.14: Topographic Uplift

(1) Changes in the topography are given by the expression $\frac{\mathrm{d}z}{\mathrm{d}t} = U$. To determine the function $z(t)$, we must integrate this expression, which amounts to writing

$$
\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= U, \\
z(t) &= U \times t + C.
\end{aligned}
$$

At $t = 0$, we have $z(t = 0) = C$, and finally $z(t) = z(t = 0) + Ut$.

(2) It can be seen on the graph that after 10 million years, the topographic uplift is between 100 and 120 m. To obtain a more precise value, we use the equation $z(t) = z(t = 0) + Ut$

by taking $t = 10^7$ years and expressing the value of speed U in m/year, that is to say, by multiplying by 10^{-3} (passage from mm to m):

$$\begin{aligned} z(t = 10^7) &= z(t = 0) + U \times 10^7 \\ &= 10 + 10^{-2} \times 10^{-3} \times 10^7 \\ &= 110 \ \text{m}. \end{aligned}$$

```
#-----------------------------------
# Topographic rising
#-----------------------------------
# Time (year)
t <- seq(0,10e6,100000)
# Rising velocity (mm/y)
U <- 1e-2
# Initial altitude (m)
z0 <- 10
z <- function(t) {
  # U (mm/y) -> U(m/y)
  U <- U * 1e-3
  return (z0 + U * t)
}

plot(t,z(t), type="l",
     xlab="Time (year)",
     ylab="Topography (m)",
        ylim=c(0,120))
```

Solution 6.15: Terrain Erosion

(1) This differential equation can be solved by writing successively

$$\begin{aligned} \frac{\mathrm{d}h}{\mathrm{d}t} &= -kh, \\ \frac{\mathrm{d}h}{h} &= -k\mathrm{d}t, \\ \int_{h_0}^{h} \frac{\mathrm{d}x}{x} &= -k \int_0^t \mathrm{d}y, \\ [\ln x]_{h_0}^{h} &= -k[y]_0^t \text{ (we have } x > 0), \\ \ln\left(\frac{h}{h_0}\right) &= -kt, \\ h &= h_0 e^{-kt}. \end{aligned}$$

(This answer could also be written directly using the formula from the text.)

(2)

```
#-----------------------------------
# Erosion
#-----------------------------------
# Time (M-years)
t <- seq(0,35,1)
# Constant (/M-years)
k <- 0.2
# Initial topography (m)
h0 <- 2000
# Topography (m)
h <- h0*exp(-k*t)
plot(t,h,type="l",
     xlab="Time (My)",
     ylab="Altitude (m)")
```

(3) A relief erosion of 50% can be written $h(t) = \frac{h_0}{2}$. We are therefore looking for a time such as

$$\begin{aligned} \frac{h_0}{2} &= h_0 e^{-kt_1}, \\ \frac{1}{2} &= e^{-kt_1}, \\ \ln\left(\frac{1}{2}\right) &= -kt_1, \\ t_1 &= -\frac{1}{k}\ln\left(\frac{1}{2}\right), \end{aligned}$$

which gives $t_1 \approx 3.46$ Myr.

Solution 6.16: Carbon-14 Dating

(1) We have here a first-order linear differential equation; $N(t)$ is therefore of the form $N(t) = Ke^{-\lambda t}$. As, moreover, $N(0) = N_0$, on the one hand, and $N(0) = Ke^0 = K$, on the other hand, we have $K = N_0$, and so $N(t) = N_0 e^{-\lambda t}$. (Fig. 6.25).

(2) The definition of the half-life gives the equality $\frac{N(5730)}{N_0} = 0.5$. This ratio is also $e^{-\lambda \times 5730}$.

Thus, we have $e^{-\lambda \times 5730} = 0.5$, from which $-\lambda \times 5730 = \ln(0.5) = -\ln 2$, and so $\lambda = \frac{\ln 2}{5730} \approx 1.2 \times 10^{-4}$ years.

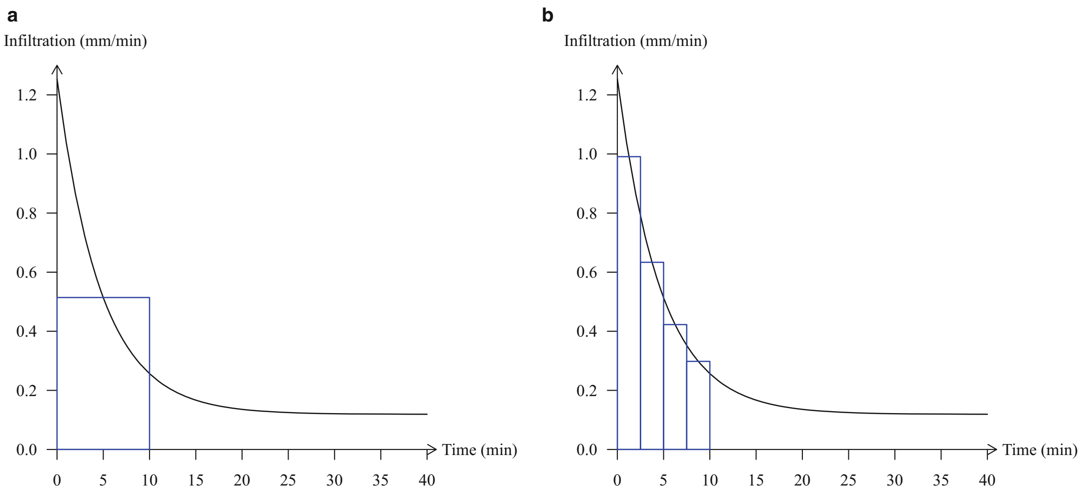

Fig. 6.25 Horton infiltration Model: the amount of water that has infiltrated over time corresponds to the area under the curve

(3) We want t so that $\frac{N(t)}{N_0} = 0.16$, which gives $e^{-\lambda t} = 0.16$, and thus (by repeating a procedure analogous to that of the preceding question) $t = \frac{\ln(0.16)}{-\lambda} = -\frac{\ln 0.16}{\ln 2} \times 5730 \approx 15{,}150$ years (when discovered in 1949).

Solution 6.17: Population Dynamics, Verhulst's Model

(1) Let K be the constant function solution of the logistic equation, then $y(t) = K$ for every t and $\frac{dy}{dt} = 0$. The logistic equation is then written $0 = K \times (a \times K + b)$. Because we assume that $K \neq 0$, so $a \times K + b = 0$, which gives $K = -\frac{b}{a}$.

(2) Xcas gives $y(t) = \frac{b\exp(bt)}{-a\exp(bt) + c_0}$. Thus, dividing by $\exp(bt)$ yields to $y(t) = \frac{b}{-a + c_0\exp(-bt)}$; then, dividing by $-a$, we obtain $y(t) = \frac{-b/a}{1 - \frac{c_0}{a}\exp(-bt)}$, which is indeed of the desired form since $K = -\frac{b}{a}$.

(3) $y(0)$ is $\frac{K}{1+\mu}$. Thus, we have $\frac{K}{1+\mu} = y_0$, which gives $\mu = \frac{K}{y_0} - 1$. $\lim_{t\to+\infty} y(t) = K$ because $\lim_{t\to+\infty} \exp(-bt) = 0$ since $b > 0$.

Solution 6.18: Model of Unit Hydrograph

(1) The combined Maillet and Nash equations lead to

$$\frac{dQ}{dt} = -\frac{Q}{\tau}.$$

We have here a differential equation that can be solved by separating the variables Q and t, which gives successively

$$\begin{aligned}
\frac{dQ}{Q} &= -\frac{dt}{\tau}, \\
\int_{Q_0}^{Q} \frac{dx}{x} &= -\frac{1}{\tau}\int_0^t dy, \\
[\ln x]_{Q_0}^{Q} &= -\frac{1}{\tau}[y]_0^t, \\
\ln Q - \ln Q_0 &= -\frac{1}{\tau}(t-0), \\
\ln\frac{Q}{Q_0} &= -\frac{t}{\tau}, \\
Q &= Q_0 e^{-\frac{t}{\tau}}
\end{aligned}$$

(one can also directly use the result of the text).

When emptying a tank, the flow rate therefore decreases exponentially with time. If the tank is initially filled with a volume V_0, then we will have $Q = \frac{V_0}{\tau} e^{-\frac{t}{\tau}}$.

(2) The flow rate at the outlet of the first tank was just calculated and is given by

$$Q_1 = \frac{V_0}{\tau} e^{-\frac{t}{\tau}}.$$

The flow rate in the second tank is given by

$$\begin{aligned}
Q_2(t) &= \int_0^t Q_1(x) Q_1(t-x) \mathrm{d}x \\
&= \int_0^t \frac{V_0}{\tau} e^{-\frac{x}{\tau}} \frac{V_0}{\tau} e^{-\frac{t-x}{\tau}} \mathrm{d}x \\
&= \frac{V_0^2}{\tau^2} \int_0^t e^{-\frac{x}{\tau}} e^{-\frac{t-x}{\tau}} \mathrm{d}x \\
&= \frac{V_0^2}{\tau^2} \int_0^t e^{-\frac{t}{\tau}} \mathrm{d}x \\
&= \frac{V_0^2}{\tau^2} e^{-\frac{t}{\tau}} \int_0^t \mathrm{d}x \\
&= \frac{V_0^2 t}{\tau^2} e^{-\frac{t}{\tau}}.
\end{aligned}$$

Using the just obtained expression $Q_2(t)$, the flow rate in the third tank is written

$$\begin{aligned}
Q_3(t) &= \int_0^t Q_1(x) Q_2(t-x) \mathrm{d}x \\
&= \int_0^t \frac{V_0}{\tau} e^{-\frac{x}{\tau}} \frac{V_0^2(t-x)}{\tau^2} e^{-\frac{t-x}{\tau}} \mathrm{d}x \\
&= \frac{V_0^3}{\tau^3} \int_0^t (t-x) e^{-\frac{t}{\tau}} \mathrm{d}x \\
&= \frac{V_0^3}{\tau^3} e^{-\frac{t}{\tau}} \int_0^t (t-x) \mathrm{d}x \\
&= \frac{V_0^3}{\tau^3} e^{-\frac{t}{\tau}} \left(t \int_0^t \mathrm{d}x - \int_0^t x \mathrm{d}x \right) \\
&= \frac{V_0^3}{\tau^3} e^{-\frac{t}{\tau}} \left(t[x]_0^t - \left[\frac{x^2}{2}\right]_0^t \right) \\
&= \frac{V_0^3 t^2}{2\tau^3} e^{-\frac{t}{\tau}}.
\end{aligned}$$

(3) We see that the more the number n of tanks increases, the more the flow at the outlet of the watershed has a reduced peak and a lagged time.

```
#-------------------------------------
# Nash's model
#-------------------------------------
# River basin's area (ha)
S <- 1
# Uniform rainfall (mm)
h <- .1
# Initial volume (m3)
# ha to m2 (x 10000)
# mm to m (/ 1 000)
V0 <- S*h*10000/1000
# Parameter (h)
tau <- 1
# Time (h)
t <- seq(0,20,0.1)
# Flow rate from 1 to n tanks
plot(0,0,type="l",lwd=1.2,
     xlab="Time (h)",ylab="Flow rate
     (m3/h)",
     xlim=c(0,20),ylim=c(0,1))
for(n in 1:10) {
  Q <- V0^n*t^(n-1)/(tau^n*factorial
  (n-1))*exp(-t/tau)
  lines(t,Q,col=n)
}
```

Solution 6.19: Opening Fractures in Karstic Zone

(1) We write F in the form $A \times \frac{1}{1+Bx}$, where

$$A = k_0 \times \left(1 - \frac{c_s}{c_{eq}}\right)^{-3} \text{ and } B = \frac{k_0}{6Dc_{eq}}.$$

We can write successively

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{mF}{\rho} \\
&= \frac{m}{\rho}A \times \frac{1}{1+Bx}, \\
(1+Bx)\mathrm{d}x &= \frac{m}{\rho}A\mathrm{d}t, \\
\int_{x_0}^{x}(1+Bu)\mathrm{d}u &= \frac{m}{\rho}A\int_0^t \mathrm{d}y, \\
[u]_{x_0}^{x} + B\left[\frac{u^2}{2}\right]_{x_0}^{x} &= \frac{m}{\rho}A[y]_0^t, \\
(x-x_0) + \frac{B}{2}(x^2 - x_0^2) &= \frac{m}{\rho}At,
\end{aligned}
$$

We then get,

$$0 = \frac{B}{2}x^2 + x - x_0\left(1 + \frac{B}{2}x_0\right) - \frac{m}{\rho}At.$$

Here we find a second-order equation of variable x and of the shape $ax^2 + bx + c = 0$, with $a = \frac{B}{2}$, $b = 1$, and $c = -x_0\left(1+\frac{B}{2}x_0\right) - \frac{m}{\rho}At$. We have

$$
\begin{aligned}
\Delta &= b^2 - 4ac \\
&= 1 + 4\frac{B}{2}\left[x_0\left(1+\frac{B}{2}x_0\right) + \frac{m}{\rho}At\right].
\end{aligned}
$$

This expression is always positive, so we have two solutions of the form $x = \frac{-b \pm \sqrt{\Delta}}{2a}$. Only the positive solution can be considered here because we are examining thickness:

$$x(t) = \frac{-1 + \sqrt{1 + 2B\left[x_0\left(1+\frac{B}{2}x_0\right) + \frac{m}{\rho}At\right]}}{B}.$$

(2) We see that fractures whose thickness is initially $d_0 = 0.2$ mm may, after “only” 2000 years, have a thickness greater than 6 m!

```
#--------------------------------------
# Karstic fractured area
#--------------------------------------
# Dissolution coefficient (mol/m2/s)
k0 <- 4e-7
# Diffusion coefficient (m2/s)
D <- 1e-9
# Calcium concentration at equilibrium
(mol/m3)
Ceq <- 2
# Concentration threshold (mol/m3)
Cs <- 0.9 * Ceq
# Atomic mass of calcite (kg/mol)
m <- 0.1001
# Volumic mass of calcite (kg/m3)
rho <- 2700
# Initial diameter of fractures (m)
d0 <- 2e-4
# Time (year)
ta <- seq(0,2000,1)
# Time (s)
t <- ta * 365*24*3600
# Parameters of second-order equation
a <- k0/(12*D*Ceq)
b <- 1
# If C > Cs
c <- -d0*(1+k0*d0/(12*D*Ceq))-
  m*k0*(1-Cs/Ceq)^{-3}*t/rho
Delta <- b^2-4*a*c
# Fracture diameter evolution (m)
d <- (-b+sqrt(Delta))/(2*a)

plot(ta,d,type="l",lwd=1.2,
     xlab="Fracture diameters (m)")
```

Appendices

Appendix A: Free Software R

This appendix is intended to summarize the use of the freeware R, which will allow the reader to plot functions, import data, and analyze those using numerical methods. This in turn will allow the reader to learn the basics of the software but, it will not replace a proper lecture or a more advanced tutorial.

We advise, for instance, the very good tutorials available on the R documentation page:

https://cran.r-project.org/manuals.html.

For geography and Earth science–specific applications, there are also many resources on the Web.

The software R was originally intended for statistical analysis and graphical representations. Packages that can be downloaded for free provide a very complete toolbox. R is freeware, which means its use, modification, and distribution are freely permitted. R has an object-oriented language, but it is more accessible than other such languages, such as C++. A very active community uses R, so it is easy to find help on the Web.

To Install R

For optimal use of R, the use of an integrated development environment (IDE) is recommended. There are many IDEs, such as Emacs (https://www.gnu.org/software/emacs/), Eclipse (https://Eclipse.org/downloads/), and RStudio (https://www.rstudio.com/). The latter is very well suited for both beginners and advanced users. It allows one, for example, to share one's working environment and to have easy access to the code (the set of instructions typed by the operator), to figures, to the help, or to the download page of packages. It also offers the possibility of coloring the syntax of the code for better visualization of its structure.

Note 61 (Attention) *R must be installed before the IDE so that the IDE automatically finds links to R.*

To install R:

- Go to the R website: https://www.r-project.org/.
- On the left, click on **CRAN** *(which stands for Comprehensive R Archive Network)* under **Download**.
- Choose a mirror site, for example the closest to your home.
- Under the **Download and Install R** tab, click on your operating system (Windows, Linux, or Mac OS).
- For Windows, choose the **basic** installation.
- Click on **Download R-x.y.z for Windows**; the number of the latest version of R is given by the numbers **x.y.z**.
- An **R-x.y.z-Win.exe** file is downloaded to your computer.
- Double click on this file to install R (if the installer asks questions that are too hard to understand, take the default option).

To install RStudio:

- Go the RStudio website: https://www.rstudio.com/.
- Click on **Download RSstudio**.

C. Fleurant, S. Bodin-Fleurant, *Mathematics for Earth Science and Geography*, Springer Textbooks in Earth Sciences, Geography and Environment, https://doi.org/10.1007/978-3-319-69242-5

- Click on **Desktop** (free version) to install RStudio on your computer.
- Choose the installer that corresponds to the operating system (Windows, Linux, Mac OS).
- An **RStudio-x.y.z.exe** file is downloaded.
- Double click on this file to install RStudio (if the installer asks questions too hard to understand, take the default option).

Good Practices

To produce code and, therefore, results of good quality , it is important to act with some strictness. These good practices aim to make your code clearer, more efficient, and, therefore, more effective (Genolini, 2015).

Choosing Your IDE

The IDE is a tool that accompanies R, so it is important to choose a good one. Some IDEs are very basic and should be avoided. Others allow substantial functionality, some of them being essential to do quality work. Some examples are given here (Fig. A.1):

- Presentation of code: some IDEs color words according to their nature (e.g., variable, function), which gives great clarity to the code;

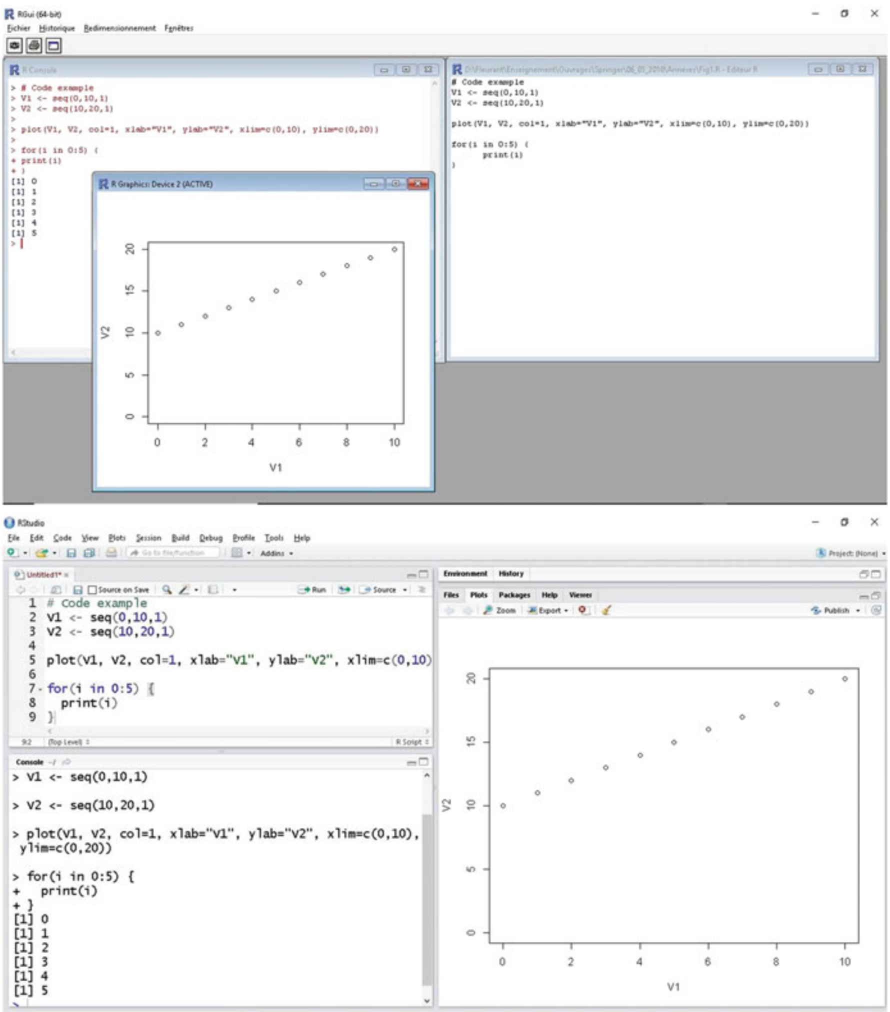

Fig. A.1 The basic editor of R (above) is much less user-friendly and functional than that of RStudio (below)

- Indentation: depending on the hierarchy of the instructions relative to each other, some IDEs make horizontal spaces, which provides a good view of the structure of the code;
- Assistance: when you are a beginner, many code errors are due to brackets or braces that have been opened and not reclosed. Some IDEs offer assistance that automatically closes any brackets or braces that have just been opened by the operator.

Simple manipulations

These are the basic building blocks of the code, so it is important to name them explicitly. This will make it easier to understand the code in its reading and make sense of it.

Example 81 *Nonexplicit variable names:*

```
tx <- c(3.2, 4.3, 0.5, 1.1)
M <- tx / length(tx)
```

In this example, the names chosen for the objects do not convey information about their nature. To understand that the variable `M` is the mean of the erosion rate, we must decode the writing.

Explicit variable names:

```
# Erosion Rate (mm/y)
ErosionRate <- c(3.2, 4.3, 0.5, 1.1)

# The mean of the erosion rates (mm/y)
MeanErosionRate <- ErosionRate /
length(ErosionRate)
```

We see here at once the link between the variable and what it means. For example, calling a datum `MeanErosionRate`, we understand that it will contain a mean value (here of the erosion rate).

It is also very important to enrich each object with a comment (preceded by the symbol `#`). This makes it possible to explain the nature of a variable and to give its unit if necessary. Generally speaking, comments should not be used sparingly as they will be very helpful in reading the code, making it meaningful and resuming it faster should the code be returned to after a long period of time.

There are no rules for writing objects. We recommend beginning each word in the name with a capital letter (when several words are joined together).

Finally, it should be noted that a compromise must also be made between an explicit name and its size: too long object names would make the code difficult to read.

Be careful not to give the same name for two different objects, and do not use a reserved instruction of R for a variable name (e.g., `T` is used for the Boolean variable `TRUE`, matrix is used to declare a matrix).

Code Writing

```
Code 1:
tx<-c(3.2,4.3,0.5,1.1);M<-.0;for(i  in
1:4){M<-M+tx[i]};
M<-M/4;print(M)
Code 2 :
# Erosion Rate (mm/y)
ErosionRate <- c(3.2, 4.3, 0.5, 1.1)

# The mean of the erosion rates (mm/y)
MeanErosionRate <- .0
for(i in 1:4) {
  MeanErosionRate <- MeanErosionRate +
ErosionRate[i]
}
MeanErosionRate <- MeanErosionRate / 4

# Print the result
print(MeanErosionRate)
```

Codes 1 and 2 produce the same result: they calculate the mean of the four erosion rates given. We prefer code 2, which is indented, commented, and structured by blocks in braces.

Online Help

Online help can be accessed directly from the console by typing `help(FunctionName)`. It gives many indications about the function of interest: how it is used, what arguments it needs, other related functions or examples of applications.

With RStudio you can also access this help in the Help window.

Using Packages

A **package** is a library of specific functions for a topic: image processing or Fourier analysis, for example. To take advantage of these functions that are not installed by default with R, two steps are necessary: package installation and loading the package.

There are several approaches to installing a new package:

- From the console or directly from the R script, type the command `install.packages("PackageName")`, where `"PackageName"` is the name of the package to install;
- From the RStudio Packages tab: find the desired package, check the package, and click *install.*

To load the previously installed package, use the function `library (PackageName)` either in the console at the beginning of the R session or directly in the script that will load the package when it is executed.

Functioning Principles

Console or R Script?

To calculate $\frac{3 \times 10^5}{0.1 \times 10^7}$, R can be used in two ways:

From the **console** you enter the following commands and press *enter* (like a simple calculator) to display the response (0.3).

```
> 3*1e5/(0.1*1e7)
[1] 0.3
```

Several points should be emphasized in writing in R:

- The sign $\times$ is written `*`;
- A power number as `10^\alpha` is written `1e\alpha`. Be careful: `10e2` has a value of 1000, while `10^2` has a value of 100. `e` already includes 10;
- Do not forget the brackets in the denominator. If one wrote

```
> 3*1e5/0.1*1e7
[1] 3e+13
```

this would mean $\frac{3 \times 10^5}{0.1} \times 10^7$.

From a **script** (File > New File > R Script), you write the following code:

```
MyCalculus <- 3 * 1e5 / ( 0.1 * 1e7 )
MyCalculus
```

Several points should be made in this code under R:

- The first line indicates that the result of the desired calculation will be stored in an object called `MyCalculus`. It can be seen that the sign `<-` is equivalent to the sign =;
- The second line asks that the value of the variable `MyCalculus` be displayed;
- The writing was spaced with respect to the one from the console. Both entries are possible. Everything is about readability.

We run the program (using the shortcut Ctrl + Alt + r, for example) and the result is as follows:

```
> MyCalculus
[1] 0.3
```

The console can both use R in a supercalculator mode and retrieve information that would have been displayed on screen earlier. Note that the console mode does not allow for keeping track of the results.

Functions

A function is a block of code that, starting from input values (called arguments), processes them and returns an output result. R contains a large number of predefined functions, for example:

```
ErosionRate <- c(3.2, 4.3, 0.5, 1.1)
mean(ErosionRate)
```

The `mean()` function, as suggested, calculates an average value. The input argument is `ErosionRate`; the function will then calculate the mean of these four values and display 2.275.

One can also create one's own functions using the reserved word function:

```
Difference <- function(x1,x2) {
    return ( x1 - x2 )
}
Difference(5,11)
```

The `Difference()` function has two input arguments, `x1` and `x2`, and returns their difference. When there are several arguments, be careful to respect their order. Here, when we use the function `Difference(5,11)`, we have `x1` = `5` and `x2` = `11`, and the result will be −6. If we reverse the order of the arguments, the result will be 6.

If the number of arguments is not correct, R returns an error message.

Obviously, the result of the function can be stored in a variable to be reused later:

```
Difference <- function(x1,x2) {
    return ( x1 - x2 )
}
Diff <- Difference(5,11)
print(abs(Diff))
```

Here we store the result of the function `Difference(5,11)` in the object `Diff` and show its value using the function `print()`.

Saving Your Script

Do not forget to save your work regularly.

When leaving R or RStudio, it is suggested that you save your working environment So you can retrieve the work did later. Your work will be stored in the RData file located in the current directory.

Data Structuring

In R, data can be structured in different types of objects: `vector`, `matrix`, and `data.frame` (there is also the list, the factor, and the array, but we will not detail them here).

These objects may contain data that could be integers, real numbers, or characters.

Vector

A vector is an object that has one dimension, that is to say, a sequence of data. It is created in R mainly in two different ways: by the function `c()` or the function `seq()`.

```
VecInteger <- c(5,-22,1,13)
VecReals <- c(1.5,6.7,2e-3,-11.6)
VecCharacters <- c("Victor","Abel","-
Martin","Jeanne")

VecTimeStep <- seq(0,100,0.1)
```

The vector `VecInteger` is a sequence of integers 5, 22, 1, and 13. The `VecReals` vector is the sequence of real numbers 1.5, 6.7, 2, and 11.6. The vector `VecCharacters` is a sequence of characters (or rather here of strings) Victor, Abel, Martin, and Jeanne. Finally, the `VecTimeStep` vector is a sequence of numbers that starts at 0 and ends at 100 with a step of 0.1. It is therefore the sequence of the real numbers 0, 0.1, 0.2,. . ., 99.8, 99.9, 100.

Note that a vector must contain data of the same type.

To access the value of a vector, the operator `[]` is used. Thus `VectReals[3]` returns the value 2 (the third value of the vector) and `VectTimeStep[20]` returns the value 19.9 (the twentieth value of the vector).

There are many functions that can be used on vectors and that therefore make it possible to calculate a value from a series of data: mean, standard deviation, search for maxima, and so forth.

Matrix

A matrix is an object that has two dimensions, that is to say, it corresponds to data organized in rows and columns. A matrix can be created in different ways.

Combining two vectors by the `rbind()` or `cbind()` function:

```
Vec1 <- c(1,2,3,4)
Vec2 <- c(5,6,7,8)

Mat1 <- cbind(Vec1,Vec2)
Mat2 <- rbind(Vec1,Vec2)
```

This results in

```
> Mat1
     Vec1 Vec2
[1,]    1    5
[2,]    2    6
[3,]    3    7
[4,]    4    8
> Mat2
     [,1] [,2] [,3] [,4]
Vec1    1    2    3    4
Vec2    5    6    7    8
```

One can also create a matrix using the reserved R word matrix:

```
Vec3 <- c(1,2,3,4,5,6,7,8,9)

Mat3 <- matrix(Vec3,nrow=3,ncol=3,by-
row=FALSE)
Mat4 <- matrix(Vec3,nrow=3,ncol=3,by-
row=TRUE)
```

We specify the number of rows (`nrow`), the number of columns (`ncol`), and the way the matrix is filled (`byrow`). When this last criterion is true (`TRUE`), the matrix is filled horizontally (per row); otherwise it is filled vertically (per column):

```
> Mat3
     [,1] [,2] [,3]
[1,]    1    4    7
[2,]    2    5    8
[3,]    3    6    9

> Mat4
     [,1] [,2] [,3]
[1,]    1    2    3
[2,]    4    5    6
[3,]    7    8    9
```

The data of a matrix are accessed by specifying its row and column numbers, as for a vector but in two dimensions. For example, for the `Mat3` matrix, the element that is in line 2 and column 3 is 8. For `Mat4` matrix, the element that is in line 3 and column 1 is 7:

```
> Mat3[2,3]
[1] 8

> Mat4[3,1]
[1] 7
```

One can also access a whole row or an entire column. The elements of line 2 for `Mat3` and those of column 1 for `Mat4` are

```
> Mat3[2,]
[1] 2 5 8

> Mat4[,1]
[1] 1 4 7
```

Note that a matrix must contain data of the same type.

As with vectors, many functions allow you to calculate a value from the elements of the matrix: average of a row or column, search of maxima over the entire matrix, and so on.

Data.frame

This is an object that makes it possible to store variables of different types (e.g., characters, numbers). In the following example, 10 numbers (sizes) are randomly generated between 1.5 m and 1.9 m. These sizes are assigned to male (M) and female (F).

```
Size <- runif(10,1.5,1.9)
Gender  <-c("M","M","F","M","F",
"F","F","M","M","F")

Table <- data.frame(Size,Gender)
```

The content of the data.frame is as follows:

```
> Table
       Size Gender
1  1.874439    M
2  1.837619    M
3  1.702383    F
4  1.535661    M
5  1.832196    F
6  1.899510    F
7  1.554595    F
8  1.830622    M
9  1.515266    M
10 1.649236    F
```

Data in a column can be accessed using the operator $ followed by the column name:

```
     > Table$Size
        [1] 1.874439 1.837619 1.702383
1.535661 1.832196 1.899510
        [7] 1.554595 1.830622 1.515266
1.649236
```

An element of this column is accessed by adding its position via the operator []:

```
      > Table$Size[3]
     [1] 1.702383
```

Importing Data

To import a text file, use the read.table() function:

```
MyData  <-  read.table("D:/Data/data1.
txt")
```

Note that you must specify the path to the text file. Another way to do this is to use the setwd() command at the very beginning of the script to set the directory where the data are located:

```
setwd("D:/Data/")
MyData <- read.table("data1.txt")
```

In addition, the path is identified by "/" and not "\", as is the case in Windows.

To import a .csv file, the read.csv() command is used in the same way. Importing Excel files requires the use of an xlsx package.

The data are exported in the same way by simply replacing read() with write().

Graphics

The main function for making graphics in R is plot(). This function requires at least two arguments: x and y-coordinates having the same dimension (that is, the same size or number of elements). In the following example, a point is plotted and then a series of four other points:

```
# Plot one single point
x <- 2.5
y <- 13
plot(x,y)

# Plot 4 points
x <- c(1.5, 2.0, 5, 0.5, -6.2)
y <- c(13, -33, 5.6, 11,8)
plot(x,y)
```

Each plot() function creates a new graphic. Thus, to add other points on an existing graph, use the function points() (for points) and lines() (for lines).

There are a large number of parameters to produce very beautiful graphics (Fig. A.2 and associated code). The scales can be set with xlim() and ylim(), the labels of the axes with xlab() and ylab(), the symbols with pch, the colors with col, and so forth.

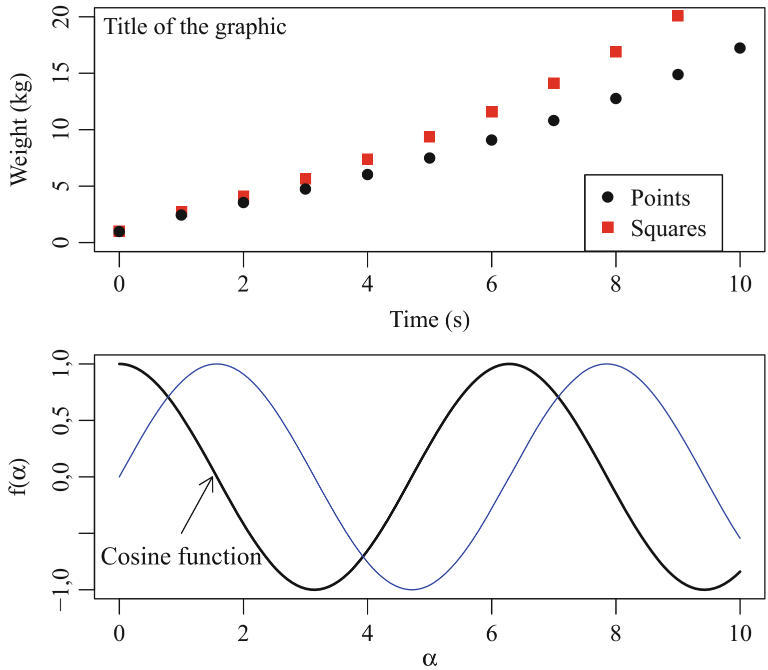

Fig. A.2 Examples of graphics easily produced in R

One can also insert text on a graphic with `text()` and text with special characters with `expression()`, add a legend with `legend()`, or draw segments (`segments()`), arrows (`arrows()`), and rectangles (`rect()`).

The position of a graph in a window is fully configurable (`par()`) and makes it possible to export high-quality graphics in jpeg (`jepg ()`) or in pdf (`pdf ()`).

A large number of plot functions are available: histogram (`hist()`), boxplot (`boxplot()`), pie (`pie ()`), in 3D (`persp ()`). The graphic possibilities are enormous; apart from the basic graphs described here, there are many graphical packages such as lattice, ggplot2, and rgl.

```
#-------------------------------------
# Some Graphics
#-------------------------------------
# Export in pdf format
pdf("Graphic.pdf", family="Times",
    width=5.5, height=5)
# Windows parameter (par): split the
windows in 2 lines
# and 2 columns (mfrow) and manage the
space between
# the graphics (mar)
par(mfrow=c(2,1),
    mar = c(4,4.5,0.5,1)
    )

# Graphic 1
x <- seq(0,10,1)
y <- exp(sqrt(x))
# pch : symbols
# col : color
# xlab : label of x axe
# ylab : label of y axe
# ylim : scale of y axe
plot(x,y,pch=15,col=2,
        xlab="Time (s)", ylab="Weight
(kg)",
     ylim=c(0,20))
# Add some points
x1 <- seq(0,10,1)
y1 <- exp(0.9*sqrt(x1))
points(x1,y1,pch=16)
# Add a title
text(1,19,"Title of the graphic")
# Add a legend
```

```
legend(7.5,6,c("Points","Squares"),
       pch=c(16,15),col=c(1,2))

# Graphic 2
x <- seq(0,10,0.01)
# type : plot a line
# lwd : width of the line
#  expression  :  manage  mathematical
symbols
plot(x,cos(x),type="l",col=1,lwd=2,
     xlab=expression(italic(alpha)),
    ylab=expression(f(italic(alpha))))
# Add a line
lines(x,sin(x),col=4,lwd=1)
# Add an arrow and text
arrows(1,-0.5,1.5,0,length=0.1)
text(1,-0.7,"Cosine function")

dev.off()
```

Appendix B: Free Software Xcas

This appendix is a brief introduction to Xcas, a free computer algebra system, and the main useful commands for mathematics used in this book.

Introduction and Overview

A formal calculation software package allows for theoretical mathematical calculations (and approximate) such as the exact solution of equations, the derivative of functions (including with parameters), and so forth.

For example, such software can solve the equation $x^2 - 7x + 4 = 0$ and $x^2 - ax + 3b = 0$, and derive the function f defined by $f(x) = \ln(x^2 - 3x + 2)$.

Xcas is a formal calculation software developed at Fourier University in Grenoble (mainly by Bernard Parisse, associate professor at the Institut Fourier, Grenoble University, France). Xcas is a real Swiss Army knife for mathematics: in addition to formal calculation, Xcas produces 2D and 3D graphics, 2D and 3D interactive geometry, spreadsheets, statistics, and programming.

This free software can be downloaded (for offline use on your browser) or used directly online (which can be useful when not working on one's own computer).

- The Xcas software (Giac / Xcas) can be downloaded by entering "Xcas" on a search engine or by using the link to the installation page of Xcas:
  https://www-fourier.ujf-grenoble.fr/~parisse/install_en.html
- An Xcas online version (less complete version) can also be used directly online (via a browser) without installation, using the following link:
  http://www-fourier.ujf-grenoble.fr/~parisse/xcasen.html
  For tablets or smartphones, use online Xcas.

Principle of Operation

Xcas Downloaded

In addition to the online help and tutorial, one can find numerous resources and aids on the Internet. The authors invite readers to refer to them to complete the elements given in what follows.

Online Xcas

Using the "OK" button generates a new line. Then change the command line to write a new instruction (Fig. B.1).

Major Command Lines

The Keyboard

In the online version (button *123+*) as well as in the downloaded version (control *Kbd* for "keyboard"), a keyboard is available with numbers and key commands and useful mathematical functions (such as sin, $\int$).

Fig. B.1 Xcas online version

The command *abc* allows access to Roman letters and the key α allows for typing Greek letters.

A Few Rules for Writing Calculations

A comma (,) is used to separate items in a list of a sequence ([1,2,3] and (1,2,3)), and a semicolon is used to separate instructions of a program.

Symbols * (respectively /, ^) are used for indicating a multiplication operation (division, a power).

Note that Xcas recognizes $2x$ (2 multiplied by x), but ax is treated by Xcas as a new variable (whose name has two letters) and not the product of a times x (for this, enter $a*x$).

Calculating Sums

One can use directly the symbol $\sum$ on the keyboard or the command *sum*.

Example 82

Type:
sum (k ^ 2, k, 1,10)
to calculate $\sum_{k=1}^{10} k^2$.

Solving Equations

The exact solution of an equation is performed using the command *solve*; a solution giving approximate values of solutions is performed with the command *fsolve*. The name of the variable must be specified (it may be omitted if the variable is x).

Example 83

solve (x ^ 3 + x ^ 2-2x-2 = 0)
and
solve (t ^ 3 + t ^ 2-2t-2 = 0, t)
give the exact solutions of the equation $x^3 + x^2 - 2x - 2 = 0$ (which are: $-\sqrt{2}$, -1 and $\sqrt{2}$).
fsolve (t ^ 3 + t ^ 2-2t-2 = 0, t)
gives the approximate solutions of the equation (namely: −1.41421356237; −1; 1.41421356237).

Functions

The most common functions are known by Xcas with explicit names: *exp* for exponential function, *sin* for the sine function...

To declare a function with one or more variables, use =

Example 84

f (x): = x ^ 3-7x + 4
*g (x) = a * x ^ 2 + 2x + b*
h (t): = ln (t ^ 2 + 3)
*k (x, y, z): = x ^ 2 * y + x * y * z*

You can ask Xcas to calculate exact values ( ) or approximate values ( ).

The ability to plot functions using Xcas was not used in this textbook since we favored the R software for this; the graphical representation of functions is of course also possible with Xcas.

Differentiation, Integration

The commands *diff* (or the keyboard command δ or postfix version ’) and *int* (or the keyboard command ∫) allow the user to calculate derivatives, partial derivatives, or integrals of expressions or functions. The answer will be of the same nature as the argument: an expression or a function.

To name the derivative function, use the designation “:= ”.

Note that you cannot write $g(x)$: = *diff*($f(x)$) because the letter x would therefore have two meanings (on the one hand, a variable associated with the function, and on the other hand, a letter used for the formal derivation). Use g = *diff*(f) or *g: = function_diff(f).*

Example 85

f (x): = ln (3x + 4)
g: = diff (f)
k (z, t): = z ^ 2-3t
diff (k (z, t), z)
diff (k (z, t), z $ 1, $ t 1)
calculates $\dfrac{\partial^2 k}{\partial z \partial t}(z, t)$.
A: = int (f (x), x, 2,3)

Indicated in this order are the function, the variable name, the lower limit of integration, and the upper limit of integration.

Differential Equations

To solve differential equations accurately, use the function *desolve*. The unknown function is by default named y, but you can use another name (see help on *desolve* for details). One can use the commands y’ and y” for the derivatives (and in this case the variable is x), or use *diff*($y(t)$,t) to select the variable. If the initial conditions are not specified, the solutions are given with arbitrary constants.

Example 86

desolve (y'-3y = 0, y)
*desolve (x ^ 2 * y '+ 2 * y = 0, y)*
*desolve ([y ''+ 3 * y '+ 2 * y = 0, y (0) = 1, y (1) = -1], y)*

If initial conditions are given, put all conditions inside brackets, in list form.

To solve differential equations in an approximate way (and obtain an approximate value of the solution at a given point), use the command *odesolve*: this command is used for differential equations of the form $y' = f(x,y)$.

Example 87

*odesolve (-x * y [x, y], [0,1], 2)*

allows one to know the approximate value of g (2), where g is the solution of the differential equation $y' = -x * y$, such as $g(0) = 1$.

Further Reading

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