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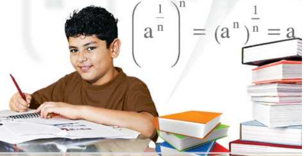
IIT Foundation Mathematics

Class – IX

$$\left(a^{\frac{1}{n}}\right)^n = \left(a^n\right)^{\frac{1}{n}} = a$$



S.K. GUPTA
ANUBHUTI GANGAL

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PREFACE AND A NOTE FOR THE STUDENTS

ARE YOU ASPIRING TO BECOME AN ENGINEER AND AN IIT SCHOLAR ?

Here is the book especially designed to motivate you, to sharpen your intellect, to develop the right attitude and aptitude, and to lay a solid foundation for your success in various entrance examinations like **IIT, EAMCET, WBJEE, MPPET, SCRA, J&K CET, Kerala PET, OJEE, Rajasthan PET, AMU, BITSAT,** etc.

SALIENT FEATURES

1. Content based on the curriculum of the classes for ***CBSE, ICSE***, ***Andhra Pradesh*** and ***Boards of School Education of Other States.***
2. Full and comprehensive coverage of all the topics.
3. Detailed synopsis of each chapter at the beginning in the form of ***'Key Concepts'***. This will not only facilitate thorough *'**Revision**'* and *'**Recall**'* of every topic but also greatly help the students in understanding and mastering the concepts besides providing a ***back-up*** to classroom teaching.
4. The books are enriched with an exhaustive range of hundreds of thought provoking objective questions in the form of solved examples and practice questions in practice sheets which not only offer a great variety and reflect the modern trends but also invite, explore, develop and put to test the ***thinking, analysing*** and ***problem-solving skills of the students***.
5. **Answers, Hints** and **Solutions** have been provided to boost up the morale and increase the confidence level.
6. **Self Assessment Sheets** have been given at the end of each chapter to help the students to assess and evaluate their understanding of the concepts and learn to attack the problems independently.

   We hope this book will be able to fulfil its aims and objectives and will be found immensely useful by the students aspiring to become top class engineers.

   Suggestions for improvement and also the feedback received from various sources would be most welcome and gratefully acknowledged.

AUTHORS

CONTENTS

Chapter 1.	Logarithms	1-1 to 1-15
Chapter 2.	Polynomials	2-1 to 2-16
Chapter 3.	Quadratic Equations	3-1 to 3-19
Chapter 4.	Inequalities	4-1 to 4-26
Chapter 5.	Relations	5-1 to 5-13
Chapter 6.	Plane Geometry–Triangles	6-1 to 6-22
Chapter 7.	Quadrilaterals	7-1 to 7-26
Chapter 8.	Permutations and Combinations	8-1 to 8-30
Chapter 9.	Probability	9-1 to 9-31
Chapter 10.	Trigonometry	10-1 to 10-25
Chapter 11.	Coordinate Geometry	11-1 to 11-28
Chapter 12.	Area and Perimeter	12-1 to 12-14
Chapter 13.	Volume and Surface Area of Solids	13-1 to 13-18

1 Logarithms

KEY FACTS

1. Definition: If a and n are positive real numbers such that $a \neq 1$ and x is real, then $a^x = n \Rightarrow x = \log_a n$.
Here x is said to be the logarithm of the number n to the base a.

Ex. $4^3 = 64 \Rightarrow \log_4 64 = 3$, $\quad 10^{-1} = \frac{1}{10} = 0.1 \Rightarrow \log_{10} 0.1 = -1$, $\quad 5^x = 4 \Rightarrow x = \log_5 4$,

$a^0 = 1 \Rightarrow \log_a 1 = 0$, $\quad a^1 = a \Rightarrow \log_a a = 1$.

2. Some Important Facts about Logarithms

- $\log_a n$ is real if $n > 0$
- $\log_a n$ is imaginary if $n < 0$
- $\log_a n$ is not defined if $n = 0$
- The logarithm of 1 to any base $\boldsymbol{a}$**,** $a > 0$ and $a \neq 1$ is zero. $\boxed{\mathbf{\log_a 1 = 0}}$
- The logarithm of any number $\boldsymbol{a}$, $a > 0$ and $a \neq 1$, to the same base is 1. $\boxed{\mathbf{\log_a a = 1}}$
- If $\boldsymbol{a}$ and $\boldsymbol{x}$ are positive real numbers, where $a \neq 1$, then $\boxed{\boldsymbol{a^{\log_a x} = x}}$

Proof. Let $\log_a x = p$. Then, $x = a^p$ (By def.) $\Rightarrow x = a^{\log_a x}$ (Substituting the value of p)

Ex. $3^{\log_3 7} = 7$, $2^{\log_2 9} = 9$, $5^{\log_5 x} = x$

- For $a > 0$, $a \neq 1$, $\boldsymbol{\log_a x_1 = \log_a x_2 \Rightarrow x_1 = x_2\ (x_1, x_2 > 0)}$
- If $a > 1$ and $x > y$, then $\log_a x > \log_a y$.
- If $0 < a < 1$ and $x > y$, then $\log_a x < \log_a y$

3. Laws of Logarithms

For $x > 0$, $y > 0$ and $a > 0$ and $a \neq 1$, any real number n

- $\boldsymbol{\log_a xy = \log_a x + \log_a y}$ **Ex.** $\log_2(15) = \log_2(5 \times 3) = \log_2 5 + \log_2 3$
- $\boldsymbol{\log_a(x/y) = \log_a x - \log_a y}$ **Ex.** $\log_2\left(\frac{3}{7}\right) = \log_2 3 - \log_2 7$
- $\boldsymbol{\log_a(x)^n = n \log_a x}$ **Ex.** $\log(2)^5 = 5\log 2$,

  $$\log\left(\frac{a^3}{b^3}\right) = \log a^3 - \log b^3 = 3\log a - 3\log b$$

- $\boldsymbol{\log_a x = \frac{1}{\log_x a}}$ **Ex.** $\log_5 2 = \frac{1}{\log_2 5}$
- $\boldsymbol{\log_{a^n} x = \frac{1}{n}\log_a x}$ **Ex.** $\log_8 7 = \log_{2^3}(7) = \frac{1}{3}\log_2 7$, $\log_{\sqrt{5}} 3 = \log_{(5)^{\frac{1}{2}}}(3) = \log_{5^{1/2}}(3) = \frac{1}{1/2}\log_5 3 = 2\log_5 3$
- $\boldsymbol{\log_{a^n} x^m = \frac{m}{n}\log_a x}$ **Ex.** $\log_{2^5} 5^4 = \frac{4}{5}\log_2 5$

Base changing formula

- $\log_a x = \log_b x . \log_a b$ **Ex.** $\log_{12}32 = \log_{16}32.\ \log_{12}16.$ (The base has been changed from 12 to 16)

$$\underset{\text{Old base}}{\uparrow} \qquad \underset{\text{New base}}{\uparrow}$$

- $x^{\log_a y} = y^{\log_a x}$ **Ex.** $3^{\log 7} = 7^{\log 3}$ (It being understood that base is same)

[**Proof**. $x^{\log_a y} \to = x^{\log_x y \,.\, \log_a x}$ (Base changing formula)

$= \left(x^{\log_x y}\right)^{\log_a x}$ (Using $n \log_a x = (\log_a x)^n$)

$= y^{\log_a x}$ (Using $x^{\log_x y} = y$.)

- $\log_a b = \dfrac{\log b}{\log a}$ (It being understood that base is same)
- If $\log_a b = x$ for all $a > 0$, $a \neq 1$, $b > 0$ and $x \in R$, then $\log_{1/a} b = -x$, $\log_a 1/b = -x$ and $\log_{1/a} 1/b = x$

4. Some Important Properties of Logarithms

- a, b, c are in G.P. $\Leftrightarrow \log_a x, \log_b x, \log_c x$ are in H.P.
- a, b, c are in G.P. $\Leftrightarrow \log_x a, \log_x b, \log_x c$ are in A.P.

5. Natural or Naperian logarithm **is denoted by $\log_e N$, where the base is e.**

Ex. $\log_e 7$, $\log_e \left(\dfrac{1}{64}\right)$, $\log_e b$, etc.

- **Common or Brigg's logarithm** is denoted by $\log_{10} N$, where the base is **10.**

  Ex. $\log_{10}5$, $\log_{10}\left(\dfrac{1}{81}\right)$, etc.

- $\log_a x$ is a decreasing function if $0 < a < 1$
- $\log_a x$ is an increasing function if $a > 1$.

6. Characteristic and Mantissa

- **Characteristic:** The integral part of the logarithm is called characteristic.

(*i*) If the number is greater than unity and there are n digits in integral part, then its characteristic = $(n - 1)$

(*ii*) When the number is less than 1, the characteristic is one more than the number of *zeroes* between the decimal point and the first significant digit of the number and is negative. It is written as $\overline{(n+1)}$ or Bar $(n + 1)$.

Ex.

Number	Characteristic	Number	Characteristic
4.1456	0	0.823	$\bar{1}$
24.8920	1	0.0234	$\bar{2}$
238.1008	2	0.000423	$\bar{4}$

7. Arithmetic Progression. **A sequence** $a_1, a_2, a_3, ..., a_n$ is said to be in arithmetic progression, when $a_2 - a_1 = a_3 - a_2 = ... = a_n - a_{n-1}$, *i.e.*, when the terms in the sequence increase or decrease by a constant quantity called the ***common difference***.

Ex. 1, 3, 5, 7, 9, ... 6, 11, 17, 23, ... – 5, –2, 1, 4, 7, ...

- **Sum of first '*n*' terms of an Arithmetic Progression**

  $$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l],$$

  where a = first term, n = number of terms, d = common difference, l = last term.

- **Sum of first "*n*" natural numbers.**

  $$S_n = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}.$$

Also, written as $\Sigma n = \frac{n(n+1)}{2}$

- Also, **if a, b, c are in A.P.** then $2b = a + c$

8. Geometric Progression : A sequence $a_1, a_2, a_3, \ldots\ldots\ldots\ldots, a_n$ is said to be in Geometric Progression when,

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots\ldots\ldots\ldots = \frac{a_n}{a_{n-1}} = r(\text{say})$$

where $a_1, a_2, a_3, \ldots\ldots\ldots\ldots$ are all non zero numbers and r is called the **common ratio.**

Ex. 3, 6, 12, 24, ... $r = 2$;

64, 16, 4, 1, $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots\ldots\ldots\ldots$ $r = \frac{1}{4}$

- **Sum of first n terms of a G.P.** $S_n = \frac{a(r^n - 1)}{(r-1)}$ if $r > 1$ $= \frac{a(1-r^n)}{(1-r)}$ if $r < 1$ $= \frac{lr - a}{r - 1}$

  where, a = first term, r = common ratio, l = last term

- **Sum of an infinite G.P.** $S_\infty = \frac{a}{1-r}$, where a = first term, r = common ratio.

- For three terms a, b, c to be in G.P., $b^2 = ac$

9. Harmonic Progression : A series of quantities $a_1, a_2, a_3, \ldots\ldots\ldots\ldots, a_n$ are said to be in H.P. when their reciprocals $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots\ldots\ldots\ldots, \frac{1}{a_n}$ are in A.P.

- When three quantities a, b, c are in H.P., then, $b = \frac{2ac}{a+c}$.

SOLVED EXAMPLES

Ex. 1. If $\log_a 5 + \log_a 25 + \log_a 125 + \log_a 625 = 10$, then find the value of a.

Sol. $\log_a 5 + \log_a 25 + \log_a 125 + \log_a 625 = 10$

$\Rightarrow \log_a (5 \times 25 \times 125 \times 625) = 10$

$\Rightarrow \log_a (5^1 \times 5^2 \times 5^3 \times 5^4) = 10$

$\Rightarrow \log_a 5^{10} = 10 \Rightarrow a^{10} = 5^{10} \Rightarrow \boldsymbol{a = 5}$. [Using $\log_a x = n \Rightarrow x = a^n$]

Ex. 2. Solve for x : $\log_{10} [\log_2 (\log_3 9)] = x$.

Sol. $\log_{10} [\log_2 (\log_3 9)] = x$

$\Rightarrow \log_2 (\log_3 9) = 10^x$

$\Rightarrow \log_2 (\log_3 3^2) = 10^x$

$\Rightarrow \log_2 (2 \log_3 3) = 10^x$

$\Rightarrow \log_2 2 = 10^x \Rightarrow 10^x = 1 = 10^0 \Rightarrow \boldsymbol{x = 0}$.

Ex. 3. Find the value of $\log_x x + \log_x x^3 + \log_x x^5 + \ldots\ldots\ldots + \log_x x^{2n-1}$.

Sol. $\log_x x + \log_x x^3 + \log_x x^5 + \ldots\ldots\ldots + \log_x x^{2n-1} = \log_x x + 3\log_x x + 5\log_x x + \ldots\ldots\ldots + (2n-1)\log_x x$

$= 1 + 3 + 5 + \ldots\ldots\ldots\ldots + (2n-1) = \frac{n}{2}[1 + (2n-1)] = \boldsymbol{n^2}$ $\left[\text{Using } \log_x x = 1 \text{ and for A.P. } S_n = \frac{n}{2}(a+l)\right]$

Ex. 4. If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$, show that $f\left(\dfrac{2x}{1+x^2}\right) = 2f(x)$.

Sol. $f\left(\dfrac{2x}{1+x^2}\right) = \log\left[\dfrac{1+\dfrac{2x}{1+x^2}}{1-\dfrac{2x}{1+x^2}}\right] = \log\left[\dfrac{1+x^2+2x}{1+x^2-2x}\right] = \log\left[\dfrac{(1+x)^2}{(1-x)^2}\right] = 2\log\left[\dfrac{1+x}{1-x}\right] = \mathbf{2f(x)}.$

Ex. 5. If $a = \log_{24}12$, $b = \log_{36}24$, $c = \log_{48}36$, then prove that $1 + abc = 2bc$.

Sol. $1 + abc = 1 + \log_{24}12.\ \log_{36}24.\ \log_{48}36 = 1 + \log_{36}12.\ \log_{48}36$

$= 1 + \log_{48}12 = \log_{48}48 + \log_{48}12$ $[\because \log_a x.\ \log_b a = \log_b x]$

$= \log_{48}(48 \times 12) = \log_{48}(24 \times 24)$

$= \log_{48}(24)^2 = 2\log_{48}24.$ ...(*i*)

Also, $2bc = 2\log_{36}24.\ \log_{48}36 = 2\log_{48}24$ ...(*ii*)

From (*i*) and (*ii*), we have RHS = LHS.

Ex. 6. Solve $\log_{2x+3}(6x^2 + 23x + 21) = 4 - \log_{3x+7}(4x^2 + 12x + 9)$.

Sol. Given, $\log_{(2x+3)}(6x^2 + 23x + 21) = 4 - \log_{(3x+7)}(4x^2 + 12x + 9)$

$\Rightarrow \log_{(2x+3)}(2x+3)(3x+7) = 4 - \log_{(3x+7)}(2x+3)^2$

$\Rightarrow \log_{(2x+3)}(2x+3) + \log_{(2x+3)}(3x+7) = 4 - 2\log_{(3x+7)}(2x+3)$

$\Rightarrow \log_{(2x+3)}(3x+7) + 2\log_{(3x+7)}(2x+3) = 4 - 1 = 3$ [Since $\log_{2x+3}(2x+3) = 1$]

$\Rightarrow \log_{(2x+3)}(3x+7) + \dfrac{2}{\log_{(2x+3)}(3x+7)} = 3$ $\left[\text{Using } \log_a x = \dfrac{1}{\log_x a}\right]$

Let $\log_{(2x+3)}(3x+7) = t$. Then,

$t + \dfrac{2}{t} = 3 \Rightarrow t^2 - 3t + 2 = 0 \Rightarrow (t-1)(t-2) = 0 \Rightarrow t = 1, 2$

$t = 1 \Rightarrow \log_{(2x+3)}(3x+7) = 1 \Rightarrow \log_{(2x+3)}(3x+7) = \log_{(2x+3)}(2x+3)$ [Replacing 1 by $\log_{(2x+3)}(2x+3)$]

$\Rightarrow 3x + 7 = 2x + 3 \Rightarrow x = -4.$

$t = 2 \Rightarrow \log_{(2x+3)}(3x+7) = 2 \Rightarrow \log_{(2x+3)}(3x+7) = \log_{(2x+3)}(2x+3)^2$

$\Rightarrow (3x+7) = (2x+3)^2 \Rightarrow 4x^2 + 9x + 2 = 0 \Rightarrow (4x+1)(x+2) = 0 \Rightarrow x = -1/4, -2$

But $x = -4$ and -2 are extraneous solutions, so $x = -\mathbf{\dfrac{1}{4}}$.

Ex. 7. If $\log_x(a-b) - \log_x(a+b) = \log_x(b/a)$, find $\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2}$. ***(CAT 2012)***

Sol. Given, $\log_x(a-b) - \log_x(a+b) = \log_x(b/a) \Rightarrow \log_x\left[\dfrac{(a-b)}{(a+b)}\right] = \log_x\left(\dfrac{b}{a}\right)$

$\Rightarrow a(a-b) = b(a+b) \Rightarrow a^2 - ab = ab + b^2$

$\Rightarrow a^2 - b^2 = 2ab \Rightarrow a^2 - 2ab - b^2 = 0 \Rightarrow \left(\dfrac{a}{b}\right)^2 - 2\left(\dfrac{a}{b}\right) - 1 = 0$

This is a quadratic equation in $\dfrac{a}{b}$ and the product of the roots is -1 *i.e*, if a/b is a root, then $\left(-\dfrac{b}{a}\right)$ is the other root. Also, sum of its roots = 2

$\therefore \left(\dfrac{a}{b}\right)^2 + \left(\dfrac{b}{a}\right)^2 = \dfrac{a^2}{b^2} + \dfrac{b^2}{a^2} = \left[\dfrac{a}{b} + \left(-\dfrac{b}{a}\right)\right]^2 + 2 = 2^2 + 2 = \mathbf{6}.$

Ex. 8. If $\log_e 2.\ \log_b 625 = \log_{10} 16.\ \log_e 10$, then find the value of b.

Sol. Given, $\log_e 2.\ \log_b 625 = \log_{10} 16.\ \log_e 10 \Rightarrow \log_e 2.\ \log_b 5^4 = \log_{10} 2^4.\ \log_e 10$

$\Rightarrow\ \log_e 2.\ 4\log_b 5 = 4\log_{10} 2.\ \log_e 10$

$\Rightarrow\ \log_b 5. = \dfrac{\log_{10} 2.\log_e 10}{\log_e 2} = \dfrac{\log_e 2}{\log_e 2} = 1\ \Rightarrow\ b^1 = 5 \Rightarrow b = \mathbf{5}.$ $[\because \log_a x.\log_x b = \log_a b]$

Ex. 9. If $(x^4 - 2x^2y^2 + y^2)^{a-1} = (x-y)^{2a}(x+y)^{-2}$, then the value of a is

(a) $x^2 - y^2$ **(b)** $\log(xy)$ **(c)** $\dfrac{\log(x-y)}{\log(x+y)}$ **(d)** $\log(x-y)$

Sol. Given, $(x^4 - 2x^2y^2 + y^2)^{a-1} = (x-y)^{2a}(x+y)^{-2}$

$\Rightarrow\ [(x^2 - y^2)^2]^{a-1} = (x-y)^{2a}(x+y)^{-2}$

$\Rightarrow\ (x-y)^{2(a-1)}(x+y)^{2(a-1)} = (x-y)^{2a}(x+y)^{-2}$

$\Rightarrow\ \dfrac{(x-y)^{2(a-1)}}{(x-y)^{2a}}.\dfrac{(x+y)^{2(a-1)}}{(x+y)^{-2}} = 1 \Rightarrow (x-y)^{-2}(x+y)^{2a} = 1$

$\Rightarrow\ \log[(x-y)^{-2}(x+y)^{2a}] = \log 1 \Rightarrow -2\log(x-y) + 2a\log(x+y) = \log 1$

$\Rightarrow\ 2a\log(x+y) = 2\log(x-y) \Rightarrow a = \mathbf{\dfrac{\log(x-y)}{\log(x+y)}}.$ [Since $\log 1 = 0$]

Ex. 10. If $\log_x a$, $a^{x/2}$ and $\log_b x$ are in GP, then x is

(a) $\log_a(\log_b a)$ **(b)** $\log_a(\log_e a) + \log_a(\log_e b)$

(c) $-\log_a(\log_a b)$ **(d)** $\log_a(\log_e b) - \log_a(\log_e a)$

Sol. If $\log_x a$, $a^{x/2}$ and $\log_b x$ are in GP, then $\left(a^{x/2}\right)^2 = (\log_b x) \times (\log_x a)$

$\Rightarrow\ a^x = \log_b a \Rightarrow \log a^x = \log(\log_b a) \Rightarrow x\log a = \log(\log_b a) \Rightarrow x\log_a a = \log_a(\log_b a)$

$\Rightarrow\ \boldsymbol{x = \log_a(\log_b a)}.$

Ex. 11. What is the least value of the expression $2\log_{10} x - \log_x(1/100)$ for $x > 1$?

Sol. $2\log_{10} x - \log_x \dfrac{1}{100} = 2\log_{10} x - \dfrac{\log_{10} 10^{-2}}{\log_{10} x}$ $\left[\text{Using } \log_a b = \dfrac{\log_x b}{\log_x a}\right]$

$$= 2\log_{10} x + \frac{2}{\log_{10} x} = 2\left(\log_{10} x + \frac{1}{\log_{10} x}\right)$$

Given, $x > 1 \Rightarrow \log_{10} x > 0$

But since AM $\geq$ GM

$$\therefore\ \left[\frac{\log_{10} x + \dfrac{1}{\log_{10} x}}{2}\right] \geq \sqrt{\log_{10} x \times \frac{1}{\log_{10} x}}$$

$$\Rightarrow\ \log_{10} x + \frac{1}{\log_{10} x} \geq 2\ \Rightarrow 2\left[\log_{10} x + \frac{1}{\log_{10} x}\right] \geq 4$$

For $x = 10$, $2[\log_{10} x + \log_{10} x] \geq 4$

Hence, the least value of $\left[\log_{10} x - \log_x \dfrac{1}{100}\right]$ is 4.

Ex. 12. If $\log_3 2$, $\log_3(2^x - 5)$ and $\log_3(2^x - 7/2)$ are in A.P., then what is the value of x ?

Sol. Given, $\log_3 2$, $\log_3(2^x - 5)$ and $\log_3(2^x - 7/2)$ are in A.P.

$$\Rightarrow\ 2\left[\log_3(2^x - 5)\right] = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$$

$\Rightarrow$ $\log_3 (2^x - 5)^2 = \log_3 [2 \times (2^x - 7/2)]$

$\Rightarrow$ $(2^x - 5)^2 = (2^{x+1} - 7) \Rightarrow 2^{2x} - 10.2^x + 25 = 2.2^x - 7$

$\Rightarrow$ $2^{2x} - 12.2^x + 32 = 0 \Rightarrow y^2 - 12y + 32 = 0$ [Let $y = 2^x$]

$\Rightarrow$ $(y - 8)(y - 4) = 0 \Rightarrow y = 8$ or $4 \Rightarrow 2^x = 8$ or $2^x = 4 \Rightarrow x =$ **3 or 2.**

Ex. 13. Let $u = (\log_2 x)^2 - 6(\log_2 x) + 12$, where x is a real number. Then the equation $x^u = 256$ has :

(*a*) No solution for x **(*b*) Exactly one solution for x**

(*c*) Exactly two distinct solutions for x **(*d*) Exactly three distinct solutions for x** ***(CAT 2004)***

Sol. Given, $u = (\log_2 x)^2 - 6(\log_2 x) + 12 = p^2 - 6p + 12$ (where $p = \log_2 x$) ...(*i*)

Also, given, $x^u = 256$

Taking log to the base 2 of both the sides, we have

$u \log_2 x = \log_2 256 = \log_2 2^8 = 8 \log_2 2 \Rightarrow u \log_2 x = 8 \Rightarrow u = \dfrac{8}{\log_2 x} = 8/p$ ...(*ii*)

From (*i*) and (*ii*) $\dfrac{8}{p} = p^2 - 6p + 12$

$\Rightarrow$ $8 = p^3 - 6p^2 + 12 \Rightarrow p^3 - 6p^2 + 12p - 8 = 0$

$\Rightarrow$ $(p - 2)^3 = 0 \Rightarrow p = 2.$ [Using $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$]

$\therefore$ $\log_2 x = 2 \Rightarrow x = 2^2 = 4$

Hence the equation $u^4 = 256$ has **exactly one solution.**

Ex. 14. If $\log_y x = (a.\ \log_z y) = (b.\ \log_x z) = ab$, then which of the following pairs of values for (a, b) is not possible ?

(*a*) $\left(-2, \dfrac{1}{2}\right)$ **(*b*) (1, 1)** **(*c*) (0.4, 2.5)** **(*d*) $\left(\pi, \dfrac{1}{\pi}\right)$** **(*e*) (2, 2)**

(CAT 2004)

Sol. Given, $\log_y x = (a.\ \log_z y) = (b \log_x z) = ab$

$\Rightarrow a = \dfrac{\log_y x}{\log_z y}$ and $b = \dfrac{\log_y x}{\log_x z}$

$$\therefore\ a \times b = \frac{\log_y x}{\log_z y} \times \frac{\log_y x}{\log_x z} = \frac{\left[\dfrac{\log x}{\log y}\right]}{\left[\dfrac{\log y}{\log z}\right]} \times \frac{\left[\dfrac{\log x}{\log y}\right]}{\left[\dfrac{\log z}{\log x}\right]} = \left(\frac{\log x}{\log y}\right)^3 = (\log_y x)^3 = (ab)^3$$

$\Rightarrow ab - a^3b^3 = 0 \Rightarrow ab(1 - a^2b^2) = 0 \Rightarrow ab = \pm 1$

$\therefore$ Only option (*e*) does not satisfy the condition, sin(2, 2) is not a possible value of (*a, b*).

PRACTICE SHEET

LEVEL–1

1. (*i*) The solution of $\log_\pi (\log_2 (\log_7 x)) = 0$ is

(*a*) 2 (*b*) π^2 (*c*) 7^2 (*d*) None of these

(*WBJEE 2008*)

Similar questions

(*ii*) $\log_{27} (\log_3 x) = \dfrac{1}{3} \Rightarrow x =$

(*a*) 3 (*b*) 6 (*c*) 9 (*d*) 27

(*EAMCET 2004*)

(*iii*) The solution of $\log_{99} (\log_2 (\log_3 x)) = 0$

(*a*) 4 (*b*) 9 (*c*) 44 (*d*) 99

(*BCECE 2006*)

2. If $x = \log_b a$, $y = \log_c b$, $z = \log_a c$, then xyz is

(*a*) 0 (*b*) 1 (*c*) abc (*d*) $a + b + c$

(*UPSEE 2003*)

3. (*i*) $7^{2 \log_7 5}$ is equal to

(*a*) 5 (*b*) 25 (*c*) $\log_7 25$ (*d*) $\log_7 35$

(*KCET 2007*)

Similar question

(*ii*) The real roots of the equation $7^{\log_7 (x^2 - 4x + 5)}$

(*a*) 1 and 2 (*b*) 2 and 3 (*c*) 3 and 4 (*d*) 3 and 4

(*DCE 2001*)

4. $\left(\dfrac{1}{\log_3 12}+\dfrac{1}{\log_4 12}\right)$ is

(*a*) 0 (*b*) $\dfrac{1}{2}$ (*c*) 1 (*d*) 2

(*WBJEE 2009*)

5. If *a, b, c* do not belong to the set {0, 1, 2, 3 .... 9}, then $\log_{10}\left(\dfrac{a+10b+10^2c}{10^{-4}a+10^{-3}b+10^{-2}c}\right)$ is equal to

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

(*EAMCET 2005*)

6. Assuming that the base is 10, the value of the expression log 6 + 2 log 5 + log 4 – log 3 – log 2 is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

7. $\log\dfrac{a^2}{bc}+\log\dfrac{b^2}{ac}+\log\dfrac{c^2}{ab}$ equals

(*a*) –1 (*b*) *abc* (*c*) 3 (*d*) 0

8. If $\log_r 6 = m$ and $\log_r 3 = n$, then what is $\log_r (r/2)$ equal to ?

(*a*) $m - n + 1$ (*b*) $m + n - 1$ (*c*) $1 - m - n$ (*d*) $1 - m + n$

(*CDS 2009*)

9. The value of $25^{(-1/4 \log_5 25)}$ is

(*a*) $\dfrac{1}{5}$ (*b*) $-\dfrac{1}{25}$ (*c*) –25 (*d*) None of these

10. If $\log_{10}x - \log_{10}\sqrt{x} = \dfrac{2}{\log_{10} x}$, find the value of *x*.

(*a*) 10 (*b*) –1 (*c*) $100, \dfrac{1}{100}$ (*d*) $\dfrac{1}{1000}$

(*CAT 2004*)

11. If $\log_4 2 + \log_4 4 + \log_4 x + \log_4 16 = 6$, then *x* is equal to

(*a*) 4 (*b*) 8 (*c*) 32 (*d*) 64

(*KCET 2006*)

12. If $2^x \cdot 3^{x+4} = 7^x$, then *x* is equal to

(*a*) $\dfrac{3\log_e 4}{\log_e 7 - \log_e 6}$ (*b*) $\dfrac{4\log_e 3}{\log_e 6 - \log_e 7}$

(*c*) $\dfrac{3\log_e 4}{\log_e 6 - \log_e 7}$ (*d*) $\dfrac{4\log_e 3}{\log_e 7 - \log_e 6}$

(*MPPET 2009*)

13. (*i*) If *n* = 1000 !, then the value of $\dfrac{1}{\log_2 n}+\dfrac{1}{\log_3 n}+\ldots+\dfrac{1}{\log_{1000} n}$ is

(*a*) 0 (*b*) 1 (*c*) 10 (*d*) 10^3

(*KCET 2009, Kerala PET 2006, DCE 2005*)

Similar question

(*ii*) If *x* = 1999!, then $\sum_{x=1}^{1999} \log_n x$ is equal to

(*a*) – 1 (*b*) 0 (*c*) 1 (*d*) $\sqrt[1999]{1999}$.

(*AMU 2003*)

14. If $\dfrac{\log x}{b-c}=\dfrac{\log y}{c-a}=\dfrac{\log z}{a-b}$, then the value of $x^{b+c} \cdot y^{c+a} \cdot z^{a+b}$ is

(*a*) 1 (*b*) 0 (*c*) *abc* (*d*) *xyz*

(*KCET 2011*)

15. If $\log_x 484 - \log_x 4 + \log_x 14641 - \log_x 1331 = 3$, then the value of *x* is

(*a*) 1 (*b*) 3 (*c*) 11 (*d*) None of these

(*DCE 2008*)

LEVEL–2

16. If $\log_{\sqrt{3}} 5 = a$ and $\log_{\sqrt{3}} 2 = b$, then $\log_{\sqrt{3}} 300$ is equal to

(*a*) $a + b + 1$ (*b*) $2(a + b + 1)$

(*c*) $2(a + b + 2)$ (*d*) $(a + b + 4)$ **(*Kerala 2007*)**

17. If $\log_7 2 = \lambda$, then the value of $\log_{49}(28)$ is

(*a*) $\dfrac{1}{2}(2\lambda+1)$ (*b*) $(2\lambda + 1)$

(*c*) $2(2\lambda + 1)$ (*d*) $\dfrac{3}{2}(2\lambda+1)$

(*WBJEE 2011*)

18. The value of *x* satisfying $\log_2(3x - 2) = \log_{\frac{1}{2}} x$ is

(*a*) – 1 (*b*) $-\dfrac{1}{3}$ (*c*) $\dfrac{1}{3}$ (*d*) 1

(*AMU 2011*)

19. $\log_3 2, \log_6 2, \log_{12} 2$ are in

(*a*) A.P. (*b*) G.P. (*c*) H.P. (*d*) None of these

(*Raj PET 2006, 2001*)

20. (*i*) The value of $\dfrac{\log_3 5 \times \log_{25} 27 \times \log_{49} 7}{\log_{81} 3}$ is

(*a*) 1 (*b*) $\dfrac{2}{3}$ (*c*) 3 (*d*) 6

(*WBJEE 2010*)

(*ii*) $\log_{4\sqrt[3]{4^2}}\left(\dfrac{1}{1024}\right)$ is equal to

(*a*) – 5 (*b*) – 3 (*c*) 3 (*d*) 5

(*COMEDK 2010*)

21. If $2^{\log_{10} 3\sqrt{3}} = 3^{k\log_{10} 2}$ then the value of *k* is :

(*a*) 1 (*b*) 1/2 (*c*) 2 (*d*) $\dfrac{3}{2}$

22. $\dfrac{1}{(\log_a bc)+1}+\dfrac{1}{(\log_b ac)+1}+\dfrac{1}{(\log_c ab)+1}$ is equal to :

(*a*) 1 (*b*) 2 (*c*) 0 (*d*) *abc*

23. What is the value of $(\log_{1/2}2)(\log_{1/3}3)(\log_{1/4}4)$ ..... $(\log_{1/1000}1000)$?

(*a*) 1 (*b*) –1 (*c*) 1 or –1 (*d*) 0

(*CDS 2007*)

24. The value of $\log_{10}\sqrt{10\sqrt{10\sqrt{10\sqrt{10}}}}$ .... to ∞ is

(*a*) 4 (*b*) 3 (*c*) 2 (*d*) 1

25. If $\log_a m = x$, then $\log_{1/a}\left(\frac{1}{m}\right)$ equals

(*a*) $\frac{1}{x}$ (*b*) $-x$ (*c*) $-\frac{1}{x}$ (*d*) x

26. Find the value of x if the base is 10 :

$5^{\log x} - 3^{\log x - 1} = 3^{\log x + 1} - 5^{\log x - 1}$

(*a*) 1 (*b*) 0 (*c*) 100 (*d*) 10

27. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, then $a^a b^b c^c$ equals :

(*a*) –1 (*b*) 0 (*c*) abc (*d*) 1

28. The value of $\log_{\sqrt{b}} a \log_{\sqrt[3]{c}} b \log_{\sqrt[4]{a}} c$ is :

(*a*) 1 (*b*) 10 (*c*) 24 (*d*) 0

29. Evaluate x if $\log_3(3+x) + \log_3(8-x) - \log_3(9x-8) = 2 - \log_3 9$

(*a*) 2 (*b*) –2 (*c*) 4 (*d*) –4

30. If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$, then

(*a*) $xyz = x + y + z + 2$ (*b*) $xyz = x + y + z + 1$

(*c*) $x + y + z = 1$ (*d*) $xyz = 1$

LEVEL–3

31. Given, $\log_a x = \frac{1}{\alpha}, \log_b x = \frac{1}{\beta}, \log_c x = \frac{1}{\gamma}$, then $\log_{abc} x$ equals :

(*a*) $\alpha\beta\gamma$ (*b*) $\frac{1}{\alpha\beta\gamma}$

(*c*) $\alpha + \beta + \gamma$ (*d*) $\frac{1}{\alpha+\beta+\gamma}$

32. If $y = \frac{1}{a^{1-\log_a x}}, z = \frac{1}{a^{1-\log_a y}}$ and $x = a^k$, then $k =$

(*a*) $\frac{1}{a^{1-\log_a z}}$ (*b*) $\frac{1}{1-\log_a z}$ (*c*) $\frac{1}{1+\log_z a}$ (*d*) $\frac{1}{1-\log_z a}$

33. Solve for x if $a > 0$ and $2\log_x a + \log_{ax} a + 3\log_{a^2 x} a = 0$

(*a*) $a^{3/2}$ (*b*) $a^{1/2}$ (*c*) $a^{3/4}$ (*d*) $a^{-4/3}$

34. Find the value of x, if $\log_2(5.2^x + 1)$, $\log_4(2^{1-x} + 1)$ and 1 are in A.P.

(*a*) $1 + \log_5 2$ (*b*) $1 - \log_2 5$ (*c*) $\log_2 10$ (*d*) $\log_2 5 + 1$

(*AIEEE 2002*)

35. If $\frac{1}{3}\log_3 M + 3\log_3 N = 1 + \log_{0.008} 5$, then

(*a*) $M^9 = \frac{9}{N}$ (*b*) $N^9 = \frac{9}{M}$

(*c*) $M^3 = \frac{3}{N}$ (*d*) $N^9 = \frac{3}{M}$

(*CAT 2003*)

36. If $\frac{\log x}{a^2+ab+b^2} = \frac{\log y}{b^2+bc+c^2} = \frac{\log z}{c^2+ca+a^2}$, then $x^{a-b}.\, y^{b-c}.\, z^{c-a} =$

(*a*) 0 (*b*) –1 (*c*) 1 (*d*) 2

37. If x, y, z are distinct positive numbers different from 1, such that $(\log_y x.\log_z x - \log_x x) + (\log_x y.\log_z y - \log_y y) + (\log_x z.\log_y z - \log_z z) = 0$ then xyz equals

(*a*) 100 (*b*) –1 (*c*) 10 (*d*) 1

38. If a, b, c be the p^{th}, q^{th}, r^{th} terms of a GP, then the value of $(q-r)\log a + (r-p)\log b + (p-q)\log c$ is :

(*a*) 0 (*b*) 1 (*c*) –1 (*d*) pqr

39. If 1, $\log_9(3^{1-x} + 2)$ and $\log_3(4.3^x - 1)$ are in A.P., then x is equal to

(*a*) $\log_4 3$ (*b*) $\log_3 4$ (*c*) $1 + \log_3 4$ (*d*) $\log_3(3/4)$

40. What is the sum, of 'n' terms in the series :

$\log m + \log\left(\frac{m^2}{n}\right) + \log\left(\frac{m^3}{n^2}\right) + \log\left(\frac{m^4}{n^3}\right) + ...$

(*a*) $\log\left[\frac{n^{(n-1)}}{m^{(n+1)}}\right]^{n/2}$ (*b*) $\log\left[\frac{m^m}{n^n}\right]^{n/2}$

(*c*) $\log\left[\frac{m^{(1-n)}}{n^{(1-m)}}\right]^{n/2}$ (*d*) $\log\left[\frac{m^{(1+n)}}{n^{(n-1)}}\right]^{n/2}$

(*CAT 2003*)

41. Find x, if $\log_{2x}\sqrt{x} + \log_{2\sqrt{x}} x = 0$:

(*a*) 1, $2^{-5/6}$ (*b*) 1, $2^{-6/5}$ (*c*) 4, –2 (*d*) None of these

42. The number of solutions satisfying the given equation

$x^{\left[(\log_3 x)^2 - \frac{9}{2}\log_3 x + 5\right]} = 3\sqrt{3}$ for $x \in R$ are :

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

43. Solve the following equations for x and y.

$\log_{100}|x+y| = \frac{1}{2}$, $\log_{10} y - \log_{10}|x| = \log_{100} 4$

(*a*) $\left(\frac{8}{3}, \frac{16}{3}\right)$, (–8, –16) (*b*) $\left(\frac{10}{3}, \frac{20}{3}\right)$, (+10, 20)

(*c*) $\left(-\frac{10}{3}, -\frac{20}{3}\right)$, (70, 20) (*d*) None of these

44. If $\log(a+c) + \log(a-2b+c) = 2\log(a-c)$, then a, b, c are in

(*a*) A.P. (*b*) G.P. (*c*) H.P. (*d*) None of these

45. If $5^{3x^2\log_{10}2} = 2^{(x+1/2)\log_{10}25}$, then the value of x is :

(*a*) –1 (*b*) 2 (*c*) $\frac{1}{2}$ (*d*) $-\frac{1}{3}$

46. The number $\log_2 7$ is :

(*a*) a prime number (*b*) a rational number

(*c*) an irrational number (*d*) an integer **(*DCE 2000*)**

47. If x, y, z are in G.P. and $(\log x - \log 2y)$, $(\log 2y - \log 3z)$ and $(\log 3z - \log x)$ are in A.P., then x, y, z are the lengths of the sides of a triangle which is :

(*a*) acute angled (*b*) equilateral
(*c*) right angled (*d*) obtuse angled

(Rajasthan PET 2006)

48. In a right-angled triangle, the sides are a, b and c with c as hypotenuse and $c - b \neq 1$, $c + b \neq 1$. Then the value of

$$\left[\frac{\log_{c+b} a + \log_{c-b} a}{2 \log_{c+b} a \times \log_{c-b} a}\right] \text{ is}$$

(*a*) –1 (*b*) $\frac{1}{2}$ (*c*) 1 (*d*) 2

(WBJEE 2010)

49. The value of

$$6 + \log_{\frac{3}{2}}\left[\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\ldots}}}}\right] \text{ is}$$

(*a*) $\frac{8}{3\sqrt{2}}$ (*b*) $\frac{4}{3}$ (*c*) 8 (*d*) 4

(IIT 2012)

50. The equation $x^{3/4(\log_2 x)^2 + (\log_2 x) - 5/4} = \sqrt{2}$ has

(*a*) at least one real solution
(*b*) exactly one irrational solution
(*c*) exactly three real solutions
(*d*) all of the above. ***(IIT 1989)***

ANSWERS

1. (*i*) (*c*) (*ii*) (*d*) (*iii*) (*b*) **2.** (*b*) **3.** (*i*) (*b*) (*ii*) (*b*) **4.** (*c*) **5.** (*d*) **6.** (*c*) **7.** (*d*) **8.** (*d*) **9.** (*a*)
10. (*c*) **11.** (*c*) **12.** (*d*) **13.** (*i*) (*b*) (*ii*) (*c*) **14.** (*a*) **15.** (*c*) **16.** (*b*) **17.** (*a*) **18.** (*d*) **19.** (*c*)
20. (*i*) (*c*) (*ii*) (*b*) **21.** (*d*) **22.** (*a*) **23.** (*b*) **24.** (*d*) **25.** (*d*) **26.** (*c*) **27.** (*d*) **28.** (*c*)
29. (*c*) **30.** (*a*) **31.** (*d*) **32.** (*b*) **33.** (*d*) **34.** (*b*) **35.** (*b*) **36.** (*c*) **37.** (*d*) **38.** (*a*)
39. (*d*) **40.** (*d*) **41.** (*b*) **42.** (*d*) **43.** (*b*) **44.** (*c*) **45.** (*d*) **46.** (*c*) **47.** (*d*) **48.** (*c*)
49. (*d*) **50.** (*c*)

HINTS AND SOLUTIONS

1. (*i*) $\log_\pi (\log_2 (\log_7 x)) = 0$

$\Rightarrow (\log_2 (\log_7 x) = \pi^0 = 1$ [Using $\log_a m = x \Rightarrow m = a^x$]

$\Rightarrow \log_7 x = 2^1 = 2 \Rightarrow x = 7^2$.

2. Hint. $x = \log_b a \Rightarrow x = \dfrac{\log_e a}{\log_e b}$, $y = \log_c b \Rightarrow y = \dfrac{\log_e b}{\log_e c}$

$z = \log_a c \Rightarrow z = \dfrac{\log_e c}{\log_e a}$.

3. (*i*) $7^{2\log_7 5} = 7^{\log_7 5^2} = 5^2$ $\left[\text{Using } a^{\log_a x} = x\right]$

(*ii*) $7^{\log_7 (x^2 - 4x + 5)} = x - 1$

$\Rightarrow x^2 - 4x + 5 = x - 1$. Now, solve.

4. Hint. $\dfrac{1}{\log_3 12} + \dfrac{1}{\log_4 12} = \log_{12} 3 + \log_{12} 4$

$\left[\text{Using } \log_a x = \dfrac{1}{\log_x a}\right]$

5. Hint. Given exp. = $\log_{10}\left\{10^4\left(\dfrac{a + 10b + 10^2 c}{a + 10b + 10^2 c}\right)\right\}$

6. Given exp. $= \log 6 + 2\log 5 + \log 4 - \log 3 - \log 2$

$= \log 6 + \log (5)^2 + \log 4 - \log 3 - \log 2$

$= \log 6 + \log 25 + \log 4 - (\log 3 + \log 2)$

$= \log (6 \times 25 \times 4) - \log (3 \times 2)$

$= \log\left(\dfrac{6 \times 25 \times 4}{3 \times 2}\right) = \log_{10} 100 = \log_{10} 10^2$

$= 2 \log_{10} 10 = 2 \times 1 = \mathbf{2}$.

7. $\log \dfrac{a^2}{bc} + \log \dfrac{b^2}{ca} + \log \dfrac{c^2}{ab} = \log\left(\dfrac{a^2}{bc} \times \dfrac{b^2}{ac} \times \dfrac{c^2}{ab}\right)$

$= \log\left(\dfrac{a^2 b^2 c^2}{a^2 b^2 c^2}\right) = \log 1 = \mathbf{0}$.

8. Given, $\log_r 6 = m$ and $\log_r 3 = n$

Since, $\log_r 6 = \log_r (2 \times 3) = \log_r 2 + \log_r 3$

$\Rightarrow \log_r 2 + \log_r 3 = m$

$\Rightarrow \log_r 2 + n = m \Rightarrow \log_r 2 = m - n$

Now, $\log_r (r/2) = \log_r r - \log_r 2 = 1 - (m - n) = \mathbf{1 - m + n}$.

9. $25^{\left[\left(-\frac{1}{4}\log_5 25\right)\right]} = 5^{[2(-1/4)\log_5 25)]}$

$= 5^{(-1/2 \log_5 25)} = 5^{\log_5 (25)^{-1/2}} = 25^{-1/2} = \dfrac{1}{5}$

$(\because\ a^{\log_a x} = x)$

10. Given, $\log_{10} x - \log_{10} \sqrt{x} = \dfrac{2}{\log_{10} x}$

$\Rightarrow \log_{10}\left(\dfrac{x}{\sqrt{x}}\right) = \dfrac{2}{\log_{10} x}$

$\Rightarrow \log_{10} \sqrt{x} = \dfrac{2}{\log_{10} x} \Rightarrow \dfrac{1}{2}\log_{10} x = \dfrac{2}{\log_{10} x}$

$\Rightarrow (\log_{10} x)^2 = 4 \Rightarrow \log_{10} x = \pm 2$

If $\log_{10} x = +2$ then $x = 10^2 = \mathbf{100}$

If $\log_{10} x = -2$ then $x = 10^{-2} = \mathbf{1/100}$.

11. Hint. Given, $\log_4 (2 \times 4 \times x \times 16) = 6 \Rightarrow \log_4 (128x) = 6$

$\Rightarrow 128x = 4^6$ [Using $\log_a x = n \Rightarrow x = a^n$]

12. Hint. $2^x \cdot 3^{x+4} = 7^x \Rightarrow \log_e (2^x \cdot 3^{x+4}) = \log_e 7^x$

Taking log to the same base on both sides

$\Rightarrow x \log 2 + (x+4) \log 3 = x \log 7$

$\Rightarrow x (\log 7 - \log 2 - \log 3) = 4 \log 3$

13. (*i*) Given, $1000! = n$. Now, $\dfrac{1}{\log_2 n} + \dfrac{1}{\log_3 n} + ... + \dfrac{1}{\log_{1000} n}$

$= \log_n 2 + \log_n 3 + ... + \log_n 1000 \quad \left[\text{Using } \dfrac{1}{\log_b a} = \log_a b\right]$

$= \log_n (2 \times 3 \times 4 \times ... \times 1000) = \log_n (1000!) = \log_n n = 1.$

14. Hint. $\dfrac{\log x}{b-c} = \dfrac{\log y}{c-a} = \dfrac{\log z}{a-b} = k(\text{suppose})$

$\Rightarrow \log_e x = k(b-c) \Rightarrow x = e^{k(b-c)}$

$\log_e y = k(c-a) \Rightarrow y = e^{k(c-a)}$

$\log_e z = k(a-b) \Rightarrow z = e^{k(a-b)}$

$\therefore\ x^{b+c} \cdot y^{c+a} \cdot z^{a+b}$

$= e^{k(b-c)(b+c)} \cdot e^{k(c-a)(c+a)} \cdot e^{k(a-b)(a+b)}$

Now, complete.

15. Hint. $\log_x 484 - \log_x 4 + \log_x 14641 - \log_x 1331 = 3$

$\Rightarrow \log_x (2^2 \times 11^2) - \log_x (2^2) + \log_x (11^4) - \log_x (11^3) = 3$

$\Rightarrow 2 \log_x 2 + 2 \log_x 11 - 2 \log_x 2 + 4 \log_x 11 - 3 \log_x 11 = 3$

$\Rightarrow 3 \log_x 11 = 3 \Rightarrow \log_x 11 = 1 \Rightarrow x^1 = 11 \Rightarrow x = 11.$

16. Hint. $\log_{\sqrt{3}} 300 = \log_{\sqrt{3}} \left[(\sqrt{3})^2 .10^2\right]$

$= 2 \log_{\sqrt{3}} \sqrt{3} + 2 \log_{\sqrt{3}} 10 = 2 + 2 \log_{\sqrt{3}} (2 \times 5)$

17. $\log_{49}(28) = \log_{7^2}(7 \times 2^2) = \log_{7^2} 7 + \log_{7^2} 2^2$

$= \dfrac{1}{2} \log_7 7 + \dfrac{2}{2} \log_7 2$

$\left[\text{Using } \log_{a^n} x = \dfrac{1}{n} \log_a x, \log_{a^n}(x^m) = \dfrac{m}{n} \log_a x\right]$

$= \dfrac{1}{2} + \lambda = \dfrac{1}{2}(2\lambda + 1).$

18. Given, $\log_2(3x-2) = \log_{\frac{1}{2}} x \Rightarrow \log_2(3x-2) = \log_{2^{-1}} x$

$\Rightarrow \log_2(3x-2) = \dfrac{1}{-1} \log_2 x \quad \left[\text{Using } \log_{a^n} x = \dfrac{1}{n} \log_a x\right]$

$\Rightarrow \log_2 (3x-2) = (-1) \log_2 x = \log_2 x^{-1}$

[Using $n \log_a x = \log_a x^n$]

$\Rightarrow (3x-2) = x^{-1} = \dfrac{1}{x} \Rightarrow 3x^2 - 2x - 1 = 0$

$\Rightarrow (3x+1)(x-1) = 0 \Rightarrow x = -\dfrac{1}{3}$ or 1

$\Rightarrow x = 1$, since $\log_2 (3x-2)$ is not defined when $x = -\dfrac{1}{3}$.

19. Since $\log_2 3 + \log_2 12 = \log_2 (3 \times 12) = \log_2 36 = \log_2 6^2$ $= 2 \log_2 6$, therefore,

$\log_2 3$, $\log_2 6$ and $\log_2 12$ in A.P.

$\Rightarrow \dfrac{1}{\log_2 3}, \dfrac{1}{\log_2 6}, \dfrac{1}{\log_2 12}$ and in H.P.

$\Rightarrow \log_3 2, \log_6 2, \log_{12} 2$ and in H.P.

$\left[\text{Using } \log_a x = \dfrac{1}{\log_x a}\right]$

20. (*i*) $\dfrac{\log_3 5 \times \log_{25} 27 \times \log_{49} 7}{\log_{81} 3} = \dfrac{\log_3 5 \times \log_{5^2} 3^3 \times \log_{7^2} 7}{\log_{3^4} 3}$

$= \dfrac{\log_3 5 \times \frac{3}{2} \log_5 3 \times \frac{1}{2} \log_7 7}{\frac{1}{4} \log_3 3}$

$\left[\text{Using } \log_{a^n} x^m = \dfrac{m}{n} \log_a x, \log_{a^n} x = \dfrac{1}{n} \log_a x\right]$

$= 3 (\log_3 5 \times \log_5 3) = 3 \times 1 = 3$ [Using $\log_b a \times \log_a b = 1$]

(*ii*) **Hint.** Since $4.\sqrt[3]{4^2} = 2^2.(2^4)^{\frac{1}{3}} = 2^{2+\frac{4}{3}} = 2^{\frac{10}{3}}$ and $1024 = 2^{10}$, therefore,

$\log_{4.\sqrt[3]{4^2}} \left(\dfrac{1}{1024}\right) = \log_{2^{10/3}} (2^{-10}) = \dfrac{-10}{\frac{10}{3}} \log_2 2$

$= -3 \times 1 = -3. \quad \left[\text{Using } \log_{a^n} x^m = \dfrac{m}{n} \log_a x\right].$

21. $2^{\log_{10} 3\sqrt{3}} = 3^{k \log_{10} 2} \Rightarrow 2^{\log_{10}(3^{3/2})} = 3^{k \log_{10} 2}$

$\Rightarrow 2^{\log_2 (3^{3/2}) \cdot \log_{10} 2} = 3^{k \log_{10} 2}$ [Using $\log_a x = \log_b x \cdot \log_a b$]

$\Rightarrow \left[2^{\log_2 (3^{3/2})}\right]^{\log_{10} 2} = (3^k)^{\log_{10} 2} \Rightarrow 2^{\log_2 3^{3/2}} = 3^k$

$\Rightarrow 3^{3/2} = 3^k \Rightarrow k = \mathbf{\dfrac{3}{2}} \quad (\because a^{\log_a x} = x)$

22. $\dfrac{1}{\log_a bc + 1} + \dfrac{1}{\log_b ac + 1} + \dfrac{1}{\log_c ab + 1}$

$= \dfrac{1}{\log_a bc + \log_a a} + \dfrac{1}{\log_b ac + \log_b b} + \dfrac{1}{\log_c ab + \log_c c}$

$= \dfrac{1}{\log_a (abc)} + \dfrac{1}{\log_b (abc)} + \dfrac{1}{\log_c (abc)}$

$= \log_{abc}(a) + \log_{abc}(b) + \log_{abc}(c) = \log_{abc} abc = 1$

$\left(\text{Using } \log_a b = \dfrac{1}{\log_b a}\right)$

23. $(\log_{1/2} 2)(\log_{1/3} 3)(\log_{1/4} 4)\ ....\ \left(\log_{\frac{1}{1000}} 1000\right)$

$= \left(\dfrac{\log 2}{\log \frac{1}{2}}\right)\left(\dfrac{\log 3}{\log \frac{1}{3}}\right)\left(\dfrac{\log 4}{\log \frac{1}{4}}\right) .... \left(\dfrac{\log 1000}{\log \frac{1}{1000}}\right)$

$= \left(\dfrac{\log 2}{-\log 2}\right)\left(\dfrac{\log 3}{-\log 3}\right)\left(\dfrac{\log 4}{-\log 4}\right) .... \left(\dfrac{\log 1000}{-\log 1000}\right)$

($\because \log \dfrac{1}{2} = \log 1 - \log 2 = 0 - \log 2 = -\log 2$ and similarly for others)

$= (-1) \times (-1) \times (-1) \times\ ....\ \times (-1) = \mathbf{-1}$

($\because$ Number of terms is odd)

24. Given exp. $= \log_{10}(10^{1/2}\ 10^{1/4}\ 10^{1/8}$ .... to $\infty)$

$$= \log_{10} 10^{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{.......... to } \infty\right)}$$

$$= \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{........ to } \infty\right). \log_{10} 10$$

[Using $\log_a x^n = n \log_a x$]

$$= \frac{1/2}{(1-1/2)} \times 1 = 1$$

$$\left(\text{Using sum of GP of infinite terms} = \frac{a}{1-r}\right)$$

25. $\log_a m = x \Rightarrow a^x = m$

$\log_{1/a} 1/m = y \Rightarrow (1/a)^y = 1/m \Rightarrow m = a^y \Rightarrow a^y = a^x \Rightarrow y = x$

26. $5^{\log x} - 3^{\log x - 1} = 3^{\log x + 1} - 5^{\log x - 1}$

$\Rightarrow 5^{\log x} - 3^{\log x} \times 3^{-1} = 3^{\log x}. 3 - 5^{\log x}. 5^{-1}$

$\Rightarrow 5^{\log x} - \frac{1}{3} \times 3^{\log x} = 3 \times 3^{\log x} - \frac{1}{5} \times 5^{\log x}$

$\Rightarrow \left(3+\frac{1}{3}\right)3^{\log x} = \left(1+\frac{1}{5}\right)5^{\log x} \Rightarrow \frac{10}{3} \times 3^{\log x} = \frac{6}{5} \times 5^{\log x}$

$\Rightarrow \frac{3^{\log x}}{5^{\log x}} = \frac{6}{5} \times \frac{3}{10} = \frac{9}{25} \Rightarrow \left(\frac{3}{5}\right)^{\log x} = \left(\frac{3}{5}\right)^2$

$\Rightarrow \log_{10} x = 2 \Rightarrow x = 10^2 = \mathbf{100}$

27. Let $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b} = k$

$\Rightarrow \log a = k(b-c), \log b = k(c-a), \log c = k(a-b)$

Now let $a^a b^b c^c = p$. Then,

$\log p = \log a^a + \log b^b + \log c^c = a \log a + b \log b + c \log c$

$= a \times k(b-c) + b \times k(c-a) + c \times k(a-b)$

$= k(ab - ac + bc - ba + ca - cb) = 0$

$\Rightarrow \log p = \log 1$ (Putting log 1 for 0)

$\Rightarrow p = 1 \Rightarrow a^a b^b c^c = \mathbf{1.}$

28. Using the formula $\log_{a^n} x^m = \frac{m}{n} \log_a x$, we have

$\log_{\sqrt{b}} a \log_{\sqrt[3]{c}} b \log_{\sqrt[4]{a}} c = \log_{b^{1/2}} a \log_{c^{1/3}} b \log_{a^{1/4}} c$

$= 2 \log_b a \times 3 \log_c b \times 4 \log_a c$

$= 24 \frac{\log a}{\log b} \times \frac{\log b}{\log c} \times \frac{\log c}{\log a} = \mathbf{24.}$

29. $\log_3 (3+x) + \log_3 (8-x) - \log_3 (9x-8) = 2 - \log_3 9$

$\Rightarrow \log_3 (3+x) + \log_3 (8-x) - \log_3 (9x-8) + \log_3 9 = 2$

$\Rightarrow \log_3 \left[\frac{(3+x)(8-x)(9)}{(9x-8)}\right] = 2$

$\Rightarrow \frac{9(24+8x-3x-x^2)}{(9x-8)} = 3^2 = 9$

$\Rightarrow -x^2 + 5x + 24 = 9x - 8 \Rightarrow x^2 + 4x - 32 = 0$

$\Rightarrow (x+8)(x-4) = 0 \Rightarrow x = -8, 4.$

Taking the positive value $x = \mathbf{4.}$

30. $x = \log_a bc \Rightarrow a^x = bc \Rightarrow a^{x+1} = abc \Rightarrow a = (abc)^{1/(x+1)}$

Similarly, $b = (abc)^{1/(y+1)}$, $c = (abc)^{1/(z+1)}$

$\therefore abc = (abc)^{\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}}$

$\Rightarrow \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1 \Rightarrow (y+1)(z+1) + (x+1)$

$(z+1) + (x+1)(y+1) = (x+1)(y+1)(z+1)$

$\Rightarrow yz + y + z + 1 + xz + x + z + 1 + xy + y + x + 1 = xyz + xy + yz + zx + x + y + z + 1$

$\Rightarrow \boldsymbol{x + y + z + 2 = xyz.}$

31. $\alpha = \frac{1}{\log_a x}, \beta = \frac{1}{\log_b x}, \gamma = \frac{1}{\log_c x}$

$\Rightarrow \alpha = \log_x a, \beta = \log_x b, \gamma = \log_x c$

$\Rightarrow \alpha + \beta + \gamma = \log_x a + \log_x b + \log_x c = \log_x (abc)$

$\Rightarrow \frac{1}{\alpha+\beta+\gamma} = \frac{1}{\log_x (abc)} = \boldsymbol{\log_{abc} x}.$

32. $y = \frac{1}{a^{1-\log_a x}} = a^{-(1-\log_a x)}$

$\Rightarrow \log_a y = \frac{1}{1-\log_a x}$ and $\log_a z = \frac{1}{1-\log_a y}$

$\therefore \log_a z = \frac{1}{1-\left(\frac{1}{1-\log_a x}\right)} = \frac{1-\log_a x}{-\log_a x}$

$\Rightarrow -\log_a z = -1 + \frac{1}{\log_a x} \Rightarrow \frac{1}{\log_a x} = 1 - \log_a z$

$\Rightarrow \log_a x = \frac{1}{1-\log_a z}$

$\Rightarrow x = a^{\frac{1}{1-\log_a z}} = a^k \Rightarrow k = \boldsymbol{\frac{1}{1-\log_a z}}.$

33. Since $\log_{ax} a = \frac{1}{\log_a ax} = \frac{1}{\log_a a + \log_a x} = \frac{1}{1+\log_a x}$ and

$\log_{a^2 x} a = \frac{1}{\log_a a^2 x} = \frac{1}{\log_a a^2 + \log_{ax} x} = \frac{1}{2\log_a a + \log_{ax} x}$

$= \frac{1}{2+\log_a x}$

Given, $2 \log_x a + \log_{ax} a + 3 \log_{a^2 x} a = 0$

$\Rightarrow \frac{2}{\log_a x} + \frac{1}{1+\log_a x} + \frac{3}{2+\log_a x} = 0$

Now let $\log_a x = t$, then $\frac{2}{t} + \frac{1}{1+t} + \frac{3}{2+t} = 0$

$\Rightarrow 2(1+t)(2+t) + t(2+t) + 3t(1+t) = 0$

$\Rightarrow 2(2 + 2t + t + t^2) + 2t + t^2 + 3t + 3t^2 = 0$

$\Rightarrow 4 + 6t + 2t^2 + 2t + t^2 + 3t + 3t^2 = 0$

$\Rightarrow 6t^2 + 11t + 4 = 0$

$\Rightarrow (3t+4)(2t+1) = 0 \Rightarrow t = -1/2, -4/3$

When $t = -1/2$, $\log_a x = -\frac{1}{2} \Rightarrow x = a^{-1/2}$

When $t = -\frac{4}{3}$, $\log_a x = -\frac{4}{3} \Rightarrow \boldsymbol{x = a^{-4/3}}$

34. Given, $\log_2(5.2^x+1)$, $\log_4(2^{1-x}+1)$, 1 are in A.P.

$\Rightarrow \log_2(5.2^x+1)+1 = 2\log_4(2^{1-x}+1)$

$\Rightarrow \log_2(5.2^x+1)+\log_2 2 = 2\log_{2^2}(2^{1-x}+1)$

$\Rightarrow \log_2(5.2^x+1).2 = 2\times\frac{1}{2}\log_2(2^{1-x}+1)$

$\left(\because \log_{a^n} x = \frac{1}{n}\log_a x\right)$

$\Rightarrow \log_2(10.2^x+2) = \log_2(2^{1-x}+1)$

$\Rightarrow 10.2^x+2 = 2^{1-x}+1 \Rightarrow 10.2^x+2 = \frac{2}{2^x}+1$

Let $2^x = a$, then

$10.\,a+2 = \frac{2}{a}+1 \Rightarrow 10a+1 = \frac{2}{a} \Rightarrow 10a^2+a-2=0$

$\Rightarrow (5a-2)(2a+1)=0 \Rightarrow a=\frac{2}{5} \Rightarrow 2^x=\frac{2}{5}$

$\left(\because 2^x>0,\text{ reject } a=-\frac{1}{2}\right)$

$\Rightarrow \log 2^x = \log\frac{2}{5}$

$\Rightarrow x\log_2 2 = \log_2 2-\log_2 5 \Rightarrow \boldsymbol{x = 1-\log_2 5}.$

35. $\frac{1}{3}\log_3 M + 3\log_3 N = 1+\log_{0.008} 5$

$\Rightarrow \log_3 M^{1/3}+\log_3 N^3 = 1+\log_{0.008}5$

$\Rightarrow \log_3 M^{1/3}N^3 = 1+\log_{0.008}5$

$\Rightarrow M^{1/3}N^3 = 3^{(1+\log_{0.008}5)}$

$\Rightarrow M^{1/3}N^3 = 3^1\,.\,3^{\log_{0.008}5}$

$\Rightarrow N^9 = \frac{27}{M}(3^{3\log_{0.008}5})$

$\Rightarrow N^9 = \frac{27}{M}\left(3^{\log_{(0.2)^3}(5^3)}\right)$

$\Rightarrow N^9 = \frac{27}{M}(3^{\log_{0.2}5})$ $\left[\because \log_{a^n} x^m = \frac{m}{n}\log_a x\right]$

$\Rightarrow N^9 = \frac{27}{M}(3^{\log_{1/5}5}) = \frac{1}{M}(27)(3^{-1}) = \boldsymbol{\frac{9}{M}}.$

36. Let each ratio $= k$ and base $= e$

$\Rightarrow \log_e x = k(a^2+ab+b^2)$

$\Rightarrow (a-b)\log_e x = k(a-b)(a^2+ab+b^2)$

$\Rightarrow \log_e x^{a-b} = k(a^3-b^3) \Rightarrow x^{a-b} = e^{k(a^3-b^3)}$

Similarly, $y^{b-c} = e^{k(b^3-c^3)}$, $z^{c-a} = e^{k(c^3-a^3)}$

$\therefore x^{a-b}\,.\,y^{b-c}\,.\,z^{c-a} = e^{k(a^3-b^3)}.e^{k(b^3-c^3)}.e^{k(c^3-a^3)}$

$= e^{k[a^3-b^3+b^3-c^3+c^3-a^3]} = e^0 = \mathbf{1}.$

37. $\log_y x\,.\,\log_z x - \log_x x = \frac{\log x}{\log y}.\frac{\log x}{\log z}-1 = \frac{(\log x)^2}{\log y.\log z}-1$

Similarly, $\log_x y\,.\,\log_z y - \log_y y = \frac{(\log y)^2}{\log x.\log z}-1$ and

$\log_x z.\,\log_y z - \log_z z = \frac{(\log z)^2}{\log x.\log y}-1$

$\therefore \text{LHS} = \frac{(\log x)^2}{\log y.\log z}-1+\frac{(\log y)^2}{\log z.\log x}-1+\frac{(\log z)^2}{\log x.\log y}-1$

$= \frac{(\log x)^3+(\log y)^3+(\log z)^3-3\log x.\log y.\log z}{\log x.\log y.\log z} = 0$

(given)

$\Rightarrow (\log x)^3+(\log y)^3+(\log z)^3-3\log x.\log y.\log z = 0$

$\Rightarrow \log x+\log y+\log z = 0$

(if $a+b+c=0$, then $a^3+b^3+c^3=3abc$)

$\Rightarrow \log xyz = 0 \Rightarrow xyz = \mathbf{1}.$

38. Let h be the first term and k be the common ratio of a GP, then

$a = hk^{p-1}$, $\quad b = hk^{q-1}$, $\quad c = hk^{r-1}$

$\therefore (q-r)\log a+(r-p)\log b+(p-q)\log c$

$= \log[hk^{p-1}]^{q-r}+\log[hk^{q-1}]^{r-p}+\log[hk^{r-1}]^{p-q}$

$= \log(h^{q-r+r-p+p-q})(k^{p-1})^{q-r}(k^{q-1})^{r-p}(k^{r-1})^{p-q}$

$= \log(h^o\,k^o) = \log 1 = \mathbf{0}.$

39. 1, $\log_9(3^{1-x}+2)$, $\log_3(4.3^x-1)$ are in A.P.

$\Rightarrow \log_3 3$, $\log_3(3^{1-x}+2)^{1/2}$, $\log_3(4.3^x-1)$ are in A.P.

$\Rightarrow 3$, $(3^{1-x}+2)^{1/2}$, (4.3^x-1) are in G.P.

$\left[\text{Since } \log_9(3^{1-x}+2) = \log_{3^2}(3^{1-x}+2) = \frac{1}{2}\log_3(3^{1-x}+2),\right.$

$\left.\text{using } \log_{a^n} x = \frac{1}{n}\log_a x = \log_3(3^{1-x}+2)^{\frac{1}{2}}\right]$

$\Rightarrow [[3^{1-x}+2]^{1/2}]^2 = 3.(4.3^x-1)$

$\Rightarrow 3^{1-x}+2 = 4.3^{x+1}-3$

$\Rightarrow 4.3^{x+1}-3^{1-x} = 5 \Rightarrow 12.3^x-\frac{3}{3^x} = 5$

Let $3^x = y$, then $12y-\frac{3}{y} = 5 \Rightarrow 12y^2-5y-3=0$

$\Rightarrow (3y+1)(4y-3) = 0 \Rightarrow y = -\frac{1}{3},\frac{3}{4}$

$\therefore$ Rejecting the negative value, we have $3^x = \frac{3}{4}$

$\Rightarrow x = \boldsymbol{\log_3\frac{3}{4}}.$

40. $S = \log m+\log\frac{m^2}{n}+\log\frac{m^3}{n^2}+\,.........\,n$ terms

$= \log\left[m\cdot\frac{m^2}{n}\cdot\frac{m^3}{n^2}.........\frac{m^n}{n^{n-1}}\right] = \log\left[\frac{m^{(1+2+3+4+........+n)}}{n^{(1+2+3+.......(n-1)}}\right]$

$= \log\left[\frac{m^{\frac{n(n+1)}{2}}}{n^{\frac{n(n-1)}{2}}}\right] = \boldsymbol{\log\left[\frac{m^{n+1}}{n^{n-1}}\right]^{n/2}}.$

41. $\log_{2x}\sqrt{x}+\log_{2\sqrt{x}}x = 0$ ...(*i*)

Let $\log_2 x = t$. Then,

$\log_{2x}\sqrt{x} = \frac{\log_2\sqrt{x}}{\log_2 2x} = \frac{\frac{1}{2}\log_2 x}{\log_2 2+\log_2 x} = \frac{t/2}{1+t}$

$$\log_{2\sqrt{x}} x = \frac{\log_2 x}{\log_2 2\sqrt{x}} = \frac{\log_2 x}{\log_2 2 + \frac{1}{2}\log_2 x} = \frac{t}{1+t/2}$$

$\therefore$ Substituting in (i), we get

$$\frac{t/2}{1+t} + \frac{t}{1+t/2} = 0 \Rightarrow \frac{t}{2+2t} + \frac{2t}{2+t} = 0$$

$\Rightarrow t(2+t) + 2t(2+2t) = 0 \Rightarrow 2t + t^2 + 4t + 4t^2 = 0$

$\Rightarrow 5t^2 + 6t = 0 \Rightarrow t(5t+6) = 0 \Rightarrow t = 0$ or $-\dfrac{6}{5}$

$\Rightarrow \log_2 x = 0 \Rightarrow x = 2^0 = 1$ and $\log_2 x = -\dfrac{6}{5} \Rightarrow x = 2^{-6/5}$

$\therefore x =$ **1 or $2^{-6/5}$**

42. Taking log of both the sides to base 3, we have,

$$\left[(\log_3 x)^2 - \frac{9}{2}\log_3 x + 5\right]\log_3 x = \log_3 3^{3/2} = \frac{3}{2} \quad (\because \log_3 3 = 1)$$

$\Rightarrow 2(\log_3 x)^3 - 9(\log_3 x)^2 + 10\log_3 x - 3 = 0$

$\Rightarrow 2y^3 - 9y^2 + 10y - 3 = 0$ (Take $\log_3 x = y$)

$\Rightarrow (y-1)(y-3)(2y-1) = 0$ (Factorising)

$\Rightarrow (\log_3 x - 1)(\log_3 x - 3)(2\log_3 x - 1) = 0$

$\Rightarrow \log_3 x = 1, \log_3 x = 3, 2\log_3 x = 1 \Rightarrow x = 3^1, x = 3^3, x^2 = 3^1$

$\Rightarrow x = (3, 27, \sqrt{3})$

$\therefore$ There are three solutions.

43. $\log_{100}|x+y| = \dfrac{1}{2} \Rightarrow |x+y| = 100^{\frac{1}{2}}$

$\Rightarrow |x+y| = 10$ as (-10 is inadmissible) ...(i)

$\log_{10} y - \log_{10}|x| = \log_{100} 4$

$$\Rightarrow \log_{10}\frac{y}{|x|} = \log_{10^2} 2^2 = \log_{10} 2$$

$$\left[\text{Using } \log_{a^n}(x^m) = \frac{m}{n}\log_a x\right.$$

$\Rightarrow \dfrac{y}{|x|} = 2 \Rightarrow y = 2|x|$ ...(ii)

Substituting the value of y from (ii) in (i), we get

$||x + 2|x|| = 10$

If $x > 0$, then $3x = 10 \Rightarrow x = \dfrac{10}{3}$

If $x < 0$, then $x = 10$.

$\therefore$ If $x = \dfrac{10}{3}$, then $y = \dfrac{20}{3}$ and if $x = 10$, $y = 20$.

44. Given, $\log(a+c) + \log(a-2b+c) = 2\log(a-c)$

$\Rightarrow \log(a+c)(a-2b+c) = \log(a-c)^2$

$\Rightarrow (a+c)(a-2b+c) = (a-c)^2$

$\Rightarrow \cancel{a^2} + ca - 2ba - 2bc + ac + \cancel{c^2} = \cancel{a^2} - 2ac + \cancel{c^2}$

$\Rightarrow 4ac = 2ba + 2bc \Rightarrow 2ac = b(a+c)$

$\therefore b = \dfrac{2ac}{a+c} \Rightarrow a, b, c$ are in H.P.

45. $5^{3x^2 \log_{10} 2} = 2^{\left(x+\frac{1}{2}\right)\log_{10} 25}$

$\Rightarrow 5^{3x^2\log_{10} 2} = 2^{\left(\frac{2x+1}{2}\right)\times 2\log_{10} 5} = 2^{(2x+1)\log_{10} 5}$

$\Rightarrow 5^{3x^2\log_{10} 2} = 2^{(2x+1)\log_2 5.\log_{10} 2}$

(using $\log_a x = \log_b x \,.\, \log_a b$)

$\Rightarrow 5^{3x^2\log_{10} 2} = [2^{\log_2 5^{(2x+1)}}]^{\log_{10} 2}$

$\Rightarrow (5^{3x^2})^{\log_{10} 2} = (5^{2x+1})^{\log_{10} 2}$ $\left[\text{Using } a^{\log_a x} = x\right]$

$\Rightarrow 3x^2 = 2x + 1 \Rightarrow 3x^2 - 2x - 1 = 0$

$\Rightarrow (x-1)(3x+1) = 0$

$\Rightarrow x = \mathbf{1, -\dfrac{1}{3}}.$

46. Let us assume $\log_2 7$ be a rational number. Then,

$\log_2 7 = \dfrac{p}{q}$, where $p, q \in I$ and $q \neq 0$

$\Rightarrow 2^{p/q} = 7 \Rightarrow 2^p = 7^q$

This is not true as 2 is even and 7 is odd.

$\therefore$ Hence our assumption that $\log_2 7$ is a rational number is wrong.

$\therefore \log_2 7$ is an irrational number.

47. x, y, z are in G.P. $\Rightarrow y^2 = xz$ ...(i)

$(\log x - \log 2y)$, $(\log 2y - \log 3z)$ and $(\log 3z - \log x)$ are in A.P.

$\Rightarrow 2(\log 2y - \log 3z) = (\log x - \log 2y) + (\log 3z - \log x)$

$\Rightarrow 3\log 2y = 3\log 3z \Rightarrow \log 2y = \log 3z \Rightarrow \mathbf{y = \dfrac{3}{2}z.}$

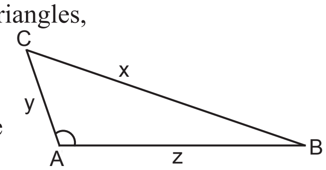
$\therefore$ Putting the value of y in (i), we have

$$\left(\frac{3}{2}z\right)^2 = xz \Rightarrow x = \frac{9}{4}z.$$

Now, by the cosine rule of triangles,

$\cos A = \dfrac{y^2 + z^2 - x^2}{2yz}$,

where x is the length of the side opposite $\angle A$.



$$\Rightarrow \cos A = \frac{\left(\frac{3}{2}z\right)^2 + z^2 - \left(\frac{9}{4}z\right)^2}{2\times\frac{3}{2}z\times z} = \frac{\frac{9}{4}z^2 + z^2 - \frac{81}{16}z^2}{3z^2}$$

$$= \frac{\frac{9}{4} + 1 - \frac{81}{16}}{3} = \frac{1}{3}\times\left[\frac{36+16-81}{16}\right] = -\frac{29}{48} < 0$$

$\because \cos A$ is less than 0, *i.e*, negative, $\angle A$ is obtused and the triangle is obtuse angled.

48. In a right angled triangle with a, b as sides and c as hypotenuse,

$c^2 = a^2 + b^2$ (Pythagoras' Theorem)

Now, given expression $= \dfrac{\log_{c+b} a + \log_{c-b} a}{2\times\log_{c+b} a\times\log_{c-b} a}$

$$= \frac{1}{2}\left[\frac{1}{\log_{c-b} a} + \frac{1}{\log_{c+b} a}\right]$$

$= \frac{1}{2}[\log_a(c-b)+\log_a(c+b)] = \frac{1}{2}[\log_a[(c-b)(c+b)]$

$= \frac{1}{2}[\log_a(c^2-b^2)] = \frac{1}{2}\log_a a^2 = \log_a a = \mathbf{1}.$

49. Let

$A = 6 + \log_{\frac{3}{2}}\left[\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}}}}}\ ....\right]$

Let $p = \sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}}}}\ ....$

$\Rightarrow p = \sqrt{4-\frac{1}{3\sqrt{2}}p} \Rightarrow p^2 = 4-\frac{1}{3\sqrt{2}}p \Rightarrow p^2+\frac{1}{3\sqrt{2}}p-4=0$

$\Rightarrow\ p = \dfrac{-\frac{1}{3\sqrt{2}} \pm \sqrt{\frac{1}{18}+16}}{2} = \dfrac{-\frac{1}{3\sqrt{2}} \pm \frac{17}{3\sqrt{2}}}{2}$

(Applying the formula for roots of Q.E.)

$\Rightarrow p = \frac{16}{3\times 2\sqrt{2}}$ or $\frac{-18}{3\times 2\sqrt{2}} = \frac{8}{3\sqrt{2}}$ or $-\frac{3}{\sqrt{2}}$

Neglecting $p = \frac{-3}{\sqrt{2}}$ as $p \geq 0$, we have $p = \frac{8}{3\sqrt{2}}$

$\therefore\ A = 6 + \log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}}\times\frac{8}{3\sqrt{2}}\right) = 6 + \log_{\frac{3}{2}}\left(\frac{4}{9}\right)$

$= 6 + \log_{\frac{3}{2}}\left(\frac{3}{2}\right)^{-2} = 6 - 2\log_{\frac{3}{2}} 3/2 = 6 - 2 = \mathbf{4}.$

50. Given, $x^{3/4(\log_2 x)^2} + \log_2 x - 5/4 = \sqrt{2}$

Taking log to the base 2 of both the sides, we have

$\left[\frac{3}{4}(\log_2 x)^2 + (\log_2 x) - 5/4\right]\log_2 x = \log_2\sqrt{2}$

$= \log_2 2^{1/2} = \frac{1}{2}\log_2 2 = \frac{1}{2}$

Let us assume $\log_2 x = a$. Then,

$\left(\frac{3}{4}a^2 + a - \frac{5}{4}\right)a = \frac{1}{2} \Rightarrow 3a^3 + 4a^2 - 5a = 2$

$\Rightarrow 3a^3 + 4a^2 - 5a - 2 = 0.$

Using hit and trial method check for $a = 1$.

$f(a) = 3a^3 + 4a^2 - 5a - 2 \Rightarrow f(1) = 3.1^3 + 4.1^2 - 5.1 - 2 = 0$

$\therefore$ $(a - 1)$ is a factor of $3a^3 + 4a^2 - 5a - 2$

$\therefore$ Now by dividing $3a^3 + 4a^2 - 5a - 2$ by $(a - 1)$, we get

$3a^3 + 4a^2 - 5a - 2 = (a - 1)(3a + 1)(a + 2) = 0$

$\Rightarrow a = 1$ or $a = -\frac{1}{3}$ or $a = -2$

$\Rightarrow \log_2 x = 1$ or $\log_2 x = -\frac{1}{3}$ or $\log_2 x = -2$

$\Rightarrow x = 2^1 = 2$ or $x = 2^{-1/3}$ or $x = 2^{-2} = \frac{1}{4}$

$\therefore$ The given equation has exactly three real solutions, wherein $x = 2^{-1/3}$ is irrational.

SELF ASSESSMENT SHEET

1. If $\log_2[\log_7(x^2 - x + 37)] = 1$, then what could be the value of x ?

(*a*) 3 (*b*) 5 (*c*) 4 (*d*) None of these

(***CAT 1997***)

2. $\log_{\sqrt{3}}\sqrt{3\sqrt{3\sqrt{3\sqrt{3}}}} =$

(*a*) $\frac{31}{32}$ (*b*) $\frac{15}{16}$ (*c*) $\frac{7}{16}$ (*d*) $\frac{15}{8}$

3. Find the sum of 'n' terms of the series :

$\log_2\left(\frac{x}{y}\right) + \log_4\left(\frac{x}{y}\right)^2 + \log_8\left(\frac{x}{y}\right)^3 + \log_{16}\left(\frac{x}{y}\right)^4 + ...$

(*a*) $\log_2\left(\frac{x}{y}\right)^{4n}$ (*b*) $n\log_2\left(\frac{x}{y}\right)$

(*c*) $\log_2\left(\frac{x^{n-1}}{y^{n-1}}\right)$ (*d*) $\frac{1}{2}\log_2\left(\frac{x}{y}\right)^{n(n+1)}$

4. If $\log_x a$, $a^{x/2}$ and $\log_b x$ are in *GP*, then x is equal to :

(*a*) $\log_a(\log_b a)$ (*b*) $\log_a(\log_e a) - \log_a(\log_e b)$

(*c*) $-\log_a(\log_a b)$ (*d*) both (*a*) and (*b*)

5. If $\frac{\log x}{l+m-2n} = \frac{\log y}{m+n-2l} = \frac{\log z}{n+l-2m}$, then xyz is equal to :

(*a*) 0 (*b*) 1 (*c*) lmn (*d*) 2

6. If $a = \log_{12} m$ and $b = \log_{18} m$, then $\frac{a-2b}{b-2a}$ equals

(*a*) $\log_3 2$ (*b*) $\log_2 3$ (*c*) 0 (*d*) 1

7. If $x = \log_{2a} a$, $y = \log_{3a} 2a$, $z = \log_{4a} 3a$, then $xyz - 2yz$ equals

(*a*) a^3 (*b*) 1 (*c*) 0 (*d*) –1

8. The sum of n terms of the series $\sum_{x=1}^{n}\log\frac{2^x}{3^{x-1}}$ is :

(*a*) $\log\left(\frac{3^{n-1}}{2^{n+1}}\right)^{n/2}$ (*b*) $\log\left(\frac{2^{n-1}}{3^{n+1}}\right)^{n/2}$

(*c*) $\log\left(\frac{3^{n+1}}{2^{n-1}}\right)^{n/2}$ (*d*) $\log\left(\frac{2^{n+1}}{3^{n-1}}\right)^{n/2}$

(***UPSEE 2011***)

9. The number of meaningful solutions of

$\log_4(x - 1) = \log_2(x - 3)$ is

(*a*) zero (*b*) 1 (*c*) 2 (*d*) 3

(***IIT 2001***)

10. The value of $\left[0.16^{\log 2.5\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+......+\infty\right)}\right]^{1/2}$ is:

(*a*) –1 (*b*) 0 (*c*) 1 (*d*) None of these

(***AMU 2009***)

SELF ASSESSMENT SHEET

1. (*c*) **2.** (*d*) **3.** (*b*) **4.** (*b*) **5.** (*b*) **6.** (*a*) **7.** (*d*) **8.** (*d*) **9.** (*b*) **10.** (*d*)

HINTS AND SOLUTIONS

1. $\log_2 [\log_7(x^2 - x + 37)] = 1$

$\Rightarrow \log_7(x^2 - x + 37) = 2^1 = 2$

$\Rightarrow x^2 - x + 37 = 7^2 = 49 \Rightarrow x^2 - x - 12 = 0$

Now solve for x.

2. Given expression $= \log_{\sqrt{3}} 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}}$

$= \log_{3^{1/2}} 3^{15/16} = \frac{15}{16} \times 2 \log_3 3 = \mathbf{\frac{15}{8}}.$

3. Given series

$= \log_2 \left(\frac{x}{y}\right) + \log_{2^2}\left(\frac{x}{y}\right)^2 + \log_{2^3}\left(\frac{x}{y}\right)^3 + \log_{2^4}\left(\frac{x}{y}\right)^4 + ...$

$= \log_2\left(\frac{x}{y}\right) + \log_2\left(\frac{x}{y}\right) + \log_2\left(\frac{x}{y}\right) + \log_2\left(\frac{x}{y}\right) + ...n \text{ terms}$

$= \log_2\left(\frac{x}{y}.\frac{x}{y}.\frac{x}{y}.\frac{x}{y}. ...... \ n \text{ terms}\right)$

$= \log_2\left(\frac{x}{y}\right)^n = \boldsymbol{n \log_2 \left(\frac{x}{y}\right)}.$

4. $\log_x a$, $a^{x/2}$, $\log_b x$ are in GP $\Rightarrow [a^{x/2}]^2 = \log_x a \,.\, \log_b x$

$\Rightarrow a^x = \frac{\log a}{\log x} . \frac{\log x}{\log b}$

$\Rightarrow a^x = \frac{\log a}{\log b} = \log_b a$

$\Rightarrow x = \log_a (\log_b a)$

$= \log a\left(\frac{\log_e a}{\log_e b}\right) = \mathbf{\log a \ (\log_e a) - \log a \ (\log_e b)}$

5. Let $\frac{\log x}{l + m - 2n} = \frac{\log y}{m + n - 2l} = \frac{\log z}{n + l - 2m} = k.$ Then

$\log x = k(l + m - 2n), \log y = k(m + n - 2l); \log z = k(n + l - 2m)$

$\Rightarrow \log x + \log y + \log z = k(l + m - 2n) + k(m + n - 2l) + k(n + l - 2m)$

$\Rightarrow \log(xyz) = 0 \Rightarrow \log(xyz) = \log 1 \Rightarrow xyz = \mathbf{1}.$

6. $\frac{a - 2b}{b - 2a} = \frac{\log_{12} m - 2\log_{18} m}{\log_{18} m - 2\log_{12} m}$

$= \frac{\frac{\log m}{\log 12} - 2\frac{\log m}{\log 18}}{\frac{\log m}{\log 18} - 2\frac{\log m}{\log 12}} = \frac{\log m \log 18 - 2\log m \log 12}{\log m \log 12 - 2 \log m \log 18}$

$= \frac{\log 18 - 2\log 12}{\log 12 - 2\log 18} = \frac{\log (3^2 \times 2) - 2\log(2^2 \times 3)}{\log(2^2 \times 3) - 2\log (3^2 \times 2)}$

$= \frac{2\log 3 + \log 2 - 4\log 2 - 2\log 3}{2\log 2 + \log 3 - 4\log 3 - 2\log 2} = \frac{-3\log 2}{-3\log 3} = \frac{\log 2}{\log 3} = \mathbf{\log_3 2.}$

7. $x = \log_{2a} a = \frac{\log a}{\log 2a}, \quad y = \log_{3a} 2a = \frac{\log 2a}{\log 3a}$

$z = \log_{4a} 3a = \frac{\log 3a}{\log 4a}$

$\therefore xyz - 2yz = \frac{\log a}{\log 2a}.\frac{\log 2a}{\log 3a}.\frac{\log 3a}{\log 4a} - 2\frac{\log 2a}{\log 3a}.\frac{\log 3a}{\log 4a}$

$= \frac{\log a}{\log 4a} - 2\frac{\log 2a}{\log 4a} = \frac{\log a - 2\log 2a}{\log 4a}$

$= \frac{\log a - \log(2a)^2}{\log 4a} = \frac{\log a/4a^2}{\log 4a} = \frac{\log(4a)^{-1}}{\log(4a)} = \frac{-1.\log 4a}{\log 4a} = \mathbf{-1.}$

8. $\sum_{x=1}^{n} \log\frac{2^x}{3^{x-1}} = \log\left(\frac{2^1}{3^0}\right) + \log\left(\frac{2^2}{3^1}\right) + \log\left(\frac{2^3}{3^2}\right) + ... + \log\left(\frac{2^n}{3^{n-1}}\right)$

$= \log\left(\frac{2^1}{3^0}.\frac{2^2}{3^1}.\frac{2^3}{3^2}. .... \frac{2^n}{3^{n-1}}\right)$

$= \log\left(\frac{2^{1+2+3+.......+n}}{3^{1+2+3+......+(n-1)}}\right) = \log\left[\frac{2^{\frac{n(n+1)}{2}}}{3^{\frac{n(n-1)}{2}}}\right] = \mathbf{\log\left[\frac{2^{n+1}}{3^{n-1}}\right]^{n/2}}$

9. $\log_4(x - 1) = \log_2(x - 3) \Rightarrow \log_{2^2}(x - 1) = \log_2(x - 3)$

$\Rightarrow \frac{1}{2}\log_2(x-1) = \log_2(x-3) \Rightarrow \log_2(x-1) = 2\log_2(x-3)$

$\left[\text{Using } \log_{a^m}(b^n) = \frac{n}{m}\log_a b\right]$

$\Rightarrow \log_2(x - 1) = \log_2(x - 3)^2$

$\Rightarrow (x - 1) = (x - 3)^2 \Rightarrow x - 1 = x^2 - 6x + 9$

$\Rightarrow x^2 - 7x + 10 = 0 \Rightarrow (x - 2)(x - 5) = 0 \Rightarrow x = 2 \text{ or } 5$

Neglecting $x = 2$ as $\log_2(x - 3)$ is defined when $x > 2$.

$\Rightarrow$ There is only one meaningful solution of the given equation.

10. $\left[(0.16)^{\log_{2.5}\left[\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+.......\infty\right]}\right]^{1/2} = \left[(0.4)^{2\log_{2.5}\left[\frac{1/3}{1-1/3}\right]}\right]^{1/2}$

$\left[\because \text{Sum of infinite GP} = \frac{\text{First term}}{1 - \text{common ratio}}\right]$

$= \left[(0.4)^{\log_{5/2}\left(\frac{1}{2}\right)}\right] = \left[(0.4)^{\log_{5/2}\left(\frac{1}{2}\right)}\right] = \left[0.4^{\log_{(2/5)^{-1}}(2)^{-1}}\right]$

$\left[\because \log_{a^m} b^n = \frac{n}{m}\log_a b\right]$

$= 0.4^{\log_{2/5} 2} = 0.4^{\log_{0.4} 2} = \mathbf{2.} \qquad \left[\text{Using } a^{\log_a x} = x\right]$

2 Polynomials

KEY FACTS

1. A function $f(x)$ of the form $f(x) = a_0 + a_1x + a_2x^2 + \text{.........} + a_nx^n$
where $a_0, a_1, a_2, \text{.........}, a_n$ are real numbers, $a_n \neq 0$ and n is a non negative integer is called a polynomial in x.
The real numbers $a_0, a_1, a_2, \text{..........}, a_n$ are called coefficients of the polynomial.

Ex. (*a*) $6x^2 - 8x + 5$ is a polynomial with integral coefficients.

(*b*) $\frac{9}{5}x^3 + \frac{4}{7}x^2 - 8$ is a polynomial with rational coefficients.

(*c*) $6x^4 - \sqrt{3}x^2 + 3\sqrt{5}$ is a polynomial with real coefficients.

2. Types of Polynomials

- **Monomial :** A polynomial having only one term as $9, \sqrt{2}x, \frac{1}{4}x^2$, etc.
- **Binomial :** A polynomial having only two terms as $4x - 5$, $6x^2 + 8x$, etc.
- **Trinomial :** A polynomial having only three terms as $4x^2 - \sqrt{2}x + \frac{1}{3}$

3. Degree of a Polynomial :

- The degree of a polynomial in one variable is the highest exponent of the variable in that polynomial.
Degree of $9x^7 - 6x^5 + 4x^3 + 8$ is **7.**
- The degree of a polynomial in more than one variable is the highest sum of the powers of the variables.
Degree of $4x^5 - 6x^2y^4 + 8 - 3xy^6$ is **1 + 6 = 7.**
- A polynomial is said to be ***linear***, ***quadratic***, ***cubic*** or ***biquadratic*** if its degree is 1, 2, 3 or 4 respectively.
- A **constant** is a polynomial of degree 0.

4. Division of a Polynomial by Another Polynomial.

If $f(x)$ and $g(x)$ are two polynomials, $g(x) \neq 0$, such that $f(x) = g(x).\ q(x) + r(x)$ where degree of $r(x) <$ degree of $f(x)$, then $f(x)$ is divided by $g(x)$, and it gives $\boldsymbol{q(x)}$ as **quotient** and $\boldsymbol{r(x)}$ as **remainder**.

Note : If $r(x) = 0$, then the divisor $g(x)$ is a factor of $f(x)$.

5. Remainder Theorem : If $\boldsymbol{f(x)}$ be any polynomial of degree ≥ 1, and a be any number, then if $f(x)$ is divided by $(x - a)$, the remainder is $\boldsymbol{f(a)}$.

Ex. (*a*) The remainder when $f(x) = (5x^2 - 4x - 1)$ is divided by $(x - 1)$ is $f(1) = 5.1^2 - 4.1 - 1 = \mathbf{0}$.

(*b*) The remainder when $f(x) = x^4 + 2x^3 - 3$ is divided by $(x + 2)$ is $f(-2) = (-2)^4 + 2.(-2)^3 - 3 = 16 - 16 - 3 = \mathbf{-3}$.

6. Factor Theorem : Let $f(x)$ be a polynomial of degree $n > 0$. If $f(a) = 0$, for any real number a, then $(x - a)$ is a factor of $f(x)$.
Conversely, if $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$.

Ex. $f(x) = x^3 - 6x^2 + 11x - 6$ is exactly divisible by $(x - 1)$ as $f(1) = 1^3 - 6.1^2 + 11.1 - 6 = \mathbf{0}$.

7. Some useful Identities :

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2 = (a+b)^2 - 4ab$
- $(a+b)(a-b) = a^2 - b^2$
- $(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$
- $(a+b)^2 - (a-b)^2 = 4ab$
- $(a+b)^3 = a^3 + b^3 + 3ab\,(a+b)$
- $(a-b)^3 = a^3 - b^3 - 3ab\,(a-b)$
- $a^3 + b^3 = (a+b)(a^2+b^2-ab)$
- $a^3 - b^3 = (a-b)(a^2+b^2+ab)$
- $(a+b+c)^2 = a^2+b^2+c^2+2\,(ab+bc+ca)$
- $(a^3+b^3+c^3) - 3abc = (a+b+c)(a^2+b^2+c^2-ab-ca-bc)$
- $a+b+c=0 \Rightarrow a^3+b^3+c^3 = 3abc$
- $(a^n - b^n)$ is divisible by $(a-b)$ for all values of n.
- $(a^n - b^n)$ is divisible by $(a+b)$ for only even values of n.
- $(a^n + b^n)$ is never divisible by $(a-b)$
- $(a^n + b^n)$ is divisible by $(a+b)$ only when n is odd.

8. **Note :** When a polynomial $f(x)$ is divided by $(x-a)$ and $(x-b)$, the respective remainders are A and B. Then, if the same polynomial is divided by $(x-a)(x-b)$, then the remainder will be :

$$\frac{A-B}{a-b}x + \frac{Ba-Ab}{a-b}.$$

9. Homogeneous Expressions : If all the terms of an algebraic expression are of the same degree, then such an expression is called homogeneous expression.

- Homogeneous expressions in (x, y) of
  Degree 1 $\rightarrow px + qy$
  Degree 2 $\rightarrow px^2 + rxy + qy^2$
  Degree 3 $\rightarrow px^3 + rx^2y + sxy^2 + qy^3$
- Homogeneous expression in (x, y, z) of
  Degree 1 $\rightarrow px + qy + rz$
  Degree 2 $\rightarrow px^2 + qy^2 + rz^2 + sxy + tyz + uzx$
  Degree 3 $\rightarrow ax^3 + by^3 + cz^3 + dx^2y + exy^2 + fy^2z + gyz^2 + hz^2x + kzx^2$.

10. Symmetric Expression : An algebraic expression $f(x, y)$ in two variables x, y is called a symmetric expression if $f(x, y) = f(y, x)$.

An algebraic expression $f(x, y, z)$ is said to be a **cyclic expression,** if $f(x, y, z) = f(y, z, x) = f(z, x, y)$

e.g. $f(a, b, c) = a(b-c) + b(c-a) + c(a-b)$

- $\sum$ **(Sigma)** is used for the **sum** of the terms of a cyclic expression.

Ex. $\sum\limits_{x,\,y,\,z} x^3(y-z) = x^3(y-z) + y^3(z-x) + z^3(x-y)$

- π **(Pi)** is used for the **product** of the terms of a cyclic expression.

Ex. $\underset{a,b,c}{\pi}\,(a-b) = (a-b)(b-c)(c-a)$

11. Horner's Method of Synthetic Division for Factorization

Ex. Divide $3x^3 - 2x^2 - 19x + 22$ by $(x-2)$

Sol.

$$\begin{array}{r|rrrr} 2 & 3 & -2 & -19 & 22 \\ \text{Multiplier} & 0 & 6 & 8 & 22 \\ & \times 2 & \times 2 & \times 2 & \\ \hline & 3 & 4 & -11 & \boxed{0} \leftarrow \text{Remainder} \end{array}$$

$\therefore f(x) = (x-2)(3x^2 + 4x - 11)$

Step 1: Write the coefficients of the descending powers of x in the first horizontal row.

Step 2: The multiplier is obtained by putting the divisor $(x-2) = 0 \Rightarrow x = 2$.

Step 3: Now below the 1st coefficient, *i.e.*, 3 in the first horizontal row, put 0 and add 3 + 0, *i.e.*, 3.

Now 3 × multiplier = 3 × 2 = 6 = 2nd element of 2nd horizontal row. –2 + 6 = 4

Now 4 × multiplier = 4 × 2 = 8 = 3rd element of 2nd horizontal row. –19 + 8 = – 11

For the last element again $-11 \times 2 = 22$ and $22 + (-22) = 0$.

The first three figures in the third row stand for the coefficients of descending powers of x of quotient and the last entry is for the remainder.

SOLVED EXAMPLES

Ex. 1. For what value of p is the coefficient of x^2 in the product $(2x-1)(x-k)(px+1)$ equal to 0 and the constant term equal to 2 ? ***(CDS 2005)***

Sol. $(2x-1)(x-k)(px+1) = (2x-1)(px^2 + x - kpx - k)$

$$= 2px^3 + 2x^2 - 2kpx^2 - 2kx - px^2 - x + kpx + k$$

$$= 2px^3 + x^2[2 - 2kp - p] - x[2k + 1 - kp] + k$$

Here constant term $= k = \mathbf{2}$.

Coefficient of $x^2 = 2 - 2kp - p = 2 - 4p - p = 2 - 5p$

Given, $2 - 5p = 0 \Rightarrow p = \mathbf{\frac{2}{5}}$.

Ex. 2. For what value of m will the expression $3x^3 + mx^2 + 4x - 4m$ be divisible by $x + 2$? ***(CDS 2005)***

Sol. $f(x) = 3x^3 + mx^2 + 4x - 4m$

$f(x)$ is divisible by $(x+2)$ if $f(-2) = 0$

Now $f(-2) = 3(-2)^3 + m(-2)^2 + 4(-2) - 4m = -24 + 4m - 8 - 4m = -32 \neq 0$

$\therefore$ No such value of m exists for which $(x+2)$ is a factor of the given expression.

Ex. 3. If $x^5 - 9x^2 + 12x - 14$ is divisible by $(x-3)$, what is the remainder ? ***(CDS 2011)***

Sol. Let $f(x) = x^5 - 9x^2 + 12x - 14$

$f(x)$ is divisible by $(x-3)$ so remainder $= f(3)$.

$\therefore f(3) = (3)^5 - 9(3)^2 + 12(3) - 14 = 243 - 81 + 36 - 14 = \mathbf{184}$.

Ex. 4. If the expressions $(px^3 + 3x^2 - 3)$ and $(2x^3 - 5x + p)$ when divided by $(x-4)$ leave the same remainder, then what is the value of p ?

Sol. Let $f(x) = px^3 + 3x^2 - 3$

$g(x) = 2x^3 - 5x + p$

When divisible by $x - 4$, the remainders for the given expressions are $f(4)$ and $g(4)$ respectively.

$f(4) = p(4)^3 + 3(4)^2 - 3 = 64p + 48 - 3 = 64p + 45$

$g(4) = 2(4)^3 - 5(4) + p = 128 - 20 + p = 108 + p$.

Given, $f(4) = g(4) \Rightarrow 64p + 45 = 108 + p \Rightarrow 63p = 63 \Rightarrow p = \mathbf{1}$.

Ex. 5. What is/are the factors of $(x^{29} - x^{24} + x^{13} - 1)$?

(*a*) $(x-1)$ only **(*b*) $(x+1)$ only** **(*c*) $(x-1)$ and $(x+1)$** **(*d*) Neither $(x-1)$ nor $(x+1)$**

(CDS 2008)

Sol. For $(x-1)$ to be a factor of the given expression, the value of expression at $x = 1$ is

$(1)^{29} - (1)^{24} + (1)^{13} - 1 = 1 - 1 + 1 - 1 = 0$

$\therefore (x-1)$ is a factor of $x^{29} - x^{24} + x^{13} - 1$

Similarly for $(x+1)$ to be the factor, the value of expression at $x = -1$ is

$(-1)^{29} - (-1)^{24} + (-1)^{13} - 1 = -1 - 1 - 1 - 1 = -4 \neq 0$

$\therefore (x+1)$ is not a factor of $x^{29} - x^{24} + x^{13} - 1$.

Hence, (*a*) is the correct option.

Ex. 6. Which one of the following is one of the factors of $x^2 (y - z) + y^2 (z - x) - z (xy - yz - zx)$?

(*a*) $(x - y)$ (*b*) $(x + y - z)$ (*c*) $(x - y - z)$ (*d*) $(x + y + z)$

(CDS 2007)

Sol. $x^2 (y - z) + y^2 (z - x) - z(xy - yz - zx)$

$$= x^2y - x^2z + y^2z - y^2x - zxy + yz^2 + z^2x$$
$$= xy(x - y - z) + z^2(x + y) - z (x^2 - y^2)$$
$$= xy(x - y - z) - z (x + y) (x - y - z) = (x - y - z) (xy - yz - zx)$$

Hence, (*c*) is the correct option.

Ex. 7. Without actual division show that $2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$.

Sol. Let $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$ and $g(x) = x^2 - 3x + 2 = x^2 - 2x - x + 2 = x(x - 2) - 1 (x - 2) = (x - 2) (x - 1)$

For $f(x)$ to be exactly divisible by $g(x)$, $(x - 1)$ and $(x - 2)$ should be the factors of $f(x)$, *i.e.*,

$f(1) = 0$ and $f(2) = 0$.

Now, $f(1) = 2. (1)^4 - 6.(1)^3 + 3.(1)^2 + 3.1 - 2 = 2 - 6 + 3 + 3 - 2 = 0$

$f(2) = 2.(2)^4 - 6(2)^3 + 3(2)^2 + 3.2 - 2 = 32 - 48 + 12 + 6 - 2 = 0.$

$\therefore$ $(x - 1)$ and $(x - 2)$ are factors of $f(x) \Rightarrow f(x)$ is exactly divisible by $g(x)$.

Ex. 8. If $a + b + c = 0$, then what is the value of $a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2$? ***(CDS 2005)***

Sol. Given, $a + b + c = 0$.

Now, $a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2 = (a^2 + b^2 + c^2)^2 - 4a^2b^2 - 4b^2c^2 - 4c^2a^2$

$$= [(a + b + c)^2 - 2ab - 2bc - 2ca]^2 - 4a^2b^2 - 4b^2c^2 - 4c^2a^2$$
$$= [0^2 - 2ab - 2bc - 2ca]^2 - 4a^2b^2 - 4b^2c^2 - 4c^2a^2$$
$$= 4a^2b^2 + 4b^2c^2 + 4c^2a^2 + 8ab^2c + 8abc^2 + 8a^2bc - 4a^2b^2 - 4b^2c^2 - 4c^2a^2$$
$$= 8ab^2c + 8abc^2 + 8a^2bc = 8abc\,(b + c + a) = 8abc.\,0 = \mathbf{0}.$$

Ex. 9. If $x = \dfrac{a-b}{a+b}, y = \dfrac{b-c}{b+c}, z = \dfrac{c-a}{c+a}$, then what is the value of $\dfrac{1+x}{1-x}.\dfrac{1+y}{1-y}.\dfrac{1+z}{1-z}$? ***(CDS 2006)***

Sol. $x = \dfrac{a-b}{a+b} \Rightarrow \dfrac{1}{x} = \dfrac{a+b}{a-b}$

$\Rightarrow \dfrac{1+x}{1-x} = \dfrac{a+b+a-b}{a+b-a+b} = \dfrac{2a}{2b} \Rightarrow \dfrac{1+x}{1-x} = \dfrac{a}{b}$ (Applying componendo and dividendo)

Similarly, $\dfrac{1+y}{1-y} = \dfrac{b}{c}, \dfrac{1+z}{1-z} = \dfrac{c}{a}$ $\therefore$ $\dfrac{1+x}{1-x}\cdot\dfrac{1+y}{1-y}\cdot\dfrac{1+z}{1-z} = \dfrac{a}{b}\cdot\dfrac{b}{c}\cdot\dfrac{c}{a} = \mathbf{1}.$

Ex. 10. If $x + y + z = 0$, then what is $\left[\dfrac{(y-z-x)}{2}\right]^3 + \left[\dfrac{(z-x-y)}{2}\right]^3 + \left[\dfrac{(x-y-z)}{2}\right]^3$ equal to ?

Sol. $\left(\dfrac{y-z-x}{2}\right)^3 + \left(\dfrac{z-x-y}{2}\right)^3 + \left(\dfrac{x-y-z}{2}\right)^3$

$$= \left(\frac{y-(z+x)}{2}\right)^3 + \left(\frac{z-(x+y)}{2}\right)^3 + \left(\frac{x-(y+z)}{2}\right)^3$$

$$= \left(\frac{y-(-y)}{2}\right)^3 + \left(\frac{z-(-z)}{2}\right)^3 + \left(\frac{x-(-x)}{2}\right)^3 \qquad (\because x + y + z = 0)$$

$$= \left(\frac{2y}{2}\right)^3 + \left(\frac{2z}{2}\right)^3 + \left(\frac{2x}{2}\right)^3 = y^3 + z^3 + x^3 = \mathbf{3xyz}. \qquad (\because a^3 + b^3 + c^3 = 3abc, \text{ if } a + b + c = 0)$$

Ex. 11. If $x^2 - 4x + 1 = 0$, then what is the value of $x^3 + \dfrac{1}{x^3}$?

Sol. $x^2 - 4x + 1 = 0$

$$\Rightarrow x = \frac{4 \pm \sqrt{16 - 4\times1\times1}}{2\times1} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$$\left[\begin{array}{l}\text{Roots quadratic eqn } ax^2 + bx + c = 0\\ = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ \text{Here } a = 1, b = -4, c = 1\end{array}\right]$$

$$\therefore \quad x^3 + \frac{1}{x^3} = (2+\sqrt{3})^3 + \frac{1}{(2+\sqrt{3})^3} = (2+\sqrt{3})^3 + \left[\frac{(2-\sqrt{3})\times 1}{(2+\sqrt{3})(2-\sqrt{3})}\right]^3 = (2+\sqrt{3})^3 + (2-\sqrt{3})^3$$

$$= 2^3 + (\sqrt{3})^3 + 3\times2\times\sqrt{3}(2+\sqrt{3}) + 2^3 - (\sqrt{3})^3 - 3\times2\times\sqrt{3}\,(2-\sqrt{3})$$

$= 8 + 18 + 8 + 18 = \mathbf{52}$. Similarly for $x = 2 - \sqrt{3}$, $x^3 + \dfrac{1}{x^3} = \mathbf{52.}$

Ex. 12. If $\dfrac{1}{y+z} + \dfrac{1}{z+x} = \dfrac{2}{x+y}$, then what is $(x^2 + y^2)$ equal to ?

Sol. $\dfrac{1}{y+z} + \dfrac{1}{z+x} = \dfrac{2}{x+y}$

$$\Rightarrow \frac{1}{y+z} - \frac{1}{x+y} = \frac{1}{x+y} - \frac{1}{z+x} \quad \Rightarrow \frac{(x+y)-(y+z)}{(y+z)(x+y)} = \frac{(z+x)-(x+y)}{(x+y)(z+x)}$$

$$\Rightarrow \frac{x-z}{y+z} = \frac{z-y}{z+x} \Rightarrow (x-z)(x+z) = (z-y)(z+y) \quad \Rightarrow x^2 - z^2 = z^2 - y^2 \Rightarrow x^2 + y^2 = \mathbf{2z^2}.$$

Ex. 13. If the sum and difference of two expressions are $5a^2 - a - 4$ and $a^2 + 9a - 10$ respectively, then what is their LCM ?

Sol. Let P and Q be the two expressions. Then,

$P + Q = 5a^2 - a - 4$ ...(*i*)

$P - Q = a^2 + 9a - 10$ ...(*ii*)

Adding (*i*) and (*ii*)

$\Rightarrow 2P = 6a^2 + 8a - 14 \Rightarrow P = 3a^2 + 4a - 7 = (a-1)(3a+7)$

From (*i*), $Q = (5a^2 - a - 4) - (3a^2 + 4a - 7) = 2a^2 - 5a + 3 = (a-1)(2a-3)$

$\therefore$ LCM of P and $Q = (a-1)(2a-3)(3a+7)$.

Ex. 14. Without actual division, show that $(x-1)^{2n} - x^{2n} + 2x - 1$ is divisible by $2x^3 - 3x^2 + x$.

Sol. Let $f(x) = (x-1)^{2n} - x^{2x} + 2x - 1$

$g(x) = 2x^3 - 3x^2 + x = x(2x^2 - 3x + 1)$

Now $g(x) = 0 \Rightarrow x[2x^2 - 3x + 1] = 0 \Rightarrow x[2x^2 - 2x - x + 1] = 0$

$\Rightarrow [2x(x-1) - 1(x-1)] = 0 \Rightarrow (2x-1)(x-1) = 0 \Rightarrow x = \dfrac{1}{2}, 1$

$\therefore$ For $f(x)$ to be exactly divisible by $g(x)$, $f\left(\dfrac{1}{2}\right) = f(1)$ should be all zero.

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}-1\right)^{2n} - \left(\frac{1}{2}\right)^{2n} + 2\times\frac{1}{2} - 1 = \left(-\frac{1}{2}\right)^{2n} - \left(\frac{1}{2}\right)^{2n} + 1 - 1 = \left(\frac{1}{2}\right)^{2n} - \left(\frac{1}{2}\right)^{2n} + 1 - 1 = 0$$

$f(1) = (1-1)^{2n} - 1^{2n} + 2\times 1 - 1 = 0 - 1 + 2 - 1 = 0.$

$\therefore$ $[(x-1)^{2n} - x^{2n} + 2x - 1]$ is completely divisible by $2x^3 - 3x^2 + x$.

Ex. 15. If the HCF of $(x^2 + x - 12)$ and $(2x^2 - kx - 9)$ is $(x - k)$, then what is the value of k ? ***(CDS 2008)***

Sol. Since $(x - k)$ is the HCF of $(x^2 + x + 12)$ and $(2x^2 - kx - 9)$

$(x - k)$ will be a factor of $2x^2 - kx - 9$

$\therefore\ 2.k^2 - k.k - 9 = 0 \Rightarrow k^2 - 9 = 0 \Rightarrow k = \pm 3$

Also, the factors of $(x^2 + x - 12) = (x + 4)(x - 3)$ $\therefore$ $\boldsymbol{k = 3}$.

PRACTICE SHEET

LEVEL–1

1. When $x^{13} + 1$ is divided by $x - 1$, the remainder is :
(*a*) 1 (*b*) –1 (*c*) 0 (*d*) 2

2. If $x^3 + 5x^2 + 10k$ leaves remainder $-2x$ when divided by $x^2 + 2$, then what is the value of k ?
(*a*) –2 (*b*) –1 (*c*) 1 (*d*) 2
(*CDS 2012*)

3. $x^4 + xy^3 + x^3y + xz^3 + y^4 + yz^3$ is divisible by :
(*a*) $(x - y)$ only (*b*) $(x^3 + y^3 + z^3)$ only
(*c*) both $(x + y)$ and $(x^3 + y^3 + z^3)$
(*d*) None of the above **(*CDS 2012*)**

4. For what value of k, will the expression $(3x^3 - kx^2 + 4x + 16)$ be divisible by $(x - k/2)$?
(*a*) 4 (*b*) –4 (*c*) 2 (*d*) 0
(*CDS 2007*)

5. When $(x^3 - 2x^2 + px - q)$ is divided by $(x^2 - 2x - 3)$, the remainder is $(x - 6)$, What are the values of p and q respectively ?
(*a*) –2, –6 (*b*) 2, – 6 (*c*) –2, 6 (*d*) 2, 6
(*CDS 2009*)

6. If $\dfrac{x^3 + ax^2 + bx + 4}{x^2 + x - 2}$ is a polynomial of degree 1 in x, then what are the values of a, b respectively ?
(*a*) –1, –4 (*b*) –1, 4 (*c*) 3, –4 (*d*) 3, 4
(*CDS 2005*)

7. When $a + b + c + 3a^{1/3}\, b^{2/3} + 3a^{2/3}\, b^{1/3}$ is divided by $a^{1/3} + b^{1/3} + c^{1/3}$, what is the remainder ?
(*a*) $3a$ (*b*) $3b$ (*c*) 0 (*d*) $c^{2/3}$
(*CDS 2005*)

8. If the polynomials $ax^3 + 4x^2 + 3x - 4$ and $x^3 - 4x + a$ leave the same remainder when divided by $(x - 3)$, the value of a is :
(*a*) 2 (*b*) –3/2 (*c*) –1 (*d*) 4

9. Let R_1 and R_2 be the remainders when the polynomials $x^3 + 2x^2 - 5ax - 7$ and $x^2 + ax^2 - 12x + 6$ are divided by $(x + 1)$ and $(x - 2)$ respectively. If $2R_1 + R_2 = 6$, the value of a is :
(*a*) –2 (*b*) 1 (*c*) –1 (*d*) 2

10. If both $(x - 2)$ and $(x - 1/2)$ are factors of $px^2 + 5x + r$, then:
(*a*) $p = 2r$ (*b*) $p + r = 0$ (*c*) $p = r$ (*d*) $p \times r = 1$

11. If the expression $ax^2 + bx + c$ is equal to 4, when $x = 0$, leaves a remainder 4 when divided by $x + 1$ and leaves a remainder 6 when divided by $x + 2$, then the values of a, b and c are respectively,
(*a*) 1, 1, 4 (*b*) 2, 2, 4 (*c*) 3, 3, 4 (*d*) 4, 4, 4

12. $f(x) = x^4 - 2x^3 + 3x^2 - ax + b$ is a polynomial such that when it is divided by $(x - 1)$ and $(x + 1)$, the remainders are respectively 5 and 19. Determine the remainder when $f(x)$ is divided by $(x - 2)$.
(*a*) 6 (*b*) 10 (*c*) 2 (*d*) 8

13. If $(x^2 - 1)$ is a factor of $ax^4 + bx^3 + cx^2 + dx + e$, then :
(*a*) $a + c + e = 0$ (*b*) $ace = 1$
(*c*) $b + d = 0$ (*d*) Both (*a*) and (*c*)

14. What is $\dfrac{x^2 - 3x + 2}{x^2 - 5x + 6} \div \dfrac{x^2 - 5x + 4}{x^2 - 7x + 12}$ equal to ?
(*a*) $\dfrac{x+3}{x-3}$ (*b*) 1 (*c*) $\dfrac{x+1}{x-1}$ (*d*) 2
(*CDS 2011*)

15. If the expression $(px^3 + x^2 - 2x - q)$ is divisible by $(x - 1)$ and $(x + 1)$, then the values of p and q respectively are ?
(*a*) 2, – 1 (*b*) –2, 1 (*c*) –2, –1 (*d*) 2, 1
(*CDS 2010*)

LEVEL–2

16. When $x^{40} + 2$ is divided by $x^4 + 1$, what is the remainder ?
(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4
(*CDS 2009*)

17. If the remainder of the polynomial $a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ when divided by $(x - 1)$ is 1, then which one of the following is correct ?
(*a*) $a_0 + a_2 + \ldots\ldots = a_1 + a_3 + \ldots\ldots$
(*b*) $a_0 + a_2 + \ldots\ldots = 1 + a_1 + a_3 + \ldots\ldots$
(*c*) $1 + a_0 + a_2 + \ldots\ldots = -(a_1 + a_3 + \ldots\ldots)$
(*d*) $1 - a_0 - a_2 - \ldots\ldots = a_1 + a_3 + \ldots\ldots$ **(*CDS 2009*)**

18. The remainder when $1 + x + x^2 + x^3 + \ldots\ldots.. + x^{1007}$ is divided by $(x - 1)$ is
(*a*) 1006 (*b*) 1008 (*c*) 1007 (*d*) 0

19. A cubic polynomial $f(x)$ is such that $f(1) = 1$, $f(2) = 2$, $f(3) = 3$ and $f(4) = 5$, then $f(6)$ equals :
(*a*) 7 (*b*) 6 (*c*) 10 (*d*) 13

20. If the polynomial $x^6 + px^5 + qx^4 - x^2 - x - 3$ is divisible by $x^4 - 1$, then the value of $p^2 + q^2$ is :
(*a*) 1 (*b*) 9 (*c*) 10 (*d*) 13
(*CDS 2001*)

21. The factors of $x^8 + x^4 + 1$ are :

(*a*) $(x^4 + 1 - x^2)(x^2 + 1 + x)(x^2 + 1 - x)$

(*b*) $(x^4 + 1 - x^2)(x^2 - 1 + x)(x^2 + 1 + x)$

(*c*) $(x^4 - 1 + x^2)(x^2 - 1 + x)(x^2 + 1 + x)$

(*d*) $(x^4 - 1 + x^2)(x^2 + 1 - x)(x^2 + 1 + x)$ **(*CDS 1999*)**

22. If the polynomial $x^{19} + x^{17} + x^{13} + x^{11} + x^7 + x^5 + x^3$ is divided by $(x^2 + 1)$, then the remainder is :

(*a*) 1 (*b*) $x^2 + 4$ (*c*) $-x$ (*d*) x

23. If $(x + k)$ is a common factor of $x^2 + px + q$ and $x^2 + lx + m$, then the value of k is

(*a*) $\frac{p+q}{l+m}$ (*b*) $\frac{p-l}{q-m}$ (*c*) $\frac{q+m}{q+l}$ (*d*) $\frac{q-m}{p-l}$

24. If $(x - 1)$ is a factor of $Ax^3 + Bx^2 - 36x + 22$ and $2^B = 64^A$, find A and B ?

(*a*) $A = 4, B = 16$ (*b*) $A = 6, B = 24$

(*c*) $A = 2, B = 12$ (*d*) $A = 8, B = 16$

25. When a polynomial $f(x)$ is divided by $(x - 3)$ and $(x + 6)$, the respective remainders are 7 and 22. What is the remainder when $f(x)$ is divided by $(x - 3)(x + 6)$?

(*a*) $\frac{-5}{3}x + 12$ (*b*) $-\frac{7}{3}x + 14$ (*c*) $-\frac{5}{3}x + 16$ (*d*) $-\frac{7}{3}x + 12$

26. If $p(x)$ is a common multiple of degree 6 of the polynomials $f(x) = x^3 + x^2 - x - 1$ and $g(x) = x^3 - x^2 + x - 1$, then which one of the following is correct ?

(*a*) $p(x) = (x - 1)^2 (x + 1)^2 (x^2 + 1)$

(*b*) $p(x) = (x - 1)(x + 1)(x^2 + 1)^2$

(*c*) $p(x) = (x - 1)^3 (x + 1)(x^2 + 1)$

(*d*) $p(x) = (x - 1)^2 (x^4 + 1)$ **(*CDS 2012*)**

27. Which one of the following is divisible by $(1 + a + a^5)$ and $(1 + a^4 + a^5)$ individually ?

(*a*) $(a^2 + a + 1)(a^3 + a^2 + 1)(a^3 + a + 1)$

(*b*) $(a^4 - a + 1)(a^3 + a^2 + 1)(a^3 + a - 1)$

(*c*) $(a^4 + a + 1)(a^3 - a^2 + 1)(a^3 + a + 1)$

(*d*) $(a^2 + a + 1)(a^3 - a^2 + 1)(a^3 - a + 1)$ **(*CDS 2005*)**

28. Consider the following statements :

1. $a^n + b^n$ is divisible by $a + b$ if $n = 2k + 1$, where k is a positive integer.
2. $a^n - b^n$ is divisible by $a - b$ if $n = 2k$, where k is a positive integer. Which of the statements given above is/are correct ?

(*a*) 1 only (*b*) 2 only

(*c*) Both 1 and 2 (*d*) Neither 1 nor 2

(*CDS 2005*)

29. If $(x - 2)$ is a common factor of the expressions $x^2 + ax + b$ and $x^2 + cx + d$, then $\frac{b-d}{c-a}$ is equal to

(*a*) -2 (*b*) -1 (*c*) 1 (*d*) 2

(*EAMCET 2004*)

30. Let $a \neq 0$ and $p(x)$ be a polynomial of degree greater than 2. If $p(x)$ leaves remainders a and $-a$, when divided respectively by $(x + a)$ and $(x - a)$, then the remainder when $p(x)$ is divided by $(x^2 - a^2)$ is

(*a*) $-2x$ (*b*) $-x$ (*c*) 0 (*d*) $2a$

(*EAMCET 2003*)

31. If $9x^2 + 3px + 6q$ when divided by $(3x + 1)$ leaves a remainder $\left(-\frac{3}{4}\right)$ and $qx^2 + 4px + 7$ is exactly divisible by $(x + 1)$, then the values of p and q respectively will be :

(*a*) $0, \frac{7}{4}$ (*b*) $-\frac{7}{4}, 0$ (*c*) Same (*d*) $\frac{7}{4}, 0$

32. What should be subtracted from $27x^3 - 9x^2 - 6x - 5$ to make it exactly divisible by $(3x - 1)$

(*a*) -5 (*b*) -7 (*c*) 5 (*d*) 7

(*CDS 2009*)

33. The values of a, b and c respectively for the expression $f(x) = x^3 + ax^2 + bx + c$, if $f(1) = f(2) = 0$ and $f(4) = f(0)$ are :

(*a*) 9, 20, 12 (*b*) –9, –20, 12

(*c*) –9, 20, –12 (*d*) –9, –20, –12

34. The remainder, when x^{200} is divided by $x^2 - 3x + 2$ is

(*a*) $(2^{200} - 1)x + (-2^{200} + 2)$

(*b*) $(2^{200} + 1)x + (-2^{200} - 2)$

(*c*) $(2^{200} - 1)x + (-2^{200} - 2)$

(*d*) 2^{100}

35. (*i*) For $a \neq b$, if $x + k$ is the HCF of $x^2 + ax + b$ and $x^2 + bx + a$, then the value of $a + b$ is equal to

(*a*) – 2 (*b*) – 1 (*c*) 0 (*d*) 2

(*Type Raj PET 2004, Kerala PET 2004*)

(*ii*) If $(x + k)$ is the HCF of $ax^2 + ax + b$ and $x^2 + cx + d$, then what is the value of k ?

(*a*) $\frac{b+d}{a+c}$ (*b*) $\frac{a+b}{c+d}$ (*c*) $\frac{a-b}{c-d}$

(*d*) None of these **(*CDS 2008*)**

LEVEL–3

36. The value of $\frac{(a-b)^3 + (b-c)^3 + (c-a)^3}{(a-b)(b-c)(c-a)}$ is :

(*a*) 1 (*b*) 3 (*c*) 1/3 (*d*) Zero

(*CDS 2000*)

37. If $a^2 = by + cz$, $b^2 = cz + ax$, $c^2 = ax + by$, then the value of $\frac{x}{a+x} + \frac{y}{b+y} + \frac{z}{c+z}$ will be:

(*a*) $a + b + c$ (*b*) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ (*c*) 1 (*d*) 0

(*CDS 2001*)

38. If $x + y + z = 0$, then $x(y - z)^3 + y(z - x)^3 + z(x - y)^3$ equals

(*a*) 0 (*b*) $y + z$ (*c*) 1 (*d*) $(z + x)^2$

39. If $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{(a+b+c)}$, where $a + b + c \neq 0$, $abc \neq 0$, what is the value of $(a + b)(b + c)(c + a)$?

(*a*) 0 (*b*) 1 (*c*) –1 (*d*) 2

(*CDS 2005*)

40. If $\dfrac{x}{(b-c)(b+c-2a)} = \dfrac{y}{(c-a)(c+a-2b)}$

$= \dfrac{z}{(a-b)(a+b-2c)}$, what is the value of $x + y + z$?

(*a*) $(a + b + c)$ (*b*) $a^2 + b^2 + c^2$

(*c*) 0 (*d*) 1 **(*CDS 2005*)**

41. If $a + b + c = 0$, then find the value of :

$\dfrac{a^2}{a^2 - bc} + \dfrac{b^2}{b^2 - ca} + \dfrac{c^2}{c^2 - ab}$

(*a*) 4 (*b*) 2 (*c*) 1 (*d*) 0

(*MAT 2005*)

42. The value of $\dfrac{(x-y)^3 + (y-z)^3 + (z-x)^3}{(x^2-y^2)^3 + (y^2-z^2)^3 + (z^2-x^2)^3}$ is :

(*a*) 1 (*b*) $[2(x+y+z)]^{-1}$

(*c*) $[(x + y)(y + z)(z + x)]^{-1}$ (*d*) 0 **(*CDS 2004*)**

43. If $a + b + c = 0$, then $a^2 + ab + b^2$ is equal to :

(*a*) $b^2 - bc + c^2$ (*b*) $c^2 - ab$

(*c*) $b^2 + bc + c^2$ (*d*) 0 **(*CDS 2004*)**

44. If $pqr = 1$, the value of $\dfrac{1}{[1+p+q^{-1}]} + \dfrac{1}{[1+q+r^{-1}]}$

$+ \dfrac{1}{[1+r+p^{-1}]}$ will be equal to :

(*a*) 1 (*b*) 0 (*c*) –1 (*d*) –2

(*CDS 2004*)

45. If $a = \dfrac{xy}{x+y}$, $b = \dfrac{xz}{x+z}$ and $c = \dfrac{yz}{y+z}$, where *a, b* and *c* are non-zero, then what is *x* equal to ?

(*a*) $\dfrac{2abc}{ac+bc-ab}$ (*b*) $\dfrac{2abc}{ab-ac+bc}$

(*c*) $\dfrac{2abc}{ab+bc+ac}$ (*d*) $\dfrac{2abc}{ab+ac-bc}$

(*CDS 2006*)

46. If $a + b + c = 0$, then what is the value of

$\dfrac{a^2+b^2+c^2}{(a-b)^2+(b-c)^2+(c-a)^2}$

(*a*) 1 (*b*) 3 (*c*) $\dfrac{1}{3}$ (*d*) 0 **(*CDS 2006*)**

47. If $x + y + z = 2s$, then what is

$(s - x)^3 + (s - y)^3 + 3(s - x)(s - y)z$ equal to :

(*a*) z^3 (*b*) $-z^3$ (*c*) x^3 (*d*) y^3

(*CDS 2007*)

48. If $x + \dfrac{1}{x} = p$, then $x^6 + \dfrac{1}{x^6}$ equals to :

(*a*) $p^6 + 6p$ (*b*) $p^6 - 6p$

(*c*) $p^6 + 6p^4 + 9p^2 + 2$ (*d*) $p^6 - 6p^4 + 9p^2 - 2$

(*CDS 2007*)

49. If $x + y + z = 0$, then what is the value of :

$\dfrac{1}{x^2+y^2-z^2} + \dfrac{1}{y^2+z^2-x^2} + \dfrac{1}{z^2+x^2-y^2}$?

(*a*) $\dfrac{1}{x^2+y^2+z^2}$ (*b*) 1

(*c*) –1 (*d*) 0 **(*CDS 2010*)**

50. If $x + y + z = 0$, then $\dfrac{x^2}{2x^2+yz} + \dfrac{y^2}{2y^2+zx} + \dfrac{z^2}{2z^2+xy} =$

(*a*) 4 (*b*) 2 (*c*) 3 (*d*) 1

51. If $(b + c - a)x = (c + a - b)y = (a + b - c)z = 2$, then $\left(\dfrac{1}{y}+\dfrac{1}{z}\right)\left(\dfrac{1}{z}+\dfrac{1}{x}\right)\left(\dfrac{1}{x}+\dfrac{1}{y}\right)$ is equals:

(*a*) $a^2b^2c^2$ (*b*) abc (*c*) a^2b^2 (*d*) $(abc)^2$

52. If $a^x = (x + y + z)^y$, $a^y = (x + y + z)^z$, $a^z = (x + y + z)^x$, then:

(*a*) $3(x + y + z) = a$ (*b*) $2a = x + y + z$

(*c*) $x + y + z = 0$ (*d*) $x = y = z = a/3$

53. If $x + \dfrac{1}{x} = a$, then what is the value of $x^3 + x^2 + \dfrac{1}{x^3} + \dfrac{1}{x^2}$?

(*a*) $a^3 + a^2$ (*b*) $a^3 + a^2 - 5a$

(*c*) $a^3 + a^2 - 3a - 2$ (*d*) $a^3 + a^2 - 4a - 2$

(*CDS 2012*)

54. If $x^{1/3} + y^{1/3} + z^{1/3} = 0$, then what is $(x + y + z)^3$ equal to ?

(*a*) 1 (*b*) 3 (*c*) $3xy$ (*d*) $27xyz$

55. $\dfrac{2a}{a+b} + \dfrac{2b}{b+c} + \dfrac{2c}{c+a} + \dfrac{(b-c)(c-a)(a-b)}{(b+c)(c+a)(a+b)}$ equals

(*a*) 0 (*b*) –1 (*c*) 3 (*d*) 2

ANSWERS

1. (*d*)	**2.** (*c*)	**3.** (*b*)	**4.** (*b*)	**5.** (*c*)	**6.** (*a*)	**7.** (*c*)	**8.** (*c*)	**9.** (*d*)	**10.** (*c*)
11. (*a*)	**12.** (*b*)	**13.** (*d*)	**14.** (*b*)	**15.** (*d*)	**16.** (*c*)	**17.** (*d*)	**18.** (*b*)	**19.** (*b*)	**20.** (*c*)
21. (*a*)	**22.** (*c*)	**23.** (*d*)	**24.** (*c*)	**25.** (*a*)	**26.** (*a*)	**27.** (*b*)	**28.** (*c*)	**29.** (*d*)	**30.** (*b*)
31. (*d*)	**32.** (*b*)	**33.** (*c*)	**34.** (*a*)	**35.** (*i*) (*b*) (*ii*) (*d*)		**36.** (*b*)	**37.** (*c*)	**38.** (*a*)	**39.** (*a*)
40. (*c*)	**41.** (*b*)	**42.** (*c*)	**43.** (*b*)	**44.** (*a*)	**45.** (*a*)	**46.** (*c*)	**47.** (*a*)	**48.** (*d*)	**49.** (*d*)
50. (*d*)	**51.** (*b*)	**52.** (*d*)	**53.** (*c*)	**54.** (*d*)	**55.** (*c*)				

HINTS AND SOLUTIONS

1. Remainder when $x^{13} + 1$ is divided by $(x – 1) = 1^{13} + 1 = \mathbf{2.}$

2.
$$\begin{array}{r|l} & x+5 \\ \hline x^2+2 & x^3 + 5x^2 \qquad + 10k \\ & \underline{\overset{}{x^3} \qquad\quad + 2x} \\ & \quad\;\; - \qquad\quad - \\ & \quad 5x^2 - 2x + 10k \\ & \quad \underline{5x^2 \qquad\quad + 10} \\ & \quad - \qquad\qquad - \\ & \quad -2x + 10k - 10 \quad = \text{Remainder} \end{array}$$

Given, $–2x + 10k – 10 = – 2x$

$\Rightarrow 10k = 10 \Rightarrow k = \mathbf{1.}$

3. Hint. $x^4 + xy^3 + x^3y + xz^3 + y^4 + yz^3$

$= (x^4 + xy^3 + xz^3) + (x^3y + y^4 + yz^3)$

$= x(x^3 + y^3 + z^3) + y(x^3 + y^3 + z^3)$

$= (x + y) (x^3 + y^3 + z^3)$

4. Let $f(x) = 3x^3 – kx^2 + 4x + 16$. Then, $f(x)$ will be divisible by $(x – k/2)$ if $f(k/2) = 0$

$\Rightarrow 3.(k/2)^3 – k.(k/2)^2 + 4(k/2) + 16 = 0$

$$\Rightarrow \frac{3k^3}{8} - \frac{k^3}{4} + \frac{4k}{2} + 16 = 0$$

$$\Rightarrow \frac{3k^3 - 2k^3 + 16k + 128}{8} = 0$$

$\Rightarrow k^3 + 16k + 128 = 0 \Rightarrow (k + 4) (k^2 – 4k + 32) = 0$

$\Rightarrow k + 4 = 0 \Rightarrow k = \mathbf{– 4.}$

5.
$$\begin{array}{r|l} & \qquad x \\ \hline x^2 - 2x - 3 & x^3 - 2x^2 + px - q \\ & x^3 - 2x^2 - 3x \\ & - \quad\;\; + \quad\;\; + \\ \hline & \quad (p+3)\,x - q \end{array}$$

Given, $(p + 3)x – q = x – 6$

$\Rightarrow p + 3 = 1$ and $q = 6$

$\Rightarrow p = \mathbf{– 2}, q = \mathbf{6}$

6.
$$\begin{array}{r|l} & \quad x + (a-1) \\ \hline x^2 + x - 2 & \; x^3 + ax^2 + \; bx + 4 \\ & \; x^3 + \; x^2 - \; 2x \\ & - \quad\; - \quad\; + \\ \hline & (a-1)\,x^2 + (b+2)x + 4 \\ & (a-1)\,x^2 + (a-1)x - 2(a-1) \\ & - \qquad\quad - \qquad\quad + \\ \hline & \qquad (b-a+3)x + (2a+2) \end{array}$$

As the given polynomial is of degree 1, the degree of the remainder should be less than 1, *i.e.*, 0, *i.e.*, the remainder has only a constant term.

$\Rightarrow b – a + 3 = 0$ and $2a + 2 = 0 \Rightarrow a = – 1$

$\therefore\ b – (–1) + 3 = 0 \Rightarrow b = – 4.$

$\therefore\ \boldsymbol{a = – 1, b = – 4.}$

7. Let $a^{1/3} = x$, $b^{1/3} = y$, $c^{1/3} = z$. Then,

$a + b + c + 3a^{1/3}\, b^{2/3} + 3a^{2/3}\, b^{1/3} = x^3 + y^3 + z^3 + 3xy^2 + 3x^2y$

and $a^{1/3} + b^{1/3} + c^{1/3} = x + y + z.$

Now $x^3 + y^3 + z^3 + 3xy^2 + 3x^2y$

$= x^3 + 3xy^2 + 3x^2y + y^3 + z^3$

$= (x + y)^3 + z^3$

$= [x + y + z][(x + y)^2 - (x + y) z + z^2].$

$\therefore$ Given expression is completely divisible by $(x + y + z)$, *i.e.*, by $a^{1/3} + b^{1/3} + c^{1/3}$.

8. Let $f(x) = ax^3 + 4x^2 + 3x – 4$

$g(x) = x^3 – 4x + a.$

Remainders when $f(x)$ and $g(x)$ are divided by $(x – 3)$ are $f(3)$ and $g(3)$ respectively. Now,

$f(3) = a.\ (3)^3 + 4.(3)^2 + 3.3 – 4$

$= 27a + 36 + 9 – 4 = 27a + 41$ ...(*i*)

$g(3) = (3)^3 – 4.(3) + a = 27 – 12 + a = 15 + a$ ...(*ii*)

Given, $f(3) = g(3)$

$\therefore\ 27a + 41 = 15 + a \Rightarrow 26a = – 26 \Rightarrow a = \mathbf{– 1.}$

9. Let $f(x) = x^3 + 2x^2 – 5ax – 7$

$\therefore\quad R_1 = f(–1) = (–1)^3 + 2.\ (–1)^2 – 5.a.\ (–1) –7$

$= –1 + 2 + 5a –7 = 5a – 6$

$g(x) = x^3 + ax^2 – 12x + 6$

$R_2 = g(2) = (2)^3 + a.(2)^2 – 12.(2) + 6$

$= 8 + 4a – 24 + 6 = 4a – 10$

Given, $2R_1 + R_2 = 6 \Rightarrow 2(5a – 6) + (4a – 10) = 6$

$\Rightarrow\quad 10a – 12 + 4a – 10 = 6$

$\Rightarrow\quad 14a – 22 = 6 \Rightarrow 14a = 28 \Rightarrow a = \mathbf{2.}$

10. Let $f(x) = px^2 + 5x + r$

Since, $(x – 2)$ and $\left(x - \dfrac{1}{2}\right)$ are the factors of $f(x)$, therefore,

$f(2) = 0$ and $f\left(\dfrac{1}{2}\right) = 0.$

$\therefore\ f(2) = p \times (2)^2 + 5 \times 2 + r = 4p + 10 + r = 0$ ...(*i*)

$$f\left(\frac{1}{2}\right) = p \times \left(\frac{1}{2}\right)^2 + 5 \times \frac{1}{2} + r = \frac{p}{4} + \frac{5}{2} + r = p + 10 + 4r = 0 \quad ...(ii)$$

(*i*) and (*ii*) $\Rightarrow 4p + 10 + r = p + 10 + 4r \Rightarrow 3p = 3r \Rightarrow \boldsymbol{p = r.}$

11. Given exp. $f(x) = ax^2 + bx + c$

$\therefore$ When $x = 0$, $a.0 + b.0 + c = 4 \Rightarrow \mathbf{c = 4.}$

The remainders when $f(x)$ is divided by $(x + 1)$ and $(x + 2)$ respectively are $f(–1)$ and $f(–2)$.

$\therefore\quad f(–1) = a.(–1)^2 + b.(–1) + c = 4$

$\Rightarrow\quad a – b + c = 4 \Rightarrow a – b + 4 = 4 \Rightarrow a – b = 0$ ...(*i*)

$f(–2) = a.(–2)^2 + b(–2) + c = 6$

$\Rightarrow\quad 4a – 2b + 4 = 6 \Rightarrow 4a – 2b = 2$ ...(*ii*)

Solving (*i*) and (*ii*) simultaneously we get, $\boldsymbol{a = 1, b = 1.}$

12. When $f(x) = x^4 – 2x^3 + 3x^2 – ax + b$ is divided by $(x – 1)$ and $(x + 1)$, the remainders are 5 and 19 respectively.

i.e., $f(1) = 5$ and $f(–1) = 19$

$\Rightarrow 1 – 2 + 3 – a + b = 5$ and $1 + 2 + 3 + a + b = 19$

$\Rightarrow –a + b = 3$ and $a + b = 13$

Adding the two equations, we get $2b = 16 \Rightarrow b = 8 \Rightarrow a = 5$

$\therefore f(x) = x^4 - 2x^3 + 3x^2 - ax + b$

$= x^4 - 2x^3 + 3x^2 - 5x + 8$

$\therefore$ Remainder, when $f(x)$ is divided by $(x - 2)$ is equal to $f(2)$

$\therefore f(2) = 2^4 - 2.2^3 + 3.2^2 - 5.2 + 8$

$= 16 - 16 + 12 - 10 + 8 = \mathbf{10.}$

13. Let $f(x) = ax^4 + bx^4 + cx^2 + dx + e$ be the given polynomial. Then, $(x^2 - 1)$ is a factor of $f(x)$.

$\Rightarrow (x - 1)(x + 1)$ is a factor of $f(x)$

$\Rightarrow (x - 1)$ and $(x + 1)$ are factors of $f(x)$

$\Rightarrow f(1) = 0$ and $f(-1) = 0$

$\Rightarrow a + b + c + d + e = 0$ and $a - b + c - d + e = 0.$

Adding and subtracting the two equations, we get

$2(a + c + e) = 0$ and $2(b + d) = 0$

$\Rightarrow a + c + e = 0$ and $b + d = 0.$

14. $\dfrac{x^2 - 3x + 2}{x^2 - 5x + 6} \div \dfrac{x^2 - 5x + 4}{x^2 - 7x + 12}$

$= \dfrac{(x-1)(x-2)}{(x-3)(x-2)} \div \dfrac{(x-4)(x-1)}{(x-3)(x-4)}$

$= \dfrac{(x-1)}{(x-3)} \div \dfrac{(x-1)}{(x-3)} = \dfrac{(x-1)}{(x-3)} \times \dfrac{(x-3)}{(x-1)} = \mathbf{1.}$

15. $px^3 + x^2 - 2x - q$ is divisible by $(x - 1)$ and $(x + 1)$

$\Rightarrow p(1)^3 + (1)^2 - 2(1) - q = 0 \Rightarrow p - q = 1$ ...(*i*)

and $p(-1)^3 + (-1)^2 - 2(-1) - q = 0 \Rightarrow p + q = 3$ ...(*ii*)

Solving (*i*) and (*ii*) $p = 2, q = 1.$

16. Put $x^4 = -1$ in $f(x) = x^{40} + 2$

Remainder $= (x^4)^{10} + 2 = (-1)^{10} + 2 = \mathbf{3.}$

17. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n$

Given, $f(1) = 1$

$\Rightarrow a_0 + a_1 + a_2 + \ldots\ldots\ldots + a_n = 1$

$\Rightarrow 1 - a_0 - a_2 - \ldots\ldots\ldots = a_1 + a_3 + \ldots\ldots\ldots .$

18. Required remainder $= f(1)$

$= 1 + 1 + 1 + 1 + 1 \ldots\ldots\ldots + 1$ (1008 times)

$= 1008 \times 1 = \mathbf{1008.}$

19. Let the cubic polynomial be :

$f(x) = ax^3 + bx^2 + cx + d.$

Given, $f(1) = 1 \Rightarrow a + b + c + d = 1$ ....(*i*)

$f(2) = 2 \Rightarrow 8a + 4b + 2c + d = 2$ ...(*ii*)

$f(3) = 4 \Rightarrow 27a + 9b + 3c + d = 3$ ...(*iii*)

$f(4) = 5 \Rightarrow 125a + 25b + 5c + d = 5$ ...(*iv*)

(*ii*) – (*i*) $\Rightarrow 7a + 3b + c = 1$ ...(*v*)

(*iii*) – (*ii*) $\Rightarrow 19a + 5b + c = 1$ ...(*vi*)

(*iv*) – (*iii*) $\Rightarrow 98a + 16b + 2c = 2$ ...(*vii*)

(*vi*) – (*v*) $\Rightarrow 12a + 2b = 0 \Rightarrow 6a + b = 0$ ...(*viii*)

(*vii*) – 2 (*vi*) $\Rightarrow 60a + 6b = 0 \Rightarrow 10a + b = 0$ ...(*ix*)

Solving (*viii*) and (*ix*), we get $a = 0 \Rightarrow b = 0$

Putting $a = 0, b = 0$ in (*v*), we, get $c = 1$

Also from (*i*), $a = 0, b = 0, c = 1 \Rightarrow d = 0.$

Putting values of a, b, c, d in $f(x) = ax^3 + bx^2 + cx + d$, we get the polynomial $f(x) = x \Rightarrow f(6) = \mathbf{6.}$

20. $f(x) = x^6 + px^5 + qx^4 - x^2 - x - 3$

$= x^4 . x^2 + p.x^4 x + q.x^4 - x^2 - x - 3$

As $(x^4 - 1)$ is a factor of $f(x)$, so putting $x^4 = 1$, we get

$x^2 + px + q - x^2 - x - 3 = 0$

$\Rightarrow (p - 1)x + (q - 3) = 0 \Rightarrow p - 1 = 0$ and $q - 3 = 0$

$\Rightarrow p = 1$ and $q = 3.$

$\therefore p^2 + q^2 = 1 + 9 = \mathbf{10.}$

21. $x^8 + x^4 + 1 = x^8 + 2x^4 + 1 - x^4$ (Adding and subtracting x^4)

$= (x^4 + 1)^2 - (x^2)^2 = (x^4 + 1 + x^2)(x^4 + 1 - x^2)$

$= [(x^4 + 2x^2 + 1) - x^2](x^4 + 1 - x^2)$

$= [(x^2 + 1)^2 - (x)^2](x^4 + 1 - x^2)$

$= (x^2 + 1 + x)(x^2 + 1 - x)(x^4 + 1 - x^2)$

22. $f(x) = x^{19} + x^{17} + x^{13} + x^{11} + x^7 + x^5 + x^3$

Putting $x^2 = -1$, we get

$f(x) = (x^2)^9.x + (x^2)^8.x + (x^2)^6.x + (x^2)^5.x + (x^2)^2.x + x^2.x$

$= (-1)^9x + (-1)^8.x + (-1)^6.x + (-1)^5.x + (-1)^2.x + (-1).x$

$= -x + x + x - x + x - x = \mathbf{-x.}$

23. Let $f(x) = x^2 + px + q$

$g(x) = x^2 + lx + m.$

Since $(x + k)$ is a common factor of $f(x)$ and $g(x)$,

$f(-k) = k^2 - pk + q = 0$

$g(-k) = k^2 - lk + m = 0$

$\Rightarrow k^2 - px + q = k^2 - lk + m$

$\Rightarrow q - m = (p - l)k \Rightarrow k = \dfrac{q - m}{p - l}$

24. Since $(x - 1)$ is a factor of $Ax^3 + Bx^2 - 36x + 22$

$\therefore A(1)^3 + B(1)^2 - 36(1) + 22 = 0$

$\Rightarrow A + B = 14$ ...(*i*)

and $2^B = 64^A \Rightarrow 2^B = (2^6)^A \Rightarrow B = 6A$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*) $A = 2, B = 12.$

25. The function $f(x)$ is not known. Here,

$a = 3, \quad b = -6$

$A = 7, \quad B = 22$ **[*Refer to Key Fact 8*]**

$\therefore$ Required remainder $= \dfrac{A - B}{a - b}x + \dfrac{Ba - Ab}{a - b}$

$= \dfrac{7 - 22}{3 - (-6)}x + \dfrac{22 \times 3 - 7 \times (-6)}{3 - (-6)} = \mathbf{\dfrac{-5}{3}x + 12}.$

26. $f(x) = x^3 + x^2 - x - 1$

$g(x) = x^3 - x^2 + x - 1$

$f(x) . g(x) = (x^3 + x^2 - x - 1) . (x^3 - x^2 + x - 1)$

$= x^6 - \cancel{x^5} + \cancel{x^4} - \cancel{x^3} + \cancel{x^5} - \cancel{x^4} + \cancel{x^3} - \cancel{x^2}$

$-x^4 + \cancel{x^3} - \cancel{x^2} + \cancel{x} - \cancel{x^3} + x^2 - \cancel{x} + 1$

$= x^6 - x^4 - x^2 + 1$

$\therefore p(x) = x^6 - x^4 - x^2 + 1$

$= x^4(x^2 - 1) - (x^2 - 1) = (x^2 - 1)(x^4 - 1)$

$= (x - 1)(x + 1)[(x^2)^2 - 1]$

$= (x - 1)(x + 1)[(x^2 - 1)(x^2 + 1)]$

$= (x-1)(x+1)[(x-1)(x+1)(x^2+1)]$
$= (x-1)^2(x+1)^2(x^2+1).$

27. The given expression has to be divided by $(1 + a + a^5)$ and $(1 + a^4 + a^5)$ individually, so highest power of a is $5 + 5 = 10$, followed by $5 + 4 = 9$, and both are positive.

In options (*a*) and (*d*) the highest power of a is 8, hence these options are not acceptable.

The only choices are (*b*) and (*c*), but in option (*c*) a^{10} is positive but a^9 is negative and in (*b*) both a^{10} and a^9 are positive. Hence (*b*) is the correct option.

28. Statement (1) is correct as for $k = 1$, $n = 2 \times 1 + 1 = 3$.

$\therefore\ a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$ which is divisible by $(a + b)$, statement (2) is also correct as for $k = 1$, $n = 2$,

$\therefore\ a^2 - b^2 = (a-b)(a+b)$ which is divisible by $(a - b)$.

29. $(x - 2)$ is a common factor of $(x^2 + ax + b)$ and $(x^2 + cx + d)$

$\Rightarrow 4 + 2a + b = 0$ ...(*i*)

and $4 + 2c + d = 0$ ...(*ii*)

$\therefore\ 2a + b = 2c + d \Rightarrow b - d = 2(c - a) \Rightarrow \dfrac{b-d}{c-a} = \mathbf{2}.$

30. Let $rx + t$ be the remainder, $q(x)$ be the quotient when $p(x)$ is divided by $x^2 - a^2$.

$\therefore\ p(x) = (x^2 - a^2).\ qx + rx + t$ ...(*i*)

Given, $p(x)$ leaves remainders a and $-a$ respectively when divided by $(x + a)$ and $(x - a)$.

$\therefore\ p(-a) = a$ and $p(a) = -a$

Putting $x = -a$ in (i), we get

$p(-a) = 0.\ q(-a) + (-ra + t)$

$\Rightarrow a = -ra + t$ ...(*ii*)

Putting $x = a$, in (*i*), we get

$p(a) = 0.q\ (a) + (ra + t)$

$\Rightarrow -a = ra + t$ ...(*iii*)

$\therefore$ Adding (*ii*) and (*iii*), we get $2t = 0 \Rightarrow t = 0 \Rightarrow r = -1$

$\therefore$ Required remainder $= rx + t = \mathbf{-x}.$

31. Given, $(9x^2 + 3px + 6q)$, when divided by $(3x + 1)$ leaves a remainder $-\dfrac{3}{4}$

$\therefore\ f(x) = 9x^2 + 3px + 6q - \left(-\dfrac{3}{4}\right) = \left(9x^2 + 3px + 6q + \dfrac{3}{4}\right)$

is exactly divisible by $(3x + 1)$

$$\therefore\ f\left(-\frac{1}{3}\right) = 0 \Rightarrow 9\left(-\frac{1}{3}\right)^2 + 3p.\left(-\frac{1}{3}\right) + 6q + \frac{3}{4} = 0$$

$$\Rightarrow 6q - p + \frac{7}{4} = 0$$

$\Rightarrow 24q - 4p + 7 = 0$ ...(*i*)

Now, the expression $g(x) = qx^2 + 4px + 7$ is exactly divisible by $x + 1$

$\Rightarrow g(-1) = 0 \Rightarrow q - 4p + 7 = 0$ ...(*ii*)

Solving equations (*i*) and (*ii*), we get $q = \mathbf{0}$, $p = \mathbf{\dfrac{7}{4}}.$

32. To make $f(x) = 27x^3 - 9x^2 - 6x - 5$ exactly divisible by $(3x - 1)$, the remainder obtained on division should be subtracted.

$$\text{Remainder} = f\left(+\frac{1}{3}\right) = 27 \times \left(\frac{1}{3}\right)^3 - 9 \times \left(\frac{1}{3}\right)^2 - 6 \times \frac{1}{3} - 5$$

$$\left(\because 3x - 1 = 0 \Rightarrow x = \frac{1}{3}\right)$$

$= 1 - 1 - 2 - 5 = \mathbf{-7}.$

33. Given, $f(x) = x^3 + lx^2 + mx + n$.

$f(1) = f(2) = 0 \Rightarrow (x - 1)$ and $(x - 2)$ are factors of $f(x)$.

Since, $f(x)$ is polynomial of degree 3, it shall have three linear factors. So, let the third factor be $(x - k)$.

Then, $f(x) = (x-1)(x-2)(x-k)$

$\Rightarrow f(x) = x^3 + lx^2 + mx + n = (x-1)(x-2)(x-k)$

Given, $f(4) = f(0)$

$\Rightarrow (4-1)(4-2)(4-k) = (-1)(-2)(-k)$

$\Rightarrow 24 - 6k = -2k \Rightarrow 4k = 24 \Rightarrow k = 6$

$\therefore\ f(x) = (x-1)(x-2)(x-6) = (x^2 - 3x + 2)(x - 6)$

$= x^3 - 9x^2 + 20x - 12$

$\therefore\ x^3 + lx^2 + mx + n = x^3 - 9x^2 + 20x - 12$

$\Rightarrow l = -9$, $m = 20$, $n = -12$.

34. Let $x^{200} = (x^2 - 3x + 2).\ Q(x) + lx + m$ ...(*i*)

where, $Q(x)$ = quotient and $(lx + m)$ is the remainder

Now $(x^2 - 3x + 2) = 0 \Rightarrow (x-1)\ (x-2) = 0 \Rightarrow x = 1, 2.$

Substituting $x = 1$ in (*i*), we have,

$1^{200} = 0.\ Q.(x) + l + m$ ...(*ii*)

Similarly, for $x = 2$,

$2^{200} = 0.\ Q(x) + 2l + m$ ...(*iii*)

$\therefore\ l + m = 1,\ 2l + m = 2^{200}$

Solving we get, $l = 2^{200} - 1$ and $m = 2 - 2^{200}$

Hence remainder $= lx + m = (2^{200} - 1)\,x + (-2^{200} + 2)$.

35. (*i*) Since $x + k$ is the HCF of the given expressions, therefore, $x = -k$ will make each expression zero.

$k^2 - ak + b = 0$ ...(*i*)

$k^2 - bk + a = 0$ ...(*ii*)

Solving (*i*) and (*ii*) by the rule of cross multiplication,

$$\frac{k^2}{-a^2 + b^2} = \frac{k}{b - a} = \frac{1}{-b + a}$$

From last two relations, $k = \dfrac{b-a}{-(b-a)} = -1$

$$\therefore\ \frac{k^2}{-a^2 + b^2} = \frac{1}{-b + a} \Rightarrow \frac{(-1)^2}{-a^2 + b^2} = \frac{1}{-b + a}$$

$$\Rightarrow \frac{1}{(b-a)(b+a)} = \frac{-1}{(b-a)} \Rightarrow a + b = -1.$$

(*ii*) **Hint.** $ak^2 - ak + b = 0$

$k^2 - ck + d = 0$

Solving by the rule of cross multiplication,

$$\frac{k^2}{-ad + bc} = \frac{k}{b - ad} = \frac{1}{-ac + a}$$

$$\Rightarrow k = \frac{b - ad}{a(1 - c)}, \frac{bc - ad}{b - ad}.$$

36. Since $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$.

So, as $(a - b) + (b - c) + (c - a) = 0$

$\Rightarrow (a - b)^3 + (b - c)^3 + (c - a)^3 = 3(a - b)(b - c)(c - a)$

$\therefore$ Given expression $= \dfrac{3(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)} = \mathbf{3.}$

37. $a^2 = by + cz \Rightarrow a^2 + ax = ax + by + cz$

$\Rightarrow a(a + x) = ax + by + cz$ ...(*i*)

Similarly, $b^2 = cz + ax \Rightarrow b(b + y) = ax + by + cz$ ...(*ii*)

and $c^2 = ax + by \Rightarrow c(c + z) = ax + by + cz$ ...(*iii*)

Hence, $\dfrac{x}{a+x} + \dfrac{y}{b+y} + \dfrac{c}{c+z}$

$= \dfrac{ax}{a(a+x)} + \dfrac{by}{b(b+y)} + \dfrac{cz}{c(c+z)}$

$= \dfrac{x.a}{ax+by+cz} + \dfrac{y.b}{ax+by+cz} + \dfrac{z.c}{ax+by+cz}$

$= \dfrac{ax+by+cz}{ax+by+cz} = \mathbf{1.}$

38. Now, $x + y + z = 0$

$\Rightarrow x = -y - z,\ y = -x - z,\ z = -x - y$

$\therefore x(y - z)^3 + y(z - x)^3 + z(x - y)^3$

$= (-y - z)(y - z)^3 + (-z - x)(z - x)^3 + (-x - y)(x - y)^3$

$= -(y + z)(y - z)^3 - (z + x)(z - x)^3 - (x + y)(x - y)^3$

$= -[(y^2 - z^2)(y - z)^2 + (z^2 - x^2)(z - x)^2 + (x^2 - y^2)(x - y)^2]$

$= -[(y^2 - z^2)(y^2 - 2yz + z^2) + (z^2 - x^2)(z^2 - 2xz + x^2)$
$+ (x^2 - y^2)(x^2 - 2xy + y^2)]$

$= -[(\cancel{y^4} - \cancel{z^4}) - 2yz(y^2 - z^2) + (\cancel{z^4} - \cancel{x^4})$
$- 2xz(z^2 - x^2) + (\cancel{x^4} - \cancel{y^4}) - 2xy(x^2 - y^2)]$

$= 2yz(y^2 - z^2) + 2xz(z^2 - x^2) + 2xy(x^2 - y^2)$

$= 2(y^3z - yz^3 + z^3x - x^3z + x^3y - xy^3)$

$= 2[x^3(y - z) + y^3(z - x) + z^3(x - y)]$

$= 2[-(y - z)(z - x)(x - y).(x + y + z)]$

$= \mathbf{0.}$ $(\because x + y + z = 0)$

39. $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{a+b+c}$

$\Rightarrow (a + b + c)\left[\dfrac{bc + ac + ab}{abc}\right] = 1$

$\Rightarrow (a + b + c)(bc + ac + ab) = abc$

$\Rightarrow abc + a^2c + a^2b + b^2c + abc + ab^2 + bc^2 + ac^2 + abc = abc$

$\Rightarrow a^2(c + b) + bc(c + b) + ab(c + b) + ac(c + b) = 0$

$\Rightarrow (b + c)(a^2 + bc + ab + ac) = 0$

$\Rightarrow (b + c)(a^2 + ab + bc + ac) = 0$

$\Rightarrow (b + c)[a(a + b) + c(a + b)] = 0$

$\Rightarrow (b + c)(a + b)(c + a) = 0.$

40. Let $\dfrac{x}{(b-c)(b+c-2a)} = \dfrac{y}{(c-a)(c+a-2b)}$

$= \dfrac{z}{(a-b)(a+b-2c)} = k.$

Then, $x = k(b - c)(b + c - 2a)$

$y = k(c - a)(c + a - 2b)$

$z = k(a - b)(a + b - 2c)$

$\therefore x + y + z = k(b - c)(b + c - 2a) + k(c - a)(c + a - 2b)$
$+ k(a - b)(a + b - 2c)$

$= k(b^2 - c^2 - 2ab + 2ca) + k(c^2 - a^2 - 2bc + 2ab)$
$+ k(a^2 - b^2 - 2ca + 2bc)$

$= k(b^2 - c^2 - 2ab + 2ca + c^2 - a^2 - 2bc + 2ab + a^2 - b^2$
$- 2ca + 2bc)$

$= k \times 0 = \mathbf{0.}$

41. $a + b + c = 0$

$\Rightarrow a^2 = (b + c)^2$ or $a = -b - c$

$\therefore$ Given expression $= \dfrac{a^2}{a^2 - bc} + \dfrac{b^2}{b^2 - ca} + \dfrac{c^2}{c^2 - ab}$

$= \dfrac{(b+c)^2}{(b+c)^2 - bc} + \dfrac{b^2}{b^2 + c(b+c)} + \dfrac{c^2}{c^2 + b(b+c)}$

$= \dfrac{(b+c)^2}{b^2 + c^2 + bc} + \dfrac{b^2}{b^2 + c^2 + bc} + \dfrac{c^2}{c^2 + b^2 + bc}$

$= \dfrac{b^2 + c^2 + 2bc + b^2 + c^2}{b^2 + c^2 + bc} = \dfrac{2(b^2 + c^2 + bc)}{(b^2 + c^2 + bc)} = \mathbf{2.}$

42. $\dfrac{(x-y)^3 + (y-z)^3 + (z-x)^3}{(x^2-y^2)^3 + (y^2-z^2)^3 + (z^2-x^2)^3}$

$= \dfrac{3(x-y)(y-z)(z-x)}{3(x^2-y^2)(y^2-z^2)(z^2-x^2)}$

$\left[\begin{array}{l}\because a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc \\ \text{Here } (x - y) + (y - z) + (z - x) = 0 \\ \text{and } (x^2 - y^2) + (y^2 - z^2) + (z^2 - x^2) = 0\end{array}\right]$

$= \dfrac{3(x-y)(y-z)(z-x)}{3(x+y)(x-y)(y+z)(y-z)(z+x)(z-x)}$

$= \dfrac{1}{(x+y)(y+z)(z+x)} = \mathbf{[(x+y)(y+z)(z+x)]^{-1}}$

43. If $a + b + c = 0$, then $a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow (a + b)(a^2 - ab + b^2) + c^3 = 3abc$

$\Rightarrow (-c)(a^2 - ab + b^2) + c^3 = 3abc$ $[\because (a + b) = -c]$

$\Rightarrow a^2 - ab + b^2 - c^2 = -3ab$

$\Rightarrow a^2 - ab + b^2 + 2ab - c^2 = -3ab + 2ab$

$= a^2 + ab + b^2 = \boldsymbol{c^2 - ab}.$

44. $\dfrac{1}{1+p+q^{-1}}+\dfrac{1}{1+q+r^{-1}}+\dfrac{1}{1+r+p^{-1}}$

$$=\frac{1}{1+p+\frac{1}{q}}+\frac{1}{1+q+\frac{1}{r}}+\frac{1}{1+r+\frac{1}{p}}$$

$$=\frac{q}{q+pq+1}+\frac{r}{r+qr+1}+\frac{p}{p+pr+1}$$

$$=\frac{q}{q+\frac{1}{r}+1}+\frac{r}{r+\frac{1}{p}+1}+\frac{p}{p+pr+1} \quad [\because pqr=1]$$

$$=\frac{qr}{qr+1+r}+\frac{pr}{pr+1+p}+\frac{p}{p+pr+1}$$

$$=\frac{qr}{\frac{1}{p}+1+r}+\frac{pr}{pr+1+p}+\frac{p}{p+pr+1}$$

$$=\frac{pqr}{1+p+pr}+\frac{pr}{pr+1+p}+\frac{p}{p+pr+1}$$

$$=\frac{pqr+pr+p}{1+p+pr}=\frac{1+pr+p}{1+p+pr}=\mathbf{1.}$$

45. Given, $c=\dfrac{yz}{y+z} \Rightarrow cy+cz=yz \Rightarrow yz-cz=cy \Rightarrow z(y-c)=cy$

$\Rightarrow z=\dfrac{cy}{y-c}$

Also $b=\dfrac{xz}{x+z} \Rightarrow z=\dfrac{bx}{x-b}$

$\therefore\ \dfrac{cy}{y-c}=\dfrac{bx}{x-b} \Rightarrow cyx-cyb=bxy-bxc$

$\Rightarrow cyx-cyb-bxy=-bxc$

$\Rightarrow -y(bx+bc-cx)=-bxc$

$\Rightarrow y=\dfrac{bxc}{bx+bc-cx}$

Now, $a=\dfrac{xy}{x+y} \Rightarrow y=\dfrac{ax}{x-a}$.

$\therefore\ \dfrac{bxc}{bx+bc-cx}=\dfrac{ax}{x-a}$

$\Rightarrow abx^2+abcx-acx^2=bx^2c-abcx$

$\Rightarrow 2abcx=x^2(bc+ac-ab) \Rightarrow x=\dfrac{2abc}{(bc+ac-ab)}$

46. $a+b+c=0 \Rightarrow (a+b+c)^2=0$

$\Rightarrow a^2+b^2+c^2+2ab+2bc+2ca=0$

$\Rightarrow a^2+b^2+c^2=-(2ab+2bc+2ca)$

Now, $\dfrac{a^2+b^2+c^2}{(a-b)^2+(b-c)^2+(c-a)^2}$

$$=\frac{a^2+b^2+c^2}{a^2+b^2-2ab+b^2+c^2-2bc+c^2+a^2-2ca}$$

$$=\frac{a^2+b^2+c^2}{2(a^2+b^2+c^2)-(2ab+2bc+2ca)}$$

$$=\frac{a^2+b^2+c^2}{2(a^2+b^2+c^2)+(a^2+b^2+c^2)}$$

$$=\frac{a^2+b^2+c^2}{3(a^2+b^2+c^2)}=\mathbf{\frac{1}{3}.}$$

47. $x+y+z=2s$

Also, $(s-x)+(s-y)+(-z)=2s-(x+y+z)$
$=2s-2s=0.$

$\Rightarrow (s-x)^3+(s-y)^3+(-z)^3-3(s-x)(s-y)(-z)=0$

$\left[\begin{array}{l}\because\ a+b+c=0\\ \Rightarrow a^3+b^3+c^3-3abc=0\end{array}\right]$

$\Rightarrow (s-x)^3+(s-y)^3+3(s-x)(s-y)(z)=z^3$

48. Given, $x+\dfrac{1}{x}=p \quad \Rightarrow \left(x+\dfrac{1}{x}\right)^2=p^2$

$\Rightarrow x^2+\dfrac{1}{x^2}+2=p^2 \quad \Rightarrow x^2+\dfrac{1}{x^2}=p^2-2$

$\Rightarrow \left(x^2+\dfrac{1}{x^2}\right)^3=(p^2-2)^3$

$\Rightarrow x^6+\dfrac{1}{x^6}+3\left(x^2+\dfrac{1}{x^2}\right)=p^6-8+6p^2(p^2-2)$

$\Rightarrow x^6+\dfrac{1}{x^6}+3(p^2-2)=p^6-8+6p^2(p^2-2)$

$\Rightarrow x^6+\dfrac{1}{x^6}=\boldsymbol{p^6-6p^4-9p^2-2}$

49. Given, $x+y+z=0 \Rightarrow x+y=-z$

$\Rightarrow x^2+y^2+2xy=z^2 \Rightarrow x^2+y^2=z^2-2xy$

$\therefore\ \dfrac{1}{x^2+y^2-z^2}=\dfrac{1}{z^2-2xy-z^2}=\dfrac{1}{-2xy}=-\dfrac{1}{2xy}$

Similarly, $\dfrac{1}{y^2+z^2-x^2}=-\dfrac{1}{2yz}$ and $\dfrac{1}{z^2+x^2-y^2}=-\dfrac{1}{2zx}$

$\therefore\ \dfrac{1}{x^2+y^2-z^2}+\dfrac{1}{y^2+z^2-x^2}+\dfrac{1}{z^2+x^2-y^2}$

$$=-\frac{1}{2xy}-\frac{1}{2yz}-\frac{1}{2zx}=-\frac{1}{2}\left[\frac{z+x+y}{xyz}\right]=\mathbf{0}$$

$[\because x+y+z=0]$

50. $x+y+z=0 \Rightarrow x=-y-z$ ...(*i*)

$y=-x-z$ ...(*ii*)

$z=-x-y$ ...(*iii*)

Now, $\dfrac{x^2}{2x^2+yz}+\dfrac{y^2}{2y^2+zx}+\dfrac{z^2}{2z^2+xy}$

$$=\frac{x^2}{x^2+x.x+yz}+\frac{y^2}{y^2+y.y+zx}+\frac{z^2}{z^2+z.z+xy}$$

$$= \frac{x^2}{x^2 + x(-y-z) + yz} + \frac{y^2}{y^2 + y(-x-z) + zx} + \frac{z^2}{z^2 + z(-x-y) + xy}$$

$$= \frac{x^2}{x^2 - xy - xz + yz} + \frac{y^2}{y^2 - yx - zy + zx} + \frac{z^2}{z^2 - zx - zy + xy}$$

$$= \frac{x^2}{x(x-y) - z(x-y)} + \frac{y^2}{y(y-x) - z(y-x)} + \frac{z^2}{z(z-x) - y(z-x)}$$

$$= \frac{x^2}{(x-y)(x-z)} + \frac{y^2}{(y-x)(y-z)} + \frac{z^2}{(z-x)(z-y)}$$

$$= -\frac{x^2}{(x-y)(z-x)} - \frac{y^2}{(x-y)(y-z)} - \frac{z^2}{(z-x)(y-z)}$$

$$= -\left[\frac{x^2(y-z) + y^2(z-x) + z^2(x-y)}{(x-y)(y-z)(z-x)}\right]$$

$$= -\left[\frac{-(x-y)(y-z)(z-x)}{(x-y)(y-z)(z-x)}\right] = -(-1) = \mathbf{1}.$$

Factorising the numerator.

51. Given, $(b + c - a)x = (c + a - b)y = (a + b - c)z = 2$

$$\Rightarrow x = \frac{2}{(b+c-a)};\ y = \frac{2}{(c+a-b)};\ z = \frac{2}{(a+b-c)}$$

$$\therefore \frac{1}{x} = \frac{b+c-a}{2};\ \frac{1}{y} = \frac{c+a-b}{2};\ \frac{1}{z} = \frac{a+b-c}{2}$$

$$\therefore \left(\frac{1}{y} + \frac{1}{z}\right) = \left(\frac{c+a-b}{2} + \frac{a+b-c}{2}\right) = \frac{2a}{2} = a$$

$$\left(\frac{1}{z} + \frac{1}{x}\right) = \left(\frac{a+b-c}{2} + \frac{b+c-a}{2}\right) = \frac{2b}{2} = b$$

$$\left(\frac{1}{x} + \frac{1}{y}\right) = \left(\frac{b+c-a}{2} + \frac{a+c-b}{2}\right) = \frac{2c}{2} = c$$

$$\therefore \left(\frac{1}{y} + \frac{1}{z}\right)\left(\frac{1}{z} + \frac{1}{x}\right)\left(\frac{1}{x} + \frac{1}{y}\right) = a \,.\, b \,.\, c = \boldsymbol{abc}.$$

52. $a^x \,.\, a^y \,.\, a^z = (x + y + z)^{x+y+z}$

$\Rightarrow a^{x+y+z} = (x + y + z)^{x+y+z}$

$\Rightarrow a = (x + y + z)$

Now, $(x + y + z)^y = a^x$ (given)

$\Rightarrow (x + y + z)^y = (x + y + z)^x \Rightarrow y = x$

Similarly, $y = z$ and $z = x$.

$$\therefore x = y = z = \frac{x+y+z}{3} = \boldsymbol{\frac{a}{3}}.$$

53. Given, $x + \frac{1}{x} = a$.

$$\text{Now, } x^3 + x^2 + \frac{1}{x^3} + \frac{1}{x^2} = \left(x^3 + \frac{1}{x^3}\right) + \left(x^2 + \frac{1}{x^2}\right)$$

$$= \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) + \left(x + \frac{1}{x}\right)^2 - 2$$

$= a^3 - 3a + a^2 - 2 = \boldsymbol{a^3 + a^2 - 3a - 2}.$

54. If $x^{1/3} + y^{1/3} + z^{1/3} = 0$, then

$(x^{1/3})^3 + (y^{1/3})^3 + (z^{1/3})^3 = 3x^{1/3}y^{1/3}z^{1/3}$

$$\left[\begin{array}{l}\because\ a + b + c = 0 \\ \Rightarrow a^3 + b^3 + c^3 = 3abc\end{array}\right]$$

$\Rightarrow x + y + z = 3x^{1/3}y^{1/3}z^{1/3}$

Now taking the cube of both the sides, we have

$(x + y + z)^3 = (3x^{1/3}y^{1/3}z^{1/3})^3 = \mathbf{27}\boldsymbol{xyz}.$

55. Given expression

$$= \frac{2a(b+c)(c+a) + 2b(a+b)(c+a) + 2c(a+b)(b+c)}{(a+b)(b+c)(c+a)} + \frac{(b-c)(c-a)(a-b)}{(b+c)(c+a)(a+b)}$$

$$= \frac{2a(bc + c^2 + ab + ac) + 2b(ac + bc + a^2 + ab) + 2c(ab + b^2 + ac + bc) + (bc - c^2 - ab + ac)(a-b)}{(b+c)(c+a)(a+b)}$$

$$= \frac{2abc + 2ac^2 + 2a^2b + 2a^2c + 2abc + 2b^2c + 2ba^2 + 2ab^2 + 2abc + 2cb^2 + 2ac^2 + 2bc^2 + abc - ac^2 - a^2b + a^2c - b^2c + bc^2 + ab^2 - abc}{(b+c)(c+a)(a+b)}$$

$$= \frac{6abc + 3ac^2 + 3a^2b + 3a^2c + 3b^2c + 3ab^2 + 3bc^2}{(b+c)(c+a)(a+b)}$$

$$= \frac{3[2abc + ac^2 + a^2b + a^2c + b^2c + ab^2 + bc^2]}{(b+c)(c+a)(a+b)}$$

$$= \frac{3(b+c)(c+a)(a+b)}{(b+c)(c+a)(a+b)} = \mathbf{3}.$$

SELF ASSESSMENT SHEET

1. $(x^n - a^n)$ is divisible by $(x - a)$

(*a*) for all values of n (*b*) for even values of n

(*c*) for odd values of n (*d*) only for prime values of n

2. If $(x + 1)$ is a factor of $x^4 + 9x^3 + 7x^2 + 9ax + 5a^2$, then :

(*a*) $a = 137$ (*b*) $5a^2 - 9a - 1 = 0$

(*c*) $5a^2 + 9a + 17 = 0$ (*d*) $a = \sqrt{131}$

(CDS 2004)

3. When $x^3 + 2x^2 + 4x + b$ is divided by $(x + 1)$, the quotient is $x^2 + ax + 3$ and the remainder is $-3 + 2b$. What are the values of a and b respectively ?

(*a*) 1, 0 (*b*) –1, 0 (*c*) 1, 1 (*d*) –1, –1

(CDS 2005)

4. If $x^3 + px + q$ and $x^3 + qx + p$ have a common factor, then which of the following is correct ?

(*a*) $p + q = 0$ (*b*) $p + q - 1 = 0$

(*c*) $(p + q + 1) = 0$ (*d*) $p - q + 1 = 0$

(*CDS 2005*)

5. $(2x - 3y)^3 + (3y - 4z)^3 + (4z - 2x)^3$ can be factorised into which one of the following ?

(*a*) $(2x + 3y + 4z)\,(2x - 3y - 4z)$

(*b*) $(2x + 3y - 4z)\,(2x - 3y + 4z)$

(*c*) $(2x - 3y)\,(3y - 4z)\,(4z - 2x)$

(*d*) $6\,(2x - 3y)\,(3y - 4z)\,(2z - x)$ **(*CDS 2005*)**

6. If $ax^3 + bx^2 + x - 6$ has $(x + 2)$ as a factor and leaves a remainder 4, when divided by $(x - 2)$, the value of a and b respectively are :

(*a*) 1, –2 (*b*) 2, 1 (*c*) 0, 2 (*d*) 1, –1

7. If $4x^2 - 6x + m$ is divisible by $x - 3$, which one of the following is the greatest divisor of m ?

(*a*) 9 (*b*) 12 (*c*) 18 (*d*) 36

(*CDS 2006*)

8. If $y = x + \dfrac{1}{x}$, then $x^4 + x^3 - 4x^2 + x + 1 = 0$ can be reduced to which one of the following ? $(x \neq 0)$

(*a*) $y^2 + y - 2 = 0$ (*b*) $y^2 + y - 4 = 0$

(*c*) $y^2 + y - 6 = 0$ (*d*) $y^2 + y + 6 = 0$

(*CDS 2006*)

9. If $x^2 = y + z$, $y^2 = z + x$, $z^2 = x + y$, then what is the value of:

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} \; ?$$

(*a*) 1 (*b*) 0 (*c*) –1 (*d*) 2

(*CDS 2007*)

10. $\dfrac{(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3}{(a - b)^3 + (b - c)^3 + (c - a)^3}$ on simplification is equal to :

(*a*) 1 (*b*) $(a - b)\,(b - c)\,(c - a)$

(*c*) $(a + b)\,(b + c)\,(c + a)$ (*d*) 0

ANSWERS

1. (*a*)	**2.** (*b*)	**3.** (*a*)	**4.** (*c*)	**5.** (*d*)	**6.** (*c*)	**7.** (*c*)	**8.** (*c*)	**9.** (*a*)	**10.** (*c*)

HINTS AND SOLUTIONS

2. Let $f(x) = x^4 + 9x^3 + 7x^2 + 9ax + 5a^2$.

If $(x + 1)$ is a factor of $f(x)$, then

$f(-1) = 0 \Rightarrow (-1)^4 + 9\,(-1)^3 + 7\,(-1)^2 + 9a\,(-1) + 5a^2 = 0$

$\Rightarrow 1 - 9 + 7 - 9a + 5a^2 = 0 \Rightarrow \mathbf{5a^2 - 9a - 1 = 0.}$

3. Let $f(x) = x^3 + 2x^2 + 4x + b$.

When divided by $(x + 1)$, the remainder $= f(-1)$

Given, remainder $= -3 + 2b$

$\therefore -3 + 2b = f(-1) = (-1)^3 + 2\,(-1)^2 + 4\,(-1) + b$

$\Rightarrow -3 + 2b = -1 + 2 - 4 + b$

$\Rightarrow -3 + 2b = -3 + b$.

This is only possible when $\boldsymbol{b = 0}$.

$\therefore f(x) = x^3 + 2x^2 + 4x$.

Now dividing $f(x)$ by $(x + 1)$, we see that

$$\begin{array}{r|l}
 & x^2 + x + 3 \\ \hline
x + 1 & x^3 + 2x^2 + 4x \\
 & x^3 + x^2 \\
 & - \quad - \\ \hline
 & \quad x^2 + 4x \\
 & \quad x^2 + x \\
 & \quad - \quad - \\ \hline
 & \qquad 3x \\
 & \qquad 3x + 3 \\
 & \qquad - \quad - \\ \hline
 & \qquad\quad -3 \\ \hline
\end{array}$$

$\therefore$ Quotient $= x^2 + ax + 3 = x^2 + x + 3 \Rightarrow a = \mathbf{1}$.

4. Let the common factor be $x - \alpha$, then $x = \alpha$ will make the given expressions zero, *i.e.*,

$$\alpha^2 + p\alpha + q = 0$$
$$\alpha^2 + q\alpha + p = 0$$

Solving by the rule of cross-multiplication, we have

$$\frac{\alpha^2}{p^2 - q^2} = \frac{\alpha}{q - p} = \frac{1}{q - p}$$

From, last two relation, $\alpha = 1$.

$\Rightarrow \alpha^2 = \dfrac{p^2 - q^2}{q - p} \Rightarrow 1 = -(p + q) \Rightarrow \boldsymbol{p + q + 1 = 0.}$

5. Since $(2x - 3y) + (3y - 4z) + (4z - 2x) = 0$, therefore,

$(2x - 3y)^3 + (3y - 4z)^3 + (4z - 2x)^3$

Take out 2 common

$= 3.(2x - 3y)(3y - 4z)\,(4z - 2x)$

$= 6.(2x - 3y)(3y - 4z)(2z - x)$.

6. Let $f(x) = ax^3 + bx^2 + x - 6$

$(x + 2)$ is a factor of $f(x) \Rightarrow f(-2) = 0$

$\therefore f(-2) = -8a + 4b - 2 - 6 = 0$

$\Rightarrow -8a + 4b - 8 = 0 \Rightarrow -2a + b = 2$ ...(*i*)

$(x - 2)$ leaves a remainder 4, when dividing $f(x) \Rightarrow f(2) = 4$

$\therefore f(2) = 8a + 4b + 2 - 6 = 4 \Rightarrow 8a + 4b - 8 = 0$

$\Rightarrow 2a + b = 2$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*) $b = \mathbf{2}$, $a = \mathbf{0}$.

7. If $f(x) = 4x^2 - 6x + m$ is divisible by $(x - 3)$, then $f(3) = 0$
$\Rightarrow 4(3)^2 - 6.3 + m = 0 \Rightarrow 36 - 18 + m = 0 \Rightarrow m = -18$.
The greatest divisor of m = **18**.

8. $x^4 + x^3 - 4x^2 + x + 1 = 0$

$$\Rightarrow x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0$$

$$\Rightarrow x^2 + \frac{1}{x^2} + x + \frac{1}{x} - 4 = 0$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 - 2 + \left(x + \frac{1}{x}\right) - 4 = 0$$

$$\Rightarrow y^2 + y - 6 = 0 \qquad \left(\because x + \frac{1}{x} = y\right)$$

9. $x^2 = y + z \Rightarrow x^2 + x = x + y + z$

$$\Rightarrow x(x + 1) = x + y + z \Rightarrow \frac{x}{x + y + z} = \frac{1}{x + 1}$$

Similarly, $\dfrac{1}{y+1} = \dfrac{y}{x + y + z}$ and $\dfrac{1}{z+1} = \dfrac{z}{x + y + z}$

$$\therefore \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = \frac{x}{x + y + z} + \frac{y}{x + y + z}$$

$$+ \frac{z}{x + y + z} = \frac{x + y + z}{x + y + z} = \mathbf{1}.$$

10. Use the identity, if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

3 Quadratic Equations

KEY FACTS

1. An equation in which the highest power of the variable is 2 is called a quadratic equation.
$ax^2 + bx + c = 0$, where a, b, c are constants is a general quadratic equation and $a \neq 0$, and $a, b, c \in R$.

2. Solving a Quadratic Equation : To find the roots of a quadratic equation is called solving a quadratic equation.

(*a*) **Method I :** ***Factorising the quadratic equation into linear factors.***

The quadratic expression $ax^2 + bx + c = 0$ can be expressed as a product of two linear factors as the degree of the algebraic expression here is **2.**

Let $ax^2 + bx + c = (mx + n)(ex + f)$, where $m \neq 0, e \neq 0$.

Then, $ax^2 + bx + c = 0 \Rightarrow (mx + n)(ex + f) = 0$

$\Rightarrow (mx + n) = 0$ or $(ex + f) = 0$

$\Rightarrow x = -\dfrac{n}{m}$ or $x = -\dfrac{f}{e}$

$\therefore$ The two roots of $ax^2 + bx + c = 0$ are $\dfrac{-n}{m}$ and $\dfrac{-f}{e}$.

(*b*) **Method II :** ***Using the formula.***

$ax^2 + bx + c = 0$ $\quad (a \neq 0)$

$\Rightarrow ax^2 + bx = -c$ $\quad$ (Transposing the constant term)

$\Rightarrow x^2 + \dfrac{b}{a}x = -\dfrac{c}{a}$ $\quad$ (Dividing by the coefficient of x^2)

$\Rightarrow x^2 + \dfrac{b}{a}x + \dfrac{b^2}{4a^2} = \dfrac{b^2}{4a^2} - \dfrac{c}{a}$ $\quad$ $\left(\text{Adding}\left(\dfrac{b}{2a}\right)^2 \text{ on both the sides to make LHS a perfect square}\right)$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \Rightarrow x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The formula

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Hence, the roots of the equation $ax^2 + bx + c = 0$ are $\boldsymbol{\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}}$ and $\boldsymbol{\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}}$

These two values are called the roots of the equation and are also called the **zeros of the function defined** by $f(x) = ax^2 + bx + c$.

3. Equations Reducible to Quadratic Equations

Type I : $ax^{2n} + bx^n + c = 0$

In this type of an equation, we put $x^n = y$. So, $ax^{2n} + bx^n + c = 0$ reduces to $ay^2 + by + c = 0$.

Now solve for y and hence for x.

Example. ***Solve : $4^{1+x} + 4^{1-x} = 10$ for x.***

Sol. $4^{1+x} + 4^{1-x} = 10 \quad \Rightarrow 4^1.4^x + 4^1.4^{-x} = 10 \quad \Rightarrow 4.4^x + \frac{4}{4^x} = 10$

$\Rightarrow 4.4^{2x} + 4 = 10 \times 4^x \quad \Rightarrow 4.4^{2x} - 10.4^x + 4 = 0.$

Let $4^x = y$. Then, the given equation becomes $4y^2 - 10y + 4 = 0$

$\Rightarrow 2y^2 - 5y + 2 = 0 \quad \Rightarrow (y-2)(2y-1) = 0 \quad \Rightarrow y = 2$ or $\frac{1}{2}$.

$\Rightarrow 4^x = 2$ or, $4^x = +\frac{1}{2} \quad \Rightarrow 2^{2x} = 2^1$ or $2^{2x} = 2^{-1} \quad \Rightarrow 2x = 1$ or $2x = -1 \Rightarrow x = \mathbf{\frac{1}{2}}$ **or** $\mathbf{\frac{-1}{2}}$.

Type II : $az + \frac{b}{z} = c$, *where a, b, c are constants.*

Example. ***If $\sqrt{\frac{2x^2+x+2}{x^2+3x+1}} + 2\sqrt{\frac{x^2+3x+1}{2x^2+x+2}} - 3 = 0$, find x.***

Sol. Let $\sqrt{\frac{2x^2+x+2}{x^2+3x+1}} = y$. Then, $y + 2 \times \frac{1}{y} - 3 = 0$

$\Rightarrow \quad y^2 - 3y + 2 = 0 \quad \Rightarrow (y-1)(y-2) = 0 \Rightarrow y = 1$ or 2

$\therefore \quad y = 1 \Rightarrow \sqrt{\frac{2x^2+x+2}{x^2+3x+1}} = 1 \quad \Rightarrow \frac{2x^2+x+2}{x^2+3x+1} = 1 \quad \Rightarrow \quad 2x^2 + x + 2 = x^2 + 3x + 1$

$\Rightarrow \quad x^2 - 2x + 1 = 0 \quad \Rightarrow (x-1)^2 = 0 \Rightarrow x = 1.$

$y = 2 \Rightarrow \sqrt{\frac{2x^2+x+2}{x^2+3x+1}} = 2 \quad \Rightarrow \frac{2x^2+x+2}{x^2+3x+1} = 4$

$\Rightarrow 2x^2 + x + 2 = 4x^2 + 12x + 4 \quad \Rightarrow 2x^2 + 11x + 2 = 0$

$\therefore x = \frac{-11 \pm \sqrt{121 - 4\times2\times2}}{2\times2} = \frac{-11 \pm \sqrt{121-16}}{4} = \frac{-11 \pm \sqrt{105}}{4}. \quad \therefore \ \mathbf{x = 1, \frac{-11 \pm \sqrt{105}}{4}}.$

Type III : *Equations of type (x + a) (x + b) (x + c) (x + d) + k = 0, where the sum of two of the quantities a, b, c, d is equal to the sum of the other two.*

Example. ***(x + 1) (x + 2) (x + 3) (x + 4) + 1 = 0***

Sol. $[(x+1)(x+4)][(x+2)(x+3)] + 1 = 0 \qquad (\because 1 + 4 = 2 + 3 = 5)$

$\Rightarrow (x^2 + 5x + 4)(x^2 + 5x + 6) + 1 = 0$

Let $x^2 + 5x = y$. Then, $(y+4)(y+6) + 1 = 0$

$\Rightarrow y^2 + 10y + 24 + 1 = 0 \Rightarrow y^2 + 10y + 25 = 0 \quad \Rightarrow (y+5)^2 = 0 \Rightarrow y = -5$

$\therefore x^2 + 5x = -5 \Rightarrow x^2 + 5x + 5 = 0. \quad \Rightarrow x = \frac{-5 \pm \sqrt{25-20}}{2} = \mathbf{\frac{-5 \pm \sqrt{5}}{2}}.$

4. Important Properties of Inequalities

1. *An inequality will still hold after each side has been increased, diminished, multiplied or divided by the same **positive** quantity.*
2. *In an inequality any term may be transposed from one side to the other if its sign is changed.*
3. *Both the sides of an inequality can be multiplied or divided by the same **negative** number **by reversing the sign of inequality.***

5. Nature of Roots. A quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, has two roots which by the quadratic formula are as under :

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}$$

The expression $\boldsymbol{b^2 - 4ac}$ is called the **discriminant.**

Examining the nature of the roots means to see what type of roots the equation has, that is, whether they are **real or imaginary, real or irrational, equal or unequal**. The nature of the roots depends entirely on the value of the discriminant $\boldsymbol{D = b^2 - 4ac}$

Thus, if a, b, c are rational, then

I. If $\boldsymbol{D = b^2 - 4ac > 0}$ (*i.e*, *positive*), the roots are **real** and **unequal.**

Also,

(a) If $\boldsymbol{D = b^2 - 4ac}$ is a **perfect square,** the roots are **rational.**

(b) If $\boldsymbol{D = b^2 - 4ac}$ is not a **perfect square,** the roots are **irrational.**

(c) If $\boldsymbol{D = b^2 - 4ac = 0}$**,** the roots are **equal,** each being equal to $\boldsymbol{\frac{-b}{2a}}$.

So, $\boldsymbol{ax^2 + bx + c = 0}$ **is a perfect square if** $\boldsymbol{D = 0}$**.**

II. If $\boldsymbol{D = b^2 - 4ac < 0}$ **(***i.e.*, **negative),** the roots are **imaginary** (*complex*).

Example. ***Examine the nature of the roots of the equations:***

(i) $\boldsymbol{2x^2 + 2x + 3 = 0}$ ***(ii)*** $\boldsymbol{2x^2 - 7x + 3 = 0}$

(iii) $\boldsymbol{x^2 - 5x - 2 = 0}$ ***(iv)*** $\boldsymbol{4x^2 - 4x + 1 = 0.}$

Sol. (i) $2x^2 + 2x + 3 = 0$ (Here, $a = 2, b = 2, c = 3$)

$\therefore$ $D = b^2 - 4ac = (2)^2 - 4 \times 2 \times 3 = 4 - 24 = \mathbf{-20 < 0}$

Hence, roots are **imaginary.**

(ii) $2x^2 - 7x + 3 = 0$ (Here, $a = 2, b = -7, c = 3$)

$\therefore$ $D = b^2 - 4ac = 49 - 24 = \mathbf{25 > 0}$ and **a perfect square**

Hence, roots are **real and rational.**

(iii) $x^2 - 5x - 2 = 0.$ (Here, $a = 1, b = -5, c = -2$)

$\therefore$ $D = b^2 - 4ac = 25 + 8 = \mathbf{33 > 0}$ and **not a perfect square**

Hence, roots are **real and irrational.**

(iv) $4x^2 - 4x + 1 = 0$ (Here, $a = 4, b = -4, c = 1$)

$\therefore$ $D = b^2 - 4ac = 16 - 16 = \mathbf{0.}$

Hence, roots are **real and equal.**

6. Sum and Product of Roots:

If the two roots of the quadratic equation $ax^2 + bx + c = 0$ obtained by the quadratic formula be denoted by α and β, then we have

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}, \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\therefore \text{ Sum of roots} = \alpha + \beta = -\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-2b}{2a} = \boldsymbol{\frac{-b}{a}}$$

$$\text{Product of roots} = \alpha\beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) \times \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b)^2 - (\sqrt{b^2-4ac})^2}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \boldsymbol{\frac{c}{a}}.$$

Thus, **Sum of roots** $= \dfrac{-\text{Coeff. of } x}{\text{Coeff. of } x^2}$ **; Product of roots** $= \dfrac{\text{Constant term}}{\text{Coeff. of } x^2}$

Thus, if α, β be the roots of the equation $6x^2 - 5x + 7 = 0$, then $\alpha + \beta = -(-5/6) = 5/6$, $\alpha\beta = 7/6$.

7. Values of the Symmetric Functions of the Roots

If α, β be the roots of a given quadratic equation and we wish to find the value of a symmetric function of α and β, **we can do so by proceeding as follows :**

Method:

Step I. *Write the values of* $\alpha + \beta$ *and* $\alpha\beta$ *from the given equation.*

Step II. *Express the given function in terms of* $\alpha + \beta$ *and* $\alpha\beta$.

Step III. *Substitute the values of* $\alpha + \beta$ *and* $\alpha\beta$ *from **step 1.***

Caution:Do not find the values of α and β separately.

The following algebraic relations can be very useful :

1. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
2. $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$
3. $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = (\alpha + \beta)\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$
4. $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) = (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta] = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
5. $\alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2) = (\alpha - \beta)[(\alpha - \beta)^2 + 3\alpha\beta] = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$
6. $\alpha^4 - \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$
7. $\alpha^4 - \beta^4 = (\alpha^2 - \beta^2)(\alpha^2 + \beta^2) = (\alpha - \beta)(\alpha + \beta)(\alpha^2 + \beta^2) = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}\,(\alpha+\beta)[(\alpha+\beta)^2 - 2\alpha\beta]$

8. Formation of Equations with given Roots :

Suppose we have to form the equation whose roots are α and β. Then, as $x = \alpha$, $x = \beta$ are the roots of the equation, so $(x - \alpha) = 0$ and $(x - \beta) = 0$

$\therefore (x - \alpha)(x - \beta) = 0$

$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow$ **x^2 – (Sum of roots) x + Product of roots = 0.**

Thus, the equation whose roots are 5 and 7 is $x^2 - (5 + 7)x + 5 \times 7 = 0 \quad \Rightarrow \; x^2 - 12x + 35 = 0$.

9. To find the condition when a relation between the two roots is given

Step I. *Let one root be* α. *Write the other root using the given relation.*

Step II. *Write the sum and product of the roots.*

Step III. *Eliminate* $\boldsymbol{\alpha}$ *from the two relations obtained in Step II.*

Ex. ***Find the condition that one root of $ax^2 + bx + c = 0$ may be four times the other.***

Sol. Let the roots be α and 4α. Then,

$$\alpha + 4\alpha = 5\alpha = -\frac{b}{a} \qquad \ldots(i)$$

$$\alpha \,.\, 4\alpha = 4\alpha^2 = \frac{c}{a} \qquad \ldots(ii)$$

From (i) $\alpha = -\dfrac{b}{5a}$

$\therefore$ From (ii) $4.\left(-\dfrac{b}{5a}\right)^2 = \dfrac{c}{a} \Rightarrow \dfrac{4b^2}{25a^2} = \dfrac{c}{a} \Rightarrow$ $\mathbf{4b^2 = 25\,ac.}$

10. Special Roots:
For a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, $a, b, c \in R$ whose roots are α and β.

(*a*) **Reciprocal roots**

If $\alpha = \dfrac{1}{\beta}$, then $\alpha\beta = 1 \Rightarrow \alpha\beta = \dfrac{c}{a} = 1 \Rightarrow$ $\boldsymbol{c = a}$

Thus the roots of a quadratic equation will be reciprocal of each other if **coefficient of x^2 = constant term.**

(*b*) **Zero roots**

Case I : When one root is zero, say $\alpha = 0$.

Then, $\alpha\beta = 0 \Rightarrow \dfrac{c}{a} = 0, \Rightarrow$ $\boldsymbol{c = 0}$ as $a \neq 0$

Case II : When both roots are zero, *i.e*, $\alpha = 0, \beta = 0$

Then, $\alpha + \beta = 0$ and $\alpha\beta = 0$

$\Rightarrow \quad -\dfrac{b}{a} = 0$ and $\dfrac{c}{a} = 0 \quad \Rightarrow$ $\boldsymbol{b = 0}$ **and** $\boldsymbol{c = 0}$ as $a \neq 0$.

(*c*) **Infinite roots**

Let α, β be the roots of $ax^2 + bx + c = 0$ ...(*i*)

Then, the equation whose roots are $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ is $cx^2 + bx + a = 0$ ...(*ii*)

Now if one root of (*ii*) is zero then $a = 0 \Rightarrow$ the corresponding root of (*i*) is $\dfrac{1}{0} = \infty$.

If both the roots of (*ii*) are zero, then $a = 0, b = 0 \Rightarrow$ both the corresponding roots of (*i*) are infinitely large.

Thus, For **one root** to be **infinite, $\boldsymbol{a = 0}$;**

For **both roots** to be **infinite, $\boldsymbol{a = 0, b = 0}$**

11. Signs of the Roots

(*a*) Positive roots : Both the roots will be positive if $\alpha + \beta$ and $\alpha\beta$ are both positive, *i.e*, $-\dfrac{b}{a}$ and $\dfrac{c}{a}$ are positive. It will be so when *b* and *a* are of opposite signs and *c* and *a* are of the same sign.

(*b*) Negative roots : Both the roots will be negative if $\alpha + \beta$ is negative and $\alpha\beta$ is positive, *i.e.*, $-\dfrac{b}{a}$ is negative and $\dfrac{c}{a}$ is positive, *i.e.*, when ***a*, *b* and *c* all have the same sign.**

(*c*) Roots of opposite signs. It will occur when $\alpha\beta$ is negative, *i.e.*, ***c* and *a* are of opposite signs.**

(*d*) Roots equal in magnitude but opposite in sign. It will occur if $\alpha + \beta = 0$, *i.e.*, $-\dfrac{b}{a} = 0$, *i.e.* $b = 0$.

For real solutions the signs of c and a should be opposite.

12. Common Roots

1. To find the condition that two quadratic equations may have one common root.

Let the two quadratic equations be $ax^2 + bx + c = 0$, $a_1 x^2 + b_1 x + c_1 = 0$ and let α be their common root. Then,

$$a\alpha^2 + b\alpha + c = 0 \qquad ...(i)$$

$$a_1\alpha^2 + b_1\alpha + c_1 = 0 \qquad ...(ii)$$

Solving them by the rule of cross-multiplication, we have, $\dfrac{\alpha^2}{bc_1 - cb_1} = \dfrac{\alpha}{ca_1 - ac_1} = \dfrac{1}{ab_1 - a_1b}$

$$\alpha^2 = \frac{bc_1 - cb_1}{ab_1 - a_1b}, \alpha = \frac{ca_1 - ca_1}{ab_1 - a_1b}$$

$$\Rightarrow \quad \frac{bc_1 - cb_1}{ab_1 - a_1b} = \frac{(ca_1 - ac_1)^2}{(ab_1 - a_1b)^2} \Rightarrow (bc_1 - cb_1)(ab_1 - a_1b) = (ca_1 - ac_1)^2$$

2. To find the condition that the two quadratic equations may have both the roots common.

Let the common roots of the equations $ax^2 + bx + c = 0$ and $a_1x^2 + b_1x + c_1 = 0$ be α and β. Then,

From first equation, $\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$

From second equation, $\alpha + \beta = -\frac{b_1}{a_1}, \alpha\beta = \frac{c_1}{a_1}$

So, $-\frac{b}{a} = -\frac{b_1}{a_1}$ and $\frac{c}{a} = +\frac{c_1}{a_1} \Rightarrow \boldsymbol{\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}}$.

13. Transformed Equations

If $f(x) = ax^2 + bx + c = 0$ is a quadratic equation, then the equation whose roots are the :

(*a*) **Reciprocals of the roots of $f(x) = 0$ is $f\left(\frac{1}{x}\right) = 0$** *i.e.*, $\frac{a}{x^2} + \frac{b}{x} + c = 0 \Rightarrow \boldsymbol{cx^2 + bx + a = 0}$.

(*b*) **Roots of $f(x) = 0$, each increased by a constant k is $f(x - k) = 0$,** *i.e.*, $\boldsymbol{a(x - k)^2 + b(x - k) + c = 0}$.

(*c*) **Roots of $f(x) = 0$, each decreased by a constant k is $f(x + k) = 0$,** *i.e.*, $\boldsymbol{a(x + k)^2 + b(x + k) + c = 0}$.

(*d*) **Roots of $f(x) = 0$ with signs changed** is $f(-x) = 0$, *i.e.* $a(-x)^2 + b(-x) + c = 0 \Rightarrow \boldsymbol{ax^2 - bx + c = 0}$.

(*e*) **Roots of $f(x) = 0$ with each multiplied by $k \neq 0$ is $f\left(\frac{x}{k}\right) = 0$** *i.e.* $a\left(\frac{x}{k}\right)^2 + b\left(\frac{x}{k}\right) + c = 0$ *i.e.*, $\boldsymbol{ax^2 + kbx + k^2c = 0}$

14. Relation between the roots of a cubic equation and its coefficients.

Let the cubic equation be $x^3 + S_1x^2 + S_2x + S_3 = 0$, where S_1, S_2, S_3 are the coefficients.

Let α, β, γ be the roots of the given cubic equation. Then,

$$\boldsymbol{S_1 = -(\alpha + \beta + \gamma), \quad S_2 = (\alpha\beta + \beta\gamma + \gamma\alpha), \quad S_3 = -(\alpha\beta\gamma)}$$

Conversely, if the roots of a cubic equation are given as $\alpha_1, \beta_1, \gamma$, then its equation can be written as :

$\boldsymbol{x^3 - S_1x^2 + S_2x - S_3 = 0}$, where

$S_1 = (\alpha + \beta + \gamma), S_2 = (\alpha\beta + \beta\gamma + \gamma\alpha)$ and $S_3 = \alpha\beta\gamma$.

15. Relation between the roots of a bi-quadratic equation (degree 4) and its coefficients.

A bi-quadratic equation, whose roots are α, β, γ and δ is

$$x^4 - S_1 x^3 + S_2 x^2 - S_2x + S_4 = 0$$

where $S_1 = \alpha + \beta + \gamma + \delta$, $S_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$, $S_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta_L$, $S_4 = \alpha\beta\gamma\delta$.

SOLVED EXAMPLES

Ex. 1. What are the roots of the equation $(a + b + x)^{-1} = a^{-1} + b^{-1} + x^{-1}$? ***(CDS 2007)***

Sol. Given, $\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x} \Rightarrow \frac{1}{a+b+x} - \frac{1}{x} = \frac{1}{a} + \frac{1}{b}$

$\Rightarrow \frac{x-(a+b+x)}{x(a+b+x)} = \frac{a+b}{ab} \quad \Rightarrow \frac{-(a+b)}{x(a+b+x)} = \frac{a+b}{ab} \quad \Rightarrow -ab = x^2 + (a+b)x$

$\Rightarrow x^2 + (a+b)x + ab = 0 \quad \Rightarrow (x+a)(x+b) = 0 \Rightarrow \boldsymbol{x = -a, -b}$.

Ex. 2. What is one of the roots of the equation $\sqrt{\frac{2x}{3-x}} - \sqrt{\frac{3-x}{2x}} = \frac{3}{2}$?

(*a*) 1 **(*b*) 2** **(*c*) 3** **(*d*) 4** ***(CDS 2008)***

Sol. Given equation is $\sqrt{\frac{2x}{3-x}} - \sqrt{\frac{3-x}{2x}} = \frac{3}{2}$

Let $\sqrt{\dfrac{2x}{3-x}} = a$. Then, the given equation reduces to $a - \dfrac{1}{a} = \dfrac{3}{2}$

$\Rightarrow\ 2(a^2 - 1) = 3a \quad \Rightarrow \quad 2a^2 - 3a - 2 = 0 \quad \Rightarrow \quad 2a^2 - 4a + a - 2 = 0$

$\Rightarrow\ 2a\,(a - 2) + 1\,(a - 2) = 0 \quad \Rightarrow \quad (2a + 1)\,(a - 2) = 0$

$\Rightarrow\ a - 2 = 0 \quad \Rightarrow\ a = 2$ or $(2a + 1) = 0 \ \Rightarrow\ a = -\dfrac{1}{2}$

$\therefore \sqrt{\dfrac{2x}{3-x}} = 2 \quad \Rightarrow\ 2x = 4(3 - x)$

$\Rightarrow 6x = 12 \ \Rightarrow\ x = 2$

or

$\sqrt{\dfrac{2x}{3-x}} = -\dfrac{1}{2} \ \Rightarrow\ \dfrac{2x}{3-x} = \dfrac{1}{4}$

$\Rightarrow 8x = 3 - x \quad \Rightarrow\ 9x = 3 \Rightarrow x = \dfrac{1}{3}.$

Hence, according to the given options, (*b*) is correct.

Ex. 3. If $3^x + 27\,(3^{-x}) = 12$, then what is the value of x ? ***(CDS 2009)***

Sol. Given, $3^x + 27\,(3^{-x}) = 12$

Let $3^x = y$. Then, $y + \dfrac{27}{y} = 12 \quad \Rightarrow\ y^2 - 12y + 27 = 0$

$\Rightarrow y^2 - 9y - 3y + 27 = 0 \quad \Rightarrow\ (y - 3)\,(y - 9) = 0 \quad \Rightarrow\ y = 3, 9$

$\Rightarrow 3^x = 3$ or $3^x = 9 \quad \Rightarrow\ x =$ **1 or 2.**

Ex. 4. What is the ratio of sum of squares of roots to the product of the roots of the equation $7x^2 + 12x + 18 = 0$? ***(CDS 2009)***

Sol. Let α, β be the roots of the equation $7x^2 + 12x + 18 = 0$.

$\left[\text{For a quadratic equation } ax^2 + bx + c = 0, \text{ sum of roots} = -\dfrac{b}{a}, \text{ product of roots} = +\dfrac{c}{a}\right]$

$\therefore\ \alpha + \beta = -\dfrac{12}{7}$ and $\alpha\beta = \dfrac{18}{7}$

$\Rightarrow (\alpha + \beta)^2 = \left(\dfrac{-12}{7}\right)^2 \Rightarrow \alpha^2 + \beta^2 + 2\alpha\beta = \dfrac{144}{49}$

$\Rightarrow\ \alpha^2 + \beta^2 = \dfrac{144}{49} - \dfrac{36}{7} = \dfrac{-108}{49}$

$\therefore$ Required ratio $= \alpha^2 + \beta^2 : \alpha\beta = \dfrac{\;\dfrac{-108}{49}\;}{18/7} = -\dfrac{6}{7} = -$ **6 : 7.**

Ex. 5. What is the value of a for which the equation $2x^2 + 2\sqrt{6}\,x + a = 0$ has equal roots ? ***(Kerala PET 2010)***

Sol. The equation $2x^2 + 2\sqrt{6}x + a = 0$ has equal roots if the discriminant $D = 0$.

$\therefore$ Here, $D = (2\sqrt{6})^2 - 4 \times (2) \times (a) = 0$ $\qquad [D = b^2 - 4ac$ for $ax^2 + bx + c = 0]$

$\Rightarrow 24 - 8a = 0 \Rightarrow a =$ **3.**

Ex. 6. Of the following quadratic equations, which is the one whose roots are 2 and – 15 ?

(*a*) $x^2 - 2x + 15 = 0$ (*b*) $x^2 + 15x - 2 = 0$ (*c*) $x^2 + 13x - 30 = 0$ (*d*) $x^2 - 30 = 0$. ***(MAT)***

Sol. Sum of roots = 2 + (– 15) = – 13; Product of roots = 2 × (– 15) = – 30.

$\therefore$ Required equation is x^2 – (sum of roots) x + product of roots = 0

$\Rightarrow$ Reqd. equation = $\mathbf{x^2 - 13x - 30 = 0.}$

Ex. 7. If one of the roots of the equation $x^2 + ax + 3 = 0$ is 3 and one of the roots of the equation $x^2 + ax + b = 0$ is three times the other root, then what is the value of b ? ***(J&K CET 2005)***

Sol. Let 3 and α be the roots of the equation $x^2 + ax + 3 = 0$

Then, sum of roots $= 3 + \alpha = -a$ ...(*i*), Product of roots $= 3\alpha = 3$ ...(*ii*)

From (*ii*) $\alpha = 1$. $\therefore$ Substituting $\alpha = 1$ in (*i*), we get $a = -4$.

$\therefore$ The second equation $x^2 + ax + b = 0$ becomes $x^2 - 4x + b = 0$.

Let β and 3β be the roots of this equation. Then, sum of roots $= \beta + 3\beta = 4 \Rightarrow 4\beta = 4 \Rightarrow \beta = 1$

and Product of roots $= \beta \times 3\beta = b \Rightarrow 3\beta^2 = b \Rightarrow \boldsymbol{b = 3}$.

Ex. 8. If α, β be the two roots of the equation $x^2 + x + 1 = 0$, then the equation whose roots are α/β and β/α is ?

(*a*) $x^2 - x - 1 = 0$ (*b*) $x^2 - x + 1 = 0$ (*c*) $x^2 + x - 1 = 0$ (*d*) $x^2 + x + 1 = 0$

(UPSEE 2005)

Sol. Let α, β be the roots of the equations $x^2 + x + 1 = 0$. Then,

Sum of roots $= \alpha + \beta = -1$, Product of roots $= \alpha\beta = 1$

Now the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is

$$x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)x + \left(\frac{\alpha}{\beta} \times \frac{\beta}{\alpha}\right) = 0.$$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{(-1)^2 - 2(1)}{1} = -1 \text{ and } \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1.$$

$\therefore$ Required equation $\boldsymbol{= x^2 + x + 1 = 0}$.

Ex. 9. If the difference in the roots of the equation $x^2 - px + q = 0$ is unity, then which one of the following is correct ?

(*a*) $p^2 + 4q = 1$ (*b*) $p^2 - 4q = 1$ (*c*) $p^2 + 4q = -1$ (*d*) $p^2 - 4q = -1$.

(CDS 2005)

Sol. Given, $x^2 - px + q = 0$

Let α, β be the roots of the given equation. Then,

$$\alpha + \beta = -\frac{(-p)}{1} = p \quad ...(i), \qquad \alpha\beta = \frac{q}{1} = q \quad ...(ii)$$

Also, $\alpha - \beta = 1$ (given) ...(*iii*)

$\therefore$ From (*i*) and (*iii*), $2\alpha = p + 1 \Rightarrow \alpha = \frac{p+1}{2}$

$\therefore$ From (*i*) and (*iii*), $2\beta = p - 1 \Rightarrow \beta = \frac{p-1}{2}$

Substituting these values of α and β in (*ii*), we have $\left(\frac{p+1}{2}\right)\left(\frac{p-1}{2}\right) = q$

$\Rightarrow \frac{p^2 - 1}{4} = q \Rightarrow p^2 - 1 = 4q \Rightarrow \boldsymbol{p^2 - 4q = 1}$.

Ex. 10. If the roots of the equation $x^2 + x + 1 = 0$ are in the ratio of $m : n$, then which one of the following relation holds ?

(*a*) $m + n + 1 = 0$ (*b*) $\frac{m}{n} + \frac{n}{m} + 1 = 0$ (*c*) $\sqrt{m} + \sqrt{n} + 1 = 0$ (*d*) $\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}} + 1 = 0$.

(CDS 2005)

Sol. $x^2 + x + 1 = 0$

$\therefore$ Roots are $= \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{3}\,i}{2}$ (where $= i = \sqrt{-1}$) $\left[\because \text{Roots} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$.

Given, $\dfrac{\dfrac{-1+\sqrt{3}\,i}{2}}{\dfrac{-1-\sqrt{3}\,i}{2}} = \dfrac{m}{n} \Rightarrow \dfrac{-1+\sqrt{3}\,i}{-1-\sqrt{3}\,i} = \dfrac{m}{n}$

$\Rightarrow \dfrac{m+n}{m-n} = \dfrac{(-1+\sqrt{3}\,i)+(-1-\sqrt{3}\,i)}{(-1+\sqrt{3}\,i)-(-1-\sqrt{3}\,i)} = \dfrac{-2}{2\sqrt{3}\,i}$ (Applying componendo and dividendo)

$\Rightarrow \dfrac{m+n}{m-n} = \dfrac{-1}{\sqrt{3}\,i} = \dfrac{i^2}{\sqrt{3}\,i} = \dfrac{i}{\sqrt{3}}$ $(\because i^2 = -1)$

$\Rightarrow \left(\dfrac{m+n}{m-n}\right)^2 = \left(\dfrac{i}{\sqrt{3}}\right)^2 \Rightarrow \dfrac{m^2+n^2+2mn}{m^2+n^2-2mn} = \dfrac{-1}{3} \Rightarrow \dfrac{(m^2+n^2+2mn)+(m^2+n^2-2mn)}{(m^2+n^2+2mn)-(m^2+n^2-2mn)} = \dfrac{-1+3}{-1-3}$

$\Rightarrow \dfrac{2(m^2+n^2)}{2\,(2mn)} = \dfrac{2}{-4} \Rightarrow \dfrac{m^2+n^2}{2mn} = \dfrac{1}{-2} \Rightarrow \dfrac{m^2+n^2}{mn} = -1 \Rightarrow \boldsymbol{\dfrac{m}{n} + \dfrac{n}{m} + 1 = 0.}$

Ex. 11. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then which one of the following is correct ?

(*a*) $a < 2$ (*b*) $2 < a < 3$ (*c*) $3 < a < 4$ (*d*) $a > 4$ *(CDS 2012)*

Sol. If the roots of the equation $x^2 - 2ax + a^2 - a - 3 = 0$ are real and less than 3, then $D \geq 0$ and $f(3) > 0$.

$\Rightarrow 4a^2 - 4(a^2 + a - 3) \geq 0$ and $(3)^2 - 2a\,(3) + a^2 + a - 3 > 0$

$\Rightarrow a^2 - a^2 - a + 3 \geq 0$ and $9 - 6a + a^2 + a - 3 > 0$

$\Rightarrow -a + 3 \geq 0$ and $a^2 - 5a + 6 > 0 \Rightarrow a - 3 \leq 0$ and $(a-2)(a-3) > 0$

$\Rightarrow a \leq 3$ and $a < 2$ or $a > 3 \Rightarrow \boldsymbol{a < 2.}$

Ex. 12. What are the number of solutions for real *x*, which satisfy the equation

$$\mathbf{2\log_2 \log_2 x + \log_{1/2} \log_2 (2\sqrt{2}x) = 1?}$$

Sol. $2\log_2 \log_2 x + \log_{1/2} \log_2 (2\sqrt{2}x) = 1$

$\Rightarrow 2\log_2 \log_2 x - \log_2 \log_2 (2\sqrt{2}x) = 1$ $(\because \log_{1/a} x = -\log_a x)$

$\Rightarrow \log_2 (\log_2 x)^2 - \log_2 (\log_2 (2\sqrt{2}x)) = \log_2 2$

$\Rightarrow \log_2 \dfrac{(\log_2 x)^2}{\log_2 (2\sqrt{2}x)} = \log_2 2 \Rightarrow \dfrac{(\log_2 x)^2}{\log_2 (2\sqrt{2}x)} = 2 \Rightarrow (\log_2 x)^2 = 2\log_2 (2\sqrt{2}x)$

$\Rightarrow (\log_2 x)^2 = 2\log_2 (2^{3/2}x) \Rightarrow (\log_2 x)^2 = 2\left[\dfrac{3}{2}\log 2x\right] = 2[3/2\ \{\log_2 2 + \log_2 x\}]$

$\Rightarrow (\log_2 x)^2 = 3 + 2\log_2 x \Rightarrow (\log_2 x)^2 - 2\log_2 x - 3 = 0$

$\Rightarrow (\log_2 x - 3)(\log_2 x + 1) = 0 \Rightarrow \log_2 x = 3$ or $\log_2 x = -1 \Rightarrow x = 2^3 = 8$ or $x = 2^{-1} = \dfrac{1}{2}$

But for $x = \dfrac{1}{2}$, $\log_2 \log_2 \left(\dfrac{1}{2}\right)$ is undefined so $\boldsymbol{x = 8}$ is the only possible value of x.

Ex. 13. If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$, then what is $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ is equal to ?

Sol. Given α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$. Then,

$$S_1 = \alpha + \beta + \gamma = -\frac{\text{coefficient of } x^2}{\text{coefficient of } x^3} = -a; \quad S_2 = \alpha\beta + \beta\gamma + \alpha\gamma = \frac{\text{coefficient of } x}{\text{coefficient of } x^3} = b$$

$$S_3 = \alpha\beta\gamma = \frac{-\text{constant term}}{\text{coefficient of } x^3} = -c$$

$$\therefore \alpha^{-1} + \beta^{-1} + \gamma^{-1} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{S_2}{S_3} = \frac{b}{-c} = -\boldsymbol{\frac{b}{c}}.$$

Ex. 14. If α, β are the roots of the equation $9x^2 + 6x + 1 = 0$, then write the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$.

Sol. As α, β are the roots of the equation $9x^2 + 6x + 1 = 0$, $\alpha + \beta = -\frac{6}{9} = -\frac{2}{3}$, $\alpha\beta = \frac{1}{9}$

$\therefore$ Required equation $= x^2 -$ (Sum of roots)x + Product of roots $= 0$

i.e., $x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \frac{1}{\alpha}\cdot\frac{1}{\beta} = 0$, *i.e.*, $x^2 - \left(\frac{\beta + \alpha}{\gamma\beta}\right)x + \frac{1}{\alpha\beta} = 0$

i.e., $x^2 - \left(\frac{-2/3}{1/9}\right)x + 9 = 0$, *i.e.*, $\mathbf{x^2 + 6x + 9 = 0}$

Alternatively,

The equation whose roots $\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$ are the reciprocals of the roots (α, β) of the equation $9x^2 + 6x + 1 = 0$ can be obtained by replacing x by $\frac{1}{x}$ in the given equation.

$\therefore$ Required equation is : $9\left(\frac{1}{x}\right)^2 + 6\left(\frac{1}{x}\right) + 1 = 0 \Rightarrow \mathbf{x^2 + 6x + 9 = 0.}$

Ex. 15. Find the values of k for which the equations $x^2 - kx - 21 = 0$ and $x^2 - 3kx + 35 = 0$ will have a common root?

Sol. Let α be the common root of both the given equations. Then α satisfies both the equations. So,

$$\alpha^2 - k\alpha - 21 = 0 \quad \text{...}(i)$$

$$\alpha^2 - 3k\alpha + 35 = 0 \quad \text{...}(ii)$$

Solving equations (*i*) and (*ii*) simultaneously, we get

$$\frac{\alpha^2}{-35k - 63k} = \frac{\alpha}{-21 - 35} = \frac{1}{-3k + k}$$

$$\left[\begin{array}{c} a_1x + b_1y + c_1 = 0;\ a_2x + b_2y + c_2 = 0 \\ \dfrac{x}{b_1c_2 - b_2c_1} = \dfrac{y}{c_1a_2 - c_2a_1} = \dfrac{1}{a_1b_2 - a_2b_1} \end{array}\right]$$

$$\Rightarrow \alpha^2 = \frac{-98k}{-2k} = 49 \text{ and } \alpha = \frac{-56}{-2k} = \frac{28}{k}$$

$$\therefore 49 = \left(\frac{28}{k}\right)^2 \Rightarrow k^2 = \frac{28 \times 28}{49} = 16 \Rightarrow k = \pm\mathbf{4.}$$

PRACTICE SHEET

1. What are the roots of the equation $\log_{10}(x^2 - 6x + 45) = 2$?

(*a*) 9, –5 (*b*) –9, 5 (*c*) 11, – 5 (*d*) –11, 5

(*CDS 2010*)

2. What is one of the values of x in the equation $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$

(*a*) $\frac{5}{13}$ (*b*) $\frac{7}{13}$ (*c*) $\frac{9}{13}$ (*d*) $\frac{11}{3}$

(*CDS 2007*)

3. What are the roots of the equation $4^x - 3.2^{x+2} + 32 = 0$?

(*a*) 1, 2 (*b*) 3, 4 (*c*) 2, 3 (*d*) 1, 3

(*CDS 2010*)

4. What are the roots of the quadratic equation $a^2 b^2 x^2 - (a^2 + b^2)x + 1 = 0$?

(*a*) $\frac{1}{a^2}, \frac{1}{b^2}$ (*b*) $-\frac{1}{a^2}, -\frac{1}{b^2}$

(*c*) $\frac{1}{a^2}, -\frac{1}{b^2}$ (*d*) $-\frac{1}{a^2}, \frac{1}{b^2}$ (***CDS 2011***)

5. If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ for $a \neq 0$ are real and equal, then the value of $a^3 + b^3 + c^3$ is :

(*a*) *abc* (*b*) 3*abc*

(*c*) 0 (*d*) None of these

(***MAT 2003***)

6. If sin θ and cos θ are the roots of the equations $ax^2 - bx + c = 0$, then which of the following is correct?

(*a*) $a^2 + b^2 + 2ac = 0$ (*b*) $a^2 - b^2 + 2ac = 0$

(*c*) $a^2 + b^2 + 2ab = 0$ (*d*) $a^2 - b^2 - 2ac = 0$.

(***CDS 2011***)

7. The roots of the quadratic equation $x^2 - 2\sqrt{3}x - 22 = 0$ are:

(*a*) imaginary (*b*) real, rational, equal

(*c*) real, rational, unequal (*d*) real, irrational, unequal

(***WBJEE 2010***)

8. Which one of the following is the equation whose roots are respectively three times the roots of the equation $ax^2 + bx + c = 0$?

(*a*) $ax^2 + 3bx + c = 0$ (*b*) $ax^2 + 3bx + 9c = 0$

(*c*) $ax^2 - 3bx + 9c = 0$ (*d*) $ax^2 + bx + 3c = 0$

(***CDS 2007***)

9. If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\alpha\beta^2 + \alpha^2\beta + \alpha\beta$ equals :

(*a*) $\frac{bc}{-a^2}$ (*b*) 0 (*c*) *abc* (*d*) $\frac{c(a-b)}{a^2}$

(***AMU 2000***)

10. For what value of *m* the ratio of the roots of the equation $12x^2 - mx + 5 = 0$ is 3 : 2 ?

(*a*) $5\sqrt{10}$ (*b*) $10\sqrt{5}$ (*c*) $25\sqrt{2}$ (*d*) $15\sqrt{5}$

(***Rajasthan PET 2002***)

11. If the roots of the equation $ax^2 + bx + c = 0$ are equal in magnitude but opposite in sign, then which one of the following is correct ?

(*a*) $a = 0$ (*b*) $b = 0$

(*c*) $c = 0$ (*d*) $b = 0, c \neq 0, a \neq 0$.

(***CDS 2005***)

12. If $2x^2 - 7xy + 3y^2 = 0$, then the value of $x : y$ is

(*a*) 3 : 2 (*b*) 2 : 3

(*c*) 3 : 1 and 1 : 2 (*d*) 5 : 6

(***MAT 2003***)

13. If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, such that $\beta = \alpha^{1/3}$, then

(*a*) $(a^3 b)^{1/4} + (ac^3)^{1/4} + a = 0$

(*b*) $(a^3 c)^{1/4} + (ac^3)^{1/4} + b = 0$

(*c*) $(a^3 b)^{1/4} + (ab^3)^{1/4} + c = 0$

(*d*) $(b^3 c)^{1/4} + (bc^3)^{1/4} + a = 0$ (***Kerala PET 2003***)

14. If the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, then *a, b, c* are in :

(*a*) AP (*b*) GP

(*c*) HP (*d*) None of these

15. If an integer *P* is chosen at random in the interval $0 \leq p \leq 5$, the probability that the roots of the equation $x^2 + px + \frac{p}{4} + \frac{1}{2} = 0$ are real is

(*a*) $\frac{2}{3}$ (*b*) $\frac{2}{5}$ (*c*) $\frac{3}{5}$ (*d*) $\frac{4}{5}$

16. Two students *A* and *B* solve an equation of the form $x^2 + px + q = 0$. *A* starts with a wrong value of *p* and obtains the roots as 2 and 6. *B* starts with a wrong value of *q* and gets the roots as 2 and –9. What are the correct roots of the equations ?

(*a*) 3 and –4 (*b*) –3 and –4 (*c*) –3 and 4 (*d*) 3 and 4

(***CDS 2012***)

17. If α and β are the roots of the equation $x^2 - 6x + 6 = 0$, what is $\alpha^3 + \beta^3 + \alpha^2 + \beta^2 + \alpha + \beta$ equal to ?

(*a*) 150 (*b*) 138 (*c*) 128 (*d*) 124

(***CDS 2011***)

18. If α and β are the roots of the equation $x^2 + px + q = 0$, then $-\alpha^{-1}$ and $-\beta^{-1}$ are the roots of which one of the following equations ?

(*a*) $qx^2 - px + 1 = 0$ (*b*) $q^2 + px + 1 = 0$

(*c*) $x^2 + px - q = 0$ (*d*) $x^2 - px + q = 0$

(***CDS 2010***)

19. The number of solution of $\log_4 (x - 1) = \log_2 (x - 3)$ is :

(*a*) 0 (*b*) 5 (*c*) 2 (*d*) 3

(***AMU 2007***)

20. The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has

(*a*) no real roots (*b*) exactly one real root

(*c*) exactly four real roots (*d*) infinite number of real roots.

(***AIEEE 2012***)

21. If $5^{56}\left(\frac{1}{5}\right)^{x}\left(\frac{1}{5}\right)^{\sqrt{x}} > 1$, then *x* satisfies :

(*a*) [0, 49) (*b*) (49, 64] (*c*) [0, 64) (*d*) [49, 64)

(***DCE 2007***)

22. The sum of the roots of the equation $\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$ is zero. What is the product of the roots of the equation ?

(*a*) $-\frac{(a+b)}{2}$ (*b*) $\frac{(a+b)}{2}$

(*c*) $-\frac{(a^2+b^2)}{2}$ (*d*) $\frac{(a^2+b^2)}{2}$ (***CDS 2010***)

23. For what value of k will the roots of the equation $kx^2 - 5x + 6 = 0$ be in the ratio 2 : 3 ?

(*a*) 0 (*b*) 1 (*c*) –1 (*d*) 2

(***CDS 2010***)

24. The number of real solutions of the equation $2|x|^2 - 5|x| + 2 = 0$ is :

(*a*) 0 (*b*) 4
(*c*) 2 (*d*) None of these

25. If p, q, r are positive and are in A.P., the roots of quadratic equation $px^2 + qx + r = 0$ are real for :

(*a*) $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$ (*b*) $\left|\frac{p}{r} - 7\right| \geq 4\sqrt{3}$
(*c*) all p and r (*d*) no p and r

26. The values of x which satisfy the expression $(5 + 2\sqrt{6})^{x^2+3} + (5 - 2\sqrt{6})^{x^2-3} = 10$ are :

(*a*) $\pm 2, \pm\sqrt{3}$ (*b*) $\pm\sqrt{2}, \pm 4$ (*c*) $\pm 2, \pm\sqrt{2}$ (*d*) $2, \sqrt{2}, \sqrt{3}$

27. If α, β, γ are the roots of the equation $2x^3 - 3x^2 + 6x + 1 = 0$, then $\alpha^2 + \beta^2 + \gamma^2$ is equal to

(*a*) $\frac{-15}{4}$ (*b*) $\frac{-9}{4}$ (*c*) $\frac{13}{4}$ (*d*) 4

(***KCET 2005***)

28. If α, β, γ are the roots of the equation $x^3 + 4x + 1 = 0$, then $(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1}$ is equal to

(*a*) 2 (*b*) 3 (*c*) 4 (*d*) 5

(***UPSEE 2003***)

29. If the roots of $x^3 - 12x^2 + 12x - 28 = 0$ are in A.P, their common difference is

(*a*) ± 3 (*b*) ± 2
(*c*) ± 1 (*d*) None of these

(***Rajasthan PET 2001***)

30. The quadratic equation whose roots are three times the roots of $3ax^2 + 3bx + c = 0$ is

(*a*) $ax^2 + bx + 3c = 0$ (*b*) $ax^2 + 3bx + c = 0$
(*c*) $ax^2 + 3bx + 3c = 0$ (*d*) $9ax^2 + 9bx + c = 0$.

(***WBJEE 2009***)

31. If α and β are the roots of the equation $ax^2 + bx + c = 0$ and if $px^2 + qx + r = 0$ has roots $\frac{1-\alpha}{\alpha}$ and $\frac{1-\beta}{\beta}$, then r equals:

(*a*) abc (*b*) $a + 2b$ (*c*) $a + b + c$ (*d*) $ab + bc + ca$.

32. The equation whose roots are the negatives of the roots of the equation $x^7 + 3x^5 + x^3 - x^2 + 7x + 2 = 0$ is :

(*a*) $x^7 + 3x^5 + x^3 - x^2 - 7x - 2 = 0$
(*b*) $x^7 + 3x^5 + x^3 - x^2 + 7x - 2 = 0$
(*c*) $x^7 + 3x^5 + x^3 + x^2 - 7x + 2 = 0$
(*d*) $x^7 + 3x^5 + x^3 + x^2 + 7x - 2 = 0$ (***EAMCET 2001***)

33. Given that α, γ are the roots of the equation $Ax^2 - 4x + 1 = 0$ and β, δ are the roots of the equation $Bx^2 - 6x + 1 = 0$, then the values of A and B respectively such that α, β, γ and δ are in H.P are :

(*a*) –5, 9 (*b*) 3/2, 5
(*c*) 3, 8 (*d*) None of these

34. Let α, β be the roots of the equation $(x - a)(x - b) = c, c \neq 0$, then the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are :

(*a*) *a, c* (*b*) *b, c* (*c*) *a, b* (*d*) $a + c, b + c$

35. If the roots of the equation $x^3 - ax^2 + bx - c = 0$ are three consecutive integers, then what is the smallest possible value of b ?

(*a*) $-\frac{1}{\sqrt{3}}$ (*b*) –1 (*c*) 0 (*d*) 1 (***CAT***)

36. If two equations $x^2 + a^2 = 1 - 2ax$ and $x^2 + b^2 = 1 - 2bx$ have only one common root, then

(*a*) $(a - b) = -1$ (*b*) $|a - b| = 1$
(*c*) $a - b = 1$ (*d*) $|a - b| = 2$ (***DCE 2004***)

37. If α, β, γ are the roots of the equation $x^3 - 3x + 11 = 0$, then the equation whose roots are $(\alpha + \beta), (\beta + \gamma), (\gamma + \alpha)$ is :

(*a*) $x^3 + 3x + 11 = 0$ (*b*) $x^3 - 3x + 11 = 0$
(*c*) $x^3 + 3x - 11 = 0$ (*d*) $x^3 - 3x - 11 = 0$

38. If α, β are the roots of $ax^2 + bx + c = 0$, and $\alpha + k, \beta + k$ are the roots of $px^2 + qx + r = 0$, then $k =$

(*a*) $-\frac{1}{2}(a/b - p/q)$ (*b*) $(a/b - p/q)$
(*c*) $\frac{1}{2}(b/a - q/p)$ (*d*) $(ab - pq)$

39. Find the value of $\sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots \infty}}}$

(*a*) –4 (*b*) 2 (*c*) 3 (*d*) 6

40. The roots of $(x - a)(x - a - 1) + (x - a - 1)(x - a - 2) + (x - a)(x - a - 2) = 0, a \in R$ are always :

(*a*) imaginary (*b*) real and distinct
(*c*) equal (*d*) rational and equal

ANSWERS

1. (*c*)	**2.** (*c*)	**3.** (*c*)	**4.** (*a*)	**5.** (*b*)	**6.** (*b*)	**7.** (*c*)	**8.** (*b*)	**9.** (*d*)	**10.** (*a*)
11. (*d*)	**12.** (*c*)	**13.** (*b*)	**14.** (*c*)	**15.** (*a*)	**16.** (*b*)	**17.** (*b*)	**18.** (*a*)	**19.** (*b*)	**20.** (*a*)
21. (*a*)	**22.** (*c*)	**23.** (*b*)	**24.** (*b*)	**25.** (*b*)	**26.** (*c*)	**27.** (*a*)	**28.** (*c*)	**29.** (*a*)	**30.** (*c*)
31. (*c*)	**32.** (*d*)	**33.** (*c*)	**34.** (*c*)	**35.** (*b*)	**36.** (*d*)	**37.** (*d*)	**38.** (*c*)	**39.** (*c*)	**40.** (*b*)

HINTS AND SOLUTIONS

1. Given, $\log_{10}(x^2 - 6x + 45) = 2 \Rightarrow x^2 - 6x + 45 = 10^2 = 100$
$\Rightarrow x^2 - 6x - 55 = 0 \Rightarrow x^2 - 11x + 5x - 55 = 0$
$\Rightarrow x(x - 11) + 5(x - 11) = 0 \Rightarrow (x + 5)(x - 11) = 0$
$\Rightarrow x = -$ **5 or 11.**

2. Let $\sqrt{\dfrac{x}{1-x}} = y$. Then, the given equation reduces to

$$y + \frac{1}{y} = \frac{13}{6} \Rightarrow 6(y^2 + 1) = 13y$$
$\Rightarrow 6y^2 - 13y + 6 = 0 \Rightarrow 6y^2 - 9y - 4y + 6 = 0$
$\Rightarrow 3y(2y - 3) - 2(2y - 3) = 0$
$\Rightarrow (3y - 2)(2y - 3) = 0 \quad \Rightarrow y = \dfrac{2}{3}$ and $\dfrac{3}{2}$

when $y = \dfrac{2}{3}, \sqrt{\dfrac{x}{1-x}} = \dfrac{2}{3} \Rightarrow \dfrac{x}{1-x} = \dfrac{4}{9}$

$\Rightarrow 9x = 4 - 4x \Rightarrow 13x = 4 \Rightarrow x = \mathbf{\dfrac{4}{13}}$

when $y = \dfrac{3}{2}, \sqrt{\dfrac{x}{1-x}} = \dfrac{3}{2} \Rightarrow \dfrac{x}{1-x} = \dfrac{9}{4}$

$\Rightarrow 4x = 9 - 9x \Rightarrow 13x = 9 \Rightarrow x = \mathbf{\dfrac{9}{13}}.$

3. $4^x - 3.2^{x+2} + 32 = 0$
$\Rightarrow 2^{2x} - 3.2^2.2^x + 32 = 0 \quad \Rightarrow 2^{2x} - 12.2^x + 32 = 0$
Let $2^x = a$. Then, $a^2 - 12a + 32 = 0$
$\Rightarrow (a - 8)(a - 4) = 0 \Rightarrow a = 8$ and 4
$\Rightarrow 2^x = 8$ and $2^x = 4 \quad \Rightarrow$ $\boldsymbol{x = 3}$ and $\boldsymbol{x = 2}$**.**

4. Let the roots of the equation $a^2b^2x^2 - (a^2 + b^2)x + 1 = 0$ be α and β. Then,

$$\alpha + \beta = \frac{a^2 + b^2}{a^2b^2} \quad ....(i), \quad \alpha\beta = \frac{1}{a^2b^2} \quad ...(ii)$$

$$\alpha - \beta = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$$
$$= \sqrt{\left(\frac{a^2 + b^2}{a^2b^2}\right)^2 - \frac{4}{a^2b^2}}$$
$$= \sqrt{\frac{a^4 + b^4 + 2a^2b^2 - 4a^2b^2}{(a^2b^2)^2}}$$
$$= \sqrt{\frac{(a^2 - b^2)^2}{(a^2b^2)^2}} = \frac{a^2 - b^2}{a^2b^2} \quad ...(iii)$$

$\therefore$ On solving (i) and (ii), we get $\alpha = \mathbf{\dfrac{1}{b^2}, \beta = \dfrac{1}{a^2}.}$

5. Given that the roots are real and equal,
$D = 0 \Rightarrow b^2 - 4ac = 0$ for $ax^2 + bx + c = 0$.
$\therefore [-2(a^2 - bc)]^2 - 4(c^2 - ab)(b^2 - ac) = 0$
$\Rightarrow 4[a^4 + b^2c^2 - 2a^2bc - c^2b^2 + ac^3 + ab^3 - a^2bc) = 0$
$\Rightarrow 4a(a^3 + b^3 + c^3 - 3abc) = 0$
$\Rightarrow \boldsymbol{a^3 + b^3 + c^3 = 3abc.}$

6. As $\sin\theta$ and $\cos\theta$ are the roots of the equation $ax^2 - bx + c = 0$.
$\therefore \sin\theta + \cos\theta = \dfrac{b}{a}$ and $\sin\theta\cos\theta = \dfrac{c}{a}$

$\Rightarrow (\sin\theta + \cos\theta)^2 = \dfrac{b^2}{a^2}$

$\Rightarrow \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = \dfrac{b^2}{a^2}$

$\Rightarrow 1 + \dfrac{2c}{a} = \dfrac{b^2}{a^2} \Rightarrow \dfrac{2c}{a} = \dfrac{b^2}{a^2} - 1 = \dfrac{b^2 - a^2}{a^2}$

$\Rightarrow 2ac = b^2 - a^2 \Rightarrow \boldsymbol{a^2 - b^2 + 2ac = 0.}$

7. Given equation is $x^2 - 2\sqrt{3}x - 22 = 0$.
Discriminant $= D = b^2 - 4ac$ (for $ax^2 + bx + c = 0$)
$$= (-2\sqrt{3})^2 - 4(-22) = 12 + 88 = 100$$
As $D > 0$ and is a perfect square, the roots are real, rational and unequal.

8. Let α, β be the roots of the equation $ax^2 + bx + c = 0$. Then,
$$\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}.$$
By the given condition, roots of the required equation are 3α and 3β.

$\therefore$ Sum of roots $= 3\alpha + 3\beta = 3(\alpha + \beta) = -\dfrac{3b}{a}$

Product of roots $= 3\alpha \,.\, 3\beta = 9\alpha\beta = \dfrac{9c}{a}$

$\therefore$ Required equation
$= x^2 -$ (sum of roots)x + (product of roots) $= 0$

$\Rightarrow x^2 - \left(\dfrac{-3b}{a}x\right) + \dfrac{9c}{a} = 0$

$\Rightarrow \boldsymbol{ax^2 + 3bx + 9c = 0.}$

9. Given, α, β are the roots of the equation $ax^2 + bx + c = 0$.
Then, $\alpha + \beta = -b/a$, $\alpha\beta = c/a$
Then, $\alpha\beta^2 + \alpha^2\beta + \alpha\beta = \alpha\beta(\alpha + \beta) + \alpha\beta = (c/a)(-b/a) + c/a$
$$= -\frac{bc}{a^2} + \frac{c}{a} = \frac{-bc + ac}{a^2} = \mathbf{\frac{c(a-b)}{a^2}.}$$

10. Given, the roots of the given equation $12x^2 - mx + 5 = 0$ are in the ratio $3 : 2$. Let the roots of the given equation be 3α and 2α. Then,

Sum of roots $= 3\alpha + 2\alpha = \dfrac{m}{12} \Rightarrow 5\alpha = \dfrac{m}{12}$ ...(i)

and $(3\alpha)(2\alpha) = \dfrac{5}{12} \Rightarrow 6\alpha^2 = \dfrac{5}{12} \Rightarrow \alpha^2 = \dfrac{5}{72}$

$\Rightarrow \alpha = \sqrt{\dfrac{5}{72}}$ ...(ii)

$\therefore$ From (i) and (ii)

$$5.\sqrt{\frac{5}{72}} = \frac{m}{12} \Rightarrow m = 60\sqrt{\frac{5}{72}} = 60.\frac{\sqrt{5}}{6\sqrt{2}} = 10\sqrt{\frac{5}{2}}$$

$$= 10\cdot\frac{\sqrt{5}}{\sqrt{2}}.\frac{\sqrt{2}}{\sqrt{2}} = \frac{10}{2}.\sqrt{10} = \mathbf{5\sqrt{10}}.$$

11. Given equation is $ax^2 + bx + c = 0$.

$\therefore$ Roots are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Given $\left[\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}\right] = -\left[\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}\right]$

$\Rightarrow -b + \sqrt{b^2 - 4ac} = b + \sqrt{b^2 - 4ac}$

$\Rightarrow 2b = 0 \Rightarrow b = 0$. but $a \neq 0, c \neq 0$.

12. $2x^2 - 7xy + 3y^2 = 0$

$\Rightarrow 2\left(\dfrac{x}{y}\right)^2 - 7\left(\dfrac{x}{y}\right) + 3 = 0$

$\therefore \dfrac{x}{y} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{7 \pm \sqrt{49 - 24}}{4} = \dfrac{7 \pm 5}{4} = 3, \dfrac{1}{2}$

$\therefore$ $x : y$ = **3 : 1 and 1 : 2.**

13. Let α, β be the roots of the equation $ax^2 + bx + c = 0$. Then,

$\alpha + \beta = -\dfrac{b}{a}$ ...(i)

$\alpha\beta = \dfrac{c}{a}$ ...(ii)

and $\beta = \alpha^{1/3}$ ...(iii)

$\therefore$ From (ii) and (iii), $\alpha.(\alpha)^{1/3} = \dfrac{c}{a} \Rightarrow \alpha^{4/3} = \dfrac{c}{a} \Rightarrow \alpha = \left(\dfrac{c}{a}\right)^{3/4}$

$\therefore \quad \beta = \left(\left(\dfrac{c}{a}\right)^{3/4}\right)^{1/3} = \left(\dfrac{c}{a}\right)^{1/4}$

$\therefore$ Putting these values of α and β in eqn. (i), we have

$\left(\dfrac{c}{a}\right)^{3/4} + (c/a)^{1/4} = -\dfrac{b}{a}$

$\Rightarrow a.\, a^{-3/4}\, c^{3/4} + a.\, a^{-1/4}\, c^{1/4} = -b$

$\Rightarrow a^{1/4}\, c^{3/4} + a^{3/4}\, c^{1/4} + b = 0$

$\Rightarrow (ac^3)^{1/4} + (a^3 c)^{1/4} + b = 0.$

14. If the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, then Discriminant (D) = 0, *i.e.*,

$\Rightarrow b^2 (c - a)^2 - 4a(b - c)\, c(a - b) = 0.$

$\Rightarrow b^2 (c^2 + a^2 - 2ac) - 4ac\, (ab - ca - b^2 + bc) = 0$

$\Rightarrow b^2c^2 + b^2 a^2 - 2ab^2c - 4a^2bc + 4a^2c^2 + 4ab^2c - 4abc^2 = 0$

$\Rightarrow a^2b^2 + b^2c^2 + 4a^2c^2 + 2ab^2c - 4a^2bc - 4abc^2 = 0$

$\Rightarrow (ab + bc - 2ac)^2 = 0 \Rightarrow ab + bc - 2ac = 0$

$\Rightarrow ab + bc = 2ac \Rightarrow \dfrac{1}{c} + \dfrac{1}{a} = \dfrac{2}{b} \Rightarrow a, b, c$ are in H.P.

15. The equation $x^2 + px + \dfrac{p}{4} + \dfrac{1}{2} = 0$ has real roots if the discriminant $D \geq 0$.

$\Rightarrow p^2 - 4\left(\dfrac{p}{4} + \dfrac{1}{2}\right) \geq 0 \Rightarrow p^2 - p - 2 \geq 0$

$\Rightarrow p^2 - 2p + p - 2 \geq 0 \Rightarrow p(p - 2) + 1\,(p - 2) \geq 0$

$\Rightarrow (p - 2)\,(p + 1) \geq 0$

$\Rightarrow (p - 2) \geq 0$ and $(p + 1) \geq 0$

$\Rightarrow p \geq 2$ or $p \leq -1$

The condition $p \leq -1$ is not admissible as $0 \leq p \leq 5$.

Now $p \geq 2 \Rightarrow p$ can take up the value 2 or 3 or 4 or 5 from the given values. {0, 1, 2, 3, 4, 5}

$\therefore$ Probability (Roots of given equation are real)

$$= \frac{\text{Number of values } p \text{ can take}}{\text{Given number of values}} = \frac{4}{6} = \mathbf{\frac{2}{3}}.$$

16. Let the roots of the quadratic equation $x^2 + px + q = 0$ be α and β. According to the given condition, A starts with a wrong value of p and obtains the roots as 2 and 6. But this time, the value of q is correct.

$\therefore \quad q$ = Product of roots = $\alpha\beta = 2 \times 6 = 12$.

According to the second condition, B starts with a wrong value of q and obtains the roots as 2 and –9. But this time, the value of p is correct.

$\therefore$ p = sum of roots = $\alpha + \beta = 2 + (-9) = -7$ ...(i)

$\therefore (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = (-7)^2 - 4.12 = 49 - 48 = 1$

$\Rightarrow \alpha - \beta = 1$ ...(ii)

$\therefore$ Solving equations (i) and (ii), we get $\boldsymbol{\alpha = -3}$ **and** $\boldsymbol{\beta = -4}$**.**

17. $\alpha + \beta = 6, \alpha\beta = 6$

$\therefore (\alpha + \beta)^2 = 6^2 \Rightarrow \alpha^2 + \beta^2 + 2\alpha\beta = 36$

$\Rightarrow \alpha^2 + \beta^2 = 36 - 2 \times 6 = 24$

Now, $\alpha^3 + \beta^3 + \alpha^2 + \beta^2 + \alpha + \beta$

$= (\alpha + \beta)\,(\alpha^2 + \beta^2 - \alpha\beta) + (\alpha^2 + \beta^2) + (\alpha + \beta)$

$= 6(24 - 6) + 24 + 6 = 6 \times 18 + 30 = \mathbf{138}.$

18. Since, α and β are the roots of the equation $x^2 + px + q = 0$,

$\therefore \alpha + \beta = -p$ and $\alpha\beta = q$

Now, equation whose roots are $-\alpha^{-1}$ and $-\beta^{-1}$ is

x^2 – (sum of the roots) x + product of the roots = 0

i.e., $x^2 - (-\alpha^{-1} - \beta^{-1})x + (-\alpha^{-1})\,(-\beta^{-1}) = 0$

$-\alpha^{-1} - \beta^{-1} = -\left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right) = -\left(\dfrac{\alpha + \beta}{\alpha\beta}\right) = \dfrac{p}{q}$

and $(-\alpha^{-1})\,(-\beta^{-1}) = \dfrac{1}{\alpha\beta} = \dfrac{1}{q}$

$\therefore$ Required equation $= x^2 - \dfrac{p}{q}x + \dfrac{1}{q} = 0$

$\Rightarrow \boldsymbol{qx^2 - px + 1 = 0}.$

19. $\log_4 (x - 1) = \log_2 (x - 3)$

$\Rightarrow \log_{2^2} (x - 1) = \log_2 (x - 3)$

$\Rightarrow \frac{1}{2}\log_2 (x-1) = \log_2 (x-3)$

$$\left[\text{Using } \log_{m^n}(x) = \frac{1}{n}\log_m x\right]$$

$\Rightarrow \log_2 (x-1) = 2\log_2 (x-3)$
$\Rightarrow \log_2 (x-1) = \log_2 (x-3)^2$
$\Rightarrow (x-1) = (x-3)^2 \Rightarrow (x-1) = x^2 - 6x + 9$
$\Rightarrow x^2 - 7x + 10 = 0 \Rightarrow (x-2)(x-5) = 0 \Rightarrow x = 2$ or 5.
$x = 2$ is inadmissible as $\log_2(x-3)$ is not defined when $x = 2$.
$\therefore$ $\boldsymbol{x = 5.}$

20. Given, $e^{\sin x} - e^{-\sin x} - 4 = 0$
Let $e^{\sin x} = y$. Then, the given equation becomes

$$y - \frac{1}{y} = 4 \Rightarrow y^2 - 4y - 1 = 0$$

$$\Rightarrow \quad y = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$$

$$\Rightarrow \quad e^{\sin x} = 2 \pm \sqrt{5} \Rightarrow \sin x = \log_e (2 \pm \sqrt{5})$$

$$\Rightarrow \quad \sin x = \log_e (2 + \sqrt{5})$$

$(\because (2-\sqrt{5}) < 0$ and so $\log_e (2-\sqrt{5})$ is not defined)

Now $(2+\sqrt{5}) > 4 \Rightarrow \log_e (2+\sqrt{5}) > 1$
But the value of $\sin x$ lies between –1 and 1, both values inclusive, so $\sin x \neq \log_e (2+\sqrt{5})$
$\therefore$ There are no possible real roots of the given equation.

21. $5^{56}\left(\frac{1}{5}\right)^x\left(\frac{1}{5}\right)^{\sqrt{x}} > 1$

$\Rightarrow 5^{56} \times 5^{-x} \times 5^{-\sqrt{x}} > 1 \Rightarrow 5^{56 - x - \sqrt{x}} > 5^0$
$\Rightarrow 56 - x - \sqrt{x} > 0 \Rightarrow x + \sqrt{x} - 56 < 0$
$\Rightarrow y^2 + y - 56 < 0$, where $y = \sqrt{x}$
$\Rightarrow (y+8)(y-7) < 0 \Rightarrow -8 < y < 7 \Rightarrow -8 < \sqrt{x} < 7$
$\Rightarrow 0 \le \sqrt{x} < 7$ as $\sqrt{x}$ cannot be negative
$\Rightarrow 0 \le x < 49 \Rightarrow x \in [0, 49)$

22. Given, $\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$

$$\Rightarrow \frac{(x+b)+(x+a)}{(x+a)(x+b)} = \frac{1}{c} \Rightarrow \frac{2x+b+a}{x^2+(a+b)x+ab} = \frac{1}{c}$$

$\Rightarrow 2cx + (a+b)c = x^2 + (a+b)x + ab$
$\Rightarrow x^2 + (a+b-2c)x + ab - ac - bc = 0$
Let α, β be the roots of this equation. Then,
$\alpha + \beta = -(a+b-2c) = 0$ (Given)
$\Rightarrow a + b = 2c$ and $\alpha\beta = ab - ac - bc = ab - (a+b)c$

$$= ab - (a+b)\frac{(a+b)}{2}$$

$$= \frac{2ab - (a^2+b^2+2ab)}{2} = -\left(\boldsymbol{\frac{a^2+b^2}{2}}\right).$$

23. Let the roots of the equation $kx^2 - 5x + 6 = 0$ be α and β.
Then, $\alpha + \beta = 5/k$ ...(*i*)
$\alpha\beta = 6/k$ ...(*ii*)
Given $\alpha/\beta = \frac{2}{3} \Rightarrow \alpha = \frac{2}{3}\beta$
$\therefore$ From (*i*) and (*ii*),

$$\frac{2}{3}\beta + \beta = \frac{5}{k} \text{ and } \frac{2}{3}\beta^2 = \frac{6}{k}$$

$$\Rightarrow \frac{5}{3}\beta = \frac{5}{k} \text{ and } \beta^2 = \frac{9}{k} \Rightarrow \beta = \frac{3}{k} \text{ and } \beta^2 = \frac{9}{k}$$

$$\Rightarrow \frac{9}{k^2} = \frac{9}{k} \Rightarrow 9k^2 - 9k = 0 \Rightarrow k(k-1) = 0 \Rightarrow k = 0 \text{ or } 1$$

But $k = 0$ does not satisfy the condition, so $\boldsymbol{k = 1.}$

24. $2|x|^2 - 5|x| + 2 = 0$
$\Rightarrow (2|x| - 1)(|x| - 2) = 0$
$\Rightarrow |x| = \frac{1}{2}, 2 \Rightarrow x = \pm\frac{1}{2}, \pm 2$
So, there are 4 solutions.

25. $\because$ p, q, r are in A.P.

$q = \frac{p+r}{2}$ $[\because p + r = 2q]$

For the real roots $q^2 - 4pr \ge 0$ [Disct. ≥ 0]

$$\Rightarrow \left(\frac{p+r}{2}\right)^2 - 4pr \ge 0$$

$\Rightarrow p^2 + r^2 - 14pr \ge 0$

$$\Rightarrow \left(\frac{p}{r}\right)^2 - 14\left(\frac{p}{r}\right) + 1 \ge 0$$

$$\Rightarrow \left(\frac{p}{r} - 7\right)^2 \ge 48 = \left|\boldsymbol{\frac{p}{r} - 7}\right| \boldsymbol{\ge 4\sqrt{3}.}$$

26. Let $y = 5 + 2\sqrt{6}$. Then $\frac{1}{y} = 5 - 2\sqrt{6}$. Thus the given expression reduces to $y^{x^2-3} + \left(\frac{1}{y}\right)^{x^2-3} = 10$

Again let $y^{x^2-3} = t$. Then,

$$t + \frac{1}{t} = 10 \Rightarrow t^2 - 10t + 1 = 0$$

$$\Rightarrow \quad t = \frac{10 \pm \sqrt{100-4}}{2} = \frac{10 \pm \sqrt{96}}{2}$$

$$= \frac{10 \pm 4\sqrt{6}}{2} = 5 \pm 2\sqrt{6}$$

$$\therefore (5+2\sqrt{6})^{x^2-3} = 5 \pm 2\sqrt{6} = (5+2\sqrt{6})^{\pm 1}$$

$\Rightarrow x^2 - 3 = 1$ or $x^2 - 3 = -1$
$\Rightarrow x^2 = 4$ or $x^2 = 2$
$\Rightarrow x = \pm 2$ or $x = \pm\sqrt{2}$

27. Given, α, β, γ are the roots of the $2x^3 - 3x^2 + 6x + 1 = 0$.

Since $S_1 = -\dfrac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3} \Rightarrow S_1 = \alpha+\beta+\gamma = -\left(\dfrac{-3}{2}\right) = \dfrac{3}{2}$

$S_2 = \dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^3} \Rightarrow S_2 = \alpha\beta+\beta\gamma+\alpha\gamma = \dfrac{6}{2} = 3$

$S_3 = -\dfrac{\text{Coefficient of constant term}}{\text{Coefficient of } x^3} \Rightarrow S_3 = \alpha\beta\gamma = -\dfrac{1}{2}$

Now, $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= \left(\dfrac{3}{2}\right)^2 - 2\times 3 = \dfrac{9}{4} - 6 = \mathbf{\dfrac{-15}{4}}.$

28. Given, α, β, γ are the roots of the equation $x^3 + 4x + 1 = 0$.

$\therefore S_1 = \alpha+\beta+\gamma = 0$ (Coefficient of $x^2 = 0$)

$S_2 = \alpha\beta+\beta\gamma+\alpha\gamma = 4$

$S_3 = \alpha\beta\gamma = -1$

$\therefore (\alpha+\beta)^{-1} + (\beta+\gamma)^{-1} + (\gamma+\alpha)^{-1}$

$= \dfrac{1}{\alpha+\beta} + \dfrac{1}{\beta+\gamma} + \dfrac{1}{\gamma+\alpha}$

$= \dfrac{1}{-\gamma} + \dfrac{1}{-\alpha} + \dfrac{1}{-\beta}$ $\left[\begin{array}{l}\because \alpha+\beta+\gamma=0\\ \Rightarrow \alpha+\beta=-\gamma, \beta+\gamma=-\alpha, \gamma+\alpha=-\beta\end{array}\right]$

$= -\left[\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}\right] = -\left[\dfrac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}\right] = -\left[\dfrac{4}{-1}\right] = \mathbf{4.}$

29. Let $(\alpha - d)$, α, $(\alpha + d)$ be the three roots of the given cubic equation $x^3 - 12x^2 + 12x - 28 = 0$

$\therefore S_1 = (\alpha - d) + \alpha + (\alpha + d) = 12$

$\Rightarrow 3\alpha = 12 \Rightarrow \alpha = 4$

and $S_3 = (\alpha - d)\,.\,\alpha.\,(\alpha + d) = 28$

$\Rightarrow (4 - d)\,.\,4.\,(4 + d) = 28$

$\Rightarrow 16 - d^2 = 7 \Rightarrow d^2 = 9 \Rightarrow d = \pm\mathbf{3}.$

30. Let α, β be the roots of the equation $3ax^2 + 3bx + c = 0$. We have to find the equation whose roots are 3α and 3β, which can be got by putting $y = 3x$ in the given equation, *i.e.*, substituting x for $\dfrac{y}{3}$ in the given equations.

$\therefore$ The required equation is : $3a\left(\dfrac{y}{3}\right)^2 + 3b\left(\dfrac{y}{3}\right) + c = 0$

$\Rightarrow \dfrac{ay^2}{3} + by + c = 0 \quad \Rightarrow \quad ay^2 + 3by + 3c = 0$

$\Rightarrow \quad \mathbf{ax^2 + 3bx + 3c = 0.}$

31. As α, β are the roots of the equation $ax^2 + bx + c = 0$, so

$\alpha + \beta = -\dfrac{b}{a}$, $\alpha\beta = \dfrac{c}{a}$

The equation whose roots are $\dfrac{1-\alpha}{\alpha}$ and $\dfrac{1-\beta}{\beta}$ can be written as :

$$x^2 - \left(\frac{1-\alpha}{\alpha} + \frac{1-\beta}{\beta}\right)x + \left(\frac{1-\alpha}{\alpha}.\frac{1-\beta}{\beta}\right) = 0$$

Now, $\dfrac{1-\alpha}{\alpha} + \dfrac{1-\beta}{\beta} = \dfrac{\beta-\alpha\beta+\alpha-\alpha\beta}{\alpha\beta} = \dfrac{\alpha+\beta-2\alpha\beta}{\alpha\beta} = \dfrac{\alpha+\beta}{\alpha\beta} - 2$

$= \dfrac{-b/a}{c/a} - 2 = -\dfrac{b}{c} - 2 = -\dfrac{b-2c}{c}$

and $\left(\dfrac{1-\alpha}{\alpha}.\dfrac{1-\beta}{\beta}\right) = \dfrac{1-\alpha-\beta+\alpha\beta}{\alpha\beta} = \dfrac{1-(\alpha+\beta)+\alpha\beta}{\alpha\beta}$

$= \dfrac{1}{\alpha\beta} - \dfrac{\alpha+\beta}{\alpha\beta} + 1 = \dfrac{a}{c} + \dfrac{b}{c} + 1 = \dfrac{a+b+c}{c}$

$\therefore$ The required equation is

$$x^2 - \left[-\frac{(b+2c)}{c}\right]x + \left[\frac{a+b+c}{c}\right] = 0$$

$\Rightarrow cx^2 + (b + 2c)x + (a + b + c) = 0$ ...(*i*)

Compairing eqn. (*i*) with the given equation $px^2 + qx + r = 0$, we get $r = \boldsymbol{a + b + c}.$

32. To find the equation whose roots are the negatives of the roots of the given equation, we replace x by $(-x)$ in the given equation.

$\therefore$ Required equation is

$(-x)^7 + 3(-x)^5 + (-x)^3 - (-x)^2 + 7(-x) + 2 = 0$

i.e, $-x^7 - 3x^5 - x^3 - x^2 - 7x + 2 = 0$

i.e., $x^7 + 3x^5 + x^3 + x^2 + 7x - 2 = 0.$

33. As α, γ are the roots of the equation $Ax^2 - 4x + 1 = 0$, so

$\alpha + \gamma = \dfrac{4}{A}$ ...(*i*)

and $\alpha\gamma = \dfrac{1}{A}$ ...(*ii*)

Given, β, δ are the roots of the equation $Bx^2 - 6x + 1 = 0$,

so $\beta + \delta = \dfrac{6}{B}$ ...(*iii*)

and $\beta\delta = \dfrac{1}{B}$ ...(*iv*)

Given, α, β, γ, δ are in H.P., so

$\beta = \dfrac{2\alpha\gamma}{\alpha+\gamma} = \dfrac{2/A}{4/A} = \dfrac{1}{2}$

$\gamma = \dfrac{2\beta\delta}{\beta+\delta} = \dfrac{2/B}{6/B} = \dfrac{1}{3}.$

Also β is the root of the equation $Bx^2 - 6x + 1 = 0$, so

$B\beta^2 - 6\beta + 1 = 0 \Rightarrow B \times \dfrac{1}{4} - 6\times\dfrac{1}{2} + 1 = 0$

$\Rightarrow \dfrac{B}{4} - 2 = 0 \Rightarrow \dfrac{B}{4} = 2 \Rightarrow B = 8$

Given, γ is the root of the equation $Ax^2 - 4x + 1 = 0$, so

$A\gamma^2 - 4\gamma + 1 = 0$

$\Rightarrow A \times \dfrac{1}{9} - 4\times\dfrac{1}{3} + 1 = 0 \quad \Rightarrow \dfrac{A}{9} - \dfrac{1}{3} = 0 \Rightarrow A = \mathbf{3.}$

A, B = **3, 8** respectively.

34. The given equation is $(x-a)(x-b)=c$

$\Rightarrow x^2-(a+b)x+(ab-c)=0$

As α, β are the roots of this equation, so

$\alpha+\beta=a+b$ and $\alpha\beta=ab-c$

Let γ, δ be the roots of the equation $(x-\alpha)(x-\beta)+c=0$

i.e., γ, δ are the roots of the equation $x^2-(\alpha+\beta)x+(\alpha\beta+c)=0$

$\therefore \quad \gamma+\delta=\alpha+\beta=a+b$ ...(*i*)

$\gamma\delta=\alpha\beta+c=ab-c+c=ab$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*) we can infer that the roots of the equation $(x-\alpha)(x-\beta)+c=0$ are ***a*** and ***b***.

35. Let the roots of the equation $x^3-ax^2+bx-c=0$ be $(\alpha-1), \alpha, (\alpha+1)$

$\therefore S_2=(\alpha-1)\alpha+\alpha(\alpha+1)+(\alpha+1)(\alpha-1)=b$

$\Rightarrow \alpha^2-\alpha+\alpha^2+\alpha+\alpha^2-1=b$

$\Rightarrow 3\alpha^2-1=b$

$\therefore$ Minimum value of $b=-1$, when $\alpha=$ **0.**

36. The given equations are written as :

$x^2+2ax+a^2-1=0$ ...(*i*)

$x^2+2bx+b^2-1=0$ ...(*ii*)

If α is the common root of both the equations, α satisfies both the equations, so,

$\alpha^2+2a\alpha+(a^2-1)=0$ ...(*iii*)

$\alpha^2+2b\alpha+(b^2-1)=0$ ...(*iv*)

Solving equations (*iii*) and (*iv*) simultaneously

$$\frac{\alpha^2}{2a(b^2-1)-2b(a^2-1)}=\frac{\alpha}{(a^2-1)-(b^2-1)}=\frac{1}{2b-2a}$$

$$\Rightarrow \frac{\alpha^2}{2ab^2-2ba^2+2(b-a)}=\frac{\alpha}{(a^2-b^2)}=\frac{1}{2(b-a)}$$

$$\Rightarrow \frac{\alpha^2}{2ab(b-a)+2(b-a)}=\frac{\alpha}{-(a+b)(b-a)}=\frac{1}{2(b-a)}$$

$$\Rightarrow \frac{\alpha^2}{2(b-a)(ab+1)}=\frac{\alpha}{-(a+b)(b-a)}=\frac{1}{2(b-a)}$$

[By the rule of cross-multiplication, the solution of two simultaneous equations :

$a_1x+b_1y+c_1=0$

$a_2x+b_2y+c_2=0$ is

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}\Bigg]$$

$\therefore \alpha^2=ab+1$ and $\alpha=-\frac{1}{2}(a+b)$

$\Rightarrow (ab+1)=\frac{1}{4}(a+b)^2$

$\Rightarrow a^2+b^2+2ab=4ab+4$

$\Rightarrow a^2+b^2-2ab=4 \Rightarrow (a-b)^2=4 \Rightarrow |a-b|=$ **2.**

37. Given equation is $x^3-3x+11=0$

If α, β, γ are the roots of the given equation, then

$S_1=\alpha+\beta+\gamma=0$

$\Rightarrow \beta+\gamma=-\alpha, \gamma+\alpha=-\beta$ and $\alpha+\beta=-\gamma$

$\therefore$ The equation whose roots are $(\alpha+\beta), (\beta+\gamma), (\gamma+\alpha)$ is the equation whose roots are $-\gamma, -\alpha, -\beta$.

$\therefore$ We can obtain the required equations by replacing x by $(-x)$ in the given equation.

$\therefore$ Required equation is $(-x)^3-3(-x)+11=0$

i.e, $-x^3+3x+11=0$

i.e., $x^3-3x-11=$ **0.**

38. As α, β are the roots of the equation $ax^2+bx+c=0$, so

$\alpha+\beta=-\frac{b}{a}, \ \alpha\beta=\frac{c}{a}$

Also, $(\alpha+x), (\beta+x)$ are the roots of the equation $px^2+qx+r=0$, then $\alpha+x+\beta+x=-\frac{q}{p}$

and $(\alpha+x)(\beta+x)=\frac{r}{p} \quad \Rightarrow \alpha+\beta+2x=-\frac{q}{p}$

$\Rightarrow \frac{-b}{a}+2x=-\frac{q}{p} \Rightarrow \boldsymbol{K=\frac{1}{2}\left(\frac{b}{a}-\frac{q}{p}\right)}$.

39. Let $x=\sqrt{6+\sqrt{6+\sqrt{6+.....\infty}}}$

$\Rightarrow x=\sqrt{6+x}$

$\Rightarrow x^2=6+x \Rightarrow x^2-x-6=0$

$\Rightarrow (x-3)(x+2)=0$

$\Rightarrow x=3$ or -2

$\Rightarrow \boldsymbol{x=3}$ as $x=-2$ does not satisfy the given equation.

40. The equation is $(x-a)(x-a-1)+(x-a-1)(x-a-2)+(x-a)(x-a-2)=0$.

Let $(x-a)=y$, then the equation becomes

$y(y-1)+(y-1)(y-2)+y(y-2)=0$

$\Rightarrow y^2-y+y^2-3y+2+y^2-2y=0$

$\Rightarrow 3y^2-6y+2=0$

$\therefore$ Discriminant $=D=b^2-4ac=36-4\times3\times2$

$=36-24=12>0$

$\therefore$ Roots are real and distinct.

SELF ASSESSMENT SHEET

1. If the sum as well as the product of roots of a quadratic equation is 9, then the equation is:

(*a*) $x^2+9x-18=0$ (*b*) $x^2-18x+9=0$

(*c*) $x^2+9x+9=0$ (*d*) $x^2-9x+9=0$.

(*CDS 2010*)

2. If one root of the equation $\frac{x^2}{a}+\frac{x}{b}+\frac{1}{c}=0$ is reciprocal of the other, then which of the following is correct ?

(*a*) $a=b$ (*b*) $b=c$ (*c*) $ac=1$ (*d*) $a=c$

(*CDS 2012*)

3. If one root of the equation $ax^2+x-3=0$ is -1, then what is the other root ?

(*a*) $\frac{1}{4}$ (*b*) $\frac{1}{2}$ (*c*) $\frac{3}{4}$ (*d*) 1

(*CDS 2010*)

4. If the equation $(a^2 + b^2)\,x^2 - 2\,(ac + bd)x + (c^2 + d^2) = 0$ has equal roots, then which one of the following is correct ?
(*a*) $ab = cd$ (*b*) $ad = bc$
(*c*) $a^2 + c^2 = b^2 + d^2$ (*d*) $ac = bd$ **(*CDS 2010*)**

5. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then what is the value of $\alpha^3 + \beta^3$?
(*a*) $\frac{b^3 + 3abc}{a^3}$ (*b*) $\frac{a^3 - b^3}{3abc}$ (*c*) $\frac{3abc - b^3}{a^3}$ (*d*) $\frac{b^3 - 3abc}{a^3}$ **(*CDS 2008*)**

6. If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of their squares, then which one of the following is correct ?
(*a*) $a^2 + b^2 = c^2$ (*b*) $a^2 + b^2 = a + b$
(*c*) $2ac = ab + b^2$ (*d*) $2c + b = 0$

7. One root of $x^2 + kx - 8 = 0$ is the square of the other, then the value of k is :
(*a*) 2 (*b*) 8 (*c*) –8 (*d*) –2 **(*CAT 1995*)**

8. Let p and q be the roots of the quadratic equation $x^2 - (\alpha - 2)x - \alpha - 1 = 0$. What is the minimum possible value of $p^2 + q^2$?
(*a*) 0 (*b*) 3 (*c*) 4 (*d*) 5 **(*CAT 2003*)**

9. If α, β, γ are the roots of $x^3 - 2x^2 + 3x - 4 = 0$, then the value of $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ is :
(*a*) –7 (*b*) –5 (*c*) –3 (*d*) 0 **(*EAMCET 2007*)**

10. If α, β are the roots of the equations $x^2 + 4x + 3 = 0$, then the equation whose roots are $2\alpha + \beta$ and $\alpha + 2\beta$ is
(*a*) $x^2 - 12x - 33 = 0$ (*b*) $x^2 - 12x + 35 = 0$
(*c*) $x^2 + 12x - 33 = 0$ (*d*) $x^2 + 12x + 35 = 0$. **(*J&K CET 2009*)**

11. The equation whose roots are twice the roots of the equation $x^2 - 3x + 3 = 0$ is
(*a*) $x^2 - 3x + 6 = 0$ (*b*) $x^2 - 4x + 8 = 0$
(*c*) $x^2 - 6x + 12 = 0$ (*d*) $x^2 - 8x + 6 = 0$

12. If $x^2 + mx + n = 0$ and $x^2 + px + q = 0$ have a common root, then the common root is
(*a*) $\frac{q - n}{m - p}$ (*b*) $\frac{q - n}{m + p}$
(*c*) $\frac{q + n}{m + p}$ (*d*) None of these

ANSWERS

1. (*d*)	**2.** (*d*)	**3.** (*c*)	**4.** (*b*)	**5.** (*c*)	**6.** (*c*)	**7.** (*d*)	**8.** (*d*)	**9.** (*a*)	**10.** (*d*)
11. (*c*)	**12.** (*a*)								

HINTS AND SOLUTIONS

1. Equation : x^2 – (Sum of roots)x + Product of roots = 0
$\Rightarrow \mathbf{x^2 - 9x + 9 = 0.}$

2. The equation can be written as $\frac{1}{a}x^2 + \frac{1}{b}x + \frac{1}{c} = 0.$
i.e., $bcx^2 + acx + ab = 0.$
Let α, 1/α be the roots of the given equation, then
product of roots $= \alpha \times \frac{1}{\alpha} = \frac{ab}{bc} \Rightarrow \frac{ab}{bc} = 1 \Rightarrow \mathbf{a = c.}$

3. Let the other root of the equation $ax^2 + x - 3 = 0$ be α.
As (–1) is a root of the given equation, it satisfies the given equation, *i.e.*, $a(-1)^2 + (-1) - 3 = 0 \Rightarrow a - 1 - 3 = 0 \Rightarrow a = 4.$
∴ The equation becomes $4x^2 + x - 3 = 0.$
Now, product of roots $= \alpha \times (-1) = -\frac{3}{4}$
$\therefore \alpha = \mathbf{\frac{3}{4}.}$

4. Equal roots ⇒ Discriminant = 0
$\Rightarrow 4(ac + bd)^2 - 4(a^2 + b^2)(c^2 + d^2) = 0$
$\Rightarrow [\cancel{a^2c^2} + \cancel{b^2d^2} + 2acbd] - [\cancel{a^2c^2} + b^2c^2 + a^2d^2 + \cancel{b^2d^2}] = 0$
$\Rightarrow -[b^2c^2 + a^2d^2 - 2acbd] = 0$
$\Rightarrow (bc - ad)^2 = 0 \Rightarrow \mathbf{bc = ad.}$

5. $\alpha + \beta = -b/a$, $\alpha\beta = c/a$.
$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$
$= (\alpha + \beta)\{(\alpha + \beta)^2 - 2\alpha\beta - \alpha\beta\}$
$= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$
$= -\frac{b}{a}\left[\frac{b^2}{a^2} - \frac{3c}{a}\right]$
$= \frac{-b^3}{a^3} + \frac{3bc}{a^2} = \mathbf{\frac{3abc - b^3}{a^3}}$

6. Let α, β be the roots of the equation $ax^2 + bx + c = 0$. Then,
$\alpha + \beta = -b/a$, $\alpha\beta = c/a$
Given, $\alpha + \beta = \alpha^2 + \beta^2$
i.e., $\alpha + \beta = (\alpha^2 + \beta)^2 - 2\alpha\beta$
$\Rightarrow \frac{-b}{a} = \frac{b^2}{a^2} - \frac{2c}{a}$
$\Rightarrow -\frac{ab}{a^2} = \frac{b^2}{a^2} - \frac{2ac}{a^2} \Rightarrow ab + b^2 = \mathbf{2ac.}$

7. Let α and α^2 be the roots of the equation $x^2 + kx - 8 = 0.$
Then, product of roots $= \alpha \,.\, \alpha^2 = -8$
$\alpha^3 = -8 \Rightarrow \alpha = -2$
∴ The root (–2) satisfies the given equation, *i.e.*,
$(-2)^2 + k\,.\,(-2) - 8 = 0$
$4 - 2k - 8 = 0 \Rightarrow -2k = 4 \Rightarrow k = \mathbf{-2.}$

8. If p and q are the roots of the equation $x^2 - (\alpha - 2)x - (\alpha + 1) = 0$.

$\therefore$ Sum of roots $= p + q = (\alpha - 2)$

Product of roots $= pq = -\alpha - 1$

$\therefore p^2 + q^2 = (p + q)^2 - 2pq$

$= (\alpha - 2)^2 + 2(\alpha + 1)$

$= \alpha^2 + 4 - 4\alpha + 2\alpha + 2 = (\alpha + 1)^2 + 5$

$p^2 + q^2$ will be minimum when $\alpha = 0$.

$\therefore$ Minimum value of $p^2 + q^2 = \mathbf{5}$.

9. Given, α, β, γ are the roots of $x^3 - 2x^2 + 3x - 4 = 0$. Then,

$S_1 = \alpha + \beta + \gamma = 2$

$S_2 = \alpha\beta + \beta\gamma + \gamma\alpha = 3$

$S_3 = \alpha\beta\gamma = 4$

Now, $(\alpha\beta + \beta\gamma + \gamma\alpha)^2$

$= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta^2\gamma + 2\beta\gamma^2\alpha + 2\alpha^2\beta\gamma$

$= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\beta + \gamma + \alpha)$

$\Rightarrow \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

$= 3^2 - 2 \times 4 \times 2 = 9 - 16 = \mathbf{-7}$.

10. Given, α, β are the roots of the quadratic equations $x^2 + 4x + 3 = 0$,

$\Rightarrow$ Sum of roots $= \alpha + \beta = -4$ ...(*i*)

Product of roots $= \alpha\beta = 3$ ...(*ii*)

Given, $2\alpha + \beta$ and $\alpha + 2\beta$ are the roots of the required equation, so

Required equation is

$$= x^2 - \underbrace{(2\alpha + \beta + \alpha + 2\beta)}_{\text{Sum of roots}}x + \underbrace{(2\alpha + \beta)(\alpha + 2\beta)}_{\text{Product of roots}} = 0$$

Sum $= 2\alpha + \beta + \alpha + 2\beta = 3\alpha + 3\beta = 3(\alpha + \beta)$

$= 3 \times -4 = -12$

Product $= (2\alpha + \beta)(\alpha + 2\beta) = 2\alpha^2 + \beta\alpha + 4\alpha\beta + 2\beta^2$

$= 2(\alpha^2 + \beta^2) + 5\alpha\beta$

$= 2(\alpha^2 + \beta^2 + 2\alpha\beta) + \alpha\beta$

$= 2(\alpha + \beta)^2 + \alpha\beta = 2 \times 16 + 3 = 35$

$\therefore$ Reqd. equation is $\mathbf{x^2 - 12x + 35 = 0}$.

11. Let α, β be the roots of the given equation $x^2 - 3x + 3 = 0$.

Given, $\alpha + \beta = +3$, $\alpha\beta = 3$

By the given condition, the roots of the required equation are 2α and 2β.

$\therefore$ Sum of roots $= 2\alpha + 2\beta = 2(\alpha + \beta) = 2 \times 3 = 6$

Product of roots $= 2\alpha \,.\, 2\beta = 4\alpha\beta = 4 \times 3 = 12$.

$\therefore$ Required equation is $\mathbf{x^2 - 6x + 12 = 0}$

12. Let α be the common root of the equations $x^2 + mx + n = 0$ and $x^2 + px + q = 0$. Then,

$\alpha^2 + m\alpha + n = 0$ ...(*i*)

$\alpha^2 + p\alpha + q = 0$ ...(*ii*)

Solving (*i*) and (*ii*) simultaneously, we get

$$\frac{\alpha^2}{mq - np} = \frac{\alpha}{n - q} = \frac{1}{p - m} \Rightarrow \alpha = \frac{n - q}{p - m} = \mathbf{\frac{q - n}{m - p}}.$$

4 Inequalities

KEY FACTS

I. An **inequality** is a statement involving quantities (variables or constants) connected to each other with signs of inequality, *i.e.*, $<, \leq, >, \geq$.

Thus, $x < y$, $x \leq y$, $x > y$ and $x \geq y$ represent inequalities.

II. Note on the use of brackets for open and closed intervals.

Sometimes the domains of variation, *i.e.*, the intervals are denoted as follows:

(*a*) **'$a < x < b$'** is denoted by (a, b) and is called the open interval of the variable x.

(*b*) **'$a \leq x \leq b$'** is denoted by $[a, b]$ and is called a closed interval of the variable x, as x can take up values 'a' and 'b' also.

(*c*) Any 'x' is denoted by $(-\infty, \infty)$. Here, it should be noted that the symbols $-\infty$, ∞ are not numbers in any sense whatsoever.

(*d*) **'$x \geq a$'** is denoted by $[a, \infty)$ and **'$x \leq b$'** is denoted by $(-\infty, b]$.

(*e*) **'$a \leq x < b$'** is denoted by $[a, b)$, **'$a < x \leq b$'** is denoted by $(a, b]$. These are called semi-closed intervals.

III. Important Properties of Inequalities.

For all $a, b, c \in R$ (*i*) ***If $a > b$ and $b > c$, then $a > c$.*** **Ex.** $4 > 2$ and $2 > -1 \Rightarrow 4 > -1$	(*ii*) ***If $a > b$, then for all c, $a + c > b + c$ and $a - c > b - c$.*** **Ex.** $7 > 4 \Rightarrow 7 + 2 > 4 + 2 \Rightarrow 9 > 6$ $7 > 4 \Rightarrow 7 - 2 > 4 - 2 \Rightarrow 5 > 2$
(*iii*) ***If $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.*** **Ex.** $4 > 2 \Rightarrow 4 \times 3 > 2 \times 3 \Rightarrow 12 > 6$ $4 > 2 \Rightarrow \frac{4}{2} > \frac{2}{2} \Rightarrow 2 > 1$	(*iv*) ***If $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.*** **Ex.** $8 > 6 \Rightarrow 8 \times -2 < 6 \times -2 \Rightarrow -16 < -12$ $8 > 6 \Rightarrow \frac{8}{-2} < \frac{6}{-3} \Rightarrow -4 < -2$
(*v*) ***If $a \neq 0$, $b \neq 0$ and $a > b > 0$, then $\frac{1}{a} < \frac{1}{b}$.*** **Ex.** $3 > 1 > 0 \Rightarrow \frac{1}{3} < 1$	(*vi*) ***If $a_1 > b_1, a_2 > b_2, \ldots\ldots\ldots\ldots, a_n > b_n$ then*** • $a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots + a_n > b_1 + b_2 + b_3 + \ldots\ldots\ldots\ldots + b_n$ • $a_1 . a_2 . a_3 \ldots\ldots\ldots\ldots a_n > b_1 . b_2 . b_3 \ldots\ldots\ldots\ldots b_n$ **Ex.** $6 > 4, 2 > 1, 4 > 3$ $\Rightarrow$ • $6 + 2 + 4 > 4 + 1 + 3$, *i.e.*, $12 > 8$ $\Rightarrow$ • $6 \times 2 \times 4 > 4 \times 1 \times 3$, *i.e.*, $48 > 12$

(*vii*) **(*a*) *If m > 0, then for a > b > 0***

- $a^m > b^m$ • $a^{-m} < b^{-m}$ • $a^{1/m} > b^{1/m}$

Ex. Let $a = 3$, $b = 2$ and $m = 3$, then

- $3^3 > 2^3$ since $27 > 8$
- $3^{-3} < 2^{-3}$, *i.e.*, $\frac{1}{3^3} < \frac{1}{2^3}$ since $\frac{1}{27} < \frac{1}{8}$
- $3^{1/3} > 2^{1/3}$

(*b*) *Similarly, if m > 0 and a < b, a > 0, b > 0, then*

- $a^m < b^m$ • $a^{-m} > b^{-m}$ • $a^{1/m} < b^{1/m}$

(*viii*) **(*a*) *If a > 1 and m > 0, then • $a^m > 1$ and • $0 < a^{-m} < 1$***

Let $a = 2$, $m = 2 \Rightarrow 2^2 = 4 > 1$ and $0 < 2^{-2} < 1$ as $0 < \frac{1}{4} < 1$

(*b*) *If 0 < a < 1 and m > 0, then $0 < a^m < 1$ and $a^{-m} > 1$*

Let $a = \frac{1}{2}$, $m = 2$, then $0 < \left(\frac{1}{2}\right)^2 < 1$ since $0 < \frac{1}{4} < 1$ and $\left(\frac{1}{2}\right)^{-2} > 1$ since $4 > 1$

(*ix*) **(*a*) *If a > 1 and m > n > 0, then $a^m > a^n$***

Ex. $a = 2$, $m = 3$, $n = 2$, then $2^3 > 2^2$

(*b*) *If 0 < a < 1 and m > n > 0 then $a^m < a^n$*

Ex. $a = \frac{1}{3}$, $m = 3$, $n = 2$, then $\left(\frac{1}{3}\right)^3 < \left(\frac{1}{3}\right)^2$ since $\frac{1}{27} < \frac{1}{9}$

(*x*) ***If 0 < a < 1 < b and r is a positive rational number, then***

- ***$0 < a^r < 1 < a^{-r}$ and • $0 < b^{-r} < 1 < b^r$***

Ex. If $0 < \frac{1}{2} < 1 < 3$ and $r = 3$, then

- $0 < \left(\frac{1}{2}\right)^3 < 1 < \left(\frac{1}{2}\right)^{-3}$ since $0 < \frac{1}{8} < 1 < 8$
- $0 < (3)^{-3} < 1 < (3)^3$ since $0 < \frac{1}{27} < 1 < 27$

(*xi*) **(*a*) *If a > 1 and x > y > 0, then $\log_a x > \log_a y$***

Ex. $a = 3$, $x = 27$, $y = 9$

$\Rightarrow \log_3 27 > \log_3 9 \Rightarrow \log_3 3^3 > \log_3 3^2 \Rightarrow 3 > 2$

(*b*) *If 0 < a < 1 and x > y > 0, then $\log_a x < \log_a y$*

Ex. $a = 1/3$, $x = 27$, $y = 9$

$\Rightarrow \log_{1/3} 27 < \log_{1/3} 9 \Rightarrow \log_{1/3} 3^3 < \log_{1/3} 3^2$

$\Rightarrow \log_{3^{-1}} 3^3 < \log_{3^{-1}} 3^2 \Rightarrow \frac{3}{-1} < \frac{2}{-1}$ since $-3 < -2$

(*xii*) **(*a*) *If a > 1, then $a^{f(x)} > a^{g(x)} \Leftrightarrow f(x) > g(x)$***

(*b*) *If 0 < a < 1, then $a^{f(x)} > a^{g(x)} \Leftrightarrow f(x) < g(x)$*

(*c*) *If a > 1, then $\log_a f(x) > \log_a g(x) \Leftrightarrow f(x) > g(x) > 0$*

(*d*) *If 0 < a < 1, then $\log_a f(x) > \log_a g(x) \Leftrightarrow 0 < f(x) < g(x)$*

(*e*) *If a > 1, then $\log_a x > p \Rightarrow x > a^p$*

Ex. $\log_4 x > 3 \Rightarrow x > 4^3 \Rightarrow x > 64$

(*f*) *If 0 < a < 1, then $\log_a x > p \Rightarrow 0 < x < a^p$*

Ex. $\log_{1/2} x > 3 \Rightarrow 0 < x < (1/2)^3$

$$\Rightarrow 0 < x < \frac{1}{8}$$

(*g*) *If 0 < a < 1, then $0 < \log_a x < p \Rightarrow a^p < x < 1$*

Ex. $0 < \log_{1/3} x < 2 \Rightarrow \left(\frac{1}{3}\right)^2 < x < 1 \Rightarrow \frac{1}{9} < x < 1$

(*xiii*) ***Recall Modulus Properties***

(*a*) $a \le |a|$

(*b*) $|ab| = |a|\,|b|$

(*c*) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

(*d*) • $|a + b| < |a| + |b|$
• $|a - b| > \big||a| - |b|\big|$

(*e*) • $|a + b| = |a| + |b|$ if $ab > 0$
• $|a - b| = |a| - |b|$ if $ab < 0$

(*xiv*) ***For any positive real number a,***

(*a*) $|x| < a \Rightarrow -a < x < a \Rightarrow x \in (-a, a)$

(*b*) $|x| \le a \Rightarrow -a \le x \le a \Rightarrow x \in [-a, a]$

Ex. $|x| < 5 \Rightarrow -5 \le x \le 5$

(*c*) $|x| > a \Rightarrow x < -a$ or $x > a \Rightarrow x \in (-\infty, -a) \cup (a, \infty)$

(*d*) $|x| \ge a \Rightarrow x \le -a$ or $x \ge a \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

Ex. $|x| < 7 \Rightarrow x < -7$ or $x > 7$

Note: *i.e.*, square brackets mean that the point inside it is inclusive.

(*xv*) ***Let r be a positive real number and a be fixed real number. Then,***

(*a*) $|x - a| < r \Rightarrow a - r < x < a + r \Rightarrow x \in (a - r, a + r)$

(*b*) $|x - a| \le r \Rightarrow a - r \le x \le a + r \Rightarrow x \in [a - r, a + r]$

Ex. $|x - 2| < 5 \Rightarrow 2 - 5 < x < 2 + 5$
$\Rightarrow -3 < x < 7$

(*c*) $|x - a| > r \Rightarrow x < a - r$ or $x > a + r \Rightarrow x \in (-\infty, a - r) \cup (a + r, \infty)$

(*d*) $|x - a| \ge r \Rightarrow x \le a - r$ or $x \ge a + r \Rightarrow x \in (-\infty, a - r] \cup [a + r, \infty)$

Ex. $|x - 3| \ge 8$
$\Rightarrow x < 3 - 8$ or $x > 3 + 8$
$\Rightarrow x < -5$ or $x > 11$

(*xvi*) ***Let a, b be positive real numbers. Then,***

(*a*) $a < |x| < b \Rightarrow x \in (-b, -a) \cup (a, b)$

(*b*) $a \le |x| \le b \Rightarrow x \in [-b, -a] \cup [a, b]$

(*c*) $a < |x - c| < b \Rightarrow x \in (-b + c, -a + c) \cup (a + c, b + c)$

(*d*) $a \le |x - c| \le b \Rightarrow x \in [-b + c, -a + c] \cup [a + c, b + c]$

To find the solution set of a quadratic inequation of the form $x^2 + ax + b < 0$ or $x^2 + ax + b > 0$, the following steps are to be followed:

Step 1. *Set inequality to zero*
Step 2. *Factorise the inequality*
Step 3. *Set each factor equal to zero to obtain critical points*
Step 4. *Place critical points on the number line*
Step 5. *Use interval method for testing*
Step 6. *Write the solution set.*

Ex. Find the solution set of $x^2 - 5x + 6 \ge 0$.

Sol. $x^2 - 5x + 6 \ge 0 \Rightarrow (x - 3)(x - 2) \ge 0$.

The critical points are 3 and 2.

Plotting 2 and 3 on the number line, we have it as shown below:

The number line is divided into three parts. Now we examine the sign of the expression $(x - 3)(x - 2)$ in each of these three intervals.

As $(x - 3)(x - 2) \ge 0$, the points 2 and 3 are included here.

When $x \le 2$, $(x - 3)(x - 2) \ge 0 \Rightarrow x \in (-\infty, 2]$

When $x \ge 3$, $(x - 3)(x - 2) > 0 \Rightarrow x \in [3, \infty)$

When $2 < x < 3$, $(x - 3)(x - 2) < 0$

$\therefore$ **$x \in (-\infty, 2] \cup [3, \infty)$**

These solutions can be further remembered as given below:

(*xvii*) ***Quadratic Inequalities or Quadratic Inequations***

If $a < b$, then

(*a*) $(x - a)(x - b) > 0$

$\Rightarrow$ $x < a$ or $x > b$

or $x \in (-\infty, a) \cup (b, \infty)$

or **$x \in R - [a, b]$**

Ex. $(x - 3)(x - 5) > 0 \Rightarrow x < 3$ or $x > 5$

or $x \in (-\infty, 3) \cup (5, \infty)$

or $x \in R - [3, 5]$

(*b*) $(x - a)(x - b) \ge 0$

$\Rightarrow$ $x \le a$ or $x \ge b$

or $x \in (-\infty, a] \cup [b, \infty)$

or **$x \in R - (a, b)$**

Ex. $(x - 2)(x + 5) \ge 0$

$\Rightarrow$ $(x - 2)(x - (-5)) \ge 0$

Here $-5 < 2$

$\therefore x \le -5$ or $x \ge 2$

or $x \in (-\infty, -5] \cup [2, \infty)$

(*c*) $(x - a)(x - b) < 0$

$\Rightarrow\ a < x < b$

$\Rightarrow\ x \in (a, b)$

Ex. $(x - 4)(x + 2) < 0$

$\Rightarrow\ (x - 4)(x - (-2)) < 0$

$\Rightarrow\ -2 < x < 4 \quad (\because -2 < 4)$

$\Rightarrow\ x \in (-2, 4)$

(*d*) $(x - a)(x - b) \le 0$

$\Rightarrow\ a \le x \le b$

$\Rightarrow\ x \in [a, b]$

Ex. $(x - 2)(x - 3) \le 0$

$\Rightarrow\ 2 \le x \le 3$

$\Rightarrow\ x \in [2, 3]$.

IV. Important Inequalities

(*a*) (*i*) If $a > 0$, then $a + \dfrac{1}{a} \ge 2$, (*ii*) If $a < 0$, then $a + \dfrac{1}{a} \le -2$

(*b*) Arithmetic-Geometric Mean Inequality

For distinct positive real numbers,

Arithmetic mean > Geometric mean, *i.e.*,

If $a_1, a_2, a_3, \ldots\ldots, a_n$ are n distinct positive real numbers, then

$$\frac{a_1 + a_2 + \ldots\ldots + a_n}{n} > (a_1, a_2 \ldots\ldots, a_n)^{1/n}, \text{ i.e., A.M.} > \text{G.M.}$$

Also if $a_1 = a_2 = a_3 \ldots\ldots = \ldots\ldots = a_n$, then A.M. = G.M.

Hence for positive real numbers, $\boxed{\text{A.M.} \ge \text{G.M.}}$

(*i*) In particular,

$$\frac{a+b}{2} \ge \sqrt{ab}, \quad \frac{a+b+c}{3} \ge \sqrt[3]{abc}, \quad \frac{a+b+c+d}{4} \ge (abcd)^{1/4}$$

(*ii*) Also **A.M. $\ge$ G.M. $\ge$ H.M.**, *i.e.*,

$$\frac{a_1 + a_2 + \ldots\ldots + a_n}{n} \ge (a_1.a_2 \ldots\ldots a_n)^{1/n} \ge \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots \frac{1}{a_n}\right)}$$

(*c*) (*i*) If the *sum of two positive quantities* a_1 and a_2 *is constant,* then their *product is greatest when* $a_1 = a_2$.

(*ii*) If the *product of these quantities is a constant,* then their *sum is least when* $a_1 = a_2$.

In general, if $a_1, a_2, \ldots\ldots, a_n$ are n positive quantities such that their sum $a_1 + a_2 + a_3 + \ldots\ldots + a_n = C$ (constant), then the product $(a_1.a_2.a_3 \ldots\ldots a_n)$ is maximum when $a_1 = a_2 = a_3 = \ldots\ldots = a_n = \dfrac{C}{n}$ and the maximum product $= \left(\dfrac{C}{n}\right)^n$.

Also, if $a_1, a_2, \ldots\ldots a_n$ are n positive quantities such that their product $a_1.a_2 \ldots\ldots a_n = P$ (constant), then their sum $(a_1 + a_2 + \ldots\ldots + a_n)$ is least when $a_1 = a_2 = a_3 = \ldots\ldots = a_n = (P)^{1/n}$ and the least sum $= n\,(P^{1/n})$.

(*d*) Weighted A.M. – G.M. inequality

If $a_1, a_2, a_3 \ldots\ldots a_n$ are n positive real numbers and $m_1, m_2, m_3 \ldots\ldots m_n$ are n positive rational numbers, then

$$\frac{m_1a_1 + m_2a_2 + \ldots\ldots + m_na_n}{m_1 + m_2 + \ldots\ldots m_n} > \left(a_1^{m_1}\ a_2^{m_2} \ldots a_3^{m_n}\right)^{\frac{1}{m_1+m_2+\ldots+m_n}}$$

Weighted A.M. > Weighted G.M.

(*e*) If $a_1, a_2, a_3, \ldots\ldots, a_n$ are unequal positive real numbers and m is a positive rational number different from 0 and 1, then

(*i*) $$\left(\frac{a_1^m + a_2^m + \ldots + a_n^m}{n}\right) < \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m \textit{ if } 0 < m < 1$$

The arithmetic mean of the *m*th powers of *n* positive quantities is less than the *m*th power of their arithmetic mean if *m* lies between 0 and 1.

(ii) $\left(\frac{a_1^m + a_2^m + ... + a_n^m}{n}\right) > \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m$ **if $m < 0$ or $m > 1$**

The arithmetic mean of the *m*th powers of *n* positive qualities is greater than the *m*th power of their arithmetic mean if $m < 0$ or $m > 1$.

SOLVED EXAMPLES

Ex. 1. Solve the following linear inequations:

(*i*) $4x - 12 \le 0$ (*ii*) $-5x + 10 < 0$ (*iii*) $7x - 35 \ge 0$ (*iv*) $11x + 9 > 53$, where $x \in W$.

Sol. (*i*) $4x - 12 \le 0 \Rightarrow 4x \le 12 \Rightarrow x \le 3$

$\therefore$ Solution set of given inequation is $\mathbf{(-\infty, 3]}$

(*ii*) $-5x + 10 < 0 \Rightarrow -5x < -10$

$\Rightarrow \frac{-5x}{-5} > \frac{-10}{-5}$ [$\because$ Dividing both sides of an inequality by a negative number reverses the inequality]

$\Rightarrow x > 2 \Rightarrow$ Solution set is $\mathbf{(2, \infty)}$

(*iii*) $7x - 35 \ge 0 \Rightarrow 7x \ge 35 \Rightarrow x \ge 5$

$\Rightarrow$ Solution set is $\mathbf{[5, \infty)}$

(*iv*) $11x + 9 > 53 \Rightarrow 11x > 44 \Rightarrow x > 4$

$\therefore$ Solution set is $\mathbf{(4, \infty)}$

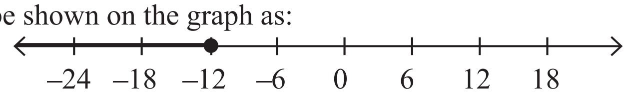
Ex. 2. Solve the inequality $4\left(\frac{1}{2} - p\right) + 7 \ge 57$ over *R* (set of real numbers).

Sol. $4\left(\frac{1}{2} - p\right) + 7 \ge 57 \Rightarrow 2 - 4p + 7 \ge 57$

$\Rightarrow -4p \ge 57 - 9 \Rightarrow -4p \ge 48 \Rightarrow 4p \le -48 \Rightarrow p \le -12$

$\therefore$ $\mathbf{p \in (-\infty, -12]}$

The solution set can be shown on the graph as:



Ex. 3. Solve the following inequation: $\frac{2x-3}{4} + 8 \ge 2 + \frac{4x}{3}; x \in R$.

Sol. $\frac{2x-3}{4} + 8 \ge 2 + \frac{4x}{3} \Rightarrow 12\left(\frac{2x-3}{4}\right) + 12 \times 8 \ge 12 \times 2 + \frac{4x}{3} \times 12$ (Multiplying each term by LCM = 12)

$\Rightarrow 3(2x - 3) + 96 \ge 24 + 16x \Rightarrow 6x - 9 + 96 \ge 24 + 16x$

$\Rightarrow 6x + 87 \ge 24 + 16x \Rightarrow 87 - 24 \ge 16x - 6x \Rightarrow 63 \ge 10x$

$\Rightarrow 10x \le 63 \Rightarrow x \le 6.3$

$\therefore$ $x \in \mathbf{(-\infty, 6.3]}$

Ex. 4. Find the solution set of $-3 < x - 2 \le 9 - 2x$; $x \in Z$ (set of integers).

Sol. $-3 < x - 2 \le 9 - 2x$

$\Rightarrow -3 < x - 2$ and $x - 2 \le 9 - 2x \Rightarrow -3 + 2 < x$ and $x + 2x \le 9 + 2$

$\Rightarrow -1 < x$ and $3x \le 11$ or $x \le \frac{11}{3} \Rightarrow -1 < x \le \frac{11}{3}$

Since $x \in Z$, so the solution set = **{0, 1, 2, 3}.**

Ex. 5. Find the range of values of x, which satisfy the inequality: $\dfrac{-1}{5} \le \dfrac{3x}{10} + 1 < \dfrac{2}{5} : x \in R$

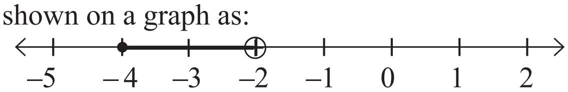
Sol. $-\dfrac{1}{5} \le \dfrac{3x}{10} + 1 < \dfrac{2}{5}$

$\Rightarrow 10 \times \left(-\dfrac{1}{5}\right) \le 10 \times \left(\dfrac{3x}{10} + 1\right) < 10 \times \dfrac{2}{5} \quad \Rightarrow -2 \le 3x + 10 < 4$

$\Rightarrow -2 \le 3x + 10$ and $3x + 10 < 4 \quad \Rightarrow -12 \le 3x$ and $3x < -6$

$\Rightarrow -4 \le x$ and $x < -2 \quad \Rightarrow -4 \le x < -2$, *i.e.*, $x \in [-4, -2)$

This solution set can be shown on a graph as:



Ex. 6. Solve the following pairs of inequations and also graph the solution set
(*i*) $2x - 9 < 7$ and $3x + 9 \le 25, x \in R$ (*ii*) $3x - 2 > 19$ or $3 - 2x \ge -7, x \in R$

Sol. (*i*) Let $A = \{x : 2x - 9 < 7, x \in R\}$

$\therefore \quad 2x - 9 < 7 \quad \Rightarrow 2x < 16 \quad \Rightarrow x < 8$

$\Rightarrow \quad A = \{x : x < 8, x \in R\}$

Let $B = \{x : 3x + 9 \le 25, x \in R\}$

$\therefore \quad 3x + 9 \le 25 \quad \Rightarrow 3x \le 16 \quad \Rightarrow x \le \dfrac{16}{3}$

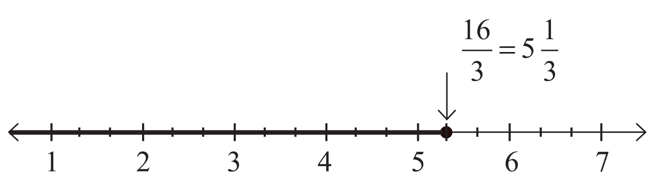
$\Rightarrow \quad B = \{x : x \le \dfrac{16}{3}, x \in R\}$

$\therefore$ Required solution set $= A \cap B$

$$= \{x : x < 8, x \in R\} \cap \left\{x : x \le \frac{16}{3}, x \in R\right\}$$

$$= \left\{x : x \le \frac{16}{3}, x \in R\right\} = x \in \left(-\infty, \frac{16}{3}\right].$$

This solution set can be shown on a graph as:



(*ii*) Let $A = \{x : 3x - 2 > 19, x \in R\}$

Then, $3x - 2 > 19 \Rightarrow 3x > 21 \Rightarrow x > 7 \Rightarrow A = \{x : x > 7, x \in R\}$

Let $B = \{x : 3 - 2x \ge -7, x \in R\}$

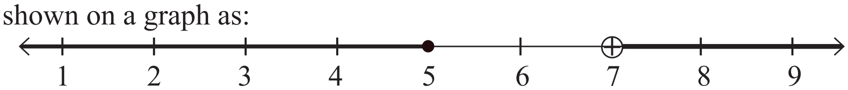
Then, $3 - 2x \ge -7 \Rightarrow 10 \ge 2x \Rightarrow 2x \le 10 \Rightarrow x \le 5 \Rightarrow B = \{x : x \le 5, x \in R\}$

Required solution set $= A$ or $B = A \cup B$

$= \{x : x > 7, x \in R\} \cup \{x : x \le 5, x \in R\}$

$= x > 7$ or $x \le 5 = x \in (7, \infty)$ or $x \in (-\infty, 5]$

This can be shown on a graph as:



Ex. 7. Solve the following inequations, $x \in R$.

(*i*) $\dfrac{1}{x-3} < 0$ (*ii*) $\dfrac{x+1}{x+2} > 1$

Sol. (*i*) $\dfrac{1}{x-3} < 0 \Rightarrow (x-3) < 0$ $[\because a/b < 0 \text{ and } a > 0 \Rightarrow b < 0]$

$\Rightarrow x < 3 \Rightarrow \boldsymbol{x \in (-\infty, 3)}$

(*ii*) $\dfrac{x+1}{x+2} > 1 \quad \Rightarrow \dfrac{x+1}{x+2} - 1 > 0 \quad \Rightarrow \dfrac{x+1-x-2}{x+2} > 0$

$\Rightarrow \dfrac{-1}{x+2} > 0 \quad \Rightarrow x + 2 < 0 \quad (\because a/b > 0, a < 0 \Rightarrow b < 0)$

$\Rightarrow x < -2 \quad \therefore \boldsymbol{x \in (-\infty, -2)}$

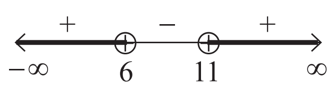
Ex. 8. Solve the following linear equations, $x \in R$

(*i*) $\dfrac{x-6}{x-11} > 0$ (*ii*) $\dfrac{x-3}{x+5} > 2$ (*iii*) $\dfrac{x-1}{x+3} \geq 2$ (*iv*) $\dfrac{5x-6}{x+6} \leq 1$

Sol. (*i*) $\dfrac{x-6}{x-11} > 0$

Equating $(x - 6)$ and $(x - 11)$ to zero, we obtain $x = 6, 11$ as the critical points.

Now we plot these points on the real number line as shown:



The real number line is divided into three regions. Now check the sign of the expression $\dfrac{x-6}{x-11}$ in all the three regions.

When $x < 6$, both numerator and denominator are negative, so $\dfrac{x-6}{x-11}$ is +ve.

When $x > 11$, both numerator and denominator are positive, so $\dfrac{x-6}{x-11}$ is +ve.

When $6 < x < 11$, the expression $\dfrac{x-6}{x-11}$ becomes –ve and hence < 0.

So, the solution set of the given inequations is the union of regions containing positive signs.

$\therefore \quad \dfrac{x-6}{x-11} > 0 \quad \Rightarrow \boldsymbol{x \in (-\infty, 6) \cup (11, \infty)}.$

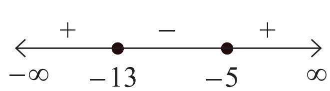
(*ii*) $\dfrac{x-3}{x+5} > 2 \quad \Rightarrow \dfrac{x-3}{x+5} - 2 > 0$

$\Rightarrow \dfrac{x-3-2x-10}{x+5} > 0 \quad \Rightarrow \dfrac{-x-13}{x+5} > 0$

$\Rightarrow \dfrac{x+13}{x+5} < 0$ (Multiplying by – 1 to make co-efficient of x positive in the expression in the numerator)

Now, putting, $(x + 13)$ and $(x + 5)$ equal to zero, we get the critical points as $x = -13, -5$. Now plot these points on the real number line as shown and divide the line into three parts.

When x lies between $-\infty$ and -13, the expression $\dfrac{x+13}{x+5}$ becomes +ve.



Similarly, when x lies between -5 and ∞, the expression becomes +ve.

The expression is negative or (<0) when x lies between –13 and –5.

Hence $\frac{x+13}{x+5} < 0 \Rightarrow x \in \mathbf{(-13, -5)}$

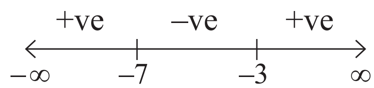
(*iii*) $\mathbf{\frac{x-1}{x+3} \geq 2} \quad \Rightarrow \frac{x-1}{x+3} - 2 \geq 0$

$\Rightarrow \frac{x-1-2x-6}{x+3} \geq 0 \quad \Rightarrow \frac{-x-7}{x+3} \geq 0 \quad \Rightarrow \frac{x+7}{x+3} \leq 0$

The critical points are –7 and –3, (on equating $x + 7 = 0$ and $x + 3 = 0$)

When $x < -7$, $\frac{x+7}{x+3}$ becomes +ve

When $x > -3$, $\frac{x+7}{x+3}$ becomes + ve



The expression $\frac{x+7}{x+3} < 0$ when x lies between –7 and –3

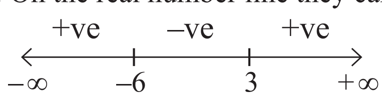
Also $\frac{x+7}{x+3} \leq 0$ when –7 is also included.

$\therefore \frac{x+7}{x+3} \leq 0 \;\; \boldsymbol{x} \in \mathbf{[-7, -3)}$

(*iv*) $\mathbf{\frac{5x-6}{x+6} \leq 1} \quad \Rightarrow \frac{5x-6}{x+6} - 1 \leq 0$

$\Rightarrow \frac{5x-6-x-6}{x+6} \leq 0 \quad \Rightarrow \frac{4x-12}{x+6} \leq 0 \quad \Rightarrow \frac{4(x-3)}{x+6} \leq 0 \quad \Rightarrow \frac{x-3}{x+6} \leq 0$

The critical points are $x = 3, -6$. On the real number line they can be shown as:



When $x < -6$ or $x > 3$, then the expresion $\frac{x-3}{x+6}$ becomes positive.

When x lies between –6 and 3 (included), *i.e.*, $-6 < x \leq 3$, $\frac{x-3}{x+6} \leq 0$.

$\Rightarrow \frac{x-3}{x+6} \leq 0 \quad \Rightarrow \quad \boldsymbol{x} \in \mathbf{(-6, 3]}$.

Ex. 9. Solve :

(*i*) $\mathbf{|2x-3| \leq \frac{1}{4}}$ (*ii*) $\mathbf{|x-4| \geq 7}$ (*iii*) $\mathbf{1 \leq |x-3| \leq 5}$

Sol. (*i*) We know that $|x-a| \leq r \quad \Rightarrow \quad (a-r) \leq x \leq (a+r)$

$\therefore \quad |2x-3| \leq \frac{1}{4} \Rightarrow \left(3 - \frac{1}{4}\right) \leq 2x \leq \left(3 + \frac{1}{4}\right) \quad \Rightarrow \quad \frac{11}{4} \leq 2x \leq \frac{13}{4}$

$\Rightarrow \frac{11}{8} \leq x \leq \frac{13}{8} \quad \Rightarrow \quad \boldsymbol{x} \in \mathbf{\left[\frac{11}{8}, \frac{13}{8}\right]}$.

(*ii*) Since $|x-a| \geq r \quad \Rightarrow x \leq a - r$ or $x \geq a + r$

$|x-4| \geq 7 \quad \Rightarrow x \leq 4 - 7$ or $x \geq 4 + 7 \quad \Rightarrow \quad x \leq -3$ or $x \geq 11$

$\Rightarrow x \in (-\infty, -3]$ or $x \in [11, \infty) \quad \Rightarrow \quad \boldsymbol{x} \in \mathbf{(-\infty, -3] \cup [11, \infty)}$.

(*iii*) Since $a \leq |x-c| \leq b \quad \Rightarrow x \in [-b+c, -a+c] \cup [a+c, b+c]$

$\therefore 1 \leq |x-3| \leq 5 \quad \Rightarrow x \in [-5+3, -1+3] \cup [1+3, 5+3] \quad \Rightarrow \quad \boldsymbol{x} \in \mathbf{[-2, 2] \cup [4, 8]}$.

Ex. 10. Solve $\dfrac{|x|-1}{|x|-2} \geq 0, x \in R, x \neq \pm 2.$

Sol. Let $|x| = y$. Then

$$\frac{|x|-1}{|x|-2} \geq 0 \Rightarrow \frac{y-1}{y-2} \geq 0$$

On equating $(y-1)$ and $(y-2)$ equal to zero, we have the critical points as $y = 1, 2$. Now using the real number line, we see that the expression $\dfrac{y-1}{y-2}$ is greater than equal to zero (positive) only when, $y \leq 1$ or $y > 2$.

$$-\infty \qquad 1 \qquad 2 \qquad \infty$$

$\Rightarrow |x| \leq 1$ or $|x| > 2$

$\Rightarrow -1 \leq x \leq 1$ or $(x < -2$ or $x > 2)$ $\left[\begin{array}{l} \because |x| \leq a \Rightarrow -a \leq x \leq a \\ |x| > a \Rightarrow x < -a \text{ or } x > a \end{array}\right]$

$\Rightarrow$ **$x \in [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$**

Ex. 11. Solve the inequation $\left|\dfrac{2}{x-4}\right| > 1, x \neq 4.$

Sol. $\left|\dfrac{2}{x-4}\right| > 1, x \neq 4$

$\Rightarrow \dfrac{2}{|x-4|} > 1 \Rightarrow 2 > |x-4| \quad \Rightarrow |x-4| < 2$

$\Rightarrow 4-2 < x < 4+2 \quad \Rightarrow 2 < x < 6$ $\qquad [|x-a| < r \Rightarrow a-r < x < a+r]$

But $x \neq 4$

$\therefore x \in (2, 4) \cup (4, 6)$

Quadratic Inequalities

Ex. 12. Solve $x^2 - 5x + 4 > 0$.

Sol. $x^2 - 5x + 4 > 0 \Rightarrow (x-1)(x-4) > 0$

Now equating $(x-1)$ and $(x-4)$ to zero, we get the critical points as 1 and 4. Plot the points 1 and 4 on the real number line and then examine the sign of the expression in the three portions of the line divided by these points.

$$-\infty \qquad 1 \qquad 4 \qquad +\infty$$

When $x < 1$, *i.e.*, $x \in (-\infty, 1)$, both the terms of the given expression are negative, hence $(x-1)(x-4) > 0$.
Similarly when $x > 4$, *i.e.*, $x \in (4, \infty)$, $(x-1)(x-4) > 0$.
When $x \in (1, 4)$, one term being +ve and other –ve, the expression $(x-1)(x-4) < 0$.
$\therefore$ For $(x-1)(x-4) > 0$, the required solution set is

$$\boldsymbol{x \in (-\infty, 1) \cup (4, \infty).}$$

Ex. 13. Find the range of values of x for which $\dfrac{x^2+x+1}{x^2+2} < \dfrac{1}{3}$, x being real.

Sol. $\dfrac{x^2+x+1}{x^2+2} - \dfrac{1}{3} < 0$

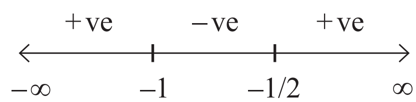
$\Rightarrow \dfrac{3x^2+3x+3-x^2-2}{3(x^2+2)} < 0 \quad \Rightarrow \dfrac{2x^2+3x+1}{3(x^2+2)} < 0$

Now we have to find the range of values in which $2x^2 + 3x + 1 < 0$ as $3(x^2 + 2)$ is +ve for all real values of x.

Now $2x^2 + 3x + 1 < 0 \quad \Rightarrow 2x^2 + 2x + x + 1 < 0$

$\Rightarrow (2x + 1)(x + 1) < 0 \quad \Rightarrow \left(x+\frac{1}{2}\right)(x + 1) < 0.$

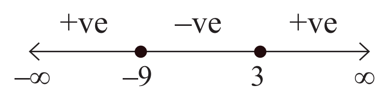
Critical points are –1 and $-\frac{1}{2}$. Plotting the critical points on the real number line and observing the sign of the expression $(2x + 1)(x + 1)$ in the intervals formed, we see that the expression $\left(x+\frac{1}{2}\right)(x+1)$ is negative or less than zero in the interval $\left(-1,-\frac{1}{2}\right)$, *i.e.*, $-1 < x < -\frac{1}{2}$.



$\boldsymbol{x \in \left(-1,-\frac{1}{2}\right)}$

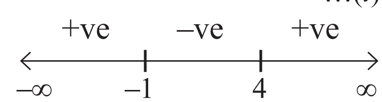
Ex. 14. Find the range of values of x which satisfy $x^2 + 6x - 27 > 0$, $-x^2 + 3x + 4 > 0$ simultaneously.

Sol. $x^2 + 6x - 27 > 0 \quad \Rightarrow \quad (x + 9)(x - 3) > 0 \quad \Rightarrow \quad x = -9, 3$



By the method of intervals we see that $(x + 9)(x - 3)$ is positive when $x < -9$ and $x > 3$

$x^2 + 6x - 27 > 0 \quad \Rightarrow \quad x \in (-\infty, -9) \cup (3, \infty)$ …(*i*)



Now $-x^2 + 3x + 4 > 0 \quad \Rightarrow \quad x^2 - 3x - 4 < 0 \Rightarrow (x - 4)(x + 1) < 0$

$\therefore \quad (x - 4)(x + 1) = 0 \quad \Rightarrow \quad x = -1, 4$

$\therefore$ The expression $(x - 4)(x + 1)$ is negative when x lies between –1 and 4.

$\therefore \quad x \in (-1, 4)$ …(*ii*)

$\therefore$ (*i*) and (*ii*) $\Rightarrow \quad x \in (3, 4)$.

Ex. 15. Prove that :

(*i*) $\boldsymbol{x+\frac{1}{x} \ge 2}$, **if $x > 0$** (*ii*) $\boldsymbol{x+\frac{1}{x} \le -2}$, **if $x < 0$** (*iii*) $\boldsymbol{\left|x+\frac{1}{x}\right| \ge 2}$, **if $x \ne 0$.**

Sol. (*i*) We have $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 \ge 0$ (Being a perfect square)

$\Rightarrow x+\frac{1}{x}-2\sqrt{x}.\frac{1}{\sqrt{x}} \ge 0 \quad \Rightarrow \quad x+\frac{1}{x}-2 \ge 0 \quad \Rightarrow \quad \boldsymbol{x+\frac{1}{x} \ge 2}$

(*ii*) Since $x < 0$, we cannot take $\sqrt{x}$ and proceed

Let $x = -a$, since x is negative and 'a' must be positive.

$\therefore \left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2 \ge 0 \quad \Rightarrow \quad a+\frac{1}{a}-2 \ge 0 \quad \Rightarrow a+\frac{1}{a} \ge 2$

Now replace a by $-x$, so $-x-\frac{1}{x} \ge 2 \Rightarrow \boldsymbol{x+\frac{1}{x} \le 2}$

(*iii*) To prove $\left|x+\frac{1}{x}\right| \ge 2$, we can prove $\left(x+\frac{1}{x}\right)^2 \ge 4$

We have $\left(x-\frac{1}{x}\right)^2 \ge 0 \quad \Rightarrow \quad x^2+\frac{1}{x^2}-2 \ge 0 \quad \Rightarrow \quad x^2+\frac{1}{x^2} \ge 2$

$\Rightarrow \quad x^2+\frac{1}{x^2}+2 \ge 4 \quad \Rightarrow \quad \boldsymbol{\left(x+\frac{1}{x}\right)^2 \ge 4}.$

Ex. 16. Prove that for any three positive reals numbers a, b, c, $a^2 + b^2 + c^2 \geq ab + bc + ca$.

Sol. $a^2 + b^2 + c^2 > ab + bc + ca$

To prove $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$

Let $S = a^2 + b^2 + c^2 - ab - bc - ca$

Then $S = \frac{1}{2}(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$

$$= \frac{1}{2}(a^2 + b^2 - 2ab + b^2 + c^2 - 2bc + c^2 + a^2 - 2ca) = \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0$$

As the RHS is the sum of squares which are positive $\Rightarrow S \geq 0$. Also the equality, *i.e.* = 0 holds when $a = b = c$.

Ex. 17. If x, y, z are all positive real numbers. Prove that $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$ **(*IIT 1984*)**

Sol. We apply the AM, GM inequality here

AM of $x, y, z = \frac{x+y+z}{3}$, GM of $x, y, z = \sqrt[3]{xyz}$

AM of $\frac{1}{x}, \frac{1}{y}, \frac{1}{z} = \frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{3}$, GM of $\frac{1}{x}, \frac{1}{y}, \frac{1}{z} = \sqrt[3]{\frac{1}{xyz}}$

$\therefore \quad \frac{x+y+z}{3} \geq \sqrt[3]{xyz}$ (Applying A.M. ≥ G.M.) …(*i*)

and $\quad \frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{3} \geq \sqrt[3]{\frac{1}{xyz}}$ …(*ii*)

Multiplying (*i*) and (*ii*), we get

$$\frac{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}{9} \geq \sqrt[3]{xyz.\frac{1}{xyz}} \quad \Rightarrow \quad \boldsymbol{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9}$$

Ex. 18. If x, y are positive real numbers such that $x + y = 1$, prove that $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) \geq 9$.

Sol. For positive real numbers, x, y applying AM ≥ GM, we have

$$\frac{x+y}{2} \geq \sqrt{xy}$$

$$\Rightarrow \frac{1}{2} \geq \sqrt{xy} \qquad (\because x + y = 1)$$

$$\Rightarrow 1 \geq 2\sqrt{xy} \quad \Rightarrow 1 \geq 4xy \quad \Rightarrow 2 \geq 8xy \quad \Rightarrow 1 + 1 \geq 8xy$$

$$\Rightarrow 1 + x + y \geq 8xy \quad \Rightarrow 1 + x + y + xy \geq 9xy \quad \Rightarrow (1 + x)(1 + y) \geq 9xy$$

$$\Rightarrow \frac{(1+x)(1+y)}{xy} \geq 9 \quad \Rightarrow \left(\frac{x+1}{x}\right)\left(\frac{y+1}{y}\right) \geq 9 \quad \Rightarrow \boldsymbol{\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) \geq 9}.$$

Ex. 19. If *a, b, c, x, y, z* are all positive, then prove that $(ab + xy) . (ax + by) \geq 4abxy$.

Sol. Applying AM - GM inequality, we have

$$\frac{ab+xy}{2} \geq \sqrt{ab.xy} \Rightarrow (ab+xy) \geq 2\sqrt{ab.xy} \quad \ldots(i)$$

and $$\frac{(ax+by)}{2} \geq \sqrt{ax.by} \Rightarrow (ax+by) \geq 2\sqrt{ab \cdot xy} \quad \ldots(ii)$$

Multiplying (*i*) and (*ii*), we get

$(ab + xy)(ax + by) \geq 4\sqrt{abxy}\sqrt{abxy} \Rightarrow$ $\mathbf{(ab + xy)(ax + by) \geq 4\ abxy.}$

Ex. 20. If *a, b, c* are all positive, $a + b + c = 1$ and $(1 - a)(1 - b)(1 - c) \geq K(abc)$, then find the value of *K*.

Sol. Given, $a + b + c = 1$, so that $a + b = 1 - c$, $b + c = 1 - a$, $c + a = 1 - b$

Now, AM $\geq$ GM $\Rightarrow (a + b) \geq 2\sqrt{ab} \Rightarrow (1 - c) \geq 2\sqrt{ab}$ …(*i*)

Similarly, $(b + c) \geq 2\sqrt{bc} \Rightarrow (1-a) \geq 2\sqrt{bc}$ …(*ii*)

$(c + a) \geq 2\sqrt{ca} \Rightarrow (1-b) \geq 2\sqrt{ca}$ …(*iii*)

Multiplying (*i*), (*ii*) and (*iii*), we get

$\Rightarrow (1 - a)(1 - b)(1 - c) \geq 8\sqrt{ab}\sqrt{bc}\sqrt{ca}$

$\Rightarrow (1 - a)(1 - b)(1 - c) \geq 8\ abc \Rightarrow$ $\mathbf{K = 8.}$

Ex. 21. Prove that 2.4.6 .... $(2n) < (n + 1)^n$.

Sol. Applying AM - GM, inequality, we have

$$\frac{2+4+6+....+2n}{n} > (2.4.6.....2n)^{1/n}$$

$$\Rightarrow \frac{1}{n}\left[\frac{n}{2}(2+2n)\right] > (2.4.6......2n)^{\frac{1}{n}} \qquad \left[\text{Sum of an AP} = \frac{n}{2}[\text{First term} + \text{Last term}]\right]$$

$\Rightarrow (n + 1) > (2.4.6......2n)^{1/n} \Rightarrow (2.4.6.....2n)^{\frac{1}{n}} < (n + 1) \Rightarrow$ $\mathbf{(2.4.6......2n) < (n + 1)^n.}$

Ex. 22. For positive real numbers *x, y, z,* prove that

$$\mathbf{2(x^3 + y^3 + z^3) \geq xy(x + y) + yz(y + z) + zx(z + x)}$$

Sol. For positive real numbers *x*, *y*, *z*, we have AM $\geq$ GM

$$\Rightarrow \frac{x^2+y^2}{2} \geq (x^2 . y^2)^{1/2} \Rightarrow x^2 + y^2 \geq 2xy$$

$\Rightarrow x^2 + y^2 - xy \geq xy$ $[\because x > y \Rightarrow x + a > y + a\ \forall\ a \in R]$

$\Rightarrow (x + y)(x^2 + y^2 - xy) \geq xy(x + y)$ $[\because x > y \Rightarrow ax > ay\ \forall\ a > 0]$

$\Rightarrow (x^3 + y^3) \geq xy(x + y)$ …(*i*)

Similarly, we can show that,

$y^3 + z^3 \geq yz(y + z)$ …(*ii*)

and $z^3 + x^3 \geq zx(z + x)$ …(*iii*)

Adding (*i*), (*ii*) and (*iii*), we have

$\Rightarrow x^3 + y^3 + y^3 + z^3 + z^3 + x^3 \geq xy(x + y) + yz(y + z) + zx(z + x)$

$\Rightarrow$ $\mathbf{2(x^3 + y^3 + z^3) \geq xy(x + y) + yz(y + z) + zx(z + x)}$

Ex. 23. Find the minimum value of $\log_n m + \log_m n$, where $m > 1$ and $n > 1$.

Sol. When $m > 1$ and $n > 1$, then, we have

$\log_n m > 0$ and $\log_m n > 0$

Now, AM > GM for positive real numbers

$$\therefore \quad \frac{1}{2}(\log_n m + \log_m n) \geq (\log_n m . \log_m n)^{1/2} \quad \Rightarrow \quad \frac{1}{2}(\log_n m + \log_m n) \geq \left(\frac{\log_e m}{\log_e n} . \frac{\log_e n}{\log_e m}\right)^{1/2}$$

$\Rightarrow \quad (\log_n m + \log_m n) \geq 2 \times 1 = 2$

$\therefore$ Minimum value of $\log_n m + \log_m n$ is 2.

Ex. 24. If a, b, c are positive real numbers, then show that $(a + 1)^7 (b + 1)^7 (c + 1)^7 > 7^7 a^4b^4c^4$. (*IIT 2004*)

Sol. LHS $= (a + 1)^7 (b + 1)^7 (c + 1)^7$

$= [(a + 1) (b + 1) (c + 1)]^7$

$= [1 + a + b + c + ab + bc + ca + abc]^7 > (a + b + c + ab + bc + ca + abc)^7$ ...(*i*)

Now using the AM, GM inequality, *i.e.,* AM $\geq$ GM, we have

$$\frac{a + b + c + ab + bc + ca + abc}{7} \geq (a.b.c.ab.bc.ca.abc)^{1/7}$$

$\Rightarrow \quad \frac{1}{7^7}(a + b + c + ab + bc + ca + abc)^7 \geq (a^4b^4c^4)$ [Raising both sides to power 7]

$\Rightarrow \quad (a + b + c + ab + bc + ca + abc)^7 \geq 7^7 (a^4 b^4 c^4)$ ...(*ii*)

From (*i*) and (*ii*)

$\boldsymbol{(a + 1)^7 (b + 1)^7 (c + 1)^7 > 7^7 a^4b^4c^4}$.

Ex. 25. If a, b, c are the sides of a ΔABC, then show that $\frac{1}{2} < \frac{ab + bc + ca}{a^2 + b^2 + c^2} \leq 1$.

Sol. We have $a > 0, b > 0, c > 0$, so

$(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$

$\Rightarrow \quad a^2 + b^2 - 2ab + b^2 + c^2 - 2bc + c^2 + a^2 - 2ca \geq 0$

$\Rightarrow \quad 2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ca \quad \Rightarrow \quad a^2 + b^2 + c^2 \geq ab + bc + ca$

$\Rightarrow \quad 1 \geq \frac{ab + bc + ca}{a^2 + b^2 + c^2} \quad \Rightarrow \quad \frac{ab + bc + ca}{a^2 + b^2 + c^2} \leq 1$ ...(*i*)

Now for the other part let us using the triangle inequality,

i.e., the sum of two sides of a triangle is greater than the third side, we have

$\therefore \quad a + b > c, \quad b + c > a, \quad c + a > b.$ Also, $a > 0, \ b > 0, \ c > 0$

$\Rightarrow \quad a + b - c > 0, \quad b + c - a > 0, \quad c + a - b > 0$

$\Rightarrow \quad a (b + c - a) + b (c + a - b) + c (a + b - c) > 0$

$\Rightarrow \quad ab + ac - a^2 + bc + ba - b^2 + ca + cb - c^2 > 0$

$\Rightarrow \quad 2 (ab + bc + ac) > a^2 + b^2 + c^2 \quad \Rightarrow \quad \frac{ab + bc + ac}{a^2 + b^2 + c^2} \geq \frac{1}{2}$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*)

$$\boldsymbol{\frac{1}{2} < \frac{ab + bc + ac}{a^2 + b^2 + c^2} \leq 1.}$$

Ex. 26. If A, B, C are the angles of an acute angled triangle, then show that $\tan A . \tan B . \tan C \geq 3\sqrt{3}$.

Sol. A, B, C being the angles of an acute angled triangle, $\tan A > 0, \tan B > 0, \tan C > 0$.

Also, $\quad A + B + C = \pi$

$\Rightarrow \quad A + B = \pi - C \quad \Rightarrow \quad \tan(A+B) = \tan(\pi - C)$

$\Rightarrow \quad \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C \quad \Rightarrow \quad \tan A + \tan B + \tan C = \tan A \,.\, \tan B \,.\, \tan C \qquad \ldots(i)$

Now applying AM $\geq$ GM, we have

$$\frac{\tan A + \tan B + \tan C}{3} \geq (\tan A \,.\, \tan B \,.\, \tan C)^{1/3}$$

$\Rightarrow \quad (\tan A + \tan B + \tan C)^3 \geq 27\,(\tan A \,.\, \tan B \,.\, \tan C)$

$\Rightarrow \quad (\tan A \,.\, \tan B \,.\, \tan C)^3 \geq 27\,(\tan A \,.\, \tan B \,.\, \tan C) \qquad$ (From (i))

$\Rightarrow \quad (\tan A \,.\, \tan B \,.\, \tan C)^2 \geq 27 \quad \Rightarrow \quad \mathbf{\tan A \,.\, \tan B \,.\, \tan C \geq 3\sqrt{3}}$**.**

Ex. 27. Show that If $m > 1$, then the sum of the mth powers of $\underset{\sim}{n}$ even numbers is greater than $n\,(n + 1)^m$.

Sol. Since $m > 1$, using the property:

AM of mth power of n quantities > mth power of the AM of n quantities, we have

$$\frac{2^m + 4^m + 6^m + \ldots. + (2n)^m}{n} > \left(\frac{2 + 4 + 6 + \ldots. + 2n}{n}\right)^m$$

$$\Rightarrow \quad \frac{2^m + 4^m + 6^m + \ldots. + (2n)^m}{n} > \frac{1}{n^m}.2^m(1 + 2 + 3 + \ldots. + n)^m$$

$$= \frac{1}{n^m}2^m.\left(\frac{n}{2}(1+n)\right)^m = \frac{1}{n^m}2^m\left(\frac{n^m}{2^m}(n+1)^m\right) = (n+1)^m$$

$\Rightarrow \quad \mathbf{2^m + 4^m + 6^m + \ldots. (2n)^m > n\,(n + 1)^m.}$

PRACTICE SHEET

1. The solution set for the inequality $2x - 10 < 3x - 15$ over the set of real numbers is

(a) (0, 5) (b) $(5, \infty)$ (c) $(-\infty, 5)$ (d) $(-5, 0)$

2. The range of values of x which satisfy the inequality $-2\frac{2}{3} \leq x + \frac{1}{3} < 3\frac{1}{3} : x \in R$ is

(a) $-3 \leq x \leq 3$ (b) $-3 \leq x < 3$

(c) $-\infty < x < 3$ (d) $3 < x < \infty$

3. $A = \{x : 11x - 5 > 7x + 3, x \in R\}$ and $B = \{x : 18x - 9 > 15 + 12x, x \in R\}$.

The range of the set $A \cap B$ is

(a) $[-\infty, 4)$ (b) (0, 4) (c) $[4, \infty]$ (d) $(-4, 4)$

4. If the inequality $\frac{6x - 5}{4x + 1} < 0$ exists over a set R of real numbers, then $x \in$

(a) $\left(\frac{1}{4}, \frac{5}{6}\right)$ (b) $\left(-\infty, \frac{1}{4}\right) \cup \left(\frac{5}{6}, \infty\right)$

(c) $\left(-\frac{1}{4}, \frac{5}{6}\right)$ (d) $\left(\frac{5}{6}, \infty\right)$

5. The solution set of $\frac{x + 3}{x - 2} \leq 2$ is

(a) $(-\infty, 2) \cup (7, \infty)$ (b) $(-\infty, 2] \cup (7, \infty)$

(c) $(-\infty, 2) \cup [7, \infty)$ (d) $(-\infty, 2] \cup [7, \infty)$

(***Kerala PET 2011***)

6. If $\frac{2x - 3}{3x - 7} > 0$ over a set of real numbers (R), then the solution set of x is

(a) $\left(\frac{3}{2}, \frac{7}{3}\right)$ (b) $\left(\frac{-3}{2}, \frac{7}{3}\right)$

(c) $\left(-\infty, \frac{3}{2}\right)$ (d) $\left(-\infty, \frac{3}{2}\right) \cup \left(\frac{7}{3}, \infty\right)$

7. The solution set of the system of inequations $\frac{x}{2x + 1} \geq \frac{1}{4}$ and $\frac{6x}{4x - 1} < \frac{1}{2}$ is

(a) $\left(-\frac{1}{2}, \frac{1}{8}\right)$ (b) $\left(\frac{-1}{8}, \frac{1}{4}\right)$

(c) $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$ (d) Null set

8. If r is real number such that $|r| < 1$ and if $a = 5\,(1 - r)$, then

(a) $-5 < a < 5$ (b) $0 < a < 10$

(c) $0 < a < 5$ (d) $-10 < a < 10$

9. For $\frac{|x - 1|}{x + 2} < 1$, x lies in the interval

(a) $(-\infty, -4)$ (b) $(-\infty, -2) \cup \left(-\frac{1}{2}, \infty\right)$

(c) $\left[-\frac{1}{2}, 1\right]$ (d) $(-\infty, 1) \cup [2, 3]$

(***AMU 2009***)

10. If $|x - 4| < 9$, then

(*a*) $|x| < 13$ (*b*) $|x - 13| < 0$

(*c*) $-5 < x < 13$ (*d*) $-13 < x < 13$

11. If $|x + 3| \geq 7$, then $x \in$

(*a*) $(-10, 4)$ (*b*) $(-\infty, -10] \cup [4, \infty)$

(*c*) $[-10, 4]$ (*d*) $[-10, \infty]$

12. If $\frac{-1}{|x|-2} \geq 1$, where $x \in R$, $x \neq \pm 2$ then the solution set of x is

(*a*) $(-\infty, 2) \cup (1, \infty)$ (*b*) $(-2, -1]$

(*c*) $[1, \infty)$ (*d*) $(-2, -1] \cup [1, 2)$

13. The set of values for which $x^3 + 1 \geq x^2 + x$ is

(*a*) $x \geq 0$ (*b*) $x \leq 0$ (*c*) $x \geq -1$ (*d*) $-1 \leq x \leq 1$

14. Solve $(|x - 1| - 3)(|x + 2| - 5) < 0$. Then

(*a*) $-7 < x < -2$ (*b*) $3 < x < 4$

(*c*) $x < -7$ and $x > 4$ (*d*) Both (*a*) and (*b*)

15. If $-x^2 + 3x + 4 > 0$, then which of the following is correct?

(*a*) $x \in (-1, 4)$ (*b*) $x \in [-1, 4]$

(*c*) $x \in (-\infty, -1) \cup (4, \infty)$ (*d*) $x \in (-\infty, 1] \cup [4, \infty)$

(*NDA/NA 2003*)

16. Which of the following values of x do not satisfy the inequality $x^2 - 3x + 2 > 0$ at all.

(*a*) $1 \leq x \leq 2$ (*b*) $-1 \geq x \geq -2$

(*c*) $0 \leq x \leq 2$ (*d*) $0 \geq x \geq -2$ **(*CAT*)**

17. What values of x satisfy $x^{2/3} + x^{1/3} - 2 \leq 0$?

(*a*) $-8 \leq x \leq 8$ (*b*) $-8 \leq x \leq 1$

(*c*) $-1 \leq x \leq 8$ (*d*) $1 \leq x \leq 8$ **(*CAT 2006*)**

18. Find the complete set of values that satisfy the inequalities $||x| - 3| < 2$ and $||x| - 2| < 3$.

(*a*) $(-5, 5)$ (*b*) $(-5, -1) \cup (1, 5)$

(*c*) $(1, 5)$ (*d*) $(-1, 1)$ **(*CAT 2012*)**

19. Find the range of real values of x for which $\frac{x-1}{4x+5} < \frac{x-3}{4x-3}$.

(*a*) $\left(-\infty, \frac{-5}{4}\right) \cup \left(\frac{3}{4}, \infty\right)$ (*b*) $\left(\frac{-3}{4}, \frac{5}{4}\right)$

(*c*) $\left(-\infty, \frac{-5}{4}\right] \cup \left[\frac{3}{4}, \infty\right)$ (*d*) $\left(\frac{-5}{4}, \frac{3}{4}\right)$

20. If x is real and $x^2 + 3x + 2 > 0$, $x^2 - 3x - 4 \leq 0$, then which of the following is correct?

(*a*) $-1 \leq x < 4$ (*b*) $2 \leq x \leq 4$

(*c*) $-1 < x \leq 1$ (*d*) $-1 \leq x < 1$ or $2 < x \leq 4$

(*NDA/NA 2008*)

21. The set of all real numbers x, for which $x^2 - |x + 2| + x > 0$, is

(*a*) $(-\infty, -2) \cup (2, \infty)$ (*b*) $(\sqrt{2}, \infty)$

(*c*) $(-\infty, -1) \cup (1, \infty)$ (*d*) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

(*IIT 2002*)

22. If $\log_e(x^2 - 16) \leq \log_e(4x - 11)$, then

(*a*) $-1 \leq x \leq 5$ (*b*) $x < -1$ or $x > 5$

(*c*) $4 < x \leq 5$ (*d*) $x < -4$ or $x > 4$

(*WBJEE 2012*)

23. If x, y, z are positive real numbers, then $(x^2 + y^2 + z^2) \geq$

(*a*) xyz (*b*) $x^3 + y^3 + z^3$

(*c*) $xy + yz + zx$ (*d*) $x + y + z$

24. If $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1.a_2.a_3 \ldots a_n = 1$, then their sum is

(*a*) a positive integer (*b*) divisible by n

(*c*) never less than n (*d*) none of these **(*IIT 1991*)**

25. If $a^2 + b^2 + c^2 = 1$, $x^2 + y^2 + z^2 = 1$, where a, b, c, x, y, z are positive reals then $ax + by + cz$ is

(*a*) ≥ 1 (*b*) ≥ 2

(*c*) ≤ 1 (*d*) None of these

26. If x, y, z are three positive numbers, then the minimum value of $\frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z}$ is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 6

(*AMU 2010*)

27. If $a_n \geq 1$ for all $n \in N$ $(n \geq 3)$, then the minimum value of $\log_{a_2} a_1 + \log_{a_3} a_2 + \log_{a_4} a_3 + \ldots + \log_{a_1} a_n$ is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) None of these

28. If a, b, c, x, y, z are all positve real numbers, then $\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right) \geq$

(*a*) 8 (*b*) 64 abc (*c*) xyz (*d*) 9

29. For positive real numbers a, b, c, the least value of $a^{\log b - \log c} + b^{\log c - \log a} + c^{\log a - \log b}$ is

(*a*) 0 (*b*) 1 (*c*) 3 (*d*) 6

30. If a, b, c, d are four distinct positive real numbers and if $3s = a + b + c + d$, then

(*a*) $abcd > 81(s - a)(s - b)(s - c)(s - d)$

(*b*) $abcd < 9(s - a)(s - b)(s - c)(s - d)$

(*c*) $abcd < 18(s - a)(s - b)(s - c)(s - d)$

(*d*) $abcd < 27(s - a)(s - b)(s - c)(s - d)$

31. If $\left(\frac{n+1}{2}\right)^n \left(\frac{2n+1}{3}\right)^n > (n!)^k$, then $k =$

(*a*) 1 (*b*) 3 (*c*) 2 (*d*) 4

32. If a, b, c be the lengths of the sides of a triangle and $(a + b + c)^3 \geq k(a + b - c)(b + c - a)(c + a - b)$, then k equals

(*a*) 1 (*b*) 3 (*c*) 8 (*d*) 27

33. Let a, b, c be the lengths of the sides of a right angled triangle, the hypotenuse having the length c, then $a + b$ is

(*a*) $> 2c$ (*b*) $< \sqrt{2}c$ (*c*) $> \sqrt{2}c$ (*d*) $< 2c$

34. If a, b, c are the sides of a triangle, then

$$\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c} \text{ is}$$

(a) ≤ 3 (b) ≥ 2 (c) ≥ 3 (d) ≤ 2

35. For three distinct positive real numbers a, b, c $(1 + a^3)(1 + b^3)(1 + c^3)$ is greater than

(a) abc (b) $(1 + abc)$ (c) $(1 + abc)^3$ (d) $(1 + abc)^2$

36. If a, b, c, d are positive reals such that $a + b + c + d = 2$, then $M = (a + b)(c + d)$ satisfies the relation

(a) $0 \leq M \leq 1$ (b) $1 \leq M \leq 2$
(c) $2 \leq M \leq 3$ (d) $3 \leq M \leq 4$ ***(IIT 2000)***

37. If $a_1, a_2, a_3 \,.......\, a_n$ are positive real numbers whose product is a fixed number 'c', then the minimum value of $a_1 + a_2 \,.....\, + a_{n-1} + 2a_n$ is

(a) $n(2c)^{1/n}$ (b) $(n + 1)c^{1/n}$ (c) $2nc^{1/n}$ (d) $(n+1)(2c)^{1/n}$ ***(IIT 2002)***

38. If a, b, c are the lengths of the sides of a non-equilateral triangle, then $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$ is

(a) $> \frac{s}{a+b+c}$ (b) $> \frac{2s}{abc}$

(c) $< \frac{2s}{abc}$

(d) None of these, where $s = \frac{a+b+c}{2}$.

39. Let a, b, c be positive numbers, then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \text{ is}$$

(a) $\geq \frac{3}{2}$ (b) ≥ 4

(c) $\leq \frac{3}{2}$ (d) None of these

40. Let a, b, c be positive numbers lying in the interval (0, 1], then $\frac{a}{1+b+ca}+\frac{b}{1+c+ab}+\frac{c}{1+a+bc}$ is

(a) < 0 (b) > 0 (c) ≤ 1 (d) ≤ -1

41. If $\log_{10}(x^3 + y^3) - \log_{10}(x^2 + y^2 - xy) \leq 2$, then the maximum value of xy for all $x \geq 0$, $y \geq 0$ is

(a) 1200 (b) 2500 (c) 3000 (d) 3500 ***(DCE 2009)***

42. The minimum value of $3^{\sin^2 x} + 3^{\cos^2 x}, x \in R$ is

(a) $\frac{1}{2}$ (b) 3 (c) 6 (d) $2\sqrt{3}$

43. When a and b are positive reals, prove that $(a^4 + b^4)(a^5 + b^5)$ is

(a) $> 2(a^9 + b^9)$ (b) $< 2(a^9 + b^9)$
(c) $= 2(a^9 + b^9)$ (d) None of these

44. The minimum value of the sum of real numbers a^{-5}, a^{-4}, $3a^{-3}$, 1, a^8 and a^{10} with $a > 0$ is

(a) $\frac{1}{2}$ (b) 6 (c) 8 (d) $\frac{10}{3}$ ***(IIT 2011)***

45. If a, b, c are the sides of a non-equilateral triangle, then the expression $(b + c - a)(c + a - b)(a + b - c) - abc$ is

(a) negative (b) non-negative
(c) positive (d) non-positive ***(IIT 1986)***

46. If a, b, c are distinct positive integers, then $ax^{b-c} + bx^{c-a} + cx^{a-b}$ is

(a) > 3 (b) $> \frac{1}{a+b+c}$
(c) < 0 (d) $> a + b + c$

47. If a, b, c are positive real numbers such that $a + b + c = p$, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ is greater than

(a) 16 (b) $9p$ (c) $\frac{9}{p}$ (d) $\frac{16}{p}$

48. If a, b, c are the lengths of the sides of a non-equilateral triangle, then, $\frac{1}{a+b-c}+\frac{1}{b+c-a}+\frac{1}{c+a-b}$ is

(a) $> \frac{9}{a+b+c}$ (b) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$
(c) Both (a) and (b) (d) None of these

49. If $a_1, a_2, \,.....,\, a_n$ are distinct positive real numbers such that $a_1 + a_2 + \,.....\, + a_n = 1$, then $\frac{1}{a_1}+\frac{1}{a_2}+....+\frac{1}{a_n}$ is

(a) $> n$ (b) $> n^2$
(c) $> \frac{1}{n}$ (d) $> n^3$

50. If A, B, C are the angles of an acute angled triangle, then $\cot {}^{A}/_{2} \,.\, \cot {}^{B}/_{2} \,.\, \cot {}^{C}/_{2} \geq$

(a) $4\sqrt{3}$ (b) 8 (c) $3\sqrt{3}$ (d) 3

ANSWERS

1. (b)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (c)	**6.** (d)	**7.** (d)	**8.** (b)
9. (b)	**10.** (c)	**11.** (b)	**12.** (d)	**13.** (c)	**14.** (d)	**15.** (a)	**16.** (a)
17. (b)	**18.** (b)	**19.** (d)	**20.** (d)	**21.** (d)	**22.** (a)	**23.** (c)	**24.** (c)
25. (c)	**26.** (d)	**27.** (d)	**28.** (d)	**29.** (c)	**30.** (a)	**31.** (c)	**32.** (d)
33. (b)	**34.** (c)	**35.** (c)	**36.** (a)	**37.** (a)	**38.** (b)	**39.** (a)	**40.** (c)
41. (b)	**42.** (d)	**43.** (b)	**44.** (c)	**45.** (a)	**46.** (d)	**47.** (c)	**48.** (c)
49. (a)	**50.** (c)						

HINTS AND SOLUTIONS

1. $2x - 10 < 3x - 15 \Rightarrow -10 + 15 < 3x - 2x$

$\Rightarrow 5 < x \Rightarrow x > 5 \Rightarrow \boldsymbol{x \in (5, \infty)}$.

2. $-2\frac{2}{3} \le x + \frac{1}{3} < 3\frac{1}{3} \Rightarrow \frac{-8}{3} \le x + \frac{1}{3} < \frac{10}{3}$

$\Rightarrow \frac{-8}{3} - \frac{1}{3} \le x < \frac{10}{3} - \frac{1}{3}$

$\Rightarrow \frac{-9}{3} \le x < \frac{9}{3} \Rightarrow \boldsymbol{-3 \le x < 3}$.

3. $A = \{x : 11x - 5 > 7x + 3, x \in R\}$

$\Rightarrow 11x - 5 > 7x + 3 \Rightarrow 4x > 8 \Rightarrow x > 2$

$\Rightarrow A = \{x : x > 2, x \in R\}$

$B = \{x : 18x - 9 \ge 15 + 12x, x \in R\}$

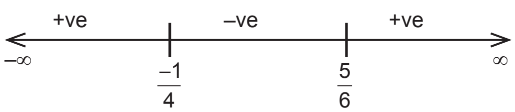
$\Rightarrow 18x - 9 \ge 15 + 12x \Rightarrow 6x \ge 24 \Rightarrow x \ge 4$

$\Rightarrow B = \{x : x \ge 4, x \in R\}$

$\therefore A \cap B = \{x : x > 2, x \in R\} \cap \{x : x \ge 4, x \in R\}$

$\Rightarrow x \ge 4 \Rightarrow \boldsymbol{x \in [4, \infty)}$.

4. $\frac{6x-5}{4x+1} < 0$ on putting $(6x - 5)$ and $(4x + 1)$ equal to zero, we get the critical points as $x = \frac{5}{6}$ and $x = -\frac{1}{4}$. On the real number line they can be shown as:



- When $x < -\frac{1}{4}$, both numerator and denominator of the expression $\frac{6x-5}{4x+1}$ are negative making the expression positive.
- When x lies between $-\frac{1}{4}$ and $\frac{5}{6}$, the expression $\frac{6x-5}{4x+1}$ becomes negative, hence less than zero.
- When $x > \frac{5}{6}$, both $(6x - 5)$ and $(4x + 1)$ are positive, making the expression $\frac{6x-5}{4x+1}$ positive and hence greater than zero.

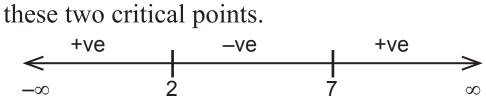
$\therefore$ For $\frac{6x-5}{4x+1} < 0$, $\boldsymbol{x \in \left(-\frac{1}{4}, \frac{5}{6}\right)}$.

5. $\frac{x+3}{x-2} \le 2 \Rightarrow \frac{x+3}{x-2} - 2 \le 0 \Rightarrow \frac{x+3-2x+4}{x-2} \le 0$

$\Rightarrow \frac{7-x}{x-2} \le 0 \Rightarrow \frac{x-7}{x-2} \ge 0$ Here $x \ne 2$

The critical points on putting $(x - 7)$ and $(x - 2)$ are obtained as equal to zero are 7 and 2.

The real number line is divided into 3 parts as shown below by these two critical points.



When $x < 2$, the expression $\frac{x-7}{x-2} > 0$.

When x lies between 2 and 7, the expression $\frac{x-7}{x-2} < 0$.

When $x \ge 7$, the expression $\frac{x-7}{x-2} \ge 0$.

$\therefore$ The inequality $\frac{x+3}{x-2} \le 2$ holds when $x < 2$ or $x \ge 7$, *i.e.*, $x \in (-\infty, 2) \cup [7, \infty)$.

6. Similar to Q. 5. Also see Solved Example 8 (*ii*).

7. $\frac{x}{2x+1} \ge \frac{1}{4}$ and $\frac{6x}{4x-1} < \frac{1}{2}$

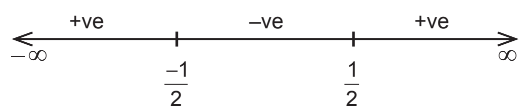
$\Rightarrow \frac{x}{2x+1} - \frac{1}{4} \ge 0$ and $\frac{6x}{4x-1} - \frac{1}{2} < 0$

$\Rightarrow \frac{4x-2x-1}{4(2x+1)} \ge 0$ and $\frac{12x-4x+1}{2(4x-1)} < 0$

$\Rightarrow \frac{2x-1}{2x+1} \ge 0$ and $\frac{8x+1}{4x-1} < 0$

Now for $\frac{2x-1}{2x+1} \ge 0$, $x \ne \frac{1}{2}$

The critical points are $\frac{-1}{2}$ and $\frac{1}{2}$ which are plotted on the real number line as shown below:

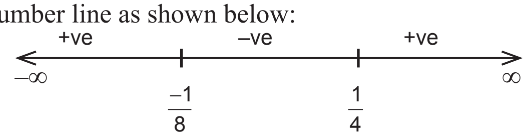


Examining the expression for the three parts of the number line, *i.e.*, $x < -\frac{1}{2}$, $\frac{-1}{2} < x \le \frac{1}{2}$, $x \ge \frac{1}{2}$, we see that $\frac{2x-1}{2x+1} \ge 0$ when $x < -\frac{1}{2}$ or $x \ge \frac{1}{2}$

$\therefore x \in \left(-\infty, -\frac{1}{2}\right) \cup \left[\frac{1}{2}, \infty\right)$ ...(*i*)

Now for $\frac{8x+1}{4x-1} < 0$, $x \ne \frac{1}{4}$

The critical points are $-\frac{1}{8}, \frac{1}{4}$ which are plotted on the real number line as shown below:



The expression $\frac{8x+1}{4x-1} < 0$, when $x \in \left(\frac{-1}{8}, \frac{1}{4}\right)$ ...(*ii*)

$\therefore$ It is clear from (i) and (ii) that the intersection of the solution sets given by (i) and (ii), *i.e.*,

$\left\{\left(-\infty, -\frac{1}{2}\right) \cup \left[\frac{1}{2}, \infty\right)\right\} \cap \left(-\frac{1}{8}, \frac{1}{4}\right) = \phi$.

8. $|r| < 1$

$\Rightarrow\ -1 < r < 1$

$\Rightarrow\ 1 > -r > -1 \Rightarrow -1 < -r < 1$

($\because\ a > b \Rightarrow ax < bx$, where $x < 0$)

$\Rightarrow\ 0 < 1 - r < 2$

(Adding 1, $c < a < b \Rightarrow c + x < a + x < b + x\ \forall\ x \in R$)

$\Rightarrow\ 0 \times 5 < 5\,(1 - r) < 5 \times 2$

($c < a < b \Rightarrow cx < ax < bx\ \forall\ x \in R,\ x > 0$)

$\Rightarrow\ 0 < 5\,(1 - r) < 10$

$\Rightarrow\ \mathbf{0 < a < 10.}$

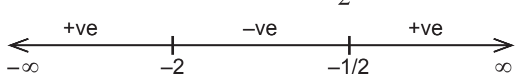
9. **Case I:** When $x - 1 < 0$, *i.e.*, $x < 1$, then $|x - 1| = -(x - 1) = 1 - x$

$\therefore$ Given expression $= \dfrac{1-x}{x+2} < 1$

$\Rightarrow\ \dfrac{1-x}{x+2} - 1 < 0 \Rightarrow \dfrac{1-x-x-2}{x+2} < 0$

$\Rightarrow\ \dfrac{-2x-1}{x+2} < 0 \Rightarrow \dfrac{2x+1}{x+2} > 0$

Now critical points are $x = -2,\ -\dfrac{1}{2}$.



Examining the expression $\dfrac{2x+1}{x+2}$ in the intervals marked by the critical points on the number line, we see that $\dfrac{2x+1}{x+2} > 0$,

when $x \in (-\infty, -2)$ or $x \in \left(-\dfrac{1}{2}, \infty\right)$. Here $x < 1$, therefore the solution set is $x \in (-\infty, -2) \cup \left(-\dfrac{1}{2}, 1\right)$ …(*i*)

Case II : When $x - 1 \geq 0$, *i.e.*, when $x \geq 1$

$|x - 1| = (x - 1)$

$\therefore$ Given expression $= \dfrac{x-1}{x+2} < 1$

$\Rightarrow\ \dfrac{x-1}{x+2} - 1 < 0 \Rightarrow \dfrac{x-1-x-2}{x+2} < 0 \Rightarrow \dfrac{-3}{x+2} < 0$

This is possible only when $x + 2 > 0 \Rightarrow x > -2$

Here $x \geq 1\ \ \therefore\ x \in [1, \infty)$ …(*ii*)

$\therefore$ From (*i*) and (*ii*) $x \in (-\infty, -2) \cup \left(-\dfrac{1}{2}, 1\right) \cup [1, \infty)$

$\Rightarrow\ \boldsymbol{x \in (-\infty, -2) \cup \left(-\dfrac{1}{2}, \infty\right)}$

10. $|x - 4| < 9\ \Rightarrow\ -9 < x - 4 < 9$

$\Rightarrow\ -9 + 4 < x < 9 + 4\ \Rightarrow\ \mathbf{-5 < x < 13.}$

11. $|x + 3| \geq 7$

$\Rightarrow\ x + 3 \leq -7$ or $x + 3 \geq 7$

$\Rightarrow\ x \leq -10$ or $x \geq 4 \Rightarrow x \in (-\infty, -10]$ or $x \in [4, \infty)$

$\Rightarrow\ \boldsymbol{x \in (-\infty, -10] \cup [4, \infty).}$

12. $\dfrac{-1}{|x|-2} \geq 1 \Rightarrow \dfrac{-1}{|x|-2} - 1 \geq 0$

$\Rightarrow\ \dfrac{-1-(|x|-2)}{|x|-2} \geq 0\ \Rightarrow\ \dfrac{1-|x|}{|x|-2} \geq 0$

$\Rightarrow\ \dfrac{|x|-1}{|x|-2} \leq 0\ \Rightarrow\ \dfrac{y-1}{y-2} \leq 0$, where $y = |x|$

Examining $\dfrac{y-1}{y-2}$ on the real number line shown below:

We see that $\dfrac{y-1}{y-2} \leq 0$ when $1 \leq y < 2$

$\Rightarrow\ 1 \leq |x| < 2\ \Rightarrow\ |x| \geq 1$ and $|x| < 2$

$\Rightarrow\ x \leq -1$ or $x \geq 1$ and $-2 < x < 2$

$\Rightarrow\ \boldsymbol{x \in (-2, -1] \cup [1, 2).}$

13. $x^3 + 1 \geq x^2 + x\ \Rightarrow\ x^3 + 1 - x^2 - x \geq 0$

$\Rightarrow\ x^3 - x^2 + 1 - x \geq 0\ \Rightarrow\ x^2(x - 1) - (x - 1) \geq 0$

$\Rightarrow\ (x^2 - 1)(x - 1) \geq 0\ \Rightarrow\ (x + 1)(x - 1)^2 \geq 0$

As $(x - 1)^2$ is +ve so the given expression is ≥ 0, when $x + 1 \geq 0 \Rightarrow \boldsymbol{x \geq -1.}$

14. $(|x - 1| - 3)(|x + 2| - 5) < 0$

The product of the two factors is < 0, *i.e.*, negative when one of the factors is positive and one negative.

Case I : $(|x - 1| - 3) > 0$ and $(|x + 2| - 5) < 0$

$\Rightarrow\ |x - 1| > 3$ and $|x + 2| < 5$

$\Rightarrow\ -(x - 1) > 3,\ (x - 1) > 3$ and $-(x + 2) < 5,\ x + 2 < 5$

$\Rightarrow\ -x > 2,\ x > 4$ and $-x < 7,\ x < 3$

$\Rightarrow\ x < -2,\ x > 4$ and $x > -7,\ x < 3$

Combining we get $-7 < x < -2$

Case II : $(|x - 1| - 3) < 0$ and $(|x + 2| - 5) > 0$

$\Rightarrow\ |x - 1| < 3$ and $|x + 2| > 5$

$\Rightarrow\ -(x - 1) < 3,\ (x - 1) < 3$ and $(x + 2) > 5,\ -(x + 2) > 5$

$\Rightarrow\ -x < 2,\ x < 4$ and $x > 3,\ -x > 7$

$\Rightarrow\ x > -2,\ x < 4$ and $x > 3,\ x < -7$

$\Rightarrow\ 3 < x < 4$

$\therefore$ **$-7 < x < -2$ and $3 < x < 4$ is the solution.**

15. $-x^2 + 3x + 4 > 0$

$\Rightarrow\ x^2 - 3x - 4 < 0\ \Rightarrow\ (x - 4)(x + 1) < 0$

$\Rightarrow\ (x - 4) < 0,\ (x + 1) > 0$ or $(x - 4) > 0,\ (x + 1) < 0$

$\Rightarrow\ x < 4,\ x > -1$ or $x > 4,\ x < -1$

$\Rightarrow\ \mathbf{-1 < x < 4.}$

16. $x^2 - 3x + 2 > 0\ \Rightarrow\ (x - 2)(x - 1) > 0$

$\Rightarrow\ (x - 2) > 0,\ (x - 1) < 0$ or $(x - 2) < 0,\ (x - 1) > 0$

$\Rightarrow\ x > 2,\ x < 1$ or $x < 2,\ x > 1\ \Rightarrow\ x < 1$ or $x > 2$

$\therefore$ No value of x which lies between these extremes, *i.e.*, 1 and 2 satisfies the inequality, *i.e.*, $1 \leq x \leq 2$ is the solution set not satisfying the inequality $x^2 - 3x + 2 > 0$ at all.

17. $x^{2/3} + x^{1/3} - 2 \le 0$

$\Rightarrow \quad y^2 + y - 2 \le 0$, where $y = x^{1/3}$

$\Rightarrow \quad (y-1)(y+2) \le 0$

$\Rightarrow \quad (y-1) \le 0, (y+2) \ge 0$ or $(y-1) \ge 0, (y+2) \le 0$

$\Rightarrow \quad y \le 1, y \ge -2$ or $y \ge 1, y \le -2$

$\Rightarrow \quad -2 \le y \le 1 \quad \Rightarrow \quad -2 \le x^{1/3} \le 1$

$\Rightarrow \quad \mathbf{-8 \le x \le 1.}$

18. Let $|x| = p$, where $p \ge 0$ ...(*i*)

So $|p-3| < 2$ and $|p-2| < 3$ ...(*ii*)

$\Rightarrow \quad 1 < p < 5$ and $-1 < p < 5$ ...(*iii*)

$(\because |x-a| < r \Rightarrow a - r < x < a + r)$

Therefore, the conditions (*i*), (*ii*) and (*iii*) are satisfied by $1 < p < 5$, *i.e.* $1 < |x| < 5$, *i.e.,* $|x| > 1$ and $|x| < 5$

i.e., $x < -1$ or $x > 1$ and $-5 < x < 5$

$\Rightarrow \quad \mathbf{x \in (-5, -1) \cup (1, 5).}$

19. $\dfrac{x-1}{4x+5} < \dfrac{x-3}{4x-3}$ or $\dfrac{x-1}{4x+5} - \dfrac{x-3}{4x-3} < 0$

$$\Rightarrow \quad \frac{(4x-3)(x-1)-(x-3)(4x+5)}{(4x+5)(4x-3)} < 0$$

$$\Rightarrow \quad \frac{(4x^2-3x-4x+3)-(4x^2-12x+5x-15)}{(4x+5)(4x-3)} < 0$$

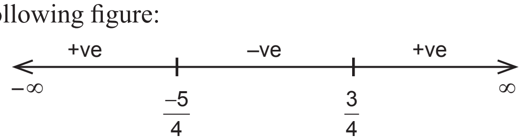
$$\Rightarrow \quad \frac{4x^2-7x+3-4x^2+7x+15}{(4x+5)(4x-3)} < 0$$

$$\Rightarrow \quad \frac{18}{(4x+5)(4x-3)} < 0$$

$\Rightarrow \quad (4x+5)(4x-3) < 0$ as $18 > 0$

$$\Rightarrow \quad 16\left(x+\frac{5}{4}\right)\left(x-\frac{3}{4}\right) < 0$$

Putting each factor equal to zero, the critical points are $-\dfrac{5}{4}, \dfrac{3}{4}$. Plotting on the real number line, we have the following figure:

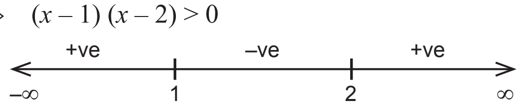


Thus the expression $16\left(x+\dfrac{5}{4}\right)\left(x-\dfrac{3}{4}\right)$ is positive when $x < -\dfrac{5}{4}$ or $x > \dfrac{3}{4}$

$\therefore$ The required range $= \left(-\dfrac{5}{4}, \dfrac{3}{4}\right)$.

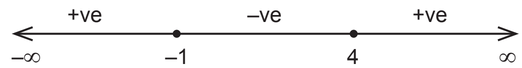
20. $x^2 - 3x + 2 > 0$

$\Rightarrow \quad (x-1)(x-2) > 0$



$\Rightarrow \quad x \in (-\infty, 1) \cup (2, \infty)$ ...(*i*)

and $x^2 - 3x - 4 \le 0 \Rightarrow (x-4)(x+1) \le 0$



$\Rightarrow \quad -1 \le x \le 4$ ...(*ii*)

$\therefore$ Combining the solution sets in (*i*) and (*ii*), we have

$\mathbf{-1 < x < 1 \text{ or } 2 < x \le 4}$

21. Case I: When $x + 2 \ge 0$, then $x \ge -2$

$\Rightarrow \quad |x+2| = x + 2$

$\therefore \quad x^2 - |x+2| + x > 0 = x^2 - (x+2) + x > 0$

$\Rightarrow \quad x^2 - 2 > 0 \quad \Rightarrow \quad x^2 > 2$

$\Rightarrow \quad x < -\sqrt{2}$ or $x > \sqrt{2}$

$\Rightarrow \quad x \in (-2, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ $[\because x \ge -2]$ ...(*i*)

Case II: When $x + 2 < 0$, then $x < -2$

$\Rightarrow \quad |x+2| = -(x+2)$

$\therefore \quad x^2 - |x+2| + x > 0 \Rightarrow x^2 + (x+2) + x > 0$

$\Rightarrow \quad x^2 + 2x + 2 > 0 \Rightarrow (x^2 + 2x + 1) + 1 > 0$

$\Rightarrow \quad (x+1)^2 + 1 > 0$, which is true for all value of x.

$\therefore \quad x < -2 \quad \Rightarrow \quad x \in (-\infty, -2)$ ...(*ii*)

From (*i*) and (*ii*)

$x \in (-2, -\sqrt{2}) \cup (\sqrt{2}, \infty) \cup (-\infty, -2)$

$\Rightarrow \quad \mathbf{x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)}$

22. $\log_e (x^2 - 16) < \log_e (4x - 11)$

$(x^2 - 16) \le 4x - 11$

$(\because a > 1, \log_a f(x) > \log_a (gx) \Rightarrow f(x) > g(x) > 0)$

$\Rightarrow \quad x^2 - 4x - 5 \le 0 \quad \Rightarrow \quad (x-5)(x+1) < 0$

$\Rightarrow \quad \mathbf{-1 \le x \le 5}$

$(\because$ If $a < b$, then $(x-a)(x-b) \le 0 \Rightarrow a \le x \le b)$

23. Since $x > 0, y > 0, z > 0$, therefore

$(x-y)^2 \ge 0, (y-z)^2 \ge 0, (z-x)^2 \ge 0$

$\Rightarrow \quad x^2 + y^2 - 2xy \ge 0$

$y^2 + z^2 - 2yz \ge 0$

$z^2 + x^2 - 2zx \ge 0$

$\Rightarrow \quad x^2 + y^2 \ge 2xy, y^2 + z^2 \ge 2yz, z^2 + x^2 \ge 2zx$

$\Rightarrow \quad x^2 + y^2 + y^2 + z^2 + z^2 + x^2 > 2xy + 2yz + 2zx$

$\Rightarrow \quad 2(x^2 + y^2 + z^2) \ge 2(xy + yz + zx)$

$\Rightarrow \quad \mathbf{x^2 + y^2 + z^2 \ge xy + yz + zx.}$

24. For positive numbers we know that AM $\ge$ GM

$$\therefore \quad \left(\frac{a_1 + a_2 + a_3 + ... + a_n}{n}\right) \ge (a_1.a_2.a_3...a_n)^{\frac{1}{n}}$$

$$\Rightarrow \quad \left(\frac{a_1 + a_2 + a_3 + ... + a_n}{n}\right) \ge 1^{\frac{1}{n}}$$

$$\Rightarrow \quad \left(\frac{a_1 + a_2 + a_3 + ... + a_n}{n}\right) \ge 1$$

$\Rightarrow \quad \mathbf{a_1 + a_2 + a_3 + ... + a_n \ge n.}$

25. Applying AM $\geq$ GM, we have

$\frac{a+x}{2} \geq \sqrt{ax} \Rightarrow a+x \geq 2\sqrt{ax}$

$\Rightarrow (a+x)^2 \geq 4ax \Rightarrow a^2+x^2+2ax \geq 4ax$

$\Rightarrow a^2+x^2 \geq 2ax$

Similarly, $b^2+y^2 \geq 2by$

$c^2+z^2 \geq 2cz$

$\therefore a^2+x^2+b^2+y^2+c^2+z^2 \geq 2ax+2by+2cz$

$\Rightarrow (a^2+b^2+c^2)+(x^2+y^2+z^2) \geq 2(ax+by+cz)$

$\Rightarrow 2 \geq 2(ax+by+cz) \Rightarrow 1 \geq ax+by+cz$

$\Rightarrow \boldsymbol{ax+by+cz \leq 1}$**.**

26. $x > 0, y > 0, z > 0$

$\Rightarrow \frac{x}{y}, \frac{y}{x}, \frac{y}{z}, \frac{z}{y}, \frac{x}{z}, \frac{z}{x}$ are all positive numbers.

$\therefore$ Applying AM – GM inequality, we have

$\Rightarrow \frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right) \geq \left(\frac{x}{y}.\frac{y}{x}\right)^{\frac{1}{2}} \Rightarrow \frac{x}{y}+\frac{y}{x} \geq 2$

Similarly, $\frac{y}{z}+\frac{z}{y} \geq 2, \; \frac{x}{z}+\frac{z}{x} \geq 2$

$\therefore \frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{x}{z}+\frac{z}{x} \geq 2+2+2=6$

$\Rightarrow \frac{y}{x}+\frac{z}{x}+\frac{x}{y}+\frac{z}{y}+\frac{x}{z}+\frac{y}{z} \geq 6$

$\Rightarrow \frac{y+z}{x}+\frac{x+z}{y}+\frac{x+y}{z} \geq 6$

$\therefore$ The minimum value of $\frac{y+z}{x}+\frac{x+z}{y}+\frac{x+y}{z}$ is **6**.

27. As $a_n \geq 1 \; \forall \; n \in N$, therefore

$\log_{a_2} a_1 \geq 0, \; \log_{a_3} a_2 \geq 0, \ldots\ldots, \log_{a_1} a_n \geq 0$

For positive numbers, AM $\geq$ GM

$\Rightarrow \frac{1}{n}[\log_{a_2} a_1 + \log_{a_3} a_2 + \ldots\ldots + \log_{a_1} a_n]$

$\geq (\log_{a_2} a_1 . \log_{a_3} a_2 \ldots\ldots \log_{a_n} a_{n-1} \; \log_{a_1} a_n)^{\frac{1}{n}}$

$\Rightarrow \frac{1}{n}[\log_{a_2} a_1 + \log_{a_3} a_2 + \ldots\ldots \log_{a_1} a_n]$

$\geq \left[\frac{\log_e a_1}{\log_e a_2}.\frac{\log_e a_2}{\log_e a_3}\ldots\ldots\frac{\log_e a_{n-1}}{\log_e a_n}\;\frac{\log_e a_n}{\log_e a_1}\right]^{\frac{1}{n}} = 1^{\frac{1}{n}} = 1$

$\therefore [\log_{a_2} a_1 + \log_{a_3} a_2 + \ldots\ldots \log_{a_1} a_n] \geq n$

The minimum value of the given sum is n, but as $n > 3$, n cannot take up the values **0, 1 or 2.**

28. a, b, c, x, y, z being all positive real numbers

$\Rightarrow \frac{x}{a}, \frac{y}{b}, \frac{z}{c}, \frac{a}{x}, \frac{b}{y}, \frac{c}{z}$ are all positive real numbers.

Now applying the AM – GM inequality, we have

$\frac{1}{3}\left[\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right] \geq \left(\frac{x}{a}.\frac{y}{b}.\frac{z}{c}\right)^{\frac{1}{3}}$ and $\frac{1}{3}\left[\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right] \geq \left(\frac{a}{x}.\frac{b}{y}.\frac{c}{z}\right)^{\frac{1}{3}}$

$\therefore \frac{1}{3}\times\frac{1}{3}\left[\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right]\left[\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right]$

$\geq \left(\frac{x}{a}.\frac{y}{b}.\frac{z}{c}\right)^{\frac{1}{3}}\left(\frac{a}{x}.\frac{b}{y}.\frac{c}{z}\right)^{\frac{1}{3}}$

$\Rightarrow \left[\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right]\left[\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right] \geq 9\left(\frac{x}{a}.\frac{y}{b}.\frac{z}{c}.\frac{a}{x}.\frac{b}{y}.\frac{c}{z}\right)^{\frac{1}{3}}$

$\geq 9.1^{\frac{1}{3}}$

$\Rightarrow \boldsymbol{\left[\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right]\left[\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right] \geq 9}$.

29. $a > 0, b > 0, c > 0 \Rightarrow \log a, \log b, \log c$ are all defined.

Also $a^{\log b - \log c}, b^{\log c - \log a}, c^{\log a - \log b}$ are all positive quantities.

$\therefore$ Applying AM $\geq$ GM, we have

$\frac{1}{3}\left[a^{\log b - \log c} + b^{\log c - \log a} + c^{\log a - \log b}\right]$

$\geq \left[a^{\log b - \log c} . b^{\log c - \log a} . c^{\log a - \log b}\right]^{\frac{1}{3}}$ ...(*i*)

Let $x = a^{\log b - \log c} . b^{\log c - \log a} . c^{\log a - \log b}$

$\Rightarrow \log x = (\log b - \log c)\log a + (\log c - \log a)\log b + (\log a - \log b)\log c$

Now, $\log_e x = 0 \Rightarrow x = e^0 = 1.$

$\therefore$ (*i*) $\Rightarrow \frac{1}{3}\left[a^{\log b - \log c} + b^{\log c - \log a} + c^{\log a - \log b}\right] \geq 1$

$\Rightarrow a^{\log b - \log c} + b^{\log c - \log a} + c^{\log a - \log b} \geq 3$

$\Rightarrow$ The least value of $a^{\log b - \log c} + b^{\log c - \log a} + c^{\log a - \log b}$ is **3.**

30. $3s = a+b+c+d \Rightarrow 3s - b - c - d = a$

$\Rightarrow a = (s-b)+(s-c)+(s-d)$

For distinct positive real numbers AM > GM

$\Rightarrow \frac{1}{3}[(s-b)+(s-c)+(s-d)] > \{(s-b)(s-c)(s-d)\}^{\frac{1}{3}}$

$\Rightarrow (s-b)+(s-c)+(s-d) > 3\{(s-b)(s-c)(s-d)\}^{\frac{1}{3}}$

$\Rightarrow a > 3\{(s-b)(s-c)(s-d)\}^{\frac{1}{3}}$ …(*i*)

Similarly, $b > 3\{(s-a)(s-c)(s-d)\}^{\frac{1}{3}}$ …(*ii*)

$c > 3\{(s-a)(s-b)(s-d)\}^{\frac{1}{3}}$ …(*iii*)

$d > 3\{(s-a)(s-b)(s-c)\}^{\frac{1}{3}}$ …(*iv*)

$\therefore$ (*i*) × (*ii*) × (*iii*) × (*iv*)

$\Rightarrow abcd > 81\{(s-a)^3(s-b)^3(s-c)^3(s-d)^3\}^{\frac{1}{3}}$

$\Rightarrow \boldsymbol{abcd > 81(s-a)(s-b)(s-c)(s-d)}$**.**

31. $\dfrac{1^2+2^2+3^2+....n^2}{n} \geq (1^2.2^2.3^2....n^2)^{\frac{1}{n}}$

$\Rightarrow \quad \dfrac{n\,(n+1)\,(2n+1)}{6n} \geq (1.2.3.....n)^{\frac{2}{n}}$

$\Rightarrow \quad \left(\dfrac{n+1}{2}\right)\left(\dfrac{2n+1}{3}\right) \geq (n!)^{2/n}$

$\left(\dfrac{n+1}{2}\right)^n \left(\dfrac{2n+1}{3}\right)^n \geq (n!)^2$

$\Rightarrow \quad K = \mathbf{2}.$

32. By triangle inequality, we have sum of lengths of two sides of $a\ \Delta >$ length of third side

$\Rightarrow \quad a+b>c,\ b+c>a,\ c+a>b$

$\Rightarrow \quad a+b-c>0,\ b+c-a>0,\ c+a-b>0$

All being positive quantities, we apply AM > GM

$$\Rightarrow \quad \frac{(a+b-c)+(b+c-a)+(c+a-b)}{3} \geq \left[(a+b-c)\,(b+c-a)\,(c+a-b)\right]^{\frac{1}{3}}$$

$\Rightarrow \quad \left(\dfrac{a+b+c}{3}\right)^3 \geq [(a+b-c)\,(b+c-a)\,(c+a-b)]$

$(a+b+c)^3 \geq 27\,(a+b-c)\,(b+c-a)\,(c+a-b)$

$\Rightarrow \quad K = \mathbf{27}.$

33. Since a, b, c are the lengths of the sides of a right triangle having hypotenuse of length c, therefore $a^2+b^2=c^2$ (Pythagoras, Theoram)

Now, $\quad a>0, b>0 \Rightarrow (a-b)^2 \geq 0$

$\Rightarrow a^2+b^2 \geq 2ab$

$\Rightarrow 2(a^2+b^2) \geq a^2+b^2+2ab$

$\Rightarrow 2\,(a^2+b^2) \geq (a+b)^2$

$\Rightarrow 2c^2 \geq (a+b)^2 \Rightarrow (a+b)^2 \leq 2c^2$

$\Rightarrow (a+b) \leq \sqrt{\mathbf{2}}\mathbf{c}.$

34. Since a, b, c are the sides of a triangle,

$a+b-c>0,\ b+c-a>0,\ c+a-b>0$

Let $x=b+c-a,\ y=c+a-b,\ z=a+b-c$

Then $x+y=2c,\ y+z=2a,\ z+x=2b.$

$$\therefore \quad \frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}$$

$$=\frac{y+z}{2x}+\frac{z+x}{2y}+\frac{x+y}{2z} = \frac{1}{2}\left[\frac{y}{x}+\frac{z}{x}+\frac{z}{y}+\frac{x}{y}+\frac{x}{z}+\frac{y}{z}\right]$$

$$=\frac{1}{2}\left[\frac{y}{x}+\frac{x}{y}+\frac{y}{z}+\frac{z}{y}+\frac{x}{z}+\frac{z}{x}\right]$$

$$\left[\left(\frac{y}{x}+\frac{x}{y}\right)\left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{x}{z}+\frac{y}{z}\right)\right] \geq 6$$

$$\left[\because \forall\ a>0,\ a+\frac{1}{a} \geq 2\right]$$

$$\therefore \quad \frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c} \geq \frac{6}{2} \geq \mathbf{3}.$$

35. $(1+a^3)\,(1+b^3)\,(1+c^3) = 1+a^3+b^3+c^3+a^3b^3+b^3c^3+c^3a^3+a^3b^3c^3 \quad ...(i)$

Now for distinct positive reals, a, b, c, AM > GM

$\dfrac{a^3+b^3+c^3}{3} > (a^3b^3c^3)^{1/3}$

$\Rightarrow \quad a^3+b^3+c^3 > 3abc \quad ...(ii)$

$(\because\ a, b, c > 0 \Rightarrow a^3, b^3, c^3 > 0)$

Also $\dfrac{a^3b^3+b^3c^3+c^3a^3}{3} > (a^3b^3.b^3c^3.c^3a^3)^{1/3}$

$\Rightarrow \quad a^3b^3+b^3c^3+c^3a^3 > 3a^2b^2c^2 \quad ...(iii)$

$\therefore$ Putting the values from (*ii*) and (*iii*) on the RHS of (*i*), we have

$(1+a^3)\,(1+b^3)\,(1+c^3) > 1+3abc+3a^2b^2c^2+a^3b^3c^3 = (1+abc)^3$

$\therefore \quad \mathbf{(1+a^3)\,(1+b^3)\,(1+c^3) > (1+abc)^3.}$

36. Let $x=a+b$, $y=c+d$. Then, $x+y=1$ and $M=xy$.

If the sum of two quantities is a constant, then their product is maximum when both the quantities are equal, *i.e.*, $x=y$

$\Rightarrow \quad x=1, y=1 \Rightarrow xy=1.$

$\therefore$ Maximum value of M is 1

Also, a, b, c, d being positive real numbers

$(a+b)\,(c+d) \geq 0$

$\therefore \quad \mathbf{0 \leq M \leq 1}$

Alternatively: For positive quantities, AM ≥ GM

$\dfrac{1}{2}\{(a+b)+(c+d)\} \geq \{(a+b)\,(c+d)\}^{1/2}$

$\Rightarrow \quad M^{\frac{1}{2}} \leq \dfrac{1}{2}\times 2 \ \Rightarrow\ M^{\frac{1}{2}} \leq 1 \ \Rightarrow\ M \leq 1$

Also, $\ M=(a+b)\,(c+d) \geq 0$

$\therefore \quad \mathbf{0 \leq M \leq 1.}$

37. For positive real numbers, AM ≥ GM

$\Rightarrow \quad \dfrac{(a_1+a_2+a_3+......+2a_n)}{n} \geq (a_1.a_2.a_3......2a_n)^{\frac{1}{n}}$

$\Rightarrow \quad (a_1+a_2+a_3+......+2a_n) \geq n(c.2)^{\frac{1}{n}} = n(2c)^{\frac{1}{n}}$

Hence the least value is $\mathbf{n(2c)^{\frac{1}{n}}}$.

38. $a>0, b>0, c>0 \Rightarrow ab>0, bc>0, ca>0$

$\therefore \quad \dfrac{(ab)^2+(bc)^2}{2} \geq \left((ab)^2(bc)^2\right)^{\frac{1}{2}}$

$\Rightarrow \quad a^2b^2+b^2c^2 \geq 2\,(a^2b^4c^2)^{1/2}$

$\Rightarrow \quad a^2b^2+b^2c^2 \geq 2\,(abc).b \quad ...(i)$

Similarly, $b^2c^2+c^2a^2 \geq 2\,(abc).c \quad ...(ii)$

$c^2a^2+a^2b^2 \geq 2(abc)a \quad ...(iii)$

(*i*) + (*ii*) + (*iii*)

$\Rightarrow \quad 2(a^2b^2+b^2c^2+c^2a^2) \geq 2\,abc\,(a+b+c)$

$\Rightarrow \quad a^2b^2+b^2c^2+c^2a^2 \geq abc\,(a+b+c)$

Dividing both sides by $a^2b^2c^2$, we have

$\Rightarrow \quad \frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2} \geq \frac{a+b+c}{abc}$

$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq \frac{2s}{abc}$

$\Rightarrow \quad \mathbf{\frac{2s}{abc} \leq \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}.$

39. We know that $a+\frac{1}{a}\geq 2 \therefore a+\frac{1}{a}+b+\frac{1}{b}+c+\frac{1}{c}\geq 2+2+2\geq 6$

$\therefore \quad \frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{x}{z}+\frac{z}{x}\geq 6$

$\therefore \quad \frac{a+b}{b+c}+\frac{b+c}{a+b}+\frac{b+c}{c+a}+\frac{c+a}{b+c}+\frac{a+b}{c+a}+\frac{c+a}{a+b}\geq 6$

$\Rightarrow \quad \frac{b+c+2a}{b+c}+\frac{a+b+2c}{a+b}+\frac{c+a+2b}{c+a}\geq 6$

$\Rightarrow \quad 1+\frac{2a}{b+c}+1+\frac{2c}{a+b}+1+\frac{2b}{c+a}\geq 6$

$\Rightarrow \quad \frac{2a}{b+c}+\frac{2c}{a+b}+\frac{2b}{c+a}\geq 3$

$\Rightarrow \quad \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq \frac{3}{2}$

Equality holds for $\mathbf{a = b = c}$.

40. $(1 + ab) = (1 – a) (1 – b) + a + b$

$\Rightarrow \quad 1 + c + ab = (1 – a) (1 – b) + a + b + c \geq a + b + c$

$\therefore \quad 1 + a + bc = (1 – b) (1 – c) + a + b + c \geq a + b + c$

$1 + b + ac = (1 – a) (1 – c) + a + b + c \geq a + b + c$

$\therefore \quad \frac{a}{1+b+ca}+\frac{b}{1+c+ab}+\frac{c}{1+a+bc}$

$\leq \frac{a}{a+b+c}+\frac{b}{a+b+c}+\frac{c}{a+b+c}$

$\left(\because\ a\geq b\Rightarrow \frac{1}{a}\leq\frac{1}{b}\right)$

$\Rightarrow \quad \frac{a}{1+b+ca}+\frac{b}{1+c+ab}+\frac{c}{1+a+bc}\leq \frac{a+b+c}{a+b+c}=1$

$\Rightarrow \quad \mathbf{\frac{a}{1+b+ca}+\frac{b}{1+c+ab}+\frac{c}{1+a+bc}\leq 1}.$

41. $\log_{10}(x^3 + y^3) – \log_{10}(x^2 + y^2 – xy) \leq 2$

$\Rightarrow \quad \log_{10}\frac{(x^3+y^3)}{(x^2+y^2-xy)}\leq 2$

$\Rightarrow \quad \log_{10}\left[\frac{(x+y)(x^2+y^2-xy)}{(x^2+y^2-xy)}\right]\leq 2$

$\Rightarrow \quad \log_{10}(x + y) \leq 2$

$\Rightarrow \quad (x + y) < 10^2 \Rightarrow x + y \leq 100$

Now $x > 0, y > 0 \Rightarrow$ AM $\geq$ GM

$\therefore \quad \frac{x+y}{2}\geq\sqrt{xy}$

$\Rightarrow \quad \frac{100}{2}\geq\frac{x+y}{2}\geq\sqrt{xy}$

$\Rightarrow \quad \sqrt{xy}\leq 50 \Rightarrow xy \leq 2500.$

Maximum value of $xy = 2500$

42. For all $x \in R$, $3^{\sin^2 x} > 0, 3^{\cos^2 x} > 0$

$\therefore$ AM $\geq$ GM

$\Rightarrow \quad \frac{1}{2}\left[3^{\sin^2 x}+3^{\cos^2 x}\right]\geq (3^{\sin^2 x}\cdot 3^{\cos^2 x})^{\frac{1}{2}}$

$\Rightarrow \quad \left[3^{\sin^2 x}+3^{\cos^2 x}\right]\geq 2(3^{\sin^2 x+\cos^2 x})^{\frac{1}{2}}$

$\Rightarrow \quad 3^{\sin^2 x}+3^{\cos^2 x}\geq 2\sqrt{3}$ $\quad (\therefore \sin^2 x + \cos^2 x = 1)$

43. Given, $x > 0, y > 0$ and $x \neq y$

$\Rightarrow \quad x^4 > 0, y^4 > 0, x^5 > 0, y^5 > 0, x^9 > 0, y^9 > 0.$

For distinct positive reals, we know that

AM > GM

$\therefore \quad \frac{1}{2}(x^4+y^4) > (x^4.y^4)^{\frac{1}{2}}$

$\Rightarrow \quad (x^4 + y^4) > 2(x^2 y^2)$ …(*i*)

Also $\frac{1}{2}(x^5+y^5) > (x^5 y^5)^{\frac{1}{2}}$

$\Rightarrow \quad (x^5 + y^5) > 2(x^{5/2} y^{5/2})$ …(*ii*)

Now multiplying (i) and (ii), we have

$(x^4 + y^4)(x^5 + y^5) > 4(x^2 y^2)(x^{5/2} y^{5/2})$

$\Rightarrow \quad (x^4 + y^4)(x^5 + y^5) > 4x^{9/2} y^{9/2}$ …(*iii*)

Also, $\frac{1}{2}(x^9+y^9) > (x^9.y^9)^{\frac{1}{2}}$

$\Rightarrow \quad (x^9 + y^9) > 2x^{9/2} y^{9/2}$

$\Rightarrow \quad 2(x^9 + y^9) > 4x^{9/2} y^{9/2}$ …(*iv*)

$\therefore$ From (*iii*) and (*iv*), we have

$\mathbf{(x^4 + y^4)(x^5 + y^5) < 2(x^9 + y^9)}$.

44. These real numbers can be taken as 8 quantities,

$a^{-5}, a^{-4}, a^{-3}, a^{-3}, a^{-3}, 1, a^8, a^{10}$.

All are positive as $a > 0$

$\therefore$ AM $\geq$ GM (as all quantities are not distinct)

$\therefore \quad \frac{1}{8}[a^{-5}+a^{-4}+a^{-3}+a^{-3}+a^{-3}+1+a^8+a^{10}]$

$\geq (a^{-5} . a^{-4} . a^{-3} . a^{-3} . a^{-3} . 1. a^8 . a^{10})^{1/8}$

$\Rightarrow \quad \frac{1}{8}[a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}] \geq 1^{1/8}$

$\Rightarrow \quad (a^{-5}+a^{-4}+a^{-3}+a^{-3}+a^{-3}+1+a^8+a^{10})\geq 8$

$\therefore$ Minimum value of the sum is **8.**

45. Let $x = b + c - a, y = c + a - b, z = a + b - c$

Since a, b, c are the sides of a non-equilateral triangle, by the triangle inequality $a + b > c, b + c > a, c + a > b$

$\Rightarrow \quad a + b - c > 0, b + c - a > 0, c + a - b > 0$

$\Rightarrow \quad z > 0, x > 0, y > 0$

Also, x, y, z are distinct.

$\therefore \quad$ AM > GM

$$c = \frac{x+y}{2} > \sqrt{xy}$$

$$b = \frac{x+z}{2} > \sqrt{xz}$$

$$a = \frac{y+z}{2} > \sqrt{yz}$$

$\therefore \quad abc > \sqrt{xy}.\sqrt{xz}.\sqrt{yz}$

$\Rightarrow \quad abc > xyz \;\Rightarrow abc > (b + c - a)(c + a - b)(a + b - c)$

$\Rightarrow \quad (b + c - a)(c + a - b)(a + b - c) - abc < 0$, *i.e.,* negative.

46. Using weighted AM – GM inequality, *i.e.*,

$$\frac{m_1a_1 + m_2a_2 + ... + m_na_n}{m_1 + m_2 + ... + m_n} \geq (a_1^{m_1}.a_2^{m_2}....a_n^{m_n})^{\frac{1}{m_1+m_2+....+m_n}},$$

we have

$$\Rightarrow \frac{ax^{b-c} + bx^{c-a} + cx^{a-b}}{a+b+c} > \left((x^{b-c})^a.(x^{c-a})^b.(x^{a-b})^c\right)^{\frac{1}{a+b+c}}$$

$$\Rightarrow \frac{ax^{b-c} + bx^{c-a} + cx^{a-b}}{a+b+c} > \{(x^{ab-ac}.x^{cb-ab}.x^{ac-bc})\}^{\frac{1}{a+b+c}}$$

$$\Rightarrow \frac{ax^{b-c} + bx^{c-a} + cx^{a-b}}{a+b+c} > (x^{ab-ac+cb-ab+ac-bc})^{\frac{1}{a+b+c}}$$

$$= (x^0)^{\frac{1}{a+b+c}} = x^0 = 1$$

$\therefore \quad \boldsymbol{ax^{b-c} + bx^{c-a} + cx^{a-b} > a + b + c.}$

47. The AM of mth powers of n positive numbers is greater than the mth power of their AM if $m < 0$ or $m > 1$. So for $m = -1$,

$$\frac{a^{-1} + b^{-1} + c^{-1}}{3} > \left(\frac{a+b+c}{3}\right)^{-1}$$

$$\Rightarrow \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) > \left(\frac{p}{3}\right)^{-1} \qquad \text{(Given: } a + b + c \equiv p\text{)}$$

$$\Rightarrow \left(\boldsymbol{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}\right) > \boldsymbol{\frac{9}{p}}.$$

48. Since a, b, c are the sides of a non-equilateral triangle, $a > 0, b > 0, c > 0$. Also by triangle inequality.

$a + b > c \Rightarrow a + b - c > 0$

$b + c > a \Rightarrow b + c - a > 0$

$c + a > b \Rightarrow c + a - b > 0$

$$\therefore \frac{(a+b-c)^{-1} + (b+c-a)^{-1} + (c+a-b)^{-1}}{3}$$

$$> \left\{\frac{(a+b-c) + (b+c-a) + (c+a-b)}{3}\right\}^{-1}$$

[$\because$ AM of the mth powers of n positive quantities is greater than the mth power of their AM if $m < 0$ or $m > 1$]

$$\therefore \quad \frac{1}{a+b-c} + \frac{1}{b+c-a} + \frac{1}{c+a-b} > 3 \times \left[\frac{a+b+c}{3}\right]^{-1}$$

$$\Rightarrow \quad \frac{1}{a+b-c} + \frac{1}{b+c-a} + \frac{1}{c+a-b} > 3 \times \frac{3}{a+b+c}$$

$$\Rightarrow \quad \frac{1}{a+b-c} + \frac{1}{b+c-a} + \frac{1}{c+a-b} > \frac{9}{a+b+c}$$

Also, $\dfrac{1}{a+b-c} + \dfrac{1}{b+c-a} = \dfrac{b+c-a+a+b-c}{\{b+(a-c)\}\{b-(a-c)\}}$

$$= \frac{2b}{b^2 - (a-c)^2} > \frac{2b}{b^2} = \frac{2}{b}$$

$$\because \quad b^2 - (a - c)^2 < b^2 \Rightarrow \frac{1}{b^2 - (a-c)^2} > \frac{1}{b^2}$$

$$\therefore \quad \frac{1}{a+b-c} + \frac{1}{b+c-a} > \frac{2}{b} \qquad \ldots(i)$$

Similarly, $\dfrac{1}{b+c-a} + \dfrac{1}{c+a-b} > \dfrac{2}{c} \qquad \ldots(ii)$

$$\frac{1}{a+b-c} + \frac{1}{c+a-b} > \frac{2}{a} \qquad \ldots(iii)$$

Adding (i), (ii) and (iii)

$$\Rightarrow \quad \frac{2}{a+b-c} + \frac{2}{b+c-a} + \frac{2}{c+a-b} > \frac{2}{a} + \frac{2}{b} + \frac{2}{c}$$

$$\Rightarrow \quad \frac{1}{a+b-c} + \frac{1}{b+c-a} + \frac{1}{c+a-b} > \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Hence (a) and (b) are both correct option.

49. AM of mth powers > mth of AM, when $m < 0$ or $m > 1$

$$\Rightarrow \quad \frac{(a_1)^{-1} + (a_2)^{-1} + (a_3)^{-1} + ... + (a_n)^{-1}}{n}$$

$$> \left(\frac{a_1 + a_2 + a_3 + ... + a_n}{n}\right)^{-1}$$

$$\Rightarrow \quad \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) > \frac{n}{a_1 + a_2 + ... + a_n}$$

$$\Rightarrow \quad \left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) > \frac{n^2}{a_1 + a_2 + ... + a_n}$$

$$\Rightarrow \quad \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} > n^2 \; [\because a_1 + a_2 + ... + a_n = 1].$$

50. A, B, C being the angles of an acute angled triangle,

$A + B + C = \pi$

$\because$ A, B, C are acute angles, $\tan A > 0$, $\tan B > 0$, $\tan C > 0$

Also, $A + B + C = \pi \quad \Rightarrow \quad \dfrac{A}{2} + \dfrac{B}{2} + \dfrac{C}{2} = \dfrac{\pi}{2}$

$\therefore \quad \cot\left(\frac{A}{2}+\frac{B}{2}\right) = \cot\left(\frac{\pi}{2}-\frac{C}{2}\right) = \tan\frac{C}{2}$

$\Rightarrow \dfrac{\cot\frac{A}{2}\cot\frac{B}{2}-1}{\cot\frac{A}{2}+\cot\frac{B}{2}} = \dfrac{1}{\cot\frac{C}{2}}$

$\Rightarrow \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$ ...(i)

$\left[\because \cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}\right]$

Also as $\cot\frac{A}{2} > 0$, $\cot\frac{B}{2} > 0$, $\cot\frac{C}{2} > 0$

Applying AM – GM inequality, we have

$\dfrac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{3} \geq \left(\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}\right)^{\frac{1}{3}}$

$\Rightarrow \left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)^3 \geq 27\left(\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}\right)$

$\Rightarrow \left(\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}\right)^3 \geq 27\left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)$

(From (i))

$\Rightarrow \left(\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}\right)^2 \geq 27$

$\Rightarrow \mathbf{\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2} \geq 3\sqrt{3}.}$

SELF ASSESSMENT SHEET

1. The number of positive integral values of m satisfying the inequalities $8m + 35 > 75$ and $5m + 18 < 53$ is

(a) 4 (b) 1 (c) 0 (d) 2

2. Given $a > 0$, $b > 0$, $a > b$ and $c \neq 0$, the inequality which is not always correct is

(a) $a - c > b - c$ (b) $\frac{a}{c^2} > \frac{b}{c^2}$

(c) $a + c > b + c$ (d) $ac > bc$

3. If x is an integer that satisfies $9 < 4x - 1 \leq 19$, then x is an element of which of the following sets?

(a) {3, 4} (b) {2, 3, 4} (c) {3, 4, 5} (d) {2, 3, 4, 5}

(NDA/NA 2008)

4. The set of all x satisfying the inequality $\frac{4x-1}{3x+1} \geq 1$ is

(a) $[2, \infty)$ (b) $\left(-\infty, -\frac{1}{3}\right) \cup [2, \infty)$

(c) $\left(-\infty, \frac{-2}{3}\right]$ (d) $\left(-\infty, -\frac{2}{3}\right] \cup [4, \infty)$

(Kerala PET 2006)

5. Find the set of values of x satisfying $\left|\frac{5-x}{3}\right| < 2$

(a) $1 < x < 11$ (b) $-1 < x < 11$

(c) $x < 11$ (d) None of these

6. The solution of $4x^2 + 4x + 1 > 0$ is

(a) All real numbers except $-\frac{1}{2}$

(b) All real numbers

(c) $(-\infty, +\infty)$

(d) None of these

7. The set of values of x for which the inequalities $x^2 - 3x - 10 < 0$, $10x - x^2 - 16 > 0$ hold simultaneously is

(a) (–2, 5) (b) (2, 8) (c) (–2, 8) (d) (2, 5)

(EAMCET 2007)

8. For three distinct positive numbers p, q and r, if $p + q + r = a$, then

(a) $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > \frac{18}{a}$ (b) $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > \frac{9}{a}$

(c) $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < \frac{3}{a}$ (d) $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < \frac{6}{a}$

9. 1.3.5.....$(2n - 1)$ is

(a) $< n$ (b) $< n^n$ (c) $> n$

(d) None of these

10. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2a_3 \ldots a_n = 1$. Then $(1 + a_1)(1 + a_2) \ldots (1 + a_n)$ is

(a) ≥ 2 (b) $\geq 2^{n-1}$ (c) $\geq 2^n$ (d) $\leq 2^{n-1}$

11. If three positive real numbers, a, b, c are such that $a + b + c = 1$, then the minimum value of $\frac{(1-a)(1-b)(1-c)}{abc}$ is

(a) 2 (b) 3 (c) 9 (d) 8

12. The minimum value of the expression

$\left(\frac{3b+4c}{a}\right)+\left(\frac{4c+a}{3b}\right)+\left(\frac{a+3b}{4c}\right)$ (a, b, c are +ve) is

(a) 1 (b) 4 (c) 6 (d) 8

(AMU 2005)

ANSWERS

1. (b) **2.** (d) **3.** (c) **4.** (b) **5.** (b) **6.** (a) **7.** (d) **8.** (b)
9. (b) **10.** (c) **11.** (d) **12.** (c)

HINTS AND SOLUTIONS

1. $8m + 35 > 75$ and $5m + 18 < 53$

$\Rightarrow 8m > 40 \quad \Rightarrow 5m < 35$

$\Rightarrow m > 5 \quad\quad \Rightarrow m < 7$

$\therefore$ There is only **one** integral value of m, *i.e.*, $m = 6$ satisfying the given inequalities.

2. $a > b \quad \Rightarrow \quad ac > bc$ when $c > 0$, but

$a > b \quad \Rightarrow \quad ac < bc$ when $c < 0$.

$\therefore$ $ac > bc$ is not always correct.

3. Given, $9 < 4x - 1 \leq 19$

$\Rightarrow 9 < 4x - 1$ and $4x - 1 \leq 19$

$\Rightarrow 9 + 1 < 4x$ and $4x \leq 19 + 1$

$\Rightarrow x > \frac{5}{2}$ and $x \leq 5$

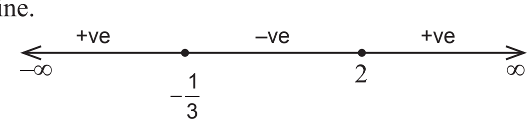
$\therefore \frac{5}{2} < x \leq 5$

$\because x \in Z \therefore x = \{3, 4, 5\}$.

4. $\frac{4x-1}{3x+1} \geq 1 \Rightarrow \frac{4x-1}{3x+1} - 1 \geq 0$

$\Rightarrow \frac{4x-1-3x-1}{3x+1} \geq 0 \Rightarrow \frac{x-2}{3x+1} \geq 0 \Rightarrow x \neq -\frac{1}{3}$

Critical points are $x = -\frac{1}{3}, 2$. Place the points on the number line.



- When $x < -\frac{1}{3}$, $\frac{x-2}{3x+1} > 0$
- When $x \geq 2$, $\frac{x-2}{3x+1} \geq 0$
- When $-\frac{1}{3} < x \leq 2$, $\frac{x-2}{3x+1}$ is –ve.

$\therefore$ $\boldsymbol{x \in \left(-\infty, -\frac{1}{3}\right) \cup [2, \infty)}$.

5. $\left|\frac{5-x}{3}\right| < 2 \Rightarrow |5 - x| < 6$

$\Rightarrow -6 < 5 - x < 6 \Rightarrow -11 < -x < 1 \Rightarrow 11 > x > -11$

($\because$ $a < b < c \Rightarrow ma > mb > mc$, when m is –ve. Here $m = \mathbf{-1}$)

$\Rightarrow \mathbf{-1 < x < 11}$.

6. $4x^2 + 4x + 1 > 0 \Rightarrow (2x + 1)^2 > 0$

$\Rightarrow (2x + 1) < 0$ or $(2x + 1) > 0$

$\Rightarrow x > -\frac{1}{2}$ or $x < -\frac{1}{2}$

$\therefore$ The inequality holds for all real numbers except $x = \mathbf{-\frac{1}{2}}$.

7. $x^2 - 3x - 10 < 0$

$\Rightarrow (x + 2)(x - 5) < 0 \Rightarrow (x - (-2))(x - 5) < 0$

$\Rightarrow x \in (-2, 5)$

(If $a < b$, then $(x - a)(x - b) < 0 \Rightarrow a < x < b$)

$10x - x^2 - 16 > 0 \Rightarrow x^2 - 10x + 16 < 0 \Rightarrow (x - 2)(x - 8) < 0$

$\Rightarrow x \in (2, 8)$

(If $a < 0$, then $(x - a)(x - b) < 0 \Rightarrow a < x < b$)

$\therefore x \in (-2, 5) \cap (2, 8)$

$\Rightarrow \boldsymbol{x \in (2, 5)}$.

8. $p > 0 \Rightarrow \frac{1}{p} > 0;\ q > 0 \Rightarrow \frac{1}{q} > 0;\ r > 0 \Rightarrow \frac{1}{r} > 0$

For distinct positive numbers, AM > GM

$$\therefore \frac{p+q+r}{3} > (pqr)^{\frac{1}{3}} \Rightarrow (p+q+r) > 3(pqr)^{\frac{1}{3}} \qquad \ldots(i)$$

Also, $\frac{1}{3}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right) > \left(\frac{1}{p}.\frac{1}{q}.\frac{1}{r}\right)^{\frac{1}{3}}$

$$\Rightarrow \left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right) > 3\left(\frac{1}{p}.\frac{1}{q}.\frac{1}{r}\right)^{\frac{1}{3}} \qquad \ldots(ii)$$

From $(i) \times (ii)$

$\Rightarrow (p + q + r)\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right) > 9(pqr)^{\frac{1}{3}} \times \left(\frac{1}{pqr}\right)^{\frac{1}{3}}$

$\Rightarrow a\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right) > 9(1)^{\frac{1}{3}} \Rightarrow \boldsymbol{\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right) > \frac{9}{a}}$.

9. Here AM > GM, as all are distinct positive numbers.

$$\left(\frac{1+3+5+....+(2n-1)}{n} > (1.3.5......(2n-1)\right)^{\frac{1}{n}}$$

$$\Rightarrow \frac{\frac{n}{2}[1+2n-1]}{n} > (1.3.5....(2n-1))^{\frac{1}{n}}$$

$\Rightarrow (1.3.5....(2n-1))^{\frac{1}{n}} < n$

$\Rightarrow 1.3.5....(2n - 1) < n^n$.

10. $\left(\frac{1+a_1}{2}\right) \geq \sqrt{a_1} \Rightarrow (1+a_1) \geq 2\sqrt{a_1}$

$\left(\frac{1+a_2}{2}\right) \geq \sqrt{a_2} \Rightarrow (1+a_2) \geq 2\sqrt{a_2}$

$\left(\frac{1+a_n}{2}\right) \geq \sqrt{a_n} \Rightarrow (1+a_n) \geq 2\sqrt{a_n}$

$\therefore (1 + a_1)(1 + a_2) ...... (1 + a_n)$

$\geq 2\sqrt{a_1} \times 2\sqrt{a_2} \times ..... \times 2\sqrt{a_n}$

$\Rightarrow (1 + a_1)(1 + a_2) .... (1 + a_n) \geq 2^n \sqrt{(a_1.a_2....a_n)}$

$= 2^n.\sqrt{1} = \mathbf{2^n}$.

11. $a + b + c = 1 \Rightarrow \; b + c = 1 - a$

$a + c = 1 - b$

$a + b = 1 - c$

Now, $a > 0$, $b > 0$, $c > 0 \Rightarrow$ AM $\geq$ GM

$\Rightarrow \quad \dfrac{b+c}{2} \geq \sqrt{bc} \Rightarrow (1-a) \geq 2\sqrt{bc}$

$\dfrac{a+c}{2} \geq \sqrt{ac} \;\Rightarrow\; (1-b) \geq 2\sqrt{ac}$

$\dfrac{a+b}{2} \geq \sqrt{ab} \;\Rightarrow\; (1-c) \geq 2\sqrt{ab}$

$\therefore \quad (1-a)(1-b)(1-c) \geq 8\sqrt{bc}.\sqrt{ac}.\sqrt{ab}$

$\Rightarrow \quad (1-a)(1-b)(1-c) \geq 8\,abc$

$\Rightarrow \quad \dfrac{(1-a)(1-b)(1-c)}{abc} \geq 8$

$\therefore$ Minimum value is **8.**

12. $\dfrac{3b+4c}{a} + \dfrac{4c+a}{3b} + \dfrac{a+3b}{4c}$

$= \dfrac{3b}{a} + \dfrac{4c}{a} + \dfrac{4c}{3b} + \dfrac{a}{3b} + \dfrac{a}{4c} + \dfrac{3b}{4c}$

$= \left(\dfrac{3b}{a} + \dfrac{a}{3b}\right) + \left(\dfrac{4c}{a} + \dfrac{a}{4c}\right) + \left(\dfrac{4c}{3b} + \dfrac{3b}{4c}\right)$

$\because \; a > 0, b > 0, c > 0$

$\therefore \; \dfrac{a}{3b}, \dfrac{3b}{a}, \dfrac{4c}{a}, \dfrac{a}{4c}, \dfrac{4c}{3b}, \dfrac{3b}{4c}$ are all greater than zero.

Now $\forall \; a > 0$, $\left(a + \dfrac{1}{a}\right) \geq 2$

$\therefore \; \left(\dfrac{3b}{a} + \dfrac{a}{3b}\right) + \left(\dfrac{4c}{a} + \dfrac{a}{4c}\right) + \left(\dfrac{4c}{3b} + \dfrac{3b}{4c}\right) \geq 2 + 2 + 2 = 6$

$\therefore$ Minimum value of the expression is **6.**

5 Relations

KEY FACTS

1. **Cartesian product:** Let A and B be two non-empty sets. Then the set of all possible ordered pairs (x, y) such that the first component x of the ordered pairs is an element of set A, and the second component y is an element of set B, is called the cartesian product of the sets A and B. It is denoted by $\boldsymbol{A \times B}$ read as **"A cross B"**.

$$\boldsymbol{A \times B = \{(a, b), a \in A \text{ and } b \in B\}}$$

Also, $n(A \times B) = n(A) \times n(B) = pq$ if set A has p elements and set B has q elements.

> **Notes: 1. The cartesian product $A \times B$ is not the same as $B \times A$.**
> In $A \times B$, the set A is named first so its elements will appear as the first components of the ordered pairs.
> In $B \times A$, the set B is named first, so its elements will appear as the first components of the ordered pairs.
> **2. If either A or B is a null set, then we define $A \times B$ to be a null set.**
> If $A = \{a, b\}$ and $B = \phi$ then $A \times B = \phi$
> **3. If either A or B is an infinite set and the other is a non-empty set, then $A \times B$ is also an infinite set.**
> **4. If A and B are two non-empty sets having n-elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.**
> **5. If $A = B$, then $A \times B = A \times A$ and is denoted by A^2.**

Examples:

Ex. 1. If $A = \{a, b\}$ and $B = \{1, 2, 3\}$, then

$A \times B = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$

$B \times A = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$.

Ex. 2. If $A = \{2, 4, 6\}$, then

$A \times A = \{(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)\}$.

2. **Relation:**

If A and B are any two non-empty sets, then any subset of $A \times B$ is defined as a relation from A to B.

For example,

Suppose $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. Then $\{(2, 3), (2, 4), (1, 3)\}$ is a relation in $A \times B$. Many more relations (subsets) can be selected at random from our product set $A \times B$.

3. **Domain, codomain and range of a relation:**

Let R be a relation from set A to set B. Then, the **set of first element** of the ordered pairs in $\boldsymbol{R}$ is called the **domain** and the **set of second elements** of the ordered pairs in $\boldsymbol{R}$ is called the **range.** The second set B is called the codomain of R.

Thus for a relation $R = \{(a, b); a, b \in R\}$,

Domain $= \{a : (a, b) \in R\}$ and Range $= \{b : (a, b) \in R\}$

For example,

If $A = \{16, 25, 36, 49\}$ and $B = \{1, 4, 5, 6\}$ and $\boldsymbol{R}$ be the relation **"is square of "** from A to B, then

$R = \{(a, b) : a = b^2, a \in A, b \in B\}$

$\therefore\ R = \{(16, 4), (25, 5), (36, 6)\}$. Then,

Domain of $R = \{16, 25, 36\}$, Range of $R = \{4, 5, 6\}$ and Codomain of $R = \{1, 4, 5, 6\}$.

4. Number of relations that are possible from a set *A* of *m* elements to another set *B* of *n* elements.

Number of elements in set $A = m$

Number of elements in set $B = n$

$\therefore$ Number of elements in set $(A \times B) = mn$

$\therefore$ Number of subsets of $(A \times B) = \mathbf{2^{mn}}$

Since every subset of $A \times B$ is a relation from A to B, therefore,

Number of relations possible from A to B = $\mathbf{2^{mn}}$

> **Remember**
>
> If $n(A) = m$
>
> $n(B) = n$
>
> Then, $\mathbf{n(A \times B) = mn}$
>
> **Number of relations possible from A to B = 2^{mn}**

> **Note: 1.** If R_1 and R_2 are two relations from A to B, then $R_1 \cup R_2$, $R_1 \cap R_2$ and $R_1 - R_2$ are also relations from A to B.
>
> **2.** Since $\phi \le A$, *i.e.*, null set is a subset of every set, ϕ is a relation from A to B. Also domain $(\phi) = \phi$ and Range $(\phi) = \phi$

5. Inverse of a relation:

For any binary relation R, a second relation can be constructed by merely interchanging first and second components in every ordered pair.

The relation thus obtained is called the inverse of the first one and designated as $\boldsymbol{R^{-1}}$. Thus,

$$\boxed{R^{-1} = \{(y, x) : (x, y) \in R\}}$$

For example,

1. Then inverse of the husband-wife relation is wife-husband relation.

2. Let $R = \{(2, 1), (3, 2), (4, 3), (4, 5)\}$. Then,

$R^{-1}\ \{(1, 2), (2, 3), (3, 4), (5, 4)\}$.

So $(R^{-1})^{-1} = R$.

6. Types of relations:

Let A be a non-empty set. Then, a relation R on A is said to be

- **Reflexive** ***if (a, a) $\in$ R for each a $\in$ A, i.e., if a R a for each a $\in$ A.***

For example, the relation "***is as strong as***" is reflexive since every member of a particular set will be as strong as himself, but the relation "***is the mother of***" is not reflexive as a person cannot be his/her own mother.

- **Symmetric** ***if (a, b) $\in$ R $\Rightarrow$ (b, a) $\in$ R for all a, b $\in$ A, i.e., if a R b $\Rightarrow$ b R a for all a, b $\in$ A.***

For example, the relation "***weighs the same as***" is symmetric as if x weighs same as y. Then y weighs same as x, but the relation "***is less than***" is not symmetric as: if x is less than y, then y is not less than x.

- **Transitive** ***if (a, b) $\in$ R, (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R for all a, b, c $\in$ A, i.e., if a R b and b R c then a R c.***

For example, the relation "***equals to***" is a transitive relation for if $x = y$ and $y = z$, then $x = z$, but the relation "***is perpendicular to***" on a set of coplanar lines is not transitive for if line a is perpendicular to line b and line b is perpendicular to line c, then line a is not perpendicular to line c.

> - **Equivalence:** ***A relation R on a set A is said to be an equivalence relation if it is reflexive, symmetric and transitive.***
>
> **Equivalence Relation** →
> **Reflexive** (aRa for each $a \in A$)
> **Symmetric** ($aRb \Rightarrow bRa$ for all $a, b \in A$)
> **Transitive** (aRb, $bRc \Rightarrow aRc$ for all $a, b, c \in A$)

For example,

(*i*) "***Equality***" is an equivalence relation because

- $x = x$ • $x = y \Rightarrow y = x$ • $x = y, y = z \Rightarrow x = z$.

(*ii*) "***Is parallel to***" on a set A of coplanar lines is an equivalence relation since: for all the lines $a, b, c \in A$.

- $a \parallel a$
- $a \parallel b \Rightarrow b \parallel a$
- $a \parallel b, b \parallel c \Rightarrow a \parallel c$.

SOLVED EXAMPLES

Ex. 1. Let R be a relation from $A = \{1, 2, 3, 4, 5, 6\}$ to $B = \{1, 3, 5\}$ which is defined as "x is less than y". Write R as a set of ordered pairs. Also state the domain, range and codomain of R.

Sol. $R = \{a, b : a < b, a \in A, b \in B\}$, where $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5\}$.

$\therefore$ $R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$

Domain of $R = \{1, 2, 3, 4\}$

Range of $R = \{3, 5\}$

Codomain of $R = \{1, 3, 5\}$.

Ex. 2. Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$. Which of the following are relations from A to B ?

(*i*) $\{(a, y), (a, z), (c, x), (d, y)\}$ **(*ii*)** $\{(a, x), (b, y), (c, x), (a, d)\}$

(*iii*) $\{(a, x), (y, d), (x, c)\}$ **(*iv*)** $\{(y, a), (z, a), (z, c), (y, d)\}$

(*v*) $\{(a, x), (x, a), (b, y), (y, b)\}$ **(*vi*)** $\{(a, x), (b, y), (c, z), z\}$

(*vii*) $\{a, b, x, y, z\}$

Sol. (*i*) Yes.

(*ii*) No, because in the ordered pair (a, d), $a \in A$ and $d \notin B$.

(*iii*) No, because in (y, d), $y \in B$.

(*iv*) No. because here the first entries in all the ordered pairs are in the set B.

(*v*) No.

(*vi*) No, because the element z is not an ordered pair.

(*vii*) No, because the elements of the set are not ordered pairs.

Ex. 3. Determine the domain and range of the following relations:

(*i*) $\{(-3, 1), (-1, 1), (1, 0), (3, 0)\}$

(*ii*) $\{(x, y)$**: x is a multiple of 3 and y is a multiple of 5**$\}$

(*iii*) $\{(x, x^2)$**: x is a prime number less than 15**$\}$

Sol. (*i*) Domain = $\{-3, -1, 1, 3\}$, Range = $\{0, 1\}$

(*ii*) Domain = $\{x : x$ is a multiple of $3\} = \{3n : n \in Z\}$

Range $= \{y : y$ is a multiple of $5\} = \{5n : n \in Z\}$

(*iii*) Relation = $\{(x, x^2) : x$ is a prime number less than $15\}$

$= \{(2, 4), (3, 9), (5, 25), (7, 49), (11, 121), (13, 169)\}$

Domain = $\{2, 3, 5, 7, 11, 13\}$, Range = $\{4, 9, 25, 49, 121, 169\}$

Ex. 4. Let N be the set of natural numbers. Describe the following relation in words giving its domain and the range. $\{(1, 1), (16, 2), (81, 3), (216, 4)\}$

Sol. The given relation stated in words is

$R = \{(x, y) : x$ is the fourth power of $y; x \in N, y \in \{1, 2, 3, 4\}\}$.

Ex. 5. I is the set of integers. Describe the following relations in words, giving its domain and range.

$\{(0, 0), (1, -1), (2, -2), (3, -3)...\}$

Sol. $R = \{(0, 0), (1, -1), (2, -2), (3, -3) ...\} = \{(x, y) : y = -x, x \in W\}$

Domain = $\{0, 1, 2, 3, ...\} = W$, Range = $\{...,-3, -2, -1, 0\}$

Ex. 6. Choose the correct option.
The relation $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ on a set $A = \{1, 2, 3\}$ is
(*a*) reflexive, transitive but not symmetric
(*b*) reflexive, symmetric but not transitive
(*c*) symmetric, transitive but not reflexive
(*d*) reflexive but neither symmetric nor transitive. **(*NDA/NA 2010*)**

Sol. $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ on set $A = \{1, 2, 3\}$.
(*i*) Since $(1, 1), (2, 2), (3, 3) \in R \Rightarrow R$ is reflexive.
(*ii*) $(1, 2) \in R$ but $(2, 1) \notin R \Rightarrow R$ is not symmetric.
(*iii*) $(1, 2) \in R$, $(2, 3) \in R$ and $(1, 3) \in R \Rightarrow R$ is transitive.
$\therefore$ Option (*a*) is the right answer.

Ex. 7. Let X be the set of all graduates in India. Elements x and y in X are said to be related, if they are graduates of the same university. Is this given relation an equivalence relation? **(*NDA/NA 2010*)**

Sol. $R = \{(x, y)\}$: x and y are graduates of same university, $x, y \in$ {All graduates of India}.
***R* is reflexive** as $(x, x) \in R$, since $\boldsymbol{x}$ and $\boldsymbol{x}$ are graduates from the same university.
***R* is symmetric** as $(x, y) \in R \Rightarrow x$ and y are graduates from the same university.
$\Rightarrow y$ and x are graduates from the same university
$\Rightarrow (y, x) \in R$.
***R* is transitive** as $(x, y) \in R$ and $(y, z) \in R$
$\Rightarrow$ x and y are graduates from the same university and y and z are graduates from the same university
$\Rightarrow$ x and z are graduates from the same university
$\Rightarrow$ $(x, z) \in R$.
R being reflexive, symmetric and transitive is an equivalence relation.

Ex. 8. Show that the relation R in the set A of all the books in a library of a school given by $R = \{(x, y)$: x and y have the same number of pages$\}$ is an equivalence relation.

Sol. Given: A = {All books in a library of a school}
$R = \{(x, y)$: x and y have the same number of pages$\}$
Reflexivity: $(x, x) \in R \Rightarrow R$ is reflexive on A
Symmetric: Since books x and y have the same number of pages, so $(x, y) \in R$.
Since books y and x have the same number of pages, so $(y, x) \in R$.
$\Rightarrow$ R is symmetric on A.
Transitivity: Books x, y, z have the same number of pages $\Rightarrow (x, y) \in R$ and $(y, z) \in R$
$\Rightarrow$ $(x, z) \in R \Rightarrow R$ is transitive on A.
Hence, R is an equivalence relation.

Ex. 9. Show that the relation '$\cong$' congruence on the set of all triangles in Euclidean plane geometry is an equivalence relation.

Sol. Reflexive :	$A \cong A$	**True**
Symmetric :	if $A \cong B$ then $B \cong A$	**True**
Transitive :	if $A \cong B$ and $B \cong C$,	
then	$A \cong C$	**True**

Therefore, the relation '$\cong$' is an equivalence relation.

Ex. 10. Show that the relation '$\subset$' with respect to sets is not an equivalence relation.

Sol. Reflexive: If A is a set, then $A \subset A$. **False**

Symmetric: If A and B are sets and $A \subset B$, then $B \subset A$. **False**

Transitive: If A, B and C are sets and if $A \subset B$ and $B \subset C$ then $A \subset C$ **True**

Hence '$\subset$' is not an equivalence relation since it possesses only the transitive property.

Ex. 11. Show that the relation "$\geq$" on the set of real numbers is not an equivalence relation.

Sol. Reflexive : $a \geq a$ **True** since $a = a$

Symmetric : If $a \geq b$, then $b \geq a$. This statement is true for the case $a = b$, but false in the other instances.

Transitive : If $a \geq b$, and $b \geq c$, then $a \geq c$. **True**

The relation "$\geq$" is not an equivalence relation since it lacks the property of symmetry.

Ex. 12. Write 'yes' if each of the following relations is an equivalence relation. If it is not, then write whether it is reflexive or symmetric or transitive.

(*i*) is parallel to **(*ii*) is perp. to** **(*iii*) is greater than**

(*iv*) is a factor of **(*v*) is a multiple of**

Sol. (*i*) *'is parallel to'* is *reflexive,* because any line is parallel to itself.

Symmetric, because if line l is parallel to line m, then line m is parallel to line l.

Transitive, because if $l \parallel m$ and $m \parallel n$ then $l \parallel n$.

Therefore,'is parallel to' is an equivalence relation.

(*ii*) No, because this relation is only symmetric.

(*iii*) No, because this relation is only transitive. It is neither reflexive nor symmetric.

(*iv*) No, because this relation is only reflexive and transitive. It is not symmetric. If x is a factor of y, and $x \neq y$, y cannot be a factor of x.

(*v*) No, because the relation is reflexive and transitive but not symmetric.

Ex. 13. If R is a relation in $N \times N$ defined by (a, b) R (c, d) if and only if $ad = bc$, show that R is an equivalence relation.

Sol. **(*i*)** ***R* is reflexive.** For all $(a, b) \in N \times N$ we have (a, b) R (a, b) because

$ab = ba$

$\Rightarrow$ R is reflexive.

(*ii*) ***R* is symmetric.** Suppose (a, b) R (c, d)

Then (a, b) R $(c, d) \Rightarrow ad = bc \Rightarrow cb = da$

$\boxed{\text{Commutivity of multiplication in } N}$

$\Rightarrow$ (c, d) R $(a, b) \Rightarrow R$ is symmetric.

(*iii*) ***R* is transitive.** Suppose (a, b) R (c, d) and (c, d) R (e, f). Then

$ad = bc$ and $cf = de$

$\Rightarrow$ $(ad)(cf) = (bc)(de)$

$\Rightarrow$ $af = be \Rightarrow (a, b)$ R $(e, f) \Rightarrow R$ is transitive.

Since R is reflexive, symmetric and transitive, therefore, R is an equivalence relation on $N \times N$.

Ex. 14. Let A = {real numbers}

Let $R = \{(a, b) : a, b \in A$ and $a - b < 5\}$

Is R an equivalence relation? Justify your answer.

Sol. **(*i*)** **Reflexive:** For all $a \in A$, $a - a = 0 < 5 \Rightarrow aRa$.

$\therefore$ R is reflexive.

(*ii*) **Symmetric:** For example let $a = 2$, $b = 8$, then $a - b = 2 - 8 = -6 < 5 \Rightarrow a = aRb$

But, $6 - a = 8 - 2 = 6 \not< 5 \Rightarrow b \not R\ a$.

Hence R is not symmetric.

(*iii*) **Transitive:** If $a = 4$, $b = 0$, $c = -4$, then $a - b = 4 - 0 = 4 < 5 \Rightarrow aRb$

and $b - c = 0 - (-4) = 4 < 5 \Rightarrow bRc$

But $a - c = 4 - (-4) = 8 < 5 \Rightarrow a \not R\ c$.

$\therefore$ R is not transitive.

$\Rightarrow$ R is not an equivalence relation.

Ex. 15. If $R = \{a, b : a - b$ is even and $a, b \in Z\}$, then R is an equivalence relation.

Sol. **Reflexive:** For all $a \in Z$, $a - a = 0$, an even integer $\Rightarrow aRa \Rightarrow R$ is reflexive

Symmetric. For all $a, b \in Z$, if $a - b$ is an even integer, then $b - a = -(a - b)$ is an even integer.

i.e., $aRb \Rightarrow bRa \Rightarrow R$ is symmetric.

Transitive. If $a - b$ is an even integer and $b - c$ is an even integer,

then $(a - b) + (b - c)$ is an even integer $\Rightarrow a - c$ is an even integer.

i.e., aRb and $bRc \Rightarrow aRc$.

Hence R is transitive.

$\Rightarrow$ R is an equivalence relation.

Ex. 16. Let Z be the set of all integers and R be the relation on Z defined as $R = \{(a, b): a, b \in Z)$ and $(a - b)$ is divisible by 5.

Prove that R is an equivalence relation.

Sol. **Reflexive:** Since 5 divides $a - a$ for all $a \in Z$, therefore, R is reflexive.

Symmetric: $(a, b) \in R \Rightarrow 5$ divides $a - b$

$\Rightarrow$ 5 divides $b - a \Rightarrow b - a \in R$

$\therefore$ R is symmetric.

Transitive: $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a - b$ and $b - c$ are both divisible by 5

$\Rightarrow$ $a - b + b - c$ is divisible by 5

$\Rightarrow$ $(a - c)$ is divisible by 5

$\Rightarrow$ $(a, c) \in R$

$\therefore$ R is transitive.

Since R is reflexive, symmetric and transitive, therefore, R is an equivalence relation.

Ex. 17. Show that the relation S in the set R of real numbers, defined as $S = \{(a, b) : a, b \in R$ and $a \le b^3\}$ is neither reflexive, nor symmetric, nor transitive.

Sol. (*i*) Since $a \le a^3$ is not true for all $a \in R$

For example, if $a = \dfrac{1}{4}$, then $a < a^3$, *i.e.,* $a \le a^3$ is not true.

So, R is not reflexive.

(*ii*) $(a, b) \in R$ need not imply that $(b, a) \in R$

For example, $(1, 2) \in R$ but $(2, 1) \in R$ because $1 \le 2^3$ but $2 \not\le 1^3$.

$\therefore$ R is not symmetric.

(*iii*) $(a, b) \in R$ and $(b, c) \in R$ need not imply that $(a, c) \in R$.

For example, $(80, 5) \in R$ and $(5, 2) \in R$ but $(80, 2) \in R$

Since, $80 < 5^3$, $5 < 2^3$ but $80 \not< 2^3$.

$\therefore$ R is not transitive.

Ex. 18. Show that the relation S in the set $A = \{x \in z : 0 \le x \le 12\}$ given by $S = \{(a, b) : a, b \in z, |a - b|$ is divisible by 4$\}$ is an equivalence relation. Find the set of all elements related to 1.

Sol. Reflexive: For all $a \in A$, $|a - b| = 0$ is divisible by $4 \Rightarrow (a, a) \in S$.

$\therefore$ S is reflexive.

Symmetric: Let $a, b \in A$. Then $(a, b) \in S \Rightarrow |a - b|$ is divisible by 4

$\Rightarrow$ $|b - a|$ is divisible by $4 \Rightarrow (b, a) \in S$

$\therefore$ S is symmetric.

Transitive: Let $a, b, c \in A$, $(a, b) \in S$ and $(b, c) \in S$

$\Rightarrow$ $|a - b|$ is divisible by 4 and $|b - c|$ is divisible by 4.

$\Rightarrow$ $(a - b)$ and $(b - c)$ are divisible by 4.

$\Rightarrow$ $(a - b) + (b - c) = (a - c)$ is divisible by 4.

$\Rightarrow$ $|a - c|$ is divisible by $4 \Rightarrow (a, c) \in S$.

$\therefore$ S is transitive.

Since S is reflexive, symmetric and transitive, therefore, S is an equivalence relation.

Ex. 19. Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b): b = a + 1\}$ is reflexive, symmetric or transitive.

Sol. Reflexive:

$R = \{(a, b) : b = a + 1\}$

$= \{(a, a + 1) : a, a + 1 \in \{1, 2, 3, 4, 5, 6\}\}$

$= \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

$\Rightarrow$ R is not reflexive since $(a, a) \notin R$ for all a.

Symmetric: R is not symmetric as $(a, b) \in R$ but $(b, a) \notin R$

Transitive: R is not transitive as $(a, b) \in R$ and $(b, c) \in R$ but $(a, c) \notin R$

e.g., $(1, 2) \in R$ $(2, 3) \in R$ but $(1, 3) \notin R$

PRACTICE SHEET

1. Let $A = \{2, 3, 4, 6\}$ and let R be a relation on A defined as $R = \{(a, b)$; $a \in A, b \in A$, a divides $B\}$ Then which of following is not R?

(*a*) $R = \{(2, 2)\}, (2, 4), (3, 6), (3, 6)$

(*b*) $R = \{(2, 2), (3, 3), (2, 4), (4, 6)\}$

(*c*) $R = \{(2, 4), (3, 3), (3, 6), (4, 4)\}$

(*d*) $R = \{(2, 2), (3, 3), (4, 4), (2, 4), (2, 6), (3, 6), (6, 6)\}$

2. Which of the following are relations from B to A where $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$?

(*i*) $\{(z, b), (y, c), (x, a)\}$ (*ii*) $\{(x, b), (y, a)\}$

(*iii*) $\{(b, y), (z, a), (x, c)\}$ (*iv*) $\{(x, a), (x, b), (y, c), (z, d)\}$

(*a*) (*i*) and (*iii*) (*b*) Only (*iv*)

(*c*) (*i*), (*ii*) and (*iv*) (*d*) All of the above

3. Given $A = \{-2, -1, 0, 1, 2\}$, which of the following relations on A have both domain and range equal to A?

(*i*) R : "is equal to"

(*ii*) R : "is the multiplicative inverse of "

(*iii*) R : "is the additive inverse of "

(*iv*) R : "is less than"

(*a*) Only (*i*) (*b*) Only (*iv*)

(*c*) All of the above (*d*) (*i*) and (*iii*)

4. The relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ is:

(*a*) Reflexive only

(*b*) Reflexive and symmetric only

(*c*) Symmetric only

(*d*) Transitive only

5. If $n(A) = 5$ and $n(B) = 7$, then the number of relations on $A \times B$ is

(*a*) 2^{25} (*b*) 2^{35} (*c*) 2^{12} (*d*) 35

(*Kerala PET 2012*)

6. Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 12), (3, 6)\}$ be a relation on set $A = \{3, 6, 9, 12\}$. The relation is

(*a*) Reflexive only

(*b*) Reflexive and symmetric only

(*c*) Reflexive and transitive only

(*d*) An equivalence relation. **(*AIEEE 2005*)**

7. Which of the following is an equivalence relation defined on set $A = \{1, 2, 3\}$

(*a*) $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

(*b*) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2)\}$

(*c*) $\{(1, 1), (2, 2), (3, 1), (1, 3), (3, 3)\}$

(*d*) $\{(1, 2), (2, 1), (1, 3), (3, 1)\}$

8. Given the relation $R = \{(1, 2), (2, 3)\}$ on the set $A = \{1, 2, 3\}$, the minimum number of ordered pairs which can be added to R to make it an equivalence relation is

(*a*) 2 (*b*) 3 (*c*) 5 (*d*) 7

(*AMU 2008*)

9. Let Z be the set of integers. Then the relation $R = \{(a, b) : a, b \in Z$ and $(a + b)$ is even$\}$ defined on Z is

(*a*) Only symmetric

(*b*) Symmetric and transitive only

(*c*) An equivalence relation

(*d*) None of the above

10. If R be a relation defined as $a\,R\,b \Leftrightarrow |a| < b$, then R is

(*a*) Reflexive only

(*b*) Symmetric only

(*c*) Transitive only

(*d*) Reflexive and transitive but not symmetric

(*Odisha JEE 2012*)

11. Let the relation R defined on the set of natural numbers N be : $R = \{(a, b) : b$ is divisible by $\forall\, a, b \in N\}$. Then R is

(*a*) Reflexive and symmetric only

(*b*) Symmetric and transitive only

(*c*) Reflexive and transitive only

(*d*) An equivalence relation

12. Let R be a relation defined on the set A of all triangles such that $R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$. Then R is

(*a*) Reflexive only (*b*) Transitive only

(*c*) Symmetric only (*d*) An equivalence relation.

13. Let R be a relation defined as a Rb if $|a - b| > 0$, then the relation is

(*a*) Reflexive only (*b*) Symmetric only

(*c*) Transitive only (*d*) Symmetric and transitive

(*VITEEE 2008*)

14. On the set R of all real numbers, *a* relation R is defined by $R = \{(a, b) : 1 + ab > 0\}$. Then R is

(*a*) Reflexive and symmetric only

(*b*) Reflexive and Transitive only

(*c*) Symmetric and Transitive only

(*d*) An equivalence relation

15. Which of the following relations is **only symmetric**?

(*a*) "less than equal to" on a set of Real numbers

(*b*) "is a multiple of " on the set of positive integers

(*c*) "is perpendicular to" on a set of a coplanar lines

(*d*) "is the father of " on a set of family members.

16. If R is a relation defined on the set of natural numbers N such that $(a, b)\,R\,(c, d)$ if and only if $a + d = b + c$, then R is

(*a*) Symmetric and transitive but not reflexive

(*b*) Reflexive and transitive but not symmetric

(*c*) Reflexive and symmetric but not transitive

(*d*) An equivalence relation

17. Let I be the set of integers and R be a relation on I defined by $R = \{(x, y) : (x - y)$ is divisible by 11, $x, y \in I\}$. Then R is

(*a*) An equivalence relation (*b*) Symmetric only

(*c*) Reflexive only (*d*) Transitive only

18. Let W denote the words in the English dictionary. Let the relation R be defined by $R = \{(x, y) \in W \times W$: the words x and y have at least one letter in common$\}$. Then R is

(a) Reflexive and transitive, not symmetric

(b) Reflexive and symmetric and not transitive

(c) Symmetric and transitive, not reflexive

(d) Reflexive, symmetric and transitive **(*AIEEE 2006*)**

19. On a set N of all natural numbers is defined the relation R by $a\,R\,b$ iff the GCD of a and b is 2, then R is

(*a*) Reflexive and Transitive

(*b*) Symmetric and Transitive

(*c*) Symmetric only

(*d*) Not reflexive, not symmetric, not transitive

(*Kerala PET 2007*)

20. Let N be the set of integers. A relation R on N is defined as $R = \{(x, y) : xy > 0, x, y \in N\}$. Then, which of the following is correct ?

(*a*) R is symmetric but not reflexive

(b) R is reflexive but not symmetric

(*c*) R is symmetric and reflexive but not transitive

(*d*) R is an equivalence relation **(*NDA/NA2007*)**

21. Let R be a relation on the set of integers given by $a = 2^k.b$ for some integer k. Then R is

(*a*) reflexive but not symmetric

(b) reflexive and transitive but not symmetric

(*c*) equivalence relation

(*d*) symmetric and transitive but not reflexive

(*Kerala CEE 2006*)

22. Consider the following relations $R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$;

$S = \left\{\left(\dfrac{m}{n}, \dfrac{p}{q}\right)\middle|\right.$ m, n, p and q are integers such that $n, q \neq 0$

and $qm = pn\}$. Then,

(*a*) R is an equivalence relation but S is not an equivalence relation
(*b*) Neither R nor S is an equivalence relation
(*c*) S is an equivalence relation but R is not an equivalence relation
(*d*) R and S are both equivalence relations

(***AIEEE 2010***)

23. Let R be a relation on the set N, defined by $\{(x, y) : 2x - y = 10\}$ then R is
(*a*) Reflexive (*b*) Symmetric
(*c*) Transitive (*d*) None of the above

(***AMU 2012***)

24. If A and B are two equivalence relations defined on a set C, then
(*a*) $A \cap B$ is an equivalence relation
(*b*) $A \cap B$ is not an equivalence relation
(*c*) $A \cup B$ is an equivalence relation
(*d*) $A \cup B$ is not an equivalence relation (***UPSEE 2011***)

25. For any two real number a and b, we defined aRb if and only if $\sin^2 a + \cos^2 b = 1$. The relation R is
(*a*) reflexive but not symmetric
(*b*) symmetric but not transitive
(*c*) transitive but not reflexive
(*d*) an equivalence relation

ANSWERS

1. (*b*)	**2.** (*c*)	**3.** (*d*)	**4.** (*c*)	**5.** (*b*)	**6.** (*c*)	**7.** (*c*)	**8.** (*d*)	**9.** (*c*)	**10.** (*c*)
11. (*c*)	**12.** (*d*)	**13.** (*d*)	**14.** (*a*)	**15.** (*c*)	**16.** (*d*)	**17.** (*a*)	**18.** (*b*)	**19.** (*c*)	**20.** (*d*)
21. (*c*)	**22.** (*c*)	**23.** (*a*)	**24.** (*a*)	**25.** (*d*)					

HINTS AND SOLUTIONS

1. The relation = $\{(2, 2), (3, 3), (2, 4), (4, 6)\}$ is not R as the one of the ordered pairs (4, 6) does not satisfy the condition "a divides b" as 4 does not divide 6.

2. The set of ordered pairs $\{(b, y), (z, a), (x, c)\}$ does not state a relation from B to A as the ordered pair (b, y) has the first element 'b' from set A, whereas it should be from set B.

3. Let us examine the domain and range of the each relation individually:

(*i*) R : "is equal to" means $R = \{(a, b) : a = b, a \in A, b \in A\}$
$\therefore$ $R = \{(-2, -2), (-1, -1), (0, 0), (1, 1), (2, 2)\}$
$\therefore$ Domain of $R = \{-2, -1, 0, 1, 2\}$ and Range of $R = \{-2, -1, 0, 1, 2\}$
Hence both are equal and equal to A.

(*ii*) R : "is the multiplicative inverse of " means $R = \{(a, b): ab = 1, a \in A, b \in A\}$
$\therefore$ $R = \{(-1, -1), (1, 1)\}$
Here domain = $\{-1, -1\}$, range = $\{-1, 1\}$.

(*iii*) R : "is the additive inverse of "means $R = \{(a, b) : a = -b, a \in A, b \in A\}$
$\therefore$ $R = \{(-2, 2), (-1, 1), (0, 0), (1, -1), (2, -2)\}$
Hence domain = $\{-2, -1, 0, 1, 2\}$ and range = $\{2, 1, 0, -1, -2\}$
Both are equal and equal to A.

(*iv*) R : "is less than " means $R = \{(a, b) : a < b, a \in A, b \in A\}$
$\therefore$ $R = \{(-2, -1), (-2, 0), (-2, 1), (-2, 2), (-1, 0), (-1, 1), (-1, 2), (0, 1), (0, 2), (1, 2)\}$
Here domain = $\{-2, -1, 0, 1\}$ and range = $\{-1, 0, 1, 2\}$.
$\therefore$ Relations given in (*i*) and (*iii*) satisfy the given condition.

4. As $(1, 1), (2, 2) \notin R$ so R is not reflexive. A relation $a\ R\ b$ is said to symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$.
Here $(1, 2) \in R$ and $(2, 1) \in R$ so it is symmetric.
Also, as $(1, 2) \in R$, but $(2, 3), (1, 3) \notin R$, so R is not transitive.

5. $n(A) = 5$ and $n(B) = 7$
$\therefore$ $n(A \times B) = 5 \times 7 = 35$
Total number of relations from A to B = Total number of subsets of $(A \times B)$ = $\mathbf{2^{35}}$.

6. Here $A = \{3, 6, 9, 12\}$ and
$R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 12), (3, 6)\}$
- $\because$ (3, 3), (6, 6), (9, 9), (12, 12) all belong to R

$\Rightarrow \{(a, a): \in R\ \forall\ a \in A\}$
$\Rightarrow R$ is reflexive
- $(3, 6) \in R$ but $(6, 3) \notin R$. Also $(6, 12) \in R$ but $(12, 6) \notin R$.

So $(a, b) \in R \not\Rightarrow (b, a) \in R$
Hence R is not symmetric
Now $(3, 6) \in R$, $(6, 12) \in R$ and $(3,12) \in R \Rightarrow (3, 12) \in R$
$(x, y) \in R, (y, z) \in R \Rightarrow (x, z) \in R$
Hence R is transitive.

7. Option (*c*) satisfies all the conditions of an equivalence relation.
- $(1, 1), (2, 2), (3, 3) \in R$

$\Rightarrow (a, a) \in R\ \forall a \in A \Rightarrow R$ is reflexive
- $(3, 1) \in R$ and $(1, 3) \in R$

$\Rightarrow (a, b) \in R \Rightarrow (b, a) \in R\ \forall\ a, b \in A \Rightarrow$ symmetric
- $(1, 3) \in R, (3, 1) \in R$ and $(1, 1) \in R \Rightarrow (a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R\ \forall\ a, b, c \in A \Rightarrow R$ is transitive.

8. $A = \{1, 2, 3\}$.
$R = \{(1, 2), (2, 3)\}$
To make R an equivalence relation, it should be:
(i) **Reflexive:** So three more ordered pairs (1, 1), (2, 2), (3, 3) should be added to R to make it reflexive.
(ii) **Symmetric :** As R contains (1, 2) and (2, 3) so two more ordered pairs (2, 1) and (3, 2) should be added to make it symmetric.
(iii) **Transitive:** $(1, 2) \in R$, $(2, 3) \in R$. So to make R

transitive (1, 3) should be added to R. Also to maintain the symmetric property (3, 1) should then be added to R.

So, $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (2, 1), (3, 2), (1, 3), (3, 1)\}$ is an equivalence relation. So minimum 7 ordered pairs are to be added.

9. R is reflexive as $(a, a) \in R$. $a + a = 2a$ is even.

R is symmetric as $(a, b) \in R \Rightarrow (b, a) \in R$ as $a + b = b + a$ = even (Commutative law)

R is transitive as $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $(a + b)$ is even and $(b + c)$ is even

$\Rightarrow$ $(a + b) + (b + c)$ is even

$\Rightarrow$ $(a + 2b + c)$ is even $\Rightarrow (a + c)$ is even as ($2b$ is even)

$\Rightarrow$ $(a, c) \in R$

$\therefore$ R is an equivalence relation on the set of integers.

10. • There exists a real number –2, such that | –2 | is not less than –2 as $2 \not\leq -2$.

Thus for all negative, real numbers x, $|x| \not\leq x$.

Hence $(x, x) \notin R\ \forall$ real numbers.

Hence R is not reflexive.

• R is not symmetric since there exist real numbers –2 and 3 such that $|-2| \leq 3$ but $|3| \not\leq -2$,

i.e., $(-2, 3) \in R \not\Rightarrow (3, -2) \in R$.

• R is transitive since $\forall$ real numbers a, b, c

$(a, b) \in R, (b, c) \in R \Rightarrow |a| \leq b$ and $|b| \leq c$

$\Rightarrow$ $|a| \leq b \leq |b| \leq c$ ($\because$ For all x, $|x| \leq x$)

$\Rightarrow$ $|a| \leq c$

$\Rightarrow$ $(a, c) \in R$

11. R is reflexive as every natural number is divisible by itself. So $(a, a) \in R$.

R is not symmetric as $(a, b) \in R$ does not imply $(b, a) \in R$, *i.e.*, if b is divisible by a, then a is not divisible by b.

For example $(4, 8) \in R$ as 8 is divisible by 4, but $(8, 4) \notin R$ as 4 is not divisible by 8.

R is transitive as $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$

$(a, b) \in R \Rightarrow b$ is divisible by $a \Rightarrow b = ma$ where $m \in N$

$(b, c) \in R \Rightarrow c$ is divisible by $b \Rightarrow c = nb \quad n \in N$

$c = n \times ma \Rightarrow c = nma$

c is divisible by $a \Rightarrow (a, c) \in R$.

12. Every triangle is similar to itself, so $(T_1, T_1) \in R \Rightarrow R$ is reflexive.

$(T_1, T_2) \in R \Rightarrow T_1 \sim T_2 \Rightarrow T_2 \sim T_1, \Rightarrow (T_2, T_1) \in R \Rightarrow R$ is symmetric

$$\left.\begin{array}{l}(T_1, T_2) \in R \Rightarrow T_1 \sim T_2 \\ (T_2, T_3) \in R \Rightarrow T_2 \sim T_3\end{array}\right\} \Rightarrow T_1 \sim T_3 \Rightarrow (T_1, T_3) \in R \Rightarrow R \text{ is}$$

transitive.

$\therefore$ R is an equivalence relation.

13. $|a - a| = |0| = 0$ so $(a, a) \notin R \Rightarrow R$ is not reflexive

$(a, b) \in R \Rightarrow |a - b| > 0 \Rightarrow |b - a| > 0 \Rightarrow (b, a) \in R$

($\because |a - b| = |b - a|$)

$\Rightarrow$ R is symmetric

$(a, b) \in R \Rightarrow |a - b| > 0$ and $(b, c) \in R \Rightarrow |b - c| > 0$ $\forall$ real numbers a, b, c.

$\therefore$ $|a - b| > 0$ and $|b - c| > 0 \Rightarrow |a - c| > 0$

$\Rightarrow$ $(a, c) \in R \ \Rightarrow\ R$ is transitive.

14. $(a, a) \in R \Rightarrow 1 + a \,.\, a = 1 + a^2 > 0\ \forall$ real numbers a

$\Rightarrow$ R is reflexive

$(a, b) \in R \Rightarrow 1 + ab > 0 \Rightarrow 1 + ba > 0 \Rightarrow (b, a) \in R$

$\Rightarrow$ R is symmetric

We observe that $\left(1, \dfrac{1}{2}\right) \in R$ and $\left(\dfrac{1}{2}, -1\right) \in R$

but $(1, -1) \notin R$ as $1 + 1 \times (-1) = 0 \not> 0$

$\Rightarrow$ R is not transitive.

15. **(*a*)** Let $a, b, c \in A$ where A is a set of real numbers.

Then $R = \{(a, b) : a \leq b, a, b \in A\}$ is :

Reflexive: $a \leq a \Rightarrow (a, a) \in R$ **(Yes)**

Symmetric: $a \leq b \Rightarrow (a, b) \in R$, but

$a \leq b \Rightarrow a < b$ or $a = b$

$\Rightarrow$ $b = a$ but $b \not< a$ so $(b, a) \notin R$ **(No)**

Transitive: $\left.\begin{array}{l}(a, b) \in R \Rightarrow a \leq b \\ (b, c) \in R \Rightarrow b \leq c\end{array}\right\} \Rightarrow a \leq c \Rightarrow (a, c)\ R$ **(Yes)**

(*b*) Let A = set of positive integers. Then,

$R = \{(a, b) : a$ is a multiple of b, $a, b \in R\}$

Reflexive: Every positive integer is a multiple itself, so,

$(a, a) \in R$ **(Yes)**

Symmetric: $(a, b) \in R \Rightarrow a$ is a multiple of $b \not\Rightarrow b$ is a multiple of $a \Rightarrow (b, a) \notin R$ **(No)**

Transitive : $(a, b) \in R \Rightarrow a$ is a multiple of b

$\Rightarrow$ $a = mb\ \forall\ m \in N$.

$(b, c) \in R \Rightarrow b$ is a multiple of $c \Rightarrow b = nc\ \forall\ n \in N$

$\therefore$ $a = m \times nc \Rightarrow a = mnc\ \forall\ m, n \in N$

$\Rightarrow$ a is a multiple of $c \Rightarrow (a, c) \in R$.

(*c*) • This relation is **not reflexive** as a line cannot be perpendicular to itself.

• If $l_1 \perp l_2$ then $l_2 \perp l_1$, therefore given relation **is symmetric**

• $l_1 \perp l_2$ and $l_2 \perp l_3 \not\Rightarrow l_1 \perp l_3$, so given relation is **not transitive.**

(*d*) • A person cannot be his own father, so relation is **not reflexive.**

• If a is father of b, then b cannot be father of a, so relation **is not symmetric.**

• If a is father of b, b is father of c, then a cannot be father of c, so relation is **not transitive.**

$\therefore$ From the given options (c) is only symmetric.

16. We can check the given properties as follows:

Reflexive: Let $(a, b) \in N \times N$. Then $(a, b) \in N$

$\Rightarrow$ $a + b = b + a$ (Communtative law of Addition)

$\Rightarrow$ $(a, b)\ R\ (b, a)$

$\Rightarrow$ $(a, b)\ R\ (a, b)$

$\Rightarrow$ R is reflexive.

Symmetric: Let $(a, b), (c, d) \in N \times N$ such that

$(a, b)\ R\ (c, d)$. Then

$(a, b)\ R\ (c, d) \Rightarrow a + d = b + c \Rightarrow b + c = a + d$

$\Rightarrow \quad c + b = a + d$ (By commutativity of addition on N)

$\Rightarrow \quad (c, d)\ R\ (a, b)$

$\therefore \quad R$ is symmetric.

Transitive : Let $(a, b), (c, d), (e, f) \in N \times N$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then,

$$\left.\begin{array}{l}(a, b)\ R\ (c, d) \Rightarrow a + d = b + c \\ (c, d)\ R\ (e, f) \Rightarrow c + f = d + e\end{array}\right\}$$

$$\Rightarrow (a + d) + (c + f) = (b + c) + (d + e)$$

$$\Rightarrow a + c = b + e \Rightarrow (a, b)\ R\ (e, f)$$

$\therefore \quad (a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$ on $N \times N$ so R is transitive.

Hence R is an equivalence relation on $N \times N$.

17. • For all $a \in I$, $a - a = 0$, which is divisible by 11.

Thus, $(a, a) \in R$ for all $a \in N \Rightarrow R$ **is reflexive**

• Let $(a, b) \in R$ $(a - b)$ is divisible by 11

$\Rightarrow \quad -(a - b)$ is divisible by 11

$\Rightarrow \quad (b - a)$ is divisible by 11

$\Rightarrow \quad (b, a) \in R$

$\Rightarrow \quad R$ **is symmetric.**

• Let $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow \quad (x - y)$ is divisible by 11 and $(y - z)$ is divisible by 11

$\Rightarrow \quad (x - y) + (y - z)$ is divisible by 11

$\Rightarrow \quad (x - z)$ is divisible by 11

$\Rightarrow \quad (x, z) \in R$

$\Rightarrow \quad R$ **is transitive**

$\therefore \quad R$ is an equivalence relation.

18. • Let $x \in W$.

$(x, x) \in R$, since the words 'x' and 'x' have all letters in common

$\Rightarrow \quad$ 'x' and 'x' have at least one letter in common

$\Rightarrow \quad$ ***R* is reflexive.**

• Let $x, y, z \in W$.

Then $(x, y) \in R \quad \Rightarrow$ 'x' and 'y' have at least one letter in common

$\Rightarrow \quad$ 'y' and 'x' have at least one letters common

$\Rightarrow \quad (y, x) \in R$

$\Rightarrow \quad$ ***R* is symmetric.**

• Let $x, y, z \in W$

Then $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow \quad$ 'x' and 'y' have at least one letter common and 'y' and 'z' have at least one letter common

which does not necessarily mean that 'x' and 'z' have at lest one letter common.

$\therefore \quad$ ***R* is not transitive**

For example, let x = 'AND', y = 'NOT', z = 'PET'

x and y have 'N' common

y and z have 'T' common

but x and z have no common letter.

19. • Let $a \in N$. Then

$(a, a) \notin R$ as the GCD of 'a' and 'a' is 'a' not 2.

R is **not reflexive**

• Let $a, b \in N$. Then,

$(a, b) \notin R \Rightarrow$ GCD of 'a' and 'b' is 2

$\Rightarrow \quad$ GCD of 'b' and 'a' is 2

$\Rightarrow \quad (b, a) \in R$

$\therefore \quad$ ***R* is symmetric**

• Let $a, b, c \in N$. Then,

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad$ GCD of a and b is 2 and GCD of b and c is 2

$\not\Rightarrow \quad$ GCD of a and c is 2

***R* is not transitive**

For example, let $a = 4$, $b = 10$, $c = 12$

GCD of $(4, 10) = 2$

GCD of $(10, 12) = 2$

But GCD of $(4, 12) = 4$.

20. $R = \{(x, y) : xy > 0\ x, y \in N\}$

• $x, x \in N \Rightarrow x^2 > 0 \Rightarrow R$ is reflexive

• $x, y \in N$ and $(x, y) \in R \Rightarrow xy > 0$

$\Rightarrow \quad yx > 0\ (y, x) \in R$

$\Rightarrow \quad R$ is symmetric

• $x, y, z \in N$ and $(x, y) \in R$, $(y, z) \in R$

$\Rightarrow \quad xy > 0$ and $yz > 0$

$\Rightarrow \quad xz > 0 \Rightarrow (x, z) \in R$

$\Rightarrow \quad R$ is transitive.

$\therefore \quad R$ is an equivalence relation.

21. Given, $a\ R\ b = a = 2^k.b$ for some integer.

Reflexive: $a\ R\ a \Rightarrow a = 2^0.a$ for $k = 0$ (an integer). **True**

Symmetric: $a\ R\ b \Rightarrow a = 2^k\ b \Rightarrow b = 2^{-k} . a$

$\Rightarrow b\ R\ a$ as $k, -k$ are both integers. **True**

Transitive: $a\ R\ b \Rightarrow a = 2^{k_1}\ b$

$b\ R\ c \Rightarrow b = 2^{k_2}\ c$

$\therefore \quad a = 2^{k_1} . 2^{k_2}c = 2^{k_1 + k_2}\ c$, $k_1 + k_2$ is an integer.

$\therefore \quad a\ R\ b, b\ R\ c \Rightarrow a\ R\ c$ **True**

$\therefore$ R is an equivalence relation.

22. $R = \{(x, y) \mid x, y \in R, x = wy, w$ is a rational number$\}$

Reflexive: $x\ R\ x \Rightarrow x = wx \Rightarrow w = 1$, (a rational number)

Hence R is reflexive.

Symmetric: $x\ R\ y \not\Rightarrow y\ R\ x$ as

$0\ R\ 1 \Rightarrow 0 = (0) . 1$ where 0 is a rational number but $1\ R\ 0 \Rightarrow 1 = (w)\ 0$ which is not true for any rational number.

$\therefore$ R is not an equivalence relation.

$$S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right) \middle| m, n, p, q \in I, n, q \neq 0 \text{ and } qm = pn\right\}.$$

Reflexive $\frac{m}{n}\ S\ \frac{m}{n} \Rightarrow mn = nm$ (**True**)

Symmetric $\frac{m}{n} S \frac{p}{q} \Rightarrow mq = pn \Rightarrow pn = mq \Rightarrow \frac{p}{q} S \frac{m}{n}$ **(True)**

Transitive $\frac{m}{n} S \frac{p}{q}$ and $\frac{p}{q} S \frac{r}{s}$

$\Rightarrow mq = pn$ and $ps = qr$

$\Rightarrow mq.ps = pn.qr \Rightarrow ms = nr$

$\Rightarrow \frac{m}{n} = \frac{r}{s} \Rightarrow \frac{m}{n} S \frac{r}{s}$ **(True)**

$\therefore$ S is an equivalence relation.

23. Given, $\{(x, y) : 2x - y = 10\}$

Reflexive, $x R x = 2x - x = 10 \Rightarrow x = 10 \Rightarrow y = 10$

$\therefore$ Point $(10, 10) \in N \Rightarrow R$ is **reflexive.**

25. Given, $a R b \Rightarrow \sin^2 a + \cos^2 b = 1$

Reflexive: $a R a \Rightarrow \sin^2 a + \cos^2 a = 1 \ \forall \ a \in R$ **(True)**

Symmetric: $a R b \Rightarrow \sin^2 a + \cos^2 b = 1$

$\Rightarrow 1 - \cos^2 a + 1 - \sin^2 b = 1$

$\Rightarrow \sin^2 b + \cos^2 a = 1$

$\Rightarrow b R a \ \forall \ a, b \in R$ **(True)**

Transitive: $a R a$ and $b R c$

$\Rightarrow \sin^2 a + \cos^2 b = 1$ and $\sin^2 b + \cos^2 c = 1$

$\therefore$ Adding these two equations we get

$\sin^2 a + \cos^2 b + \sin^2 b + \cos^2 c = 2$

$\Rightarrow \sin^2 a + \cos^2 c = 1 \quad \Rightarrow a R c$ **(True)**

$\therefore$ R is an equivalence relation.

SELF ASSESSMENT SHEET

1. Let $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8\}$. Then $R = \{(1, 5), (1, 7), (2, 6)\}$ is a relation from set A to B defined as :

(*a*) $R = \{(a, b) : a, b$ are odd$\}$
(*b*) $R = \{(a, b) : a, b$ are even$\}$
(*c*) $R = \{(a, b) : a, b$ are primes$\}$
(*d*) $R = \{(a, b) : b/a$ is odd$\}$

2. The range of the relation R defined by $R = \{(x + 1, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}\}$ is

(*a*) $\{1, 2, 3, 4, 5, 6\}$ (*b*) $\{5, 6, 7, 8, 9, 10\}$
(*c*) $\{6, 7, 8, 9, 10, 11\}$ (*d*) $\{0, 1, 2, 3, 4, 5\}$

3. The relation "is parallel to" on a set S of all straight lines in a plane is :

(*a*) Symmetric only
(*b*) Reflexive and Transitive only
(*c*) Transitive only
(*d*) An equivalence relation

4. Let R be the relation in the set $\{1, 2, 3, 4\}$ given by $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$. Then R is

(*a*) Reflexive and symmetric but not transitive
(*b*) Reflexive and transitive but not symmetric
(*c*) Symmetric and transitive but not reflexive
(*d*) An equivalence relations

5. Let $A = \{2, 3, 5, 6\}$. Then, which of the following relations is transitive only?

(*a*) $R = \{(2, 3), (6, 6)\}$
(*b*) $R = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 3), (3, 5), (2, 6)\}$.
(*c*) $R = \{(2, 6), (6, 2)\}$
(*d*) $R = \{(2, 3), (3, 5), (2, 5)\}$.

6. Let $A = \{(2, 5, 11)\}$, $B = \{3, 6, 10\}$ and R be a relation from A to B defined by $R = \{(a, b) : a$ and b are co-prime$\}$. Then R is

(*a*) $\{(2, 3), (2, 6), (5, 10), (5, 6)\}$
(*b*) $\{(2, 6), (2, 10), (5, 10)\}$
(*c*) $\{(2, 3), (5, 3), (5, 6), (11, 3), (11, 6), (11, 10)\}$
(*d*) $\{(2, 10), (5, 3), (5, 6), (11, 10)\}$.

7. If R is a relation on a finite set A having n elements, then the number of relations on A is

(*a*) n^2 (*b*) 2^n (*c*) n^n (*d*) 2^{n^2}

(*AMU 2006*)

8. The relation 'is less than' on a set of natural numbers is

(*a*) Only reflexive (*b*) Only symmetric
(*c*) Only transitive (*d*) An equivalence relation.

9. Let R be *a* relation on the set of all real numbers R $R = \{(a, b) \in R \times R : a^2 + b^2 = 1\}$, then R is

(*a*) Equivalence (*b*) Only transitive
(*c*) Only symmetric (*d*) None of these

10. Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (1, 1), (2, 3)\}$ be a relation on A.

What minimum number of ordered pairs must be added with the elements of R so that it may become transitive?

(*a*) 1 (*b*) 2 (*c*) 0 (*d*) 3

ANSWERS

1. (*d*) **2.** (*b*) **3.** (*d*) **4.** (*b*) **5.** (*d*) **6.** (*c*) **7.** (*d*) **8.** (*c*) **9.** (*c*) **10.** (*a*)

HINTS AND SOLUTIONS

1.
- Since $(2, 6) \in R$, the relation "*a* and *b* are odd" does not exist.
- Since $(1, 5)$ and $(1, 7) \in R$, the relation "*a* and *b* are even" does not exist.
- None of the ordered pairs in R are prime numbers.
- $5/1 = 5$, $7/1 = 7$, $6/2 = 3$, quotients being all odd numbers, the relation b/a is odd exists.

2. The range of the given relation is defined by the second element, *i.e.*, $(x + 5)$ in the ordered pair $(x + 1, x + 5)$ defining the relation.

$\because$ $x \in \{0, 1, 2, 3, 4, 5\}$, therefore Range = $\{0 + 5, 1 + 5, 2 + 5, 3 + 5, 4 + 5, 5 + 5\} = \{5, 6, 7, 8, 9, 10\}$.

3. Let $R = \{(x, y)$: line x is parallel to line y, $x\ y \in$ set of coplanar straight lines$\}$.

- Every line is parallel to itself. So, if $x \in S$, then $(x, x) \in R$

$\Rightarrow$ R is reflexive

- If $(x, y) \in R \Rightarrow x \,||\, y \Rightarrow y \,||\, x \Rightarrow (y, x) \in R$

$\Rightarrow$ R is symmetric

- $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow$ $x \,||\, y$ and $y \,||\, z$

$\Rightarrow$ $x \,||\, z \Rightarrow (x, z) \in R$

$\Rightarrow$ R is transitive

$\therefore$ R being reflexive, symmetric and transitive, it is an equivalence relation.

4. Let $A = \{1, 2, 3, 4\}$

- $\because$ (1, 1), (2, 2), (3, 3) and (4, 4) $\in R \Rightarrow R$ is reflexive
- $\because$ (1, 2) $\in R$ but (2, 1) $\notin R$; (1, 3) $\in R$ and (3, 1) $\notin R$; (3, 2) $\in R$ and (2, 3) $\notin R \Rightarrow R$ is not symmetric
- (1, 3) $\in R$ and (3, 2) $\in R$ and (1, 2) $\in R$

$\Rightarrow$ R is transitive.

7. Set A has n elements $\Rightarrow n(A) = n$

$\Rightarrow$ $A \times A$ has $n \times n = n^2$ elements

$\therefore$ Number of relations on A = Number of subsets of $A \times A$ = $\mathbf{2^{n^2}}$

8. Let N be the set of natural numbers. Then

$R = \{(a, b) : a < b, a, b \in N\}$

A natural number is not less than itself

$\Rightarrow$ $(a, a) \notin R$ where $a \in N$

$\Rightarrow$ R is not reflexive

- $\forall$ $a, b \in N$, $(a, b) \in R \Rightarrow a < b \not\Rightarrow b < a \Rightarrow (b, a) \notin R$

$\Rightarrow$ R is not symmetric.

- $\forall$ $a, b, c \in N$, $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a < b$ and $b < c$

$\Rightarrow$ $a < c$ $(a, c) \in R$

$\Rightarrow$ R is transitive.

9. • $\forall$ $a \in R$, $a^2 + a^2 \neq 1 \Rightarrow (a, a) \notin R \Rightarrow R$ is not reflexive.

For example, $0^2 + 0^2 = 0$, $1^2 + 1^2 = 1$, $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$ and so on.

- $\forall$ $a, b \in R$, $(a, b) \in R \Rightarrow a^2 + b^2 = 1 \Rightarrow b^2 + a^2 = 1$

$\Rightarrow$ $(b, a) \in R$

$\Rightarrow$ R is symmetric

- $\forall$ $a, b, c \in R$, $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a^2 + b^2 = 1$ and $b^2 + c^2 = 1$ which does not necessarily mean

$a^2 + c^2 = 1 \Rightarrow (a, c) \notin R$

For example, let $a = 0$, $b = 1$, $c = 0$

$0^2 + 1^2 = 1$ and $1^2 + 0^2 = 1$

But $\mathbf{0^2 + 0^2 \neq 1}$.

10. For the relation R to become transitive:

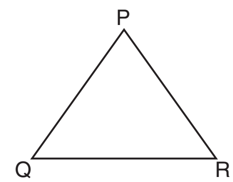
(1, 2) $\in R$ and (2, 3) $\in R$ should imply (1, 3) $\in R$

$\therefore$ Minimum one ordered pair (1, 3) should be added to R.

6 Plane Geometry –Triangles

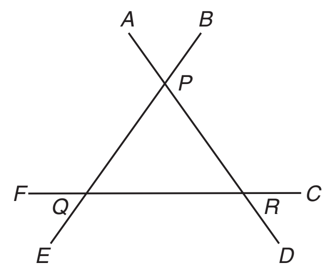
KEY FACTS

I. Definitions



A triangle is a three sided closed figure formed by three non-collinear points.

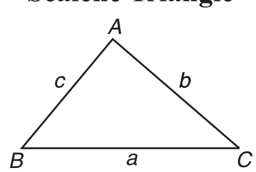
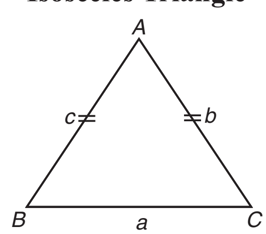
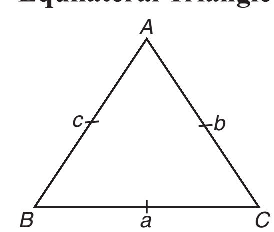
The three points P, Q and R in the given figure are called the **vertices,** line segments joining the three vertices, *i.e.*, PQ, QR and PR are called the **sides** and $\angle P$, $\angle Q$ and $\angle R$ are the **interior angles** of the triangle.



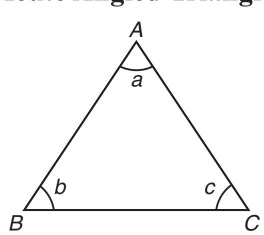
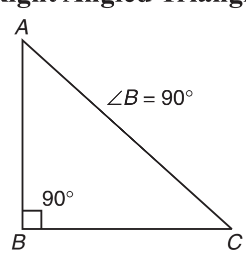
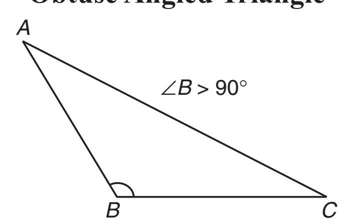
If the sides of a triangle are produced as shown in the given diagram, then the angles $\angle PRC$, $\angle QRD$, $\angle PFQ$, $\angle RQE$, $\angle QPA$ and $\angle RPB$ are the exterior angles of ΔABC.

II. Types of Triangles:

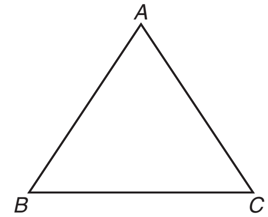
a. By sides:

Scalene Triangle	Isosceles Triangle	Equilateral Triangle
 $a \neq b \neq c$ (All the sides are unequal)	 (At least two sides are equal. Here, $AB = AC$) ***Angles opposite equal sides are also equal,*** *i.e.*, $\angle C = \angle B$.	 $a = b = c$ (All sides are equal) ***All angles are equal to 60º***

b. By angles:

Acute Angled Triangle	Right Angled Triangle	Obtuse Angled Triangle
 All angles are acute, *i.e.*, $\angle A < \angle 90º$, $\angle B < 90º$, $\angle C < 90º$	 One of the angles is a right angle. The other two are ***complementary*** to each other	 One of the angles is an obtuse angle.

III. Some Important Properties of Triangles:

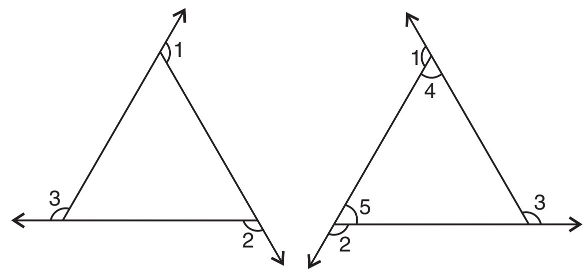


a. ***The sum of the three interior angles of a triangle is always 180º, i.e.***, $\angle BAC + \angle ABC + \angle BCA = 180°$.

b. (*i*) ***If the sides of a triangle are produced in order then, the sum of the three (ordered) exterior angles of a triangle is 360º, i.e.***, **in both the figures,** $\angle 1 + \angle 2 + \angle 3 = 360°$

(*ii*) ***The measure of an exterior angle is equal to the sum of the measures of the interior opposite angles, i.e.***, **in figure (ii)** $\angle 3 = \angle 4 + \angle 5$.

(*iii*) ***The measure of an exterior angle is greater than the measure of each of the interior opposite angles, i.e.***, **in figure (ii)** $\angle 3 > \angle 4$ **and** $\angle 3 > \angle 5$.



(*iv*) ***The sum of the measure of exterior angle at a vertex and its adjacent interior angle is 180º.***

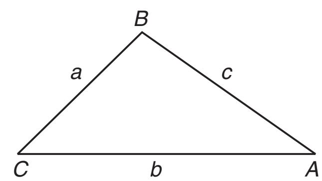
IV. Triangle Inequalities:

(*i*) ***Sum of any two sides of a triangle is always greater than the third side.***

(*ii*) ***The difference of any two sides is always less than the third side.***

(*iii*) ***If two sides of a triangle are not equal, then the angle opposite to the greater side is greater and vice versa.***

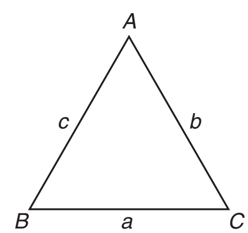
(*iv*) Let a, b, c be the three sides of a triangle ΔABC where $AB = c$ is the longest side (say). Then,



- if $c^2 < a^2 + b^2$, then the triangle is acute angled.
- if $c^2 = a^2 + b^2$, then the triangle is right angled.
- if $c^2 > a^2 + b^2$, then the triangle is obtuse angled.

V. Sine Rule:

In a ΔABC, if a, b, c be the three sides opposite to the angles A, B and C respectively, then



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

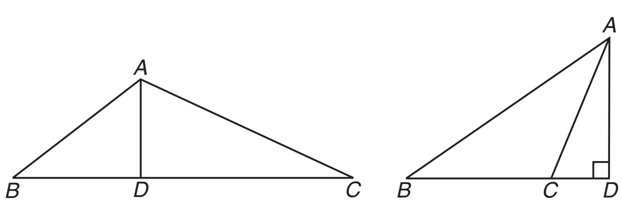
VI. Cosine Rule:

In a ΔABC, if a, b, c be the sides opposite to the angles A, B and C respectively, then

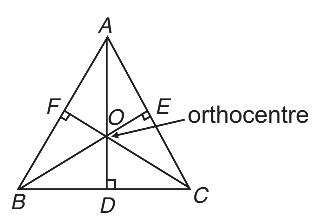
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

VII. Some Important Definitions:

(*i*) **Altitude** ***of a triangle is the perpendicular drawn from a vertex to the opposite side*** (produced if necessary). Every triangle has three altitudes.

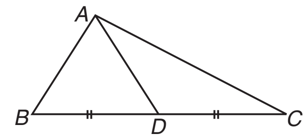


AD is the altitude on side BC.



Orthocentre ***is the point of intersection of the three altitudes of a triangle***. Hence, O is the orthocentre of ΔABC, where $\angle BOC = 180º - \angle A$, $\angle AOB = 180º - \angle C$, $\angle COA = 180º - \angle B$.

(*ii*) **Median** ***is the straight line segment joining the mid-point of any side to the opposite vertex.*** Every triangle has three medians and a median bisects the area of a Δ.

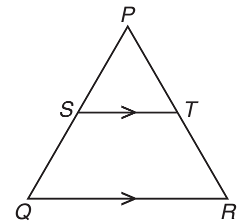


VIII. Important Theorems on Triangles

1. **Basic Proportionality Theorem (BPT):** ***Any line parallel to one side of a triangles divides the other two sides proportionally.***

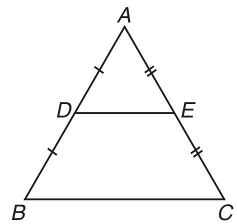
   Thus, if ST is drawn parallel to side QR of ΔPQR, then

$$\frac{PS}{SQ} = \frac{PT}{TR} \quad \text{or} \quad \frac{PS}{PQ} = \frac{PT}{PR}$$



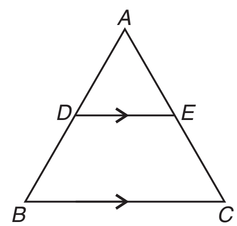
2. **Mid-point Theorem:** ***The line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to one-half of it.***

   Thus, if D and E are the midpoints of sides AB and AC respectively of ΔABC, then $DE \parallel BC$ and $DE = \frac{1}{2}BC$.



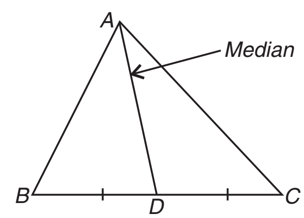
   Converse of Mid-point Theorem: ***The straight line drawn through the mid-point of one side of a triangle parallel to another side bisects the third side.***

   Thus, a line drawn through D, the mid-point of side AB of ΔABC, parallel to BC bisects AC, *i.e.*, E is the mid-points of AC, *i.e.*, $AE = EC$.



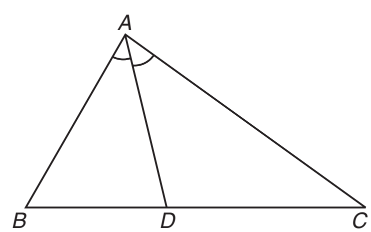
3. **Apollonius Theorem:** ***In a triangle, the sum of the squares of any two sides of a triangle is equal to twice the sum of the square of the median to the third side and square of half the third side,***

   i.e., $AB^2 + AC^2 = 2\left(AD^2 + \left(\frac{BC}{2}\right)^2\right) = 2(AD^2 + BD^2) = 2(AD^2 + CD^2)$.

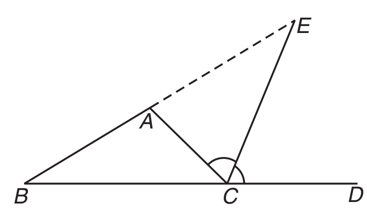


4. **Interior Angle Bisector Theorem:** ***In a triangle, the angle bisector of an angle divides the opposite side to the angle in the ratio of the remaining two sides.***

   Thus, if AD is the internal bisector of $\angle A$ of ΔABC, then $\frac{BD}{CD} = \frac{AB}{AC}$ and $BD \times AC = DC \times AB = AD^2$



5. **External Angle Bisector Theorem:** In a triangle, the angle bisector of any exterior angle of a triangle divides the side opposite to the external angle in the ratio of the remaining two sides, *i.e.*, If CE is the bisector of external angle ACD, then $\frac{BE}{AE} = \frac{BC}{AC}$.

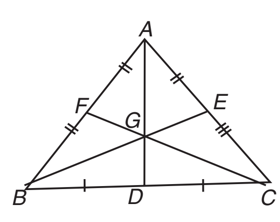


   ∴ Area (ΔABD) = Area (ΔACD) = $\frac{1}{2}$ Area (ΔABC).

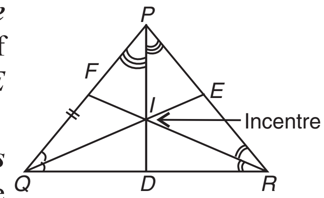
   Centroid ***is point of intersection or point of concurrence of the three medians of a triangle***. Also it divides each median in the ratio 2:1 (vertex : base)

   G is the centroid of ΔABC, *i.e.*, the point of concurrence of medians AD, BE and CF. Also,

$$\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = \frac{2}{1}$$

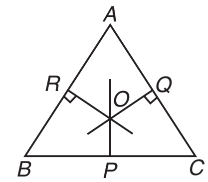


6. **Angle Bisector:** ***A line segment from a vertex of triangle which bisects the angle at the vertex is called the angle bisector***. PD, QE and RF are angle bisectors of $\angle P$, $\angle Q$ and $\angle R$ respectively of ΔPQR such that $\angle QPD = \angle RPD$, $\angle PQE = \angle RQE$ and $\angle PRF = \angle QRF$.



Incentre ***is the point of intersection of the angle bisectors of a triangle and it is equidistant from all the three sides of the triangle***, *i.e.*, perpendicular distance between the side and incentre is equal for all the three sides.

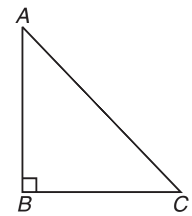
7. **Perpendicular Bisector:** ***A line segment bisecting a side of a triangle at right angles*** *is* ***called a perpendicular bisector***. The point of concurrence of the perpendicular bisectors is called the **circumcentre**. Here O is the circumcentre of ΔABC. The circumcentre is equidistant from the vertices of a triangle, *i.e.*, $OA = OB = OC$.



Note: The orthocentre, centroid, incentre and circumcentre coincide in case of an equilateral triangle.

8. **Pythagoras' Theorem:** ***The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.***

Here ΔABC is right angled at C. So AC is the hypotenuse. Hence, according to Pythagoras, Theorem $AC^2 = AB^2 + BC^2$.



Converse of Pythagoreas' Theorem: ***If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the triangle is right angled.***

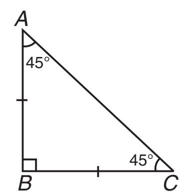
If a, b, c, be the sides opposite to $\angle A$, $\angle B$ and $\angle C$ respectively of a ΔABC and $c^2 = a^2 + b^2$, then ΔABC is right angled at C.

Such numbers or triplets a, b, c which satisfy the condition $a^2 + b^2 = c^2$ are called Pythagorean triplets. Some examples of Pythagorean triplets are: (3, 4, 5) ; (5, 12, 13), (7, 24, 25), etc.

9. **45º – 45º – 90º triangle:** ***If the angles of a triangle are 45º, 45º and 90º, then the hypotenuse (the longest side) is $\sqrt{2}$ times any smaller side, i.e.,***

In ΔABC, $\angle B = 90^\circ$, $\angle A = \angle C = 45^\circ$, then

$\boldsymbol{AB = BC}$ **and** $\boldsymbol{AC = \sqrt{2}\, AB}$ **or** $\boldsymbol{\sqrt{2}\, BC}$.



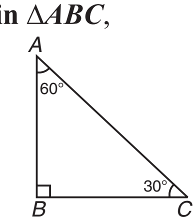
The converse of the above theorem: If the ratio of the sides of a triangle is $1 : 1 : \sqrt{2}$, then it is a 45º – 45º – 90º triangle.

10. **30º – 60º – 90º triangle:** ***If the angles of a triangle are 30º, 60º and 90º, then the side opposite to the 30º angle is half of the hypotenuse and side opposite to 60º is $\frac{\sqrt{3}}{2}$ times the hypotenuse, i.e.,*** **in ΔABC**,

where $\angle B = 90^\circ$ and $\angle C = 30^\circ$, $\angle A = 60^\circ$, then

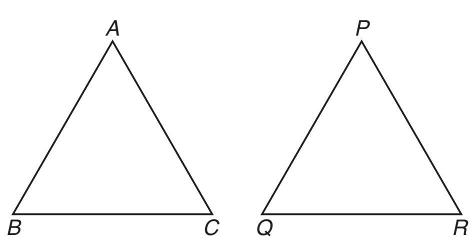
$\boldsymbol{AB = \frac{1}{2}\, AC}$ **and** $\boldsymbol{BC = \frac{\sqrt{3}}{2}\, AC}$**,** then

i.e., the ratio of the sides is $1 : \sqrt{3} : 2$. The converse of the above also holds true.



IX. Congruency of Triangles: Two triangles are congruent to each other

(*i*) ***if each of the three sides and three angles of one triangle are equal to the respective sides and angles of the other.***



$\Delta ABC \cong \Delta PQR$ if

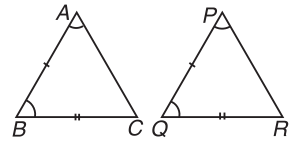
$AB = PQ$ and $\angle A = \angle P$

$AC = PR$ and $\angle B = \angle Q$

$BC = QR$ and $\angle C = \angle R$

Tests of Congruency:

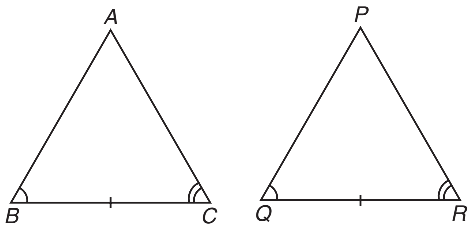
(*i*) **SAS Axiom (Side-angle-side):** ***If the two sides and the included angle of one triangle are respectively equal to the two sides and the included angle of the other, the triangles are congruent.***



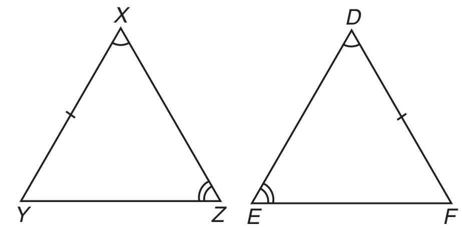
$AB = PQ$
$AC = PR$
$BC = QR$

Note: The two equal sides must be opposite to angles which are known to be equal.

(*ii*) **ASA or AAS Axiom (Two angles, corresponding side):** ***If two angles and one side of a triangle are respectively equal to two angles and the corresponding side of the other triangles, the triangles are congruent. The side may be in included side.***

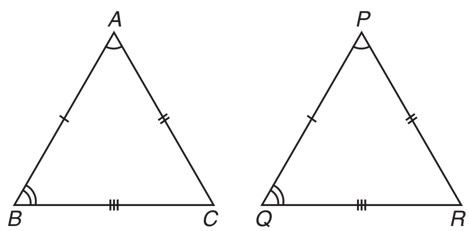


$\angle B = \angle Q$
$\angle C = \angle R$
$BC = QR$
$\Delta ABC \cong \Delta PQR$ (ASA)



$\angle X = \angle D$
$\angle Z = \angle E \quad \therefore\ \Delta XYZ \cong \Delta DEF$ (AAS)
$XY = DF$

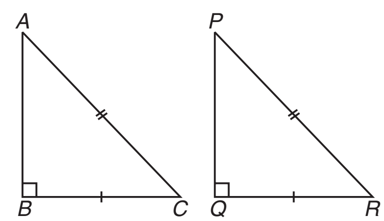
(*iii*) **SSS Axiom (Three sides):** ***If three sides of one triangle are respectively equal to the corresponding three sides of the other triangle, the triangles are congruent :***



$$\left.\begin{aligned} AB &= PQ \\ AC &= PR \\ BC &= QR \end{aligned}\right\} \Rightarrow \quad \Delta ABC \cong \Delta PQR \ \text{(SSS)}$$

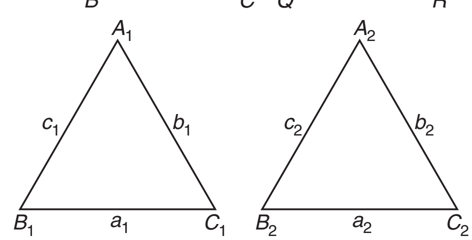
(*iv*) **RHS Axiom (Right angle-Hypotenuse-side):** ***If the hypotenuse and one side of a right angled triangle are respectively equal to the hypotenuse and corresponding side of the other right angled triangle, the two triangles are congruent.***

$\therefore \quad AC = PR$, $BC = QR$ and $\angle B = \angle Q = 90^\circ \Rightarrow \Delta ABC \cong PQR \quad (RHS)$



X. Similarity of Triangles:

Two triangles are said to be similar, if their corresponding angles are equal, and their corresponding sides are proportional.



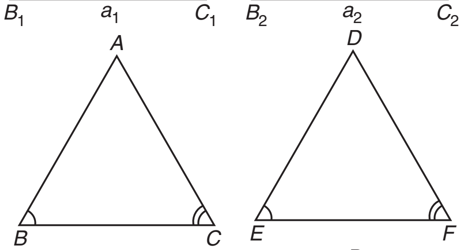
Thus, $\Delta A_1B_1C_1$ is similar to $\Delta A_2B_2C_2$ or $\Delta A_1B_1C_1 \sim \Delta A_2B_2C_2$ if

(*i*) $\angle A_1 = \angle A_2$; $\angle B_1 = \angle B_2$; $\angle C_1 = \angle C_2$

(*ii*) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = k$

Tests for Similarity:

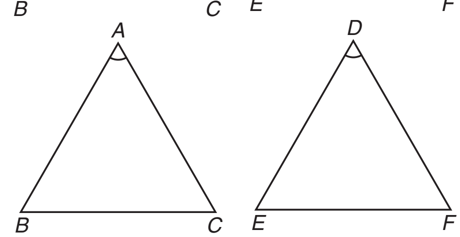
(*i*) **A-A axiom of similarily:** ***If two angles of one triangle are equal to the corresponding two angles of the other triangle, then the triangles are said to be similar.***



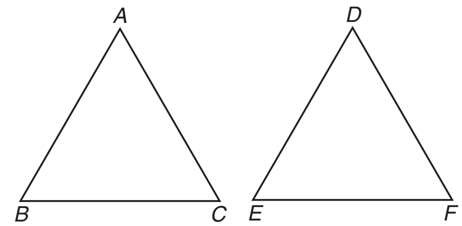
Note : If two pairs of corresponding angles in two triangles are equal, then the third pair will obviously be equal.

$\angle ABC = \angle DEF$, $\angle ACB = \angle DFE \Rightarrow \Delta ABC \sim \Delta DEF$. (*AA similarity*)

(*ii*) **S-A-S axiom of similarity:** ***If two triangles have a pair of corresponding angles equal and the sides including them proportional, then the triangles are similar.***



In Δs ABC and DEF, $\angle A = \angle D$, $\dfrac{AB}{DE} = \dfrac{AC}{DF} = k$, then $\Delta ABC \sim \Delta DEF$.

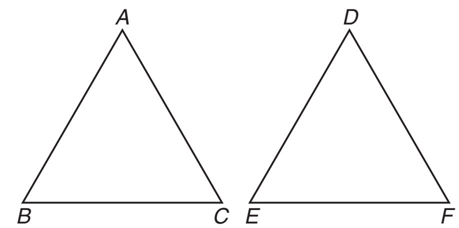


(*iii*) S-S-S axiom of similarity: ***If two triangles have their pairs of corresponding sides proportional, then the triangles are similar.***

If $\dfrac{BC}{EF} = \dfrac{AB}{DE} = \dfrac{AC}{DF}$, then, $\Delta ABC \sim \Delta DEF$

XI. Properties of Similar Triangles:

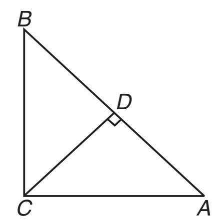
For a pair of similar triangles,



1. **Ratio of sides** ***= Ratio of Altitudes***
   = Ratio of Medians
   = Ratio of angle bisectors
   = Ratio of in-radii
   = Ratio of circums-radii.
2. **Ratio of areas =** ***Ratio of squares of corresponding sides.***

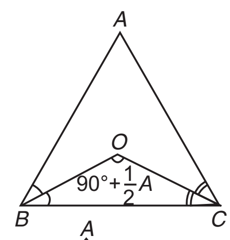
$$\frac{\text{Area }(\Delta ABC)}{\text{Area }(\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{AC^2}{DF^2} = \frac{BC^2}{EF^2}$$

3. **In a right angled triangle*, the triangles on each side of the altitude drawn from the vertex to the right angle to the hypotenuse are similar to the original triangle and to each other too*.**
   ***i.e.*, $\Delta BCA \sim \Delta BDC \sim \Delta CDA$.**

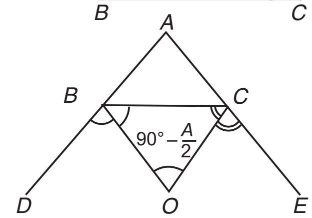


XII. Some Useful Results for a Triangle:

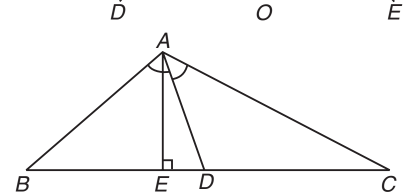
1. In a ΔABC, if the bisectors of $\angle ABC$ and $\angle ACB$ meet at O, then $\boldsymbol{\angle BOC = 90^\circ + \frac{1}{2}\angle A}$.



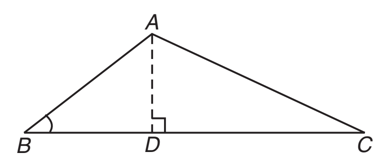
2. In a ΔABC, if the sides AB and AC are produced to D and E respectively and the bisectors of $\angle DBC$ and $\angle ECB$ intersect at O, then $\boldsymbol{\angle BOC = 90^\circ - \frac{\angle A}{2}}$.



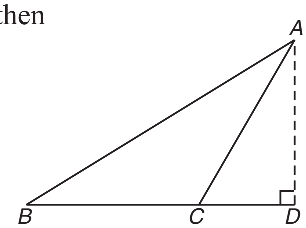
3. In a ΔABC, if AD is the angle bisector of $\angle BAC$ and $AE \perp BC$, then $\boldsymbol{\angle DAE = \frac{1}{2}\angle ABC - \angle ACB}$.



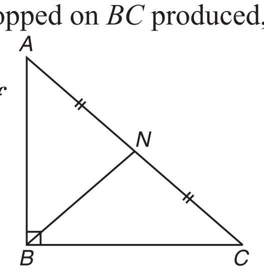
4. In an acute angled triangle, ABC, AD is the perpendicular dropped on BC, then $\boldsymbol{AC^2 = AB^2 + BC^2 - 2BD.DC}$.



5. In an obtuse angled triangle, if AD is the perpendicular dropped on BC produced, then $\boldsymbol{AC^2 = AB^2 + BC^2 + 2BD.DC}$.

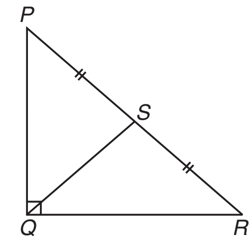


6. **In a right angled Δ,** ***the median to the hypotenuse is half the hypotenuse.***
   BN is the median from *B* on *AC* such that $AN = NC$.
   Then, $\boldsymbol{BN = \frac{1}{2}AC = AN = NC}$.



7. If in a right angled ΔPQR, $\angle Q = 90^\circ$, PR is the hypotenuse, and a perpendicular QS is dropped on the hypotenuse from the right angle vertex Q, then

(*i*) $QS = \dfrac{PQ \times QR}{PR}$ (*ii*) $\dfrac{1}{QS^2} = \dfrac{1}{PQ^2} + \dfrac{1}{QR^2}$



XIII. Area Formulae for a Triangle:

1. **General Formula: Area of a triangle = $\dfrac{1}{2} \times$ base $\times$ height**

2. • **Area of a scalene triangle = $\sqrt{s(s-a)(s-b)(s-c)}$**, where a, b, c are the sides of the triangle and s is the semi-perimeter of the triangle, *i.e.*, $s = \dfrac{a+b+c}{2}$.

   • Also, **Area of a $\Delta = r \times s = \dfrac{abc}{4R}$**, where $r \to$ in radius, $R \to$ circumradius.

3. **Area of a right angled triangle = $\dfrac{1}{2} \times$ base $\times$ height.**

   For a right angled triangle,

   (*i*) **Inradius = $\dfrac{AB + BC - AC}{2}$**, where ΔABC is rt $\angle d$ at $\angle B$

   (*ii*) **Inradius = $\dfrac{\text{Area}}{\text{Semi-perimeter}}$**

   (*iii*) **Circum radius = $\dfrac{\text{Hypotenuse}}{2}$**

4. **Area of an isosceles triangle = $\dfrac{b}{4}\sqrt{4a^2 - b^2}$**, where a is the length of one of the equal sides, b is the length of third side.

5. **For an equilateral triangle,**

   Area = $\dfrac{\sqrt{3}}{4}$ (side)^2, Inradius = $\dfrac{\text{Side}}{2\sqrt{3}}$, Circumradius = $\dfrac{\text{Side}}{\sqrt{3}}$

   $\therefore$ **Circumradius = 2 × Inradius.**

> **Note :** (*i*) *For the given perimeter of a triangle*, **the area of an equilateral triangle is maximum.**
> (*ii*) *For the given area of a triangle*, **the perimeter of an equilateral triangle is minimum.**

6. Two triangles on equal (or same bases and lying between same parallel lines have equal area.)

SOLVED EXAMPLES

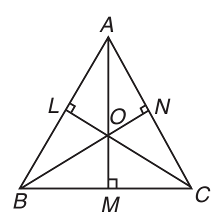
Ex. 1. Let O be any point inside a triangle ABC. Let L, M and N be the points on AB, BC and CA respectively, where perpendicular from O meet these lines. Show that :

$AL^2 + BM^2 + CN^2 = AN^2 + CM^2 + BL^2$

Sol. Join O to A, B and C

In right Δs OAL, OBM and OCN.

$$\left.\begin{aligned} OL^2 + AL^2 &= OA^2 \quad \ldots(i) \\ OM^2 + BM^2 &= OB^2 \quad \ldots(ii) \\ ON^2 + CN^2 &= OC^2 \quad \ldots(iii) \end{aligned}\right\} \left(\begin{array}{c}\text{Pythagoras'} \\ \text{Theoram}\end{array}\right)$$



$\therefore$ Adding (*i*), (*ii*) and (*iii*), we get

$OL^2 + AL^2 + OM^2 + BM^2 + ON^2 + CN^2 = OA^2 + OB^2 + OC^2$

$AL^2 + BM^2 + CN^2 = (OA^2 + OB^2 + OC^2) - (OL^2 + OM^2 + ON^2)$ ...(1)

Similarly in right Δs OAN, OBL and OMC,

$ON^2 + AN^2 = OA^2$ ...(*iv*)

$OL^2 + BL^2 = OB^2$ ...(*v*)

$OM^2 + CM^2 = OC^2$ ...(*vi*)

$\therefore$ Adding (*iv*), (*v*) and (*vi*)

$ON^2 + AN^2 + OL^2 + BL^2 + OM^2 + CM^2 = OA^2 + OB^2 + OC^2$

$\Rightarrow$ $AN^2 + BL^2 + CM^2 = (OA^2 + OB^2 + OC^2) - (ON^2 + OL^2 + OM^2)$ ...(2)

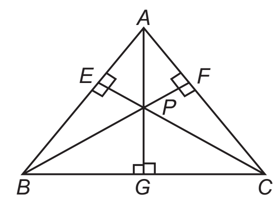
From (1) and (2)

$AL^2 + BM^2 + CN^2 = AN^2 + BL^2 + CM^2$

Ex. 2. A point is selected at random inside an equilateral triangle. From this point a perpendicular is dropped to each side. Prove that the sum of these perpendiculars is independent of the location of the given point.

Sol. Let P be any point in the equilateral ΔABC with each side = S units.

Let PE, PF and PG be the lengths of the perpendiculars from P on AB, AC and BC respectively. Then.



Area of ΔABC = Area (ΔPAB) + Area (ΔPBC) + Area (ΔPAC)

$$= \frac{1}{2} \times AB \times PE + \frac{1}{2} \times AC \times PF + \frac{1}{2} \times BC \times PG$$

$$= \frac{1}{2} \times S \times (PE + PF + PG)$$

where $AB = BC = CA = S$

Also, let h be the height be ΔABC, then

Area of $\Delta ABC = \frac{1}{2} \times S \times h$

$\therefore$ $\frac{1}{2} \times S \times h = \frac{1}{2} \times S \times (PE + PF + PG)$

$\Rightarrow$ $h = (PE + PF + PG)$

$\Rightarrow$ Sum of the perpendiculars = height of equilateral Δ = a constant

$\Rightarrow$ Sum of perpendiculars is independent of location of P.

Ex. 3. In a ΔABC, the lengths of sides BC, CA and AB are a, b and c respectively. Median AD drawn from A is perpendicular to BC. Express b in terms of a and c.

Sol. By Appolonius theorem, in ΔABC,

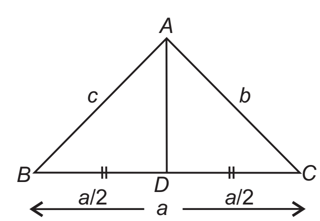
$AB^2 + AC^2 = 2\,(AD^2 + BD^2)$

$c^2 + b^2 = 2\,(c^2 - \frac{a^2}{4} + \frac{a^2}{4})$ ($\because$ In rt. ΔABD, $AD = \sqrt{AB^2 - BD^2}$)

$\Rightarrow$ $b^2 + c^2 = a^2 - 2c^2$

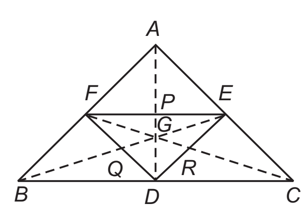
$\Rightarrow$ $b^2 = a^2 - 3c^2$

$\Rightarrow$ $b = \sqrt{a^2 - 3c^2}$



Ex. 4. If the medians AD, BE and CF of ΔABC meet at G, prove that G is the centroid of ΔDEF also.

Sol. Since D and E are the mid-points of sides BC and AC of ΔABC, therefore,



$DE \parallel BA$ and $DE = \frac{1}{2} BA$ (By mid-point theorem)

$\Rightarrow$ $DE \parallel FA$ and $DE = FA$ ($\because$ $FA = \frac{1}{2}$ BA)

Also, $DF \parallel AC$ and $DF = \frac{1}{2} AC$ (By mid-point theorem)

$\Rightarrow$ $DF \parallel AE$ and $DF = AE$.

$\therefore$ $DEAF$ is a parallelogram, whose diagonals AD and FE intersect at P.

Since the diagonals of a parallelogram bisect each other, therefore, $AP = PD$ and $FP = PE$

$\Rightarrow$ P is the mid-point of FE

$\Rightarrow$ DP is the median of ΔDEF.

Similarly it can be shown that, $FDEC$ is a parallelogram and hence R is the mid-point of ED and hence FR is the median of ΔDEF.

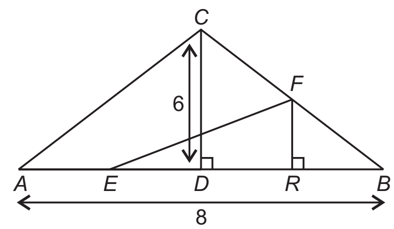
$\therefore$ Medians DP and FR intersect at point G, where G is the centroid of ΔDEF.

Ex. 5. *ABC* is an acute angled triangle. *CD* is the altitude through *C*. If *AB* = 8 units, *CD* = 6 units, find the distance between the mid-points of *AD* and *BC*.

Sol. Let E be the mid-point of AD and F the mid-point of BC.

Draw $FR \perp AB$.

$\because$ $CD \perp AB \Rightarrow FR \parallel CD$.



Also, F being the mid-point of BC and $FR \parallel CD \Rightarrow R$ is the mid-point of BD (By converse of mid-point theorem).

Now by basic proportionality theorem, $FR = \frac{1}{2}\ CD = FR = 3$ units.

Also, $ER = ED + DR = \frac{1}{2}(AD + DB) = \frac{1}{2} \times AB\ = 4$ units.

$\therefore$ In ΔFER, $EF = \sqrt{FR^2 + ER^2} = \sqrt{9+16} = \sqrt{25} = \mathbf{5}$ **units.**

Ex. 6. In the given figure, $\angle MON$ = $\angle MPO$ = $\angle NQO$ = 90º and *OQ* is the bisector of $\angle MON$ and *QN* = 10, *OR* = 40/7. Find *OP*. (*CDS 2012*)

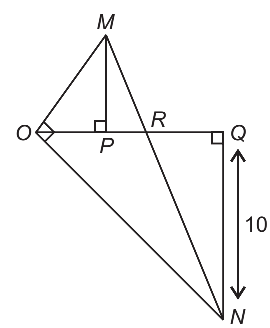
Sol. In ΔOMP, $\angle MOP = 45º$ (OQ bisects $\angle MON$)

$\therefore$ $\angle OMP = 45º$.

$\Rightarrow$ $OP = PM = x$ (say) (sides opp. equal $\angle s$ are equal)

Also in OQN, $\angle QON = 45º$ (OQ bisects $\angle MON$)

$\Rightarrow$ $\angle QNO = 45º \Rightarrow OQ = ON = 10$ (sides opp. equal $\angle s$ are equal)



$\therefore$ $QR = OQ - OR = 10 - \frac{40}{7} = \frac{30}{7}$.

$\Delta PMR \sim \Delta QNS$, since $\angle MPR = \angle NQR = 90º$ and $\angle MRP = \angle QRP$ (vert. opp. $\angle s$)

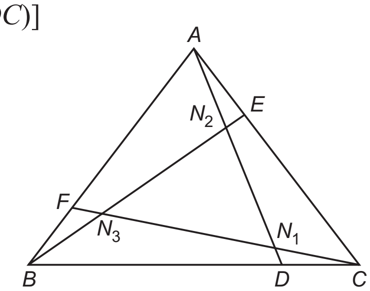
$\Rightarrow$ $\frac{PM}{PR} = \frac{QN}{QR}$ $\Rightarrow$ $\frac{x}{\frac{40}{7} - x} = \frac{10}{\frac{30}{7}} = \frac{7}{3}$

$\Rightarrow$ $\frac{7x}{40 - 7x} = \frac{7}{3}$ $\Rightarrow$ $21x = 280 - 49x \Rightarrow x = \mathbf{4.}$

Ex. 7. In the figure *CD*, *AE* and *BF* are one-third of their respective sides. It is given that $AN_2 : N_2N_1 : N_1D$ = 3 : 3 : 1 and similarly for the lines *BE* and *CF*. Show that the area of $\Delta N_1 N_2 N_3$ is $\frac{1}{7}$ Area (ΔABC).

Sol. Ar $(\Delta N_1 N_2 N_3)$ = Area (ΔABC) – [(Area (ΔCBF) + Area (ΔABE) + Area (ΔADC)]

+ (Area (ΔN_1CD) + Area (N_2AE) + Area (ΔN_3FB))



$\because$ $AE = \frac{1}{3} AC$ and height of ΔABE and ΔABC is same, keeping AC as the base,

Area of $\Delta ABE = \frac{1}{3}$ Area (ΔABC) (Area $= \frac{1}{2} \times b \times h$)

Similarly, Area (ΔCBF) = Area (ΔADC) = $\frac{1}{3}$ Area (ΔABC)

In ΔCBF, $FN_3 : N_3N_1 : N_1C = 3 : 3 : 1 \Rightarrow$ Area of $\Delta N_1CD = \frac{1}{7}$ Area (ΔBFC)

$\Rightarrow$ Area (ΔN_1CD) = $\frac{1}{7} \times \frac{1}{3}$ Area (ΔABC) = $\frac{1}{21}$ Area (ΔABC)

Similarly, Area (ΔN_2AE) = Area ($\Delta N_3 FB$) = $\frac{1}{21}$ Area (ΔABC)

$\therefore$ Area ($\Delta N_1N_2N_3$) = Area (ΔABC) – $3 \cdot \frac{1}{3}$ Area (ΔABC) + $3 \cdot \frac{1}{21}$ Area (ΔABC) = $\mathbf{\frac{1}{7}}$ **Area (ΔABC)**

Ex. 8. In the diagram AB and AC are the equal sides of an isosceles triangle ABC, in which is inscribed equilateral triangle DEF. Designate angle BFD by a, angle ADE by b and angle FEC by c. Then show that $a = \frac{b+c}{2}$.

Sol. For ΔBDF, ext $\angle ADF = \angle B + a$ ...(i)

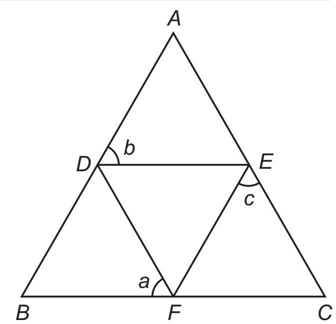
$\Rightarrow$ $b + 60^\circ = \angle B + a$ ...(ii)

Similarly $a + 60^\circ = c + \angle C$

$\therefore$ Eq (i) – Eq (ii) $\Rightarrow b - a = a - c + \angle B - \angle C$

$\because$ $AB = AC \Rightarrow \angle B = \angle C$ (Isosceles Δ property)

$\therefore$ $b - a = a - c \Rightarrow b + c = 2a \Rightarrow a = \mathbf{\frac{b+c}{2}}$.



Ex. 9. In the given figure, line DE is parallel to line AB. $CD = 3$ while $DA = 6$. Which of the following must be true?

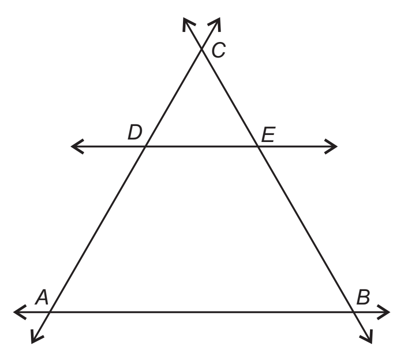
I. $\Delta CDE \sim \Delta CAB$ **II. $\frac{\text{Area of } \Delta CDE}{\text{Area of } \Delta CAB} = \left(\frac{CD}{CA}\right)^2$** **III. If $AB = 4$, then $DE = 2$**

Sol. I. Since $DE \parallel AB$, $\angle CDE = \angle CAB$, $\angle CED = \angle CBA$ } Corresponding angles

$\therefore$ $\Delta CDE \sim \Delta CAB$ (AA similarly)

II. Since ratio of the areas of similar triangles is the square of the ratio of the corresponding sides.

$\therefore$ $\frac{\text{Area }(\Delta CDE)}{\text{Area }(\Delta CAB)} = \left(\frac{CD}{CA}\right)^2$



III. $CA = CD + DA = 3 + 6 = 9$.

$\therefore$ $\Delta CDE \sim \Delta CAB \Rightarrow \frac{CD}{CA} = \frac{DE}{AB}$

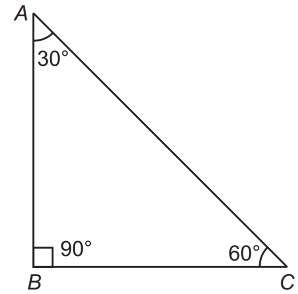
$\Rightarrow \frac{3}{9} = \frac{DE}{4} \Rightarrow DE = \frac{12}{9} = \frac{4}{3}$

$\therefore$ If $AB = 4$, then $DE = \frac{4}{3}$

$\therefore$ I and II are true and III is false.

Ex. 10. If the angles of a triangle are in the ratio 1 : 2 : 3, then find the ratio of the corresponding opposite sides.

Sol. If the angles are in the ratio 1 : 2 : 3, then the angles of the triangle are 30º, 60º, 90º. Therefore, the triangle is a right angled triangle.



The side ratios opposite to the angles 30º, 60º and 90º is $BC : AB : AC$, *i.e.*, $AC \sin 30º; AC \sin 60º : AC$,

i.e., $-\frac{1}{2} \times AC : \frac{\sqrt{3}}{2} \times AC : AC$, *i.e.*, $\mathbf{1 : \sqrt{3} : 2.}$

Ex. 11. The angles of a triangle are in the ratio 8 : 3 : 1. What is the ratio of the longest side of the triangle to the next longest side?

Sol. The angles are $\frac{8}{12} \times 180º, \frac{3}{12} \times 180º, \frac{1}{12} \times 180º$ *i.e.*, 120º, 45º and 15º.

Also we know that the longest side is opposite the greatest angle and so on.

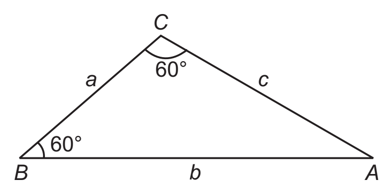
$\therefore$ Let the longest side opposite the greatest angle 120º be x and let the next longest side opposite angle 45º be y. Then, by the sine rule

$$\frac{\sin 120º}{x} = \frac{\sin 45º}{y}$$

$$\Rightarrow \quad \frac{x}{y} = \frac{\sin 120º}{\sin 45º} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{\mathbf{6}}}{\mathbf{2}}.$$

Ex. 12. In ΔABC, $a = 2x$, $b = 3x + 2$, $c = \sqrt{12}$ and $\angle c = 60º$. Find x.

Sol. Here we use the law of cosines. So $c^2 = a^2 + b^2 - 2ab \cos C$



$\Rightarrow \quad 12 = (2x)^2 + (3x + 2)^2 - 2\,(2x)\,(3x + 2) \cos 60º$

$\Rightarrow \quad 12 = 4x^2 + 9x^2 + 12x + 4 - (12x^2 + 8x) \times \frac{1}{2}$

$\Rightarrow \quad 7x^2 + 8x - 8 = 0$

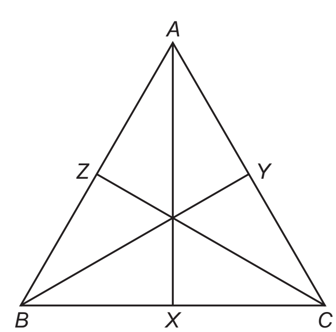
$$x = \frac{-8 \pm \sqrt{64 + 224}}{14} = \frac{-8 \pm \sqrt{288}}{14} \left(\text{Recall that roots of a quadratic eq. } ax^2 + bx + c \text{ are } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right).$$

Since the side of a triangle must be positive, therefore,

$$x = \frac{-8 + \sqrt{288}}{14} \simeq \mathbf{0.64.}$$

Ex. 13. The bisectors of the angles of a triangle *ABC* meet *BC*, *CA* and *AB* at *X*, *Y* and *Z* respectively. Prove that $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$

Sol. We make use of the bisector theorem here, *i.e.*, the bisector (internal or external) of an angle of triangle divides the opposite side in the ratio of the sides containing the angle.



AX bisects $\angle A \Rightarrow \frac{BX}{XC} = \frac{AB}{AC}$ ...(*i*)

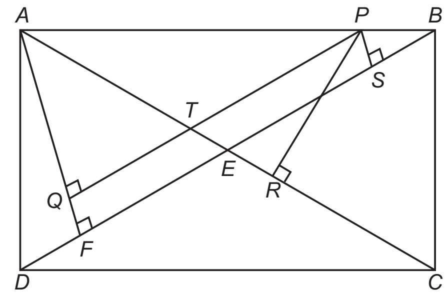
BY bisects $\angle B \Rightarrow \frac{CY}{YA} = \frac{BC}{BA}$ ...(*ii*)

CZ bisects $\angle C \Rightarrow \frac{AZ}{ZB} = \frac{CA}{CB}$ ...(*iii*)

$\therefore$ Multiplying (*i*), (*ii*) and (*iii*), we get

$$\frac{BX}{XC}.\frac{CY}{YA}.\frac{AZ}{ZB} = \frac{AB}{AC}.\frac{BC}{BA}.\frac{CA}{CB} = \mathbf{1}.$$

Ex. 14. As given in the diagram alongside, P is any point on AB, $PS \perp BD$ and $PR \perp AC$. $AF \perp BD$ and $PQ \perp AF$. Then show that $PR + PS$ is equal to AF.



Sol. $\Delta PTR \sim \Delta ATQ$ (AA similarity)

($\because \angle ATQ = \angle PTR$ (vert. opp. $\angle s$) and $\angle PRT = \angle AQT = 90°$)

$\therefore \dfrac{PR}{AQ} = \dfrac{PT}{AT}$ …(*i*)

$\therefore AF \perp BD$ and $PQ \perp AF \Rightarrow PQ \parallel BD \Rightarrow \angle TPA = \angle PBS$ …(*ii*)

Also, diagonals of a rectangle are equal and bisect each other

$\Rightarrow EA = EB \Rightarrow \angle EAB = \angle EBA \Rightarrow \angle TAP = \angle PBS$ …(*iii*)

$\therefore$ From (*i*) and (*ii*), $\angle TPA = \angle TAP \Rightarrow PT = AT$ (Isosceles triangle property)

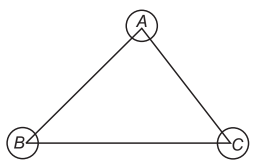
$\Rightarrow PR = AQ$

Also, $PQ \parallel BD$ and $PS \perp BD$ and $QF \perp BD \Rightarrow PS = QF$

$\therefore PR + PS = AQ + QF = AF$.

PRACTICE SHEET

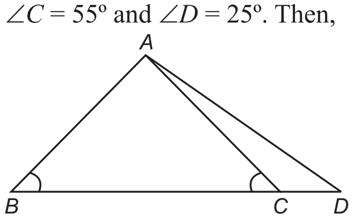
1. In the given figure, what is the sum of the angles formed around A, B, C except at angles of ΔABC.



(*a*) 360° (*b*) 720°
(*c*) 900° (*d*) 1000°

(CDS 2010)

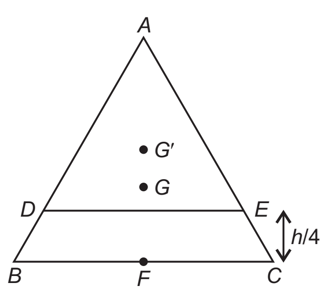
2. In the given figure, $\angle B = \angle C = 55°$ and $\angle D = 25°$. Then,



(*a*) $BC < CA < CD$
(*b*) $BC > CA > CD$
(*c*) $BC < CA$, $CA > CD$
(*d*) $BC > CA$, $CA < CD$

3. In-centre of a triangle lies in the interior of:
(*a*) An isosceles triangle only
(*b*) Any triangle
(*c*) An equilateral triangle only
(*d*) A right triangle only

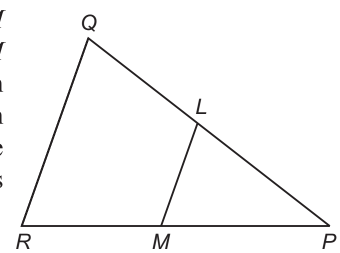
4. G is the centroid of ΔABC with height h units. If a line DE parallel to BC cuts ΔABC at a height $h/4$ from BC, find the distance GG' in terms of AG if G' is the centroid of ΔADE.



(*a*) $\frac{1}{2}AG$ (*b*) $\frac{3}{4}AG$
(*c*) $\frac{1}{4}AG$ (*d*) $\frac{2}{3}AG$

5. In an equilateral triangle if a, b and c denote the lengths of the perpendicular from A, B and C respectively on the opposite sides, then
(*a*) $a > b > c$ (*b*) $a > b < c$
(*c*) $a < b < c$ (*d*) $a = b = c$

6. In the given figure, LM is parallel to QR. If LM divides the ΔPQR such that the area of trapezium $LMRQ$ is two times the area of ΔPLM, then what is $\dfrac{PL}{LQ}$ equal to?



(*a*) $\frac{1}{\sqrt{2}}$ (*b*) $\frac{1}{\sqrt{3}}$ (*c*) $\frac{1}{2}$ (*d*) $\frac{1}{3}$

(CDS 2011)

7. If the medians of two equilateral triangles are in the ratio 3 : 2, then what is the ratio of their sides?
(*a*) 1 : 1 (*b*) 2 : 3 (*c*) 3 : 2 (*d*) $\sqrt{3} : \sqrt{2}$

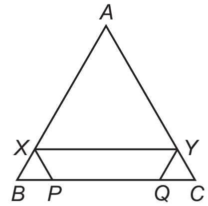
(CDS 2009)

8. If A is the area of the right angled triangle and b is one of the sides containing the right angle, then what is the length of the altitude on the hypotenuse?

(*a*) $\dfrac{2Ab}{\sqrt{b^4+4A^2}}$ (*b*) $\dfrac{2A^2b}{\sqrt{b^4+4A^2}}$

(*c*) $\dfrac{2Ab^2}{\sqrt{b^4+4A^2}}$ (*d*) $\dfrac{2A^2b^2}{\sqrt{b^4+A^2}}$ ***(CDS 2008)***

9. In the given figure, ABC is an equilateral triangle of side length 30 cm. XY is parallel to BC, XP is parallel to AC and YQ is parallel to AB. If $(XY + XP + YQ)$ is 40 cm, then what is PQ equal to?



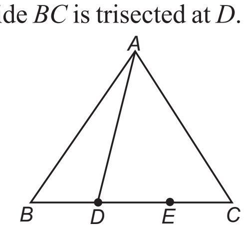
(*a*) 5 cm (*b*) 12 cm
(*c*) 15 cm (*d*) None of these **(*CDS 2010*)**

10. If AD is the median of ΔABC, then
(*a*) $AB^2 + AC^2 = 2AD^2 + 2BD^2$
(*b*) $AB^2 + AC^2 = 2AD^2 + BD^2$
(*c*) $AB^2 + AC^2 = AD^2 + BD^2$
(*d*) $AB^2 + AC^2 = AD^2 + 2BD^2$

11. Let ABC be a triangle of area 16 cm^2. XY is drawn parallel to BC dividing AB in the ratio 3 : 5. If BY is joined, then the area of triangle BXY is
(*a*) 3.5 cm^2 (*b*) 3.7 cm^2 (*c*) 3.75 cm^2 (*d*) 4.0 cm^2

12. From a point O in the interior of a ΔABC if perpendiculars OD, OE and OF are drawn to the sides BC, CA and AB respectively, then which of the following statements is true?
(*a*) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$
(*b*) $AB^2 + BC^2 = AC^2$
(*c*) $AF^2 + BD^2 + CE^2 = OA^2 + OB^2 + OC^2$
(*d*) $AF^2 + BD^2 + CE^2 = OD^2 + OE^2 + OF^2$

13. In an equilateral triangle ABC, the side BC is trisected at D. Then AD^2 is equal to



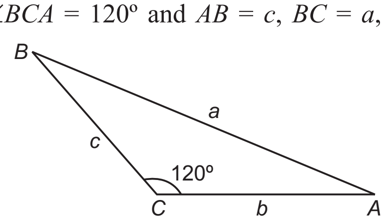
(*a*) $\frac{9}{7}AB^2$ (*b*) $\frac{7}{9}AB^2$
(*c*) $\frac{3}{4}AB^2$ (*d*) $\frac{4}{5}AB^2$

14. In a ΔABC, $AB = 10$ cm, $BC = 12$ cm and $AC = 14$ cm. Find the length of median AD. If G is the centroid, find the length of GA.
(*a*) $\frac{5}{3}\sqrt{7}, \frac{5}{9}\sqrt{7}$ (*b*) $5\sqrt{7}, 4\sqrt{7}$
(*c*) $\frac{10}{\sqrt{3}}, \frac{8}{3}\sqrt{7}$ (*d*) $4\sqrt{7}, \frac{8}{3}\sqrt{7}$

15. If a, b, c are the sides of a triangle and $a^2 + b^2 + c^2 = bc + ca + ab$, then the triangle is:
(*a*) equilateral (*b*) isosceles
(*c*) right-angled (*d*) obtuse angled
(*CAT 2000*)

16. H is the orthocentre of ΔABC whose altitudes are AD, BE and CF. Then the orthocentre of ΔHBC is
(*a*) F (*b*) E (*c*) A (*d*) D

17. In the given figure, $\angle BCA = 120°$ and $AB = c$, $BC = a$, $AC = b$. Then



(*a*) $c^2 = a^2 + b^2 + ba$
(*b*) $c^2 = a^2 + b^2 - ba$
(*c*) $c^2 = a^2 + b^2 - 2ba$
(*d*) $c^2 = a^2 + b^2 + 2ab$

18. In the ΔABC, $AB = 2$ cm, $BC = 3$ cm and $AC = 4$ cm. D is the middle-point of AC. If a square is constructed on the side BD, what is the area of the square?
(*a*) 4.5 cm^2 (*b*) 2.5 cm^2 (*c*) 6.35 cm^2 (*d*) None of these
(*CDS 2009*)

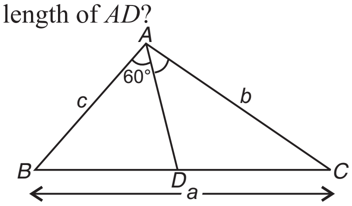
19. In the given figure, (not drawn to scale), P is a point on AB such that $AP : PB = 4 : 3$. PQ is parallel to AC and QD is parallel to CP. In ΔARC, $\angle ARC = 90°$ and in ΔPQS, $\angle PSQ = 90°$. The length of $QS = 6$ cm. What is the ratio of $AP : PD$?
(*a*) 10 : 3 (*b*) 2 : 1 (*c*) 7 : 3 (*d*) 8 : 3
(*CAT 2003*)

20. In ΔLMN, LO is the median. Also LO is the bisector of $\angle MLN$. If $LO = 3$ cm, and $LM = 5$ cm, then find the area of ΔLMN.
(*a*) 12 cm^2 (*b*) 10 cm^2 (*c*) 4 cm^2 (*d*) 6 cm^2
(*CAT 2009*)

21. A point within an equilateral triangle whose perimeter is 30 m is 2 m from one side and 3 m from another side. Find its distance from third side.
(*a*) $\sqrt{5} - 3$ (*b*) $5\sqrt{3} - 5$ (*c*) $5\sqrt{5} - 3$ (*d*) $5\sqrt{3} - 3$

22. A city has a park shaped as a right angled triangle. The length of the longest side of this park is 80 m. The Mayor of the city wants to construct three paths from the corner point opposite to the longest side such that these paths divide the longest side into four equal segments. Determine the sum of the squares of the lengths of the three paths.
(*a*) 4000 m (*b*) 4800 m (*c*) 5600 m (*d*) 6400 m
(*XAT 2012*)

23. In a triangle ABC, AD is the angle bisector of $\angle BAC$ and $\angle BAD = 60°$. What is the length of AD?

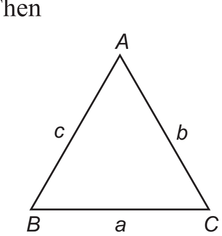


(*a*) $\frac{b+c}{bc}$ (*b*) $\frac{bc}{b+c}$
(*c*) $\sqrt{b^2 + c^2}$ (*d*) $\frac{(b+c)^2}{bc}$

24. In a ΔABC, the internal bisector of angle A meets BC at D. If $AB = 4$, $AC = 3$ and $\angle A = 60°$, then the length of AD is
(*a*) $2\sqrt{3}$ (*b*) $\frac{12\sqrt{3}}{7}$ (*c*) $\frac{15\sqrt{3}}{8}$ (*d*) $\frac{6\sqrt{3}}{7}$
(*CAT 2002*)

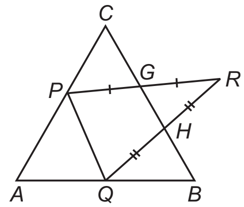
25. Suppose the medians PP' and QQ' of ΔPQR intersect at right angles. If $QR = 3$ and $PR = 4$, then the length of side PQ is
(*a*) $\sqrt{3}$ (*b*) $\sqrt{2}$ (*c*) $\sqrt{5}$ (*d*) $\sqrt{6}$

26. In the given triangle ABC, the length of sides AB and AC is same (*i.e.*, $b = c$) and $60° < A < 90°$. Then



(*a*) $b < a < b\sqrt{3}$
(*b*) $c < a < c\sqrt{2}$
(*c*) $b < a < 2b$
(*d*) $\frac{c}{3} < a < 3c$

27. In the given figure, P and Q are the mid-points of AC and AB. Also, $PG = GR$ and $HQ = HR$. What is the ratio of the Area of ΔPQR : Area of ΔABC



(*a*) 1 : 2 (*b*) 2 : 3
(*c*) 3 : 4 (*d*) 3 : 5

28. The lengths of the sides a, b, c of a ΔABC are connected by the relation $a^2 + b^2 = 5c^2$. The angle between medians drawn to the sides 'a' and 'b' is

(*a*) 60º (*b*) 45º (*c*) 90º (*d*) None of these

29. ABC is a triangle with $\angle BAC = 60º$. A point P lies on one-third of the way from B to C and AP bisects $\angle BAC$. $\angle APC$ equals

(*a*) 90º (*b*) 45º (*c*) 60º (*d*) 120º

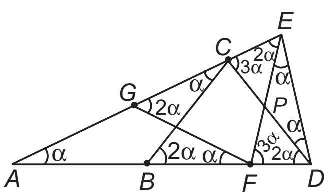
(XAT 2007)

30. A rectangle inscribed in a triangle has its base coinciding with the base ***b*** of the triangle. If the altitude of the triangle is ***h***, and the altitude ***x*** of the rectangle is half the base of the rectangle, then

(*a*) $x = \frac{1}{2}h$ (*b*) $x = \frac{bh}{h+b}$

(*c*) $x = \frac{bh}{2h+b}$ (*d*) $x = \sqrt{\frac{hb}{2}}$ ***(FMS 2011)***

31. In the given figure $AB = BC = CD = DE = EF = FG = GA$. Then $\angle DAE$ is approximately:



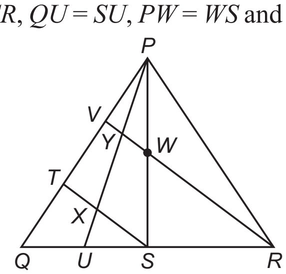
(*a*) 15º (*b*) 20º
(*c*) 30º (*d*) 25º

(CAT 2000)

32. If ABC is a triangle in which $\angle B = 2\ \angle C$. D is a point on side BC such that AD bisects $\angle BAC$ and $AD = CD$, then $\angle BAC =$

(*a*) 62º (*b*) 72º (*c*) 76º (*d*) 84º

33. In the figure shown here, $QS = SR$, $QU = SU$, $PW = WS$ and $ST \parallel RV$. What is the value of $\frac{\text{Area of } \Delta PSX}{\text{Area of } \Delta PQR}$?



(*a*) $\frac{1}{5}$ (*b*) $\frac{1}{3}$

(*c*) $\frac{1}{6}$ (*d*) $\frac{1}{7}$

34. In a ΔABC, AD, BE and CF are the medians drawn from the vertices A, B and C respectively. Then study the following statements and choose the correct option.

I. $3(AB + BC + AC) > 2(AD + BE + CF)$
II. $3(AB + BC + AC) < 2(AD + BE + CF)$
III. $3(AB + BC + AC) < 4(AD + BE + CF)$
IV. $3(AB + BC + AC) > 4(AD + BE + CF)$

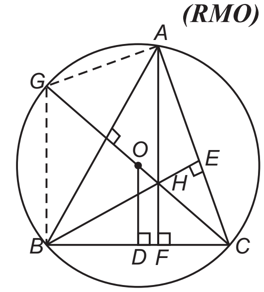
(*a*) I and IV are true (*b*) I and III are true
(*c*) II and IV are true (*d*) II and III are true

35. In a ΔABC, $AB = AC$. P and Q are points on AC and AB respectively such that $CB = BP = PQ = QA$. Then $\angle AQP =$

(*a*) $\frac{2\pi}{7}$ (*b*) 3π (*c*) $\frac{5\pi}{7}$ (*d*) $\frac{4\pi}{7}$

(RMO)

36. In a triangle, the ratio of the distance between a vertex and the orthocentre and the distance of the circumcentre from the side opposite the vertex is



(*a*) 3 : 1 (*b*) 4 : 1
(*c*) 2 : 1 (*d*) $\sqrt{2} : 1$

37. In a ΔABC, angle A is twice angle B. Then,

(*a*) $a^2 = b(a + c)$ (*b*) $a^2 = \sqrt{bc}$
(*c*) $a^2 = b(b + c)$ (*d*) $a^2 = b + c$

38. Let ABC be a triangle. Let D, E, F be points respectively on segments BC, CA, AB such that AD, BE and CF concur at point K. Suppose $BD/DC = BF/FA$ and $\angle ADB = \angle AFC$, then

(*a*) $\angle ABE = \angle CAD$ (*b*) $\angle ABE = \angle AFC$
(*c*) $\angle ABE = \angle FKB$ (*d*) $\angle ABE = \angle BCF$

(RMO 2011)

39. Let ABC be a triangle in which $AB = AC$ and let I be its in-centre. Suppose $BC = AB + AI$. $\angle BAC$ equals.

(*a*) 45º (*b*) 90º (*c*) 60º (*d*) 75º

(RMO 2009)

40. In a ΔABC, let D be the mid-point of BC. If $\angle ADB = 45º$ and $\angle ACD = 30º$. then $\angle BAD$ equals.

(*a*) 45º (*b*) 60º (*c*) 30º (*d*) 15º

(RMO 2005)

ANSWERS

1. (*c*)	**2.** (*d*)	**3.** (*b*)	**4.** (*c*)	**5.** (*d*)	**6.** (*b*)	**7.** (*c*)	**8.** (*a*)	**9.** (*d*)	**10.** (*a*)
11. (*c*)	**12.** (*a*)	**13.** (*b*)	**14.** (*d*)	**15.** (*a*)	**16.** (*c*)	**17.** (*a*)	**18.** (*b*)	**19.** (*c*)	**20.** (*a*)
21. (*b*)	**22.** (*c*)	**23.** (*b*)	**24.** (*b*)	**25.** (*c*)	**26.** (*b*)	**27.** (*a*)	**28.** (*c*)	**29.** (*d*)	**30.** (*c*)
31. (*d*)	**32.** (*b*)	**33.** (*a*)	**34.** (*b*)	**35.** (*c*)	**36.** (*c*)	**37.** (*c*)	**38.** (*a*)	**39.** (*b*)	**40.** (*c*)

HINTS AND SOLUTIONS

1. $\angle A = 360^\circ - \text{Ext. } \angle A$

$\angle B = 360^\circ - \text{Ext. } \angle B$

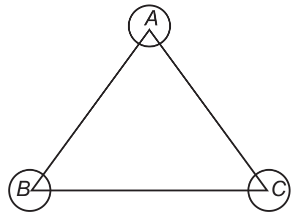
$\angle C = 360^\circ - \text{Ext. } \angle C$

We know,

$\angle A + \angle B + \angle C = 180^\circ$

$\Rightarrow$ $360^\circ - \text{Ext. } \angle A + 360^\circ - \text{Ext. } \angle B + 360^\circ - \text{Ext. } \angle C = 180^\circ$

$\Rightarrow$ Ext. $\angle A$ + Ext. $\angle B$ + Ext. $\angle C = 1080^\circ - 180^\circ$ = $\mathbf{900^\circ}$**.**



2. $\angle B = \angle C = 55^\circ \Rightarrow AC = AB$

$\Rightarrow \angle BAC = 180^\circ - (\angle B + \angle C) = 180^\circ - 110^\circ = 70^\circ$

Now in ΔACD, $\angle ACD = 180^\circ - 55^\circ = 125^\circ$

$\therefore\ \angle CAD = 180^\circ - (125^\circ + 25^\circ) = 30^\circ$

$\Rightarrow \angle CAD > \angle CDA \Rightarrow CD > AC$

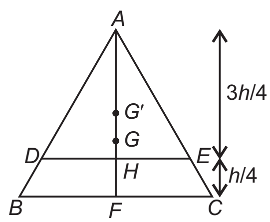
($\because$ In a given triangle, the greater angle has greater side opposite to it)

Also $\angle BAC > \angle ABC \Rightarrow BC > AC$

$\therefore$ $\boldsymbol{BC > CA}$ **and** $\boldsymbol{CA < CD}$**.**

4. AF is the median to BC from A in ΔABC. G is the centroid of ΔABC

$\Rightarrow AG = \dfrac{2}{3}AF \Rightarrow AF = \dfrac{3}{2}AG.$



Now AH is the median to DE from A in ΔADE. G' is the centroid of ΔADE

$\Rightarrow AG' = \dfrac{2}{3}AH$

Since $DE \parallel BC$, $AH = \dfrac{3}{4}AF$

(By basic proportionality theorem)

$\therefore\ AG' = \dfrac{2}{3} \times \dfrac{3}{4}AF = \dfrac{1}{2}AF = \dfrac{1}{2} \times \dfrac{3}{2}AG = \dfrac{3}{4}AG$

$\therefore\ GG' = AG - AG' = AG - \dfrac{3}{4}AG = \dfrac{\mathbf{1}}{\mathbf{4}}\boldsymbol{AG.}$

6. $\because\ LM \parallel QR$

$\left.\begin{array}{l}\angle PLM = \angle PQR \\ \angle PML = \angle PRQ\end{array}\right\}$ *Corresponding* $\angle s$

$\Rightarrow \Delta PQR \sim \Delta PLM$

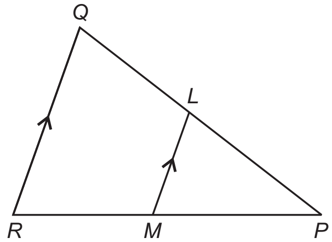
$\therefore\ \dfrac{\text{Area of } \Delta PLM}{\text{Area of } \Delta PQR} = \dfrac{PL^2}{PQ^2}$ …(i)

Area of ΔPQR = Area of ΔPLM + Area of trap. $LMRQ$

Given, Area of trapezium $LMRQ$ = 2 Area of ΔPLM

$\therefore$ Area of ΔPQR = Area of ΔPLM + 2 Area of ΔPLM

= 3 Area of ΔPLM



$\therefore$ From (i), we have

$\dfrac{\text{Area of } \Delta PLM}{3 \times \text{Area of } \Delta PLM} = \dfrac{PL^2}{PQ^2}$

$\Rightarrow \dfrac{PL^2}{PQ^2} = \dfrac{1}{3} \Rightarrow \dfrac{PL}{PQ} = \dfrac{\mathbf{1}}{\boldsymbol{\sqrt{3}}}.$

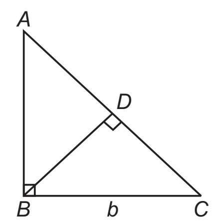
7. Median of an equilateral triangle = $\dfrac{\sqrt{3}}{2} \times \text{side}$

Let the sides of the two equilateral triangles be a_1 and a_2 respected. Then,

$\dfrac{\frac{\sqrt{3}}{2}a_1}{\frac{\sqrt{3}}{2}a_2} = \dfrac{3}{2} \Rightarrow \dfrac{a_1}{a_2} = \dfrac{\mathbf{3}}{\mathbf{2}}.$

8. Let ABC be the given right angled triangle, right angled at B. Let $BC = b$.

Then, Area of $\Delta ABC = \dfrac{1}{2} \times BC \times AB$



$\Rightarrow\ A = \dfrac{1}{2} \times b \times AB \Rightarrow AB = \dfrac{2A}{b}$

In ΔABC, $AC^2 = AB^2 + BC^2$

$AC^2 = \dfrac{4A^2}{b^2} + b^2$

$\Rightarrow\ AC = \sqrt{\dfrac{4A^2}{b^2} + b^2}$

Again in ΔABC,

$A = \dfrac{1}{2} \times AC \times BD$

$\Rightarrow\ A = \dfrac{1}{2} \times \sqrt{\dfrac{4A^2 + b^4}{b^2}} \times BD$

$\Rightarrow\ BD = \dfrac{\boldsymbol{2Ab}}{\boldsymbol{\sqrt{4A^2 + b^4}}}.$

9. $AB \parallel YQ \Rightarrow \angle XBP = \angle YQC$ (corresponding angles) and

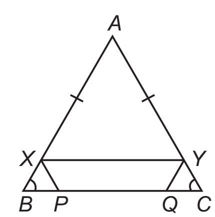
$XP \parallel AC \Rightarrow \angle XPB = \angle YCQ$

ΔABC being an equilateral triangle,

$\angle B = \angle C = 60^\circ$

$\Rightarrow \angle XPB = \angle YQC = 60°$

$\Rightarrow \angle XPB = \angle QYC = 60^\circ$



$\Rightarrow \Delta XBP$ and ΔYQC are equilateral triangles.

Now, $XY \parallel BC \Rightarrow \dfrac{AX}{AB} = \dfrac{XY}{BC} \Rightarrow AX = XY$ ($\because AB = BC$)

Also, $XY + XP + YQ = 40 \Rightarrow AX + XB + YQ = 40$

($\because AX = XY, XP = XB$)

$\Rightarrow AB + YQ = 40 \Rightarrow YQ = 40 - AB = 40 - 30 = 10$

$\therefore\ XP = YQ = 10$ cm.

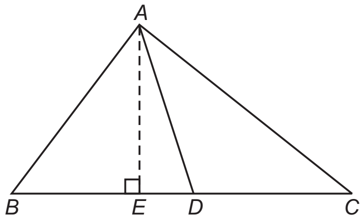
$\Rightarrow BP = QC = 10$ cm (ΔXBP and ΔYQC are equilateral)

$\Rightarrow PQ = BC - (BP + QC) = 30 - 10 - 10 =$ **10 cm.**

10. Let AD be the median of ΔABC

$\therefore BD = DC$

Let AE be the perpendicular from A on BC. Then,



In rt. ΔABE,

$AB^2 = BE^2 + AE^2$

$= (BD - ED)^2 + AE^2$

$= BD^2 + \underbrace{ED^2 + AE^2} - 2BD.ED$

$= BD^2 + AD^2 - 2BD.ED$ …(i)

In rt. ΔACE,

$AC^2 = AE^2 + EC^2$

$= AE^2 + (ED + DC)^2$

$= \underbrace{AE^2 + ED^2} + DC^2 + 2ED.DC$

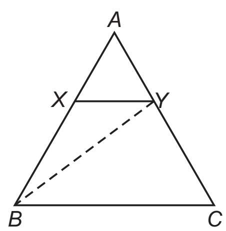
$= AD^2 + DC^2 + 2ED.DC$

$= AD^2 + BD^2 + 2ED.BD$ ($\because BD = DC$)…(ii)

Adding (i) and (ii), we get

$\boldsymbol{AB^2 + AC^2 = 2AD^2 + 2BD^2}$.

11. Let the areas of ΔAXY and ΔBXY be A_1 and A_2 respectively. The heights of the ΔAXY and ΔBXY are equal, so



$$\frac{A_1}{A_2} = \frac{\frac{1}{2} \times AX \times \text{height}}{\frac{1}{2} \times BX \times \text{height}}$$

$$\Rightarrow \frac{A_1}{A_2} = \frac{AX}{BX} = \frac{3}{5} \quad \ldots(i)$$

Also, $\Delta AXY \sim \Delta ABC$

$$\Rightarrow \frac{\text{Area of } \Delta AXY}{\text{Area of } \Delta ABC} = \left(\frac{AX}{AB}\right)^2$$

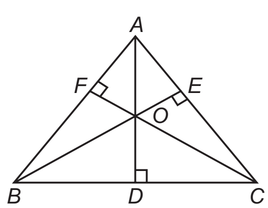
$$\Rightarrow \frac{A_1}{16} = \left(\frac{3x}{3x+5x}\right)^2 \Rightarrow \frac{A_1}{16} = \frac{9}{64}$$

$$\Rightarrow A_1 = \frac{9}{64} \times 16 = \frac{9}{4} \quad \ldots(ii)$$

$$\therefore \text{From } (i) \text{ and } (ii)\ A_2 = \frac{5}{3} \times A_1 = \frac{5}{3} \times \frac{9}{4} = \frac{15}{4} = \mathbf{3.75\ cm^2}.$$

12. Join OA, OB and OC.

By Pythagoras, theorem,



In ΔAOF, $AF^2 = AO^2 - OF^2$ …(i)

In ΔBOD, $BD^2 = BO^2 - OD^2$ …(ii)

In ΔCOE, $CE^2 = CO^2 - OE^2$ …(iii)

Adding (i), (ii) and (iii), we get

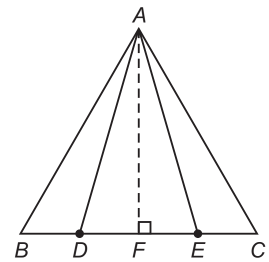
$$AF^2 + BD^2 + CE^2 = AO^2 - OF^2 + BO^2 - OD^2 + CO^2 - OE^2$$
$$= \underbrace{AO^2 - OE^2} + \underbrace{BO^2 - OF^2} + \underbrace{CO^2 - OD^2}$$
$$= \boldsymbol{AE^2 + BF^2 + CD^2}.$$

13. ABC being an equilateral triangle, $AB = BC = AC$.

Also BC being trisected at D

$$\Rightarrow \quad BD = \frac{1}{3}BC \quad \ldots(i)$$

Let AF be drawn perpendicular to $BC \Rightarrow BF = FC$.



Also E is given as the other point of trisection, so $BD = DE = EC$

Also, $BD = 2DF$ …(ii)

Now $AB^2 = BF^2 + AF^2$ and

$AD^2 = DF^2 + AF^2$

Now $AB^2 = BF^2 + AF^2$

$$\Rightarrow AB^2 = \left(\frac{1}{2}BC\right)^2 + AF^2$$

$$\Rightarrow AB^2 = \frac{1}{4}BC^2 + AF^2 \quad (\because BC = AB)$$

$$\Rightarrow AB^2 = \frac{1}{4}AB^2 + AF^2 \Rightarrow AF^2 = \frac{3}{4}AB^2 \quad \ldots(iii)$$

Also, $AD^2 = DF^2 + AF^2$

$$= \left(\frac{1}{2} \times \frac{1}{3}BC\right)^2 + AF^2 \quad (\text{From } (i) \text{ and } (ii))$$

$$= \left(\frac{1}{6}BC\right)^2 + AF^2$$

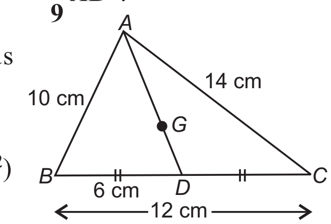
$$= \frac{1}{36}BC^2 + AF^2$$

$$= \frac{1}{36}AB^2 + \frac{3}{4}AB^2$$

($\because BC = AB$ and putting the value from (iii))

$$= \frac{28}{36}AB^2 \Rightarrow AD^2 = \frac{\mathbf{7}}{\mathbf{9}}\boldsymbol{AB^2}.$$

14. Applying the Apollosnius theorem,



$AB^2 + AC^2 = 2(BD^2 + AD^2)$

$\Rightarrow 100 + 196 = 2\,(36 + AD^2)$

$\Rightarrow 2AD^2 = 296 - 72 = 224$

$\Rightarrow AD^2 = 112 \Rightarrow AD = \mathbf{4\sqrt{7}}$

As G is the centroid, so $\dfrac{AG}{GD} = \dfrac{2}{1}$

$$\Rightarrow AG = \frac{2}{3}AD = \frac{2}{3} \times 4\sqrt{7} = \frac{\mathbf{8}}{\mathbf{3}}\boldsymbol{\sqrt{7}}.$$

15. $a^2 + b^2 + c^2 = ab + bc + ca$

$\Rightarrow a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0$

$\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

Sum of perfect squares = 0 $\Rightarrow$ Each term of the sum is zero

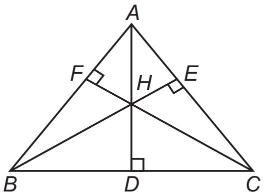
$\Rightarrow (a - b) = 0 = (b - c) = (c - a)$

$\Rightarrow a = b = c$

$\Rightarrow$ The triangle is equilateral.

16. H is the orthocentre of ΔABC

$\Rightarrow AD \perp BC$, $BE \perp CA$, $CF \perp AB$



In ΔBHC,

$AD \perp BC \Rightarrow HD \perp BC$

$CF \perp AB \Rightarrow BF \perp HC$ (Produced)

$\Rightarrow$ The altitudes HD and BF of ΔHBC intersect in A.

$\Rightarrow A$ is the orthocentre of ΔHBC.

17. By the cosine rule, we have,

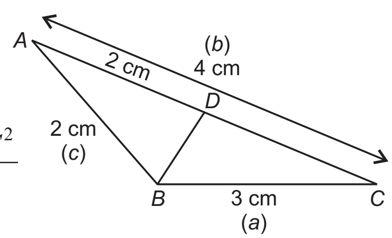
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\Rightarrow \cos 120^\circ = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\Rightarrow -\frac{1}{2} = \frac{a^2 + b^2 - c^2}{2ab}$$

$\Rightarrow \boldsymbol{c^2 = a^2 + b^2 + ab}$**.**

18. In ΔABC,



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{AC^2 + AB^2 - BC^2}{2 \times AC \times AB}$$

$$= \frac{4^2 + 2^2 - 3^2}{2 \times 4 \times 2}$$

$$= \frac{16 + 4 - 9}{16} = \frac{11}{16}$$

In ΔBAD,

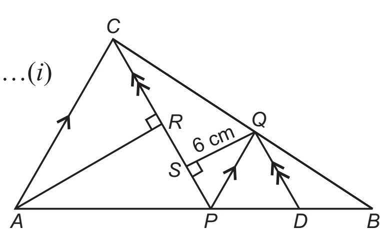
$$\cos A = \frac{AD^2 + AB^2 - BD^2}{2 \times AD \times AB}$$

$$\Rightarrow \frac{11}{16} = \frac{4 + 4 - BD^2}{2 \times 2 \times 2} = \frac{8 - BD^2}{8}$$

$\Rightarrow 11 = 16 - 2BD^2 \Rightarrow 2BD^2 = 5 \Rightarrow BD^2 =$ **2.5**

$\therefore$ Area of square on $BD = (BD)^2 = 2.5\ \text{cm}^2$.

19. $PQ \parallel AC$



$$\Rightarrow \frac{CQ}{QB} = \frac{AP}{PB} = \frac{4}{3} \quad \ldots(i)$$

Also, $QD \parallel CP$

$$\Rightarrow \frac{PD}{DB} = \frac{CQ}{QB} = \frac{4}{3}$$

(From (i))

$$\frac{PD}{DB} = \frac{4}{3} \Rightarrow \frac{PD}{PB} = \frac{PD}{PD + DB} = \frac{4}{4+3} = \frac{4}{7} \quad \ldots(ii)$$

$\therefore$ From (i) and (ii)

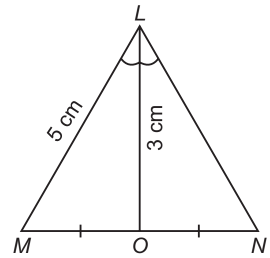
$$\frac{AP}{PB} = \frac{4}{3} \text{ and } \frac{PB}{PD} = \frac{7}{4}$$

$$\therefore \frac{AP}{PB} \times \frac{PB}{PD} = \frac{4}{3} \times \frac{7}{4} \Rightarrow \frac{AP}{PD} = \frac{7}{3} \Rightarrow AP : PD = \mathbf{7 : 3.}$$

20. LO being the median on MN,

$MO = ON$

Also, LO being the internal bisector of $\angle MLN$,



$$\frac{LM}{LN} = \frac{MO}{ON} = 1$$

(By bisector theorem)

$\Rightarrow LM = LN$

$\Rightarrow \Delta LMN$ is isosceles triangle.

Now $\Delta LOM \cong \Delta LON$ (By SSS)

$\Rightarrow \angle LOM = \angle LON = 90^\circ$ (cpct)

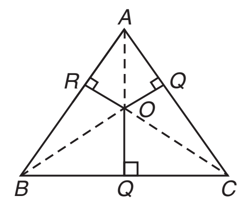
$\therefore$ In ΔLOM, $MO = \sqrt{LM^2 - LO^2}$

$= \sqrt{25 - 9} = \sqrt{16} = 4$ cm .

Area of $\Delta LMN = \frac{1}{2} \times MN \times LO$

$= \frac{1}{2} \times 8 \times 3\ \text{cm}^2$ $=$ **12 cm²**.

21. Given, ABC is an equilateral triangle such that $AB = BC = CA = 10$ m



If O is any point in the ΔABC, then

Area of ΔABC

= Area (ΔOAB) + Area (ΔOAC) + Area (ΔOBC)

$$= \frac{1}{2} \times AB \times OR + \frac{1}{2} \times AC \times OP + \frac{1}{2} \times AC \times OQ$$

$$= \frac{1}{2} \times AB \times (OR + OP + OQ) \qquad (\because AB = BC = CA)$$

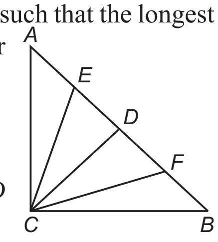
$$= \frac{1}{2} \times 10 \times (2 + 3 + OQ)$$

$\because$ Area of an equilateral $\Delta = \frac{\sqrt{3}}{4}(\text{side})^2$

$$\therefore \frac{\sqrt{3}}{4} \times (10)^2 = \frac{1}{2} \times 10 \times (5 + OQ)$$

$\Rightarrow 5\sqrt{3} = 5 + OQ \Rightarrow OQ =$ $\mathbf{5\sqrt{3} - 5.}$

22. CE, CD and CF are the required paths such that the longest side of the park AB is divided into four equal segments.



$AE = ED = DF = FB = 20$ m.

Let $AC = b$, $BC = a$

Then, by Apollonius theorem in ΔACD

$AC^2 + CD^2 = 2\,(CE^2 + 20^2)$

$$\Rightarrow \frac{1}{2}(b^2 + CD^2) = CE^2 + 20^2 \quad \ldots(i)$$

Similarly in ΔCDB, $(CB^2 + CD^2) = 2\,(CF^2 + 20^2)$

$$\Rightarrow CF^2 + 20^2 = \frac{1}{2}(a^2 + CD^2) \quad \ldots(ii)$$

Adding (i) and (ii),

$$CE^2 + CF^2 + 2 \times 20^2 = \frac{1}{2}(a^2 + b^2 + 2 \times CD^2)$$

$$\Rightarrow CE^2 + CF^2 = \frac{1}{2}(80^2 + 2 \times 40^2) - 2 \times 20^2$$

$(\because AC^2 + CB^2 = AB^2 \Rightarrow a^2 + b^2 = 80^2)$

($\because$ $CD = 40$, $\because$ line joining the vertex at the right $\angle$ to the mid-point of the hypotenuse is half the hypotenuse)

$$\Rightarrow CE^2 + CF^2 = \frac{1}{2}(6400 + 3200) - 800$$

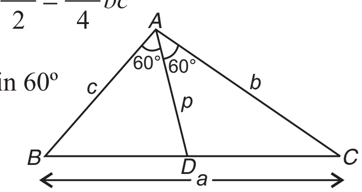
$$= \mathbf{4000.}$$

Now $CE^2 + CF^2 + CD^2 = 4000 + 40^2 = \mathbf{5600.}$

23. Let $AD = p$

Area of $\Delta ABC = \frac{1}{2}bc \;\sin \angle BAC = \frac{1}{2}bc \;\sin 120^\circ$

$$= \frac{1}{2}bc \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}bc$$



Area of $\Delta BAD = \frac{1}{2}cp \;\sin 60^\circ$

$$= \frac{\sqrt{3}}{4}cp$$

Area of $\Delta CAD = \frac{1}{2}bp \;\sin 60^\circ = \frac{\sqrt{3}}{4}bp$

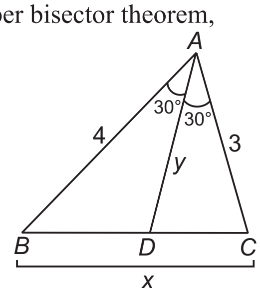
Now Area (ΔABC) = Area (ΔBAD) + Area (ΔCAD)

$$\Rightarrow \frac{\sqrt{3}}{4}bc = \frac{\sqrt{3}}{4}cp + \frac{\sqrt{3}}{4}bp \Rightarrow bc = p(b+c) \Rightarrow p = \mathbf{\frac{bc}{b+c}.}$$

24. Let $BC = x$ and $AD = y$, then as per bisector theorem,

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{4}{3}$$

$\therefore$ $BD = \frac{4x}{7}$ and $DC = \frac{3x}{7}$



Now in ΔABD using cosine rule,

$$\cos 30^\circ = \frac{AB^2 + AD^2 - BD^2}{2 \times AB \times AD}$$

$$= \frac{16 + y^2 - \frac{16x^2}{49}}{2 \times 4 \times y} \Rightarrow \frac{\sqrt{3}}{2} = \frac{16 + y^2 - \frac{16x^2}{49}}{2 \times 4 \times y}$$

$$\Rightarrow 4\sqrt{3y} = 16 + y^2 - \frac{16x^2}{49} \quad \ldots(i)$$

In ΔACD using the cosine rule,

$$\cos 30^\circ = \frac{AC^2 + AD^2 - CD^2}{2 \times AC \times AD}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{9 + y^2 - \frac{9x^2}{49}}{2 \times 3 \times y}$$

$$\Rightarrow 3\sqrt{3y} = 9 + y^2 - \frac{9x^2}{49} \quad \ldots(ii)$$

Also in ΔABC, $\cos 60^\circ = \frac{AB^2 + AC^2 - BC^2}{2 \times AB \times AC}$

$$\Rightarrow \frac{1}{2} = \frac{16 + 9 - BC^2}{2 \times 4 \times 3} \Rightarrow 12 = 25 - BC^2$$

$\Rightarrow BC^2 = 13 \Rightarrow x^2 = 13$

$\therefore$ Subtracting eqn (ii) from (i), we get

$$\sqrt{3}\,y = 7 - \frac{7x^2}{49} = 7 - \frac{7 \times 13}{49} = 7 - \frac{13}{7} = \frac{49 - 13}{7} = \frac{36}{7}$$

$$\Rightarrow y = \frac{36}{7 \times \sqrt{3}} = \mathbf{\frac{12\sqrt{3}}{7}}.$$

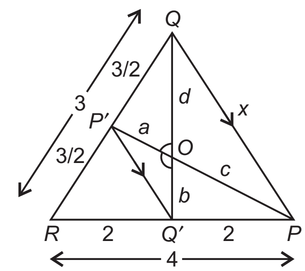
25. Join $P'Q'$.

P', Q' being the mid-points of QR and PR respectively, we have $P'Q' \parallel PQ$ and $P'Q' = \frac{1}{2}PQ$

(By the mid-point theoram)

Let $OP' = a$, $OQ' = b$, $OP = c$, $OQ = d$ and $PQ = x$

Then, $P'Q' = \frac{1}{2}x$.



$\therefore$ In rt. Δ $OP'Q'$,

$$a^2 + b^2 = \frac{x^2}{4} \quad \ldots(i)$$

In rt. Δ $OP'Q$,

$$a^2 + d^2 = \frac{9}{4} \quad \ldots(ii)$$

In rt. ΔOQP,

$$c^2 + d^2 = x^2 \quad \ldots(iii)$$

In rt. Δ $OQ'P$,

$$b^2 + c^2 = 4 \quad \ldots(iv)$$

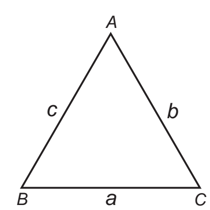
$\therefore$ Eq (i) – Eq (ii) + Eq (iii) – Eq (iv)

$$\Rightarrow a^2 + b^2 - (a^2 + d^2) + (c^2 + d^2) - (b^2 + c^2) = \frac{x^2}{4} - \frac{9}{4} + x^2 - 4$$

$$\Rightarrow 0 = \frac{5x^2}{4} - \frac{25}{4} \Rightarrow \frac{5x^2}{4} = \frac{25}{4} \Rightarrow x^2 = 5 \Rightarrow \mathbf{x = \sqrt{5}.}$$

26. When $\angle A = 60^\circ$ and $b = c$, then ΔABC is equilateral

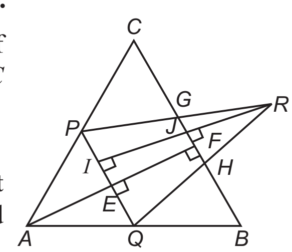
$\Rightarrow a = b = c$



when $A = 90^\circ$, then ΔABC is an isosceles right angled triangle and $a = \sqrt{2}b$ or $a = \sqrt{2}c$.

$\therefore$ $60^\circ < A < 90^\circ = \mathbf{c < a < \sqrt{2}c.}$

27. P and Q being the mid-points of AC and AB respectively, $PQ \parallel BC$ and $PQ = \frac{1}{2}BC$



Let $AF \perp BC$ be drawn such that it intersects PQ and BC in E and F respectively.

$$PQ \parallel BC \Rightarrow \frac{AE}{EF} = \frac{AP}{PC} = 1 \Rightarrow AE = EF$$

Also let $RI \perp PQ$ be drawn such that it intersect BC and PQ in J and I respectively. G and H being the mid-points of sides PR and RQ of ΔPQR, $GH \parallel PQ$ and $GH = \frac{1}{2}PQ$

(By midpoint Theorem)

Also, $GH \parallel PQ \Rightarrow \dfrac{RJ}{JI} = \dfrac{RG}{GP} = 1 \Rightarrow RJ = JI$

(By Basic Proportionality Theorem)

But $EF = JI$

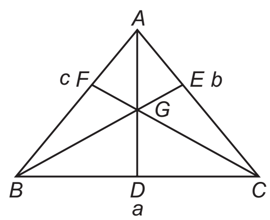
$\therefore\ AE = EF = RJ = JI$

$\therefore\ AF = AE + EF = RJ + JI = RI = h$ (say)

Then, $\dfrac{\text{Area }(\Delta PQR)}{\text{Area }(\Delta ABC)} = \dfrac{\frac{1}{2} \times PQ \times h}{\frac{1}{2} \times BC \times h} = \dfrac{PQ}{BC} = \mathbf{\dfrac{1}{2}}.$

28. AD being the median to BC,

$AB^2 + AC^2 = 2\,(BD^2 + AD^2)$ (Apollonius Th.)



$\Rightarrow c^2 + b^2 = 2\left(\dfrac{a^2}{4} + AD^2\right)$

$\Rightarrow 2c^2 + 2b^2 = 4\left(\dfrac{a^2}{4} + AD^2\right)$

$\Rightarrow 4AD^2 = 2c^2 + 2b^2 - a^2$

$\Rightarrow AD = \dfrac{1}{2}\sqrt{2c^2 + 2b^2 - a^2}$

Now G divides AD in ratio 2 : 1 $\quad \therefore\ AG = \dfrac{2}{3} AD$

$$\Rightarrow \qquad AG^2 = \frac{4}{9} AD^2 = \frac{4}{9} \times \frac{1}{4}(2c^2 + 2b^2 - a^2)$$

$$= \frac{1}{9}(2c^2 + 2b^2 - a^2)$$

Similarly, $\quad GB^2 = \dfrac{1}{9}(2c^2 + 2a^2 - b^2)$

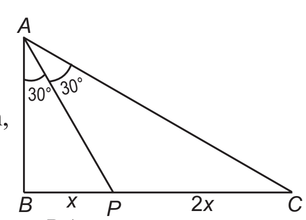
$$AG^2 + GB^2 = \frac{1}{9}\ [2c^2 + 2b^2 - a^2 + 2c^2 + 2a^2 - b^2]$$

$$= \frac{1}{9}\ [a^2 + b^2 + 4c^2]$$

$$= \frac{1}{9}[5c^2 + 4c^2] = \frac{9c^2}{9} = c^2 = BC^2$$

$\Rightarrow AG \perp GB \Rightarrow$ Angle between the medians drawn to sides $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\mathbf{90^o}$.

29. Let $BP = x$. Then $PC = 2x$



$\because\ AP$ bisects $\angle BAC$,

By the angle bisector theorem,

$\dfrac{AB}{AC} = \dfrac{BP}{PC} = \dfrac{1}{2}$

Using the sine formula, $\dfrac{AC}{\sin B} = \dfrac{BA}{\sin C}$

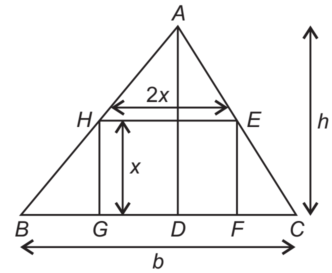
$\Rightarrow \dfrac{\sin C}{\sin B} = \dfrac{BA}{AC} = \dfrac{1}{2}$

$\Rightarrow \dfrac{\sin C}{\sin B} = \dfrac{\frac{1}{2}}{1} \Rightarrow \sin C = \dfrac{1}{2}$ and $\sin B = 1$

$\Rightarrow \angle C = 30°$ and $\angle B = 90°$

$\therefore\ $ In ΔAPC, $APC = 180^o - (30^o + 30^o) = \mathbf{120^o}$.

30. Let AD be the height of the given triangle ABC, where $AD = h$ and $BC = b$ Also let height HG of rectangle $EFGH$ equal x so that $HE = GF = 2x$



Now $\Delta BGH \sim \Delta BDA$

$\Rightarrow \dfrac{BG}{BD} = \dfrac{HG}{AD}$

$\Rightarrow \dfrac{BG}{BD} = \dfrac{x}{h} \Rightarrow BG = \dfrac{x}{h} BD \qquad \ldots(i)$

Also, $\Delta CFE \sim \Delta CDA$

$\Rightarrow \dfrac{CF}{CD} = \dfrac{EF}{AD} = \dfrac{x}{h}$

$\Rightarrow CF = \dfrac{x}{h} CD \qquad \ldots(ii)$

$BG + CF = BC - GF = b - 2x \qquad \ldots(iii)$

$\therefore$ From (i), (ii) and (iii)

$b - 2x = \dfrac{x}{h}(BD + CD) \quad \Rightarrow \quad b - 2x = \dfrac{x}{h} BC = \dfrac{x}{h} b$

$\Rightarrow bh - 2xh = xb \Rightarrow bh = xb + 2xh = x(b + 2h)$

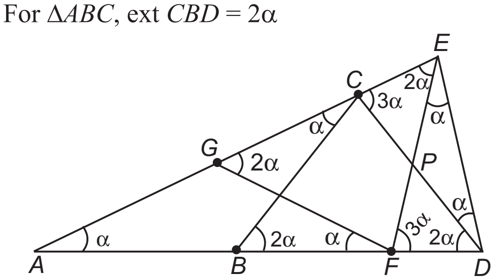
$\Rightarrow x = \boldsymbol{\dfrac{bh}{b + 2h}}.$

31. Let $\angle EAD = \alpha$. Then,

In ΔABC, $AB = BC \Rightarrow \angle BCA = \alpha$ ⎤ sides opp. equal

In ΔAGF, $AG = GF \Rightarrow \angle AFG = \alpha$ ⎦ angles are equal

$\therefore$ For ΔABC, ext $CBD = 2\alpha$



In ΔCBD, $CB = CD \Rightarrow \angle CDB = 2\alpha$

For ΔAFG, ext $\angle FGC = 2\alpha$

$\because\ $ In ΔGFE, $GF = EF \Rightarrow \angle FEG = \angle FGE = 2\alpha$

For ΔEAF, ext. $\angle EFD = 3\alpha$

$\because\ EF = ED\ \therefore\ \angle EDF = \angle EFD = 3\alpha \Rightarrow \angle EDP = \alpha$

For ΔCAD, ext. $\angle DCE = 3\alpha$

In ΔECD, $DC = ED \Rightarrow \angle DEC = \angle DCE = 3\alpha$

$\Rightarrow \angle FED = \angle DEC - \angle FEC = 3\alpha - 2\alpha = \alpha.$

$\therefore\ $ In ΔEFD, $\alpha + 3\alpha + 3\alpha = 180^o$

$\Rightarrow 7\alpha = 180^o \Rightarrow \alpha = \dfrac{180^o}{7} = 26^o$ or approximately 25^o.

32. In ΔABC, let BP bisect $\angle ABC$

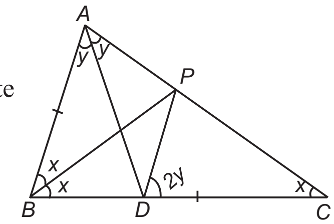
Let $\angle C = x \Rightarrow \angle B = 2x$

$\therefore\ \angle PBC = \angle ABP = x$

In ΔPBC,

$\angle PBC = \angle PCB = x$

$\Rightarrow$ $PC = PB$ (sides opposite equal angles are equal)



Now in ΔAPB and ΔBPC

$AB = CD$ (Given)

$PB = PC$ (Proved above)

$\angle ABP = \angle DCP = x$

$\therefore$ $\Delta APB \cong \Delta DPC$

$\Rightarrow$ $\angle BAP = \angle PDC = 2y$ and $AP = DP$

$\therefore$ In ΔAPD, $AP = DP \Rightarrow \angle PDA = \angle PAD = y$

$\therefore$ $\angle DPA = 180^\circ - 2y$ …(i)

Also from ΔDPC, $\angle DPC = 180^\circ - (x + 2y)$ …(ii)

$\therefore$ From (i) and (ii), $\angle DPA + \angle DPC = 180^\circ$

$\Rightarrow 180^\circ - 2y + 180^\circ - (x + 2y) = 180^\circ$

$\Rightarrow x + 4y = 180^\circ$ …(1)

Also in ΔABC, $\angle A + \angle B + \angle C = 180^\circ$

$2y + 2x + x = 180^\circ$

$3x + 2y = 180^\circ$ …(2)

$\therefore$ (2) – 3 × (1) $\Rightarrow 3x + 2y - (3x + 12y) = 180^\circ - 3 \times 180^\circ$

$\Rightarrow -10y = -360^\circ \Rightarrow y = 36^\circ$

$\therefore$ $\angle BAC = 2y = 2 \times 36^\circ = \mathbf{72^\circ}$.

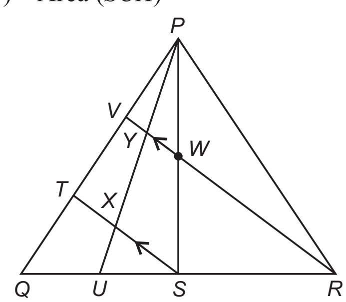
33. Area (ΔPSX) = Area (PUS) – Area (SUX)

In ΔPXS, $WY \parallel SX$

$\Rightarrow \dfrac{PY}{YX} = \dfrac{PW}{WS} = 1$

(Given, $PW = WS$)

$\Rightarrow PY = YX$



In ΔRUY,

$SX \parallel RY = \dfrac{UX}{XY} = \dfrac{US}{SR} = \dfrac{1}{2}$

($\because$ $QS = SR$ and $QV = US$)

$= UX = \dfrac{1}{2}(XY)$

$\therefore$ In ΔPUS, $UX = \dfrac{1}{2}XY = \dfrac{1}{2}\left(\dfrac{1}{2}PX\right) = \dfrac{1}{4}PX$

$PU = UX + PX = \dfrac{1}{4}PX + PX = \dfrac{5}{4}PX$

$\therefore$ $\dfrac{\text{Area of } \Delta SUX}{\text{Area of } \Delta PUS} = \dfrac{1}{5}$

Now Area of $\Delta PUS = \dfrac{1}{4}$(Area of ΔPQR)

Area of $\Delta SUX = \dfrac{1}{4} \times \dfrac{1}{5} \times$ Area of ΔPQR

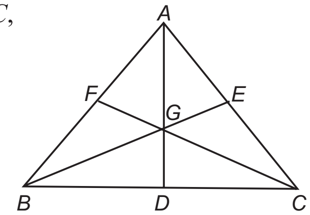
$= \dfrac{1}{20}$ Area of ΔPQR

$$\therefore \frac{\text{Area of } \Delta PSX}{\text{Area of } \Delta PQR} = \frac{\frac{1}{4}\text{Area of } \Delta PQR - \frac{1}{20}\text{Area of } \Delta PQR}{\text{Area of } \Delta PQR}$$

$$= \frac{5-1}{20} = \mathbf{\frac{1}{5}}.$$

34. G being the centroid of ΔABC,

$\dfrac{AG}{GD} = \dfrac{BG}{GE} = \dfrac{CG}{GF} = \dfrac{2}{1}$



In ΔABD,

$AB + BD > AD$

$\Rightarrow AB + \dfrac{BC}{2} > AD$ …(i)

In ΔBEC,

$BC + CE > BE \Rightarrow BC + \dfrac{AC}{2} > BE$ …(ii)

In ΔAFC,

$AC + AF > CF \Rightarrow AC + \dfrac{AB}{2} > CF$ …(iii)

Adding (i), (ii) and (iii), we get

$AB + BC + AC + \dfrac{BC}{2} + \dfrac{AC}{2} + \dfrac{AB}{2} > AD + BE + CF$

$\Rightarrow \dfrac{2AB + 2BC + 2AC + BC + AC + AB}{2} > AD + BE + CF$

$\Rightarrow$ $3(AB + BC + AC) > 2\,(AD + BE + CF) \Rightarrow$ ***I* is true**

Also, in $\Delta\, BGC$,

$BG + GC > BC$

$\left(\because \dfrac{BG}{GE} = \dfrac{2}{1} \text{ and } \dfrac{CG}{GF} = \dfrac{2}{1} \Rightarrow BG = \dfrac{2}{3}BE \text{ and } CG = \dfrac{2}{3}CF\right)$

$\Rightarrow \dfrac{2}{3}BE + \dfrac{2}{3}CF > BC \Rightarrow 2BE + 2CF > 3BC$ …(iv)

Similarly in ΔBGA,

$BG + GA > AB$

$\Rightarrow \dfrac{2}{3}BE + \dfrac{2}{3}AD > AB \Rightarrow 2BE + 2AD > 3AB$ …(v)

In ΔCGA,

$CG + GA > AC$

$\Rightarrow \dfrac{2}{3}CF + \dfrac{2}{3}AD > AC$

$\Rightarrow 2CF + 2AD > 3AC$ …(vi)

Adding (iii), (iv) and (v), we get

$2BE + 2CF + 2BE + 2AD + 2CF + 2AD$

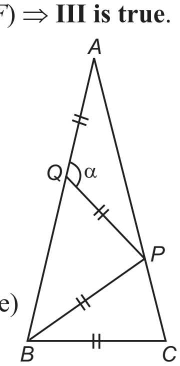
$> 3BC + 3AB + 3AC$

$\Rightarrow$ $4(AD + BE + CF) > 3(AB + BC + AC)$

$\Rightarrow$ $3(AB + BC + AC) < 4(AD + BE + CF) \Rightarrow$ **III is true**.

35. Let $\angle AQP = \alpha$

In ΔAQP, $AQ = QP$



$\Rightarrow \angle QAP = \angle QPA = \dfrac{1}{2}(\pi - \alpha)$

$= \dfrac{\pi}{2} - \dfrac{\alpha}{2}$

$\angle PQB = \pi - \alpha$

(AQB being a straight line)

In ΔPQB,

$PQ = PB \Rightarrow \angle PBQ = \angle PQB = \pi - \alpha$

$\therefore$ $\angle QPB = \pi - 2\,(\pi - \alpha) = 2\alpha - \pi$

In ΔABC,

$\angle BAC + \angle ABC + \angle ACB = \pi$

$\Rightarrow \angle QPA + \angle ACB + \angle ACB = \pi$

($\because \angle QAP = \angle BAC, AB = AC \Rightarrow \angle ACB = \angle ABC$)

$\Rightarrow \frac{\pi}{2} - \frac{\alpha}{2} + 2\angle ACB = \pi$

$\Rightarrow 2\angle ACB = \pi - \frac{\pi}{2} + \frac{\alpha}{2} = \frac{\pi}{2} + \frac{\alpha}{2} \Rightarrow \angle ACB = \frac{\pi}{4} + \frac{\alpha}{4}$

$\therefore$ In ΔBPC, $BP = BC$

$\Rightarrow \angle BPC = \angle BCP = \angle ACB = \frac{\pi}{4} + \frac{\alpha}{4}$

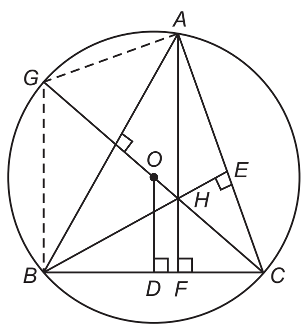
Now APC being a straight line,

$\angle APQ + \angle BPQ + \angle BPC = \pi$

$\Rightarrow \frac{\pi}{2} - \frac{\alpha}{2} + 2\alpha - \pi + \frac{\pi}{4} + \frac{\alpha}{4} = \pi$

$\Rightarrow \frac{7\alpha}{4} = \frac{5\pi}{4} \Rightarrow \alpha = \mathbf{\frac{5\pi}{7}}$.

36. Let ABC be the given triangle whose circumcentre is O. Produce CO to meet the circle at $G \Rightarrow COG$ is the diameter of the circle.



$\Rightarrow \angle GAC = \angle GBC = 90^\circ$ (Angles in a semicircle)

Let AF and BE be the perpendiculars from vertex A and B respectively on sides BC and AC.

$\therefore$ H is the orthocentre of ΔABC. We need to find the ratio $AH : OD$, where OD is the perpendicular distance of the circumcentre O from side BC.

$OD \perp BC \Rightarrow BD = DC$ (Perpendicular from the centre of the circle bisects the chord)

Also, $GB \perp BC$ and $OD \perp BC \Rightarrow OD \parallel GB$.

$\therefore$ In ΔBGC, by the midpoint theorem, O and D being the mid-points of sides GC and BC, $OD \parallel GB$ and $OD = \frac{1}{2}GB$

From (i)

Now GA and BE are both perpendiculars to AC

$\Rightarrow GA \parallel BE \Rightarrow GA \parallel BH$

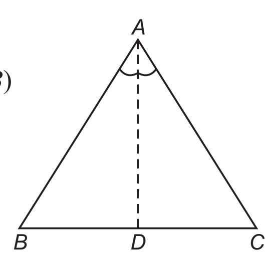
Also, $GB \parallel AF \Rightarrow GB \parallel AH$

$\Rightarrow GAHB$ is a parallelogram

$\Rightarrow GB = AH = 2OD$

$\therefore \frac{AH}{OD} = \frac{2OD}{OD} = \mathbf{\frac{2}{1}}$.

37. In Δs ABC and DAC



$\angle ABC = \angle DAC$

($\because AD$ bisects $\angle A$ and $\angle A = 2\angle B$)

Also, $\angle ACB = \angle DCA$

$\Rightarrow \Delta ABC \sim DAC$ (AA similarity)

$\frac{AC}{DC} = \frac{BC}{AC} \Rightarrow AC^2 = BC.DC$

$\Rightarrow DC = \frac{AC^2}{BC} = \frac{b^2}{a}$ …(i)

(In a ΔABC, $AB = c$, $BC = a$, $AC = b$)

Also, AD being the angle bisector of $\angle A$,

$\frac{BD}{CD} = \frac{AB}{AC} = \frac{c}{b}$

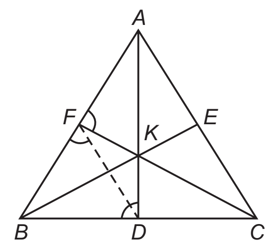
$\therefore \frac{BD}{CD} + 1 = \frac{c}{b} + 1$

$\Rightarrow \frac{BD + CD}{CD} = \frac{c + b}{b}$

$\Rightarrow \frac{BC}{CD} = \frac{C + b}{b} \Rightarrow \frac{a}{CD} = \frac{c + b}{b} \Rightarrow CD = \frac{ab}{b + c}$ …(ii)

$\therefore$ From (i) and (ii), $\frac{b^2}{a} = \frac{ab}{b + c} \Rightarrow \boldsymbol{a^2 = b(b + c)}$.

38. Since $\frac{BD}{DC} = \frac{BF}{FA}$



By basic proportionality theorem, we have $FD \parallel AC$.

Also, $\angle BDK = \angle ADB = \angle AFC$

$= 180^\circ - \angle BFK$

$\therefore \angle BDK + \angle BFK = 180^\circ$

$\Rightarrow BDKF$ is a cyclic quadrilateral

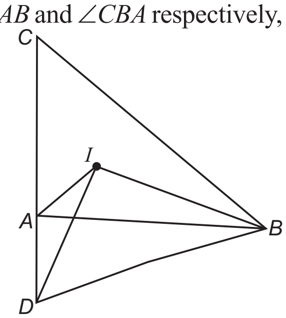
$\Rightarrow \angle FBK = \angle FDK$

(Angles in the same segment)

$\therefore \angle \boldsymbol{ABE} = \angle FBK = \angle FDK = \angle FDA = \angle \boldsymbol{DAC}$

($\because FD \parallel AC$, alt. $\angle s$ are equal)

39. AI and BI are the bisectors of $\angle CAB$ and $\angle CBA$ respectively,



$\therefore$ In ΔABC,

$\angle A + \angle B + \angle C = 180^\circ$

$\Rightarrow \angle A + \angle B = 180^\circ - \angle C$

$\frac{\angle A}{2} + \frac{\angle B}{2} = 90^\circ - \frac{\angle C}{2}$

Also, in ΔAIB,

$\angle AIB = 180^\circ - \left(\frac{\angle A}{2} + \frac{\angle B}{2}\right)$

$= 180^\circ - \left(90^\circ - \frac{\angle C}{2}\right) = 90^\circ + \frac{\angle C}{2}$

Extend CA to D such that $AD = AI$.

Then, $BC = AB + AI \Rightarrow BC = CA + AD = CD$

(By hypothesis, $AB = AC$, $AD = AI$)

$\Rightarrow \angle CDB = \angle CBD = 90^\circ - \frac{\angle C}{2}$

(Since in ΔCDB, $CD = CB$ and $\angle C + \angle D + \angle B = 180°$)

Thus, $\angle AIB + \angle ADB = 90^\circ + \frac{\angle C}{2} + 90^\circ - \frac{\angle C}{2} = 180^\circ$

$\Rightarrow AI\ BD$ is a cyclic quadrilateral

Also $\angle ADI = \angle ABI = \frac{\angle B}{2}$ (angles in the same segment)

$\therefore$ In ΔDAI, $\angle DAI$, $= 180^\circ - 2\ (\angle ADI) = 180^\circ - \angle B$

($\because AD = AI \Rightarrow \angle ADI = \angle AID$)

Thus, $\angle CAI = B \Rightarrow A = 2B$ (CAD is a straight angle)

Since, $AC = AB \Rightarrow \angle B = \angle C$

$\therefore$ In ΔABC, $\angle A + \angle B + \angle C = 180^\circ$

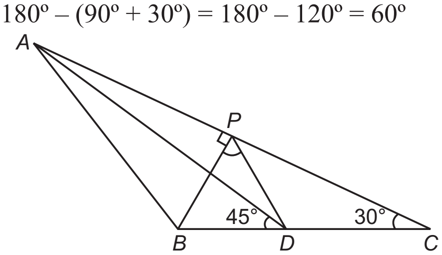
$\Rightarrow 4\angle B = 180^\circ \Rightarrow \angle B = 45^\circ \Rightarrow \angle A =$ **90º.**

40. Draw $BP \perp AC$ and join P to D.

In ΔBPC,

$\angle PBC = 180^\circ - (\angle BPC + \angle BCP)$

$= 180^\circ - (90^\circ + 30^\circ) = 180^\circ - 120^\circ = 60^\circ$



In ΔBPC,

$$\sin 30^\circ = \frac{BP}{BC} \Rightarrow \frac{BP}{BC} = \frac{1}{2} \Rightarrow BP = \frac{1}{2}BC = BD$$

Now in ΔBPD, $BP = BD \Rightarrow \angle BDP = \angle PBD = 60^\circ$

$\Rightarrow \angle BPD = 60^\circ \Rightarrow \Delta BPD$ is equilateral

$\therefore$ $PB = PD$ and $\angle ADP = 60^\circ - 45^\circ = 15^\circ$

In ΔADC, ext. $\angle ADB = \angle ACD + \angle DAC$

$45^\circ = 30^\circ + \angle DAC \Rightarrow \angle DAC = 15^\circ$

$\therefore$ In ΔAPD, $\angle ADP = \angle PAD \Rightarrow PD = PA$

We have $PD = PA = PB$

$\Rightarrow$ P is the circumcentre of ΔADB as circumcentre is equidistant from the vertices of a Δ.

$$\Rightarrow BAD = \frac{1}{2}\angle BPD$$

($\because$ Angle subtended by an are at the centre of a circle is half the angle subtended by the same are at any other point on the remaining part of the circumference.)

$$\Rightarrow BAD = \frac{1}{2} \times 60^\circ = \mathbf{30^\circ.}$$

7 Quadrilaterals

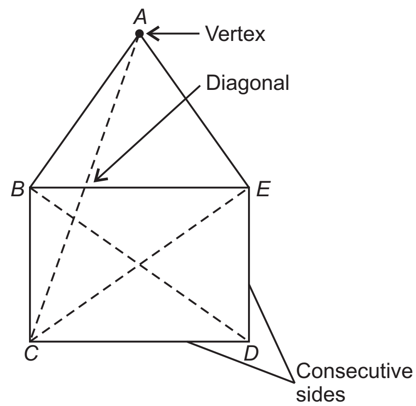
KEY FACTS

1. **Polygon:** A closed plane figure bounded by line segments is called a polygon. Poly means 'many', so a polygon is a closed many sided figure.  are all polygons.

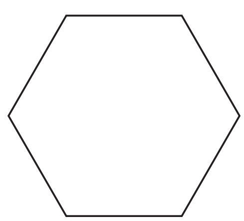
   Polygons are classified as follows with reference to the number of sides they have:

5 sides	6 sides	7 sides	8 sides	9 sides	10 sides	12 sides
pentagon	**hexagon**	**heptagon**	**octagon**	**nonagon**	**decagon**	**dodecagon**

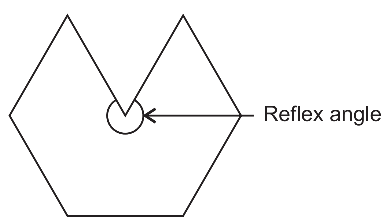
2. **Some important terms used with polygons**



   1. **Vertex:** It is the point where two adjacent sides of a polygon meet.
      For example, *A*, *B*, *C*, *D*, *E*.
   2. **Consecutive sides:** Are those sides which have a vertex in common, *viz*., *AB*, *BC*; *BC*, *CD*; *CD*, *DE*; *DE EA*.
   3. **Diagonal:** The line joining any two non-adjacent or non-consecutive vertices of a polygon, *viz*., *AC*, *BD*, *CE*, *BE*.
   4. **Perimeter:** The sum of the lengths of all sides of a polygon, *i.e.*, *AB* + *BC* + *CD* + *DE* + *EA*.
   5. **Convex polygon:** In a convex polygon each interior angle is less than 180°.

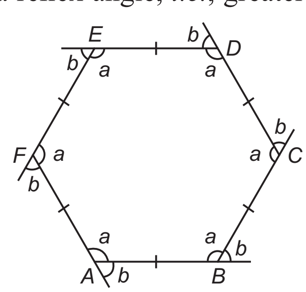


Convex Polygon



Concave Polygon

   6. **Concave polygon:** In this type of a polygon at least one of the interior angles is a reflex angle, *i.e.*, greater than 180°. It is also called a **re-entrant polygon.**
   7. **Regular polygon:** A polygon is regular if:
      (***i***) it is convex
      (***ii***) all of its sides are equal
      (***iii***) all of its interior angles are equal
      (***iv***) all of its exterior angles are equal.



The figure shows a regular hexagon *ABCDEF*, with all equal interior angles marked **'*a*'** and all exterior angles marked **'*b*'**. The sides are equal, *i.e.*, *AB* = *BC* = *CD* = *DE* = *EF* = *FA*.

Some common examples of a regular polygon are: *an equilateral triangle, a square.*

8. **Sum of the angles of a polygon:**

   a. **Sum of the interior angles** of a convex polygon of ***n*** **sides = (2*n* – 4)rt. ∠*s* = (2n – 4) × 90°**

   b. If the sides of a convex polygon are produced in order, **the sum of the exterior angles so formed is 4 rt. ∠*s*, *i.e.*, 360°.**

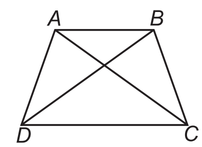
9. **In a regular polygon:** ***a.*** Each exterior angle = $\dfrac{360°}{\text{Number of sides}}$

   b. Each interior angle = 180° – Exterior angle

   $= \dfrac{\mathbf{(2\boldsymbol{n}-4)\times 90°}}{\boldsymbol{n}}$, where n is the number of sides.

3. Quadrilateral:

A quadrilateral is a plane figure bounded by four straight line segments.

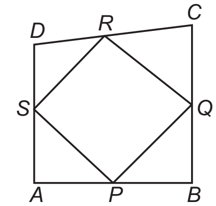


Ex. *ABCD* is a quadrilateral. The four line segments bounding it, *i.e.*, ***AB*, *BC*, *CD* and *DA*** are called its **sides**. ***A*, *B*, *C*, *D*** are the **vertices** and ***AC*** and ***BD*,** the line segments joining the opposite vertices are the **diagonals** of the given quadrilateral.

Properties:

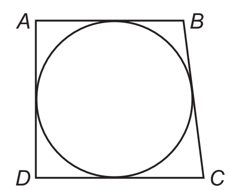
1. Sum of the four interior angles = 360°.
2. The figure formed by joining the mid-points of the adjacent sides of a quadrilateral is a parallelogram.

   Ex. *PQRS* is a parallelogram.



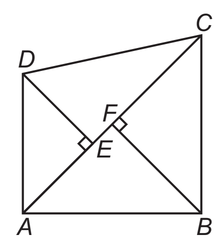
3. The sum of the opposite sides of a quadrilateral circumscribed about a circle is always equal.

   $AB + DC = AD + BC$



4. Area of quadrilateral = $\dfrac{1}{2}$ × one of the diagonals × sum of the perpendiculars drawn on the diagonal from opposite vertices

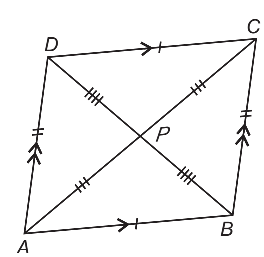
   Ex. Area of quad. $ABCD = \dfrac{1}{2} \times AC \times (BF + DE)$



4. Special Types of Qudrilaterals and their Properties

PARALLELOGRAM:

A quadrilateral in which opposite sides are equal and parallel is called a parallelogram.

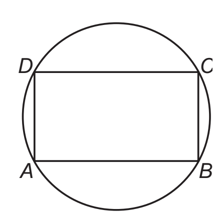


Properties:

1. *Both pairs of opposite sides are equal, i.e., AB = DC, AD = BC.*
2. *Opposite angles are equal, i.e., ∠A = ∠C, ∠B = ∠D.*
3. *Sum of any two adjacent angles is 180°, i.e.,*

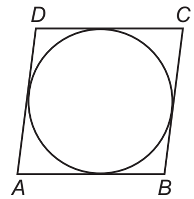
   ∠A + ∠B = ∠B + ∠C = ∠C + ∠D = ∠D + ∠A = 180°.
4. *Diagonals bisect each other, i.e., AP = PC, BP = PD.*
5. *Each diagonal divides a parallelogram into two congruent triangles, i.e.,*

   ΔABC ≅ ΔADC, ΔDAB ≅ ΔBCD.
6. *In case of a parallelogram, the diagonals need not be of equal length, need not bisect at right angles and need not bisect the angles at the vertices.*
7. *Lines joining the mid-points of the adjacent sides of a parallelogram is a parallelogram.*
8. *The parallelogram that is inscribed in a circle is a rectangle. ABCD is a rectangle.*



Property 8

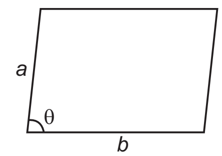
9. *The parallelogram that is circumscribed about a circle is a rhombus.*
   ABCD is a rhombus.



Property 9

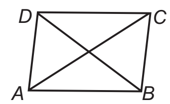
10. **1. Area of a parallelogram = *base* × *height***

    2. Area of a parallelogram = *product of any two adjacent sides* × *sine of included angle*
    = $ab \sin\theta$



11. Perimeter of a parallelogram = 2 (Sum of any two adjacent sides)

12. $AC^2 + BD^2 = 2\,(AB^2 + BC^2)$



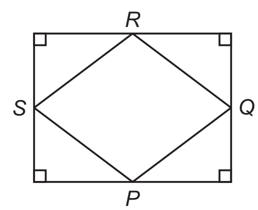
13. Parallelograms that lie on the same base and between the same parallel lines are equal in area.

RECTANGLE:

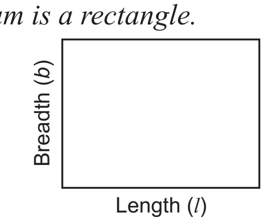
A rectangle is a parallelogram in which all the four angles at the vertices are right angles, *i.e.*, = 90°.

Properties:

1. *Opposite sides are equal and parallel.*
2. *All angles are each equal to 90°.*
3. *Diagonals are* **equal** *and bisect each other but are not necessarily at right angles.*



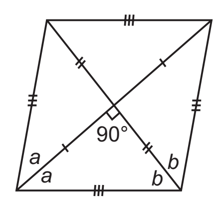
4. *The figure formed by joining the mid-points of the adjacent sides of a rectangle is a* **rhombus.**
   Thus, *PQRS* is a rhombus.
5. *The quadrilateral formed by the intersection of the angle bisectors of a parallelogram is a rectangle.*
6. **Area of a rectangle = length × breadth = $l \times b$**
7. **Diagonals of a rectangle = $\sqrt{\text{length}^2 + \text{breadth}^2} = \sqrt{l^2 + b^2}$**
8. **Perimeter of a rectangle = 2(length + breadth) = ($l + b$)**



RHOMBUS:

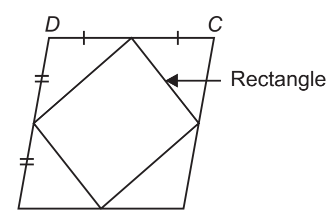
A parallelogram with all sides equal (adjacent sides equal) is called a rhombus.

Properties:



Properties 1-5

1. *Opposite sides are parallel and all sides are equal.*
2. *Opposite angles are equal.*
3. *Diagonals bisect each other at right angles but they are not necessarily equal.*
4. *Diagonals bisect the vertex angles.*
5. *Sum of any two adjacent angles is 180°.*
6. *Figure formed by joining the mid-points of the adjacent sides of a rhombus is a rectangle.*



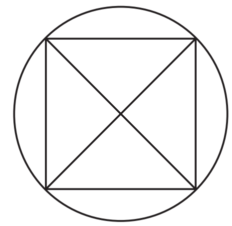
Property 6

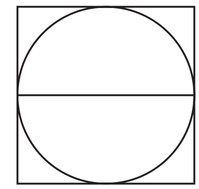
7. • **Area of rhombus = base × height**
   • **Area of rhombus = $\frac{1}{2}$ × product of diagonals**
   • **Area of rhombus = Product of adjacent sides × sine of included angle.**

SQUARE:

A parallelogram whose all sides are equal and whose all angles are each equal to right angle is a square. Thus each square is a parallelogram, a rectangle and a rhombus.

Properties:

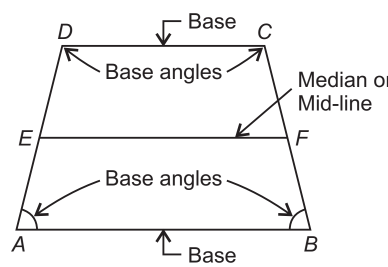

Property 4


Property 5

1. *All sides are equal and parallel.*
2. *All angles are each equal to 90°.*
3. *Diagonals are equal and bisect each other at right angles.*
4. *Diagonal of an inscribed square is equal to the diameter of the inscribing circle.*
5. *Side of the circumscribed square is equal to the diameter of the inscribed circle.*
6. The figure formed by joining the mid-points of the adjacent sides of a square is a square.
7. **Area of a square =** $(\text{side})^2 = \dfrac{(\text{diagonal})^2}{2}$
8. **Diagonal = side** $\sqrt{2}$
9. **Perimeter = 4 × side**

TRAPEZIUM:

A quadrilateral which has only one pair of opposite sides parallel and other pair of opposite sides not parallel is called a trapezium or a trapezoid.

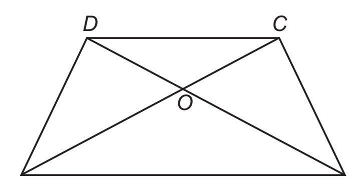


Trapezium

- The parallel sides ***AB*** and ***DC*** of the trapezium *ABCD* shown in the given figure are called the **bases of the trapezium.**
- The pair of angles that contain the same base, *i.e.*, $\angle A$ **and** $\angle B$**;** $\angle C$ **and** $\angle D$ are called **base angles.**
- The line joining the mid-points of the non-parallel sides is called the mid-line or median, *i.e.*, here is *EF*.
- *If the non parallel sides of a trapezium are equal, it is an* **isosceles trapezium.**

Properties:

1. The median is half the sum of the parallel sides, *i.e.*, $EF = \frac{1}{2}(AB + DC)$.
2. In case of an isosceles trapezium, the diagonals are also equal to each other, *i.e.*, $AC = BD$
3. Diagonals intersect each other proportionally in the ratio of lengths of parallel sides.
   $$\frac{AO}{OC} = \frac{BO}{OD} = \frac{AB}{CD}$$



Isosceles Trapezium (AC = BD)

4. By joining the mid-points of the adjacent sides of a trapezium four similar triangles are obtained.
5. If a trapezium is inscribed in a circle (cyclic trapezium), then it is an isosceles trapezium.
6. Area of a trapezium = $\frac{1}{2}$ × height × (sum of parallel sides)
7. Also, $AC^2 + BD^2 = BC^2 + AD^2 + 2AB \cdot CD$

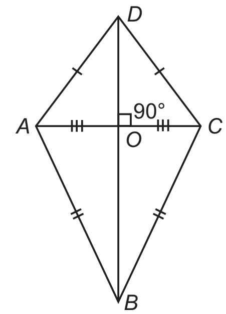
KITE:

A quadrilateral in which two pairs of adjacent sides are equal is called a kite.

Properties:

1. *Adjacent sides are equal, i.e.,* $AD = DC$; $AB = BC$.

2. *Shorter diagonal is bisected by the longer diagonal, i.e., $OA = OC$.*
3. *The diagonals are perpendicular to each other, i.e., $AC \perp BD$.*
4. *The angles at the vertices of the shorter diagonal are equal, i.e., $\angle A = \angle C$.*
5. *The longer diagonal, i.e., BD here, divides the kite into two congruent triangles, i.e., $\Delta ABD \cong \Delta DBC$.*
6. **Area of a kite = $\frac{1}{2}$ × product of diagonals.**



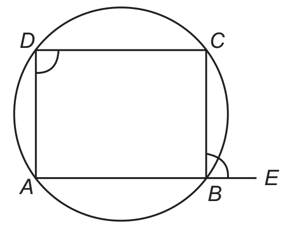
Kite

CYCLIC QUADRILATERAL:

A quadrilateral whose all four vertices lie on a circle is called cyclic quadrilateral.

Properties:

1. *The sum of pair of opposite angles of a cyclic quadrilateral is 180°, i.e.,* $\angle DAB + \angle BCD = 180°$; $\angle ADC + \angle ABC = 180°$
2. If a side of a cyclic quadrilateral is produced, then the exterior angle so formed is equal to the interior opposite angle, *i.e.*, $\angle CBE = \angle ADC$.
3. • **Area of a cyclic quadrilateral = $\sqrt{(s-a)(s-b)(s-c)(s-d)}$,** where s is the semi-perimeter, i.e., $s = \frac{a+b+c+d}{2}$, and a, b, c, d denote the lengths of the sides of the quadrilateral.
   • **Area of a cyclic quadrilateral in which a circle can be inscribed = $\sqrt{a \times b \times c \times d}$,** where a, b, c and d are the lengths of the sides of the quadrilateral.



Cyclic Quadrilateral

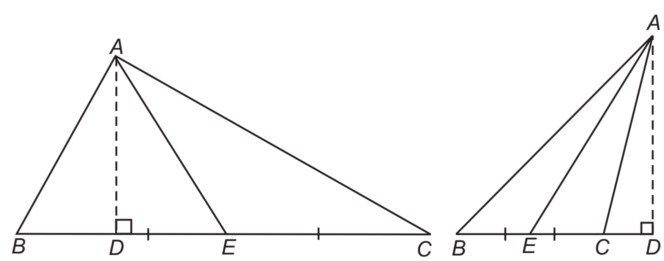
5. Apollonius' Theorem:

In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side, together with twice the square on the median which bisects the third side.

ΔABC is the given triangle where the median AE is drawn from A to E.

According to the given theorem,

$$AB^2 + AC^2 = 2BE^2 + 2AE^2$$



SOLVED EXAMPLES

Ex. 1. *ABCD* is a parallelogram. *AB* is produced to *E* such that *BE* = *AB*. Prove that *ED* bisects *BC*.

Sol. $ABCD$ is a parallelogram.

$\Rightarrow AB \parallel CD \Rightarrow AE \parallel CD$, CB is the transversal $\Rightarrow \angle 3 = \angle 4$ (alt. $\angle s$ are equal)

Now in Δs OCD and OBE,

$CD = AB = BE$

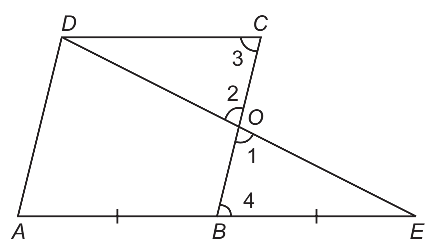
$\angle 2 = \angle 1$ (Vertically opposite $\angle s$)

$\angle 3 = \angle 4$ (Proved above)

$\therefore \Delta OCD \cong \Delta OBE$ (AAS)

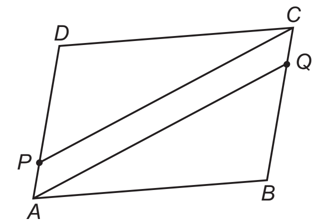
$\Rightarrow BO = OC \Rightarrow O$ is the mid-point of BC

$\Rightarrow ED$ bisects BC.



Ex. 2. *ABCD* is a parallelogram. *P* is a point on *AD* such that $AP = \frac{1}{3}AD$ and *Q* is a point on *BC* such that $CQ = \frac{1}{3}BC$. Prove that *AQCP* is a parallelogram.

Sol. *ABCD* is a parallelogram



$\Rightarrow AD = BC$ and $AD \parallel BC$

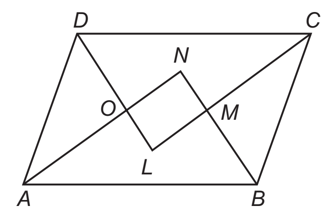
$\Rightarrow \frac{1}{3}AD = \frac{1}{3}BC$ and $AD \parallel BC$

$\Rightarrow AP = CQ$ and $AP \parallel CQ$

$\Rightarrow$ *APCQ* is a parallelogram.

Ex. 3. Prove that the angle bisectors of a parallelogram form a rectangle.

Sol. Let *ABCD* be the given parallelogram, whose angle bisectors form the quadrilateral *LMNO* which needs to be proved a rectangle.



$AB \parallel DC$ and *AD* is the transversal

$\Rightarrow \angle A + \angle D = 180°$ (Co-interior angles)

$\Rightarrow \frac{1}{2}\angle A + \frac{1}{2}\angle D = 90°$ (*AO* and *DO* are angle bisectors of $\angle A$ and $\angle D$ respectively)

$\Rightarrow \angle DAO + \angle ADO = 90°$

$\therefore$ In ΔADO,

$\angle DAO + \angle ADO + \angle AOD = 180°$

$\Rightarrow \angle AOD = 180° - 90° = 90°$

$\Rightarrow \angle NOL = \angle AOD = 90°$ (vert. opp. $\angle s$)

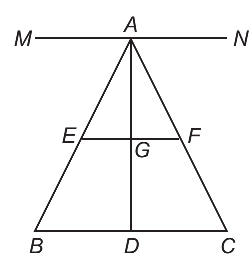
Similarly, we can show,

$\angle ONM = \angle NML = \angle MLO = 90°$

Hence, $\angle MNO$ is a rectangle.

Ex. 4. Prove that any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.

Sol. Given, a triangle *ABC* with *E* and *F* respectively as mid-points of *AB* and *AC*. The line *AD* from vertex *A* meets *EF* in *G* and we are to prove that $AG = GD$.



Draw a line *MAN* through *A* parallel to *BC*.

Since, *E* and *F* are mid-points of *AB* and *AC* respectively, $EF \parallel BC$. (Mid pt. Theorem)

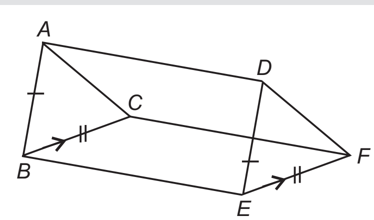
Given, $MAN \parallel BC$

$\Rightarrow MAN \parallel EF \parallel BC$

Thus, by the intercept theorem, the intercepts *AE* and *EB* made by the transversal *AB* on the parallel lines *MAN*, *EF* and *BC* are equal, so the intercepts made by the transversal *AD* on these three parallel lines will also be equal, *i.e.*, $AG = GD$.

Ex. 5. In ΔABC and ΔDEF, $AB = DE$, $AB \parallel DE$, $BC = EF$ and $BC \parallel EF$. Vertices *A*, *B* and *C* are joined to vertices *D*, *E* and *F* respectively. Show that $\Delta ABC \cong \Delta DEF$.

Sol. Given: $AB = DE$, $AB \parallel DE$



As one pair of opposite sides is equal and parallel

$\Rightarrow$ *ABED* is a parallelogram.

Given: $BC = EF$ and $BC \parallel EF$

$\Rightarrow$ *BCFE* is a parallelogram

Now, $ABED$ is a parallelogram $\Rightarrow$ $AD \parallel BE$ and $AD = BE$ ...(*i*)

$BCFE$ is a parallelogram $\Rightarrow$ $CF \parallel BE$ and $CF = BE$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*),

$AD = CF$ and $AD \parallel CF$

$\Rightarrow$ $ADFC$ is a parallelogram.

$\Rightarrow$ $AC = DF$.

Now in Δs ABC and DEF

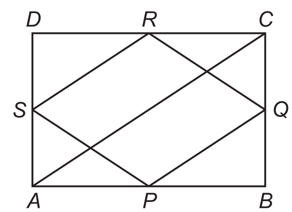
$AB = DE$ (Given), $BC = EF$ (Given) and $AC = DF$ (Proved above)

$\therefore$ $\Delta ABC \cong \Delta DEF$ (SSS)

Ex. 6. If $ABCD$ is a rectangle and P, Q, R and S are the mid-points of the sides AB, BC, CD and DA respectively, then quadrilateral $PQRS$ is a rhombus.

Sol. $ABCD$ is the given rectangle. Join AC. In ΔABC, P and Q are the mid-points of sides AB and BC respectively.

$\therefore$ By mid-point theorem,



$PQ \parallel AC$ and $PQ = \frac{1}{2} AC$ ...(*i*)

In ΔADC, R and S are the mid-points of CD and AD respectively

$\therefore$ $SR \parallel AC$ and $SR = \frac{1}{2} AC$ ...(*ii*)

From (*i*) and (*ii*), we get

$PQ \parallel SR$ and $PQ = SR \Rightarrow PQRS$ is a parallelogram

$ABCD$ is a rectangle, $AD = BC \Rightarrow \frac{1}{2} AD = \frac{1}{2} BC \Rightarrow AS = BQ$

$\therefore$ In Δs APS and BPQ

$AS = BQ$ (Proved above)

$AP = PB$ (P is mid-point of AB)

$\angle SAP = \angle QBP = 90°$

$\therefore$ $\Delta APS \cong \Delta BPQ$ (SAS)

$\Rightarrow$ $PS = PQ$

$\therefore$ $PQRS$ is a parallelogram with adjacent sides equal

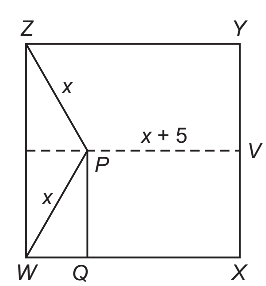
$\Rightarrow$ $PQRS$ is a rhombus.

Ex. 7. $WXYZ$ is a square of side length 30. V is a point on XY and P is a point inside the square with PV perpendicular to XY. $PW = PZ = PV - 5$. Find PV.

Sol. $PZ = PW$

$\Rightarrow$ P lies half-way between ZY and WX. Let $PZ = PW = x \Rightarrow PV = x + 5$

$\because$ $PV \perp XY$ (Given)



$\therefore$ V is also the mid-point of XY

$\Rightarrow$ $VX = VY = 15$

$WQ = WX - QX = WX - PV = 30 - (5 + x) = 25 - x.$

$PQ = 15$

$\therefore$ $PW = \sqrt{WQ^2 + PQ^2}$

$\Rightarrow$ $x = \sqrt{(25 - x)^2 + 15^2}$

$\Rightarrow \quad x^2 = 625 - 50x + x^2 + 225$

$\Rightarrow \quad 50x = 850 \quad \Rightarrow \quad x = 17$

$\therefore \quad PV = x + 5 = \mathbf{22}.$

Ex. 8. *ABCD* is a trapezium in which side *AB* is parallel to side *DC* and *E* is the mid-point of side *AD*. If *F* is a point on side *BC* such that segment *EF* is parallel to side *DC*. Prove that $EF = \frac{1}{2}(AB + DC)$.

Sol. Let *ABCD* be the given trapezium in which $AB \parallel DC$. In ΔADC, *E* is the mid-point of *AD* and $EG \parallel DC$

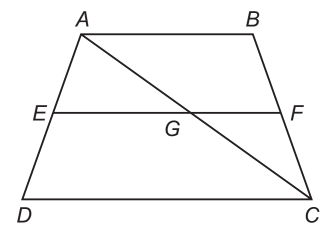
$\Rightarrow$ *G* is the mid-point of *AC*

$\Rightarrow EG = \frac{1}{2} DC$ ...(*i*)

($\therefore$ Line segment joining the mid-points of to sides of a triangle is parallel to the third side and half of it).

Also, $AB \parallel DC,\ EF \parallel DC \Rightarrow EF \parallel AB \Rightarrow GF \parallel AB$

$\therefore$ In ΔABC, $GF \parallel AB$ and *G* is the mid-point of *AC*



$\Rightarrow$ *F* is the mid-point of *BC*

$\Rightarrow GF = \frac{1}{2} AB$ ...(*ii*)

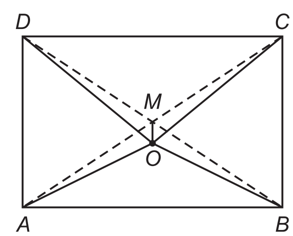
(By mid-point theoram)

$\therefore$ From (*i*) and (*ii*), $GE + GF = \frac{1}{2} DC + \frac{1}{2} AB$

$\Rightarrow \ \boldsymbol{EF = \frac{1}{2}(AB + DC)}.$

Ex. 9. A point *O* in the interior of a rectangle *ABCD* is joined with each of the vertices *A*, *B*, *C* and *D*. Then, show that $OA^2 + OC^2 = OB^2 + OD^2$.

Sol. As the diagonals of a rectangle are equal and bisect each other, $AM = DM$



Applying Applonius theorem,

In ΔAOC, $OA^2 + OC^2 = 2(OM^2 + AM^2)$

In ΔBOD, $OB^2 + OD^2 = 2(OM^2 + DM^2)$

$\because \quad AM = DM$

$\therefore \quad \boldsymbol{OA^2 + OC^2 = OB^2 + OD^2}.$

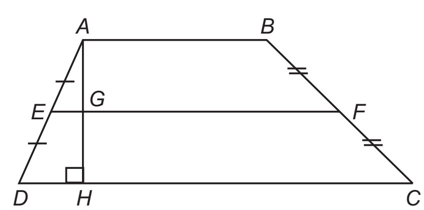
Ex. 10. The median of a trapezoid cuts the trapezoid into two regions whose areas are in the ratio 1 : 2. Compute the ratio of the smaller base of the trapeziod to the larger base.

(Median of a trapezoid means the line joining the mid-points of the 2 non-parallel sides of the trapezoid)

Sol. *ABCD* is the given trapezoid, whose median or midline is *EF.*

Let *AH* be drawn perpendicular to *DC*

$EG \parallel DH$ $\quad (\because EF \parallel DC)$



$\therefore$ By Basic Proportionality Theorem,

$\frac{AE}{ED} = \frac{AG}{GH} \Rightarrow \frac{AG}{GH} = 1$ $\quad (\because E$ is the mid-point of $AD)$

$\Rightarrow AG = GH$

Let $AG = GH = h$ and $AB = x$, $EF = y$ and $CD = z$.

Then, Area (*ABFE*) $= h/2\ (x + y)$

Area (*EFCD*) $= h/2\ (y + z)$

$$\text{Area }(ABCD) = h(x+z)$$

Now given, $\dfrac{\text{Area }(ABFE)}{\text{Area }(EFCD)} = \dfrac{1}{2} \Rightarrow \dfrac{h/2\ (x+y)}{h/2\ (y+z)} = \dfrac{1}{2}$

$\Rightarrow \quad 2x + 2y = y + z \Rightarrow 2x + y = z$ ...(*i*)

Also, $\dfrac{\text{Area }(ABFE)}{\text{Area }(ABCD)} = \dfrac{1}{3} \Rightarrow \dfrac{h/2\ (x+y)}{h(x+z)} = \dfrac{1}{3}$

$\Rightarrow \quad 3x + 3y = 2x + 2z \Rightarrow x + 3y = 2z$ ...(*ii*)

$\Rightarrow$ From (*i*) and (*ii*)

$$x + 3y = 2(2x+y) \Rightarrow x + 3y = 4x + 2y$$

$\Rightarrow \quad 3x = y$

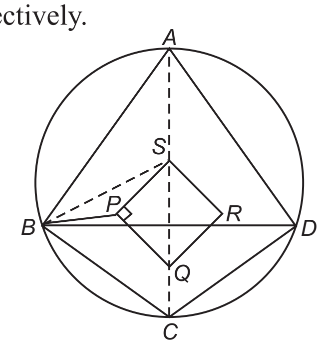
$\Rightarrow$ Putting in (*i*), we get

$$2x + 3x = z \Rightarrow 5x = z \Rightarrow \boldsymbol{x : z = 1 : 5.}$$

Ex. 11. Let *ABCD* be a cyclic quadrilateral. Show that the incentres of the triangles *ABC*, *BCD*, *CDA* and *DAB* form a rectangle.

Sol. We know that incentre is the point of concurrence of the internal bisectors of the angles of a triangle.

Let P, Q, R and S be the incentres of the triangles ABC, BCD, DAC and DAB respectively.



$\Rightarrow$ P is point of intersection of $\angle BAC$ and $\angle ABC$

In ΔAPB, $\angle APB = 180° - [\angle PAB + \angle PBA]$

$\angle APB = 180° - \left[\dfrac{\angle BAC}{2} + \dfrac{\angle ABC}{2}\right]$ ...(*i*)

But in ΔABC, $\angle ABC + \angle BAC + \angle BCA = 180°$

$\Rightarrow \dfrac{1}{2}(\angle ABC + \angle BAC) = 90° - \dfrac{\angle BCA}{2}$ ...(*ii*)

From (*i*) and (*ii*), $\angle APB = 180° - \left[90° - \dfrac{\angle BCA}{2}\right] = 90° + \dfrac{\angle BCA}{2}$ ...(*iii*)

Similarly, in ΔASB, $\angle ASB = 180° - [\angle SAB + \angle SBA]$

$\Rightarrow \angle ASB = 180° - \left[\dfrac{\angle DAB}{2} + \dfrac{\angle ABD}{2}\right]$ ...(*iv*)

Also in ΔADB,

$\angle ADB + \angle DAB + \angle ABD = 180°$

$\Rightarrow \dfrac{\angle DAB}{2} + \dfrac{\angle ABD}{2} = 90° - \dfrac{\angle ADB}{2}$ ...(*v*)

$\therefore$ From (*iv*) and (*v*)

$$\angle ASB = 180° - \left(90° - \frac{\angle ADB}{2}\right) = 90° + \frac{\angle ADB}{2} \quad \ldots(vi)$$

But $\angle BCA = \angle ADB$ (Angles in the same segment are equal)

$\therefore$ From (*iii*) and (*vi*)

$$\angle APB = \angle ASB$$

$\Rightarrow$ $ASPB$ is a cyclic quadrilateral.

$\Rightarrow \quad \angle SPB + \angle SAB = 180°$ (Opposite angles of a cyclic quadrilateral are supp.)

$\Rightarrow \quad \angle SPB = 180° - \angle SAB$

$\angle SPB = 180° - \angle A/2$ ($\because$ SA bisects $\angle A$)

Similarly, $\angle BPQ = 180° - C/2$

$$\Rightarrow \qquad \angle SPQ = 360° - (\angle SPB + \angle BPQ)$$

$$= 360° - \left(180° - \frac{\angle A}{2} + 180° - \frac{\angle C}{2}\right)$$

$$= \frac{\angle A + \angle C}{2} = \frac{180°}{2} = 90° \qquad (\because ABCD \text{ is cylic quadrilateral})$$

Thus, in quadrilateral $SPQR$, $\angle P = 90°$

Similarly it can be shown that $\angle S = \angle Q = \angle R = 90°$

$\Rightarrow$ Quadrilateral is a rectangle.

Ex. 12. Let *ABCD* be a quadrilateral. Let *X* and *Y* be the mid-points of *AC* and *BD* respectively and the lines through *X* and *Y* respectively parallel to *BD* and *AC* meet in *O*. Let *P*, *Q*, *R*, *S* be the mid-points of *AB*, *BC*, *CD* and *DA* respectively. Prove that:

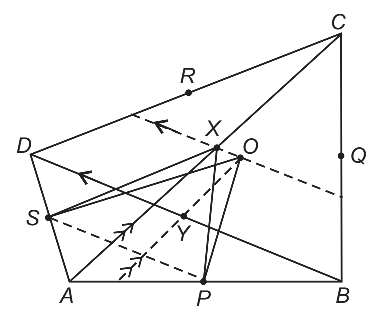
(*a*) Quadrilateral *APOS* and *APXS* have the same area.

(*b*) The areas of quadrilaterals *APOS*, *BQOP*, *CROQ* and *DSOR* are all equal.

Sol. *ABCD* is the given quadrilateral.

P, Q, R and S are the mid-points of sides AB, BC, CD and DA respectively.

$OX \parallel BD$ and $OY \parallel AC$



(*a*) In ΔABD, the line joining the mid-points P and S of sides AB and AD respectively is parallel to the third side.

$\therefore$ $PS \parallel BD$.

Now $PS \parallel BD$ and $OX \parallel BD \Rightarrow OX \parallel PS$.

$\because$ Area of triangles on the same base and between the same parallel lines are equal,

So, Area (ΔPXS) = Area (ΔPOS)

$\Rightarrow$ Area (ΔPAS) + Area (ΔPXS) = Area (ΔPAS) + Area (ΔPOS)

$\Rightarrow$ **Area (*APXS*) = Area (*APOS*)**

(*b*) Area ($APXS$) = Area (ΔAPX) + Area (ΔASX)

$$= \frac{1}{2}\text{Area } (\Delta ABX) + \frac{1}{2}\text{Area } (\Delta AXD)$$

($\because$ XP is the median of ΔAXB and XS is the median of ΔADX and a median divides a triangle into two triangles of equal area)

$\because$ X is the mid-point of AC, XB is the median of ΔABC and XD is the median of ΔADC.

$$\text{So, Area } (APXS) = \frac{1}{2}\times\frac{1}{2}\text{Area } (\Delta ABC) + \frac{1}{2}\times\frac{1}{2}\text{Area } (\Delta ADC)$$

$$= \frac{1}{4}[\text{Area } (\Delta ABC) + \text{Area } (\Delta ADC)]$$

$$= \frac{1}{4}\text{Area (quad. } ABCD)$$

Now in part (a), we have proved Area ($APXS$) = Area ($APOS$)

$$\Rightarrow \text{Area } (APOS) = \frac{1}{4}\text{Area (quad. } ABCD)$$

Similarly, it can be shown by symmetry that

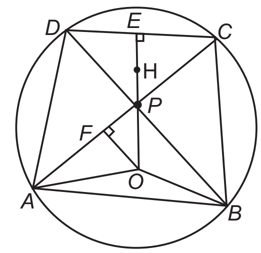
$$\text{Area } (BPOQ) = \text{Area } (CROQ) = \text{Area } (DSOR) = \frac{1}{4}\text{Area } (ABCD)$$

$\Rightarrow$ Area ($APOS$) = Area ($BPOQ$) = Area ($CROQ$) = Area ($DSOR$).

Ex. 13. The diagonals *AC* and *BD* of a cyclic quadrilateral *ABCD* intersect at *P*. Let *O* be the circumcentre of ΔAPB and *H* be the orthocentre of ΔCPD. Show that the points *H*, *P*, *O* are collinear.

Sol. Let O be the circumcentre (point of intersection of the perpendicular bisectors of the sides of a Δ) of ΔAPB

Join A to O and draw $OF \perp AP$



O being the circumcentre of ΔAPB, it is equidistant from the three vertices A, B and P

$\therefore$ $AO = OP \Rightarrow \Delta AOP$ is isosceles $\Rightarrow OF$ bisects $\angle AOP$

$\Rightarrow$ $\angle FOP = \frac{1}{2}\angle AOP$ ...(i)

Also, O being equidistant form the three vertices, we can consider a circle passing through the vertices A, P and B with O as its centre.

Then, $\angle ABP = \frac{1}{2}\angle AOP$ ...(ii) (Angle subtended by chord AP at the centre is twice the angle subtended at any other part of the circle)

Also, $\angle ABD = \angle ACD$ (Angles in the same segment)

$\Rightarrow$ $\angle ABP = \angle PCD$ ...(iii)

$\therefore$ From (i), (ii) and (iii)

$\angle FOP = \angle PCD$

Now in ΔFOP and ΔEPC

$\angle FOP = \angle PCE$ (Proved above)

$\angle OFP = \angle PEC = 90°$

$\Rightarrow$ $\angle FOP \sim \angle EPC$ (AA similarity)

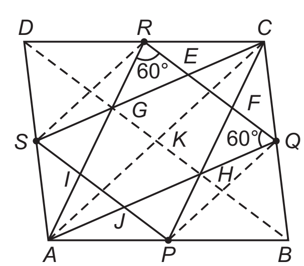
$\Rightarrow$ $\angle FPO = \angle EPC$, which can only be equal if they are vertically opposite angles.

$\Rightarrow$ EO is a straight line.

$\Rightarrow$ H, P and O are collinear.

Ex. 14. Let *ABCD* be a convex quadrilateral. *P*, *Q*, *R* and *S* are the mid-points of *AB*, *BC*, *CD* and *DA* respectively such that the triangles *AQR* and *CSP* are equilateral. Prove that *ABCD* is a rhombus and determine its angles. (*RMO 2005*)

Sol. In ΔDCB, by mid-point theoram,



$QR = \frac{1}{2}BD$

Also, in ΔABD, by mid-point theoram,

$PS = \frac{1}{2}BD$

$\Rightarrow$ $QR = PS$.

Also, given ΔAQR and ΔCSP are equilateral Δs and $QR = PS$

$\Rightarrow$ ΔAQR is congruent to ΔCSP

$\Rightarrow$ $AR = RQ = AQ = CS = CP = PS$ ...(i)

Also, $\Delta CEF \sim \Delta CSP \Rightarrow \angle CEF = 60°$ (ΔCSP is equilateral)

ΔAQR being equilateral $\Rightarrow \angle AQR = 60°$

$\therefore$ $\angle CEF = \angle AQR \Rightarrow$ alternate angles are equal $\Rightarrow$ $CS \parallel QA$

So, $CS = QA$ (from (i)) and $CS \parallel QA \Rightarrow CSAQ$ is a parallelogram

$\Rightarrow$ $SA \parallel CQ$ and $SA = CQ \Rightarrow AD \parallel BC$ and $AD = BC \Rightarrow ABCD$ is a parallelogram.

Let diagonals AC and BD bisect each other at K. Then,

$$DK = \frac{BD}{2} = QR = CS = AR.$$

$\Rightarrow$ In ΔADC, medians, AR, DK and CS are all equal

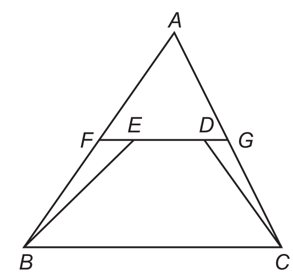
$\Rightarrow$ ΔADC is equilateral $\Rightarrow AD = AB$.

$\therefore$ $ABCD$ is a parallelogram with equal adjacent sides

$\Rightarrow$ $ABCD$ is a rhombus.

Ex. 15. If a triangle and a convex quadrilateral are drawn on the same base and no part of the quadrilateral is outside the triangle, show that the perimeter of the triangle is greater than the perimeter of the quadrilateral. **(*SAT 2000*)**

Sol. Let ABC be the given triangle and $BCDE$, the given quadrilateral.



Produce ED to meet AB in F and AC in G.

Then, in ΔBEF, $BE < FE + BF$ (Triangle inequality)

In ΔDGC, $DC < DG + GC$ (Triangle inequality)

Now perimeter of quadrilateral $BEDC$

$= BE + ED + DC + BC$

$< BF + \underbrace{FE + ED + DG} + GC + BC$

$= BF + FG + GC + BC < \underbrace{BF + AF} + \underbrace{AG + GC} + BC$ ($\because$ In ΔAFG, $FG < AF + AG$)

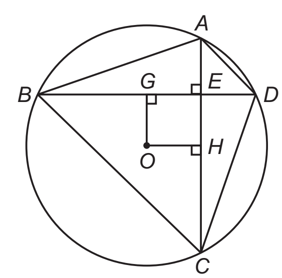
$= AB + AC + BC$

$=$ Perimeter of ΔABC.

$\therefore$ Perimeter of quad. $BEDC <$ Perimeter of ΔABC.

Ex. 16. *ABCD* is a cyclic quadrilateral such that $AC \perp BD$. *AC* meet *BD* at *E*. Prove that $EA^2 + EB^2 + EC^2 + ED^2 = 4R^2$, where *R* is the radius of the circle.

Sol. Let O be the centre of the circle, circumscribing the cyclic quadrilateral $ABCD$.



Let OG and OH be the perpendiculars from O on BD and AC respectively.

Then OG bisects BD, *i.e.*, $BG = GD$

and OH bisects AC, *i.e.*, $AH = HC$

($\because$ The perpendicular from the centre of the chord, bisects the chord)

Now, $BE^2 + ED^2 = (BG + GE)^2 + (GD - GE)^2$

$= BG^2 + GE^2 + 2BG \cdot GE + GD^2 + GE^2 - 2GD \cdot GE$

$= 2BG^2 + 2GE^2$ ...(*i*) ($\because BG = GD$)

Also, $AE^2 + EC^2 = (AH - EH)^2 + (CH + EH)^2$

$= AH^2 + EH^2 - 2AH \cdot EH + CH^2 + EH^2 - 2CH \cdot EH$

$= 2AH^2 + 2EH^2$ ...(*ii*) ($\because AH = CH$)

$\therefore$ $EA^2 + EB^2 + EC^2 + ED^2 = 2AH^2 + 2EH^2 + 2BG^2 + 2GE^2$

$= 2(AH^2 + GO^2 + BG^2 + OH^2)$ ($\therefore EH = GO, GE = OH$)

$= 2(AO^2 + BO^2) = 2(R^2 + R^2) = \mathbf{4R^2}$.

PRACTICE SHEET

LEVEL–1

1. Which one of the following statements is correct?
If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is:
(*a*) A rhombus but not a square
(*b*) A square but not a rhombus
(*c*) Either a rhombus or a square
(*d*) A rectangle but not a square **(*CDS 2005*)**

2. If the diagonals of a quadrilateral are equal and bisect each other at right angles, then the quadrilateral is a
(*a*) rectangle (*b*) square
(*c*) rhombus (*d*) trapezium **(*CDS 2012-II*)**

3. If the bisectors of the angles A and B of a quadrilateral $ABCD$ meet at O, then $\angle AOB$ is equal to :
(*a*) $\angle C + \angle D$ (*b*) $\frac{1}{2}(\angle C + \angle D)$
(*c*) $\frac{1}{2}\angle C + \frac{1}{3}\angle D$ (*d*) $\frac{1}{3}\angle C + \frac{1}{2}\angle D$
(*CDS 2004*)

4. If two parallel lines are intersected by a transversal, then the bisectors of the interior angles form which one of the following?
(*a*) Rectangle (*b*) Square
(*c*) Rhombus (*d*) Parallelogram
(*CDS 2006*)

5. $ABCD$ is a cyclic quadrilateral such that $\angle A + \angle B = 2(\angle C + \angle D)$. If $\angle C > 30°$, then which one of the following is correct?
(*a*) $\angle D \geq 90°$ (*b*) $\angle D < 90°$ (*c*) $\angle D \leq 90°$ (*d*) $\angle D > 90°$
(*CDS 2006*)

6. In a parallelogram $ABCD$, M is the mid-point of BD and BM bisects $\angle B$. Then what is $\angle AMB$ equal to?
(*a*) 45° (*b*) 60° (*c*) 90° (*d*) 120°
(*CDS 2006*)

7. If $ABCD$ is a rhombus, then
(*a*) $AC^2 + BD^2 = 6AB^2$ (*b*) $AC^2 + BD^2 = 5AB^2$
(*c*) $AC^2 + BD^2 = 4AB^2$ (*d*) $AC^2 + BD^2 = 3AB^2$

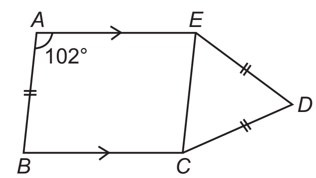
8. In a trapezium $ABCD$, AB is parallel to CD and the diagonals intersect each other at O. In this case, the ratio OA/OC is equal to:
(*a*) $\frac{OB}{OD}$ (*b*) $\frac{BC}{CD}$ (*c*) $\frac{AD}{AB}$ (*d*) $\frac{AC}{BD}$

9. $ABCD$ is a trapezium where AB and CD are non-parallel sides. If the vertices A, B, C and D are concyclic, then
(*a*) AB is also parallel to CD
(*b*) $AB = \frac{1}{2}CD$
(*c*) $\frac{1}{2}AB = CD$
(*d*) $AB = CD$

10. X, Y are the mid-points of opposite sides AB and DC of a parallelogram $ABCD$. AY and DX are joined intersecting in P. CX and BY are joined intersecting in Q. Then $PXQY$ is
(*a*) rectangle (*b*) rhombus
(*c*) square (*d*) parallelogram

11. If the sum of the diagonals of a rhombus is 12 cm and its perimeter is $8\sqrt{5}$ cm, then the lengths of the diagonals are:
(*a*) 6 cm and 6 cm (*b*) 7 cm and 5 cm
(*c*) 8 cm and 4 cm (*d*) 9 cm and 3 cm

12. In the given figure $AE = BC$ and $AE \parallel BC$ and the three sides AB, CD and ED are equal in length. If $\angle A = 102°$, then find the measure of $\angle BCD$.



(*a*) 138° (*b*) 162°
(*c*) 88° (*d*) 150°

13. If a rectangle and a parallelogram are equal in area and have the same base and are situated on the same side, and the ratio of the perimeter of the rectangle and that of the parallelogram is k, then
(*a*) $k > 1$ (*b*) $k < 1$
(*c*) $k = 1$ (*d*) Cannot be determined

14. $ABCD$ is a parallelogram. The diagonals AC and BD intersect at the point O. If E, F, G and H are the mid-points of AO, DO, CO and BO respectively, then the ratio of $(EF + FG + GH + HE)$ to $(AD + DC + CB + BA)$ is
(*a*) 1 : 1 (*b*) 1 : 2 (*c*) 1 : 3 (*d*) 1 : 4

15. A cyclic parallelogram having unequal adjacent sides is necessarily a:
(*a*) Square (*b*) Rectangle (*c*) Rhombus (*d*) Trapezium
(*MAT*)

LEVEL–2

16. In a quadrilateral $ABCD$, $\angle B = 90°$ and $AD^2 = AB^2 + BC^2 + CD^2$. Then $\angle ACD$ is equal to:
(*a*) 90° (*b*) 60° (*c*) 30° (*d*) None of these
(*MAT 2003*)

17. In a cyclic quadrilateral $PQRS$, $\angle P$ is double its opposite angle and difference between the other two angles is one-third of $\angle P$. The minimum difference between any two angles of this quadrilateral is
(*a*) 30° (*b*) 10° (*c*) 20° (*d*) 40°
(*MAT 2010*)

18. $ABCD$ is a square. P, Q, R and S are points on the sides AB, BC, CD and DA respectively such that $AP = BQ = CR = DS$. What is $\angle SPQ$ equal to?
(*a*) 30° (*b*) 45° (*c*) 60° (*d*) 90°
(*CDS 2010*)

19. The middle points of the parallel sides AB and CD of a parallelogram $ABCD$ are P and Q respectively. If AQ and CP divide the diagonal BD into three parts BX, XY and YD, then which one of the following is correct?

(*a*) $BX \neq XY \neq YD$ (*b*) $BX = YD \neq XY$
(*c*) $BX = XY = YD$ (*d*) $XY = 2BX$ **(*CDS 2010*)**

20. Let $ABCD$ be a parallellogram. Let m and n be positive integers such that $n < m < 2n$. Let $AC = 2\ mn$ and $BD = m^2 - n^2$ and $AB = \dfrac{(m^2 + n^2)}{2}$

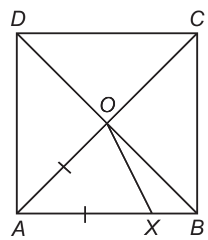
Statement I. $AC > BD$.

Statement II. $ABCD$ is a rhombus.

Which one of the following is correct in respect of the above statements?

(*a*) Both the statement I and II are true and statement II is the correct explanation of statement I.
(*b*) Both the statements I and II are true but statement II is not the correct explanation of statement I.
(*c*) Statement I is true but statement II is false.
(*d*) Statement II is true but statement I is false.

21. In the given figure, $ABCD$ is a square in which $AO = AX$. What is $\angle XOB$?



(*a*) 22.5° (*b*) 25°
(*c*) 30° (*d*) 45° **(*CDS 2009*)**

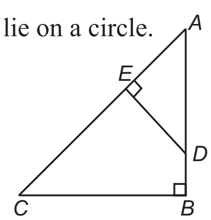
22. If $PQRS$ is trapezium such that $PQ > RS$ and L, M are the mid-points of the diagonals PR and QS respectively then what is LM equal to?

(*a*) $\dfrac{PQ}{2}$ (*b*) $\dfrac{RS}{2}$
(*c*) $\dfrac{PQ + RS}{2}$ (*d*) $\dfrac{PQ - RS}{2}$ **(*CDS 2006*)**

23. In the given figure, $\angle ABC = \angle AED = 90°$. Consider the following statements:

I. ABC and ADE are similar triangles.

II. The four points B, C, E and D may lie on a circle.

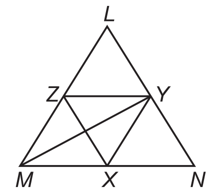


(*a*) Only I
(*b*) Only II
(*c*) Both I and II
(*d*) Neither I nor II

24. Let X be any point within a square $ABCD$. On AX a square $AXYZ$ is described such that D is within it. Which one of the following is correct?

(*a*) $AX = DZ$ (*b*) $\angle ADZ = \angle BAX$
(*c*) $AD = DZ$ (*d*) $BX = DZ$ **(*CDS 2012*)**

25. In the given figure, YZ is parallel to MN, XY is parallel is LM and XZ is parallel to LN. Then MY is



(*a*) Median of ΔLMN
(*b*) The anguler bisector of $\angle LMN$
(*c*) Perpendicular to LN
(*d*) Perpendicular bisector of LN.

26. The locus of a point in rhombus $ABCD$ which is equidistant from A and C is

(*a*) a fixed point on diagonal BD
(*b*) diagonal BD
(*c*) diagonal AC
(*d*) None of the above

27. $ABCD$ is a square. The diagonals AC and BD meet at O. Let K, L be the points on AB such that $AO = AK$, $BO = BL$. If $\theta = \angle LOK$, then what is the value of $\tan \theta$?

(*a*) $\dfrac{1}{\sqrt{3}}$ (*b*) $\sqrt{3}$ (*c*) 1 (*d*) $\dfrac{1}{2}$

(*CDS 2008*)

28. $ABCD$ is a trapezium in which $AB \parallel DC$ and $AD = BC$. If P, Q, R and S be respectively the mid-points of BA, BD, CD and CA, then $PQRS$ is a

(*a*) Rhombus (*b*) Rectangle
(*c*) Parallelogram (*d*) Square

29. A rigid square plate $ABCD$ of unit side rotates in its own plane about the middle-point of CD until the new position of A coincides with the old position of B. How far is the new position of B from the old position of A?

(*a*) 4 units (*b*) $5\sqrt{5}$ units
(*c*) $\dfrac{4\sqrt{5}}{5}$ units (*d*) $4\sqrt{5}$ units **(*RMO*)**

30. A trapezium $ABCD$ in which $AB \parallel CD$ is inscribed in a circle with centre O. Suppose the diagonals AC and BD of the trapezium intersect at M and $OM = 2$. If $\angle AMB = 60°$, the difference between the lengths of the parallel sides is:

(*a*) 2 (*b*) $\sqrt{3}$ (*c*) $3\sqrt{3}$ (*d*) $2\sqrt{3}$

31. $ABCD$ is a trapezium with AB and CD as parallel sides. The diagonals intersect at O. The area of the triangle ABO is $\boldsymbol{p}$ and that of triangle CDO is $\boldsymbol{q}$**.** The area of the trapezium is:

(*a*) $\sqrt{p} + \sqrt{q}$ (*b*) $\dfrac{1}{3}\left(\sqrt{p} + \sqrt{q}\right)^3$
(*c*) $\dfrac{1}{2}\left(\sqrt{p} + \sqrt{q}\right)^2$ (*d*) $\left(\sqrt{p} + \sqrt{q}\right)^2$

32. $PQRS$ is a rectangle in which $PQ = 2PS$. T and U are the mid-points of PS and PQ respectively. QT and US intersect at V. Find the ratio of the area of quadrilateral $QRSV$ to the area of triangle PQT.

(*a*) 4 : 1 (*b*) 8 : 3 (*c*) 5 : 2 (*d*) 7 : 4

33. A rhombus has sides of length 1 and area $\dfrac{1}{2}$. Find the angle between the two adjacent sides of the rhombus.

(*a*) 60° (*b*) 75° (*c*) 45° (*d*) 30°

34. In the given figure $ABCD$ is a quadrilateal whose diagonals intersect at O. $\angle AOB = 30°$, $AC = 24$ and $BD = 22$. The area of quadrilateral $ABCD$ is:

(*a*) 132 (*b*) 264 (*c*) 528 (*d*) 66

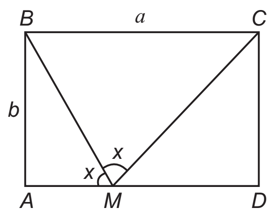
35. $ABCD$ is a square and AOB is an equilateral triangle. What is the value of $\angle DOC$?

(*a*) 120° (*b*) 150° (*c*) 125° (*d*) 80°

36. In a trapezium $ABCD$, AB is parallel to CD, BD is perpendicular to AD. AC is perpendicular to BC. If $AD = BC = 15$ cm and $AB = 25$ cm, then the area of the trapezium is

(*a*) 192 cm^2 (*b*) 232 cm^2 (*c*) 162 cm^2 (*d*) 172 cm^2

37. $ABCD$ is a rectangle with $BC = a$, $AB = b$ and $a > b$. If M is a point on AD such that $\angle BMA = \angle BMC$, then MD is equal to:



(*a*) $\sqrt{a^2 + b^2}$

(*b*) $\sqrt{a^2 - b^2}$

(*c*) $\sqrt{ab}$

(*d*) $a - b$

38. Let $ABCD$ be a square. M, N, R are the points on AB, BC and CD respectively such that $AM = BN = CR$. If $\angle MNR$ is a right angle, then $\angle MRN$ is equal to

(*a*) 30° (*b*) 45° (*c*) 60° (*d*) 75°

39. If for a regular pentagon $ABCDE$, the lines AD and BE intersect at points P, then $\angle BAD$ and $\angle APE$ respectively are

(*a*) 36° and 72° (*b*) 54° and 108°
(*c*) 72° and 108° (*d*) 36° and 108°

LEVEL–3

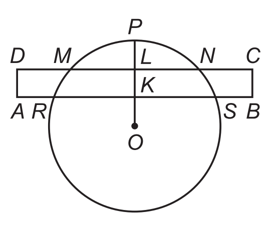
40. Let $ABCD$ be a parallelogram. P is any point on the side AB. If DP and CP are joined in such a way that they bisect the angles ADC and BCD respectively, then DC is equal to

(*a*) CB (*b*) $2CB$ (*c*) $3CB$ (*d*) $4CB$

41. The adjacent sides of a parallelogram are $2a$ and a. If the angle between them is 60°, then one of the diagonals of the parallelogram is

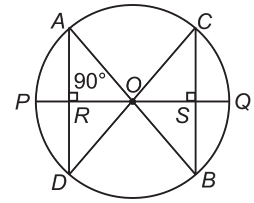
(*a*) $3a$ (*b*) $\sqrt{5}a$ (*c*) $2a$ (*d*) $\sqrt{3}a$

42. In the given figure, O is the centre of the circle. The radius OP bisects a rectangle $ABCD$ at right angles. $DM = NC = 2$ cm and $AR = SB = 1$ cm, $KS = 4$ cm and $OP = 5$ cm. What is the area of the rectangle?



(*a*) 8 cm^2 (*b*) 10 cm^2 (*c*) 12 cm^2 (*d*) 16 cm^2

43. In the given figure O is the centre of the circle. $\angle AOD = 120°$. If the radius of the circle be 'r', then find the sum of the areas of quadrilateral $AODP$ and $OBQC$



(*a*) $\frac{\sqrt{3}}{2} r^2$ (*b*) $3\sqrt{3}\, r^2$

(*c*) $\sqrt{3}\, r^2$ (*d*) $2\sqrt{3}\, r^2$

44. Let $ABCD$ be a quadrilateral with $\angle CBD = 2\angle ADB$, $\angle ABD = 2\angle CDB$ and $AB = CB$. Then which of the following statement hold true.

I $AD = CD$

II $AD = BD$

III $BEDF$ is a parallelogram

IV $AB = CD$

(*a*) I and III only (*b*) All of above
(*c*) I, II and IV (*d*) I, III and IV

(*Canadian Mathematical Olympiad 2000*)

45. A square sheet of paper $ABCD$ is so folded that B falls on the mid-point M of CD. The crease will divide BC in the ratio

(*a*) 7 : 4 (*b*) 5 : 3 (*c*) 8 : 5 (*d*) 4 : 1

46. $ABCD$ is a square. P, Q, R, S are the mid-points of AB, BC, CD and DA respectively. By joining AR, BS, CP, DQ, we get a quadrilateral which is a

(*a*) trapezium (*b*) rectangle (*c*) square (*d*) rhombus

47. Perpendiculars are drawn from the vertex of the obtuse angles of a rhombus to its sides. The length of each perpendicular is equal to a units. The distance between their feet being equal to b units. The area of the rhombus is

(*a*) $\dfrac{\sqrt{a^2 + b^2}}{2\sqrt{b^2 - a^2}}$ (*b*) $\dfrac{2ab}{2\sqrt{b^2 - a^2}}$

(*c*) $\dfrac{ab^2}{2\sqrt{b^2 - a^2}}$ (*d*) $\dfrac{2a^2b^2}{2\sqrt{b^2 - a^2}}$

48. In a trapezoid $ABCD$, side BC is parallel to side AD. Also, the lengths of the sides AB, BC, CD and AD are 8, 2, 8 and 10 units respectively. Find the radius of the circle that passes through all four of the points A, B, C and D?

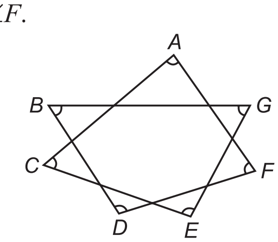
(*a*) $2\sqrt{3}$ (*b*) $2\sqrt{5}$ (*c*) $2\sqrt{11}$ (*d*) $2\sqrt{7}$

49. The length of the midline of a trapezoid equals 4 cm and the base angles are 40° and 50°. The length of the bases if the distance of their mid-points equals 1 cm is equal to

(*a*) 5 cm, 3 cm (*b*) 4 cm, 3 cm
(*c*) 7 cm, 4 cm (*d*) 6 cm, 5 cm

50. In the figure, find the value of

$\angle A + \angle B + \angle C + \angle D + \angle E + \angle F$.



(*a*) 120°

(*b*) 720°

(*c*) 360°

(*d*) 540°

ANSWERS

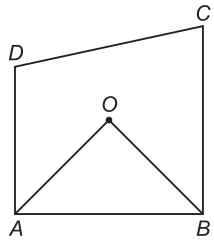
1. (*c*)	**2.** (*b*)	**3.** (*b*)	**4.** (*a*)	**5.** (*b*)	**6.** (*c*)	**7.** (*c*)	**8.** (*a*)	**9.** (*d*)	**10.** (*d*)
11. (*c*)	**12.** (*b*)	**13.** (*a*)	**14.** (*b*)	**15.** (*b*)	**16.** (*a*)	**17.** (*b*)	**18.** (*d*)	**19.** (*c*)	**20.** (*b*)
21. (*a*)	**22.** (*d*)	**23.** (*c*)	**24.** (*d*)	**25.** (*a*)	**26.** (*a*)	**27.** (*c*)	**28.** (*a*)	**29.** (*c*)	**30.** (*d*)
31. (*d*)	**32.** (*b*)	**33.** (*d*)	**34.** (*a*)	**35.** (*b*)	**36.** (*a*)	**37.** (*b*)	**38.** (*b*)	**39.** (*c*)	**40.** (*b*)
41. (*d*)	**42.** (*b*)	**43.** (*c*)	**44.** (*a*)	**45.** (*b*)	**46.** (*c*)	**47.** (*c*)	**48.** (*d*)	**49.** (*a*)	**50.** (*d*)

HINTS AND SOLUTIONS

3. In ΔAOB, $\angle AOB = 180° - \left(\frac{1}{2}\angle A + \frac{1}{2}\angle B\right)$ ...(*i*)

(Angle sum property of a Δ, OA bisects $\angle A$ and OB bisects $\angle B$)

In quadrilateral $ABCD$,



$\angle A + \angle B + \angle C + \angle D = 360°$

$\Rightarrow \angle C + \angle D = 360° - (\angle A + \angle B)$

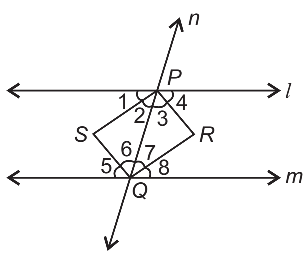
$\Rightarrow \frac{1}{2}(\angle C + \angle D)$

$= 180° - \frac{1}{2}(\angle A + \angle B)$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*)

$$\angle AOB = \frac{1}{2}(\angle C + \angle D).$$

4. Let l and m be the two given parallel lines and n the transversal intersecting them at points P and Q respectively. PS, PR and QS, QR are the bisectors of the interior angles at P and Q respectively.



Given, $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, $\angle 5 = \angle 6$, $\angle 7 = \angle 8$

Also, $\angle 1 + \angle 2 = \angle 7 + \angle 8$, (alternate angles)

$\Rightarrow 2\angle 2 = 2\angle 7 \Rightarrow \angle 2 = \angle 7$ ...(*i*)

Also, $\angle 3 + \angle 4 = \angle 5 + \angle 6$, (alternate angles)

$\Rightarrow 2\angle 3 = 2\angle 6 \Rightarrow \angle 3 = \angle 6$ ...(*ii*)

Adding (*i*) and (*ii*), we get $\angle 2 + \angle 3 = \angle 6 + \angle 7$

$\Rightarrow \angle SPR = \angle SQR$

Also, $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 180°$ (St. $\angle$)

$\Rightarrow 2(\angle 2 + \angle 3) = 180° \Rightarrow \angle 2 + \angle 3 = 90°$

$\Rightarrow \angle SPR = \angle SQR = 90°$

Also, $\angle 1 + \angle 2 + \angle 5 + \angle 6 = 180°$

(co-int. $\angle s$ are supplementary)

$\Rightarrow 2(\angle 2 + \angle 6) = 180° \Rightarrow \angle 2 + \angle 6 = 90°$

$\therefore$ In ΔPSQ, $\angle PSQ = 180° - (\angle 2 + \angle 6)$

$= 180° - 90° = 90°$

Similarly, $\angle PRQ = 90°$.

As all four angles of $PRQS$ are each $= 90° \Rightarrow PRQS$ is a rectangle.

5. In a cyclic quadrilateral $ABCD$,

$\Rightarrow \angle A + \angle C = 180° \Rightarrow \angle A = 180° - \angle C$ ...(*i*)

$\Rightarrow \angle B + \angle D = 180° \Rightarrow \angle B = 180° - \angle D$ ...(*ii*)

Also, given that

$\angle A + \angle B = 2(\angle C + \angle D)$

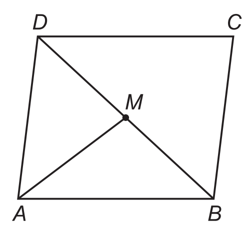
$\Rightarrow 180° - \angle C + 180° - \angle D = 2(\angle C + \angle D)$

$\Rightarrow 360° = 3(\angle C + \angle D)$

$\Rightarrow \angle C + \angle D = 120°$

Given $\angle C > 30° \Rightarrow \angle D = 120° - \angle C \Rightarrow \boldsymbol{\angle D < 90°}$.

6. If BM bisects $\angle B$, then AM bisects $\angle A$ as diagonals of a parallelogram bisect each other and here M is the point of intersection of the diagonals AC and BD. Also, diagonals of a prallelogram bisect the angles at the vertices they join.



$\Rightarrow \angle A + \angle B = 90°$

(Sum of adjacent angles of a parallelogram)

$\Rightarrow \frac{1}{2}(\angle A + \angle B) = 90°$

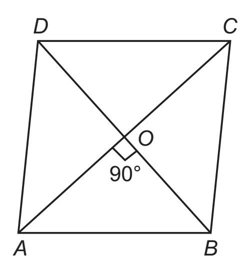
$\Rightarrow \angle MAB + \angle MBA = 90°$

In ΔAMB, $\angle MAB + \angle MBA + \angle AMB = 180°$

$\Rightarrow 90° + \angle AMB = 180° \Rightarrow \angle AMB = \mathbf{90°}$.

7. Diagonals of a rhombus bisect each other at right angles.

$\therefore$ In ΔAOB,



$OA^2 + OB^2 = AB^2$

($\because \angle AOB = 90°$)

$\Rightarrow \left(\frac{1}{2}AC\right)^2 + \left(\frac{1}{2}BD\right)^2 = AB^2$

$\Rightarrow \frac{1}{4}AC^2 + \frac{1}{4}BD^2 = AB^2$

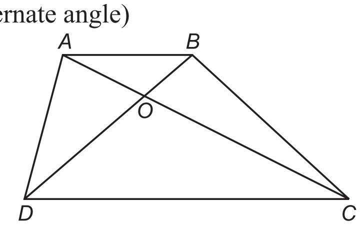
$\Rightarrow AC^2 + BD^2 = 4AB^2$.

8. In ΔAOB and ΔCOD,

$\left.\begin{aligned}\angle BAO &= \angle DCO\\ \angle ABO &= \angle CDO\end{aligned}\right\}$ (alternate angle)

$\angle AOB = \angle COD$

(vert. opp. $\angle s$)



$\Rightarrow \Delta AOB \sim \Delta COD$

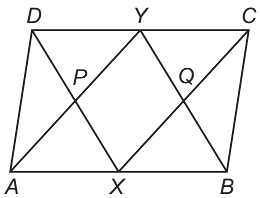
$\therefore \frac{OA}{OB} = \frac{OC}{OD}$

$\Rightarrow \dfrac{OA}{OC} = \dfrac{\mathbf{OB}}{\mathbf{OD}}$.

9. If a trapezium is inscribed in a circle, *i.e.*, it is a cyclic trapezium, then it is an isosceles trapezium, *i.e.*, non-parallel sides are equal $\Rightarrow AB = CD$.

10. Since X and Y are the mid-points of AB and DC respectively.

$AX = \frac{1}{2}AB$ and $CY = \frac{1}{2}DC$



But $AB = DC$

$\Rightarrow \frac{1}{2}AB = \frac{1}{2}DC$

$\Rightarrow AX = CY$

Also, $AB \parallel DC \Rightarrow AX \parallel CY$

Since a pair of opposite sides, AX and YC are equal and parallel, $AXCY$ is a parallelogram. Similarly, we can show that $BXDY$ is a parallelogram.

Now, $AXCY$ is a parallelogram.

$\Rightarrow AY \parallel CX$ (opp. sides are parallel)

$\Rightarrow PY \parallel QX$ ...(*i*)

Also, $BXDY$ is a parallelogram $\Rightarrow BY \parallel XD$

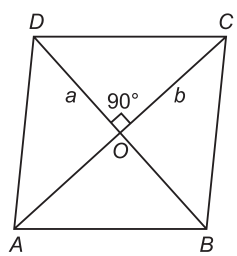
$\Rightarrow QY \parallel PX$ ...(*ii*)

Thus, in quadrilateral $PYQX$, $PY \parallel QX$ and $QY \parallel PX$.

When both pair of opposite sides are parallel in a quadrilateral, it is a prallelogram.

11. Each side of the rhombus

$= \dfrac{8\sqrt{5}}{4} = 2\sqrt{5}$



Let $OD = a$ and $OC = b$ Then, in ΔCOD,

$a^2 + b^2 = (2\sqrt{5})^2 = 20$ ...(*i*)

Also, $2a + 2b = 12$

$\Rightarrow a + b = 6$ ...(*ii*)

$\therefore$ From (*i*), $(a + b)^2 - 2ab = 20$

$\Rightarrow 36 - 2ab = 20 \Rightarrow 2ab = 16 \Rightarrow ab = 8$ ...(*iii*)

Now from (*ii*) and (*iii*), we get $a = 4$, $b = 2$

$\therefore$ Diagonals are 8 cm and 4 cm.

12. $AE \parallel BC$ and $AE = BC \Rightarrow AECB$ is a parallelogram.

(For a parallelogram one pair of opposite sides can be shown equal and parallel)

$\therefore \angle BCE = \angle BAE = 102°$

(Opposite angles of a parallelogram are equal)

$AB = EC$ (Opposite sides of a parallelogram)

$ED = CD = EC$ ($\because AB = ED = CD$)

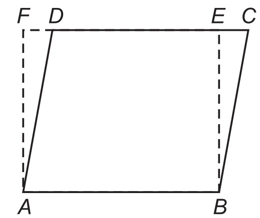
$\therefore \Delta EDC$ is equilateral $\Rightarrow \angle ECD = 60°$

$\therefore \angle BCD = \angle BAE + \angle ECD = 102° + 60° = 162°$.

13. $ABCD$ and $ABEF$ are parallelogram and rectangle respectively on the same base AB having equal perpendicular height. Therefore, the area of both the figures is equal.

Now, $AD > AF$ and $BC > BE$

(hypotenuse > perpendicular)

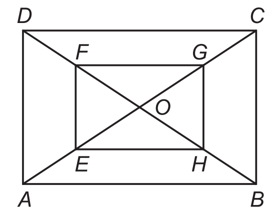


$\therefore 2(AB + BC) > 2(AB + BE)$

$\Rightarrow$ Perimeter of parallelogram > Perimeter of rectangle

$\Rightarrow \boldsymbol{k > 1}$.

14. By mid-point theorem, in Δs ODA, OCD, OBC, OBA respectively



$\dfrac{EF}{AD} = \dfrac{FG}{DC} = \dfrac{GH}{CB} = \dfrac{HE}{BA} = \dfrac{1}{2}$

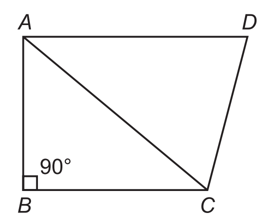
If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = ...,$

then each ratio is equal to $\dfrac{a + c + e + ...}{b + d + f + ...}$.

$\Rightarrow \dfrac{(EF + FG + GH + HE)}{(AD + DC + CB + BA)} = \dfrac{\mathbf{1}}{\mathbf{2}}$.

15. In a cyclic parallelogram each angle is equal to 90° as, sum of opposite angles is 180° and opposite angles are equal. So it is definitely either a square or a rectangle. As the given cyclic parallelogram has unequal adjacent sides, it is a rectangle.

16. Given,



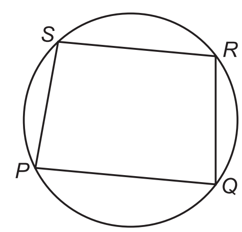
$AD^2 = AB^2 + BC^2 + CD^2$

$= AC^2 + CD^2$

(In ΔABC, $AB^2 + BC^2 = AC^2$)

$\Rightarrow \angle ACD = \mathbf{90°}$.

17. Given, $\angle P = 2\angle R$ ...(*i*)



Also, $\angle P + \angle R = 180°$ ...(*ii*)

$\angle Q + \angle S = 180°$ ...(*iii*)

$\therefore$ From (*i*) and (*ii*)

$\Rightarrow \angle R = 60°$

$\Rightarrow \angle P = 120°$

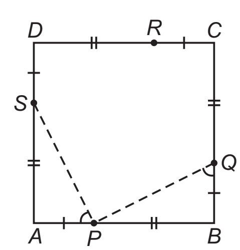
Also, given $\angle Q - \angle S = \frac{1}{3}\angle P = \frac{1}{3} \times 120° = 40°$ ...(*iv*)

Solving (*iii*) and (*iv*) simultaneously, we get

$\angle Q = 110°$, $\angle S = 70°$

$\therefore$ Minimum difference between any two angles of the quadrilateral is 10°.

18. In Δs SAP and PBQ,



$\left.\begin{array}{l} AS = BP \\ AP = BQ \end{array}\right\}$ Given

$\angle A = \angle B = 90°$

$\therefore \Delta SAP \cong \Delta PBQ$ (*SAS*)

$\Rightarrow SP = PQ$

$\angle SPA = \angle BQP = x$ (say)

and $\angle PSA = \angle QPB = y$ (say)

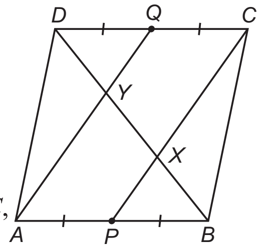
In ΔSAP or ΔPBQ

$x + y = 90°$

$\therefore\ \angle SPQ = 180° - (\angle SPA + \angle BPQ)$

$= 180° - (x + y) = 180° - 90° = \mathbf{90}°.$

19. $ABCD$ is a parallelogram.



$\Rightarrow AB = CD$ and $AB \parallel CD$

$\Rightarrow AP = QC$ and $AP \parallel QC$

$\Rightarrow APCQ$ is a parallelogram.

$\Rightarrow AQ \parallel PC$

$\therefore$ By intercept theorem in ΔDXC,

we have $\dfrac{DY}{YX} = \dfrac{DQ}{QC} = 1$

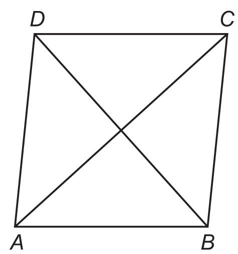
$\Rightarrow DY = YX$

Also using intercept theorem in ΔABY,

$\dfrac{BX}{XY} = \dfrac{BP}{PA} = \dfrac{1}{1} \Rightarrow BX = XY.$

$\therefore\ BX = XY = DY.$

20. Given, $AC = 2$ mn



$BD = m^2 - n^2$

$AB = \dfrac{m^2 + n^2}{2}$

By Apollonius theorem, we know that

$AC^2 + BD^2 = 2(AB^2 + BC^2)$

$\Rightarrow (2\,mn)^2 + (m^2 - n^2)^2 = 2\left\{\dfrac{1}{4}(m^2 + n^2)^2 + BC^2\right\}$

$\Rightarrow 4\,m^2n^2 + m^4 + n^4 - 2m^2n^2 = \dfrac{1}{2}(m^2 + n^2)^2 + 2BC^2$

$\Rightarrow (m^2 + n^2)^2 = \dfrac{1}{2}(m^2 + n^2)^2 + 2BC^2$

$\Rightarrow 2BC^2 = \dfrac{1}{2}(m^2 + n^2)^2 \Rightarrow BC^2 = \dfrac{(m^2 + n^2)^2}{4}$

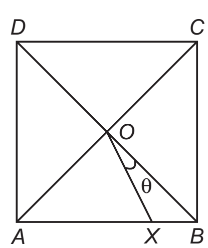
$\Rightarrow BC = \dfrac{m^2 + n^2}{2} = AB$

$\Rightarrow ABCD$ is a rhombus as adjacent sides are equal.

Let $AC > BD \Rightarrow 2mn > m^2 - n^2$

$\Rightarrow (m + n)^2 > 2m^2$, which is always true for every positive integer m, n where $n < m < 2n$.

21. Let $\angle XOB = \theta$.



In ΔOXB,

$\angle XOB + \angle XBO + \angle OXB = 180°$

$\Rightarrow \theta + 45° + \angle OXB = 180°$

$\Rightarrow \angle OXB = 180° - 45° - \theta = 135° - \theta.$

Also, $\angle OXB + \angle OXA = 180°$

$\Rightarrow \angle OXA = 180° - \angle OXB = 180° - (135° - \theta) = 45° + \theta.$

In ΔOAX, $AO = OX$

$\Rightarrow \angle OXA = \angle AOX = 45° + \theta$

Also, we know that diagonals of a square intersect each other at right angles, so

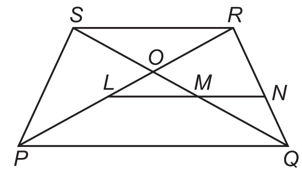
$\Rightarrow \angle AOX + \angle XOB = 90°$

$\Rightarrow 45° + \theta + \theta = 90°$

$\Rightarrow 2\theta = 45° \Rightarrow \theta = \mathbf{22.5°}.$

22. Produce LM to N which intersects side RQ at N.

By mid-point theorem,



In ΔPQR, $LN \parallel PQ$

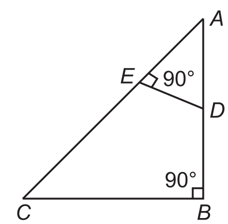
and $LN = \dfrac{1}{2} PQ$ and

in ΔQRS, $MN \parallel RS$

and $MN = \dfrac{1}{2} RS$

$\therefore\ LM = LN - MN = \mathbf{\dfrac{1}{2}\ (PQ - RS)}.$

23. In Δs ABC and ΔAED



$\angle ABC = \angle AED = 90°$

$\angle A = \angle A$

$\therefore\ \angle ABC \sim \angle AED$ (AA similarly)

Also, $\angle DEC = 180° - 90° = 90°.$

In quadrilateral $EDBC$

$\angle E + \angle D + \angle C + \angle B = 360° \Rightarrow \angle C + \angle D = 180°$

$\therefore$ Opposite angles are supplementary, $EDBC$ is cylic quadrilateral.

24. In ΔABX and ΔADZ,

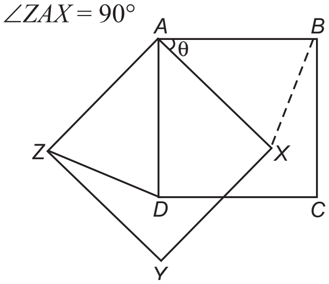
$AB = AD$ (Sides of square $ABCD$)

$AX = AZ$ (Sides of suqare $AXYZ$)

Let $\angle BAX = \theta$

$\therefore\ \angle XAD = 90º - \theta$

Also, $AXYZ$ being a square, $\angle ZAX = 90°$



$\Rightarrow \angle ZAD + \angle XAD = 90°$

$\Rightarrow \angle ZAD = 90° - \angle XAD$

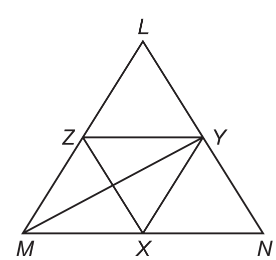
$= 90° - (90° - \theta) = \theta$

$\Rightarrow \angle BAX = \angle ZAD$

$\therefore\ \Delta ABX \cong \Delta ADZ$

$\Rightarrow BX = DZ.$

25. Since $ZY \parallel MN$ and $ZX \parallel YN$,



$XNYZ$ is a parallelogram

$\therefore\ ZX = YN$ ...(i)

Also, $ZX \parallel YN$ and $XY \parallel ZL$

($\because XY \parallel ZM$)

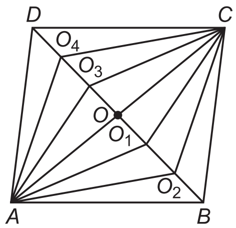
$\Rightarrow XYLZ$ is a parallelogram

$\therefore\ ZX = LY$ ...(ii)

From (i) and (ii), $YN = LY$

$\Rightarrow MY$ is the median of ΔLMN.

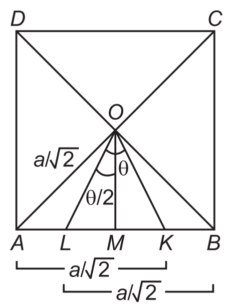
26. In a rhombus, diagonals bisect each other. Suppose in the given rhombus $ABCD$, diagonals bisect each other at point O. Then, the distances of O from the four vertices A, B, C



and D are equal. Thus, even if we take fixed point on diagonal BD as O_1, O_2, O_3, O_4, ..., they all are equidistant from the vertices A and C (by property of congruent triangles). Hence the locus of a point in rhombus $ABCD$ which is equidistant from A and C is a fixed point on diagonal BD.

Alternatively, since diagonals of a rhombus bisect each at rt. $\angle s$, therefore, BD is the single bisector of AC, and therefore, all points equidistant from A and C lie on it.

27. Let each side of the square be a.



Then, $AC = a\sqrt{2}$ and $AO = OC = \dfrac{a}{\sqrt{2}}$

$AM = \dfrac{a}{2}$, $AK = AO = \dfrac{a}{\sqrt{2}}$

$LM = \dfrac{a}{\sqrt{2}} - \dfrac{a}{2}$ and $OM = \dfrac{a}{2}$

In ΔOML, $\tan\dfrac{\theta}{2} = \dfrac{LM}{OL}$

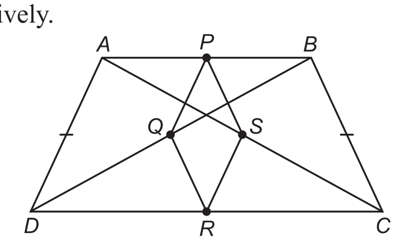
$$= \frac{\dfrac{a}{\sqrt{2}} - \dfrac{a}{2}}{\dfrac{a}{2}} = \frac{\dfrac{\sqrt{2}-1}{2}}{\dfrac{1}{2}} = \sqrt{2}-1$$

Now, $\tan\theta = \dfrac{2\tan\theta/2}{1-\tan^2\theta/2}$ $\left(\because \tan 2x = \dfrac{2\tan x}{1-\tan^2 x}\right)$

$$= \frac{2(\sqrt{2}-1)}{1-(\sqrt{2}-1)^2} = \frac{2(\sqrt{2}-1)}{1-3+2\sqrt{2}} = \frac{2(\sqrt{2}-1)}{2\sqrt{2}-2}$$

$\Rightarrow \tan\theta = 1.$

28. In ΔBDC, Q and R are the mid-points of BD and CD respectively.



$\therefore$ $QR \parallel BC$ and $QR = \dfrac{1}{2}BC$

Similarly, in ΔABC, $PS \parallel BC$ and $PS = \dfrac{1}{2}BC$

$\therefore$ $PS = QR$

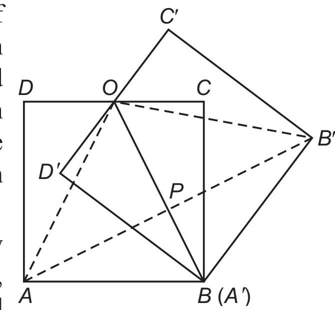
Also in ΔABD, $PQ \parallel AD$ and $PQ = \dfrac{1}{2}AD$

In ΔADC, $SR \parallel AD$ and $SR = \dfrac{1}{2}AD$

$\therefore$ $PQ = SR$

($AD = BC \Rightarrow PS = QR = PQ = SR \Rightarrow PQRS$ is a rhombus.

29. Let O be the mid-point of CD. Since the new position of A coincides with the old position of B, the rotation is in the counter-clockwise direction about O, through the angle AOB.



Let B', C', D' be the new positions of B, C, D, respectively. Also, let OB and AB' intersect at point P. We can also see that $\angle AOB = \angle BOB'$

Also, $OA = OB = OB'$. This means that OB is the angle bisector of $\angle AOB'$ of isosceles triangle AOB'.

This implies that $OP \perp AB'$ and OP bisects AB'

$\Rightarrow$ $AP = PB'$

Area of $\Delta AOB = \dfrac{1}{2}$ (Area of square $ABCD$)

$= \dfrac{1}{2}$ square units ... (ii)

But area of $\Delta AOB = \dfrac{1}{2} \times AP \times OB$ ...(ii) ($\because$ $AP \perp OP$)

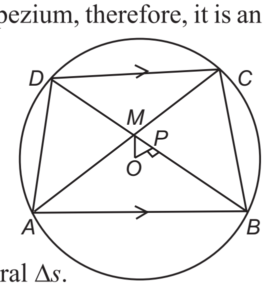
$$OB = \sqrt{OC^2 + CB^2} = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

$\therefore$ From (i) and (ii),

$$\frac{1}{2} = \frac{1}{2} \times AP \times \frac{\sqrt{5}}{2}$$

$\Rightarrow AP = \dfrac{2}{\sqrt{5}}$ $\Rightarrow$ $AB' = 2AP = 2 \times \dfrac{2}{\sqrt{5}} = \dfrac{4}{\sqrt{5}} = \mathbf{\dfrac{4\sqrt{5}}{5}}.$

30. Since the trap. ABCD is cyclic trapezium, therefore, it is an isosceles trapezium.



It is given that,

$\angle AMB = 60°$

$\Rightarrow \angle CMD = 60°$ (vert. opp. $\angle s$)

Since $AM = BM$ and $DM = CM$,

(*Property of Isos. trap.*)

ΔAMB and so ΔCMD are equilateral Δs.

Draw $OP \perp BD$. Now OM bisects $\angle AMB$ and so $\angle OMP = 30°$

In ΔOMP,

$\sin 30° = \dfrac{OP}{OM}$ $\Rightarrow$ $OP = \dfrac{OM}{2} = \dfrac{2}{2} = 1$

$\therefore \angle OPM = 90°$, $\therefore$ $PM = \sqrt{OM^2 - OP^2} = \sqrt{2^2 - 1^2} = \sqrt{3}$

$\therefore$ $AB - CD = AM - CD = BM - MD$

($\because$ $AB = AM$ and $CD = MD$)

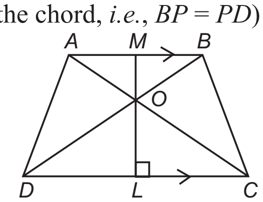
$= (BP + PM) - (PD - PM) = BP + PM - BP + PM$

($\because$ Perpendicular drawn from the centre bisects the chord, *i.e.*, $BP = PD$)

$= 2\,PM = 2 \times \sqrt{3} = \mathbf{2\sqrt{3}}.$

31. $\angle BDC = \angle DBA$

(alternate angles)



$\angle ACD = \angle CAB$

(alternate angles)

$\angle DOC = \angle AOB$ (vert. opp. $\angle s$)

$\Rightarrow \Delta DCO \sim \Delta BAO$

$\Rightarrow \dfrac{\text{Area of } \Delta ABO}{\text{Area of } \Delta DCO} = \dfrac{AB^2}{DC^2}$ (By property of similar triangles)

$\Rightarrow \dfrac{p}{q} = \dfrac{AB^2}{DC^2} \quad \Rightarrow \quad \dfrac{AB}{DC} = \dfrac{\sqrt{p}}{\sqrt{q}}$

$\Rightarrow \; AB = \sqrt{p} \cdot k$ and $DC = \sqrt{q} \cdot k$ for some constant k.

Area of $\Delta DOC = \dfrac{1}{2} \times DC \times OL$

$\Rightarrow q = \dfrac{1}{2} \times \sqrt{q} \cdot k \times OL \quad \Rightarrow OL = \dfrac{2q}{\sqrt{q} \cdot k} = \dfrac{2\sqrt{q}}{k}$

Area of $\Delta AOB = \dfrac{1}{2} \times AB \times OM$

$\Rightarrow p = \dfrac{1}{2} \times \sqrt{p} \cdot k \times OM \quad \Rightarrow \quad OM = \dfrac{2q}{\sqrt{p} \cdot k} = \dfrac{2\sqrt{p}}{k}$

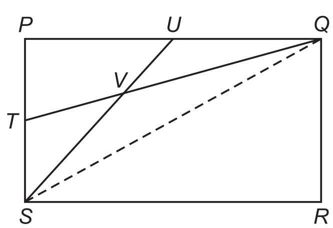
Area of a trapezium = $\dfrac{1}{2}$ × height of trapezium × sum of parallel sides

$= \dfrac{1}{2} \times (OL + OM) \times (DC + AB)$

$= \left(\dfrac{\dfrac{2\sqrt{q}}{k} + \dfrac{2\sqrt{p}}{k}}{2} \right) \left(\sqrt{q}\, k + \sqrt{p}\, k \right)$

$= \boldsymbol{\left(\sqrt{p} + \sqrt{q} \right)^2}$**.**

32. U and T are mid-points of PQ and PS respectively.



$\Rightarrow SU$ and QT are medians of ΔPSQ.

$\Rightarrow \; V$ is the centroid of ΔPSQ.

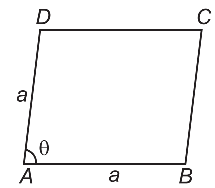
$\therefore \dfrac{\text{Area of quad. } QRSV}{\text{Area of } \Delta PQT} = \dfrac{\text{Area of } \Delta QRS + \text{Area of } \Delta QSV}{\dfrac{\text{Area of } \Delta PQS}{2}}$

$= \dfrac{\text{Area of } \Delta PQS + \dfrac{\text{Area of } \Delta PQS}{3}}{\dfrac{\text{Area of } \Delta PQS}{2}}$

$\left[\begin{array}{l} \because QS \text{ divides rectangle } PQRS \text{ into} \\ \quad \text{two equal } \Delta s \; PQS \text{ and } QRS. \\ \text{Also, Area of } \Delta QSV = \text{Area of } \dfrac{\Delta PQS}{3} \end{array} \right]$

$= \dfrac{\dfrac{4}{3}}{\dfrac{1}{2}} = \boldsymbol{\dfrac{8}{3}}$**.**

33. Let acute the angle between adjacent sides of the rhombus be θ and let each side of the rhombus = $\boldsymbol{a}$ units. Then,



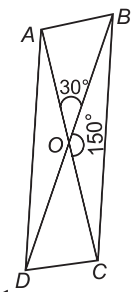
Area of rhombus $ABCD = a^2 \sin \theta$

Given, area = $\dfrac{1}{2}$ and $a = 1$

$\therefore \; \dfrac{1}{2} = 1 \times \sin \theta$

$\Rightarrow \quad \sin \theta = \dfrac{1}{2} \quad \Rightarrow \quad \theta =$ **30°.**

34. Area of quad. $ABCD$



= Area (ΔAOB) + Area (ΔBOC) + Area (ΔCOD) + Area (ΔDOA)

$= \dfrac{1}{2} \cdot OA \cdot OB \sin 30° + \dfrac{1}{2} \; OB \cdot OC \sin 150°$

$+ \dfrac{1}{2} \; OC \cdot OD \sin 30° + \dfrac{1}{2} \; OD \cdot OA \sin 150°$

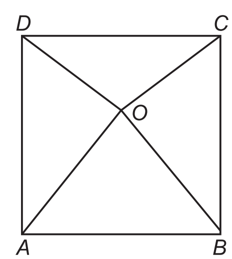
$= \dfrac{1}{4} \; OA \cdot OB + \dfrac{1}{4} \; OB \cdot OC + \dfrac{1}{4} \; OC \cdot OD + \dfrac{1}{4} \; OD \cdot OA$

$\left(\because \sin 30° = \sin 150° = \dfrac{1}{2} \right)$

$= \dfrac{1}{4} \; (OA + OC)\,(OB + OD) = \dfrac{1}{4} \times AC \times BD$

$= \dfrac{1}{4} \times 24 \times 22 =$ **132.**

35. ΔAOB is equilateral



$\Rightarrow AO = OB = AB$ and

$\angle OAB = \angle OBA = \angle AOB = 60°$

Also, $ABCD$ being a square,

$AB = BC = CD = AD$

$\Rightarrow \quad AO = AD \quad$ and $\quad BO = BC$

$\angle DAO = \angle CBO = 30° \quad (\because \; \angle OAB = \angle OBA = 60°)$

In ΔADO, $AO = AD \quad \Rightarrow \quad \angle ADO = \angle AOD$

$= \dfrac{1}{2} \; (180° - \angle DAO) = \dfrac{1}{2} \; (180° - 30°) = 75°$

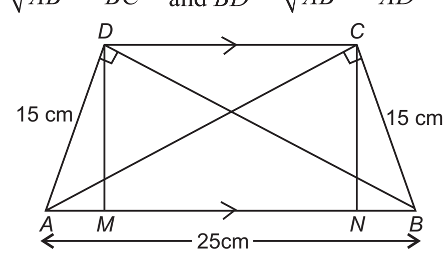
Similarly, $\angle BOC = 75°$.

$\therefore \; \angle DOC = 360° - (\angle AOB + \angle AOD + \angle BOC)$

$= 360° - (60° + 75° + 75°) = 360° - 210° =$ **150°.**

36. By Pythagoras' Theorem,

$AC = \sqrt{AB^2 - BC^2}$ and $BD = \sqrt{AB^2 - AD^2}$



$\Rightarrow \; AC = BD = \sqrt{625 - 225} = \sqrt{400} = 2$ cm

(By prop.of isos. trap. AC = BD)

Now, Area of $\Delta DAB = \frac{1}{2} \times AD \times BD$

Also, Area of $\Delta DAB = \frac{1}{2} \times DM \times AB$

$\therefore\ AD \times BD = DM \times AB$

$\Rightarrow\ DM = \frac{AD \times BD}{AB} = \frac{15 \times 20}{25} = 12$ cm

Also, $CN = DM = 12$ cm

$AM = \sqrt{AD^2 - DM^2} = \sqrt{15^2 - 12^2} = \sqrt{225 - 144}$

$= \sqrt{81} = 9$ cm

Also, $BN = AM = 9$ cm

$MN = AB - (AM + BN) = 25 - (9 + 9) = 25 - 18 = 7$ cm

$\Rightarrow CD = MN = 7$ cm

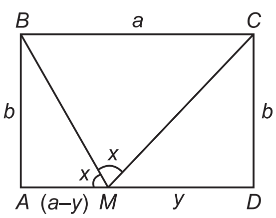
$\therefore$ Area of trapezium $ABCD = \frac{1}{2} \times DM \times (AB + CD)$

$= \frac{1}{2} \times 12 \times (25 + 7) = \frac{1}{2} \times 12 \times 32 =$ **192 cm²**.

37. Let $\angle BMA = \angle BMC = x$

Then $\angle CMD = \pi - 2x$

Let $MD = y$. Then $AM = (a - y)$



In ΔCMD,

$\tan(\pi - 2x) = \frac{b}{y}$ ...(*i*)

In ΔBAM,

$\tan x = \frac{b}{a - y}$ ...(*ii*)

From (*i*), we have $\tan(\pi - 2x) = -\tan 2x = \frac{b}{y}$

$\Rightarrow -\frac{2\tan x}{1 - \tan^2 x} = \frac{b}{y}$

$\Rightarrow -\frac{2b/(a-y)}{1 - \left(b/(a-y)\right)^2} = \frac{b}{y}$

$\Rightarrow -\frac{2b(a-y)}{(a-y)^2 - b^2} = \frac{b}{y}$

$\Rightarrow -\frac{2ba - 2by}{a^2 + y^2 - 2ay - b^2} = \frac{b}{y}$

$\Rightarrow -2bya + 2by^2 = ba^2 + by^2 - 2aby - b^3$

$\Rightarrow by^2 = b(a^2 - b^2)$

$\Rightarrow\ y = \boldsymbol{\sqrt{a^2 - b^2}}$**.**

38. Let each side of the square $= x$ units

Let $AM = BN = CR = y$ units

$\Rightarrow MB = CN = DR$

$= (x - y)$ units

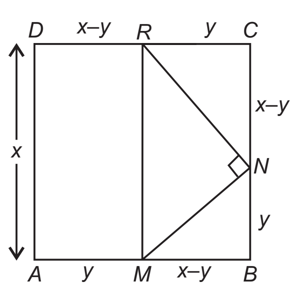
In Δs MBN and NCR

$MB = NC = (x - y)$

$BN = CR = y$

$\angle MBN = \angle NCR = 90°$

$\Rightarrow \Delta MBN \cong \Delta NCR$



$\Rightarrow MN = NR$

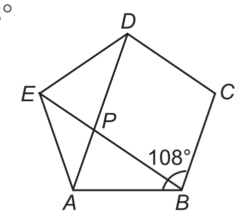
Given, $\angle MNR = 90°$

$\Rightarrow\ \Delta MNR$ is an isosceles right angled triangle.

$\Rightarrow \angle MRN = \angle NMR =$ **45°.**

39. Sum of the interior angles of a regular pentagon = 540°

$\Rightarrow$ Each interior angle $= \frac{540°}{5} = 108°$



As $AD \parallel BC$

$\angle BAD + \angle ABC = 180°$ (co-int. $\angle s$)

$\Rightarrow \angle BAD = 180° - 108° =$ **72°**

$\Rightarrow \angle PAE = 108° - 72° = 36°$

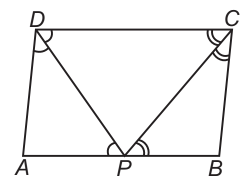
Also, applying $BE \parallel DC$, $\angle BED = 72° \Rightarrow \angle PEA = 36°$

$\therefore\ \angle APE = 180° - (\angle PAE + \angle PEA)$

(Angle sum property of a Δ)

$= 180° - (36° + 36°) = 180° - 72° =$ **108°.**

40. Given, DP bisects $\angle CDA$



$\Rightarrow \angle CDP = \angle ADP$

$\Rightarrow \angle APD = \angle CDP$

(alt. $\angle s$ $CD \parallel AB$)

$\Rightarrow AD = AP$ (Isosceles Δ property)

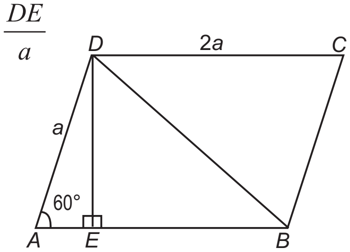
Similarly, $\angle CPB = \angle PCD = \angle PCB$

$\Rightarrow BP = BC$

$DC = AB = AP + BP = AD + BC =$ **2*BC*.**

41. In ΔDAE,

$\cos 60° = \frac{AE}{a}$, $\sin 60° = \frac{DE}{a}$



$\Rightarrow\ AE = \frac{1}{2}a,\ DE = \frac{\sqrt{3}}{2}a$

$\therefore\ BE = BA - AE$

$= 2a - \frac{1}{2}a = \frac{3a}{2}$

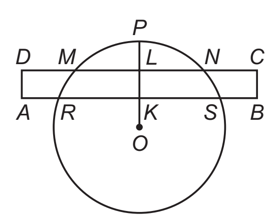
$\therefore$ In right angled ΔBDE,

$BD = \sqrt{BE^2 + DE^2} = \sqrt{\left(\frac{3a}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}a\right)^2}$

$= \sqrt{\frac{9a^2}{4} + \frac{3a^2}{4}} = \sqrt{\frac{12a^2}{4}} = \boldsymbol{\sqrt{3}a}$**.**

42. $OS = OP = ON$ = radii of the circle = 5 cm

$(OS)^2 = (OK)^2 + (KS)^2$



$\Rightarrow\ 25 = (OK)^2 + 16$

$\Rightarrow\ (OK)^2 = 25 - 16 = 9$

$\Rightarrow\ OK = 3$ cm

$LC = KB$

$\Rightarrow LN + NC = KS + SB$

$\Rightarrow\ LN + 2 = (4 + 1)$ cm $\Rightarrow\ LN = 3$ cm

Similarly, $(ON)^2 = (OL)^2 + (LN)^2$

$\Rightarrow\ 25 = (OL)^2 + 9$

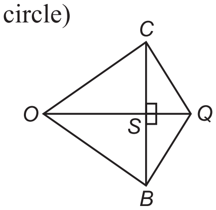
$\Rightarrow \quad (OL)^2 = 16 \quad \Rightarrow \quad OL = 4$ cm

$\therefore \quad LK = OL - OK = (4 - 3)$ cm $= 1$ cm

Also, $\quad AB = DC = 10$ cm

$\therefore$ Area of rectangle $ABCD = AB \times BC = 10$ cm $\times$ 1 cm $=$ **10 cm²**.

43. Let $OQ = OB = OC = r$ (radius of the circle)

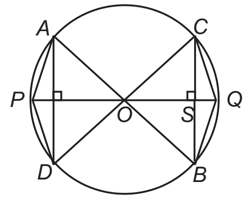


Given, $\angle AOD = \angle BOC = 120°$

$\therefore \angle BOQ = \angle COQ = 60°$

$\Rightarrow \dfrac{SB}{OB} = \sin 60° = \dfrac{\sqrt{3}}{2}$

$\Rightarrow SB = \dfrac{r\sqrt{3}}{2}$



$\therefore BC = 2\,SB = r\sqrt{3}$

$\therefore$ Area of quadrilateral $BQCO$

$= \dfrac{1}{2} \times$ product of diagonals

$= \dfrac{1}{2} \times BC \times OQ$

$= \dfrac{1}{2} \times r\sqrt{3} \times r = \dfrac{r^2\sqrt{3}}{2}$ cm^2

$\therefore$ Area of both the quadrilaterals

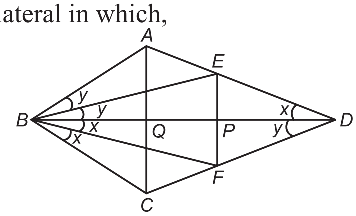
$= 2 \times \dfrac{r^2\sqrt{3}}{2} = \boldsymbol{r^2\sqrt{3}}$ **cm²**.

44. Given, $ABCD$ is a quadrilateral in which,

(1) $AB = BC$

(2) $\angle ABD = 2\angle BDC$

(3) $\angle DBC = 2\angle ADB$



Let BE and BF be the bisectors of $\angle ABD$ and $\angle CBD$ respectively.

Then, $\angle ABE = \angle EBD = \angle BDF = y$ and

$\angle CBF = \angle FBD = \angle BDE = x$

Now, $\angle EDB = \angle FBD$ and $\angle EBD = \angle BDF$

$\Rightarrow$ alternate angles are equal $\Rightarrow ED \parallel BF$ and $BE \parallel FD$

$\Rightarrow BEDF$ is a parallelogram.

Let EF and BD cut each other at P and AC and BD cut each other at Q

$\therefore$ diagonals of a parallelogram bisect each other,

$EP = PF$

Now by the bisector theoram

$\dfrac{AE}{ED} = \dfrac{AB}{BD} = \dfrac{BC}{BD} = \dfrac{CF}{FD}$ $\quad (\because AB = BC)$

$\Rightarrow \dfrac{AE}{ED} = \dfrac{CF}{FD}$

$\Rightarrow$ By the converse of Basic proportionality theoram,

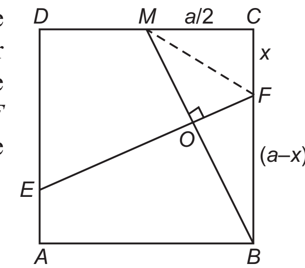
$EF \parallel AC$

$\therefore EP = PF \quad \Rightarrow \quad AQ = QC$

$\therefore \dfrac{AB}{AQ} = \dfrac{BC}{QC} \quad \Rightarrow \quad BQ$ bisects $\angle ABC \quad \Rightarrow \quad x = y$

$\Rightarrow \Delta ABD \cong \Delta BCD \quad \Rightarrow \quad$ **$AD = CD$.**

45. By physically folding the square sheet of paper as per the conditions stated in the question, we can see that EF is the crease along which the fold is made. Also,



$MF = FB$ and $MB \perp EF$.

We need to find $BF : FC$.

Let each side of the square be a units and $CF = x$ units

Then, $BF = (a - x)$ units and $MF = (a - x)$ units.

In ΔMFC, $MF^2 = MC^2 + FC^2$

$\Rightarrow FB^2 = MC^2 + FC^2$ $\quad$ (Since $MF = FB$)

$\Rightarrow (a - x)^2 = (a/2)^2 + x^2$

$\Rightarrow a^2 + x^2 - 2ax = a^2/4 + x^2$

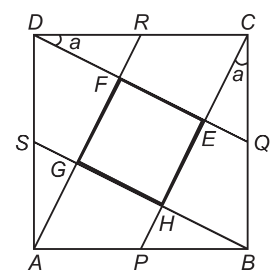
$\Rightarrow 2ax = a^2 - \dfrac{a^2}{4} = \dfrac{3a^2}{4}$

$\Rightarrow x = \dfrac{3a}{8} \quad \Rightarrow \quad a - x = a - \dfrac{3a}{8} = \dfrac{5a}{8}$

$\therefore BF : FC = (x - a) : x$

$= \dfrac{5a}{8} : \dfrac{3a}{8} =$ **5 : 3.**

46. The quadrilateral formed by joining AR, BS, CP and DQ is $EFGH$.



Now, $\Delta DCQ \cong \Delta PBC$ as,

$CD = BC$ $\quad$ (sides of a square)

$\angle DCQ = \angle PBC = 90°$

$CQ = PB$

(Half of sides of a square)

$\therefore \angle CDQ = \angle BCP = a$ (say)

Then, $\angle DQC = \angle EQC = 90° - a$

$\therefore$ In ΔEQC, $\angle CEQ = 180° - (\angle QCE + \angle CQE)$

$= 180° - (a + 90° - a) = 90°$

$\Rightarrow \angle FEH = 90°$

Similarly, $\angle EFG = \angle FGH = \angle GHE = 90°$

$\Rightarrow$ Vertex angles of quadrilateral $EFGH$ are 90° each

$\Rightarrow EFGH$ is a rectangle.

Now, $\Delta DRF \cong \Delta CQE \quad \Rightarrow \quad RF = EQ,\ DF = EC$

Following the same concept,

$BH = CE = AG = DF$ and

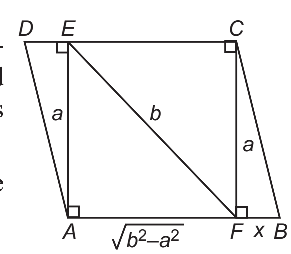
$RF = EQ = PH = GS$

$\therefore EF = FG$ $\quad$ (Adjacent sides of $EFGH$ are equal)

$\therefore$ A rectangle whose adjacent sides are equal is a square

$\Rightarrow EFGH$ is a square.

47. Let AE and CF be the perpendiculars drawn from vertex A and vertex C respectively on sides CD and AB.



$\Rightarrow E$ and F are the feet of the perpendiculars on the sides.

Given, $AE = CF = a$ and $EF = b$.

$$AF = EC = \sqrt{b^2 - a^2}$$

$\therefore$ Area of $AFCE$ = AE × AF = $a \times \sqrt{b^2 - a^2}$

Let $FB = x$. Then, $CB = \sqrt{a^2 + x^2}$

As $ABCD$ is a rhombus,

$$CB = AB$$

$$\Rightarrow \quad \sqrt{a^2 + x^2} = \sqrt{b^2 - a^2} + x$$

$$\Rightarrow \quad a^2 + x^2 = b^2 - a^2 + x^2 + 2x\sqrt{b^2 - a^2}$$

$$\Rightarrow \quad 2a^2 - b^2 = 2x\sqrt{b^2 - a^2}$$

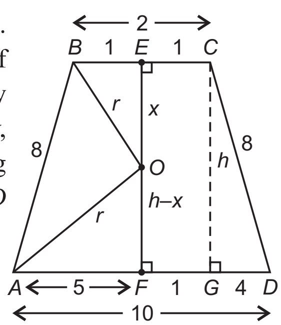
$$\Rightarrow x = \frac{2a^2 - b^2}{2\sqrt{b^2 - a^2}}$$

$\therefore$ Area of rhombus $ABCD$ = base × height = $AB \times AE$

$$= AB \times a = \left(\sqrt{b^2 - a^2} + \frac{2a^2 - b^2}{2\sqrt{b^2 - a^2}}\right) \times a$$

$$= \left(\frac{2(b^2 - a^2) + 2a^2 - b^2}{2\sqrt{b^2 - a^2}}\right) a = \frac{\boldsymbol{ab^2}}{\boldsymbol{2\sqrt{b^2 - a^2}}}.$$

48. Let $ABCD$ be the given trapezoid. Let E and F be the mid-points of the sides BC and AD respectively of trapezoid $ABCD$. By symmetry, the centre of the circle passing through the points A, B, C and D lies on EF. Let the centre be O.



Then, $OB = OA = r$ (say) (radii of the circle)

Let h be the height of the altitude EF of the trapezoid $ABCD$. Then, as we can see in the diagram, in ΔCGD,

$$h^2 + 4^2 = (8)^2 \quad \Rightarrow \quad h^2 = 64 - 16 = 48 \quad \Rightarrow \quad h = 4\sqrt{3}$$

Now let $EO = x$, so $OF = h - x = 4\sqrt{3} - x$

In ΔOBE, $OB = r = \sqrt{OE^2 + BE^2} = \sqrt{x^2 + 1}$ ...(*i*)

In ΔOAF, $OA = r = \sqrt{OF^2 + AF^2} = \sqrt{(4\sqrt{3} - x)^2 + 5^2}$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*)

$$\sqrt{x^2 + 1} = \sqrt{(4\sqrt{3} - x)^2 + 5^2}$$

$$\Rightarrow x^2 + 1 = 48 - 8\sqrt{3}x + x^2 + 25$$

$$\Rightarrow 8\sqrt{3}x = 72 \quad \Rightarrow \quad x = \frac{72}{8\sqrt{3}} = 3\sqrt{3}$$

$$\therefore r = \sqrt{x^2 + 1} = \sqrt{(3\sqrt{3})^2 + 1} = \sqrt{28}$$

$$\Rightarrow r = \mathbf{2\sqrt{7}}.$$

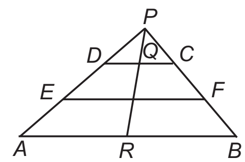
49. Let $ABCD$ be the given trapezoid, whose base angles $\angle A$ and $\angle B$ equal 40° and 50°. Also mid-line EF = 4 cm.

Extend AD and BC to meet at P.

Now in ΔAPB,

$\angle APB = 180° - (40° + 50°) = 90°$

Also, $\Delta ABP \sim \Delta DCP$



$$\Rightarrow \frac{AB}{DC} = \frac{AP}{DP} = \frac{BP}{CP} \quad ...(i)$$

Let Q and R be the mid-points of DC and AB respectively.

$\therefore$ $DC = 2DQ$ and $AB = 2AR$

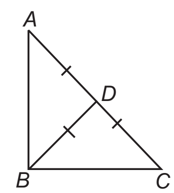
$\therefore$ From (*i*)

$$\frac{AB}{DC} = \frac{AP}{DP} = \frac{BP}{CP} \quad \Rightarrow \quad \frac{2\,AR}{2\,DQ} = \frac{AP}{DP} = \frac{BP}{CP}$$

$$\Rightarrow \frac{AR}{DQ} = \frac{AP}{DP} = \frac{BP}{CP}$$

$\Rightarrow \Delta PDQ$ and ΔPAR are similar $\Rightarrow$ P, Q, R are collinear.

Now, as ΔAPB and ΔDPC are right angled Δs



$AR = RB = PR$ and $DQ = QC = PQ$.

[If D is the mid-pt. of hyp. AC, then $BD = AD = CD$]

$$\therefore \quad \frac{AB}{2} - \frac{DC}{2} = AR - DQ = PR - PQ = QR = 1$$

$$\Rightarrow AB - CD = 2 \quad ...(i)$$

$$\text{Also, } \frac{AB + CD}{2} = 4 \quad \Rightarrow \quad AB + CD = 8 \quad ...(ii)$$

Solving (*i*) and (*ii*) simultaneously, we get

AB = **5** and CD = **3.**

50. Sum of the exterior angles of a polygon = 360°

Sum of all the angles marked **'*x*'**

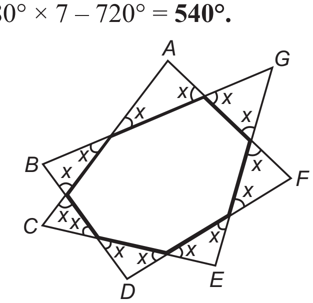
= 2 × Sum of ext. $\angle s$ of the polygon

= 2 × 360° = 720°

$\therefore$ $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G$

= Sum of angles of the 7 triangles – 720°

= 180° × 7 – 720° = **540°.**



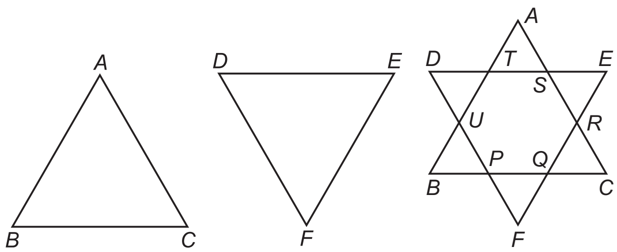
SELF ASSESSMENT SHEET

1. Let ABC be a right angled triangle with AC as its hypotenuse. Then,

(*a*) $AC^3 > AB^3 + AC^3$ (*b*) $AC^2 > AB^2 + BC^2$

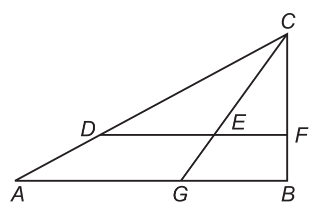
(*c*) $AC > AB + BC$ (*d*) None of these

2. There are two congruent triangles each with area 198 cm^2. Triangle DEF is placed over triangle ABC in such a way that the centroid of both the triangles coincide with each other and $AB \parallel DE$ as shown in the figure forming a star. What is the area of the common region $PQRSTU$?



(*a*) 49 cm^2 (*b*) 66 cm^2 (*c*) 132 cm^2 (*d*) 162 cm^2

3. In the given figure, $DF \parallel AB$, $BC \perp AB$, $BC = 5$, $BG = 4$, $BA = 12$, $DA = 3$. Then CE is:



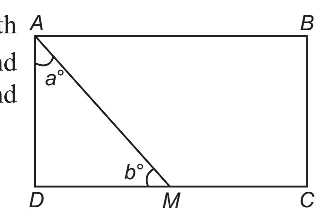
(*a*) 5.44

(*b*) 0.54

(*c*) 4.9

(*d*) 0.42

4. If $ABCD$ is a rectangle with $AB = 4$ and $BC = 2\sqrt{3}$ and M is the mid-point of CD, find $b - a$.



(*a*) 15° (*b*) 20°

(*c*) 45° (*d*) 30°

5. OD, OE and OF are the perpendicular bisectors to the three sides of the triangle ABC. What is the relationship between $m\,\angle BAC$ and $m\,\angle BOC$?

(*a*) $m\,\angle BAC = 180° - m\,\angle BOC$

(*b*) $m\,\angle BOC = 2m\,\angle BAC$

(*c*) $m\,\angle BOC = 90° + \frac{1}{2}\,m\,\angle BAC$

(*d*) $m\,\angle BOC = 90° + \frac{1}{2}\,m\,\angle BAC$

6. In a ΔABC, $CA = CB$. On CB square $BCDE$ is constructed away from the triangle. If x is the number of degrees in angle DAB, then

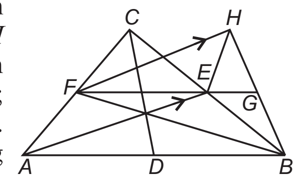
(*a*) x depends upon ΔABC

(*b*) x is independent of the triangle

(*c*) x may equal $\angle CAD$

(*d*) x is greater than 45° but less than 90°.

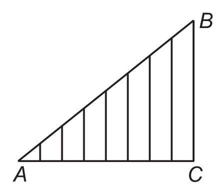
7. Given triangle ABC with medians AE, BF, CD; FH parallel and equal in length to AE; BH and HE are drawn; FE extended to meet BH in G. Which one of the following statements is not necessarily correct?



(*a*) $HE = HG$ (*b*) $BH = DC$

(*c*) $FG = \frac{3}{4}AB$ (*d*) FG is the median of ΔBFH.

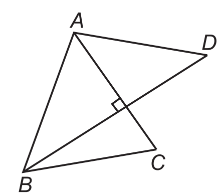
8. Side AC of a right triangle ABC is divided into 8 equal parts. Seven line segments parallel to BC are drawn to AB from the points of division. If $BC = 10$, then the sum of the lengths of the seven line segments is



(*a*) 33 (*b*) 34

(*c*) 35 (*d*) 45

9. Given $AB = AC = BD$, $BD \perp AC$ in the figure shown alongside, the sum of the measures of angles C and D is



(*a*) 120° (*b*) 140°

(*c*) 130° (*d*) 135°

10. Constructed externally on the sides AB, AC of ΔABC are equilateral triangle ABX and ACY. If P, Q, R are the mid-points of AX, AY and BC respectively, then ΔPQR is

(*a*) right angle (*b*) equilateral

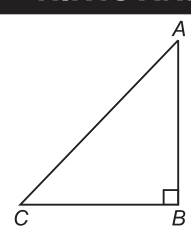
(*c*) isosceles (*d*) None of these

ANSWERS

1. (*a*)	**2.** (*c*)	**3.** (*c*)	**4.** (*d*)	**5.** (*b*)	**6.** (*b*)	**7.** (*a*)	**8.** (*c*)	**9.** (*d*)	**10.** (*b*)

HINTS AND SOLUTIONS

1. In a right angled triangle, hypotenuse is the longest side, so in ΔABC, $AC > BC$ and $AC > AB$



$AC^3 = AC^2 \cdot AC$

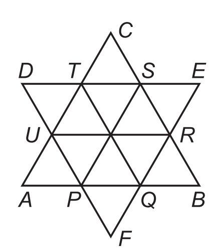
$= (AB^2 + BC^2) \cdot AC$

$= AB^2 \cdot AC + BC^2 \cdot AC$

$> AB^2 \cdot AB + BC^2 \cdot BC \quad \because \; AC > BC, AC > AB$

$\Rightarrow$ $\boldsymbol{AC^3 > AB^3 + BC^3}$.

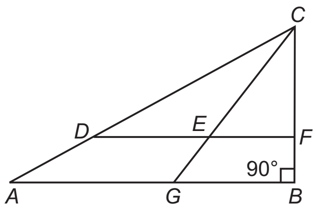
2. There are 12 similar triangles in the figure forming a star, each with equal area. But a larger triangle ABC (or DEF) has only 9 smaller triangles. Out of the 9 triangles only 6 triangles are common.



$\therefore$ Area of common region $PQRSTU$

$= \frac{6}{9} \times 198 \text{ cm}^2 = \mathbf{132\ cm^2.}$

3. By Pythagoras' Theorem,



$GC^2 = (BC)^2 + (GB)^2$

$= 25 + 16 = 41$

$\Rightarrow GC = \sqrt{41}$

$AC^2 = (BC)^2 + (AB)^2$

$= 24 + 144 = 169$

$\Rightarrow AC = 13$

$\therefore DC = AC - AD = 13 - 3 = 10$

$\Delta DEC \sim \Delta AGC$ (AA similarly $\because DF \parallel AB$)

$\therefore \frac{DC}{AC} = \frac{CE}{GC}$

$\Rightarrow \frac{10}{13} = \frac{CE}{\sqrt{41}} \Rightarrow CE = \frac{10\sqrt{41}}{13} = \mathbf{4.9}$ **(approx)**

4. $CD = AB = 4 \Rightarrow DM = 2$

$AD = BC = 2\sqrt{3}$

We know that for ΔADM, $\angle ADM = 90°$, $AD = 2\sqrt{3}$, $DM = 2$

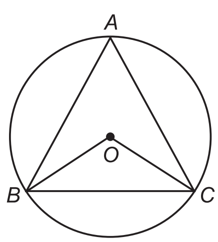
$\Rightarrow AM = \sqrt{AD^2 + DM^2} = \sqrt{12+4} = \sqrt{16} = 4$

Now By the $30° - 60° - 90°$ triangle theorem, the side half the hypotenuse is opposite the 30° angle and the side $\frac{\sqrt{3}}{2}$ times the hypotenuse is opposite the 60° angle.

$\therefore \Delta ADB$ is a $30° - 60° - 90°$ triangle.

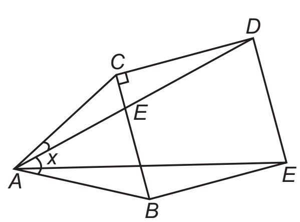
$\Rightarrow b = 60°, a = 30° \Rightarrow b - a = 60° - 30° = \mathbf{30°.}$

5. O being the point of intersection of the perpendicular bisectors of the sides of ΔABC, it is the circumcentre of ΔABC.



$\therefore \angle BOC = 2\angle BAC$ (Angle subtended by an arc at the centre of a circle is twice the angle subtended by it at any other point on the remaining part of the circumference)

6. In ΔABC, $CA = CB$



$\Rightarrow \angle CBA = \angle CAB = \angle A$

$\therefore \angle ACB = 180° - 2\angle A$

$\therefore \angle ACD = 180° - 2\angle A + 90°$

$= 270° - 2\angle A$

$\because AC = AB = CD$

$\Rightarrow AC = CD$

$\Rightarrow$ In ΔCAD, $\angle CDA = \angle CAD$

$= \frac{1}{2}(180° - (270° - 2\angle A)) = \angle A - 45°.$

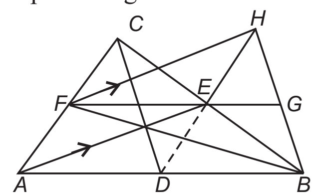
$= \angle A - x$ ($\therefore \angle CAE = \angle CAB - \angle EAB = \angle A - x$)

$\Rightarrow x = 45°$

So X is independent of the triangle.

7. $FH = AE$ and $FH \parallel AE$

$\Rightarrow AEHF$ is a parallelogram.



$\Rightarrow EH \parallel AC$, when extended meets AB in D. In congruent Δs ACD and HDB, $DC = BH$ (corresponding sides)

$\therefore$ (b) is true.

Now $FG = FE + EG$

$= \frac{1}{2}AB + \frac{1}{2}DB$ (By mid-point theorem in Δs ABC and HDB respectively)

$= \frac{1}{2}AB + \frac{1}{2} \times \frac{1}{2}AB$ ($\because D$ is the mid-point of AB)

$= \frac{3}{4}AB$

(c) is true.

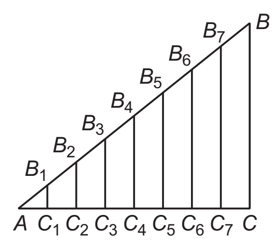
(d) $FE \parallel AB$, when extended to $G \Rightarrow EG \parallel AB$

$\Rightarrow G$ is the mid-point of HB (By converse of mid-point theorem)

$\therefore$ In ΔBFH, FG is the median, so (d) is also true.

(e) Cannot be proved true with the given information.

8. All the triangles formed with A as vertex of AB_1C_1, AB_2C_2, AB_3C_3, AB_4C_4, AB_5C_5, AB_6C_6, AB_7C_7, and ABC are similar.



Let h_k be the length of the side parallel to BC at a distance $k/8$. AC from A,

where $k = 1, 2, ..., 7$, *i.e.*,

$B_1C_1 = h_1 = \frac{AC}{8}$, $B_2C_2 = h_2 = \frac{2}{8}AC$, $B_3C_3 = h_3 = \frac{3}{8}AC$,

............., $B_7C_7 = h_7 = \frac{7}{8}AC$

$\therefore \Delta AB_1C_1 \sim \Delta ABC$, their sides are proportional

$\Rightarrow \frac{B_1C_1}{AC_1} = \frac{BC}{AC} \Rightarrow \frac{h_1}{1/8\,AC} = \frac{10}{AC}$

Similarly, $\Delta AB_2C_2 \sim \Delta ABC$, $\Delta AB_3C_3 \sim \Delta ABC$, ..., $\Delta AB_7C_7 \sim \Delta ABC$

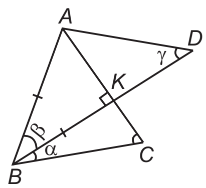
$\therefore$ In general, $\frac{h_k}{k/8\,AC} = \frac{10}{AC} \Rightarrow h_k = \frac{10\,k}{8}$

$\therefore$ Required sum $= h_1 + h_2 + h_3 + ... + h_7$

$= \frac{10}{8}(1 + 2 + 3 + ... + 7) = \frac{10}{8} \times \frac{7}{2} \times (1 + 7) = 35$

$\left(\because \text{Sum of } n \text{ terms} = \frac{n}{2}(a + l)\right)$

9. Let AC and BD intersect at point K and $\angle ABK = \beta$, $\angle KBC = \alpha$, $\angle ADK = \gamma$.



Then, In rt. ΔAKB, $\angle BAK = 90° - \beta$

Also, in rt. ΔAKD, $\angle DAK = 90° - \gamma$

$\therefore AB = BD$

$\therefore \angle BAD = \angle BDA$

$\Rightarrow (90° - \beta) + (90° - \gamma) = \gamma$

$\Rightarrow 180° - \beta - \gamma = \gamma$

$\Rightarrow 2\gamma = 180° - \beta \quad \Rightarrow \quad \gamma = 90° - \beta/2.$ ...(*i*)

In rt. ΔBKC, $\angle C = 90° - \alpha = \angle ABC = \alpha + \beta$ ($\because AB = AC$)

$\therefore 90° - \alpha = \alpha + \beta \quad \Rightarrow \quad 2\alpha = 90° - \beta$

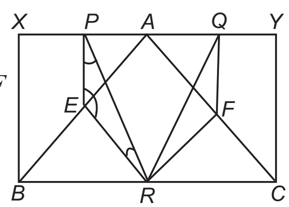
$\Rightarrow \alpha = 45° - \beta/2.$ ...(*ii*)

$\angle C + \angle D = (90° - \alpha) + \gamma = 90° - (45° - \beta/2) + 90° - \beta/2$

From (*i*) and (*ii*)

$= 45° + \beta/2 + 90° - \beta/2 =$ **135°.**

10. Let E and F be the mid-points of AB and AC respectively. Then, by mid-point theorem,



In $\Delta BAC,$

$ER \parallel AC$ and $ER = \frac{1}{2}AC = AF$

In $\Delta CAB,$

$FR \parallel AB$ and $FR = \frac{1}{2}AB = EA$

$\therefore AERF$ is a parallelogram.

$\therefore \Delta ABX$ being an equilateral Δ and $AERF$ a parallelogram, $PE = AE = RF.$

Similarly for ΔACY and ||gm $AERF$

$QF = AF = ER$

Also, $\angle PER = \angle PEA + \angle AER$

$= 60° + \angle AER$ ...(*i*)

(opp. $\angle s$ of a prallelogram are equal)

$= 60° + \angle AFR = \angle QFR$ ...(*ii*)

$\therefore PE = RF$, $QF = ER$ and $\angle PER = \angle QFR$ ...(*iii*)

$\Rightarrow \Delta EPR \cong \Delta QRF$

$\Rightarrow PR = QR$ and $\angle EPR = \angle QRF$

Also, $\angle PRQ = \angle ERF - \angle ERP - \angle QRF$ ...(*iv*)

$= 180° - \angle AER - \angle ERP - \angle QRF$

$= 180° - \angle AER - \angle ERP - \angle EPR$ (From (*iv*)

$= 180° - \angle AER - (\angle ERP + \angle EPR)$

$= 180° - \angle AER - (180° - \angle PER)$

$= \angle PER - \angle AER =$ **60°.**

Also, $PR = QR \quad \Rightarrow \quad \angle RPQ = \angle RQP = \frac{1}{2}(180° - \angle PRQ)$

$= \frac{1}{2}(180° - 60°) = 60°.$

$\Rightarrow \Delta PQR$ is equilateral.

8 Permutations and Combinations

KEY FACTS

I. Fundamental Principle of Counting

(*a*) Multiplication: Let us understand this basic principle with the help of the following examples:

Ex. 1. Suppose you have 3 shirts and 4 pairs of pants. In how many possible ways can you dress up by wearing a shirt and a pair of pants?

In the above case, you can wear any of the 3 shirts and after wearing one of these shirts any of these pairs of pants with it. If we label the shirts as S_1, S_2, S_3 and the pants as P_1, P_2, P_3 and P_4, then the different ways of dressing up can be as under:

$S_1 P_1$	$S_2 P_1$	$S_3 P_1$
$S_1 P_2$	$S_2 P_2$	$S_3 P_2$
$S_1 P_3$	$S_2 P_3$	$S_3 P_3$
$S_1 P_4$	$S_2 P_4$	$S_3 P_4$

Total number of ways = 12 = **3 × 4.**

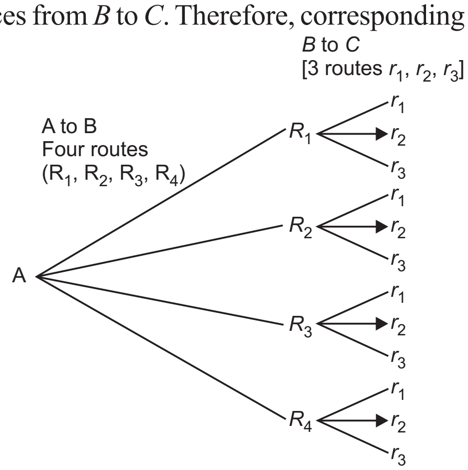
In this illustration, we **multiply** the number of ways in which you can wear a shirt and the number of ways in which you can wear a pair of pants.

Ex. 2. There are 4 different routes between cities *A* and *B*, and 3 different routes between cities *B* and *C*. How many different routes are there from city *A* to city *C* by way of city *B*?

Obviously, one can go from city *A* to city *B* by any of the 4 routes, *i.e.*, in 4 ways. After having gone to *B* by any of the different 4 routes, one can go to city *C* by any of the three routes.

Thus, corresponding to one route taken, from *A* to *B*, he has 3 choices from *B* to *C*. Therefore, corresponding to 4 routes, there are 12 choices in all.

Therefore, he can go from *A* to *C* via *B*, in 4 × 3 = 12 ways as depicted in the given tree diagram.



The possible routes taken are :

R_1r_1	R_2r_1	R_3r_1	R_4r_1
R_1r_2	R_2r_2	R_3r_2	R_4r_2
R_1r_3	R_2r_3	R_3r_3	R_4r_3

Total number of ways =12.

Thus, the examples discussed above illustrate the use of a general principle, called the product rule or the fundamental principle of counting, which is stated below.

If one operation can be performed in m ways, and if corresponding to each of the m ways of performing this operation, there are n ways of performing a second operation, then the number of ways of performing the two operations together is m × n. (This AND That).

Suppose that the first operation is performed in any one of the *m* ways, the second operation can then be performed in *n* ways and with the particular first operation, we can associate any one of the *n* ways of performing the second operation. This means that if the first operation could have been performed only in this one way, there would have been $1 \times n$, *i.e.*, *n* ways of performing both the operations. But it is given that the first operation can be performed in *m* ways and there are *n* ways of performing the second operation for every one way of performing the first operation. Therefore, there are $\boldsymbol{m \times n}$ ways of performing both the operations.

Generalisation. The above principle can be extended to the case in which the different operations can be performed in *m, n, p*,. .....ways. In this case, the number of ways of performing all the operations together would be $\boldsymbol{m \times n \times p}$**.......**

Ex. 3. There are 10 buses running between two towns *X* and *Y*. In how many ways can a man go from *X* to *Y* and return by a different bus?

Sol. The man can go from *X* to *Y* in 10 ways and as he is not to return by the same bus that he took while going, corresponding to each of the 10 ways of going, there are 9 ways of returning. Hence the total number of ways in which he can go to *Y* and be back is $10 \times 9 = 90$.

Ex. 4. How many different numbers of three digits can be formed with the digits 1, 2, 3, 4, 5, no digit is being repeated?

Sol. The unit's place can be filled with either of these 5 digits and so the unit's place can be filled in 5 ways. The ten's place can be filled in 4 ways corresponding to each way of filling up the unit's place, for we can have any digit here except the one used in the unit's place. Similarly, the hundredth's place can be filled in 3 ways as here we have any of the remaining three digits. Therefore, there are $5 \times 4 \times 3 = 60$ ways of forming a number of three digits with the five given digits.

(*b*) **Addition:** If there are two jobs which can be performed independently in *m* and *n* ways respectively, then either of the two jobs can be performed in $m + n$ ways.

For example: Suppose you have 3 full-sleeve and 4 half-sleeve shirts. Since you have the choice of wearing any of these shirts, you can wear one shirt in $3 + 4 = 7$ ways. If in addition, you have 5 T-shirts, then you can wear one of them in $3 + 4 + 5 = 12$ ways.

The above example illustrates one way of counting, which we may call the sum rule and applies when **one event** has to happen out of given disjoint events.

In this case, we use the word '***or***' between various jobs and the meaning of '***or***' is ***addition.***

II. Permutations

Def. : ***Each of the different arrangements which can be made by taking some or all of a number of things at a time is called a permutation.***

Notation: The number of permutations of *n* things taken *r* at a time is denoted by ^nP_r or ***P(n, r).*** The letter *P* is an abbreviation of the word 'permutation'.

Thus $^6P_4$ denotes the number of permutations or arrangements of 6 things taken 4 at a time.

The value of ^nP_r

To find the number of permutations of *n* different things, taken *r* at a time or to determine ^nP_r.

The number of permutations of *n* things taken *r* at a time will be the same as the number of ways in which *r* blank places can be filled up with *n* given things.

As the first place can be filled in by any one of the *n* things so there are *n* ways of filling up the first place.

After having filled in the first place by any one of the *n* things, there are $(n - 1)$ things left. Hence the second place can be filled in $(n - 1)$ ways. Now, as for every one way of filling up the first place, there are $(n - 1)$ ways of filling up the second place, so the first two places can be filled in $n(n - 1)$ ways.

After having filled in the first two places in any one of the above ways, there are $(n-2)$ things left and so the third place can be filled in $(n-2)$ ways. Now for every one way of filling up the first two places, there are $(n-2)$ ways of filling up the third place and so the first three places can be filled up in $n(n-1)(n-2)$ ways.

It may be observed that

(*a*) At every stage the number of factors is equal to the number of places filled up.

(*b*) Every factor is by one less than its preceding factor.

Position of the object	1st	2nd	...	$(r-1)$th	rth
Number of ways	n	$n-1$	...	$n-(r-2)$	$n-(r-1)$

Hence the number of ways of filling up all the r places, *i.e.*, the number of permutations of n different things taken r at a time is $n(n-1)(n-2)$......r factors

$$= n(n-1)(n-2)\dots(n-\overline{r-1})$$

Hence $\boxed{{}^nP_r = n(n-1)(n-2)\dots(n-r+1)}$

Thus, ${}^7P_2 = 7 \times 6$; ${}^{10}P_4 = 10 \times 9 \times 8 \times 7$; ${}^{20}P_3 = 20 \times 19 \times 18$.

Cor. : The number of permutations of n things taken all at a time is

$$\mathbf{{}^nP_n = n(n-1)(n-2)\dots\dots 3.\,2.1.} \qquad \text{[Putting } n \text{ for } r\text{]}$$

Ex. 1. In how many ways can 5 persons occupy 3 vacant seats?

Sol. Total number of ways $= {}^5P_3 = 5 \times 4 \times 3 = 60$.

Ex. 2. If $^{12}P_r = 1320$, find r.

Sol. ${}^{12}P_r = 12 \times 11 \times$ ...... to r factors $\Rightarrow 1320 = 12 \times 11 \times 10 \quad \therefore r = 3$.

III. Factorial Notation

The product of n natural numbers from 1 to n is denoted by n ! or $\underline{|n}$ and is read as factorial n.

Thus, $n!$ *or* $\underline{|n} = 1.\,2.\,3 \dots\dots (n-1)\,.\,n$

taken out
$n! = n(n-1)!$
One less

$4\,! = 1 \times 2 \times 3 \times 4 = 24$; $6\,! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$

$(n-1)! = 1 \times 2 \times 3 \dots\dots (n-1)$.

It is easily seen that $8\,! = 8 \times (7\,!)$.

IV. Values of ^nP_r in terms of Factorial Notation

$$^nP_r = n(n-1)(n-2)\dots(n-r+1) = \frac{n(n-1)\dots(n-r+1)}{\underline{|(n-r)}}\,.\,\underline{|(n-r)}$$

$$= \frac{n(n-1)(n-2)\dots(n-r+1)\,.\,(n-r)(n-r-1)\dots 3.\,2.1}{\underline{|(n-r)}}$$

$$= \frac{n(n-1)(n-2)\dots 3.\,2.1}{\underline{|(n-r)}} = \frac{\underline{|n}}{\underline{|(n-r)}}$$

$$\therefore \qquad \boxed{{}^nP_r = \frac{\underline{|n}}{\underline{|(n-r)}}}$$

Thus, $\quad {}^{18}P_5 = \dfrac{\underline{|18}}{\underline{|13}}$

Cor. 1. Putting $r = 0$, $^nP_0 = \dfrac{\underline{|n}}{\underline{|n}} = 1$

Cor. 2. Value of 0 !.

Putting $r = n$, ${}^nP_n = \dfrac{n!}{(n-n)} = \dfrac{n!}{0!}$; But ${}^nP_n = n!$ $\therefore n! = \dfrac{n!}{0!}$ $\boxed{\therefore \mathbf{0\,! = 1}}$

> **Note.** In fact, 0 ! is meaningless but in order to avoid contradiction in the results, we suppose that 0 ! = 1.

Ex. Find the value of *n* if $^nP_{13} : {}^{n+1}P_{12} = \dfrac{3}{4}$.

Sol. Here, ${}^nP_{13} = \dfrac{n!}{(n-13)!}$ and ${}^{n+1}P_{12} = \dfrac{(n+1)!}{(n+1-12)!} = \dfrac{(n+1)!}{(n-11)!}$ $\left[\text{Using } {}^nP_r = \dfrac{n!}{(n-r)!}\right]$

$${}^nP_{13} : {}^{n+1}P_{12} = \frac{n!}{(n-13)!} \times \frac{(n-11)!}{(n+1)!} = \frac{3}{4} \Rightarrow \frac{n!}{(n-13)!} \times \frac{(n-11)\,(n-12)\,.\,(n-13)!}{(n+1)\,.n!} = \frac{3}{4}$$

$$\Rightarrow \frac{(n-11)(n-12)}{(n+1)} = \frac{3}{4} \Rightarrow 4n^2 - 95n + 525 = 0 \text{ or } (n-15)\,(4n-35) = 0$$

$\therefore$ $n = 15$ (Rejecting the fractional value of n).

V. Restricted Permutations

Type I.

Ex. 1. In how many of the permutations of 10 things taken 4 at a time will

(*i*) one thing always occur, (*ii*) never occur?

Sol. (*i*) Keeping aside the particular thing which will always occur, the number of permutations of 9 things taken 3 at a time is ${}^9P_3$. Now this particular thing can take up any one of the four places and so can be arranged in 4 ways. Hence the total number of permutations = ${}^9P_3 \times 4 = 9 \times 8 \times 7 \times 4 = 2016$.

(*ii*) Leaving aside the particular thing which has never to occur, the number of permutations of 9 things taken 4 at a time is ${}^9P_4 = 9 \times 8 \times 7 \times 6 = 3024$.

Ex. 2. In how many of the permutations of *n* things taken *r* at a time will 5 things

(*i*) always occur, (*ii*) never occur?

Sol. (*i*) Keeping aside the 5 things, the number of permutations of $(r-5)$ things taken out of $(n-5)$ things is ${}^{n-5}P_{r-5}$. Now these 5 things can be arranged in r places in ${}^rP_5$ ways. Hence, the total number of permutations is ${}^rP_5 \times {}^{n-5}P_{r-5}$.

(*ii*) Total number of permutations = ${}^{n-5}P_r = \dfrac{(n-5)!}{(n-r-5)!}$

Type II. *When certain things are not to occur together.*

Case I. When the number of things not occurring together is two.

> **Procedure**
> 1. *Find the total number of permutations –when no restriction is imposed on the manner of arrangement.*
> 2. *Then find the number of permutations when the two things occur together.*
> 3. *The difference of the two results gives the number of permutations in which the two things do not occur together.*

Ex. 3. Prove that the number of ways in which *n* books can be placed on a shelf when two particular books are never together is (*n* – 2) × (*n* – 1)!.

Sol. Regarding the two particular books as one book, there are $(n-1)$ books now which can be arranged in ${}^{n-1}P_{n-1}$, *i.e.*, $(n-1)!$ ways. Now, these two books can be arranged amongst themselves in 2 ! ways. Hence the total number of permutations in which these two books are placed together is $2\,!\,.\,(n-1)!$. The number of permutations of n books without any restriction is $n\,!$.

Therefore, the number of arrangements in which these two books never occur together

$= n\,! - 2\,!\,.(n-1)\,! = n\,.\,(n-1)\,! - 2\,.\,(n-1)\,! = (n-2)\,.\,(n-1)\,!$

Case II. When the number of things not occurring together is more than two.

Ex. 4. In how many ways can 6 boys and 4 girls be arranged in a straight line so that no two girls are ever together?

Sol. The seating arrangement may be done as desired in two operations.

(*i*) First we fix the positions of 6 boys. Their positions are indicated by $B_1, B_2, \ldots, B_6$.

$\times\ B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6\ \times$

This can be done in $6\,!$ ways.

(*ii*) Now if the positions of girls are fixed at places (including those at the two ends) shown by the crosses, the four girls will never come together. In any one of these arrangements there are 7 places for 4 girls and so the girls can sit in ${}^7P_4$ ways.

Hence the required number of ways of seating 6 boys and 4 girls under the given condition

$= {}^7P_4 \times 6! = 7 \times 6 \times 5 \times 4 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 =$ **604800.**

Type III. *Formation of numbers with digits.*

Ex. 5. Suppose the six digits 1, 2, 4, 5, 6, 7 are given to us and we have to find the total number of numbers with no repetition of digits which can be formed under different conditions.

Sol. 1. **There is no restriction.** The number of 6-digit numbers.

$= {}^6P_6 = 6\,! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$

2. **Numbers in which a particular digit occupies a particular place.**

Suppose we have to form numbers in which 5 always occurs in the ten's place. In this case, the ten's place is fixed and the remaining five places can be filled in by the remaining 5 digits in ${}^5P_5$, *i.e.*, $5\,! = 120$ ways.

The number of numbers in which 5 occurs in the ten's place = 120.

3. **Numbers divisible by a particular number.** Suppose we have to form numbers which may be divisible by 2. These numbers will have 2 or 4 or 6 in the unit's place. Thus the unit's place can be filled in 3 ways. After having filled up the unit's place in any one of the above ways, the remaining five places can be filled in ${}^5P_5 = 5\,! = 120$ ways.

$\therefore$ The total number of numbers divisible by $2 = 120 \times 3 = 360$.

4. **Numbers having particular digits in the beginning and the end.** Suppose we have to form numbers which begin with 1 and end with 5. Here, the first and the last places are fixed and the remaining four places can be filled in $4\,!$, *i.e.*, 24 ways by the remaining four digits.

Therefore, the total number of numbers beginning with 1 and ending with 5 = 24.

Note. If the numbers could have 1 or 5 in the beginning or the end, the number would have been $2\,!\,.\,4\,!$, *i.e.*, 48.

5. **Numbers which are smaller than or greater than a particular number.** Suppose we have to form numbers which are greater than 4,00,000. In these numbers, there will be 4 or a digit greater than 4, *i.e.* 5, 6 or 7 in the lac's place. Thus this place can be filled in 4 ways. The remaining 5 places can then be filled in $5\,! = 120$ ways.

$\therefore$ The total number of numbers $= 4 \times 120 = 480$.

Type IV. *Word Building*

The following cases may arise:

1. No letter may be repeated.

2. Some letters may be repeated.

3. There may be a particular letter in the beginning or the end.

4. Some letters may occur together.

Vowels occur together   Consonants occur together

Illustration. Suppose the word 'PENCIL' is given to us and we have to form words with the letters of this word.

Sol. 1. There is no restriction on the arrangement of the letters.

The six different letters can be arranged in $^6P_6 = 6\,! = 720$ ways.

Hence the total number of words formed = **720.**

2. All words begin with a particular letter.

Suppose all words begin with E. The remaining 5 places can be filled with remaining 5 letters in $5\,!$ = **120 ways.**

3. All words begin and end with particular letters.

Suppose all words begin with L, and end with P. The remaining 4 places can then be filled in $4\,!$ ways.

$\therefore$ The total number of words formed = $4\,!$ = **24.**

> **Note.** If the words were to begin or end with E or L, these two positions could have been filled in $^2P_2 = 2$ ways. Hence the number of words in this case would have been $= 2 \times 24 = 48$.

4. N is always next to E.

P	EN	C	I	L

Since N is always next to E, therefore 'EN' is considered to be one letter.

$\therefore$ Required no. of permutations = $5! = 5 \times 4 \times 3 \times 2 \times 1$ = **120.**

5. Vowels occur together: The vowels are E and I. Regarding them as one letter, the 5 letters can be arranged in $5\,! = 120$ ways. These two vowels can be arranged amongst themselves in $2\,! = 2$ ways.

$\therefore$ The total number of words = 2×120 = **240.**

6. Consonants occur together: Regarding these consonants as one letter the three letters E, I, ($PNCL$) can be arranged in $3\,!$, *i.e.*, 6 ways. The letters $PNCL$ can be arranged among themselves in $4\,! = 24$ ways.

$\therefore$ The number of words in which consonants occur together = 6×24 = **144.**

7. Vowels occupy even places.

×	×	×	×	×	×
1	2	3	4	5	6

There are 6 letters and 3 even places. E can be placed in any one of the three even places in $^3P_1$, *i.e.*, 3 ways. Having placed E in any one of these places, I can be placed in any one of the remaining 2 places in $^2P_1$, *i.e.*, 2 ways. Thus, the vowels can occupy even places in $2 \times 3 = 6$ ways. After the vowels have been placed the remaining 4 letters can take up their positions in $^4P_4$, *i.e.*, 24 ways.

$\therefore$ The total number of words = 6×24 = **144.**

VI. Permutations of Alike Things

The number of permutations of n things taken all at a time where p of the things are alike and of one kind, q others are alike and of another kind, r others are alike and of another kind, and so on is

$$\boxed{x = \frac{n!}{p!q!r!..}}$$

Illustration: In how many ways can the letters of the word 'INDIA' be arranged ?

Sol. The word contains 5 letters of which 2 are 'I's.

The number of words possible = $\frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \mathbf{60.}$

VII. Permutations of Repeated Things

The number of permutations of n different things taken r at a time, when each thing may occur any number of times is n^r.

Suppose r places are to be filled with n things. The first place can be filled in n ways and when this has been filled up in any one of these ways, the second place can also be filled in n ways for the thing occupying the first place may occupy the second place also. Thus the first two places can be filled in $n \times n = n^2$ ways. Similarly the third place can also be filled in n ways.

Arguing in the same manner, we conclude that the r places can be filled in $n \times n \times n.....r$ times, *i.e.* n^r ways.

Illustration: In how many ways can 3 prizes be distributed among 4 boys, when

(*i*) no boy gets more than one prize;

(*ii*) a boy may get any number of prizes;

(*iii*) no boy gets all the prizes.

Sol. (*i*) The first prize can be given to any of the four boys. Then, the second prize can be given to any of the three boys. Lastly, the third prize can be given to any one of the remaining 2 boys.

$\therefore$ The number of ways in which all the 3 prizes can be given = $4 \times 3 \times 2 =$ **24.**

(*ii*) In this case, each of the three prizes can be given in 4 ways since a boy can receive any number of prizes.

$\therefore$ The number of ways in which all the prizes can be given = $4 \times 4 \times 4 = 4^3 =$ **64.**

(*iii*) Since anyone of the 4 boys can get all the prizes, therefore, the number of ways in which a boy gets all the 3 prizes = 4.

$\therefore$ Number of ways in which a boy does not get all the prizes = $64 - 4 =$ **60.**

VIII. Circular Permutations

Method I. If we have to arrange the four letters *A, B, C, D,* two of the arrangements would be *ABCD*, *DABC* which are two distinct arrangements. Now, if these arrangements are written along the circumference of a circle (Fig. 1 next page), the two arrangements are one and the same. Thus, we conclude that circular permutations are different only when the relative order of the objects is changed otherwise they are the same. Thus the arrangements in Fig. 2 are different.

As the number of circular permutations depends on the relative positions of objects, we *fix the position of one object and then arrange the remaining* $(n - 1)$ *objects in all possible ways. This can be done in* $(n - 1)\,!$ *ways.*

Illustration: 20 persons were invited for a party. In how many ways can they and the host be seated around a circular table? In how many of these ways will two particular persons be seated on either side of the host? (*IIT*)

Sol. There is 1 host and 20 guests, they are to be seated around a circular table.

(*i*) Let us fix the seat of one person, say the host, the 20 guests will be seated around the circular table in 20 ! ways, [or, $(n - 1)! = (21 - 1)\,! = 20!$]

(*ii*) The two particular persons can be seated on either side of the host in 2 ways and for each way of their taking seats, the remaining 18 persons can be seated around the circular table in 18 ! ways

Hence the number of ways of seating two particular persons on either side of the host = $2 \times 18\,!$. [or, $2! \times (19 - 1)!$, considering the host and two particular persons as one entity.]

IX. Clockwise and Counter-Clockwise Permutations

We have two types of circular permutations:

(*i*) Those in which counter-clockwise and clockwise are distinguishable. Thus while seating 4 persons *A, B, C, D* around a table, the following permutations are considered different. (Fig. 1)

(*ii*) Those in which counter-clockwise and anti-clockwise are not distinguishable. (Fig. 2)

Thus, (*a*) while forming a garland of roses or jasmine, the following arrangements are not disturbed if we turn the garland over.

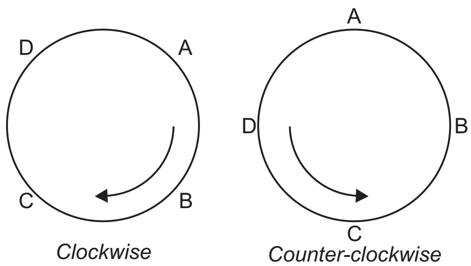


Fig. 1

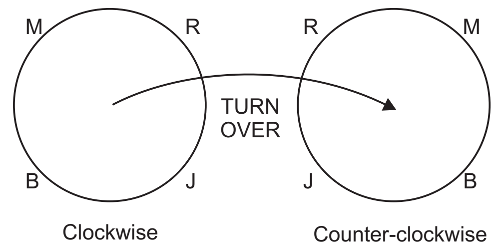


Fig. 2

(*b*) The distinction between clockwise and anti-clockwise is ignored when a number of people have to be seated around a table so as not to have the same neighbours.

Points to remember (Circular permutations)

We can **summarise** the above discussion as under:

1. Number of circular arrangements (permutations) of n different things = $(n - 1)!$
2. Number of circular arrangements of n different things when clockwise and anti-clockwise arrangements are not different = $\frac{1}{2}(n-1)!$
3. Number of circular permutations of n different things taken r at a time when **clockwise and anti-clockwise arrangements are taken as different** is $\frac{{}^nP_r}{r}$.
4. Number of circular permutations of n different things, taken r at a time, when clockwise and anti-clockwise arrangements are not different = $\frac{{}^nP_r}{2r}$.

Ex. Find the number of ways in which (*i*) *n* different beads, (*ii*) 10 different beads can be arranged to form a necklace.

Sol. Fixing the position of one bead, the remaining beads can be arranged in $(n - 1)$! ways. As there is no distriction between the clockwise and anti-clockwise arrangements, the total number of ways in which 10 different beads can be arranged = $\frac{(10-1)!}{2} = \frac{1}{2}(9\,!)$

X. Combinations

Def.: ***Each of the different groups or selections which can be made by taking some or all of a number of things at a time (irrespective of the order) is called a combination.***

By the number of combinations of n things taken r at a time is meant the number of groups of r things which can be formed from the n things. The same is denoted by the symbol $\boldsymbol{{}^nC_r}$ or $\boldsymbol{C(n, r)}$ or $\boldsymbol{\binom{n}{r}}$.

Value of ${}^nC_r$

Each combination consists of r different things which can be arranged among themselves in r ! ways. Therefore, the number of arrangements for all the ${}^nC_r$ combinations is ${}^nC_r \times r$!. This is equal to the permutations of n different things taken r at a time.

$\therefore \qquad {}^nC_r \times r! = {}^nP_r$

$$\therefore \quad {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n(n-1)(n-2)...(n-r+1)}{1.2.3\,...r}$$

$${}^nP_r = \frac{n!}{(n-r)} \qquad \boxed{\therefore\ {}^n\boldsymbol{C}_r = \frac{\boldsymbol{n!}}{\boldsymbol{r!(n-r)!}}}$$

Some Important Properties of Combinations

1. ${}^nC_r = \dfrac{{}^nP_r}{r!}$
2. ${}^nC_0 = {}^nC_n = 1$
3. ${}^nC_r = {}^nC_{n-r} \quad (0 \le r \le n)$
4. ${}^nC_x = {}^nC_y \Rightarrow x + y = n$ or $x = y\ (x, y \in W)$
5. ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$
6. If n is even, the greatest value of ${}^nC_r = {}^nC_m$ where $m = \dfrac{n}{2}$
7. If n is odd, the greatest value of ${}^nC_r = {}^nC_m$, where, $m = \dfrac{n-1}{2}$ or $m = \dfrac{n+1}{2}$
8. ${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_r = 2^n$
9. ${}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n - 1$
10. ${}^nC_0 + {}^nC_2 + {}^nC_4 + ... = 2^{2n}$

The proof of Formula 5 above is given below:

$$\boxed{{}^n\boldsymbol{C}_r + {}^n\boldsymbol{C}_{r-1} = {}^{n+1}\boldsymbol{C}_r\ (1 \le r \le n) \quad (\textit{Pascal's Rule})}$$

$$\begin{aligned} {}^nC_r + {}^nC_{r-1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} \\ &= n!\left\{\frac{(n-r+1)+r}{r!.(n-r+1)!}\right\} = \frac{(n+1)n!}{r!.(n-r+1)} = \frac{(n+1)!}{r!.(n-r+1)!} = {}^{n+1}C_r. \end{aligned}$$

Another form. $C(n, r) + C(n, r-1) = C(n+1, r)$,

i.e., $C(n, r-1) = C(n+1, r) - C(n, r)$

Ex. Find the values of $^6C_3$ and $^{30}C_{28}$.

Sol. (*i*) ${}^6C_3 = \dfrac{6\times5\times4}{1\times2\times3} = 20.$

(*ii*) ${}^{30}C_{28} = {}^{30}C_{30-28} = {}^{30}C_2 = \dfrac{30\times29}{1\times2} = 435.$

XI. Miscellaneous Types

Type I. *Total number of combinations.*

To find the total number of combinations of n dissimilar things taking any number of them at a time.

Case I. When all things are different.

Each thing may be disposed of in two ways. It may either be included or rejected.

$\therefore$ The total number of ways of disposing of all the things = $2 \times 2 \times 2 \times ......n$ times = 2^n

But this includes the case in which all the things are rejected.

Hence the total number of ways in which *one* or *more* things are taken = $2^n - 1$.

Cor. $2^n - 1$ is also the total number of the combinations of n things taken 1, 2, 3,...... or n at a time. Hence, ${}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n - 1$.

Ex. There are 5 questions in a question paper. In how many ways can a boy solve one or more questions?

Sol. The boy can dispose of each question in two ways. He may either solve it or leave it. Thus the number of ways of disposing of all the questions = 2^5.

But this includes the case in which he has left all the questions unsolved.

Hence the total number of ways of solving the paper = $2^5 - 1 = 31$.

Case II. When all things are not different.

Suppose, out of $(p + q + r +...)$ *things, p are alike of one kind, q are alike of a second kind, r alike of a third kind, and the rest different.*

Out of p things we may take 0,1, 2, 3...... or p. Hence they may be disposed of in $(p + 1)$ ways.

Similarly, q alike things may be disposed of in $(q + 1)$ and r alike things in $(r + 1)$ ways. The t different things may be disposed of in 2^t ways.

This includes that case in which all are rejected.

$\therefore$ The total number of selections = $(p + 1)(q + 1)(r + 1)\, 2^t - 1$.

Ex. Prove that from the letters of the sentence, 'Daddy did a deadly deed', one or more letters can be selected in 1919 ways.

Sol. In the given sentence, there are 9 *d's* ; 3 *a's* ; 3 *e's* ; 2 *y's* ; 1 *i*; and 1 *l*.

$\therefore$ The total number of selections = $(9 + 1)(3 + 1)(3 + 1)(2 + 1)(1 + 1)(1 + 1) - 1$

$= 10 \times 4 \times 4 \times 3 \times 2 \times 2 - 1 = 1919$.

Type II. *Division into groups.*

To find the number of ways in which $p + q$ things can be divided into two groups containing p and q things respectively.

Every time when a set of p things is taken, a second set of q things is left behind. Hence the required number of ways = the number of combinations of $(p + q)$ things taken p at a time

$$= {}^{p+q}C_p = \frac{(p+q)!}{p!q!}$$

Cor. 1. Generalisation. The number of ways in which $p + q + r$ things can be divided into three groups containing p, q and r things respectively

$$= {}^{p+q+r}C_p \times {}^{q+r}C_q \times {}^rC_r = \frac{(p+q+r)!}{p!(q+r)!} \times \frac{(q+r)!}{q!r!} \times 1 = \frac{(p+q+r)!}{p!q!r!}$$

Similarly the result can be extended to the case of dividing a given number of things into more than three groups.

Cor. 2. The number of ways in which $3p$ things can be divided equally into three *distinct* groups is $\frac{(3p)!}{(p!)^3}$. $(q = p, r = p)$

Cor. 3. The number of ways in which $3p$ things can be divided into three *identical* groups is $\frac{(3p)!}{3!(p!)^3}$.

Ex. 1. In how many ways can 15 things be divided into 3 groups containing 8, 4 and 3 things respectively?

Sol. The number of ways = $\frac{(15)!}{8!4!3!} = 225225$.

Ex. 2. In how many ways can 18 different books be divided equally among 3 students?

Sol. The required number of ways = $\frac{(18)!}{(6!)^3}$

SOLVED EXAMPLES

Ex. 1. Each section in first year of plus two course has exactly 30 students. If there are 3 sections, in how many ways can a set of 3 student representatives be selected from each section?

Sol. 1st representative can be selected from first section in 30 ways.

2nd representative can be selected from second section in 30 ways.

3rd representative can be selected from third section in 30 ways.

$\therefore$ Required number of ways = 30 × 30 × 30 = **27000.**

Ex. 2. How many numbers are there between 100 and 1000 such that every digit is either 2 or 9?

Sol. Any number between 100 and 1000 is of 3 digits. The unit's place can be filled by 2 or 9 in 2 ways. Similarly ten's place can be filled in 2 ways.

The hundred's place can also be filled in 2 ways.

$\therefore$ Required no. of numbers = 2 × 2 × 2 = **8.**

Ex. 3. How many odd numbers less than 1000 can be formed using the digits 0, 2, 5, 7 repetition of digits are allowed?

Sol. Since the required numbers are less than 1000 therefore, they are 1-digit, 2-digit or 3-digit numbers.

One-digit numbers. Only two odd one-digit numbers are possible, namely, **5** and **7.**

Two-digit numbers. For two-digit odd numbers the unit place can be filled up by 5 or 7 *i.e.*, in two ways and ten's place can be filled up by 2, 5 or 7 (not 0) in 3 ways.

$\therefore$ No. of possible 2-digit odd numbers = 2 × 3 = **6.**

Three-digit numbers. For three-digit odd numbers, the unit place can be filled up by 5 or 7 in 2 ways. The ten's place can be filled up by any one of the digits 0, 2, 5, 7 in 4 ways. The hundred's place can be filled up by 2, 5 or 7 (not 0) in 3 ways.

$\therefore$ No. of possible 3-digit numbers = 2 × 4 × 3 = 24

Hence total number of odd numbers = 2 + 6 + 24 = **32.**

Ex. 4. If $\dfrac{(2n)!}{3!(2n-3)!} : \dfrac{n!}{2!(n-2)!} = 44:3$, find n.

Sol. Given, $\dfrac{(2n)!}{3!(2n-3)!} : \dfrac{n!}{2!(n-2)!} = 44:3$

$$\Rightarrow \frac{(2n)(2n-1)(2n-2)(2n-3)!}{3\times 2!.(2n-3)!} \times \frac{2!(n-2)!}{n(n-1)(n-2)!} = \frac{44}{3}$$

$$\Rightarrow \frac{(2n)(2n-1)(2n-2)}{3} \times \frac{1}{n(n-1)} = \frac{44}{3}$$

$$\Rightarrow \frac{4(n)(2n-1)(n-1)}{3} \times \frac{1}{n(n-1)} = \frac{44}{3}$$

$\Rightarrow 4(2n-1) = 44 \quad \Rightarrow \quad 8n - 4 = 44 \quad \Rightarrow \quad 8n = 48 \quad \Rightarrow \quad n = \mathbf{6}$

Ex. 5. How many numbers greater than 5000 can be formed with the digits 3, 5, 7, 8, 9 no digit being repeated?

Sol. Obviously with the given 5-digits, numbers greater than 5000 are either 4-digit numbers having 5 or 7 or 8 or 9 at the thousand's place or 5-digit numbers.

Since the thousand's place in the required 4-digit numbers cannot take 3 as a value, we have 4 options for thousand's place, the remaining 4 out of 5 (1 earning the one used up at thousand's place) for hundred's place, 3 for tens' place and 2 for one's place.

(**Note:** Repetition of digits is not allowed)

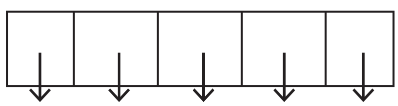
Four-Digit Numbers

Thousands Place 5, 7, 8 or 9	**Hundreds Place** Any of the 4 remaining digits	**Tens Place** Any of the remaining 3 digits	**Ones Place** Any of the remaining 2 digits
4 ways	4 ways	3 ways	2 ways

$\therefore$ Number of 4-digit numbers greater than 5000

That can be formed with given digits = $4 \times 4 \times 3 \times 2 = 96$

For the 5-digit numbers, the various places can be filled up as shown:



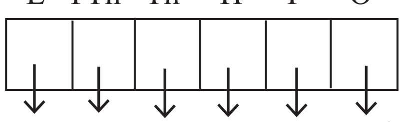
$\therefore$ Number of 5-digit numbers = $5 \times 4 \times 3 \times 2 \times 1 = 120$.

$\therefore$ Total required numbers = 96 + 120 = **216.**

Ex. 6. How many 6-digit numbers can be formed from the digits 1, 2, 4, 5, 6, 7 (no digit being repeated) which are divisible by 5?

Sol. A number is divisible by 5 only if its unit's digit is 5 or 0.

So we fix the unit's digit as 5 and fill up the remaining 5 places with the remaining 5 digits in the following way:



Note : Repetition of digits is not allowed

$\therefore$ Required number of numbers divisible by 5 = $5 \times 4 \times 3 \times 2 \times 1 =$ **120.**

Ex. 7. How many numbers can be formed by using any number of the digits 3, 1, 0, 5, 7, 2, 9, no digit being repeated in any number?

Sol. The number of single digit numbers is ${}^7P_1$.

The permutations of 7 digits taken 2 at a time are ${}^7P_2$. But ${}^6P_1$ of these have zero in the ten's place and so reduce to one digit numbers.

Hence the number of two-digit numbers is ${}^7P_2 - {}^6P_1$

Similarly the number of the three-digit numbers is ${}^7P_3 - {}^6P_2$ and so on.

$\therefore$ The total number required

$$= {}^7P_1 + ({}^7P_2 - {}^6P_1) + ({}^7P_3 - {}^6P_2) + ({}^7P_4 - {}^6P_3) + ({}^7P_5 - {}^6P_4) + ({}^7P_6 - {}^6P_5) + ({}^7P_7 - {}^6P_6) = \mathbf{11743.}$$

Ex. 8. How many different numbers can be formed with the digits 1, 3, 5, 7, 9, when taken all at a time, and what is their sum?

Sol. The total number of numbers = 5 ! = 120. Suppose we have 9 in the unit's place. We will have 4 ! = 24 such numbers. The number of numbers in which we have 1, 3, 5 or 7 in the unit's place is also 4 ! = 24 in each case.

Hence the sum of the digits in the unit's place in all the 120 numbers

$= 24\ (1 + 3 + 5 + 7 + 9) = 600.$

The number of numbers when we have any one of the given digits in ten's place is also 41=24 in each case.

Hence the sum of the digits in the ten's place

$= 24\ (1 + 3 + 5 + 7 + 9)$ tens $= 600$ tens $= 600 \times 10.$

Proceeding similarly, the required sum

= 600 units + 600 tens + 600 hundreds + 600 thousands + 600 ten thousands

$= 600(1 + 10 + 100 + 1000 + 10000) = 600 \times 11111 =$ **6666600.**

Ex. 9. Find the number of words that can be formed by taking all the letters of the word "COMBINE", such that the vowels occupy odd places. ***(WBJEE 2010)***

Sol. There are 7 letters in the word *COMBINE* of which 4 are consonants and 3 are vowels (*O*, *I*, *E*).

There are 4 odd places in a 7 letter word, so the number of ways 3 vowels can be arranged in 4 places $= {}^4P_3$

After arranging the 3 vowels, there are 4 places left (one at odd position and 3 at even positions). So the 4 consonants can be arranged in these 4 places in 4! ways.

$\therefore$ Required number of words $= {}^4P_3 \times 4! = 4 \times 4 \times 3 \times 2 \times 1 =$ **96.**

Ex. 10. How many ways are there to arrange the letter in the word GARDEN with the vowels in alphabetical order? **(*AIEEE 2004*)**

Sol. The word *GARDEN* contains 6 letters — 4 consonants (*G*, *R*, *D*, *N*) and 2 vowels *A*, *E*. The 4 consonants can be arranged in 6 places in ${}^6P_4$ ways. In each of these arrangements two places will remain blank in which the first place will be filled by *A* and the place following it by *E* as vowels have to be in alphabetical order.

This can always be done in only 1 way, *i.e.*, *E* following *A*

$\therefore$ Required number of ways $= {}^6P_4 \times 1 = \dfrac{6!}{(6-4)!} = \dfrac{6!}{2!} \times 1 = 6 \times 5 \times 4 \times 3 =$ **360.**

Ex. 11. How many signals can be made by hoisting 2 blue, 2 red and 5 yellow flags on a pole at the same time?

Sol. The number of signals $= \dfrac{9!}{2!2!5!} = \dfrac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 2 \times 5 \times 4 \times 3 \times 2 \times 1} =$ **756.**

Ex. 12. A coin is tossed 6 times. In how many different ways can we obtain 4 heads and 2 tails?

Sol. Whether we toss a coin 6 times or toss 6 coins at a time, the number of arrangements will be the same.

$\therefore$ The number of arrangements of 4 heads and 2 tails out of 6 is $\dfrac{6!}{4!2!} = 15.$

Ex. 13. How many numbers can be formed with digits 1, 2, 3, 4, 3, 2, 1, so that odd digits always occupy the odd places? **(*IIT*)**

Sol. The odd digits having two 1's alike and two 3's alike can be arranged in four odd places in $\dfrac{4!}{2!2!} = 6$ ways.

The three even digits having two 2's alike can be arranged in the three even places in $\dfrac{3!}{2!} = 3$ ways.

$\therefore$ The number of numbers $= 6 \times 3 =$ **18.**

Ex. 14. There are 3 copies each of 4 different books. Find the number of ways of arranging them on a shelf. **(*IIT*)**

Sol. Total number of books $= 3 \times 4 = 12$

Each of the 4 different titles has 3 copies each

$\therefore$ Required number of ways of arranging them on a shelf $= \dfrac{12!}{3!3!3!3!} = \dfrac{12!}{(3!)^4} =$ **369600.**

Ex. 15. Find the number of arrangements of the letters of the word 'BANANA' in which the two *N's* do not appear adjacently. **(*IIT Prel. 2002*)**

Sol. Considering the two *N's* as one letter, the number of letters to be arranged = 5.

Therefore, the number of arrangements = $\dfrac{5!}{3!} = 20$ ($\because$ *A* repeated 3 times)

Total number of arrangements if there were no restriction imposed = $\dfrac{6!}{3!\,2!} = 60.$

(*A* repeated 3 times and *N* repeated 2 times)

$\therefore$ Required number of arrangements = 60 – 20 = **40.**

Ex. 16. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?

Sol. A million is a 7-digit number. So any number greater than 1 million will contain all the seven digits. Since the digit 2 occurs twice and digit 3 occurs thrice and the rest are different, therefore, number of possible numbers which can be formed with the given seven digits = $\dfrac{7!}{(2!)\,(3!)} = 420.$

These possible numbers include those which have 0 at the millions place. Keeping 0 fixed at the millions place, the remaining 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are different can be arranged in $= \dfrac{6!}{(2!)(3!)} = 60.$ ways.

$\therefore$ Number of numbers greater than 1 million made from the given digits = 420 – 60 = **360.**

Ex. 17. In how many ways can the letters of the word 'ARRANGE' be arranged such that the two *r's* do not occur together?

Sol. There are two *a's*, two *r's* in the word 'arrange', therefore the number of arrangements = $\dfrac{7!}{2!\,2!} = 1260.$ ...(1)

The number of arrangements in which the two *r's* occur together = $\dfrac{6!}{2!} = 360.$ ...(2)

$\therefore$ The number of arrangements in which two *r's* do not occur together = (1) – (2) = 1260 – 360 = **900.**

Ex. 18. If the letters of the word 'AGAIN' be arranged in a dictionary, what is the fiftieth word?

Sol. In a dictionary, the words are arranged in an alphabetical order.

(*i*) Starting with *A*, the remaining 4 letters *G*, *A*, *I*, *N* can be arranged in 4! = 24 ways. These are the first 24 words.

(*ii*) Then, starting with *G*, the remaining letters *A*, *A*, *I*, *N* can be arranged in $\dfrac{4!}{2!} = 12$ ways. Thus, there are 12 words starting will *G*.

(*iii*) Now, the words will start with *I*. Starting with *I*, the remaining letters *A*, *G*, *A*, *N* can be arranged in $\dfrac{4!}{2!} = 12$ ways. So, there are 12 words, which start with *I*.

(*iv*) Thus, so far, we have constructed 24 + 12 + 12, *i.e.*, 48 words. The 49th word will start with *N* and is *NAAGI*. Hence, the 50th word is ***NAAIG.***

Ex. 19. The letters of the word 'RANDOM' are written in all possible ways and these words are written out as in a dictionary. Find the rank of the word 'RANDOM'.

Sol. The words in a dictionary are written in an alphabetical order, which, here is *ADMNOR*.

(*i*) Starting with *A*, the remaining letters *D*, *M*, *N*, *O*, *R* can be arranged in 5! = 120 ways. So, there are 120 words starting with *A*.

(*ii*) Similarly, number of words starting with *M* = 120, starting with *N* = 120, and starting with *O* =120.

(*iii*) Now, the number of words starting with R is also 120. Out of these words, one word is *RANDOM*. First, we find the words starting with *RAD* and *RAM*.

Number of starting with $RAD = 3! = 6$

Number of starting with $RAM = 3! = 6$

(*iv*) Thus, so far, $120 + 120 + 120 + 120 + 120 + 6 + 6 = 612$ words have been constructed.

(*v*) Now, the words starting with *RAN* will appear. Their number is also $= 3! = 6$. One of these words is the word *RANDOM* itself.

The first word beginning with *RAN* is *RANDMO*. It is 613th word and so the next word is *RANDOM*. *RANDOM* is the 614th word.

Hence, rank of *RANDOM* = **614.**

Ex. 20. How many numbers of 3-digits can be formed with the digits 1, 2, 3, 4, 5 when digits may be repeated?

Sol. Since repetition is allowed, each of the 3 places in a 3-digit number can be filled in 5 ways.

H	T	O
5 ways	**5** ways	**5** ways

$\therefore$ Required number of 3-digit numbers $= 5 \times 5 \times 5 = 5^3 = 125$

Ex. 21. How many numbers each containing four digits can be formed, when a digit may be repeated any number of times?

Sol. There are in all 10 digits, including zero. As the first digit of the number cannot be zero, so it can be chosen in 9 ways. Again, as a digit may occur any number of times in a number, the second, third and fourth digits of the numbers can be any one of the ten digits and so each of the remaining three places can be filled in 10 ways. Hence the total number of 4-digit numbers $= 9 \times 10^3 = 9000$.

Verification. All the 4-digit numbers will be between 1000 and 9999 and so their number is $9999 - 999 = 9000$.

Ex. 22. Eight different letters of an alphabet are given. Words of 4 letters from these are formed. Find the number of such words with at least one letter repeated. **(*EAMCET 2006*)**

Sol. If any letter can be used any number of times, then the number of words of 4 Letters with 8 different letters is $8 \times 8 \times 8 \times 8 = 8^4 = 4096$

Number of words of 4 letters with at least one letter repetition not allowed $= {}^8P_4 = 8 \times 7 \times 6 \times 5 = 1680$

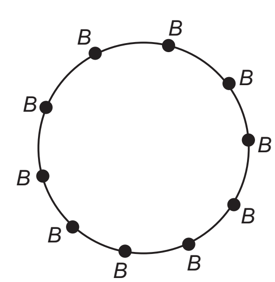
$\therefore$ Number of 4 letter words with at least one letter repeated is $8^4 - {}^8P_4 = 4096 - 1680 =$ **2416.**

Ex. 23. (*a*) In how many ways can a party of 4 boys and 4 girls be seated at a circular table so that no 2 boys are adjacent

(*b*) In how many ways can 10 boys and 5 girls sit around a circular table, so that no two girls sit together?

Sol. (*a*) Let the girls first take up their seats. They can sit in 3 ! ways. When they have been seated, then there remain 4 places for the boys each between two girls. Therefore the boys can sit in 4 ! ways. Therefore there are 3 ! × 4 !, *i.e.*, 144 ways of seating the party.

(*b*) Let B, denote the position of the boys around the table. 10 boys can be seated around the table in 9 ! ways.



There are 10 spaces between the boys, which can be occupied by 5 girls in ${}^{10}P_5$ ways. Hence,

Total number of ways $= 9! \times {}^{10}P_5 = \mathbf{\dfrac{9!\,10!}{5!}}$.

Ex. 24. A round table conference is to be held between delegates of 20 countries. In how many ways can they be seated if two particular delegates are
(*i*) always together, (*ii*) never together?

Sol. (*i*) Let D_1 and D_2 be the two particular delegates. Considering D_1 and D_2 as one delegate, we have 19 delegates in all. 19 delegates can be seated round a circular table in (19 – 1)! = 18 ! ways.
But two particular delegates can seat themselves in 2! (D_1 D_2 or D_2 D_1) ways.
Hence, the total number of ways = 18! × 2! = 2 (18!)

(*ii*) To find the number of ways in which two particular delegates never sit together, we subtract the number of ways in which they sit together from the total number of ways of seating 20 persons *i.e.*, (20 – 1)! = 19! ways.
Hence the total number of ways in this case = 19! – 2 (18!) = 19 (18!) – 2 (18!) = **17 (18!).**

Ex. 25. Find the number of ways in which 10 different flowers can be strung to form a garland so that 4 particular flowers are never separated.

Sol. Consider 4 particular flowers as one flower. Then, we have 7 flowers which can be strung to form a garland in (7 – 1)! = 6! ways. But 4 particular flowers can be arranged in 4! ways.

Hence, the required number of ways = $\frac{1}{2}(6!\times 4!) = \mathbf{8640.}$

Ex. 26. In how many ways can 7 persons sit around a table so that all shall not have the same neighbours in any two arrangements.

Sol. 7 persons can sit around a table in 6 ! ways but as each person will have the same neighbours in clockwise and anti-clockwise arrangements, the required number = $\frac{1}{2}$ 6! = **360.**

Ex. 27. If $^nC_{r-1} = 36$, $^nC_r = 84$ and $^nC_{r+1} = 126$, then find the value of *r*. ***(DCE 2000, MPPET 2009, IIT, Pb CET)***

Sol. $\frac{^nC_r}{^nC_{r-1}} = \frac{84}{36} \Rightarrow \frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{84}{36} \Rightarrow \frac{n-r+1}{r} = \frac{7}{3}$ ...(*i*)

Also, $\frac{^nC_{r+1}}{^nC_{r-1}} = \frac{126}{84} \Rightarrow \frac{n!}{(r=1)!(n-r-1)} \times \frac{r!(n-r)!}{n!} = \frac{3}{2} \Rightarrow \frac{n-r}{r+1} = \frac{3}{2}$ ...(*ii*)

From (*ii*) $2n - 2\underline{r} = 3r + 3 \Rightarrow n = \frac{5r+3}{2!}$

Substituting in (*i*), we get $\frac{\frac{5r+3}{2} - r + 1}{r} = \frac{7}{3} \Rightarrow \boldsymbol{r = 3.}$

Ex. 28. In how many ways can 4 persons be selected from amongst 9 persons? How many times will a particular person be always selected?

Sol. The number of ways in which 4 persons can be selected from amongst 9 persons

$= {}^9C_4 = \frac{9\times 8\times 7\times 6}{1\times 2\times 3\times 4} = 126.$

The number of ways in which a particular person is always to be selected

$= {}^8C_3 = \frac{8\times 7\times 6}{1\times 2\times 3} = \mathbf{56.}$

Ex. 29. Find the number of diagonals that can be drawn by joining the angular points of a
(*i*) heptagon
(*ii*) a polygon of 20 sides. **(*J&K CET 2009*)**

Sol. (*i*) A heptagon has seven angular points and seven sides.
The join of two angular points is either a side or a diagonal.

The number of lines joining the angular points = $^7C_2 = \frac{7 \times 6}{1 \times 2} = 21.$

But the number of sides = 7 $\quad \therefore$ The number of diagonals = 21 – 7 = **14.**

(*ii*) Number of diagonals in a polygon of n sides = $^nC_2 - n$. Here, $n = 20$

$\therefore$ Required number of diagonals = $^{20}C_2 - 20 = \frac{20 \times 19}{2 \times 1} - 20 = \mathbf{170.}$

Ex. 30. A committee of 4 is to be selected from amongst 5 boys and 6 girls. In how many ways can this be done so as to include (*i*) exactly one girl, (*ii*) at least one girl?

Sol. (*i*) In this case we have to select one girl out of 6 and 3 boys out of 5.
The number of ways of selecting 3 boys = $^5C_3 = {^5C_2} = 10$.
The number of ways of selecting one girl = $^6C_1 = 6$.
$\therefore$ The required committee can be formed in 6 × 10 = 60 ways.

(*ii*) The committee can be formed with
(*a*) one boy and three girls, or (*b*) 2 boys and 2 girls,
or (*c*) 3 boys and one girl, or (*d*) 4 girls alone.

The committee can be formed in (*a*) $^5C_1 \times {^6C_3}$ ways ;

The committee can be formed in (*b*) $^5C_2 \times {^6C_2}$ ways ;

The committee can be formed in (*c*) $^5C_3 \times {^6C_1}$ ways ;

The committee can be formed in (*d*) $^6C_4$ ways.

Hence the required number of ways of forming the committee

$= {^5C_1} \times {^6C_3} + {^5C_2} \times {^6C_2} + {^5C_3} \times {^6C_1} + {^6C_4}$ = 100 + 150 + 60 + 15 = **325 ways.**

(*ii*) **Method II.** Required ways = (Committees of 4 out of 11 without any restriction) – (Committees in which no girl is included) = $^{11}C_4 - {^5C_4}$ = **325.**

Ex. 31. A student is to answer 10 out of 13 questions in an examination such that he must choose at least 4 from the first five questions. Find the number of choices available to him. **(*AIEEE 2003*)**

Sol. Two cases are possible:
(*i*) Selecting 4 out of first five questions and 6 out of remaining 8 questions

$\therefore$ Number of choices in this case = $^5C_4 \times {^8C_6} = {^5C_1} \times {^8C_2} = \frac{5 \times 8 \times 7}{1 \times 2} = 140$

(*ii*) Selecting 5 out of first five questions and 5 out of remaining 8 questions.

$\Rightarrow$ Number of choices = $^5C_5 \times {^8C_5} = 1 \times {^8C_3} = \frac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56.$

$\therefore$ Total number of choices = 140 + 56 = **196.**

Ex. 32. How many different words, each containing 2 vowels and 3 consonants, can be formed with 5 vowels and 17 consonants?

Sol. Two vowels can be selected in $^5C_2$ ways.

Three consonants can be selected in $^{17}C_3$ ways.

$\therefore$ 2 vowels and 3 consonants can be selected in $^5C_2 \times {}^{17}C_3$ ways.

Now, each group of 2 vowels and 3 consonants can be arranged in 5 ! ways.

$\therefore$ The total number of words = $^5C_2 \times {}^{17}C_3 \times 5\,! =$ **816000.**

PRACTICE SHEET

1. How many words (with or without meaning) can be formed from three distinct letters of the English alphabet?

(*a*) 17576 (*b*) 15600
(*c*) 14400 (*d*) None of these

2. Two persons enter a railway compartment where there are 6 vacant seats. In how many different ways can they seat themselves?

(*a*) 36 (*b*) 11 (*c*) 30 (*d*) 12

3. How many integers of four digits can be formed with the digits 0, 1, 3, 5, 6 (assuming no repetitions)?

(*a*) 96 (*b*) 120 (*c*) 15 (*d*) 625

4. There are 12 true-false questions in an examination. How many sequences of answers are possible ?

(*a*) 12! (*b*) 120 (*c*) 2^{12} (*d*) $2^{12} - 2$

5. How many seven-digit phone numbers are possible if 0 and 1 cannot be used as the first digit and the first three digits cannot be 555, 411 or 936?

(*a*) 412560 (*b*) 7970000 (*c*) 797000 (*d*) 362880

6. There are 5 roads leading to a town from a village. The number of different ways in which a villager can go to the town and return back is:

(*a*) 5 (*b*) 10 (*c*) 25 (*d*) 20

(*AMU 2002*)

7. Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.

(*a*) 120 (*b*) 20 (*c*) 320 (*d*) 240

8. How many numbers are there between 100 and 1000 such that at least one of their digits is 7?

(*a*) 900 (*b*) 648 (*c*) 729 (*d*) 252

9. How many numbers divisible by 5 and lying between 4000 and 5000 can be formed from the digits 4, 5, 6, 7 and 8?

(*a*) 625 (*b*) 125 (*c*) 25 (*d*) 20

10. How many three-digit odd numbers can be formed by using the digits 1, 2, 3, 4, 5, 6 if repetition of digits is not allowed?

(*a*) 72 (*b*) 108 (*c*) 40 (*d*) 60

11. The number of all 3-digit numbers in each of which the sum of the digits is even is

(*a*) 450 (*b*) 375 (*c*) 365 (*d*) 250

12. Find the total number of ways in which 20 balls can be put in 5 boxes so that the first box contains just one ball.

(*a*) 20^5 (*b*) 5^{20} (*c*) 20×4^{19} (*d*) 4^{20}

13. Find the number of ways in which five large books, four medium sized books and three small books can be placed on a shelf so that all books of the same size together

(*a*) $5 \times 4 \times 3$ (*b*) $5! \times 4! \times 3!$
(*c*) $3 \times 5! \times 4! \times 3!$ (*d*) $3! \times 5! \times 4! \times 3!$

(*VITEE 2006*)

14. If $P(5, r) = 2P(6, r-1)$, find r.

(*a*) 2 (*b*) 4
(*c*) 3 (*d*) Cannot be determined

15. If $^{2n+1}P_n : {}^{2n-1}P_n = 3 : 5$, then n equals

(*a*) 2 (*b*) 3 (*c*) 6 (*d*) 4

(*Kerala PET 2009*)

16. 6 different letters of an alphabet are given. Words with four letters are formed from these given letters. Determine the number of words which have at least one letter repeated.

(*a*) 1296 (*b*) 996 (*c*) 936 (*d*) 360

17. How many eight-distinct letter words can be formed with the letters of the word "COURTESY" beginning with *C* and ending with *Y*?

(*a*) 576 (*b*) 640 (*c*) 336 (*d*) 720

18. How many different words can be formed from with the letters of the word "RAINBOW" so that the vowels occupy odd places.

(*a*) 676 (*b*) 336 (*c*) 576 (*d*) 144

19. How many words can be formed from the letters of the word "DAUGHTER" so that the vowels always come together?

(*a*) 2880 (*b*) 4320 (*c*) 3600 (*d*) 3200

20. How many words can be formed from the letters of the word "SUNDAY" so that the vowels never come together?

(*a*) 480 (*b*) 360 (*c*) 400 (*d*) 720

21. There are 5 boys and 3 girls. In how many ways can they stand in a row so that no two girls are together?

(*a*) 36000 (*b*) 25600 (*c*) 14400 (*d*) 15600

22. Find how many arrangements can be made with the letters of the word "MATHEMATICS" in which the vowels occur together?

(*a*) 10080 (*b*) 120960 (*c*) 14400 (*d*) 36500

23. All the words that can be formed using the letters A, H, L, U, R are written as in a dictionary (no alphabet is repeated). Then the rank of the word RAHUL is

(*a*) 71 (*b*) 72 (*c*) 73 (*d*) 74

(*Kerala PET 2008*)

24. The letters of the word 'COCHIN' are permuted and all the permutations are arranged in alphabetical order as in English dictionary. The number of words that appear before the word COCHIN is

(*a*) 48 (*b*) 96 (*c*) 192 (*d*) 360

(*IIT 2007*)

25. Four friends have 7 shirts, 6 pants and 8 ties. In how many ways can they wear them?

(*a*) 336000 (*b*) 33600 (*c*) 124800 (*d*) 508032000

26. If all the L's occur together and also all I's occur together, when the letters of the word 'HALLUCINATION' are permuted, then the number of such arrangements of letters is:

(*a*) $\dfrac{7!}{2!2!2!2!}$ (*b*) $\dfrac{11!}{2!2!}$

(*c*) $\dfrac{13!}{2!2!2!2!}$ (*d*) $\dfrac{11!}{2!2!2!2!}$

27. In how many ways can the letters of the word "AFLATOON" be arranged if the consonants and vowels must occupy alternate places?

(*a*) 176 (*b*) 144 (*c*) 136 (*d*) 288

28. How many different numbers greater than 60000 can be formed with the digits 0, 2, 2, 6, 8?

(*a*) 144 (*b*) 48 (*c*) 24 (*d*) 288

29. There are 5 red, 4 white and 3 blue marbles in a bag. They are drawn one by one and arranged in a row. Assuming that all the 12 marbles are drawn, determine the number of different arrangements.

(*a*) 20772 (*b*) 27270 (*c*) 22707 (*d*) 27720

30. How many natural numbers not exceeding 4321 can be formed with the digits 1, 2, 3, 4 if repetition is allowed?

(*a*) 123 (*b*) 113 (*c*) 222 (*d*) 313

31. In how many ways can 8 people sit around a circular Table?

(*a*) 5040 (*b*) 40320 (*c*) 20160 (*d*) 2520

32. A committee of 11 members sit at a round table. In how many ways can they be seated if the "President" and the "Secretary" choose to sit together.

(*a*) $\dfrac{10!}{2!}$ (*b*) $\dfrac{9!}{2!}$ (*c*) $9! \times 2!$ (*d*) $\dfrac{11!}{2!}$

33. In how many ways can 7 men and 7 women sit on a round table such that no two women sit together?

(*a*) $2(7!)$ (*b*) $(6!)^2$ (*c*) $7! \times 6!$ (*d*) $(7!)^2$

(*BITSAT 2007*)

34. In how many ways can 6 gentlemen and 3 ladies be seated round a table so that every gentleman may have a lady by his side?

(*a*) 360 (*b*) 1440 (*c*) 720 (*d*) 1260

35. There are 6 numbered chairs placed around a circular table. 3 boys and 3 girls want to sit on them such that neither of two boys nor two girls sit adjacent to each other. How many such arrangements are possible?

(*a*) 44 (*b*) 72 (*c*) 48 (*d*) 12

36. Find the number of ways in which 10 different flowers can be strung to form a garland so that three particular flowers are always together.

(*a*) $\dfrac{9! \times 3!}{2}$ (*b*) $\dfrac{7! \times 3!}{2}$ (*c*) $\dfrac{8! \times 3!}{2}$ (*d*) $7! \times 2!$

37. If $C(2n, 3) : C(n, 2) = 12 : 1$ find n.

(*a*) 2 (*b*) 5 (*c*) 4 (*d*) 3

38. If ${}^nC_r : {}^nC_{r+1} = 1 : 2$ and ${}^nC_{r+1} : {}^nC_{r+2} = 2 : 3$ determine the values of n and r.

(*a*) $n = 10, r = 6$ (*b*) $n = 12, r = 6$

(*c*) $n = 10, r = 4$ (*d*) $n = 14, r = 4$

39. In how many ways can a committee of five persons be formed out of 8 members when a particular member is taken every time?

(*a*) 35 (*b*) 42 (*c*) 50 (*d*) 56

40. In how many ways can a team of 11 players be selected from 14 players when two of them play as goalkeepers only?

(*a*) 112 (*b*) 132 (*c*) 91 (*d*) 182

41. 15 men in a room shake hands with each other, then the total number of handshakes is

(*a*) 90 (*b*) 105 (*c*) 110 (*d*) 115

(*Rajasthan PET 2002*)

42. A man has 6 friends. Number of different ways, he can invite 2 or more for dinner is

(*a*) 28 (*b*) 57 (*c*) 71 (*d*) 96

(*Orissa JEE 2011*)

43. There are 5 professors and 6 students out of whom a committee of 2 professors and 3 students is to be formed such that a particular student is excluded.

(*a*) 100 (*b*) 150 (*c*) 200 (*d*) 125

44. Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included?

(*a*) 15 (*b*) 25 (*c*) 20 (*d*) 10

45. A committee of 5 persons is to be formed out of 6 gents and 4 ladies. In how many ways can this be done if at most two ladies are included?

(*a*) 168 (*b*) 156 (*c*) 186 (*d*) 165

46. At an election, a voter may vote for any number of candidates, not greater than the number to be elected. There are 10 candidates and 4 are to be elected. If a voter votes for at least one candidate, then the number of ways he can vote is

(*a*) 385 (*b*) 1110 (*c*) 5040 (*d*) 6210

(*AIEEE 2006*)

47. In a chess tournament, where participants were to play one game with one another, two players fell ill, having played 6 games each without playing among themselves. If the total number of games played was 117, then the number of participants at the beginning was:

(*a*) 15 (*b*) 16 (*c*) 17 (*d*) 18

(*AMU 2005*)

48. 7 relatives of a man comprise 4 ladies and 3 gentlemen; his wife also has 7 relatives; 3 of them are ladies and 4 gentlemen. In how many ways can they invite in a dinner party, 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives?

(*a*) 265 (*b*) 375 (*c*) 395 (*d*) 485

(*IIT 1985*)

49. A person invites 15 guests for dinner and wishes to arrange them at two round tables that can accommodate 8 persons and 7 persons respectively. In how many ways can he arrange the guests?

(*a*) $7! \times 6!$ (*b*) $2 \times 8! \times 7!$

(*c*) $\frac{15!}{56}$ (*d*) $15C_2 \times \frac{13!}{56}$

50. How many diagonals can be drawn in a polygon of 15 sides?

(*a*) 16 (*b*) 60 (*c*) 80 (*d*) 90

(*J&K CET 2009*)

51. There are 18 points in a plane such that no three of them are in the same line except five points which are collinear. The number of triangles formed by these points is:

(*a*) 805 (*b*) 806 (*c*) 813 (*d*) 816

(*Rajasthan PET 2007*)

52. In how many ways can a mixed doubles game be arranged from amongst 8 married couples if no husband and wife play in the same game?

(*a*) 840 (*b*) 240

(*c*) 480 (*d*) None of these

53. In how many way can a committee of 4 women and 5 men be chosen from 9 women and 7 men, if Mr. A refuse to serve on the committee if Ms. B is a member?

(*a*) 1608 (*b*) 1860 (*c*) 1680 (*d*) 1806

54. If m parallel lines in a plane are intersected by a family of n parallel lines, find the number of parallelograms formed?

(*a*) m^n (*b*) $(m + 1)(n + 1)$

(*c*) $\frac{(m-n)}{n!}$ (*d*) $\frac{mn(n-1)(m-1)}{4}$

55. In how many ways can a pack of 52 cards be divided equally among four players in order?

(*a*) $(52!)^4$ (*b*) $4 \times (13!)$

(*c*) $\frac{52!}{(13!)^4}$ (*d*) None of these

56. In how many ways can a pack of 52 cards be divided into 4 sets, three of them having 16 cards each and the fourth just 4 cards?

(*a*) $16! \times 52!$ (*b*) $\frac{52!}{(16!)^3}$

(*c*) $\frac{52!}{(3!)^{16}}$ (*d*) $\frac{52!}{(16!)^3 \times (3!)}$

57. 7 persons enter an elevator on the ground floor of a 11 storey hotel. Any one of them can leave the elevator at any of the 10 floors. The number of ways in which the 7 persons can leave the elevator, if each one of them can leave it at any of the ten floors is

(*a*) $10!$ (*b*) 10^7 (*c*) $7!$ (*d*) 7^{10}

58. Eighteen quests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three on the other side. Determine the number of ways in which the seating arrangement can be made?

(*a*) ${}^{18}C_4 \times {}^{14}C_3 \times 9! \times 2$ (*b*) ${}^{11}C_5 \times {}^{6}C_6 \times 9! \times 2!$

(*c*) ${}^{11}C_5 \times {}^{6}C_6 \times 9! \times 9!$ (*d*) ${}^{18}C_4 \times {}^{14}C_3 \times 9! \times 2!$

59. A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour.

(*a*) 452 (*b*) 524 (*c*) 425 (*d*) 254

60. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five. In how many different ways can we place all the balls so that no box remains empty?

(*a*) 6 (*b*) 36 (*c*) 90 (*d*) 150

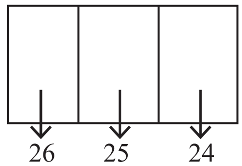
(*IIT 1981, 2012*)

ANSWERS

1. (*b*)	**2.** (*c*)	**3.** (*a*)	**4.** (*c*)	**5.** (*b*)	**6.** (*c*)	**7.** (*c*)	**8.** (*d*)	**9.** (*c*)	**10.** (*d*)
11. (*a*)	**12.** (*c*)	**13.** (*d*)	**14.** (*c*)	**15.** (*d*)	**16.** (*c*)	**17.** (*d*)	**18.** (*c*)	**19.** (*b*)	**20.** (a)
21. (*c*)	**22.** (*b*)	**23.** (*c*)	**24.** (*b*)	**25.** (*d*)	**26.** (*b*)	**27.** (*d*)	**28.** (*c*)	**29.** (*d*)	**30.** (*d*)
31. (*a*)	**32.** (*c*)	**33.** (*c*)	**34.** (*b*)	**35.** (*b*)	**36.** (*b*)	**37.** (*b*)	**38.** (*d*)	**39.** (*a*)	**40.** (*b*)
41. (*b*)	**42.** (*b*)	**43.** (*a*)	**44.** (*b*)	**45.** (*c*)	**46.** (*a*)	**47.** (*c*)	**48.** (*d*)	**49.** (*c*)	**50.** (*d*)
51. (*b*)	**52.** (*a*)	**53.** (*d*)	**54.** (*d*)	**55.** (*c*)	**56.** (*d*)	**57.** (*b*)	**58.** (*c*)	**59.** (*c*)	**60.** (*d*)

HINTS AND SOLUTIONS

1. There are 26 letters in the English alphabet, so for words with three distinct letters, the first place has 26 choices of letters, the next has remaining 25 choices and the third place has 24 choices.



$\therefore$ Total number of words $= 26 \times 25 \times 24$

$= \mathbf{15600}$

2. The first person can sit on any of the six vacant seats.

$\therefore$ Number of ways to seat first person = 6

Now the second person can sit on any of the remaining five seats.

So number of ways to seat second person = 5

$\therefore$ Total number of ways in which both can be seated

$= 6 \times 5 = \mathbf{30.}$

3. The thousands' place cannot be occupied by 0, as the number will then become three digit, so there are 4 choices for thousands' place. As repetition is not allowed, so leaving the digit occupying thousands' place, there are again 4 choices for hundreds' place (including 0)

Th	H	T	U
4	4	3	2

(Excluding 0)

Similarly for Tens' place there 3 choices, for units' place 2 choices.

$\therefore$ Total number of 4-digit numbers without repetition of digits $= 4 \times 4 \times 3 \times 2 = \mathbf{96.}$

4. For each of the 12 questions, there are 2 ways of answering **'true'** or **'false'.** Hence, total number of sequences of answers possible

$= \underbrace{2 \times 2 \times 2 \times \ldots\ldots \times 2}_{12 \text{ times}} = \mathbf{2^{12}.}$

5. For the first digit there are 8 choices (out of 10 digits) as 0 and 1 cannot be used. Since repetition can be done, the 2nd digit and the 3rd digit have 10 choices each. So, the first three digits can be filled in $(8 \times 10 \times 10 - 3)$ ways (We need to exclude the numbers 555, 411 and 936 also from first three digits)

The last four digits of the telephone number can be filled in $(10 \times 10 \times 10 \times 10)$ ways.

$\therefore$ Total number of seven digit phone numbers

$= (8 \times 10 \times 10 - 3) \times 10 \times 10 \times 10 \times 10 = \mathbf{7970000.}$

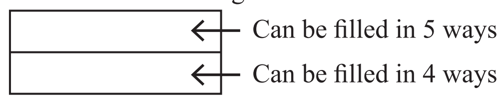
6. Number of ways of going from the village to the town = Number of roads between them = 5

Since all the 5 roads can be used for going back from town to the village, so number of ways of returning = 5.

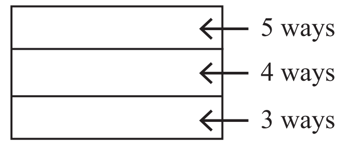
$\therefore$ Total number of ways of going to the town and coming back to the village $= 5 \times 5 = \mathbf{25.}$

7. At least 2 flags in order means that a signal may consist of either 2 flags, 3 flags, 4 flags in succession one below the other.

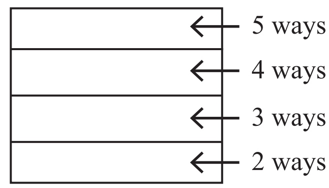
Total number of signals = Number of 2 flag signals



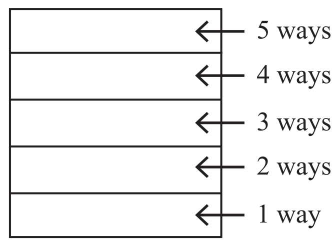
+ Number of 3 flag signals



+ Number of 4 flag signals



+ Number of 5 flag signals



$= 5 \times 4 + 5 \times 4 \times 3 + 5 \times 4 \times 3 \times 2 + 5 \times 4 \times 3 \times 2 \times 1$

$= 20 + 60 + 120 + 120 = \mathbf{320.}$

8. A number between 100 and 1000 has 3-digits

$\therefore$ Total number of 3-digit numbers having at least one of their digits as 7 = (Total number of 3-digit numbers) – (Total number of 3-digit numbers in which 7 does not appear at all)

(*a*) **Total number of 3-digit numbers:**

Clearly repetition of digits is allowed.

The hundred's place can be filled in 9 ways, *i.e.*, using any of the digits 1 to 9 excluding zero.

The ten's and the unit's place can be filled in 10 ways using all the digits from 0 – 9 in each place.

$\therefore$ Total number of 3-digit numbers $= 9 \times 10 \times 10 = 900$

(*b*) **Total number of 3-digit numbers in which 7 does not appear at all:**

Here, the digits to be used are 0, 1, 2, 3, 4, 5, 6, 8, 9 *i.e.*, 9 in number.

Now, the hundred's place can be filled in 8 ways (excluding 0), the tens' and ones' place can be filled in 9 ways each.

$\therefore$ Total number of 3-digit numbers in which 7 does not appear, $= 8 \times 9 \times 9 = 648$

$\therefore$ From (a) and (b), we have, **total number of 3-digit numbers having at least one of their digits as 7 = 900 – 648 = 252.**

9. A number between 4000 and 5000 will have a 4 in the thousand's place and since the number has to be divisible by 5, it will have 5 at unit's place. The hundreds' and tens' place each can be filled with any of the five digits. So,

Th	H	T	O
1	5	5	1

Total number of required numbers = 1 × 5 × 5 × 1 = **25.**

10. For a number to be odd, we should have 1 or 3 or 5 at the unit's place. So there are 3 ways of filling the unit's place.

As repetition of digits is not allowed, the ten's place can be filled in 5 ways with any of the remaining 5-digts and the hundred's place can be filled in 4 ways by the remaining 4-digits. So,

Required number of three-digit odd numbers = 3 × 5 × 4 = **60.**

11. Out of the 10 digits 0, 1, 2, 3, ..., 9, five digits, *i.e.*, 0, 2, 4, 6, 8 are even and five digits, *i.e.*, 1, 3, 5, 7 and 9 are odd.

The sum of the three digits D_1, D_2, D_3 of the number $D_1 D_2 D_3$ will be even if :

(i) All the three digits are even:

$\therefore$ Number of such numbers = 4 × 5 × 5 = 100

(Here the position D_1 cannot be occupied by 0)

(ii) One of the digits is even and the rest two are odd :

(a) D_1 is even, D_2 is odd, D_3 is odd

$\therefore$ Number of such numbers = 4 × 5 × 5 = 100

(Again since $D_1 \neq 0$, so only 4 choices for D_1)

(b) D_1 is odd, D_2 is even, D_3 is odd

$\therefore$ Number of such numbers = 5 × 5 × 5 = 125

(c) D_1 is odd, D_2 is odd, D_3 is even

$\therefore$ Number of such numbers = 5 × 5 × 5 = 125

$\therefore$ Total number of required numbers

= 100 + 100 + 125 + 125 = **450.**

12. One ball can be put in first box in 20 ways because we can put any one of the twenty balls in the first box. Now, remaining 19 balls are to be put into remaining 4 boxes. This can be done in 4^{19} ways, because there are 4 choices for each ball. Hence, the required number of ways = $\mathbf{20 \times 4^{19}}$.

13. Let us consider all the books of same size as one entity. Now there are three different entities which have to be arranged on the shelf and this can be done is 3! ways.

Also the five large books can be arranged amongst themselves in 5! ways, four medium books can be arranged amongst themselves in 4! ways and the three small books can be arranged amongst themselves in 3! ways. So,

Required number of ways of arranging the books so that all same sized books are together = **3! × 5! × 4! × 3!**

14. $P(5, r) = 2.\ P(6, r–1)$

$$\Rightarrow \quad {}^5P_r = 2 \times {}^6P_{r-1}$$

$$\Rightarrow \quad \frac{\underline{|5}}{\underline{|5-r}} = 2 \times \frac{\underline{|6}}{\underline{|6-(r-1)}}$$

$$\Rightarrow \quad \frac{\underline{|5}}{\underline{|5-r}} = \frac{2 \times 6 \times \underline{|5}}{\underline{|7-r}}$$

$$\Rightarrow \quad \frac{\underline{|5}}{\underline{|5-r}} = \frac{12 \times \underline{|5}}{(7-r)\,(6-r)\underline{|5-r}}$$

$$\Rightarrow \quad 1 = \frac{12}{(7-r)\,(6-r)} \quad \Rightarrow \quad (7-r)\,(6-r) = 12$$

$$\Rightarrow \quad (7-r)\,(6-r) = 4 \times 3$$

$$\Rightarrow \quad 7-r = 4 \text{ and } 6-r = 3 \quad \Rightarrow \quad \boldsymbol{r = 3.}$$

15. ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$

$$\Rightarrow \quad \frac{(2n+1)\,!}{(2n+1-(n-1))\,!} : \frac{(2n-1)\,!}{(2n-1-n)\,!} = 3:5$$

$$\Rightarrow \quad \frac{(2n+1)\,!}{(n+2)\,!} \times \frac{(n-1)\,!}{(2n-1)\,!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)\,(2n)\,\cancel{(2n-1)}\,!}{(n+2)\,(n+1)\,(n)\,\cancel{(n-1)}} \times \frac{\cancel{(n-1)}\,!}{\cancel{(2n-1)}\,!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{4n^2 + 2n}{n(n^2 + 3n + 2)} = \frac{3}{5}$$

$$\Rightarrow \quad (4n^2 + 2n)\,5 = 3 \times (n^3 + 3n^2 + 2n)$$

$$\Rightarrow \quad 20n^2 + 10n = 3n^3 + 9n^2 + 6n$$

$$\Rightarrow \quad 3n^3 - 11n^2 - 4n = 0$$

$$\Rightarrow \quad n(3n^2 - 11n - 4) = 0$$

$$\Rightarrow \quad n(3n^2 - 12n + n - 4) = 0$$

$$\Rightarrow \quad n(3n(n-4) + 1(n-4)) = 0$$

$$\Rightarrow \quad n = 0 \text{ or } 4 \text{ or } -\frac{1}{3}$$

Ignoring $n = 0$, $-\frac{1}{3}$ as these are in admissible values, we have $\boldsymbol{n = 4.}$

16. The number of 4-letter words which can be formed from 6 letters when one or more of the letters is repeated = 6 × 6 × 6 × 6 = 1296

Number of 4-letter words which can be formed from the given 6 letters when none of the letters is repeated

= Number of arrangements of 6 letters taken 4 at a time

$$= {}^6P_4 = \frac{6\,!}{(6-4)\,!} = \frac{6\,!}{2\,!} = 6 \times 5 \times 4 \times 3 = 360$$

$\therefore$ Number of 4 letter words which have at least one of their letters repeated = 1296 – 360 = **936.**

17. The first place will always be filled by C and the last place will always be filled with Y. The remaining six places can be filled by the remaining 6 letters in ${}^6P_6$ ways.

$\therefore$ Total number of words beginning with C and ending with $Y = 1 \times 1 \times {}^6P_6 = 6\,! = \mathbf{720}$**.**

18. There are 7 letters in the word RAINBOW out of which 4 are consonants and 3 are vowels.

$\underline{1} \quad 2 \quad \underline{3} \quad 4 \quad \underline{5} \quad 6 \quad \underline{7}$

In order that the vowels may occupy odd places, we first of all arrange any 3 consonants in even places in ${}^4P_3$ ways and then the odd places can be filled by 3 vowels and the remaining 1 consonant in ${}^4P_4$ ways. So,

Required number of words = ${}^4P_3 \times {}^4P_4 = 24 \times 24 = \mathbf{576}$**.**

19. There are eight letters in the word "DAUGHTER" including three vowels (A, U, E) and 5 consonants (D, G, H, T, R)

If the vowels are to be together, we consider them as one letter, so the 6 letters now (5 consonants and 1 vowels entity) can be arranged in ${}^6P_6 = 6\,!$ ways. Also corresponding to each of these arrangements, the 3 vowels can be arranged amongst themselves in 3 ! ways.

$\therefore$ Required number of words = $6\,! \times 3\,!$

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 = \mathbf{4320}$**.**

20. Total number of words in which vowels are never together

= Total number of words that can be formed with the letters of the word SUNDAY – Number of words in which vowels are always together

There are 6 letters in the word SUNDAY. So,

Total number of words formed by using all 6 letters $= {}^6P_6 = 6\,! = 720$.

Considering the vowels U and A as one unit, we have 5 letters that can be arranged in ${}^5P_5$ ways and also corresponding to each of these arrangement of 5 letters the vowels can be arranged amongst themselves in 2 ! ways.

So, number of words in which vowels are always together $= 5\,! \times 2\,! = \mathbf{240}$**.**

Hence, total number of words in which vowels are never together $= 720 - 240 = \mathbf{480}$**.**

21. First of all we arrange the 5 boys in 5 ! ways.

Then we arrange the 3 girls in the remaining 6 places between the 5 boys and on the extreme in ${}^6P_3$ ways.

× B × B × B × B × B ×

$\therefore$ Required number of ways = $5\,! \times {}^6P_3$

$= 120 \times 6 \times 5 \times 4 = \mathbf{144000}$**.**

22. There are 11 letters in the word "MATHEMATICS" out of which 4 are vowels and the rest 7 are consonants.

Let the four vowels be written together. $\boxed{\text{A A E I}}$ M, T, H, M, T, C, S

Consider the four vowels as one as unit, then these 8 letters (7 consonants and the vowel unit) can be permuted in $\frac{8\,!}{2\,!2\,!} = 10080$ ways. (There are two pairs of same letters AA and MM)

Corresponding to each of these permutations, the 4 vowels can be arranged among themselves in $\frac{4\,!}{2\,!} = 12$ ways.

$\therefore$ Required number of word in which vowels occur together

$= \frac{8\,!}{2\,!2\,!} \times \frac{4\,!}{2\,!} = 10080 \times 12 = \mathbf{120960}$.

23. The words coming before RAHUL in the dictionary will have A, H, L or R as their first letters.

I. When A is the first letter, the rest of the 4 letters H, L, R, U can fill the next 4 places in 4 !

$\therefore$ Number of words beginning with $A = 24$

II. When H is the first letter, the rest of the 4 letters A, L, R, U can fill the next 4 places in 4!

Number of words beginning with H = 24

Similarly, the number of words beginning with L = 24

Now among the words having R as their first letter, there is only one word which comes before RAHUL and that is RAHLU

Thus there are (24 × 3) + 1 = 73 words before RAHUL.

24. To determine the rank of the word COCHIN when the letters of the word 'COCHIN' are permuted and all permutations arranged in alphabetical order, we see that all words begin with C. So we have the following cases when permuted words are in alphabetical.

I. First two letters are CC

Then the number of ways of arranging the remaining 4 letters 0, H, I, N in remaining 4 places 4 !

$\therefore$ Number of word having CC has first two letters = 24

II. First two letters are CH (next to CC in alphabetical order)

Here also, as in case I,

Number of words with CH as first two letters = 4 ! = 24

Similarly, the number of words with CI and CN as their first two letters will be 24 each.

The first word with CO as first two letters is COCHIN

$\therefore$ 24 + 24 + 24 + 24 = **96** words appear before COCHIN.

25. 7 shirts can be worn by 4 friends in ${}^7P_4$ ways.

Similarly, 6 pants and 8 ties can be worn by 4 friends in ${}^6P_4$ and ${}^8P_4$ ways respectively.

$\therefore$ Total number of ways in which 7 shirts, 6 pants and 8 ties can be worn by 4 friends = ${}^7P_4 \times {}^6P_4 \times {}^8P_4$

$= \frac{7\,!}{3\,!} \times \frac{6\,!}{2\,!} \times \frac{8\,!}{4\,!}$

$= (7 \times 6 \times 5 \times 4) \times (6 \times 5 \times 4 \times 3) \times (8 \times 7 \times 6 \times 5)$

$= 840 \times 360 \times 1680 =$ **508032000.**

26. In the word 'HALLUCINATION' there are total 13 letters out of which 7 are consonants and 6 are vowels. Also there are 2L's, 2N's, 2A's and 2I's.

If L's occur together and I's occur together, then we consider them as one unit each. So, the arrangement can be written as:

$\boxed{L, L}$, $\boxed{I, I}$, H, A, A, U, C, N, N, T, O

Now these 11 letters (9 letters and 2 units of L and I) can be arranged in $\frac{11!}{2!2!}$ ways. ($\because$ There are 2A's and 2N's)

27. There are 8 letters in the word AFLATOON out of which 4 are vowels, *i.e.*, A, A, O, O and 4 are consonants, *i.e.*, F, L, T, N.

There are 2 ways in which the 4 consonants and 4 vowels occupy alternate places:

V C V C V C V C or CV CV CV CV

where V–vowel, C–consonant.

The four vowels can be arranged in 4 places in $\frac{4!}{2!2!}$ ways $= 6$ ways

The four consonants can be arranged in their 4 places in 4! ways = 24 ways.

Also, there are 2 cases, so,

$\therefore$ Required number of permutations $= 6 \times 24 \times 2 =$ **288.**

28. Numbers greater than 60000 will have either 6 or 8 in the TTh place and will consist of 5-digits.

If the digit 6 occupies the TTh place, the remaining 4 places can be occupied in $\frac{4!}{2!}$ ways. ($\because$ There are two 2's)

Number of numbers beginning with 6 $= \frac{4!}{2!} = 12$

Similarly, number of numbers beginning with 8 $= \frac{4!}{2!} = 12$

$\therefore$ Number of different numbers greater than 60000 formed with digits 0, 2, 2, 6, 8

$= 12 + 12 =$ **24.**

29. There are 5 + 4 + 3 = 12 marbles of which 5 are red (alike), 4 are white (alike) and 3 are blue (alike).

$\therefore$ Required number of arrangements $= \frac{12!}{5! \times 4! \times 3!}$

$= \frac{\cancel{12} \times 11 \times 10 \times 9 \times \cancel{8}^{4} \times 7 \times \cancel{6} \times \cancel{5!}}{\cancel{5!} \times \cancel{4 \times 3} \times \cancel{2} \times 1 \times \cancel{3 \times 2} \times 1}$

$=$ **27720.**

30. As there are 4 digits 1, 2, 3, 4 and repetition of digits is allowed.

Total number of 1-digit numbers = 4

Total number of 2-digit numbers $= 4 \times 4 = 16$

Total number of 3-digit numbers $= 4 \times 4 \times 4 = 64$

Number of 4-digit numbers beginning with 1 $= 4 \times 4 \times 4 = 64$ ($\because$ The first place is occupied by 1)

Number of 4-digit numbers beginning with 2 $= 4 \times 4 \times 4 = 64$

Number of 4-digit numbers beginning with 3 $= 4 \times 4 \times 4 = 64$

Number of 4-digit numbers beginning with 41 $= 4 \times 4 = 16$

Number of 4-digit numbers beginning with 42 $= 4 \times 4 = 16$

Number of 4-digit numbers beginning with 431 = 4

Number of 4-digit numbers beginning with 432 = 1 (4321 only)

$\therefore$ Total number of 4-digit numbers $= 64 + 64 + 64 + 16 + 16 + 4 + 1 = 229$

$\therefore$ Total number of natural numbers not exceeding 4321 $= 4 + 16 + 64 + 229 =$ **313.**

31. As is known in case of circular permutations, we keep one place fixed, so 8 people can sit around a circular table in (8 – 1) ! ways = 7! ways = 5040 ways.

32. Considering the president and secretary as one member, we have now 10 members in all. These 10 members can be seated round the circular table in (10 – 1)! = 9! ways.

Also the president and secretary can seat themselves in 2! ways **PS or SP**

$\therefore$ Required number of ways of seating = **9! × 2!.**

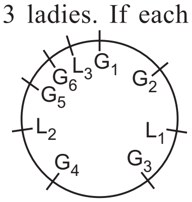
33. 7 men can seat themselves in a round table in (7 – 1)! = 6! ways.

Now there are 7 places vacant between these 7 men.

$\therefore$ 7 women can seat themselves in these 7 places in 7 ! ways.

$\therefore$ Total number of required arrangement where no two women sit together = **6! × 7!.**

34. There are 9 people, 6 gentlemen and 3 ladies. If each gentleman has to have a lady by his side, the seating arrangement can be done as shown below:



This can be done in 5! (Gentlemen) × 3! (Ladies) = 720 ways.

The arrangements can also be made in opposite direction. So,

Total number of required arrangements = 2 × 720 = **1440.**

35. Since the chairs are numbered, so they are distinguishable. Therefore 3 boys can be arranged on 3 alternate chairs in 3! ways. 3 girls can be arrenged in 3! ways

Also, the girls can be seated before the boys.

Total number of required ways = 3! × 3! + 3! × 3!

$= 2 \times (3!)^2$

36. Consider the three particular flowers as one flower. Then we have (10 – 3) + 1 = 8 flowers which can be strung in the garland.

Thus the garland can be formed in (8 – 1)!, *i.e.*, 7! ways

But the 3 particular flowers can be arranged amongst themselves in 3! ways.

$\therefore$ Required number of ways $= \frac{1}{2}$ (7! × 3!)

37. Given ${}^{2n}C_3 : {}^{n}C_2 = 12 : 1$

$$\Rightarrow \frac{{}^{2n}C_3}{{}^{n}C_2} = \frac{12}{1} \Rightarrow \frac{\frac{(2n)!}{(2n-3)!\,3!}}{\frac{n!}{(n-2)!\,2!}} = \frac{12}{1}$$

$$\Rightarrow \frac{(2n)!}{n!} \times \frac{(n-2)!\,2!}{(2n-3)!\,3!} = \frac{12}{1}$$

$$\Rightarrow \frac{(2n)(2n-1)\,(2n-2)\,(2n-3)!}{n(n-1)\,(n-2)!} \times \frac{(n-2)! \times 2!}{(2n-3)! \times 3 \times 2!} = \frac{12}{1}$$

$$\Rightarrow \frac{(2n)\,(2n-1)\,(2n-2)}{3n\,(n-1)} = \frac{12}{1}$$

$$\Rightarrow \frac{4(\cancel{n})\,(2n-1)\,\cancel{(n-1)}}{3\cancel{n}\,\cancel{(n-1)}} = \frac{12}{1}$$

$\Rightarrow 8n - 4 = 36 \quad \Rightarrow \quad 8n = 40 \quad \Rightarrow \boldsymbol{n = 5.}$

38. Given, ${}^{n}C_r : {}^{n}C_{r+1} = 1 : 2$ and ${}^{n}C_{r+1} : {}^{n}C_{r+2} = 2 : 3$

$$\Rightarrow \frac{n!}{r!(n-r)!} : \frac{n!}{(r+1)!\,(n-r-1)!} = 1:2 \text{ and}$$

$$\frac{n!}{(r+1)!\,(n-r-1)!} : \frac{n!}{(r+2)!\,(n-r-2)!} = 2:3$$

$$\Rightarrow \frac{n!}{r!\,(n-r)\,(n-r-1)!} \times \frac{(r+1)r!\,(n-r-1)!}{n!} = \frac{1}{2}$$

and $$\frac{n!}{(r+1)!\,(n-r-1)\,(n-r-2)!} \times \frac{(r+2)\,(r+1)!\,(n-r-2)!}{n!} = \frac{2}{3}$$

$$\Rightarrow \frac{(r+1)}{n-r} = \frac{1}{2} \text{ and } \frac{r+2}{n-r-1} = \frac{2}{3}$$

$\Rightarrow 2r + 2 = n - r \quad$ and $\quad 3r + 6 = 2n - 2r - 2$

$\Rightarrow n - 3r = 2 \quad$ and $\quad 2n - 5r = 8$

Solving the two simultaneous equations, we get
$\boldsymbol{n = 14, r = 4.}$

39. When a particular member is taken every time, then we need to choose the remaining 4 members from 7 members in ${}^{7}C_4$ ways.

$\therefore$ Required number of ways $= {}^{7}C_4 = \frac{7!}{4!\,3!} = \frac{7 \times 6 \times 5}{3 \times 2} = \mathbf{35.}$

40. As each team of 11 players has one goalkeeper and 10 team members, and out of 14 players there are 2 goalkeepers and 12 team members.

So the number of ways in which a team of 11 can be selected

$= {}^{12}C_{10} \times {}^{2}C_1 = \frac{12!}{10! \times 2!} \times 2$

$= \frac{12 \times 11}{2} \times 2 = \mathbf{132.}$

41. Since every person in the room shakes hand with every other person.

So, total number of handshakes = Number of ways of selecting 2 men out of 15 men

$= {}^{15}C_2 = \frac{15!}{13! \times 2!} = \frac{15 \times 14}{2} = \mathbf{105.}$

42. He can write 2 or more friends out of 6 friends in the given number of ways:

Invite (*i*) 2 friends out of 6 friends
or (*ii*) 3 friends out of 6 friends
or (*iii*) 4 friends out of 6 friends
or (*iv*) 5 friends out of 6 friends
or (*v*) 6 friends out of 6 friends.

$\therefore$ Total number of ways of inviting 2 or more friends

$= {}^{6}C_2 + {}^{6}C_3 + {}^{6}C_4 + {}^{6}C_5 + {}^{6}C_6$

$= \frac{6!}{4!\,2!} + \frac{6!}{3!\,3!} + \frac{6!}{4!\,2!} + \frac{6!}{5!\,1!} + \frac{6!}{6!}$

$= \frac{6 \times 5}{2} + \frac{6 \times 5 \times 4}{3 \times 2} + \frac{6 \times 5}{2} + 6 + 1$

$= 15 + 20 + 15 + 6 + 1 = \mathbf{57.}$

43. If a particular student is excluded, then the committee has to be chosen as: 2 professors from 5 professors and 3 students from 5 students (as one student is excluded)

$\therefore$ Total number of ways of forming the committee

$= {}^{5}C_2 \times {}^{5}C_3$

$= \frac{5!}{2!\,3!} \times \frac{5!}{3!\,2!}$

$= \frac{5 \times 4}{2} \times \frac{5 \times 4}{2} = \mathbf{100.}$

44. If at least one woman has to be included then the committee can be formed as:

- 1 woman and 2 men
- 2 women and 1 man

$\therefore$ Number of ways of forming the committee

$= {}^{2}C_1 \times {}^{5}C_2 + {}^{2}C_2 \times {}^{5}C_1$

$= 2 \times 10 + 1 \times 5 = 20 + 5 = \mathbf{25.}$

45. A committee of 5 persons, including at most two ladies can be constituted in the following ways :

I. Selecting 5 gents only out of 6 gents.

II. Selecting 4 gents out of 6 gents and 1 lady out of 4 ladies.

III. Selecting 3 gents out of 6 gents and 2 ladies out of 4 ladies.

$\therefore$ Total number of ways of forming the committee

$= {}^{6}C_5 + {}^{6}C_4 \times {}^{4}C_1 + {}^{6}C_3 \times {}^{4}C_2$

$= 6 + \frac{6 \times 5}{2} \times 4 + \frac{6 \times 5 \times 4}{3 \times 2} \times \frac{4 \times 3}{2}$

$= 6 + 60 + 120 = \mathbf{186.}$

46. As there are 4 candidates to be elected and the voter votes for at least one candidate, the voter can vote for 1 or 2 or 3 or 4 candidates out of 10 candidates.

$\therefore$ Total number of ways in which the voter can vote

$= {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{4}C_4$

$= 10 + \frac{10\times 9}{2\times 1} + \frac{10\times 9\times 8}{3\times 2\times 1} + \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}$

$= 10 + 45 + 120 + 210 =$ **385.**

47. Let the number of participants in the beginning be ***n***.

Total number of games played by the 2 players who fell ill $= 6 + 6 = 12$

Number of remaining players is $(n - 2)$

Number of games played by $(n-2)$ players with each other $= {}^{n-2}C_2$

$\therefore$ Total number of games played $= {}^{n-2}C_2 + 12$

Given ${}^{n-2}C_2 + 12 = 117$

$\Rightarrow {}^{n-2}C_2 = 105 \quad \Rightarrow \frac{n-2!}{(n-4)!\,2!} = 105$

$\Rightarrow \frac{(n-2)(n-3)}{2} = 105 \Rightarrow n^2 - 5n - 204 = 0$

$\Rightarrow (n-17)(n+12)\,0 \Rightarrow n = 17$ (Neglecting –ve value)

48. The 6 people (3 ladies and 3 gentlemen) to be invited can be selected in the given ways:

(i) 3 ladies from man's relatives and 3 gentlemen from wife's relatives, *i.e.*,

No. of ways $= {}^4C_3 \times {}^3C_0 \times {}^3C_0 \times {}^4C_3 = 16$

(ii) 2 ladies and 1 gentleman from man's relatives and 1 lady and 2 gentlemen from wife's relatives, *i.e.*,

No. of ways $= {}^4C_2 \times {}^3C_1 \times {}^3C_1 \times {}^4C_2$

$= \frac{4\times 3}{2} \times 3\times 3\times \frac{4\times 3}{2} = 324$

(iii) 1 lady and 2 gentlemen from man's relatives and 2 ladies and 1 gentleman from wife's relatives, *i.e.*,

No. of ways $= {}^4C_1 \times {}^3C_2 \times {}^3C_2 \times {}^4C_1 = 4\times 3\times 4\times 3$
$= 144$

(iv) 3 gentlemen from man's relatives and 3 ladies from wife's relatives, *i.e.*,

No. of ways $= {}^4C_0 \times {}^3C_3 \times {}^3C_3 \times {}^4C_0 = 1\times 1 = 1$

$\therefore$ Total number of ways of selecting the relatives for dinner party $= 16 + 324 + 144 + 1 =$ **485.**

49. First the 15 guests have to be divided into two groups of 8 persons and 7 persons.

$\therefore$ Number of ways of dividing the guests $= \frac{15!}{8!\,7!}$

Now 8 persons can be seated in one round table in $(8-1)!$ $= 7!$ ways

Also 7 persons can be seated in another round table in $(7-1)! = 6!$ ways

$\therefore$ Number of ways of arranging the guests $= \frac{15!}{8!\,\cancel{7!}} \times \cancel{7!} \times 6!$

$= \frac{15!}{8\times 7} = \mathbf{\frac{15!}{56}}$.

50. Number of diagonals for a n-sided closed polygon

$= {}^nC_2 - n = \frac{n!}{(n-2)!\,2!} - n$

$= \frac{n(n-1)\,(n-2)!}{(n-2)!\times 2} - n = \frac{n(n-1)-2n}{2}$

$= \frac{n^2 - n - 2n}{2} = \frac{n-3n}{2} = \frac{n(n-3)}{2}$

$\therefore$ Number of diagonals for a 15–sided polygon

$= \frac{15\times(15-3)}{2} = \frac{15\times 12}{2} =$ **90.**

51. Number of triangles that can be drawn with all 18 points in the plane $= {}^{18}C_3$

But 5 points are collinear

$\therefore$ Required number of triangles $= {}^{18}C_3 - {}^5C_3$

$= \frac{18!}{15!\,3!} - \frac{5!}{2!\,3!} = \frac{18\times 17\times 16}{3\times 2} - \frac{5\times 4}{2}$

$= 816 - 10 =$ **806.**

52. We can choose 2 men out of 8 men in ${}^8C_2$ ways. Since no husband and wife are to play in the same game, two women out of the remaining 6 women can be chosen in ${}^6C_2$ ways.

If M_1, M_2, W_1, W_2 are respectively the two men and two women chosen for the teams then the teams can be formed as $(M_1, W_1)\ (M_2, W_2)$ or $(M_1, W_2)\ (M_2, W_1)$, *i.e.*, in 2 ways. Hence, total number of ways of arranging the game $= {}^8C_2 \times {}^6C_2 \times 2$

$= \frac{8\times 7}{2} \times \frac{6\times 5}{2} \times 2 =$ **840.**

53. The required committee can be chosen in the following ways:

I. When Ms. B is a member and Mr. A refuses to serve.

[Selection has to be made of 3 women from 8 women as Ms B is already there and 5 men from 6 men as Mr A is excluded]

II. When Ms B is not a member and Mr. A can serve.

(Selection is to made of 4 women from 8 women as Ms. B is excluded and 5 men from 7 men)

$\therefore$ Total number of ways in which the committee can be formed.

$= {}^8C_3 \times {}^6C_5 + {}^8C_4 \times {}^7C_5$

$= \frac{8\times 7\times 6}{3\times 2} \times 6 + \frac{8\times 7\times 6\times 5}{4\times 3\times 2} \times \frac{7\times 6}{2}$

$= 56 \times 6 + 70 \times 21 = 336 + 1470 =$ **1806.**

54. A parallelogram is formed by choosing two straight lines from the set of ***m*** parallel lines and choosing two straight lines from the set of ***n*** parallel lines.

$\therefore$ Number of parallelograms formed $= {}^mC_2 \times {}^nC_2$

$$= \frac{m(m-1)}{2} \times \frac{n(n-1)}{2} = \frac{mn\,(m-1)\,(n-1)}{4}$$

55. This means that first player gets 13 cards, then second player gets 13 cards, then third player gets 13 cards and then fourth gets 13 cards.

$\therefore$ First player gets 13 cards in ${}^{52}C_{13}$ ways

Second player gets 13 cards in ${}^{39}C_{13}$ ways

Third player gets 13 cards in ${}^{26}C_{13}$ ways

Fourth player gets 13 cards in ${}^{13}C_{13}$ ways

$\therefore$ Total number of ways $= {}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13}$

$$= \frac{52!}{39!13!} \times \frac{39!}{26!13!} \times \frac{26!}{13!13!} \times 1 = \frac{52!}{(13!)^4}$$

Alternatively, 52 cards can be divided equally among 4 players in $\frac{52!}{(13!)^4 \times 4!}$ ways.

But as here order is important, hence required number of ways

$$= \frac{52! \times 4!}{(13!)^4 \times 4!} = \frac{52!}{(13!)^4}$$

56. First player can get 16 cards in ${}^{52}C_{16}$ ways

Second player can get 16 cards in ${}^{36}C_{16}$ ways

Third player can get 16 cards in ${}^{20}C_{16}$ ways

Fourth player can get 4 cards in ${}^{4}C_{4}$ ways

But the first three sets can be interchanged in 3! ways

$\therefore$ Required number of ways

$$= {}^{52}C_{16} \times {}^{36}C_{16} \times {}^{20}C_{16} \times {}^{4}C_{4} \times \frac{1}{3!}$$

$$= \frac{52!}{(16!)^3 \times 3!}$$

57. The first person can leave the elevator in any of the 10 floors

$\therefore$ Number of ways 1st person can leave the elevator = 10

Similarly, for each of the remaining 6 persons, the number of ways each one can leave the elevator = 10

$\therefore$ All the 7 persons can leave the elevator in

$$= \underbrace{10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10}_{7 \text{ terms}} = \mathbf{(10)^7} \text{ ways.}$$

58. As four particular guests want to sit on a particular side say (S_1) and three others on the other side, say, (S_2). So we are left with 11 guests out of which we choose 5 for side S_1 from remaining 11 in ${}^{11}C_5$ ways and 6 from remaining 6 for side S_2 in ${}^6C_6$ ways.

Also the 9 persons on each side can be arranged amongst themselves in 9! ways.

$\therefore$ Required number of ways of making the seating arrangement $= \mathbf{{}^{11}C_5 \times {}^6C_6 \times 9! \times 9!}$

59. 6 balls consisting of at least two balls of each colour from 5 red and 6 white balls can be made in the following ways:

(*a*) Selecting 2 red balls out of 5 red balls and 4 white balls out of 6, *i.e.*,

$$\text{Number of ways} = {}^5C_2 \times {}^6C_4 = \frac{5 \times 4}{2} \times \frac{6 \times 5}{2}$$

$$= 10 \times 15 = 150$$

(*b*) Selecting 3 red balls out of 5 red balls and 3 white balls out of 6, *i.e.*,

$$\text{Number of ways} = {}^5C_3 \times {}^6C_3 = \frac{5 \times 4}{2} \times \frac{6 \times 5 \times 4}{3 \times 2}$$

$$= 10 \times 20 = 200$$

(*c*) Selecting 4 red balls out of 5 red balls and 2 white balls out of 6, *i.e.*,

$$\text{Number of ways} = {}^5C_3 \times {}^6C_2 = 5 \times \frac{6 \times 5}{2} = 5 \times 15 = 75$$

$\therefore$ Total number of ways of selecting at least two balls of each colour = 150 + 200 + 75 = **425.**

60. As no box has to be empty, each box should have at least 1 ball.

Let the boxes be labelled as B_1, B_2 and B_3. Then the distribution of the balls can be shown as:

B_1	B_2	B_3
1	1	3
1	2	2
1	3	1
2	1	2
2	2	1
3	1	1

$\therefore$ Number of ways of placing the balls in different boxes so that no box remains empty

$$= {}^5C_1 \times {}^4C_1 \times {}^3C_3 + {}^5C_1 \times {}^4C_2 \times {}^2C_2 + {}^5C_1 \times {}^4C_3 \times {}^1C_1 + {}^5C_2 \times {}^3C_1 \times {}^2C_2 + {}^5C_2 \times {}^3C_2 \times {}^1C_1 + {}^5C_3 \times {}^2C_1 \times {}^1C_1$$

$$= 5 \times 4 \times 1 + 5 \times \frac{4 \times 3}{2} \times 1 + 5 \times 4 \times 1 + \frac{5 \times 4}{2} \times 3 \times 1 + \frac{5 \times 4}{2} \times 3 \times 1 + \frac{5 \times 4}{2} \times 2 \times 1$$

$= 20 + 30 + 20 + 30 + 30 + 20 =$ **150.**

SELF ASSESSMENT SHEET

1. How many integers greater than 999 but not greater than 4000 can be formed with the digits 0, 1, 2, 3 and 4 if repetition of digits is allowed?

(*a*) 499 (*b*) 500 (*c*) 375 (*d*) 376

(*CAT 2008*)

2. Consider the word RADAR. Whichever way you read it, from left to right or right to left, you get the same word. Such a word is known as a **palindrome.** Find the maximum possible number of 5–letter palindromes (meaningful or non-meaningful)

(*a*) 676 (*b*) 78 (*c*) 17576 (*d*) 130

3. How many arrangements can be formed out of the letters of the word EXAMINATION so that vowels always occupy odd places?

(*a*) 72000 (*b*) 86400 (*c*) 10800 (*d*) 64000

(*SNAP 2007*)

4. A number lock consists of 3 rings each marked with 10 different numbers. In how many cases the lock cannot be opened?

(*a*) 3^{10} (*b*) 10^3 (*c*) 30 (*d*) 999

(*SNAP 2008*)

5. A six digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5 if repetition of digits is not allowed. The total number of ways this can be done is:

(*a*) 120 (*b*) 240 (*c*) 600 (*d*) 720

(*Kerala PET*)

6. How many different nine-digit numbers can be formed from the number 223355888 by rearranging its digits so that odd digits occupy even positions?

(*a*) 16 (*b*) 36 (*c*) 60 (*d*) 96

(*IIT 2000*)

7. In how many ways can the eight directors, the vice-chairman and chairman of a firm be seated at a round table, if the chairman has to sit between the vice-chirman and the director?

(*a*) 9! × 2 (*b*) 2 × 8! (*c*) 2 × 7! (*d*) None of these

(*CAT*)

8. A total of 28 handshakes was exchanged at the conclusion of a party. Assuming that each participants was equally polite toward all the others, the number of people present was

(*a*) 14 (*b*) 28 (*c*) 56 (*d*) 8

(*FMS 2011*)

9. Out of 6 ruling and 5 opposition party members, 4 are to be selected for a delegation. In how many ways can it be done so as to include at least one opposition member?

(*a*) 300 (*b*) 315 (*c*) 415 (*d*) 410

(*JMET 2011*)

10. A team of 8 players is to be chosen from a group of 12 players. Out of the eight players one is to be elected as captain and another as vice-captain. In how many ways can this be done?

(*a*) 27720 (*b*) 13860 (*c*) 6930 (*d*) 495

(*NDA/NA 2010*)

11. There are 10 points on a line and 11 points on another line, which are parallel to each other. How many triangles can be drawn taking the vertices on any of the line?

(*a*) 1050 (*b*) 2550 (*c*) 150 (*d*) 1045

(*CAT*)

12. In how many ways is it possible to choose a white square and a black square on a chess board so that the squares must not lie in the same row or column ?

(*a*) 56 (*b*) 896 (*c*) 60 (*d*) 768

(*CAT 2002*)

13. In a chess competition involving some boys and girls of a school, every student has to play exactly one game with every other student. It was found that in 45 games both the players were girls and in 190 games both were boys. The number of games in which one player was a boy and the other was a girl is

(*a*) 200 (*b*) 216 (*c*) 235 (*d*) 256

(*CAT 2005*)

14. The value of $\sum_{r=1}^{n} \frac{{}^nP_r}{r!}$ is

(*a*) 2^n (*b*) $2^n - 1$ (*c*) 2^{n-1} (*d*) $2^n + 1$

(*IIFT 2007*)

15. In how many ways can four letters of the word 'SERIES' be arranged?

(*a*) 24 (*b*) 42 (*c*) 84 (*d*) 102

(*IIFT 2010*)

ANSWERS

1. (*d*) **2.** (*c*) **3.** (*c*) **4.** (*d*) **5.** (*c*) **6.** (*c*) **7.** (*b*) **8.** (*d*) **9.** (*b*) **10.** (*a*) **11.** (*d*) **12.** (*d*) **13.** (*a*) **14.** (*b*) **15.** (*d*)

HINTS AND SOLUTIONS

1. The smallest number in the required series is 1000 and the greatest is 4000 (The only number to start with 4)

∴ The thousands' place can be filled with any of the 3-digits 1, 2 and 3.

The next three places (hundred's, ten's and unit's place) can take any of the five values 0 or 1 or 2 or 3 or 4.

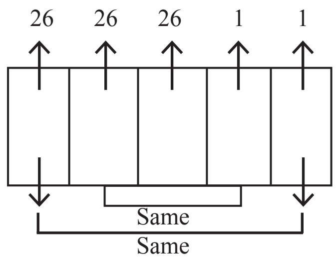
∴ Number of numbers from 100 to 3999

= 3 × 5 × 5 × 5 = 375

As 4000 is also included, so required number = **376.**

2. The first letter can be chosen in 26 ways as there are 26 letters in the English alphabet.

Having chosen the first letter, we have 26 choices for the second letter also.



Similarly for the third letter also, we have 26 choices.

The fourth letter has to be same as the second and the fifth letter has to be same as the first letter so 1 choice for each.

$\therefore$ Maximum possible number of 5-letter palindromes

$= 26 \times 26 \times 26 \times 1 \times 1 =$ **17576.**

3. There are 11 letters in the word EXAMINATION of which there are 2A's, 2I's.

Also there are 5 consonants and 6 vowels.

The 6 vowels can be arranged in 6 odd places in $\dfrac{6!}{2!\,2!}$ ways (2A's, 2I's)

The 5 consonants can be arranged in 5 even possible $= \dfrac{5!}{2!}$ ways (2N's)

$\therefore$ Total number of arrangements possible $= \dfrac{6!}{2!\times 2!} \times \dfrac{5!}{2!}$

$= \dfrac{6\times 5\times 4\times 3}{2} \times 5 \times 4 \times 3 =$ **10800.**

4. The first ring can be marked with any of the 10 numbers, *i.e.*, Number of ways of marking the first ring = 10

Similarly the second and third ring can also be marked with any of the 10 numbers.

$\therefore$ Number of number combinations that can appear on the 3 rings $= 10 \times 10 \times 10 = 1000$

The lock can be opened with a single combination only.

$\therefore$ Number of cases in which the lock cannot be opened $= 1000 - 1 =$ **999.**

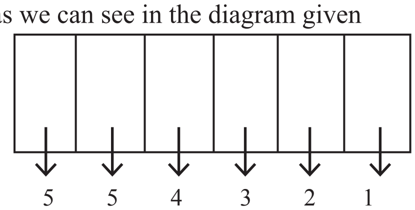
5. As repetition of digits is not allowed and the six digit number has to be formed using all the 6-digits 0, 1, 2, 3, 4, 5 implies that all the six digits will be used in making the number.

Sum of the 6-digits = 0 + 1 + 2 + 3 + 4 + 5 = 15

Hence any 6-digit number formed with these 6 digits will be divisible by 3.

The first place from the left can be filled with any of the non zero numbers.

Now, as we can see in the diagram given



The second place with remaining 5 numbers (including 0), the third with remaining 4, the fourth with remaining 3 and so on.

$\therefore$ Number of ways in which all the six places can be filled

= Number of ways of forming the 6-digit number divisible by 3

$= 5 \times 5 \times 4 \times 3 \times 2 \times 1 =$ **600.**

6. There are 9-digits in the given number of which there are 4 odd-digits (Two 3's, Two 5's) and 5 even digits (Two 2's, Three 8's).

In case of a 9-digit number there are 5 odd positions and 4 even positions

The 4 odd digts can occupy the 4 even positions in $\dfrac{4!}{2!\,2!}$ ways (Two 3's, Two 5's)

— X — X — X — X — $\begin{pmatrix} \text{X Odd} \\ \text{— Even} \end{pmatrix}$

Also the 5 even digits can occupy the 5 odd positions in $\dfrac{5!}{3!\,2!}$ ways (Three 8's, Two 2's)

$\therefore$ Required number of 9-digit numbers $= \dfrac{4!}{2!\,2!} \times \dfrac{5!}{3!\,2!}$

$= 6 \times 10 =$ **60.**

7. Let the vice-chairman and chairman form 1 unit along with 8 directors. So now we have 9 units to arrange in a circle which can be done in (9 – 1)! ways = 8! ways.

D C VC D

or

D VC C D

Also the chairman and vice chairman can be arranged amongst themselves in 2 different ways.

$\therefore$ Required number of ways = **2 × 8!.**

8. Let the number of people present be n. Then if n people shake hands with one another, number of handshakes $= {}^nC_2$.

$\Rightarrow\ {}^nC_2 = 28$

$\Rightarrow\ \dfrac{n!}{(n-2)!\,2!} = 28 \Rightarrow \dfrac{n(n-1)}{2} = 28$

$\Rightarrow\ n(n-1) = 56 \quad \Rightarrow n(n-1) = 8 \times 7 \quad \Rightarrow n =$ **8.**

9. 4 members of the delegation can be selected in the following ways:

I. 1 opposition member and 3 Ruling party members, *i.e.*,

Number of ways of this selection $= {}^5C_1 \times {}^6C_3$

II. 2 opposition members and 2 ruling party members, *i.e.*,

Number of ways of this selection $= {}^5C_2 \times {}^6C_2$

III. 3 opposition members and 1 ruling party member, *i.e.*,

Number of ways of this selection $= {}^5C_3 \times {}^6C_1$

IV. 4 opposition members, *i.e.*,

Number of ways of this selection $= {}^5C_4$.

$\therefore$ Total number of ways for required selection

$= {}^5C_1 \times {}^6C_3 + {}^5C_2 \times {}^6C_2 + {}^5C_3 \times {}^6C_1 + {}^5C_4$

$= 5 \times \frac{6 \times 5 \times 4}{3 \times 2} + \frac{5 \times 4}{2} \times \frac{6 \times 5}{2} + \frac{5 \times 4}{2} \times 6 + 5$

$= 100 + 150 + 60 + 5 =$ **315.**

10. Number of ways to choose 8 players from 12 players

$= {}^{12}C_8 = \frac{12!}{8!\,4!} = 495$

Number of ways to choose a captain and a vice captain

$= {}^{8}C_1 \times {}^{7}C_1$

$= 8 \times 7 = 56$

$\therefore$ Required number of ways $= 495 \times 56 =$ **27720.**

11. For a triangle, two points on one line and one on the other has to be chosen.

No. of ways = 2 points from 10 points and 1 point from 11 points **or** 1 point from 10 points and 2 points from 11 points

$= {}^{10}C_2 \times {}^{11}C_1 + {}^{10}C_1 \times {}^{11}C_2$

$= \frac{10 \times 9}{2} \times 11 + 10 \times \frac{11 \times 10}{2} = 495 + 550 =$ **1045.**

12. There are 32 black and 32 white squares on a chess board. Then number of ways of choosing one white and one black square on the chose board

$= {}^{32}C_1 \times {}^{32}C_1 = 32 \times 32 = 1024$

There are 8 rows and 8 columns on a chess board each containing 4 white squares and 4 black squares.

$\therefore$ Number of ways to choose a white and a black square from the same row $= {}^{4}C_1 \times {}^{4}C_1 \times 8 = 128$

Similarly, number of ways to choose a white and a black square from the same column $= {}^{4}C_1 \times {}^{4}C_1 \times 8 = 128$

$\therefore$ Number of ways of choosing a white square and a black square on the chess board so that squares do not lie in the same row or column

$= 1024 - (128 + 128) = 1024 - 256 =$ **768.**

13. Let the number of **girls** be $\boldsymbol{x}$ and let the number of **boys** be $\boldsymbol{y}$.

Then, as in 45 games 2 girls played against each other,

${}^{x}C_2 = 45$ ...(*i*)

Similarly, ${}^{y}C_2 = 190$ ...(*ii*)

${}^{x}C_2 = 45 \Rightarrow \frac{x!}{(x-2)!\,2!} = 45 \Rightarrow x(x-1) = 90$

$\Rightarrow x(x-1) = 10 \times 9 \Rightarrow x =$ **10**

${}^{y}C_2 = 190 \Rightarrow \frac{y(y-1)}{2} = 190 \Rightarrow y(y-1) = 380$

$\Rightarrow y(y-1) = 20 \times 19 \Rightarrow y =$ **20.**

$\therefore$ There are 10 girls and 20 boys.

Hence, the total number of games in which one player is a girl and other player is a boy $= 10 \times 20 =$ **200.**

14. $\sum_{r=1}^{n} \frac{{}^{n}P_r}{r!} = \sum_{r=1}^{n} \frac{n!}{(n-r)!\,r!} = \sum_{r=1}^{n} {}^{n}C_r$

$= {}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 + \ldots + {}^{n}C_{n-1} + {}^{n}C_n$

We know that,

${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \ldots + {}^{n}C_{n-1} + {}^{n}C_n = 2^n$

$\therefore \sum_{r=1}^{n} {}^{n}C_r = 2^n - {}^{n}C_0 = \mathbf{2^n - 1.}$

15. The given word SERIES has 2 S and 2 E and the rest are distinct. The number of ways of arranging the 4 letters of word are as follows:

I. 4 letters are distinct, *i.e.*, S, E, R, I

$\therefore$ Number of ways of arranging these distinct 4 letters $= 4! = 24$

II. 2 letters are same and 2 are distinct, *i.e.*

SSRI, SSRE, SSIE, EERI, EERS, EEIS

$\therefore$ Number of ways of arranging letters in this way

$= \frac{4!}{2!} \times 6 = 72$

III. Two are same of one kind and two are same of other kind

$\therefore$ Number of ways of above arrangement $= \frac{4!}{2!\,2!} = 6$

$\therefore$ Total number of ways $= 24 + 72 + 6 =$ **102.**

9 Probability

KEY FACTS

1. Probability: It is the chance of happening of an event when measured quantitatively.

2. Trial: It is the performance of an experiment, such as throwing a dice or tossing a coin.

3. Random Experiment. A random experiment is an experiment in which

(*i*) all the outcomes of the experiment are known in advance and

(*ii*) the exact outcome of any specific performance of the experiment is unpredictable, i.e., not known in advance.

For example,

(*i*) Tossing a coin is a random experiment as :

(*a*) Assuming that the coin does not land on the edge, there are only two possible outcomes of tossing– a head or a tail.

(*b*) But on preforming the experiment we are not sure whether we shall get a head or a tail.

(*ii*) Drawing a card from a well shuffled pack of 52 cards is a random experiment as:

(*a*) Total number of outcomes is 52 since there are 52 cards.

(*b*) We do not know in advance which card is drawn in a specific draw.

4. Event: It is something that happens, whose probability we may want to measure, as drawing a spade from a pack of 52 cards.

5. Occurrence of an Event: If a die* is thrown and the desired event is getting an even number on the top, then the event A is ***said to have occurred*** if the number appearing on top of the die is either 2 or 4 or 6. If 1 or 3 or 5 appears on the top, then event ***A has not occurred.***

6. Sample Space is the set of all possible outcomes of a random experiment generally denoted by ***S***. For example, when a die is thrown, $S = \{1, 2, 3, 4, 5, 6,\}$. Each element of a sample space is called ***a sample point.***

7. Elementary or Simple Event. An event containing only a single sample point is called an *elementary event*.

For example,

(*i*) If a coin is tossed, getting head or tail are two elementary events.

(*ii*) If a die is thrown, getting anyone of 1, 2, 3, 4, 5 or 6 on the top are six elementary events.

8. Compound or Composite Event. An event containing more than one sample point is called a compound or composite event. It is a ***subset of the sample spaces.***

For example,

(*i*) If a die is thrown, then the statement "getting an odd number on the top" is a compound event containing the sample points 1, 3 and 5.

Let A denote the given event, then $A = \{1, 3, 5\}$

Sample space $S = \{1, 2, 3, 4, 5, 6\}$

Clearly $A \subset S$

---

* The plural of die is **dice.**

(*ii*) If two coins are tossed simultaneously then the sample space S is given by :

$S = \{(HH), (HT), (TH), (TT)\}$

If A be the event that "one head and one tail" turn up, then

$A = \{(HT), (TH)\}$

9. Sure Event: Let S be the sample space of a random experiment since S is a subset of itself, it also represents an event. Since all the outcomes of a random experiment belong to sample space S, the event S represents a *sure event.*

10. Impossible Event: If we assume that S is the sample space of a random experiment then ϕ (null set) is a subset of S.

As no outcome of the experiment belong to ϕ, the event ϕ can never occur when the experiment is performed. Hence ϕ is an impossible event.

11. Probability of a **sure event** is **1** and the probability of an **impossible event** is 0.

Probability is never less than 0 and greater than 1.

12. Equally Likely Events. When two or more events have an equal chance of happening, then they are called equally likely.

Ex. (*i*) In a toss of a coin, both head and tail are equally likely to occur.

(*ii*) In a throw of a die, all the six faces (1, 2, 3, 4, 5, 6) are equally likely to occur.

13. Complementary Event. Let S be the sample space associated with a random experiment and let E be an event. The event of non-occurrence of E is called the complementary event of E.

It is represented by $\overline{E}$ or E'. $\overline{E}$ is also known as **not *E*.**

Ex. If in a throw of a die, getting a prime number is a favourable event A, then the event of getting a number that is not prime is called the complementary event of A.

If $A = \{2, 3, 5\}$, then $\overline{A} = \{1, 4, 6\}$

14. Exhaustive Events. The total number of possible outcomes of a random experiment is called exhaustive events.

Ex. (*i*) In tossing a coin, exhaustive events are 2 (Head or tail)

(*ii*) In throwing a die, the exhaustive number of cases is 6, since any of the faces marked with 1, 2, 3, 4, 5 or 6 may come uppermost.

(*iii*) In drawing 4 cards from a well shuffled pack of 52 cards, the exhaustive number of cases is $52\ C_4$, since 4 cards can be drawn out of 52 cards in $52\ C_4$ ways.

15. Mutually Exclusive Events. The events are said to be mutually exclusive if they cannot occur simultaneously in a single draw. Events such as tossing a head or a tail with a coin, drawing a queen or a jack from a pack of cards, throwing an even number or an odd number with a dice are all *mutually exclusive events. Here,* ***the occurrence of an event rules out the happening of the other event in the same experiment, i.e,.***

If we toss a coin, we can never get a head and a tail in the same toss.

Ex : Suppose a card is drawn from a pack of cards and we define events as below :

A : The card drawn is a spade ;

B : The card drawn is a heart ;

C : The card drawn is a king.

A and B are mutually exclusive, since if a card is a spade, it cannot be a heart, *i.e*, if A occurs, B cannot occur and *vice versa*. On the other hand, events A and C are not mutually exclusive as the card can be the king of spades.

Similarly for B and C, the card can be king of hearts.

16. Algebra of Events. Let A, B and C be any two events associated with a random experiment whose sample space is S. Then,

(*i*) $A \cup B$. **(Union of *A* and *B*)** is the event that occurs if A occurs or B occurs or both A and B occur.

(*ii*) $A \cap B$. **(Intersection of *A* and *B*).** It is the event set which contains all sample points or outcomes which the two events A and B have in common.

Ex: (In a throw of a die), A : Event of getting an odd number
B : Event of getting a prime number
$\Rightarrow A = \{1, 3, 5\}, B = \{2, 3, 5\}$. Then, $A \cap B = \{3, 5\}$.

(*iii*) $\overline{A}$, **(Complement of an event A).** It is the set of all sample points of sample spaces that are not contained in A.

In the toss of a coin, if A is getting a tail, then $\overline{A}$ is getting a head.

(*iv*) **($A \cup B \cup C$)** is the event that occurs when at least one of the events A, B or C occurs

(*v*) **($A \cap B \cap C$)** is the event that occurs when all the three events A, B and C occur.

(*vi*) ***As mutually exclusive events*** cannot occur together, if events A and B are mutually exclusive, then $A \cap B = \phi$ since A and B have nothing in common.

(*vii*) ***Mutually exclusive and exhaustive events.***

Let S be the sample space associated with a random experiment. If $E_1, E_2, ..., E_n$ are **mutually exclusive** elementary events associated with the random experiment, then $Ei \cap Ej = \phi$ for all $i \neq j$ and $E_1 \cup E_2 \cup E_3 ... \cup E_n = S$

(Since **exhaustive** means the total number of possible outcomes) Therefore, **an event and its complementary event are both mutually exclusive and exhaustive since:**

$A \cap \overline{A} = \phi$ and $A \cup \overline{A} = S$.

Ex. Let 1 ball be drawn from a bag containing 12 balls of which 4 balls are white, 4 are red and 4 are green.

Let A **:** Event-ball drawn is white
B **:** Event-ball drawn is red
C **:** Event-ball drawn is green.

It is obvious that one of the three events must occur as the ball drawn is either white or red or green.

This means that A, B and C form a mutually exclusive and exhaustive set of events.

$\therefore A \cap B = \phi, B \cap C = \phi, A \cap C = \phi$ and $A \cup B \cup C = S$.

17. Generally probability examples involve coins, dice and a pack of cards.

We give below the sample spaces (S) associated with some common events.

(*i*) **Toss of a coin:**

$S = \{H, T\} \Rightarrow n(S) = 2$ where ***n*(Event)** is **number of outcomes.**

(*ii*) **Toss of two coins simultaneously:**

$S = \{HH, HT, TH, TT\} \Rightarrow n(S) = 4 = (2 \times 2)$

(*iii*) **Toss of three coins simultaneously :**

$S = \{HHH, HHT, HTT, TTT, THH, TTH, THT, HTH\}$

$\Rightarrow n(S) = 8 = (2 \times 2 \times 2)$

(*iv*) **Throw of a single die:**

$S = \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6$

(*v*) **Throw of two dice together :**

$$\boldsymbol{S} = \left\{\begin{matrix} (1,1) & (1,2) & ... & (1,6) \\ (2,1) & (2,2) & ... & (2,6) \\ (3,1) & (3,2) & ... & (3,6) \\ (4,1) & (4,2) & ... & (4,6) \\ (5,1) & (5,2) & ... & (5,6) \\ (6,1) & (6,2) & ... & (6,6) \end{matrix}\right\} \Rightarrow n(S) = 36 = (6 \times 6)$$

(*vi*) **Drawing a card from a normal pack of 52 cards :**

Here S = All 52 cards. $\Rightarrow n(S) = 52$.

Note : A pack of cards consists of **52 cards**, divided in **4 suits**, **2 red** (*Hearts and Diamonds*) and **2 black** (*Spades and Clubs*).

Each of the suits has 13 cards bearing the values 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, Queen, King and Ace. The Jack, Queen and King are called **picture** or **face cards.** Therefore there are 12 face cards in all.

18. Definition of Probability: ***If in a random experiment there are*** **n** ***mutually exclusive and equally likely elementary events and*** **m** ***of them are favourable to an event A, then the probability P of happening of A denoted by P(A) is defined as the ratio*** $\frac{m}{n}$.

i.e, $$P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \frac{n(A)}{n(S)},$$

where $n(A)$ = number of sample points in event A

$n(S)$ = number of sample points in sample space S.

If $P(A)$ = probability of occurrence of A, then

$P(\bar{A})$ = probability of failure of A or non-occurrence of A = **1 – $P(A)$ as $P(A) + P(\bar{A}) = 1$**

Note : The probabilities of mutually exclusive and exhaustive events always adds up to 1.

19. Odds in favour of an event and Odds against an event.

Probabilities are often expressed in terms of ***"odds".*** Suppose an outcome ***A*** **occurs** ***m*** **times** out of ***n*** **equally outcomes** in a random experiment, then ***A*** **does not occur (*n* – *m*) times.** Thus,

$$\text{Odds in favour of } A = \frac{\text{Number of times } A \text{ occurs}}{\text{Number of times } A \text{ does not occur}}$$

$$= \frac{m}{n-m} = m : (n-m)$$

$$\text{Odds against } A = \frac{\text{Number of times } A \text{ does not occur}}{\text{Number of times } A \text{ occurs}}$$

$$= \frac{n-m}{m} = (n-m) : m$$

Thus, • Odds in favour of an event = Probability (success) : Probability (failure)

• Odds against an event = Probability (failure) : Probability (success)

Ex : What are the odds in *favour* of and against the event A : "getting a 5 in a single throw of die".

Here, $A = \{5\}$ and sample space $S = \{1, 2, 3, 4, 5, 6\}$

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{6}$ $\Rightarrow P(\text{not } A) = 1 - \frac{1}{6} = \frac{5}{6}$

$\therefore$ Odds in favour of $A = P(A) : P(\bar{A}) = 1/6 : 5/6 = 1 : 5$

Odds against $A = P(\bar{A}) : P(A) = 5/6 : 1/6 = 5 : 1$

20. Addition Theorem of Probability

(*a*) **For Two Events.** If A and B are two events associated with a random experiment, then

$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

$\Rightarrow$ $P(A \text{ or } B) = P(A) + P(B) — P(A \text{ and } B)$

Corollary 1: If A and B are **mutually exclusive events**, then,

$P(A \cap B) = 0$, therefore

$\mathbf{P(A \cup B) = P(A) + P(B)}$

Corollary 2: $\mathbf{P(A \text{ or } B)} \le P(\underline{A}) + P(B)$

Given, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Since, $P(A \cap B)$ is greater than or equal to 0,

$P(A \text{ or } B) < P(A) + P(B)$

Equality in the above result holds when A and B are mutually exclusive as $P(A \cap B) = 0$.

(*b*) **For three events**

If A, B and C be any three events associated with a random experiment, then

$$\boldsymbol{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)}$$

Corollary 1: If A, B and C are mutually exclusive events, then

$P(A \cap B) = P(B \cap C) = P(A \cap C) = P(A \cap B \cap C) = 0$

$\therefore$ $\boldsymbol{P(A \cup B \cup C) = P(A) + P(B) + P(C)}$

Note : (*i*) If A and B are two events such that $A \subseteq B$, then $P(A) \leq P(B)$

(*ii*) If E is an event associated with a random experiment, then $0\ P(E) \leq 1$

21. Independent and Dependent Events.

An event E is said to be **independent** of another event F, when the actual happening of F does not influence in any degree the probability of the happening of E. If the probability of the happening of E is dependent or influenced by the previous happening then E is said to be dependent on F.

Illustration :

Imagine a bag containing eight balls – 5 of which are red and 3 of other colours. If you pick out two balls from the bag at random one after the other, what is the probability that both will be red.

The probability will be different depending on whether or not you put the first ball back in the bag before you pick the second ball.

Case I : If the first ball picked is replaced back in the bag, then the second event is **independent** of first as first draw of a red ball has no effect on the second draw.

Case II : If the first ball picked is red and you do not put it back, there will be one less ball to pick from. In this way the probability of picking a second red ball depends on whether the first ball was red, hence a case of **dependent events.**

22. Probability of Independent Events – Multiplication Rule

Study the following situations.

1. Umesh tosses a coin and Preeti rolls a die. Umesh knows that the probability of his coin landing “head” is $\frac{1}{2}$.

The probability of Preeti’s die showing a six is $\frac{1}{6}$.

What is the probability of getting a head and a six ?

If we draw the sample space diagram, we see that there are 12 possible outcomes, and

		1	2	3	4	5	6
Coin	H						H6
	T						

$$P\,(H \text{ and } 6) = \frac{1}{12}$$

$$P\,(H) = \frac{1}{2}, P(6) = \frac{1}{6}$$

$$P\,(H \text{ and } 6) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

Whether or not the coin lands “head” has no effect or whether the die shows a ‘six’ or any score has no effect.

$P\,(H \text{ and } 6) = P\,(H) \times P\,(6)$

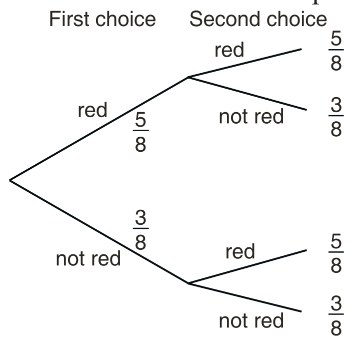
If two events, A and B, are independent $P\,(A \text{ and } B) = P\,(A) \times P\,(B)$

The two events, getting a **“head”** and scoring a **‘six’**, are **independent.**

The above rule is called the **multiplication** or the **‘AND’ rule** and is used to find the probabilities of joint occurrence of independent events.

2. In the situation described in point 22, the tree diagram to show the probability of picking two red balls, is as under:

Independent events – first ball replaced before second ball is picked



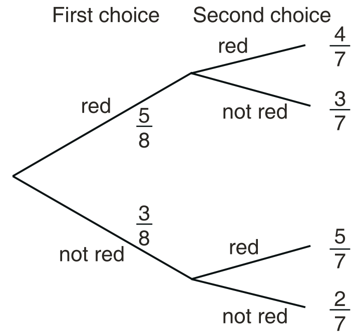
Here the probability of two red balls being chosen is:

$\frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$

or:

P (red) × P (red) = P (red and red)

Dependent events – first ball **not** replaced before second ball is picked



If a red ball is picked first and not replaced there are only seven balls left to choose from. Only four of these are red, so for the second ball:

P (red) = $\frac{4}{7}$

and P (not red) = $\frac{3}{7}$

If a 'not red' ball is picked first and not replaced there are still seven balls left to choose from, but now the five left are red, so for the second ball:

P (red) = $\frac{5}{7}$ and P (not red) = $\frac{2}{7}$

So when the first ball is not replaced, the probability of two reds being picked is:

$\frac{5}{8} \times \frac{4}{7} = \frac{20}{56} = \frac{5}{14}$

23. Multiplication Theorem on Probability

Statement I. ***If two events A and B are independent, then probability that they will both occur is equal to the product of their individual probabilities.***

i.e. $\quad \boldsymbol{P(A \text{ and } B) = P(A) \times P(B)}$ ['AND' rule]

In set notation, $\quad \boldsymbol{P(A \cap B) = P(A) \times P(B)}$

Note. If A, B, C are independent events, then

$$\boldsymbol{P(A \text{ and } B \text{ and } C) = P(A) \times P(B) \times P(C)}$$

In set notation, $\boldsymbol{P(A \cap B \cap C) = P(A) \,.\, P(B) \,.\, P(C)}$.

In general, if $A_1, A_2, \ldots\ldots A_n$ are n independent events, then

$P(A_1 \text{ and } A_2 \text{ and } A_3 \text{ and } \ldots\ldots A_n) = P(A_1) \times P(A_2) \times P(A_3) \times \ldots\ldots \times P(A_n)$

In set notation, $P(A_1 \cap A_2 \cap A_3 \ldots\ldots \cap A_n) = P(A_1) \times P(A_2) \ldots\ldots \times P(A_n)$

Finding probabilities of simultaneous occurrence of two independent events.

Method. ***Use the relation P (A ∩ B) = P (A) . P (B).***

Ex. 1. Two dice are thrown. Find the probability of getting an odd number on the first die and a multiple of 3 on the other.

Sol. Let A : Getting an odd number on first die; B : Getting a multiple of 3 on second die, then

$$A = \{1, 3, 5\}, B = \{3, 6\} \quad \therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}, P(B) = \frac{2}{6} = \frac{1}{3}$$

The events A and B are independent

$\therefore$ Required probability = $P(A) \,.\, P(B) = \frac{1}{2} \times \frac{1}{3} = \boldsymbol{\frac{1}{6}}$.

Ex. 2. A class consists of 80 students, 25 of them are girls and 55 boys. 10 of them are rich and 20 are fair complexioned. What is the probability of selecting a fair complexioned rich girl?

Sol. Let $P(A)$ = Probability of selecting a fair complexioned person. Then $P(A) = \frac{20}{80} = \frac{1}{4}$

Let $P(B)$ = Probability of selecting a rich person. Then $P(B) = \frac{10}{80} = \frac{1}{8}$

Let $P(C)$ = Probability of selecting a girl. Then $P(C) = \frac{25}{80} = \frac{5}{16}$

Since the events A, B, C are independent, so probability of selecting a fair complexioned rich girl

$$= P(A) \,.\, P(B) \,.\, P(C) = \frac{1}{4} \times \frac{1}{8} \times \frac{5}{16} = \frac{5}{512} = \mathbf{0.009}.$$

3. Formal Definition of Independence

Events A and B are independent if $P(A \cap B) = P(A).\, P(B)$, otherwise they are dependent.

Caution. Any physical description of the events will not tell us if they are independent or not.

Proving the independence or dependence of events.

Ex. 1. A coin is tossed thrice and all eight outcomes are assumed equally likely. Find whether the events E and F are independent or not ?
E : the number of heads is odd.
F : the number of tails is odd.

Sol. When a coin is tossed three times, the sample space is given by
$S = [HHH, HHT, HTH, THT, THH, HTT, TTH, TTT]$
$E = \{HHH, HTT, THT, TTH\}$, $F = \{TTT, HTH, THH, HHT\}$

$$E \cap F = \phi$$

$$P(E) = \frac{4}{8} = \frac{1}{2},\quad P(F) = \frac{4}{8} = \frac{1}{2},\ P(E \cap F) = \phi$$

$$P(E) \,.\, P(F) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \neq P(E \cap F)$$

$\therefore$ E and F are not independent events.

Ex. 2. A fair coin is tossed three times. Let A, B and C be defined as follows:
A = {first toss is heads}, B = {second toss is heads} and
C = {exactly two heads are tossed in a row}
Check the independence of (*i*) A and B, (*ii*) B and C and (*iii*) C and A.

Sol. The sample space is $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$$A = \{HHH, HHT, HTH, HTT\},\ B = \{HHH, HHT, THH, THT\} \text{ and } C = \{HHT, THH\}$$

Also, $$A \cap B = \{HHH, HHT\},\ B \cap C = \{HHT, THH\},\ C \cap A = \{HHT\}$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{8} = \frac{1}{2},\ P(B) = \frac{n(B)}{n(S)} = \frac{4}{8} = \frac{1}{2},\ P(C) = \frac{n(C)}{n(S)} = \frac{2}{8} = \frac{1}{4},$$

$$P(A \cap B) = \frac{n[(A \cap B)]}{n(S)} = \frac{2}{8} = \frac{1}{4}. \text{ Similarly, } P(B \cap C) = \frac{2}{8} = \frac{1}{4} \text{ and } P(C \cap A) = \frac{1}{8}.$$

Now, $P(A) \,.\, P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(A \cap B)$, therefore, A and B are independent events.

$P(B) \,.\, P(C) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8} \neq P(B \cap C)$ which is $\frac{1}{4}$. Therefore, B and C are not independent events.

$P(C) \,.\, P(A) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} = P(C \cap A)$. Therefore, C and A are independent events.

Three events A, B and C are independent if:

(*i*) $P(A \cap B) = P(A).\, P(B)$, $P(A \cap C) = P(A).\, P(C)$ and $P(B \cap C) = P(B).\, P(C)$, *i.e.*, if the events are pairwise independent and

(*ii*) $P(A \cap B \cap C) = P(A) \,.\, P(B) \,.\, P(C)$.

The example given below shows that condition (*ii*) does not follow from condition (*i*). In other words, three events may be pairwise independent but not independent themselves.

Ex. 3. Let a pair of fair coins be tossed. Here $S = \{HH, HT, TH, TT\}$. Consider the events A = {heads on the first coin} = $\{HH, HT\}$, B = {heads on the second coin} = $\{HH, TH\}$, C = {heads on exactly one coin} = $\{HT, TH\}$

Sol. Then $$P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2} \text{ and}$$

$$P(A \cap B) = P(\{HH\}) = \frac{1}{4},\ P(A \cap C) = P(\{HT\}) = \frac{1}{4}$$

$$P(B \cap C) = P(\{TH\}) = \frac{1}{4},\ (A \cap B \cap C) = \phi$$

$\therefore$ $$P(A \cap B \cap C) = P(\phi) = 0 \neq P(A).\ P(B).\ P(C)$$

Thus condition (*i*) is satisfied, *i.e.*, the events are pairwise independent. But condition (*ii*) is not satisfied and so the three events are not independent.

Ex. 4. Let A and B be two possible events of an experiment such that $P(A) = 0.4$, $P(A \cup B) = 0.7$ and $P(B) = p$. (*i*) For what value of p are A and B mutually exclusive ? (*ii*) For what value of p are A and B independent ? (*AP 1996*)

Sol. (*i*) Since A and B are mutually exclusive, therefore, $A \cap B = \phi$.

Substituting in $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, we have $0.7 = 0.4 + p - 0 \Rightarrow p = 0.3$

(*ii*) Since A and B are independent, therefore, $P(A \cap B) = P(A) . P(B)$

i.e., $P(A) + P(B) - P(A \cup B) = P(A) . P(B)$

Substituting the given values we have $0.4 + p - 0.7 = 0.4 \times p \Rightarrow \boldsymbol{p = 0.5}$**.**

Also, we have the following :

1. If p_1, p_2, p_3 .... p_n are the probabilities that certain events happen, the probability of **failing of all** these events is given by
$$\boldsymbol{P = (1 - p_1)(1 - p_2) \ldots\ldots (1 - p_n)}$$
2. **Probability of the occurrence of AT LEAST ONE of the several independent events of a random experiment.** If p_1, p_2, p_3, ... p_n are the probabilities that certain events happen, then the probability that at least one of these events must happen is
$$\boldsymbol{1 - (1 - p_1)(1 - p_2)(1 - p_3) \ldots\ldots (1 - p_n).}$$
3. **Probability of the occurrence of exactly one of the two events.**
Let A and B be two events associated with a random experiment and let p_1, and p_2 be the probabilities of their occurrence respectively, then
$$P(A) = p_1, P(B) = p_2 \quad \therefore \quad P(\overline{A}) = 1 - P(A) = 1 - p_1 = q_1 \text{ (say)}$$
$$P(\overline{B}) = P(\text{non-occurrence of } B) = 1 - P(B) = 1 - p_2 = q_2 \text{ (say)}$$
Since only one of the two events A and B is to occur, therefore, there are two possibilities
(*i*) A occurs and B does not (*ii*) B occurs and A does not
Since these are mutually exclusive events, therefore,
Reqd. probability = $P(A\overline{B}) + P(\overline{A}B) = P(A) . P(\overline{B}) + P(B) . P(\overline{A}) = \boldsymbol{p_1 q_2 + p_2 q_1}$.

Ex. 1. A husband and a wife appear in an interview for two vacancies in the same post. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that

(*i*) Both of them will be selected **(*ii*) Only one of them will be selected**

(*iii*) None of them will be selected **(*iv*) At least one of them will be selected.**

Sol. Let A : Event of husband being selected

B : Event of wife being selected

Then $P(A) = \frac{1}{7}$, $P(B) = \frac{1}{5}$

$P(\bar{A}) = 1 - \frac{1}{7} = \frac{6}{7}, P(\bar{B}) = 1 - \frac{1}{5} = \frac{4}{5}$

(*i*) P(Both are selected) = $P(A) \times P(B) = \frac{1}{7} \times \frac{1}{5} = \frac{1}{35}$

(*ii*) P(only one is selected) = P(A selected) × P(B not selected) + P(A not selected) × P(B selected)

$$= P(A) \times P(\bar{B}) + P(\bar{A}) \times P(B) = \frac{1}{7} \times \frac{4}{5} + \frac{6}{7} \times \frac{1}{5} = \frac{4}{35} + \frac{6}{35} = \frac{10}{35} = \mathbf{\frac{2}{7}}$$

(*iii*) P(none selected) = $P(\bar{A}) \times P(\bar{B}) = \frac{6}{7} \times \frac{4}{5} = \frac{24}{35}$

(*iv*) P(at least one selected) = 1 – P(none selected) = $1 - \frac{24}{35} = \mathbf{\frac{11}{35}}$

4. Addition theorem for independents events:

If $A_1, A_2, ..., A_n$ are n independent events associated with a random experiment, then

$$P(A_1 \cup A_2 \cup ..... A_n) = 1 - P(\overline{A_1})\, P(\overline{A_2})\, ...\, P(\overline{A_n})$$

5. Repeated events

If the probability that an event will happen is p, the chance that it will happen in any succession of r trials is p^r.

The probability of non-occurrence is $q = 1 - p$. Therefore, for n repeated non-occurrence of the event we have the probability $q^r = (1 - p)^r$.

Ex. 2. A machine manufactured by a firm consists of two parts A and B. But of 100 A's manufactured, 9 are likely to be defective and out of 100 B's manufactured 5 are likely to be defective. Find the probability that a machine manufactured by the firm is free of any defect. Give your answer, rounded off to two places of decimal.

Sol. Let E = {part A is defective}, F = {part B is defective}.

Then, $P(E) = \frac{9}{100}, P(F) = \frac{5}{100}$ $\therefore$ $P(\bar{E}) = 1 - \frac{9}{100} = \frac{91}{100}$ and $P(\bar{F}) = 1 - \frac{5}{100} = \frac{95}{100}$

Since E and F are independent, therefore, $\bar{E}$ and $\bar{F}$ are also independent.

Now P (none is defective) = $P(\bar{E}) \,.\, P(\bar{F}) = \frac{91}{100} \times \frac{95}{100}$ = 0.91 × 0.95 = 0.8645 = **0.86** to 2 d.p.

Finding probability of occurrence of at least one event for independent events.

Ex. 1. The probability that Dimpu gets scholarship is 0.9 and Pintu will get is 0.8. What is the probability that at least one of them gets the scholarship. (*Pb 1995 C*)

Sol. Let A be the event that Dimpu gets scholarship and B the event that Pintu gets scholarship. It is given that $P(A) = 0.9$, $P(B) = 0.8$.

The probability that none of them gets the scholarship = (1 – 0.9) (1 – 0.8) = 0.1 × 0.2 = 0.02

$\therefore$ The probability that at least one of them gets the scholarship = 1 – 0.02 = **0.98.**

Ex. 2. A can solve 90% of the problems given in a book and B can solve 70%. What is the probability that at least one of them will solve a problem selected at random from the book ?

Sol. Let E be the event that A solve the problem and F the event that B solves the problem.

Then $P(E) = \frac{90}{100} = \frac{9}{10}, P(F) = \frac{70}{100} = \frac{7}{10}, P(\bar{E}) = 1 - \frac{9}{10} = \frac{1}{10}, P(\bar{F}) = 1 - \frac{7}{10} = \frac{3}{10}$

$\therefore$ Probability that no one solves the problem $P(\bar{E}) \,.\, P(\bar{F}) = \frac{1}{10} \times \frac{3}{10} = \frac{3}{100}$.

Hence, probability that at least one of them will solve a problem = $1 - \frac{3}{100} = \frac{97}{100} = \mathbf{0.97.}$

Ex. 3. A bag contains 5 white, 7 red and 4 black balls. If four balls are drawn one by one with replacement, what is the probability that none is white.

Sol. Let A, B, C, D denote the events of not getting a white ball in first, second, third and fourth draw respectively. Since the balls are drawn with replacement, therefore, A, B, C, D are independent events such that $P(A) = P(B) = P(C) = P(D)$.

Since out of 16 balls, 11 are not white, therefore, $P(A) = 11/16$.

$\therefore$ Required probability = $P(A) \,.\, P(B) \,.\, P(C) \,.\, P(D)$

$$= \frac{11}{16} \times \frac{11}{16} \times \frac{11}{16} \times \frac{11}{16} = \left(\mathbf{\frac{11}{16}}\right)^{\mathbf{4}}.$$

Ex. 4. A Police man fires six bullets on a decoit. The probability that the decoit will be killed by one bullet is 0.6. What is the probability that the decoit is still alive ?

Sol. Let A_i be the event that the decoit is killed by the ith bullet $(1 \le i \le 6)$. Then $\overline{A_i}$ is the event that the decoit is not killed, $\therefore$ $P(A_i) = 0.6$ and $P(\overline{A_i}) = 1 - 0.6 = 0.4$

$\therefore$ Probability of the decoit being still alive

$= P(\overline{A_1})\, P(\overline{A_2})\, P(\overline{A_3}) P(\overline{A_4})\, P(\overline{A_5})\, P(\overline{A_6}) = 0.4 \times 0.4 \times 0.4 \times 0.4 \times 0.4 \times 0.4 = \mathbf{0.004096.}$

Note. Using simpler notation, we may write success : Bullet will hit the decoit

$p = 0.6 \Rightarrow q = 1 - p = 1 - 0.6 = 0.4$

Required probability = $qqqqqq = (q)^6 = \mathbf{(0.4)^6}$.

Problems solved by using both the addition and multiplication theorems of probability.

Ex. 1. A bag contains 6 black and 3 white balls. Another bag contains 5 black and 4 white balls. If one ball is drawn from each bag, find the probability that these two balls are of the same colour.

Sol. Let W_1 and W_2 denote the events of drawing a white ball from the first and one from the second bag respectively. Let B_1 and B_2 denote the events of drawing black balls from the two bags in the same order. Then

P (both balls are white) = P [white ball from (I bag) and white ball from II bag]

$= P(W_1) \,.\, P(W_2) = \frac{3}{9} \times \frac{4}{9} = \frac{12}{81}$ [‘AND’ Theorem]

Similarly, P (both balls are black) = $\frac{6}{9} \times \frac{5}{9} = \frac{30}{81}$. [‘AND’ Theorem]

$\therefore$ The probability of getting two balls of the same colour [‘OR’ Theorem]

$= \frac{12}{81} + \frac{30}{81} = \frac{42}{81} = \mathbf{\frac{14}{27}}$. (By addition theorem for mutually exclusive events.

Ex. 2. Two balls are drawn at random from a bag containing 3 white, 3 red, 4 green and 4 black balls, one by one without replacement. Find the probability that both the balls are of different colours.

Sol. Given, 3 white (3 W), 3 red (3 R), 4 green (4 G), 4 black (4 B) balls

Total no. of balls = 3 + 3 + 4 + 4 = 14

Two balls are to be drawn, one by one without replacement.

There are 4 possibilities.

	First Ball	Second Ball
Case I	White	Not white
Case II	Red	Not red
Case III	Green	Not green
Case IV	Black	Not black

Since all cases are mutually exclusive, therefore

$$\text{Reqd. probability} = \frac{3}{14}\times\frac{11}{13}+\frac{3}{14}\times\frac{11}{13}+\frac{4}{14}\times\frac{10}{13}+\frac{4}{14}\times\frac{10}{13}=\frac{33+33+40+40}{14\times13}=\frac{146}{182}=\mathbf{\frac{73}{91}}.$$

Ex. 3. The probability of student A passing examination is $\frac{3}{7}$ and of student B passing is $\frac{5}{7}$. Assuming the two events "A passes", "B passses" as independent, find the probability of (*i*) only A passing the examination (*ii*) only one of them passing the examination.

Sol. $p_1 = P(A) = \frac{3}{7},\ p_2 = P(B) = \frac{5}{7}\ \therefore\ q_1 = P(\bar{A}) = 1 - P(A) = 1 - \frac{3}{7} = \frac{4}{7}.$

$q_2 = P(\bar{B}) = 1 - P(B) = 1 - \frac{5}{7} = \frac{2}{7}$

(*i*) P (only A passes) = $p_1 q_2 = \frac{3}{7}\times\frac{2}{7} = \mathbf{\frac{6}{49}}$. P (only B passes) = $q_1 p_2 = \frac{4}{7}\times\frac{5}{7} = \frac{20}{49}$

(*ii*) P (only one of them passes) = $p_1 q_2 + q_1 p_2 = \frac{3}{7}\times\frac{2}{7}+\frac{4}{7}\times\frac{5}{7} = \mathbf{\frac{26}{49}}$.

Ex. 4. Two cards are drawn from a well shuffled pack of 52 cards one after another without replacement. Find the probability that one of these is an ace and the other is a queen of the opposite shade.

Sol. Probability of drawing an ace in the first draw = $\frac{4}{52}$.

Probability of drawing a queen of opposite shade in the second draw = $\frac{2}{51}$.

Probability of drawing a queen in the first draw = $\frac{4}{52}$.

Probability of drawing an ace of opposite shade in the second draw = $\frac{2}{51}$.

$\therefore$ Required probability = $\frac{4}{52}\times\frac{2}{51}+\frac{4}{52}\times\frac{2}{51} = \mathbf{\frac{4}{663}}$. ['AND' and 'OR' Theorems]

Ex. 5. *A* can hit a target three times in five shots, *B* two times in five shots and *C* three times in four shots. They fire a volley. What is the probability that two shots hit the target ?

Sol. $P(A) = \frac{3}{5}, P(B) = \frac{2}{5}$ and $P(C) = \frac{3}{4}$. In order that two shots may hit the target the following three mutually exclusive events are possible,

(*i*) A and B hit the target and not C. (*ii*) B and C hit the target and not A.

(*iii*) A and C hit the target and not B.

If p_1, p_2, p_3 are the probabilities of these three events then

$$p_1 = P(A).\ P(B)\ .\ P(\text{not } C) = \frac{3}{5}\times\frac{2}{5}\times\left(1-\frac{3}{4}\right) = \frac{3}{5}\times\frac{2}{5}\times\frac{1}{4} = \frac{6}{100}$$

$$p_2 = P(B)\times P(C)\times P(\text{not } A) = \frac{2}{5}\times\frac{3}{4}\times\left(1-\frac{3}{5}\right) = \frac{2}{5}\times\frac{3}{4}\times\frac{2}{5} = \frac{12}{100}$$

$$p_3 = P(A) \times P(C) \times P(\text{not } B) = \frac{3}{5} \times \frac{3}{4} \times \left(1 - \frac{2}{5}\right) = \frac{3}{5} \times \frac{3}{4} \times \frac{3}{5} = \frac{27}{100}$$

$\therefore$ Required probability $= p_1 + p_2 + p_3 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} = \frac{45}{100} = \mathbf{\frac{9}{20}}$.

Ex. 6. The probability of A, B, C solving a problem are $\frac{1}{3}, \frac{2}{7}$ and $\frac{3}{8}$ respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them will solve it.

Sol. Let E_1, E_2, E_3 be the events that the problem is solved by A, B, C respectively and let p_1, p_2, p_3 be corresponding probabilities. Then,

$$p_1 = P(E_1) = \frac{1}{3},\ p_2 = P(E_2) = \frac{2}{7},\ p_3 = P(E_3) = \frac{3}{8},\ q_1 = P(\overline{E_1}) = 1 - \frac{1}{3} = \frac{2}{3},$$

$$q_2 = P(\overline{E_2}) = 1 - \frac{2}{7} = \frac{5}{7},\ q_3 = P(\overline{E_3}) = 1 - \frac{3}{8} = \frac{5}{8}.$$

The problem will be solved by exactly one of them if it happens in the following mutually exclusive ways:

(1) A solves and B, and C do not solve;

(2) B solves and A, and C do not solve;

(3) C solves and A, and B do not solve;

Required probability $= p_1 q_2 q_3 + q_1 p_2 q_3 + q_1 q_2 p_3$

$$= \frac{1}{3} \times \frac{5}{7} \times \frac{5}{8} + \frac{2}{3} \times \frac{2}{7} \times \frac{5}{8} + \frac{2}{3} \times \frac{5}{7} \times \frac{3}{8} = \frac{25}{168} + \frac{5}{42} + \frac{5}{28} = \mathbf{\frac{25}{56}}.$$

MORE SOLVED EXAMPLES

Ex. 1. If two coins are tossed once, what is the probability of getting at least one head ?

Sol. When two coins are tossed once, there are four possible outcomes, *i.e.*, $S = \{HH, HT, TH, TT\}$

$\therefore$ Total number of outcomes $= n(S) = 4$

Let A : Event of getting at least one head

$\Rightarrow A = \{HH, HT, TH\} \Rightarrow n(A) = 3$

$\therefore P(A) = \frac{n(A)}{n(S)} = \mathbf{\frac{3}{4}}$.

Ex. 2. Two unbiased dice are rolled. Find the probability of getting a multiple of 2 on one die and a multiple of 3 on the other die ?

Sol. When two unbiased dice are rolled, the possible out comes are

$$S = \begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) & (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) & (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) & (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{bmatrix}$$

$\therefore n(S) = 36$

Let A : getting a multiple of 2 on one die and a multiple of 3 on the other die.

$\Rightarrow A = \{(2, 3), (2, 6), (4, 3), (4, 6), (6, 3), (6, 6), (3, 2), (3, 4), (3, 6), (6, 2), (6, 4)\}$

$\Rightarrow n(A) = 11$

$\therefore P(A) = \frac{n(A)}{n(S)} = \mathbf{\frac{11}{36}}$.

Ex. 3. In an examination there are 3 multiple choice questions and each question has 4 choices. If a student randomly selects answer for all the three questions. What is the probability that the student will not answer all the 3 questions correctly ? ***(NDA/NA 2011)***

Sol. Probability of selecting a correct choice for a question $= \dfrac{1}{4}$ (∵ Out of 4 choices only one is correct)

∴ Probability of answering all the three questions correctly $= \dfrac{1}{4} \times \dfrac{1}{4} \times \dfrac{1}{4} = \dfrac{1}{64}$

⇒ Probability of not answering all the three questions correctly $= 1 - \dfrac{1}{64} = \mathbf{\dfrac{63}{64}}$.

Ex. 4. What is the probability of getting 53 Sundays or 53 Tuesdays or 53 Thursdays in a non–leap year ? ***(Orissa JEE 2010)***

Sol. A non-leap year consists of 365 days. Therefore in a non-leap year there are 52 complete weeks and 1 day over which can be one of the seven days of the week.

Possible outcomes $n(S) = 7$ = {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}.

∴ Number of possible outcomes $n(S) = 7$ (As there are seven days in a week)

Let A : Getting the extra day as Sunday or Tuesday or Thursday

⇒ A = {Sunday, Tuesday, Thursday}

⇒ $n(A) = 3$

∴ $P(A) = \dfrac{n(A)}{n(S)} = \mathbf{\dfrac{3}{7}}$.

Ex. 5. An integer is chosen at random from the first two hundred positive integers. What is the probability that the integer chosen is divisible by 6 or 8 ?

Sol. As there are 200 integers, total number of exhaustive, mutually exclusive and equally likely cases, *i.e*, $n(S) = 200$

Let A : Event of integer chosen from 1 to 200 being divisible by 6

⇒ $n(A) = 33 \quad \left(\dfrac{200}{6} = 33\dfrac{1}{3}\right)$

Let B : Event of integer chosen from 1 to 200 being divisible by 8

⇒ $n(B) = \dfrac{200}{8} = 25$

$A \cap B$: Event of integer chosen from 1 to 200 being divisible by both 6 and 8, *i.e*, divisible by 24

(∵ LCM of 6 and 8 = 24)

⇒ $n(A \cap B) = 8 \quad \left(\dfrac{200}{24} = 8(\text{approx})\right)$

∴ $P(A \cup B) = P(\text{Integer chosen is divisible by 6 or 8})$

$$= P(A) + P(B) - P(A \cap B) = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} = \frac{33}{200} + \frac{25}{200} - \frac{8}{200} = \frac{50}{200} = \mathbf{\frac{1}{4}}.$$

Ex. 6. There are three events E_1, E_2 and E_3, one of which is must and only one can happen. The odds are 7 to 4 against E_1 and 5 to 3 against E_2. Find the odds against E_3. ?

Sol. Since one and only one of the three events E_1, E_2 and E_3 can happen, *i.e*, they are mutually exclusive.

Therefore, $P(E_1) + P(E_2) + P(E_3) = 1$ ...(*i*)

Odds against E_1 are 7 : 4 ⇒ Odds in favour of E_1 are 4 : 7

⇒ $P(E_1) = \dfrac{4}{4+7} = \dfrac{4}{11}$ ...(*ii*)

Odds against E_2 are 5 : 3 ⇒ Odds in favour of E_2 are 3 : 5

⇒ $P(E_2) = \dfrac{3}{3+5} = \dfrac{3}{8}$ ...(*iii*)

$\therefore$ From (i), (ii) and (iii)

$$\frac{4}{11}+\frac{3}{8}+P(E_3)=1 \quad\Rightarrow\quad P(E_3)=1-\left(\frac{4}{11}+\frac{3}{8}\right)=1-\left(\frac{32+33}{88}\right)=1-\frac{65}{88}=\frac{23}{88}$$

$\therefore$ Odds against E_3 are $\dfrac{1-P(E_3)}{P(E_3)}=\dfrac{1-\frac{23}{88}}{\frac{23}{88}}=\dfrac{65/88}{23/88}=$ **65 : 23.**

Ex. 7. Find the probability that the three cards drawn from a pack of 52 cards are all black ?

Sol. Number of ways in which three cards can be drawn from a pack of 52 cards $n(S)={}^{52}C_3$.

Let A : Event of drawing all the three cards as black

Then, $n(A)={}^{26}C_3$ ($\because$ There are 26 black cards)

$$\therefore\ P(A)=\frac{n(A)}{n(S)}=\frac{{}^{26}C_3}{{}^{52}C_3}=\frac{26\times25\times24}{52\times51\times50}=\mathbf{\frac{2}{17}}.$$

Ex. 8. Four cards are drawn from a full pack of cards. Find the probability that :

(a) there is one card of each suit

(b) all the four are spades, and one of them is a king

(c) at least one of the four cards is an ace.

Sol. 4 cards can be drawn from a pack of cards in ${}^{52}C_4$ ways

$\therefore$ Exhaustive number of cases $= n(S)={}^{52}C_4$

(a) There are 4 suits, each containing 13 cards.

Let A : Event of drawing one card from each suit

$\Rightarrow$ Favourable number of cases $= n(A) = 13C_1\times13C_1\times13C_1\times13C_1$

$$\therefore\ P(A)=\frac{n(A)}{n(S)}=\frac{{}^{13}C_1\times{}^{13}C_1\times{}^{13}C_1\times{}^{13}C_1}{{}^{52}C_4}=\frac{13\times13\times13\times13}{\frac{52\times51\times50\times49}{4\times3\times2\times1}}\qquad \because\left[{}^{n}C_r=\frac{\lfloor n}{\lfloor n-r\,\lfloor r}\right]$$

$$=\mathbf{\frac{2197}{20825}}$$

(b) Let A : Drawing 4 spade cards of which one is king of spades.

Then, Favourable number of cases $= n(A)={}^{12}C_3\times1$

($\because$ There is only one king of spades and the rest of the three spades we draw from remaining 12 spade cards)

$n(S)={}^{52}C_4$

$$\therefore\quad P(A)=\frac{n(A)}{n(S)}=\frac{{}^{12}C_3\times1}{{}^{52}C_4}=\frac{\frac{12\times11\times10}{3\times2\times1}}{\frac{52\times51\times50\times49}{4\times3\times2\times1}}=\frac{12\times11\times10\times4}{52\times51\times50\times49}=\mathbf{\frac{44}{54145}}$$

(c) Let A : Drawing at least one ace.

Now since there are 4 aces in the pack of 52 cards, therefore, the number of ways of drawing 4 cards so that no card is an ace $={}^{48}C_4$

$\therefore$ Probability of drawing four cards so that none is an ace

$$P(\bar{A})=\frac{{}^{48}C_4}{{}^{52}C_4}=\frac{48\times47\times46\times45}{52\times51\times50\times49}=\frac{38916}{54145}$$

[Here $\bar{A}$ denotes the complement of event A, $i.e$, non-happening of event A]

$$\therefore\quad P(A)=1-P(\bar{A})=1-\frac{38916}{54145}=\frac{15229}{54145}$$ ($\because P$(Event) + P(complement of event) = 1)

Ex. 9. Among 15 players, 8 are batsman and 7 are bowlers. Find the probability that a team is chosen of 6 batsman and 5 bowlers ? ***(UPSEE 2002)***

Sol. The chosen consists of players (6 + 5).

$\therefore$ Number of ways of selecting 11 players out of 15 players = $n(S) = {}^{15}C_{11}$

Let A : Event of choosing 6 batsmen of 8 batsmen and 5 bowlers of 7 bowlers

Then, $n(A) = {}^8C_6 \times {}^7C_5$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{{}^8C_6 \times {}^7C_5}{{}^{15}C_{11}} = \frac{\frac{8\times7}{2}\times\frac{7\times6}{2}}{\frac{15\times14\times13\times12}{4\times3\times2\times1}} = \frac{8\times7}{2}\times\frac{7\times6}{2}\times\frac{4\times3\times2\times1}{15\times14\times13\times12} = \mathbf{\frac{28}{65}}.$$

Ex. 10. A bag contains 7 white, 5 black and 4 red balls. Four balls are drawn without replacement. Find the probability that at least three balls are black. ?

Sol. Let A : Event of getting at least 3 black balls

Then $n(A) = {}^5C_3 \times {}^{11}C_1 + {}^5C_4$ ($\because$ Besides 5 black balls, there are 11 other balls)

(3 black + others) (4 black)

$$= \frac{5\times4}{2} \times 11 + 5 = 115$$

Total numbers of ways in which 4 balls can be drawn from (7 + 5 + 4) = 16 balls

$$n(S) = {}^{16}C_4 = \frac{16\times15\times14\times13}{4\times3\times2\times1} = 1820$$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{115}{1820} = \mathbf{\frac{23}{364}}.$$

Ex. 11. Four boys and three girls stand in a queue for an interview. What is the probability that they will be in alternate positions ?

Sol. Total number of ways of arranging 4 boys and 3 girls, *i.e.*, 7 people in a queue (row) = $n(S) = 7!$

Let A : Event in which the 4 boys and 3 girls occupy alternate position.

This is possible when the 4 boys occupy the 4 odd places standing form position 1 and the girls fill in the three even places between then as :

1		3		5		7
B	G	B	G	B	G	B
	2		4		6	

$\therefore$ 4 boys can be arranged in a queue in 4! ways

3 girls can fill the gaps in the queue in 3! ways.

$$\therefore \quad n(A) = 4! \times 3!$$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{4!\times3!}{7!} = \frac{4\times3\times2\times1\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1} = \mathbf{\frac{1}{35}}.$$

Ex. 12. A bag contains 30 tickets numbered from 1 to 30. Five tickets are drawn at random and arranged in ascending order. Find the probability that the third number is 20.

Sol. Total number of ways in which 5 tickets can be drawn = $n(S) = {}^{30}C_5$.

The tickets are arranged in the form $T_1, T_2, T_3\ (= 20), T_4, T_5$

Where $T_1, T_2 \in \{1, 2, 3, \ldots, 19\}$ and $T_4, T_5 \in \{21, 22, \ldots, 30\}$

$\therefore$ Number of favourable cases = ${}^{19}C_2 \times 1 \times {}^{10}C_2$

$$\therefore \text{Required probability} = \frac{{}^{19}C_2 \times {}^{10}C_2}{{}^{30}C_5} = \frac{19\times18}{2}\times\frac{10\times9}{2}\times\frac{5\times4\times3\times2\times1}{30\times29\times28\times27\times26} = \mathbf{\frac{285}{5278}}.$$

Ex. 13. Find the probability that a two digit number formed by the digit 1, 2, 3, 4 and 5 is divisible by 4. ***(UPSEE)***

Sol. The two digit numbers can be formed by putting any of 5 digits at the one 's place and also one of the 5 digits at ten's place. So,

Total number of 2–digit numbers that can be formed using these 5–digits = $5 \times 5 = 25$

The 2–digit numbers formed by 1, 2, 3, 4 and 5 that are divisible by 4 are {12, 24, 32, 44, 52}, *i.e*, 5 in number.

$\therefore$ Required probability = $\frac{5}{25} = \mathbf{\frac{1}{5}}$.

Ex. 14. A five digit number is formed by the digits 0, 1, 2, 3, 4 (without repetition). Find the probability that the number formed is divisible by 4 ?

Sol. Without repetition, a five -digit number can be formed using the five digits in 5! ways ($5 \times 4 \times 3 \times 2 \times 1$)

Out of these 5! numbers, 4! numbers will be starting with digit 0. (**0** (fixed) $4 \times 3 \times 2 \times 1$)

$\therefore$ Total number of 5–digit numbers that can be formed using the digits 0, 1, 2, 3, 4 = 5! – 4! = 120 – 24 = **96.**

$\Rightarrow$ No. of exhaustive cases = $n(S) = 96$

Now, only these numbers are divisible by 4 in which the numbers formed by the last two digits is divisible by 4.

Thus, the numbers ending in 04, 12, 20, 24, 32 and 40 will be divisible by 4.

(*i*) If the numbers end in 04, the remaining three numbers 1, 2, 3, can be arranged in 3! ways = 6 ways

(*ii*) If the numbers end in 20, the remaining three numbers, *i.e.*, 1, 4, 3 can be arranged in 3! ways = 6 ways

(*iii*) If the numbers end in 40, the remaining three numbers, *i.e.*, 1, 2, 3 can be arranged in 3! ways = 6 ways

(*iv*) If the numbers end in 12, the remaining three numbers, *i.e.*, 0, 3, 4 can be arranged 3! ways, but these cases in which 0 is the extreme left digit one to be discorded. The number of such cases is 2!.

$\therefore$ Number of numbers ending in 12 = 3! – 2! = 4

(*v*) Similarly the numbers ending in 24 and 32 are 4 each.

$\therefore$ Total number of favourable cases, *i.e.*, number of 5–digit numbers divisible by 4

= 6 + 6 + 6 + 4 + 4 + 4 = 30

$\therefore$ Required probability = $\frac{30}{96} = \mathbf{\frac{5}{16}}$.

Ex. 15. There are 10 persons who are to be seated around a circular table. Find the probability that two particular persons will always sit together.

Sol. Total number of ways in which 10 person can sit around a circular table = 9!

($\because$ We shall keep one place fixed and the rest of the 9 places will be filled in ($9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$) ways as there is no repetition in such case. The first variable place can be filled by any of the remaining 9 person, second place by any of the remaining 8 persons and so on.)

If two particular persons (say *A* and *B*) sit together, then the number of ways in which 9 person (considering the pair of *A* and *B* as one unit) can sit round the table = 2! × 8!. (Here *A* and *B* can interchange places amongst themselves in 2! ways)

$\therefore$ Favourable number of cases = 2 × 8!

$\therefore$ Required probability = $\frac{2 \times 8!}{9!} = \frac{2 \times 8!}{9 \times 8!} = \mathbf{\frac{2}{9}}$.

Ex. 16. The letters of the word 'SOCIETY' are placed at random in a row. What is the probability that three vowels come together ?

Sol. There are 7 letters in the word SOCIETY.

$\therefore$ Total number of ways of arranging all the 7 letters = $n(S)$ = 7!. When the case of three vowels being together is taken, then the three vowels are considered as one unit, so the number of ways in which 5 letters (SCTY–4, IEO–1) can be arranged = 5!

Also the 3 vowels can be arranged amongst themselves in 3! ways

$\therefore$ Total number of favourable cases = 5! × 3!

$\therefore$ Required probability = $\dfrac{5! \times 3!}{7!} = \mathbf{\dfrac{1}{7}}$

Ex. 17. There are *n* letters and *n* addressed envelopes. If the letters are placed in the envelopes at random, what is the probability that all the letters are not placed in the right envelope ?

Sol. Total number of ways of placing n letters in n envelopes = $n!$

All the letters can be placed correctly in only 1 way

$\therefore$ Probability of placing all the letters in the right envelopes = $\dfrac{1}{n!}$

$\therefore$ Probability that all the letters are not placed in the right envelope = $1 - \dfrac{1}{n!}$.

Ex. 18. In a group there are 3 women and 3 men. 4 people are selected at random from this group. Find the probability that 3 women and 1 man or 1 woman and 3 men may be selected ?

Sol. A : Selected 3 women and 1 man

B : Selected 1 women and 3 men

S : Selected 4 people from 6 people (3 + 3)

Then $n(A) = {}^3C_3 \times {}^3C_1$, $n(B) = {}^3C_1 \times {}^3C_3$, $n(S) = {}^6C_4$

$\therefore$ Required probability = $P(A) + P(B) = \dfrac{n(A)}{n(S)} + \dfrac{n(B)}{n(S)} = \dfrac{{}^3C_3 \times {}^3C_1}{{}^6C_4} + \dfrac{{}^3C_1 \times {}^3C_3}{{}^6C_4} = \dfrac{2 \times 1 \times 3}{15} = \mathbf{\dfrac{2}{5}}$.

Ex. 19. Two cards are drawn from a pack of 52 cards. What is the probability that either both are red or both are kings ?

Sol. Let S : Drawing 2 cards out of 52 card

A : Drawing 2 red cards

B : Drawing 2 kings

$A \cup B$: Drawing 2 red cards or 2 kings

$\therefore$ $n(S) = {}^{52}C_2$

$n(A) = {}^{26}C_2$ ($\because$ There are 26 red cards)

$n(B) = {}^4C_2$ ($\because$ There are 4 kings)

But there are 2 red kings, so

$A \cap B$: Drawing 2 red kings

$\Rightarrow$ $n(A \cap B) = {}^2C_2$.

$\therefore$ Required probability = $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} = \frac{{}^{26}C_2}{{}^{52}C_2} + \frac{{}^4C_2}{{}^{52}C_2} - \frac{{}^2C_2}{{}^{52}C_2}$$

$$= \frac{26 \times 25}{52 \times 51} + \frac{4 \times 3}{52 \times 51} - \frac{2}{52 \times 51} = \frac{660}{2652} = \mathbf{\frac{55}{221}}.$$

Ex. 20. One bag contains 3 black and 4 white balls and the other bag contains 4 black and 3 white balls. A die is rolled. If two or five comes up, a ball is taken from the first bag, otherwise a ball is drawn from the second bag. Find the probability of choosing a white ball ? ***(NMOC)***

Sol. Let A : Getting 2 or 5

B : Getting white ball from first bag

C : Getting white ball from second bag.

$\therefore$ $P(A) = \dfrac{2}{6} = \dfrac{1}{3}$ $\Rightarrow P(\bar{A}) = 1 - P(A) = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$\therefore$ Required probability = First bag white ball + Second bag white ball

$$= P(A)\,P(B) + P(\bar{A})\,P(C) = \frac{1}{3}\times\frac{4}{7}+\frac{2}{3}\times\frac{3}{7}=\frac{4+6}{21}=\mathbf{\frac{10}{21}}.$$

Ex. 21. An urn contains 3 white and 5 blue balls and a second urn contains 4 white and 4 blue balls. If one ball is drawn from each urn, what is the probability that they will be of the same colour ?

Sol. Let E : Event of drawing both the balls of same colour from the two urns

E_1 : Getting 1 white ball from the first urn and 1 white ball from the second urn

E_2 : Getting 1 blue ball from the first urn and 1 blue ball from the second urn

Then, $P(E) = P(E) + P(E_2)$ ($\because$ The two events E_1 and E_2 are mutually exclusive)

$= \{P(1 \text{ white from 1st urn}) \times P(1 \text{ white from 2nd urn})\} + \{P(1 \text{ blue from 1st urn}) \times P(1 \text{ blue from 2nd urn})\}$

$= \frac{3}{8}\times\frac{4}{8}+\frac{5}{8}\times\frac{4}{8}$ ($\because$ Event of drawing a ball from one urn is independent of drawing a ball from other urn)

$= \frac{12}{64}+\frac{20}{64}=\frac{32}{64}=\mathbf{\frac{1}{2}}.$

Ex. 22. A speaks truth in 75% and B in 80% of the cases. In what per cent of the cases are they likely to contradict each other in narrating the same incident ? ***(AIEEE 2004, IIT)***

Sol. Let E_1 : Event that A speak the truth

E_2 : Event that B speaks the truth

Then, $\bar{E}_1$: Event that A tells a lie

$\bar{E}_2$: Event that B tells a lie

E_1, E_2 and so $\bar{E}_1, \bar{E}_2$ are independent events.

$\therefore$ A and B will contradict each other in the following two mutually exclusive ways

(i) A speaks the truth , B tells a lie

(ii) A tells a lie, B speaks the truth

Now, $P(E_1) = \frac{75}{100}=\frac{3}{4} \Rightarrow P(\bar{E}_1)=1-\frac{3}{4}=\frac{1}{4}$

$P(E_2) = \frac{80}{100}=\frac{4}{5} \Rightarrow P(\bar{E}_2)=1-\frac{4}{5}=\frac{1}{5}$

$\therefore$ Required probability = $P(E_1)\times P(\bar{E}_2)+P(\bar{E}_1)\times P(E_2) = \frac{3}{4}\times\frac{1}{5}+\frac{1}{4}\times\frac{4}{5}=\mathbf{\frac{7}{20}}.$

$\therefore$ % of cases in which A and B contradict each other = $\left(\frac{7}{20}\times 100\right)\% = \mathbf{35\%}.$

Ex. 23. A problem in mathematics is given to four students *A*, *B*, *C* and *D*. Their chances of solving the problem are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$ respectively. What is the probability that the problem will be solved ?

Sol. Given, $P(A) = \frac{1}{2} \Rightarrow P(\bar{A}) = P(A \text{ not solving the problem}) = 1 - \frac{1}{2}=\frac{1}{2}$

$P(B) = \frac{1}{3} \Rightarrow P(\bar{B}) = 1-\frac{1}{3}=\frac{2}{3}$

$P(C) = \frac{1}{4} \Rightarrow P(\bar{C}) = 1-\frac{1}{4}=\frac{3}{4}$

$P(D) = \frac{1}{5} \Rightarrow P(\bar{D}) = 1-\frac{1}{5}=\frac{4}{5}$

$\therefore$ P(problem will not be solved) = $P(\bar{A}) \times P(\bar{B}) \times P(\bar{C}) \times P(\bar{D})$ $= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5}$

$\therefore$ P(problem will be solved) $= 1 - \frac{1}{5} = \mathbf{\frac{4}{5}}$.

Ex. 24. *A* and *B* throw a coin alternately till one of them gets a 'head' and wins the game. Find their respective probabilities of winning .

Sol. Let ***A*** : Event of A getting a head

$\Rightarrow$ $\bar{A}$: Event of A not getting a head

$\therefore$ $P(A) = \frac{1}{2}$ and $P(\bar{A}) = 1 - \frac{1}{2} = \frac{1}{2}$

Similarly, ***B*** : Event of B getting a head

$\bar{B}$: Event of B not getting a head.

So $P(B) = \frac{1}{2}$ and $P(\bar{B}) = 1 - \frac{1}{2} = \frac{1}{2}$.

Let A start the game. $A's$ wins if he throws a head in the 1st throw or 3rd throw or 5th throw or throw and so on.

Probability of A's winning in 1st throw = $P(A) = \frac{1}{2}$

Probability of A's winning in 3rd throw = P(not A). P(not B). $P(A) = P(\bar{A}) \times P(\bar{B}) \times P(A)$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^3$$

Similarly, probability of A's winning in 5th throw = $P(\bar{A}) \times P(\bar{B}) \times P(\bar{A}) \times P(\bar{B}) \times P(A)$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^5 \text{ and so on.}$$

Since all these events are mutually exclusive,

$$p(A \text{ winning the game first}) = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^7 \ldots = \frac{1}{2}\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \ldots\right]$$

$$= \frac{1}{2}\left[\frac{1}{1 - \left(\frac{1}{2}\right)^2}\right] = \frac{1}{2} \times \frac{1}{3/4} = \frac{1}{2} \times \frac{4}{3} = \mathbf{\frac{2}{3}}.$$

($\because$ For an infinite GP, $S_\infty = \frac{\text{First term}}{1 - \text{common ratio}}$)

Since A and B are mutually exclusive events, as either of them will win, P(B winning the game first) $= 1 - \frac{2}{3} = \mathbf{\frac{1}{3}}$.

PRACTICE SHEET

1. What is the probability that a number selected at random from the set of numbers $\{1, 2, 3, \ldots, 100\}$ is a perfect cube?

(*a*) $\frac{1}{25}$ (*b*) $\frac{2}{25}$ (*c*) $\frac{3}{25}$ (*d*) $\frac{4}{25}$

2. Three fair coins are tossed simultaneously. Find the probability of getting more heads than the number of tails.

(*a*) $3/8$ (*b*) $7/8$ (*c*) $5/8$ (*d*) $1/2$

3. Three identical dice are rolled. The probability that same number will appear on each of them is

(*a*) $\frac{1}{6}$ (*b*) $\frac{1}{12}$ (*c*) $\frac{1}{18}$ (*d*) $\frac{1}{36}$

(*DCE 2001*)

4. Two dice are rolled simultaneously. The probability of getting a multiple of 2 on one dice and a multiple of 3 on the other is

(*a*) $\frac{15}{36}$ (*b*) $\frac{25}{52}$ (*c*) $\frac{11}{36}$ (*d*) $\frac{5}{6}$

5. One card is drawn from a pack of 52 cards, each of the 52 cards being equally likely to be drawn. Find the probability that the card drawn is red and a king.

(*a*) $\frac{1}{13}$ (*b*) $\frac{1}{52}$ (*c*) $\frac{1}{26}$ (*d*) $\frac{5}{6}$

6. If two dice are rolled simultaneously, the probability that the product of the numbers thrown will be greater than 16 is

(*a*) $\frac{1}{6}$ (*b*) $\frac{1}{9}$ (*c*) $\frac{2}{9}$ (*d*) $\frac{5}{36}$

7. Counters marked 1, 2, 3 are placed in a bag and one is drawn and replaced. The operation is repeated three times. The chance of obtaining a total of 6 is

(*a*) $\frac{2}{9}$ (*b*) $\frac{7}{27}$ (*c*) $\frac{13}{27}$ (*d*) $\frac{14}{27}$

(***AMU 2000***)

8. A bag contains 7 red and 5 green balls. The probability of drawing all four balls as red balls, when four balls are drawn at random is

(*a*) $\frac{14}{99}$ (*b*) $\frac{7}{99}$ (*c*) $\frac{7}{35}$ (*d*) $\frac{4}{12}=\frac{1}{3}$

9. From a group of 3 man and 2 women, two person are selected at random. Find the probability that at least one women is selected.

(*a*) $\frac{1}{5}$ (*b*) $\frac{7}{10}$ (*c*) $\frac{2}{5}$ (*d*) $\frac{5}{6}$

10. Five horses are in a race. Mr *A*. Selects two of the horses at random and bets on them. The probability that Mr *A* selected the winning horse is

(*a*) $\frac{1}{5}$ (*b*) $\frac{2}{5}$ (*c*) $\frac{3}{5}$ (*d*) $\frac{4}{5}$

11. Two dice are rolled once. Find the probability of getting an even number on the first die, or a total of 7.

(*a*) $\frac{5}{12}$ (*b*) $\frac{1}{4}$ (*c*) $\frac{7}{12}$ (*d*) $\frac{5}{9}$

12. A die is thrown twice. What is the probability that at least one of the two throws comes up with the number 3 ?

(*a*) $\frac{11}{12}$ (*b*) $\frac{11}{36}$ (*c*) $\frac{7}{12}$ (*d*) $\frac{1}{6}$

13. Two cards are drawn at random from a well-shuffled pack of 52 cards. What is the probability that either both are black or both are kings ?

(*a*) $\frac{55}{112}$ (*b*) $\frac{55}{221}$ (*c*) $\frac{33}{221}$ (*d*) $\frac{33}{112}$

14. A card is drawn at random from a well-shuffled pack of 52 cards. Find the probability of getting a red card or a diamond or a jack ?

(*a*) $\frac{10}{13}$ (*b*) $\frac{15}{26}$ (*c*) $\frac{41}{52}$ (*d*) $\frac{7}{13}$

15. A natural number is chosen at random from amongst the first 300. What is the probability that the number chosen is a multiple of 2 or 3 or 5 ?

(*a*) $\frac{1}{10}$ (*b*) $\frac{11}{15}$ (*c*) $\frac{4}{150}$ (*d*) $\frac{17}{30}$

16. A drawer contains 5 brown and 4 blue socks well mixed. A man reaches the drawer and pulls out 2 socks at random. What is the probability that they match ?

(*a*) $\frac{2}{9}$ (*b*) $\frac{1}{3}$ (*c*) $\frac{4}{9}$ (*d*) $\frac{5}{9}$

(***J&K CET 2009***)

17. If three natural numbers from 1 to 100 are selected randomly, then the probability that all are divisible by both 2 and 3 is

(*a*) $\frac{2}{105}$ (*b*) $\frac{3}{767}$ (*c*) $\frac{4}{1155}$ (*d*) $\frac{5}{1224}$

(***IIT 2004***)

18. 6 boys and 6 girls are sitting in a row randomly. The probability that boys and girls sit alternately is

(*a*) $\frac{1}{36}$ (*b*) $\frac{1}{462}$ (*c*) $\frac{5}{126}$ (*d*) $\frac{1}{231}$

19. A four digit number is formed by the digits 1, 2, 3, 4 with no repetition. The probability that the number is odd is

(*a*) $\frac{1}{2}$ (*b*) $\frac{1}{3}$ (*c*) $\frac{1}{4}$ (*d*) $\frac{1}{6}$

(***UPSEE 2005***)

20. Probability of all 4–digit numbers having all the digits same is

(*a*) $\frac{1}{4}$ (*b*) $\frac{9}{1000}$ (*c*) $\frac{1}{1000}$ (*d*) $\frac{1}{100}$

21. The letters of the word 'NATIONAL' are arranged at random. What is the probability that the last letter will be T ?

(*a*) $\frac{1}{4}$ (*b*) $\frac{5}{8}$ (*c*) $\frac{1}{8}$ (*d*) $\frac{3}{8}$

22. The probability that in the random arrangement of the letters of the word 'UNIVERSITY' the two I's do not come together is

(*a*) $\frac{1}{5}$ (*b*) $\frac{4}{5}$ (*c*) $\frac{1}{10}$ (*d*) $\frac{9}{10}$

(***UPSEE 2001***)

23. An urn contains nine balls, of which three are red, four are blue and two are green. Three balls are drawn at random without replacement from the urn. The probability that the three balls have different colours is

(*a*) $\frac{1}{3}$ (*b*) $\frac{2}{7}$ (*c*) $\frac{1}{21}$ (*d*) $\frac{2}{23}$

(***AIEEE 2010***)

24. The probability of India winning a test match against Westindies is $\frac{1}{2}$. Assuming independence from match to match, the probability that in a match series, India's second win occurs at the third test is

(*a*) $\frac{1}{2}$ (*b*) $\frac{2}{3}$ (*c*) $\frac{1}{4}$ (*d*) $\frac{2}{5}$

(***IIT 1993***)

25. A pack of cards contains 4 aces, 4 kings, 4 queens and 4 jacks. Two cards are drawn at random. The probability that at least one them is an ace is

(*a*) $\frac{1}{5}$ (*b*) $\frac{1}{9}$ (*c*) $\frac{3}{16}$ (*d*) $\frac{9}{20}$

(***Rajasthan PET 2005***)

26. A bag contains 5 green and 7 red balls, out of which two balls are drawn at random. What is the probability that they are of the same colour ?

(*a*) $\frac{5}{11}$ (*b*) $\frac{7}{22}$ (*c*) $\frac{16}{33}$ (*d*) $\frac{31}{66}$

(***EAMCET 2007***)

27. Two players A and B play a game by alternately drawing a card from a well-shuffled pack of playing cards, replacing the card each time after draw. The first one to draw a queen wins the game. If A begins the game, then the probability of his winning the game is

(*a*) $\frac{13}{25}$ (*b*) $\frac{7}{26}$ (*c*) $\frac{3}{17}$ (*d*) $\frac{15}{52}$

28. 6 boys and 6 girls are seated in a row. Probability that all the boys sit together is

(*a*) $\frac{1}{102}$ (*b*) $\frac{1}{112}$ (*c*) $\frac{1}{122}$ (*d*) $\frac{1}{132}$

(***WB JEE 2011***)

29. The probability that A hits a target is $\frac{1}{3}$ and the probability that B hits it is $\frac{2}{5}$. What is the probability that the target will be hit, if each one of A and B shoots the target ?

(*a*) $\frac{5}{6}$ (*b*) $\frac{3}{5}$ (*c*) $\frac{11}{15}$ (*d*) $\frac{1}{6}$

30. A coin and six faced die, both unbiased are thrown simultaneously. The probability of getting a tail on the coin and an even number on the die is

(*a*) $\frac{1}{2}$ (*b*) $\frac{3}{4}$ (*c*) $\frac{1}{4}$ (*d*) $\frac{2}{3}$

31. An anti-aircraft gun can take a maximum of four shots at any plane moving away from it. The probabilities of hitting the plane at 1st, 2nd, 3rd and 4th shots are 0.4, 0.3, 0.2 and 0.1 respectively. What is the probability that at least one shot hits the plane ?

(*a*) 0.0024 (*b*) 0.3024 (*c*) 0.6976 (*d*) 0.9976

(***IIT 1981***)

32. A bag contains 4 red and 3 black balls. A second bag contains 2 red and 4 black balls. One bag is selected at random. From the selected bag, one ball is drawn. Find the probability that the ball drawn is red.

(*a*) $\frac{23}{42}$ (*b*) $\frac{19}{42}$ (*c*) $\frac{7}{32}$ (*d*) $\frac{16}{39}$

33. If from each of the three boxes containing 3 blue and 1 red balls, 2 blue and 2 red balls, 1 blue and 3 red balls, one ball is drawn at random, then the probability that 2 blue and 1 red ball will be drawn is :

(*a*) $\frac{13}{32}$ (*b*) $\frac{27}{32}$

(*c*) $\frac{19}{32}$ (*d*) None of these

34. A bag contains 5 green and 11 blue balls and the second one contains 3 green and 7 blue balls. Two balls are drawn from one of the bags. What is the probability that they are of different colours ?

(*a*) $\frac{5}{32}$ (*b*) $\frac{1}{10}$ (*c*) $\frac{111}{240}$ (*d*) $\frac{1}{7}$

35. A management institute has six senior professors and four junior professors. Three professors are selected at random for a government project. The probability that at least one of the junior professors would get selected is

(*a*) $5/6$ (*b*) $2/3$ (*c*) $1/5$

(*d*) $1/6$ (*e*) None of these (***XAT 2007***)

36. A card is drawn at random from a well shuffled pack of 52 cards

X : The card drawn is black or a king

Y : The card drawn is a club or heart or a jack

Z : The card drawn is an ace or a diamond or a queen

Then, which of the following is correct.

(*a*) $P(X) > P(Y) > P(Z)$ (*b*) $P(X) \geq P(Y) = P(Z)$

(*c*) $P(X) = P(Y) > P(Z)$ (*d*) $P(X) = P(Y) = P(Z)$

(***IIFT 2009***)

37. The game of "chuck-a-luck" is played at carnivals in some parts of Europe. Its rules are as follows : You pick a number from 1 to 6 and the operator rolls three dice. If the number you picked comes up an all the three dice, the operator pays you ₹ 3, if it comes up on two dice, you are paid ₹ 2; and if it comes on just one die, you are paid ₹ 1. Only if the number you picked does not come up at all, you pay the operator ₹ 1. The probability that you will win money playing in this game is :

(*a*) 0.52 (*b*) 0.753

(*c*) 0.42 (*d*) None of these

(***IIFT 2008***)

38. The odds against certain event are 5:2 and the odds in favour of another in dependent event are 6:5. The probability that at least one of the event will happen is

(*a*) $\frac{12}{77}$ (*b*) $\frac{25}{77}$ (*c*) $\frac{52}{77}$ (*d*) $\frac{65}{77}$

(***IIFT 2005***)

39. If the chance that a vessel arrives safely at a port is $\frac{9}{10}$, then what is the chance that out of 5 vessels expected at least 4 will arrive safely ?

(*a*) $\frac{(14 \times 9^4)}{10^5}$ (*b*) $\frac{(15 \times 9^5)}{10^4}$ (*c*) $\frac{(14 \times 9^3)}{10^4}$ (*d*) $\frac{(14 \times 9^6)}{10^5}$

(***JMET 2011***)

40. A group of $2n$ boys and $2n$ girls is divided at random into two equal batches. The probability that each batch will have equal number of boys and girls is

(*a*) $\frac{1}{2}n$ (*b*) $\frac{1}{4}n$ (*c*) $\frac{({}^{2n}C_n)^2}{{}^{4n}C_{2n}}$ (*d*) ${}^{2n}C_n$

(***ATMA 2008***)

41. A box contains 100 balls numbers from 1 to 100. If three balls are selected at random and with replacement from the box, what is the probability that the sum of the three numbers on the balls selected from the box will be odd ?

(*a*) $\frac{3}{8}$ (*b*) $\frac{1}{2}$ (*c*) $\frac{3}{4}$ (*d*) $\frac{1}{4}$

(***ATMA 2005***)

42. Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle with these vertices is equilateral equals :

(*a*) $\frac{1}{6}$ (*b*) $\frac{1}{3}$

(*c*) $\frac{1}{10}$ (*d*) None of these

43. A die is rolled three times. The probability of getting a larger number than the pervious number each time is:

(*a*) $\frac{5}{72}$ (*b*) $\frac{5}{54}$ (*c*) $\frac{13}{216}$ (*d*) $\frac{1}{18}$

(*MPPET 2011*)

44. A fair dice is thrown twenty times. The probability that on the tenth throw the fourth six appears is

(*a*) $\frac{{}^{20}C_{10} \times 5^6}{6^{20}}$ (*b*) $\frac{120 \times 5^7}{6^{10}}$

(*c*) $\frac{84 \times 5^6}{6^{10}}$ (*d*) None of these

(*Odisha JEE 2013*)

45. A bag contains *a* white and *b* black balls. Two players *A* and *B* alternately draw a ball from the bag replacing the ball each time after the draw till one of them draws a white ball and wins the game. *A* begins the game. If the probability of *A* winning the game is 3 times that of *B*, then the ratio *a* : *b* is

(*a*) 1 : 1 (*b*) 1 : 2

(*c*) 2 : 1 (*d*) None of these

(*BITSAT 2009*)

46. A determinant of second order is made with the elements 0, 1. What is the probability that the determinant is positive?

(*a*) $\frac{7}{12}$ (*b*) $\frac{11}{12}$ (*c*) $\frac{3}{16}$ (*d*) $\frac{15}{16}$

(*AMU 2009*)

47. Out of $3n$ consecutive natural numbers 3 natural numbers are chosen at random without replacement. The probability that the sum of the chosen numbers in divisible by 3 is

(*a*) $\frac{n(3n^2 - 3n + 2)}{2}$ (*b*) $\frac{(3n^2 - 3n + 2)}{2(3n-1)(3n-2)}$

(*c*) $\frac{3n^2 - 3n + 2}{(3n-1)(3n-2)}$ (*d*) $\frac{n(3n-1)(3n-2)}{3(n-1)}$

(*DCE 2009*)

48. A bag contains $2n + 1$ coins. It is known that *n* of these coins have a head on both sides, whereas the remaining $(n + 1)$ coins are fair. A coin is picked up at random from the bag and tossed. If the probability that the toss results in a head is $\frac{31}{42}$, then *n* is equal to

(*a*) 10 (*b*) 11 (*c*) 12 (*d*) 13

(*EAMCET 2013*)

49. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answer just by guessing is

(*a*) $\frac{17}{3^5}$ (*b*) $\frac{13}{3^5}$ (*c*) $\frac{11}{3^5}$ (*d*) $\frac{10}{3^5}$

(*IIT JEE Main 2013*)

50. If the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs is α, E_2 occurs is β, only E_3 occurs is γ. Let the probability *p* that none of the event E_1, E_2 or E_3 occurs satisfy the equations $(\alpha - 2\beta)\, p = \alpha\beta$ and $(\beta - 3\gamma)\, p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval (0, 1).

Then, $\frac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3}$ is equal to

(*a*) 3 (*b*) 5 (*c*) 6 (*d*) 8

(*IIT JEE Advance 2013*)

ANSWERS

1. (*a*)	**2.** (*d*)	**3.** (*d*)	**4.** (*c*)	**5.** (*c*)	**6.** (*c*)	**7.** (*b*)	**8.** (*b*)	**9.** (*b*)	**10.** (*b*)
11. (*c*)	**12.** (*b*)	**13.** (*b*)	**14.** (*d*)	**15.** (*b*)	**16.** (*c*)	**17.** (*c*)	**18.** (*b*)	**19.** (*a*)	**20.** (*c*)
21. (*c*)	**22.** (*b*)	**23.** (*b*)	**24.** (*c*)	**25.** (*d*)	**26.** (*d*)	**27.** (*a*)	**28.** (*d*)	**29.** (*b*)	**30.** (*c*)
31. (*c*)	**32.** (*b*)	**33.** (*a*)	**34.** (*c*)	**35.** (*a*)	**36.** (*c*)	**37.** (*c*)	**38.** (*c*)	**39.** (*a*)	**40.** (*c*)
41. (*d*)	**42.** (*c*)	**43.** (*b*)	**44.** (*c*)	**45.** (*c*)	**46.** (*c*)	**47.** (*c*)	**48.** (*a*)	**49.** (*c*)	**50.** (*c*)

HINTS AND SOLUTIONS

1. Let us assume *S* as the sample space in all questions. *S* means the set denoting the total number of outcomes possible.

Let $S = \{1, 2, 3, \ldots, 100\}$ be the sample space. Then,

$n(S) = 100$

Let *A* : Event of selecting a cube from the given set *S*.

$\Rightarrow A = \{18, 27, 64\}$

$\Rightarrow n(A) = 4$

$\therefore$ Required probability $P(A) = \frac{n(A)}{n(S)} = \frac{4}{100} = \frac{1}{25}$

2. Let *S* be the sample space. Then,

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$\Rightarrow n(S) = 8$

Let *A* : Event of getting more heads than number of tails.

Then, $A = \{HHH, HHT, HTH, THH\}$

$\Rightarrow n(A) = 4$

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{4}{8} = \frac{1}{2}$.

3. Total number of outcomes when three identical dice are rolled, $n(S) = 6 \times 6 \times 6 = 216$

Let A : Event of rolling same number or each dice

$\Rightarrow A = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6)\}$

$\Rightarrow n(A) = 6$

$\therefore \quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{6}{216} = \dfrac{\mathbf{1}}{\mathbf{36}}.$

4. Total number of outcomes when two identical dice are rolled, $n(S) = 6 \times 6 = 36$

Let A : Event of rolling a multiple of 2 on one die and a multiple of 3 on the other die

$\Rightarrow \quad A = \{(2, 3), (2, 6), (4, 3), (4, 6), (6, 3), (6, 6), (3, 2), (6, 2), (3, 4), (6, 4), (3, 6)\}$

$\Rightarrow \quad n(A) = 11$

$\therefore \quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{\mathbf{11}}{\mathbf{36}}.$

5. There is a total of 52 cards $n(S) = 52$

Let A : Event of drawing a red king

Since there are only two red kings in the pack, $n(A) = 2$

$\therefore \quad P(A) = \dfrac{2}{52} = \dfrac{\mathbf{1}}{\mathbf{26}}.$

6. Let S be the sample space of drawing a counter three times and replacing it each time.

Then, $n(S) = 3 \times 3 \times 3 = 27$

Let A : Event of obtaining a total of 6 in the three draws of counters.

Then, $A = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1), (2, 2, 2)\}$

$\Rightarrow n(A) = 7$

$\therefore \quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{\mathbf{7}}{\mathbf{27}}.$

8. There are $(7 + 5) = 12$ balls in the bag.

4 balls can be drawn at random from 12 balls in ${}^{12}C_4$ ways.

$\therefore n(S) = {}^{12}C_4 = \dfrac{\lfloor 12}{\lfloor 8 \lfloor 4} = \dfrac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 495$

Let A : Event of drawing all four balls as red.

4 red balls can be drawn from 7 red balls in ${}^{7}C_4$ ways

$\therefore \quad n(A) = {}^{7}C_4 = \dfrac{\lfloor 7}{\lfloor 3 \lfloor 4} = \dfrac{7 \times 6 \times 5}{3 \times 2} = 35$

$\therefore \quad$ Required probability $= \dfrac{n(A)}{n(S)} = \dfrac{35}{495} = \dfrac{\mathbf{7}}{\mathbf{99}}$

9. Total number of ways of selecting 2 persons at random out of 5 persons $= {}^5C_2$

$\therefore \quad n(S) = {}^5C_2 = \dfrac{\lfloor 5}{\lfloor 3 \lfloor 2} = \dfrac{5 \times 4}{2 \times 1} = \mathbf{10}$

Let A : Event of selecting at least one woman.

$\Rightarrow \quad A$ = Selecting 1 woman + 1 man or selecting 2 women

$\Rightarrow \quad n(A) = {}^2C_1 \times {}^3C_1 + {}^2C_2 = 2 \times 3 + 1 = 7$

$\therefore \quad$ Required probability $= \dfrac{n(A)}{n(S)} = \dfrac{\mathbf{7}}{\mathbf{10}}.$

10. As each horse has equal chance of winning the race,

Number of ways in which one of the five horses wins the race $= {}^5C_1$

$\therefore \quad n(S) = {}^5C_1 = \dfrac{\lfloor 5}{\lfloor 4 \lfloor 1} = 5$

To find the chance that Mr A selects the winning horses, it is essential that one of the two horses selected by him wins the race.

E : Mr A selecting the winning horse.

$\Rightarrow \quad n(E) = {}^2C_1 = 2$

$\therefore \quad$ Required probability $= \dfrac{n(E)}{n(S)} = \dfrac{\mathbf{2}}{\mathbf{5}}.$

11. Total number of ways in which 2 dice are rolled $= 6 \times 6 = 36$

$\Rightarrow \quad n(S) = 36$

Let A : Event of rolling an even number of 1st dice

B : Event of rolling a total of 7

$\Rightarrow \quad A = \{(2, 1), (2, 2)\ldots, (2, 6), (4, 1), (4, 2) \ldots, (4, 6), (6, 1), (6, 2), \ldots, (6, 6)\}$

and $B = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

$\Rightarrow n(A) = 3 \times 6 = 18, n(B) = 6$

$\Rightarrow \quad A \cap B$: Event of rolling an even number on 1st dice and a total of 7

$\Rightarrow A \cap B = \{(2, 5), (4, 3), (6, 1)\} \Rightarrow n(A \cap B) = 3$

$\therefore P(A) = \dfrac{n(A)}{n(S)} = \dfrac{18}{36}; P(B) = \dfrac{n(B)}{n(S)} = \dfrac{6}{36}; P(A \cap B) = \dfrac{3}{36}$

$\Rightarrow \quad P(A \cup B) = P$(Event of rolling an even number 1st dice or a total of 7)

$= \quad P(A) + P(B) - P(A \cap B)$

$= \quad \dfrac{18}{36} + \dfrac{6}{36} - \dfrac{3}{36} = \dfrac{21}{36} = \dfrac{\mathbf{7}}{\mathbf{12}}.$

12. Let S = total ways in which two dice can be rolled

$\Rightarrow \quad n(S) = 6 \times 6 = 36$

Let A : Event of throwing 3 with 1st dice,

B : Event of throwing 3 with 2nd dice.

Then, $A = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}$

$\Rightarrow \quad n(A) = 6$

$B = \{(1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3)\} \Rightarrow n(B) = 6$

$A \cap B = \{(3, 3)\} \Rightarrow n(A \cap B) = 1$

$\therefore \quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{6}{36}, P(B) = \dfrac{n(B)}{n(S)} = \dfrac{6}{36},$

$P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{1}{36}$

$\therefore$ Required probability = P(Throwing a 3 with at least one of the dice)

$= P(A) + P(B) - P(A \cap B)$

$= \dfrac{6}{36} + \dfrac{6}{36} - \dfrac{1}{36} = \mathbf{\dfrac{11}{36}}.$

13. S : Drawing 2 cards out of 52 cards

$\Rightarrow \quad n(S) = {}^{52}C_2 = \dfrac{\lfloor 52}{\lfloor 50 \lfloor 2} = \dfrac{52 \times 51}{2} = 1326$

A : Event of drawing 2 black cards out of 26 black cards

$\Rightarrow \quad n(A) = {}^{26}C_2 = \dfrac{26 \times 25}{2} = 325$

B : Event of drawing 2 kings out of 4 kings

$\Rightarrow \quad n(B) = {}^{4}C_2 = \dfrac{\lfloor 4}{\lfloor 2 \lfloor 2} = 6$

$\Rightarrow \quad A \cap B$: Event of drawing 2 black kings

$\Rightarrow \quad n(A \cap B) = {}^{2}C_2 = 1$

$\therefore \quad P(A) = \dfrac{325}{1326}, P(B) = \dfrac{6}{1326}, P(A \cap B) = \dfrac{1}{1326}$

$\therefore \quad P$(Both cards are black or both kings)

$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \dfrac{325}{1326} + \dfrac{6}{1326} - \dfrac{1}{1326} = \dfrac{330}{1326} = \mathbf{\dfrac{55}{221}}.$

14. Here $n(S) = 52$

Let A, B, C be the events of getting a red card, a diamond and a jack respectively.

$\because$ There are 26 red cards, 13 diamonds and 4 jacks,

$n(A) = 26, n(B) = 13, n(C) = 4$

$\Rightarrow \quad n(A \cap B) = n(\text{red and diamond}) = 13,$

$n(B \cap C) = n(\text{diamond and jack}) = 1,$

($\because$ There is only jack of diamonds)

$n(A \cap C) = n(\text{red and jack}) = 2$

($\because$ There are two red jacks)

$n(A \cap B \cap C) = n(\text{red, diamond, jack}) = 1$

$\therefore P(A) = \dfrac{26}{52}, P(B) = \dfrac{13}{52}, P(C) = \dfrac{4}{52}, P(A \cap B) = \dfrac{13}{52},$

$P(B \cap C) = \dfrac{1}{52}, P(A \cap C) = \dfrac{2}{52}, P(A \cap B \cap C) = \dfrac{1}{52}$

$\therefore \quad P$(a red card or a diamond or a jack) $= P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

$= \dfrac{26}{52} + \dfrac{13}{52} + \dfrac{4}{52} - \left(\dfrac{13}{52} + \dfrac{1}{52} + \dfrac{2}{52}\right) + \dfrac{1}{52}$

$= \dfrac{44}{52} - \dfrac{16}{52} = \dfrac{28}{52} = \mathbf{\dfrac{7}{13}}.$

15. $n(S) = 300$

Let A : Event of getting a number divisible by 2

B : Event of getting a number divisible by 3

C : Event of getting a number divisible by 5

$\therefore \; A \cap B$: Event of getting a number divisible by both 2 and 3, *i.e.*, 6

$B \cap C$: Event of getting a number divisible by both 3 and 5, *i.e*, 15

$A \cap C$: Event of getting a number divisible by both 2 and 5, *i.e.*, 10

$A \cap B \cap C$: Event of getting a number divisible by all 2, 3 and 5, *i.e.*, 30

Then, $n(A) = 150, n(B) = 100, n(C) = 60$

$n(A \cap B) = 50, n(B \cap C) = 20, n(A \cap C) = 30,$

$n(A \cap B \cap C) = 10$

$\Rightarrow \quad P(A) = \dfrac{150}{300}, P(B) = \dfrac{100}{300}, P(C) = \dfrac{60}{300}$

$P(A \cap B) = \dfrac{50}{300} \; P(B \cap C) = \dfrac{20}{300}, P(A \cap C) = \dfrac{30}{300},$

$P(A \cap B \cap C) = \dfrac{10}{300}$

$\therefore \quad P$(Number divisible by 2 or 3 or 5) $= P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - (P(A \cap B) - P(B \cap C) - P(A \cap C)) + P(A \cap B \cap C)$

$= \dfrac{150}{300} + \dfrac{100}{300} + \dfrac{60}{300} - \left(\dfrac{50}{300} + \dfrac{20}{300} + \dfrac{30}{300}\right) + \dfrac{10}{300}$

$= \dfrac{320}{300} - \dfrac{100}{300} = \dfrac{220}{300} = \mathbf{\dfrac{11}{15}}.$

16. $n(S)$ = Total number of ways in which 2 socks can be drawn out of 9 socks (5 brown and 4 blue socks)

$= \quad {}^{9}C_2 = \dfrac{\lfloor 9}{\lfloor 7 \lfloor 2} = \dfrac{9 \times 8}{2} = \mathbf{36}$

Let A : Event of drawing 2 socks of same colour

$\Rightarrow \quad A$ = Drawing 2 brown socks out of 5 brown socks or Drawing 2 blue sock out of 4 blue socks

$\Rightarrow \quad n(A) = {}^{5}C_2 + {}^{4}C_2 = \dfrac{\lfloor 5}{\lfloor 3 \lfloor 2} + \dfrac{\lfloor 4}{\lfloor 2 \lfloor 2}$

$= \dfrac{5 \times 4}{2} + \dfrac{4 \times 3}{2} = 10 + 6 = 16$

$\therefore \quad$ Required probability $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{16}{36} = \mathbf{\dfrac{4}{9}}.$

17. Let $n(S)$ = Number of ways of selecting 3 numbers from 100 numbers

$= {}^{100}C_3$

Let E : Event of selecting three numbers divisible by both 2 and 3 from numbers 1 to 100

= Event of selecting three multiples of 6 from 1 to 100

Selecting 3 numbers from

= {6,12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96}

$\Rightarrow \quad n(E) = {}^{16}C_3$

$\therefore \quad P$ (Selecting 3 numbers exactly divisible by 6 between 1 and 100) $= \dfrac{n(E)}{n(S)} = \dfrac{{}^{16}C_3}{{}^{100}C_3} = \dfrac{16 \times 15 \times 14}{100 \times 99 \times 98} = \mathbf{\dfrac{4}{1155}}.$

18. Let S be the sample space.

Then, $n(S)$ = Number of ways in which 6 boys and 6 girls can sit in a row = 12!

Let E : Event of 6 girls and 6 boys sitting alternately.

Then, the 6 girls and 6 boys can be arranged in alternate position in two ways.

Ist way : We start with a boy. Then the arrangement is :

$B\,G \quad B\,G \quad B\,G \quad B\,G \quad B\,G \quad B\,G$

$\therefore$ Number of ways of arranging 6 boys in 6 places = 6!

Number of ways of arranging 6 girls in 6 places = 6!

$\therefore$ Number of ways of arranging 6 boys and 6 girls in alternate places = 6! × 6!

Similarly **IInd way :** Here we start with a girl. Then the arrangement is $G\,B \quad G\,B \quad G\,B \quad G\,B \quad G\,B \quad G\,B$

$\therefore$ Number of ways of arranging 6 boys and 6 girls alternately this way

= 6! × 6!

$\therefore$ $n(E)$ = 6! × 6! + 6! × 6!

= 2 × 6! × 6!

$$\therefore \quad n(E) = \frac{n(E)}{n(S)} = \frac{2\times 6!\times 6!}{12!}$$

$$= \frac{2\times 6\times 5\times 4\times 3\times 2\times 1}{12\times 11\times 10\times 9\times 8\times 7} = \mathbf{\frac{1}{462}}.$$

19. Let S be the sample space

Then $n(S)$ = Number of four digit numbers that can be formed with out repetition

1 of 4 digits	1 of 3 digits	1 of 2 digits	1 of 1 digit
Th	H	T	O

= 4! = 4 × 3 × 2 × 1 ways = 24 ways.

Let E : Event of forming odd 4 digit numbers with the digits 1, 2, 3, 4 without repetition.

Then, the **ones place** can be filled by the odd digits 1 or 3, in 2 ways.

The **thousands place** can be filled with the remaining 3 digits in 3 ways

The **hundreds place** can be filled with the remaining 2 digits in 2 ways

The **tens place** can be filled with the remaining 1 digit in 1 way

$\therefore$ $n(E)$ = 2 × 3 × 2 × 1 = 12

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{12}{24} = \mathbf{\frac{1}{2}}.$$

20. Let S be the sample space where 4 digit numbers are formed using digits 0 to 9 repetition. Then,

$n(S)$ = Total number of 4 digit number formed

= 9 × 10 × 10 × 10 = 9000

($\because$ The thousands place rest can only be accuried by the 9 non-zero digits, while the rest of the three places, *i.e.*, hundreds, tens and ones can each be filled in 10 ways)

Let E : Event of forming four digit numbers having all the digits same.

Then $n(E)$ = 9 × 1 × 1 × 1 = 9

($\because$ For one place (thousands place) we choose the number from the non-zero 9 digits and for the rest of the places, the same digit is repeated)

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{9}{9000} = \mathbf{\frac{1}{1000}}$$

21. Let S be the sample space.

Then $n(S)$ = Total number of ways in which the letters of word NATIONAL can be arranged

$$= \frac{8!}{2!\,2!} \qquad (\because \text{There are } 2A\text{'s and } 2N\text{'s in 8 letters})$$

Let E : Event of arranged letters of the word NATIONAL, so that the last position is occupied by letter 'T'.

Here position of T is fixed, so

$n(E)$ = Number of ways in which the rest of the 7 letters can be arranged in 7 places.

$$= \frac{7!}{2!\,2!} \qquad (\because \text{There are } 2A\text{'s and } 2N\text{'s})$$

$$\therefore \text{Required probability} = \frac{n(E)}{n(S)} = \frac{\frac{7!}{2!\,2!}}{\frac{8!}{2!\,2!}} = \frac{7!}{8!} = \frac{7!}{8\times 7!} = \mathbf{\frac{1}{8}}.$$

22. Let S be the sample space. Then,

$n(S)$ = Total number of ways in which the letters of the word 'UNIVERSITY' can be arranged

$$= \frac{10!}{2!} \qquad (\because \text{There are } 2I\text{'s})$$

Let E : Event of arranging the letters of the word 'UNIVERSITY' such that 2 I's remain together.

Not considering the two I's, the remaining 8 letters of 'UNIVERSITY' can be arranged in 8! ways. There seven places between these 8 letters and two on extreme ends which can be occupied by the two I's, so that they are not together.

—X—X—X—X—X—X—X—X—

The places marked '–' can be filled with I's.

$\therefore$ Now 2I's can be arranged in any of the 2 out of 9 places

$$\text{in } {}^9C_2 \text{ ways} = \frac{\underline{|9}}{\underline{|7}\times\underline{|2}} = \frac{9\times 8}{2} = 36 \text{ ways.}$$

$\therefore$ $n(E)$ = 8! × 36

$$\therefore \quad P(E) = \frac{8!\times 36}{10!/2!} = \frac{\cancel{8!}\times 36\times 2!}{10\times 9\times \cancel{8!}} = \mathbf{\frac{4}{5}}.$$

23. Let S be the sample space having 9 balls ($3R + 4B + 2G$)

Then $n(S)$ = Total number of ways in which 3 balls can be drawn out of the 9 balls

$$= \frac{9\times 8\times 7}{3\times 2} = \mathbf{84}$$

Let A : Event of drawing three balls of different colours from the urn.

$\Rightarrow$ A = Event of drawing 1 red ball out of 3 red balls, 1 blue ball out of 4 blue balls and 1 green ball out of 2 green balls

$\Rightarrow$ $n(A) = {}^3C_1 \times {}^4C_1 \times {}^2C_1 = 3 \times 4 \times 2 = 24.$

$\therefore$ $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{24}{84} = \mathbf{\dfrac{2}{7}}.$

24. Let A = Event that India wins the match. Then,

$\overline{A}$ = Event that India loses the match.

$P(A) = \dfrac{1}{2}$ and $P(\overline{A}) = 1 - \dfrac{1}{2} = \dfrac{1}{2}$ $(\because P(A) + P(\overline{A}) = 1)$

P(Third match is India's second winning match)

= P(India wins the 1st match, loses 2nd match, wins 3rd match) + P(India loses 1st match, wins 2nd match, wins 3rd match)

(or)

$\Rightarrow$ Required probability = $P(A\,\overline{A}\,A) + P(\overline{A}\,A\,A)$

$= P(A) \times P(\overline{A}) \times P(A) + P(\overline{A}) \times P(A) \times P(A)$

$= \dfrac{1}{2} \times \dfrac{1}{2} \times \dfrac{1}{2} + \dfrac{1}{2} \times \dfrac{1}{2} \times \dfrac{1}{2} = \dfrac{1}{8} + \dfrac{1}{8} = \dfrac{2}{8} = \mathbf{\dfrac{1}{4}}.$

25. Let S be the sample space for drawing 2 cards out of 4 aces, 4 kings, 4 queens and 4 jacks *i.e*, 16 cards.

Then $n(S) = {}^{16}C_2$

P(Drawing at least one ace) = 1 – P(Drawing no ace)

Let E : Event of drawing no aces in the 2 drawn cards

$\Rightarrow n(E) = {}^{12}C_2$(Cards leaving aces = 16 – 4 – 12)

$\therefore$ $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{{}^{12}C_2}{{}^{16}C_2} = \dfrac{12 \times 11}{16 \times 15} = \dfrac{11}{20}$

$\therefore$ P(drawing at least one ace) $= 1 - \dfrac{11}{20} = \mathbf{\dfrac{9}{20}}.$

26. Total number of balls in the bag = 12 (5 Green + 7 Red)

Let S be the sample space of drawing 2 balls out of 12 balls. Then

$n(S) = {}^{12}C_2 = \dfrac{12 \times 11}{2} = 66$

$\therefore$ Let A : Event of drawing two red balls

$\Rightarrow$ $n(A) = {}^7C_2 = \dfrac{7 \times 6}{2} = 21$

B : Event of drawing two green balls

$\Rightarrow$ $n(B) = {}^5C_2\ \dfrac{5 \times 4}{2} = 10$

$\therefore$ P(Event of drawing 2 balls of same colour) = P(Drawing two red balls) or P(drawing two green balls)

$= P(A) + P(B) = \dfrac{n(A)}{n(S)} + \dfrac{n(B)}{n(S)} = \dfrac{21}{66} + \dfrac{10}{66} = \mathbf{\dfrac{31}{66}}.$

27. Let E : Event of drawing a queen in a single draw the pack of 52 cards.

As there are 4 queens in a pack of 52 cards,

$P(E) = \dfrac{4}{52} = \dfrac{1}{13}$

$P(\overline{E})$ = P(not drawing a queen) $= 1 - P(E) = 1 - \dfrac{1}{13} = \dfrac{12}{13}$

$\therefore$ P(A wins) = P(A draws a queen in the 1st or 3rd or 5th or ...draws with B not drawing the queen in the 2nd or 4th or 6th or ...draws)

$= P(E \text{ or } \overline{E}\,\overline{E}\,E \text{ or } \overline{E}\,\overline{E}\,\overline{E}\,\overline{E}\,E \text{ or } ...)$

$= P(E) + P(\overline{E}) \,.\, P(\overline{E}) \,.\, P(E) + P(\overline{E})\, P(\overline{E})\, P(\overline{E}) \,.\, P(\overline{E}) \,.\, P(E) + ...\infty$

$= \dfrac{1}{13} + \left(\dfrac{12}{13} \times \dfrac{12}{13} \times \dfrac{1}{13}\right) + \left(\dfrac{12}{13} \times \dfrac{12}{13} \times \dfrac{12}{13} \times \dfrac{12}{13} \times \dfrac{1}{13}\right) + ...\infty$

$= \dfrac{1}{13}\left[1 + \left(\dfrac{12}{13}\right)^2 + \left(\dfrac{12}{13}\right)^4 + ...\infty\right]$

$= \dfrac{1}{13}\left[\dfrac{1}{1 - \left(\dfrac{12}{13}\right)^2}\right] = \dfrac{1}{13}\left[\dfrac{1}{\dfrac{(169 - 144)}{169}}\right]$

$= \dfrac{1}{13} \times \dfrac{169}{25} = \mathbf{\dfrac{13}{25}}.$

[Sum of a $G.P$ with infinite terms = $\dfrac{a}{1 - r}$

where a = 1st term, r = common ratio.]

28. Let S be the sample space for arranging 6 boys and 6 girls in a row. Then, $n(S)$ = 12!

If all 6 boys are to sit together, then consider the 6 boys as one entity. Now the remaining, *i.e.*, 6 girls can be arranged in a row in 6! ways. There are 5 places between the 6 girls and 2 on the extreme ends, where the entity of 6 boys can be placed, *i.e.*, for the single entity of 6 boys, we have 7 places where they can be arranged in 7 ways and also amongst themselves they can be arranged in 6! ways.

$\therefore$ No. of ways of arranging 6 boys and 6 girls in a row where the 6 boys are together = 6! × 7 × 6!

$\therefore$ Required probability

$= \dfrac{6! \times 7 \times 6!}{12!} = \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2}{12 \times 11 \times 10 \times 9 \times 8 \times 7} = \mathbf{\dfrac{1}{132}}.$

29. Let A : Event that A hits the target

B : Event that B hits the target

Given, $P(A) = \dfrac{1}{3}$

$\Rightarrow$ P(A not hitting the target) = $P(\overline{A}) = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$P(B) = \dfrac{2}{5} \Rightarrow P$($B$ not hitting the target) = $P(\overline{B}) = 1 - \dfrac{2}{5} = \dfrac{3}{5}$

$\therefore$ P(Target is hit) = P(A hits) × P(B does not hit) + P(A does not hit) × P(B hits) + P(A hits) × P(B hits)

$= P(A) \times P(\overline{A}) + P(\overline{A}) \times P(B) + P(A) \times P(B)$

$= \dfrac{1}{3} \times \dfrac{3}{5} + \dfrac{2}{3} \times \dfrac{2}{5} + \dfrac{1}{3} \times \dfrac{2}{5} = \dfrac{3}{15} + \dfrac{4}{15} + \dfrac{2}{15} = \dfrac{9}{15} = \mathbf{\dfrac{3}{5}}.$

30. Let A : Event of getting a tail on the coin

B : Event of getting an even number on the die.

Then, $P(A) = \frac{1}{2}$

$P(B) = \frac{3}{6} = \frac{1}{2}$ as $B = \{2, 4, 6\}$

A and B being independent events,

P(Getting tail on the coin and even number on the die)

$= P(A \cap B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \mathbf{\frac{1}{4}}.$

31. Let A, B, C, D be the events that 1st, 2nd, 3rd and 4th shots hits the plane respectively. Then,

$P(A) = 0.4, P(B) = 0.3, P(C) = 0.2, P(D) = 0.1$

$\Rightarrow P(\text{not } A) = 1 - P(A), P(\text{not } B) = 1 - P(B)$

$= 1 - 0.4 = 0.6 \qquad = 1 - 0.3 = 0.7$

$P(\text{not } C) = 1 - P(C), \quad P(\text{not } D) = 1 - P(D)$

$= 1 - 0.2 = 0.8 \qquad = 1 - 0.1 = 0.9$

$\therefore$ P(None of the shots hits the plane)

$= P(\text{not } A) \times P(\text{not } B) \times P(\text{not } C) \times P(D)$

$= 0.6 \times 0.7 \times 0.8 \times 0.9 = 0.3024$

(As A, B, C and D are independent events)

P(At least one shot hits the plane)

$= 1 - P$(None hits the plane)

$= 1 - 0.3024 = \mathbf{0.6976}.$

32. A red ball can be selected in two mutually exclusive ways.

(i) Selecting bag I and then drawing a red ball from it

(ii) Selecting bag II and them drawing a red ball from it

$\therefore$ P(red ball) = P(Selecting bag I) × P(Red ball from bag I) + (selecting bag II) × P(Red ball from bag II)

$= \frac{1}{2} \times \frac{4}{7} + \frac{1}{2} \times \frac{2}{6} = \frac{2}{7} + \frac{1}{6} = \mathbf{\frac{19}{42}}.$

33. The three boxes B_1, B_2 and B_2 contain the different coloured balls as follows :

	Blue	Red
Box 1	3	1
Box 2	2	2
Box 3	1	3

There can be three mutually exclusive cases of drawing 2 blue balls and 1 red balls in the ways as given :

	Box 1	Box 2	Box 3
Case I	1 Blue	1 Blue	1 Red
Case II	1 Blue	1 Red	1 Blue
Case III	1 Red	1 Blue	1 Blue

$\therefore$ P(Drawing 2 blue and 1 red ball)

$$= P(\overset{B_1}{\text{Blue}}, \overset{B_2}{\text{Blue}}, \overset{B_3}{\text{Red}}) + P(\overset{B_1}{\text{Blue}}, \overset{B_2}{\text{Red}}, \overset{B_3}{\text{Blue}}) + P(\overset{B_1}{\text{Red}}, \overset{B_2}{\text{Blue}}, \overset{B_3}{\text{Blue}})$$

$$= \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} + \frac{3}{4} \times \frac{2}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4}$$

$$= \frac{18}{64} + \frac{6}{64} + \frac{2}{64} = \frac{26}{64} = \mathbf{\frac{13}{32}}.$$

34. P(Drawing of two balls of different colours from one of the bags)

= P(choosing the 1st bag) × P(Drawing 1 green out 5 green and 1 out of 11 blue balls) + P(choosing the 2nd bag) × P(Drawing 1 green out of 3 green balls and 1 out of 7 blue balls)

P(choosing 1st bag or 2nd bag) $= \frac{1}{2}$ ($\because$ There are 2 bags)

P(choosing 1 green and 1 blue ball from bag 1)

$$= \frac{{}^5C_1 \times {}^{11}C_1}{{}^{16}C_2} = \frac{5 \times 11 \times 2}{16 \times 15} = \frac{11}{24}$$

P(choosing 1 green and 1 blue ball from bag 2)

$$= \frac{{}^3C_1 \times {}^7C_1}{{}^{10}C_2} = \frac{3 \times 7 \times 2}{10 \times 9} = \frac{7}{15}$$

$\therefore$ Required probability $= \frac{1}{2} \times \frac{11}{24} + \frac{1}{2} \times \frac{7}{15}$

$$= \frac{11}{48} + \frac{7}{30} = \frac{55 + 56}{240} = \mathbf{\frac{111}{240}}.$$

35. P(At least one junior professor is selected)

= P(Selecting 1 Junior) × P(Selecting 2 Seniors) + P(Selecting 2 Junior) × P(Selecting 1 Senior) + P(Selecting all 3 Juniors)

$\therefore$ Required probability $= \frac{{}^4C_1 \times {}^6C_2}{{}^{10}C_3} + \frac{{}^4C_2 \times {}^6C_1}{{}^{10}C_3} + \frac{{}^4C_3}{{}^{10}C_3}$

$$= \frac{\cancel{4} \times \cancel{6} \times \cancel{5} \times \cancel{3} \times \cancel{2}}{\cancel{10} \times \cancel{9} \times \cancel{8} \times \cancel{2}} + \frac{\cancel{4} \times \cancel{3} \times \cancel{6} \times \cancel{3} \times \cancel{2}}{\cancel{2} \times 10 \times \cancel{9} \times \cancel{8} \times \cancel{2}} + \frac{\cancel{4} \times \cancel{3} \times \cancel{2}}{10 \times \cancel{9} \times \cancel{8}}$$

$$= \frac{1}{2} + \frac{3}{10} + \frac{1}{30} = \frac{15 + 9 + 1}{30} = \frac{25}{30} = \mathbf{\frac{5}{6}}.$$

36. $P(X) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$

($\because$ There are 26 black cards, 4 kings and 2 black kings)

$P(Y) = \frac{13}{52} + \frac{13}{52} + \frac{4}{52} - \frac{1}{52} - \frac{1}{52} = \frac{28}{52}$

($\because$ There are 13 clubs, 13 hearts and 4 jacks, 1 jack of clubs and 1 jack of hearts)

$P(Z) = \frac{4}{52} + \frac{13}{52} + \frac{4}{52} - \frac{1}{52} - \frac{1}{52} = \frac{19}{52}$

($\because$ There are 4 aces, 13 diamonds, 4 queens, 1 ace of diamond, 1 queen of diamond)

$\therefore$ $\boldsymbol{P(X) = P(Y) > P(Z)}.$

37. P(Particular number comes on the dice) = $\frac{1}{6}$

($\because$ there are in all 6 numbers)

P(particular number does not come on the dice) = $1 - \frac{1}{6} = \frac{5}{6}$

As there are 3 dices, so,

P(Picked number does not come in any of dice) = $\left(\frac{5}{6}\right)^3$

P(You lose money) = $\left(\frac{5}{6}\right)^3 = \frac{125}{216} \approx 0.58$

$\therefore$ P(Winning) = 1 – 0.58 = **0.42**

38. Given, odds against Event 1 = 5 : 2

$\Rightarrow$ P(Event 1 not happening) = $\frac{5}{5+2} = \frac{5}{7}$

Odds in favour of Event 2 = 6 : 5

$\Rightarrow$ P(Event 2 happens) = $\frac{6}{6+5} = \frac{6}{11}$

$\Rightarrow$ P(Event 2 not happening) = $1 - \frac{6}{11} = \frac{5}{11}$

$\therefore$ P(None of the events happen) = $\frac{5}{7} \times \frac{5}{11} = \frac{25}{77}$

($\because$ Both event are independent)

$\Rightarrow$ P(At least one event happens) = $1 - \frac{25}{77} = \mathbf{\frac{52}{77}}$.

39. P(At least 4 vessels arrive safely)

= P(exactly four vessels arrive safely)

+ P(All five vessels arrive safely)

$$= {}^5C_4 \times \left(\frac{9}{10}\right)^4 \times \left(\frac{1}{10}\right) + \left(\frac{9}{10}\right)^5$$

$$= \frac{5}{10} \times \left(\frac{9}{10}\right)^4 + \left(\frac{9}{10}\right)^5 = \left(\frac{\mathbf{9}}{\mathbf{10}}\right)^4 \left(\frac{\mathbf{5}}{\mathbf{10}} + \frac{\mathbf{9}}{\mathbf{10}}\right) = \frac{\mathbf{14 \times 9^4}}{\mathbf{10^5}}.$$

40. Total number of boys and girls = $2n + 2n = 4n$

Since, there are two equal batches, each batch has $2n$ members

$\therefore$ Let S (Sample space) : Selecting one batch out of 2

$\Rightarrow S$: Selecting $2n$ members out of $4n$ members.

$\Rightarrow n(S) = {}^{4n}C_{2n}$

If each batch has to have equal number of boys and girls, each batch should have n boys and n girls.

Let E : Event that each batch has 'n' boys and 'n' girls

$\Rightarrow$ $n(E) = {}^{2n}C_n \times {}^{2n}C_n = ({}^{2n}C_n)^2$

$\therefore$ Required probability = $\frac{n(E)}{n(S)} = \frac{\mathbf{({}^{2n}C_n)^2}}{\mathbf{{}^{4n}C_{2n}}}$.

41. The box contains 100 balls numbered from 1 to 100. Therefore, there are 50 even and 50 odd numbered balls. The sum of the three numbers drawn will be odd, if all three are odd or one is even and 2 are odd.

$\therefore$ Required probability = P(odd) × P(odd) × P(odd) + P(even) × P(odd) × P(odd)

$$= \frac{{}^{50}C_1}{{}^{100}C_1} \times \frac{{}^{50}C_1}{{}^{100}C_1} \times \frac{{}^{50}C_1}{{}^{100}C_1} + \frac{{}^{50}C_1}{{}^{100}C_1} \times \frac{{}^{50}C_1}{{}^{100}C_1} + \frac{{}^{50}C_1}{{}^{100}C_1}$$

$$= 2 \times \frac{{}^{50}C_1 \times {}^{50}C_1 \times {}^{50}C_1}{{}^{100}C_1 \times {}^{100}C_1 \times {}^{100}C_1}$$

$$= 2 \times \frac{50 \times 50 \times 50}{100 \times 100 \times 100} = \mathbf{\frac{1}{4}}.$$

42. Let S be the sample space.

Then $n(S)$ = Number of triangles formed by selecting any three vertices of 6 vertices of a regular hexagon

$$= {}^6C_3 = \frac{6 \times 5 \times 4}{3 \times 2} = 20.$$

Let A : Event that the selected three vertices form an equilateral triangle.

Then $n(A) = 2$

(As only two equilateral triangles are formed from the vertices of a regular hexagon)

$\therefore$ Required probability = $\frac{n(A)}{n(S)} = \frac{2}{20} = \mathbf{\frac{1}{10}}$.

43. Total number of ways three die can be rolled

= 6 × 6 × 6

= 216

A larger number than the previous number can be got in the three throws as

(1, 2, 3), (1, 2, 4), (1, 2, 5) (1, 2, 6), (1, 3, 4), (1, 3, 5) (1, 3, 6), (1, 4, 5) (1, 4, 6) (1, 5, 6) (2, 3, 4), (2, 3, 5), (2, 3, 6) (2, 4, 5), (2, 4, 6), (2, 5, 6), (3, 4, 5), (3, 4, 6) (3, 5, 6), (4, 5, 6).

$\therefore$ Total number of favourable cases = 20

$\therefore$ Required probability = $\frac{20}{216} = \mathbf{\frac{5}{54}}$

44. In the first nine throws we should have three sixes and six non-sixes and a six in the tenth throw and thereafter whatever face appears, it doesn't matter.

$\therefore$ Required probability

$$= {}^9C_3 \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^6 \times \frac{1}{6} \times \underbrace{1 \times 1 \times 1 \ldots\ldots \times 1}_{10 \text{ times}} = \frac{\mathbf{84 \times 5^6}}{\mathbf{6^{10}}}$$

45. Let W denote the event of drawing a white ball at any draw and B that of drawing a black ball.

Then, $P(W) = \frac{a}{a+b}$, $P(B) = \frac{b}{a+b}$

$\therefore$ P (A wins the game) = P (W or BBW or $BBBBW$ or ......)

= $P(W) + P(B) \,.\, P(B) \,.\, P(W) + P(B) \,.\, P(B) \,.\, P(B) \,.\, P(B) \,.\, P(W) + \ldots\ldots$

= $P(W)\,(1 + (P(B))^2 + (P(B))^4 + \ldots\ldots)$

$$= \frac{P(W)}{1 - (P(B))^2} = \frac{\frac{a}{a+b}}{1 - \frac{b^2}{(a+b)^2}} = \frac{a\,(a+b)}{a^2 + 2ab} = \frac{a+b}{a+2b}$$

$\therefore$ P (B wins the game) = 1 – P (A wins the game)

$= 1 - \frac{(a+b)}{(a+2b)} = \frac{b}{a+2b}$

According to the given condition,

$\frac{a+b}{a+2b} = 3 \cdot \frac{b}{a+2b} \Rightarrow a = 2b \Rightarrow a : b = \mathbf{2 : 1.}$

46. Total number of determinants that can be formed using 0 and 1 = 16 (4 × 4)

The positive determinants are $\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$, *i.e.*, 3 in number.

$\therefore$ Required probability = $\mathbf{\frac{3}{16}}$.

47. In $3n$ consecutive natural numbers,

(*i*) n numbers are of the form $3p$

(*ii*) n numbers are of the form $3p + 1$

(*iii*) n numbers are of the form $3p + 2$

For the sum of the chosen 3 numbers to be divisible by 3, we can proceed in two ways:

(*i*) Either we can select all the three numbers from any one of the set or

(*ii*) We can select one number from each set.

$\therefore$ Favourable number of cases = ${}^nC_3 + {}^nC_3 + {}^nC_3 + ({}^nC_1 \times {}^nC_1 \times {}^nC_1)$

$= 3 \times \left(\frac{\underline{|n}}{\underline{|n-3}\,\underline{|3}} \right) + n^3 = 3 \times \left(\frac{n(n-1)(n-2)}{6} \right) + n^3$

Total number of selections

$= {}^{3n}C_3 = \frac{\underline{|3n}}{\underline{|3n-3}\,\underline{|3}} = \frac{3n(3n-1)(3n-2)}{6}$

$\therefore$ Required probability $= \dfrac{\frac{3n(n-1)(n-2)}{6} + n^3}{\frac{3n(3n-1)(3n-2)}{6}}$

$= \dfrac{\frac{3n\left[(n-1)(n-2) + 2n^2\right]}{6}}{\frac{3n(3n-1)(3n-2)}{6}}$

$= \frac{n^2 - 3n + 2 + 2n^2}{(3n-1)(3n-2)} = \mathbf{\frac{3n^2 - 3n + 2}{(3n-1)(3n-2)}}$.

48. As $(n + 1)$ coins are fair

P (Tossing a tail) $= \dfrac{\frac{n+1}{2}}{2n+1} = \frac{n+1}{2(2n+1)}$

$\therefore$ P (Tossing a head) $= 1 - \frac{n+1}{2(2n+1)}$

$= \frac{4n + 2 - n - 1}{2(2n+1)} = \frac{3n+1}{4n+2}$

Given, $\frac{3n+1}{4n+2} = \frac{31}{42}$

$\Rightarrow 126n + 42 = 124n + 62 \Rightarrow 2n = 20 \Rightarrow n = \mathbf{10.}$

49. Probability of guessing a correct answer $= \frac{1}{3}$

Probability of guessing an incorrect answer $= \frac{2}{3}$

$\therefore$ Probability of guessing 4 or more correct answers

$= {}^5C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right) + {}^5C_5 \left(\frac{1}{3}\right)^5 = 5 \times \frac{2}{3^5} + \frac{1}{3^5} = \mathbf{\frac{11}{3^5}}$.

50. Let x, y, z be the probabilities of happening of events E_1, E_2 and E_3 respectively. Then,

$\alpha = P$ (occurrence of E_1 only) $= x(1-y)(1-z)$

$\beta = P$ (occurrence of E_2 only) $= (1-x)y(1-z)$

$\gamma = P$ (occurrence of E_3 only) $= (1-x)(1-y)z$

$p = P$ (not occurrence of E_1, E_2, E_3) $= (1-x)(1-y)(1-z)$.

$\therefore (\alpha - 2\beta)p = \alpha\beta$

$\Rightarrow [x(1-y)(1-z) - 2(1-x)y(1-z)](1-x)(1-y)(1-z)$
$= x(1-y)(1-z)(1-x)y(1-z)$

$\Rightarrow (1-z)[x(1-y) - 2y(1-x)] = xy(1-z)$

$\Rightarrow x - xy - 2y + 2xy = xy$

$\Rightarrow x = 2y$ ...(*i*)

Also, $(\beta - 3\gamma)p = 2\beta\gamma$

$\Rightarrow y = 3z$ ...(*ii*)

$\therefore x = 6z$ (From (*i*) and (*ii*))

$\Rightarrow \frac{x}{z} = 6 \Rightarrow \frac{P(E_1)}{P(E_3)} = \mathbf{6.}$

SELF ASSESSMENT SHEET

1. Thirty days are in September, April, June and November. Some months are of thirty one days. A month is chosen at random. Then its probability of having exactly three days less than a maximum of 31 day is

(*a*) $\frac{15}{16}$ (*b*) 1 (*c*) $\frac{3}{48}$ (*d*) None of these

(*SNAP 2007*)

2. There are four hotels in a town. If 3 men check into the hotels in a day, then what is the probability that each checks into a different hotel ?

(*a*) $\frac{6}{7}$ (*b*) $\frac{1}{8}$ (*c*) $\frac{3}{8}$ (*d*) $\frac{5}{9}$

(*JMET 2011*)

3. A basket contains 2 blue, 4 red, 3 green and 5 black balls. If 4 balls are picked at random, what is the probability that either 2 are blue and 2 are red or 2 are green and 2 are black ?

(*a*) $\frac{3}{7}$ (*b*) $\frac{3}{91}$ (*c*) $\frac{5}{1001}$ (*d*) $\frac{36}{1001}$

(*MBA CET Mah. 2007*)

4. There are three events *A*, *B* and *C*, one of which must and can only happen. If the odds one 8 : 3 against *A*, 5 : 2 against *B*, the odds against *C* must be

(*a*) 13 : 7 (*b*) 3 : 2 (*c*) 43 : 34 (*d*) 43 : 77

(*ATMA 2008*)

5. Three letters are randomly selected from the 26 capital letters of the English alphabet. What is the probability that letters *A* will not be included in the choice ?

(*a*) $\frac{1}{12}$ (*b*) $\frac{23}{26}$ (*c*) $\frac{12}{13}$ (*d*) $\frac{25}{26}$

(*NDA/NA 2009*)

6. *A* and *B* are two mutually exclusive and exhaustive events with $P(B) = 3P(A)$. What is the value of $P(\overline{B})$?

(*a*) $\frac{3}{4}$ (*b*) $\frac{1}{4}$ (*c*) $\frac{1}{3}$ (*d*) $\frac{2}{3}$

(*NDA/NA 2009*)

7. Two decks of playing cards are well shuffled and 26 cards are randomly distributed to a player. Then, the probability that the player gets all distinct cards is

(*a*) ${}^{52}C_{26} \mid {}^{104}C_{26}$ (*b*) $2 \times {}^{52}C_{26} \mid {}^{104}C_{26}$

(*c*) $2^{13} \times {}^{52}C_{26} \mid {}^{104}C_{26}$ (*d*) $2^{26} \times {}^{52}C_{26} \mid {}^{104}C_{26}$.

(*WBJEE 2012*)

8. An article manufactured by a company consists of two parts *X* and *Y*. In the process of manufacture of the part *X*, 9 out of 100 parts may be defective. Similarly, 5 out of 100 are likely to be defective in the manufacture of part *Y*. Calculate the probability that the assembled product will not be defective.

(*a*) 0.6485 (*b*) 0.6565 (*c*) 0.8645 (*d*) None of these

9. If *n* integers taken at random are multiplied together, then the probability that the last digit of the product is 1, 3, 7 or 9 is

(*a*) $\frac{2^n}{5^n}$ (*b*) $\frac{4^n - 2^n}{5^n}$ (*c*) $\frac{4^n}{5^n}$ (*d*) None of these

(*UPSEE 2013*)

10. A letter is taken out at random from 'ASSISTANT' and another is taken out from 'STATISTICS'. The probability that they are same letters is

(*a*) $\frac{1}{45}$ (*b*) $\frac{13}{90}$ (*c*) $\frac{19}{90}$ (*d*) None of these

(*AMU 2007*)

ANSWERS

1. (*d*)	**2.** (*c*)	**3.** (*d*)	**4.** (*c*)	**5.** (*b*)	**6.** (*b*)	**7.** (*d*)	**8.** (*c*)	**9.** (*a*)	**10.** (*c*)

HINTS AND SOLUTIONS

1. The month having 3 days less than 31 days has 28 days, *i.e*, it is the month of February.

$P(\text{Choosing February}) = \mathbf{\frac{1}{12}}$

2. Total number of ways of checking in the 4 hotels by 3 men $= 4 \times 4 \times 4 = \mathbf{4^3}$.

Number of ways in which each man checks into a different hotel $= 4 \times 3 \times 2$

(As for the 1st person, there are 4 choices, for 2nd remaining 3, and for the 3rd remaining 2).

$\therefore$ P(Each person checks into a different hotel)

$$= \frac{4 \times 3 \times 2}{4 \times 4 \times 4} = \mathbf{\frac{3}{8}}.$$

3. Required probability $= \dfrac{P(\text{2 blue balls}) \times P(\text{2 red balls})}{P(\text{4 balls out of 14 ball})} + \dfrac{P(\text{2 green balls}) \times P(\text{2 black balls})}{P(\text{4 out of 14 balls})}$

$$= \frac{{}^2C_2 \times {}^4C_2}{{}^{14}C_4} + \frac{{}^3C_2 \times {}^5C_2}{{}^{14}C_4}$$

$$= \frac{\frac{4 \times 3}{2}}{\frac{14 \times 13 \times 12 \times 11}{4 \times 3 \times 2 \times 1}} + \frac{\frac{3 \times 5 \times 4}{2}}{\frac{14 \times 13 \times 12 \times 11}{4 \times 3 \times 2}}$$

$$= \frac{6}{1001} + \frac{30}{1001} = \mathbf{\frac{36}{1001}}$$

4. Given, odds against $A = 8 : 3$

$$\Rightarrow P(\text{not } A) = \frac{8}{8+3} = \frac{8}{11} \quad \Rightarrow \quad P(A \text{ happens}) = \frac{3}{11}$$

Odds against $B = 5 : 2$

$$\Rightarrow \quad P(\text{not } B) = \frac{5}{5+2} = \frac{5}{7}$$

$$\Rightarrow \quad P(B \text{ happens}) = \frac{2}{7}$$

As out of *A*, *B* and *C*, one and only one can happen, so

$P(A) + P(B) + P(C) = 1$

$P(C) = 1 - (P(A) + P(B))$

$$= 1 - \left(\frac{3}{11} + \frac{2}{7}\right) = 1 - \left(\frac{21 + 22}{77}\right) = 1 - \frac{43}{77} = \frac{34}{77}$$

$$P(\text{not } C) = 1 - \frac{34}{77} = \frac{43}{77}$$

$$\text{odds against } C = \frac{P(not\ c)}{P(c)} = \frac{43/77}{34/77} = \mathbf{43 : 34.}$$

5. Total number of ways in which 3 letters can be selected from 26 letters $= {}^{26}C_3$.

If *A* is not to be included in the choice, there are 25 letters left, so number of ways in which 3 letters can be selected without including $A = {}^{25}C_3$.

$$\therefore \text{ Required probability} = \frac{{}^{25}C_3}{{}^{26}C_3} = \frac{25 \times 24 \times 23}{26 \times 25 \times 24} = \mathbf{\frac{23}{26}}.$$

6. Since A and B are mutually exclusive and exhaustive events, therefore,

$P(A \cap B) = 0$, $P(A \cup B) = 1$

We know that, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow 1 = P(A) + 3\,P(A)$ (Given $P(B) = 3P(A)$)

$$\Rightarrow \quad 4P(A) = 1 \Rightarrow P(A) = \frac{1}{4}$$

$$\therefore \quad P(B) = 1 - \frac{1}{4} = \frac{3}{4} \qquad (\because P(A) + P(B) = 1)$$

$$\Rightarrow \quad P(\overline{B}) = 1 - P(B) = 1 - \frac{3}{4} = \mathbf{\frac{1}{4}}.$$

7. Since there are 52 distinct cards in a deck and each distinct card is 2 in number.

$\therefore$ 2 decks will also contain only 52 distinct cards, two each.

$\therefore$ Probability that the player gets all distinct cards

$$= \mathbf{\frac{{}^{52}C_{26} \times 2^{26}}{{}^{104}C_{26}}}.$$

8. Required probability $= P(X$ not defective and Y not defective$)$

$= P(\overline{X}) \times P(\overline{Y})$

$= (1 - P(X))\,(1 - P(Y))$

$$= \left(1 - \frac{9}{100}\right)\left(1 - \frac{5}{100}\right)$$

$$= \frac{91}{100} \times \frac{95}{100} = \frac{8645}{10000} = \mathbf{0.8645}.$$

9. In any number the last digits can 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. Therefore, the last digit of each number can be chosen in 10 ways.

$\therefore$ Exhaustive number of ways $= 10^n$. If the last digit be 1, 3, 7 or 9, none of the numbers can be even or end in 0 or 5. That is, there is a choice of only 4 digits for the last digit of each of these n numbers.

So, favourable number of ways $= 4^n$

$$\therefore \quad \text{Required probability} = \frac{4^n}{10^n} = \mathbf{\frac{2^n}{5^n}}.$$

10. In the words 'ASSISTANT' and STATISTICS', 'N' and 'C' are the uncommon letters.

The same letters are A, I, S and T whose numbers in both the words are as follows:

		A	I	S	T
ASSISTANT	$\rightarrow$	2	1	3	2
STATISTICS	$\rightarrow$	1	2	3	3

$$\text{Probability of choosing '}A\text{'} = \frac{{}^2C_1}{{}^9C_1} \times \frac{{}^1C_1}{{}^{10}C_1} = \frac{2}{9} \times \frac{1}{10} = \frac{1}{45}$$

$$\text{Probability of choosing '}I\text{'} = \frac{1}{{}^9C_1} \times \frac{{}^2C_1}{{}^{10}C_1} = \frac{1}{9} \times \frac{2}{10} = \frac{1}{45}$$

$$\text{Probability of choosing '}S\text{'} = \frac{{}^3C_1}{{}^9C_1} \times \frac{{}^3C_1}{{}^{10}C_1} = \frac{3}{9} \times \frac{3}{10} = \frac{1}{10}$$

$$\text{Probability of choosing '}T\text{'} = \frac{{}^2C_1}{{}^9C_1} \times \frac{{}^3C_1}{{}^{10}C_1} = \frac{2}{9} \times \frac{3}{10} = \frac{1}{15}$$

$$\therefore \quad \text{Required probability} = \frac{1}{45} + \frac{1}{45} + \frac{1}{10} + \frac{1}{15} = \mathbf{\frac{19}{90}}.$$

10 Trigonometry

KEY FACTS

I. The common systems of measuring angles are:

(*a*) **Sexagesimal System:** Here, one complete revolution is divided into 360 equal parts, each called a **degree**, a degree is divided into sixty equal parts, each called a **second**. Thus,

One complete revolution = 360°

$1° = 60'$ (minutes)

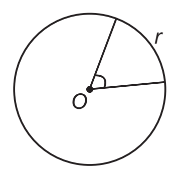
$1' = 60''$ (seconds)

(*b*) **Circular System:** The unit of measuring an angle in this system is **radian**. A **radian** is the measure of the central angle subtended by an arc equal in length to the radius of the circle. **1 radian** is written as $\mathbf{1^c}$.

II. π radians = 180 degrees

(*a*) **To convert an angle in radians to its equivalent in degrees, multiply the number of radians by $\dfrac{180°}{\pi}$.**

$\therefore \dfrac{\pi}{3}$ radians $= \dfrac{\pi}{3} \times \dfrac{180°}{\pi} = 60°$

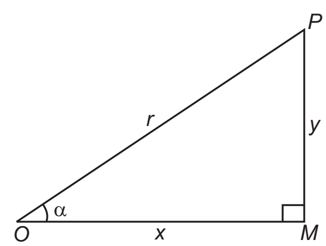


(*b*) **To convert an angle in degrees to its equivalent in radians, multiply the number of degrees by $\dfrac{\pi}{180°}$.**

$\therefore 45° = \dfrac{\pi}{180°} \times 45° = \dfrac{\pi}{4}$ rad.

III. Trigonometrical functions (or Ratios)

In a right angled triangle *OMP*, where $\angle OMP = 90°$, $\angle POM = \alpha$, base $OM = x$, perpendicular $PM = y$ and hypotenuse $OP = r$,



(*a*) $\sin\alpha = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{y}{r}$

(*b*) $\cos\alpha = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{x}{r}$

(*c*) $\tan\alpha = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{y}{x}$

(*d*) $\text{cosec}\,\alpha = \dfrac{1}{\sin\alpha} = \dfrac{\text{Hypotenuse}}{\text{Prepedicular}} = \dfrac{r}{y}$

(*e*) $\sec\alpha = \dfrac{1}{\cos\alpha} = \dfrac{\text{Hypotenuse}}{\text{Base}} = \dfrac{r}{y}$

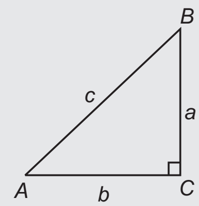
(*f*) $\cot\alpha = \dfrac{1}{\tan\alpha} = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{x}{y}$

Note:

1. In the right $\angle d\ \Delta ABC$, it can be easily seen that,

(*a*) $\sin A = \dfrac{a}{c} = \cos B$ **(*b*)** $\cos A = \dfrac{b}{c} = \sin B$

(*c*) $\tan A = \dfrac{a}{b} = \cot B$ **(*d*)** $\tan B = \dfrac{b}{a} = \cot A$



2. $(\sin\theta)^2 = \sin^2\theta$, $(\cos\theta)^3 = \cos^3\theta$ read as sine square θ and cos cube θ respectively.

IV. Fundamental relations between trigonometric functions

1. Reciprocal relations

(*i*) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} \Rightarrow \operatorname{cosec}\theta \times \sin\theta = 1$ (*ii*) $\sec\theta = \dfrac{1}{\cos\theta} \Rightarrow \sec\theta \times \cos\theta = 1$

(*iii*) $\cot\theta = \dfrac{1}{\tan\theta} \Rightarrow \cot\theta \times \tan\theta = 1$

2. Quotient relations

(*i*) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (*ii*) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

3. Square relations (Pythagorean Identities)

(*i*) $\sin^2\theta + \cos^2\theta = 1$

(*ii*) $\sec^2\theta = 1 + \tan^2\theta$ or $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta - 1 = \tan^2\theta$

(*iii*) $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ or $\operatorname{cosec}^2\theta - \cot^2\theta = 1$ or $\operatorname{cosec}^2\theta - 1 = \cot^2\theta$

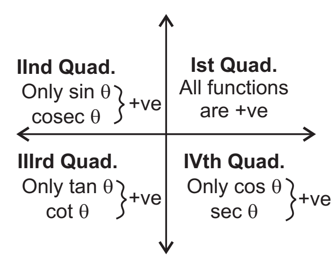
V. Signs of trigonometric functions

The sign of a particular *t*-function is any quadrant can be remembered by the phrase. "All – sin – tan – cos" (**A**dd **S**ugar **T**o **C**offee).

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

Ist quad· 2nd quad· 3rd quad· 4th quad

The *t*-function stated in the given phrase corresponding to the particular quadrant along with its reciprocal is positive in that quadrant and the rest are negative.



VI. Values of trigonometric ratios of some standard angles

θ / *t*-function	**0°**	**30° or $\frac{\pi}{3}$**	**45° or $\frac{\pi}{4}$**	**60° or $\frac{\pi}{6}$**	**90° or $\frac{\pi}{2}$**	**180° or π**
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined	0
cosec θ	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	undefined
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined	–1
cot θ	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	undefined

VII. Trigonometric functions of angles of any magnitude

The following formulae are helpful in reducing functions of any larger angle to function of smaller angles.

	$-\theta$	**$90° - \theta$**	**$90° + \theta$**	**$180° - \theta$**	**$180° + \theta$**	**$270° - \theta$**	**$270° + \theta$**	**$360° - \theta$**	**$360° + \theta$**
sin	– sin	cos θ	cos θ	sin θ	– sin θ	– cos θ	– cos θ	– sin θ	sin θ
cos	cos θ	sin θ	– sin θ	– cos θ	– cos θ	– sin θ	sin θ	cos θ	cos θ
tan	– tan θ	cot θ	– cot θ	– tan θ	tan θ	cot θ	– cot θ	– tan θ	tan θ
cosec	– cosec θ	sec θ	sec θ	cosec θ	– cosec θ	– sec θ	– sec θ	– cosec θ	cosec θ
sec	sec θ	cosec θ	– cosec θ	– sec θ	– sec θ	– cosec θ	cosec θ	sec θ	sec θ
cot	– cot θ	tan θ	– tan θ	– cot θ	cot θ	tan θ	– tan θ	cot θ	cot θ

	$2n\pi - \theta$ or $n \cdot 360° - \theta$	**$2n\pi + \theta$ or $n \cdot 360° + \theta$**
sin	–sin θ	sin θ
cos	cos θ	cos θ
tan	–tan θ	tan θ

VIII. Some important formulae

(*a*) Sum Formulae

(*i*) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(*ii*) $\cos(A+B) = \cos A \cos B + \sin A \sin B$

(*iii*) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(*iv*) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

(*v*) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

(*b*) Difference formulae

(*i*) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(*ii*) $\cos(A-B) = \cos A \cos B + \sin A \sin \mathrm{B}$

(*iii*) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(*iv*) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(*c*) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

(*d*) $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

$\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

$\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

$\cos C - \cos D = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

(*e*) Product formulae

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \sin B = \sin(A+B) + \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-\mathrm{B})$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

(f) Double angle formulae

- $\sin 2A = 2 \sin A \cos A$

  $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$

- $\cos 2A = \cos^2 A - \sin^2 A$

  $\cos 2A = 1 - 2\sin^2 A$

  $\cos 2A = 2\cos^2 A - 1$

  $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

- $2\sin^2 A = 1 - \cos 2A$

  $\Rightarrow \sin^2 A = \dfrac{1-\cos 2A}{2}$

- $2\cos^2 A = 1 + \cos 2A$

  $\Rightarrow \cos^2 A = \dfrac{1+\cos 2A}{2}$

- $\tan^2 A = \dfrac{\sin^2 A}{\cos^2 A} = \dfrac{1-\cos 2A}{1+\cos 2A}$

(g) Half angle formulae

- $\sin A = 2\sin A/2 \cos A/2$

  $\sin A = \dfrac{2\tan A/2}{1-\tan^2 A/2}$

- $\cos A = \cos^2 A/2 - \sin^2 A/2$

  $\cos A = 1 - 2\sin^2 A/2$

  $\cos A = 2\cos^2 A/2 - 1$

  $\cos A = \dfrac{1-\tan^2 A/2}{1+\tan^2 A/2}$

- $\tan A = \dfrac{2\tan A/2}{1-\tan^2 A/2}$

- $\sin A/2 = \pm\sqrt{\dfrac{1-\cos A}{2}}$

- $\cos A/2 = \pm\sqrt{\dfrac{1-\cos A}{2}}$

- $\tan A/2 = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$

(h) Triple-angle formulae

- $\sin 3A = 3\sin A - 4\sin^3 A$

- $\cos 3A = 4\cos^3 A - 3\cos A$

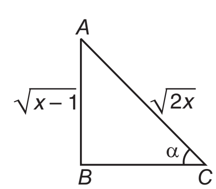
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

- $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

SOLVED EXAMPLES

Ex. 1. If α is an acute angle and $\sin\alpha = \sqrt{\dfrac{x-1}{2x}}$, then what is tan α equal to? ***(CDS 2010)***

Sol. In ΔABC where $\angle B = 90°$, $\sin\alpha = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{\sqrt{x-1}}{\sqrt{2x}} = \dfrac{AB}{AC}$



$\therefore\ BC^2 = AC^2 - AB^2 = 2x - (x-1) = x + 1$

$\Rightarrow BC = \sqrt{x+1}$

$\therefore\ \tan\alpha = \dfrac{\text{Prependicular}}{\text{Base}} = \dfrac{AB}{BC} = \dfrac{\sqrt{x-1}}{\sqrt{x+1}}$.

Ex. 2. What is the value of the expression

$$3\tan^2\frac{\pi}{6} + \frac{4}{3}\cos^2\frac{\pi}{6} - \frac{1}{2}\cot^3\frac{\pi}{4} - \frac{2}{3}\sin^2\frac{\pi}{3} + \frac{1}{8}\sec^4\frac{\pi}{3}\ ?$$

Sol. $3\tan^2\frac{\pi}{6}+\frac{4}{3}\cos^2\frac{\pi}{6}-\frac{1}{2}\cot^3\frac{\pi}{4}-\frac{2}{3}\sin^2\frac{\pi}{3}+\frac{1}{8}\sec^4\frac{\pi}{3}$

$= 3\tan^2 30° + \frac{4}{3}\cos^2 30° - \frac{1}{2}\cot^3 45° - \frac{2}{3}\sin^2 60° + \frac{1}{8}\sec^4 60°$

$= 3\times\left(\frac{1}{\sqrt{3}}\right)^2+\frac{4}{3}\times\left(\frac{\sqrt{3}}{2}\right)^2-\frac{1}{2}\times(1)^3-\frac{2}{3}\times\left(\frac{\sqrt{3}}{2}\right)^2+\frac{1}{8}\times(2)^4$

$= 3\times\frac{1}{3}+\frac{4}{3}\times\frac{3}{4}-\frac{1}{2}-\frac{2}{3}\times\frac{3}{4}+\frac{1}{8}\times 16 = 1+1-\frac{1}{2}-\frac{1}{2}+2=\mathbf{3}.$

Ex. 3. If $A + B = 90°$, then what is the value of $\sqrt{\sin A \sec B - \sin A \cos B}$?

Sol. Given $A + B = 90°$

$\therefore\ \sqrt{\sin A\sec B-\sin A\cos B}=\sqrt{\sin A\sec(90°-A)-\sin A\cos(90°-A)}$

$=\sqrt{\sin A.\text{cosec}\,A-\sin A.\sin A}=\sqrt{1-\sin^2 A}=\sqrt{\cos^2 A}=\mathbf{\cos A}.$

Ex. 4. If $\sin(A+B+C)=1$, $\tan(A-B)=\frac{1}{\sqrt{3}}$ and $\sec(A+C)=2$, then find the values of the angles A, B and C in degrees.

Sol. $\sin(A+B+C)=1 \Rightarrow \sin(A+B+C)=\sin 90° \Rightarrow A+B+C=90°$ ...(*i*)

$\tan(A-B)=\frac{1}{\sqrt{3}} \Rightarrow \tan(A-B)=\tan 30° \Rightarrow A-B=30°$ ...(*ii*)

$\sec(A+C)=2 \Rightarrow \sec(A+C)=\sec 60° \Rightarrow A+C=60°$ ...(*iii*)

Eq (*i*) – Eq (*iii*) $\Rightarrow B=30°$

$\therefore$ From eqn (*ii*), $A-30°=30° \Rightarrow A=60°$

$\therefore$ From eqn (*i*), $60°+30°+C=90° \Rightarrow C=0° \Rightarrow A=\mathbf{60°}, B=\mathbf{30°}, C=\mathbf{0°}.$

Ex. 5. If $\sec\theta=\sqrt{2}$ and $\frac{3\pi}{2}<\theta<2\pi$, find the value of $\frac{1+\tan\theta+\text{cosec}\,\theta}{1+\cot\theta-\text{cosec}\,\theta}$.

Sol. $\sec\theta=\sqrt{2} \Rightarrow \cos\theta=\frac{1}{\sqrt{2}}$

$\therefore \sin\theta=\pm\sqrt{1-\cos^2\theta}=\pm\sqrt{1-\frac{1}{2}}=\pm\frac{1}{\sqrt{2}}$

Since θ lies in the fourth quadrant, so $\sin\theta$ is –ve and $\cos\theta$ is +ve.

$\therefore\ \sin\theta=-\frac{1}{\sqrt{2}},\ \text{cosec}\,\theta=-\sqrt{2}$

$\tan\theta=\frac{\sin\theta}{\cos\theta}=-\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{1}=-1 \Rightarrow \cot\theta=-1$

$\therefore\ \frac{1+\tan\theta+\text{cosec}\,\theta}{1+\cot\theta-\text{cosec}\,\theta}=\frac{1-1-\sqrt{2}}{1-1+\sqrt{2}}=\mathbf{-1}.$

Ex. 6. What is the value of $\cot(-870°)$?

Sol. $\cot(-870°) = -\cot 870°$ $(\because \cot(-\theta)=-\cot\theta)$

$= -\cot(720°+150°) = -\cot(2\times 360°+150°)$

$= -\cot 150° = -\cot(90° + 60°)$ $(\because \cot(n.360° + \theta) = \cot\theta,\ n \in N)$

$= -(-\tan 60°)$ $(\because \cot(90° + \theta) = -\tan\theta)$

$= \tan 60° = \mathbf{\sqrt{3}}$.

Ex. 7. What is the value of cos 480º sin 150º + sin 600º cos 390º? **(*Kerala PET 2006*)**

Sol. $\cos 480° \sin 150° + \sin 600° \cos 390°$

$= \cos(360° + 120°) \sin(180° - 30°) + \sin(2 \times 360° - 120°) \cos(360° + 30°)$

$= \cos 120° \sin 30° - \sin(120°) \cos 30°$

$= -\cos 60° \sin 30° - \sin(180° - 60°) \cos 30°$

$= -\cos 60° \sin 30° - \sin 60° \cos 30°$

$= -(\sin 30° \cos 60° + \sin 60° \cos 30°)$

$= -\sin(30° + 60°)$ $(\because \sin(A + B) = \sin A \cos B + \cos A \sin B)$

$= -\sin 90° = \mathbf{-1}$.

$$\left[\begin{array}{l} \because \cos(360° + \theta) = \cos\theta \\ \sin(n.360° - \theta) = -\sin\theta,\ n \in N \\ \sin(180° - \theta) = \sin\theta \\ \sin(180° + \theta) = -\sin\theta \\ \cos 120° = \cos(180° - 60°) = -\cos 60° \end{array}\right]$$

Ex. 8. If $x = y\cos\left(\dfrac{2\pi}{3}\right) = z\cos\left(\dfrac{4\pi}{3}\right)$, then what is $xy + yz + zx$ equal to? ***(NDA/NA 2011)***

Sol. $x = y\cos 120° = z\cos 240°$

$\Rightarrow\ x = y\cos(180° - 60°) = z\cos(180° + 60°)$

$\Rightarrow\ x = -y\cos 60° = -z\cos 60°$ $(\because \cos(180° - \theta) = -\cos\theta = \cos(180° + \theta))$

$\Rightarrow\ x = -\dfrac{1}{2}y = -\dfrac{z}{2}$ $\Rightarrow\ 2x = -y = -z$

$\Rightarrow\ \dfrac{x}{\frac{1}{2}} = \dfrac{y}{(-1)} = \dfrac{z}{(-1)} = k$ $\Rightarrow\ x = \dfrac{k}{2}, y = -k, z = -k$

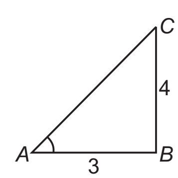
$\therefore\ xy + yz + zx = \left(\dfrac{k}{2}\right)(-k) + (-k)(-k) + (-k)\left(\dfrac{k}{2}\right) = \dfrac{-k^2}{2} + k^2 - \dfrac{k^2}{2} = \mathbf{0}$.

Ex. 9. If A lies in the third quadrant and $3\tan A - 4 = 0$, then what is the value of $5\sin 2A + 3\sin A + 4\cos A$?

Sol. $3\tan A - 4 = 0$

$\Rightarrow\ 3\tan A = 4 \Rightarrow \tan A = \dfrac{4}{3}$

$\Rightarrow\ \tan A = \dfrac{4}{3} \Rightarrow \dfrac{BC}{AB} = \dfrac{4}{3}$ $\Rightarrow AC = \sqrt{AB^2 + BC^2} = \sqrt{9 + 16} = \sqrt{25} = \mathbf{5}$.



$\therefore$ A being in the third quadrant, $\sin A$ and $\cos A$ are negative

So, $\sin A = -\dfrac{4}{5}$ and $\cos A = -\dfrac{3}{5}$.

$\therefore$ Given expression $5\sin 2A + 3\sin A + 4\cos A$

$= 5 \times 2\sin A\cos A + 3\sin A + 4\cos A$

$= 10 \times \left(\dfrac{-4}{5}\right) \times \left(\dfrac{-3}{5}\right) + 3 \times \left(\dfrac{-4}{5}\right) + 4 \times \left(\dfrac{-3}{5}\right)$

$= \dfrac{24}{5} - \dfrac{12}{5} - \dfrac{12}{5} = \mathbf{0}$.

Ex. 10. Show that $\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left\{\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right\}=1$.

Sol. $\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left\{\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right\}$

$= \cos(270°+x)\cos(360°+x)\{\cot(270°-x)+\cot(360°+x)\}$

$= \sin x\cos x\{\tan x+\cot x\}$

$$\left[\begin{array}{l}\because \cos(360°+\theta)=\cos\theta\\ \cot(360°+\theta)=\cot\theta\\ \cos(270°+\theta)=+\sin\theta\\ \cot(270°-\theta)=\tan\theta\end{array}\right]$$

$= \sin x\cos x\left[\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right] = \sin x\cos x\left[\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\right] = \mathbf{1}.$

Ex. 11. What is the value of $\frac{\cos 15°+\cos 45°}{\cos^3 15°+\cos^3 45°}$? ***(NDA/NA 2013)***

Sol. $$\frac{\cos 15°+\cos 45°}{\cos^3 15°+\cos^3 45°} = \frac{\cos 15°+\cos 45°}{(\cos 15°+\cos 45°)(\cos^2 15°+\cos^2 45°-\cos 45°\cos 15°)}$$

($\because$ $a^3+b^3=(a+b)(a^2-ab+b^2)$)

$$= \frac{1}{(\cos^2 15°+\cos^2 45°-\cos 45°\cos 15°)} \quad \ldots(i)$$

$$\left[\begin{array}{l}\cos 15° = \cos(45°-30°) = \cos 45°\cos 30°+\sin 45°\sin 30°\\ \qquad = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}\end{array}\right]$$

$$\therefore \frac{\cos 15°+\cos 45°}{\cos^3 15°+\cos^3 45°} = \frac{1}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2-\frac{1}{\sqrt{2}}\times\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)} \quad \text{(From } (i))$$

$$= \frac{1}{\frac{3+1+2\sqrt{3}}{8}+\frac{1}{2}-\left(\frac{\sqrt{3}+1}{4}\right)} = \frac{1}{\frac{4+4+2\sqrt{3}-2\sqrt{3}-2}{8}} = \frac{8}{6} = \mathbf{\frac{4}{3}}.$$

Ex. 12. What is the value of $\sqrt{3}$ cosec 20° – sec 20° ?

Sol. $\sqrt{3}$ cosec 20° – sec 20°

$$= \frac{\sqrt{3}}{\sin 20°}-\frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20°-\sin 20°}{\sin 20°\cos 20°}$$

$$= \frac{4}{2\sin 20°\cos 20°}\left(\frac{\sqrt{3}}{2}\cos 20°-\frac{1}{2}\sin 20°\right)$$ (Multiplying numerator and denominator by 4)

$$= \frac{4}{\sin 40°}(\sin 60°\cos 20°-\cos 60°\sin 20°)$$ ($\because$ $\sin 2\theta = 2\sin\theta\cos\theta$)

$$= \frac{4}{\sin 40°}(\sin(60°-20°))$$ ($\because$ $\sin(A-B)=\sin A\cos B-\cos A\sin B$)

$$= \frac{4}{\sin 40°}.\sin 40° = \mathbf{4}.$$

Ex. 13. Given, 3 sin θ + 4 cos θ = 5, then what is 3 cos θ – 4 sin θ equal to?

Sol. $3 \sin \theta + 4 \cos \theta = 5$

$\Rightarrow (3 \sin \theta + 4 \cos \theta)^2 = 25 \Rightarrow 9 \sin^2 \theta + 16 \cos^2 \theta + 24 \sin \theta \cos \theta = 25$

$\Rightarrow 9(1 - \cos^2 \theta) + 16 (1 - \sin^2 \theta) + 24 \sin \theta \cos \theta = 25$ $\qquad (\because \sin^2 \theta + \cos^2 \theta = 1)$

$\Rightarrow 9 - 9 \cos^2 \theta + 16 - 16 \sin^2 \theta + 24 \sin \theta \cos \theta = 25$

$\Rightarrow 9 \cos^2 \theta + 16 \sin^2 \theta - 24 \sin \theta \cos \theta = 0$

$\Rightarrow (3 \cos \theta - 4 \sin \theta)^2 = 0 \Rightarrow 3 \cos \theta - 4 \sin \theta = \mathbf{0.}$

Ex. 14. If tan $x = b/a$, then what is the value of a cos 2x + b sin 2x? **(*UPSEAT 2007, AMU 2002*)**

Sol. Given $\tan x = b/a$

$$a \cos 2x + b \sin 2x = a\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right) + b\left(\frac{2\tan x}{1+\tan^2 x}\right) = \frac{a(1-\tan^2 x) + b(2\tan x)}{(1+\tan^2 x)}$$

$$= \frac{a(1-b^2/a^2) + b(2.b/a)}{1+b^2/a^2} = \frac{a\left(\frac{a^2-b^2}{a^2}\right) + \frac{2b^2}{a}}{\frac{a^2+b^2}{a^2}}$$

$$= \frac{\frac{a^2-b^2}{a} + \frac{2b^2}{a}}{\frac{a^2+b^2}{a^2}} = \frac{\frac{a^2-b^2+2b^2}{a}}{\frac{a^2+b^2}{a^2}} = \frac{a^2+b^2}{a} \cdot \frac{a^2}{a^2+b^2} = \boldsymbol{a.}$$

Ex. 15. If θ and φ are angles in the first quadrant such that tan θ = $\frac{1}{7}$ and sin φ = $\frac{1}{\sqrt{10}}$, then show that θ + 2φ = 45°.

Sol. Since θ and φ lie in the Ist quadrant, sin θ, sin φ; cos θ, cos φ and tan θ, tan φ are all positive.

$$\Rightarrow \sin \phi = \frac{1}{\sqrt{10}} \Rightarrow \cos \phi = \sqrt{1-\sin^2 \phi} = \sqrt{1-\frac{1}{10}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{\frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}}} = \frac{1}{3} \quad \therefore \quad \tan 2\phi = \frac{2\tan \phi}{1-\tan^2 \phi} = \frac{2 \times \frac{1}{3}}{1-\frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{3}{4}$$

Also, given $\tan \theta = \frac{1}{2}$

$$\therefore \tan(\theta + 2\phi) = \frac{\tan \theta + \tan 2\phi}{1 - \tan \theta \tan 2\phi} \qquad \left(\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right)$$

$$= \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7} \cdot \frac{3}{4}} = \frac{\frac{4+21}{28}}{1-\frac{3}{28}} = \frac{\frac{25}{28}}{\frac{25}{28}} = 1$$

$\Rightarrow \tan(\theta + 2\phi) = \tan 45° \Rightarrow \theta + 2\phi = \mathbf{45°.}$

Ex. 16. If $A + B + C = \frac{\pi}{2}$, then find sin 2A + sin 2B + sin 2C.

Sol. $\sin 2A + \sin 2B + \sin 2C$

$= (\sin 2A + \sin 2B) + 2 \sin C \cos C$

$= 2 \sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) + 2 \sin C \cos C$ $\quad\left(\text{Using } \sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}\right)$

$= 2 \sin (A + B) \cos (A - B) + 2 \sin C \cos C$ $\quad\left(\text{Given } A+B+C = \frac{\pi}{2} \Rightarrow A+B = \frac{\pi}{2} - C\right)$

$= 2 \sin\left(\frac{\pi}{2} - C\right) \cos (A - B) + 2 \sin C \cos C$

$= 2 \cos C \cos (A - B) + 2 \sin C \cos C$ $\quad$ (Using $\sin (90° - \theta) = \cos \theta$)

$= 2 \cos C (\cos (A - B) + \sin C)$

$= 2 \cos C \left(\cos (A - B) + \sin \left\{\frac{\pi}{2} - (A+B)\right\}\right)$

$= 2 \cos C (\cos (A - B) + \cos (A + B))$

$= 2 \cos C \left(2 \cos\left(\frac{A-B+A+B}{2}\right)\cos\left(\frac{A-B-A-B}{2}\right)\right)$ $\quad\left(\text{Using } \cos C + \cos D = 2\cos \frac{C+D}{2} \cos \frac{C-D}{2}\right)$

$= 2 \cos C (2 \cos A \,.\, \cos (-B)) =$ **$4 \cos A \cos B \cos C$** $\quad$ (Using $\cos (-\theta) = \cos \theta$)

Ex. 17. If $\alpha + \beta = \frac{\pi}{2}$ and $\beta + \gamma = \alpha$, then show that tan α = tan β = 2tan γ. ***(IIT 2003)***

Sol. $\alpha + \beta = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{2} - \beta \Rightarrow \tan \alpha = \tan\left(\frac{\pi}{2} - \beta\right)$

$\Rightarrow \tan \alpha = \cot \beta \Rightarrow \tan \alpha \tan \beta = 1.$ …(i)

Now, $\beta + \gamma = \alpha \Rightarrow \gamma = \alpha - \beta$

$\Rightarrow \tan \gamma = \tan (\alpha - \beta) \Rightarrow \tan \gamma = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

$\Rightarrow \tan \gamma = \frac{\tan \alpha - \tan \beta}{1+1} = \frac{\tan \alpha - \tan \beta}{2}$ $\quad$ ($\because \tan \alpha \tan \beta = 1$)

$\Rightarrow 2 \tan \gamma = \tan \alpha - \tan \beta \Rightarrow$ $\mathbf{\tan \alpha = \tan \beta + 2 \tan \gamma}$.

Ex. 18. If $\cos 2B = \frac{\cos (A+C)}{\cos (A-C)}$, then show that tan A, tan B and tan C are in G.P.

Sol. $\cos 2B = \frac{\cos (A+C)}{\cos (A-C)}$

$\Rightarrow \frac{1 - \tan^2 B}{1 + \tan^2 B} = \frac{\cos A \cos C - \sin A \sin C}{\cos A \cos C + \sin A \sin C}$

$\Rightarrow \frac{1 - \tan^2 B}{1 + \tan^2 B} = \frac{1 - \tan A \tan C}{1 + \tan A \tan C}$ $\quad$ (On dividing the numerator and denominator of RHS by $\cos A \cos C$)

$\Rightarrow 1 + \tan A \tan C - \tan^2 B - \tan A \tan^2 B \tan C = 1 + \tan^2 B - \tan A \tan C - \tan A \tan^2 B \tan C$

$\Rightarrow 2 \tan A \tan C = 2 \tan^2 B \Rightarrow \tan A \,.\, \tan C = \tan^2 B \Rightarrow \tan A, \tan B, \tan C$ are in *G.P.*

Ex. 19. If $\cos x = \dfrac{2\cos y - 1}{2 - \cos y}$, where $x \in (0, \pi)$, then show that $\tan\dfrac{x}{2}\cot\dfrac{y}{2} = \sqrt{3}$. ***(Manipal Engineering 2010)***

Sol. $\cos x = \dfrac{2\cos y - 1}{2 - \cos y}$

$$\Rightarrow \frac{1-\tan^2 x/2}{1+\tan^2 x/2} = \frac{2\left[\dfrac{1-\tan^2 y/2}{1+\tan^2 y/2}\right]-1}{2-\left[\dfrac{1-\tan^2 y/2}{1+\tan^2 y/2}\right]} \quad \Rightarrow \frac{1-\tan^2 x/2}{1+\tan^2 x/2} = \frac{2\left(1-\tan^2 y/2\right)-\left(1+\tan^2 y/2\right)}{2\left(1+\tan^2 y/2\right)-\left(1-\tan^2 y/2\right)}$$

$$\Rightarrow \frac{1-\tan^2 x/2}{1+\tan^2 x/2} = \frac{1-3\tan^2 y/2}{1+3\tan^2 y/2}$$

$$\Rightarrow 1 + 3\tan^2 y/2 - \tan^2 x/2 - 3\tan^2 x/2\,\tan^2 y/2 = 1 - 3\tan^2 y/2 + \tan^2 x/2 - 3\tan^2 x/2\,\tan^2 y/2$$

$$\Rightarrow 6\tan^2 y/2 = 2\tan^2 x/2 \quad \Rightarrow \tan^2 x/2\,.\,\frac{1}{\tan^2 y/2} = 3 \quad \Rightarrow \mathbf{\tan x/2\,.\cot y/2 = \sqrt{3}}.$$

Ex. 20. If A, B and C are the angles of a triangle such that sec $(A - B)$, sec A and sec $(A + B)$ are in arithmetic progression then show that $2\sec^2 A = \sec^2\dfrac{B}{2}$. ***(UPSEEE 2011)***

Sol. Since sec $(A - B)$, sec A , sec $(A + B)$ are in A.P., therefore

$$\sec A = \frac{\sec(A-B)+\sec(A+B)}{2} = \frac{1}{2}\left[\frac{1}{\cos(A-B)}+\frac{1}{\cos(A+B)}\right]$$

$$= \frac{\cos(A+B)+\cos(A-B)}{2\cos(A-B)\cos(A+B)} = \frac{2\cos A\cos B}{2\,[(\cos A\cos B+\sin A\sin B)(\cos A\cos B-\sin A\sin B)]}$$

$$= \frac{\cos A\cos B}{(\cos^2 A\cos^2 B-\sin^2 A\sin^2 B)} = \frac{\cos A\cos B}{(\cos^2 A\cos^2 B)-(1-\cos^2 A)(1-\cos^2 B)}$$

$$\Rightarrow \sec A = \frac{\cos A\cos B}{[\cancel{\cos^2 A\cos^2 B} - 1 + \cos^2 A + \cos^2 B - \cancel{\cos^2 A\cos^2 B}]}$$

$$\Rightarrow \cos^2 A + \cos^2 B - 1 = \cos^2 A\cos B \Rightarrow \cos^2 A(1-\cos B) = 1-\cos^2 B$$

$$\Rightarrow \cos^2 A = 1+\cos B \Rightarrow \cos^2 A = 2\cos^2\frac{B}{2} \Rightarrow \sec^2 A = \frac{1}{2}\sec^2\frac{B}{2} \Rightarrow \mathbf{2\sec^2 A = \sec^2\frac{B}{2}}.$$

Ex. 21. If $\tan\theta + \sec\theta = 4$, then what is the value of $\sin\theta$? ***(NDA/NA 2012)***

Sol. Given, $\tan\theta + \sec\theta = 4$

$$\Rightarrow \frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta} = 4 \quad \Rightarrow \frac{1+\sin\theta}{\cos\theta} = 4$$

$$\Rightarrow \frac{\sin^2\dfrac{\theta}{2}+\cos^2\dfrac{\theta}{2}+2\sin\theta/2\cos\theta/2}{(\cos^2\theta/2-\sin^2\theta/2)} = 4$$

(Using the formulas $\sin^2\theta + \cos^2\theta = 1$, $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2 - \sin^2\theta$)

$$\Rightarrow \frac{\left(\sin\dfrac{\theta}{2}+\cos\dfrac{\theta}{2}\right)^2}{\left(\cos\dfrac{\theta}{2}+\sin\dfrac{\theta}{2}\right)\left(\cos\dfrac{\theta}{2}-\sin\dfrac{\theta}{2}\right)} = 4$$

Alternatively,

Given, $\tan\theta + \sec\theta = 4$ ...(*i*)

$\sec^2\theta = 1 + \tan^2\theta$

$\Rightarrow \sec^2\theta - \tan^2\theta = 1$ ...(*ii*)

eq. (*ii*) ÷ eq. (*i*)

$\Rightarrow \sec\theta - \tan\theta = \dfrac{1}{4}$ ...(*iii*)

eq. (*i*) + eq. (*iii*)

$\Rightarrow 2\sec\theta = \dfrac{17}{4} \Rightarrow \sec\theta = \dfrac{17}{8}$

$\Rightarrow \cos\theta = \dfrac{8}{17}$

Now, use $\sin\theta = \sqrt{1-\cos^2\theta}$

$$\Rightarrow \frac{(\sin\theta/2+\cos\theta/2)}{\cos\theta/2-\sin\theta/2}=4 \qquad \Rightarrow \frac{1+\tan\theta/2}{1-\tan\theta/2}=4$$

$$\Rightarrow 1+\tan\ \theta/2 = 4-4\tan\ \theta/2 \qquad \Rightarrow 5\tan\theta/2 = 3 \Rightarrow \tan\ \theta/2 = 3/5$$

$$\therefore \sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2}=\frac{2\times 3/5}{1+9/25}=\frac{6/5}{34/25}=\frac{30}{34}=\mathbf{\frac{15}{17}.}$$

Ex. 22. Find the value of tan $\left(7\frac{1^\circ}{2}\right)$.

Sol. $$\tan 7\frac{1}{2}^\circ = \frac{\sin 7\frac{1}{2}^\circ}{\cos 7\frac{1}{2}^\circ}=\frac{\sin^2 7\frac{1}{2}^\circ}{\sin 7\frac{1}{2}^\circ\cos 7\frac{1}{2}^\circ}=\frac{2\sin^2 7\frac{1}{2}^\circ}{2\sin 7\frac{1}{2}^\circ\cos 7\frac{1}{2}^\circ}$$

$$=\frac{1-\cos 15^\circ}{\sin 15^\circ} \qquad \text{(Using } 1-\cos 2\theta = 2\sin^2\theta,\ \sin 2\theta = 2\sin\theta\cos\theta)$$

$$=\frac{1-\cos(45^\circ-30^\circ)}{\sin(45^\circ-30^\circ)}=\frac{1-(\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ)}{\sin 45^\circ\cos 30^\circ-\sin 30^\circ\sin 45^\circ}$$

$$=\frac{1-\left(\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\right)}{\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{2}\times\frac{1}{\sqrt{2}}}=\frac{1-\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}-1}{2\sqrt{2}}}=\frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$$

$$=\frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}=\frac{2\sqrt{6}-3-\sqrt{3}+2\sqrt{2}-\sqrt{3}-1}{3-1}$$

$$=\frac{2\sqrt{6}-2\sqrt{3}-4-2\sqrt{2}}{2}=\mathbf{\sqrt{6}-\sqrt{3}-\sqrt{2}-2.}$$

Ex. 23. If tan $\left(\frac{\pi}{4}+\theta\right)$ + tan $\left(\frac{\pi}{4}-\theta\right)$ = p sec 2θ, then find the value of p.

Sol. $$\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)=p\sec 2\theta$$

$$\Rightarrow \frac{\tan\pi/4+\tan\theta}{1-\tan\pi/4\,.\tan\theta}+\frac{\tan\pi/4-\tan\theta}{1+\tan\pi/4\,.\tan\theta}=p\sec 2\theta \qquad \left(\because \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}\right)$$

$$\Rightarrow \frac{1+\tan\theta}{1-\tan\theta}+\frac{1-\tan\theta}{1+\tan\theta}=p\sec\theta \qquad \left(\because \tan\frac{\pi}{4}=\tan 45^\circ=1\right)$$

$$\Rightarrow \frac{(1+\tan\theta)^2+(1-\tan\theta)^2}{1-\tan^2\theta}=p\sec^2\theta$$

$$\Rightarrow \frac{1+2\tan\theta+\tan^2\theta+1-\ 2\tan\theta+\tan\theta}{1-\tan^2\theta}=p\sec 2\theta$$

$$\Rightarrow \frac{2(1+\tan^2\theta)}{1-\tan^2\theta}=p\sec 2\theta \quad\Rightarrow \frac{2}{\cos 2\theta}=p\sec 2\theta \qquad \left(\text{Using } \cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$\Rightarrow 2\sec 2\theta = p\sec 2\theta \Rightarrow \boldsymbol{p=2.}$$

Ex. 24. Prove that $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$.

Sol. LHS $= \dfrac{(\sin 7A + \sin A) + (\sin 5A + \sin 3A)}{(\cos 7A + \cos A) + (\cos 5A + \cos 3A)}$

$$= \frac{2\sin\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right) + 2\sin\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}{2\cos\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right) + 2\cos\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}$$

$$\left(\text{Using } \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right);\ \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right)$$

$$= \frac{2\sin 4A\cos 3A + 2\sin 4A\cos A}{2\cos 4A\cos 3A + 2\cos 4A\cos A} = \frac{2\sin 4A(\cos 3A + \cos A)}{2\cos 4A(\cos 3A + \cos A)} = \mathbf{\tan 4A.}$$

Ex. 25. If $\cos\alpha + \cos\beta + \cos\gamma = 0$, then prove that $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 12\cos\alpha\cos\beta\cos\gamma$.

Sol. $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = (4\cos^3\alpha - 3\cos^3\alpha) + (4\cos^3\beta - 3\cos\beta) + (4\cos^3\gamma - 3\cos\gamma)$

$= 4(\cos^3\alpha + \cos^3\beta + \cos^3\gamma) - 3(\cos\alpha + \cos\beta + \cos\gamma)$

$= 4(\cos^3\alpha + \cos^3\beta + \cos^3\gamma) - 3 \times 0 = 4(\cos^3\alpha + \cos^3\beta + \cos^3\gamma)$

$= 4 \times 3\cos\alpha\cos\beta\cos\gamma$ $\qquad (\because a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc)$

$= \mathbf{12\cos\alpha\cos\beta\cos\gamma}$.

Ex. 26. If $A = \dfrac{41\pi}{12}$, then what is the value of $\dfrac{1 - 3\tan^2 A}{3\tan A - \tan^3 A}$.

Sol. $\because \tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

$$\therefore \frac{1 - 3\tan^2 A}{3\tan A - \tan^3 A} = \frac{1}{\tan 3A} = \frac{1}{\tan 3 \times \frac{41\pi}{12}} = \frac{1}{\tan\frac{41\pi}{4}}$$

$$= \frac{1}{\tan\left(10\pi + \frac{\pi}{4}\right)} = \frac{1}{\tan\frac{\pi}{4}} = \mathbf{1.} \qquad [\text{Using } \tan(2n\pi + \theta) = \tan\theta]$$

Ex. 27. If $\sec\alpha$ and $\operatorname{cosec}\alpha$ are the roots of the equation $x^2 - px + q = 0$, then prove that $p^2 = q(q + 2)$.

(Kerala PET 2007)

Sol. Sum of roots $= \sec\alpha + \operatorname{cosec}\alpha = p$

$\Rightarrow \dfrac{1}{\cos\alpha} + \dfrac{1}{\sin\alpha} = p \Rightarrow \dfrac{\sin\alpha + \cos\alpha}{\sin\alpha\cos\alpha} = p$ ...(*i*)

Product of roots $= \sec\alpha \,.\, \operatorname{cosec}\alpha = q \quad \Rightarrow \dfrac{1}{\sin\alpha\cos\alpha} = q$ ...(*ii*)

Now $p^2 = \left(\dfrac{\sin\alpha + \cos\alpha}{\sin\alpha\cos\alpha}\right)^2 = \dfrac{\sin^2\alpha + \cos^2\alpha + 2\sin\alpha\cos\alpha}{\sin\alpha^2\cos^2\alpha}$

$\Rightarrow p^2 = \dfrac{1 + 2\sin\alpha\cos\alpha}{\sin^2\alpha\cos^2\alpha} = \dfrac{1}{\sin^2\alpha\cos^2\alpha} + \dfrac{2}{\sin\alpha\cos\alpha} = q^2 + 2q = \boldsymbol{q(q + 2)}.$

Ex. 28. If θ and ϕ are acute angles such that $\sin\theta = \frac{1}{2}$ and $\cos\phi = \frac{1}{3}$, than $\theta + \phi$ lies in

(*a*) $\left]\frac{\pi}{3}, \frac{\pi}{2}\right[$ (*b*) $\left]\frac{2\pi}{3}, \frac{5\pi}{3}\right[$ (*c*) $\left]\frac{\pi}{2}, \frac{2\pi}{3}\right[$ (*d*) $\left]\frac{5\pi}{6}, \pi\right[$ ***(IIT 2004)***

Sol. $\sin\theta = \frac{1}{2} \Rightarrow \sin\theta = \sin\frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6}$ ($\because$ θ and ϕ are acute angles lying in the first quadrant) …(*i*)

Now $\cos\phi = \frac{1}{3} \Rightarrow 0 < \cos\phi < \frac{1}{2} \Rightarrow \cos\frac{\pi}{2} < \cos\phi < \cos\frac{\pi}{3} \Rightarrow \frac{\pi}{2} < \phi < \frac{\pi}{3}$ …(*ii*)

$\therefore$ From (*i*) and (*ii*) $\frac{\pi}{2} + \frac{\pi}{6} < \theta + \phi < \frac{\pi}{3} + \frac{\pi}{6}$

$\Rightarrow \frac{\pi}{2} < \theta + \phi < \frac{2\pi}{3} \Rightarrow \theta + \phi$ lies in the open interval $\left]\frac{\pi}{2}, \frac{2\pi}{3}\right[$.

Hence (*c*) is the correct option.

Ex. 29. The angle *A* lies in the third quadrant and it satisfies the equation 4 ($\sin^2 x + \cos x$) = 1. What is the measure of angle *A*? ***(NDA/NA 2010)***

Sol. $4\sin^2 x + 4\cos x = 1$

$\Rightarrow 4\sin^2 x + 4\cos x - 1 = 0 \quad \Rightarrow \quad 4(1 - \cos^2 x) + 4\cos x - 1 = 0$

$\Rightarrow -4\cos^2 x + 4\cos x + 3 = 0 \quad \Rightarrow \quad 4\cos^2 x - 4\cos x - 3 = 0$

$\Rightarrow 4\cos^2 x - 6\cos x + 2\cos x - 3 = 0 \quad \Rightarrow \quad 2\cos x(2\cos x - 3) + 1(2\cos x - 3) = 0$

$\Rightarrow (2\cos x + 1)(2\cos x - 3) = 0$

$\Rightarrow \cos x = -\frac{1}{2}$ and $\cos x = \frac{3}{2}$ (not possible)

Now $\cos x = -\frac{1}{2}$

Since *A* lies in the third quadrant and $\cos A = -\frac{1}{2}$, therefore,

$\cos A = \cos(180° + 60°) = \cos 240° \Rightarrow A = \mathbf{240°}$.

Ex. 30. Find the value of $\cos^2 76 + \cos^2 16 - \cos 76° \cos 16°$. ***(EAMCET)***

Sol. $\cos^2 76° + \cos^2 16° - \cos^2 76° \cos 16°$

$$= \frac{1 + \cos(76° \times 2)}{2} + \frac{1 + \cos(16° \times 2)}{2} - \frac{1}{2}[\cos(76° - 16°) + \cos(76° - 16°)]$$

[$\because$ $2\cos^2\theta = 1 + \cos 2\theta$, $\cos(A + B) + (A - B) = 2\cos A \cos B$]

$$= \frac{1}{2}[1 + \cos 152° + 1 + \cos 32° - \cos 92° + \cos 60°]$$

$$= \frac{1}{2}\left[\left(2 - \frac{1}{2}\right) + \cos 152° + \cos 32° - \cos 92°\right]\left[\because \cos 60° = \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{3}{2} + 2\cos\left(\frac{152° + 32°}{2}\right)\cos\left(\frac{152° - 32°}{2}\right) - \cos 92°\right]$$

$$= \frac{1}{2}\left[\frac{3}{2} + 2\cos 92° \cos 60° - \cos 92°\right]\left(\because \cos C + \cos D = 2\cos\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)\right)$$

$$= \frac{1}{2}\left[\frac{3}{2} + \cos 92° - \cos 92°\right] = \mathbf{\frac{3}{4}}. \quad \left(\because \cos 60° = \frac{1}{2}\right)$$

PRACTICE SHEET

1. If $\cot\theta = \dfrac{2xy}{x^2 - y^2}$, then what is $\cos\theta$ equal to?

(*a*) $\dfrac{x^2 - y^2}{x^2 + y^2}$ (*b*) $\dfrac{x^2 + y^2}{x^2 - y^2}$

(*c*) $\dfrac{2xy}{x^2 + y^2}$ (*d*) $\dfrac{2xy}{\sqrt{x^2 + y^2}}$ **(*CDS 2009*)**

2. If $\sec\theta = \dfrac{13}{5}$, then what is the value of $\dfrac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta}$?

(*a*) 1 (*b*) 3 (*c*) 2 (*d*) 4

(*CDS 2007*)

3. If cosec $A = 2$, then the value of $\dfrac{1}{\tan A} + \dfrac{\sin A}{1 + \cos A}$ is

(*a*) $\sqrt{2} - 1$ (*b*) $\sqrt{3} + 2$

(*c*) 0 (*d*) 2

4. The value of $4\cot^2 45° - \sec^2 60° + \sin^2 60° + \cos^2 90°$ is

(*a*) $\dfrac{1}{2}$ (*b*) $\dfrac{3}{4}$ (*c*) 1 (*d*) $\dfrac{11}{16}$

5. Evaluate $\dfrac{1}{4}\left(\cot^4 30° - \text{cosec}^4 60°\right) + \dfrac{3}{2}\left(\sec^2 45° - \tan^2 30°\right) - 5\cos^2 60°$.

(*a*) $\dfrac{55}{18}$ (*b*) $\dfrac{11}{6}$ (*c*) $\dfrac{7}{4}$ (*d*) 0

6. If $2x^2 \cos 60° - 4\cot^2 45° - 2\tan 60° = 0$, what is the value of x?

(*a*) 2 (*b*) 3 (*c*) $\sqrt{3} - 1$ (*d*) $\sqrt{3} + 1$

(*CDS 2007*)

7. The cotangent of the angles $\pi/3$, $\pi/4$ and $\pi/6$ are in

(*a*) A.P (*b*) G.P. (*c*) H.P.

(*d*) None of these **(*AMU*)**

8. The value of cos1°, cos 2°... cos100° is

(*a*) –1 (*b*) 0 (*c*) 1

(*d*) None of these **(*AIEEE 2002*)**

9. If $\pi = 22/7$, then a unit radian is approximately equal to

(*a*) 57° 16′ 22″ (*b*) 57° 15′ 22″

(*c*) 57° 16′ 20″ (*d*) 57° 15′ 20″ **(*CDS 2007*)**

10. What is the value of

$$\frac{5\sin 75°\sin 77° + 2\cos 13°\cos 15°}{\cos 15°\sin 77°} - \frac{7\sin 81°}{\cos 90}?$$

(*a*) –1 (*b*) 0 (*c*) 1 (*d*) 2

(*CDS 2009*)

11. What is the value of

$\sin^2 15° + \sin^2 20° + \sin^2 25° + \ldots\ldots + \sin^2 75°$?

(*a*) $\tan^2 15° + \tan^2 20° + \tan^2 25° + \ldots\ldots + \tan^2 75°$

(*b*) $\cos^2 15° + \cos^2 20° + \cos^2 25° + \ldots\ldots + \cos^2 75°$

(*c*) $\cot^2 15° + \cot^2 20° + \cot^2 25° + \ldots\ldots + \cot^2 75°$

(*d*) $\sec^2 15° + \sec^2 20° + \sec^2 25° + \ldots\ldots + \sec^2 75°$.

12. If $\tan (x^2 - 8x + 60)° = \cot (6x - 5)°$, what is one of the values of x?

(*a*) 7 (*b*) 8 (*c*) 9 (*d*) 10.

(*CDS 2005*)

13. What is the value of $\sec (90 - \theta)°.\ \sin\theta \sec 45°$?

(*a*) 1 (*b*) $\dfrac{\sqrt{3}}{2}$ (*c*) $\sqrt{2}$ (*d*) $\sqrt{3}$

(*CDS 2012*)

14. If $x + y = 90°$, then what is the value of $\left(1 + \dfrac{\tan x}{\tan y}\right)\sin^2 y$?

(*a*) 0 (*b*) 1/2 (*c*) 1 (*d*) 2

(*CDS 2006*)

15. If $\dfrac{\tan 26° + \tan 19°}{x(1 - \tan 26° \tan 19°)} = \cos 60°$, then the value of x is

(*a*) 1 (*b*) $\sqrt{2}$ (*c*) 2 (*d*) $\sqrt{3}$

16. The value of cot 105° is

(*a*) $\sqrt{3} - 2$ (*b*) $2 - \sqrt{3}$

(*c*) $\sqrt{2} + 3$ (*d*) $\sqrt{3} + 2$ **(*Odisha JEE*)**

17. sin 120° cos 150° – cos 240° sin 330° is equal to

(*a*) $-\left(\dfrac{\sqrt{3} + 1}{4}\right)$ (*b*) –1

(*c*) 1 (*d*) $\dfrac{2}{3}$ **(*EAMCET 2006*)**

18. If A, B, C, D are the successive angles of a cyclic quadrilateral, then what is $\cos A + \cos B + \cos C + \cos D$ equal to:

(*a*) 4 (*b*) 2 (*c*) 1 (*d*) 0

(*CDS 2011*)

19. What is the value of tan (–1575°)?

(*a*) 1 (*b*) $\dfrac{1}{2}$ (*c*) 0 (*d*) –1

(*NDA/NA 2009*)

20. The value of $\dfrac{\sin 300° \tan 330° \sec 420°}{\tan 135° \sin 210° \sec 315°}$ is equal to

(*a*) $\dfrac{1}{\sqrt{3}}$ (*b*) $\dfrac{1}{\sqrt{2}}$ (*c*) $\sqrt{2}$ (*d*) $\sqrt{3}$

21. cos 1° + cos 2° + cos 3° + ... + cos180° is equal to

(*a*) –1 (*b*) 0 (*c*) 1 (*d*) 2

(*KCET 2003*)

22. What is $\dfrac{\sin^6\theta - \cos^6\theta}{\sin^2\theta - \cos^2\theta}$ equal to?

(*a*) $\sin^4\theta - \cos^4\theta$ (*b*) $1 - \sin^2\theta\cos^2\theta$

(*c*) $1 + \sin^2\theta\cos^2\theta$ (*d*) $1 - 3\sin^2\theta\cos^2\theta$

(*CDS 2011*)

23. If $\dfrac{\cos x}{1+\operatorname{cosec} x} + \dfrac{\cos x}{\operatorname{cosec} x - 1} = 2,$ which one of the following is one of the values of x ?

(*a*) $\pi/2$ (*b*) $\pi/3$ (*c*) $\pi/4$ (*d*) $\pi/6$

(*CDS 2009*)

24. If $x = a(1 + \cos\theta\cos\phi)$, $y = b(1 + \cos\theta\sin\phi)$ and $z = c(1 + \sin\theta)$, then which of the following is correct?

(*a*) $\left(\dfrac{x-a}{a}\right)^2 + \left(\dfrac{y-b}{b}\right)^2 + \left(\dfrac{z-c}{c}\right)^2 = 1$

(*b*) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

(*c*) $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$

(*d*) $\dfrac{(x-a)^2}{a} + \dfrac{(y-b)^2}{b} + \dfrac{(z-c)^2}{c} = 1$ **(*CDS 2005*)**

25. If $\sin^4 x + \sin^2 x = 1$ then what is 1 are the value of $\cot^4 x + \cot^2 x$?

(*a*) $\cos^2 x$ (*b*) $\sin^2 x$ (*c*) $\tan^2 x$ (*d*) 1

(*CDS 2006*)

26. If $a\sin\theta + b\cos\theta = c$, what is/are the values of $(a\cos\theta - b\sin\theta)$?

(*a*) $c - a + b$ (*b*) $c - b + a$

(*c*) $\pm\sqrt{a^2 + b^2 - c^2}$ (*d*) $\pm\sqrt{c^2 + b^2 - a^2}$

(*CDS 2005*)

27. What is the value of $\sin A\cos A\tan A + \cos A\sin A\cot A$?

(*a*) $\sin^2 A + \cos A$ (*b*) $\sin^2 A + \tan^2 A$

(*c*) $\sin^2 A + \cot^2 A$ (*d*) $\operatorname{cosec}^2 A - \cot^2 A$

28. $\cos\alpha\sin(\beta-\gamma) + \cos\beta\sin(\gamma-\alpha) + \cos\gamma\sin(\alpha-\beta)$ is equal to

(*a*) 0 (*b*) $1/2$

(*c*) 1 (*d*) $4\cos\alpha\cos\beta\cos\gamma$

(*EAMCET*)

29. If $\sin\theta$, $\cos\theta$ and $\tan\theta$ are in geometric progression, then $\cot^6\theta - \cot^2\theta$ is equal to

(*a*) $\dfrac{1}{2}$ (*b*) 1 (*c*) 2 (*d*) 3

30. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then the value of $\cos\left(\theta - \dfrac{\pi}{4}\right)$ is

(*a*) $\dfrac{1}{2\sqrt{2}}$ (*b*) $\dfrac{1}{\sqrt{2}}$ (*c*) $\dfrac{1}{2}$ (*d*) $\dfrac{1}{4}$

(*EAMCET 2013*)

31. If $3\sin x + 4\cos x = 5$, then $6\tan\dfrac{x}{2} - 9\tan^2\dfrac{x}{2}$ is equal to

(*a*) 0 (*b*) 1 (*c*) 3 (*d*) 4

(*EAMCET 2012*)

32. $\cos^2\alpha + \cos^2(\alpha - 120°) + \cos^2(\alpha + 120°)$ is equal to

(*a*) 0 (*b*) $\dfrac{1}{2}$ (*c*) 1 (*d*) $\dfrac{3}{2}$

(*MPPET*)

33. The expression $\dfrac{\tan A}{1-\cot A} + \dfrac{\cot A}{1-\tan A}$ can be written as

(*a*) $\sin A\cos A + 1$ (*b*) $\sec A\operatorname{cosec} A + 1$

(*c*) $\tan A + \cot A$ (*d*) $\sec A + \operatorname{cosec} A$

(*IIT 2013*)

34. The value of $\tan 5\theta$ is

(*a*) $\dfrac{\tan^5\theta + 10\tan^3\theta - 5\tan\theta}{5\tan^4\theta - 10\tan^2\theta + 1}$

(*b*) $\dfrac{5\tan\theta + 10\tan^3\theta - \tan^5\theta}{1 + 10\tan^2\theta - 5\tan^4\theta}$

(*c*) $\dfrac{\tan^5\theta - 10\tan^3\theta - 5\tan\theta}{5\tan^4\theta + 10\tan^2\theta + 1}$

(*d*) $\dfrac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

35. If α and β are such that $\tan\alpha = 2\tan\beta$, then what is $\sin(\alpha + \beta)$ equal to?

(*a*) 1 (*b*) $2\sin(\alpha - \beta)$

(*c*) $\sin(\alpha - \beta)$ (*d*) $3\sin(\alpha - \beta)$

(*NDA/NA 2007*)

36. If $\sin x + \operatorname{cosec} x = 2$, then $\sin^n x + \operatorname{cosec}^n x$ is equal to

(*a*) 2 (*b*) 2^n (*c*) 2^{n-1} (*d*) 2^{n-2}

(*AMU 2008*)

37. The value of $\cos\dfrac{2\pi}{15}.\cos\dfrac{4\pi}{15}.\cos\dfrac{8\pi}{15}.\dfrac{16\pi}{15}$ is equal to:

(*a*) $\dfrac{1}{16}$ (*b*) $\dfrac{1}{32}$ (*c*) $\dfrac{1}{64}$ (*d*) $\dfrac{1}{8}$

(*Manipal Engineering 2010*)

38. If $\alpha + \beta + \gamma = \pi$, then the value of $\sin^2\alpha + \sin^2\beta - \sin^2\gamma$ is equal to

(*a*) $2\sin\alpha$ (*b*) $2\sin\alpha\cos\beta\sin\gamma$

(*c*) $2\sin\alpha\sin\beta\cos\gamma$ (*d*) $2\sin\alpha\sin\beta\sin\gamma$

(*Manipal Engineering 2012*)

39. What is the value of $\cos 15°$?

(*a*) $\dfrac{1}{2}\left(\sqrt{2-\sqrt{3}}\right)$ (*b*) $\dfrac{1}{2}\left(\sqrt{2+\sqrt{3}}\right)$

(*c*) $\sqrt{2}+\sqrt{3}$ (*d*) $\sqrt{2}-\sqrt{3}$

(*NDA/NA 2008*)

40. What is $\sqrt{2+\sqrt{2+\sqrt{2+\cos 4A}}}$ equal to?

(*a*) $\cos A$ (*b*) $\cos (2A)$ (*c*) $2\cos A/2$ (*d*) $\sqrt{2\cos A}$

(*NDA/NA 2008*)

41. If $\cos x \neq 1$, then what is $\dfrac{\sin x}{1+\cos x}$ equal to?

(*a*) $-\cot\dfrac{x}{2}$ (*b*) $\cot\dfrac{x}{2}$ (*c*) $\tan\dfrac{x}{2}$ (*d*) $-\tan\dfrac{x}{2}$

(*NDA/NA 2010*)

42. The value of $\dfrac{\sin 70° + \cos 40°}{\cos 70° + \sin 40°}$ is equal to

(*a*) $\dfrac{1}{2}$ (*b*) 0 (*c*) $\sqrt{3}$ (*d*) $\dfrac{\sqrt{3}}{2}$

43. If $\cos 2\alpha = \dfrac{3\cos 2\beta - 1}{3 - \cos 2\beta}$, then $\tan \alpha$ is equal to

(*a*) $\dfrac{\tan\beta}{\sqrt{2}}$ (*b*) $\tan \beta$ (*c*) $\dfrac{\tan^2\beta}{\sqrt{2}}$ (*d*) $\sqrt{2}\tan\beta$

(*Odisha JEE 2006*)

44. $\dfrac{\sin 5\theta}{\sin\theta}$ is equal to

(*a*) $16\cos^4\theta - 12\cos^2\theta - 1$

(*b*) $16\cos^4\theta - 12\cos^2\theta + 1$

(*c*) $16\cos^4\theta + 12\cos^2\theta - 1$

(*d*) $16\cos^4\theta + 12\cos^2\theta + 1$ **(*EAMCET 2001*)**

45. If $\sin A + \sin B = a$, $\cos A - \cos B = b$, then the value of $\cos (A - B)$ is

(*a*) $\dfrac{2ab}{a^2+b^2}$ (*b*) $\dfrac{2ab}{a^2-b^2}$

(*c*) $\dfrac{a^2-b^2}{a^2+b^2}$ (*d*) $\dfrac{a^2+b^2}{a^2-b^2}$

46. The value of $\sin 16° + \cos 16°$ is

(*a*) $\dfrac{1}{\sqrt{3}}\left(\sqrt{2}\cos 1° + \sin 1°\right)$ (*b*) $\dfrac{1}{\sqrt{2}}\left(\cos 1° + \sqrt{3}\sin 1°\right)$

(*c*) $\dfrac{1}{\sqrt{3}}\left(\cos 1° + \sqrt{2}\sin 1°\right)$ (*d*) $\dfrac{1}{\sqrt{2}}\left(\sqrt{3}\cos 1° + \sin 1°\right)$

47. $\left[1+\cos\dfrac{\pi}{8}\right]\left[1+\cos\dfrac{3\pi}{8}\right]\left[1+\cos\dfrac{5\pi}{8}\right]\left[1+\cos\dfrac{7\pi}{8}\right]$ is equal to

(*a*) $\dfrac{1}{8}$ (*b*) $\dfrac{1}{2}$ (*c*) $\dfrac{1+\sqrt{2}}{2\sqrt{2}}$ (*d*) $\cos\dfrac{\pi}{8}$

(*AIEEE 2002, DCE 2003*)

48. Evaluate:

$\cos 2(\theta + \phi) + 4\cos(\theta + \phi)\sin\theta\sin\phi + 2\sin^2\phi$

(*a*) $\sin 2\theta$ (*b*) $\cos 2\theta$ (*c*) $\sin 3\theta$ (*d*) $\cos 3\theta$

(*Odisha CET 2004*)

49. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then what is $\cot (A - B)$ equal to?

(*a*) $\dfrac{1}{y}-\dfrac{1}{x}$ (*b*) $\dfrac{1}{x}-\dfrac{1}{y}$ (*c*) $\dfrac{1}{x}+\dfrac{1}{y}$ (*d*) $-\dfrac{1}{x}-\dfrac{1}{y}$

(*NDA/NA 2011*)

50. $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$ is equal to

(*a*) 0 (*b*) $2\tan\alpha$ (*c*) $\cot\alpha$ (*d*) $\tan 16\alpha$

(*IIT*)

ANSWERS

1. (*c*)	**2.** (*b*)	**3.** (*d*)	**4.** (*b*)	**5.** (*a*)	**6.** (*d*)	**7.** (*b*)	**8.** (*b*)	**9.** (*a*)
10. (*b*)	**11.** (*b*)	**12.** (*a*)	**13.** (*c*)	**14.** (*c*)	**15.** (*c*)	**16.** (*a*)	**17.** (*b*)	**18.** (*d*)
19. (*a*)	**20.** (*c*)	**21.** (*a*)	**22.** (*b*)	**23.** (*c*)	**24.** (*a*)	**25.** (*d*)	**26.** (*c*)	**27.** (*d*)
28. (*a*)	**29.** (*b*)	**30.** (*a*)	**31.** (*b*)	**32.** (*d*)	**33.** (*b*)	**34.** (*d*)	**35.** (*d*)	**36.** (*a*)
37. (*a*)	**38.** (*c*)	**39.** (*b*)	**40.** (*c*)	**41.** (*c*)	**42.** (*c*)	**43.** (*d*)	**44.** (*b*)	**45.** (*c*)
46. (*d*)	**47.** (*a*)	**48.** (*b*)	**49.** (*c*)	**50.** (*c*)				

HINTS AND SOLUTIONS

1. $\cot\theta = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{BC}{AB}$ (where $\angle ACB = \theta$)

$\therefore\ BC = (2xy)\,k$

$AB = (x^2 - y^2)\,k$

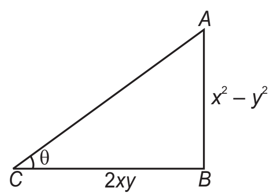
$\Rightarrow AC^2 = AB^2 + BC^2$

$= k^2\,((x^2 + y^2)^2 + (2xy)^2)$

$= k^2\,[x^4 + y^4 - 2x^2y^2 + 4x^2y^2]$

$= k^2\,[x^4 + y^4 + 2x^2y^2] = k^2\,(x^2 + y^2)^2$

$\Rightarrow AC = k\,(x^2 + y^2)$



$\therefore\ \cos\theta = \dfrac{\text{Base}}{\text{Hyp.}} = \dfrac{BC}{AC} = \dfrac{\mathbf{2xy}}{\mathbf{x^2 + y^2}}$.

2. Given, $\sec\theta = \dfrac{13}{5} \Rightarrow \sec^2\theta = \dfrac{169}{25}$

$\Rightarrow\ 1 + \tan^2\theta = \dfrac{169}{25}$ $(\because \sec^2\theta - \tan^2\theta = 1)$

$\Rightarrow\ \tan^2\theta = \dfrac{169}{25} - 1 = \dfrac{144}{25} \Rightarrow \tan\theta = \dfrac{12}{5}$

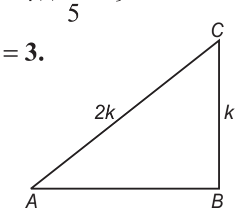
$$\therefore \quad \frac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta} = \frac{\dfrac{2\sin\theta}{\cos\theta} - 3}{\dfrac{4\sin\theta}{\cos\theta} - 9}$$

(On dividing each term of numerator and denominator by $\cos\theta$)

$$= \frac{2\tan\theta - 3}{4\tan\theta - 9} = \frac{2 \times \dfrac{12}{5} - 3}{4 \times \dfrac{12}{5} - 9}$$

$$= \frac{24 - 15}{48 - 45} = \frac{9}{3} = \mathbf{3.}$$

3. $\operatorname{cosec} A = \dfrac{\text{Hypotenuse}}{\text{Perpendicular}} = \dfrac{2}{1}$



$\Rightarrow \; AC = 2k, BC = k$

$\Rightarrow \; AB^2 = \sqrt{AC^2 - BC^2} = \sqrt{4k^2 - k^2} = \sqrt{3k^2} = k\sqrt{3}$

$\therefore \; \tan A = \dfrac{\text{Perp.}}{\text{Base}} = \dfrac{BC}{AB} = \dfrac{k}{k\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\sin A = \dfrac{1}{\operatorname{cosec} A} = \dfrac{1}{2}$

$\cos A = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{AB}{AC} = \dfrac{k\sqrt{3}}{2k} = \dfrac{\sqrt{3}}{2}$

$$\therefore \quad \frac{1}{\tan A} + \frac{\sin A}{1 + \cos A} = \frac{1}{1/\sqrt{3}} + \frac{1/2}{1 + \sqrt{3}/2}$$

$$= \frac{\sqrt{3}}{1} + \frac{1/2}{\dfrac{2+\sqrt{3}}{2}} = \frac{\sqrt{3}}{1} + \frac{1}{2+\sqrt{3}}$$

$$= \frac{2\sqrt{3} + 3 + 1}{2 + \sqrt{3}} = \frac{2\sqrt{3} + 4}{2 + \sqrt{3}} = \frac{2\left(\sqrt{3} + 2\right)}{2 + \sqrt{3}} = \mathbf{2.}$$

4. $4\cot^2 45° - \sec^2 60° + \sin^2 60° + \cos^2 90°$

$= 4(\cot 45°)^2 - (\sec 60°)^2 + (\sin 60°)^2 + (\cos 90°)^2$

$= 4 \times 1^2 - (2)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2 + (0)^2$

$= 4 - 4 + \dfrac{3}{4} + 0 = \mathbf{\dfrac{3}{4}.}$

5. $\dfrac{1}{4}(\cot^4 30° - \operatorname{cosec}^4 60°)$

$$+ \frac{3}{2}(\sec^2 45° - \tan^2 30°) - 5\cos^2 60°$$

$$= \frac{1}{4}\left[(\cot 30°)^4 - (\operatorname{cosec} 60°)^4\right]$$

$$+ \frac{3}{2}\left[(\sec 45°)^2 - (\tan 30°)^2\right] - 5(\cos 60°)^2$$

$$= \frac{1}{4}\left[\left(\sqrt{3}\right)^4 - \left(\frac{2}{\sqrt{3}}\right)^4\right] + \frac{3}{2}\left[\left(\sqrt{2}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right] - 5\left(\frac{1}{2}\right)^2$$

$$= \frac{1}{4}\left(9 - \frac{16}{9}\right) + \frac{3}{2}\left(2 - \frac{1}{3}\right) - 5 \times \frac{1}{4}$$

$$= \frac{1}{4}\left[\frac{81 - 16}{9}\right] + \frac{3}{2}\left[\frac{6 - 1}{3}\right] - \frac{5}{4}$$

$$= \frac{1}{4} \times \frac{65}{9} + \frac{3}{2} \times \frac{5}{3} - \frac{5}{4} = \frac{65}{36} + \frac{5}{2} - \frac{5}{4}$$

$$= \frac{65 + 90 - 45}{36} = \frac{110}{36} = \mathbf{\frac{55}{18}.}$$

6. $2x^2 \cos 60° - 4\cot^2 45° - 2\tan 60° = 0$

$\Rightarrow \; 2x^2 \times \dfrac{1}{2} - 4 \times (1)^2 - 2 \times \sqrt{3} = 0$

$\Rightarrow \; x^2 - 4 - 2\sqrt{3} = 0 \;\Rightarrow\; x^2 = 4 + 2\sqrt{3}$

$\Rightarrow \; x^2 = 3 + 1 + 2\sqrt{3} \;\Rightarrow\; x^2 = (\sqrt{3})^2 + (1)^2 + 2\sqrt{3}$

$\Rightarrow \; x^2 = \left(\sqrt{3} + 1\right)^2 \Rightarrow x = \mathbf{\sqrt{3} + 1.}$

7. $\cot\dfrac{\pi}{3} = \cot 60° = \dfrac{1}{\sqrt{3}}, + \cot\dfrac{\pi}{4} = \cot 45° = 1,$

$\cot\dfrac{\pi}{6} = \cot 30° = \sqrt{3}$

We can see that

$$\left(\cot\frac{\pi}{3}\right).\left(\cot\left(\frac{\pi}{6}\right)\right) = \frac{1}{\sqrt{3}}.\sqrt{3} = 1 = \left(\cot\frac{\pi}{4}\right)^2$$

$\therefore$ cot $\pi/3$, cot $\pi/4$, cot $\pi/6$ are in G.P

8. $\because \cos 90° = 0$

$\therefore \quad \cos 1°. \cot 2°. .... \cos 90° . \cos 91° ..... . \cos 100° = \mathbf{0.}$

9. π radians = 180°

$\Rightarrow \; 1 \text{ radian} = \dfrac{180}{\pi} \text{ degree} = \dfrac{180}{22} \times 7 \text{ degree}$

$= \dfrac{630}{11} \text{ degree} = 57\dfrac{3}{11} \text{ degree}$

$= 57° + \dfrac{3}{11} \times 60 \text{ min} = 57° + \dfrac{180}{11} \text{ min}$

$= 57° + 16\dfrac{4}{11} \text{ min}$

$= 57° + 16' + \dfrac{4}{11} \times 60 \text{ s}$

$= 57° + 16' + 21.8''$

$= \mathbf{57° \; 16' \; 22''}$ (approx).

10. $\dfrac{5\sin 75° \sin 77° + 2\cos 13° \cos 15°}{\cos 15° \sin 77°} - \dfrac{7\sin 81°}{\cos 9°}$

$$= \frac{5\sin(90° - 15°)\sin 77° + 2\cos(90° - 77°)\cos 15°}{\cos 15° \sin 77°}$$

$$- \frac{7\sin(90° - 9°)}{\cos 9°}$$

$= \dfrac{5\cos 15°\sin 77° + 2\sin 77°\cos 15°}{\cos 15°\sin 77°} - \dfrac{7\sin 9°}{\cos 9°}$

$= \dfrac{7\cos 15°\sin 77°}{\cos 15°\sin 77°} - \dfrac{7\cos 9°}{\cos 9°} = 7 - 7 = \mathbf{0.}$

11. $\sin^2 15° + \sin^2 20° + \sin^2 25° + \ldots\ldots + \sin^2 75°$

$= \sin^2 (90° - 75°) + \sin^2 (90° - 70°) + \ldots + \sin^2 (90° - 15°)$

$= \cos^2 75° + \cos^2 70° + \ldots\ldots + \cos^2 \mathbf{15°}$

12. $\tan (x^2 - 8x + 60)° = \cot (6x - 5)°$

$\Rightarrow \tan (x^2 - 8x + 60)° = \tan [90 - (6x - 5)]°$

$\Rightarrow (x^2 - 8x + 60)° = 90° - (6x - 5)°$

$\Rightarrow x^2 - 8x + 60 = 90 - 6x + 5 \Rightarrow x^2 - 2x - 35 = 0$

$\Rightarrow (x - 7)(x + 5) = 0 \Rightarrow x = 7, -5 \Rightarrow x = \mathbf{7.}$

13. $\sec (90 - \theta)° \sin\theta \sec 45°$

$= \operatorname{cosec}\theta \sin\theta.\ (\sqrt{2}) = \dfrac{1}{\sin\theta}.\sin\theta.\sqrt{2} = \sqrt{\mathbf{2}}.$

14. $\left(1 + \dfrac{\tan x}{\tan y}\right).\sin^2 y$

$= \left(1 + \dfrac{\tan(90° - y)}{\tan y}\right).\sin^2 y \quad (\because x + y = 90°)$

$= (1 + \cot y.\cot y).\sin^2 y$

$= (1 + \cot^2 y).\sin^2 y = \operatorname{cosec}^2 y.\sin^2 y = \mathbf{1.}$

15. $\dfrac{\tan 26° + \tan 19°}{x(1 - \tan 26°\tan 19°)} = \cos 60°$

$= \dfrac{\tan 26° + \tan 19°}{1 - \tan 26°\tan 19°} = x\cos 60°$

$= \tan (26° + 19°) = x \times \dfrac{1}{2}$

$\left[\because \tan (A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}\right]$

$= \tan 45° = x/2 \Rightarrow x/2 = 1 \Rightarrow x = \mathbf{2.}$

16. $\cot 105° = \cot (90° + 15°) = -\tan 15°$

$= -\tan (45° - 30°) = -\left[\dfrac{\tan 45° - \tan 30°}{1 + \tan 45°\tan 30°}\right]$

$\left(\because \tan (A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}\right)$

$= -\left[\dfrac{1 - \dfrac{1}{\sqrt{3}}}{1 + 1.\dfrac{1}{\sqrt{3}}}\right] = -\dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$

$= -\dfrac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)} = -\dfrac{4 - 2\sqrt{3}}{2} = \sqrt{\mathbf{3}} \mathbf{- 2.}$

17. $\sin 120°\cos 150° - \cos 240°\sin 330°$

$= \sin (180° - 60°)\cos (180° - 30°)$

$- \cos (180° + 60°)\sin (360° - 30°)$

$= (\sin 60°).(-\cos 30°) - (-\cos 60°)(-\sin 30°)$

$= \dfrac{\sqrt{3}}{2} \times -\dfrac{\sqrt{3}}{2} - \left(\dfrac{-1}{2}\right)\left(\dfrac{-1}{2}\right) = -\dfrac{3}{4} - \dfrac{1}{4} = -\dfrac{4}{4} = \mathbf{-1.}$

18. In a cyclic quadrilateral, the sum of the opposite angles is 180°.

$\Rightarrow A + C = 180°$ and $B + D = 180°$

$\therefore \cos A + \cos B + \cos C + \cos D$

$= \cos A + \cos B + \cos (180° - C) + \cos (180° - B)$

$= \cos A + \cos B - \cos A - \cos B$

$= \mathbf{0.}$

19. $\tan (-1575°)$

$= -\tan (1575°) \qquad (\because \tan(-\theta) = -\tan\theta)$

$= -\tan (4 \times 360° + 135°)$

$= -\tan (135°) \qquad (\because \tan (n.360° + \theta) = \tan\theta)$

$= -\tan (90° + 45°) \qquad (\because \tan (90° + \theta) = -\cot\theta)$

$= -(-\cot 45°) = \cot 45° = \mathbf{1.}$

20. $\dfrac{\sin 300°\tan 330°\sec 420°}{\tan 135°\sin 210°\sec 315°}$

$= \dfrac{\sin(360° - 60°)\tan(360° - 30°)\sec(360° + 60°)}{\tan(180° - 45°)\sin(180° + 30°)\sec(360° - 45°)}$

$= \dfrac{(-\sin 60°)(-\tan 30°)(\sec 60°)}{(-\tan 45°)(-\sin 30°)(\sec 45°)}$

$= \dfrac{\sin 60° \times \tan 30° \times \sec 60°}{\tan 45° \times \sin 30° \times \sec 45°}$

$= \dfrac{\sqrt{3}/2 \times 1/\sqrt{3} \times 2}{1 \times 1/2 \times \sqrt{2}} = \sqrt{\mathbf{2}}\mathbf{.}$

21. $\cos 1° + \cos 2° + \cos 3° + \ldots\ldots + \cos 180°$

$= (\cos 1° + \cos 179°) + (\cos 2° + \cos 178°) +$

$\ldots\ldots + (\cos 89° + \cos 91°) + \cos 90° + \cos 180°$

$= [\cos 1° + \cos (180° - 1°)] + [\cos 2° + \cos (180° - 2°)] +$

$\ldots\ldots + [\cos 89° + \cos (180° - 91°)] + \cos 90° + \cos 180°$

$= (\cos 1° - \cos 1°) + (\cos 2° - \cos 2°) +$

$\ldots\ldots + (\cos 89° - \cos + 89°) + 0 + (-1) = \mathbf{-1.}$

$(\because \cos (180° - \theta) = -\cos\theta)$

22. $\dfrac{\sin^6\theta - \cos^6\theta}{\sin^2\theta - \cos^2\theta} = \dfrac{(\sin^2\theta)^3 - (\cos^2\theta)^3}{\sin^2\theta - \cos^2\theta}$

$= \dfrac{(\sin^2\theta - \cos^2\theta)(\sin^4\theta + \cos^4\theta + \sin^2\theta\cos^2\theta)}{(\sin^2\theta - \cos^2\theta)}$

(Using $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$)

$= \sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta$

$= (\sin^2\theta + \cos^2\theta)^2 - \sin^2\theta\cos^2\theta$

$= \mathbf{1 - \sin^2\theta\cos^2\theta} \qquad (\because \sin^2\theta + \cos^2\theta = 1)$

23. $\dfrac{\cos x}{1+\operatorname{cosec} x}+\dfrac{\cos x}{\operatorname{cosec} x-1}=2$

$\Rightarrow \dfrac{\cos x(\operatorname{cosec} x-1)+\cos x(1+\operatorname{cosec} x)}{\operatorname{cosec}^2 x-1}=2$

$\Rightarrow \dfrac{2\cos x\operatorname{cosec} x}{\cot^2 x}=2$

$\Rightarrow \dfrac{\cos x}{\sin x}\times\dfrac{\sin^2 x}{\cos^2 x}=1$

$\Rightarrow \tan x=1 \Rightarrow \tan x=\tan\dfrac{\pi}{4} \Rightarrow x=\boldsymbol{\dfrac{\pi}{4}}.$

24. Given, $x=a(1+\cos\theta\cos\phi)$

$\Rightarrow \dfrac{x}{a}=1+\cos\theta\cos\phi \quad \Rightarrow \dfrac{x}{a}-1=\cos\theta\cos\phi$

$\Rightarrow \dfrac{x-a}{a}=\cos\theta\cos\phi$ ...(*i*)

Similarly, $y=b(1+\cos\theta\sin\phi)$

$\Rightarrow \dfrac{y-b}{b}=\cos\theta\sin\phi$ ...(*ii*)

$z=c(1+\sin\theta) \Rightarrow \dfrac{z-c}{c}=\sin\theta$ ...(*iii*)

Squaring eqns (*i*), (*ii*) and (*iii*) and adding, we get

$\left(\dfrac{x-a}{a}\right)^2+\left(\dfrac{y-b}{b}\right)^2+\left(\dfrac{z-c}{c}\right)^2$

$=\cos^2\theta\cos^2\phi+\cos^2\theta\sin^2\phi+\sin^2\theta$

$=\cos^2\theta(\cos^2\phi+\sin^2\phi)+\sin^2\theta$

$=\cos^2\theta+\sin^2\theta=\mathbf{1}.$

25. $\sin^4 x+\sin^2 x=1$

$\Rightarrow \sin^4 x=1-\sin^2 x$

$\Rightarrow \sin^4 x=\cos^2 x$ ...(*i*)

$\therefore \cot^4 x+\cot^2 x$

$=\cot^2 x(1+\cot^2 x)=\cot^2 x \cdot \operatorname{cosec}^2 x$

$=\dfrac{\cos^2 x}{\sin^2 x\cdot\sin^2 x}=\dfrac{\cos^2 x}{\sin^4 x}$

$=\dfrac{\cos^2 x}{\cos^2 x}=\mathbf{1}.$ [Using (*i*)]

26. Given, $a\sin\theta-b\cos\theta=c$

$(a\cos\theta-b\sin\theta)^2$

$=a^2\cos^2\theta-2ab\cos\theta\sin\theta+b^2\sin^2\theta$

$=a^2(1-\sin^2\theta)-2(a\sin\theta)(b\cos\theta)+b^2(1-\cos^2\theta)$

$=a^2-a^2\sin^2\theta-2(a\sin\theta)(b\cos\theta)+b^2-b^2\cos^2\theta$

$=a^2+b^2-[a^2\sin^2\theta+2(a\sin\theta)(b\cos\theta)+b^2\cos^2\theta]$

$=a^2+b^2-(a\sin\theta+b\cos\theta)^2=a^2+b^2-c^2$

$\Rightarrow (a\cos\theta-b\sin\theta)=\pm\boldsymbol{\sqrt{a^2+b^2-c^2}}.$

27. $\sin A\cos A\tan A+\cos A\sin A\cot A$

$=\sin A\cos A\cdot\dfrac{\sin A}{\cos A}+\cos A\sin A\dfrac{\cos A}{\sin A}$

$=\sin^2 A+\cos^2 A=1=\boldsymbol{\operatorname{cosec}^2 A-\cot^2 A}.$

28. $\cos\alpha\sin(\beta-\gamma)+\cos\beta\sin(\gamma-\alpha)+\cos\gamma\sin(\alpha-\beta)$

$=\cos\alpha[\sin\beta\cos\gamma-\cos\beta\sin\gamma]$
$\quad+\cos\beta[\sin\gamma\cos\alpha-\cos\alpha\sin\beta]$
$\quad+\cos\gamma[\sin\alpha\cos\beta-\cos\alpha\sin\beta]$

$=\cos\alpha\sin\beta\cos\gamma-\cos\alpha\cos\beta\sin\gamma$
$\quad+\cos\beta\sin\gamma\cos\alpha-\cos\beta\cos\alpha\sin\gamma$
$\quad+\cos\gamma\sin\alpha\cos\beta-\cos\gamma\cos\alpha\sin\beta$

$=\mathbf{0}.$

29. $\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P.

$\Rightarrow \sin\theta\times\tan\theta=\cos^2\theta \Rightarrow \sin\theta\cdot\dfrac{\sin\theta}{\cos\theta}=\cos^2\theta$

$\Rightarrow \sin^2\theta=\cos^3\theta$ …(*i*)

Now, $\cot^6\theta-\cot^2\theta=\dfrac{\cos^6\theta}{\sin^6\theta}-\dfrac{\cos^2\theta}{\sin^2\theta}$

$=\dfrac{(\cos^3\theta)^2}{\sin^6\theta}-\dfrac{\cos^2\theta}{\sin^2\theta}$

$=\dfrac{\sin^4\theta}{\sin^6\theta}-\dfrac{\cos^2\theta}{\sin^2\theta}$ (By (*i*))

$=\dfrac{1}{\sin^2\theta}-\dfrac{\cos^2\theta}{\sin^2\theta}$

$=\dfrac{1-\cos^2\theta}{\sin^2\theta}=\dfrac{\sin^2\theta}{\sin^2\theta}=\mathbf{1}.$

30. $\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$

$\Rightarrow \tan(\pi\cos\theta)=\tan(\pi/2-\pi\sin\theta)$

$\Rightarrow \pi\cos\theta=\dfrac{\pi}{2}-\pi\sin\theta$

$\Rightarrow \pi(\cos\theta-\sin\theta)=\dfrac{\pi}{2} \quad \Rightarrow \cos\theta+\sin\theta=\dfrac{1}{2}$

$\Rightarrow \dfrac{1}{\sqrt{2}}\cos\theta+\dfrac{1}{\sqrt{2}}\sin\theta=\dfrac{1}{2\sqrt{2}}$

(Multiplying both sides by $\dfrac{1}{\sqrt{2}}$)

$\Rightarrow \cos\dfrac{\pi}{4}\cos\theta+\sin\dfrac{\pi}{4}\sin\theta=\dfrac{1}{2\sqrt{2}}$

$\Rightarrow \cos(\pi/4-\theta)=\boldsymbol{\dfrac{1}{2\sqrt{2}}}$

$(\because \cos(A-B)=\cos A\cos B+\sin A\sin B)$

31. $3\sin x+4\cos x=5$

$\Rightarrow 3\left(\dfrac{2\tan x/2}{1+\tan^2 x/2}\right)+4\left(\dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)=5$

$\left(\because \sin x=\dfrac{2\tan x/2}{1+\tan^2 x/2}, \cos x=\dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)$

$\Rightarrow \dfrac{6\tan x/2+4-4\tan^2 x/2}{1+\tan^2 x/2}=5$

$\Rightarrow 6\tan x/2+4-4\tan^2 x/2=5+5\tan^2 x/2$

$\Rightarrow 6\tan x/2-9\tan^2 x/2=\mathbf{1}.$

32. $\cos^2\alpha + \cos^2(\alpha - 120°) + \cos^2(\alpha + 120°)$

$= \cos^2\alpha + \{\cos\alpha \ \cos 120° + \sin\alpha \ \sin 120°\}^2 + \{\cos\alpha \ \cos 120° - \sin\alpha \ \sin 120°\}^2$

($\because \cos(A+B) = \cos A \cos B - \sin A \ \sin B$)

$$= \cos^2\alpha + \left\{-\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha\right\}^2 + \left\{-\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha\right\}^2$$

$$\left(\cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2}\right)$$

$$= \cos^2\alpha + \frac{1}{4}\cos^2\alpha + \frac{3}{4}\sin^2\alpha - \frac{\sqrt{3}}{2}\sin\alpha\cos\alpha + \frac{1}{4}\cos^2\alpha + \frac{3}{4}\sin^2\alpha + \frac{\sqrt{3}}{2}\sin\alpha\cos\alpha$$

$$= \cos^2\alpha + \frac{1}{4}\cos^2\alpha + \frac{1}{4}\cos^2\alpha + \frac{3}{4}\sin^2\alpha + \frac{3}{4}\sin^2\alpha$$

$$= \frac{3}{2}\cos^2\alpha + \frac{3}{2}\sin^2\alpha = \frac{3}{2}(\cos^2\alpha + \sin^2\alpha) = \mathbf{\frac{3}{2}}.$$

33. $\dfrac{\tan A}{1-\cot A} + \dfrac{\cot A}{1-\tan A}$

$$= \frac{\sin A}{\cos A} \times \frac{1}{1 - \frac{\cos A}{\sin A}} + \frac{\cos A}{\sin A} \times \frac{1}{1 - \frac{\sin A}{\cos A}}$$

$$= \frac{\sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A} + \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A - \sin A}$$

$$= \frac{\sin^2 A}{\cos A(\sin A - \cos A)} - \frac{\cos^2 A}{\sin A(\sin A - \cos A)}$$

$$= \frac{1}{(\sin A - \cos A)}\left[\frac{\sin^3 A - \cos^3 A}{\cos A \sin A}\right]$$

$$= \frac{(\sin A - \cos A)(\sin^2 A + \sin A\cos A + \cos^2 A)}{(\sin A - \cos A)\cos A \sin A}$$

$$= \frac{1 + \sin A\cos A}{\cos A \sin A} \qquad (\because \sin^2 A + \cos^2 A = 1)$$

$= \mathbf{\sec A \operatorname{cosec} A + 1.}$

34. $\tan 5\theta = \tan(2\theta + 3\theta) = \dfrac{\tan 2\theta + \tan 3\theta}{1 - \tan 2\theta \tan 3\theta}$

$$\left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right]$$

$$= \frac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}}{1 - \left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)\cdot\left(\dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)}$$

$$= \frac{2\tan\theta(1 - 3\tan^2\theta) + (3\tan\theta - \tan^3\theta)(1 - \tan^2\theta)}{(1-\tan^2\theta)(1-3\tan^2\theta) - (2\tan\theta)(3\tan\theta - \tan^3\theta)}$$

$$= \frac{2\tan\theta - 6\tan^3\theta + 3\tan\theta - \tan^3\theta - 3\tan^3\theta + \tan^5\theta}{1 - \tan^2\theta - 3\tan^2\theta + 3\tan^4\theta - 6\tan^2\theta + 2\tan^4\theta}$$

$$= \mathbf{\frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}}.$$

35. Given, $\tan\alpha = 2\tan\beta \Rightarrow \dfrac{\frac{\sin\alpha}{\cos\alpha}}{\frac{\sin\beta}{\cos\beta}} = 2$

$$\Rightarrow \frac{\sin\alpha \ \cos\beta}{\cos\alpha \ \sin\beta} = \frac{2}{1}$$

Using componendo and dividendo, we get

$$\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta} = \frac{2+1}{2-1}$$

$$\Rightarrow \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{3}{1} \Rightarrow \sin(\alpha+\beta) = \mathbf{3\sin(\alpha-\beta).}$$

36. $\sin x + \operatorname{cosec} x = 2$ ...(*i*)

Squaring both the sides, we have

$\sin^2 x + \operatorname{cosec}^2 x + 2 = 4$ ...(*ii*)

$\Rightarrow \sin^2 x + \operatorname{cosec}^2 x = 2$, *i.e.,* for $n = 2$, $\sin^n x + \cos^n x = 2$

Cubing both the sides of (*i*), we have

$\sin^3 x + \operatorname{cosec}^3 x + 3\sin x \ \operatorname{cosec} x(\sin x + \operatorname{cosec} x) = 8$

$\Rightarrow \sin^3 x + \operatorname{cosec}^3 + 3 \times 2 = 8$

$\Rightarrow \sin^3 x + \operatorname{cosec}^3 = 8 - 6 = 2,$

i.e., for $n = 3$, $\sin^n x + \cos^n x = 2$

Squaring both the sides of (*ii*), we have

$\sin^4 x + \operatorname{cosec}^4 x + 2 = 4$

$\Rightarrow \sin^4 x + \operatorname{cosec}^4 x = 2$, *i.e.,* for $n = 4$, $\sin^n x + \cos^n x = 2$

Proceeding in the same way, we see that

$\sin^n x + \operatorname{cosec}^n x = 2$ for all $n \in N$.

37. $\cos\dfrac{2\pi}{15}\cdot\cos\dfrac{4\pi}{15}\cdot\cos\dfrac{8\pi}{15}\cdot\cos\dfrac{16\pi}{15}$

$$= \frac{1}{2\sin\frac{2\pi}{15}}\cdot\left(2\sin\frac{2\pi}{15}\cdot\cos\frac{2\pi}{15}\right)\cos\frac{4\pi}{15}\cdot\cos\frac{8\pi}{15}\cdot\cos\frac{16\pi}{15}$$

$$\left(\text{On multiplying and dividing the expression by } 2\sin\frac{2\pi}{15}\right)$$

$$= \frac{1}{2\sin\frac{2\pi}{15}}\cdot\sin\frac{4\pi}{15}\cdot\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cdot\cos\frac{16\pi}{15}$$

($\because \sin 2A = 2\sin A\cos A$)

$$= \frac{1}{4\sin\frac{2\pi}{15}}\cdot\left(2\sin\frac{4\pi}{15}\cdot\cos\frac{4\pi}{15}\right)\cos\frac{8\pi}{15}\cdot\cos\frac{16\pi}{15}$$

$$= \frac{1}{4\sin\frac{2\pi}{15}}\cdot\sin\frac{8\pi}{15}\cdot\cos\frac{8\pi}{15}\cdot\cos\frac{16\pi}{15}$$

$= \dfrac{1}{8\sin\dfrac{2\pi}{15}} \cdot \left(2\sin\dfrac{8\pi}{15} \cdot \cos\dfrac{8\pi}{15}\right) \cdot \cos\dfrac{16\pi}{15}$

$= \dfrac{1}{8\sin\dfrac{2\pi}{15}} \cdot \sin\dfrac{16\pi}{15} \cdot \cos\dfrac{16\pi}{15}$

$= \dfrac{1}{16\sin\dfrac{2\pi}{15}} \cdot \sin\dfrac{32\pi}{15}$

$= \dfrac{1}{16\sin\dfrac{2\pi}{15}} \cdot \left(\sin 2\pi + \dfrac{2\pi}{15}\right)$ (Note the step)

$= \dfrac{1}{16\sin\dfrac{2\pi}{15}} \cdot \sin\dfrac{2\pi}{15}$ $(\because \sin(360° + \theta) = \sin\theta)$

$= \mathbf{\dfrac{1}{16}}$.

38. $\sin^2\alpha + \sin^2\beta - \sin^2\gamma$

$= \sin^2\alpha + \sin(\beta+\gamma)\sin(\beta-\gamma)$

[Using $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$]

$= \sin^2\alpha + \sin(\pi-\alpha)\sin(\beta-\gamma)$ $(\because \alpha+\beta+\gamma = \pi)$

$= \sin^2\alpha + \sin\alpha\sin(\beta-\gamma)$ $(\because \sin(180° - \theta) = \sin\theta)$

$= \sin\alpha\,[\sin\alpha + \sin(\beta-\gamma)]$

$= \sin\alpha\,[\sin(\pi-(\beta+\gamma)) + \sin(\beta-\gamma)]$

$= \sin\alpha\,[\sin(\beta+\gamma) + \sin(\beta-\gamma)]$

$= \sin\alpha\,[\sin\beta\cos\gamma + \cos\beta\sin\gamma + \sin\beta\cos\gamma - \cos\beta\sin\gamma]$

$(\because \sin(A+B) = \sin A\cos B + \cos A\sin B)$

$= \mathbf{2\sin\alpha\sin\beta\cos\gamma}$.

39. We know that $\cos 2\theta = 2\cos^2\theta - 1$

Putting $\theta = 15°$ in the above expression,

$\cos 30° = 2\cos^2 15° - 1$

$\Rightarrow \dfrac{\sqrt{3}}{2} + 1 = 2\cos^2 15°$

$\Rightarrow \cos^2 15° = \dfrac{\sqrt{3}+2}{4} \Rightarrow \cos 15° = \mathbf{\dfrac{1}{2}\sqrt{\sqrt{3}+2}}$.

40. $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 4A}}}$

$= \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 4A)}}}$

$= \sqrt{2+\sqrt{2+\sqrt{4\cos^2 2A}}}$ $\left(\begin{array}{l}\because \cos 2A = 2\cos^2 A - 1\\ \text{or } 1+\cos 2A = 2\cos^2 A\end{array}\right)$

$= \sqrt{2+\sqrt{2+2\cos 2A}}$

$= \sqrt{2+\sqrt{2(1+\cos 2A)}}$

$= \sqrt{2+\sqrt{2(2\cos^2 A)}} = \sqrt{2+2\cos A}$

$= \sqrt{2(1+\cos A)} = \sqrt{2\times 2\cos^2\dfrac{A}{2}}$

$= \sqrt{4\cos^2 A/2} = \mathbf{2\cos A/2}$.

41. $\dfrac{\sin x}{1+\cos x} = \dfrac{2\sin x/2\cos x/2}{1+2\cos^2 x/2 - 1}$

$(\because \sin 2\theta = 2\sin\theta\cos\theta \text{ and } \cos 2\theta = 2\cos^2 - 1)$

$= \dfrac{2\sin x/2\cos x/2}{2\cos^2 x/2} = \dfrac{\sin x/2}{\cos x/2} = \mathbf{\tan x/2}$.

42. $\dfrac{\sin 70° + \cos 40°}{\cos 70° + \sin 40°} = \dfrac{\sin 70° + \cos(90° - 50°)}{\cos 70° + \sin(90° - 50°)}$

$= \dfrac{\sin 70° + \sin 50°}{\cos 70° + \cos 50°}$ $\left[\begin{array}{l}\because \sin(90° - \theta) = \cos\theta\\ \cos(90° - \theta) = \sin\theta\end{array}\right]$

$= \dfrac{2\sin\left(\dfrac{70°+50°}{2}\right)\cos\left(\dfrac{70°-50°}{2}\right)}{2\cos\left(\dfrac{70°+50°}{2}\right)\cos\left(\dfrac{70°-50°}{2}\right)}$

$\left[\begin{array}{l}\because \sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)\\ \cos A + \cos B = 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)\end{array}\right]$

$= \dfrac{2\sin 60°\cos 10°}{2\cos 60°\cos 10°} = \tan 60° = \mathbf{\sqrt{3}}$.

43. $\tan^2\alpha = \dfrac{\sin^2\alpha}{\cos^2\alpha} = \dfrac{2\sin^2\alpha}{2\cos^2\alpha} = \dfrac{1-\cos 2\alpha}{1+\cos 2\alpha}$

$(\because 1 - \cos 2\theta = 2\sin^2\theta,\ 1 + \cos 2\theta = 2\cos^2\theta)$

$= \dfrac{1 - \dfrac{3\cos 2\beta - 1}{3 - \cos 2\beta}}{1 + \dfrac{3\cos 2\beta - 1}{3-\cos 2\beta}} = \dfrac{3 - \cos 2\beta - 3\cos 2\beta + 1}{3 - \cos 2\beta + 3\cos 2\beta - 1}$

$= \dfrac{4 - 4\cos 2\beta}{2 + 2\cos 2\beta} = \dfrac{4(1-\cos 2\beta)}{2(1+\cos 2\beta)}$

$= 2\left(\dfrac{\sin^2\beta)}{2\cos^2\beta)}\right) = 2\tan^2\beta$

$\therefore$ $\tan^2\alpha = 2\tan^2\beta \Rightarrow \tan\alpha = \mathbf{\sqrt{2}\tan\beta}$.

44. $\dfrac{\sin 5\theta}{\sin\theta} = \dfrac{\sin(2\theta+3\theta)}{\sin\theta}$

$= \dfrac{1}{\sin\theta}\{\sin 2\theta\cos 3\theta + \cos 2\theta\sin 3\theta\}$

$= \dfrac{1}{\sin\theta}\Big\{2\sin\theta\cos\theta\left(4\cos^3\theta - 3\cos\theta\right)$

$+ \left(2\cos^2\theta - 1\right)\left(3\sin\theta - 4\sin^3\theta\right)\Big\}$

$(\because \cos 3A = 4\cos^3 A - 3\cos A,\ \sin 3A = 3\sin A - 4\sin^3 A)$

$= \{2\cos\theta(4\cos^3\theta - 3\cos\theta) + (2\cos^2\theta - 1)(3 - 4\sin^2\theta)\}$.

(Note the step)

$= 8\cos^4\theta - 6\cos^2\theta + 6\cos^2\theta - 3 - 8\cos^2\theta\sin^2\theta + 4\sin^2\theta$

$= 8\cos^4\theta - 3 - 8\cos^2\theta(1 - \cos^2\theta) + 4(1 - \cos^2\theta)$

$(\because \sin^2\theta + \cos^2\theta = 1)$

$= 8\cos^4\theta - 3 - 8\cos^2\theta + 8\cos^4\theta + 4 - 4\cos^2\theta$

$= \mathbf{16\cos^4\theta - 12\cos^2\theta + 1}$.

45. $\sin A + \sin B = a$

$\Rightarrow\ 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = a \qquad ...(i)$

$\cos A - \cos B = b$

$\Rightarrow\ 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right) = b$

$\Rightarrow\ 2\sin\left(\frac{A+B}{2}\right)\sin\left[-\left(\frac{A-B}{2}\right)\right] = b$

$\Rightarrow\ -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) = b \qquad ...(ii)$

$(\because \sin(-\theta) = -\sin\theta)$

Dividing eqn (*ii*) by eqn (*i*), we get

$$\frac{-\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)} = \frac{b}{a}$$

$\Rightarrow\ -\tan\left(\frac{A-B}{2}\right) = b/a \Rightarrow \tan\left(\frac{A-B}{2}\right) = -b/a$

$$\therefore\ \cos(A-B) = \frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)}$$

$\left(\because \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$$= \frac{1-b^2/a^2}{1+b^2/a^2} = \boldsymbol{\frac{a^2-b^2}{a^2+b^2}}.$$

46. $\sin 16° + \cos 16° = \sin(15° + 1°) + \cos(15° + 1°)$

$= (\sin 15° \cos 1° + \cos 15° \sin 1°)$
$\qquad + (\cos 15° \cos 1° - \sin 15° \sin 1°)$

$= \sin 1° (\cos 15° - \sin 15°) + \cos 1° (\sin 15° + \cos 15°)$

Now, $\cos 15° = \cos(45° - 30°)$

$= \cos 45° \cos 30° + \sin 45° \sin 30°$

$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

and $\sin 15° = \sin(45° - 30°)$

$= \sin 45° \cos 30° - \cos 45° \sin 30°$

$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\therefore \sin 16° + \cos 16°$

$= \sin 1°\left(\frac{\sqrt{3}+1}{2\sqrt{2}} - \frac{\sqrt{3}-1}{2\sqrt{2}}\right) + \cos 1°\left(\frac{\sqrt{3}-1}{2\sqrt{2}} + \frac{\sqrt{3}+1}{2\sqrt{2}}\right)$

$= \sin 1°\left(\frac{2}{2\sqrt{2}}\right) + \cos 1°\left(\frac{2\sqrt{3}}{2\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}}\sin 1° + \frac{\sqrt{3}}{\sqrt{2}}\cos 1° = \boldsymbol{\frac{1}{\sqrt{2}}\left(\sin 1° + \sqrt{3}\cos 1°\right)}.$

47. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$

$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\left(\pi-\frac{3\pi}{8}\right)\right)\left(1+\cos\left(\pi-\frac{\pi}{8}\right)\right)$

$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$

$(\because \cos(\pi-\theta) = -\cos\theta)$

$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right) = \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$

$= \sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8} \qquad \left(\because \sin\frac{3\pi}{8} = \sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right) = \cos\frac{\pi}{8}\right)$

$= \frac{1}{4}\left(4\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right) = \frac{1}{4}\left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2$

$= \frac{1}{4}\left(\sin\frac{\pi}{4}\right)^2 \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^2 = \boldsymbol{\frac{1}{8}}.$

(Using, $2\sin\theta\cos\theta = \sin 2\theta$)

48. $\cos 2(\theta+\phi) + 4\cos(\theta+\phi)\sin\theta\sin\phi + 2\sin^2\phi$

$= \{\cos 2\theta\cos 2\phi - \sin 2\theta\sin 2\phi\}$
$\qquad + 4\{(\cos\theta\cos\phi - \sin\theta\sin\phi)\sin\theta\sin\phi\} + 2\sin^2\phi$

$= \{(1-2\sin^2\theta)(1-2\sin^2\phi) - 2\sin\theta\cos\theta \cdot 2\sin\phi\cos\phi\}$
$\qquad + [4\cos\theta\cos\phi\sin\theta\sin\phi - 4\sin^2\theta\sin^2\phi] + 2\sin^2\phi$

$= 1 - 2\sin^2\theta - 2\sin^2\phi + 4\sin^2\theta\sin^2\phi$
$\quad - 4\sin\theta\sin\phi\cos\theta\cos\phi + 4\cos\theta\cos\phi\sin\theta\sin\phi$
$\qquad - 4\sin^2\theta\sin^2\phi + 2\sin^2\phi$

$= 1 - 2\sin^2\theta = \mathbf{\cos 2\theta}.$

49. $\tan A - \tan B = x$

$\Rightarrow\ \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B} = x$

$\Rightarrow\ \frac{\sin A\cos B - \sin B\cos A}{\cos A\cos B} = x$

$\Rightarrow\ \frac{\sin(A-B)}{\cos A\cos B} = x$

$\Rightarrow\ \frac{1}{x} = \frac{\cos A\cos B}{\sin(A-B)} \qquad \ldots(i)$

$\cot B - \cot A = y$

$\Rightarrow\ \frac{\cos B}{\sin B} - \frac{\cos A}{\sin A} = y$

$\Rightarrow\ \frac{\cos B\sin A - \cos A\sin B}{\sin B\sin A} = y$

$\Rightarrow\ \frac{\sin(A-B)}{\sin A\sin B} = y \Rightarrow \frac{1}{y} = \frac{\sin A\sin B}{\sin(A-B)} \qquad ...(ii)$

Adding (*i*) and (*ii*), we get

$\frac{1}{x} + \frac{1}{y} = \frac{\cos A\cos B + \sin A\sin B}{\sin(A-B)}$

$\Rightarrow\ \frac{\cos(A-B)}{\sin(A-B)} = \mathbf{\cot(A-B)}.$

50. $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \dfrac{8}{\tan 8\alpha}$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \dfrac{8}{\dfrac{2\tan 4\alpha}{1-\tan^2 4\alpha}}$ $\left(\because \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}\right)$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \dfrac{4(1-\tan^2 4\alpha)}{\tan 4\alpha}$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \dfrac{4}{\tan 4\alpha} - 4\tan 4\alpha$

$= \tan\alpha + 2\tan 2\alpha + \dfrac{4}{\tan 4\alpha}$

$= \tan\alpha + 2\tan 2\alpha + \dfrac{4}{\dfrac{2\tan 2\alpha}{1-\tan^2 2\alpha}}$

$= \tan\alpha + 2\tan 2\alpha + \dfrac{2(1-\tan^2 2\alpha)}{\tan 2\alpha}$

$= \tan\alpha + \dfrac{2}{\tan 2\alpha} = \tan\alpha + \dfrac{2}{\dfrac{2\tan\alpha}{1-\tan^2\alpha}}$

$= \tan\alpha + \dfrac{1-\tan^2\alpha}{\tan\alpha} = \dfrac{1}{\tan\alpha} = \mathbf{\cot\alpha.}$

SELF ASSESSMENT SHEET

1. If q cosec $\theta = p$ and θ is acute, then what is the value of $\sqrt{p^2-q^2}\tan\theta$?

(*a*) p (*b*) q (*c*) pq (*d*) $\sqrt{p^2+q^2}$

2. If $0 < x < 45°$ and $45° < y < 90°$, then which one of the following is correct?

(*a*) $\sin x = \sin y$ (*b*) $\sin x < \sin y$
(*c*) $\sin x > \sin y$ (*d*) $\sin x \le \sin y$

3. If $\tan^2 y\ \ \text{cosec}^2 x - 1 = \tan^2 y$, then which one of the following is correct?

(*a*) $x - y = 0$ (*b*) $x = 2y$
(*c*) $y = 2x$ (*d*) $x - y = 1°$

4. If α and β the complementary angles, then what is $\sqrt{\text{cosec}\,\alpha\,.\,\text{cosec}\,\beta}\left(\dfrac{\sin\alpha}{\sin\beta}+\dfrac{\cos\alpha}{\cos\beta}\right)^{-1/2}$ equal to

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) None of there

(*CDS 2011*)

5. What is $\left(\dfrac{\sec 18°}{\sec 144°}+\dfrac{\text{cosec}\,18°}{\text{cosec}\,144°}\right)$ equal to?

(*a*) sec 18° (*b*) cosec 18°
(*c*) – sec 18° (*d*) – cosec 18°

6. If an angle α is divided into two parts A and B such that $A - B = x$ and $\tan A : \tan B = 2:1$, then what is $\sin x$ equal to?

(*a*) $3\sin\alpha$ (*b*) $\dfrac{2\sin\alpha}{3}$

(*c*) $\dfrac{\sin\alpha}{3}$ (*d*) $2\sin\alpha$ **(*NDA/NA 2011*)**

7. If $\cos\theta = \dfrac{8}{17}$ and θ lies in the first quadrant, then the value of $\cos(30° + \theta) + \cos(45° - \theta) + \cos(120° - \theta)$ is

(*a*) $\dfrac{23}{17}\left(\dfrac{\sqrt{3}-1}{2}-\dfrac{1}{\sqrt{2}}\right)$ (*b*) $\dfrac{23}{17}\left(\dfrac{\sqrt{3}+1}{2}-\dfrac{1}{\sqrt{2}}\right)$

(*c*) $\dfrac{23}{17}\left(\dfrac{\sqrt{3}-1}{2}+\dfrac{1}{\sqrt{2}}\right)$ (*d*) $\dfrac{23}{17}\left(\dfrac{\sqrt{3}+1}{2}+\dfrac{1}{\sqrt{2}}\right)$

(*DCE 2008*)

8. Let A, B, C be the angles of a plain triangle $\tan(A/2) = \dfrac{1}{3}$, $\tan\left(\dfrac{B}{2}\right) = \dfrac{2}{3}$. Then $\tan(C/2)$ is equal to

(*a*) $\dfrac{2}{9}$ (*b*) $\dfrac{1}{3}$ (*c*) $\dfrac{7}{9}$ (*d*) $\dfrac{2}{3}$

9. If $\dfrac{1+\cos A}{1-\cos A} = \dfrac{m^2}{n^2}$, then $\tan A$ is equal to

(*a*) $\pm\dfrac{2mn}{m^2-n^2}$ (*b*) $\pm\dfrac{2mn}{m^2+n^2}$

(*c*) $\dfrac{m^2+n^2}{m^2-n^2}$ (*d*) $\dfrac{m^2-n^2}{m^2+n^2}$

(*J&K CET 2007*)

10. If α, β, γ, are in A.P, then $\dfrac{\sin\alpha-\sin\gamma}{\cos\gamma-\cos\alpha}$. is equal to

(*a*) $\sin\alpha$ (*b*) $\cos\alpha$ (*c*) $\tan\beta$ (*d*) $\cot\beta$

ANSWERS

1. (*b*) **2.** (*b*) **3.** (*a*) **4.** (*b*) **5.** (*a*) **6.** (*c*) **7.** (*c*) **8.** (*c*) **9.** (*a*) **10.** (*d*)

HINTS AND SOLUTIONS

1. q cosec $\theta = p$

$\therefore \quad \sqrt{p^2 - q^2}\,\tan\theta$

$= \sqrt{q^2\,\text{cosec}^2\theta - q^2}\,.\tan\theta = \sqrt{q^2\,(\text{cosec}^2\theta - 1)}.\,\tan\theta$

$= \sqrt{q^2\cot^2\theta}.\,\tan\theta = q\cot\theta\,.\,\tan\theta = \boldsymbol{q}.$

2. Since sin θ in cosec from θ = 0° to θ = 90°, *i.e.*, where $0° < \theta < 90°$ then $0 < \sin\theta < 1$. $\therefore$ **$\sin x < \sin y$.**

3. $\tan^2 y\ \text{cosec}^2 x - 1 = \tan^2 y$

$\Rightarrow \tan^2 y\ \text{cosec}^2 x - \tan^2 y = 1$

$\Rightarrow \tan^2 y\ (\text{cosec}^2 x - 1) = 1 \Rightarrow \tan^2 y\,.\cot^2 x = 1$

$\Rightarrow \cot^2 x = \cot^2 y \Rightarrow x = y \Rightarrow \boldsymbol{x - y = 0}.$

4. Given $\alpha + \beta = 90°$

$$\therefore \sqrt{\text{cosec}\,\alpha\,.\,\text{cosec}\,\beta}\left(\frac{\sin\alpha}{\sin\beta} + \frac{\cos\alpha}{\cos\beta}\right)^{-1/2}$$

$$= \frac{1}{(\sin\alpha\sin\beta)^{1/2}}\left(\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\beta\cos\beta}\right)^{-1/2}$$

$$= \frac{1}{(\sin\alpha\sin\beta)^{1/2}}\left(\frac{\sin(\alpha+\beta)}{\sin\beta\cos\beta}\right)^{-1/2}$$

$$= \frac{1}{(\sin\alpha\sin\beta)^{1/2}}\left(\frac{\sin 90°}{\sin\beta\cos(90° - \alpha)}\right)^{-1/2}$$

$$= \frac{1}{(\sin\alpha\sin\beta)^{\frac{1}{2}}}\left(\frac{1}{\sin\beta\sin\alpha}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{(\sin\alpha\sin\beta)^{\frac{1}{2}}} \times (\sin\alpha\sin\beta)^{\frac{1}{2}} = \mathbf{1}.$$

5. $\dfrac{\sec 18°}{\sec 144°} + \dfrac{\text{cosec}\,18°}{\text{cosec}\,144°}$

$$= \frac{\sec 18°}{\sec(180° - 36°)} + \frac{\text{cosec}\,18°}{\text{cosec}\,(180° - 36°)}$$

$$= -\frac{\sec 18°}{\sec 36°} + \frac{\text{cosec}\,18°}{\text{cosec}\,36°} \quad \left(\begin{array}{l}\because \sec(180° - \theta) = -\sec\theta \\ \text{cosec}\,(180° - \theta) = \text{cosec}\theta\end{array}\right)$$

$$= \frac{\sin 36°}{\sin 18°} - \frac{\cos 36°}{\cos 18°} = \frac{\sin 36°\cos 18° - \cos 36°\sin 18°}{\sin 18°\cos 18°}$$

$(\because \sin(A - B) = \sin A\cos B - \cos A\sin B)$

$$= \frac{\sin(36° - 18°)}{\sin 18°\cos 18°} = \frac{\sin 18°}{\sin 18°\cos 18°} = \frac{1}{\cos 18°} = \mathbf{sec\ 18°}.$$

6. $\alpha = A + B$

$x = A - B$

$\Rightarrow \; A = \dfrac{\alpha + x}{2}$ and $B = \dfrac{\alpha - x}{2}$

$$\text{Given, } \frac{\tan A}{\tan B} = \frac{2}{1} \Rightarrow \frac{\tan\left(\frac{\alpha + x}{2}\right)}{\tan\left(\frac{\alpha - x}{2}\right)} = \frac{2}{1}$$

$$\Rightarrow \frac{\sin\left(\frac{\alpha + x}{2}\right).\cos\left(\frac{\alpha - x}{2}\right)}{\cos\left(\frac{\alpha + x}{2}\right).\sin\left(\frac{\alpha - x}{2}\right)} = \frac{2}{1}$$

$$\Rightarrow \frac{2\sin\left(\frac{\alpha + x}{2}\right)\cos\left(\frac{\alpha - x}{2}\right)}{2\cos\left(\frac{\alpha + x}{2}\right)\sin\left(\frac{\alpha - x}{2}\right)} = \frac{2}{1}$$

$$\Rightarrow \frac{\sin\alpha + \sin x}{\sin\alpha - \sin x} = \frac{2}{1}$$

$$\left(\begin{array}{l}\because\ 2\sin\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right) = \sin C + \sin D \\ \text{and } 2\cos\left(\frac{C + D}{2}\right)\sin\left(\frac{C - D}{2}\right) = \sin C - \sin D\end{array}\right)$$

$\Rightarrow \; \sin\alpha + \sin x = 2\sin\alpha - 2\sin x$

$\Rightarrow \; 3\sin x = \sin\alpha \Rightarrow \boldsymbol{\sin x = \dfrac{\sin\alpha}{3}}.$

7. $\cos(30° + \theta) + \cos(45° - \theta) + \cos(120° - \theta)$

$= \cos 30°\cos\theta - \sin 30°\sin\theta + \cos 45°\cos\theta + \sin 45°\sin\theta + \cos 120°\cos\theta + \sin 120°\sin\theta$

$$\left(\cos\theta = \frac{8}{17} \Rightarrow \sin\theta = \sqrt{1 - \left(\frac{8}{17}\right)^2} = \sqrt{\frac{289 - 64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}\right)$$

$$= \frac{\sqrt{3}}{2}\times\frac{8}{17} - \frac{1}{2}\times\frac{15}{17} + \frac{1}{\sqrt{2}}\times\frac{8}{17} + \frac{1}{\sqrt{2}}\times\frac{15}{17} + \left(\frac{-1}{2}\right)\times\frac{8}{17} + \frac{\sqrt{3}}{2}\times\frac{15}{17}$$

$$\left[\begin{array}{l}\because\ \cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2} \\ \sin 120° = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}\end{array}\right]$$

$$= \frac{8}{17}\left(\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} - \frac{1}{2}\right) + \frac{15}{17}\left(\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} - \frac{1}{2}\right)$$

$$= \frac{8}{17}\left(\frac{\sqrt{3} - 1}{2} + \frac{1}{\sqrt{2}}\right) + \frac{15}{17}\left(\frac{\sqrt{3} - 1}{2} + \frac{1}{\sqrt{2}}\right)$$

$$= \mathbf{\frac{23}{17}\left(\frac{\sqrt{3} - 1}{2} + \frac{1}{\sqrt{2}}\right)}.$$

8. $A + B + C = 180°$ (Angle sum property of a triangle)

$\Rightarrow\ A + B = 180° - C$

$$\Rightarrow\ \frac{A+B}{2} = 90° - \frac{C}{2}$$

$$\Rightarrow\ \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(90° - \frac{C}{2}\right)$$

$$\Rightarrow\ \frac{\tan A/2 + \tan B/2}{1 - \tan A/2 \tan B/2} = \cot C/2$$

$$\Rightarrow\ \frac{\frac{1}{3} + \frac{2}{3}}{1 - \frac{1}{3} \times \frac{2}{3}} = \cot C/2 \ \Rightarrow\ \frac{1}{1 - \frac{2}{9}} = \cot C/2$$

$$\Rightarrow\ \frac{9}{7} = \cot C/2 \ \Rightarrow\ \mathbf{\tan C/2 = 7/9}.$$

9. $$\frac{1 + \cos A}{1 - \cos A} = \frac{m^2}{n^2}$$

$$\Rightarrow\ \frac{2\cos^2 A/2}{2\sin^2 A/2} = \frac{m^2}{n^2}$$

($\because$ $1 + \cos 2A = 2\cos^2 A$, $1 - \cos^2 A = 2\sin^2 A$)

$$\Rightarrow\ \tan^2 A/2 = \frac{n^2}{m^2} \Rightarrow \tan A/2 = \pm\frac{n}{m}$$

$$\therefore\ \tan A = \frac{2\tan A/2}{1 - \tan^2 A/2} = \pm\frac{\frac{2n}{m}}{1 - \frac{n^2}{m^2}}$$

$$= \pm\frac{\frac{2n}{m}}{\frac{m^2 - n^2}{m^2}} = \pm\frac{\mathbf{2nm}}{\mathbf{m^2 - n^2}}.$$

10. Since α, β and γ are in A.P.

$\alpha + \gamma = 2\beta$

$$\Rightarrow \beta = \frac{\alpha + \gamma}{2}$$

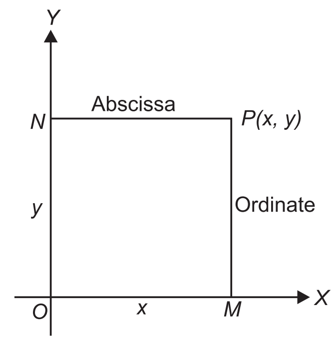
$$\therefore\ \frac{\sin\alpha - \sin\gamma}{\cos\gamma - \cos\alpha} = \frac{2\cos\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right)}{2\sin\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right)}$$

$$= \cot\left(\frac{\alpha+\gamma}{2}\right) = \mathbf{\cot\beta}.$$

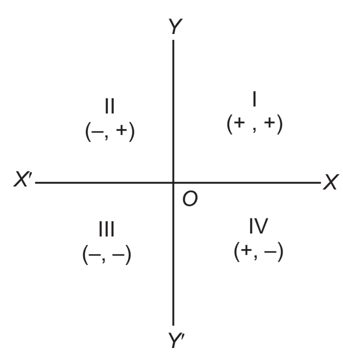
11 Coordinate Geometry

KEY FACTS

1. **Co-ordinate Geometry** is that branch of geometry in which two numbers, called co-ordinates are used to indicate the position of a point in a plane and which make use of algebraic methods in the study of geometric figures.
2. **Rectangular Axes:** The position of a point in a plane is fixed by selecting the axes of reference which are formed by combining the two number scales at right angles to each other so that their zero points coincide. The **horizontal** number scale is called the ***x*-axis** and the **vertical** number scale, the ***y*-axis**. The point where the two scales cross each other is the **origin**. The two together are called **rectangular axes.**
3. **Co-ordinates (Abscissa and Ordinate):** The position of each point of the plane is determined with reference to the rectangular axes by means of a pair of numbers called co-ordinates which are distances of the point from the respective axes. The distance of the point from the *y*-axis is called the ***x*-coordinate** or **abscissa** and the distance of the point from the *x*-axis is called ***y*-coordinate** or **ordinate.**

   The position of point P in the plane with respect to the co-ordinate axes is represented by the ordered pair **(x, y)**, where x is **abscissa** and y is the **ordinate**.
   - **Co-ordinate of the origin are (0, 0).**
   - The co-ordinates of any point on the *x*-axis are **$(x, 0)$** and the co-ordinates of any point of the *y*-axis are **$(0, y)$**
4. **Quadrants:** The co-ordinate axes separate the plane into four regions called the quadrants. By custom the quadrants are numbered I, II, III, IV, in the counter clockwise direction as shown in the figure.

   (*i*) For distances along the *x*-axis, positive values are measured to the right of the origin and negative values to its left.

   (*ii*) For distances along the *y*-axis, positive values are measured upward above *x*-axis and negative values downward below *x*-axis.

   Study the table given below for clear understanding.

Region	Quadrant Name	Value of x and y	Sign of (x, y)
XOY	1st quadrant	$x > 0, y > 0$	(+, +)
YOX′	2nd quadrant	$x < 0, y > 0$	(–, +)
X′OY′	3rd quadrant	$x < 0, y < 0$	(–, –)
XOY′	4th quadrant	$x > 0, y < 0$	(+, –)

5. **Distance Formula:**

   (***a***) The distance d between any two points say $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

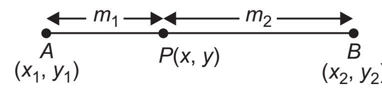
$$\Rightarrow \quad d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \Rightarrow d = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(*b*) The distance of a point $P(x_1, y_1)$ form the origin

$$= \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2} = \sqrt{x_1^2 + y_1^2}$$

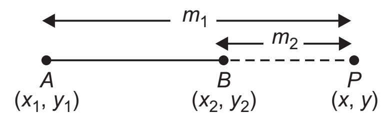
6. Section Formula:

(*a*) **Internal division:** If $P(x, y)$ divides the line segment formed by the joining of the points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ internally in the ratio $m_1 : m_2$. Then



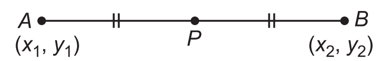
$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2} \quad \text{and} \quad y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

(*b*) **External division:** If $P(x, y)$ divides the line segment formed by the joining of the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $m_1 : m_2$, then



$$x = \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2} \quad \text{and} \quad y = \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}$$

(*c*) **Mid-point formula:** Putting the ratio as $m_1 : m_2 = 1 : 1$, the **co-ordinates of the mid-point** are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

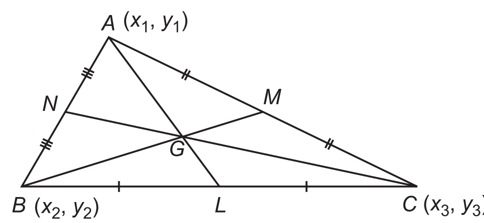


7. Some Important Points Relating to a Triangle:

(*a*) **Centroid of a triangle (*Point of intersection of medians of a* Δ).**

If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of a triangle, then the co-ordinates of its centroid are

$$\left(\frac{(x_1 + x_2 + x_3)}{3}, \frac{(y_1 + y_2 + y_3)}{3}\right)$$

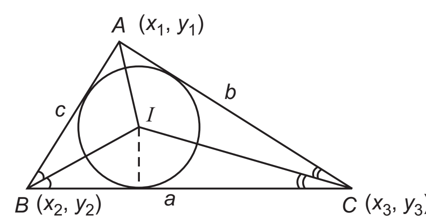


(Centroid G divides median from the vertex side in the ratio $2:1$, *i.e.*, $AG : GL = 2 : 1$)

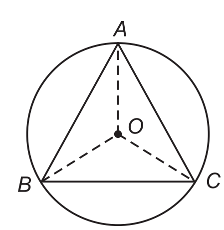
(*b*) **Incentre (*Point of concurrence of the internal bisectors of the angles of a* Δ).**

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle ABC such that $BC = a$, $CA = b$ and $AB = c$, then the co-ordinates of the incentre are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$$

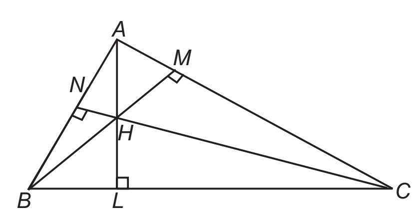


(*c*) **Circumcentre.** If 'O' is the circumcentre of ΔABC, then $OA = OB = OC$ and OA is called the circumradius. So, to find the circumcentre of ΔABC, we use the relation $OA = OB = OC$. This gives two simultaneous liner equations and their solution provides the co-ordinates of the circumcentre.

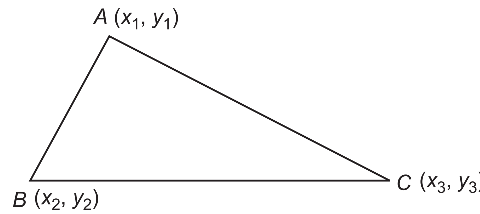


(*d*) **Orthocentre (*Point of concurrence of the altitudes of a* Δ).**

To determine the orthocentre, first we find equations of lines passing through the vertices and perpendicular to the opposite sides. Solving any two of these three equations we get the co-ordinates of the orthocentre.



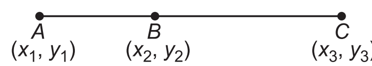
8. **(*a*) Area of a triangle:** The area of a triangle ABC those vertices are given as $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$



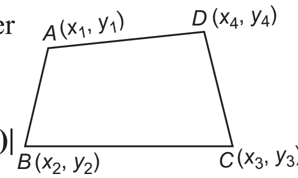
$$= \frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

Condition of collinearity of three points: If three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) be on a straight line, the area of the triangle formed by them is zero, *i.e.*

$$\mathbf{\frac{1}{2}[x_1\,(y_2 - y_3) + x_2\,(y_3 - y_1) + x_3\,(y_1 - y_2)] = 0.}$$



(*b*) Area of a quadrilateral: The area of the quadrilateral where vertices taken in order $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ can be calculated by the formula:

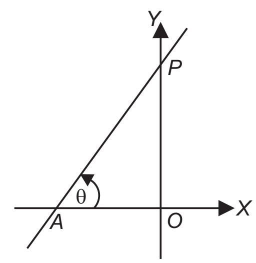


Area of quad. $ABCD = \frac{1}{2}\,|(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)|$

THE STRAIGHT LINE

9. Slope or Gradient of a Straight Line: ***The slope (or gradient) of a line is the tangent of the angle which the part of the line above the x-axis makes with the positive direction of the x-axis. The slope of a line is indicated by the letter 'm'.***

Thus, if AP be the portion of the line above the x-axis and $\angle XAP = \theta$, then $\mathbf{m = \tan\theta}$.



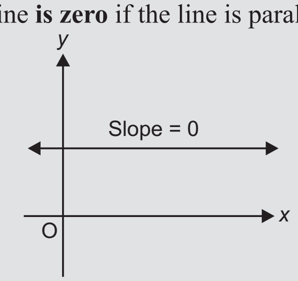
Notes:

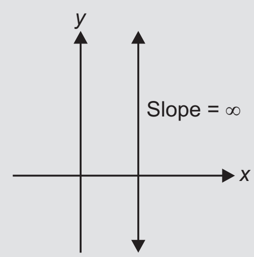
(*a*) θ is measured positively, *i.e.*, in the anti-clockwise direction from 0º to 180º.

(*b*) The slope of a line is **positive** if **it makes an acute angle** in the anti-clockwise direction with the x-axis.

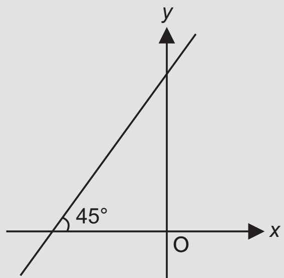
(*c*) The slope of the line is **negative** if the **line makes an obtuse angle** in the anti-clockwise direction with the x-axis.

(*d*) The slope of a line **is zero** if the line is parallel to the x-axis.

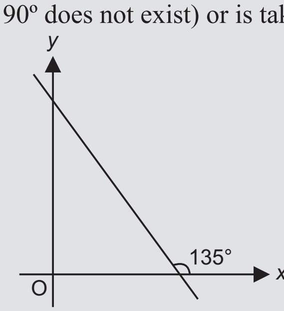




(*e*) The slope of a line parallel to the y-axis is not defined (tan 90º does not exist) or is taken loosely as ∞.



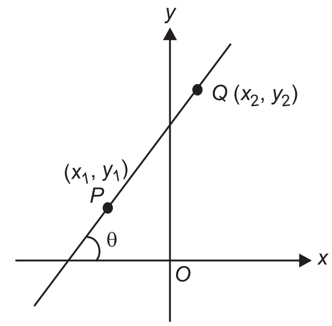
(*i*)



(*ii*)

(*f*) (*i*) If $\theta = 45º$, then $m = 45º = 1$ (*ii*) If $\theta = 135º$, then $m = \tan 135º = -1$

Hence, slope of a line making equal angles with the axes is given by $\boxed{\mathbf{m = \pm 1.}}$



10. Slope of a line joining two points (x_1, y_1) and (x_2, y_2) is given by

$$\boldsymbol{m} = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\textbf{(difference of ordinates)}}{\textbf{(difference of abscissas)}}$$

11. Slope of a line whose equation is $ax + by + c = 0$ is given by

$$m = \frac{-a}{b} = \frac{-\text{ coefficient of } x}{\text{cofficient of } y}$$

12. Angle between two lines whose slopes are given: The angle θ between two lines whose slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\text{difference of slopes}}{1 + \text{product of slopes}}\right|$$

Particular Cases:

(*i*) **Two lines are parallel** when $m_1 = m_2$, *i.e.*, their slopes are equal.

(*ii*) **Two lines are perpendicular** when $m_1 m_2 = -1$, *i.e.*, $m_2 = -\dfrac{1}{m_1}$.

13. Equation of a Straight Line (Different Forms):

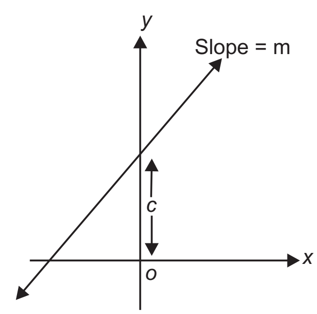


Fig. for (*d*)

(*a*) *Equation of x-axis is* $\boldsymbol{y = 0}$ *and that of y-axis is* $\boldsymbol{x = 0}$*.*

(*b*) *Equation of a line parallel to x-axis is* $\boldsymbol{y = b}$*, where b is the distance of the line from the x-axis.*

(*c*) *Equation of a line perpendicular to y-axis is* $\boldsymbol{x = a}$*, where a is the distance of the line from the y-axis.*

(*d*) *Equation of a line whose slope is* ***m*** *and which cuts off an* ***intercept 'c'*** *from the y-axis is* $\boldsymbol{y = mx + c}$*.*

In particular, equation of a line whose slope is ***m*** *and which passes through the origin is* $\boldsymbol{y = mx}$*.*

(*e*) *The equation of a line whose slope is* ***m*** *and which passes through the* (x_1, y_1) *is* $\boldsymbol{y - y_1 = m(x - x_1)}$*.*

(*f*) *The equation of a line passing through two points* (x_1, y_1) *and* (x_2, y_2) *is* $(y - y_1) = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

(*g*) **Intercept form.** The equation of a line cutting off intercepts **'*a*'** and **'*b*'** respectively from the x-axis and y-axis is $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

The length *PQ* of the line intercepted between the co-ordinates axis = $\sqrt{a^2 + b^2}$.

(*h*) **Perpendicular form:** The equation of the line on which the length of the perpendicular from the origin is '*p*' and the perpendicular makes an angle 'α' with the x-axis is $\boldsymbol{x \cos\alpha + y \sin\alpha = p}$.

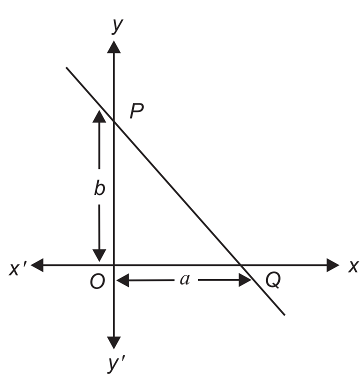


Fig. for (*g*)

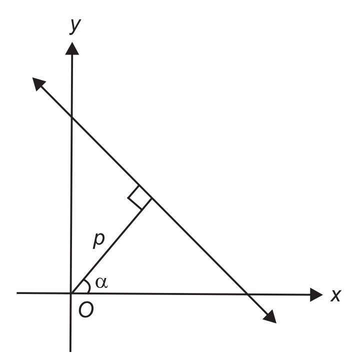


Fig. for (*h*)

14. General Equation of a Straight Line: The general equation of a straight line is $\boldsymbol{ax + by + c = 0}$, where $a, b, c \in R$ and a and b are not simultaneously zero.

(*a*) **To transform $ax + by + c = 0$ to slope-intercept form, *i.e.*, $y = mx + c$.**

$$ax + by + c = 0 \Rightarrow y = \frac{-a}{b}x - \frac{c}{b} \Rightarrow \textbf{Slope} = \frac{-a}{b} \text{ and } y\text{-intercept} = \frac{-c}{b}$$

(*b*) To transform $ax + by + c = 0$ to intercept form $\left(\frac{x}{a}+\frac{y}{b}=1\right)$

$$ax + by + c = 0 \quad \Rightarrow \quad ax + by = -c \quad \Rightarrow \quad \frac{a}{-c}x+\frac{b}{-c}y=1 \quad \Rightarrow \quad \frac{x}{(-c/a)}+\frac{y}{(-c/b)}=1$$

(*c*) To transform $ax + by + c = 0$ to perpendicular form ($x \cos \alpha + y \sin \alpha = p$)

- Divide throughout by $\sqrt{a^2+b^2}$, *i.e.,*

$$ax + by + c = 0 \quad \Rightarrow \quad \frac{a}{\sqrt{a^2+b^2}}x+\frac{b}{\sqrt{a^2+b^2}}y+\frac{c}{\sqrt{a^2+b^2}}=0.$$

- Transpose the constant team to the right hand side and make it positive by changing the sign if necessary.

i.e., $\frac{-a}{\sqrt{a^2+b^2}}x+\frac{-b}{\sqrt{a^2+b^2}}y=\frac{c}{\sqrt{a^2+b^2}}$

where $\cos \alpha = \frac{-a}{\sqrt{a^2+b^2}}$, $\sin \alpha = \frac{-b}{\sqrt{a^2+b^2}}$, $p = \frac{c}{\sqrt{a^2+b^2}}$

15. Identical Lines:

If two equation $ax + by + c = 0$ and $a'x + b'y + c' = 0$ represent the same straight line, then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$, *i.e.,*

ratio of coefficients of x = ratio of coefficients of y = ratio of constant terms

16. Position of Two Points Relative to a Line:

Two points lie on the same side or opposite sides of the given line $ax + by + c = 0$ according as $ax + by + c$ is of the same sign or opposite signs when the co-ordinates of the given points are substituted successively for x and y in $ax + by + c$.

Example: To check whether the points (2, 3) and (1, 3) are on the same side or on opposite sides of the line $x - 2y = -3$.

The equation of the line can be written as $x - 2y + 3 = 0$ ...(*i*)

Substituting co-ordinates (2, 3) on the left side of (*i*), we get

$$2 - 2(3) + 3 = 2 - 6 + 3 = -1$$

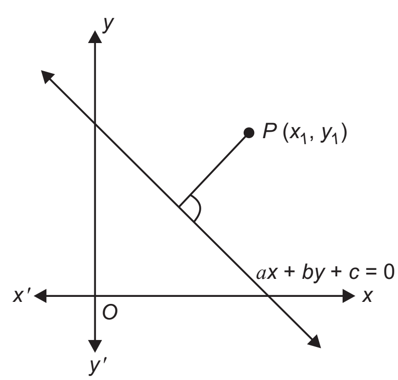
Substituting co-ordinates (1, 3) on the left side of (*i*), we get

$$1 - 2(3) + 3 = 1 - 6 + 4 = -2$$

Since both the results are of the same sign, the points lie on the same side of the line.

17. Intersection of Straight Lines:

The co-ordinates of the point or points of intersection of two lines or two curves satisfy the equations of both of them and so their co-ordinates are obtained by solving their equations simultaneously.



18. Perpendicular distance of a point (x_1, y_1) from the line $ax + by + c = 0$ is given by

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$$

19. Distance between two parallel lines:

Let $ax + by + c = 0$ ...(*i*)

and $ax + by + c' = 0$ be the given parallel lines.

To find the distance between them, choose a point on any one of the lines and find the perpendicular distance from this point on the other line.

A convenient point can be obtained by letting $x = 0$ in $ax + by + c = 0$ and getting $y = \frac{-c}{b}$. Therefore, the point would be $(0, -c/b)$

$\therefore$ Hence required distance between the lines $= \frac{\left|a.0+b.\left(\frac{-c}{b}\right)+c'\right|}{\sqrt{a^2+b^2}} = \frac{|c'-c|}{\sqrt{a^2+b^2}}$.

20. If $u = 0$ and $v = 0$ be two given lines, then the line through their intersection is $\boldsymbol{u + kv = 0}$.

21. Equation of a **line parallel to a given line** $\boldsymbol{ax + by + c = 0}$ is $\boldsymbol{ax + by + k = 0}$, where k is a constant.

22. Equation of a **line perpendicular to a given line** $\boldsymbol{ax + by + c = 0}$ is $\boldsymbol{bx - ay + k = 0}$.

Help-Line

$$ax + by + c = 0 \xrightarrow[\text{obtained on changing the constant term}]{\text{Eqn. of parallel line is}} \boldsymbol{ax + by + k = 0}$$

$$ax + by + c = 0 \xrightarrow[\text{obtained on interchanging the coeffts, changing the sign of one of the } x - y \text{ terms and changing the constant}]{\text{Eqn. of perp. line is}} \boldsymbol{bx - ay + k = 0}$$

SOLVED EXAMPLES

Ex. 1. If the points (2, 1) and (1, –2) are equidistant from the point (x, y), show that $x + 3y = 0$.

Sol. (x, y) is equidistant from the points (2, 1) and (1, –2)

$\Rightarrow$ Distance between (x, y) and (2, 1) = Distance between (x, y) and (1, –2)

$\Rightarrow \sqrt{(x-2)^2 + (y-1)^2} = \sqrt{(x-1)^2 + (y+2)^2} \Rightarrow x^2 - 4x + 4 + y^2 - 2y + 1 = x^2 - 2x + 1 + y^2 + 4y + 4$

$\Rightarrow -4x + 2x - 2y - 4y = 0 \Rightarrow -2x - 6y = 0 \Rightarrow \boldsymbol{x + 3y = 0}$

Ex. 2. Find the co-ordinates of the points on the x-axis which are at a distance of 10 units from the point (– 4, 8)?

Sol. Let the co-ordinates of any point on the x-axis be $(x, 0)$. Then distance between $(x, 0)$ and (– 4, 8) is 10 units.

$\Rightarrow \sqrt{(x+4)^2 + (0-8)^2} = 10 \Rightarrow x^2 + 8x + 16 + 64 = 100 \Rightarrow x^2 + 8x - 20 = 0$

$\Rightarrow (x + 10)(x - 2) = 0 \Rightarrow x = -10$ or 2

$\therefore$ The required points are **(– 10, 0) and (2, 0).**

Ex. 3. Prove that the points (1, –1), $\left(\frac{-1}{2}, \frac{1}{2}\right)$ and (1, 2) are the vertices of an isosceles triangle.

Sol. Let $P(1, -1)$, $Q\left(\frac{-1}{2}, \frac{1}{2}\right)$ and $R(1, 2)$ be the vertices of the ΔPQR.

Then, $PQ = \sqrt{\left(\frac{-1}{2} - 1\right)^2 + \left(\frac{1}{2} + 1\right)^2} = \sqrt{\frac{9}{4} + \frac{9}{4}} = \sqrt{\frac{18}{4}} = \frac{3\sqrt{2}}{2}$

$QR = \sqrt{\left(1 + \frac{1}{2}\right)^2 + \left(2 - \frac{1}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{9}{4}} = \sqrt{\frac{18}{4}} = \frac{3\sqrt{2}}{2}$

$PR = \sqrt{(1-1)^2 + (2+1)^2} = \sqrt{9} = 3$

$\because$ $PQ = QR$, the triangle PQR is isosceles.

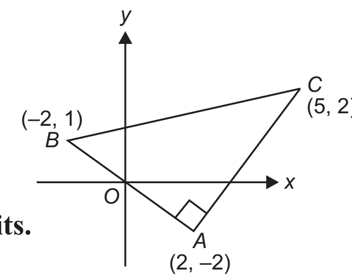
Ex. 4. Prove that the points (2, –2), (–2, 1) and (5, 2) are the vertices of a right angled triangle. Also find the length of the hypotenuse and the area of the triangle.

Sol. Let $A \equiv (2, -2)$, $B \equiv (-2, 1)$, $C \equiv (5, 2)$. Then,

$AB = \sqrt{(-2-2)^2 + (1+2)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

$BC = \sqrt{(5+2)^2 + (2-1)^2} = \sqrt{49 + 1} = \sqrt{50} = 5\sqrt{2}$

$AC = \sqrt{(5-2)^2 + (2+2)^2} = \sqrt{9+16} = \sqrt{25} = 5$



$\Rightarrow$ $AB^2 + AC^2 = BC^2$ as **25 + 25 = 50**

$\therefore$ ΔABC is right angled at A and hyp. $BC = 5\sqrt{2}$ units

Area of $\Delta ABC = \frac{1}{2}$ × base × height $= \frac{1}{2} \times AB \times AC = \frac{1}{2} \times 5 \times 5$ = **12.5 sq. units.**

Ex. 5. Let *P*(–3, 2), *Q*(–5, –5), *R*(2, –3) and *S*(4, 4) be four points in a plane. Then show that *PQRS* is a rhombus. Is it a square ? Also find the area of the rhombus.

Sol. $P \equiv (-3, 2)$, $Q \equiv (-5, -5)$, $R \equiv (2, -3)$, $S \equiv (4, 4)$

$\therefore$ $PQ = \sqrt{(-5+3)^2 + (-5-2)^2} = \sqrt{4+49} = \sqrt{53}$

$QR = \sqrt{(2+5)^2 + (-3+5)^2} = \sqrt{49+4} = \sqrt{53}$

$RS = \sqrt{(4-2)^2 + (4+3)^2} = \sqrt{4+49} = \sqrt{53}$

$PS = \sqrt{(4+3)^2 + (4-2)^2} = \sqrt{49+4} = \sqrt{53}$

$\because$ $PQ = QR = RS = PS$, therefore $PQRS$ is a rhombus. For $PQRS$ to be a square, diagonals PR and QS should be equal.

$PR = \sqrt{(2+3)^2 + (-3-2)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$

$QS = \sqrt{(4+5)^2 + (4+5)^2} = \sqrt{81+81} = \sqrt{162} = 9\sqrt{2}$

As $PR \neq QS$, so $PQRS$ is not a square.

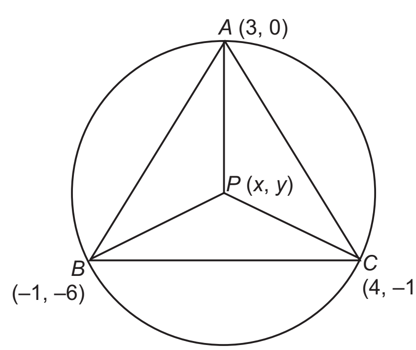
Area of rhombus $= \frac{1}{2}$ × (Product of length of diagonals) $= \frac{1}{2} \times 5\sqrt{2} \times 9\sqrt{2}$ = **45 sq. units.**

Ex. 6. Find the co-ordinates of the circumcentre of the triangle whose vertices are (3, 0), (–1, –6) and (4, –1). Also find its circum-radius.

Sol. The circumcentre of a triangle is equidistant from the vertices of a triangle.

Let $A(3, 0)$, $B(-1, -6)$, and $C(4, -1)$ be the vertices of ΔABC and $P(x, y)$ be the circumcentre of this triangle.

Then, $PA = PB = PC \Rightarrow PA^2 = PB^2 = PC^2$



Now, $PA^2 = PB^2 \Rightarrow (x-3)^2 + (y-0)^2 = (x+1)^2 + (y+6)^2$

$\Rightarrow x^2 - 6x + 9 + y^2 = x^2 + 2x + 1 + y^2 + 12y + 36$

$\Rightarrow -6x + 9 - 2x - 1 - 12y - 36 = 0$

$\Rightarrow 8x + 12y + 28 = 0 \Rightarrow 2x + 3y = -7$ ...(*i*)

$PB^2 = PC^2 \Rightarrow (x+1)^2 + (y+6)^2 = (x-4)^2 + (y+1)^2$

$\Rightarrow x^2 + 2x + 1 + y^2 + 12y + 36 = x^2 - 8x + 16 + y^2 + 2y + 1$

$\Rightarrow 10x + 10y + 20 = 0 \Rightarrow x + y = -2$ ...(*ii*)

Solving eqn (*i*) and eqn (*ii*) simultaneously, we have

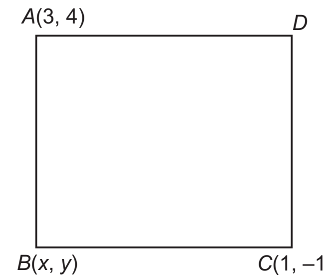
$(i) - 2 \times (ii) \Rightarrow (2x + 3y) - (2x + 2y) = -7 + 4 \Rightarrow y = -3 \quad \Rightarrow x - 3 = -2 \quad \Rightarrow x = 1$

$\therefore$ Co-ordinates of circumcentre $P \equiv$ **(1, –3)**

$\therefore$ Circumradius $= PA = PB = PC = \sqrt{(1-3)^2 + (-3-0)^2} = \sqrt{4+9} = \sqrt{\mathbf{13}}$ **units.**

Ex. 7. Let the opposite angular points of a square be (3, 4) and (1, – 1), Find the co-ordinates of the remaining angular points.

Sol. Let $ABCD$ be the given square and let $A \equiv (3, 4)$ and $C \equiv (1, -1)$. Also let $B \equiv (x, y)$.



$ABCD$ being a square,

$AB = BC \Rightarrow AB^2 = BC^2, \quad \angle ABC = 90°$

$\Rightarrow \left(\sqrt{(x-3)^2+(y-4)^2}\right)^2 = \left(\sqrt{(x-1)^2+(y+1)^2}\right)^2$

$\Rightarrow x^2 - 6x + 9 + y^2 - 8y + 16 = x^2 - 2x + 1 + y^2 + 2y + 1$

$\Rightarrow 4x + 10y - 23 = 0 \Rightarrow x = \frac{23-10y}{4}$ ...(*i*)

Also, in ΔABC, $\angle B = 90° \Rightarrow AB^2 + BC^2 = AC^2$

$\Rightarrow (x-3)^2 + (y-4)^2 + (x-1)^2 + (y+1)^2 = (3-1)^2 + (4+1)^2$

$\Rightarrow x^2 - 6x + 9 + y^2 - 8y + 16 + x^2 - 2x + 1 + y^2 + 2y + 1 = 4 + 25$

$\Rightarrow 2x^2 + 2y^2 - 8x - 6y + 27 = 29 \Rightarrow 2x^2 + 2y^2 - 8x - 6y - 2 = 0 \Rightarrow x^2 + y^2 - 4x - 3y - 1 = 0$ ...(*ii*)

Now substitute the value of x from (*i*), we have

$\left(\frac{23-10y}{4}\right)^2 + y^2 - (23-10y) - 3y - 1 = 0 \Rightarrow (23-10y)^2 + 16y^2 - 16(23-10y) - 48y - 16 = 0$

$\Rightarrow 529 - 460y + 100y^2 + 16y^2 - 368 + 160y - 48y - 16 = 0 \Rightarrow 116y^2 - 348y + 145 = 0$

$\Rightarrow 4y^2 - 12y + 5 = 0 \Rightarrow (2y-1)(2y-5) = 0 \Rightarrow y = \frac{1}{2}$ or $\frac{5}{2}$

Putting $y = \frac{1}{2}$ and $\frac{5}{2}$ respectively in (*i*), we get $x = \frac{9}{2}$ and $x = -\frac{1}{2}$.

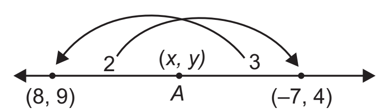
$\therefore$ The required vertices of the square are $\left(\mathbf{\frac{9}{2}, \frac{1}{2}}\right)$ **and** $\left(\mathbf{-\frac{1}{2}, \frac{5}{2}}\right)$

Ex. 8. Find the coordinates of the point which divides the join of the points (8, 9) and (–7, 4) internally in the ratio 2 : 3.

Sol. Let the co-ordinates of the point of internal division A be (x, y). Then,

$x = \frac{2\times(-7)+3\times 8}{2+3} = \frac{-14+24}{5} = \frac{10}{5} = 2$

$y = \frac{2\times 4+3\times 9}{2+3} = \frac{8+27}{5} = \frac{35}{5} = 7$



$\therefore$ Co-ordinates of the point for internal division are **(2, 7).**

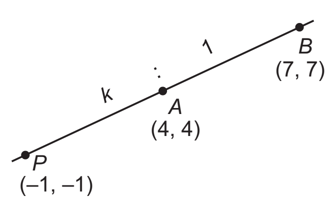
Ex. 9. In what ratio is the line joining the points *A*(4, 4) and *B*(7, 7) divided by *P*(–1, –1)?

Sol. Let P divide AB in the ratio $k : 1$.

Then, co-ordinates of P are $\left(\frac{7k+4}{k+1}, \frac{7k+4}{k+1}\right)$

But $P \equiv (-1, -1)$

$\therefore \frac{7k+4}{k+1} = -1 \Rightarrow 7k + 4 = -k - 1 \Rightarrow 8k = -5 \Rightarrow k = -\frac{5}{8}$

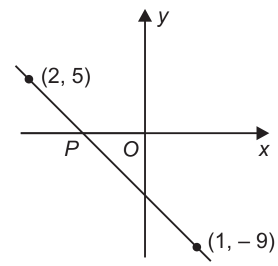


$\because$ k is negative, it means that the division is external.

$\therefore$ AB is divided by P externally in the ratio $\frac{5}{8} : 1$, *i.e.* **5 : 8.**

Ex. 10. Find the ratio in which the *x*-axes divides the line joining the points (–2, 5) and (1, –9) ?

Sol. Let the x-axis divide the line joining the points (–2, 5) and (1, –9) in the ratio $k : 1$. Let the point of division on x-axis is P



Then,

$$x = \frac{k-2}{k+1}, y = \frac{-9k+5}{k+1}$$

As this is a point on the x–axis, $y = 0$

$$\therefore \quad \frac{-9k+5}{k+1} = 0 \quad \Rightarrow 9k = 5 \quad \Rightarrow k = \frac{5}{9}$$

k being positive, the division is internal.

$\therefore$ x-axis divides the given line internally in the ratio **5 : 9.**

Ex. 11. Find the point of trisection of the line segment joining the points (1, 2) and (11, 9) ?

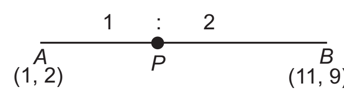
Sol. Let $A(1, 2)$ and $B(11, 9)$ be the given points. Let the points of trisection be P and Q. Then,

$AP = PQ = QB = k$ (say)

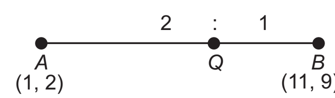
$\Rightarrow \quad AQ = AP + PQ = 2k$ and $PB = PQ + QB = 2k$

$\therefore \quad AP : PB = k : 2k = 1 : 2$ and $AQ : QB = 2k : k = 2 : 1$

$\Rightarrow$ P divides AB internally in the ratio 1 : 2 and Q divides AB internally in the ratio 2 : 1.



$$\therefore \quad \text{Coordinates of } P \text{ are } \left[\frac{1\times 11 + 2\times 1}{1+2}, \frac{1\times 9 + 2\times 2}{1+2}\right], i.e\left(\frac{13}{3}, \frac{13}{3}\right)$$



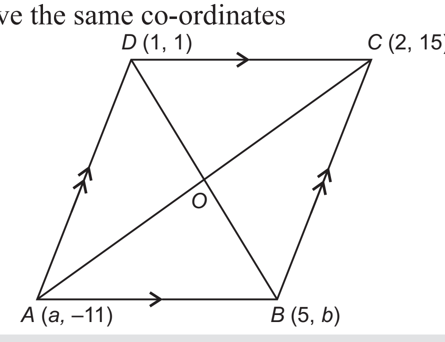
$$\text{Coordinates of } Q \text{ are } \left[\frac{2\times 11 + 1\times 1}{2+1}, \frac{2\times 9 + 1\times 2}{2+1}\right], i.e\left(\frac{23}{3}, \frac{20}{3}\right)$$

Hence, the two points of trisection are $\left(\mathbf{\frac{13}{3}, \frac{13}{3}}\right)$ and $\left(\mathbf{\frac{23}{3}, \frac{20}{3}}\right)$**.**

Ex. 12. If the points *A*(*a*, –11), *B*(5, *b*), *C*(2, 15) and *D*(1, 1) are the vertices of a parallelogram *ABCD*, find the values of *a* and *b*.

Sol. $ABCD$ is a parallelogram, if the mid-points of diagonals AC and BD have the same co-ordinates

($\because$ Diagonals of a parallelogram bisect each other)



$$\text{Co-ordinates of mid-point of } AC \text{ are } \left(\frac{a+2}{2}, \frac{-11+15}{2}\right) = \left(\frac{a+2}{2}, 2\right)$$

$$\text{Co-ordinates of mid-point of } BD \text{ are } \left(\frac{5+1}{2}, \frac{b+1}{2}\right) = \left(3, \frac{b+1}{2}\right)$$

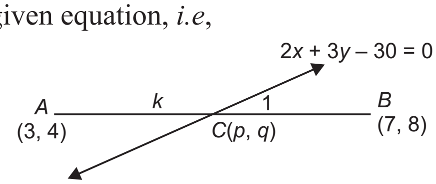
Here, $\frac{a+2}{2} = 3$ and $2 = \frac{b+1}{2}$ $\Rightarrow$ $\boldsymbol{a = 4, b = 3}$**.**

Ex. 13. Determine the ratio in which 2*x* +3*y* – 30 = 0 divides the join of *A*(3, 4) and *B*(7, 8) and at what point?

Sol. Let the line $2x + 3y - 30 = 0$ divide the join of $A(3, 4)$ and $B(7, 8)$ at point $C(p, q)$ in the ratio k **:** 1. Then,

$$p = \frac{7k+3}{k+1}, q = \frac{8k+4}{k+1}$$

As the point C lies on the line $2x + 3y - 30 = 0$, it satisfies the given equation, *i.e*,



$$2\times\left(\frac{7k+3}{k+1}\right) + 3\left(\frac{8k+4}{k+1}\right) - 30 = 0$$

$\Rightarrow \quad 14k + 6 + 24k + 12 - 30k - 30 = 0$

$$\Rightarrow \quad 8k - 12 = 0 \quad \Rightarrow k = \frac{12}{8} = \frac{3}{2}$$

$\therefore$ The line $2x + 3y - 30 = 0$ divides the line joining $A(3, 4)$ and $B(7, 8)$ in the ratio $\frac{3}{2}:1$, *i.e.* **3 : 2 at *C*.**

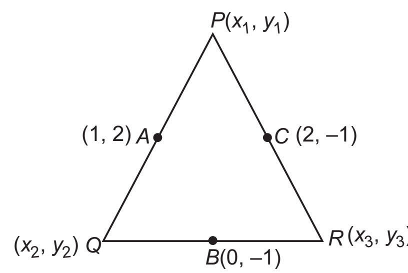
Now the co-ordinates of C are $\left(\dfrac{7\times\frac{3}{2}+3}{3/2+1}, \dfrac{8\times\frac{3}{2}+4}{3/2+1}\right) = \left(\mathbf{\dfrac{27}{5}, \dfrac{32}{5}}\right)$.

Ex. 14. The co-ordinates of mid-points of sides of a triangle are (1, 2), (0, –1) and (2, –1). Find its centroid.

Sol. Let $A(1, 2)$, $B(0, -1)$ and $C(2, -1)$ be the mid-points of the sides PQ, QR and RP of the triangle PQR.

Let the co-ordinates of P, Q and R be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively.

Then, by the mid-point formula.



$\dfrac{x_1 + x_2}{2} = 1$ and $\dfrac{y_1 + y_2}{2} = 2 \quad \Rightarrow x_1 + x_2 = 2$ and $y_1 + y_2 = 4 \quad$ ...(*i*)

$\dfrac{x_1 + x_3}{2} = 2$ and $\dfrac{y_1 + y_3}{2} = -1 \quad \Rightarrow x_1 + x_3 = 4$ and $y_1 + y_3 = -2 \quad$ ...(*ii*)

$\dfrac{x_2 + x_3}{2} = 0$ and $\dfrac{y_2 + y_3}{2} = -1 \quad \Rightarrow x_2 + x_3 = 0$ and $y_2 + y_3 = -2 \quad$ ...(*iii*)

From (*i*), (*ii*) and (*iii*)

$x_1 + x_2 + x_1 + x_3 + x_2 + x_3 = 2 + 4 + 0$ and $y_1 + y_2 + y_1 + y_3 + y_2 + y_3 = 4 + (-2) + (-2)$

$\Rightarrow x_1 + x_2 + x_3 = 3$ and $y_1 + y_2 + y_3 = 0 \quad$ ...(*iv*)

Now, $(x_1 + x_2 + x_3) - (x_1 + x_2) = 3 - 2 \Rightarrow x_3 = 1$ and $(y_1 + y_2 + y_3) - (y_1 + y_2) = 0 - 4 \Rightarrow y_3 = -4$

$(x_1 + x_2 + x_3) - (x_1 + x_3) = 3 - 4 \Rightarrow x_2 = -1$ and $(y_1 + y_2 + y_3) - (y_1 + y_3) = 0 - (-2) \Rightarrow y_2 = 2$

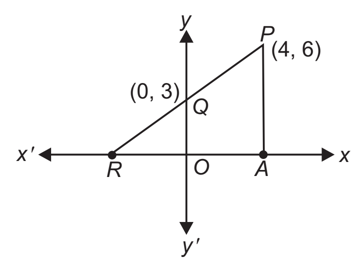
$(x_1 + x_2 + x_3) - (x_2 + x_3) = 3 - 0 \Rightarrow x_1 = 3$ and $(y_1 + y_2 + y_3) - (y_2 + y_3) = 0 - (-2) \Rightarrow y_1 = 2$

$\therefore$ The vertices of the ΔPQR are $P(3, 2)$, $Q(-1, 2)$ and $R(1, -4)$

Hence, co-ordinates of centroid of $\Delta PQR = \left(\dfrac{3+(-1)+1}{3}, \dfrac{2+2+(-4)}{3}\right)$ = **(1, 0).**

Ex. 15. In the adjoining figure, *P* and *Q* have co-ordinates (4, 6)and (0, 3) respectively. Find (*i*) the co-ordinates of *R* (*ii*) Area of quadrilateral *OAPQ*.

Sol. Let $OR = x$ units



(*i*) $\Delta QOR \sim \Delta PAR$

$\Rightarrow \dfrac{PA}{AR} = \dfrac{QO}{OR} \Rightarrow \dfrac{6}{x+4} = \dfrac{3}{x} \Rightarrow \dfrac{x}{x+4} = \dfrac{3}{6} \Rightarrow \dfrac{x}{x+4} = \dfrac{1}{2} \Rightarrow 2x = x + 4 \Rightarrow x = 4$

$\therefore$ Co-ordinates of R are $(-4, 0)$

(*ii*) Quadrilateral $OAPQ$ is a trapezium.

Area of trapezium $OAPQ = \dfrac{1}{2}(OQ + AP) \times OA = \dfrac{1}{2}(3 + 6) \times 4 = \dfrac{1}{2}\times 9\times 4$ = **18 sq. units.**

Ex. 16. Find the co-ordinates of the in-centre of the triangle whose vertices are (–36, 7), (20, 7) and (0, –8).

Sol. Let $A(-36, 7)$, $B(20, 7)$ and $C(0, -8)$ be the vertices of the given triangle.

Then, $a = BC = \sqrt{(0-20)^2 + (-8-7)^2} = \sqrt{400+225} = \sqrt{625} = 25$

$b \; AC = \sqrt{(0+36)^2 + (-8-7)^2} = \sqrt{1296+225} = \sqrt{1521} = 39$

$c \; AB = \sqrt{(20+36)^2 + (7-7)^2} = \sqrt{56^2} = 56.$

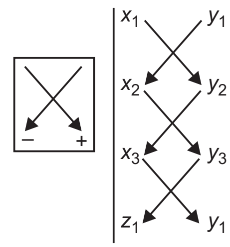
The co-ordinates of the incentre of the ΔABC are

$$\left[\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right] = \left(\frac{25(-36)+39(20)+56\times 0}{25+39+56}, \frac{25(7)+39(7)+56(-8)}{25+39+56}\right)$$

$$= \left(\frac{-900+780}{120}, \frac{175+273-448}{120}\right) = \left(\frac{-120}{120}, \frac{0}{120}\right) = \mathbf{(-1, 0)}$$

Ex. 17. Find the area of a triangle whose vertices are (1, 3), (2, 4) and (5, 6).

Sol. Let $A(x_1, y_1) \equiv (1, 3)$, $B(x_2, y_2) \equiv (2, 4)$, $C(x_3, y_3) \equiv (5, 6)$ be the vertices of ΔABC



Area of $\Delta ABC = \frac{1}{2}\,|\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}|$

$$= \frac{1}{2}\,|\{1(4-6)+2(6-3)+5(3-4)\}| = \frac{1}{2}\,|\{-2+6-5\}| = \mathbf{\frac{1}{2}\ sq.\ units.}$$

Ex. 18. Find the area of the quadrilateral whose vertices are (3, 4), (0, 5), (2, –1) and (3, –2).

Sol. Let $A(x_1, y_1) = (3, 4)$, $B(x_2, y_2) \equiv (0, 5)$, $C(x_3, y_3) \equiv (2, -1)$and $D(x_4, y_4) \equiv (3, -2)$ be the vertices of quadrilateral $ABCD$.

Area of quad. $ABCD = \frac{1}{2}\,|\{(x_1 y_2 - x_2 y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)\}|$

$$= \frac{1}{2}\,|\{(3\times 5 - 0\times 4) + (0\times(-1) - 2\times 5) + (2\times(-2) - 3\times(-1)) + (3\times 4 - 3\times(-2))\}|$$

$$= \frac{1}{2}\,|\{(15-0)+(0-10)+(-4+3)+(12+6)\}|$$

$$= \frac{1}{2}\,|\{15-11+0+18\}| = \frac{1}{2}\times 22 = \mathbf{11\ sq.\ units.}$$

Ex. 19. Show that the points (a, $b + c$), (b, $c + a$), (c, $a + b$) are collinear.

Sol. For three points to be collinear, the area of the triangle formed by the three points should be zero.

$\therefore$ Area of Δ formed by the given three points

$$= \frac{1}{2}\,[a((c+a)-(a+b)) + b((a+b)-(b+c)) + c((b+c)-(c+a))]$$

$$= \frac{1}{2}\,[a(c-b)+b(a-c)+c(b-a)] = \frac{1}{2}\,[ac-ab+ba-bc+cb-ca] = \mathbf{0.}$$

Hence (a, $b + c$), (b, $c + a$) and (c, $a + b$) are collinear.

Ex. 20. The area of a triangle is 5. Two of its vertices are (2, 1) and (3, –2). The third vertex is (x, y) where $y = x + 3$. Find the co-ordinates of the third vertex.

Sol. Let $A \equiv (x, y)$, $B \equiv (2, 1)$, $C \equiv (3, -2)$

Area of $\Delta ABC = \frac{1}{2}\,|\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}|$

$$= \frac{1}{2}\,|\,x(1+2)+2(-2-y)+3(y-1)\,| = \frac{1}{2}\,|\,3x-4-2y+3y-3\,| = \frac{1}{2}\,|\,3x+y-7\,|$$

Given $\frac{1}{2}\,|\,3x+y-7\,| = 5$

$\Rightarrow |\,3x+y-7\,| = 10 \Rightarrow 3x+y-7 = 10$ or $-(3x+y-7) = 10 \Rightarrow 3x+y = 17$ or $3x+y = -3$

Case I. $3x + y = 17$. Also given $y = x + 3$

$\therefore\ 3x + x + 3 = 17 \Rightarrow 4x = 14 \Rightarrow x = \frac{7}{2} \Rightarrow y = \frac{7}{2} + 3 = \frac{13}{2}$

Case II. $3x + y = -3$, $y = x + 3$

$\therefore\ 3x + x + 3 = -3 \Rightarrow 4x = -6 \Rightarrow x = \frac{-3}{2} \Rightarrow y = \frac{-3}{2} + 3 = \frac{3}{2}$

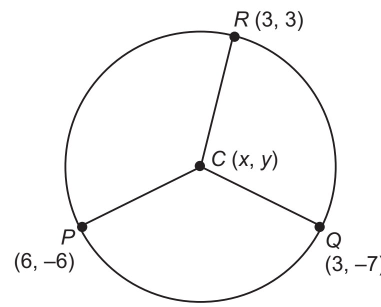
$\therefore$ Co-ordinates of A are $\left(\frac{7}{2}, \frac{13}{2}\right)$ **or** $\left(\frac{-3}{2}, \frac{3}{2}\right)$

Ex. 21. Find the centre of a circle passing through the points (6, –6), (3, –7) and (3, 3).

Sol. Let $C(x, y)$ be the centre of the circle passing through the points $P(6, -6)$, $Q(3, -7)$ and $R(3, 3)$

Then, $PC = QC = RC$(Being radius of the same circle)

$PC^2 = QC^2$



$\Rightarrow (x-6)^2 + (y+6)^2 = (x-3)^2 + (y+7)^2$

$\Rightarrow x^2 - 12x + 36 + y^2 + 12y + 36 = x^2 - 6x + 9 + y^2 + 14y + 49$

$\Rightarrow -12x + 12y + 6x - 14y + 72 - 58 = 0$

$\Rightarrow -6x - 2y + 14 = 0 \Rightarrow 3x + y - 7 = 0$ ...(*i*)

Also, $QC^2 = RC^2$

$\Rightarrow (x-3)^2 + (y+7)^2 = (x-3)^2 + (y-3)^2$

$\Rightarrow x^2 - 6x + 9 + y^2 + 14y + 49 = x^2 - 6x + 9 + y^2 - 6y + 9$

$\Rightarrow 14y + 6y = 9 - 49 \Rightarrow 20y = -40 \Rightarrow y = -2$ ...(*ii*)

Putting $y = -2$ in (*i*), we get

$3x + (-2) - 7 = 0 \Rightarrow 3x = 9 \Rightarrow x = 3$

$\therefore$ The centre is **(3, –2).**

Ex. 22. Find the slope and inclination of the line which passes through the points (1, 2) and (5, 6) ?

Sol. Slope $(m) = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{6-2}{5-1} = \frac{4}{4} = 1$

Also slope $(m) = \tan\theta$, where θ is the inclination of the line to the positive direction of the x-axis in the anti-clockwise direction.

$\tan\theta = 1 \quad \Rightarrow \theta = \tan^{-1}1 = \mathbf{45^o}$.

Ex. 23. Find the value of *a*, if the line passing through (–5, –8) and (3, 0) is parallel to the line passing through (6, 3) and (4, *a*).

Sol. Two lines are parallel if their slopes are equal

$\therefore\ \frac{0-(-8)}{3-(-5)} = \frac{a-3}{4-6} \Rightarrow \frac{8}{8} = \frac{a-3}{-2} \quad \Rightarrow \quad a - 3 = -2 \Rightarrow \boldsymbol{a} = \mathbf{1}.$

Ex. 24. Without using Pythagoras' theorem, show that the points *A* (0, 4), *B*(1, 2) and *C*(3, 3) are the vertices of a right angle triangle.

Sol. Slope of $AB = \frac{2-4}{1-0} = -2$, Slope of $BC = \frac{3-2}{3-1} = \frac{1}{2}$

Slope of $AC = \frac{3-4}{3-0} = -\frac{1}{3}$

Slope of $AB \times$ Slope of $BC = -2 \times \frac{1}{2} = -1$

$\therefore\ AB \perp BC$, *i.e*, $\angle B = 90^o \quad \Rightarrow \quad \Delta ABC$ is a right angled.

Ex. 25. If the points $(x, 1)$, $(1, 2)$ and $(0, y + 1)$ are collinear show that $\frac{1}{x}+\frac{1}{y}=1$.

Sol. There are two ways to prove it.

1st way: Area of triangle formed by the given points = 0 if they are collinear.

$$\therefore \quad \frac{1}{2}\,[x(2-(y+1)) + 1((y+1)-1) + 0(1-2)] = 0$$

$$\Rightarrow \quad \frac{1}{2}\,[2x - xy - x + y + 0] = 0 \quad \Rightarrow x + y - xy = 0 \quad \Rightarrow \quad x + y = xy \quad \Rightarrow \quad \frac{x}{xy}+\frac{y}{xy}=1 \quad \Rightarrow \quad \boldsymbol{\frac{1}{y}+\frac{1}{x}=1.}$$

2nd way: Slope of the lines formed by joining these points are equal.

Slope of line joining $(x, 1)$, $(1, 2)$ = $\frac{(2-1)}{1-x}$

Slope of line joining $(1, 2)$, $(0, y + 1)$ = $\frac{y+1-2}{0-1}$

$$\Rightarrow \quad \frac{1}{1-x}=\frac{y-1}{-1} \Rightarrow -1 = (1-x)\,(y-1) \quad \Rightarrow \quad -1 = y - xy - 1 + x \Rightarrow x + y = xy \Rightarrow \boldsymbol{\frac{1}{x}+\frac{1}{y}=1.}$$

Ex. 26. What is the equation of the line having the y-intercept -1 and parallel to the line $y = 5x - 7$?

Sol. Comparing $y = 5x - 7$ with $y = mx + c$, the slope of given line = m = 5

$\therefore$ Equation of a line parallel to $y = 5x - 7$ having y-intercept = -1 is $\boldsymbol{y = 5x - 1.}$

Ex. 27. Find the equation of the straight line passing through the point (4, 5) and perpendicular to $3x - 2y + 5 = 0$.

Sol. $3x - 2y + 5 = 0 \Rightarrow -2y = -3x - 5 \Rightarrow y = \frac{3}{2}x + \frac{5}{2}$

On comparing with $y = mx + c$, we see that slope of given line = $\frac{3}{2}$

As the required line is perpendicular to the given line,

Slope of required line = $\frac{-2}{3}$

$\therefore$ Equation of required line: $(y - 5) = \frac{-2}{3}\,(x - 4)$

$\Rightarrow 3(y - 5) = -2x + 8 \quad \Rightarrow 3y - 15 = -2x + 8 \quad \Rightarrow \boldsymbol{3y + 2x - 23 = 0}$

Ex. 28. The line segment joining $P(5, -2)$ and $Q(9, 6)$ is divided in the ratio 3 : 1 by a point A on it. Find the equation of a line through the point A parallel to the line $x - 3y + 4 = 0$.

Sol. Co-ordinates of A are $\left(\frac{3\times 9+1\times 5}{3+1}, \frac{3\times 6+1\times -2}{3+1}\right) = \left(\frac{32}{4}, \frac{16}{4}\right)$, *i.e.* (8, 4)

Now, $x - 3y + 4 = 0 \Rightarrow -3y = -x - 4 \Rightarrow y = \frac{x}{3} + \frac{4}{3}$

$\therefore$ Slope of given line = $\frac{1}{3}$

$\Rightarrow$ Slope of required line = $\frac{1}{3}$ (Since lines are parallel)

$\therefore$ Equation of line through (8, 4) with slope $\frac{1}{3}$ is

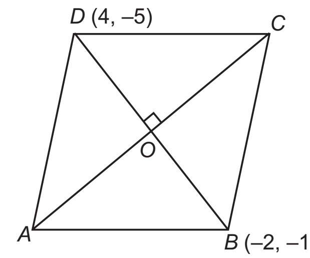
$$(y - 4) = \frac{1}{3}\,(x - 8) \qquad [\text{Using, } y - y_1 = m\,(x - x_1)]$$

$\Rightarrow 3y - 12 = x - 8 \quad \Rightarrow \boldsymbol{3y - x = 4.}$

Ex. 29. (–2, –1) and (4, –5) are the co-ordinates of vertices *B* and *D* respectively of rhombus *ABCD*. Find the equation of the diagonal *AC*.

Sol. Diagonals of a rhombus bisect each other at right angles $\Rightarrow$ Co-ordinates of mid-points of AC and BD are equal

$\therefore \quad 0 \equiv \left(\frac{4+(-2)}{2}, \frac{-5+(-1)}{2}\right) = (1, -3)$



Slope of $BD = \frac{-5+1}{4+2} = \frac{-4}{6} = \frac{-2}{3}$

$\because \quad AC \perp BD$, Slope of $AC = \frac{3}{2}$

$\therefore$ Equation of AC, passing through O (1, –3) having slope $\frac{3}{2}$ is

$$y + 3 = \frac{3}{2}(x-1)$$

$\Rightarrow 2y + 6 = 3x - 3 \Rightarrow \mathbf{2y = 3x - 9.}$

Ex. 30. The line $x - 4y = 6$ is the perpendicular bisector of the segment *AB* and the co-ordinates of *B* are (1, 3). Find the co-ordinates of *A*.

Sol. Let the co-ordinates of A be (h, k).

Now, slope of given line, *i.e.*, $x - 4y = 6$, *i.e.*, $y = \frac{x}{4} + \frac{-6}{4}$ is $\frac{1}{4}$

As $AB \perp$ given line, product of their slopes = –1

Slope of $AB = \frac{3-k}{1-h}$

$\Rightarrow \frac{3-k}{1-h} \times \frac{1}{4} = -1 \Rightarrow 3 - k = -4(1-h) \Rightarrow 3 - k = -4 + 4h \Rightarrow 4h + k = 7$ ...(*i*)

Mid-point of $AB = \left(\frac{h+1}{2}, \frac{k+3}{2}\right)$

This mid-point also lies on the line $x - 4y = 6$

$\therefore \quad \left(\frac{h+1}{2}\right) - 4\left(\frac{k+3}{2}\right) = 6$

$\Rightarrow h + 1 - 4(k+3) = 12 \quad \Rightarrow \quad h - 4k - 11 = 12 \quad \Rightarrow \quad h - 4k = 23$ ...(*ii*)

Now solving (*i*) and (*ii*) simultaneously, we have

From (*i*), $k = 7 - 4h$, putting the value of k in (*ii*), we have

$h - 4(7 - 4h) = 23 \Rightarrow h - 28 + 16h = 23 \quad \Rightarrow \quad 17h = 51 \Rightarrow h = 3$

$\therefore \quad k = 7 - 12 = -5$

$\therefore$ Coordinates of A are (h, k) *i.e.,* **(3, –5).**

Ex. 31. What is the equation of the straight line passing through the point (4, 3) and making intercepts on the co-ordinates axes whose sum is –1 ?

Sol. Let the x-intercept = a. Then y-intercept = $-1 - a$

The equation of the required line is $\frac{x}{a} + \frac{y}{-1-a} = 1$

Given, it passes through (4, 3), so,

$\frac{4}{a} + \frac{3}{-1-a} = 1 \Rightarrow -4 - 4a + 3a = -a - a^2 \quad \Rightarrow \quad a^2 - 4 = 0 \quad \Rightarrow (a+2)(a-2) = 0 \quad \Rightarrow a = -2$, or 2.

When $a = 2$, required line is $\frac{x}{2} - \frac{y}{3} = 1$

When $a = -2$, required line is $\frac{x}{-2} + \frac{y}{3} = 1$

Ex. 32. Find the equation of the line through the point (3, 2) which makes an angle of 45° with the line $x - 2y = 3$? ***(Kerala PET 2007)***

Sol. Given line: $x - 2y = 3 \Rightarrow y = \frac{x}{2} - \frac{3}{2}$ ...(*i*)

$\therefore$ Its slope $= m_1 = \frac{1}{2}$

Let m_2 be the slope of line through (3, 2). Since this line is inclined at 45° to line (*i*),

$$\tan 45^\circ = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{1}{2} - m_2}{1 + \frac{1}{2} m_2}\right|$$

$$\Rightarrow \frac{\frac{1}{2} - m_2}{1 + \frac{1}{2} m_2} = \pm 1 \Rightarrow \frac{1 - 2m_2}{2 + m_2} = \pm 1 \quad \Rightarrow \quad 1 - 2m_2 = 2 + m_2 \quad \text{or} \quad 1 - 2m_2 = -2 - m_2$$

$$\Rightarrow 3m_2 = -1 \quad \text{or} \quad -m_2 = -3 \quad \Rightarrow \quad m_2 = -\frac{1}{3} \quad \text{or} \quad m_2 = 3$$

Since the line passes through (3, 2), the equation of the required line is

$$(y - 2) = -\frac{1}{3}(x - 3) \quad \text{or} \quad (y - 2) = 3(x - 3)$$

$$\Rightarrow 3y - 6 = -x + 3 \quad \text{or} \quad y - 2 = 3x - 9 \quad \Rightarrow \quad \mathbf{3y + x - 9 = 0} \quad \text{or} \quad \mathbf{y - 3x + 7 = 0.}$$

Ex. 33. If p is the length of the perpendicular drawn from the origin to the line $\frac{x}{a} + \frac{y}{b} = 1$, then show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$ **(*NDA/NA 2011*)**

Sol. $\because$ Length of perpendicular from point (x_1, y_1) to line $ax + by + c = 0 = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

$\therefore$ Length of perpendicular from (0, 0) to $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{\left|\frac{1}{a} \times 0 + \frac{1}{b} \times 0 - 1\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = p$

$$\Rightarrow \frac{|-1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = p \Rightarrow \frac{1}{p} = \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} \Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

Ex. 34. Find the perpendicular distance between the lines $9x + 40y - 20 = 0$ and $9x + 40y + 21 = 0$.

Sol. Putting $x = 0$ in equation of one of the lines say $9x + 40y - 20 = 0$, we get $y = \frac{1}{2}$.

$\therefore$ *A* point on $9x + 40y - 20 = 0$ is $\left(0, \frac{1}{2}\right)$

$\therefore$ Distance of $\left(0, \frac{1}{2}\right)$ from $9x + 40y + 21 = 0$ is $\frac{\left|9 \times 0 + 40 \times \frac{1}{2} + 21\right|}{\sqrt{9^2 + 40^2}} = \frac{|41|}{\sqrt{1681}} = \frac{41}{41} = \mathbf{1.}$

Ex. 35. Show that the equation of the parallel line midway between the parallel lines

$$\boldsymbol{ax + by + c_1 = 0 \text{ and } ax + by + c_2 = 0 \text{ is } ax + by + \frac{c_1 + c_2}{2} = 0.}$$

Sol. The two given lines are $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$.

Any line parallel to these two lines and midway between them is

$$ax + by + c = 0 \quad \text{...}(i)$$

Putting $x = 0$, $y = -\frac{c}{b}$ is a point on line (i)

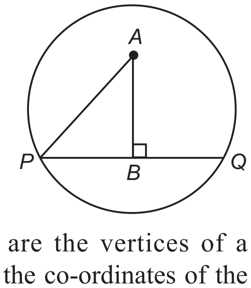
It is equidistant from the given lines but in opposite directions, so

$$\frac{\left|a \times 0 + b \times -\frac{c}{b} + c_1\right|}{\sqrt{a^2 + b^2}} = -\frac{\left|a \times 0 + b \times -\frac{c}{b} + c_2\right|}{\sqrt{a^2 + b^2}} \quad \Rightarrow \quad -c + c_1 = c - c_2 \Rightarrow c = \frac{c_1 + c_2}{2}.$$

$\therefore$ Required equation is $\boldsymbol{ax + by + \frac{c_1 + c_2}{2} = 0}$.

PRACTICE SHEET

1. The point whose abscissa is equal to its ordinate and which is equidistant from $A(–1, 0)$ and $B(0, 5)$ is
   (a) (1, 1) (b) (2, 2) (c) (–2, –2) (d) (3, 3) **(*NDA/NA 2013*)**
2. What is the perimeter of the triangle with the vertices $A(–4, 2)$, $B(0, –1)$ and $C(3, 3)$?
   (a) $7 + 3\sqrt{2}$ (b) $10 + 5\sqrt{2}$ (c) $11 + 6\sqrt{2}$ (d) $5 + \sqrt{2}$ **(*NDA/NA 2012*)**
3. The point P is equidistant from $A(1, 3)$, $B(–3, 5)$ and $C(5, –1)$. Then PB is equal to :
   (a) $5\sqrt{2}$ (b) 5 (c) $5\sqrt{5}$ (d) $5\sqrt{10}$
4. If $M(x, y)$ is equidistant from $A(a + b, b – a)$ and $B(a – b, a + b)$, then
   (a) $bx = ay$ (b) $ax = by$ (c) $a = b$ (d) $x = y$
5. The points (a, a), $(–a, –a)$ and $(-\sqrt{3a}, +\sqrt{3a})$ are the vertices of.......triangle whose area is.......
   (a) Isosceles, $2\sqrt{2}a^2$ sq. units
   (b) Equilateral, $2\sqrt{3}a^2$ sq. units
   (c) Scalene, $4\sqrt{3}a^2$ sq. units
   (d) None of these
6. The coordinates of the circumcentre of the triangle whose vertices are (8, 6), (8, –2) and (2, –2).
   (a) (5, 2) (b) (–2, –5) (c) (0, 0) (d) (5, 0)
7. Name the quadrilateral $ABCD$, the coordinates of whose vertices are $A(3, 5)$, $B(1, 1)$, $C(5, 3)$ and $D(7, 7)$.
   (a) Square (b) Rhombus (c) Rectangle (d) Trapezium
8. The vertices $A(4, 5)$, $B(7, 6)$, $C(4, 3)$ and $D(1, 2)$ from the quadrilateral $ABCD$ whose special name is
   (a) Rectangle (b) Square
   (c) Parallelogram (d) Rhombus
9. $ABCD$ is a rectangle formed by the points $A(–1, –1)$, $B(–1, 4)$, $C(5, 4)$ and $D(5, –1)$. P, Q, R and S are mid-points of AB, BC, CD and DA respectively. The quadrilateral $PQRS$ is a
   (a) Square (b) Rectangle
   (c) Rhombus (d) None of these
10. The points $P(12, 8)$, $Q(–2, a)$ and $R(6, 0)$ are the vertices of a right angled triangle where $\angle R = 90º$. The value of a is
    (a) 5 (b) – 6 (c) – 5 (d) 6
11. Point $A(5, 1)$ is the centre of the circle with radius 13 units. $AB \perp$ chord PQ. B is (2, –3). The length of chord PQ is

    (a) 10 units (b) 20 units
    (c) 12 units (d) 24 units
12. If $A(3, 5)$, $B(– 5, – 4)$, $C(7, 10)$ are the vertices of a parallelogram taken in order, then the co-ordinates of the fourth vertex are:
    (a) (10, 12) (b) (12, 17) (c) (15, 19) (d) (18, 21) **(*IIT 1998*)**
13. The points $A(2, 3)$, $B(3, 5)$, $C(7, 7)$ and $D(5, 6)$ are such that:
    (a) A, B, C and D are collinear
    (b) $ABCD$ is a parallelogram
    (c) D lies inside ΔABC
    (d) D lies on the boundary of ΔABC **(*AMU 2007*)**
14. (0, –1) and (0, 3) are the two opposite vertices of a square. The other two vertices are:
    (a) (0, 1), (0, – 3) (b) (2, 1) (– 2, 1)
    (c) (3, – 1), (0, 0) (d) (2, 2) (1, 1) **(*KCET 2005*)**
15. Find the coordinates of the point which divides externally the join of the points (3, 4) and (– 6, 2) in the ratio 3 : 2.
    (a) (–24, –2) (b) (–24, –4) (c) (21, 8) (d) (24, 2)
16. In what ratio is the line joining the points (2, –3) and (5, 6) divided by the x-axis.
    (a) 1 : 3 (b) 1 : 2 (c) 2 : 1 (d) 3 : 1

17. The ratio in which the line $3x + 4y = 7$ divides the line joining the points $(-2, 1)$ and $(1, 2)$ is

(*a*) 2 : 3 (*b*) 4 : 5 (*c*) 7 : 3 (*d*) 9 : 4

(*Gujarat CET 2008*)

18. Determine the ratio in which the point $P(m, 6)$ divides the join of $A(-4, 3)$ and $B(2, 8)$. Also find the value of m.

(*a*) $2 : 5\ ;\ m = 3$ (*b*) $3 : 2\ ;\ m = -\frac{2}{5}$

(*c*) $1 : 2\ ;\ m = \frac{1}{4}$ (*d*) $2 : 1\ ;\ m = \frac{2}{3}$

19. For an equilateral triangle ΔABC with vertices, $A(1, 2)$, $B(2, 3)$, its incentre is $\left(\frac{9+\sqrt{3}}{6}, \frac{15-\sqrt{3}}{6}\right)$. The coordinates of vertex C are:

(*a*) $\left(\frac{3-\sqrt{3}}{2}, \frac{5+\sqrt{3}}{2}\right)$ (*b*) $\left(\frac{3-\sqrt{3}}{6}, \frac{5+\sqrt{3}}{6}\right)$

(*c*) $\left(\frac{3+\sqrt{3}}{2}, \frac{5-\sqrt{3}}{2}\right)$ (*d*) $\left(\frac{3+\sqrt{3}}{6}, \frac{5-\sqrt{3}}{6}\right)$

(*Gujarat CET 2007*)

20. If the co-ordinates of the mid-points of the sides of a triangle are $(1, 1)$, $(2, -3)$, $(3, 4)$, find its incentre.

(*a*) $(0, 0)$

(*b*) $\left(\frac{2\sqrt{13}+20\sqrt{2}}{\sqrt{13}+\sqrt{17}+5\sqrt{2}}, \frac{8\sqrt{13}-6\sqrt{17}}{\sqrt{13}+\sqrt{17}+5\sqrt{2}}\right)$

(*c*) $\left(\frac{2\sqrt{2}+20\sqrt{5}}{\sqrt{2}+\sqrt{5}+\sqrt{13}}, \frac{8\sqrt{2}-10\sqrt{13}}{\sqrt{2}+\sqrt{5}+\sqrt{13}}\right)$

(*d*) $(-2, 2)$

21. If $(0, 0)$ and $(2, 0)$ are the two vertices of a triangle whose centroid is $(1, 1)$, then the area of the triangle is:

(*a*) 1 (*b*) $\frac{\sqrt{3}}{2}$ (*c*) 2 (*d*) 3

(*Kerala PET 2002*)

22. If the three points $(k, 2k)$, $(2k, 3k)$ and $(3, 1)$ are collinear then k is equal to

(*a*) -2 (*b*) $-\frac{1}{2}$ (*c*) $\frac{1}{2}$ (*d*) $\frac{3}{2}$

23. If the points $A(1, 2)$, $B(0, 0)$ and $C(a, b)$ are collinear, then

(*a*) $a = b$ (*b*) $a = 2b$ (*c*) $a = -b$ (*d*) $2a = b$

24. The vertices of a ΔABC are $A(4, 6)$, $B(1, 5)$ and $C(7, 2)$. A line is drawn to intersect sides AB and AC at D and E respectively such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$. Then ratio, $\frac{\text{Area of } \Delta ADE}{\text{Area of } \Delta ABC}$ is

(*a*) 1 : 2 (*b*) 1 : 4 (*c*) 1 : 8 (*d*) 1 : 16

25. If the area of the quadrilateral whose angular points A, B, C, D taken in order are $(1, 2)$, $(-5, 6)$, $(7, -4)$ and $(-2, k)$ be zero, then the value of k is

(*a*) 1 (*b*) 3 (*c*) -3 (*d*) -1

26. A, B, C are the points $(-1, 5)$, $(3, 1)$ and $(5, 7)$ respectively. D, E, F are the mid-points of BC, CA and AB respectively. Then, Area of ΔABC : Area of ΔDEF is equal to:

(*a*) 2 : 1 (*b*) 3 : 1 (*c*) 4 : 1 (*d*) 8 : 1

27. The line $3x + 4y - 24 = 0$ cuts the x-axis at A and y-axis at B. Then the in centre of the ΔAOB, where O is the origin is

(*a*) $(1, 2)$ (*b*) $(2, 2)$ (*c*) $(2, 12)$ (*d*) $(12, 12)$

(*Rajasthan PET 2006*)

28. The slope of a line perpendicular to the line which passes through the points $(-k, h)$ and $(b, -f)$ is

(*a*) -1 (*b*) $\frac{f-h}{b+k}$ (*c*) $\frac{b+k}{f+h}$ (*d*) $\frac{-b+k}{f-h}$

29. The medians AD and BE of the triangle with vertices $A(0, b)$, $B(0, 0)$ and $C(a, 0)$ are mutually perpendicular if

(*a*) $b = -\sqrt{2}a$ (*b*) $a = \pm\sqrt{2}b$

(*c*) $b = \sqrt{2}a$ (*d*) $a = b$

30. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is

(*a*) $2x - 9y - 11 = 0$ (*b*) $2x - 9y - 7 = 0$

(*c*) $2x + 9y - 11 = 0$ (*d*) $2x + 9y + 7 = 0$

(*IIT 2000*)

31. What are the co-ordinates of the foot of the perpendicular from the point $(2, 3)$ on the line $x + y - 11 = 0$?

(*a*) $(2, 9)$ (*b*) $(5, 6)$ (*c*) $(-5, 6)$ (*d*) $(6, 5)$

(*NDA/NA 2011*)

32. What is the equation of the line joining the origin with the point of intersection of the lines $4x + 3y = 12$ and $3x + 4y = 12$?

(*a*) $x + y = 1$ (*b*) $x - y = 1$ (*c*) $3y = 4x$ (*d*) $x = y$

(*NDA/NA 2011*)

33. If $(-5, 4)$ divides the line segment between the co-ordinate axes in the ratio 1 : 2, then what is its equation ?

(*a*) $8x + 5y + 20 = 0$ (*b*) $5x + 8y - 7 = 0$

(*c*) $8x - 5y + 60 = 0$ (*d*) $5x - 8y + 57 = 0$.

(*NDA/NA 2010*)

34. If $x \cos\theta + y \sin\theta = 2$ is perpendicular to the line $x - y = 3$, then what is the value of θ ?

(*a*) $\pi/6$ (*b*) $\pi/4$ (*c*) $\pi/2$ (*d*) $\pi/3$

(*NDA/NA 2009*)

35. The co-ordinates of P and Q are $(-3, 4)$ and $(2, 1)$ respectively. If PQ is extended to R such that $PR = 2QR$, then what are the co-ordinates of R ?

(*a*) $(3, 7)$ (*b*) $(2, 4)$ (*c*) $\left(-\frac{1}{2}, \frac{5}{2}\right)$ (*d*) $(7, -2)$

(*NDA/NA 2007*)

36. If the points with the co-ordinates $\{a, ma\}$, $\{b, (m + 1)b\}$, $\{c, (m + 2)c\}$ are collinear, then which of the following is correct ?

(*a*) a, b, c are in A.P. only for all m

(*b*) a, b, c are in G.P. only for all m

(*c*) a, b, c are in H.P. only for all m

(*d*) a, b, c are in A.P. only for all $m = 1$ **(*NDA/NA 2007*)**

37. The line through the points (4, 3) and (2, 5) cuts off intercepts of lengths λ and μ on the axes. Which one of the following is correct ?

(*a*) $\lambda > \mu$ (*b*) $\lambda < \mu$ (*c*) $\lambda > -\mu$ (*d*) $\lambda = \mu$

(*NDA/NA 2008*)

38. The equation of the line bisecting the join of (3, – 4) and (5, 2) and having its intercepts on the x-axis and y-axis in the ratio 2 : 1 is

(*a*) $2x - y = 9$ (*b*) $x - 2y = 4$ (*c*) $2x + y = 7$ (*d*) $x + 2y = 2$

(*KCET 2005*)

39. A line passes through the point of intersection of the lines $100x + 50y - 1 = 0$ and $75x + 25y + 3 = 0$ and makes equal intercepts on the axes. Its equation is

(*a*) $25x - 25y + 6 = 0$ (*b*) $5x - 5y + 3 = 0$

(*c*) $25x + 25y - 4 = 0$ (*d*) $5x + 5y - 7 = 0$

(*Kerala PET 2008*)

40. The image of the origin with reference to the line $4x + 3y - 25 = 0$ is

(*a*) (– 4, –3) (*b*) (– 3, 4) (*c*) (8, 6) (*d*) (6, – 8)

41. What is the angle between the lines whose equations are: $3x + y - 7 = 0$ and $x + 2y + 9 = 0$.

(*a*) 45º (*b*) $\tan^{-1}\left(\frac{5}{2}\right)$ (*c*) 135º (*d*) $\tan^{-1}\left(\frac{3}{4}\right)$

42. Find the equation of the line which passes through the point of intersection of the lines $2x - y + 5 = 0$ and $5x + 3y - 4 = 0$ and is perpendicular to the line $x - 3y + 21 = 0$.

(*a*) $2x + y + 10 = 0$ (*b*) $3x + y + 21 = 0$

(*c*) $3x + y = 0$ (*d*) $3y - x + 21 = 0$

43. What is the acute angle between the lines $Ax + By = A + B$ and $A(x - y) + B(x + y) = 2B$?

(*a*) 45º (*b*) $\tan^{-1}\left(\frac{A}{\sqrt{A^2 + B^2}}\right)$

(*c*) $\tan^{-1}\left(\frac{B}{\sqrt{A^2 + B^2}}\right)$ (*d*) 60º **(*NDA/NA 2007*)**

44. The line L is given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point (13, 32). The line K is parallel to L and has the equation $\frac{x}{c} + \frac{y}{3} = 1.$ Then the distance between L and K is

(*a*) $\sqrt{17}$ (*b*) $\frac{17}{\sqrt{15}}$ (*c*) $\frac{23}{\sqrt{15}}$ (*d*) $\frac{23}{\sqrt{17}}$

(*AIEEE 2010*)

45. A straight line passes through the points (a, 0) and (0, b). The length of the line segment contained between the axes is 13 and the product of the intercepts is 60. Find the equation of the straight line

(*a*) $5x + 12y = 60$ (*b*) $7x - 12y = 50$

(*c*) $5x + 12y + 60 = 0$ (*d*) Both (*a*) and (*c*)

46. Find the equation of the straight line with a positive gradient which passes through the point (–5, 0) and is at a perpendicular distance of 3 units from the origin.

(*a*) $3x + 4y - 15 = 0$ (*b*) $4x - 3y + 15 = 0$

(*c*) $3x - 4y + 15 = 0$ (*d*) $3y - x + 10 = 0$

47. A straight line is parallel to the lines $3x - y - 3 = 0$ and $3x - y + 5 = 0$ and lies between them. Find its equation if its distances from these lines are in the ratio 3 : 5.

(*a*) $3x - y + 10 = 0$ (*b*) $3x - y = 0$

(*c*) $3x - y = 0$ (*d*) $3y - x - 10 = 0$

48. The point A(0, 0), B(1, 7) and C(5, 1) are the vertices of a triangle. Find the length of the perpendicular from A to BC and hence the area of the $\Delta\ ABC$.

(*a*) $\frac{17}{\sqrt{84}}$; 17 sq. units (*b*) $\frac{17}{\sqrt{84}}$; 13 sq. units

(*c*) $\frac{17}{\sqrt{13}}$; 13 sq. units (*d*) $\frac{17}{\sqrt{13}}$; 17 sq. units

49. If p be the length of the perpendicular from the origin on the straight line $ax + by = p$ and $b = \frac{\sqrt{3}}{2},$ then what is the angle between the perpendicular and the positive direction of x ?

(*a*) 30º (*b*) 45º (*c*) 60º (*d*) 90º

(*NDA/NA 2007*)

50. The orthocentre of a triangle whose vertices are (0, 0), (3, 0) and (0, 4) is

(*a*) (2, 1) (*b*) (–1, 0) (*c*) (0, 1) (*d*) (0, 0)

(*Rajasthan PET*)

ANSWERS

1. (*b*)	**2.** (*b*)	**3.** (*d*)	**4.** (*a*)	**5.** (*b*)	**6.** (*a*)	**7.** (*b*)	**8.** (*c*)	**9.** (*c*)	**10.** (*d*)
11. (*d*)	**12.** (*c*)	**13.** (*d*)	**14.** (*b*)	**15.** (*a*)	**16.** (*b*)	**17.** (*d*)	**18.** (*b*)	**19.** (*c*)	**20.** (*b*)
21. (*d*)	**22.** (*a*)	**23.** (*d*)	**24.** (*d*)	**25.** (*b*)	**26.** (*c*)	**27.** (*b*)	**28.** (*c*)	**29.** (*b*)	**30.** (*d*)
31. (*b*)	**32.** (*d*)	**33.** (*c*)	**34.** (*b*)	**35.** (*d*)	**36.** (*c*)	**37.** (*d*)	**38.** (*d*)	**39.** (*c*)	**40.** (*c*)
41. (*a*)	**42.** (*c*)	**43.** (*a*)	**44.** (*d*)	**45.** (*d*)	**46.** (*c*)	**47.** (*b*)	**48.** (*d*)	**49.** (*c*)	**50.** (*d*)

HINTS AND SOLUTIONS

1. Let the point be P whose abscissa = ordinate = a.

$\therefore P \equiv (a, a)$

Given, $PA = PB$

$\Rightarrow (a+1)^2 + a^2 = a^2 + (a-5)^2$

$\Rightarrow 2a^2 + 2a + 1 = 2a^2 - 10a + 25$

$\Rightarrow 12a = 24 \Rightarrow a = 2.$

$\therefore$ The point is **(2, 2).**

2. Perimeter of $\Delta ABC = AB + BC + CA$

$$= \sqrt{(0+4)^2 + (-1-2)^2} + \sqrt{(3-0)^2 + (3+1)^2} + \sqrt{(3+4)^2 + (3-2)^2}$$

$$= \sqrt{16+9} + \sqrt{9+16} + \sqrt{49+1}$$

$$= \sqrt{25} + \sqrt{25} + \sqrt{50} = 5 + 5 + 5\sqrt{2} = \mathbf{10 + 5\sqrt{2}}.$$

3. Let $P \equiv (x, y)$. Then, $PA = PB$ and $PB = PC$.

$\therefore PA^2 = PB^2 \Rightarrow (x-1)^2 + (y-3)^2 = (x+3)^2 + (y-5)^2$

$\Rightarrow x^2 - 2x + 1 + y^2 - 6y + 9 = x^2 + 6x + 9 + y^2 - 10y + 25$

$\Rightarrow -8x + 4y - 24 = 0 \Rightarrow 2x - y + 6 = 0$ ...(*i*)

$PB^2 = PC^2 \Rightarrow (x+3)^2 + (y-5)^2 = (x-5)^2 + (y+1)^2$

$\Rightarrow x^2 + 6x + 9 + y^2 - 10y + 25 = x^2 - 10x + 25 + y^2 + 2y + 1$

$\Rightarrow 16x - 12y + 8 = 0 \Rightarrow 4x - 3y + 2 = 0$ ...(*ii*)

From (*i*), $y = 2x + 6$. Putting in (*ii*), we have

$4x - 3(2x + 6) + 2 = 0$

$\Rightarrow 4x - 6x - 18 + 2 = 0 \Rightarrow -2x - 16 = 0 \Rightarrow x = -8$

$\therefore y = 2 \times (-8) + 6 = -10.$

$\therefore PB = \sqrt{(-8+3)^2 + (-10-5)^2}$

$$= \sqrt{(-5)^2 + (-15)^2} = \sqrt{25 + 225} = \sqrt{250} = \mathbf{5\sqrt{10}}.$$

4. Given, $AM = BM$

$\Rightarrow AM^2 = BM^2$

$\Rightarrow [x - (a+b)]^2 + [y - (b-a)]^2 = [x - (a-b)]^2 + (y - (a+b))^2$

$$\Rightarrow \cancel{x^2} - 2(a+b)x + \cancel{(a+b)^2} + \cancel{y^2} - 2(b-a)y + \cancel{(b-a)^2}$$

$$= \cancel{x^2} - 2(a-b)x + \cancel{(a-b)^2} + \cancel{y^2} - 2(a+b)y + \cancel{(a+b)^2}$$

$$\Rightarrow -\cancel{2ax} - 2bx - \cancel{2by} + 2ay = -\cancel{2ax} + 2bx - 2ay - \cancel{2by}$$

$\Rightarrow -4bx = -4ay \Rightarrow$ **$bx = ay$.**

5. Let $A(a, a)$, $B(-a, -a)$ and $C\ (-\sqrt{3}a, \sqrt{3}a)$ be the vertices of ΔABC. Then,

$$AB = \sqrt{(a+a)^2 + (a+a)^2} = \sqrt{4a^2 + 4a^2}$$

$$= \sqrt{8a^2} = 2\sqrt{2}a$$

$$BC = \sqrt{(-a+\sqrt{3}a)^2 + (-a-\sqrt{3}a)^2}$$

$$= \sqrt{a^2 - 2\sqrt{3}a + 3a^2 + a^2 + 2\sqrt{3}a + 3a^2} = \sqrt{8a^2} = 2\sqrt{2}a$$

$$AC = \sqrt{(a+\sqrt{3}a)^2 + (a-\sqrt{3}a)^2}$$

$$= \sqrt{a^2 + 2\sqrt{3}a + 3a^2 + a^2 - 2\sqrt{3}a + 3a^2} = \sqrt{8a^2} = 2\sqrt{2}a$$

$\because AB = BC = AC$, ΔABC is equilateral.

$$\text{Area} = \frac{\sqrt{3}}{4}\ (\text{side})^2 = \frac{\sqrt{3}}{4} \times (2\sqrt{2}a)^2$$

$$= \frac{\sqrt{3}}{4} \times 8a^2 = \mathbf{2\sqrt{3}a^2}.$$

6. Let $A(8, 6)$, $B(8, -2)$ and $C(2, -2)$ be the vertices of the given triangle and $P(x, y)$ be the circum-centre of this triangle. Then, $PA^2 = PB^2 = PC^2$

Now, $PA^2 = PB^2 \Rightarrow (x-8)^2 + (y-6)^2 = (x-8) + (y+2)^2$

$\Rightarrow x^2 - 16x + 64 + y^2 - 12y + 36 = x^2 - 16x + 64 + y^2 + 4y + 4$

$\Rightarrow 16y = 32 \Rightarrow y = 2.$

Now, $PB^2 = PC^2 \Rightarrow (x-8)^2 + (y+2)^2 = (x-2)^2 + (y+2)^2$

$\Rightarrow x^2 - 16x + 64 + y^2 + 4y + 4 = x^2 - 4x + 4 + y^2 + 4y + 4$

$\Rightarrow 12x = 60 \Rightarrow x = 5.$

$\therefore$ Co-ordinates of the circumcentre are **(5, 2).**

7. $AB = \sqrt{(3-1)^2 + (5-1)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

$$BC = \sqrt{(1-5)^2 + (1-3)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

$$CD = \sqrt{(5-7)^2 + (3-7)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

$$AD = \sqrt{(3-7)^2 + (5-7)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

$$AC = \sqrt{(3-5)^2 + (5-3)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$BD = \sqrt{(1-7)^2 + (1-7)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$$

Now, $AB = BC = CD = AD \Rightarrow$ All sides are equal

Also, $AC \neq BD \Rightarrow$ Diagonals are not equal.

$\Rightarrow ABCD$ is a **rhombus.**

8. $AB = \sqrt{(4-7)^2 + (5-6)^2} = \sqrt{9+1} = \sqrt{10}$

$$BC = \sqrt{(7-4)^2 + (6-3)^2} = \sqrt{9+9} = 3\sqrt{2}$$

$$CD = \sqrt{(4-1)^2 + (3-2)^2} = \sqrt{9+1} = \sqrt{10}$$

$$AD = \sqrt{(4-1)^2 + (5-2)^3} = \sqrt{9+9} = 3\sqrt{2}$$

$$AC = \sqrt{(4-4)^2 + (5-3)^2} = \sqrt{4} = 2$$

$BD = \sqrt{(7-1)^2 + (6-2)^2} = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$

$AB = CD$, $BC = AD$ and $AC \neq BD \Rightarrow$ opposite sides are equal and diagonals are not equal.

$\Rightarrow$ $ABCD$ is a **parallelogram.**

9. Co-ordinates of P are $\left(\dfrac{-1-1}{2}, \dfrac{-1+4}{2}\right)$ *i.e.* $\left(-1, \dfrac{3}{2}\right)$

Co-ordinates of Q are $\left(\dfrac{-1+5}{2}, \dfrac{4+4}{2}\right)$, *i.e*, (2, 4)

Co-ordinates of R are $\left(\dfrac{5+5}{2}, \dfrac{4-1}{2}\right)$, *i.e*, $\left(5, \dfrac{3}{2}\right)$

Co-ordinates of S are $\left(\dfrac{-1+5}{2}, \dfrac{-1-1}{2}\right)$, *i.e*, (2, – 1)

Now, $PQ = \sqrt{(2+1)^2 + \left(4 - \dfrac{3}{2}\right)^2} = \sqrt{3^2 + \left(\dfrac{5}{2}\right)^2}$

$= \sqrt{9 + \dfrac{25}{4}} = \sqrt{\dfrac{36+25}{4}} = \sqrt{\dfrac{61}{4}}$

$QR = \sqrt{(5-2)^2 + \left(\dfrac{3}{2} - 4\right)^2} = \sqrt{3^2 + \left(\dfrac{-5}{2}\right)^2}$

$= \sqrt{9 + \dfrac{25}{4}} = \sqrt{\dfrac{36+25}{4}} = \sqrt{\dfrac{61}{4}}$

$RS = \sqrt{(2-5)^2 + \left(-1 - \dfrac{3}{2}\right)^2} = \sqrt{(-3)^2 + \left(\dfrac{-5}{2}\right)^2}$

$= \sqrt{9 + \dfrac{25}{4}} = \sqrt{\dfrac{36+25}{4}} = \sqrt{\dfrac{61}{4}}$

$SP = \sqrt{(-1-2)^2 + \left(\dfrac{3}{2} + 1\right)^2} = \sqrt{(-3)^2 + \left(\dfrac{5}{2}\right)^2}$

$= \sqrt{9 + \dfrac{25}{4}} = \sqrt{\dfrac{36+25}{4}} = \sqrt{\dfrac{61}{4}}$

$\Rightarrow$ $PQ = QR = RS = SP \Rightarrow$ All sides are equal

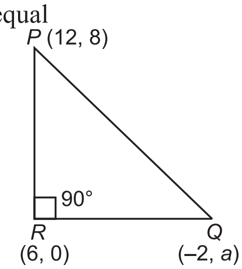
Also, $PR = \sqrt{(5+1)^2 + \left(\dfrac{3}{2} - \dfrac{3}{2}\right)^2} = \sqrt{36} = 6$

$SQ = \sqrt{(2-2)^2 + (4+1)^2} = \sqrt{25} = 5$

$\Rightarrow$ $PR \neq SQ \Rightarrow$ Diagonals are not equal

$\Rightarrow$ $PQRS$ is a **rhombus.**

10. ΔPQR is right angle at R



$\Rightarrow$ $PR^2 + RQ^2 = PQ^2$

(Pythagoras' Theorem)

$\Rightarrow$ $(6-12)^2 + (0-8)^2 + (-2-6)^2 + (a-0)^2 = (-2-12)^2 + (a-8)^2$

$\Rightarrow$ $36 + 64 + 64 + a^2 = 196 + a^2 - 16a + 64$

$\Rightarrow$ $16a = 196 - (64 + 36) = 196 - 100 = 96$

$\Rightarrow$ $\mathbf{a = 6.}$

11. $AB \perp$ chord $PQ \Rightarrow AB$ bisects chord $PQ \Rightarrow PQ = 2PB$.

$AB = \sqrt{(2-5)^2 + (-3-1)^2} = \sqrt{(-3)^2 + (-4)^2}$

$= \sqrt{9+16} = \sqrt{25} = 5$

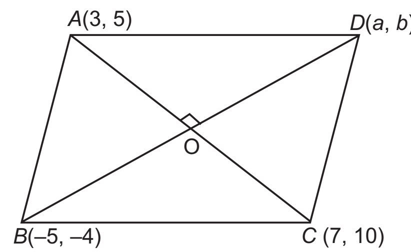
AP = radius of circle = 13

$\therefore$ By Pythagoras' Theorem, $PB = \sqrt{AP^2 - AB^2}$

$= \sqrt{169 - 25} = \sqrt{144} = 12$ units

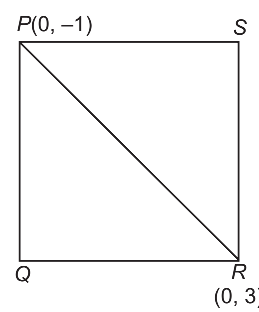
$\therefore$ $PQ = 2 \times PB =$ **24 units.**

12. Let $\Delta(a, b)$ be the fourth vertex of the parallelogram $ABCD$. The diagonals of a parallelogram bisect each other at point O (say), so the diagonals AC and BD have the same mid-point.



13. $\because$ Mid-point of $BC = \left(\dfrac{7+3}{2}, \dfrac{7+5}{2}\right)$, *i.e.*, (5, 6), we can easily show that D lies on the boundary of ΔABC.

14. Let $PQRS$ be the required square and $P(0, -1)$ and $R(0, 3)$ be its two opposite vertices.



Length of diagonal PR

$= \sqrt{(0-0)^2 + (3+1)^2} = \sqrt{16} = 4$

$\therefore$ Length of each side $= \dfrac{\text{diagonal}}{\sqrt{2}}$

$= \dfrac{4}{\sqrt{2}} = \mathbf{2\sqrt{2}.}$

Let the co-ordinates of another vertex of the square say Q be (a, b)

$\therefore$ Its distance from vertex P should be equal to its distance from vertex R.

$\therefore$ $\sqrt{(a-0)^2 + (b+1)^2} = \sqrt{(a-0)^2 + (b-3)^2}$

$\Rightarrow$ $a^2 + b^2 + 2b + 1 = a^2 + b^2 - 6b + 9$

$\Rightarrow$ $8b = 8 \Rightarrow b = 1$

Also, this distance $QP = 2\sqrt{2}$

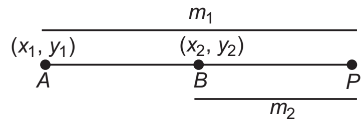
$\therefore$ $\sqrt{(a-0)^2 + (b+1)^2} = 2\sqrt{2}$

$\Rightarrow a^2 + b^2 + 2b + 1 = 8 \Rightarrow a^2 + 4 = 8 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2.$

$\therefore$ The other two vertices of the square are (+2, 1) and (–2, 1).

15. Co-ordinates of the point of external division are

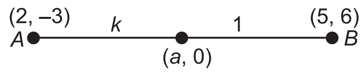
$\left(\dfrac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \dfrac{m_1 y_2 - m_2 y_1}{m_1 - m_2}\right)$, *i.e.*,



$\therefore$ Required point $= \left(\dfrac{3\times -6 - 2\times 4}{3-2}, \dfrac{3\times 2 - 2\times 4}{3-2}\right)$

$= \left(\dfrac{-24}{1}, \dfrac{-2}{1}\right)$, *i.e.*, **(–24, –2).**

16. Any point on the x-axis is $(a, 0)$.



Let the point $(a, 0)$ divide the join of $A(2, -3)$ and $B(5, 6)$ in the ratio $k : 1$. Then the co-ordinates of the point of division are $\left(\dfrac{5k+2}{k+1}, \dfrac{6k-3}{k+1}\right)$

Now, $\dfrac{5k+2}{k+1} = a$ and $\dfrac{6k-3}{k+1} = 0$

$\therefore\ 6k - 3 = 0 \Rightarrow k = \dfrac{1}{2}$

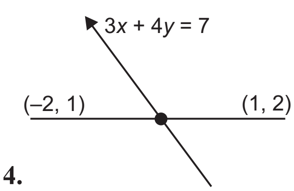
$\therefore$ Required ratio is $k : 1 \Rightarrow \dfrac{1}{2} : 1 =$ **1 : 2.**

17. Let the line $3x + 4y = 7$ divide the join of $(-2, 1)$ and $(1, 2)$ in the ratio $k : 1$. Then the co-ordinates of the point of division are : $\left(\dfrac{k-2}{k+1}, \dfrac{2k+1}{k+1}\right)$

This point lies on the line $3x + 4y = 7$, so satisfies the equation of the given line, *i.e.*, $3\left(\dfrac{k-2}{k+1}\right) + 4\left(\dfrac{2k+1}{k+1}\right) = 7$

$\Rightarrow\ 3k - 6 + 8k + 4 = 7k + 7$

$\Rightarrow\ 4k = 9 \Rightarrow k = \dfrac{9}{4}$.



$\therefore$ Required ratio $= \dfrac{9}{4} : 1 =$ **9 : 4.**

18. Let $P(m, 6)$ divides AB in the ratio $k : 1$.

Then co-ordinates of P are $\left(\dfrac{2k-4}{k+1}, \dfrac{8k+3}{k+1}\right)$

Given, co-ordinates of P are $(m, 6) \Rightarrow \dfrac{2k-4}{k+1} = m$ ...(*i*)

and $\dfrac{8k+3}{k+1} = 6 \Rightarrow 8k + 3 = 6k + 6 \Rightarrow 2k = 3 \Rightarrow k = \dfrac{3}{2}$

$\therefore$ Required ratio is $\dfrac{3}{2} : 1 =$ **3 : 2**

Now, $\dfrac{2k-4}{k+1} = m \Rightarrow \dfrac{2\times\frac{3}{2} - 4}{\frac{3}{2}+1} = \dfrac{3-4}{5/2} = \dfrac{-2}{5}$

$\therefore\ m = \mathbf{\dfrac{-2}{5}}$.

19. Let the co-ordinates of vertex C are (x, y).

Since the incentre and centroid of an equilateral triangle coincide, co-ordinates of centroid of

$\Delta ABC \equiv \left(\dfrac{9+\sqrt{3}}{6}, \dfrac{15-\sqrt{3}}{6}\right)$

$\therefore\ \dfrac{x+1+2}{3} = \dfrac{9+\sqrt{3}}{6}$ and $\dfrac{2+3+y}{3} = \dfrac{15-\sqrt{3}}{6}$

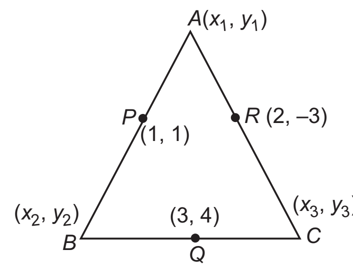
$\Rightarrow\ 2(x+3) = 9 + \sqrt{3}$ and $2(5+y) = 15 - \sqrt{3}$

$\Rightarrow\ 2x = 3 + \sqrt{3}$ and $10 + 2y = 15 - \sqrt{3}$

$\Rightarrow\ x = \dfrac{3+\sqrt{3}}{2}$ and $2y = 5 - \sqrt{3} \Rightarrow y = \dfrac{5-\sqrt{3}}{2}$

$\therefore$ Required co-ordinates are $\left(\mathbf{\dfrac{3+\sqrt{3}}{2}, \dfrac{5-\sqrt{3}}{2}}\right)$.

20. Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of $\Delta\, ABC$ the coordinates who mid-points are given as $P(1, 1)$, $Q(3, 4)$ and $R(2, -3)$



Now, $\dfrac{x_1+x_2}{2} = 1$ and $\dfrac{y_1+y_2}{2} = 1$ ...(*i*)

$\dfrac{x_2+x_3}{2} = 3$ and $\dfrac{y_2+y_3}{2} = 4$ ...(*ii*)

$\dfrac{x_1+x_3}{2} = 2$ and $\dfrac{y_1+y_3}{2} = -3$ ...(*iii*)

$\therefore\ \dfrac{x_1+x_2+x_2+x_3+x_1+x_3}{2} = 6$

and $\dfrac{y_1+y_2+y_2+y_3+y_1+y_3}{2} = 2$

$\Rightarrow\ x_1 + x_2 + x_3 = 6$ and $y_1 + y_2 + y_3 = 2$

Now, $(x_1 + x_2 + x_3) - (x_1 + x_2) = 6 - 2 \Rightarrow x_3 = 4$

and $(y_1 + y_2 + y_3) - (y_1 + y_2) = 2 - 2 \Rightarrow y_3 = 0$

$(x_1 + x_2 + x_3) - (x_2 + x_3) = 6 - 6 \Rightarrow x_1 = 0$

and $(y_1 + y_2 + y_3) - (y_2 + y_3) = 2 - 8 \Rightarrow y_1 = -6$

$(x_1 + x_2 + x_3) - (x_1 + x_3) = 6 - 4 \Rightarrow x_2 = 2$

and $(y_1 + y_2 + y_3) - (y_1 + y_3) = 2 + 6 \Rightarrow y_2 = 8$

$\therefore\ A(0, -6)$, $B(2, 8)$, $C(4, 0)$ are the vertices of $\Delta\, ABC$

Now, $a = BC = \sqrt{(4-2)^2 + (0-8)^2} = \sqrt{4+64}$

$= \sqrt{68} = 2\sqrt{17}$

$b = AC = \sqrt{(4-0)^2 + (0+6)^2} = \sqrt{16+36} = \sqrt{52} = 2\sqrt{13}$

$c = AB = \sqrt{(2-0)^2 + (8+6)^2} = \sqrt{4+196} = \sqrt{200} = 10\sqrt{2}$

$\therefore$ Co-ordinates of incentre of $\Delta\, ABC$ are

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

$$= \left(\frac{2\sqrt{17}\times 0+2\sqrt{13}\times 2+10\sqrt{2}\times 4}{2\sqrt{17}+2\sqrt{13}+10\sqrt{2}}, \frac{2\sqrt{17}\times -6+2\sqrt{13}\times 8+10\sqrt{2}\times 0}{2\sqrt{17}+2\sqrt{13}+10\sqrt{2}}\right)$$

$$= \left(\frac{4\sqrt{13}+40\sqrt{2}}{2\sqrt{17}+2\sqrt{13}+10\sqrt{2}}, \frac{-12\sqrt{17}+16\sqrt{13}}{2\sqrt{17}+2\sqrt{13}+10\sqrt{2}}\right)$$

$$= \left(\mathbf{\frac{2\sqrt{13}+20\sqrt{2}}{\sqrt{17}+\sqrt{13}+5\sqrt{2}}, \frac{8\sqrt{13}-6\sqrt{17}}{\sqrt{17}+\sqrt{13}+5\sqrt{2}}}\right).$$

21. Let (x, y) be the co-ordinates of the third vertex of the triangle.

Then $\dfrac{0+2+x}{3}=1$ and $\dfrac{0+0+y}{3}=1$

$\Rightarrow$ $2+x=3$ and $y=3$

$\Rightarrow$ $x=1, y=3.$

$\therefore$ Co-ordinates of vertices of the triangle are (0, 0), (2, 0) and (1, 3).

$\therefore$ Area of triangle $= \dfrac{1}{2}\,[0(0-3)+2(3-0)+1(0-0)]$

$\left[\because \text{Area of } \Delta = \dfrac{1}{2}\,(x_1(y_2-y_3)+x_2(y_2-y_3)+x_3(y_1-y_2)\right]$

$= \dfrac{1}{2}[0+6+0] = \dfrac{6}{2} = \mathbf{3.}$

22. For three points to be collinear, area of the triangle formed by the three points should be equal to zero, *i.e.*

$\dfrac{1}{2}\,[k(3k-1)+2k(1-2k)+3(2k-3k)]=0$

$\Rightarrow$ $\dfrac{1}{2}\,[3k^2-k+2k-4k^2-3k]=0$

$\Rightarrow$ $k^2+2k=0$ $\Rightarrow$ $k=0$ or -2

Neglecting $k=0$, as then $(k, 2k)$ and $(2k, 3k)$ will be the same point, we take $k=-2$.

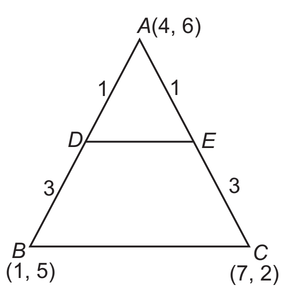
23. Points $A(1, 2)$, $B(0, 0)$ and $C(a, b)$ are collinear if area of $\Delta\, ABC = 0$.

$\Rightarrow$ $\dfrac{1}{2}\,[1(0-b)+0\,(b-2)+a\,(2-0)]=0$

$\Rightarrow$ $-b+2a=0 \Rightarrow \boldsymbol{2a=b}.$

24. Given $\dfrac{AD}{AB}=\dfrac{1}{4}\Rightarrow\dfrac{AD}{DB}=\dfrac{1}{3}$,

i.e., D divides AB internally in the ratio 1 : 3.



$\therefore$ Co-odinates of D are

$\left(\dfrac{1+12}{1+3}, \dfrac{5+18}{1+3}\right)$ *i.e.,* $\left(\dfrac{13}{4}, \dfrac{23}{4}\right)$

Also, $\dfrac{AE}{AC}=\dfrac{1}{4}\Rightarrow\dfrac{AC}{EC}=\dfrac{1}{3}$, *i.e., E* divides AC internally in the ratio 1 : 3.

Co-ordinates of E are $\left(\dfrac{7+12}{1+3}, \dfrac{2+18}{1+3}\right)$, *i.e.,* $\left(\dfrac{19}{4}, 5\right)$

Now, area of $\Delta\, ABC = \dfrac{1}{2}\,[4(5-2)+1(2-6)+7(6-5)]$

$= \dfrac{1}{2}\,[12-4+7] = \dfrac{15}{2}$ sq. units

Area of $\Delta\, ADE$, where $A(4, 6)$, $D\left(\dfrac{13}{4}, \dfrac{23}{4}\right)$, $E\left(\dfrac{19}{4}, 5\right)$ is

$\dfrac{1}{2}\left[4\left(\dfrac{23}{4}-5\right)+\dfrac{13}{4}(5-6)+\dfrac{19}{4}\left(6-\dfrac{23}{4}\right)\right]$

$= \dfrac{1}{2}\left[4\times\dfrac{3}{4}+\dfrac{13}{4}\times -1+\dfrac{19}{4}\times\dfrac{1}{4}\right] = \dfrac{1}{2}\left[3-\dfrac{13}{4}+\dfrac{19}{16}\right]$

$= \dfrac{1}{2}\left[\dfrac{48-52+19}{16}\right] = \dfrac{15}{`32}$ sq. units

$\therefore$ $\dfrac{\text{Area of } \Delta ADE}{\text{Area of } \Delta ABC} = \dfrac{15/32}{15/2} = \mathbf{1:16}.$

25. Area of quadrilateral whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) taken in order is

$\dfrac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3)+(x_4y_1-x_1y_4)]$

$= \dfrac{1}{2}[(1\times 6)-(-5)\times 2]+[(-5\times -4)-(7\times 6)]+[(7\times k)-(-2)\times(-4)]+[(-2\times 2)-(1\times k)]$

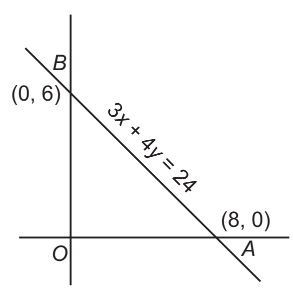
$= \dfrac{1}{2}\,[(6+10)+(20-42)+(7k-8)+(-4-k)]$

$= \dfrac{1}{2}\,[16-22+7k-8-4-k]$

$= \dfrac{1}{2}\,[-18+6k]=0 \Rightarrow 6k=18 \Rightarrow \boldsymbol{k=3.}$

26. Type Q. 24.

27. The line $3x+4y-24=0$ cuts the axis at A. To obtain the co-ordinates of A put $y=0$, as on x-axis, $y=0$.



$\therefore$ $A \equiv (8, 0)$ ...(*i*)

Also, on y-axis, $x=0$, therefore

$B \equiv (0, 6)$ ...(*ii*)

$\therefore$ The three vertices of $\Delta\, AOB$ are $A(8, 0)$, $O(0, 0)$, $B(0, 6)$

$\therefore$ $a = OB = 6$, $b = OA = 8$,

$c = AB = \sqrt{(0-8)^2+(6-0)^2} = \sqrt{64+36} = \sqrt{100} = 10.$

$\therefore$ Co-ordinates of the incentre

$= \left(\dfrac{ax_1+bx_2+cx_3}{a+b+c}, \dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$

$= \left(\dfrac{6\times 8+8\times 0+10\times 0}{6+8+10}, \dfrac{6\times 0+8\times 6+10\times 0}{6+8+10}\right)$

$= \left(\dfrac{48}{24}, \dfrac{48}{24}\right) = \mathbf{(2, 2)}.$

28. Let the slope of the lin passing through the points $(-k, h)$ and $(b, -f)$ be m_1. Then $m_1 = \dfrac{-f-h}{b+k} = -\left(\dfrac{f+h}{b+k}\right)$ $\left[\text{Slope} = \dfrac{y_2 - y_1}{x_2 - x_1}\right]$

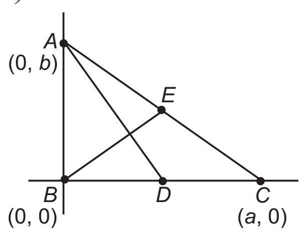
$\therefore$ Slope of line perpendicular to the given line $= -\dfrac{1}{m_1}$

$$= \frac{-1}{-\left(\dfrac{f+h}{b+k}\right)} = \left(\frac{\boldsymbol{b+k}}{\boldsymbol{f+h}}\right)$$

29. Let D be the mid-point of BC.

Then $D \equiv \left(\dfrac{a+0}{2}, \dfrac{0}{2}\right)$ *i.e.* $\left(\dfrac{a}{2}, 0\right)$

Let E be the mid-point of AC, then



$E \equiv \left(\dfrac{a+0}{2}, \dfrac{0+b}{2}\right) = \left(\dfrac{a}{2}, \dfrac{b}{2}\right)$

$AD \perp BE$ if slope of $AD \times$ Slope of $BE = -1$ ...(*i*)

Slope of $AD = \dfrac{0-b}{a/2-0} = \dfrac{-b}{a/2} = \dfrac{-2b}{a}$

Slope of $BE = \left(\dfrac{b/2-0}{a/2-0}\right) = \dfrac{b/2}{a/2} = \dfrac{b}{a}$.

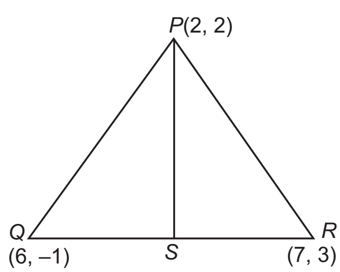
$\therefore$ From (*i*), $\dfrac{-2b}{a} \times \dfrac{b}{a} = -1$

$\Rightarrow$ $2b^2 = a^2 \Rightarrow a = \pm\sqrt{2}\boldsymbol{b}$.

30. PS being the median of ΔPQR, S is the mid-point of QR, *i.e.*,

Coordinates of $S \equiv$

$\left(\dfrac{6+7}{2}, \dfrac{-1+3}{2}\right) = \left(\dfrac{13}{2}, 1\right)$



Slope of line parallel to PS = Slope of PS

$$= \frac{1-2}{\dfrac{13}{2}-2} = \frac{-1}{9/2} = \frac{-2}{9}.$$

$\therefore$ Equation of a line parallel to PS passing through the point $(1, -1)$ is

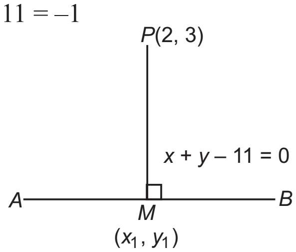
$(y+1) = \dfrac{-2}{9}(x-1)$, *i.e.*, $9y + 9 = -2x + 2$

$\Rightarrow$ $\boldsymbol{2x + 9y + 7 = 0}$.

31. Let the foot of the perpendicular be $M(x_1, y_1)$

Slope of line AB, *i.e.*, $y = -x + 11 = -1$

Slope of line $PM = \dfrac{y_1 - 3}{x_1 - 2}$



Now, $PM \perp AB$

$\Rightarrow \left(\dfrac{y_1 - 3}{x_1 - 2}\right) \times -1 = -1$

$\Rightarrow$ $y_1 - 3 = x_1 - 2 \Rightarrow x_1 - y_1 = -1$ ...(*i*)

Also, (x_1, y_1) lies on AB

$\Rightarrow x_1 + y_1 - 11 = 0$

$\Rightarrow x_1 + y_1 = 11$ ...(*ii*)

Adding (*i*) and (*ii*), we get $2x_1 = 10 \Rightarrow x_1 = 5$

Putting x_1 in (*ii*), we get $y_1 = 6$.

$\therefore$ Required foot of the perpendicular M is **(5, 6).**

32. The equations of the given lines are:

$4x + 3y = 12$ ...(*i*)

$3x + 4y = 12$ ...(*ii*)

Solving the simultaneous equations (*i*) and (*ii*), we get

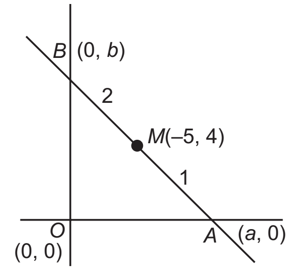
$x = \dfrac{12}{7}, y = \dfrac{12}{7}$

$\therefore$ Point of the intersection of the given lines is $\left(\dfrac{12}{7}, \dfrac{12}{7}\right)$

Now equation of the line passing through $(0, 0)$ and $\left(\dfrac{12}{7}, \dfrac{12}{7}\right)$ is

$$y - 0 = \left(\frac{\dfrac{12}{7} - 0}{\dfrac{12}{7} - 0}\right)(x - 0), \textit{ i.e.,} \quad \boldsymbol{y = x.}$$

33. Let AB be the given line between the co-ordinates axes $x = 0, y = 0$.



Let $A(a, 0)$ and $B(0, b)$ be two points of the given line on the co-ordinates axes. $(-5, 4)$ divides AB in the ratio $1 : 2$.

$\therefore$ Co-ordinates of point of division are $\left(\dfrac{1 \times 0 + 2 \times a}{1+2}, \dfrac{1 \times b + 2 \times 0}{1+2}\right)$, *i.e.*, $\left(\dfrac{2a}{3}, \dfrac{b}{3}\right)$

Given, $\dfrac{2a}{3} = -5$, and $\dfrac{b}{3} = 4$

$\Rightarrow$ $a = \dfrac{-15}{2}, b = 12$

$\therefore$ The points A and B are $\left(\dfrac{-15}{2}, 0\right)$ and $(0, 12)$ respectively.

Equation of line AB : $(y - 0) = \dfrac{12 - 0}{0 - \left(\dfrac{-15}{2}\right)}\left(x + \dfrac{15}{2}\right)$

$\Rightarrow$ $y = \dfrac{12 \times 2}{15}\left(x + \dfrac{15}{2}\right)$

$= \dfrac{24}{15 \times 2}(2x + 15) = \dfrac{4}{5}(2x + 15)$

$\Rightarrow$ $5y = 8x + 60 \Rightarrow \boldsymbol{8x - 5y + 60 = 0}$ is the equation of AB.

34. $x \cos\theta + y \sin\theta = 2$

$\Rightarrow$ $y \sin\theta = 2 - x\cos\theta \Rightarrow y = -x\cot\theta + 2$

$\Rightarrow$ Slope of this line $= -\cot\theta$

Also, given $x - y = 3 \Rightarrow y = x + 3 \Rightarrow$ Slope of this line = 1

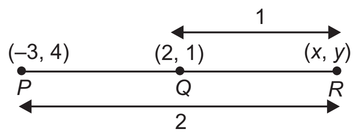
$\therefore$ These lines are perpendicular, $-\cot\theta \times 1 = -1$

$\Rightarrow \quad \cot\theta = 1 \Rightarrow \cot\theta = \cot\dfrac{\pi}{4} \Rightarrow \theta = \boldsymbol{\dfrac{\pi}{4}}$

35. Let the co-ordinates of R be (x, y).

As can be easily seen, it is a point of external division

Also, $PR = 2QR$



$\Rightarrow R$ divides the join of P and Q externally in the ratio 2:1.

$\therefore \quad x = \dfrac{2\times 2 - 1\times -3}{2-1},\ y = \dfrac{2\times 1 - 1\times 4}{2-1}$

$\Rightarrow \quad x = 4 + 3 = 7$ and $y = 2 - 4 = -2$.

$\therefore$ Co-ordinates of R are **(7, –2).**

36. As the points $A(a, ma)$, $B[b, (m+1)b]$ and $C[c, (m+2)c]$ are collinear.

Area of $\Delta\, ABC$ should be equal to zero.

$\Rightarrow \quad \dfrac{1}{2}\,[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$

$\Rightarrow \quad \dfrac{1}{2}\,[a\,\{(m+1)\,b - (m+2)\,c\} + b\,\{(m+2)\,c - ma\} + c\,\{ma - (m+1)b\}] = 0$

$\Rightarrow \quad mab + ab - mac - 2ac + mbc + 2bc - mab + mac - mbc - bc = 0$

$\Rightarrow \quad ab - 2ac + 2bc - bc = 0 \Rightarrow ab + bc = 2ac$

$\Rightarrow \quad b = \dfrac{2ac}{a+c}$

$\therefore$ ***a, b, c are harmonic progression (H.P.) for all m.***

37. Let the equation of the line in the intercept from be

$\dfrac{x}{\lambda} + \dfrac{y}{\mu} = 1$

Since it passes through (4, 3) and (2, 5)

$\dfrac{4}{\lambda} + \dfrac{3}{\mu} = 1 \qquad ...(i)$

and $\dfrac{2}{\lambda} + \dfrac{5}{\mu} = 1 \qquad ...(ii)$

Multiplying (ii) by 2 and subtracting from (i), we get

$\left(\dfrac{4}{\lambda} + \dfrac{3}{\mu}\right) - \left(\dfrac{4}{\lambda} + \dfrac{10}{\mu}\right) = 1 - 2$

$\Rightarrow \quad \dfrac{-7}{\mu} = -1 \Rightarrow \mu = 7$

Putting $\mu = 7$ in (i), we get $\dfrac{4}{\lambda} + \dfrac{3}{7} = 1$

$\Rightarrow \quad \dfrac{4}{\lambda} = 1 - \dfrac{3}{7} = \dfrac{4}{7} \Rightarrow \lambda = 7$

$\therefore \quad \boldsymbol{\lambda = \mu = 7.}$

38. Let the required equation make intercept on x-axis $= 2a$

$\Rightarrow$ intercept made on y-axis $= a$

$\therefore$ Eqn of the given line in the intercept from:

$\dfrac{x}{2a} + \dfrac{y}{a} = 1 \qquad ...(i)$

Since the line given by (i) bisects the join of (3, – 4) and (5, 2), the mid-point of the line joining (3, – 4) and (5, 2) lies on (i).

Mid- point of join of (3, – 4) and (5, 2) $= \left(\dfrac{3+5}{2}, \dfrac{-4+2}{2}\right)$

$= (4, -1)$

Putting (4, –1) in (i), we get

$\dfrac{4}{2a} + \dfrac{-1}{a} = 1 \Rightarrow \dfrac{4-2}{2a} = 1 \Rightarrow \dfrac{2}{2a} = 1 \Rightarrow a = 1.$

$\therefore$ Required equation of line : $\dfrac{x}{2\times 1} + \dfrac{y}{1} = 1 \Rightarrow \boldsymbol{x + 2y = 2.}$

39. The point of intersection of the given lines can be obtained by solving the equations of the two lines simultaneously.

$100x + 50y = 1 \qquad ...(i)$

$75x + 25y = -3 \qquad ...(ii)$

Eqn (i) – 2 × Eqn (ii)

$\Rightarrow \quad (100x + 50y) - (150x + 50y) = 1 - (-6)$

$\Rightarrow \quad -50x = 7 \Rightarrow x = \dfrac{-7}{50}$

Putting the value of x in (i), we get

$100 \times \dfrac{-7}{50} + 50y = 1 \Rightarrow 50y = 1 + 14 = 15 \Rightarrow y = \dfrac{15}{50} = \dfrac{3}{10}$

Let the required line make x-intercept = y-intercept = a.

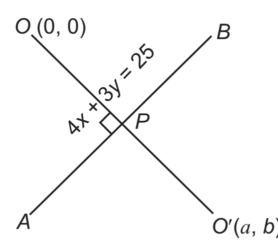
Then eqn of required line is $\dfrac{x}{a} + \dfrac{y}{a} = 1 \Rightarrow x + y = a$

Since it passes through $\left(\dfrac{-7}{50}, \dfrac{3}{10}\right)$, therefore

$\Rightarrow \quad \dfrac{-7}{50} + \dfrac{3}{10} = a \Rightarrow \dfrac{-7+15}{50} = a \Rightarrow a = \dfrac{8}{50} = \dfrac{4}{25}$

$\therefore$ Eqn of required line: $x + y = \dfrac{4}{25} \Rightarrow$ **25 + 25y – 4 = 0.**

40. Let AB be the given line $4x + 3y = 25$



Let $O'(a, b)$ be the image of O in the given line AB.

Let $O\,O'$ cut AB in point P. Also $OP \perp AB$ and P is the mid-point of OO'.

$\therefore$ Co-ordinates of P are $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$

$\because$ P lies on AB, $4\times\dfrac{a}{2} + 3\times\dfrac{b}{2} - 25 = 0$

$\Rightarrow \quad 4a + 3b = 50 \qquad ...(i)$

Also, $OO' \perp AB \Rightarrow$ slope of OO' × slope of $AB = -1$

$\Rightarrow \quad \left(\dfrac{b-0}{a-0}\right)\times\dfrac{-4}{3} = -1 \qquad \left(\because \text{Line } AB : y = \dfrac{-4x}{3} + \dfrac{25}{3}\right)$

$\Rightarrow \quad \frac{b}{a} = \frac{3}{4} \Rightarrow 3a = 4b$ ...(*ii*)

From (*ii*), $a = \frac{4b}{3}$. Putting this value of a in (*i*), we get

$$4 \times \frac{4b}{3} + 3b = 50$$

$\Rightarrow \quad 16b + 9b = 150 \Rightarrow 25b = 150 \Rightarrow b = 6$

$\Rightarrow \quad a = \frac{4 \times 6}{3} = 8$

$\therefore$ The image of the point $O(0, 0)$ in the line $4x + 3y - 25 = 0$ is **(8, 6).**

41. $3x + y - 7 = 0 \Rightarrow y = -3x + 7 \Rightarrow$ Slope $(m_1) = -3$

$x + 2y + 9 = 0 \Rightarrow y = \frac{-x}{2} - \frac{9}{2} \Rightarrow$ Slope $(m_2) = -\frac{1}{2}$

If θ is the angle between the given lines, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-3 - \left(\frac{1}{2}\right)}{1 + (-3)\left(-\frac{1}{2}\right)}\right| = \left|\frac{-\frac{5}{2}}{1 + \frac{3}{2}}\right|$$

$$= \left|\frac{-\frac{5}{2}}{\frac{5}{2}}\right| = 1$$

$\therefore \quad \theta = $ **45°.**

42. The equations of the two lines whose point of intersection is needed are:

$2x - y = -5$ ...(*i*)

$5x + 3y = 4$ ...(*ii*)

$3 \times$ Eqn (*i*) + Eqn (*ii*) $\Rightarrow (6x - 3y) + (5x + 3y) = -15 + 4$

$\Rightarrow \quad 11x = -11 \Rightarrow x = -1$

Putting $x = -1$ in (*i*), we get $-2 - y = -5 \Rightarrow y = 3$.

$\therefore$ Point of intersection is $(-1, 3)$.

Slope of line $x - 3y + 21 = 0$, *i.e.*, $y = \frac{x}{3} + 7$ is $\frac{1}{3}$.

$\Rightarrow$ Slope of line perpendicular to line $x - 3y + 21 = 0$ is -3 $[\because m_1 \times m_2 = -1]$

$\therefore$ Equation of a line through $(-1, 3)$ with slope -3 is $y - 3 = -3(x + 1) \Rightarrow y - 3 = -3x - 3 \Rightarrow$ $\mathbf{3x + y = 0.}$

43. The equations of the given lines are:

$Ax + By = A + B \Rightarrow By = -Ax + (A + B)$

$\Rightarrow \quad y = -\frac{A}{B}x + \frac{(A + B)}{B}$ ...(*i*)

and $A(x - y) + B(x + y) = 2B \Rightarrow (A + B)x + (B - A)y = 2B$

$\Rightarrow \quad y = \frac{-(A + B)}{(B - A)}x + \frac{2B}{(B - A)}$ ...(*ii*)

$\therefore$ Slope of line (*i*) $= m_1 = -\frac{A}{B}$

Slope of line (*ii*) $= m_2 = -\frac{(A + B)}{B - A}$

Let θ be the angle between both the lines, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{A}{B} + \left(\frac{A + B}{B - A}\right)}{1 + \frac{A}{B}\left(\frac{A + B}{B - A}\right)}\right|$$

$$= \left|\frac{-A(B - A) + B(A + B)}{B(B - A) + A(A + B)}\right|$$

$$= \left|\frac{-AB + A^2 + BA + B^2}{B^2 - BA + A^2 + AB}\right| = \left|\frac{A^2 + B^2}{A^2 + B^2}\right| = 1$$

$\therefore \quad \tan\theta = 1 \Rightarrow \theta = \tan^{-1}(1) =$ **45°.**

44. The given lines are:

$L : \frac{x}{5} + \frac{y}{b} = 1$ ...(*i*)

$K : \frac{x}{c} + \frac{y}{3} = 1$ ...(*ii*)

Since line L passes through (13, 32),

$$\frac{13}{5} + \frac{32}{b} = 1 \Rightarrow \frac{32}{b} = 1 - \frac{13}{5} = \frac{-8}{5} \Rightarrow b = \frac{32 \times 5}{-8} = -20.$$

$\therefore$ Line L is $\frac{x}{5} - \frac{y}{20} = 1$, *i.e.*, $y = 4x - 20$

$\Rightarrow$ Slope of line $L = 4$

As $L \parallel K$, slope of line K = Slope of line L.

Eqn of line K can be written as $y = \frac{-3x}{c} + 3$

$\therefore$ Slope of $K = -\frac{3}{c}$.

Given, $\frac{-3}{c} = 4 \Rightarrow c = \frac{-3}{4}$

$\therefore$ Equation of line K : $\left(\frac{-4}{3}\right)x + \frac{y}{3} = 1 \Rightarrow 4x - y + 3 = 0.$

$\therefore$ Distance between the lines $L \equiv 4x - y - 20 = 0$ and $K \equiv 4x - y + 3 = 0$ is

$$d = \left|\frac{3 - (-20)}{\sqrt{4^2 + (-1)^2}}\right| = \frac{\mathbf{23}}{\sqrt{\mathbf{17}}}.$$

(Distance between parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $d = \frac{|c_2 - c_1|}{\sqrt{a^2 + b^2}}$)

45. Since the line passes through $A(a, 0)$ and $B(0, b)$, it makes intercepts a and b on x-axis and y-axis respectively. Let the equation of this line in the intercept from be $\frac{x}{a} + \frac{y}{b} = 1$

By the given condition, $AB = \sqrt{a^2 + b^2} = 13$ and $ab = 60$

$\therefore \quad a^2 + b^2 = 169 \Rightarrow a^2 + b^2 + 2ab = 169 + 120$

$\Rightarrow (a + b)^2 = 289 \Rightarrow a + b = \pm 17$ ...(*i*)

Also, $a^2 + b^2 - 2ab = 169 - 120 \Rightarrow (a - b)^2 = 49$

$\Rightarrow a - b = \pm 7.$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*) $a = 12, b = 5$ and $a = -12, b = -5$

$\therefore$ The required equations of the straight line are:

$\frac{x}{12}+\frac{y}{5}=1$ and $\frac{x}{-12}+\frac{y}{-5}=1$

$\Rightarrow$ $5x + 12y = 60$ and $\mathbf{5x + 12y + 60 = 0.}$

46. Let $(m > 0)$ be the gradient (slope) of the required line. Then, Equation of any line through $(-5, 0)$ having slope $= m$ is

$y - 0 = m(x - (-5))$ or $mx - y + 5m = 0$ ...(i)

Its perpendicular distance from origin is 3

$$\Rightarrow \frac{\pm|m.0-0+5m|}{\sqrt{m^2+(-1)^2}}=3 \Rightarrow |5m| = 3\sqrt{m^2+1}$$

$$\left(\because \text{Distance of } (x_1, y_1) \text{ from line } ax+by+c=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}\right)$$

$\Rightarrow$ $25m^2 = 9(m^2 + 1) \Rightarrow 16m^2 = 9 \Rightarrow m = \frac{3}{4}$ ($\because m$ is +ve)

$\therefore$ Required equation: $y = \frac{3}{4}(x+5)$

$\Rightarrow$ $\mathbf{3x - 4y + 15 = 0.}$

47. Given lines are $3x - y - 3 = 0$ and $3x - y + 5 = 0$.

Line parallel to the given lines can be written as

$3x - y + c = 0$ ...(i)

Let us taken a point, say, (0, c) on (i)

(Putting $x = 0$ in (i), we get $y = c$)

$$\therefore \frac{\text{Distance of } (0, c) \text{ from } 3x-y-3}{\text{Distance of } (0, c) \text{ from } 3x-y+5}=\frac{3}{5}$$

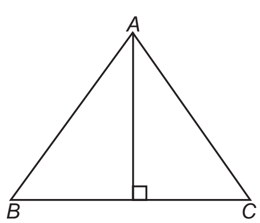
$$\Rightarrow \frac{\frac{|3\times 0-c-3|}{\sqrt{3^2+1^2}}}{\frac{|3\times 0-c+5|}{\sqrt{3^2+1^2}}}=\frac{3}{5} \Rightarrow \frac{c+3}{-c+5}=\frac{3}{5}$$

$\Rightarrow$ $5c + 15 = -3c + 15 \Rightarrow 8c = 0 \Rightarrow c = 0.$

Substituting $c = 0$ in (i), the required equation is

$\mathbf{3x - y = 0.}$

48. Equation of line BC :



$y - 7 = \frac{1-7}{5-1}(x-1)$

$\Rightarrow$ $y - 7 = \frac{-6}{4}(x-1)$

$\Rightarrow$ $2(y - 7) = -3(x - 1)$

$\Rightarrow$ $2y - 14 = -3x + 3 \Rightarrow 3x + 2y - 17 = 0.$

$\therefore$ Distance of perpendicular from $A(0, 0)$ on $BC \equiv 3x + 2y - 11 = 0$ is

$$\frac{|3\times 0+2\times 0-17|}{\sqrt{3^2+2^2}}=\frac{17}{\sqrt{13}}$$

Area of $\Delta ABC = \frac{1}{2} \times BC \times$ perpendicular distance

$BC = \sqrt{(5-1)^2+(1-7)^2} = \sqrt{4^2+6^2} = \sqrt{16+36}$

$= \sqrt{52} = 2\sqrt{13}$

$\therefore$ Required area $= \frac{1}{2}\times\frac{17}{\sqrt{13}}\times 2\sqrt{13} = 17$ sq. units.

49. Since p is the length of perpendicular from origin on the straight line $ax + by - p = 0$.

$p = \frac{|a.0+b.0-p|}{\sqrt{a^2+b^2}}$

$\Rightarrow 1 = \sqrt{a^2+b^2} \Rightarrow 1 = a^2 + \frac{3}{4}$ $\left(\because b=\frac{\sqrt{3}}{2}\right)$

$\Rightarrow$ $a^2 = \frac{1}{4} \Rightarrow a = \frac{1}{2}.$

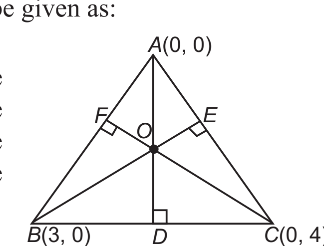
$\therefore$ Equation of the straight line is $\frac{1}{2}x+\frac{\sqrt{3}}{2}y=p$

$\Rightarrow$ $x \cos 60° + y \sin 60° = p$

Hence required angle is 60°, which is the angle between the perpendicular and the positive direction of x-axis.

50. Let the vertices of ΔABC be given as:

$A(0, 0)$, $B(3, 0)$ and $C(0, 4)$



The orthocentre O is the point of intersection of the altitudes drawn from the vertices of ΔABC on the opposite sides.

Slope of $BC = \frac{4-0}{0-3}=\frac{-4}{3}$

$\therefore$ Slope of $AD \perp BC = \frac{3}{4}$ (Slope of $BC \times$ Slope of $AD = -1$)

Equation of line through $A(0, 0)$ having slope $\frac{3}{4}$ is

$y - 0 = \frac{3}{4}(x-0) \Rightarrow 4y = 3x$ ...(i)

Similarly, slope of $AC = \frac{4-0}{0-0}=\infty$

$\therefore$ Slope of line $BE \perp AC = \frac{1}{\infty}=0$

$\therefore$ Equation of BE : $(y - 0) = 0\,(x - 3) \Rightarrow y = 0$ ...(ii)

From (i), $x = 0$

Hence, the orthocentre is **(0, 0).**

SELF ASSESSMENT SHEET

1. If the sum of the squares of the distances of the point (x, y) from the points $(a, 0)$ and $(-a, 0)$ be $2b^2$, then which of the following is correct ?

(a) $x^2 + a^2 = b^2 + y^2$ (b) $x^2 + a^2 = 2b^2 - y^2$

(c) $x^2 - a^2 = b^2 + y^2$ (d) $x^2 + a^2 = b^2 - y^2$

(*NDA/NA 2011*)

2. If the points $A(1, 2)$, $B(2, 4)$ and $C(3, a)$ are collinear, what is the length of BC ?

(a) $\sqrt{2}$ units (b) $\sqrt{3}$ units (c) $\sqrt{5}$ units (d) 5 units

3. The mid-point of the line joining the points (–10, 8) and (–6, 12) divides the line joining the points (4, –2) and (– 2, 4) in the ratio

(*a*) 1 : 2 internally (*b*) 1 : 2 externally
(*c*) 2 : 1 internally (*d*) 2 : 1 externally

(*Kerala PET 2007*)

4. The two vertices of a triangle are (2, –1), (3, 2) and the third vertex lies on the line $x + y = 5$. The area of the triangle is 4 units. The third vertex is

(*a*) (0, 5) or (1, 4) (*b*) (5, 0) or (4, 1)
(*c*) (5, 0) or (1, 4) (*d*) (0, 5) or (4, 1)

5. What is the equation of the straight line which passes through (3, 4) and the sum of whose *x*-intercept and *y*-intercept is 14 ?

(*a*) $4x + 3y = 24$ (*b*) $x + y = 14$
(*c*) $4x - 3y = 0$ (*d*) $3x + 4y = 25$

(*NDA/NA 2013*)

6. What is the equation of the line passing through (2, –3) and parallel to the *y*-axis ?

(*a*) $y = -3$ (*b*) $y = 2$ (*c*) $x = 2$ (*d*) $x = -3$

(*NDA/NA 2011*)

7. The acute angle which the perpendicular from the origin on the line $7x - 3y = 4$ makes with the *x*-axis is:

(*a*) zero (*b*) positive but not $\pi/4$
(*c*) negative (*d*) $\pi/4$ **(*NDA/NA2012*)**

8. A straight line passes through the points (5, 0) and (0, 3). The length of the perpendicular from the point (4, 4) on the line is:

(*a*) $\frac{\sqrt{17}}{2}$ (*b*) $\frac{\sqrt{17}}{2}$ (*c*) $\frac{15}{\sqrt{34}}$ (*d*) $\frac{17}{2}$

(*NDA/NA 2013*)

9. What is the acute angle between the two straight lines $y = (2 - \sqrt{3})\ x + 5$ and $y = (2 + \sqrt{3})\ x - 7$?

(*a*) 60° (*b*) 45° (*c*) 30° (*d*) 15°

10. What is the product of the perpendiculars from the two points $\left(\pm\sqrt{b^2 - a^2}, 0\right)$ to the line $ax \cos\phi + by \sin\phi = ab$?

(*a*) a^2 (*b*) b^2 (*c*) ab (*d*) a/b

11. If (–2, 6) is the image of the point (4, 2) with respect to the line $L = 0$, then L is equal to

(*a*) $2x + 3y - 5$ (*b*) $3x - 2y + 5$
(*c*) $4x + 6y - 5$ (*d*) $6x - 4y + 5$

12. The straight line $ax + by + c = 0$ and the co-ordinate axes form an isosceles triangle under which of the following conditions ?

(*a*) $|a| = |b|$ (*b*) $|a| = |c|$
(*c*) $|b| = |c|$ (*d*) None of these

ANSWERS

1. (*d*) **2.** (*c*) **3.** (*d*) **4.** (*c*) **5.** (*a*) **6.** (*c*) **7.** (*c*) **8.** (*b*) **9.** (*a*) **10.** (*a*)
11. (*b*) **12.** (*a*)

HINTS AND SOLUTIONS

1. Let $P(x, y)$, $A(a, 0)$ and $B(-a, 0)$ be the given points. Then, $PA^2 + PB^2 = 2b^2$

$\Rightarrow (x - a)^2 + (y - 0)^2 + (x + a)^2 + (y - 0)^2 = 2b^2$

$\Rightarrow x^2 - 2ax + a^2 + y^2 + x^2 + 2ax + a^2 + y^2 = 2b^2$

$\Rightarrow x^2 + a^2 + y^2 = b^2$

$\Rightarrow \boldsymbol{x^2 + a^2 = b^2 - y^2}$**.**

2. Area of $\Delta ABC = 0$ for collinearity of A, B, C.

$\Rightarrow \frac{1}{2}[1(4 - a) + 2(a - 2) + 3(2 - 4)] = 0$

$\Rightarrow 4 - a + 2a - 4 + 6 - 12 = 0$

$\Rightarrow a - 6 = 0 \Rightarrow a = 6.$

$\therefore$ Point $C \equiv (3, 6)$

$\Rightarrow BC = \sqrt{(3-2)^2 + (6-4)^2}$

$= \sqrt{1 + 4} = \sqrt{5}$ **units.**

3. The mid-point of the line joining the points (–10, 8) and (– 6, 12) is $\left(\frac{-10 + (-6)}{2}, \frac{8 + 12}{2}\right)$, *i.e.*, (–8, 10).

Let (–8, 10) divide the join of (4, –2) and (–2, 4) in the ratio $k : 1$.

Then, $-8 = \frac{-2k + 4}{k + 1}$ and $10 = \frac{4k - 2}{k + 1}$

$\Rightarrow -8k - 8 = -2k + 4 \Rightarrow -6k = 12 \Rightarrow k = -2$

Since the value of k is negative, it is a case of **external division** and the ratio is **2 : 1.**

4. Let the third vertex of the triangle be $P(a, b)$.

Since it lies on the line $x + y = 5$, $a + b = 5$ ...(*i*)

Also, given area of triangle formed by the points (2, –1), (3, 2) and (a, b) = 4 units

$\Rightarrow \frac{1}{2}[2(2 - b) + 3(b + 1) + a(-1 - 2)] = \pm 4$

$\Rightarrow [4 - 2b + 3b + 3 - 3a] = \pm 8$

$\Rightarrow -3a + b = 1$ ...(*ii*) or $-3a + b = -15$ ...(*iii*)

Eqn (*i*) – Eqn (*ii*) $\Rightarrow (a + b) - (-3a + b) = 5 - 1$

$\Rightarrow 4a = 4 \Rightarrow a = 1 \Rightarrow b = 4$

Eqn (*i*) – Eqn (*iii*) $\Rightarrow (a + b) - (-3a + b) = 5 + 15$

$\Rightarrow$ $4a = 20 \Rightarrow a = 5 \Rightarrow b = 0$.

$\therefore$ The points are **(1, 4)** and **(5, 0).**

5. Let the x-intercept $= a$. Then, y-intercept $= 14 - a$

$\therefore$ Eqn of the straight line is $\frac{x}{a} + \frac{y}{14-a} = 1$

Since it passes through (3, 4), so

$$\frac{3}{a} + \frac{4}{14-a} = 1$$

$\Rightarrow$ $3(14 - a) + 4a = a(14 - a)$

$\Rightarrow$ $42 - 3a + 4a = 14a - a^2$

$\Rightarrow$ $a^2 - 13a + 42 = 0$

$\Rightarrow$ $(a - 7)(a - 6) = 0 \Rightarrow a = 7$ or 6.

$\therefore$ Eqn is $\frac{x}{7} + \frac{y}{7} = 1 \Rightarrow \boldsymbol{x + y = 7}$

or $\frac{x}{6} + \frac{y}{8} = 1 \Rightarrow 8x + 6y = 48 \Rightarrow \boldsymbol{4x + 3y = 24.}$

6. Slope of y-axis = tan 90º ($\because y$-axis $\perp x$-axis)

$\therefore$ Equation of line passing through (2, –3) parallel to y-axis is $(y + 3) = \tan 90^\circ (x - 2)$

$\Rightarrow$ $(y + 3) = \infty (x - 2) \Rightarrow (x - 2) = \frac{1}{\infty}(y + 3) = 0 \Rightarrow \boldsymbol{x = 2.}$

7. As the line from the origin is perpendicular to the line $7x - 3y = 4$, so its slope $= \frac{-1}{\text{slope of } 7x - 3y = 4}$

Slope of $7x - 3y - 4 = \frac{7}{3}$

$\therefore$ Slope of line from origin $= \frac{-1}{+7/3} = \frac{-3}{7}$

$\therefore$ $\tan\theta = \frac{-3}{7}$, where θ is the angle the line makes with the +ve direction of x-axis

$\Rightarrow$ $\theta = \tan^{-1}\left(\frac{-3}{7}\right) = -\tan^{-1}\left(\frac{3}{7}\right)$

8. Equation of the line through the points (5, 0) and (0, 3)

$$y - 0 = \frac{3-0}{0-5}(x - 5)$$

$\Rightarrow$ $y = \frac{-3}{5}(x - 5)$

$\Rightarrow$ $5y + 3x - 15 = 0$

$\therefore$ Distance of perpendicular from point (4, 4) on the line $5y + 3x - 15 = 0$ is $\left|\frac{5 \times 4 + 3 \times 4 - 15}{\sqrt{5^2 + 3^2}}\right|$

$= \frac{|20 + 12 - 15|}{\sqrt{25 + 9}} = \frac{17}{\sqrt{34}}$ units. $= \sqrt{\frac{\mathbf{17}}{\mathbf{2}}}$ **units.**

9. The two lines are:

$y = \left(2 - \sqrt{3}\right)x + 5$ ...(i)

$y = \left(2 + \sqrt{3}\right)x - 7$ ...(ii)

Slope of line (i), $m_1 = 2 - \sqrt{3}$

Slope of line (ii), $m_2 = 2 + \sqrt{3}$

If θ is the angle between the two lines, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{(2-\sqrt{3}) - (2+\sqrt{3})}{1 + (2-\sqrt{3})(2+\sqrt{3})}\right|$$

$$= \left|\frac{-2\sqrt{3}}{1+1}\right| = \sqrt{3}$$

$\therefore$ $\theta = \tan^{-1}(\sqrt{3}) = \mathbf{60^\circ}$.

10. Given, $ax\cos\phi + by\sin\phi - ab = 0$.

$\therefore$ Perpendicular distance of the given line from $\left(+\sqrt{b^2 - a^2}, 0\right)$

$$d_1 = \left|\frac{a\sqrt{b^2 - a^2}\cos\phi - ab}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\right|$$

$\therefore$ Perpendicular distance of the given line from $\left(-\sqrt{b^2 - a^2}, 0\right)$

$$d_2 = \left|\frac{-a\sqrt{b^2 - a^2}\cos\phi - ab}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\right|$$

$$\therefore\ d_1 d_2 = \left|-\frac{[a^2(b^2 - a^2)\cos^2\phi - a^2b^2]}{a^2\cos^2\phi + b^2\sin^2\phi}\right|$$

$$= \left|-\frac{a^2(-b^2\cos^2\phi - a^2\cos^2\phi)}{a^2\cos^2\phi + b^2\sin^2\phi}\right| = \boldsymbol{a^2.}$$

11. Let the equation of the L be $y = mx + c$

Since O' (–2, 6) is the image of the point O (4, 2) in line $L = 0$, the mid-point of OO', *i.e.*, $\left(\frac{-2+4}{2}, \frac{6+2}{2}\right)$, *i.e.*, (1, 4) will lie on the given line.

Also, $L \perp OO'$, so

Slope of L × Slope of $OO' = -1$

$\Rightarrow$ $m \times \left(\frac{6-2}{-2-4}\right) = -1 \Rightarrow m = \frac{-1}{\frac{-2}{3}} = \frac{3}{2}$

$\therefore$ Equation of L is $y = \frac{3}{2}x + c$

$\because$ It passes through (1, 4)

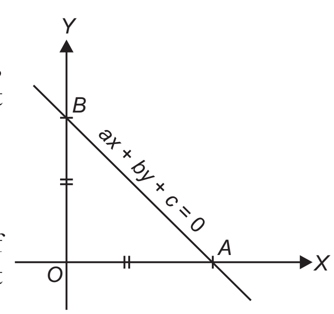
$4 = \frac{3}{2} \times 1 + c \Rightarrow c = 4 - \frac{3}{2} = \frac{5}{2}$.

$\therefore$ Required equation is $y = \frac{3}{2}x + \frac{5}{2}$

$\Rightarrow$ $3x - 2y + 5 = 0$

$\therefore$ $\boldsymbol{L = 3x - 2y + 5}$

12. The equation of line AB, *i.e.*, $ax + by + c = 0$ in intercept form is $ax + by = -c$



$\Rightarrow$ $\frac{x}{(-c/a)} + \frac{y}{(-c/b)} = 1$

$\Delta\, AOB$ is isosceles Δ if $OA = OB$, *i.e.*, x-intercept $= y$ – intercept

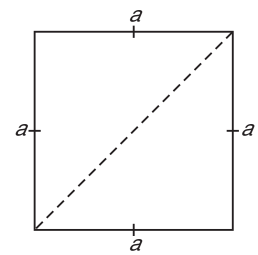
$\Rightarrow$ $\frac{-c}{a} = \frac{-c}{b} \Rightarrow \frac{1}{a} = \frac{1}{b} \Rightarrow \boldsymbol{|a| = |b|.}$

12 Area and Perimeter

KEY FACTS

1. SQUARE

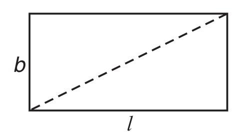
- **Perimeter of a square of side a = $4a$ = 4 × side**
- **Diagonal of a square = $a\sqrt{2}$ = side$\sqrt{2}$**
- **Area of a square = a^2 = side^2**
- **Also, Area of a square = $a^2 = \dfrac{(a\sqrt{2})^2}{2} = \dfrac{(\text{Diagonal})^2}{2}$**



2. RECTANGLE

- **Area of a rectangle = length × breadth = $l \times b$**
- **Perimeter of a rectangle = 2 (length + breadth) = 2 ($l + b$)**
- **Diagonal of a rectangle = $\sqrt{\text{length}^2 + \text{breadth}^2}$**

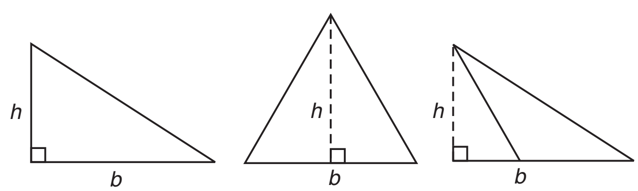
  $= \sqrt{l^2 + b^2}$



3. TRIANGLE

- **Given base and height**

  Area of a triangle = $\dfrac{1}{2}$ × base (b) × height (h), where the height of a triangle is the perpendicular distance from a vertex to the base of the triangle.

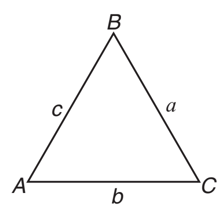


- **Given length of the three sides**

  (Heron's formula) :

  Area = $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{1}{2}(a + b + c)$,

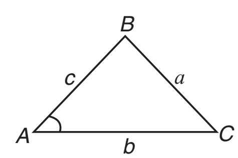
  where a, b and c are the lengths of the sides and s is the semi-perimeter of the triangle.



- **Given side, angle, side**

  If we are given the lengths of two sides of a triangle and the size of the angle between them, then

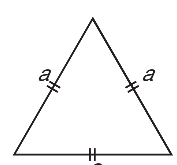
  Area of triangle = $\dfrac{1}{2}$ $ab \sin C$ or **$\dfrac{1}{2}$ $bc \sin A$** or **$\dfrac{1}{2}$ $ac \sin B$**



- **Equilateral triangle**

  Area = $\dfrac{\sqrt{3}}{4}$ (side)^2

  Altitude (h) = $\dfrac{\sqrt{3}}{2}$ side; **Perimeter = 3 × side**



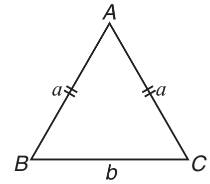
Equilateral Triangle

- **Isosceles triangle**

  Area $= \dfrac{1}{4} \times b \times \sqrt{4a^2 - b^2}$

  where each of equal sides = a, other side = b.

  Perimeter = $2a + b$



Isosceles Triangle

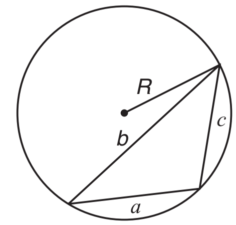
- **Circumscribed circle of a triangle**

  If a, b, c are the sides of a triangle and R the radius of the circumscribed circle, then

  Area of triangle $= \dfrac{a \times b \times c}{4R}$

  or **Radius of circumscribed circle (R)** $= \dfrac{a \times b \times c}{4(\text{Area of triangle})}$

  For an equilateral triangle, **circum-radius (R)** $= \dfrac{a \times a \times a}{4 \cdot \dfrac{\sqrt{3}}{4} a^2} = \dfrac{a}{\sqrt{3}} = \dfrac{\text{side}}{\sqrt{3}}$



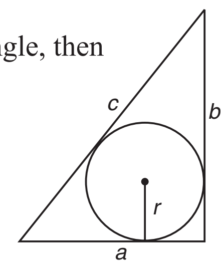
Circumscribed Circle

- **Circle inscribed in a triangle**

  If A = area of the given triangle, r = radius of inscribed circle, a, b, c are sides of the triangle, then

  $$s = \frac{a+b+c}{2} \text{ and } A = r.s \text{ or } r = \frac{A}{s} = \frac{\text{Area}}{\text{Semi-perimeter}}$$

  For an equilateral triangle, **in-radius** $= \dfrac{\dfrac{\sqrt{3}}{4} \cdot a^2}{\dfrac{3a}{2}} = \dfrac{a}{2\sqrt{3}} = \dfrac{\text{side}}{2\sqrt{3}}$

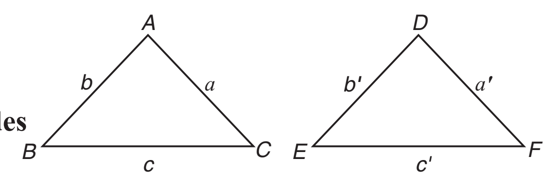


Inscribed Circle

- **For similar triangles, *ABC* and *DEF***
  - **Ratio of areas** = area (ΔABC) : area (ΔDEF)

    $= a^2 : a'^2$ or $b^2 : b'^2$ or $c^2 : c'^2$

    = **Ratio of square of corresponding sides**
  - **Ratio of perimeters = Ratio of corresponding sides.**



- **Area of a triangle, given its medians**

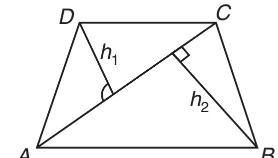
  Area $= \dfrac{4}{3}\sqrt{s_m(s_m - m_1)(s_m - m_2)(s_m - m_3)}$,

  where m_1, m_2, m_3 are the medians of the triangle and $s_m = \dfrac{m_1 + m_2 + m_3}{2}$ [See Q. 3 in Practice Sheet]

4. QUADRILATERAL

- **Area** $= \dfrac{1}{2}$ **× diagonal × sum of perpendiculars from the opposite vertices on it**

  $= \dfrac{1}{2} \times d \times (h_1 + h_2)$, where $AC = d$

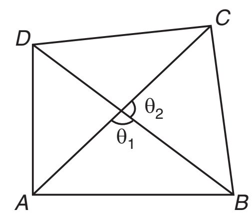


Quadrilateral

- **Area** $= \dfrac{1}{2}$ **× product of diagonals × sine of angle between them**

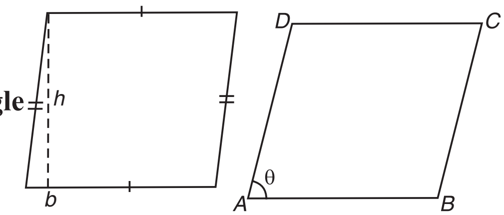
  $= \dfrac{1}{2} d_1 d_2 \sin \theta_1 = \dfrac{1}{2} d_1 d_2 \sin \theta_2$

  where $AC = d_1$ and $BD = d_2$



5. PARALLELOGRAM

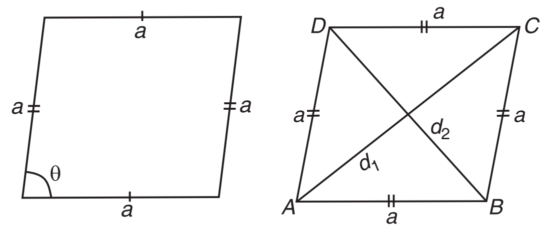
- **Area = base (b) × height (h)**
- **Area = product of any two adjacent sides × sine of included angle**
  $= AB.AD.\sin\theta$
- **Perimeter = 2 × (sum of any two adjacent sides)**



Parallelogram

6. RHOMBUS

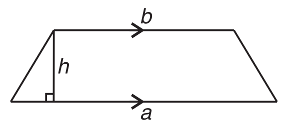
- **Area $= \frac{1}{2}$ × (Product of diagonals) $= \frac{1}{2}\, d_1 d_2$**
- **Area = Product of any two adjacent sides × sine of included angle $= a^2 \sin\theta$**
- **Perimeter $= 4a = 4$ × side**



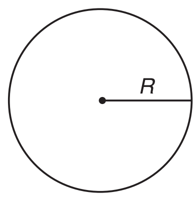
Rhombus

7. TRAPEZIUM

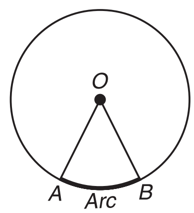
- **Area $= \frac{1}{2}$ × (sum of parallel sides) × height $= \frac{1}{2}(a+b)h$**
- **Perimeter = Sum of all four sides**



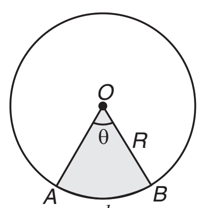
8. CIRCLE (radius = R)



- **Area $= \pi r^2$**
- **Circumference $= 2\pi R$**
- **Length of an arc (l) $= AB = 2\pi R\left(\frac{\theta}{360^\circ}\right)$**



- **Area of a sector $OAB = \pi R^2\left(\frac{\theta}{360^\circ}\right) = \frac{1}{2}$ (arc length × R)**
- **Perimeter of a sector $= l + 2R$**



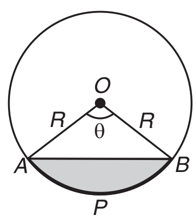
Sector

- **Area of segment APB = Area of sector $OAPB$ – Area of ΔOAB**

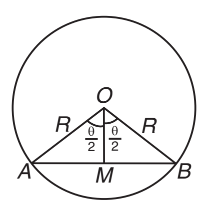
$$= \pi R^2\left(\frac{\theta}{360^\circ}\right) - \frac{R^2}{2}\sin\theta$$

- **Perimeter of segment = Arc APB + Chord AB**

$$= 2R\left[\frac{\pi\theta}{360^\circ} + \sin\left(\frac{\theta}{2}\right)\right]$$



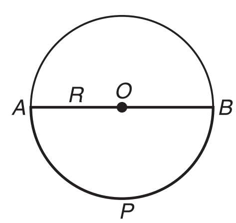
segment



$$AM = R\sin\frac{\theta}{2},\ AB = 2\,AM = 2R\sin\frac{\theta}{2}$$

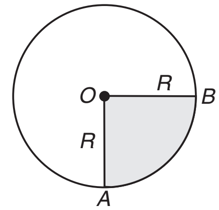
9. SEMI-CIRCLE

- **Area $= \frac{\pi R^2}{2}$ (R = radius)**
- **Perimeter = Arc APB + Diameter $AB = \pi R + 2R$**



Semi-circle

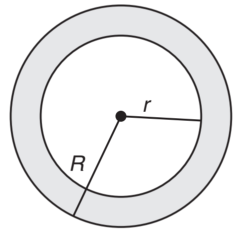
10. QUADRANT



Quadrant

- **Area =** $\dfrac{\pi R^2}{4}$ **(R = radius)**
- **Perimeter =** $\dfrac{1}{4}$**(circumference)** $+ 2R = \dfrac{1}{4} \times 2\pi R + 2R = \dfrac{1}{2}\pi R + 2R$

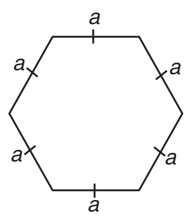
11. Area of a ring or annulus



Annulus

$= \pi(R^2 - r^2)$, where R = outer radius, r = inner radius.

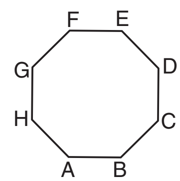
12. Area of a regular hexagon



Regular Hexagon

$= \dfrac{3\sqrt{3}}{2} a^2 = \dfrac{3\sqrt{3}}{2}$ **side**^{2}

Perimeter = 6a = 6 × side



Regular octagon

13. Area of a regular octagon $= 2a^2(1 + \sqrt{2})$, where a = side.

SOLVED EXAMPLES

Ex. 1. If the sides of a triangle are 3 cm, 4 cm and 5 cm, then what is the radius of the circum-circle?

Sol. Semi-perimeter of triangle (s) = $\dfrac{3+4+5}{2}$ cm = 6 cm

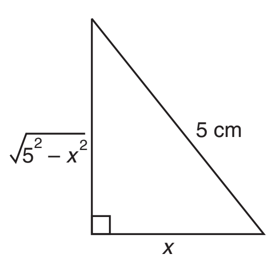
$\therefore$ Area of triangle $A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{6 \times 3 \times 2 \times 1}$ cm^2 = 6 cm^2

$\therefore$ Radius of circum-circle $= \dfrac{abc}{4(\text{Area of } \Delta)} = \dfrac{3 \times 4 \times 5}{4 \times 60}$ cm = **2.5 cm**

Ex. 2. If the length of hypotenuse of a right angled triangle is 5 cm and its area is 6 sq cm, then what are the lengths of the remaining sides? (*CDS 2001*)

Sol. Let one of the remaining sides be x cm.

Then, other side = $\sqrt{5^2 - x^2}$ cm



$\therefore$ Area $= \dfrac{1}{2} \times x \times \sqrt{25 - x^2} = 6$

$\Rightarrow x\sqrt{25 - x^2} = 12 \Rightarrow x^2(25 - x^2) = 144$

$\Rightarrow 25x^2 - x^4 = 144 \Rightarrow x^4 - 25x^2 + 144 = 0$

$\Rightarrow (x^2 - 16)(x^2 - 9) = 0$

$\Rightarrow x^2 = 16$ or $x^2 = 9 \Rightarrow x =$ **4 or 3**

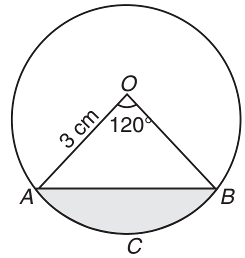
$\therefore$ The two sides are 4 cm and 3 cm.

Ex. 3. *ACB* is an arc of a circle of radius 3 cm. It subtends an angle of 120º at the centre as shown in the figure. What is the area of the shaded portion *ACB*?

Sol. Area of shaded portion ACB = Area of sector $OACB$ – Area of ΔOAB

$$= \left\{\frac{120}{360}\times\pi\,(3)^2\right\} - \left\{\frac{1}{2}\times 3\times 3\sin 120^\circ\right\}$$

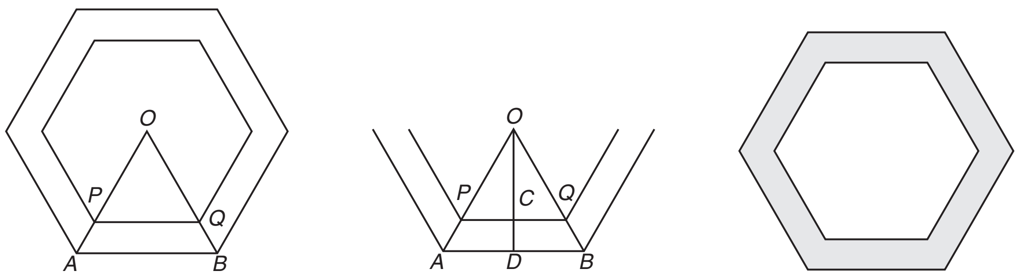
$$= 3\pi - \frac{9}{2}\times\frac{\sqrt{3}}{2} = \left(\mathbf{3\pi - \frac{9\sqrt{3}}{4}}\right)\mathbf{cm^2}.$$



Ex. 4. There are two concentric hexagons. Each of the side of both the hexagons are parallel. Each side of internal regular hexagon is 8 cm. What is the area of the shaded region, if the distance between the corresponding parallel sides is $2\sqrt{3}$ cm?

Sol. In the second figure, ΔOPQ is an equilateral triangle as a regular hexagon is divided into six equilateral triangles.

$\therefore\ \angle OPC = 60^\circ$



$$\Rightarrow \frac{OC}{OP} = \sin 60^\circ \Rightarrow OC = OP\sin 60^\circ \Rightarrow OC = 8\times\frac{\sqrt{3}}{2}\text{ cm} = 4\sqrt{3}\text{ cm.}$$

Now in similar Δs OPC and OAD, $\frac{OC}{OD} = \frac{OP}{OA} \Rightarrow \frac{4\sqrt{3}}{6\sqrt{3}} = \frac{8}{OA} \Rightarrow OA = 12$ cm $\Rightarrow AB = OA = 12$ cm

($\because\ OD = OC + 2\sqrt{3} = 4\sqrt{3}+2\sqrt{3} = 6\sqrt{3}$ cm)

$\therefore$ Required area = Area of outer hexagon – Area of inner hexagon

$$= \frac{3\sqrt{3}}{2}(12^2 - 8^2)\text{ cm}^2 = \mathbf{120\sqrt{3}\ cm^2.}$$ [Area of a regular hexagon = $\frac{3\sqrt{3}}{2}$ side^2]

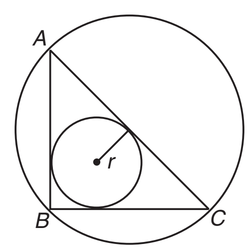
Ex. 5. ΔABC is an isosceles right angled triangle. If r is its in radius and R its circum radius, then what is $\frac{R}{r}$ equal to? **(*CDS 2007*)**

Sol. Let ABC be the right $\angle d$ isosceles triangle, where $AB = BC = x$ and $\angle ABC = 90^\circ$.

$$\therefore\quad AC = \sqrt{AB^2 + BC^2} = \sqrt{x^2+x^2} = x\sqrt{2}$$

$$s = \text{Semi-perimeter of }\Delta ABC = \frac{x+x+x\sqrt{2}}{2} = \frac{2x+x\sqrt{2}}{2} = \frac{x\sqrt{2}(\sqrt{2}+1)}{2}$$

$$A = \text{Area of }\Delta ABC = \frac{1}{2}\times x\times x = \frac{x^2}{2}$$



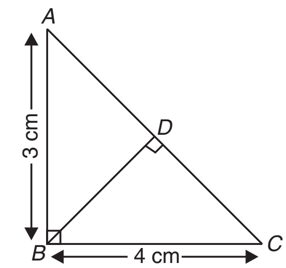
$$\therefore\ \text{In-radius of }\Delta ABC\ (r) = \frac{\text{Area}}{\text{Semi-perimeter}} = \frac{x^2/2}{x\sqrt{2}(1+\sqrt{2})/2} = \frac{x}{\sqrt{2}(1+\sqrt{2})}$$

$$\text{Circum radius of }\Delta ABC\ (R) = \frac{abc}{4\times Area} = \frac{x\times x\times x\sqrt{2}}{4\times x^2/2} = \frac{x\sqrt{2}}{2} = \frac{x}{\sqrt{2}}$$

$$\therefore\ \text{Required ratio} = \frac{R}{r} = \frac{x/\sqrt{2}}{x/\sqrt{2}\left(1+\sqrt{2}\right)} = \mathbf{1+\sqrt{2}.}$$

Ex. 6. In a right-angled triangle *ABC, D* is the foot of the perpendicular from *B* on the hypotenuse *AC.* If *AB* = 3 cm and *BC* = 4 cm, what is the area of ΔABD?

Sol. Area of $\Delta ABC = \frac{1}{2} \times 3 \times 4 \text{ cm}^2 = 6 \text{ cm}^2$. Also, $AC = \sqrt{3^2 + 4^2} = 5$ cm.



$\therefore$ Area of $\Delta ABC = \frac{1}{2} \times BD \times AC \Rightarrow 6 = \frac{1}{2} \times BD \times 5 \Rightarrow BD = \frac{12}{5}$ cm.

Now in ΔABD, $AD = \sqrt{AB^2 - BD^2} = \sqrt{3^2 - \left(\frac{12}{5}\right)^2} = \sqrt{9 - \frac{144}{25}}$

$= \sqrt{\frac{225-144}{25}} = \sqrt{\frac{81}{25}} = \frac{9}{5}$ cm

$\therefore$ Area of $\Delta ABD = \frac{1}{2} \times AD \times BD = \frac{1}{2} \times \frac{9}{5} \times \frac{12}{5} = \mathbf{\frac{54}{25}}$ **cm²**.

Ex. 7. One side of an equilateral triangle is 24 cm. The mid-points of its sides are joined to form another triangle whose mid-points are in turn joined to form still another triangle. This process continues indefinitely. Find the sum of the perimeters of all the triangles. ***(SNAP 2011)***

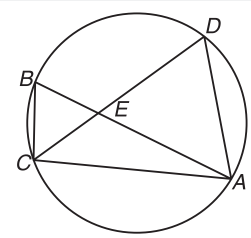
Sol. Perimeter of the largest (outermost) equilateral triangle = 3 × 24 = 72 cm.
Now, the perimeter of the triangle formed by joining the midpoints of a given triangle will be half the perimeter of the original triangle.
$\therefore$ Required sum = 72 + 36 + 18 + ... upto infinite terms

This is an infinite *GP*, where first term $\boldsymbol{a} = 72$ and common ratio $\boldsymbol{r} = \frac{1}{2}$.

$\therefore$ Required sum $= \frac{a}{1-r} = \frac{72}{1-\frac{1}{2}} = \frac{72}{\frac{1}{2}} = 72 \times 2 =$ **144 cm.**

Ex. 8. In the adjoining figure, points *A, B, C* and *D* lie on a circle. *AD* = 24 and *BC* = 12. What is the ratio of the area of the triangle *CBE* to that of triangle *ADE?* ***(CAT)***

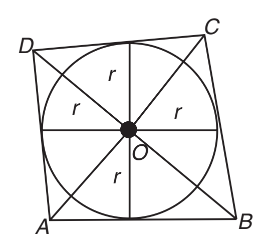
Sol. $AD = 24$, $BC = 12$.



In ΔCBE and ΔADE,
$\angle CBA = \angle CDA$, $\angle BCE = \angle DAE$ (Angles in the same segment are equal)
$\angle BEC = \angle DEA$ (vertical opposite angles are equal)
$\Rightarrow \Delta BCE$ and ΔDEA are similar Δs with sides in the ratio 1 : 2.
$\therefore$ Ratio of areas = Ratio of square of sides = $1^2 : 2^2 =$ **1 : 4**

Ex. 9. If a circular sheet of perimeter $2\pi r$ touching each side of a given quadrilateral sheet of perimeter $2p$ is cut off from the quadrilateral, then what is the area of the remaining sheet?

Sol. Let *ABCD* be the quadrilateral from which a circular sheet is cut off touching each side of the quadrilateral.
Also, given $AB + BC + CD + DA = 2p$ ...(*i*)
Circumference of the circle = $2\pi r \Rightarrow$ Radius of circle = r



$\therefore$ Area of quadrilateral = Area of ($\Delta OAB + \Delta OBC + \Delta OCD + \Delta ODA$)

$= \frac{1}{2} r\,(AB) + \frac{1}{2} r\,(BC) + \frac{1}{2} r\,(CD) + \frac{1}{2} r\,(DA)$

$= \frac{1}{2} r\,(AB + BC + CD + DA) = \frac{1}{2} r.\, 2p = pr.$

$\therefore$ Required remaining area = Area of quadrilateral – Area of circle = $pr - \pi r^2 = \mathbf{r(p - \pi r)}$.

PRACTICE SHEET

1. The diagonal of a square A is $(a + b)$. The diagonal of a square whose area is twice the area of square A is.

(*a*) $2(a + b)$ (*b*) $2(a + b)^2$ (*c*) $\sqrt{2}\,(a + b)$ (*d*) $\sqrt{2}\,(a - b)$

(*SSC 2002*)

2. The lengths of the perpendiculars drawn from any point in the interior of an equilateral triangle to the respective sides are p_1, p_2 and p_3. The length of each side of the triangle is

(*a*) $\frac{2}{\sqrt{3}}(p_1 + p_2 + p_3)$ (*b*) $\frac{1}{3}(p_1 + p_2 + p_3)$

(*c*) $\frac{1}{\sqrt{3}}(p_1 + p_2 + p_3)$ (*d*) $\frac{4}{\sqrt{3}}(p_1 + p_2 + p_3)$

(*SSC 2004*)

3. The lengths of three medians of a triangle are 9 cm, 12 cm and 15 cm. The area (in sq. cm) of this triangle is

(*a*) 24 (*b*) 72 (*c*) 48 (*d*) 144

(*SSC 2012*)

4. The perimeter of a triangular field is 240 m. If two of its sides are 78 m and 50 m, then what is the length of the perpendicular on the side of length 50 m from the opposite vertex?

(*a*) 43 m (*b*) 52.2 m (*c*) 67.2 m (*d*) 70 m

(*CDS 2011*)

5. What is the radius of a circle inscribed in a triangle having side lengths 35 cm, 44 cm and 75 cm?

(*a*) 3 cm (*b*) 4 cm (*c*) 5 cm (*d*) 6 cm

(*CDS 2010*)

6. The percentage increase in the area of a triangle if each of its side is doubled is

(*a*) 200% (*b*) 50% (*c*) 300% (*d*) 100%

7. If the area of a circle, inscribed in an equilateral triangle is 4π cm^2, then what is the area of the triangle?

(*a*) $12\sqrt{3}$ cm^2 (*b*) $9\sqrt{3}$ cm^2

(*c*) $8\sqrt{3}$ cm^2 (*d*) 18 cm^2 **(*CDS 2008*)**

8. A circle is inscribed in an equilateral triangle of side a. What is the area of any square inscribed in this circle?

(*a*) $\frac{a^2}{3}$ (*b*) $\frac{a^2}{4}$ (*c*) $\frac{a^2}{6}$ (*d*) $\frac{a^2}{8}$

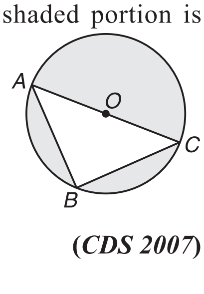
(*CDS 2007*)

9. If the distance from the vertex to the centroid of an equilateral triangle is 6 cm, then what is the area of the triangle?

(*a*) 24 cm^2 (*b*) $27\sqrt{3}$ cm^2 (*c*) 12 cm^2 (*d*) $12\sqrt{3}$ cm^2

(*CDS 2007*)

10. In the given figure, the area of the shaded portion is $(\pi - 1)R^2$, where R is the radius of the circle with centre at O. What is one of the acute angles of ΔABC ?



(*a*) 30º (*b*) 45º

(*c*) 60º (*d*) Cannot be determined.

(*CDS 2007*)

11. The area of an isosceles triangle is a, when the angle included between the two equal sides is 60º. What will be the area if the angle included between the two equal sides becomes 120º? (Keeping the length of equal sides same as before)

(*a*) $a/2$ (*b*) a (*c*) $3a/2$ (*d*) $2a$

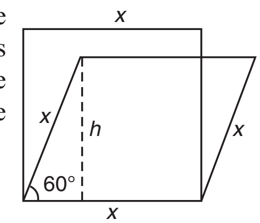
(*CDS 2006*)

12. If O is the centre of the circle and chord $AB = OA$ and the area of triangle $AOB = 4\sqrt{3}$ cm^2, then the area of the circle will be

(*a*) 4π cm^2 (*b*) 16π cm^2

(*c*) $24\sqrt{3}$ cm^2 (*d*) $24\pi\sqrt{3}$ cm^2 **(*CDS*)**

13. A square and a rhombus have the same base. If the rhombus is inclined at 60º, find the ratio of the area of the square to the area of the rhombus.



(*a*) $2\sqrt{3}:3$ (*b*) $1:\sqrt{3}$

(*c*) $\sqrt{3}:2$ (*d*) None of these

14. Find the area of regular octagon with each side 'a' cm.

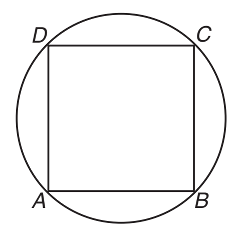
(*a*) $2a^2(1 + \sqrt{2})$ (*b*) $\sqrt{2}a\,(1 + \pi)$

(*c*) $a^2(\sqrt{2} + 2)$ (*d*) None of these

15. What is the ratio of the area of a circum circle of equilateral triangle to the area of the square with same side length as equilateral triangle?

(*a*) $\pi : \sqrt{3}$ (*b*) $\sqrt{3} : 2$ (*c*) $1 : \sqrt{2}$ (*d*) $\pi : 3$

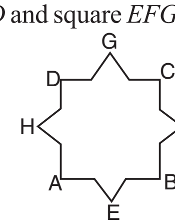
16. $ABCD$ is a square of side a cm. AB, BC, CD and AD are all chords of circles with equal radii each. If the chords subtend an angle of 120º at their respective centres, find the total area of the given figure, where arcs are a part of the circle.



(*a*) $\left[a^2 + 4\left[\frac{\pi a^2}{9} - \frac{a^2}{3\sqrt{2}}\right]\right]$ (*b*) $\left[a^2 + 4\left(\frac{\pi a^2}{9} - \frac{a^2}{4\sqrt{3}}\right)\right]$

(*c*) $[9a^2 - 4\pi + 3\sqrt{3}\,a^2]$ (*d*) None of these

17. $ABCD$ is a square. Another square $EFGH$ with the same area is placed on the square $ABCD$ such that the point of intersection of diagonals of square $ABCD$ and square $EFGH$ coincide and the sides of the square $EFGH$ are parallel to the diagonals of square $ABCD$. Thus a new figure is formed as shown in the figure. What is the area enclosed by the given figure, if each side of the square is 4 cm?



(*a*) $32(2 - \sqrt{2})$ (*b*) $16\left(\frac{3 + \sqrt{2}}{2 + \sqrt{2}}\right)$

(*c*) $32\left(\frac{2 + \sqrt{2}}{3 + \sqrt{2}}\right)$ (*d*) None of these

18. Find the area of an equilateral triangle inscribed in a circle circumscribed by a square made by joining the mid-points of the adjacent sides of a square of side 'a'.

(*a*) $\dfrac{3a^2}{16}$ (*b*) $\dfrac{3\sqrt{3}a^2}{16}$

(*c*) $\dfrac{3a^2(\pi-12)}{4}$ (*d*) $\dfrac{3\sqrt{3}a^2}{32}$

19. A goat is tethered to one end of a rope of length 20m, while the other end is fixed at the centre of a large circular field. There is a square elevated platform with sides of 10 m on the field such that one corner of the elevated square platform councides with centre of the circular field. If the goat is unable to mount the square elevated platform, what is the total area the goat will be able to graze?

(*a*) $3 \times 100\ \pi$ (*b*) $3.5 \times 100\ \pi$

(*c*) $4 \times 100\ \pi$ (*d*) $100(3\pi + 1)$

(***JMET 2009***)

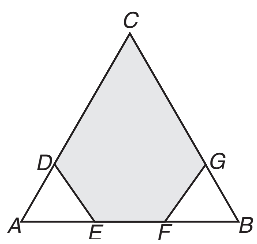
20. A semi-circle of diameter 14 cm has three chords of equal length connecting the two end points of the diameter so as to form a trapezoid inscribed within a semicircle. What is the value of the area enclosed by the trapezoid?

(*a*) $\left(\dfrac{157\times\sqrt{3}}{4}\right)$ cm^2 (*b*) $(49 \times \sqrt{3}\ )$ cm^2

(*c*) $\left(\dfrac{147\times\sqrt{3}}{4}\right)$ cm^2 (*d*) $\left(\dfrac{100}{\sqrt{3}}\right)$ cm^2

(***JMET 2009***)

21. In the given figure, ABC is a triangle in which $CDEFG$ is a pentagon. Triangles ADE and BFG are equilateral triangles each with side 2 cm and $EF = 2$ cm. Find the area of the pentagon.



$AE = EF = FB$

(*a*) $8\sqrt{3}$ cm^2

(*b*) $7\sqrt{3}$ cm^2

(*c*) $15\sqrt{3}$ cm^2

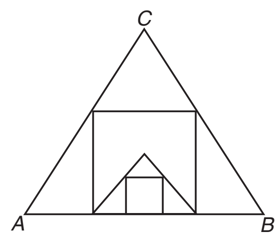
(*d*) 11.28 cm^2

22. The hypotenuse of an isosceles right-angled triangle is q. If we describe equilateral triangles (outwards) on all its three sides, then the total area of the re-entrant hexagon thus obtained is.

(*a*) $q^2\ (\sqrt{3}\ + 2)$ (*b*) $q^2\ \dfrac{\left(2\sqrt{3}+1\right)}{4}$

(*c*) $\dfrac{q^2(4\sqrt{3}-1)}{4}$ (*d*) $\dfrac{q^2(5\sqrt{3}-2)}{4}$

23. In the adjoining figure, ABC is an equilateral triangle inscribing a square of maximum possible area. Again in this squares there is an equilateral triangle whose side is same as that of the square. Further the smaller equilateral triangle inscribes a square of maximum possible area. What is the area of the innermost square if each side of the outermost triangle be 0.01 m ?



(*a*) $(873 - 504\sqrt{3})$ cm^2

(*b*) $(738 - 504\sqrt{3})$ cm^2

(*c*) $(873 - 405\sqrt{2})$ cm^2

(*d*) None of these

24. A piece of paper is in the shape of a right-angled triangle and is cut along a line that is parallel to the hypotenuse, leaving a smaller triangle. There was 35% reduction in the length of the hypotenuse of the triangle. If the area of the original triangle was 34 square inches before the cut, what is the area (in square inches) of the smaller triangle ?

(*a*) 16.665 (*b*) 16.565

(*c*) 15.465 (*d*) 14.365

(***CAT 2004***)

ANSWERS

1. (*c*)	**2.** (*a*)	**3.** (*b*)	**4.** (*c*)	**5.** (*d*)	**6.** (*c*)	**7.** (*a*)	**8.** (*c*)	**9.** (*b*)	**10.** (*b*)
11. (*b*)	**12.** (*b*)	**13.** (*a*)	**14.** (*a*)	**15.** (*d*)	**16.** (*b*)	**17.** (*a*)	**18.** (*d*)	**19.** (*b*)	**20.** (*c*)
21. (*b*)	**22.** (*b*)	**23.** (*a*)	**24.** (*d*)						

HINTS AND SOLUTIONS

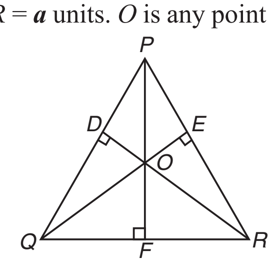
1. Area of new square = 2 (Area of the square A)

$$= 2\times\frac{(\text{diagonal})^2}{2} = 2\times\frac{(a+b)^2}{2} = (a+b)^2$$

$\therefore$ Each side of new square = $(a + b)$

$\Rightarrow$ Diagonal of new square = side $\sqrt{2}$ = $\boldsymbol{(a + b)\sqrt{2}.}$

2. Let each side of equilateral ΔPQR = $\boldsymbol{a}$ units. O is any point in the interior of ΔPQR



$\Rightarrow OD = p_1$, $OE = p_2$ and $OF = p_3$ are perpendiculars on sides PQ, PR and QR respectively.

$\therefore$ Area of ΔPQR

$= \text{Area of } \Delta OPQ + \text{Area of } \Delta OPR + \text{Area of } \Delta OQR$

$= \frac{1}{2} \times a \times p_1 + \frac{1}{2} \times a \times p_2 + \frac{1}{2} \times a \times p_3$

$= \frac{a}{2}(p_1 + p_2 + p_3)$

$\Rightarrow \frac{\sqrt{3}}{4}a^2 = \frac{a}{2}(p_1 + p_2 + p_3) \Rightarrow a = \mathbf{\frac{2}{\sqrt{3}}(p_1 + p_2 + p_3)}.$

3. Here $s_m = \frac{9+12+15}{2} = 18$ cm, where lengths of medians are $m_1 = 9$ cm, $m_2 = 12$ cm, $m_3 = 15$ cm.

$\therefore$ Area of triangle $= \frac{4}{3}\sqrt{18(18-9)(18-12)(18-15)}$ cm^2

[Refer to key facts]

$= \frac{4}{3}\sqrt{18 \times 9 \times 6 \times 3}$ cm^2 $= \frac{4}{3} \times 9 \times 6$ cm^2 $=$ **72 cm²**.

4. Given $2s = a + b + c \Rightarrow 240 = 78 + 50 +$ Third side

$\Rightarrow$ Third side $= 240$ m $- 128$ m $= 112$ m.

$\therefore$ Area of $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{120 \times (120-78) \times (120-50) \times (120-112)}$

$= \sqrt{120 \times 42 \times 70 \times 8} = 1680$

Also, Area of $\Delta = \frac{1}{2} \times b \times h$

$\therefore \frac{1}{2} \times 50 \times h = 1680 \Rightarrow h = \frac{1680}{25} =$ **67.2 m.**

5. Let $a = 35$ cm, $b = 44$ cm, $c = 75$ cm. Then

$s = \frac{a+b+c}{2} = \frac{35+44+75}{2}$ cm $= 77$ cm

$\therefore$ Area of triangle $= \sqrt{77(77-35)(77-44)(77-75)}$cm^2

$= \sqrt{77 \times 42 \times 33 \times 2}$ cm^2

$= (7 \times 11 \times 6)$ cm$^2 = 462$ cm^2

$\therefore$ Radius of incircle $= \frac{\text{Area}}{\text{semi-perimeter}} = \frac{462}{77}$ cm $=$ **6 cm.**

6. Let a, b, c be the sides of the original triangle and s be its semi-perimeter.

Then, $s = \frac{1}{2}(a+b+c)$

Let s_1 be the semi-perimeter of the new triangle. Then,

$s_1 = \frac{1}{2}(2a + 2b + 2c) = (a+b+c) = 2s$

$\therefore$ Area A of original $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ and

Area A_1 of new $\Delta = \sqrt{s_1(s_1-2a)(s_1-2b)(s_1-2c)}$

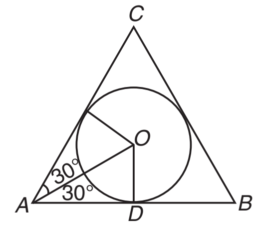
$= \sqrt{2s(2s-2a)(2s-2b)(2s-2c)}$

$= 4\sqrt{s(s-a)(s-b)(s-c)} = 4A.$

$\therefore$ Percentage increase in area $= \left(\frac{A_1 - A}{A} \times 100\right)\%$

$= \frac{(4A - A)}{A} \times 100\% =$ **300%.**

7. Since area of circle $= 4\pi$



$\Rightarrow \pi r^2 = 4\pi \Rightarrow r = 2$ cm

$\therefore$ In ΔOAD,

$\tan 30° = \frac{OD}{AD} \Rightarrow \frac{1}{\sqrt{3}} = \frac{2}{AD}$

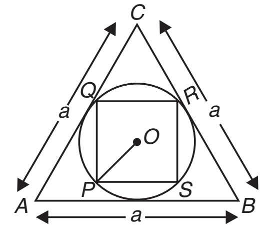
$\Rightarrow AD = 2\sqrt{3}$ cm

$\therefore AB = 2AD = 4\sqrt{3}$ cm

$\therefore$ Area of equilateral $\Delta ABC = \frac{\sqrt{3}}{4}(AB)^2$

$= \frac{\sqrt{3}}{4}(4\sqrt{3})^2 =$ $\mathbf{12\sqrt{3}\ cm^2}$.

8. If 'a' is length of the side of ΔABC, then



Area of $\Delta ABC = \frac{\sqrt{3}}{4}a^2$

semi-perimeter of $\Delta ABC = \frac{3a}{2}$

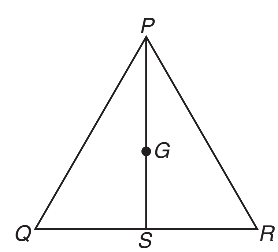
$\therefore$ Radius of in-circle $= \frac{\text{Area}}{\text{semi-perimeter}} = \frac{\sqrt{3}}{4}a^2 \times \frac{2}{3a} = \frac{a}{2\sqrt{3}}$

$\therefore$ Diagonal of square $PQRS$ = Diameter of incircle.

$= 2 \times \frac{a}{2\sqrt{3}} = \frac{a}{\sqrt{3}}$

$\therefore$ Area of square $= \frac{(\text{diagonal})^2}{2} = \frac{(a/\sqrt{3})^2}{2} = \mathbf{\frac{a^2}{6}}$.

9. Let G be the centroid of ΔPQR.



Then, $PG = 6$ cm.

Now, $\frac{PG}{GS} = \frac{2}{1} \Rightarrow GS = 3$ cm

$\therefore PS = PG + GS = 9$ cm ...(i)

$\therefore$ If a is the length of a side of ΔPQR, then ΔPQR being equilateral, $PS \perp QR$

$\therefore$ Altitude $PS = \frac{\sqrt{3}}{2}a = 9$ (From (i))

$\Rightarrow a = \frac{9 \times 2}{\sqrt{3}} = 6\sqrt{3}$ cm

$\therefore$ Area of equilateral $\Delta PQR = \frac{\sqrt{3}}{4}(a)^2$

$= \frac{\sqrt{3}}{4} \times (6\sqrt{3})^2$ cm^2 $=$ $\mathbf{27\sqrt{3}\ cm^2}$.

10. Area of triangle ABC

= Area of circle – Area of shaded region

$= \pi R^2 - (\pi - 1)R^2 = R^2$

But Area of $\Delta ABC = \frac{1}{2} \times AB \times AC$

($\angle ABC = 90°$, angle in a semicircle is a right angle)

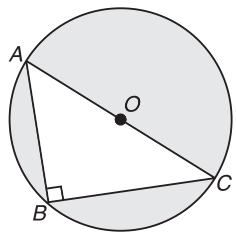
$\Rightarrow \frac{1}{2} \times AB \times BC = R^2 \Rightarrow AB \times BC = 2R^2$ ...(*i*)

In ΔABC, $AC^2 = AB^2 + BC^2$

$\Rightarrow (2R)^2 = AB^2 + BC^2$

$\Rightarrow 2 \times AB \times BC = AB^2 + BC^2$

[Using (*i*)]



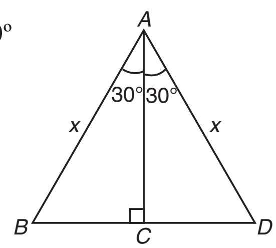
$\Rightarrow AB^2 + BC^2 - 2AB\ BC = 0$

$\Rightarrow (AB - BC)^2 = 0$

$\Rightarrow AB = BC = \sqrt{2}\ R$

$\therefore \tan A = \frac{BC}{AB} = \frac{\sqrt{2}R}{\sqrt{2}R} = 1 \Rightarrow \angle \boldsymbol{A} = \mathbf{45°}.$

11. In isosceles ΔABC, $\frac{BC}{x} = \sin 30°$



$\Rightarrow BC = x \sin 30°$

$\Rightarrow BD = 2x \sin 30°$

and $\frac{AC}{AB} = \cos 30°$

$\Rightarrow AC = x \cos 30°$

$\therefore$ Area of $\Delta ABC = a = \frac{1}{2} \times BD \times AC$

$= \frac{1}{2} \times 2x \sin 30° \times x \cos 30°$

$= x^2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} x^2 = a$ (given)

$\Rightarrow \quad x^2 = \frac{4}{\sqrt{3}} a$

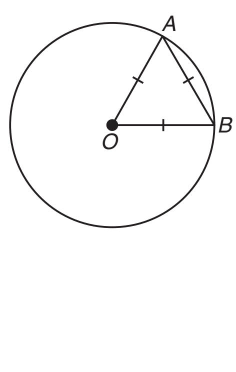
Now angle between equal sides = 120°

$\therefore$ Area of triangle $= \frac{1}{2} x^2 \sin 120°$

[Using Area of $\Delta = \frac{1}{2} AB.AD \sin 120°$]

$= \frac{1}{2} \times \left(\frac{4a}{\sqrt{3}}\right)^2 \times \frac{\sqrt{3}}{2} = \boldsymbol{a}.$

12. $AB = OA \Rightarrow AB = OA = OB$ (radii of circle are equal)



$\Rightarrow \Delta AOB$ is equilateral.

$\therefore$ If 'r' is the radius of the circle,

then area of $\Delta AOB = \frac{\sqrt{3}}{4} \text{side}^2$

$\Rightarrow \frac{\sqrt{3}}{4}(r)^2 = 4\sqrt{3}$ (given)

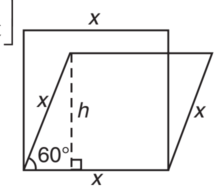
$\Rightarrow r^2 = 16 \Rightarrow r = 4$

$\therefore$ Area of circle $= \pi r^2 = \mathbf{16\pi\ cm^2}.$

13. In the given rhombus, $\frac{h}{x} = \sin 60° \Rightarrow \frac{h}{x} = \frac{\sqrt{3}}{2} \Rightarrow h = \frac{x\sqrt{3}}{2}$

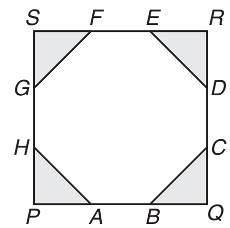
$\therefore$ Required ratio $= \frac{\text{Area of square}}{\text{Area of rhombus}}$

$= \frac{x^2}{x \times \frac{x\sqrt{3}}{2}} \left[\begin{array}{r}\text{Using, Area of rhombus} \\ = \text{base} \times \text{height}\end{array}\right]$



$= \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = \mathbf{2\sqrt{3} : 3}$

14. Let $ABCDEFGH$ be the regular octagon of side a cm.



Now if we produce the sides of the octagon on both the sides, we get a square $PQRS$.

Given, $BC = DE = FG = HA = a$ cm.

Also,

$BQ = QC = DR = RE = FS = SG = HP = PA = \frac{a}{\sqrt{2}}$

($\because$ BQC, DRE, FSG, HPA are rt. $\angle d$ Δs)

$\therefore$ Side of square $= \frac{a}{\sqrt{2}} + a + \frac{a}{\sqrt{2}} = a(1 + \sqrt{2})$ cm.

$\therefore$ Area of square $= a^2(1 + \sqrt{2})^2$ cm^2 $= a^2(3 + 2\sqrt{2})$ cm^2

Each of the shaded Δs, APH, BCQ, DER, FGS is an isosceles right angled Δ, whose area $= \frac{1}{2} \times \frac{a}{\sqrt{2}} \times \frac{a}{\sqrt{2}}$ cm^2 $= \frac{a^2}{4}$ cm^2

$\therefore$ Total area of shaded region $= 4 \times \frac{a^2}{4} = a^2$ cm^2

$\therefore$ Area of octagon = Area of square $PQRS$ – Total area of shaded isosceles Δs $= a^2(3 + 2\sqrt{2}) - a^2$

$= \mathbf{2a^2(1 + \sqrt{2})\ cm^2}.$

15. Let each side of the equilateral Δ be $\boldsymbol{a}$ units.

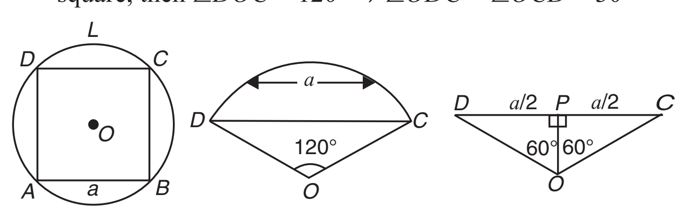
Then, circumradius of the circle $= \frac{\text{side}}{\sqrt{3}} = \frac{a}{\sqrt{3}}$ units

$\therefore$ Area of circumcircle $= \pi\left(\frac{a}{\sqrt{3}}\right)^2 = \frac{\pi a^2}{3}$ sq units

Area of square of side $\boldsymbol{a}$ units $= a^2$ sq units

$\therefore$ Required ratio $= \frac{\pi a^2}{3} \Big/ a^2 = \frac{\pi}{3} = \boldsymbol{\pi} : \mathbf{3}.$

16. As shown in the given figures, if 'a' is each side of the square, then $\angle DOC = 120° \Rightarrow \angle ODC = \angle OCD = 30°$



(*i*) (*ii*) (*iii*)

Now in fig. (*iii*), $\frac{PC}{OC} = \sin 60^\circ \Rightarrow \frac{a/2}{OC} = \frac{\sqrt{3}}{2}$

$\Rightarrow$ $OC = \frac{a}{\sqrt{3}}$, *i.e.*, radius of arc CD

Also, $OP = \frac{a}{2} \cot 60^\circ = \frac{a}{2\sqrt{3}}$

Now area of $\Delta ODC = \frac{1}{2} \times CD \times OP = \frac{1}{2} \times a \times \frac{a}{2\sqrt{3}} = \frac{a^2}{4\sqrt{3}}$

Area of sector $DOC = \pi \times \left(\frac{a}{\sqrt{3}}\right)^2 \times \frac{120}{360} = \frac{\pi a^2}{9}$

$\left[\text{Using Area of sector} = \pi R^2 \left(\frac{\theta}{360^\circ}\right)\right]$

$\therefore$ Area of segment DLC

= Area of sector DOC – Area of ΔDOC

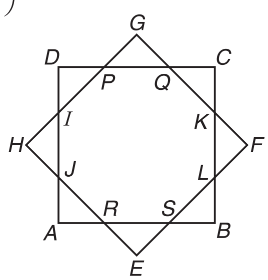
$= \frac{\pi a^2}{9} - \frac{a^2}{4\sqrt{3}}$

$\therefore$ Total area of the figure

= Area of square + Total area of 4 segments

$= \boldsymbol{a^2 + 4 \times \left(\frac{\pi a^2}{9} - \frac{a^2}{4\sqrt{3}}\right)}$.

17. As is seen in the given figure, the sides of one square are parallel to the diagonals of another square. Also, square *ABCD* and *EFGH* have same area.



$\Rightarrow$ Sides of square *ABCD* and square *EFGH* are 4 cm each.

Let $DP = a$ units

As $DP = PG = GQ = QC = a$ units and $\angle G = 90^\circ$

$\therefore$ $PQ = a\sqrt{2}$ units

$\therefore$ $DC = DP + PQ + QC = \left(a + a\sqrt{2} + a\right)$ units

$= a\left(2 + \sqrt{2}\right)$ units

Area of $\Delta\ PGQ = \frac{1}{2} \times a \times a = \frac{a^2}{2}$ sq units.

Area of all triangles outside square $ABCD = 4 \times \frac{a^2}{2}$

$= 2a^2$ sq units

Also, side of square $= 4 \Rightarrow a\left(2 + \sqrt{2}\right) = 4 \Rightarrow a = \frac{4}{2 + \sqrt{2}}$

$\therefore$ Total area of the four Δs outside square *ABCD*

$= 2 \times \left(\frac{4}{2 + \sqrt{2}}\right)^2 = \frac{2 \times 16}{(4 + 4\sqrt{2} + 2)}$

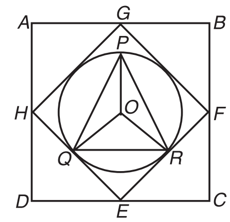
$= \frac{16}{3 + 2\sqrt{2}} \times \frac{3 - 2\sqrt{2}}{3 - 2\sqrt{2}} = 16\ (3 - 2\sqrt{2})$

$\therefore$ Total area of the figure = Area of square *ABCD* + Area of the four Δs outside *ABCD*

$= 16 + 16\ (3 - 2\sqrt{2})$

$= 16\ (4 - 2\sqrt{2}) = 32\ \left(2 - \sqrt{2}\right)$ cm².

18. Let $AB = a$ be the side of the outermost square.



Then $AG = AH = \frac{a}{2}$

$\Rightarrow$ $GH = \sqrt{\frac{a^2}{4} + \frac{a^2}{4}} = \frac{a}{\sqrt{2}}$

$\therefore$ Diameter of circle $= \frac{a}{\sqrt{2}}$

$\Rightarrow$ Radius of circle $= \frac{a}{2\sqrt{2}}$

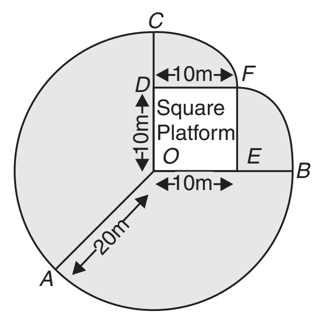
If O is the centre of the circle, then $\angle POQ = 120^\circ$.

$\therefore$ Area of $\Delta POQ = \frac{1}{2} \times PO \times OQ \times \sin 120^\circ$

$= \frac{1}{2} \times \frac{a}{2\sqrt{2}} \times \frac{a}{2\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}a^2}{32}$

$\therefore$ Area of $\Delta PQR = 3$ (Area of ΔPOQ) $= \frac{3\sqrt{3}a^2}{32}$.

19. The goat can graze the area *CABEODC* + quarter circle *CDF* + quarter circle *FEB*

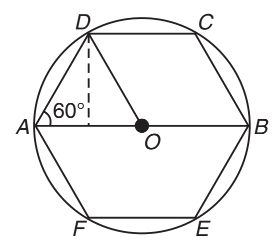


$= \frac{3}{4} \times \pi \times (20)^2 + \frac{1}{4} \times \pi \times (10)^2 + \frac{1}{4} \times \pi \times (10)^2$

$= 300\pi + 25\pi + 25\pi$

$= \mathbf{3.5 \times 100\pi}$.

20. Take the trapezoid *ABCD* in the semi-circle with centre O such that $AD = DC = CB$.



Now, complete the circle and draw an identical trapezoid in the other semicircle also. Then, *ADCBEF* is a regular hexagon.

$\Rightarrow$ $\angle DAO = 60^\circ$

(ΔDAO is an equilateral triangle)

$\Rightarrow$ $DA = AO = 7$ cm.

($\because AB = 14$ cm)

$\therefore$ Area of regular hexagon $= \frac{3\sqrt{3}}{2} \times (\text{side})^2$

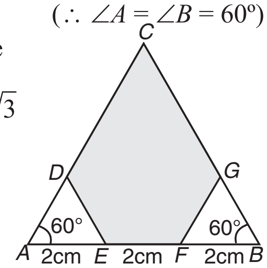
$= \frac{3\sqrt{3}}{2} \times 49$ cm²

$\Rightarrow$ Area of trapezoid $= \frac{1}{2} \times$ Area of hexagon

$= \frac{1}{2} \times \frac{3\sqrt{3}}{2} \times 49 = \mathbf{\frac{147\sqrt{3}}{4}}$ **cm².**

21. $AB = 6$ cm, $\angle C = 60^\circ$ $(\therefore \angle A = \angle B = 60^\circ)$

$\therefore$ ΔABC is an equilateral triangle



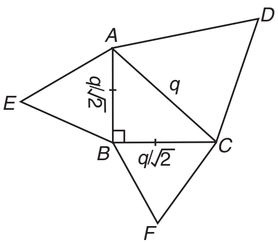
Area of $\Delta ABC = \frac{\sqrt{3}}{4} \times (6)^2 = 9\sqrt{3}$

Area of $(\Delta ADE + \Delta BFG)$

$= 2 \times \left(\frac{\sqrt{3}}{4} \times (2)^2\right) = 2\sqrt{3}$

$\therefore$ Area of pentagon = $9\sqrt{3} - 2\sqrt{3} = \mathbf{7\sqrt{3}}$ **cm²**.

22. $AC = q$, $\angle ABC = 90^\circ$



$\Rightarrow q = \sqrt{AB^2 + BC^2}$

$\Rightarrow q = \sqrt{2x^2}$

$\Rightarrow q^2 = 2x^2 \Rightarrow x = \dfrac{q}{\sqrt{2}}$

$\therefore$ Area of the re-entrant hexagon

= Sum of areas of $(\Delta ABC + \Delta ADC + \Delta BFC + \Delta AEB)$

$$= \frac{1}{2} \times \frac{q}{\sqrt{2}} \times \frac{q}{\sqrt{2}} + \frac{\sqrt{3}}{4} q^2 + \frac{\sqrt{3}}{4}\left(\frac{q}{\sqrt{2}}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{q}{\sqrt{2}}\right)^2$$

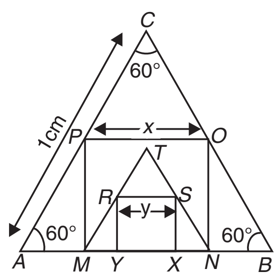
$$= \frac{q^2}{4} + \frac{\sqrt{3}}{4}q^2 + \frac{\sqrt{3}}{8}q^2 + \frac{\sqrt{3}q^2}{8} = \mathbf{\frac{q^2}{4}(2\sqrt{3}+1)}.$$

23. Since $\angle CPO = \angle COP = 60^\circ$, therefore, PCO is also an equilateral triangle.

Let each side of the square $MNOP$ be x cm.

Then $PC = CO = PO = x$ cm

Then in ΔPAM,



$\dfrac{PM}{PA} = \sin 60^\circ$

$\Rightarrow \dfrac{x}{PA} = \dfrac{\sqrt{3}}{2} \Rightarrow PA = \dfrac{2x}{\sqrt{3}}$

$\therefore AC = AP + PC = \dfrac{2x}{\sqrt{3}} + x$

Given $\dfrac{2x}{\sqrt{3}} + x = 0.01$ m = 1 cm

$\Rightarrow x = \dfrac{\sqrt{3}}{(2+\sqrt{3})}$ cm = $\sqrt{3}(2-\sqrt{3})$ cm ...(*i*)

(On rationalising the denominator)

Now, let each side of the square $RSXY$ be y. Then $RT = y$

($\because$ RTS is an equilateral triangle)

$\therefore$ In ΔRYM, $\dfrac{RY}{RM} = \sin 60^\circ$

$\Rightarrow \dfrac{y}{RM} = \dfrac{\sqrt{3}}{2} \Rightarrow RM = \dfrac{2y}{\sqrt{3}}$

$\therefore MT = MR + RT = \dfrac{2y}{\sqrt{3}} + y = \dfrac{(2+\sqrt{3})}{\sqrt{3}} y$

Given $MT = x$, then

$x = \left(\dfrac{2+\sqrt{3}}{\sqrt{3}}\right) y \Rightarrow y = \dfrac{\sqrt{3}x}{2+\sqrt{3}}$

But from (*i*), $x = \sqrt{3}(2-\sqrt{3})$

$\therefore y = \dfrac{\sqrt{3}\sqrt{3}(2-\sqrt{3})}{2+\sqrt{3}} = \dfrac{3.(2-\sqrt{3})}{(2+\sqrt{3})} \cdot \dfrac{(2-\sqrt{3})}{(2-\sqrt{3})}$

$\Rightarrow y = 3(2-\sqrt{3})^2 = 3(7-4\sqrt{3})$

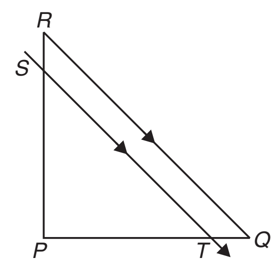
$\therefore$ Area of the inner most square = y^2

$= \left(3(7-4\sqrt{3})\right)^2$

$= 9(49 + 48 - 56\sqrt{3})$

$\mathbf{= (873 - 504\sqrt{3})}$ **cm²**.

24. Given, $ST \parallel RQ$



$\therefore \dfrac{\text{Area of } \Delta SPT}{\text{Area of } \Delta RPQ} = \dfrac{ST^2}{RQ^2}$

Also, given $ST = \left(1 - \dfrac{35}{100}\right) RQ$

$= (0.65)\, RQ$

$\therefore \dfrac{ST}{RQ} = 0.65 \Rightarrow \left(\dfrac{ST}{RQ}\right)^2 = 0.4225$

$\Rightarrow \dfrac{\text{Area of } \Delta SPT}{\text{Area of } \Delta RPQ} = 0.4225 \Rightarrow \dfrac{\text{Area of } \Delta SPT}{34} = 0.4225$

$\Rightarrow$ Area of $\Delta SPT = 0.4225 \times 34$

SELF ASSESSMENT SHEET

1. An equilateral triangle is cut from its three vertices to form a regular hexagon. What is the percentage of area wasted?

(*a*) 20% (*b*) 50% (*c*) 33.33% (*d*) 66.66%

2. The area of a square and circle is same and the perimeter of square and equilateral triangle is same, then the ratio between the area of circle and area of equilateral triangle is:

(*a*) $\pi : 3$ (*b*) $9 : 4\sqrt{3}$

(*c*) $4 : 9\sqrt{3}$ (*d*) None of these

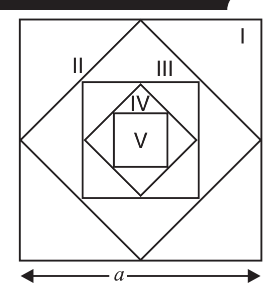
3. The figure given besides shows five squares inside one another by joining the midpoints of the outer square. The side of square I is a cm. The total area of the five squares is:

(*a*) $\dfrac{(4\sqrt{2}-1)a^2}{(4\sqrt{2}+1)}$

(*b*) $\dfrac{(4\sqrt{2}+1)a^2}{4(\sqrt{2}-1)}$

(*c*) $\dfrac{31}{16}a^2$

(*d*) $(7+3\sqrt{2})a^2$



4. Three circles of radius a, b, c touch each other externally. The area of the triangle formed by joining their centres is:

(*a*) $\sqrt{(a+b+c)abc}$ (*b*) $(a+b+c)\sqrt{ab+bc+ca}$

(*c*) $ab + bc + ca$ (*d*) None of these

(*SSC 2013*)

5. If the lengths of a sides of a triangle are in the ratio 4 : 5 : 6 and the in-radius of the triangle is 3 cm, then the altitude of the triangle corresponding to the largest side as the base is:

(*a*) 8 cm (*b*) 7.5 cm (*c*) 6 cm (*d*) 10 cm

(***SSC 2013***)

6. Find the ratio of the diameter of the circles inscribed in and circumscribing an equilateral triangle to its height?

(*a*) 1 : 2 : 1 (*b*) 2 : 4 : 3 (*c*) 1 : 3 : 4 (*d*) 3 : 2 : 1

7. In an isosceles triangle, the measure of each of equal sides is 10 cm and the angle between them is 45º. The area of the triangle is:

(*a*) 25 cm^2 (*b*) $\frac{25}{2}\sqrt{2}$ cm^2

(*c*) $25\sqrt{2}$ cm^2 (*d*) $25\sqrt{3}$ cm^2

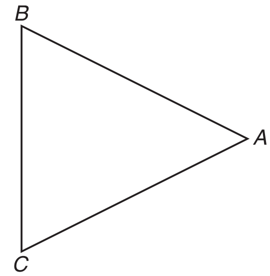
(***SSC 2006***)

8. Inside a triangular park, there is a flower bed forming a similar triangle. Around the flower bed runs a uniform path of such a width that the sides of the park are exactly double of the corresponding sides of the flower bed. The ratio of the area of the path to the flower bed is:

(*a*) 1 : 1 (*b*) 1 : 2 (*c*) 1 : 3 (*d*) 3 : 1

(***SNAP 2007***)

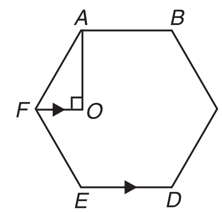
9. A cow is tethered at corner A by a rope. Neither the cow nor the rope is allowed to enter ΔABC. $\angle A = 30º$, $AB = AC =$ 10 m and $BC = 6$ cm. What is the area that can be grazed by the cow if the length of the rope is 8 m.



(*a*) $133\frac{1}{6}\pi$ sq. m (*b*) 121π sq. m

(*c*) 132π sq. m (*d*) $^{176}/_{3}\,\pi$ sq. m (***CAT***)

10. In the figure given alongside, *ABCDEF* is a regular hexagon and $\angle AOF = 90º$. *FO* is parallel to *ED*. What is the ratio of the area of triangle *AOF* to that of the hexagon *ABCDEF*?



(*a*) $\frac{1}{12}$ (*b*) $\frac{1}{6}$ (*c*) $\frac{1}{24}$ (*d*) $\frac{1}{18}$

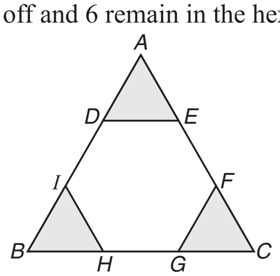
(***CAT***)

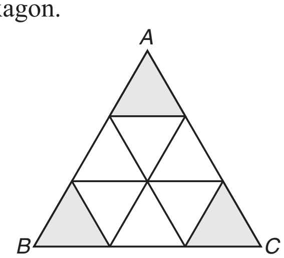
ANSWERS

1. (*c*) **2.** (*b*) **3.** (*c*) **4.** (*a*) **5.** (*b*) **6.** (*b*) **7.** (*c*) **8.** (*d*) **9.** (*d*) **10.** (*a*)

HINTS AND SOLUTIONS

1. When an equilateral triangle is cut from its three vertices to form a regular hexagon then out of the 9 equilateral triangles that form ΔABC, three triangle, ΔADE, ΔFCG, ΔIHB are cut off and 6 remain in the hexagon.





This means that $\frac{1}{3}$ rd of the area has been removed or wasted to get the hexagon.

$\therefore$ Area wasted $= \left(\frac{1}{3}\times 100\right)\% =$ **33.33%.**

2. Let each side of the square = *a* cm. Then,

Area of square = a^2 cm^2

Also, let *r* be the radius of the circle. Then, $\pi r^2 = a^2$

Let each side of the equilateral triangle = *b* cm. Then

$3b = 4a \Rightarrow b = \frac{4a}{3}$.

$\therefore$ Area of equilateral triangle

$$= \frac{\sqrt{3}}{4}b^2 = \frac{\sqrt{3}}{4}\times\left(\frac{4a}{3}\right)^2 = \frac{\sqrt{3}}{4}\times\frac{16a^2}{9} = \frac{4\sqrt{3}a^2}{9}$$

$\therefore$ Required ratio between area of circle and area of equilateral Δ is $a^2 : \frac{4\sqrt{3}a^2}{9}$ $=$ **$9 : 4\sqrt{3}$**.

3. Side of the square I = *a*

$\therefore$ Area of square I = a^2

Side of square II $= \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}$

$$= \sqrt{\frac{a^2}{4}+\frac{a^2}{4}} = \sqrt{\frac{2a^2}{4}} = \frac{a}{\sqrt{2}}$$

$\therefore$ Area of square II $= \left(\frac{a}{\sqrt{2}}\right)^2 = \frac{a^2}{2}$

Side of square III $= \sqrt{\left(\frac{a}{2\sqrt{2}}\right)^2 + \left(\frac{a^2}{2\sqrt{2}}\right)^2}$

$$= \sqrt{\frac{a^2}{8}+\frac{a^2}{8}} = \sqrt{\frac{a^2}{4}} = \frac{a}{2}$$

$\therefore$ Area of square III $= \frac{a^2}{4}$

Side of square IV $= \sqrt{\left(\frac{a}{4}\right)^2 + \left(\frac{a}{4}\right)^2}$

$$= \sqrt{\frac{a^2}{16} + \frac{a^2}{16}} = \sqrt{\frac{2a^2}{16}} = \frac{a}{2\sqrt{2}}$$

$\therefore$ Area of square IV $= \left(\frac{a}{2\sqrt{2}}\right)^2 = \frac{a^2}{8}$

Side of square V $= \sqrt{\left(\frac{a}{4\sqrt{2}}\right)^2 + \left(\frac{a}{4\sqrt{2}}\right)^2}$

$$= \sqrt{\frac{a^2}{32} + \frac{a^2}{32}} = \sqrt{\frac{2a^2}{32}} = \frac{a}{4}$$

$\therefore$ Area of square V $= \left(\frac{a}{4}\right)^2 = \frac{a^2}{16}$

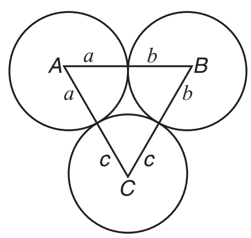
$\therefore$ Sum of the areas of the five squares

$$= a^2 + \frac{a^2}{2} + \frac{a^2}{4} + \frac{a^2}{8} + \frac{a^2}{16}$$

$$= a^2\left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right] = \mathbf{\frac{31}{16}\, a^2.}$$

4. As shown in the figure, $AB = a + b$, $BC = b + c$, $CA = a + c$

$\therefore$ Area of $\Delta ABC = \sqrt{s(s - AB)(s - BC)(s - CA)}$



where, $s = \frac{1}{2}(AB + BC + CA)$

$$= \frac{a + b + b + c + c + a}{2}$$

$= a + b + c$

$\therefore$ Area of ΔABC

$$= \sqrt{(a+b+c)\,[(a+b+c)-(a+b)][(a+b+c)-(b+c)]\,[(a+b+c)-(c+a)]}$$

$$= \sqrt{\mathbf{(a + b + c)\,.\,a.\,b.\,c}}$$

5. Area of a triangle $= \frac{1}{2} \times$ base $\times$ height

$=$ In-radius $\times$ semi-perimeter of the Δ $\left[\text{Using } r = \frac{\Delta}{s}\right]$

Let the sides of triangle be $4x$, $5x$ and $6x$ respectively.
Given: In-radius = 3 cm

Therefore, $3 \times \frac{(4x + 5x + 6x)}{2} = \frac{1}{2} \times 6x \times h$, where h is the height.

$\Rightarrow \quad 3 \times \frac{15x}{2} = \frac{6x}{2} \times h \Rightarrow h = \frac{45}{6} = \mathbf{7.5\ cm.}$

6. For an equilateral triangle of side ***a*** units,

In-radius $= \frac{a}{2\sqrt{3}}$ units

$\Rightarrow$ Diameter of inscribed circle $= \frac{a}{\sqrt{3}}$ units

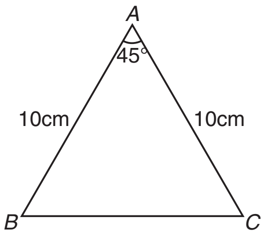
Circumradius $= \frac{a}{\sqrt{3}}$ units

$\Rightarrow$ Diameter of circumscrible circle $= \frac{2a}{\sqrt{3}}$ units

Height $= \frac{\sqrt{3}}{2} a$ units.

$\therefore$ Required ratio $= \frac{a}{\sqrt{3}} : \frac{2a}{\sqrt{3}} : \frac{\sqrt{3}}{2} a = 2a : 4a : 3a = \mathbf{2 : 4 : 3.}$

7. ΔABC is an isosceles triangle with $AB = AC = 10$ cm. $\angle A = 45°$



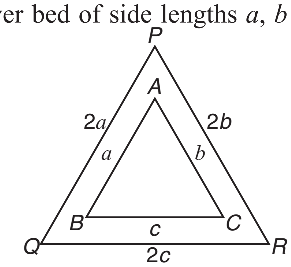
$\therefore$ Area of ΔABC

$$= \frac{1}{2} \times 10 \times 10 \times \sin 45°$$

$\left[\text{Using } \Delta = \frac{1}{2} bc \sin A\right]$

$$= \frac{50}{\sqrt{2}} = \frac{50}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \mathbf{25\sqrt{2}\ cm^2}.$$

8. Let ABC be the triangular flower bed of side lengths a, b and c respectively. Then



Area of ΔABC

$$= \sqrt{s(s - a)(s - b)(s - c)}$$

where $s = \frac{a + b + c}{2}$

Now according to the given condition,

ΔPQR forms the park with side lengths $2a$, $2b$, $2c$.

$\therefore$ Area of $\Delta PQR = \sqrt{s'(s' - 2a)(s' - 2b)(s' - 2c)}$

where $s' = \frac{2a + 2b + 2c}{2} = a + b + c = 2s$

$\therefore$ Area of $\Delta PQR = \sqrt{2s(2s - 2a)(2s - 2b)(2s - 2c)}$

$= 4\sqrt{s(s - a)(s - b)(s - c)}$

$= 4.$ Area of ΔABC.

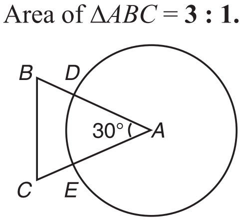
$\therefore$ Area of path $=$ Area of ΔPQR – Area of ΔABC

$= 4 \times$ Area of ΔABC – Area of ΔABC

$= 3$ (Area of ΔABC)

$\therefore$ Reqd. Ratio $=$ Area of Path : Area of $\Delta ABC = \mathbf{3 : 1.}$

9. Given : $AB = AC = 10$ m, $BC = 6$ m and $\angle A = 30°$. The area that the cow can graze is the area of the circle with centre A and radius 8 m. But the cow is not allowed to enter the sector ADE formed inside ΔABC with $\angle A = 30°$ and $AD = AE = 8$ cm.



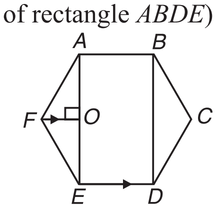
$\therefore$ Area grazed $=$ Area of circle – Area of sector ADE

$$= \pi r^2 - \pi r^2 \cdot \frac{30°}{360°} = \pi r^2\left(1 - \frac{30°}{360°}\right)$$

$$= \pi \times 64 \times \frac{11}{12} = \mathbf{\frac{176}{3}\pi\ \ sq.\ cm}$$

10. Area of $\Delta AOF = \frac{1}{2} \times OF \times AO$

$\therefore$ Area of hexagon $ABCDEF = 4 \times$ (Area of ΔAOF + Area of rectangle $ABDE$)



$= 2OF \times AO + ED \times AE$

$= 2OF \times AO + 2OF \times 2AO$

$= 6(OF \times AO)$

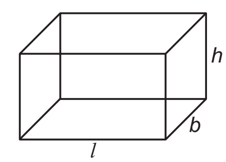
$\therefore$ Required ratio $= \frac{1/2}{6} = \mathbf{\frac{1}{12}.}$

13 Volume and Surface Area of Solids

KEY FACTS

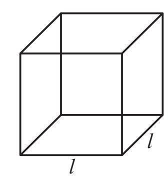
1. **Parallelopiped:** A solid bounded by three pairs of parallel plane surfaces is called a parallelopiped. The plane surfaces are known as the faces of the parallelopiped.

   (*i*) **Cuboid :** A parallelopiped whose faces are rectangles and adjacent faces are perpendicular is called a cuboid.

   - **Volume of a cuboid = ($l \times b \times h$) cu. units,** where l = length, b = breadth, h = height
   - **Whole surface of cuboid = 2($lb + bh + lh$) sq. units**
   - **Diagonal of a cuboid** = $\sqrt{l^2 + b^2 + h^2}$ **units**
   - **Area of 4 walls = 2($l + b$)h sq. units**

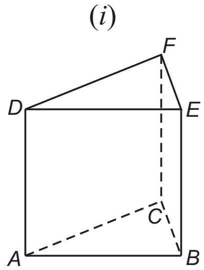
   (*ii*) **Cube :** It is a special type of cuboid whose length, breadth and height are all equal.

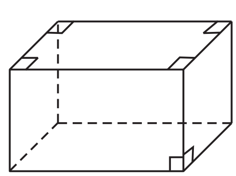
   - **Volume of cube = (edge)^3 = l^3 cu. units**
   - **Whole surface area of cube = 6(edge)^2 = 6l^2 sq. units**
   - **Diagonal =** $\sqrt{3}\, l$ **units**

2. **Prism :** A prism is a polyhedron with two parallel faces called **bases.** The other faces are always **parallelograms.** The prism is named by the shape of its base.

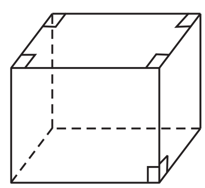
(*i*) (*ii*) (*iii*) (*iv*)



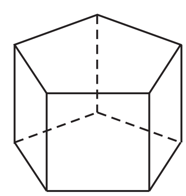
Triangular Prism



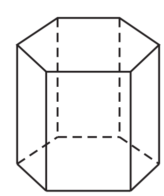
Rectangular Prism



Square Prism



Pentagonal Prism



Hexagonal Prism

- **Surface Area = 2 × Area of base shape + Perimeter of base shape × height**
- **Volume = Area of base shape × height of prism**

In case of a triangular prism, the area of the base triangles can be found out by :

(*i*) $A = \frac{1}{2} \times b \times h$, if base ($b$) and altitude ($h$) of the triangle are known.

(*ii*) $A = \sqrt{s(s-a)(s-b)(s-c)}$, if all three sides a, b, c are known where $s = \frac{a+b+c}{2}$

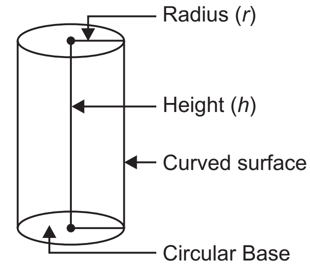
(*iii*) $A = \frac{\sqrt{3}}{4} a^2$, if the bases are equilateral triangles of side 'a'.

A prism is a **regular prism** if its bases are regular figures, *i.e.*, with sides equal, *i.e.*, equilateral triangle, square, regular hexagon, etc.

Note : Area of a regular hexagon = $\frac{\sqrt{3}}{2}$ (edge)^2.

3. Cylinder

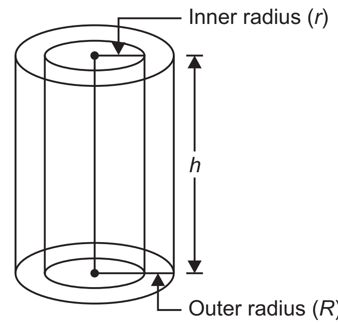
(*a*) Right circular cylinder: It is a solid generated by the revolution of a rectangle about one of its sides that remains fixed. So, a right circular cylinder is a closed solid that has two parallel circular bases connected by a curved surface.



Here r = Radius of base of cylinder, h = Perpendicular distance between bases.

- **Curved surface area of the cylinder** = Perimeter of base × height = $\mathbf{2\pi r\, h}$ **sq. units**
- **Total surface area** = Curved surface area + Area of bases = $2\pi rh + 2\pi r^2 = \mathbf{2\pi r\,(h + r)}$ **sq. units**
- **Volume** = Area of base × Height = $\mathbf{\pi r^2 h}$ **cu. units**

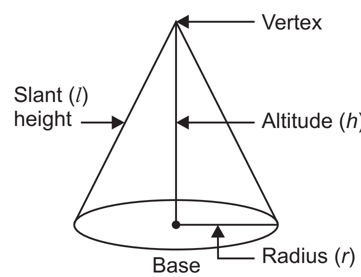
(*b*) Volume and surface area of a hollow cylinder: A solid bounded by two coaxial cylinders of same height and different radii is called a hollow cylinder.



For a hollow cylinder, whose inner radius = r, outer radius = R, perpendicular height = h.

- **Curved (Lateral) surface area** = External curved surface area + Internal curved surface area = $2\pi Rh + 2\pi rh = \mathbf{2\pi h\,(R + r)}$ **sq. units**
- **Area of bases = $\mathbf{2\pi\,(R^2 - r^2)}$ sq. units**
- **Total surface area** = $2\pi h\,(R + r) + 2\pi\,(R^2 - r^2)$ $= \mathbf{2\pi(R + r)(h + R - r)}$ **sq. units**
- **Volume of material** = External volume – Internal volume used in making the cylinder $= \mathbf{\pi R^2 h - \pi r^2 h = \pi h(R^2 - r^2)}$ **cu. units**

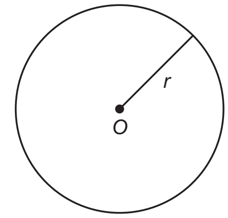
4. Right circular cone is a solid that has a circular base which is connected to its vertex by a curved surface. It is generated by the revolution of a right angled triangle about one of its sides containing the right angle as the axis.



If r = radius of base of the cone,
h = perpendicular distance between the vertex and the base, then

- **Volume = $\mathbf{\frac{1}{3}\pi r^2 h}$ cu. units**
- **Curved surface area = $\mathbf{\pi r l = \pi r\sqrt{h^2 + r^2}}$** ($l$ = slant height distance of any point on the circumference of the circle and the vertex)
- **Total surface area = $\mathbf{\pi r l + \pi r^2 = \pi r(l + r)}$** $= \pi r\,(\sqrt{h^2 + r^2} + r)$

5. Sphere: It is a solid geometric figure generated by the revolution of a semicircle about its diameter, *i.e.*, it is a round solid figure, with every point on its surface equidistant from the centre.



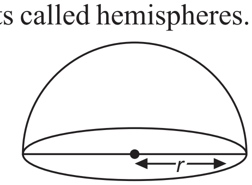
For a sphere with r as the radius,

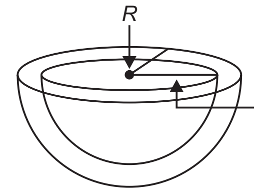
- **Volume = $\mathbf{\frac{4}{3}\pi r^3}$ cu. units**
- **Surface area = $\mathbf{4\pi r^2}$ sq. units**
- **Volume of a hollow sphere = $\mathbf{\frac{4}{3}\pi(R^3 - r^3)}$ cu. units,** where R is the external radius and r, the internal radius.

6. Hemisphere : Any plane that contains the centre of the sphere divides it into two equal parts called hemispheres.

For a hemisphere with r as radius,

- **Volume = $\mathbf{\frac{2}{3}\pi r^3}$ cu. units**

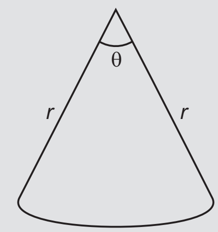




- **Curved surface area = $2\pi r^2$ sq. units**
- **Total surface area = $2\pi r^2 + \pi r^2 = 3\pi r^2$ sq. units**
- **Volume of a hollow hemisphere** = $\frac{2}{3}\pi(R^3 - r^3)$, where R = external radius and r = internal radius.
- **Curved surface area of a hollow hemisphere = $2\pi(R^2 + r^2)$**
- **Total surface area of a hollow hemisphere = $2\pi(R^2 + r^2) + \pi(R^2 - r^2)$**

Notes :

1. Curved surface area of a cone, when sector of a circle is converted into a cone = $\frac{\theta}{360^\circ} \times \pi r^2$, where θ is the sector angle, and r, the bounding radii.



2. Volume of water that flows out through a pipe = (Cross section area × Speed × Time)

3. When a solid is melted and converted into another solid, then the volume of both the solids remain the same, assuming there is no wastage in conversion.

4. Number of new solids obtained by recasting = $\frac{\text{Volume of the solid that is melted}}{\text{Volume of the solid that is made}}$.

SOLVED EXAMPLES

Ex. 1. How many small cubes each of 96 cm^2 surface area can be formed from the material obtained by melting a larger cube of 384 cm^2 surface area ? (***MAT 2007***)

Sol. Let the edge of the bigger cube be a cm and the edge of the smaller cube be b cm. Then,

$$6a^2 = 384 \quad \text{and} \quad 6b^2 = 96$$

$$\Rightarrow \quad a^2 = 64 \quad \text{and} \quad b^2 = 16$$

$$\Rightarrow \quad a = 8 \text{ cm} \quad \text{and} \quad b = 4 \text{ cm}$$

$\Rightarrow$ Volume of bigger cube = $a^3 = (8)^3$ cm^3 = 512 cm^3 and
Volume of smaller cube = $b^3 = (4)^3$ cm^3 = 64 cm^3.

$\therefore$ Total number of smaller cubes = $\frac{512}{64}$ = **8.**

Ex. 2. The diagonals of the three faces of a cuboid are x, y, z respectively. What is the volume of the cuboid ?

(*a*) $\frac{xyz}{2\sqrt{2}}$

(*b*) $\frac{\sqrt{(y^2+z^2)(z^2+x^2)(x^2+y^2)}}{2\sqrt{2}}$

(*c*) $\frac{\sqrt{(y^2+z^2-x^2)(z^2+x^2-y^2)(x^2+y^2-z^2)}}{2\sqrt{2}}$

(*d*) None of the above (***CDS 2010***)

Sol. Let the length, breadth and height of the cuboid be l, b and h respectively. Then,

$$l^2 + b^2 = x^2 \quad ...(i)$$

$$b^2 + h^2 = y^2 \quad ...(ii)$$

$$h^2 + l^2 = z^2 \quad ...(iii)$$

Adding eqn. (i), (ii) and (iii), we get

$$2(l^2 + b^2 + h^2) = x^2 + y^2 + z^2$$

$$\Rightarrow \quad l^2 + b^2 + h^2 = \frac{1}{2}(x^2 + y^2 + z^2) \quad ...(iv)$$

$\therefore$ From eqns. (i), (ii), (iii) and (iv), we get

$$h^2 = \frac{1}{2}(x^2 + y^2 + z^2) - x^2 \qquad \text{[Eqn. } (iv) - \text{Eqn. } (i)]$$

$$= \frac{1}{2}(y^2 + z^2 - x^2) \quad \Rightarrow \quad h = \sqrt{\frac{y^2 + z^2 - x^2}{2}}$$

Similarly, $l = \sqrt{\frac{z^2 + x^2 - y^2}{2}}$ and $b = \sqrt{\frac{x^2 + y^2 - z^2}{2}}$

$\therefore$ Volume of the cuboid $= lbh = \sqrt{\frac{(z^2 + x^2 - y^2)(x^2 + y^2 - z^2)(y^2 + z^2 - x^2)}{2 \times 2 \times 2}}$

$$= \mathbf{\frac{1}{2\sqrt{2}} \sqrt{(y^2 + z^2 - x^2)(z^2 + x^2 - y^2)(x^2 + y^2 - z^2)}}$$

Ex. 3. The area of the curved surface and the area of the base of a right circular cylinder are *a* square cm and *b* square cm respectively. The height of the cylinder is

(*a*) $\frac{2a}{\sqrt{\pi b}}$ cm (*b*) $\frac{a\sqrt{b}}{2\sqrt{\pi}}$ cm (*c*) $\frac{a}{2\sqrt{\pi b}}$ cm (*d*) $\frac{a\sqrt{\pi}}{2\sqrt{b}}$ cm (*SSC 2012*)

Sol. Let r and b be the radius of the base and height of the cylinder respectively. Then,

$CSA = 2\pi rh = a$ and Area of base $= \pi r^2 = b$

$2\pi rh = a \Rightarrow 4\pi^2 r^2 h^2 = a^2 \Rightarrow 4\pi b h^2 = a^2 \Rightarrow h^2 = \frac{a^2}{4\pi b} \Rightarrow h = \mathbf{\frac{a}{2\sqrt{\pi b}}}$ **cm.**

Ex. 4. A semicircular thin sheet of a metal of diameter 28 cm is bent and an open conical cup is made. What is the capacity of the cup ? (*CDS 2010*)

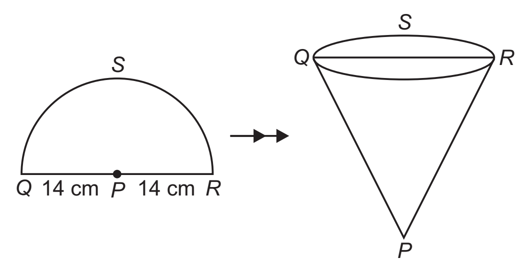
Sol. The radius (r) of the semicircle = 14 cm

Also, for the conical cup (slant height) $l = r = 14$ cm

Let R be the radius of the base of the conical cup. Then,

Circumference of the base of the cone

= Circumference of the semicircle



$\Rightarrow \quad 2\pi R = \pi r \Rightarrow 2R = r \Rightarrow R = \frac{r}{2} = \frac{14}{2} = 7$ cm

Let h be the vertical height of the cone. Then,

$$l^2 = R^2 + h^2 \Rightarrow h^2 = l^2 - R^2 = (14)^2 - 7^2 = 196 - 49 = 147$$

$\Rightarrow \quad h = \sqrt{147} = 7\sqrt{3}$ cm

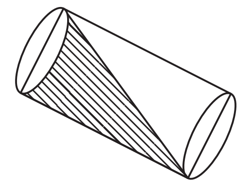
$\therefore$ Capacity of the cup $= \frac{1}{3}\pi R^2 h = \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 7\sqrt{3}$ cu cm $= \mathbf{\frac{1078}{3}\sqrt{3}}$ **cu cm.**

Ex. 5. A cylinder is filled to $\frac{4}{5}$th of its volume. It is, then tilted so that the level of water coincides with one edge of its bottom and top edge of the opposite side. In the process 30 cc of the water is spilled. What is the volume of the cylinder ? (*MAT 2008*)

Sol. Volume of the cylinder $= \pi r^2 h$. After tilting, by the given condition, cylinder is half full of water.

$\therefore \quad \frac{4}{5}\pi r^2 h - 30 = \frac{1}{2}\pi r^2 h \Rightarrow \left(\frac{4}{5} - \frac{1}{2}\right)\pi r^2 h = 30$

$\Rightarrow \frac{3}{10}\pi r^2 h = 30 \Rightarrow \pi r^2 h = \frac{30 \times 10}{3}$ cc $=$ **100 cc.**



Cylinder after tilting

Ex. 6. A sphere and a cube have the same surface area. What is the ratio of the square of volume of the sphere to the square of volume of the cube ? (*CDS 2010*)

Sol. By the given condition, surface area of sphere = surface area of cube

$\Rightarrow \quad 4\pi r^2 = 6a^2$ (where r = radius of sphere, a = edge of cube)

$\Rightarrow \quad \left(\frac{r}{a}\right)^2 = \frac{3}{2\pi}$

$$\therefore \quad \frac{(\text{Volume of sphere})^2}{(\text{Volume of cube})^2} = \frac{\left(\frac{4}{3}\pi r^3\right)^2}{(a^3)^2} = \frac{16}{9}\pi^2\left[\left(\frac{r}{a}\right)^3\right]^2$$

$$= \frac{16}{9}\pi^2\left[\left(\frac{r}{a}\right)^2\right]^3 = \frac{16}{9}\pi^2\left(\frac{3}{2\pi}\right)^3 = \frac{16}{9}\pi^2 \times \frac{27}{8\pi^3} = \boldsymbol{\frac{6}{\pi}}.$$

Ex. 7. A rectangular paper 11 cm by 8 cm can be exactly wrapped to cover the curved surface of a cylinder of height 8 cm . What is the volume of the cylinder ?

Sol. Area of the curved surface = Area of the rectangle = (11 × 8) cm^2 = 88 cm^2.

$\Rightarrow 2\pi rh = 88 \text{ cm} \Rightarrow 2 \times \frac{22}{7} \times r \times 8 = 88 \Rightarrow r = \frac{88 \times 7}{44 \times 8} = \frac{7}{4}$ cm.

$\therefore$ Volume of the cylinder $= \pi r^2 h = \frac{22}{7} \times \frac{7}{4} \times \frac{7}{4} \times 8 = \mathbf{77\ cm^3}$.

Ex. 8. A cylinder is within the cube touching all the vertical faces. A cone is inside the cylinder. If their heights are same with the same base, find the ratio of their volumes.

Sol. Let the length of each edge of the cube be x units. Then,

V_1 = Volume of the cube = x^3 cubic units.

Since a cylinder is within the cube and it touches all the faces of the cube,

radius of the cylinder = $\frac{x}{2}$ units

height of the cylinder = x units

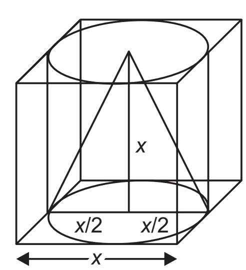
$\therefore \quad V_2$ = Volume of the cylinder $= \pi r^2 h = \frac{22}{7} \times \frac{x^2}{4} \times x$ cubic units

$= \frac{11}{14}x^3$ cubic units.

Also, the cone stands on the same base as the cylinder and has the same height.

So, $\quad V_3$ = Volume of the cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \frac{11}{14}x^3$ cubic units $= \frac{11}{42}x^3$ cubic units



$\therefore$ Required ratio $= V_1 : V_2 : V_3 = a^3 : \frac{11}{14}a^3 : \frac{11}{42}a^3 = \mathbf{42 : 33 : 11}$.

Ex. 9. Water is following at the rate of 5 km/hr through a pipe of diameter 14 cm into a rectangular tank which is 50 m long and 44 m wide. Determine the time in which the level of water in the tank will rise by 7 cm. $\left(\text{Take } \pi = \frac{22}{7}\right)$ **(*SSC CPO 2009*)**

Sol. Rate of flow = 5 km/hr = 5000 m/hour.

$\Rightarrow$ Length of cylinder for water flowing in one hour = 5000 m.

Radius = 7 cm = $\frac{7}{100}$ m.

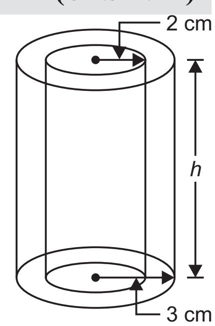
$\therefore$ Volume of water flowing through the pipe per hour = $\pi r^2 h$

$$= \frac{22}{7} \times \frac{7}{100} \times \frac{7}{100} \times 5000 \text{ m}^3 = 77 \text{ m}^3.$$

Volume of water to be filled in the rectangular tank = $(50 \times 44 \times 7/100)$ m^3 = 154 m^3

$\therefore$ Reqd. time = $\frac{154}{77}$ = **2 hours.**

Ex. 10. The outer and inner diameters of a circular pipe are 6 cm and 4 cm respectively. If its length is 10 cm, then what is the total surface area in sq cm ? **(*CDS 2011*)**



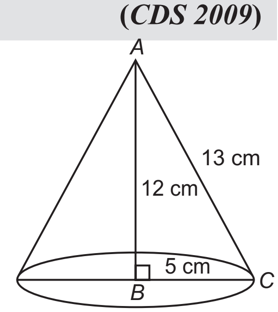
Sol. Outer radius (R) = 3 cm, Inner radius (r) = 2 cm, Height (h) = 10 cm.

$\therefore$ Total surface area of the pipe = Total area of both the bases + Outer *CSA* + Inner *CSA*

(*CSA* = Curves surface Area)

$$= 2\pi(R^2 - r^2) + 2\pi Rh + 2\pi rh$$
$$= 2\pi(R + r)(R - r) + 2\pi h(R + r)$$
$$= 2\pi(R + r)(R - r + h) = 2\pi(3 + 2)(3 - 2 + 10) \text{ cm}^2$$
$$= 2\pi(5)(11) \text{ cm}^2 = \mathbf{110\,\pi \text{ cm}^2.}$$

Ex. 11. A right ΔABC with sides 5 cm, 12 cm and 13 cm is revolved about the side 12 cm. What is the volume of the solid so obtained ? **(*CDS 2009*)**



Sol. Here, radius (r) of cone = 5 cm, vertical height (h) of the cone = 12 cm.

$\therefore$ Volume of the cone = $\frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi \times 5^2 \times 12 \text{ cu cm}$$

$= \mathbf{100\,\pi}$ **cu cm.**

Ex. 12. A sphere is cut into two equal halves and both the halves are painted from all the sides. The radius of the sphere is r unit and the rate of painting is ₹ 8 per sq. unit. What is the total cost of painting the two halves of the sphere in rupees ? **(*CDS 2008*)**

Sol. Total surface area of the two halves = $3\pi r^2 + 3\pi r^2 = 6\pi r^2$ sq units

$\therefore$ Cost of painting the two halves = $6\pi r^2 \times$ ₹ 8 = ₹ **48 πr^2.**

Ex. 13. A hemispherical bowl is made of steel 0.5 cm thick. The inside radius of the bowl is 4 cm. What is the volume of steel used in making the bowl ? (*MAT 2003*)

Sol. Inner radius of the bowl (r) = 4 cm

Outer radius of the bowl (R) = (4 + 0.5) cm = 4.5 cm

$\therefore$ Volume of steel used in making the bowl $= \frac{2}{3}\pi(R^3 - r^3) = \frac{2}{3}\times\frac{22}{7}((4.5)^3 - 4^3)$

$$= \frac{2}{3}\times\frac{22}{7}\times(91.125 - 64) = \frac{2}{3}\times\frac{22}{7}\times 27.125 = \mathbf{56.83\ cm^3.}$$

Ex. 14. The volumes of two spheres are in the ratio 64 : 27. Find the difference of their surface areas, if the sum of their radii is 7 cm.

Sol. Let the radii of the two spheres be r_1 cm and r_2 cm respectively and their respective volumes be V_1 and V_2. Then,

$$\frac{V_1}{V_2} = \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r_2^3} = \frac{64}{27} \Rightarrow \frac{r_1^3}{r_2^3} = \frac{64}{27} \Rightarrow \left(\frac{r_1}{r_2}\right)^3 = \left(\frac{4}{3}\right)^3 \Rightarrow \frac{r_1}{r_2} = \frac{4}{3} \Rightarrow r_1 = \frac{4}{3}r_2.$$

Given, $r_1 + r_2 = 7 \Rightarrow \frac{4}{3}r_2 + r_2 = 7 \Rightarrow \frac{7r_2}{3} = 7 \Rightarrow r_2 = \mathbf{3\ cm.}\ \therefore\ r_1 = \frac{4}{3}\times 3 = \mathbf{4\ cm}$

$\therefore$ Difference in the surface areas of the two spheres $= 4\pi r_2^2 - 4\pi r_1^2$

$$= 4\pi(r_2^2 - r_1^2) = 4\times\frac{22}{7}\times(16-9) = 4\times\frac{22}{7}\times 7\ \text{cm}^2 = \mathbf{88\ cm^2.}$$

Ex. 15. The base of a right prism is an equilateral triangle with a side 6 cm and its height is 18 cm. Find its volume, lateral surface area and total surface area ?

Sol. Volume of a right prism = Area of base × height.

Since the base is an equilateral triangle of side 6 cm,

Area of base $= \frac{\sqrt{3}}{4}\times(\text{side})^2 = \left(\frac{\sqrt{3}}{4}\times 6^2\right)\text{cm}^2 = \frac{\sqrt{3}}{4}\times 36\ \text{cm}^2 = 9\sqrt{3}\ \text{cm}^2$

$\therefore$ Volume $= (9\sqrt{3}\times 18)\ \text{cm}^3 = \mathbf{162\sqrt{3}\ cm^3}$

Lateral surface area = Perimeter of the base × Height

= (6 + 6 + 6) cm × 18 cm = 18 cm × 18 cm = **324 cm^2**

Total surface area = Lateral surface area + Area of ends (bases)

$= (324 + 2 \times 9\sqrt{3})\ \text{cm}^2 = (324 + 18\sqrt{3})\ \text{cm}^2 = (324 + 31.176)\ \text{cm}^2 = \mathbf{355.176\ cm^2.}$

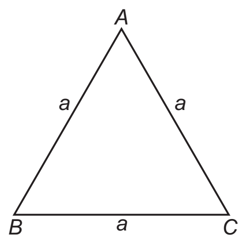
Ex. 16. The base of a right prism is a trapezium. The lengths of the parallel sides are 8 cm and 14 cm and the distance between the parallel sides is 8 cm. If the volume of the prism is 1056 cm^3, then what is the height of the prism ? (*SSC 2011*)

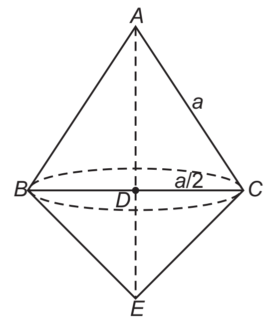
Sol. Volume of the prism = Area of the base × height

$$\Rightarrow \quad 1056 = \frac{1}{2}(8+14)\times 8\times h \Rightarrow h = \frac{1056\times 2}{22\times 8} = \mathbf{12\ cm.}$$

Ex. 17. An equilateral triangle with side a is revolved about one of its sides as axis. What is the volume of the solid of revolution thus obtained ? **(*CDS 2006*)**

Sol. When an equilateral triangle is revolved about one of its sides say BC then a double cone is generated, whose vertical radius = $a/2$ cm and slant height = a cm





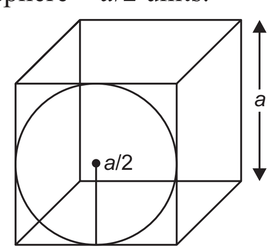
$\therefore$ Height of the cone = $\sqrt{a^2-(a/2)^2} = \sqrt{a^2-\frac{a^2}{4}} = \sqrt{\frac{3a^2}{4}} = \frac{a}{2}\sqrt{3}$ cm

Hence, volume of the solid = $2 \times \frac{1}{3}\pi r^2 h = 2 \times \frac{1}{3} \times \pi \times \frac{a}{2} \times \left(\frac{\sqrt{3}a}{2}\right)^2 = \frac{2}{3} \times \pi \times \frac{a}{2} \times \frac{3a^2}{4} = \boldsymbol{\frac{\pi a^3}{4}}$ **cm^3.**

Ex. 18. There are two identical cubes. Out of one cube, a sphere of maximum volume (V_S) is cut off. Out of the second cube, a cone of maximum volume (V_C) is cut such that its base lies on one of the faces of the cube. Which one of the following is correct ?

(*a*) $V_S = V_C$ **(*b*) $V_S = 2V_C$** **(*c*) $2V_S = 3V_C$** **(*d*) $3V_S = 4V_C$** **(*CDS 2006*)**

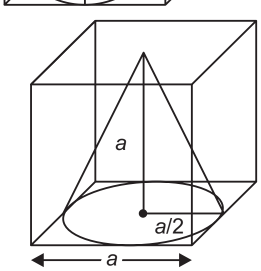
Sol. Let the side of the cube be a units. Since sphere is cut off from the cube, radius of the sphere = $a/2$ units.



$\therefore$ Volume of the sphere $= \frac{4}{3}\pi\left(\frac{a}{2}\right)^3$

$\Rightarrow \quad V_S = \frac{4}{3} \cdot \frac{\pi a^3}{8} = \frac{\pi a^3}{6}$ cu. units ...(*i*)

Since the cone is cut off from an identical cube, radius of base of cone = $a/2$ units height of cone = a units.



$\therefore$ Volume of cone $V_C = \frac{1}{3}\pi\left(\frac{a}{2}\right)^2 .a = \frac{1}{3}\pi . \frac{a^2}{4} .a = \frac{\pi a^3}{12}$ ...(*ii*)

From eqn. (*i*) and (*ii*), we get $V_S \times \frac{1}{2} = V_C \Rightarrow \boldsymbol{V_S = 2\,V_C}$**.**

Ex. 19. In a sphere of radius 2 cm a cone of height 3 cm is inscribed. What is the ratio of volumes of the cone and sphere ?

Sol. Given, radius of sphere (R) = 2 cm. Height of cone (h) = 3 cm.

Point O is the centre of the sphere.

$\therefore \quad OA = OB = R, AC = h$

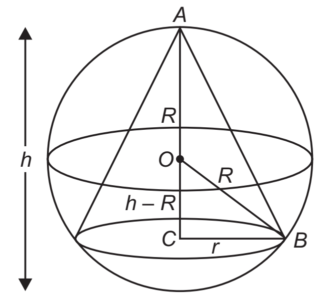
$\Rightarrow \quad OC = AC - OA = h - R$

Hence, in *rt.* $\Delta\, ORC$, $r^2 = R^2 - (h - R)^2$, where r is the radius of the cone

$$= 2^2 - (3-2)^2 = 4 - 1 = 3$$

$$\therefore \quad r = \sqrt{3}.$$

$$\therefore \quad \frac{\text{Volume of cone}}{\text{Volume of sphere}} = \frac{\frac{1}{3}\pi r^2 h}{\frac{4}{3}\pi R^3} = \frac{(\sqrt{3})^2 \times 3}{4 \times 2^3} = \mathbf{9 : 32.}$$



Ex. 20. A square has its side equal to the radius of the sphere. The square revolves round a side to generate a surface of total area *S*. If A be the surface area of the sphere, which one of the following is correct ?

(*a*) *A* = 3*S*   (*b*) *A* = 2*S*   (*c*) *A* = *S*   (*d*) *A* < *S*   (*CDS 2007*)

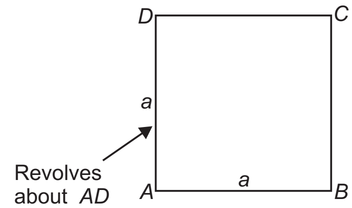
Sol. Let side of the square = a units. Then, radius of the sphere = a units

Surface area of sphere (A) = $4\pi a^2$

Since the square revolves round one of its sides, a right circular cylinder is generated whose height and radius are a units and a units respectively. So,

Total surface area of the cylinder (S) = $2\pi r\,(r + h) = 2\pi a(a + a) = 4\pi a^2$

$\therefore \quad A = S.$



Ex. 21. A cylindrical rod of iron whose height is eight times its radius is melted and cast into spherical balls each of half the radius of the cylinder. What is the number of such spherical balls ? (*SSC 2011*)

Sol. Let the radius of the base of the cylinder be r units. Height = $8r$ units

Its volume = $\pi r^2 \times 8r = 8\pi r^3$ cu. units

Radius of sphere = $r/2$ units $\quad \therefore$ Its volume = $\frac{4}{3}\pi\left(\frac{r}{2}\right)^3 = \frac{\pi r^3}{6}$ cu units.

$$\therefore \text{ Number of spherical balls} = \frac{\text{Volume of cylinder}}{\text{Volume of sphere}} = \frac{8\pi r^3}{\pi r^3} \times 6 = \mathbf{48.}$$

Ex. 22. A hemispherical bowl has its external diameter equal to 10 cm and its thickness is 1 cm. What is the whole surface area of the bowl ? (*CDS 2005*)

Sol. External radius of hemispherical bowl = 5 cm

Internal radius of the bowl = (5 – 1) cm = 4 cm

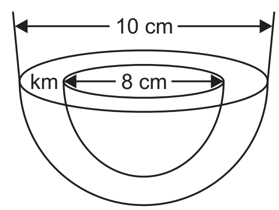
Surface area of external portion = $2\pi(5)^2 = 50\,\pi$ sq. cm

Surface area of internal portion = $2\pi(4)^2 = 32\,\pi$ sq. cm

Area of the top circular portion = $\pi(5^2 - 4^2) = 9\pi$ sq. cm

$\therefore$ Total surface area of the bowl = $50\pi + 32\pi + 9\pi = 91\pi$ sq. cm = $\left(91 \times \frac{22}{7}\right)$ sq. cm

$= 13 \times 22$ sq. cm = $\mathbf{286\ cm^2}$.

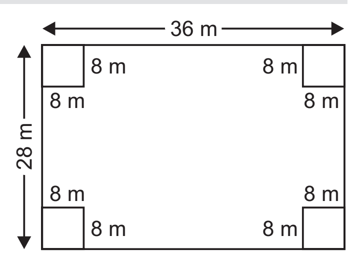


Ex. 23. A metallic sheet is of rectangular shape with dimensions 28m × 36m. From each of its corners, a square is cut off so as to make an open box. The volume of the box is *x* m³, when the length of the square is 8 m. What is the value of *x* ? (*MAT 2003*)

Sol. After cutting the squares from all the four corners, the dimensions of the box are 20 m, 12 m, 8 m.

$\therefore$ Volume of the box = $(20 \times 12 \times 8)\ \text{m}^3$

$= \mathbf{1920\ m^3}$.



PRACTICE SHEET

1. Two cans have the same height equal to 21 cm. One can is cylindrical, the diameter of whose base is 10 cm. The other can has a square base of side 10 cm. What is the difference in their capacities ?

(*a*) 350 cm^3 (*b*) 250 cm^3
(*c*) 450 cm^3 (*d*) 300 cm^3 **(*MAT 2010*)**

2. Water flows in a tank 150 m × 100 m at the base, through a pipe whose cross-section is 2 dm by 1.5 dm, at a speed of 15 km per hour. In what time will the water be 3 metre deep ?

(*a*) 100 hour (*b*) 120 hour
(*c*) 140 hour (*d*) 150 hour **(*SSC FCI 2012*)**

3. A rectangular box has dimensions x, y and z units, where $x < y < z$. If one dimension is only increased by one unit, then the increase in volume is

(*a*) Greatest when x is increased
(*b*) Greatest when y is increased
(*c*) Greatest when z is increased
(*d*) The same regardless of which dimension is increased.

4. If three cubes of copper, each with an edge 6 cm, 8 cm and 10 cm respectively are melted to form a single cube, then what is the diagonal of the new cube ?

(*a*) 18.8 cm (*b*) 22.8 cm
(*c*) 20.8 cm (*d*) 24.8 cm **(*MAT 2012*)**

5. The length and width of a swimming pool are 50 metres and 15 metres respectively. If the depth of the swimming pool at one end is 10 metres and at the other end 20 metres, then find the volume of water in the swimming pool ?

(*a*) 10000 m^3 (*b*) 11250 m^3
(*c*) 15000 m^3 (*d*) 8000 m^3

6. A cone of height 7 cm and base radius 1 cm is carved from a cuboidal block of wood 10 cm × 5 cm × 2 cm. Assuming $\pi = \frac{22}{7}$. What is the percentage of wood wasted in the process ?

(*a*) $92\frac{2}{3}\%$ (*b*) $46\frac{1}{3}\%$
(*c*) $53\frac{2}{3}\%$ (*d*) $7\frac{1}{3}\%$ **(*SSC 2002*)**

7. A cylindrical container of height 14 m and base 12 m contains oil. The oil is to be transferred to one cylindrical can, one conical can and a spherical can. The base radius of all the containers is same. The height of the conical can is 6m. While pouring some oil is dropped and hence only $\frac{3}{4}$th of cylindrical can could be filled. How much oil is dropped ?

(*a*) 54 π m^3 (*b*) 36 π m^3
(*c*) 46 π m^3 (*d*) 50 π m^3 **(*MAT 2010*)**

8. A hemispherical bowl is 176 cm round the brim. Supposing it to be half full, how many persons may be served from it in hemispherical glasses 4 cm in diameter at the top ?

(*a*) 1372 (*b*) 1272
(*c*) 1172 (*d*) 1472 **(*MAT 2009*)**

9. The length, breadth and height of a rectangular parallelopiped are in the ratios 6 : 3 : 1. If the surface area of a cube is equal to the surface area of this parallelopiped, then what is the ratio of the volume of the cube to the volume of the parallelopiped ?

(*a*) 1 : 1 (*b*) 5 : 4
(*c*) 7 : 5 (*d*) 3 : 2 **(*CDS 2010*)**

10. A copper wire 4 mm in diameter is evenly wound about a cylinder whose length is 24 cm and diameter 20 cm so as to cover the surface. Find the length and weight of the wire assuming the specific gravity to be 8.88 gm/cm^3.

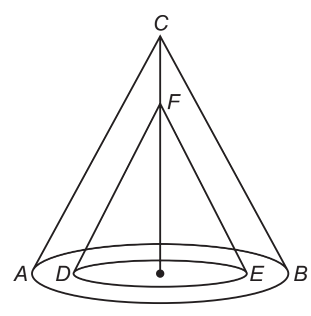
(*a*) 1100 π cm, 545π^2 gm
(*b*) 1200 π cm, 441π^2 gm
(*c*) 1200 π cm, 426.24 π^2 gm
(*d*) 1400 π cm, 426.24 π^2 gm

11. A sector of a circle of radius 15 cm has the angle 120º. It is rolled up so that the two bounding radii are joined together to form a cone. The volume of the cone is

(*a*) $(250\sqrt{2}\,\pi)$ cm^3 (*b*) $[(500\sqrt{2})\pi/3]$ cm^3
(*c*) $[(250\sqrt{2})\,\pi/3]$ cm^3 (*d*) $[(1000\sqrt{2})\,\pi/3]$ cm^3

(*MAT 2010*)

12. The radius and height of a right solid circular cone (*ABC*) are respectively 6 cm and $2\sqrt{7}$ cm. A coaxial cone (*DEF*) of radius 3 cm and height $\sqrt{7}$ cm is cut out of the cone as shown in the given figure. What is the whole surface area of the solid thus formed ?



(*a*) 96 π cm^2 (*b*) 87 π cm^2
(*c*) 60 π cm^2 (*d*) 36 π cm^2 **(*CDS 2006*)**

13. A right circular cylinder and a right circular cone have equal bases and equal volumes. But the lateral surface area of the right circular cone is 15/8 times the lateral surface area of the right circular cylinder. What is the ratio of radius to height of the cylinder ?

(*a*) 3 : 4 (*b*) 9 : 4
(*c*) 15 : 8 (*d*) 8 : 15 **(*CDS 2007*)**

14. From a wooden cylindrical block, whose diameter is equal to its height, a sphere of maximum possible volume is carved out. What is the ratio of the utilised wood to that of the wasted wood ?

(*a*) 2 : 1 (*b*) 1 : 2
(*c*) 2 : 3 (*d*) 3 : 2 **(*CDS 2008*)**

15. The volume of a cube is numerically equal to sum of its edges. What is the total surface area in square units ?

(*a*) 12 (*b*) 36
(*c*) 72 (*d*) 144 **(*CDS 2012*)**

16. The base in a right prism is an equilateral triangle of side 8 cm and the height of the prism is 10 cm. The volume of the prism is

(*a*) $150\sqrt{3}$ cubic cm (*b*) $300\sqrt{3}$ cubic cm
(*c*) $320\sqrt{3}$ cubic cm (*d*) $160\sqrt{3}$ cubic cm **(*SSC 2012*)**

17. The base of a right triangular prism is an equilateral triangle. If the height is halved and each side of the base is doubled, find the ratio of the volume of the original prism to the new prism.

(*a*) 1 : 1 (*b*) 1 : 2
(*c*) 3 : 2 (*d*) 2 : 3

18. What is the volume of a right prism standing on a triangular base of sides 5 cm, 5 cm and 8 cm whose lateral surface area is 828 cm^2 ?

(*a*) 680 cm^3 (*b*) 552 cm^3
(*c*) 1008 cm^3 (*d*) 728 cm^3

19. A sphere, a cylinder and a cone respectively are of the same radius and same height. Find the ratio of their curved surfaces.

(*a*) 3 : 2 : 1 (*b*) $4 : \sqrt{3} : 4$
(*c*) $4 : 4 : \sqrt{5}$ (*d*) $3 : 3 : \sqrt{5}$

20. The sum of the radii of two spheres is 10 cm and the sum of their volumes is 880 cm^3. What will be the product of their radii ?

(*a*) 21 (*b*) $26\frac{1}{3}$
(*c*) $33\frac{1}{3}$ (*d*) 70 **(*SSC 2005*)**

21. If S denotes the area of the curved surface of a right circular cone of height h end semi-vertical angle α, then S equals

(*a*) $\pi h^2 \tan^2\alpha$ (*b*) $\frac{1}{3}\pi h^2 \tan^2\alpha$
(*c*) $\pi h^2 \sec\alpha \tan\alpha$ (*d*) $\frac{1}{3}\pi h^2 \sec\alpha \tan\alpha$

22. The base of a right prism is a pentagon whose sides are in the ratio $1:\sqrt{2}:\sqrt{2}:1:2$ and its height is 10 cm. If the longest side of the base be 6 cm, the volume of the prism is

(*a*) 270 cm^3 (*b*) 360 cm^3
(*c*) 540 cm^3 (*d*) None of these

23. If three cylinders of radius r and height h are placed vertically such that the curved surface of each cylinder touches the curved surfaces of the other two cylinders tangentially, then the volume of the air space left between the three cylinders is

(*a*) $hr^2(3 + \pi)$ (*b*) $hr^2\left(\sqrt{3} + \frac{\pi}{2}\right)$
(*c*) $hr^2\left(\sqrt{3} - \frac{\pi}{2}\right)$ (*d*) $hr^2(\sqrt{3} - \pi)$

24. If two rectangular sheets each of dimensions $2x$ and $2y$ form the curved surfaces of two different cylinders, then the ratio of the quotient of the volumes $\left(\frac{V_1}{V_2}\right)$ to that of the areas of the curved surfaces $\left(\frac{S_1}{S_2}\right)$ of the two cylinders is

(*a*) $\frac{x^2}{y}:1$ or $\frac{y^2}{x}:1$ (*b*) $\frac{2x}{y}:1$ or $\frac{2y}{x}:1$
(*c*) $\frac{3x}{y}:1$ or $\frac{3y}{x}:1$ (*d*) $\frac{x}{y}:1$ or $\frac{y}{x}:1$

25. A sphere and a right circular cone of same radius have equal volumes. By what percentage does the height of the cone exceed its diameter ?

(*a*) 50 % (*b*) $66\frac{1}{3}$%
(*c*) 100 % (*d*) 43.75%

26. If the base of right rectangular prism remains constant and the measures of the lateral edges are halved, then its volume will be reduced by :

(*a*) 50% (*b*) 33.33%
(*c*) 66.66% (*d*) None of these

27. A cylindrical rod of iron whose radius is one-fourth of its height is melted and cast into spherical balls of the same radius as that of the cylinder. What is the number of spherical balls ?

(*a*) 2 (*b*) 3
(*c*) 4 (*d*) 5 **(*CDS 2011*)**

28. The diameter of a solid mettalic right circular cylinder is equal to its height. After culting out the largest possible solid sphere S from this cylinder, the remaining material is recast to form a solid sphere S_1. What is the ratio of the radius of the sphere S to that of the sphere S_1.

(*a*) $1 : 2^{\frac{1}{3}}$ (*b*) $2^{\frac{1}{3}} : 1$
(*c*) $2^{\frac{1}{3}} : 3^{\frac{1}{3}}$ (*d*) $3^{\frac{1}{2}} : 2^{\frac{1}{2}}$ **(*CDS 2007*)**

29. A right circular solid cone of maximum possible volume is cut off from a solid metallic right circular cylinder of volume V. The remaining metal is melt and recast into four identical solid spheres. What is the volume of each sphere ?

(*a*) $\frac{V}{12}$ (*b*) $\frac{V}{9}$

(*c*) $\frac{V}{8}$ (*d*) $\frac{V}{6}$ **(*CDS 2006*)**

30. A solid right circular cylinder of radius 8 cm and height 2 cm is melted and cast into a right circular cone of height 3 times that of the cylinder. Find the curved surface of the cone ?

(*a*) 54 π cm^2 (*b*) 80 π cm^2

(*c*) 72 π cm^2 (*d*) 77 π cm^2.

31. A spherical metal of radius 10 cm is melted and made into 1000 smaller spheres of equal sizes. In this process the surface area of the metal is increased by

(*a*) 1000 times (*b*) 100 times

(*c*) No change (*d*) None of these

(*XAT 2012*)

32. A tank internally measuring 150 cm × 120 cm × 100 cm has 1281600 cm^3 of water in it. Porous bricks are placed in the water until the tank is full upto to its brim. Each brick absorbs one-tenth of its volume of water. How many bricks of dimensions 20 cm × 6 cm × 4 cm can be put in the tank without spilling over the water.

(*a*) 1100 (*b*) 1200

(*c*) 1150 (*d*) 1250

(*e*) None of these **(*XAT 2010*)**

33. A child consumed an ice-cream of inverted right-circular conical shape from the top and left only 12.5% of the cone for her mother. If the height of the ice-cream cone was 8 cm, what was the height of the remaining ice-cream cone ?

(*a*) 2.5 cm (*b*) 3.0 cm

(*c*) 3.5 cm (*d*) 4.0 cm **(*JMET 2009*)**

34. 27 drops of water form a big drop of water. If the radius of each smaller drop is 0.2 cm, then what is the radius of the bigger drop ?

(*a*) 0.4 cm (*b*) 0.6 cm

(*c*) 0.8 cm (*d*) 1.0 cm **(*CDS 2007*)**

35. The curved surface of a cylinder is developed into a square whose diagonal is $2\sqrt{2}$ cm. The area of the base of the cylinder (in cm^2) is

(*a*) 3π (*b*) $\frac{1}{\pi}$

(*c*) π (*d*) 6π

ANSWERS

1. (*c*)	**2.** (*a*)	**3.** (*a*)	**4.** (*c*)	**5.** (*b*)	**6.** (*d*)	**7.** (*b*)	**8.** (*a*)	**9.** (*d*)	**10.** (*c*)
11. (*c*)	**12.** (*b*)	**13.** (*b*)	**14.** (*a*)	**15.** (*c*)	**16.** (*d*)	**17.** (*b*)	**18.** (*b*)	**19.** (*c*)	**20.** (*b*)
21. (*c*)	**22.** (*a*)	**23.** (*c*)	**24.** (*d*)	**25.** (*c*)	**26.** (*a*)	**27.** (*b*)	**28.** (*b*)	**29.** (*d*)	**30.** (*b*)
31. (*d*)	**32.** (*b*)	**33.** (*d*)	**34.** (*b*)	**35.** (*b*)					

HINTS AND SOLUTIONS

1. Required difference in capacities $= \frac{22}{7} \times (5)^2 \times 21 \sim (10)^2 \times 21$

$= (1650 \sim 2100)$ cm^3

$= \mathbf{450\ cm^3}$**.**

2. Time required (in hours) $= \dfrac{\text{Volume of water in the tank}}{\text{Volume of water flowing through the pipe per hour}}$

$= \dfrac{150 \times 100 \times 3}{0.2 \times 0.15 \times 15000} =$ **100 hours**

3. We can check by an example. Let $x = 2$, $y = 3$, $z = 4$. Then

Volume of box = 2 × 3 × 4 = 24 cu. units.

If we increase x by 1 unit, keeping the other same *i.e*, $x = 3$, then new volume = 3 × 3 × 4 = 36 cu. units.

If we increase y by 1 unit keeping the others same, *i.e.*, $y = 4$, then new volume = 2 × 4 × 4 = 32 cu. units

If we increase z by 1 unit keeping the other same, *i.e.*, $z = 5$, then new volume = 2 × 3 × 5 = 30 cu. units.

Hence the increase in volume is greatest, when x increases. You can check with other values also.

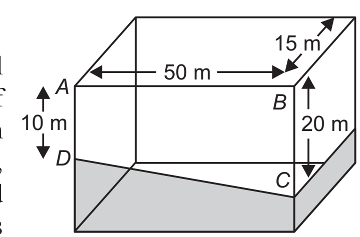
4. Let the edge of the single cube be 'a' cm. Then, total volume melted = Volume of cube formed

$\Rightarrow$ $(6)^3 + (8)^3 + (10)^3 = a^3$

$\Rightarrow$ $a^3 = 216 + 512 + 1000 = 1728 \Rightarrow a = 12$ cm.

$\therefore$ Diagonal of the new cube $= \sqrt{3}\,a = (\sqrt{3} \times 12)$ cm

= **20.8 cm** (approx.)



5. If we take the vertical crossection of the face of the swimming pool, then it is a trapezium $ABCD$, with parallel sides AD and BC respectively of lengths 10 m and 20 m and distance between parallel sides as 50 m. The width of the swimming pool is 15 m.

$\therefore$ Volume of water in the swimming pool

= Area of vertical cross section ($ABCD$) × width of swimming pool

$= \left[\frac{1}{2}(10 + 20) \times 50\right] \times 15$ m^3

= (15 × 50 × 15) m^3 = **11250 m^3.**

6. Volume of cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\times\frac{22}{7}\times 1\times 7 = \frac{22}{3}$ cu. cm

Volume of cubical block = (10 × 5 × 2) cm^3 = 100 cm^3

$\therefore$ % Wastage of wood $= \dfrac{\left(100 - \frac{22}{3}\right)}{100}\times 100$

$= \frac{278}{3}\% = \mathbf{92\frac{2}{3}\%}.$

7. Volume of oil = $\pi \times (6)^2 \times 14 = 504\,\pi\ m^3$

Volume of conical can $= \frac{1}{3}\times\pi\times(6)^2\times 6 = 72\pi\ m^3$

Volume of spherical can $= \frac{4}{3}\times\pi\times(6)^3 = 288\pi\ m^3$

Remaining oil = Vol. of oil – Vol. of oil in (conical can + spherical can)

$= 504\,\pi - (72\pi + 288\pi) = 144\,\pi m^3$

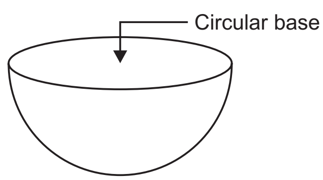
Volume of the cylindrical can = $\pi \times (6)^2 \times h = 144\,\pi$

$\Rightarrow$ $h = 4$ m.

As only $\frac{3}{4}$th of the cylindrical can could be filled,

Vol. of oil dropped $= \frac{1}{4}\times\pi\times(6)^2\times 4 = \mathbf{36\,\pi\ m^3}.$

8. Given, perimeter of the circular base of the cup = 176 cm. If r is the radius of the hemispherical bowl, then $2\pi r = 176$



$\Rightarrow$ $r = \frac{176\times 7}{22\times 2} = 28$ cm.

Total number of persons who can be served

$= \dfrac{\frac{1}{2}\times \text{Volume of hemispherical bowl}}{\text{Volume of one hemispherical glass}}$

$= \dfrac{\frac{1}{2}\times\frac{2}{3}\times\pi\times(28)^3}{\frac{2}{3}\times\pi\times(2)^3} = \frac{2744}{2} = \mathbf{1372.}$

9. Let the length, breadth and height of the rectangular parallelopiped be $6x$, $3x$ and x.

Let the side of the cube be a.

$\therefore$ By the given condition,

Surface area of a cube

= Surface area of rectangular parallelopiped

$6(a)^2 = 2(6x \times 3x + 3x \times x + x \times 6x)$

$\Rightarrow$ $6a^2 = 2(18x^2 + 3x^2 + 6x^2) \Rightarrow 6a^2 = 54x^2 \Rightarrow a^2 = 9x^2$

$\Rightarrow$ $a = 3x$

Now, Volume of cube : Volume of rectangular parallelopiped

$= a^3 : (6x \times 3x \times x) = (3x)^3 : 18x^3 = 27 : 18 = \mathbf{3 : 2.}$

10. One round of wire covers 4 mm $= \frac{4}{10}$cm in thickness of the surface of the cylinder

Length of the cylinder = 24 cm

$\therefore$ Number of rounds to cover 24 cm $= \frac{24}{4/10} = \frac{24\times 10}{4} = 60$

Diameter of the cylinder = 20 cm

$\Rightarrow$ Radius of cylinder = 10 cm

Length of the wire in completing one round

$= 2\pi r = 2\pi \times 10$ cm $= 20\,\pi$ cm.

$\therefore$ Length of the wire in covering the whole surface

= Length of the wire in completing 60 rounds

$= (20\,\pi \times 60)$ cm $= 1200\,\pi$ cm

Radius of copper wire = 2 mm $= \frac{2}{10}$ cm

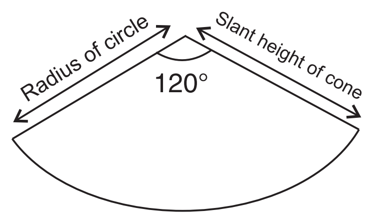
$\therefore$ Volume of wire $= \left(\pi\times\frac{2}{10}\times\frac{2}{10}\times 1200\,\pi\right) cm^3 = 48\pi^2\ cm^3$

So, weight of wire = $(48\pi^2 \times 8.88)$ gm $= \mathbf{426.24\ \pi^2\ gm.}$

11. Curved surface area of cone = Area of the sector of circle

$\Rightarrow$ $\pi rl = \pi R^2 \times \frac{120}{360}$

$(\because\ l = R)$



$\therefore$ $r = 15 \times \frac{120}{360} = 5$ cm

$\therefore$ $h = \sqrt{225 - 25} = 10\sqrt{2}$ cm

Volume of the cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\times\pi\times 25\times 10\sqrt{2}$

$= \mathbf{\frac{250\sqrt{2}\,\pi}{3}\ cm^3}$

12. Radius and height of cone ABC are 6 cm and $2\sqrt{7}$ cm respectively.

$\therefore$ Slant height of cone $ABC = \sqrt{6^2 + (2\sqrt{7})^2}$

$= \sqrt{36 + 28} = \sqrt{64} = 8$ cm

$\therefore$ Curved surface area of cone $ABC = \pi \times 6 \times 8 = 48\pi\ cm^2$

Also radius and height of cone DEF are 3 cm and $\sqrt{7}$ respectively

$\therefore$ Slant height of cone $DEF = \sqrt{3^2 + (\sqrt{7})^2} = \sqrt{9 + 7}$

$= \sqrt{16} = 4$ cm.

$\therefore$ Curved surface area of cone $DEF = \pi \times 3 \times 4 = 12\,\pi\,cm^2$

$\therefore$ Whole surface area of remaining solid

= Curved surface area of cone ABC + Area of base

+ Curved surface area of cone DEF

$= 48\,\pi + \pi\,(6^2 - 3^2) + 12\pi$

$= 48\pi + \pi(36 - 9) + 12\pi = \mathbf{87\,\pi\ cm^2.}$

13. Let r and h be the radius and height of the cone and r and H be the radius and height of the cylinder.

$\therefore$ Volume of cone $= \frac{1}{3}\pi r^2 h$

Volume of cylinder $= \pi r^2 H$

By the given condition, $\frac{1}{3}\pi r^2 h = \pi r^2 H$

$\Rightarrow h = 3H$

Lateral surface area of cone $= \pi rl$

Lateral surface area of cylinder $= 2\pi r H$

By the given condition, $\pi rl = \frac{15}{8} \times 2\pi r H$

$\Rightarrow l = \frac{15}{4}H \Rightarrow l^2 = \frac{225}{16}H^2$

$\Rightarrow r^2 + h^2 = \frac{225}{16}H^2$

$\Rightarrow r^2 + 9H^2 = \frac{225}{16}H^2$ [Using $h = 3H$]

$\Rightarrow r^2 = \frac{225}{16}H^2 - 9H^2 = \frac{81}{16}H^2$

$\Rightarrow \frac{r^2}{H^2} = \frac{81}{16} \Rightarrow \frac{r}{H} = \frac{9}{4} \Rightarrow r : H = \mathbf{9 : 4}$.

14. Let r be the radius of the cylindrical block, then height of the block $= 2r$.

Volume of the block $= \pi(r^2)(2r) = 2\pi r^3$

A sphere of maximum possible volume is carved out of the cylindrical block, so the radius of the sphere $= r$.

$\therefore$ Volume of sphere $= \frac{4}{3}\pi r^3$ = Volume of utilised wood

$\therefore$ Volume of wasted wood $= 2\pi r^3 - \frac{4}{3}\pi r^3 = \frac{2\pi r^3}{3}$

$\therefore$ Required ratio $= \frac{4}{3}\pi r^3 : \frac{2}{3}\pi r^3 = \mathbf{2 : 1}$.

15. Let a be the length of an edge of a cube. Then,

$a^3 = 12a$ ($\because$ A cube has 12 edges)

$\Rightarrow a^2 = 12 \quad \Rightarrow \quad a = 2\sqrt{3}$

$\therefore$ Total surface area of cube $= 6a^2 = 6 \times (2\sqrt{3})^2 = \mathbf{72.}$

16. Volume of a prism = Base Area × Height

$$= \left(\frac{\sqrt{3}}{4} \times (8)^2 \times 10\right) \text{cm}^3$$

$$= \left(\frac{\sqrt{3}}{4} \times 64 \times 10\right) \text{cm}^3 = \mathbf{160\sqrt{3}\ cm^3}.$$

17. Let each side of the base of the original prism be a units and the height of the prism be h units. Then

Required ratio $= \frac{\text{Vol. of original prism}}{\text{Vol. of new prism}}$

$$= \frac{\frac{\sqrt{3}}{4} \times (a)^2 \times h}{\frac{\sqrt{3}}{4} \times (2a)^2 \times h/2} = \frac{2a^2 h}{4a^2 h} = \mathbf{1 : 2.}$$

18. Lateral surface area of a prism = Perimeter of base × Height

$\Rightarrow 840 = (5 + 5 + 8) \times$ Height

$\Rightarrow$ Height $= \frac{840}{18} = 46$ cm.

Semi perimeter of the triangular base $= \frac{18}{2} = 9$ cm

$\therefore$ Area of triangle $= \sqrt{9(9-5)(9-5)(9-8)}$ (Herons Formula)

$= \sqrt{9 \times 4 \times 4 \times 1} = 12 \text{ cm}^2$

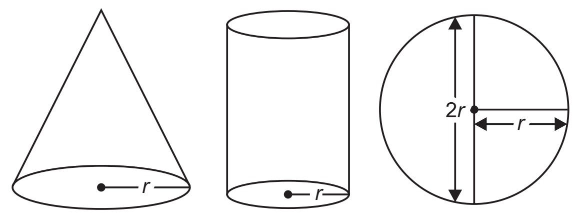
$\therefore$ Required volume of prism = Area of base × Height

$= (12 \times 46) \text{ cm}^3 = \mathbf{552\ cm^3.}$

19. Let r be the common radius of the sphere, cone and cylinder. Then, Height of the cone = Height of the cylinder = Height of the sphere $= 2r$

Let l be the slant height of the cone. Then,

$l = \sqrt{h^2 + r^2} = \sqrt{4r^2 + r^2} = \sqrt{5r^2} = r\sqrt{5}$



Now, CSA of sphere $= 4\pi r^2$

CSA of cylinder $= 2\pi r.\ 2r = 4\pi r^2$

CSA of cone $= \pi rl = \pi r.\ \sqrt{5}\ r = \sqrt{5}\ \pi r^2$

$\therefore$ Required ratio $= 4\pi r^2 : 4\pi r^2 : \sqrt{5}\ \pi r^2 = \mathbf{4 : 4 : \sqrt{5}}$.

20. Given, $r_1 + r_2 = 10$ ...(i)

and $\frac{4}{3}\pi r_1^3 + \frac{4}{3}\pi r_2^3 = 880$

$\Rightarrow \frac{4}{3}\pi(r_1^3 + r_2^3) = 880 \Rightarrow r_1^3 + r_2^3 = \frac{880 \times 3 \times 7}{4 \times 22} = 210$ ...(ii)

Taking the cube of both the sides of eqn. (i), we have $(r_1 + r_2)^3 = 1000 \Rightarrow r_1^3 + r_2^3 + 3r_1 r_2 (r_1 + r_2) = 1000$

$\Rightarrow 210 + 3r_1 r_2 (10) = 1000$

$\Rightarrow 30r_1 r_2 = 1000 - 210 = 790 \Rightarrow r_1 r_2 = \frac{790}{30} = \mathbf{26\frac{1}{3}}$.

21. $S = \pi rl$, where r = radius, l = slant height, h = height

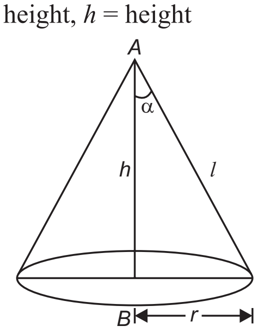
Now $\frac{r}{h} = \tan\alpha \Rightarrow r = h\tan\alpha$

$l = \sqrt{h^2 + r^2}$

$= \sqrt{h^2 + h^2\tan^2\alpha}$

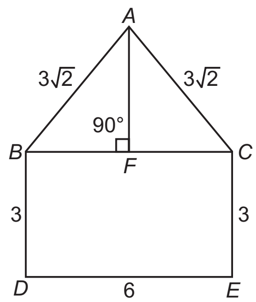
$= \sqrt{h^2(1 + \tan^2\alpha)}$

$= \sqrt{h^2\sec^2\alpha} = h\sec\alpha$



$\therefore S = \pi.\ h\tan\alpha.\ h\sec\alpha = \boldsymbol{\pi h^2 \sec\alpha \tan\alpha.}$

22. The base of the prism is a pentagon as shown.



If the longest side is 6 cm, then the other sides are 3 cm, $3\sqrt{2}$ cm, $3\sqrt{2}$ cm, 3 cm

Since ABC is an isosceles triangle, $AF \perp BC$ and $\angle AFB = 90°$.

$\therefore \quad BF = FC = 3$ cm

In $\quad \Delta ABF$, $AF = \sqrt{(3\sqrt{2})^2 - 3^2} = \sqrt{18 - 9} = \sqrt{9} = 3.$

$\therefore$ Total area of the base

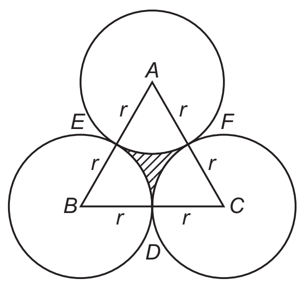
$= $ Area of ΔABC + Area of rect. $BCDE$

$= \left(\frac{1}{2} \times 6 \times 3 + 6 \times 3\right) \text{cm}^2 = (9 + 18) \text{ cm}^2 = 27 \text{ cm}^2.$

$\therefore \quad$ Volume of prism = Area of base × height

$= (27 \times 10) \text{ cm}^3 = \mathbf{270 \text{ cm}^3}.$

23. The bases of the three cylinders when placed as given are as shown in the figure :



Let the radius of the base of each cylinder = r cm.

We are required to find the volume of air.

Space left between the cylinders = **Area of shaded portion × height of cylinder**

Now, area of shaded portion

= Area of ΔABC – Sum of areas of sectors of the three bases

ΔABC, as can be seen is an equilateral triangle of side $2r$.

$\therefore \quad$ Area of $\Delta\, ABC = \frac{\sqrt{3}}{4} \times (2r)^2 = \sqrt{3} r^2$

Area of (sector AEF + sector BED + sector CFD)

($\because$ sector angles $\angle A = \angle B = \angle C = 60°$)

$= 3 \times \frac{60°}{360°} \times \pi r^2 = \frac{\pi r^2}{2}$

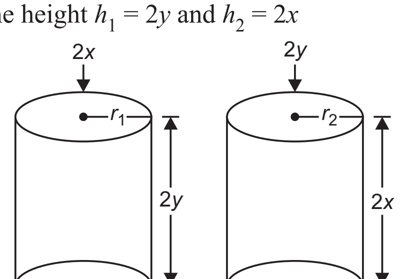
$\therefore$ Required volume $= \left(\sqrt{3}\, r^2 - \frac{\pi}{2} r^2\right) h = \left(\boldsymbol{\sqrt{3} - \frac{\pi}{2}}\right) \boldsymbol{r^2 h}.$

24. Let the radii of the two cylinders be r_1 and r_2 respectively.

Then, $2\pi r_1 = 2x$ and $2\pi r_2 = 2y$

$\Rightarrow \; r_1 = \frac{x}{\pi} \quad$ and $\quad r_2 = \frac{y}{\pi}$

Also, the height $h_1 = 2y$ and $h_2 = 2x$



$\therefore \quad$ If V_1, V_2 and S_1, S_2 be the respective volumes and curved surface areas of these cylinders, then,

$$V_1 = \pi r_1^2\, h_1 = \pi \left(\frac{x}{\pi}\right)^2 2y = \frac{2x^2 y}{\pi}$$

$$V_2 = \pi r_2^2\, h_2 = \pi \left(\frac{y}{\pi}\right). 2x = \frac{2y^2 x}{\pi}$$

$$S_1 = 2\pi r_1 h_1 = 2\pi \left(\frac{x}{\pi}\right) .2y = 4xy$$

$$S_2 = 2\pi r_2 h_2 = 2\pi \left(\frac{y}{\pi}\right) .2x = 4xy$$

$\therefore \quad \frac{V_1}{V_2} : \frac{S_1}{S_2} = \boldsymbol{\frac{x}{y} : 1.}$

25. Let r be the radius of the cone and sphere.

Let h be the height of the cone.

Then, according to the question,

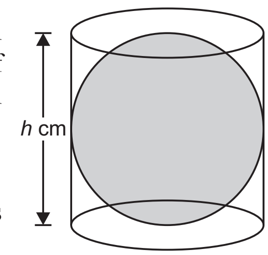
$\frac{4}{3} \pi r^3 = \frac{1}{3} \pi r^2 h \Rightarrow h = \mathbf{4}\boldsymbol{r}$

$\therefore \quad$ Required % difference $= \left(\frac{4r - 2r}{2r} \times 100\right)\% = \mathbf{100\%}.$

26. Volume of a prism = Area of base × height.

Since, the base area is constant and the height is halved, so the volume will also be halved, *i.e.* volume will be reduced by 50%.

27. Let the height of the cylindrical iron rod be h cm. Then, Radius of iron rod = Radius of spherical ball



$= \frac{h}{4}$ cm.

$\therefore \quad$ Number of spherical balls

$= \frac{\text{Volume of cylindrical rod}}{\text{Volume of spherical ball}}$

$= \frac{\pi \times \left(\frac{h}{4}\right)^2 \times h}{\frac{4}{3} \times \pi \times \left(\frac{h}{4}\right)^3} = \mathbf{3.}$

28. Let the height of the cylinder = h cm.

Then, radius of cylinder = radius of sphere = $\frac{h}{2}$ cm.

$\therefore \quad$ Volume of remaining material

= Volume of cylinder – Volume of sphere

$= \pi \left(\frac{h}{2}\right)^2 .h - \frac{4}{3}\pi \left(\frac{h}{2}\right)^3 = \frac{\pi h^3}{4} - \frac{\pi h^3}{6} = \frac{\pi h^3}{12}$

Let the recasted solid sphere have radius R cm. Then, volume of recasted sphere = Volume of remaining material.

$\Rightarrow \quad \frac{4}{3} \pi R^3 = \frac{\pi h^3}{12} \Rightarrow R^3 = \frac{h^3}{16} \Rightarrow R = \frac{h}{2.(2^{1/3})}$

$\therefore \quad$ Required ratio $= r : R = h/2 : h/2(2^{1/3}) = \mathbf{2^{1/3} : 1.}$

29. Volume of remaining metal

= Volume of cylinder – Volume of cone

$= \pi r^2 h - \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^2 h = \frac{2}{3}V$

$\therefore$ Volume of 4 spheres $= \frac{2}{3}V$

$\Rightarrow$ Volume of one sphere $= \frac{\frac{2}{3}V}{4} = \boldsymbol{\frac{V}{6}}.$

30. For cylinder, radius $(r_1) = 8$ cm, height $(h_1) = 2$ cm

For cone, radius $(r_2) = ?$, height $(h_2) = 6$ cm

Now, Volume of cone = Volume of cylinder

$\Rightarrow \frac{1}{3}\pi r_2^2 h_2 = \pi r_1^2 h_1 \Rightarrow \frac{1}{3}\pi r_2^2 \times 6 = \pi \times 8 \times 8 \times 2$

$\Rightarrow r_2^2 = \frac{\pi \times 8 \times 8 \times 2 \times 3}{\pi \times 6} = 64 \Rightarrow r_2 = 8$ cm

Curved surface of cone $= \pi r_2 l = \pi r_2 \sqrt{h_2^2 + r_2^2}$

$= \pi \times 8 \times \sqrt{36 + 64} = (\pi \times 8 \times 10)\text{ cm}^2 = \mathbf{80\pi\ cm^2}.$

31. Volume of bigger sphere $= \frac{4}{3} \times \pi \times (10)^3\text{ cm}^3 = \frac{4000}{3}\pi\text{ cm}^3$

Volume of each smaller sphere $= \frac{\frac{4000}{3}}{1000}\text{ cm}^3 = \frac{4}{3}\pi\text{ cm}^3$

$\therefore$ If r is the radius of each smaller sphere, then

$\frac{4}{3}\pi r^3 = \frac{4}{3}\pi \Rightarrow r = 1$ cm.

Surface area of bigger sphere $= 4 \times \pi \times (10)^2\text{ cm}^2 = 400\pi\text{ cm}^2$

Surface area of 1000 smaller spheres $= 1000 \times (4 \times \pi \times 1)\text{ cm}^2 = 4000\pi\text{ cm}^2$

So, the total surface area increases by $\frac{4000\pi - 400\pi}{400\pi}$ times

= **9 times.**

32. Volume of a brick = 20 cm × 6 cm × 4 cm = 480 cm^3.

Water absorbed by one brick $= \left(\frac{1}{10} \times 480\right)\text{cm}^3 = 48\text{ cm}^3$.

Let x bricks be placed in the water.

Then, x bricks absorb $48x$ cm^3 of water.

$\therefore$ Vol. of water + Volume of x bricks – Volume of absorbed water = Volume of water tank

$\Rightarrow 1281600 + 480x - 48x = 150 \times 120 \times 100$

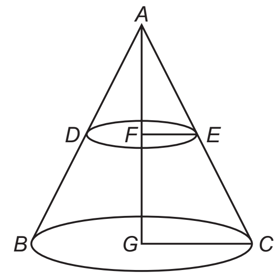
$\Rightarrow 432x = 1800000 - 1281600 = 518400$

$\Rightarrow x = \mathbf{1200}.$

33. Let ADE be the remaining portion of the cone. Then, from similar Δs AFE and AGC

$\frac{AF}{AG} = \frac{FE}{GC} \Rightarrow FE = \frac{AF.GC}{AG}$

$\Rightarrow r_1 = \frac{hr}{8}$



where r_1 = radius of cone ADE, h = height of cone ADE and r = radius of filled cone ABC

Now, volume of smaller cone

$ADE = \frac{1}{3}\pi r_1^2 h = \frac{1}{3}\pi\left(\frac{hr}{8}\right)^2 .h$

$= \frac{1}{3} \times \pi \times \frac{h^3 r^2}{64}$

Volume of bigger cone $ABC = \frac{1}{3}\pi r^2 \times 8$

Given, $\frac{1}{3} \times \pi \times \frac{h^3 r^2}{64} = 12.5\%$ of $\left(\frac{8}{3}\pi r^2\right)$

$\Rightarrow \frac{\pi}{3} \times \frac{h^3 r^2}{64} = \frac{125}{1000} \times \frac{8}{3}\pi r^2 \Rightarrow h^3 = 64 \Rightarrow h = \mathbf{4}.$

34. By the given condition,

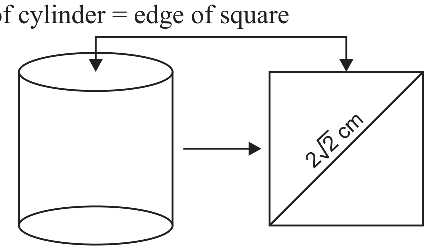
27 × Volume of smaller drops = Volume of bigger drop

$\Rightarrow 27 \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3$

$\Rightarrow 27 \times \frac{4}{3}\pi(0.2)^3 = \frac{4}{3}\pi R^3 \Rightarrow R^3 = (3 \times 0.2)^3 \Rightarrow \mathbf{R = 0.6\,cm.}$

35. Each edge of the square $= \frac{1}{\sqrt{2}} \times 2\sqrt{2} = 2$cm.

If 'r' is the radius of the cylinder, then circumference of base of cylinder = edge of square



$\Rightarrow 2\pi r = 2 \quad \Rightarrow r = \frac{1}{\pi}$

$\therefore$ Area of base $= \pi r^2 = \pi \times \left(\frac{1}{\pi}\right)^2 = \mathbf{\frac{1}{\pi}\ cm^2}.$

SELF ASSESSMENT SHEET

1. If S is the total surface area of a cube and V is its volume, then which of the following is correct ?

(*a*) $V^3 = 216\,S^2$ (*b*) $S^3 = 216\,V^2$

(*c*) $S^3 = 6V^2$ (*d*) $S^2 = 36\,V^3$ **(*CDS 2011*)**

2. The volume of a certain rectangular solid is 8 cm^3. Its total surface area is 32 cm^2 and its three dimensions are in geometric progression. The sum of the lengths in cm of all the edges of this solid is

(*a*) 28 (*b*) 32

(*c*) 36 (*d*) 44

(*e*) 40

3. The volume of the metal of a cylindrical pipe is 748 cm^3. The length of the pipe is 14 cm and its external radius is 9 cm. What is its thickness ? (Take $\pi = 22/7$)

(*a*) 1 cm (*b*) 5.2 cm

(*c*) 2.3 cm (*d*) 3.7 cm **(*SSC 2008*)**

4. A sphere and a cone have equal bases. If their heights are also equal, the ratio of their curved surface will be :

(*a*) $1 : \sqrt{3}$ (*b*) $4 : \sqrt{5}$

(*c*) 4 : 1 (*d*) $\sqrt{5} : 1$ **(*SSC 2012*)**

5. The magnitude of the volume of a closed right circular cylinder of unit height divided by the magnitude of the total surface area of the cylinder (*r* being the radius of the cylinder) is equal to

(*a*) $\frac{1}{2}\left(1+\frac{1}{r}\right)$ (*b*) $\frac{1}{2}\left(1+\frac{1}{r+1}\right)$

(*c*) $\frac{1}{2}\left(1-\frac{1}{r}\right)$ (*d*) $\frac{1}{2}\left(1-\frac{1}{r+1}\right)$

6. Three identical balls fit snugly into a cylindrical can. The radius of the spheres is equal to the radius of the can and the balls just touch the bottom and the top of the can. What fraction of the volume of the can is taken up by the balls ?

(*a*) $\frac{1}{3}$ (*b*) $\frac{3}{7}$

(*c*) $\frac{2}{3}$ (*d*) $\frac{1}{4}$

7. The total surface area of a right triangular prism of height 4 cm is $72\sqrt{3}$ cm^2. If the base of the prism is an equilateral triangle, find its volume.

(*a*) $36\sqrt{3}$ cm^3 (*b*) $42\sqrt{3}$ cm^3

(*c*) $48\sqrt{3}$ cm^3 (*d*) $54\sqrt{3}$ cm^3

8. A right triangular prism of height 18 cm and of base sides 5 cm, 12 cm and 13 cm is transformed into another right triangular prism on a base of sides 9 cm, 12 cm and 15 cm. Find the height of the new prism and the change in the whole surface area.

(*a*) 10 cm, 120 cm^2 (*b*) 8 cm, 132 cm^2

(*c*) 10 cm, 132 cm^2 (*d*) 8 cm, 120 cm^2

9. There are two prisms, one has equilateral triangle as a base and the other has a regular hexagon as a base. If both the prisms have equal height and volume, the ratio of length of each side of the equilateral triangle to the length of each side of the regular hexagon is

(*a*) $\sqrt{3} : 1$ (*b*) $\sqrt{6} : 1$

(*c*) 3 : 2 (*d*) $2 : \sqrt{3}$

10. A spherical iron shell with external diameter 21 cm weighs $22775\frac{5}{21}$ grams. Find the thickness of the shell if the metal weighs 10 gms per cu cm.

(*a*) 3 cm (*b*) 1 cm

(*c*) 2 cm (*d*) 2.5 cm.

ANSWERS

1. (*b*)	**2.** (*b*)	**3.** (*a*)	**4.** (*b*)	**5.** (*d*)	**6.** (*c*)	**7.** (*c*)	**8.** (*c*)	**9.** (*b*)	**10.** (*c*)

HINTS AND SOLUTIONS

1. Let the side of the cube be *x* units. Then,

$S = 6x^2$ and $V = x^3$

$\therefore$ $S^3 = 6^3 . (x^2)^3 = 6^3 . (x^3)^2 = \mathbf{216\ V^2}$.

2. Let the edges of the solid be a, ar, ar^2. Then,

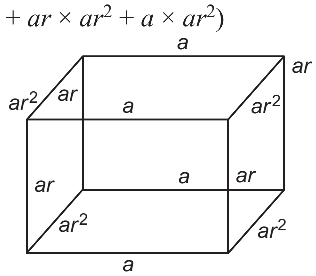
Volume $= a \times ar \times ar^2 = a^3r^3 = (ar)^3$.

Given $(ar)^3 = 8 \Rightarrow ar = 2$

Also, surface area $= 2(a \times ar + ar \times ar^2 + a \times ar^2)$

$= 2(a^2r + a^2r^3 + a^2r^2)$

$= 2ar\,(a + ar + ar^2)$



Given,

$2ar\,(a + ar + ar^2) = 32$

$\Rightarrow$ $4(a + ar + ar^2) = 32$

$\Rightarrow$ Sum of lengths of all edges = **32.**

3. Let the thickness of the pipe be *a* cm.

External Radius = 9 cm.

$\therefore$ Internal radius = $(9 - a)$ cm.

Given, Volume of metal = 748 cm^3

$\Rightarrow$ $(\pi \times 9^2 \times 14) - (\pi \times (9 - a)^2 \times 14) = 748$

$\Rightarrow$ $\pi \times 14 \times [81 - (81 + a^2 - 18a)] = 748$

$\Rightarrow$ $-a^2 + 18a = \frac{748}{\pi \times 14} = \frac{748 \times 7}{22 \times 14} = 17$

$\Rightarrow$ $a^2 - 18a + 17 = 0$

$\Rightarrow$ $(a - 17)(a - 1) = 0$

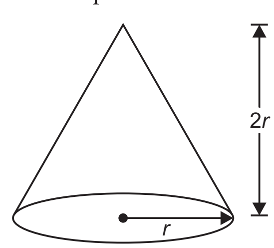
$\Rightarrow$ $\boldsymbol{a = 1}$ **or 17,** but $a = 17$ is inadmissible.

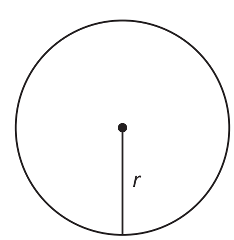
4. Let r = radius of cone and hemisphere.

Then, h (height of cone) $= 2r$

$\therefore$ Slant height (l) of cone $= \sqrt{h^2 + r^2} = \sqrt{5r^2} = r\sqrt{5}$

$\therefore$ Required ratio $= 4\pi r^2 : \pi r l = 4\pi r^2 : \pi \times r \times r\sqrt{5} = \mathbf{4 : \sqrt{5}}$.





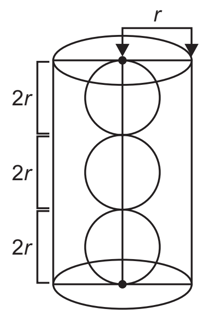
5. Volume of cylinder = $\pi r^2 h = \pi r^2$ $\quad (\because h = 1)$

Total surface area of cylinder = $2\pi r(r + h) = 2\pi r(r + 1)$

$\therefore$ Required magnitude = $\dfrac{\pi r^2}{2\pi r(r+1)} = \dfrac{r}{2(r+1)}$

$= \mathbf{\dfrac{1}{2}\left[1 - \dfrac{1}{r+1}\right].}$

6. If r is the radius of the cylinder, then its height $(h) = 3 \times 2r = 6r$.



$\therefore$ Volume of cylinder = $\pi r^2 \times 6r = 6\pi r^3$

Volume of the three balls

$= 3 \times \dfrac{4}{3}\pi r^3 = 4\pi r^3$

$\therefore$ Required fraction of volume

$= \dfrac{4\pi r^3}{6\pi r^3} = \mathbf{\dfrac{2}{3}}.$

7. Let each side of the base of the prism be a cm.

Total surface area = $72\sqrt{3}$ cm^2

$\Rightarrow$ (Perimeter of the base × Height) + 2 (Area of base) $= 72\sqrt{3}$

$\Rightarrow\ 3a \times 4 + 2\left(\dfrac{\sqrt{3}}{4}a^2\right) = 72\sqrt{3}$

$\Rightarrow\ \sqrt{3}a^2 + 24a - 144\sqrt{3} = 0 \Rightarrow a^2 + 8\sqrt{3}a - 144 = 0$

$\Rightarrow\ (a + 12\sqrt{3})(a - 4\sqrt{3}) = 0$

$\Rightarrow\ a = -12\sqrt{3}$ or $4\sqrt{3} \Rightarrow a = 4\sqrt{3}$ as $a > 0$

$\therefore$ Volume of the prism = Area of the base × Height

$= \dfrac{\sqrt{3}}{4} \times (4\sqrt{3})^2 \times 4 \text{ cm}^2 = \mathbf{48\sqrt{3}\ cm^2.}$

8. For the first prism, $a = 5$ cm, $b = 12$ cm, $c = 13$ cm

$\Rightarrow\ s = \dfrac{a+b+c}{2} = \dfrac{5+12+13}{2} = 15$ cm

Area of the base = $\sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{15 \times 10 \times 3 \times 2}$ cm^2 = 30 cm^2

Let V_1 be the volume of the prism. Then,

V_1 = Area of the base × height

$\Rightarrow\ V_1 = (30 \times 18)$ cm^3 = 540 cm^3

Let S_1 be the total surface area of the prism. Then

S_1 = Lateral surface area + 2 (Area of base)

= (Perimeter of the base × height) + 2 (Area of base)

= $(30 \times 18 + 2 \times 30)$ cm^2 = 600 cm^2

Let h be the height of new prism. Then, for the new prism

$a = 9$ cm, $b = 12$ cm, $c = 15$ cm $\Rightarrow s = \dfrac{9+12+15}{2} = 18$ cm

Area of the base = $\sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{18 \times 9 \times 6 \times 3}$ cm^2 = 54 cm^2

$\therefore\ V_2$ = Volume of new prism = $(54 \times h)$ cm^3

$V_1 = V_2 \Rightarrow 54 \times h = 540 \Rightarrow h = 10$ cm.

Let S_2 be the total surface area of the new prism. Then,

$S_2 = (36 \times 10 + 2 \times 54)$ cm^2 = 468 cm^2

Change in the whole surface area = $S_1 - S_2$

= (600 – 468) cm^2

= **132 cm^2.**

9. Let the length of each side of the equilateral triangle be a cm. Also, let the length of each side of the regular hexagon be b cm.

Area of equilateral triangle = $\dfrac{\sqrt{3}}{4}a^2$

Area of regular hexagon = $6 \times \dfrac{\sqrt{3}}{4}b^2 = \dfrac{3\sqrt{3}}{2}b^2$

Since, volume of prism with equilateral triangle base

= Volume of prism with hexagonal base

$\Rightarrow\ \dfrac{\sqrt{3}}{4}a^2 \times h = \dfrac{3\sqrt{3}}{2}b^2 \times h$

$\Rightarrow\ \dfrac{a^2}{b^2} = \dfrac{3\sqrt{2}}{2} \div \dfrac{\sqrt{3}}{4} = \dfrac{6}{1} \Rightarrow\ \mathbf{a : b = \sqrt{6} : 1}.$

10. Let the internal radius of the shell be r cm.

$\therefore$ Internal volume of the shell = $\dfrac{4}{3}\pi r^3$ cu. cm

External radius of the shell = $\dfrac{21}{2}$ cm.

External volume of the shell = $\dfrac{4}{3} \times \dfrac{22}{7} \times \dfrac{21}{2} \times \dfrac{21}{2} \times \dfrac{21}{2}$ cu. cm

= 4851 cu. cm.

Weight of 1 cu. cm of metal = 10g.

$\therefore$ Volume of the metal in the shell = $22775\dfrac{5}{21} \times \dfrac{1}{10}$ cu. cm

$= \dfrac{478280}{21} \times \dfrac{1}{10}$ cu. cm $= \dfrac{47828}{21}$ cu. cm

$\therefore$ Internal vol. of the shell = $4851 - \dfrac{47828}{21} = \dfrac{54043}{21}$ cu. cm

$\Rightarrow\ \dfrac{4}{3}\pi r^3 = \dfrac{54043}{21} \Rightarrow r^3 = \dfrac{54043 \times 3 \times 7}{21 \times 4 \times 22} = 614.125$

$\Rightarrow\ r = 8.5$ cm

$\therefore$ Thickness of shell = 10.5 cm – 8.5 cm = **2 cm**.