Trishna's

IIT JEE SUPER COURSE IN Mathematics

Algebra I

ALWAYS LEARNING

PEARSON

Super Course in Mathematics

ALGEBRA I

for IIT-JEE

Volume 1

Trishna Knowledge Systems

A division of
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PEARSON

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Preface

The IIT-JEE, the most challenging amongst national level engineering entrance examinations, remains on the top of the priority list of several lakhs of students every year. The brand value of the IITs attracts more and more students every year, but the challenge posed by the IIT-JEE ensures that only the *best* of the aspirants get into the IITs. Students require thorough understanding of the fundamental concepts, reasoning skills, ability to comprehend the presented situation and exceptional problem-solving skills to come on top in this highly demanding entrance examination.

The pattern of the IIT-JEE has been changing over the years. Hence an aspiring student requires a step-by-step study plan to master the fundamentals and to get adequate practice in the various types of questions that have appeared in the IIT-JEE over the last several years. Irrespective of the branch of engineering study the student chooses later, it is important to have a sound conceptual grounding in Mathematics, Physics and Chemistry. A lack of proper understanding of these subjects limits the capacity of students to solve complex problems thereby lessening his/her chances of making it to the top-notch institutes which provide quality training.

This series of books serves as a source of learning that goes beyond the school curriculum of Class XI and Class XII and is intended to form the backbone of the preparation of an aspiring student. These books have been designed with the objective of guiding an aspirant to his/her goal in a clearly defined step-by-step approach.

- **Master the Concepts and Concept Strands!**
  This series covers all the concepts in the latest IIT-JEE syllabus by segregating them into appropriate units. The theories are explained in detail and are illustrated using solved examples detailing the different applications of the concepts.
- **Let us First Solve the Examples—Concept Connectors!**
  At the end of the theory content in each unit, a good number of "Solved Examples" are provided and they are designed to give the aspirant a comprehensive exposure to the application of the concepts at the problem-solving level.
- **Do Your Exercise—Daily!**
  Over 200 unsolved problems are presented for practice at the end of every chapter. Hints and solutions for the same are also provided. These problems are designed to sharpen the aspirant's problem-solving skills in a step-by-step manner.
- **Remember, Practice Makes You Perfect!**
  We recommend you work out ALL the problems on your own – both solved and unsolved – to enhance the effectiveness of your preparation.

A distinct feature of this series is that unlike most other reference books in the market, this is not authored by an individual. It is put together by a team of highly qualified faculty members that includes IITians, PhDs etc from some of the best institutes in India and abroad. This team of academic experts has vast experience in teaching the fundamentals and their application and in developing high quality study material for IIT-JEE at T.I.M.E. (Triumphant Institute of Management Education Pvt. Ltd), the number 1 coaching institute in India. The essence of the combined knowledge of such an experienced team is what is presented in this self-preparatory series. While the contents of these books have been organized keeping in mind the specific requirements of IIT-JEE, we are sure that you will find these useful in your preparation for various other engineering entrance exams also.

We wish you the very best!

CHAPTER 1 PREREQUISITES

CHAPTER OUTLINE

SET OF REAL NUMBERS

One of the most important sets in our study is the set of real numbers and this set is denoted by R. Real numbers are classified into Rational and Irrational numbers.

Rational Numbers

A number, which can be expressed in the form $\frac{p}{q}$, *where p and q are integers, and q is not equal to 0 (written as* $q \neq 0$*) is called a rational number.*

The set of rational numbers is denoted by Q.

For e.g., 5, $\frac{3}{7}$, $\frac{-13}{29}$, 3.81, 1.6666..., 4.123123... are rational numbers.

Note that in the above, 3.81 can be expressed as $\frac{381}{100}$, 1.6666... is a recurring decimal, and can be expressed as $\frac{5}{3}$.

Representing recurring decimals in the rational form

(i) Let, x = 1.66666....
Then 10 x = 16.66666....
$\therefore \quad 9x = 15$
$\Rightarrow \quad x = \frac{15}{9}$ or $\frac{5}{3}$

(ii) Let, x = 0.7252525....
Then 1000 x = 725.252525....
and 10 x = 7.252525....
Subtracting, we get,
990 x = 718
$\Rightarrow \quad x = \frac{718}{990}$ or $\frac{359}{495}$

Integers

All integers are rational numbers and are classified into negative integers, zero and positive integers. The set of integers is denoted by Z.

$$Z = \{ \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

The set of positive integers i.e., the set {1, 2, 3, 4, ...} is called the set of natural numbers and is denoted by N.

$$N = \{1, 2, 3, 4, \ldots\}$$

Positive integers (or natural numbers) can be classified into prime numbers and composite numbers.

A positive number, which does not have a factor other than 1 and itself, is called a prime number. 2, 3, 5, 7, 11,13,.... are examples of prime numbers.

A positive number, which is not a prime number, is called a composite number. 4, 12, 24, 162,.... are examples of composite numbers.

Irrational Numbers

Numbers, which are not rational, are called irrational numbers.

For e.g., $\sqrt{3}$, $-\sqrt{7}$, π, e are irrational numbers.

π represents the ratio of the circumference of a circle to its diameter and is equal to 3.1415.... and e is called the exponential number and it lies between 2 and 3 and is approximately equal to 2.718.

The important characteristic of an irrational number is that it will have non-terminating and non recurring decimal representation.

To sum up, any real number with a terminating or with a recurring decimal is a rational number while a number with a non-terminating and non recurring decimal is an irrational number.

Between any two real numbers there is an infinite number of rational numbers as well as irrational numbers.

SURDS

A surd is an irrational number, which is the nth root of a rational number.

If x is a positive rational number and nth root of x i.e., $\sqrt[n]{x}$ or $x^{1/n}$ is irrational, then $\sqrt[n]{x}$ is a surd.

$\sqrt{5}$ and $\sqrt[3]{19}$ are irrational numbers and are surds.

But, $16^{1/4} = 2 \in Q \Rightarrow \sqrt[4]{16}$ is not a surd.

Other examples of surds are $(3\sqrt{2}+7\sqrt{3})$, $\sqrt[4]{15}-\sqrt[4]{17}$, and $(\sqrt{2}+\sqrt{3}+2\sqrt{5})$.

In the above example, $(3\sqrt{2}+7\sqrt{3})$ represents a binomial surd while $(\sqrt{2}+\sqrt{3}+2\sqrt{5})$ represents a trinomial surd.

Equality of surds

Let a and c be rational numbers and $\sqrt{b}$ and $\sqrt{d}$ be surds (i.e., $\sqrt{b}$ and $\sqrt{d}$ are irrational numbers).

If $a+\sqrt{b}=c+\sqrt{d}$, then, $a = c$, $b = d$

For example,

(i) If $x+\sqrt{y}=7+\sqrt{6} \Rightarrow x = 7, y = 6$

(ii) If $x-\sqrt{y}=2-5\sqrt{3} \Rightarrow x = 2, \sqrt{y}=5\sqrt{3}$ or $y = 75$

Rationalizing factor

If a and b are rational numbers such that $\sqrt{a}$ and $\sqrt{b}$ are surds, the binomial surd $(\sqrt{a}+\sqrt{b})$ on multiplication by $(\sqrt{a}-\sqrt{b})$ gives us $(a - b)$ which is a rational number. $(\sqrt{a}-\sqrt{b})$ is said to be the rationalizing factor (or the conjugate factor) of $(\sqrt{a}+\sqrt{b})$ and vice versa.

Consider the following examples:

(i) $(\sqrt{13}-2\sqrt{2})$ is the rationalizing factor of $(\sqrt{13}+2\sqrt{2})$, since, $(\sqrt{13}-2\sqrt{2})(\sqrt{13}+2\sqrt{2}) = 5 \in Q$

(ii) $\left(\sqrt{\frac{15}{7}}+\sqrt{\frac{17}{3}}\right)$ is the rationalizing factor of $\left(\sqrt{\frac{15}{7}}-\sqrt{\frac{17}{3}}\right)$,

since, $\left(\sqrt{\frac{15}{7}}+\sqrt{\frac{17}{3}}\right)\left(\sqrt{\frac{15}{7}}-\sqrt{\frac{17}{3}}\right)=\frac{15}{7}-\frac{17}{3}=\frac{-74}{21}$

= a rational number

(iii) $\left[3^{2/3}+5^{2/3}-(15)^{1/3}\right]$ is the rationalizing factor of $\left(3^{1/3}+5^{1/3}\right)$, since on multiplication and using the identity $(a + b)(a^2 + b^2 - ab) = a^3 + b^3$, we obtain the product as $\left(3^{1/3}\right)^3+\left(5^{1/3}\right)^3=3+5=8 \in Q$

CONCEPT STRANDS

Concept Strand 1

If $\frac{1}{k(13-2\sqrt{42})}=1$, find the value of k.

Solution

$$k=\frac{1}{13-2\sqrt{42}}\times\frac{13+2\sqrt{42}}{13+2\sqrt{42}}$$

$$=13+2\sqrt{42}=\left(\sqrt{7}+\sqrt{6}\right)^2$$

$\therefore$ $k = 13 + 2\sqrt{42}$.

Concept Strand 2

Find the rationalizing factor of

$\left(\sqrt[8]{3}-\sqrt[8]{2}\right)\left(\sqrt[4]{3}+\sqrt[4]{2}\right)\left(\sqrt{3}+\sqrt{2}\right)$

Solution

The given expression contains surds of the 8th order (i.e., 8th root), 4th order and 2nd order.

When a pair of conjugate surds are multiplied, the order of the product will be half that of the multiplicands; i.e., $\left(\sqrt[4]{a}+\sqrt[4]{b}\right)\left(\sqrt[4]{a}-\sqrt[4]{b}\right)=\left(\sqrt{a}-\sqrt{b}\right)$

The factor with the 8th order surd is $\left(\sqrt[8]{3}-\sqrt[8]{2}\right)$; its conjugate is $\left(\sqrt[8]{3}+\sqrt[8]{2}\right)$; their product is $\left(\sqrt[4]{3}-\sqrt[4]{2}\right)$

When the above product is multiplied by $\left(\sqrt[4]{3}+\sqrt[4]{2}\right)$, the product is $\left(\sqrt{3}-\sqrt{2}\right)$

When the above result is multiplied by $\left(\sqrt{3}+\sqrt{2}\right)$, the result is 3 – 2 = 1, which is rational.

Hence $\left(\sqrt[8]{3}+\sqrt[8]{2}\right)$ is the rationalizing factor.

Roots of surds

Square roots of expressions of the form $(a+\sqrt{b}),(a-\sqrt{b})$, $(a+\sqrt{b}+\sqrt{c}+\sqrt{d})$, $(a-\sqrt{b}-\sqrt{c}+\sqrt{d})$ where, a, b, c, d are rational and $\sqrt{b},\sqrt{c},\sqrt{d}$ are irrational are obtained as follows:

$\sqrt{a+\sqrt{b}}$ will be of the form $\sqrt{x}+\sqrt{y}$ where, x, y are rational. i.e., $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$

Squaring both sides, $a+\sqrt{b}=(x+y)+2\sqrt{xy} \Rightarrow x+y = a$ and $4xy = b$

We can solve for x and y.

Similarly, $\sqrt{a-\sqrt{b}}$ will be of the form $(\sqrt{x}-\sqrt{y})$, $\sqrt{a+\sqrt{b}+\sqrt{c}+\sqrt{d}}$ will be of the form $\sqrt{x}+\sqrt{y}+\sqrt{z}$ and $\sqrt{a-\sqrt{b}-\sqrt{c}+\sqrt{d}}$ will be of the form $\sqrt{x}-\sqrt{y}-\sqrt{z}$ where, x, y, z are rational.

We illustrate the above ideas, by two examples.

CONCEPT STRANDS

Concept Strand 3

Find $\sqrt{10-2\sqrt{21}}$.

Solution

Let $\sqrt{10-2\sqrt{21}}=\sqrt{x}-\sqrt{y}$,

Squaring, $10-2\sqrt{21}=x+y-2\sqrt{xy}$

$\Rightarrow$ $x+y=10,\ 2\sqrt{xy}=2\sqrt{21}$

$(x-y)^2=(x+y)^2-4xy=100-84=16$

$\Rightarrow$ $x-y=\pm 4$

Taking $(x-y)=4$, and using $x+y=10$, we get $x=7$, $y=3$.

Therefore, $\sqrt{10-2\sqrt{21}}=\sqrt{7}-\sqrt{3}$

Note that $x-y=-4$, leads to $x=3, y=7$ and $\sqrt{x}-\sqrt{y}=\sqrt{3}-\sqrt{7}$ is negative. ($\sqrt{a}$ where a is positive means the positive value x such that $x^2=a$.).

Therefore $\sqrt{10-2\sqrt{21}}$ cannot represent $\sqrt{3}-\sqrt{7}$.

Concept Strand 4

Find $\sqrt{36+4\sqrt{35}+2\sqrt{15}+4\sqrt{21}}$.

Solution

Let the above square root be $\sqrt{x}+\sqrt{y}+\sqrt{z}$.

On squaring and using the identity $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$, we have,

$36+4\sqrt{35}+2\sqrt{15}+4\sqrt{21}$

$=(x+y+z)+2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}$

$\Rightarrow$ $x+y+z=36;\ 2\sqrt{xy}=4\sqrt{35};\ 2\sqrt{yz}=2\sqrt{15};\ 2\sqrt{zx}=4\sqrt{21}$

$\Rightarrow$ $x+y+z=36,\ xy=140,\ yz=15,\ zx=84$

Solving for x, y, z we get $x=28, y=5, z=3$

or $\sqrt{36+4\sqrt{35}+2\sqrt{15}+4\sqrt{21}}=(2\sqrt{7}+\sqrt{5}+\sqrt{3})$

Concept Strand 5

Find the cube root of

$\left(3\sqrt{3}\right)\left(2\sqrt{2}\right)+7\sqrt{7}+\left(3\sqrt{3}\right)\left(\sqrt{2}\right)\left(\sqrt{7}\right)\left(\sqrt{3}\sqrt{2}+\sqrt{7}\right)$

Solution

It can be noticed that the given expression contains,

(a) three terms and

(b) some of the terms are the cubes of $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{7}$

As, $(a+b)^3$, which is equal to $a^3+b^3+3ab(a+b)$ and has the above two properties, it can be useful to check whether the given expression can be rewritten in the form of $a^3+b^3+3ab(a+b)$.

$$\left(3\sqrt{3}\right)\left(2\sqrt{2}\right)+\left(7\sqrt{7}\right)+\left(3\sqrt{3}\right)\left(\sqrt{2}\right)\left(\sqrt{7}\right)\left[\left(\sqrt{3}\right)\left(\sqrt{2}\right)+\sqrt{7}\right]$$

$$=\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3+\left(\sqrt{7}\right)^3+(3)\left(\sqrt{3}\times\sqrt{2}\right)\left(\sqrt{7}\right)\left[\left(\sqrt{3}\times\sqrt{2}\right)+\sqrt{7}\right]$$

$$=\left[\left(\sqrt{3}\times\sqrt{2}\right)+\sqrt{7}\right]^3=\left(\sqrt{6}+\sqrt{7}\right)^3$$

Hence, cube root of the given expression is $\left(\sqrt{6}+\sqrt{7}\right)$.

H.C.F (OR G.C.D) AND L.C.M

H.C.F stands for highest common factor, G.C.D stands for greatest common divisor, and L.C.M stands for least common multiple.

H.C.F

H.C.F (or G.C.D) is the largest factor of two or more given numbers (here, positive integers)

For example,

(i) H.C.F of the set of numbers {28, 21, 98} is 7.
(since 7 is the largest factor of all the three numbers)

(ii) H.C.F of {3, 16} is 1.
In (ii) above, 3 and 16 are said to be relatively prime or coprimes. They do not have any common factor other than 1.

L.C.M

L.C.M of two or more numbers is the least number, which is divisible by each of these numbers without leaving a remainder.

For example,

(i) L.C.M of the set of numbers {16, 24, 21} is 336.
[Here, 336 is the smallest number which is divisible by 16, 24 and 21 without a remainder]

(ii) L.C.M of {5, 12, 7, 11} is $5 \times 12 \times 7 \times 11 = 4620$
[Observe that the set of numbers 5, 12, 7, 11 are relatively prime]

Results

(i) If a and b are two numbers (positive integers), ab = L.C.M of (a, b) × H.C.F of (a, b).

(ii) H.C.F of two or more polynomial expressions is the polynomial expression of highest degree, which divides each of them without remainder.
For example, H.C.F of $\{x^2 - 5x + 6, (x - 2)^2(x - 1), (x^2 + x - 6)\}$ is $(x - 2)$.

(iii) L.C.M of two or more polynomial expressions is the polynomial expression of the lowest degree, which is divisible by each of these without remainder.
For example, L.C.M of $\{x(x + 1)(x - 4), (x^2 + 4x + 3), x^3 + x^2\}$ is $x^2(x + 1)(x + 3)(x - 4)$

(iv) H.C.F of two or more fractions

$$= \frac{\text{H.C.F of the numerators of these fractions}}{\text{L.C.M of the denominators of these fractions}}$$

For example, H.C.F of $\left\{\frac{2}{3}, \frac{16}{63}, \frac{24}{27}\right\}$ is given by

$$\frac{\text{H.C.F of }(2,16,24)}{\text{L.C.M of }(3,63,27)} = \frac{2}{189}$$

(v) L.C.M of two or more fractions

$$= \frac{\text{L.C.M of the numerators of these fractions}}{\text{H.C.F of the denominators of these fractions}}$$

For example, L.C.M of $\left\{\frac{5}{9}, \frac{40}{63}, \frac{32}{147}\right\}$ is given by

$$\frac{\text{L.C.M of }(5,40,32)}{\text{H.C.F of }(9,63,147)} = \frac{160}{3}$$

CONCEPT STRANDS

Concept Strand 6

There are a certain number of soldiers in a field. If the soldiers are arranged in rows of 8 or 15 or 20, one soldier is left out. If the soldiers are arranged in rows of 9 or 13, four soldiers only are left out. Find the number of soldiers in the field.

Solution

Number of soldiers in the field = LCM (8, 15, 20)c + 1 = (120c + 1), where c is a constant

Number of soldiers in the field = LCM (9, 13)k + 4 = 117k + 4 where, k is a constant.

Hence 120c + 1 = 117k + 4.

The above equation is satisfied when k = 1 an c = 1

Thus, the number of soldiers in the field = 1(120) + 1 = 121.

Concept Strand 7

Find the largest two-digit number, which divides 378 and 542, leaving the same remainder in each case.

Solution

Let the largest number that divides 378 and 542 leaving the same remainder, say r, be d.

Hence $378 = dq_1 + r$ and $542 = dq_2 + r$, where q_1 and q_2 are the quotients when d divides 378 and 542 respectively.

$\therefore$ $378 - r = dq_1$ and $542 - r = dq_2$.

It is clear from the above two equations, d would be the HCF of $378 - r$ and $542 - r$.

As the remainder is same in both the cases, $(542 - r) - (378 - r) = d(q_1 - q_2)$

$\Rightarrow$ $164 = d(q_2 - q_1)$

Since 164 can be written as 164×1 or 82 x 2, the largest possible two-digit divisor is 82

Concept Strand 8

The LCM of the fractions 1/5, 4/15 and 8/25 is how many times their HCF?

Solution

HCF of 1/5, 4/15 and 8/25 = 1/75

LCM of 1/5, 4/15 and 8/25 = 8/5

$$\text{Hence } \frac{(\text{LCM of } 1/5,\ 4/15 \text{ and } 8/25)}{\text{HCF of } (1/5,\ 4/15 \text{ and } 8/25)}$$

$$= \frac{8/5}{1/75} = \frac{8}{5}\text{x}75 = 120$$

RATIO, PROPORTION AND VARIATION

Ratio

The ratio of two numbers x and y is denoted by x : y and is equal to $\frac{x}{y}$. Generally, two like quantities can be compared by first expressing them in terms of the same unit and then finding their ratio.

We cannot have the ratio of two unlike quantities. For example, it is impossible to form the ratio of 5 days and 7 metres.

Often, the idea of ratio is extended to include three or more numbers. We then express the relative magnitudes of the three numbers in the form of a ratio. Suppose, the ratio of a to b is 4 to 5 and that the ratio of b to c is 5 to 9. Then we say that a is to b is to c as 4 is to 5 is to 9 and we write a : b : c = 4 : 5 : 9. This gives us $\frac{a}{4} = \frac{b}{5} = \frac{c}{9}$.

Proportion

When b and d are $\neq 0$, if the ratios $\frac{a}{b}$ and $\frac{c}{d}$ are equal, i.e., $\frac{a}{b} = \frac{c}{d}$ or a : b = c : d, then we say that a, b, c, d are in proportion.

In a proportion a and d are called extremes, b and c are called the means. If $\frac{a}{b} = \frac{b}{c}$ (i.e., $b^2 = ac$), b is called the mean proportional to a and c and c is called the third proportional to a and b.

If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \ldots..$, a, b, c, d, …. are said to be in continued proportion.

Results

(i) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$, $\frac{a-b}{b} = \frac{c-d}{d}$ and $\frac{a+b}{a-b} = \frac{c+d}{c-d}$. (Componendo-dividendo)

(ii) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots$then each of the above ratios $= \frac{pa + qc + re + \ldots\ldots..}{pb + qd + rf + \ldots\ldots}$, where, p, q, r, … are any set of constants.

CONCEPT STRANDS

Concept Strand 9

Divide Rs 1150 into four parts such that a third of the first part, a fifth of the second part, an eighth of the third part and a seventh of the fourth part are all equal.

Solution

Let a, b, c, d be the four parts. Then, a + b + c + d = 1150.

Also, $\frac{a}{3} = \frac{b}{5} = \frac{c}{8} = \frac{d}{7} = k$ (say)

$\Rightarrow$ a = 3k, b = 5k, c = 8k, d = 7k.

$\therefore$ 3k + 5k + 7k + 8k = 1150 $\Rightarrow$ k = 50

$\Rightarrow$ a = 150, = 250, c = 400 and d = 350.

Concept Strand 10

If $\frac{a+b}{a-b} = \frac{c+d}{c-d} = 3$, find $\frac{la+mc+2n}{lb+md+n}$

Solution

$\frac{a+b}{a-b} = \frac{c+d}{c-d} = 3$, applying componendo-dividendo

we get, $\frac{a}{b} = \frac{c}{d} = \frac{4}{2} = 2$

$\Rightarrow \frac{la+mc+2n}{lb+md+n} = \frac{a}{b} = \frac{c}{d} = 2$

Variation

When two quantities x and y are connected by the relation y = kx; where, k is a positive non-zero constant, we say that y varies directly as x (y directly proportional to x) and symbolically written as $y \propto x$.

For example, in uniform motion, distance travelled is directly proportional to time taken.

Let a quantity u be always equal to a constant times the product of two quantities x and y. This relation is expressed by saying that u varies jointly as x and y, written as $u \propto xy$ or u = kxy, where k is a non-zero constant.

For example, force applied on a body varies jointly as the mass and acceleration of the body.

If the product of two quantities, say x and y is always a constant, i.e., xy = k, we say that y is inversely proportional to x. (Here, $y \propto \frac{1}{x}$).

For example at constant temperature, pressure is inversely proportional to volume.

CONCEPT STRAND

Concept Strand 11

If a : b = 3 : 7, what is the value of $\frac{4a+5b}{2a+2b}$?

Solution

$\frac{a}{b} = \frac{3}{7}$

$$\frac{4a+5b}{2a+2b} = \frac{4\left(\frac{a}{b}\right)+5}{2\left(\frac{a}{b}\right)+2} = \frac{\frac{12}{7}+5}{\frac{6}{7}+2} = \frac{47}{20}$$

Alternate method

Substituting the values of a and b as 3k and 7k respectively in $\frac{4a+5b}{2a+2b}$, we get $\frac{12k+35k}{6k+14k} = \frac{47k}{20k} = \frac{47}{20}$

CONCEPT STRANDS

Concept Strand 12

The ratio of the number of boys to the number of girls in a school is 7 : 3. If an additional 15 girls were to join the class, the ratio of the number of boys to the number of girls would become 2 : 3. What is the initial number of girls in the class?

Solution

Number of boys = 7x ; Number of girls = 3x

$$\frac{7x}{3x+15} = \frac{2}{3}$$

$21x = 6x + 30$

$15x = 30 \Rightarrow x = 2$

$\therefore$ Number of girls = 6

Concept Strand 13

The kinetic energy of a body is directly proportional to the square of its speed when the mass is kept constant and is proportional to mass when its speed is kept constant. A body with a mass of 2 kg and a speed of 10 m/s has a kinetic energy of 100 joules. What is the kinetic energy of a body whose mass is 20 kg and speed is 1 m/s?

Solution

Let K, M and S respectively be the kinetic energy, mass and speed of the body.

Given:

$K \propto S^2$ (when M is kept constant) and $K \propto M$ (when S is kept constant)

$\Rightarrow$ $K \propto MS^2 \Rightarrow K = CMS^2$, where C is the constant of proportionality.

Given that when M = 2 kg, S = 10 m/s, K = 100 joules

$\Rightarrow$ $100 = C \times 2 \times 10^2$

$\Rightarrow$ $C = \frac{1}{2} \Rightarrow K = \frac{1}{2} MS^2$

When M = 20 kg and S = 1 m/s

$K = 1/2 \times 20 \times 1^2 = 10$

$\therefore$ A body of mass 20 kg moving with a speed of 1 m/s has a kinetic energy of 10 joules.

INDICES

Indices are numbers of the form a^x, where a is called the base and x is called the index. Below listed are the rules that govern the operations with indices.

Laws of indices

(i) For all values of a other than zero, $a^0 = 1$

(ii) If n is a positive integer, $\frac{1}{a^n}$ is written as a^{-n}

(iii) If n is a positive integer and x is a real number such that $x^n = a$, we write $x = a^{1/n}$ or $\sqrt[n]{a}$ (called the nth root of a)

(iv) If m and n are rational numbers,

(a) $a^m \times a^n = a^{m+n}$

(b) $\frac{a^m}{a^n} = a^{m-n}$ and

(c) $(a^m)^n = a^{mn}$

CONCEPT STRANDS

Concept Strand 14

Find the value of x, which satisfies the equation

$$\sqrt[3]{\left(\frac{5}{4}\right)^{x+2}} = \frac{4096}{15625}$$

Solution

The given equation can be written as

$$\left(\frac{5}{4}\right)^{\frac{x+2}{3}} = \left(\frac{15625}{4096}\right)^{-1}$$

$$\Rightarrow \left(\frac{5}{4}\right)^{\frac{x+2}{3}} = \left(\frac{5^6}{4^6}\right)^{-1}$$

$$\text{i.e., } \left(\frac{5}{4}\right)^{\frac{x+2}{3}} = \left(\frac{5}{4}\right)^{-6} \Rightarrow \frac{x+2}{3} = -6 \text{ or } x = -20$$

Concept Strand 15

If $a^2 + 2 = 3^{2/3} + 3^{-2/3}$, $a > 0$, find the value of $3a^3 + 9a$.

Solution

$$a^2 + 2 = 3^{2/3} + 3^{-2/3}$$

$$\text{or } a^2 + 2 = 3^{2/3} + \frac{1}{3^{2/3}}$$

$$\Rightarrow 3^{2/3}a^2 + 2 \times 3^{2/3} = 3^{4/3} + 1$$

$$\text{or } 3^{2/3}a^2 = \left(3^{2/3} - 1\right)^2$$

Taking positive square root we get $3^{1/3}a = \left(3^{2/3} - 1\right)$,

Cubing, $3a^3 = \left(3^{2/3} - 1\right)^3 = 8 - 3 \times 3^{2/3}3^{1/3}a$

$\Rightarrow$ $3a^3 + 9a = 8$.

Concept Strand 16

Solve the simultaneous equations: $9^{x-y} = 81$, $9^{x+y} = 729$

Solution

$9^{x-y} = 81 \Rightarrow x - y = 2$ — (1)

$9^{x+y} = 729 \Rightarrow x + y = 3$ — (2)

Solving (1) and (2)

$$\Rightarrow x = \frac{5}{2}, y = \frac{1}{2}.$$

Concept Strand 17

If $a = xy^{p-1}$, $b = xy^{q-1}$ and $c = xy^{r-1}$, prove that $a^{q-r}b^{r-p}c^{p-q} = 1$

Solution

$$a^{q-r}b^{r-p}c^{p-q} = x^{q-r}y^{(q-r)(p-1)}x^{r-p}y^{(r-p)(q-1)}x^{p-q}y^{(p-q)(r-1)}$$
$$= x^{q-r+r-p+p-q}y^{qp-q-rp+r+rq-r-pq+p+pr-p-qr+q} = x^0y^0 = 1.$$

Concept Strand 18

Find the value of z in terms of x and y if $3^x = 2^y = 6^z$. Given $x \neq y \neq z$

Solution

Let, $3^x = 2^y = 6^z = k$

$\Rightarrow$ $3 = k^{1/x}$; $2 = k^{1/y}$; $6 = k^{1/z}$

$\Rightarrow$ $(3 \times 2) = k^{1/x} \times k^{1/y}$

$\Rightarrow$ $6 = k^{(1/x + 1/y)} = k^{1/z}$

$$\Rightarrow \frac{1}{z} = \frac{1}{x} + \frac{1}{y}$$

In the case of the given data, $k \neq 0$, $k \neq 1$, $k \neq -1$.

$$\Rightarrow z = \frac{xy}{x+y}$$

Hence, as bases are equal, equating the powers, we get

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

LOGARITHMS

If N is a positive number and there exists a number x such that $a^x = N$, where $a > 0$ and $\neq 1$, then x is called the logarithm of N to the base a and is written as $x = \log_a N$.

For example,

$\log_2 16 = 4$, since $2^4 = 16$;

$\log_9 243 = \frac{5}{2}$, since $9^{5/2} = 243$;

$\log_4 \left(1/64\right) = -3$, since $4^{-3} = \frac{1}{64}$

$\log_3\left(\frac{1}{243}\right) = -5$, since $\frac{1}{243} = \frac{1}{3^5} = 3^{-5}$

$\log_{0.2}(0.008) = 3$, since $0.008 = (0.2)^3$

Note that $\log_a 1 = 0$ to any base a and $\log_a a = 1$.

Logarithms to the base 10 and e

Logarithms to the base 10 are called common logarithms. Logarithms to the base e (e is called the exponential number and it is an irrational number lying between 2 and 3) are called natural logarithms. $\log_e N$ is sometimes written as ln N.

The following results hold good.

Results

(i) $\log_a MN = \log_a M + \log_a N$

(ii) $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

(iii) $\log_a(M^n) = n\log_a M$

(iv) $\log_a N = \log_b N \times \log_a b$

(v) $\log_b a \times \log_a b = 1$

(vi) $2\log_{a^2} N = \log_a N$

(vii) $a^{\log_a N} = N$

(viii) If N_1 and N_2 are positive and if $\log_a N_1 > \log_a N_2$, then,
$N_1 > N_2$ if $a > 1$;
$N_1 < N_2$ if $0 < a < 1$;
i.e., if the base a is greater than 1, $\log_a N$ increases as N increases and if the base a is less than 1, $\log_a N$ decreases as N increases.

Proofs

Let $\log_a M = x$

$\Rightarrow$ $M = a^x$ and $\log_a N = y \Rightarrow N = a^y$

(i) $MN = a^x a^y = a^{x+y}$
$\Rightarrow \log_a MN = x + y$

(ii) $\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y}$
$\Rightarrow \log_a\left(\frac{M}{N}\right) = x - y$

(iii) $M^n = (a^x)^n = a^{xn}$
$\Rightarrow \log_a(M^n) = nx$

(iv) Let $\log_b N = u$
$\Rightarrow b^u = N$ and $\log_a b = v$
$\Rightarrow a^v = b$
$\therefore$ $N = (a^v)^u = a^{uv}$
$\Rightarrow \log_a N = uv$
The results (v) to (vii) may similarly be derived.

(viii) If the base a is > 1, a^x increases as x increases. Therefore, if $\log_a N_1 > \log_a N_2$ where, $a > 1$, if follows that $N_1 > N_2$.
If the base a is < 1, a^x decreases as x increases. Therefore, if $\log_a N_1 > \log_a N_2$ where, $a < 1$, follows that $N_1 < N_2$
Note: In examples worked out below when no mention about the base of the logarithm is made, it is understood that the results hold good for any base.

CONCEPT STRANDS

Concept Strand 19

Find logarithm of 5 to the base $5\sqrt{5}$.

Solution

Let $\log_{5\sqrt{5}} 5 = x$

$\Rightarrow$ $5 = (5\times\sqrt{5})^x$

$\Rightarrow$ $5^{\frac{3x}{2}} = 5$

$\Rightarrow$ $\frac{3x}{2} = 1 \Rightarrow x = \frac{2}{3}$

$\therefore$ $\log_{5\sqrt{5}} 5 = \frac{2}{3}$

Concept Strand 20

Find logarithm of $8\sqrt{32}$ to the base 2.

Solution

$\log_2(8\sqrt{32}) = \log_2 8 + \log_2\sqrt{32}$ (Product Rule)

$$= \log_2 2^3 + \frac{1}{2}\log_2 2^5 = 3\log_2{}^2 + \frac{5}{2}\log_2{}^2 = 3 + \frac{5}{2} = \frac{11}{2}$$

Concept Strand 21

If $x = \log_a 2$ and $y = \log_a 3$ show that $\log_a 72 + \log_a 24 = 6x + 3y$.

Solution

$72 = 2^3 \times 3^2$

$\therefore \quad \log_a(72) = 3\log_a 2 + 2\log_a 3 = 3x + 2y$

$\log_a 24 = \log_a 3 + 3\log_a 2 = y + 3x$

$\therefore \quad \log_a 72 + \log_a 24 = 6x + 3y$

Concept Strand 22

Find the value of $3\log_{10}2 + 2\log_{10}3 - 2\log_{10}6 + \log_{10}5$.

Solution

$3\log_{10}2 + 2\log_{10}3 - 2\log_{10}6 + \log_{10}5$

$\Rightarrow \quad \log_{10}2^3 + \log_{10}3^2 - \log_{10}6^2 + \log_{10}5$

$$\Rightarrow \quad \log\left(\frac{2^3 \times 3^2 \times 5}{6^2}\right) = \log_{10}10 = 1$$

Concept Strand 23

If $3^{2x+1} . 4^{x-1} = 36$, find the value of x?

Solution

$3^{2x+1}.4^{x-1} = 36$

$$\Rightarrow \quad 3^{2x}.3.\frac{4^x}{4} = 36$$

$$\Rightarrow \quad 9^x . 4^x = \frac{36 \times 4}{3}$$

$\Rightarrow \quad (36)^x = 48$

$\therefore \quad x \log 36 = \log 48$

$\therefore \quad x = \log_{36}48$

Concept Strand 24

If $\log_3 x . \log_9 3 . \log_2 9 = 5$. Find the value of x.

Solution

$$\log_3 x . \frac{\log 3}{\log 9} \times \frac{\log 9}{\log 2} = 5$$

$\log_3 x . \log_2 3 = 5$

$\therefore \quad \log_2 x = 5$

$\Rightarrow \quad x = 2^5 = 32$

Concept Strand 25

If $x = 1 + \log_a(bc)$, $y = 1 + \log_b(ca)$, $z = 1 + \log_c(ab)$ prove that $xyz = xy + yz + zx$.

Solution

$x = 1 + \log_a(bc)$

$= \log_a a + \log_a(bc)$

$= \log_a(abc)$

Similarly $y = \log_b(abc)$ and $z = \log_c(abc)$

$$\therefore \quad \frac{1}{x} = \log_{abc}(a),\ \frac{1}{y} = \log_{abc}(b),\ \frac{1}{z} = \log_{abc}(c)$$

$$\therefore \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \log_{abc}a + \log_{abc}b + \log_{abc}c$$

$$\frac{yz + xz + xy}{xyz} = \log_{abc}(abc)$$

$$\frac{xy + yz + xz}{xyz} = 1$$

$\therefore \quad xy + yx + xz = xyz$

Concept Strand 26

Find the value of x which satisfies the equation $\log 2 + \log(x+2) - \log(3x - 5) = \log 3$.

Solution

From the given relation we obtain

$$\frac{2(x+2)}{3x-5} = 3 \Rightarrow 2x + 4 = 9x - 15 \Rightarrow x = \frac{19}{7}$$

Concept Strand 27

If $\log\left(\frac{x+y}{5}\right) = \frac{1}{2}(\log x + \log y)$ show that $\frac{x}{y} + \frac{y}{x} = 23$.

Solution

We have, from the given relation $\frac{x+y}{5} = \sqrt{xy}$

Squaring both sides,

$(x + y)^2 = 25xy$

or $x^2 + y^2 = 23\,xy$

or $\frac{x}{y} + \frac{y}{x} = 23$

Concept Strand 28

What is the value of x, if $y = \log_7 \log_7 x$ and $5^{(y/\log_7 5)} = 2$?

Solution

$$5^{\frac{\log_7(\log_7 x)}{\log_7 5}} = 5^{\log_5(\log_7 x)}$$

$\Rightarrow \log_7 x = 2 \Rightarrow x = 49$

Concept Strand 29

If $\log_{12} 27 = a$, what is the value of $\log_6 16$?

Solution

$\log_{12} 27 = a$

$$\frac{3\log 3}{\log 3 + 2\log 2} = a \quad \left[\because \log_b a = \frac{\log a}{\log b}\right]$$

$$\log 3 = \frac{2a\log 2}{3-a} \qquad \text{— (1)}$$

$$\log_6 16 = \frac{4\log 2}{\log 2 + \log 3} = \frac{4\log 2}{\log 2 + \frac{2a\log 2}{3-a}}$$

$$\log_6 16 = \frac{4(3-a)}{3+a}$$

Concept Strand 30

Find the real number x satisfying $\log_{1/2}(2x+1) < -3$.

Solution

Logarithm is defined only if $2x + 1 > 0 \Rightarrow x > \frac{-1}{2}$.

Now, $\log_{1/2}(2x+1) < -3$

$\Rightarrow \log_{1/2}(2x+1) < \log_{1/2} 8$

$\Rightarrow 2x + 1 > 8$ (since the base is less than 1). Hence $x > \frac{7}{2}$

$\therefore$ All real numbers greater than $\frac{7}{2}$ satisfy the given condition.

Concept Strand 31

Prove that $\log_b a \times \log_c b \times \log_d c \times \log_a d = 1$.

Solution

$$\log_b a \times \log_c b \times \log_d c \times \log_a d = \log_c a \times \log_d c \times \log_a d$$
$$= \log_d a \times \log_a d = 1$$

Concept Strand 32

Show that $\log 2 + 2\log\frac{125}{64} - 5\log\frac{16}{5} = 11\log 5 - 31\log 2$.

Solution

LHS $= \log 2 + 2\log 5^3 - 2\log 2^6 - 5\log 2^4 + 5\log 5$

$= \log 2 + 6\log 5 - 12\log 2 - 20\log 2 + 5\log 5$

$= -31\log 2 + 11\log 5 =$ RHS

Concept Strand 33

Solve the equation $\log_3 2x + \frac{1}{\log_7 3} = \log_3 5$.

Solution

Equation may be rewritten as $\log_3 2x + \log_3 7 = \log_3 5$

$\Rightarrow \log_3(14x) = \log_3 5 \Rightarrow 14x = 5$

or $x = \frac{5}{14}$

Common logarithms

As mentioned earlier, logarithms of numbers to the base 10 are called common logarithms. Since we use the decimal system for the representation of numbers it will be most appropriate to use common logarithms for the computation of products, quotients, exponentiation of numbers.

We now explain how the common logarithm of a number can be found by using logarithm table (or log-table)

Characteristic and Mantissa

Examine the following:

$10^4 = 1000 \Rightarrow \log_{10} 10000 = 4$

$10^3 = 100 \Rightarrow \log_{10} 1000 = 3$

$10^2 = 100 \Rightarrow \log_{10} 100 = 2$

$10^1 = 10 \Rightarrow \log_{10} 10 = 1$

$10^0 = 1 \Rightarrow \log_{10} 1 = 0$

$10^{-1} = 0.1 \Rightarrow \log_{10} 10^{-1} = -1$

$10^{-2} = 0.01 \Rightarrow \log_{10} 10^{-2} = -2$

$10^{-3} = 0.001 \Rightarrow \log_{10} 10^{-3} = -3$

$10^{-4} = 0.0001 \Rightarrow \log_{10} 10^{-4} = -4$

We infer from the above that logarithm of a number between 1000 and 10000 lies between 3 and 4, i.e., if N is a number between 1000 and 10000, log N = 3 + a decimal. (a decimal means a number between 0 and 1)

Logarithm of a number between 100 and 1000 lies between 2 and 3 (= 2+ a decimal) and so on.

We may present out findings in a tabular form given below:

Table 1.1

Numbers	Logarithm
between 1000 and 10000	3 + a decimal
between 100 and 1000	2 + a decimal
between 10 and 100	1 + a decimal
between 1 and 10	0 + a decimal
between 0.1 and 1	−1 + a decimal
between 0.01 and 0.1	−2+ a decimal
between 0.001 and 0.01	−3 + a decimal

Thus, the logarithm of any positive number consists of two parts:

(i) integral part (ii) decimal part

The integral part of the logarithm is called its characteristic. The decimal part of the logarithm is called its mantissa. Mantissa is always positive.

For example, $\log 400 = \underset{\text{Characteristic}}{2} \;.\; \underbrace{6021}_{\text{Mantissa}}$

Method to find the characteristic and mantissa of a number

(i) Characteristic of the logarithm of a number greater than 1 is positive and is one less than the number of digits to the left of the decimal point.

(ii) Characteristic of the logarithm of a number less than 1 is negative and its numerical value is one more than the number of zeros to the right of the decimal point between the decimal point and the first non zero digit (first significant digit).

For example,

Characteristic of log 7.93 is 0

Characteristic of log 249.18 is 2

Characteristic of log 1749628 is 6

Characteristic of log 0.79 is −1 (written as $\bar{1}$)

Characteristic of log 0.0004165 is −4 (written as $\bar{4}$)

We observe that the characteristic of the logarithm of a number can be written down by inspection.

Mantissa of the logarithm of a number (which is a positive decimal) is read from log table.

For example, from the log table, we get

mantissa of log 2125 = 0.3273

mantissa of log 21.25 = 0.3273

mantissa of log 2.125 = 0.3273

mantissa of log (0.0002125) = 0.3273

Note that the mantissa of the logarithm of a number depends only on the sequence of the significant digits in that number (read from left to right). We therefore have

$\log 2125 = 3.3273$

$\log 21.25 = 1.3273$

$\log 2.125 = 0.3273$

$\log (0.0002125) = -4 + 0.3273$ which is written as $\bar{4}.3273$

$= -4 + 0.3273 = -3.6727$

Since four figure log tables are used normally for computation purposes, reading the mantissa of a number with 5 or more significant digits, it has to be rounded off to the nearest number with 4 significant digits.

Let us consider the following three examples.

(i) If the number is 32.271, the Characteristic of log 32.271 = 1. For reading the mantissa of the logarithm, we round off the number to 3227 and from the log table, the mantissa is found as 0.5088.

$\therefore$ log 32.271 = 1.5088

(ii) If the number is 719.67:
log 719.67 = 2 + mantissa corresponding to 7197= 2.8571

(iii) If the number is 246.45
log 246.45 = 2 + mantissa corresponding to 2464

Antilogarithms

If log 549.6 = 2.7401, then 549.6 is called the antilogarithm of 2.7401 (i.e., $10^{2.7401} = 549.6$)

Suppose we want to compute the value of the product N_1N_2 using logarithms. Let $\log N_1N_2 = \log N_1 + \log N_2 = x$(say)

The value of N_1N_2 is the number whose logarithm to the base 10 is x, i.e., we have to find the antilogarithm of x.

Suppose x = 1.8652

First leave the characteristic 1 and note that the mantissa is 0.8652. From the antilog table, the reading corresponding to 0.8652 is obtained as 7331. Since the characteristic is 1, antilog (1.8652) = 7.331 × 10 =73.31.

Again, let x = −5 + 0.3196 (written as $\bar{5}.3196$). Therefore, characteristic is −5. Referring to the anti log table, reading corresponding to 0.3196, is 2087. This gives x as 2.087×10^{-5} or 0.00002087

CONCEPT STRANDS

Concept Strand 34

Given $\log_{10}2 = 0.3010$, find the number of digits in 2^{18}.

Solution

Let $x = 2^{18}$

$\log_{10} x = 18 \log_{10}2 = 18 \times 0.3010 = 5.418$

$\Rightarrow$ Number of digits in x (i.e., in 2^{18}) is
= characteristic + 1 = 5 + 1 = 6

Concept Strand 35

Given $\log_{10}2 = 0.3010$ and $\log_{10}3 = 0.4771$, find the number of digits in 36^{100}.

Solution

Let $x = 36^{100}$

$\log_{10}x = 100 \log_{10}36$

$= 100[\log_{10}9 + \log_{10}4] = 100[2\log 3 + 2\log 2]$

$= 200[0.4771 + 0.3010]$

$= 200 \times 0.7781 = 155.62$

$\Rightarrow$ Number of digits in 36^{100} is given by 155 + 1 = 156

Concept Strand 36

If $\log_{10} 1234 = 3.0913$, $\log_{10} 769874 = 5.88642$, find the value of $\sqrt[8]{0.000000001234}$.

Solution

Let $\sqrt[8]{0.000000001234} = x$

$\therefore \log_{10} x = \frac{1}{8}[\log_{10} 1234 - 12] = -1.11358$

$\log_{10} x = \bar{2}.88642$ — (1)

$\log_{10} 769874 = 5.88642$ — (2)

$\therefore$ Subtracting 7 form both sides, $(\log_{10}769874) - 7 = \bar{2}.88642$

As $7 = \log_7 10^7$ and $\log p - \log q = \log \frac{p}{q}$, the equation becomes:

$\log_{10} 0.0769874 = \bar{2}.88642$

$\therefore$ x = 0.0769874

POLYNOMIALS

A polynomial of degree n (n is a positive integer) in x is of the form

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$, where $a_0, a_1, a_2, \dots, a_n \in R$

If the above polynomial is denoted by P(x), P(x) = 0 represents the corresponding polynomial equation of degree n. $a_0, a_1, a_2, \dots, a_n$ are called the coefficients of the polynomial P(x).

The value of the variable x satisfying P(x) = 0, is called a root of the polynomial equation or a zero of the polynomial P(x). An nth degree polynomial has exactly n zeros and if $x_1, x_2, x_3, \dots, x_n$ are the zeros then, $P(x) = a_0 (x - x_1)(x - x_2) \dots (x - x_n)$.

n = 1 → Linear polynomial. General form is ax + b.

A linear polynomial has exactly one zero (or one root)

$n = 2 \rightarrow$ Quadratic polynomial. General form is $ax^2 + bx + c$.

A quadratic polynomial has exactly 2 zeros (or two roots)

$n = 3 \rightarrow$ Cubic polynomial. General form is $ax^3 + bx^2 + cx + d$.

A cubic polynomial has exactly 3 zeros (or three roots) and so on.

Remainder theorem

If P(x) is divided by (x − a), then the remainder is P(a). [P(a) denotes the value of P(x) when x is replaced by a in it.]

If P(a) = 0, (x − a) is a factor of P(x) or (x − a) divides P(x) exactly.

CONCEPT STRANDS

Concept Strand 37

Find the remainder when the polynomial $P(x) = 2x^4 + x^3 + 6x^2 - 4x + 7$ is divided by (x + 3)

Solution

Remainder $= P(-3) = 2(-3)^4 + (-3)^3 + 6(-3)^2 - 4(-3) + 7 = 208$

Concept Strand 38

Show that (x − 4) is a factor of $(x^3 + 2x^2 - 25x + 4)$.

Solution

Let $P(x) = x^3 + 2x^2 - 25x + 4$.

Then,

$P(4) = 4^3 + 2 \times 4^2 - 25 \times 4 + 4 = 0$

$\Rightarrow$ (x − 4) is a factor of P(x).

Concept Strand 39

If x − 1 is a factor of $2x^5 + kx^4 + 3x^3 - 4x^2 + 6x + 7$, then find k.

Solution

Let $f(x) = 2x^5 + kx^4 + 3x^3 - 4x^2 + 6x + 7$

As x − 1 is a factor of f(x), f(1) = 0.

$f(1) = 2(1)^5 + k(1)^4 + 3(1)^3 - 4(1)^2 + 6(1) + 7 = 0$, by remainder theorem.

$\Rightarrow 2 + k + 3 - 4 + 6 + 7 = k + 14 = 0$

$\therefore k = -14$

Concept Strand 40

If x − 1 is a factor of $x^2 - ax + b$ and the remainder when $x^2 - ax + b$ is divided by x + 1 is 6, find the values of a and b.

Solution

Let $f(x) = x^2 - ax + b$.

As x − 1 is a factor of $x^2 - ax + b$, f(1) = 0

$\Rightarrow 1 - a + b = 0$

$\Rightarrow a - b = 1$ — (1)

Also the remainder when f(x) is divided by x + 1 is 6.

$\Rightarrow 1 + a + b = 6$

$\Rightarrow a + b = 5$ — (2)

Solving (1) and (2) $\Rightarrow$ a = 3 and b = 2

Concept Strand 41

Find the remainder when 2^{34} is divided by 5.

Solution

In the division, since the numerator is in terms of power of 2, the denominator should also be expressed in terms of power of 2 i.e., as $(2^2 + 1)$. Now, as the denominator is in terms of 2^2, the numerator should also be rewritten in terms of 2^2 as $(2^2)^{17}$. The problem reduces to finding the remainder when $(2^2)^{17}$ is divided by $2^2 - (-1)$. This remainder, as per the Remainder Theorem is $(-1)^{17} = -1$; and −1 + 5 = 4 (the divisor is added to get a positive remainder).

MODULUS

Modulus of a real number denotes the numerical value of that number.

Modulus of a number, say x, is denoted by |x| and

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

For example,

(i) $|3.83| = 3.83$

(ii) $|-2.65| = 2.65$

(iii) $|x - 4| = \begin{cases} x-4, & x \geq 4 \\ 4-x, & x < 4 \end{cases}$

(iv) $|x| < 1$ means x lies between −1 and 1 or $-1 < x < 1$

INEQUALITIES

A real number say a is either positive (written as $a > 0$) or negative (written as $a < 0$) or zero (written as $a = 0$).

The symbol > means 'greater than' while the symbol ≥ means greater than or equal to. The symbol < means 'less than' while the symbol ≤ means less than or equal to.

For any two non zero real numbers a and b, $a > b$ if $a - b$ is positive and $a < b$ if $a - b$ is negative.

Results

(i) If $a > b$ and $b > c$, then $a > c$
If $a < b$ and $b < c$, then $a < c$

(ii) If $a > b$ then $a + x > b + x$ and $a - x > b - x$
If $a < b$ then $a + x < b + x$ and $a - x < b - x$
(Here, x is positive)

(iii) If $a > b$ then $ax > bx$ if $x > 0$
$ax < bx$ if $x < 0$
If $a < b$ then $ax < bx$ if $x > 0$
$ax > bx$ if $x < 0$

(iv) If $a > b$, then $\frac{1}{a} < \frac{1}{b}$ if a and b are of the same sign and $\frac{1}{a} > \frac{1}{b}$ if a and b are of opposite signs.
If $a < b$ then $\frac{1}{a} > \frac{1}{b}$ if a and b are of the same sign and $\frac{1}{a} < \frac{1}{b}$ if a and b are of opposite signs.

(v) If $a_1 > b_1, a_2 > b_2, \ldots\ldots\ldots a_n > b_n$, then $a_1 + a_2 + \ldots\ldots\ldots\ldots + a_n > b_1 + b_2 + \ldots\ldots + b_n$
If $a_1 < b_1, a_2 < b_2, \ldots\ldots\ldots a_n < b_n$, then $a_1 + a_2 + \ldots\ldots\ldots\ldots + a_n < b_1 + b_2 + \ldots\ldots + b_n$
If $a_1 > b_1$, $a_2 > b_2, \ldots\ldots\ldots a_n > b_n$, and $a_1, a_2, \ldots\ldots, a_n, b_1, b_2, \ldots\ldots b_n$ are all positive, then $a_1 a_2 \ldots\ldots\ldots a_n > b_1 b_2 \ldots\ldots\ldots b_n$

Note

If $a > b$ and $c > d$, it does not always follow that $a - c > b - d$

For example, we have $5 > 2$ and $9 > 4$. How ever, $(5 - 9)$ is not greater than $(2 - 4)$

Again, $13 > 2$ and $7 > 1$, $(13 - 7)$ is greater than $(2 - 1)$

CONCEPT STRANDS

Concept Strand 42

Find the value of x for which the expression, $-|x - 3| + \frac{21}{2}$ is maximum.

Solution

$-|x - 3| + \frac{21}{2}$ has the maximum value of $\frac{21}{2}$ when $|x - 3| = 0$
i.e., when $x = 3$.

Concept Strand 43

For every $x > 0$, show that $x + \frac{1}{x} \geq 2$.

Solution

We know that $(x - 1)^2 \geq 0$ for every x, and equality holds when $x = 1$

$\Rightarrow$ $x^2 - 2x + 1 \geq 0$, dividing by x $(x > 0)$, we get

$$x - 2 + \frac{1}{x} \geq 0$$

$\Rightarrow$ $x + \frac{1}{x} \geq 2$ and equality holds when $x = 1$.

Concept Strand 44

Show that $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$.

Solution

$$a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$$

$$= \frac{1}{2}(a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ac + a^2)$$

$$= \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2] \geq 0,$$

and equality holds when $a = b = c$.

Concept Strand 45

Solve the inequality $x + 2\,|x - 2| < 5$.

Solution

When $x < 2$,

$x + 2\,|x - 2| < 5$

$\Rightarrow$ $x - 2(x - 2) < 5$

$\Rightarrow$ $-x + 4 < 5$

$\Rightarrow$ $-x < 1 \Rightarrow x > -1$

$\therefore$ Solution is $(-1, 2)$ — (1)

When $x \geq 2$,

$x + 2\,|x - 2| < 5$

$\Rightarrow$ $x + 2(x - 2) < 5$

$\Rightarrow$ $3x - 4 < 5 \Rightarrow 3x < 9$

$\Rightarrow$ $x < 3$

$\therefore$ Solution is $[2, 3)$ — (2)

Combining (1) and (2), solution is, $x \in (-1, 3)$

ARITHMETIC MEAN, GEOMETRIC MEAN

If a and b are two positive numbers $\frac{a+b}{2}$ is called the arithmetic mean of a and b and $\sqrt{ab}$ is called the geometric mean of a and b.

Arithmetic mean (A.M) of a and b $= \frac{a+b}{2}$

Geometric mean (G.M) of a and b $= \sqrt{ab}$

$$\text{A.M} - \text{G.M} = \frac{a+b}{2} - \sqrt{ab}$$

$$= \frac{a + b - 2\sqrt{ab}}{2}$$

$$= \frac{(\sqrt{a} - \sqrt{b})^2}{2} \geq 0 \text{ always}$$

This means that, A.M of any two distinct positive numbers is always greater than their G.M. i.e., A.M. > G.M.

If the numbers a and b are equal, i.e., if $a = b$, then A.M = G.M = a.

We will be dealing with the generalization of the above result in a later unit.

DETERMINANTS

The four numbers a_1, b_1, a_2, b_2, written in two rows and two columns as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ determinant of second order. It stands for the value $(a_1b_2 - a_2b_1)$ i.e.,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = (a_1b_2 - a_2b_1).$$

For example,

(i) The value of the determinant $\begin{vmatrix} 3 & 8 \\ -1 & 4 \end{vmatrix}$ is $(3 \times 4) - (8 \times -1) = 20$.

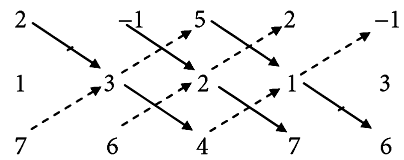
(ii) The value of $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$.

We can define a determinant of 3rd order as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ where, $a_1, b_1, \ldots c_3$ are numbers.

The value of this determinant can be computed by a simple method called "*Method of Sarus*".

For example, the value of $\begin{vmatrix} 2 & -1 & 5 \\ 1 & 3 & 2 \\ 7 & 6 & 4 \end{vmatrix}$ is computed as follows.

We rewrite first and second columns to the right of the determinant as shown:



(first follow the dark line and then the dotted line)

The answer is $2 \times 3 \times 4 + -1 \times 2 \times 7 + 5 \times 1 \times 6 - (7 \times 3 \times 5 + 6 \times 2 \times 2 + 4 \times 1 \times -1) = -85$.

We hasten to add that this method works only for third order determinants.

Properties of determinants and the general methods of expansion will be dealt with in "Matrices and Determinants".

IMPORTANT RESULTS IN GEOMETRY

Triangles

Let ABC be a triangle.

(i) Centroid

Let D, E, F represent the mid points of the sides BC, CA and AB respectively. Then, AD, BE and CF are called the medians of the triangle. The medians of a triangle are concurrent. (i.e. all the three medians pass through a point). The point of concurrence is called the centroid of the triangle and this point is denoted by G.

Also, the centroid divides each median internally in the ratio 2:1

i.e., $\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = \frac{2}{1}$ (Refer Fig. 1.1)

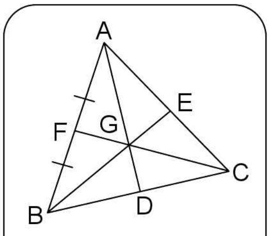


Fig. 1.1

(ii) Orthocentre

Let AX, BY, CZ be the perpendiculars from the vertices A, B, C of the triangle to the sides BC, CA, AB respectively. Then, AX, BY, CZ are called the altitudes of the triangle. The altitudes of a triangle are concurrent. The point of concurrence is called the orthocentre of the triangle, denoted by O. (Refer Fig. 1.2)

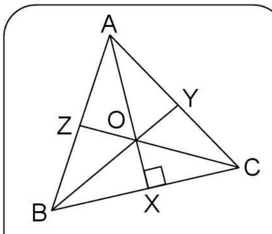


Fig. 1.2

(iii) Circumcentre

The perpendicular bisectors of the sides of a triangle are concurrent.

The point of concurrence is called the circum centre of the triangle and is denoted by S. (Refer Fig. 1.3)

SA = SB = SC = circum radius

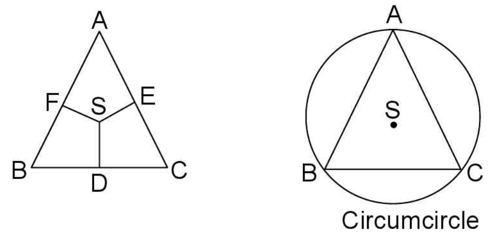


Fig. 1.3

The circle with centre at S and radius equals SA (or SB or SC) is called the circum circle of the triangle. This circle passes through the vertices A, B, C of the triangle.

(iv) Incentre

Let AD_1, BE_1, CF_1 represent the internal bisectors of the angles A, B, C of the triangle. The internal bisectors of a triangle are concurrent. The point of concurrence is called the incentre of the triangle and is denoted by I (Refer Fig. 1.4)

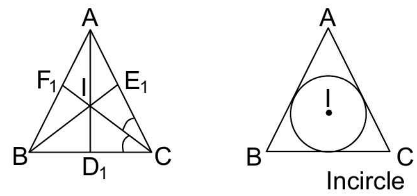


Fig. 1. 4

The circle with I as centre and radius equals the perpendicular from I to any of the sides will touch the sides of the triangle.

This circle is called the incircle of the triangle and its radius is called the inradius of the triangle.

Also, $\frac{BD_1}{D_1C} = \frac{AB}{AC}$; $\frac{CE_1}{E_1A} = \frac{BC}{BA}$ and $\frac{AF_1}{F_1B} = \frac{CA}{CB}$

i.e., internal bisectors of the angles of a triangle divide the opposite sides in the ratio of the sides containing that angle.

(v) Circumcentre S, centroid G and ortho centre O of a triangle are collinear and G divides SO in the ratio 1 : 2. i.e., SG:GO = 1:2.

Again, the perpendicular distance of S from any side is equal to half the distance of O from the opposite vertex, where S is the circum centre and O is the ortho centre of the triangle. Referring to Fig. 1.5, $SD = \frac{1}{2}AO$.

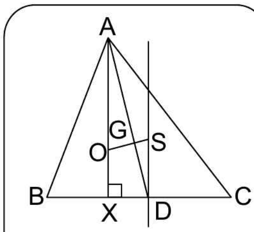
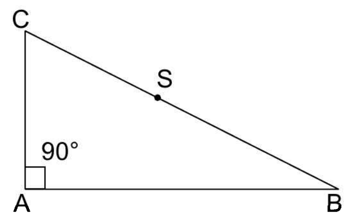


Fig. 1. 5

(Observe that both SD and AO are parallel to each other, both being perpendiculars to the side BC)

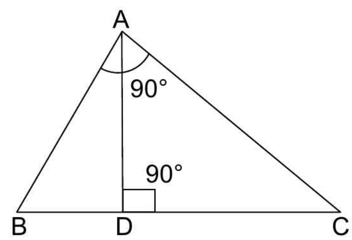
(vi) In the case of an equilateral triangle, the centroid G, orthocentre O, circum centre S and the incentre I coincide (or they are one and the same point)

(vii) Consider a right angled triangle ABC right angled at A.



The orthocentre of the triangle is at A and its circumcentre S is the mid point of the hypotenuse BC. Also, circum radius of the triangle = $\frac{1}{2}BC = \frac{1}{2}\sqrt{AB^2 + AC^2}$

(viii) ABC is a right angled triangle right angled at A. AD is the altitude through A.



Then, (a) $BD.DC = AD^2$

(b) $BD.BC = AB^2$

and (c) $CD.CB = AC^2$

(ix) Cyclic Quadrilateral

Let ABCD be a quadrilateral whose vertices lie on a circle (or the quadrilateral is circumscribed by a circle). ABCD is known as a cyclic quadrilateral. AC and BD are its diagonals. (Refer Fig. 1.6)

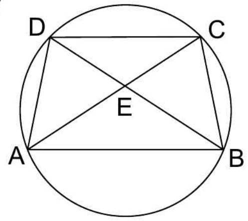


Fig. 1.6

We have $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$

Another property of the cyclic quadrilateral is, $AB.CD + AD.BC = AC.BD$

This is known as Ptolemy's theorem.

(x) Similarity of triangles

If two triangles ABC and DEF are similar, then

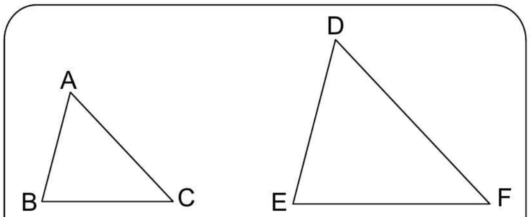


Fig. 1.7

(i) they are equiangular,
i.e., $\angle A = \angle D$; $\angle B = \angle E$; $\angle C = \angle F$

(ii) their corresponding sides are proportional

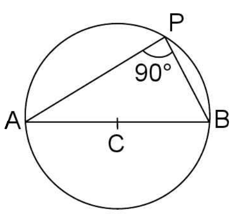
i.e., $\dfrac{BC}{EF} = \dfrac{AB}{DE} = \dfrac{AC}{DF}$

(iii) their areas are proportional to the squares of the corresponding sides

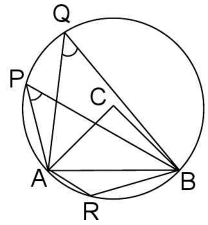
i.e., $\dfrac{\text{Area of ABC}}{\text{Area of DEF}} = \dfrac{BC^2}{EF^2} = \dfrac{AB^2}{DE^2} = \dfrac{AC^2}{DF^2}$

Circles

(i) AB is diameter of a circle whose centre is at C. Let P represent any point on the circle. Then, $\angle APB = 90°$



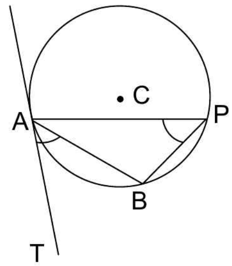
(ii) AB is a chord of the circle whose centre is at C
Let P, Q, R denote points on the circle as shown in the figure. Then, $\angle APB = \angle AQB$



$\angle ACB = 2\angle APB$

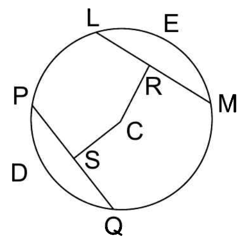
$\angle ARB = 180° - \angle APB$

(iii) AT is the tangent at A to a circle centered at C. AB is a chord of the circle. $\angle BAT = \angle APB$.

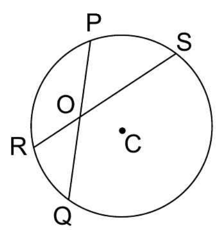


(iv) C is the centre of a circle. LM and PQ are two chords of the circle such that LM = PQ. CR and CS are perpendiculars from the centre C to the chords LM and PQ. Then, CR = CS

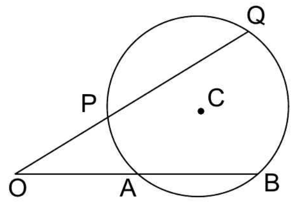
Conversely, if CR = CS, chord PQ = chord LM. Also, arc PDQ = arc LEM.



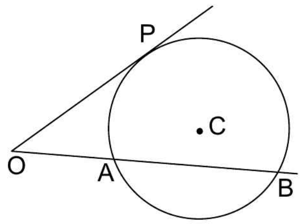
(v) Let PQ and RS be two chords of a circle centered at C. Let these chords intersect at O. Then, $PO.OQ = RO.OS$



(vi) OAB, OCD are lines through a point O where O is outside the circle centered at C. Then, OA.OB = OP.OQ



(vii) OP is a tangent drawn from a point O to a circle centered at C. OAB is a line through O meeting the circle at A and B. Then, $OA.OB = OP^2$



We may use the section formula to obtain the coordinates of the centriod and incentre of a triangle whose vertices are at (x_1,y_1), (x_2,y_2) and (x_3,y_3). Let ABC be the triangle.

To find the coordinates of the Centroid of a triangle

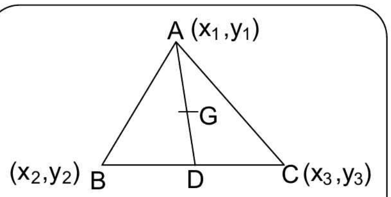


Fig. 1. 8

We know that the centroid divides each median of the triangle internally in the ratio 2:1.

AD is the median through A.

Since D is the midpoint of BC, coordinates of D are given by $\left(\dfrac{x_2+x_3}{2}, \dfrac{y_2+y_3}{2}\right)$

We have AG:GD = 2:1

Using the section formula, Coordinates of G are

$$\left(\frac{2\left(\dfrac{x_2+x_3}{2}\right)+x_1}{3}, \frac{2\dfrac{(y_2+y_3)}{2}+y_1}{3}\right)$$

i.e., $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$

To find the coordinates of the Incentre of a triangle

AD_1 and BE_1 are the internal bisectors of angles A and B. The incentre I is the point of intersection of the internal bisectors.

Let the sides BC, CA, AB of the triangle be denoted by a, b and c respectively. Then, we have $\dfrac{BD_1}{D_1C} = \dfrac{AB}{AC} = \dfrac{c}{b}$

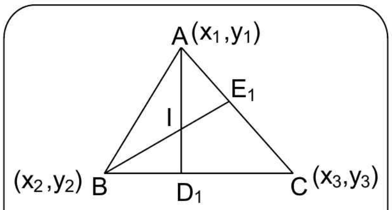


Fig. 1. 9

or D_1 is the point dividing BC internally in the ratio c:b. By section formula, D_1 is $\left(\dfrac{cx_3+bx_2}{b+c}, \dfrac{cy_3+by_2}{b+c}\right)$

We have $\dfrac{BD_1}{D_1C} = \dfrac{c}{b} \Rightarrow \dfrac{BD_1}{BD_1+D_1C} = \dfrac{c}{(c+b)}$

$$\Rightarrow \frac{BD_1}{BC} = \frac{c}{(c+b)} \Rightarrow BD_1 = \frac{c.BC}{(c+b)}.$$

From triangle ABD_1, since BI is the internal bisector of B, $\dfrac{AI}{ID_1} = \dfrac{BA}{BD_1} = \dfrac{c}{\left(\dfrac{ac}{b+c}\right)} = \dfrac{(b+c)}{a}$

Using section formula, the coordinates of I are $\left(\dfrac{ax_1+bx_2+cx_3}{a+b+c}, \dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$.

Area of the triangle whose vertices are given

Let $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ represent the vertices of a triangle ABC. Draw AL, BM, CN perpendiculars to the x-axis.

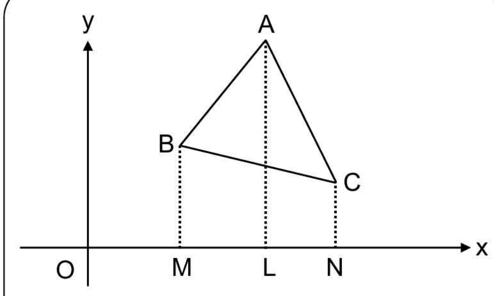


Fig. 1.10

If we denote the area of the triangle ABC by the symbol Δ (pronounced as 'delta'),

Δ = Area of trapezium ABML + Area of trapezium ALNC – Area of trapezium BMNC

$= \frac{1}{2}(AL + BM)\times ML + \frac{1}{2}(AL + CN)\times LN - \frac{1}{2}(BM + CN)\times MN$

$= \frac{1}{2}(y_1 + y_2)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$

$= \frac{1}{2}[x_1(y_1 + y_2 - y_1 - y_3) + x_2(y_2 + y_3 - y_1 - y_2) + x_3(y_1 + y_3 - y_2 - y_3)]$

$\therefore \quad \Delta = \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$

We may also represent the area as $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

(i.e., the formula for the area of a triangle is expressed as a determinant of 3rd order).

For example, the area of the triangle whose vertices are (1,–3), (5, 2) and (3, 4) is given by

$$\Delta = \frac{1}{2}[1(2-4)+5(4+3)+3(-3-2)] = 9$$

(i) Let one of the vertices of the triangle be the origin. Area of the triangle OAB where O is the origin and A and B have coordinates (x_1,y_1) and (x_2,y_2) is given by $\frac{1}{2}[x_1y_2 - x_2y_1]$

(ii)

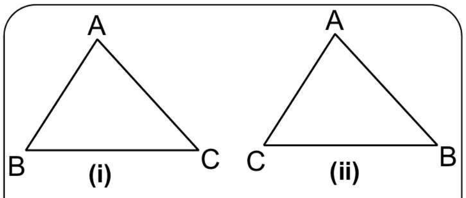


Fig. 1. 11

If the order of the given points is as in (i) of Fig. 1.11, the computation of the area Δ using the formula gives a positive value while if it is as in (ii) of Fig. 1.11, the computation gives a negative value. In the latter case, we take the numerical value (or absolute value or modulus) as the measure of the area.

(iii) If the three points are collinear (i.e., the three points lie on the same line), it is clear that the area of the triangle formed by these three points is zero. Therefore, to see whether three points in a plane are collinear we check whether the area of the triangle formed by these points is zero.

CONCEPT STRANDS

Concept Strand 46

Prove that the triangle whose vertices are (2, 4), (5, 1) and (6, 5) is an isosceles triangle.

Solution

Let the vertices be A, B, C. Let A be (2, 4), B be (5, 1) and C be (6, 5).

The triangle is isosceles if two of its sides are equal. We shall compute AB^2, BC^2 and CA^2

$AB^2 = 9 + 9 = 18$, $BC^2 = 1 + 16 = 17$, $CA^2 = 16 + 1 = 17$

Hence BC = CA.

Concept Strand 47

Prove that the triangle whose vertices are (–9, –16), (2, 6), (–6, 10) is right angled triangle.

Solution

If the vertices are denoted by A, B, C, we have $AB^2 = 605$, $BC^2 = 80$ and $CA^2 = 685$.

Clearly, $AB^2 + BC^2 = CA^2$.

Concept Strand 48

Show that points (11, 3), (–13, –7), (–1, –15) and (4, –14) lie on a circle whose centre lies at the point (–1, –2).

Solution

Let C_1 denote the centre of the circle. If the points are to lie on the circumference, the distances of these points from C_1 must be equal.

If these points are denoted by A, B, C, D, we have $C_1A^2 = 169 = C_1B^2 = C_1C^2 = C_1D^2$.

Concept Strand 49

The centre of a circle is at (3, 4) and one point on the circle is (8, 6). Find the coordinates of the other end of the diameter through this point

Solution

If (x, y) represents the coordinates of the other end of the diameter through (8, 6) it is clear that (3, 4), which is the centre of the circle must be the middle point of the line joining (x, y) and (8, 6)

We have $\dfrac{x+8}{2} = 3$, $\dfrac{y+6}{2} = 4$,

$\Rightarrow$ x = –2, y = 2.

Concept Strand 50

The midpoints of the sides of a triangle are (–1, –2), (6, 1) and (3, 5). Find the coordinates of its vertices.

Solution

Let A, B, C be the vertices of the triangle and D, E, F the midpoints of the sides BC, CA, AB.

Let G be the centroid of the triangle ABC. It is also the centroid of triangle DEF.

$$G \text{ is } \left(\frac{-1+6+3}{3}, \frac{-2+1+5}{3}\right) = \left(\frac{8}{3}, \frac{4}{3}\right).$$

G divides AD in the ratio 2 : 1. If A is (x, y), $\dfrac{-2+x}{3} = \dfrac{8}{3}$ and $\dfrac{-4+y}{3} = \dfrac{4}{3}$.

This gives x = 10, y = 8. or A is (10, 8).

If B is (x, y), $\dfrac{12+x}{3} = \dfrac{8}{3}, \dfrac{2+y}{3} = \dfrac{4}{3} \Rightarrow$ B is (–4, 2).

Similarly, we get C as (2, –6).

Concept Strand 51

Show that the three points (–1, 6), (–10, 12), (–16, 16) are collinear.

Solution

Area of the triangle formed by the three points = 0, by using the formula for area.

Concept Strand 52

Find the area of the quadrilateral whose vertices are (–1, –5), (2, –3), (1, 2) and (–2, 4).

Solution

If the vertices are denoted by A, B, C, D,

Area of the quadrilateral ABCD = the sum of the areas of the triangles ABC and ACD

$$\frac{17}{2} + \frac{25}{2} = 21 .$$

Concept Strand 53

Find the values of k so that the three points (k, 2k), (2k, 3k) and (3, 1) are collinear.

Solution

Area of triangle formed by the three points = $-k^2 - 2k = 0$, as the points are collinear.

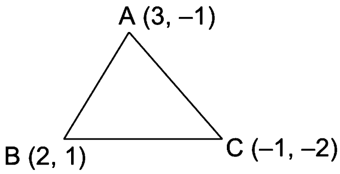
This gives k = 0 or –2, k = 0 is trivial. Thus, k = –2.

Concept Strand 54

Obtain the coordinates of the incentre of the triangle whose vertices are (3, –1), (2, 1) and (–1, 2).

Solution

Let A,B,C represent the vertices of the triangle. The sides of the triangle are given by



$$a = BC = \sqrt{(2+1)^2 + (1-2)^2} = \sqrt{10}$$

$$b = CA = \sqrt{(-1-3)^2 + (2+1)^2} = 5$$

$$c = AB = \sqrt{(3-2)^2 + (-1-1)^2} = \sqrt{5}$$

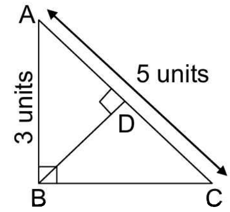
Using the formula for the incentre, its coordinates are

$$\left(\frac{\sqrt{10}\times 3 + 5\times 2 + \sqrt{5}\times -1}{(\sqrt{10}+5+\sqrt{5})}, \frac{\sqrt{10}\times(-1) + 5\times 1 + \sqrt{5}\times 2}{\sqrt{10}+5+\sqrt{5}}\right)$$

$$= \left(\frac{3\sqrt{10}+10-\sqrt{5}}{5+\sqrt{10}+\sqrt{5}}, \frac{5+2\sqrt{5}-\sqrt{10}}{5+\sqrt{10}+\sqrt{5}}\right)$$

Concept Strand 55

Find the length of side AD in the figure given below.



Solution

$BC^2 = AC^2 - AB^2 = 5^2 - 3^2 = 16 \Rightarrow BC = 4$ units

Let AD be x units.

$AB^2 - AD^2 = BC^2 - DC^2$

$3^2 - x^2 = 4^2 - (5 - x)^2$

Solving the above equation we obtain x = 1.8 units.

Alternate method:

As per the diagram, $\angle ABC = 90°$ and BD is perpendicular to the hypotenuse.

Hence, $AB^2 = AD.\ AC$

$\Rightarrow$ $3^2 = 5.AD$

$\Rightarrow$ $AD = 1.8$

LOCUS—EQUATION OF A LOCUS

Locus means the path (or curve or a surface) traced out by a point, which moves in a plane (or in space) satisfying some given conditions.

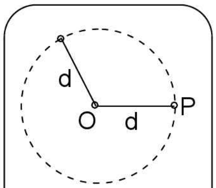


Fig. 1.12

For example, the locus of a point, which moves in a plane such that it is always at a constant distance from a fixed point in the plane, is a circle with the fixed point as its centre. [Refer Fig. 1.12]

Again, the locus of a point, which moves in a plane such that it is at equal distances from two fixed points in the plane, is the perpendicular bisector of the straight line joining the two fixed points. [Refer Fig. 1.13]

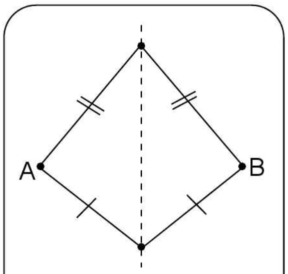


Fig. 1.13

Equation of a locus is the relation satisfied by the x coordinate and y coordinate of a point on the locus.

To find the equation of a locus:

Step I: Assume that P(x, y) is a point on the locus.

Step II: Translate the given conditions (under which the point moves in the plane) into an algebraic relation between x and y. (This means that every point on the locus is such that its x coordinate and y coordinate satisfy this relation.)

We say that this relation represents the equation of that locus (or equation of that path or equation of that curve).

CONCEPT STRANDS

Concept Strand 56

Find the equation of the locus of a point, which is equidistant from the points $(1, -3)$ and $(3, 8)$.

Solution

If $P(x, y)$ is a point on the locus, we have $(x - 1)^2 + (y + 3)^2 = (x - 3)^2 + (y - 8)^2$

This gives the equation of the locus as $4x + 22y - 63 = 0$.

Concept Strand 57

Find the equation of the locus of a point, which moves such that it is always at a distance of 5 units form the point $(3, 6)$.

Solution

If $P(x, y)$ is a point on the locus, we have $(x - 3)^2 + (y - 6)^2 = 25$.

The equation of locus is $x^2 + y^2 - 6x - 12y + 20 = 0$

Concept Strand 58

Obtain the equation of the locus of a point, which moves such that its distance from $(-2, 0)$ is twice its distance from $(3, 3)$.

Solution

If $P(x, y)$ is a point on the locus, we have $(x + 2)^2 + y^2 = 4[(x - 3)^2 + (y - 3)^2]$

This simplifies to $3(x^2 + y^2) - 28x - 24y + 68 = 0$.

We may be able to plot these loci on a graph paper by choosing points (x, y) satisfying the relations. It is obvious that when the conditions under which the point moves are different, we get different loci. That is, the equations of these loci will be different.

Identification of the nature of the locus (whether it is a straight line or a circle or some other known curve) is possible by examining the structure of its equation.

Equations of loci, which are straight lines, have one particular structure while equations of loci, which are circles, will have another structure.

TRIGONOMETRY FUNDAMENTALS

Definition of an angle

An angle is generated by rotating a ray about a point (called the vertex or pole) from some initial position (called initial side) to some terminal position (called terminal side). The amount of rotation gives the measure of the angle.

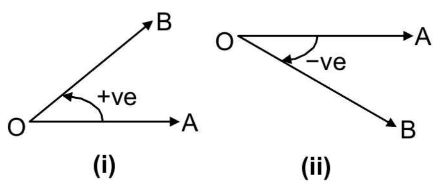


Fig. 1.14

When the rotation is in the counter clockwise sense(or anticlockwise sense), the measure of the angle is taken as positive (i.e., a positive sign is associated with the angle: refer (i) of Fig. 1.14); and, if the rotation is clockwise, the measure of the angle is taken as negative (i.e., a negative sign is associated with the angle: refer (ii) of Fig. 1.14).

We denote the angles by the letters θ, α, γ, A, B, C,.... (which are the measures of the angles in some units).

It may also be noted that angles having the same initial and terminal sides (known as coterminal angles) may have different measures [Refer Fig. 1.15]

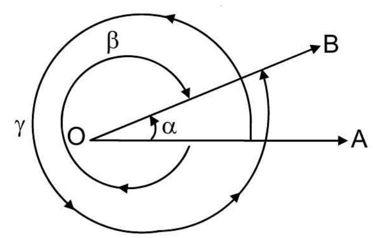


Fig. 1.15

Consider the following examples:

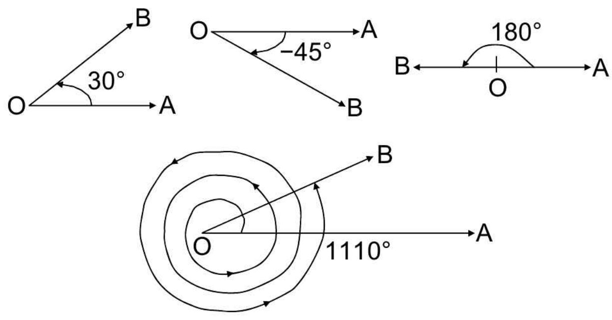


Fig. 1.16

Units of measurement of an angle

One of the units of measurement of an angle is "degree". One degree (denoted by 1°) is defined as the measure of the angle formed by joining the centre of a circle to the extremities of an arc of the circle whose length is $\frac{1}{360}$ of its circumference. A degree is divided into 60 equal parts called minutes of arc and a minute of arc is divided into 60 equal parts called seconds of arc (not to be confused with minutes of time and seconds of time)

1 right angle = 90 degrees = 90°

1° (1 degree) = 60 minutes of arc = 60'

1' (1 minute of arc) = 60 seconds of arc = 60"

There is another unit of measurement of an angle called the "circular measure" or "radian measure". This is the unit of measurement of the angle used for all theoretical purposes.

Definition of radian measure

The angle subtended at the centre of a circle by an arc equal in length to the radius of the circle is called one radian (denoted by 1^c). If r is the radius of the circle, and length of arc AB = r (Refer Fig. 1.17), then $\angle AOB = 1^c$.

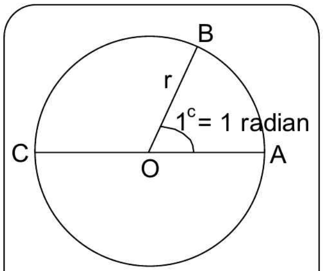


Fig. 1. 17

AOC is a diameter of the circle with centre O and radius r. The length of the arc varies as the angle subtended by the arc at the centre of the circle. Since the arc AC subtends an angle 180° at the centre and length of arc AC = πr (half the circumference), we have

π radians = 180° or

1 radian = $\frac{180^\circ}{\pi}$ ≈ 57° 17' (π = 3.14159) (≈ symbol means "approximately equal to")

1 degree ≈ 0.0175 radians

It may be observed from the above that radian measure of an angle is independent of the radius of the circle.

Below given are conversions of a few standard measures of angles.

(i) $90^\circ = \frac{\pi}{2}$ radians

(ii) $45^\circ = \frac{\pi}{4}$ radians

(iii) $30^\circ = \frac{\pi}{6}$ radians

(iv) $\frac{\pi}{5}$ radians = 36°

(v) $\frac{-5\pi}{12}$ radians = −75°

(vi) $\frac{\pi}{3}$ radians = 60°

Area of a sector

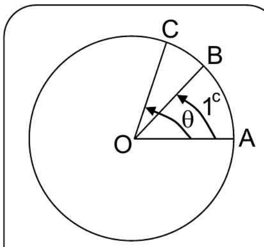


Fig. 1. 18

Let arc AC subtend angle θ at the centre, and arc AB subtend angle 1 radian at the centre.

We have, $\frac{\text{length of arc AC}}{\theta} = \frac{\text{length of arc AB}}{1} = \frac{r}{1}$

⇒ length of arc AC = r θ

Again, $\dfrac{\text{area of sector AOC}}{\theta} = \dfrac{\text{area of the circle}}{2\pi}$

$$= \frac{\pi r^2}{2\pi} = \frac{r^2}{2}$$

$\Rightarrow$ area of the sector AOC $= \dfrac{1}{2} r^2 \theta$.

Definitions of trigonometric functions (or circular functions) of an angle

We are already familiar with the definitions for the six trigonometric ratios sine θ (written as $\sin\theta$); cosine θ (written as $\cos\theta$); tangent θ (written as $\tan\theta$); cosecant θ (written as $\text{cosec}\theta$); secant θ (written as $\sec\theta$) and cotangent θ (written as $\cot\theta$); for an acute angle θ (i.e., $0 < \theta < 90°$ or $0 < \theta < \frac{\pi}{2}$) and also the relations existing between them.

For the purpose of defining these ratios we used a right-angled triangle in which one of the angles was θ. We are now going to define these ratios for any angle θ (θ need not be restricted to an acute angle, Also, θ can have positive or negative measure). These definitions are such that they automatically hold good for acute angles as well. We also call these as trigonometric functions or circular functions.

We take two mutually perpendicular straight lines XOX' and YOY' intersecting at O. This represents the rectangular Cartesian coordinate system where, XOX' is the x-axis and YOY' is the y-axis and O is the origin.

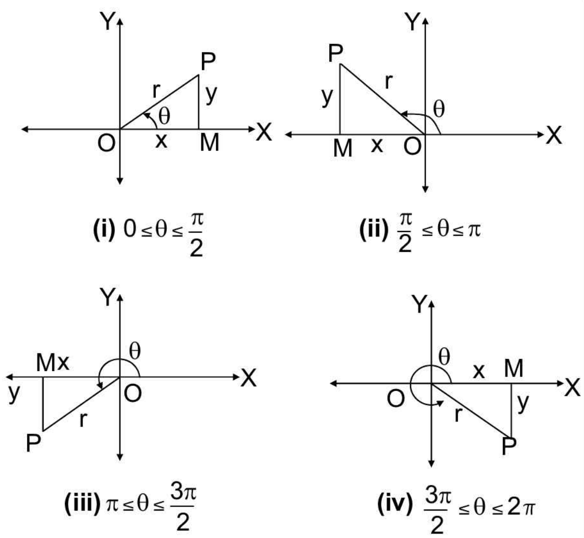


Fig. 1.19

An angle is said to be in standard position if its vertex is at the origin O and its initial side coincides with OX, the positive direction of the x-axis.

Let a ray OP start from OX and trace out $\angle XOP$ $(= \theta)$. Then, the terminal side will be in one of the four quadrants. The angle θ can be either in the radian measure or in the degree measure. It is the usual practice to write $\theta°$ if the angle is expressed in degree measure. Hence, angle θ means it is in radian measure.

If the terminal side is in the first quadrant and the rotation is in the counter clock wise sense (positive sense), $\angle XOP$ (denoted by θ) will be lying between 0 and $\frac{\pi}{2}$ [Refer (i) of Fig. 1.19].

Similarly, $\angle XOP$ will be lying between $\frac{\pi}{2}$ and π if the terminal side is in the second quadrant; $\angle XOP$ will be lying between π and $\frac{3\pi}{2}$ if the terminal side is in the third quadrant and $\angle XOP$ will be lying between $\frac{3\pi}{2}$ and 2π if the terminal side is in the fourth quadrant [Refer (ii), (iii) and (iv) of Fig. 1.19]

If the rotation of the ray is in the clockwise sense [refer (i) Fig. 1.20] we will have the angle θ lying between $\frac{-\pi}{2}$ and 0; $-\pi$ and $\frac{-\pi}{2}$; $\frac{-3\pi}{2}$ and $-\pi$; and -2π and $\frac{-3\pi}{2}$ according as the terminal side is in the fourth, third, second or first quadrants respectively.

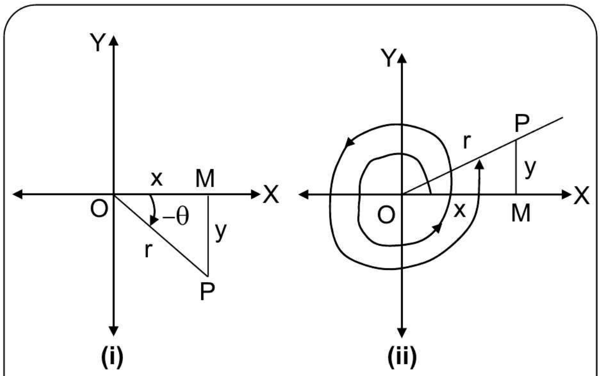


Fig. 1. 20

Suppose the ray OP completes n rotations in the positive sense and occupies a position in one of the quadrants, then the measure of the angle generated is $2n\pi + \angle XOP$ [Refer (ii) of Fig. 1.20].

If (x, y) represents the coordinates of P (referred to X'OX and Y'OY as the axes of coordinates) and if OP is denoted by r,

$$r = \sqrt{OM^2 + PM^2} = \sqrt{x^2 + y^2}.$$

OP is called the radial distance of P and it is always taken as positive.

The six trigonometric functions (or circular functions) of θ are defined in the following manner.

$$\sin\theta = \frac{\text{ordinate of P}}{\text{OP}} = \frac{MP}{OP} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{abscissa of P}}{\text{OP}} = \frac{OM}{OP} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{ordinate of P}}{\text{abscissa of P}} = \frac{MP}{OM} = \frac{y}{x},\ x \neq 0$$

$$\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{r}{y},\ y \neq 0$$

$$\sec\theta = \frac{OP}{OM} = \frac{r}{x},\ x \neq 0$$

$$\cot\theta = \frac{OM}{MP} = \frac{x}{y},\ y \neq 0$$

Observation 1

Following are the conventions regarding the signs of the coordinates of a point P in a rectangular Cartesian system.

If P lies in the first quadrant, both x and y are positive;

If P lies in the second quadrant x is negative, y is positive;

If P lies in the third quadrant, both x and y are negative;

and if P lies in the fourth quadrant, x is positive and y is negative.

Recall that OP is always taken as positive.

We therefore see that ,

For angles θ in the first quadrant (where the terminal side OP is in the first quadrant) all the circular functions are positive;

For angles θ in the second quadrant (where the terminal side OP is in the second quadrant) sinθ and its reciprocal cosecθ are positive while all the other circular functions are negative;

For angles θ in the third quadrant (where the terminal side OP is in the third quadrant) tanθ and its reciprocal cotθ are positive while all the other circular functions are negative;

Lastly, for angles θ in the fourth quadrant (where the terminal side OP is in the fourth quadrant) cosθ and its reciprocal secθ are positive while all the other circular functions are negative.

For example, suppose θ = 48°, the terminal side is in the first quadrant. ∴ All the circular functions of θ are positive.

Suppose θ = −77°, the terminal side OP will be in the fourth quadrant. This means that cos (−77°) and sec (−77°) are positive, while the other four functions are negative.

Suppose θ = 2020°, the terminal side OP corresponding to this angle is in the third quadrant (after 5 rotations about the vertex O in the counter clockwise sense) and therefore, tan 2020° and cot 2020° are positive while the other four functions are negative.

The signs of the circular functions of any angle θ corresponding to the quadrant in which θ lies can be easily remembered with the help of the following diagram.

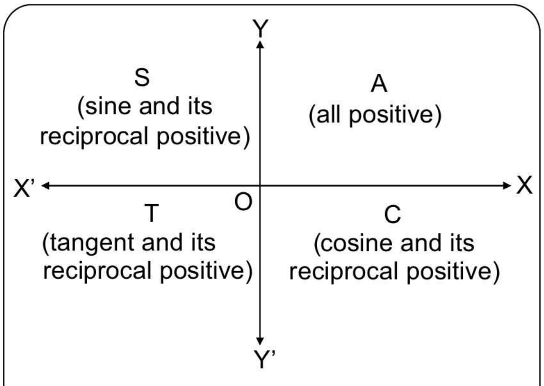


Fig. 1. 21

Observation 2

The ratios $\frac{x}{r}$ and $\frac{y}{r}$ are independent of the position P on the terminal side.

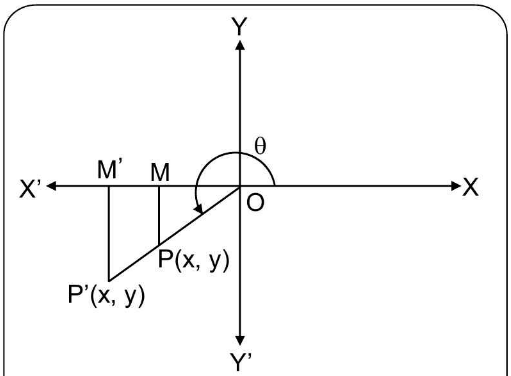


Fig. 1.22

Referring to Fig. 1.22, let P' be another point on the terminal side such that OP' = r' where, $\angle XOP = \theta$ (for illustration purpose θ is taken in the third quadrant). If the coordinates of P' are (x', y'), we have, from similar triangles OPM and OP'M', $\frac{OM}{OP} = \frac{OM'}{OP'}$.

Since P and P' are in the same quadrant, x and x' have the same sign. Therefore, it is clear that $\frac{x}{r} = \frac{x'}{r'}$. The same is the case for the other ratios $\frac{PM}{OP}$ and $\frac{P'M'}{OP'}$ i.e., $\frac{y}{r} = \frac{y'}{r'}$. This means that the circular functions of an angle θ depend on the angle θ only.

Observation 3

From the definitions we deduce the following relations between the circular functions, $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

In a similar manner, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$ and $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$.

Again, we have

$(\sin\theta)^2 + (\cos\theta)^2$ (written as $\sin^2\theta + \cos^2\theta$)

$$= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{y^2 + x^2}{r^2} = \frac{MP^2 + OM^2}{r^2} = \frac{OP^2}{r^2} = 1$$

or $\sin^2\theta + \cos^2\theta = 1$ — (1)

Note that $(\sin\theta)^n$ is written as $\sin^n\theta$, $(\cos\theta)^n$ is written as $\cos^n\theta$, $(\tan\theta)^n$ is written as $\tan^n\theta$ and so on.

From (1), $\cos^2\theta = 1 - \sin^2\theta$; $\sin^2\theta = 1 - \cos^2\theta$.

Again, from the definitions, we deduce that

$1 + \tan^2\theta = \sec^2\theta$ — (2)

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$ —(3)

From (2) and (3), $\sec^2\theta - \tan^2\theta = 1$, $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

Observation 4

Since $\frac{y}{r} = \sin\theta, \frac{x}{r} = \cos\theta$, we have $x = r\cos\theta$, $y = r\sin\theta$.

$\therefore \quad x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$.

This means that the point P(x, y) always lies on the curve $x^2 + y^2 = r^2$ which is a circle. In other words, any point on the circle $x^2 + y^2 = r^2$ (circle with centre at origin and radius equals r) can be represented in the form ($r\cos\theta$, $r\sin\theta$). This is the reason why we call the trigonometric functions as circular functions.

If r = 1, the circle $x^2 + y^2 = 1$ is called the unit circle. In this case, any point on the circle is represented by ($\cos\theta$, $\sin\theta$).

The following table gives the circular functions of a few standard angles.

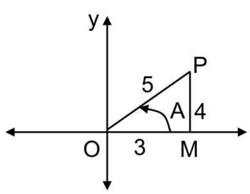
Table 1.2

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	Cosec θ	$\sec\theta$	$\cot\theta$
0	0	1	0	Not defined	1	Not defined
$\frac{\pi}{6}$ or 30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$\frac{\pi}{4}$ or 45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$ or 60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{2}$ or 90°	1	0	Not defined	1	Not defined	0

CONCEPT STRANDS

Concept Strand 59

Given $\cos A = \frac{3}{5}$ and A is in the first quadrant find the other trigonometric functions of A



Solution

A being in the first quadrant, all the trigonometric functions are positive, Referring to the figure

$$PM = \sqrt{5^2 - 3^2} = 4.$$

$\sin A = \frac{4}{5}$, $\tan A = \frac{4}{3}$, $\operatorname{cosec} A = \frac{5}{4}$, $\sec A = \frac{5}{3}$ and $\cot A = \frac{3}{4}$.

Concept Strand 60

Given θ is in the third quadrant and $\tan\theta = \frac{5}{12}$, find $\frac{2\sin\theta - 3\cos\theta}{5\sin\theta + 7\cos\theta}$.

Solution

Since θ is the third quadrant, $\tan\theta$ and $\cot\theta$ are positive while all the other functions are negative.

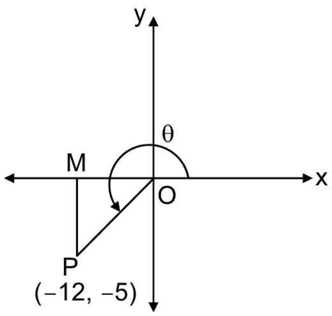
$OP^2 = 5^2 + 12^2 = 13^2$

Giving OP = 13

$\therefore \quad \sin\theta = \frac{-5}{13}, \cos\theta = \frac{-12}{13}$

$$\frac{2\sin\theta - 3\cos\theta}{5\sin\theta + 7\cos\theta} = \frac{2\times\frac{-5}{13} - 3\times\frac{-12}{13}}{5\times\frac{-5}{13} + 7\times\frac{-12}{13}}$$

$$= \frac{-26}{109}$$



Concept Strand 61

For any angle θ prove that the following relations:

(a) $\cos^2\theta\tan^2\theta + \sin^2\theta\cot^2\theta = 1$.

(b) $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

(c) $(\sin\theta + \cos\theta)(\tan\theta + \cot\theta) = \sec\theta + \operatorname{cosec}\theta$

(d) $\frac{\cos\theta}{1+\sin\theta} + \frac{\cos\theta}{1-\sin\theta} = 2\sec\theta$

(e) $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$

Solution

(a) L.H.S $= \cos^2\theta \times \left(\frac{\sin\theta}{\cos\theta}\right)^2 + \sin^2\theta \times \left(\frac{\cos\theta}{\sin\theta}\right)^2$

$$= \cos^2\theta \times \frac{\sin^2\theta}{\cos^2\theta} + \sin^2\theta \times \frac{\cos^2\theta}{\sin^2\theta}$$

$$= \sin^2\theta + \cos^2\theta$$

= R.H.S

(b) L.H.S $= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$

$= 1 + 2\sin\theta\cos\theta$

= R.H.S

(c) L.H.S $= (\sin\theta + \cos\theta)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$

$$= (\sin\theta + \cos\theta)\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)$$

$$= \frac{(\sin\theta + \cos\theta)}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta} + \frac{1}{\cos\theta} = \operatorname{cosec}\theta + \sec\theta$$

= R.H.S.

(d) L.H.S $= \frac{\cos\theta(1-\sin\theta) + \cos\theta(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$

$$= \frac{2\cos\theta}{(1-\sin^2\theta)} = \frac{2\cos\theta}{\cos^2\theta}$$

$= 2\sec\theta$

= R.H.S

(e) L.H.S $= \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$

$$= \frac{1+\cos\theta}{\sqrt{1-\cos\theta}\sqrt{1+\cos\theta}},$$

(on multiplying numerator and denominator by $\sqrt{1+\cos\theta}$)

$$= \frac{1+\cos\theta}{\sqrt{1-\cos^2\theta}} = \frac{1+\cos\theta}{\sqrt{\sin^2\theta}}$$

$$= \frac{1+\cos\theta}{\sin\theta}$$

$= \operatorname{cosec}\theta + \cot\theta$

= R.H.S

Concept Strand 62

Eliminate θ between the relations $x = a\sec\theta$, $y = b\tan\theta$.

Solution

By elimination of θ, we mean, we have to obtain a relation independent of θ by using the two given relations.

We know that $\sec^2\theta - \tan^2\theta = 1$.

$\sec\theta = \dfrac{x}{a}$ and $\tan\theta = \dfrac{y}{b}$

$\therefore \quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \sec^2\theta - \tan^2\theta = 1$

$\therefore \quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is the result of eliminating θ between two given relation.

This relation is called the eliminant.

Concept Strand 63

If $x\sin^2 A + y\cos^2 A = z$, show that $\tan^2 A = \dfrac{y-z}{z-x}$.

Solution

From the given relation, using $\sin^2 A = 1 - \cos^2 A$, we obtain $x(1-\cos^2 A) + y\cos^2 A = z$ giving $\cos^2 A = \dfrac{(z-x)}{(y-x)}$.

Since $\tan^2 A = \sec^2 A - 1$, we obtain, by substitution,

$$\tan^2 A = \left(\frac{y-x}{z-x}\right) - 1 = \left(\frac{y-z}{z-x}\right)$$

FUNDAMENTAL COUNTING PRINCIPLE

A, B, C are three cities and a sales representative has to perform journeys from A to B and then, from B to C in connection with his sales promotion work.

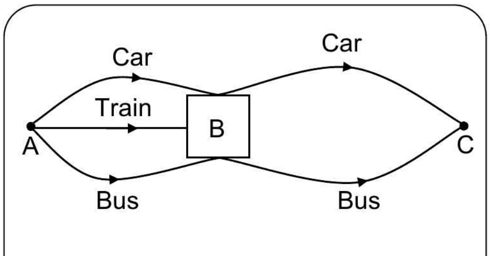


Fig . 1.23

Suppose he can perform the journeys from A to B by train, car or bus and from B to C by car or bus. In other words, he has 3 ways of performing the first leg of his journey (i.e. by car or by bus or by train) and has 2 ways of performing the second leg (i.e., by car or by bus) (refer Fig. 1.23).

It is clear that he can perform these two tasks in succession in 3×2 or 6 ways. This is an illustration of the fundamental counting principle.

If a certain operation can be performed in m different ways and having performed this, a second operation can be performed in n different ways, then the two operations can be performed in succession in mn different ways.

This principle can be generalized for any number of successive operations.

CONCEPT STRANDS

Concept Strand 64

How many different four digit numbers can be formed using the digits 2, 3, 5, 6, 7, 8 and 9 such that no digit occurs more than once in the numbers thus formed?

Solution

×	×	×	×
1000's place	100's place	10's place	unit's place

Forming a four digit number means filling the four places above using the given digits, i.e., we need to perform 4 tasks.

(i) Let us first fill the 1000's place. Since we have 7 digits, any of these digits can be used to fill this place. $\therefore$ We have 7 ways of filling the 1000's place.
(ii) After filling the 1000's place, we are left with 6 digits and therefore we can fill the 100's place in 6 ways.
(iii) Similarly the 10's place can be filled in 5 ways and
(iv) finally, the unit's place can be filled in 4 ways.

By invoking the fundamental counting principle, all the four tasks can be performed in succession in $7 \times 6 \times 5 \times 4 = 840$ ways.

$\Rightarrow$ We can form 840 four digit numbers using the given digits.

Concept Strand 65

In how many ways can 5 prizes be given away to 3 boys when each boy is eligible for one or more prizes?

Solution

Let the prizes be P_1, P_2, P_3, P_4 and P_5. P_1 can be dealt in 3 ways i.e., it can be given away to any of the 3 boys as each boy is eligible for one or more prizes. P_2 and infact each of P_3, P_4, P_5 can be given away in 3 ways.

Now using the fundamental theorem of counting, the 5 prizes can be given away in $3 \times 3 \times 3 \times 3 \times 3 = 3^5 = 243$ ways.

Concept Strand 66

In how many ways can one arrange 5 books on one book shelf?

Solution

The first of the books can be any one of the 5.

Once we place a book in the first place, the second book can be any one of the remaining 4.

Likewise till the fifth book.

$\therefore$ Total number of ways of arranging books $= 5 \times 4 \times 3 \times 2 \times 1 = 120$.

CHAPTER 2

QUADRATIC EQUATIONS AND EXPRESSIONS

CHAPTER OUTLINE

The second-degree polynomial equation $ax^2 + bx + c = 0$, where, $a \neq 0$ represents a quadratic equation.

a, b, c are called the coefficients of the equation. We assume the coefficients to be real numbers i.e., a, b and c $\in$ R.

A root of the quadratic equation is a value of x, which satisfies the equation. A quadratic equation has exactly two roots.

ROOTS OF THE QUADRATIC EQUATION

The two roots of the above quadratic equation are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$. These roots are usually denoted by α and β.

We see that,

$\alpha + \beta$ = sum of the roots $= \dfrac{-b}{a} = \dfrac{-\text{coefficient of } x}{\text{coefficient of } x^2}$

and

$\alpha\,\beta$ = product of the roots $= \dfrac{c}{a} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$.

If α and β are the roots of a quadratic equation,

$(x-\alpha)(x-\beta) = 0$ or $x^2 - (\alpha+\beta)x + \alpha\beta = 0$

or x^2 – (sum of the roots) x + (product of the roots) = 0

Recall that $(x-\alpha)$ and $(x-\beta)$ are called the factors of the quadratic polynomial $ax^2 + bx + c$, if α and β are the roots of the corresponding quadratic equation.

For example, the roots of the quadratic equation $2x^2 - 7x + 5 = 0$ are given by

$$x = \frac{7 \pm \sqrt{49-40}}{4} = \frac{7 \pm 3}{4} = \frac{5}{2} \text{ and } 1$$

Also, for the quadratic equation, $3x^2 - 4x + 7 = 0$,

Sum of the roots $= \dfrac{-(-4)}{3} = \dfrac{4}{3}$; Product of the roots $= \dfrac{7}{3}$.

CONCEPT STRANDS

Concept Strand 1

Find the roots of the quadratic equation $10x^2 - x - 3 = 0$.

Solution

Using the formula for the roots, $x = \dfrac{1 \pm \sqrt{1+120}}{20} = \dfrac{1 \pm 11}{20}$

$\Rightarrow$ Roots are $\dfrac{3}{5}$ and $-\dfrac{1}{2}$.

Alternatively, we may use method of factorization.

$10x^2 - x - 3 = 10x^2 - 6x + 5x - 3$,

(since $10 \times (-3) = -30 = (-6) \times 5$

and $(-6) + 5 = -1$)

$= 2x(5x-3) + 1(5x-3)$

$= (2x+1)(5x-3)$

$\Rightarrow$ Roots are $\dfrac{3}{5}$ and $-\dfrac{1}{2}$.

Concept Strand 2

Find the equation whose roots are $\dfrac{-1}{2}$ and $\dfrac{3}{4}$.

Solution

If α and β are the roots of a quadratic equation, then the quadratic equation is given by

x^2 – (sum of the roots) x + (product of the roots) = 0

$$\Rightarrow \quad x^2 - \left(\frac{-1}{2} + \frac{3}{4}\right)x + \left(\frac{-1}{2} \times \frac{3}{4}\right) = 0$$

$$\text{Or} \quad x^2 - \frac{1}{4}x - \frac{3}{8} = 0$$

$$\Rightarrow \quad 8x^2 - 2x - 3 = 0$$

SYMMETRIC FUNCTIONS

Using the sum and the product of the roots of a given quadratic equation we can compute the values of symmetric functions of the roots like $(\alpha^2 + \beta^2)$, $\left(\frac{1}{\alpha^3} + \frac{1}{\beta^3}\right)$ and so on.

By a symmetric function of the roots α and β we mean an expression in α and β which is unaltered if α and β are interchanged.

For example, $\alpha^4 + \beta^4 + \alpha^2\beta^2 (\alpha + \beta)$ is a symmetric expression in α and β, since, if we interchange α and β, the expression remains the same. On the other hand, the expression $(\alpha^3 + \beta^2)$ is not a symmetric expression in α and β. It becomes $\beta^3+\alpha^2$ if α and β are interchanged.

Symmetric functions of the roots, α, β is expressible in terms of the sum $(\alpha + \beta)$ and product $\alpha\beta$.

The following identities will be useful:

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\,\alpha\beta$

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\,\alpha\beta\,(\alpha + \beta)$

(iii) $\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\,\alpha\beta\,(\alpha - \beta)$

(iv) $$\frac{\alpha+1}{\alpha-1} + \frac{\beta+1}{\beta-1} = \frac{(\alpha+1)(\beta-1) + (\beta+1)(\alpha-1)}{(\alpha-1)(\beta-1)}$$

$$= \frac{2(\alpha\beta - 1)}{1 - (\alpha + \beta) + \alpha\beta}$$

CONCEPT STRANDS

Concept Strand 3

If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, find an expression for $\alpha^2 - \beta^2$ in terms of a, b and c.

Solution

We know that,

$\alpha + \beta$ = sum of the roots = $\frac{-b}{a}$ and

$\alpha\,\beta$ = product of the roots = $\frac{c}{a}$.

Required expression = $\alpha^2 - \beta^2 = (\alpha - \beta)\,(\alpha + \beta)$

$$= \left(\frac{-b}{a}\right)\left(\pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}\right)$$

$$= \pm\frac{b}{a}\sqrt{\left(\frac{-b}{a}\right)^2 - \frac{4c}{a}}$$

Simplifying further, we get,

$$\alpha^2 - \beta^2 = \pm\frac{b}{a^2}\sqrt{b^2 - 4ac}.$$

Concept Strand 4

If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, $c \neq 0$, form the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Solution

We have $\alpha + \beta = \frac{-b}{a}$

and $\alpha\,\beta = \frac{c}{a}$.

Therefore, $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-b}{c}$ and

$$\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \frac{1}{\alpha\beta} = \frac{a}{c}$$

The required equation is x^2 − (sum of the roots) x + product of the roots = 0

i.e., $x^2 + \frac{b}{c}x + \frac{a}{c} = 0$ or $cx^2 + bx + a = 0$.

Concept Strand 5

If α and β are the roots of the equation $5x^2 + 7x - 3 = 0$, form the equation whose roots are α^3 and β^3.

Solution

We have, $\alpha + \beta = \frac{-7}{5}$ and $\alpha\beta = \frac{-3}{5}$.

Sum of the roots of the required equation = $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$$= \left(\frac{-7}{5}\right)^3 - 3\left(\frac{-3}{5}\right)\left(\frac{-7}{5}\right) = -\frac{658}{125}$$

Similarly,

Product of the roots of the required equation $= \alpha^3\beta^3 = (\alpha\beta)^3$

$$= \left(\frac{-3}{5}\right)^3 = -\frac{27}{125}.$$

$\therefore$ The required equation is $x^2 + \frac{658}{125}x - \frac{27}{125} = 0$

or $125x^2 + 658x - 27 = 0$.

NATURE OF ROOTS OF A QUADRATIC EQUATION

In order to study the nature of the roots of a quadratic equation, let us solve the following four equations, using the formula for the roots.

Consider the following examples:

(i) $3x^2 - 14x - 5 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } a = 3, b = -14, c = -5$$

$$x = \frac{14 \pm \sqrt{256}}{6} = 5 \text{ or } -\frac{1}{3}.$$

(ii) $x^2 - 3x - 7 = 0$

Roots are given by $x = \frac{3 + \sqrt{37}}{2}$ and $\frac{3 - \sqrt{37}}{2}$.

(iii) $9x^2 - 6x + 1 = 0$

Roots are given by $x = \frac{1}{3}$ and $\frac{1}{3}$.

(iv) $x^2 + 2x + 2 = 0$

Roots are given by $x = \frac{-2 + \sqrt{-4}}{2}$ and $\frac{-2 - \sqrt{-4}}{2}$.

On an analysis of the nature of the roots of the four equations, it can be seen that the nature of roots of the equations depends on $(b^2 - 4ac)$. This quantity is known as the 'discriminant' of the equation.

In example (i), the discriminant is (positive and) a perfect square the roots are real, distinct and rational.

In example (ii), the discriminant is positive but not a perfect square the roots are real, distinct but irrational.

In example (iii), the discriminant is zero, the roots are real and equal.

In example (iv), the discriminant is negative and we come across a number wherein the square root of a negative number is involved.

Such a number does not exist in the set of real numbers. Therefore, we have to enlarge the real number system to accommodate such numbers of the form 'square root of a negative number'.

We call them 'complex numbers'.

INTRODUCTION TO COMPLEX NUMBERS

A number of the form $x + iy$ where x and y are real numbers (i.e., $x, y \in R$, the set of real numbers) and i stands for $\sqrt{-1}$ (or i is such that $i^2 = -1$) is called a complex number.

If z denotes this complex number $z = x + iy$, x is called the real part of z denoted by $\text{Re}(z)$. y is called the imaginary part of z denoted by $\text{Im}(z)$

Consider the following examples:

(i) $z = 2 + 3i$

Real part = 2

Imaginary part = 3

(ii) $z = -1 - 3\sqrt{2}i$

Real part = -1

Imaginary part = $-3\sqrt{2}$

(iii) $z = 4$

$\text{Re}(z) = 4$

$\text{Im}(z) = 0$

(iv) $z = -5i$

$\text{Re}(z) = 0$

$\text{Im}(z) = -5$

(v) $z = \sqrt{3} - 7i$

$Re(z) = \sqrt{3}$

$Im(z) = -7$

(vi) $z = 0$

Real part = 0 = Imaginary part.

The set of complex numbers is denoted by C.

If $y = 0$, z is real.

If $x = 0$, z is said to be purely imaginary.

This clearly shows that the set of real numbers R is a subset of the set of complex numbers, or $R \subset C$.

Algebra of complex numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ represent two complex numbers.

(i) Equality

$z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$

(ii) Addition

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

(iii) Multiplication by a real number

If k is a real number, $kz_1 = kx_1 + iky_1$

(iv) Subtraction

$z_1 - z_2 = z_1 + (-1)z_2 = (x_1 - x_2) + i(y_1 - y_2)$.

(v) Multiplication of two complex numbers

$z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

The multiplication rule is such that we may treat z_1z_2 as the product of the two factors $(x_1 + iy_1)$ and $(x_2 + iy_2)$. We use the ordinary rule for multiplication of two algebraic expressions and replace i^2 by (-1).

(vi) Complex Conjugate

If $z = x + iy$, the complex conjugate of z, denoted by $\bar{z}$ is defined as $\bar{z} = x - iy$.

For example, if $z = (5 + 7i)$, $\bar{z} = (5 - 7i)$

if $z = -4i$, $\bar{z} = 4i$

if $z = 7$, $\bar{z} = 7 = z$

We observe that $z\bar{z} = (x + iy)(x - iy) = (x^2 + y^2)$, which is real and positive. Also, conjugate of $\bar{z}$ is z. We say that z and $\bar{z}$ constitute a conjugate pair.

We will be having a detailed study of complex numbers and its applications in another module.

Referring to example (iv) under "Nature of roots of a quadratic equation", we note that the two roots of the equation $x^2 + 2x + 2 = 0$ are complex numbers. Roots are given by $x = \frac{-2 + \sqrt{-4}}{2}$ and $\frac{-2 - \sqrt{-4}}{2}$, i.e., $\frac{-2 + 2i}{2}$ and $\frac{-2 - 2i}{2}$ or $(-1 + i)$ and $(-1 - i)$. Also note that the two roots form a conjugate pair.

In a quadratic equation $ax^2 + bx + c = 0$ with real coefficients, if $b^2 - 4ac < 0$, the roots are complex in nature and they occur in conjugate pairs.

To sum up the above observations, if D denotes the discriminant $(b^2 - 4ac)$ of the quadratic equation $ax^2 + bx + c = 0$, [where, a, b, c are rational], then the nature of the roots of the quadratic equation will be as given in the table below:

Table 2.1

Nature of D	Nature of the roots
D positive	real and distinct (different)
D positive and is a perfect square	real, distinct and rational
D positive but is not a perfect square	real, distinct and irrational or, the roots are of the form $p \pm \sqrt{q}$, where p and q are rational
D zero	real and equal
D negative	complex or the roots are of the form $p \pm iq$

If we consider the graph of $y = ax^2 + bx + c$, the real roots of the quadratic equation $ax^2 + bx + c = 0$ are the x-coordinates of the points of intersection of the graph with the x-axis $(y = 0)$.

We can therefore have a graphical illustration of the results given in the above table. The curve $y = f(x) = ax^2 + bx + c$ is shown in the following graphs.

Graphical Illustration

Case (i): D > 0

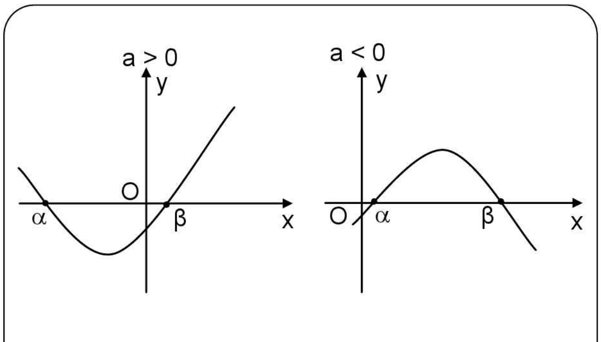


Fig. 2.1

Case (ii): D = 0

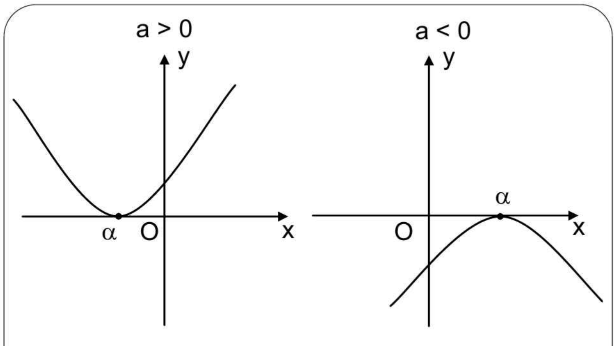


Fig. 2.2

Case (iii): D < 0

There are no real roots for $ax^2 + bx + c = 0$, i.e., there are no x-intercepts for the curve $y = ax^2 + bx + c$.

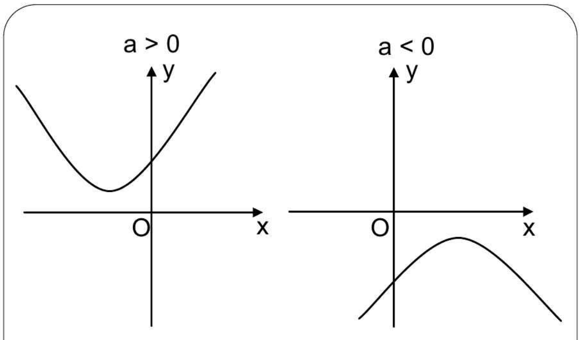


Fig. 2.3

We hasten to add that if the coefficients of a quadratic equation are not rational, and if D is greater than zero but not a perfect square, the roots of the equation will not be of the form $p \pm \sqrt{q}$, where p and q are rational.

Consider the equation $x^2 - 5x + \left(3 + \sqrt{3}\right) = 0$.

Its roots are given by $x = \dfrac{5 \pm \sqrt{25 - \left(12 + 4\sqrt{3}\right)}}{2}$

$$= \frac{5 \pm \left(2\sqrt{3} - 1\right)}{2}$$

$$= 2 + \sqrt{3} \text{ or } 3 - \sqrt{3}$$

Again, if the coefficients of a quadratic equation are not real, then its roots will not be of the form $p \pm iq$, where p, q are real and i stands for $\sqrt{-1}$.

Consider the equation $x^2 - 4x + (1 + 4i) = 0$

It can be verified that i and (4 – i) satisfy the above equation and therefore, the roots are i and (4 – i). However, these do not form a conjugate pair.

COMMON ROOTS

Consider the quadratic equations $x^2 - 5x + 6 = 0$ and $2x^2 + x - 10 = 0$. We observe that x = 2 satisfies both equations. That is, x = 2 is a common root of the two quadratic equations. We say that these two equations have a common root.

Suppose the two quadratic equations $Ax^2 + Bx + C = 0$ and $A'x^2 + B'x + C' = 0$ have a common root, we shall obtain the condition to be satisfied by the coefficients.

Let α be the common root. Then α will satisfy the two equations:

$A\alpha^2 + B\alpha + C = 0$ and

$A'\alpha^2 + B'\alpha + C' = 0$.

Treating these as simultaneous equations in α^2 and α and solving,

$$\alpha^2 = \frac{(BC' - B'C)}{(AB' - A'B)} \quad \alpha = \frac{(CA' - C'A)}{(AB' - A'B)}$$

(assuming $AB' - A'B \neq 0$)

Therefore, the condition that the two equations have a common root is that

$$(CA' - C'A)^2 = (AB' - A'B)\,(BC' - B'C)$$

If the above condition is satisfied, the common root is given by $\dfrac{(CA' - C'A)}{(AB' - A'B)}$.

Suppose two quadratic equations have two common roots.

Let us consider $x^2 - 8x + 15 = 0$ and $3x^2 - 24x + 45 = 0$. It is obvious that the coefficients of the two equations are in proportion. So, two quadratic equations $Ax^2 + Bx + C = 0$ and $A'x^2 + B'x + C' = 0$ have two common roots if, $\dfrac{A}{A'} = \dfrac{B}{B'} = \dfrac{C}{C'} = k$.

If $Ax^2 + Bx + C = 0$ and $A'x^2 + B'x + C' = 0$ have a common root, then

$$(CA' - C'A)^2 = (AB' - A'B)\,(BC' - B'C) \text{ where, } AB' - A'B \neq 0$$

If $Ax^2 + Bx + C = 0$ and $A'x^2 + B'x + C' = 0$ have two common roots, then

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}.$$

So, when there are two common roots, $Ax^2 + Bx + C = k\,(A'x^2 + B'x + C')$.

This means that the two curves $y = Ax^2 + Bx + C$ and $y = A'x^2 + B'x + C'$ have the same intercepts on the x-axis, but represent different loci.

The graphs of $y = x^2 - 8x + 15$ and $y = 3x^2 - 24x + 45$ depict this clearly.

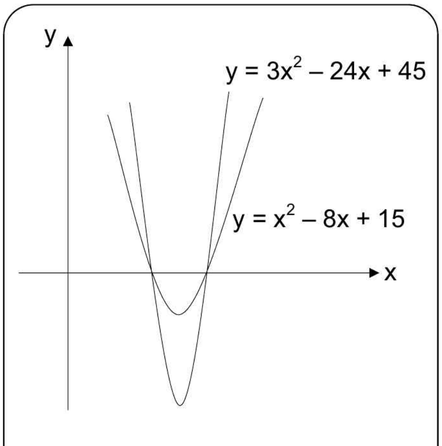


Fig. 2.4

CONCEPT STRAND

Concept Strand 6

Find the value of λ if the two equations $x^2 - 5x + \lambda = 0$ and $2x^2 - 5x - 3 = 0$, have a common root.

Solution

The condition for the two equations to have a common root is

$$(CA' - C'A)^2 = (AB' - A'B)\,(BC' - B'C),$$

i.e., $(2\lambda + 3)^2 = (-5 + 10)\,(15 + 5\lambda)$

$\Rightarrow$ $4\lambda^2 + 12\lambda + 9 = 75 + 25\lambda$

or $4\lambda^2 - 13\lambda - 66 = 0$

$\Rightarrow$ $4\lambda^2 - 24\lambda + 11\lambda - 66 = 0$

Similar to cross multiplication rule:

$$\begin{matrix} -5 & \lambda & 1 & -5 \\ -5 & -3 & 2 & -5 \end{matrix}$$

or $(4\lambda + 11)\,(\lambda - 6) = 0$

$\Rightarrow$ λ can take values $-\dfrac{11}{4}$ or 6.

Note that if $\lambda = -\dfrac{11}{4}$, the common root is $-\dfrac{1}{2}$ and if $\lambda = 6$, the common root is 3.

QUADRATIC EXPRESSIONS

We are interested in discussing the sign of the quadratic expression $(ax^2 + bx + c)$ for real values of x.

Let $Q = ax^2 + bx + c$ (where a, b, c are real). $Q = 0$ represents the corresponding quadratic equation.

Q may be written as $a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right]$.

Case 1:

Let the discriminant $(b^2 - 4ac) \le 0$.

It is clear that the expression inside the square bracket is always non-negative (positive or zero) for all real values of x. Therefore, when $b^2 - 4ac < 0$, the sign of Q for real x is the same as that of a. (Refer Fig. 2.3)

When $b^2 - 4ac = 0$, $Q \ge 0$, if $a > 0$ and $Q \le 0$, if $a < 0$. (Refer Fig. 2.2), in particular $Q = 0$ when $x = \frac{-b}{2a}$.

Case 2:

Let the discriminant $(b^2 - 4ac) > 0$. (Refer Fig. 2.1)

If the roots of $Q = 0$ are denoted by α and β, then α and β are real and distinct, and Q can be expressed as $Q = a(x - \alpha)(x - \beta)$. If $\alpha < \beta$ and we represent the roots on the real line, the sign of Q will be as shown below.

Clearly, when x takes a value beyond α and β (i.e., when $x < \alpha$ or $x > \beta$), the sign of Q is the same as that of a.

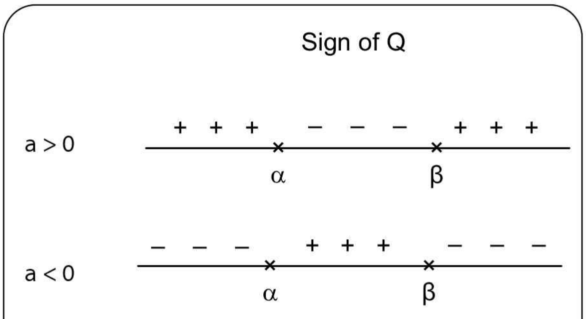


Fig. 2.5

When x takes a value between α and β (i.e., when $\alpha < x < \beta$), the sign of Q is opposite to that of a. (Refer Fig. 2.5)

To sum up, for a quadratic expression $Q = ax^2 + bx + c$

(i) If $b^2 - 4ac < 0$, Q has the same sign as that of a for all real values of x.

(ii) If $b^2 - 4ac = 0$, Q has the same sign as that of a, for all real values of x except $x = \frac{-b}{2a}$.

(iii) If $b^2 - 4ac > 0$, and if α and β are the roots of the corresponding quadratic equation $Q = 0$, Q has the same sign as that of a, for values of x taken beyond the roots α and β and Q has the sign opposite to that of a, for values of x taken between the roots α and β.

CONCEPT STRAND

Concept Strand 7

Find the range of values of x for which the given expressions are positive.

(i) $x^2 - 3x + 7$
(ii) $3x^2 - 7x + 4$
(iii) $-x^2 + 2x - 10$

Solution

(i) Discriminant of $x^2 - 3x + 7$ is $-19 < 0$

Coefficient of x^2 is positive.

$\therefore$ $x^2 - 3x + 7$ is positive for all real values of x.

(ii) Discriminant of $3x^2 - 7x + 4$ is $1 > 0$ and therefore it has zeros 1 and $\frac{4}{3}$.

$\therefore$ $3x^2 - 7x + 4$ is positive for values of x lying beyond 1 and $\frac{4}{3}$ and it is negative for values of x lying between 1 and $\frac{4}{3}$.

(iii) Discriminant of $-x^2 + 2x - 10$ is $-36 < 0$ and the coefficient of x^2 is also negative.

$\therefore$ $-x^2 + 2x - 10$ is always negative for all real values of x.

$\Rightarrow$ $-x^2 + 2x - 10$ is neither zero nor positive for any real value of x.

MAXIMUM AND MINIMUM VALUES OF A QUADRATIC EXPRESSION

Let $Q = ax^2 + bx + c$ (where a, b, c are real) represent a quadratic expression.

Q can be expressed as $a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right]$

Case 1: $(b^2 - 4ac) < 0$

In this case, the expression inside the square bracket is always positive and its minimum value is attained for $x=\frac{-b}{2a}$. The minimum value of the expression inside the square bracket is therefore $-\left(\frac{b^2-4ac}{4a^2}\right)$. Hence, if $a > 0$, Q is minimum at $x=\frac{-b}{2a}$ and the minimum value of Q is $\frac{-(b^2-4ac)}{4a}$ (Refer Fig. 2.3)

If $a < 0$, Q is maximum at $x=\frac{-b}{2a}$ and the maximum value of Q is $\frac{-(b^2-4ac)}{4a}$ (Refer Fig. 2.3)

Case 2: $(b^2 - 4ac) = 0$

If $a > 0$, the minimum value of Q is zero and is attained for $x=\frac{-b}{2a}$ and if $a < 0$, the maximum value of Q is zero and is attained at $x=\frac{-b}{2a}$. (Refer Fig. 2.2)

Case 3: $(b^2 - 4ac) > 0$

Since $\left(x+\frac{b}{2a}\right)^2$ is always non-negative, the expression inside the square bracket above has the minimum value $-\left(\frac{b^2-4ac}{4a^2}\right)$ when $x=\frac{-b}{2a}$.

Therefore, if $a > 0$, Q has the minimum value $\frac{-(b^2-4ac)}{4a}$ when $x=\frac{-b}{2a}$. If $a < 0$, Q has the maximum value $\frac{-(b^2-4ac)}{4a}$ when $x=\frac{-b}{2a}$. (Refer Fig. 2.1).

It is interesting to note that the extreme values (maximum or minimum values) of Q are attained at $x=\frac{-b}{2a}=\frac{1}{2}(\alpha+\beta)$, where α and β are the roots of the corresponding quadratic equation $Q = 0$. And the extreme value is $\frac{-(b^2-4ac)}{4a}=\frac{-D}{4a}$.

CONCEPT STRANDS

Concept Strand 8

Find the minimum value of $3x^2 - x - 8$ and find where it is attained.

Solution

Let $Q = 3x^2 - x - 8$

$$= 3\left[x^2-\frac{1}{3}x-\frac{8}{3}\right] = 3\left[\left(x-\frac{1}{6}\right)^2-\frac{1}{36}-\frac{8}{3}\right]$$

$\Rightarrow$ Q takes the minimum value when the expression inside the square is zero.

$\therefore$ Minimum point of Q is $x=\frac{1}{6}$ and minimum value of $Q=\frac{-97}{12}$.

Concept Strand 9

Find the maximum/minimum values of the following expressions

(i) $2x - 5x^2 + 3$
(ii) $6x - 3 + 7x^2$

Solution

(i) $Q = 2x - 5x^2 + 3$ has a maximum value. (since, coefficient of x^2 is negative)

$$Q = -5\left[x^2 - \frac{2x}{5} - \frac{3}{5}\right] = -5\left[\left(x - \frac{1}{5}\right)^2 - \frac{16}{25}\right]$$

$\therefore$ Maximum value of $Q = \frac{16}{5}$

(ii) $Q = 6x - 3 + 7x^2$ has a minimum value. (since coefficient of x^2 is positive)

$$= 7\left[x^2 + \frac{6x}{7} - \frac{3}{7}\right] = 7\left[\left(x + \frac{3}{7}\right)^2 - \frac{30}{49}\right]$$

$\therefore$ Minimum value of $Q = -\frac{30}{7}$

POLYNOMIAL EQUATION OF DEGREE n

The equation, $a_0x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$ is a polynomial equation in x of degree n. $a_0, a_1, a_2, \ldots\ldots a_n$ are called the coefficients of the equation and are real or complex.

When n = 1, we get a linear equation. $2x + 3 = 0$ is a linear equation.

When n = 2, we get a quadratic equation and when n = 3 we get a cubic equation etc.

Results

(i) Any nth degree polynomial equation has exactly n roots, real or complex.

(ii) Relations between the roots and coefficients.

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ represent the roots of the above polynomial equation, we have

Sum of the roots

$$= \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n = \frac{-a_1}{a_0} = (-1)^1 \frac{a_1}{a_0}$$

Sum of the products of the roots taken two at a time

$$= \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \ldots\ldots = (-1)^2 \frac{a_2}{a_0} = \frac{a_2}{a_0}$$

Sum of the products of the roots taken three at a time

$$= a1\ a2\ a3 + \ldots\ldots = (-1)3 \frac{a_3}{a_0} = -\frac{a_3}{a_0}$$

....

Product of all the n roots $= \alpha_1\alpha_2\ldots\alpha_n = (-1)^n \frac{a_n}{a_0}$

For example if α, β, γ represent the roots of the equation $x^3 - x + 1 = 0$, then $\alpha + \beta + \gamma = 0$; $\alpha\beta + \beta\gamma + \gamma\alpha = -1$; $\alpha\beta\gamma = -1$.

CONCEPT STRAND

Concept Strand 10

If α, β, γ represent the roots of the equation $2x^3 - 5x^2 + x + 9 = 0$, find the value of $\alpha^2 + \beta^2 + \gamma^2$.

Solution

Since α, β, γ are the roots $\alpha + \beta + \gamma = \frac{5}{2}$; $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{1}{2}$;

$\alpha\beta\gamma = -\frac{9}{2}$.

We know that $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$\therefore \quad \alpha^2 + \beta^2 + \gamma^2 = \left(\frac{5}{2}\right)^2 - 2\left(\frac{1}{2}\right)$$

$$= \frac{21}{4}$$

SUMMARY

Symmetric functions of the roots

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta}$

(iv) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$

Nature of the roots

(i) If $b^2 - 4ac > 0$ and is a perfect square, roots are real, distinct and rational.

(ii) If $b^2 - 4ac > 0$ and is not a perfect square, roots are real, distinct and irrational.
For quadratic equations with rational coefficients, irrational roots occur in conjugate pairs.

(iii) If $b^2 - 4ac = 0$, roots are real and equal.

(iv) If $b^2 - 4ac < 0$, roots are complex.
For quadratic equations with real coefficients, complex roots occur in conjugate pairs.

Common roots of two given equations $ax^2 + bx + c = 0$, $a'x^2 + b'x + c' = 0$

(i) Both roots are common if $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

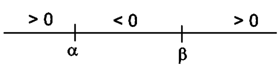
(ii) Condition for one common root : $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$

Quadratic expression: $ax^2 + bx + c$

a > 0

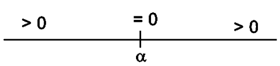
If $b^2 - 4ac > 0$ then the quadratic expression is positive for values of x beyond the roots of the corresponding equation and negative between the roots.

$ax^2 + bx + c$



If $b^2 - 4ac = 0$ quadratic expression is positive except for $x = \dfrac{-b}{2a}$. At $x = \dfrac{-b}{2a}$, the value of the expression is zero.

$ax^2 + bx + c$

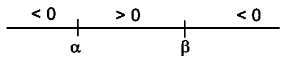


If $b^2 - 4ac < 0$ quadratic expression is positive for all values of x. $\Rightarrow ax^2 + bx + c > 0$ for $x \in R$

a < 0

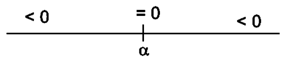
If $b^2 - 4ac > 0$ then the quadratic expression is negative for values of x beyond the roots of the corresponding equation and positive between the roots.

$ax^2 + bx + c$



If $b^2 - 4ac = 0$ quadratic expression is negative except for $x = \dfrac{-b}{2a}$. At $x = \dfrac{-b}{2a}$, the value of the expression is zero.

$ax^2 + bx + c$



If $b^2 - 4ac < 0$ quadratic expression is negative for all values of x. $\Rightarrow ax^2 + bx + c < 0$ for $x \in R$

Maximum and minimum value of the quadratic expression: $ax^2 + bx + c$

(i) If $a > 0$, quadratic expression has minimum value and its minimum value is given by $\dfrac{-(b^2 - 4ac)}{4a}$ at $x = \dfrac{-b}{2a}$

(ii) If $a < 0$, quadratic expression has maximum value and its maximum value is given by $\dfrac{-(b^2 - 4ac)}{4a}$ at $x = \dfrac{-b}{2a}$

General nth degree polynomial equation

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$,

If $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are the roots, of $f(x) = 0$ then

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n = (-1)^1 \frac{a_1}{a_0}$$

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots = (-1)^2 \frac{a_2}{a_0}$$

$$\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 \ldots = (-1)^3 \frac{a_3}{a_0}$$

....

$$\alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

CONCEPT CONNECTORS

Connector 1: Discriminate the nature of the roots of the following equations:

(a) $3x^2 - x + 6 = 0$

(b) $x^2 + 6x - 5 = 0$

(c) $4x^2 - 7x + 3 = 0$

(d) $25x^2 - 20x + 4 = 0$

Solution: (a) Discriminant = 1 – 72 < 0, Roots are complex.

(b) Discriminant = 36 + 20 > 0, not a perfect square: Roots are real, distinct and irrational.

(c) Discriminant = 49 – 48 > 0, Perfect square ⇒ Roots are real, distinct, rational.

(d) Discriminant = 400 – 400 = 0 ⇒ Roots are real and equal.

Connector 2: Prove that the roots of the equation $ax^2 - (a + b + c)x + (b + c) = 0$, $a \neq 0$, $a, b, c \in Q$ are rational.

Solution: Discriminant $= (a + b + c)2 - 4a(b + c)$

$$= (a + b)^2 + c^2 + 2c(a + b) - 4ab - 4ac = (a - b)^2 + c^2 - 2c(a - b)$$

$$= (a - b - c)^2 = \text{a perfect square.}$$

∴ The roots are real, distinct and rational if $a \neq (b + c)$ and the roots are real, equal and rational if $a = b + c$.

Aliter:

Sum of the coefficients of the given equation is zero ⇒ one of the roots is 1 which is rational i.e., the other root is also rational. ($\because$ a, b, c $\in$ Q).

Connector 3: If λ is a root of the equation $4x^2 + 2x - 1 = 0$, prove that the other root is $4\lambda^3 - 3\lambda$.

Solution: Since l is a root of the given equation, $4\lambda^2 + 2\lambda - 1 = 0$.

If β is the other root, $\lambda + \beta = -\dfrac{1}{2}$ or $\beta = -\lambda - \frac{1}{2}$.

Hence we have to establish the result

$4\lambda^3 - 3\lambda = -\lambda - \frac{1}{2}$

Now, $4\lambda^3 - 3\lambda = \lambda(4\lambda^2 + 2\lambda - 1) - \dfrac{1}{2}(4\lambda^2 + 2\lambda - 1) - \lambda - \dfrac{1}{2} = -\lambda - \dfrac{1}{2}$

Result follows.

Connector 4: If α and β are the roots of the equation $2x^2 - x + 5 = 0$, find $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}$.

Solution:

$$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{(\alpha^3 + \beta^3)}{\alpha\beta} = \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{\alpha\beta} = \frac{\left(\frac{1}{2}\right)^3 - 3\left(\frac{5}{2}\right)\left(\frac{1}{2}\right)}{\left(\frac{5}{2}\right)} = -\frac{29}{20}$$

Connector 5: Find real numbers x such that $x^2 + 4|x| - 4 = 0$.

Solution: $x < 0$: The equation is $x^2 - 4x - 4 = 0$ giving $x = 2 - \sqrt{8}$

($x = 2 + \sqrt{8}$ cannot be taken as x is assumed to be < 0)

$x > 0$: The equation is $x^2 + 4x - 4 = 0$ giving $x = -2 + \sqrt{8}$

($x = -2 - \sqrt{8}$ cannot be taken as x is assumed to be > 0)

The two values of x are $2 - \sqrt{8}$ and $-2 + \sqrt{8}$

Connector 6: If α and β are the roots of $2x^2 - 3x - 6 = 0$, form the equation whose roots are $\alpha^2 + 2$ and $\beta^2 + 2$.

Solution: Sum of the roots of the required equation = $a^2 + b^2 + 4$

$$= (\alpha + \beta)^2 - 2\alpha\beta + 4 = \frac{49}{4}.$$

Product of the roots of the required equation = $(\alpha^2 + 2)(\beta^2 + 2)$

$$= \alpha^2\beta^2 + 2(\alpha^2 + \beta^2) + 4 = \frac{59}{2}$$

The required equation is $x^2 - \left(\frac{49}{4}\right)x + \left(\frac{59}{2}\right) = 0$

or $4x^2 - 49x + 118 = 0$

Connector 7: If α and β are the roots of $ax^2 + bx + c = 0$ and α', $-\beta$ are the roots of $a'x^2 + b'x + c' = 0$ show that α, α' are the roots of $\left[\frac{b}{a} + \frac{b'}{a'}\right]^{-1} x^2 + x + \left[\frac{b}{c} + \frac{b'}{c'}\right]^{-1} = 0$

Solution: We have

$$\alpha + \beta = {}^{-b}\!/\!_{a}, \quad \alpha\beta = {}^{c}\!/\!_{a}$$

$$\alpha' - \beta = {}^{-b'}\!/\!_{a'}, \quad -\alpha'\beta = {}^{c'}\!/\!_{a'}$$

giving $\alpha + \alpha' = -\left(\frac{b}{a} + \frac{b'}{a'}\right)$

Also, $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{-b}{c}$, $\frac{-1}{\beta} + \frac{1}{\alpha'} = \frac{-b'}{c'}$.

These two, on addition, give

$$\frac{1}{\alpha} + \frac{1}{\alpha'} = -\left(\frac{b}{c} + \frac{b'}{c'}\right). \text{ Hence } \alpha\alpha' = \frac{\left({}^{b}\!/\!_{a}\right) + \left({}^{b'}\!/\!_{a'}\right)}{\left({}^{b}\!/\!_{c}\right) + \left({}^{b'}\!/\!_{c'}\right)}$$

The required equation is $x^2 + \left[\left(\frac{b}{a}\right) + \left(\frac{b'}{a'}\right)\right]x + \left[\frac{\left(\frac{b}{a}\right) + \left(\frac{b'}{a'}\right)}{\left(\frac{b}{c}\right) + \left(\frac{b'}{c'}\right)}\right] = 0$

$\Rightarrow$ Result follows.

Connector 8: If α and β are the roots of the equation $3x^2 - 2x + 6 = 0$, form the equation whose roots are $\frac{\alpha + 1}{\alpha - 1}, \frac{\beta + 1}{\beta - 1}$.

Solution: Let x represent a root of the given equation, then $y = \frac{x + 1}{x - 1}$ represents a root of the required equation.

We have $x = \frac{y + 1}{y - 1}$.

The required equation is $3\left(\frac{y + 1}{y - 1}\right)^2 - 2\left(\frac{y + 1}{y - 1}\right) + 6 = 0$

or $7y^2 - 6y + 11 = 0$. or $7x^2 - 6x + 11 = 0$

Connector 9: If α and β are the roots of the equation $x^2 + 5x - 5 = 0$, find $\dfrac{1}{(\alpha+1)^3} + \dfrac{1}{(\beta+1)^3}$.

Solution: Let us find the equation whose roots are $\alpha + 1, \beta + 1$.

If y represents a root of the required equation, $y = x + 1$ where, $x = \alpha, \beta$.
The required equation is $(y - 1)^2 + 5(y - 1) - 5 = 0$ or $y^2 + 3y - 9 = 0$

The equation whose roots are u, v, where $u = \dfrac{1}{(\alpha+1)}, v = \dfrac{1}{(\beta+1)}$ is $9x^2 - 3x - 1 = 0$.

We want $u^3 + v^3$.

$$u^3 + v^3 = (u + v)^3 - 3uv\,(u + v) = \left(\frac{3}{9}\right)^3 + \frac{3}{9} \times \frac{3}{9} = \frac{4}{27}$$

Connector 10: The roots of the equation $ax^2 + bx + c = 0$ are in the ratio $l : m$. Prove that $ac\,(l + m)^2 = b^2lm$.

Solution: Given $\dfrac{\alpha}{\beta} = \dfrac{l}{m}$

$$\frac{-b}{a} = \alpha + \beta = \alpha\left(1 + \frac{m}{l}\right)$$

$$\frac{c}{a} = \alpha\beta = \alpha\left(\frac{\alpha\, m}{l}\right)$$

Eliminating α, result follows.

Connector 11: Obtain the set of all values of m for which $mx^2 - 6mx + 5m + 1 > 0$ for all real x.

Solution: For $m = 0$, the expression $= 1 > 0$.

Now, for $m \neq 0$, $m\left(x^2 - 6x + 5 + \dfrac{1}{m}\right) = m\left[(x-3)^2 + \left(\dfrac{1}{m}\right) - 4\right]$

If $m > 0$, the expression is positive if $m < \dfrac{1}{4}$.

If $m < 0$, the expression may assume positive and negative values for real x

Thus, the answer is $0 \le m < \dfrac{1}{4}$.

Connector 12: Show that the expression $\dfrac{(x^2-4)(x^2+3x+2)\,(x^2-x-2)+10}{(x^2+5x+7)} > 0$ for all real values of x.

Solution: Denominator > 0 for all real x, since the discriminant is negative and coefficient of $x^2 > 0$.
Numerator $= (x + 2)^2 (x + 1)^2 (x - 2)^2 + 10 > 0$ for all real x.
Result follows.

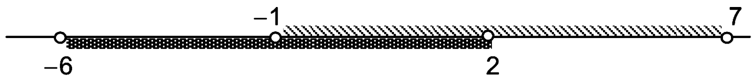
Connector 13: Find the range of values of a satisfying $-3 < \dfrac{(x^2+ax-2)}{(x^2-x+1)} < 2$.

Solution: Since $x^2 - x + 1 > 0$ for all real x, the given set of inequalities may be expressed as $-3(x^2 - x + 1) < x^2 + ax - 2 < 2\,(x^2 - x + 1)$
On simplification, we obtain

$4x^2 + (a - 3)\,x + 1 > 0$	and	$x^2 - (a + 2)\,x + 4 > 0$ for which
$(a - 3)^2 - 16 < 0$	and	$(a + 2)^2 - 16 < 0$

or $-4 < a - 3 < 4$ and $-4 < a + 2 < 4$

$\therefore$ a lies between -1 and 7 as well as between -6 and 2.



The range of values for a is $(-1, 2)$.

Connector 14: Solve the inequality $|x - 1| - |x| + |2x + 3| > 2x + 4$.

Solution: Let R1 represent the region $x < \frac{-3}{2}$; R2 represent the region $\frac{-3}{2} \le x < 0$; R_3 represent the region $0 \le x < 1$ and R_4 represent the region $x \ge 1$.

Solution in R_1

Inequality reduces to $1 - x - (-x) - 2x - 3 > 2x + 4$

This reduces to $4x + 6 < 0$ or $x < \frac{-3}{2}$. Solution is the set $\left(-\infty, \frac{-3}{2}\right)$.

Solution in R_2

$1 - x - (-x) + 2x + 3 > 2x + 4$ or $2x + 4 > 2x + 4$, which is absurd.

No solution in R 2.

Solution in R_3

$1 - x - x + 2x + 3 > 2x + 4$ or $x < 0$, which is a contradiction.

No solution in R 3.

Solution in R_4

$x - 1 - x + 2x + 3 > 2x + 4 \Rightarrow 2 > 4$ which is a contradiction.

$\therefore$ No solution in R4

Combining, the solution is $x \in \left(-\infty, \frac{-3}{2}\right)$.

Connector 15: Find the values of x which satisfy the inequality $\frac{(x - 2)}{(x + 2)} > \frac{(2x - 3)}{(4x - 1)}$.

Solution: We have to solve $\frac{x - 2}{x + 2} - \frac{2x - 3}{4x - 1} > 0$

$$\frac{(x - 2)(4x - 1) - (x + 2)(2x - 3)}{(x + 2)(4x - 1)} > 0 \text{ or } \frac{2(x^2 - 5x + 4)}{(x + 2)(4x - 1)} > 0.$$

Case 1:

Both Numerator and denominator are positive.

x should lie beyond 1 and 4, as $x^2 - 5x + 4 = (x - 4)(x - 1) > 0$ and

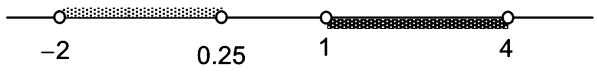
x should lie beyond -2 and $\frac{1}{4}$.

Solution set is $x \in (-\infty, -2) \cup \left(\frac{1}{4}, 1\right) \cup (4, \infty)$.

Case 2:

Both Numerator and denominator are negative.

x should lie between 1 and 4 and x should lie between -2 and $\frac{1}{4}$. Since these 2 sets are disjoint, there is no x satisfying both.



Answer is: $x \in (-\infty, -2) \cup \left(\frac{1}{4}, 1\right) \cup (4, \infty)$

Connector 16: For real values of x, prove that the value of the expression $\frac{(11x^2 + 12x + 6)}{(x^2 + 4x + 2)}$ cannot lie between –5 and 3.

Solution: Let $y = \frac{11x^2 + 12x + 6}{x^2 + 4x + 2}$

$\Rightarrow$ $x^2(y - 11) + (4y - 12)x + 2y - 6 = 0$

Since x is real, $(4y - 12)^2 - 4(y - 11)(2y - 6) \geq 0$

On simplification, we obtain $y^2 + 2y - 15 \geq 0$

$(y + 5)(y - 3) \geq 0$

y should lie beyond -5 and 3.

Connector 17: Show that the second degree expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is expressible as the product of two linear expressions in x and y.

(or)

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ can be written in the form

$(\ell x + my + n)(\ell' x + m'y + n')$, if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Solution: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = ax^2 + 2(hy + g)x + by^2 + 2fy + c$

Right hand side can be considered as a quadratic expression in x.

The second-degree expression in x and y is expressible as the product of linear expressions in x and y, only if the discriminant of the above quadratic expression is a perfect square.

i.e., $[4(hy + g)^2 - 4a(by^2 + 2fy + c)]$ or $[(hy + g)^2 - a(by^2 + 2fy + c)]$ must be a perfect square.

i.e., $(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)$ must be a perfect square – (1)

(1) may be considered as a quadratic expression in y. (1) will be a perfect square only if its discriminant is zero.

This means that $4(gh - af)^2 - 4(h^2 - ab)(g^2 - ac) = 0$

$\Rightarrow$ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, if $a \neq 0$

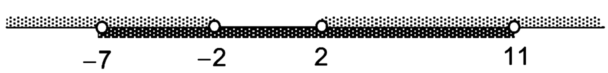
Connector 18: Find the least negative integer satisfying $x^2 - 4x - 77 < 0$ and $x^2 > 4$.

Solution:

$x\ 2 - 4x - 77 < 0$ and $x\ 2 - 4 > 0$

$(x - 11)(x + 7) < 0$ and $(x + 2)(x - 2) > 0$

$\Rightarrow$ $-7 < x < 11$ and $x < -2$ or $x > 2$



Set of values x satisfying both conditions is $-7 < x < -2$ or $2 < x < 11$

$\therefore$ The least negative integer is $x = -6$.

Connector 19: Solve: $\log_2(4^{x+1} + 4)\log_2(4^x + 1) = \log_{1/2}\left(\frac{1}{8}\right)$.

Solution: Equation may be rewritten as

$$\log_2[4(4^x + 1)] \times \log_2[4^x + 1] = \log_{\frac{1}{2}}\left(\frac{1}{8}\right) = 3$$

Let $\log_2(4^x + 1) = y$, we get $(2 + y)\,y = 3 \Rightarrow y^2 + 2y - 3 = 0$

$\Rightarrow$ $y = -3$ or 1

i.e., $\log_2\left(4^x + 1\right) = -3$ or $\log_2\left(4^x + 1\right) = 1$

These give $4^x + 1 = \frac{1}{8}$ or $4^x + 1 = 2$

4^x cannot be negative. The first option is inadmissible.

$\therefore$ $4^x = 1$ or $x = 0$.

Connector 20: Solve for x, y, given $\frac{x^2}{y} + \frac{y^2}{x} = \frac{9}{2}$, $x + y = 3$

Solution: The first equation may be written as $\frac{x^3 + y^3}{xy} = \frac{9}{2}$

$$\Rightarrow \quad \frac{(x + y)^3 - 3xy(x + y)}{xy} = \frac{9}{2}$$

$$\Rightarrow \quad \frac{27 - 9xy}{xy} = \frac{9}{2} \text{ (since } x + y = 3)$$

$\Rightarrow$ $27xy = 54$ or $xy = 2$

Since $x + y = 3$, $xy = 2$, the solutions are $x = 1, y = 2$ or $x = 2, y = 1$

If α, β are roots of a quadratic equation $ax^2 + bx + c = 0$, then $\alpha + \beta = \frac{-b}{a}$, $\alpha\beta = \frac{c}{a}$.

The quadratic equation whose roots are α and β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

TOPIC GRIP

Subjective Questions

1. Determine the nature of roots of the following equations.
   (i) $4x^2 - 6x = 0$
   (ii) $x^2 - 8x + 9 = 0$
   (iii) $x^2 - 22x + 121 = 0$
   (iv) $x^2 - 4x + 5 = 0$
   (v) $x^3 - 7x^2 + 16x - 10 = 0$
2. Obtain the value of k for which the equation $x(x + k) = 12k^2 + 3k - x$, has equal roots.
3. If α and β are the roots of the equation $3x^2 - 4x - 9 = 0$, form the equation whose roots are $\frac{\alpha+3}{\alpha-3}, \frac{\beta+3}{\beta-3}$.
4. If α and β are the roots of the equation $x^2 - 7x + 1 = 0$, obtain the equation whose roots are α^5 and β^5.
5. (i) Solve the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$.
   (ii) Solve the equation $(x - 4)(x - 9)(x - 11)(x - 6) + 16 = 0$.
6. Solve $4^x - 4^{\sqrt{x}+1} = 3 \times 2^{x+\sqrt{x}}$.
7. Solve the equation $2x^2 + |x - 2| - 5 = 0$.
8. For what values of the parameter m, the inequality $\left|\frac{x^2+mx+1}{x^2+x+1}\right| < 3$ is satisfied for all real values of x.
9. Find the interval in which k should lie so that the roots of the equation $2x^2 - kx + 8 = 0$ are between -1 and 4.
10. Find all the integral values of λ for which the quadratic equation $(x - \lambda)(x + 6) + 5 = 0$ has integral roots.

Straight Objective Type Questions

***Directions:* This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

11. If α and β are the roots of the equation $x^2 + 4x - 7 = 0$, the equation whose roots are $\frac{\alpha}{1+\alpha}$ and $\frac{\beta}{1+\beta}$ is
    (a) $2x^2 + 3x + 7 = 0$ (b) $10x^2 - 18x + 7 = 0$ (c) $2x^2 - 3x - 7 = 0$ (d) $10x^2 + 18x - 7 = 0$
12. The ratio of the roots of the equation $x^2 - 7x + 1 = 0$ is the same as the ratio of the roots of the equation $\lambda x^2 + 4x - 5 = 0$. The value of λ equals
    (a) $\frac{16}{245}$ (b) $\frac{7}{45}$ (c) $\frac{-245}{16}$ (d) $\frac{-16}{245}$
13. Number of positive roots of the equation $(x - 1)(x - 2)(x - 3)(x - 4) = 15$ is
    (a) 0 (b) 1 (c) 2 (d) 3
14. If $x^2 + px - 5 = 0$ and $x^2 + qx + 5 = 0$, $p \neq q$ have a common root, the value of $(p^2 - q^2)$ is
    (a) -20 (b) 20 (c) 10 (d) -10

15. The integer k for which the inequality $x^2 - 2(4k - 1)x + 15k^2 - 2k - 7 > 0$ is valid for real x is

(a) 2 (b) 3 (c) 4 (d) 5

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

16. Statement 1

The roots of the equation $x^2 + 7x + 12 = 0$ are both negative.

and

Statement 2

If a, b, c are of the same sign then both the roots of $ax^2 + bx + c = 0$ are negative.

17. Statement 1

Number of real roots of the equation $x^2 - |x| - 2 = 0$ is 2.

and

Statement 2

A quadratic equation $ax^2 + bx + c = 0$ has two and only two roots.

18. Statement 1

If the roots of the quadratic equation $ax^2 + bx + c = 0$ lie between -2 and 2, then both $(4a + 2b + c)$ and $(4a - 2b + c)$ must be positive.

and

Statement 2

If α and β are the real roots of the equation $ax^2 + bx + c = 0$, then $(ax^2 + bx + c)$ will have the same sign as that of a if x is chosen as a number lying beyond α and β.

19. Statement 1

Minimum value of the quadratic expression $(x^2 + x + 4)$ is $\frac{15}{4}$.

and

Statement 2

Minimum or maximum value of the quadratic expression $(ax^2 + bx + c)$ where a, b, c are real occurs at $x = \frac{\alpha + \beta}{2}$ where α and β are the roots of the equation $ax^2 + bx + c = 0$.

20. Statement 1

If $\frac{\log_e (x - e)^2}{7x^2 - 8x + 105} > 0$ then x $[e - 1, e + 1]$.

and

Statement 2

$7x^2 - 8x + 105 > 0$ for all real x.

Linked Comprehension Type Questions

Directions: **This section contains 2 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

If α, β, γ are the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, $\alpha + \beta + \gamma = -\frac{b}{a}$, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$ and $\alpha\,\beta\,\gamma = -\frac{d}{a}$.
Let p, q, r denote the roots of the cubic equation $2x^3 - x + 4 = 0$

21. The value of $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ is

(a) $\frac{-1}{4}$ (b) $\frac{1}{4}$ (c) $\frac{-1}{2}$ (d) $\frac{1}{2}$

22. The value of $p^3 + q^3 + r^3$ is

(a) -12 (b) 12 (c) -6 (d) 6

23. The cubic equation whose roots are $(p + q)(q + r)$, $(q + r)(r + p)$, $(r + p)(p + q)$, is

(a) $x^3 + 2x^2 - x + 4 = 0$ (b) $2x^3 + x^2 - 4 = 0$ (c) $2x^3 + x^2 - 8 = 0$ (d) $2x^3 - x^2 + 8 = 0$

Passage II

Consider the polynomial equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots.. + a_{n-1}x + a_n = 0$$

(a) If the coefficients $a_0, a_1, a_2, \ldots, a_n$ are rational and $(p + \sqrt{q})$ where, p and q are rational, is a root of the equation, then, $(p - \sqrt{q})$ is also a root of the equation.

(b) If the coefficients $a_0, a_1, a_2, \ldots, a_n$ are real, and if $(\alpha + i\beta)$ where α, β are real is a complex root of the equation, then $(\alpha - i\beta)$ is also a root of the equation.

24. Given that $(3 - \sqrt{10})$ is a root of the equation $x^4 - 8x^3 + 16x^2 - 28x - 5 = 0$, the sum of the reciprocals of the squares of the roots of the equation is equal to

(a) $\frac{944}{25}$ (b) $\frac{956}{25}$

(c) $\frac{38}{25}$ (d) $\frac{44}{25}$

25. Given that $(2+i\sqrt{3})$ is a root of the equation $2x^4 - 5x^3 - 3x^2 + 41x - 35 = 0$, the other three roots are such that

(a) one is complex and the other two are rational
(b) all are complex
(c) one is complex and the other two are irrational
(d) one is complex and other two are real and equal

26. Two roots of the equation $x^4 - 6x^3 + 18x^2 - 30x + 25 = 0$ are of the form $(a + ib)$ and $(b + ia)$ where, a and b are real. Then, $(a^3 + b^3)$ equals

(a) 28 (b) 18 (c) 9 (d) -9

Multiple Correct Objective Type Questions

***Directions:* Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

27. The equation $(1 - p^2)x^2 + 2px - 1 = 0$ has both its roots in the interval (0, 2), if

(a) $p < -1$ (b) $-\frac{1}{2} < p < 1$ (c) $p > \frac{3}{2}$ (d) $p > 3$

28. The equations $px^2 + qx + r = 0$ and $qx^2 + rx + p = 0$ $(r \neq 0)$ have a common root, if

(a) $p + q + r = 0$

(b) the equation$(pq - r^2)x^2 + 2(qr - p^2)x + (rp - q^2) = 0$ has equal roots

(c) $p^3 + q^3 + r^3 = pq + qr + rp + pqr$

(d) $\frac{p^3 + q^3 + r^3}{pqr} = 3$

29. If the equation $2(\log_3 x)^2 - |\log_3 x| + k = 0$ has four solutions, if

(a) $k = \frac{1}{16}$ (b) $k < \frac{1}{8}$ (c) $0 < k < \frac{1}{8}$ (d) $k > 0$

Matrix-Match Type Question

***Directions:* Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

30.

Column I	Column II
(a) The equation $2x^2 + (k - 1)x + 2 = 0$ has two real roots if k lies in the interval	(p) $R - \{-2, 0, 2\}$
(b) The equation $kx^2 + 9x - k - 2 = 0$ where, k is real has complex roots if k lies in the interval	(q) $\left(\frac{-1}{2}, 0\right)$
(c) The equation $x^2 + 3x - k^2 = 0$ has irrational roots if integer k belongs to	(r) No real values for k exist
(d) The equation $kx^2 - 2x + 1 + 2k = 0$ has one root positive and the other root negative if k lies in the interval	(s) $(-\infty, -3] \cup [5, \infty)$

IIT ASSIGNMENT EXERCISE

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

31. The roots of the equation $t + \frac{1}{t} = 6$ are

(a) $(2\sqrt{2}, 2)$ (b) $3 \pm 2\sqrt{2}$ (c) $1 \pm 2\sqrt{2}$ (d) $1 + 2\sqrt{3}$

32. Roots of the equation $7^{x+1} + 7^{1-x} = 50$ are

(a) $\{1, 1\}$ (b) $\{1, -2\}$ (c) $\{-1, 2\}$ (d) $\{-1, 1\}$

33. If the roots of $x^2 + (k - 1)x = 2k + 1$ are equal, value of k is

(a) 5, 1 (b) 5, −1 (c) −5, −1 (d) −5, 1

34. The values of 'k' such that the roots of the equation $3x^2 + (k^2 - k - 2)x - 17 = 0$ are equal and opposite in sign are

(a) $\{-1, 2\}$ (b) $\{1, 2\}$ (c) $\{1, -2\}$ (d) $\{-1, -2\}$

35. The number of real roots of the equation $x^3 - \frac{1}{x^3} + 4\left(x - \frac{1}{x}\right) = 0, x \neq 0$ is

(a) 3 (b) 0 (c) 1 (d) 2

36. If one root of the equation $ix^2 - 2(i + 1)x + 2 - i = 0$ is $2 - i$, then the other root is

(a) $-i$ (b) $2 + i$ (c) $2 - i$ (d) i

37. If the roots of the equation $(x - b)(x - c) + (x - c)(x - a) + (x - a)(x - b) = 0$ are equal then

(a) $a + b + c = 0$ (b) $a - b + c = 0$ (c) $a + b - c = 0$ (d) $a = b = c$

38. The roots of the equation $(a + c - b)x^2 + 2cx + (b + c - a) = 0$ (where a, b, c are rational number and $a \neq b$) are

(a) complex (b) distinct and rational (c) real but irrational (d) equal

39. If p and q are the roots of $x^2 + 2px + q - 6 = 0$, the value of p equals

(a) 1, − 2 (b) 1, 2 (c) −1, 2 (d) −1, −2

40. If $pq > 0$ and the roots of $\ell x^2 + mx + m = 0$ are in the ratio p:q then $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} - \sqrt{\frac{m}{\ell}}$ equals

(a) 1 (b) 0 (c) $\sqrt{\frac{\ell}{m}}$ (d) $\sqrt{\frac{m}{\ell}}$

41. If α, β are the roots of the equation $k^2 - 5k + 6 = 0$ then the equation with roots $\alpha + 1$ and $\beta + 1$ is

(a) $2k^2 - 6k + 10 = 0$ (b) $k^2 + 5k + 7 = 0$ (c) $k^2 - 7k + 12 = 0$ (d) $k^2 + 7k - 12 = 0$

42. If α, β are the roots of the equation $x^2 - px + 36 = 0$ and $\alpha^2 + \beta^2 = 9$, then the value of p is

(a) ± 3 (b) ± 6 (c) ± 8 (d) ± 9

43. If α and β are the roots of the equation $x^2 - 6x + 2 = 0$, the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is

(a) $x^2 + 16x + 1 = 0$ (b) $x^2 - 16x - 1 = 0$ (c) $x^2 - 16x + 1 = 0$ (d) $x^2 - 16x - 2 = 0$

44. If α and β are the roots of the equation $x^2 - 3x + 4 = 0$, the equation whose roots are $\alpha^2 + \alpha + 1$ and $\beta^2 + \beta + 1$ is
(a) $x^2 - 6x + 19 = 0$ (b) $x^2 + 6x - 37 = 0$ (c) $x^2 + 6x - 19 = 0$ (d) $x^2 - 6x + 37 = 0$

45. If α_1 and β_1 are the roots of the equation $3x^2 - 2x - 5 = 0$ and α_2 and β_2 are the roots of the equation $2x^2 + x - 7 = 0$, the equation whose roots are $(\alpha_1\alpha_2 + \beta_1\beta_2)$ and $(\alpha_1\beta_2 + \alpha_2\beta_1)$ is
(a) $36x^2 - 12x + 911 = 0$ (b) $9x^2 - 91x + 11 = 0$ (c) $36x^2 + 12x - 911 = 0$ (d) $9x^2 + 91x - 19 = 0$

46. If α and β are the roots of the equation $7\left(\frac{5x}{3x-2}\right) - 3\left(\frac{3x-2}{5x}\right) = 4$, $\frac{1}{\alpha} + \frac{1}{\beta}$ is
(a) $\frac{19}{3}$ (b) $-\frac{19}{22}$ (c) $\frac{25}{3}$ (d) $\frac{19}{22}$

47. If α and β are the roots of the equation $x^2 + 4x - 4 = 0$, $\frac{1}{(\alpha-2)^3} + \frac{1}{(\beta-2)^3}$ is
(a) $\frac{-5}{8}$ (b) -320 (c) 320 (d) $\frac{5}{8}$

48. If α and β are the roots of the equation $3x^2 - 7x - 5 = 0$, the equation whose roots are $\alpha^2 + 2\beta\alpha$ and $\beta^2 + 2\beta\alpha$ is
(a) $9x^2 - 19x - 280 = 0$ (b) $27x^2 - 57x - 415 = 0$ (c) $27x^2 + 57x - 415 = 0$ (d) $9x^2 - 19x + 280 = 0$

49. The number of solutions of the equation $x^2 - 5|x| + 4 = 0$ is
(a) 1 (b) 2 (c) 3 (d) 4

50. The number of real solutions of the equation $5^{1+x} + 5^{1-x} = \log_{10} 25$; $x \in R$ is
(a) 2 (b) 0 (c) 4 (d) 1

51. The equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ has
(a) no solution (b) one solution
(c) two solutions (d) more than two solutions

52. The sum of the roots of the equation $at^2 + bt + c = 0$ is equal to the sum of the squares of their reciprocals. Then
(a) $\frac{b}{c} + \frac{c}{a} = \frac{2a}{b}$ (b) $\frac{a^3}{b^3} = 1$ (c) $ab^2 + bc^2 = 2a$ (d) $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$

53. If $x = 2 + 2^{2/3} + 2^{1/3}$ then the value of $x^3 - 6x^2 + 6x$ is
(a) 1 (b) 2 (c) 3 (d) 4

54. The roots of the equation $(p + 2\sqrt{q})^{x^2-4x+1} + (p - 2\sqrt{q})^{x^2-4x+1} = 2p$ where, $p^2 - 4q = 1$ are
(a) $\pm\sqrt{2}, 2 \pm \sqrt{2}$ (b) $0, 4, 2 \pm \sqrt{2}$ (c) $\pm\sqrt{20}, \pm 2$ (d) $0, \pm 2$

55. The values of x satisfying the equation $8x^{3/2n} - 8x^{-3/2n} = 63$ are
(a) 2^n and $\frac{1}{2^n}$ (b) 2^{3n} and $\frac{1}{2^{3n}}$ (c) 2^{2n} and $\frac{1}{2^{2n}}$ (d) 2^{2n} and $\frac{1}{2^{3n}}$

56. The number of real roots of $(3 - x)^4 + (5 - x)^4 = 16$ is:
(a) 0 (b) 2 (c) 3 (d) None of these

57. Number of solutions of $e^{\cos x} + 7e^{-\cos x} = 6$, $0 < x < \frac{\pi}{2}$ is
(a) 0 (b) 1 (c) 2 (d) ∞

58. Number of real roots of the equation $|x^2 + x - 6| + 2|x| - 4 = 0$ is
(a) 0 (b) 1 (c) 2 (d) 3

59. The number of negative roots of the equation $2x^5 + 7x^3 + 6x^2 + 21 = 0$ is
(a) 5 (b) 3 (c) 1 (d) 0

60. The roots of the equation $(a+\sqrt{b})^{x^2-15} + (a-\sqrt{b})^{x^2-15} = 2a$, where $a^2 - b = 1$ are
(a) $\pm 6, \pm\sqrt{20}$ (b) $\pm 3, \pm\sqrt{5}$ (c) $\pm 4, \pm\sqrt{14}$ (d) $\pm 2, \pm 3$

61. If the equations $k(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6k(2x^2 + 1) + px + 4x^2 - 2 = 0$ have both the roots common, then p is
(a) $\dfrac{3k-1}{3k+1}$ (b) $k(r^2 - 2)$ (c) $2r$ (d) $\dfrac{k}{r}$

62. The values of λ for which the equations $3x^2 - 2\lambda x - 4 = 0$ and $x^2 - 4\lambda x + 2 = 0$ have a common root are
(a) $\dfrac{1}{2}, \dfrac{-1}{2}$ (b) $\dfrac{1}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}$ (c) $\dfrac{1}{\sqrt{5}}, \dfrac{-1}{\sqrt{5}}$ (d) $\dfrac{1}{4}, -\dfrac{1}{4}$

63. If the equations $2x^3 + kx - 4 = 0$ and $6x^4 + 3kx^2 + 2 = 0$ have a common root, k equals
(a) $\dfrac{433}{18}$ (b) $\dfrac{-433}{18}$ (c) $\dfrac{18}{433}$ (d) $\dfrac{-18}{433}$

64. If every pair among the equations $x^2 + px + qr = 0$; $x^2 + qx + rp = 0$ and $x^2 + rx + pq = 0$ has exactly one common root, then the sum of the three common roots is
(a) $p^2 + q^2 + r^2$ (b) 0 (c) $pq + qr + rp$ (d) pqr

65. For $a \neq b$, if the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ have a common root, then the value of $(a + b)$ is
(a) -1 (b) 0 (c) 1 (d) 2

66. If there is a common positive root for the equations $x^2 - x - 6 = 0$ and $ax^2 + 3x + 9 = 0$, the value of a is
(a) -2 (b) 2 (c) 3 (d) -3

67. The solution of the inequality $\dfrac{x-2}{x+3} > \dfrac{2x-5}{4x-1}$ is
(a) $x \in \left(-3, \dfrac{1}{4}\right)$ (b) $x > -3$ (c) $x < -3$ or $x > \dfrac{1}{4}$ (d) $x < \dfrac{1}{4}$

68. The set of values of a for which the quadratic expression $(a + 11)x^2 + 2(a - 3)x - a > 0$ for all real x is given by
(a) $(-11, \infty)$ (b) $(-\infty, -11)$ (c) $(-11, 0)$ (d) no real value

69. The range of values of x for which $2^{2x^2-10x+3} + 6^{x^2-5x+1} \geq 3^{2x^2-10x+3}$ holds good is
(a) $\dfrac{5-\sqrt{21}}{2} \leq x \leq \dfrac{5+\sqrt{21}}{2}$ (b) $\dfrac{\sqrt{5}-\sqrt{21}}{2} < x < \dfrac{\sqrt{5}+\sqrt{21}}{2}$
(c) $\dfrac{\sqrt{5}-21}{2} \leq x \leq \dfrac{\sqrt{5}+21}{2}$ (d) $5-\sqrt{21} \leq x \leq 5+\sqrt{21}$

70. The solution of the inequalities $x^2 + x - 2 > 0$ and $4 - x^2 - 3x > 0$ is
(a) $x > 1$ (b) $x > 1$ or $x < -4$ (c) $-4 < x < 1$ (d) $-4 < x < -2$

71. The values of m for which the expression $(m^2 - 2)x^2 + 2(m + 3)x - 7 < 0$ for all real x are given by
(a) $(-\sqrt{2}, \sqrt{2})$ (b) $\left(\dfrac{-5}{4}, \dfrac{1}{2}\right)$ (c) $\left(\dfrac{1}{2}, \sqrt{2}\right)$ (d) $\left(\dfrac{-5}{4}, -\sqrt{2}\right)$

72. The set of values of x satisfying the inequalities $x^2 + 3x + 10 > 0$; $x^2 - x - 6 \leq 0$ and $2x^2 + 7x - 9 \geq 0$ is
(a) $[-2, 3]$ (b) $(1, 3)$ (c) $(-\infty, 1]$ (d) $[1, 3]$

73. If $y = \dfrac{x+3}{2x^2+x+3}$, $x \in R$, y lies in the interval

(a) $\left[\dfrac{-1}{23}, 1\right]$ (b) $[1, \infty)$ (c) $\left[-\infty, \dfrac{-1}{23}\right] \cup [1, \infty]$ (d) $[-1, 1]$

74. Range of the function $f(x) = \dfrac{x^2-2x+4}{x^2+2x+4}$, $x \in R$ is

(a) $\left[\dfrac{1}{3}, 3\right]$ (b) $\left(\dfrac{1}{3}, 3\right)$ (c) $\left[\dfrac{1}{3}, 3\right)$ (d) $\left(\dfrac{1}{3}, 3\right]$

75. If x is real, the expression $\dfrac{x^2+7x-5}{x-3}$ takes all values which do not lie between p and q. Then p and q are

(a) 3, 5 (b) −3, 3 (c) 3, 23 (d) 3, 6

76. If x satisfies the equation $2^{\sin^2 x} + 5 \times 2^{\cos^2 x} = 7$ where, $-\pi < x < \pi$, $2\sin^2 x - 5\sin x + 4$ equals

(a) 1 (b) −1 (c) 11 (d) (a) or (c)

77. If $x = \alpha$, $y = \beta$ satisfy the equations $2^{x+y} = 6^y$ and $3^{x-1} = 2^{y+1}$, $\alpha + \beta$ equals

(a) $\log\left(\frac{3}{2}\right)$ (b) $\log_{3/2} 6$ (c) $\log_6 3$ (d) $\log_3 6$

78. Sum of the solutions of the equation $5\{x\} = x + 2[x]$ where, [ ] denotes the greatest integer function and { x } denotes the fractional part of x, is

(a) $-\dfrac{7}{4}$ (b) $\dfrac{4}{7}$ (c) $\dfrac{7}{4}$ (d) 0

79. The value of $\sqrt{15-2\sqrt{15-2\sqrt{15-2\sqrt{15 \ldots \infty}}}}$ is

(a) 6 (b) 5 (c) 4 (d) 3

80. If $xy = 28$, $yz = 18$, $zx = 14$ and x, y, z are > 0, the value of $x + y + z$ is equal to

(a) $\dfrac{41}{9}$ (b) $\dfrac{41}{3}$ (c) $\dfrac{21}{4}$ (d) $\dfrac{21}{8}$

81. The roots of the equation $k^4 - 7k^2 - 8 = 0$ are

(a) $(\pm i, \pm\sqrt{2})$ (b) $(\pm 2, \pm 3)$ (c) $(\pm i, \pm 2\sqrt{2})$ (d) $(\pm 1, \pm 2)$

82. The roots of the equation $5^{2x} + (-30)5^x + 125 = 0$ are

(a) {1, 2} (b) {5,10} (c) {5, 26} (d) {4, 5}

83. If $\sin\alpha$ and $\cos\alpha$ are the roots of $px^2 + qx + r = 0$ then

(a) $p^2 - q^2 + 2pr = 0$ (b) $p^2 + q^2 - 2pr = 0$ (c) $(p+r)^2 = q^2 - r^2$ (d) $(p-r)^2 = q^2 - r^2$

84. If the roots of $at^2 + bt + c = 0$ are reciprocals of each other, then,

(a) $a = 0$ (b) $b = 0$ (c) $b = c$ always (d) $a = c$ always

85. The quadratic equation whose roots are $\dfrac{1}{1+\sqrt{3}}$ and $\dfrac{1}{1-\sqrt{3}}$ is

(a) $k^2 + 3k + 1 = 0$ (b) $2k^2 + 2k - 1 = 0$

(c) $k^2 + 6k + 2 = 0$ (d) $3k^2 + 6k + 5 = 0$

86. The polynomial $2x^2 - 5x - 3$ is divisible by

(a) $\left(x+\frac{1}{2}\right)$ (b) $(x + 3)$ (c) $(x + 4)$ (d) $(x - 5)$

87. Roots of the equation $\log_2(n^2 - 4n + 5) = n - 2$ are

(a) {4, 5} (b) {2, –3} (c) {2, 3} (d) {3, 5}

88. The equation $x^2 - 3kx + 2e^{\log_e k^2} - 1 = 0$ has real roots such that the product of roots is 7 if the value of k is

(a) 1 (b) 2 (c) 3 (d) 4

89. The equation formed by multiplying the roots of $ax^2 + bx + c = 0$ by two is $3x^2 + 6x + 4c = 0$, then

(a) $a = b$ (b) $a = -b$ (c) $a = c$ (d) $a = 4c$

90. The equation formed by decreasing each root of $ax^2 + bx + c = 0$ by 1 is $2x^2 + 8x + 2 = 0$. Then,

(a) $a = -b$ (b) $b = -c$ (c) $c = -a$ (d) $b = a + c$

91. The coefficient of x in the quadratic equation $x^2 + px + q = 0$ was taken as 17 in place of 13, then its roots were found to be –2 and –15. The roots of the original equation are

(a) {–10, 3} (b) {–10, –3} (c) {10, –3} (d) {10, 3}

92. If the equation $t^2 - 5t + p = 0$ has roots α and β, then the value of p such that $\alpha = 4\beta$ is

(a) 1 (b) 2 (c) 3 (d) 4

93. If the equation $a^2t^2 - act + a = 0$ has roots α and β, then the value of $(\alpha - \beta)^2$ is

(a) $\frac{3ac + a^2}{2c}$ (b) $\frac{2a + c}{a^2}$ (c) $\frac{4a^2 + c^2}{2a}$ (d) $\frac{c^2 - 4a}{a^2}$

94. If α and β are the roots of the equation $3x^2 - x + 8 = 0$, the equation whose roots are $\frac{1}{\alpha + 2}$ and $\frac{1}{\beta + 2}$ is

(a) $22x^2 - 13x - 3 = 0$ (b) $13x^2 - 22x - 3 = 0$ (c) $22x^2 - 13x + 3 = 0$ (d) $22x^2 + 13x - 3 = 0$

95. If α and β are the roots of $ax^2 + bx + c = 0$, then the value of $\frac{\alpha^3 + \beta^3}{\alpha^2 + \beta^2}$ is

(a) $\frac{3abc - b^3}{a(b^2 - 2ac)}$ (b) $\frac{3abc - b^3}{a^3}$ (c) $\frac{3abc - b^3}{b^2 - 2ac}$ (d) $\frac{3abc - bc^3}{a^2}$

96. If the roots of the equation $ax^2 + bx + c = 0$ are of the form $\frac{\alpha}{\alpha - 1}$ and $\frac{\alpha + 1}{\alpha}$, then the value of $(a + b + c)^2$ is

(a) $(b^2 - 4ac)$ (b) $(b^2 - 2ac)$ (c) $(4b^2 - 2ac)$ (d) $(2b^2 - ac)$

97. The difference between the roots of $(t + 3)^2 + 9(t - 1) + k = 0$ is 5. Then the value of k is

(a) 25 (b) –35 (c) –50 (d) 50

98. The sum of the roots of the equation $ax^2 + b|x| + c = 0$ is

(a) $\frac{-b}{a}$ (b) 0 (c) $\frac{-2b}{a}$ (d) None of these

99. The number of real roots of the equation $(x - 1)^2 + (x - 2)^2 + (x - 3)^2 = 0$ is

(a) 2 (b) 1 (c) 0 (d) 3

100. The number of real solutions of $x - \frac{1}{x^2 - 25} = 5 - \frac{1}{x^2 - 25}$ is

(a) 0 (b) 2 (c) 1 (d) ∞

101. The condition that the roots of $px^2 - px + q = 0$ are in the ratio p : q is

(a) $p + q = 0$ (b) $2p - q = 0$ (c) $2p + q = 0$ (d) $p - q = 0$

102. If $2 + i\sqrt{3}$ is a root of $x^2 + px + q = 0$, where p, q are real then (p, q) is

(a) (2, 3) (b) $(-2, \sqrt{3})$ (c) (−4, 7) (d) (4, 3)

103. If the roots of the equation $(n - p)(n - q) = r$ are ℓ and m, then the roots of the equation $(n - \ell)(n - m) + r = 0$ are

(a) ℓ and m (b) p and q (c) (p − r) and (q − r) (d) $(\ell - r)$ and (m − r)

104. If the roots of the equation $\frac{a}{x - a} + \frac{b}{x - b} = 1$ are equal in magnitude but opposite in sign then

(a) $a - b = 0$ (b) $a + b = 1$ (c) $a - b = 1$ (d) $a + b = 0$

105. The condition for the equations $a_1t^2 + b_1t + c_1 = 0$ and $a_2t^2 + b_2t + c_2 = 0$ to have a common root is

(a) $a_1c_2 - a_2c_1 = b_1b_2$ (b) $a_1a_2 = b_1b_2 - c_1c_2$

(c) $(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$ (d) $(b_1c_2 - b_2c_1)^2 = 2(a_1b_2 + a_2b_1)$

106. The value of the parameter 'a' for which the equations $(1 - 2a)x^2 - 6ax - 1 = 0$ and $ax^2 - x + 1 = 0$ have one root in common is

(a) $\left\{\frac{1}{2}, \frac{2}{9}\right\}$ (b) $\left\{0, -\frac{3}{4}, \frac{2}{9}\right\}$ (c) $\left\{\frac{2}{9}\right\}$ (d) $\left\{\frac{1}{2}, 0\right\}$

107. The value of $\frac{5}{9x^2 + 6x + 11}$ is maximum when x is equal to

(a) 0 (b) $\frac{-1}{2}$ (c) $\frac{-1}{3}$ (d) $\frac{1}{3}$

108. The value of $x^2 + 2bx + c$ is positive for all real values of x, if

(a) $b^2 - 4c > 0$ (b) $b^2 - 4c < 0$ (c) $c^2 < b$ (d) $b^2 < c$

109. Let $f(x) = x^2 - 5x + 6$. Then, which of the following assertions is true?

(a) $f(x) > 0$ for all x (b) $f(x) > 0$, when $2 < x < 5$

(c) $f(x) > 6$, when $x > 0$ (d) $f(x) < 6$, when $0 < x < 5$

110. The values of x^2 satisfying the equation $x^4 - 5x^2 + 6 = 0$ are

(a) real and distinct (b) both positive (c) imaginary (d) both (a) and (b)

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

111. **Statement 1**

The expression $Q = -x^2 + 4x - 9$ is always negative for all real values of x.

and

Statement 2

The expression $(ax^2 + bx + c)$ is positive for all real values of x if $(b^2 - 4ac)$ is negative.

112. **Statement 1**

Roots of the equation $4x^3 + 7x^2 + 7x + 4 = 0$ are -1 and $\dfrac{3 \pm \sqrt{55}}{8}$.

and

Statement 2

$x = -1$ is a root of the cubic equation $ax^3 + bx^2 + bx + a = 0$.

113. Consider the quadratic expression $Q = -x^2 + 8x + 7$.

Statement 1

Both roots of the equation $Q - k = 0$ are complex if $k > 23$.

and

Statement 2

Maximum value of Q is 23.

Linked Comprehension Type Questions

Directions: **This section contains 1 paragraph. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Let α and β denote the roots of the equation $x^2 - 10x + 6 = 0$
Let S_n where n a positive integer stand for $\alpha^n + \beta^n$

114. $S_{n+2} =$

(a) $10S_{n+1} - 6S_n$ (b) $10S_{n+1} + 6S_n$ (c) $6S_{n+1} - 10S_n$ (d) $6S_{n+1} + 10S_n$

115. $S_4 =$

(a) 8848 (b) 9482 (c) 8200 (d) 7672

116. $S_4 - S_2S_3 =$

(a) 72792 (b) −72729 (c) −64488 (d) 64488

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

117. α and β are the roots of the equation $x^2 - 3x - 16 = 0$. Then

(a) $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ are the roots of the equation $16x^2 - 41x + 16 = 0$

(b) $\dfrac{\alpha+2}{\alpha-2}, \dfrac{\beta+2}{\beta-2}$ are the roots of the equation $9x^2 - 20x + 3 = 0$

(c) minimum value of the expression $(x^2 - 3x - 16)$ is $\dfrac{-73}{4}$

(d) The roots of the equation $(3x + 2)^2 - 3(3x + 2) - 16 = 0$ are $\dfrac{\alpha-2}{3}$ and $\dfrac{\beta-2}{3}$

118. Solution set of $\log_{|\sin x|}(x^2 - 8x + 23) > \dfrac{3}{\log_2 |\sin x|}$ is contained in

(a) $(3, 5)$ (b) $(-3, 3) - \left\{\pm\dfrac{\pi}{2}, 0\right\}$ (c) $(3, 5) - \left\{\pi, \dfrac{3\pi}{2}\right\}$ (d) $(-5, 8)$

119. For all real values of λ, the equation $ax^2 + (b - \lambda)x + (a - b - \lambda) = 0$ where, $a \neq 0$ has real roots. Then, which of the following are true?

(a) $a = b$ (b) $b < a < 0$ (c) $b > a > 0$ (d) $a > b > 0$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

120.

	Column I		Column II
(a)	Value of k for which $25^x - k \times 5^x - k + 3 \leq 0$ for at least one real value of x	(p)	8
(b)	Value of k for which $4x^2 - 4kx + 1 = 0$ has one root less than $\frac{1}{3}$ and the other root greater than $\frac{1}{3}$, is	(q)	-1
(c)	Value of k for which $x^3 + 2x^2 + 4x + k = 0$ has two of its roots equal in magnitude but opposite in sign	(r)	2.5
(d)	If α, β are the roots of the equation $Q = x^2 - (k - 2)x - k - 1 = 0$ and m denotes the minimum value of Q, value of k for which $\alpha^2 + \beta^2 + 4m = 0$ is	(s)	3

ADDITIONAL PRACTICE EXERCISE

Subjective Questions

121. α and β are the roots of the equation $ax^2 + 2bx + 3c = 0$. Prove that the relation satisfied by a, b and c if $\beta = \alpha^2$ is $3ac(a + 3c) = 18abc - 8b^3$

122. Prove that $(B^2 - 4AC) / A^2$ is invariant for the set of quadratic equations $Ax^2 + Bx + C = 0$ (A, B, C assuming different values) for which the difference between the two roots is a constant.

123. If the roots of the quadratic equation $\dfrac{a}{x + a^2 - a} + \dfrac{b}{x + b^2 - b} = 1$ are equal in magnitude but opposite in sign, prove that $(a^2 + b^2) = 2(a + b)$.

124. If the ratio of the roots of the equation $ax^2 + bx + c = 0$ is the same as that of the roots of the equation $a_1x^2 + b_1x + c_1 = 0$ prove that $\dfrac{b^2}{ac} = \dfrac{b_1^2}{a_1c_1}$.

125. The coefficients of the quadratic equations $ax^2 + bx + c = 0$ and $a_1x^2 + b_1x + c_1 = 0$ are rational. If the above equations have one and only one root in common, prove that both $(b^2 - 4ac)$ and $(b_1^2 - 4a_1c_1)$ must be perfect squares.

126. If $px + qy = 1$ and $rx^2 + sy^2 = 1$ have only one solution, prove that $\dfrac{p^2}{r} + \dfrac{q^2}{s} = 1$ and $x = \dfrac{p}{r},\ y = \dfrac{q}{s}$.

127. If there is a common root for the equations $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ and this common root is not equal to zero, prove that the other roots satisfy the equation $x^2 + ax + bc = 0$

128. If $k > 1$, show that the expression $\dfrac{y^2 - 2y + k^2}{y^2 + 2y + k^2}$ for all $y \in R$ lies between $\left(\dfrac{k-1}{k+1}\right)$ and $\left(\dfrac{k+1}{k-1}\right)$.

129. Prove that $\left|\dfrac{12x}{4x^2 + 9}\right| \le 1$ for $x \in R$ and equality holds if $|x| = \dfrac{3}{2}$.

130. The equation $x^2 + 12xy + 4y^2 + 4x + 8y + 20 = 0$ has real solutions (x, y). Prove that x should lie beyond –2 and 1, while y should lie beyond – 1 and $\dfrac{1}{2}$.

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

131. Solution of the equation $\log_5(5^{1/x} + 125) = \log_5 6 + 1 + \dfrac{1}{2x}$ is

(a) $\dfrac{-1}{4}, \dfrac{1}{2}$ (b) $\dfrac{1}{4}, \dfrac{1}{2}$ (c) $\dfrac{-1}{2}, \dfrac{1}{6}$ (d) $2, \dfrac{-1}{2}$

132. Number of real solutions of the system of equations $yx^{\log_y x} = x^{2.5}$; $\log_3 y \log_y (y - 2x) = 1$ is

(a) 0 (b) 1 (c) 2 (d) (–3, 5)

133. The integral values of 'a' for which the roots of the equation $ax^2 + (2a - 1)x + (a - 2) = 0$ are rational, is

(a) $2n + 1, n \in N$ (b) $n(n + 1), n \in N$ (c) $n(n + 3), n \in N$ (d) $(2n + 1), n \in N$

134. Number of quadratic equations which are unchanged by squaring their roots,

(a) 0 (b) 1 (c) 3 (d) 4

135. The solution set of the inequality $\dfrac{9}{|x - 5| - 3} \geq |x - 2|$,

(a) $[-1, 2)$ (b) $(-1, 1)$ (c) $\left(8, 5 + 3\sqrt{2}\right]$ (d) $[-1-2) \cup (8, 5 + 3\sqrt{2}]$

136. The range of values of x for which $\log_{\frac{x+4}{2}}\left\{\log_2\left(\dfrac{3x - 1}{3 + x}\right)\right\} < 0$, is

(a) $(-4, 2)$ (b) $(-1, 1)$ (c) $(0, 2) \cup (2, 7)$ (d) $(-4, -3) \cup (2, 7)$

137. The equation $\sqrt{2x - 3} - \sqrt{5x - 6} + \sqrt{3x - 5} = 0$ has

(a) one solution (b) two solutions (c) four solutions (d) no solution

138. The equation $3^{x-1} + 5^{x-1} = 34$ has

(a) no solution (b) one solution (c) two solutions (d) three solutions

139. If $3\cos 2\theta + 4\sin 2\theta = k$ has θ_1 and θ_2 as solutions, $\tan\theta_1 + \tan\theta_2$ equals

(a) $\dfrac{8}{(k + 3)}$ (b) $\dfrac{k - 3}{k + 3}$ (c) $\dfrac{-8}{k + 3}$ (d) $\dfrac{4}{(k + 3)}$

140. When a > 1, the solutions of $\left(a + \sqrt{a^2 - 1}\right)^{x^2 - 2x} + \left(a - \sqrt{a^2 - 1}\right)^{x^2 - 2x} = 2a$ are

(a) independent of a (b) all negative (c) all equal (d) all positive

141. If the equation $(3x)^2 + (27 \times 3^{1/p} - 15)x + 4 = 0$ has equal roots then p is

(a) 0 (b) 2 (c) $-\dfrac{1}{2}$ (d) 1

142. If the equation $x^2 - 15 - m(2x - 8) = 0$ has equal roots then the value of m can be

(a) 15 or 8 (b) 0 or 2 (c) 4 or 8 (d) 5 or 3

143. The roots of the equation $(1 - a^2)(x + a) - 2a(1 - x^2) = 0$ are

(a) $1 - a^2$ and $1 + a^2$ (b) $2a$ and $\dfrac{-1 + a^2}{a}$

(c) a and $\dfrac{-1 + a^2}{2a}$ (d) $a, \dfrac{-(1 + a^2)}{2a}$

144. If $a \in Z$ and the equation $(x - a)(x - 10) + 1 = 0$ has integral roots, then the values of a are

(a) 10, 8 (b) 12, 10 (c) 12, 8 (d) 9, 11

145. The number of roots of the equation $(x - 1)^2 - 5|x - 1| + 6 = 0$ is

(a) 2 (b) 3 (c) 4 (d) 1

146. If α and β are the roots of the equation $2x^2 - 6x + 1 = 0$, the equation whose roots are $\frac{1}{\alpha - 3}, \frac{1}{\beta - 3}$ is

(a) $2x^2 + 6x + 1 = 0$ (b) $x^2 - 6x - 2 = 0$ (c) $x^2 + 6x + 2 = 0$ (d) $x^2 - 6x + 2 = 0$

147. The set of values of x satisfying the inequality $|x - 1| + |x - 2| \geq 6$ is

(a) $\left(-\infty, \frac{-3}{2}\right] \cup \left[\frac{9}{2}, \infty\right)$ (b) $\left(\frac{-3}{2}, \frac{9}{2}\right)$ (c) $\left(-\infty, \frac{3}{2}\right]$ (d) $(1, 2)$

148. If one root of the equation $ax^2 + bx + c = 0$ be the square of the other, then

(a) $a^3 + bc(b + c) = 3abc$ (b) $b^3 + ac\,(a + c) = 3abc$ (c) $c^3 + ab\,(a + b) = 3abc$ (d) $b^3 - ac(a + c) = 2\,abc$

149. The values of k for which both the roots of the equation $x^2 - 2kx + k^2 - 2k + 6 = 0$ are greater than 2 are given by

(a) $k \geq 3$ (b) $2 < k \leq 3$ (c) $k > 2$ (d) $k \in R$

150. If α and β are the roots of the equation $p^2x^2 + (px + 2)(x + p) + 2 = 0$, the value of $\alpha^2\beta^2 + (\alpha\beta + 2)(\alpha + \beta) - 1$ is

(a) 3 (b) 2 (c) -1 (d) -3

151. Roots of the equation $(b + c - 2a)x^2 + 2(c + a - 2b)x + (a + b - 2c) = 0$ where a, b, c are not equal are

(a) real and distinct (b) real and equal (c) imaginary (d) cannot be determined

152. The sum of the positive solutions of the equation $x^{x\sqrt{x}} = (x\sqrt{x})^x$ is

(a) $\frac{3}{4}$ (b) $\frac{9}{4}$ (c) $\frac{13}{4}$ (d) $\frac{11}{4}$

153. If the difference between the roots of the equation $2x^2 + 3ax + 2b = 0$ is equal to the difference between the roots of the equation $2x^2 + 3bx + 2a = 0$, where $a \neq b$, then $a + b =$

(a) $\frac{-4}{9}$ (b) $\frac{16}{3}$ (c) $\frac{-16}{9}$ (d) $\frac{-8}{9}$

154. All values of the parameter a, for which the inequality $a.9^x + 4(a-1)3^x + a > 1$ is satisfied for all real values of x is

(a) $(-\infty, 1)$ (b) $(-1, 1)$ (c) $[1, \infty)$ (d) $(-4, \infty)$

155. The real roots of the equation $7^{\log_7(x^2-4x+5)} = (x - 1)$ are

(a) 1 and 2 (b) 2 and 3 (c) 3 and 4 (d) 4 and 5

156. One root of $k^2 - 9k + 14 = 0$ exceeds the other root by a certain number, the number is

(a) 2 (b) 4 (c) 7 (d) 5

157. If α and β are the roots of the equation $x^2 + x + 1 = 0$, then the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$ is

(a) 2 (b) 1 (c) -1 (d) 0

158. Roots of the equation $|n^2 - n - 6| = n + 2$ are

(a) $(-2, 1, 4)$ (b) $(0, 2, 4)$ (c) $(0, 1, 4)$ (d) $(-2, 2, 4)$

159. If α and β are the roots of the equation $x^2 - x - 1 = 0$, then the equation whose roots are $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$ and $(\alpha^3\beta^2 + \beta^3\alpha^2)$ is

(a) $x^2 + x + 1 = 0$ (b) $x^2 - 11x + 10 = 0$ (c) $x^2 + x - 10 = 0$ (d) $x^2 + 10x + 1 = 0$

160. If α and β are the roots of the equation $8x^2 - 3x + 27 = 0$ then the value of $\left(\frac{\alpha^2}{\beta}\right)^{1/3} + \left(\frac{\beta^2}{\alpha}\right)^{1/3}$ is

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) $\frac{7}{2}$ (d) 4

161. If the ratio of roots of the equation $ax^2 + 2bx + c = 0$ is the same as the ratio of the roots of $px^2 + 2qx + r = 0$, then

(a) $\frac{b}{ac} = \frac{q}{pr}$ (b) $\frac{b^2}{ac} = \frac{q^2}{pr}$ (c) $\frac{2b}{ac} = \frac{q^2}{pr}$ (d) $\frac{b}{ac} = \frac{q^2}{pr}$

162. The number of quadratic equations that can be formed by taking coefficients from different numbers {2, 3, 6, 7}, such that they all have real roots is

(a) 2 (b) 8 (c) 6 (d) 4

163. a and b are rational numbers such that $a < 1$ and $b^2 + 4$ is not a perfect square. If the equations $x^2 - 2x + a = 0$ and $x^2 + bx - 1 = 0$ have at least one common root, then the point (a, b) lies on

(a) $y = 3x + 4$ (b) $x + y + 3 = 0$ (c) $y = x$ (d) $y = -x$

164. The roots of the equation $a^3x^3 + 3x^2 - cx + 8 = 0$, $a \neq 0$ are such that the square of one root is equal to the product of the other two. Then

(a) $a = c$ (b) $2a^2 + 3a + c = 0$ (c) $ac + 6 = 0$ (d) $a^3 + a^2 - ac + 8 = 0$

165. The sum of the squares of the roots of the equation $x^2 - (\sin\alpha - 2)x - (1 + \sin\alpha) = 0$ is maximum when α is

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{2}$ (d) $\frac{\pi}{6}$

166. The number of points of intersection of the curves $y = 7x^2 - 4x + 5$ and $y = 3 \sin x$ is

(a) 4 (b) 2 (c) 1 (d) 0

167. The set of values of x satisfying the inequalities $x^2 + x - 2 \geq 0$; $2x^2 - 9x - 5 \leq 0$ and $x^2 - 5x + 6 \geq 0$ is given by

(a) $x \leq -2$ or $x \geq 4$ (b) $-2 \leq x \leq 1$ or $x \geq 5$ (c) $1 \leq x \leq 2$ or $3 \leq x \leq 5$ (d) $\frac{-1}{2} \leq x \leq 5$

168. The number of real roots of the equation $3x^5 + 10x^3 + 30x + 7 = 0$ is

(a) 1 (b) 0 (c) 3 (d) 5

169. One root of the equation $x^2 - (\lambda + 1)x + \lambda^2 + \lambda - 8 = 0$ is greater than 2 and the other is less than 2. Then λ lies between

(a) -2 and 3 (b) 3 and 5 (c) 0 and 1 (d) 1 and 2

170. The equation $\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1$ has

(a) no solution (b) one solution

(c) two solutions (d) infinitely many solutions.

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

171. Statement 1

The domain of the function $y = \sqrt{x^2 - 4x - 5}$ is $(-\infty, -1] \cup [5, \infty)$.

and

Statement 2

The graph of $y = ax^2 + bx + c$ never crosses the x-axis if $b^2 - 4ac$ is negative.

172. Statement 1

The roots of the equation $6x^2 - 5x + 1 = 0$ are both less than 1.

and

Statement 2

The roots of an equation $ax^2 + bx + c = 0$ lie in the interval (0, 1) if $ac + bc + c^2 > 0$.

173. Statement 1

Quadratic expression $x^2 - 6x - 7 < 0$ when $-1 \le x \le 7$.

and

Statement 2

If $b^2 - 4ac < 0$, then the quadratic expression $ax^2 + bx + c$ has the same sign as that of a for any real value of x.

174. Statement 1

Roots of the quadratic equation $x^2 - 4x + 7 = 0$ are complex.

and

Statement 2

If the discriminant of a quadratic equation with real coefficients is less than zero, then its roots are complex.

175. Statement 1

If $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$ have a common factor, then a is 0 or 24.

and

Statement 2

If two equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have a common root then
$(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (a_1c_2 - a_2c_1)^2$

176. Statement 1

The number of pairs (x, y) where x, y are real and satisfying
$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$ is 2

and

Statement 2

The quadratic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is resolvable into two linear factors if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and $h^2 \ge ab$

177. Statement 1

The roots of $x^2 + x + 1 = 0$ are complex.

and

Statement 2

The roots of $ax^2 + bx + c = 0$ are complex if $ac > 0$ and b lies between $-2\sqrt{ac}$ and $2\sqrt{ac}$.

178. Statement 1.

The root of the equation $3 \times 2^{2x} - 2^x - 4 = 0$ is $\log_2\left(\frac{4}{3}\right)$

and

Statement 2

If a, b, c are rational and $b = a + c$, then the quadratic equation $ax^2 + bx + c = 0$ has rational roots.

179. Statement 1

The roots of the cubic equation $2x^3 + 5x^2 + 5x + 2 = 0$ are $-1, \frac{-3}{4} \pm \frac{i\sqrt{7}}{4}$

and

Statement 2

If α is a root of the cubic equation $ax^3 + bx^2 + bx + a = 0$, $\frac{1}{\alpha}$ is also a root of the equation

180. Statement 1

A polynomial equation of the lowest degree with rational coefficients having $2 + \sqrt{5}$ as a root is of degree 2.

and

Statement 2

In a polynomial equation with rational coefficients, irrational roots occur in conjugate pairs.

Linked Comprehension Type Questions

***Directions:* This section contains 3 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

Let $a_0 \neq 0, a_1, a_2, \ldots\ldots a_n$ be real. Then the equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n = 0$ has n roots, equal or unequal real or non-real. We have the following relations connecting the roots and the coefficients.

$$\sum \alpha_i = -\frac{a_1}{a_0} = -\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}$$

$$\sum \alpha_i\alpha_j = (-1)^2\frac{a_2}{a_0} = \frac{\text{coefficient of } x^{n-2}}{\text{coefficient of } x^n}$$

....

$$\sum \alpha_i\alpha_j\ldots\alpha_k = \text{sum of the products of the roots taken r at a time}$$

$$= (-1)^r\frac{a_r}{a_0} = (-1)^r\frac{\text{coefficient of } x^{n-r}}{\text{coefficient of } x^n}$$

....

$$\alpha_1\alpha_2\ldots\alpha_n = \text{product of the roots} = (-1)^n\frac{a_n}{a_0} = (-1)^n\frac{\text{constant term}}{\text{coefficient of } x^n}$$

181. For the cubic equation $ax^3 + bx^2 + cx + d = 0$ the sum of the squares of the roots is

(a) $\frac{b^2}{a^2}$ (b) $\frac{b^2 - 2ac}{a^2}$ (c) $\frac{b^2 + 2ac}{a^2}$ (d) $(a + b + c)^2$

182. If α, β, γ are the roots of $x^3 - 4x^2 + x + 9 = 0$, then

(a) α, β, γ are all integers
(b) all of α, β, γ are not integers
(c) one of the roots = 0
(d) None of the above

183. The cubic equation whose roots are $\sqrt{5}, -\sqrt{5}, \frac{7}{2}$ is

(a) $2x^3 + 7x^2 + 10x + 35 = 0$
(b) $2x^3 - 7x^2 - 10x + 35 = 0$
(c) $x^3 - 7x^2 - 10x + 35 = 0$
(d) $2x^3 - 7x^2 + 10x - 35 = 0$

Passage II

A nursing home is interested in studying the temperature in °F when a patient is administered a medicine. A nursing home is interested in studying the temperature in °F when a patient is administered a medicine. The equation $y = -8x^2 + 64x - 120$, $x > 0$ gives the relationship between

y: the excess of temperature above the normal body temperature 98. 4°F measured in °F and x: the time in hours

184. When is the temperature at its peak? What is its value then?
(a) 4 hr, 106.4° F (b) 3 hr, 102.4° F (c) 2 hr, 104.4°F (d) 1 hr, 98°F

185. When is the temperature normal?
(a) 3 hr from the start (b) 3 hr and 5 hrs from the start
(c) 5 hr from the start (d) 4 hr after start

186. The temperature between 3 hr and 4 hs is
(a) decreasing (b) increasing
(c) stationary (d) partly decreasing, partly increasing

Passage III

A recycling centre recorded the oil in tonnes remaining in an oil tank after they opened the valve to drain it. They calculated that the quadratic function $y = 3x^2 - 28x + 60$ where, x is the hours elapsed and y the tonnes of remaining oil, approximated the above.

187. Suppose the valve was opened at 8 am, when will the tank have 28 tonnes of oil remaining?
(a) 8 h 3 min 26 sec am (b) 8 h 3 min 36 sec am
(c) 9 am (d) 1 h 20 min after 8 am

188. Due to some technical snag the operation had to stop after 2 hours. The oil remaining in the tank at that time is
(a) 8 tonnes (b) 21.6 tonnes
(c) 81.6 tonnes (d) 16 tonnes

189. When the machine started working again, 44 tonnes of more oil was poured in. How long will it take further to have the tank emptied?
(a) 3 hours 20 minutes more (b) 1 hours 10 minutes more
(c) 2 hours 15 minutes more (d) it is not possible to empty the tank

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

190. The equation $x^3 - px^2 + qx - r^2 = 0$ has two of its roots equal in magnitude but opposite in sign. Then
(a) $q^2 = pr$ (b) $r^2 = pq$
(c) the equation $px^2 + 2rx + q = 0$ has equal roots (d) the equation $px^2 + 2qx + r = 0$ has equal roots

191. A root of the equation $x^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$ is
(a) 1 (b) 2 (c) $\dfrac{1}{4}$ (d) $\dfrac{1}{\sqrt[3]{2}}$

192. Given that α, γ are the roots of the equation $Ax^2 - 4x + 1 = 0$ and β, δ the roots of the equation $Bx^2 - 6x + 1 = 0$ and that $\alpha, \beta, \gamma, \delta$ are in HP then
(a) $A = 3$ (b) $B = -8$ (c) $A = -3$ (d) $B = 8$

193. Let Q denote the quadratic expression $2x^2 - 5x - 3$. Then,

(a) The roots of the equation $Q + \lambda = 0$ will be equal if $\lambda = \frac{49}{8}$

(b) The roots of the equation $Q + \lambda = 0$ will be non-real if l $\lambda > 7$

(c) Both roots of the equation $2(x - k)^2 - 5(x - k) - 3 = 0$ will be positive for $k > \frac{1}{2}$

(d) Both roots of the equation $2(x + k)^2 - 5(x + k) - 3 = 0$ will be negative for $k > 3$

194. For real p, q, r, $p \neq q$ the equation $x^2 - (p + q)x + r^2 = 0$ has equal roots, if

(a) $2r = p + q$
(b) $p^3 + q^3 + 8r^3 = 6pqr$
(c) $p^3 + q^3 - 8r^3 + 6pqr = 0$
(d) $2p = q + r$

195. If α and β are the roots of the equation $2x^2 + 5x + 7 = 0$, then

(a) $(\alpha + 1)$ and $(\beta + 1)$ are the roots of the equation $2x^2 + x + 4 = 0$

(b) $(2\alpha - 1)$ and $(2\beta - 1)$ are the roots of the equation $x^2 + 7x + 20 = 0$

(c) $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of the equation $7x^2 + 5x + 2 = 0$

(d) α^2 and β^2 are the roots of the equation $4x^2 + 3x + 49 = 0$

196. Consider $f(x) = \dfrac{x^2 + 4x + 3}{x^2 + x + 1}$

(a) $f(x)$ is defined for all $x \in R$

(b) $f(x) \geq 0$ for all $x \in R$

(c) $f(x) \geq 0$ in $x \in (-\infty, -3] \cup [-1, \infty)$

(d) $f(x)$ can take values only in the interval $\left[\dfrac{4 - 2\sqrt{7}}{3}, \dfrac{4 + 2\sqrt{7}}{3}\right]$

197. Consider $f(x) = 6x^2 - 5x - 6$. Which of the following statements are true?

(a) $f(x)$ attains its minimum value at $x = \frac{5}{12}$

(b) The product of the zeros of $f(x)$ is (-1)

(c) The graph of the curve $y = f(x)$ does not intersect the x-axis

(d) Domain of $f(x)$ is R

Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

198. Given $1 < a, c < 5$ and $2b = a + c$

Column I	**Column II**
Quadratic Equation	**Nature of the roots**
(a) $(x - a)(x - c) = 0$	(p) Equal roots
(b) $x[x + (a + c)] + b^2 = 0$	(q) Sum of the roots equals 2b
(c) $(x - a)(x - c) + (2b + 1)x = 0$	(r) Roots lie in [1, 5]
(d) $2x^2 - (a + c)x + 5(a + c) = 0$	(s) Complex roots

199. $ax^2 + bx + c < 0$ for all $x > 10$

Column I **Quadratic Equation**		**Column II**
(a) $ax^2 + bx + 1 = 0$	(p)	Real and distinct roots
(b) $-ax^2 + bx - 4 = 0$	(q)	Complex roots
(c) $x^2 + x - a = 0$; where $\lvert a \rvert > \frac{1}{4}$	(r)	The corresponding quadratic expression is positive between the roots
(d) $ax^2 - b\lvert x \rvert + 5 = 0$	(s)	The corresponding quadratic expression is negative between the roots

200.

Column I **Equation**		**Column II** **Number of solutions**
(a) $3^{\cos^2 x} + 5\left(3^{\sin^2 x}\right) = 8$	(p)	0
(b) $3^{6x} + 7(3^{4x}) = 8(3^{5x})$	(q)	1
(c) $7^{\log_7 (x^2 - 4x + 5)} = (x - 1) = 80$	(r)	2
(d) $\lvert y - 2 \rvert^{\log_4 x - \log_x 16} = (y - 2)^7$ where, y is a real number for which $y^2 - 3y + 2 > 0$	(s)	infinitely many

SOLUTIONS

ANSWER KEYS

Topic Grip

1. (i) Distinct Rational Roots
 (ii) Irrational roots
 (iii) Equal roots
 (iv) Complex
 (v) One real root and two complex roots.

2. $k = -\frac{1}{7}$

3. $x^2 + 12x + 5 = 0$

4. $x^2 - 15127x + 1 = 0$

5. (i) $1, \frac{c(a-b)}{a(b-c)}$

 (ii) $\frac{15 \pm \sqrt{41}}{2}, \frac{15 \pm \sqrt{17}}{2}$

6. 4

7. $-1, \frac{3}{2}$

8. $-1 < m < 5$

9. $[8, 10)$

10. $0, -12$

11. (b)	**12.** (d)	**13.** (c)
14. (a)	**15.** (b)	**16.** (c)
17. (b)	**18.** (d)	**19.** (a)
20. (a)	**21.** (b)	**22.** (c)

23. (c)
24. (a)
25. (a)
26. (c)
27. (c), (d)
28. (a), (b), (d)
29. (a), (c)
30. (a) → (s)
(b) → (r)
(c) → (p), (q), (s)
(d) → (q)

IIT Assignment Exercise

31. (b)	**32.** (d)	**33.** (c)
34. (a)	**35.** (d)	**36.** (a)
37. (d)	**38.** (b)	**39.** (c)
40. (b)	**41.** (c)	**42.** (d)
43. (c)	**44.** (d)	**45.** (c)
46. (a)	**47.** (a)	**48.** (b)
49. (d)	**50.** (b)	**51.** (a)
52. (a)	**53.** (b)	**54.** (b)
55. (c)	**56.** (b)	**57.** (b)
58. (b)	**59.** (c)	**60.** (c)
61. (c)	**62.** (b)	**63.** (b)
64. (b)	**65.** (a)	**66.** (a)
67. (c)	**68.** (d)	**69.** (a)
70. (d)	**71.** (b)	**72.** (d)
73. (a)	**74.** (a)	**75.** (c)
76. (d)	**77.** (b)	**78.** (c)
79. (d)	**80.** (b)	**81.** (c)
82. (a)	**83.** (a)	**84.** (d)
85. (b)	**86.** (a)	**87.** (c)
88. (b)	**89.** (a)	**90.** (b)
91. (b)	**92.** (d)	**93.** (d)
94. (c)	**95.** (a)	**96.** (a)
97. (d)	**98.** (b)	**99.** (c)
100. (a)	**101.** (c)	**102.** (c)
103. (b)	**104.** (d)	**105.** (c)
106. (b)	**107.** (c)	**108.** (d)
109. (d)	**110.** (d)	**111.** (c)
112. (d)	**113.** (a)	**114.** (a)

115. (d)
116. (c)
117. (b), (c), (d)
118. (a), (c), (d)
119. (a), (b), (c)
120. (a) → (p), (r), (s)
(b) → (p), (r), (s)
(c) → (p)
(d) → (q)

Additional Practice Exercise

131. (b)	**132.** (b)	**133.** (b)
134. (d)	**135.** (d)	**136.** (d)
137. (a)	**138.** (b)	**139.** (a)
140. (a)	**141.** (c)	**142.** (d)
143. (d)	**144.** (c)	**145.** (c)
146. (c)	**147.** (a)	**148.** (b)
149. (a)	**150.** (d)	**151.** (a)
152. (c)	**153.** (c)	**154.** (c)
155. (b)	**156.** (d)	**157.** (a)
158. (d)	**159.** (b)	**160.** (b)
161. (b)	**162.** (c)	**163.** (b)
164. (c)	**165.** (c)	**166.** (d)
167. (c)	**168.** (a)	**169.** (a)
170. (d)	**171.** (b)	**172.** (c)
173. (d)	**174.** (a)	**175.** (a)
176. (d)	**177.** (a)	**178.** (b)
179. (b)	**180** (a)	**181.** (b)
182. (b)	**183.** (b)	**184.** (a)
185. (b)	**186.** (b)	**187.** (d)
188. (d)	**189.** (a)	

190. (b), (c)
191. (b), (c), (d)
192. (a), (d)
193. (a), (b), (c), (d)
194. (a), (b), (c)
195. (a), (b), (c), (d)
196. (a), (c), (d)
197. (a), (b), (d)
198. (a) → (q), (r)
(b) → (p)
(c) → (s)
(d) → (s)
199. (a) → (p), (r)
(b) → (p), (s)
(c) → (q)
(d) → (p), (r)
200. (a) → (s)
(b) → (r)
(c) → (q)
(d) → (r)

HINTS AND EXPLANATIONS

Topic Grip

1. (i) $4x^2 - 6x = 0$

$D = 36 - 0 = 36 > 0$ and 36 is a perfect square

Roots are real, distinct and rational.

(ii) $x^2 - 8x + 9 = 0$

$D = 64 - 36 = 28 > 0$ but not a perfect square

Roots are real, distinct and irrational.

(iii) $x^2 - 22x + 121 = 0$

$D = 484 - 484 = 0$

Roots are real and equal

(iv) $x^2 - 4x + 5 = 0$

$\Rightarrow$ $D = 16 - 20 < 0$

Roots are complex.

(v) $x^3 - 7x^2 + 16x - 10 = 0$

Clearly, $x = 1$ is a root

$\therefore$ By factorizing $(x - 1)(x^2 - 6x + 10) = 0$

$\Rightarrow$ $x^2 - 6x + 10 = 0$

$D = 36 - 40 < 0$

$\therefore$ one root is real and other two roots are complex.

2. $x^2 + (k + 1)x - (12k^2 + 3k) = 0$

Discriminant = 0 for equal roots.

i.e., $(k + 1)^2 + 4(12k^2 + 3k) = 0$

$\Rightarrow$ $49k^2 + 14k + 1 = 0 \Rightarrow (7k + 1)^2 = 0$

$\Rightarrow$ $k = -\dfrac{1}{7}$

3. Let $y = \dfrac{x+3}{x-3}$ where, $x = \alpha, \beta$.

Then, $x = \dfrac{3(y+1)}{(y-1)}$

substituting in $3x^2 - 4x - 9 = 0$, the equation whose roots are $y = \dfrac{\alpha+3}{\alpha-3}, \dfrac{\beta+3}{\beta-3}$ is

$$\frac{3\times 9(y+1)^2}{(y-1)^2} - \frac{4\times 3(y+1)}{(y-1)} - 9 = 0$$

$\Rightarrow$ $9(y + 1)^2 - 4(y^2 - 1) - 3(y - 1)^2 = 0$

$\Rightarrow$ $y^2 + 12y + 5 = 0$

or $x^2 + 12x + 5 = 0$

4. We have $\alpha + \beta = 7, \alpha\beta = 1$

$\alpha^5 + \beta^5 = (\alpha^3 + \beta^3)(\alpha^2 + \beta^2) - \alpha^3\beta^2 - \beta^3\alpha^2$

$= [(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)][(\alpha + \beta)^2 - 2\alpha\beta] - \alpha^2\beta^2(\alpha + \beta)$

$= [7^3 - 3 \times 7][7^2 - 2] - 7 = 15127$

$\alpha^5\beta^5 = (\alpha\beta)^5 = 1$

The required equation is $x^2 - 15127x + 1 = 0$

5. (i) Clearly, $x = 1$ is a root, since

$a(b - c) + b(c - a) + c(a - b) = 0$

Product of the roots $= \dfrac{c(a-b)}{a(b-c)}$

If β is the other root,

i.e., $1 \times \beta = \dfrac{c(a-b)}{a(b-c)}$ or the roots are $1, \dfrac{c(a-b)}{a(b-c)}$

(ii) Equation can be written as

$(x^2 - 15x + 44)(x^2 - 15x + 54) + 16 = 0$

$\Rightarrow$ $y(y + 10) + 16 = 0$ where $y = x^2 - 15x + 44$

$\Rightarrow$ $y^2 + 10y + 16 = 0 \Rightarrow y = -8, -2$

$y = -8 \Rightarrow x^2 - 15x + 52 = 0$

$\Rightarrow$ $x = \dfrac{15 \pm \sqrt{17}}{2}$

$y = -2 \Rightarrow x^2 - 15x + 46 = 0$

$\Rightarrow$ $x = \dfrac{15 \pm \sqrt{41}}{2}$

6. Let $a = 2^x, b = 2^{\sqrt{x}}$

The equation reduces to $a^2 - 4b^2 = 3ab$

$\Rightarrow a^2 - 3ab - 4b^2 = 0$

$\Rightarrow (a + b)(a - 4b) = 0$

$\Rightarrow a = -b$ or $a = 4b$

Since both a and b are positive, $a \neq -b$

Therefore, $a = 4b$ or $2^x = 4 \times 2^{\sqrt{x}} = 2^{2+\sqrt{x}}$

$\Rightarrow x = \sqrt{x} + 2 \Rightarrow (\sqrt{x})^2 - \sqrt{x} - 2 = 0$

Giving $\sqrt{x} = 2$ or -1 $\sqrt{x} \neq -1$

Solution is $\sqrt{x} = 2$ or $x = 4$

7. Clearly, $x = 2$ is not a solution.

Case 1: $x < 2$

$2x^2 + 2 - x - 5 = 0$

$\Rightarrow$ $2x^2 - x - 3 = 0$

$\Rightarrow$ $2x^2 - 3x + 2x - 3 = 0$

$\Rightarrow$ $(x + 1)(2x - 3) = 0$

$\Rightarrow$ $x = -1, 3/2$

Since both the above values of x satisfy $x < 2$, they are solutions of the given equation.

Case 2: $x > 2$

$2x^2 + x - 5 - 2 = 0 \Rightarrow 2x^2 - x - 7 = 0$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{57}}{4}$$

Both the values of x do not satisfy $x > 2$.

$\therefore$ The solutions of the equation are $-1, 3/2$

8. The inequality is equivalent to $-3 < \dfrac{x^2+mx+1}{x^2+x+1} < 3$

Since $x^2+x+1 = \left(x+\dfrac{1}{2}\right)^2 + \dfrac{3}{4} > 0$ for all real x

We have $-3(x^2 + x + 1) < x^2 + mx + 1$

$< 3(x^2 + x + 1)$

$\therefore$ $4x^2 + (m + 3)x + 4 > 0$ — (1)

and $2x^2 - (m - 3)x + 2 > 0$ — (2)

Since the coefficient of x^2 in LHS of (1) = 4 > 0 the inequality (1) will be valid for all x if $(m + 3)^2 - 64 < 0$ i.e., if $(m + 11)(m - 5) < 0$

or $-11 < m < 5$ — (3)

Since the coefficient of x^2 in LHS of equation (2) is 2 >0

The inequality (2) will be valid if $(m - 3)^2 - 16 < 0$

i.e., if $(m + 1)(m - 7) < 0$ or $-1 < m < 7$ — (4)

The conditions of (3) and (4) will hold simultaneously if $-1 < m < 5$.

9. Let $f(x) = 2x^2 - kx + 8$

Since the roots of the equation are real,

$k^2 - 64 \geq 0$

$\Rightarrow$ k should be beyond -8 and 8 — (1)

Since the coefficient of x^2 is > 0 and the roots are to lie between -1 and 4, $f(-1) > 0$ and $f(4) > 0$.

Also the minimum value of f(x) occurs at $x = \dfrac{k}{4}$ and therefore, $\dfrac{k}{4}$ must lie between -1 and 4.

$f(-1) > 0$ gives $k > -10$ — (2)

$f(4) > 0$ gives $k < 10$ — (3)

and $-1 < \dfrac{k}{4} < 4$ gives $-4 < k < 16$ — (4)

from (1), (2), (3), (4) we get $8 < k < 10$ when $k = 8$, the roots are each equal 2 which lies between -1 and 4. Hence, $k \in [8, 10)$

10. We have $(x - \lambda)(x + 6) = -5$

Possible choices are $x - \lambda = 5, x + 6 = -1$

or $x - \lambda = -5, x + 6 = 1$

Giving $\lambda = -12$ or $\lambda = 0$

11. Let $y = \dfrac{x}{1+x} \Rightarrow xy + y = x \Rightarrow x(y-1) = -y$

$$\Rightarrow x = \frac{y}{1-y};$$

Substituting for x in the given equation

$$\Rightarrow \quad \frac{y^2}{(1-y)^2} + \frac{4y}{1-y} - 7 = 0$$

$\Rightarrow$ $10y^2 - 18y + 7 = 0$

Or the required equation is $10x^2 - 18x + 7 = 0$

12. Let $k\alpha$ and α be the roots of $x^2 - 7x + 1 = 0$

$k\alpha + \alpha = 7, k\alpha^2 = 1$

$$\Rightarrow \left(\frac{7}{k+1}\right)^2 = \frac{1}{k} \Rightarrow \frac{(k+1)^2}{k} = 49$$

Similarly, for the second equation, we get

$$\left[\frac{-4}{\lambda(k+1)}\right]^2 = \frac{-5}{\lambda k} \Rightarrow \frac{(k+1)^2}{k} = \frac{-16\lambda}{5\lambda^2} = \frac{-16}{5\lambda}$$

We have $\dfrac{-16}{5\lambda} = 49 \Rightarrow \lambda = \dfrac{-16}{245}$

13. The equation may be rewritten as $(x^2 - 5x + 4)(x^2 - 5x + 6) = 15$

Let $y = x^2 - 5x + 4$.

Then we have $y(y + 2) - 15 = 0$

$\Rightarrow$ $y = -5, 3$;

$y = -5 \Rightarrow x^2 - 5x + 9 = 0$

$\Rightarrow$ No real roots

$y = 3 \Rightarrow x^2 - 5x + 1 = 0$

$\Rightarrow$ Two positive roots

14. Let α denote the common root

Then,

$$\frac{\alpha^2}{5p+5q} = \frac{\alpha}{-10} = \frac{1}{q-p}$$

$$\alpha = \frac{10}{(p-q)}$$

$$\alpha^2 = \frac{-5(p+q)}{(p-q)}$$

$$\Rightarrow \frac{-5(p+q)}{(p-q)} = \frac{100}{(p-q)^2}$$

$$\Rightarrow -(p+q) = \frac{20}{p-q}$$

$$\Rightarrow (p^2-q^2) = -20$$

15. Let $f(x) = x^2 - 2(4k-1)x + 15k^2 - 2k - 7 > 0$

Then $f(x) > 0 \Rightarrow$ Discriminant < 0

$4(4k-1)^2 - 4(15k^2 - 2k - 7) < 0$

$16k^2 - 8k - 15k^2 + 2k + 1 + 7 < 0$

$k^2 - 6k + 8 < 0 \Rightarrow (k-4)(k-2) < 0$

$2 < k < 4, \quad k = 3$

16. The roots of $x^2 + 7x + 12 = 0$ are -3 and -4. So Statement 1 is true.

Consider the equation $x^2 + x + 4 = 0$

Although a, b, c are positive the roots are complex.

Statement 2 is false.

17. Statement 2 is true

Statement 1

Let $x > 0$

$\Rightarrow x^2 - x - 2 = 0 \Rightarrow x = 2$ is a solution.

Let $x < 0$

$\Rightarrow x^2 + x - 2 = 0 \Rightarrow x = -2$ is a solution

$x = 0$ does not satisfy the given equation.

Equation has 2 roots. However, it does not follow from statement 2

Choice (b)

18. Statement 2 is true

Statement 1 is false since the sign of $(4a + 2b + c)$ and $(4a - 2b + c)$ depends on the sign of a. Also, other conditions have to be satisfied for the roots to lie between -2 and 2

Choice (d)

19. Statement 2 is true

Statement 1

$\alpha + \beta = -1$

Hence, using statement 2, the minimum value of $x^2 + x + 4$ is

$$\left(\frac{-1}{2}\right)^2 - \frac{1}{2} + 4 = \frac{15}{4}$$

Choice (a)

20. Statement 2

For $7x^2 - 8x + 105 = 0$, its discriminant

$= 64 - 7 \times 4 \times 105 < 0$

$\therefore \quad 7x^2 - 8x + 105 > 0 \ \forall \ x \in R$

$\therefore$ Statement 2 is true.

Statement 1

$$\frac{\log_e (x-e)^2}{7x^2 - 8x + 105} > 0 \Rightarrow (x-e)^2 > 1$$

(using R)

$\Rightarrow \quad x - e > 1 \quad$ or $\quad x - e < -1$

$\Rightarrow \quad x > e + 1 \quad$ or $\quad x < e - 1$

$\therefore$ Statement 1 is true and follows from Statement 2

$\Rightarrow$ Choice (a)

21 and 23

Passage I

We have

$$p + q + r = 0$$

$$pq + qr + rp = \frac{-1}{2}$$

$$pqr = -2$$

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{qr + rp + pq}{pqr} = \frac{-1}{2(-2)} = \frac{1}{4}$$

$p^3 + q^3 + r^3 - 3pqr$

$= (p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp)$

since $p + q + r = 0$, $p^3 + q^3 + r^3 = 3pqr$

$= 3 \times -2 = -6$

$$(p+q)(q+r) = (-r)(-p) = pr = \frac{pqr}{q} = \frac{-2}{q}$$

$$(q+r)(r+p) = (-p)(-q) = pq = \frac{pqr}{r} = \frac{-2}{r}$$

$$(r+p)(p+q) = \frac{pqr}{p} = \frac{-2}{p}$$

We have to form the equation whose roots are

$\frac{-2}{p}, \frac{-2}{q}, \frac{-2}{r}$

Let $y = \frac{-2}{x}$ where $x = p, q, r$.

The required equation is obtained by replacing x by $\frac{-2}{y}$ in $2x^3 - x + 4 = 0$

$\Rightarrow \quad 2\left(\frac{-2}{y}\right)^3 - \left(\frac{-2}{y}\right) + 4 = 0$

$\Rightarrow \quad -16 + 2y^2 + 4y^3 = 0$

$\Rightarrow \quad 2x^3 + x^2 - 8 = 0$

Passage II

24. Since the coefficients of the equation are rational, $\left(3 + \sqrt{10}\right)$ is also a root of the equation. Factors corresponding to the two roots above are $\left(x - 3 - \sqrt{10}\right)$ and $\left(x - 3 - \sqrt{10}\right)$

Their product is $(x - 3)^2 - 10$ or $(x^2 - 6x - 1)$

Dividing $(x^4 - 8x^3 + 16x^2 - 28x - 5)$ by $(x^2 - 6x - 1)$ we get the quotient as $x^2 - 2x + 5$

$$x^2 - 2x + 5 = 0$$

gives $x = 1 \pm 2i$

Roots of the equation are

$$3 \pm \sqrt{10} \text{ and } 1 \pm 2i$$

Sum of the squares of the reciprocals of the roots

$$= \frac{1}{\left(3 - \sqrt{10}\right)^2} + \frac{1}{\left(3 + \sqrt{10}\right)^2} + \frac{1}{(1 + 2i)^2} + \frac{1}{(1 - 2i)^2}$$

$$= \left(3 + \sqrt{10}\right)^2 + \left(3 - \sqrt{10}\right)^2 + \frac{(1 - 2i)^2}{25} + \frac{(1 + 2i)^2}{25}$$

$$= 2(9 + 10) + \frac{2}{25}\{1 - 4\} = 38 - \frac{6}{25} = \frac{944}{25}$$

25. Since the coefficients of the equation are real, $\left(2 - i\sqrt{3}\right)$ is also a root of the equation. Product of the factors corresponding to the roots $2 \pm i\sqrt{3}$ is $(x - 2)^2 + 3$ or $x^2 - 4x + 7$.

Dividing $(2x^4 - 5x^3 - 3x^2 + 41x - 35)$ by $(x^2 - 4x + 7)$, the quotient is $(2x^2 + 3x - 5)$

$2x^2 + 3x - 5 = 0$ has 2 rational roots, since the discriminant is a perfect square.

26. Since the coefficients of the equation are real, $(a - ib)$ and $(b - ia)$ are the other two roots.

sum of the roots $= 2a + 2b = 6$

$\Rightarrow \quad a + b = 3$

Product of the roots $= (a^2 + b^2)^2 = 25$

$\Rightarrow \quad a^2 + b^2 = 5$

$\Rightarrow \quad (a + b)^2 - 2ab = 5 \quad \Rightarrow 9 - 2ab = 5$

$2ab = 4 \quad \Rightarrow ab = 2$

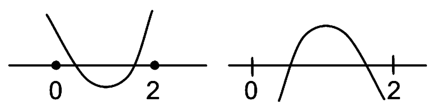
$a = 2, \; b = 1$

or

$a = 1, \; b = 2$

$a^3 + b^3 = 8 + 1 = 9$

27.



Let $f(x) = (1 - p^2)x^2 + 2px - 1$

Both the roots of $f(x) = 0$ are in $(0, 2)$

$\Rightarrow$ $D \geq 0$ and coefficient of x^2, $f(0)$ and $f(2)$ have the same sign.

$\Rightarrow$ $4p^2 + 4(1 - p^2) \geq 0$, $(1 - p^2) f(0) > 0$ and $(1 - p^2) f(2) > 0$

$\Rightarrow$ $4 \geq 0$ (which is true always), $(1 - p^2)(-1) > 0$ and $(1 - p^2)[4(1 - p^2) + 4p - 1] > 0$

$\Rightarrow$ $(p^2 - 1) > 0$ and $(1 - p^2)(-4p^2 + 4p + 3) > 0$

$\Rightarrow$ p lies beyond -1 and $+1$ and $(p + 1)(p - 1)(2p - 3)(2p + 1) > 0$

$\Rightarrow$ $(p < -1$, or $p > 1)$ and $(p < -1$,

or $\left(p < -1, \text{ or } -\frac{1}{2} < p < 1 \text{ or } p > \frac{3}{2}\right)$

$\Rightarrow$ $p < -1$ or $p > \frac{3}{2}$ — (A)

Also, $x = \frac{\alpha + \beta}{2} = \frac{-p}{(1 - p^2)}$

Must lie between 0 and 2

i.e., $0 < \frac{-p}{1 - p^2} < 2$

$\Rightarrow 0 < \frac{p}{p^2 - 1} < 2$

This gives

$p > 1$ or $-1 < p < 0$ and $\frac{p}{(p^2 - 1)} - 2 < 0$ — (1)

The second inequality is

$$\left.\begin{array}{l}\dfrac{2p^2-p-2}{p^2-1}>0\\ \Rightarrow p<-1 \text{ or } p>\dfrac{1+\sqrt{17}}{4}\end{array}\right\} \quad \text{— (2)}$$

OR

$$\frac{1-\sqrt{17}}{4}<p<1$$

Combining (1) and (2) we get $p > \dfrac{1+\sqrt{17}}{4}$ or $\dfrac{1-\sqrt{17}}{4}<p<0$

From (A) and (3)

$\Rightarrow \quad p>\dfrac{3}{2}$

Choices (c) and (d) are true

28. The equations, $px^2 + qx + r = 0$ and $qx^2 + rx + p = 0$ have a common root.

$\Rightarrow \quad (qr - p^2)^2 = (rp - q^2)(pq - r^2)$ — (1)

$\Rightarrow \quad p(p^3 + q^3 + r^3) = 3p^2qr$

$\Rightarrow \quad p^3 + q^3 + r^3 = 3pqr \quad (\because p \neq 0)$ — (2)

$\Rightarrow \quad \dfrac{p^3+q^3+r^3}{pqr} = 3 \quad (\because p, q, r \neq 0)$

$\therefore$ (d) is true and (c) is not.

(2) can be written as

$$0 = p^3 + q^3 + r^3 - 3pqr$$

$$= \frac{(p+q+r)}{2}\left[(p-q)^2+(q-r)^2+(r-p)^2\right]$$

$\Rightarrow \quad p + q + r = 0 \quad$ or $p = q = r$

$\therefore$ (a) is true.

Considering (b)

Discriminant $= 4(qr - p^2)^2 - 4(pq - r^2)(rp - q^2)$

$= 4p\{p^3 + q^3 + r^3 - 3pqr\}$

$= 0$

(b) is true

Choices (a), (b), (d)

29. $t = |\log_3 x|$

$2t^2 - t + k = 0$

Discriminant > 0

$\Rightarrow \quad 1 - 8k > 0 \quad \Rightarrow k < \dfrac{1}{8}$

$t = |\log_3 x| > 0$

The roots of (1) are positive

$\therefore$ product of the roots is > 0

$\dfrac{k}{2}>0 \quad \Rightarrow k > 0$

Hence $0 < k < \dfrac{1}{8}$

Note that $\dfrac{1}{16} \in \left(0, \dfrac{1}{8}\right)$

Choices (a), (c)

30. (a) $(k - 1)^2 - 16 \geq 0$

$(k - 1)$ lies beyond -4 and $+4$

$k - 1 \leq -4$ or $k - 1 \geq 4$

$\Rightarrow \quad k \leq -3$ or $k \geq 5$

$\Rightarrow \quad k \in (-\infty, -3\,] \cup [5, \infty)$

(b) $81 + 4k(k + 2) < 0$

$4k^2 + 8k + 81 < 0$

$\Rightarrow$ No real value for k exists

(c) $(9 + 4k^2)$ should not be a perfect square.

$9 + 4k^2 = a^2 \Rightarrow a^2 - 4k^2 = 9 \Rightarrow (a - 2k)(a + 2k) = 9$

$\Rightarrow a - 2k = 1$ or 3 and $a + 2k = 9$ or 3

$a - 2k = 3 \Rightarrow k = 0$, $a - 2k = 1 \Rightarrow k = 2$ or -2.

$k^2 \neq 4, 0$

$k \neq \pm 2, 0$

$k \in (R - \{-2, 0, 2\}$.

(d) $4 - 4k(1 + 2k) > 0$

and $\dfrac{1+2k}{k}<0$

$\Rightarrow \quad 1 - k - 2k^2 > 0$

$2k^2 + k - 1 < 0$

$2k^2 + 2k - k - 1 < 0$

$2k(k + 1) - 1(k + 1) < 0$

k lies between -1 and $\dfrac{1}{2}$ — (1)

$\dfrac{1+2k}{k}<0$

gives $k\in\left(\dfrac{-1}{2},0\right)$ — (2)

combining (1) and (2)

$= k \in \left(\dfrac{-1}{2}, 0\right)$

IIT Assignment Exercise

31. $t^2 - 6t + 1 = 0$

$$\Rightarrow t = \frac{6 \pm \sqrt{36-4}}{2} = \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}.$$

32. By trial methods –1 and +1 satisfy the equation.

33. $b^2 - 4ac = 0$

$\Rightarrow (k-1)^2 + 4(2k+1) = 0$

$k^2 - 2k + 1 + 8k + 4 \Rightarrow k^2 + 6k + 5 = 0$

$k = -5, -1.$

34. Sum of the roots = 0

$k^2 - k - 2 = 0 \Rightarrow k = 2, -1.$

35. $x^3 - \dfrac{1}{x^3} + 4\left(x - \dfrac{1}{x}\right)$

$$= \left(x - \frac{1}{x}\right)^3 + 3\left(x - \frac{1}{x}\right) + 4\left(x - \frac{1}{x}\right) = 0$$

$$\Rightarrow \left(x - \frac{1}{x}\right)\left[\left(x - \frac{1}{x}\right)^2 + 7\right] = 0$$

$$\Rightarrow \left(x - \frac{1}{x}\right) = 0$$

$\Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$

The number of real solutions = 2

36. Coefficients of the equation are not real and hence roots will not be complex conjugates.

Product of the roots $= \dfrac{2-i}{i} = \alpha(2-i)$

$$\Rightarrow \alpha = \frac{1}{i} = -i$$

37. The equation can be rewritten as

$3x^2 - 2(a+b+c)x + (ab+bc+ca) = 0$

For equal roots discriminant = 0

$\Rightarrow 4(a+b+c)^2 - 12(ab+bc+ac) = 0$

$\Rightarrow a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0$

$\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

$\Rightarrow a = b = c.$

38. Discriminant $= 4c^2 - 4(a+c-b)(b+c-a)$

$= 4\{c^2 - [c^2 - (a-b)^2]\}$

$= 4[a-b]^2$ which is a perfect square.

So the roots must be distinct and rational.

39. Sum of the roots $p + q = -2p$

$3p + q = 0 \Rightarrow q = -3p$

Product of the roots $pq = q - 6$

$\therefore\ p(-3p) = -3p - 6$

$3p^2 - 3p - 6 = 0 \Rightarrow p^2 - p - 2 = 0$

$(p-2)(p+1) = 0 \Rightarrow p = -1, 2$

40. Let the roots of the equation are $p\alpha$, $q\alpha$.

$$p\alpha + q\alpha = -\frac{m}{\ell} \qquad \text{— (1)}$$

$$p\alpha . q\alpha = \frac{m}{\ell} \Rightarrow pq\alpha^2 = \frac{m}{\ell} \qquad \text{— (2)}$$

$$\Rightarrow \alpha = \frac{-(p+q)}{pq}$$

$$\therefore \text{ From (2), } \frac{(p+q)^2}{pq} = \frac{m}{\ell}$$

$$\Rightarrow \frac{p}{q} + \frac{q}{p} + 2 = \frac{m}{\ell}$$

$$\left(\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}}\right)^2 = \frac{m}{\ell}$$

$$\Rightarrow \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} = \pm\sqrt{\frac{m}{\ell}}$$

Since $\sqrt{\dfrac{p}{q}} + \sqrt{\dfrac{q}{p}}$ cannot be negative.

$$\therefore \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} - \sqrt{\frac{m}{\ell}} = 0$$

41. $(\alpha + 1) + (\beta + 1) = \alpha + \beta + 2 = 5 + 2 = 7$

$(\alpha + 1)(\beta + 1) = \alpha\beta + \alpha + \beta + 1 = 6 + 5 + 1 = 12$

$x^2 - 7x + 12 = 0.$

42. $\alpha + \beta = p$, $\alpha\beta = 36$

$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta = p^2 \ (\because \alpha^2 + \beta^2 = 9)$

$= 9 + 2 . 36 = p^2$

$p^2 = 81$

$\Rightarrow p = \pm 9.$

43. $\alpha + \beta = 6; \alpha\beta = 2$

Sum of roots $= \dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta}$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{36-4}{2} = 16.$$

$$\text{Product} = \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1.$$

$\therefore$ Equation is $x^2 - 16x + 1 = 0$.

44. $\alpha + \beta = 3, \alpha\beta = 4$

$\alpha^2 + \alpha + 1 = \alpha(\alpha + \beta) - \alpha\beta + \alpha + 1$

$= 3\alpha - 4 + \alpha + 1 = 4\alpha - 3$

Similarly, $\beta^2 + \beta + 1 = 4\beta - 3$.

Sum of the roots $= 4(\alpha + \beta) - 6 = 6$

Product of the roots $= (4\alpha - 3)(4\beta - 3)$

$= 16\alpha\beta + 9 - 12(\alpha + \beta)$

$= 64 + 9 - 36 = 37$

$\therefore$ The equation is $x^2 - 6x + 37 = 0$

45. Sum of the roots $= (\alpha_1\alpha_2 + \beta_1\beta_2) + (\alpha_1\beta_2 + \alpha_2\beta_1)$

$$= \alpha_1(\alpha_2 + \beta_2) + \beta_1(\alpha_2 + \beta_2)$$

$$= \alpha_1\left(\frac{-1}{2}\right) + \beta_1\left(\frac{-1}{2}\right)$$

$$= \frac{-1}{2}(\alpha_1 + \beta_1) = \frac{-1}{2} \times \frac{2}{3} = \frac{-1}{3}$$

Product of the roots

$$= (\alpha_1\alpha_2 + \beta_1\beta_2)(\alpha_1\beta_2 + \alpha_2\beta_1)$$

$$= \alpha_1^2\alpha_2\beta_2 + \beta_1^2\alpha_2\beta_2 + \alpha_2^2\alpha_1\beta_1 + \beta_2^2\alpha_1\beta_1$$

$$= \alpha_2\beta_2\left(\alpha_1^2 + \beta_1^2\right) + \alpha_1\beta_1\left(\alpha_2^2 + \beta_2^2\right)$$

$$= \frac{-7}{2}\left[\frac{4}{9} - 2 \times \frac{-5}{3}\right] - \frac{5}{3}\left[\frac{1}{4} - 2 \times \frac{-7}{2}\right] = \frac{-911}{36}$$

The equation is $x^2 + \frac{x}{3} - \frac{911}{36} = 0$, or

$$36x^2 + 12x - 911 = 0$$

46. Let $t = \frac{5x}{3x - 2}$ $t \neq 0$

Equation reduces to

$$7t - \frac{3}{t} = 4$$

$$7t^2 - 4t - 3 = 0$$

$$t = 1, -\frac{3}{7}$$

$$t = 1 \Rightarrow \frac{5x}{3x - 2} = 1$$

$\Rightarrow$ $5x = 3x - 2$

$x = -1$

$$t = \frac{-3}{7} \Rightarrow \frac{5x}{3x - 2} = \frac{-3}{9}$$

$35x = -9x + 6$

$44x = 6 \Rightarrow x = \frac{3}{22}$

$\alpha = -1, \beta = \frac{3}{22}$

$$\frac{1}{\alpha} + \frac{1}{\beta} = -1 + \frac{22}{3} = \frac{19}{3}$$

47. $x^2 + 4x - 4 = 0$

Equation whose roots are $\alpha - 2, \beta - 2$, is

$(x + 2)^2 + 4(x + 2) - 4 = 0$

$x^2 + 8x + 8 = 0$

$\Rightarrow$ Equation whose roots $\frac{1}{\alpha - 2}, \frac{1}{\beta - 2}$ is

$8x^2 + 8x + 1 = 0$ — (1)

Let $\lambda = \frac{1}{\alpha - 2}, \mu = \frac{1}{\beta - 2}$

λ and μ are the roots of (1)

we want $\lambda^3 + \mu^3$

$$\lambda^3 + \mu^3 = (\lambda + \mu)^3 - 3\lambda\mu(\lambda + \mu)$$

$$= (-1)^3 - 3 \times \frac{1}{8} \times (-1)$$

$$= -1 + \frac{3}{8} = -\frac{5}{8}$$

48. We have $\alpha + \beta = \frac{7}{3}, \alpha\beta = \frac{-5}{3}$

Sum of the roots of the required equation

$= \alpha^2 + \beta^2 + 4\alpha\beta = (\alpha + \beta)^2 + 2\alpha\beta$

$$= \frac{49}{9} - \frac{10}{3} = \frac{19}{9}$$

Product of the roots of the required equation

$= (\alpha^2 + 2\beta\alpha)(\beta^2 + 2\beta\alpha)$

$= \alpha^2\beta^2 + 4\beta^2\alpha^2 + 2\beta\alpha(\alpha^2 + \beta^2)$

$= 5\alpha^2\beta^2 + 2\beta\alpha[(\alpha + \beta)^2 - 2\alpha\beta]$

$= \alpha^2\beta^2 + 2\beta\alpha(\alpha + \beta)^2$

$$= \frac{25}{9} - 2 \times \frac{5}{3} \times \frac{49}{9}$$

$$= \frac{75 - 490}{27} = -\frac{415}{27}$$

$27x^2 - 57x - 415 = 0$

49. For $x \geq 0$ $\quad x^2 - 5x + 4 = 0 \quad \Rightarrow x = 4, 1$

For $x \geq 0$ $\quad x^2 + 5x + 4 = 0 \quad \Rightarrow x = -1, -4$

50. Let $5^x = y$

The equation reduces to

$$5y + \frac{5}{y} = \log_{10}\left(\frac{100}{4}\right)$$

$$= 2 - \log_{10}4$$

$$= k(\text{say})$$

$5y^2 - ky + 5 = 0$

Discriminant of the quadratic $= k^2 - 100$

Since $k < 2$, $k^2 - 100 < 0$

Roots are complex.

$\Rightarrow$ Number of roots $= 0$

OR

Since AM $\geq$ GM

$$\frac{5^{1+x} + 5^{1-x}}{2} \geq \left(5^{1+x}.5^{1-x}\right)^{\frac{1}{2}}$$

$\Rightarrow$ $5^{1+x} + 5^{1-x} \geq 10$

i.e., LHS is greater than or equal to 10 but RHS is less than 2. $\Rightarrow$ No solution.

51. $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$

On squaring,

$$x + 1 + x - 1 - 2\sqrt{x+1}\sqrt{x-1} = 4x - 1$$

$\Rightarrow$ $2\sqrt{x^2 - 1} = 2x - 1$

Again squaring,

$4(x^2 - 1) = (2x - 1)^2 = 4x^2 - 4x + 1$

$\Rightarrow$ $-4 = -4x + 1$

$x = \frac{5}{4}$ which does not satisfy the given equation.

$\therefore$ no solution

52. $\alpha + \beta = \dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} = \dfrac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$

$$\frac{-b}{a} = \frac{\left(\frac{-b}{a}\right)^2 - 2\frac{c}{a}}{\frac{c^2}{a^2}} \Rightarrow \frac{-b}{a} = \frac{b^2 - 2ac}{c^2}$$

$$\Rightarrow \frac{-c}{a} = \frac{b^2 - 2ac}{bc} \Rightarrow \frac{-c}{a} = \frac{b}{c} - \frac{2a}{b}$$

$$\Rightarrow \frac{b}{c} + \frac{c}{a} = 2.\frac{a}{b}$$

53. $x = 2 + 2^{2/3} + 2^{1/3}$

$x - 2 = 2^{2/3} + 2^{1/3}$

Cubing on both sides,

$x^3 - 6x^2 + 12x - 8$

$= 4 + 2 + 3 \times 2^{4/3} \times 2^{1/3} + 3 \times 2^{2/3} \times 2^{2/3}$

$= 6 + 3 \times 2^{5/3} + 3 \times 2^{4/3} = 6 + 3 \times 2[2^{2/3} + 2^{1/3}]$

$\therefore$ $x^3 - 6x^2 + 12x - 8 = 6 + 6(x - 2)$

Or $x^3 - 6x^2 + 6x = 2$

54. Since $\left(p + 2\sqrt{q}\right)\left(p - 2\sqrt{q}\right) = p^2 - 4q = 1$,

$$p - 2\sqrt{q} = \frac{1}{p + 2\sqrt{q}}$$

Let $\left(p + 2\sqrt{q}\right)^{x^2 - 4x + 1}$ be denoted by y.

Then, we have $y + \dfrac{1}{y} = 2p$

$\Rightarrow$ $y^2 - 2py + 1 = 0$ (since y cannot be equal to zero)

$\Rightarrow$ $y = \dfrac{2p \pm \sqrt{4p^2 - 4}}{2} = p \pm \sqrt{p^2 - 1}$,

$= p \pm \sqrt{4q}$, since $p^2 - 4q = 1$

$= p \pm 2\sqrt{q}$

$y = p + 2\sqrt{q}$ means

$$\left(p + 2\sqrt{q}\right)^{x^2 - 4x + 1} = p + 2\sqrt{q}$$

$\Rightarrow$ $x^2 - 4x + 1 = 1$

$\Rightarrow$ $x^2 - 4x = 0$ or $x = 0, 4$

$y = p - 2\sqrt{q}$ means

$$\left(p + 2\sqrt{q}\right)^{x^2 - 4x + 1} = p - 2\sqrt{q}$$

$$= \left(p + 2\sqrt{q}\right)^{-1}$$

$\Rightarrow$ $x^2 - 4x + 1 = -1$

$\Rightarrow$ $x^2 - 4x + 2 = 0$ or $x = 2 \pm \sqrt{2}$

Solutions are $x = 0, 4, 2 \pm \sqrt{2}$

55. Let $y = x^{3/2n}$

$\Rightarrow$ $8y - \dfrac{8}{y} = 63$

$\Rightarrow$ $8y^2 - 63y - 8 = 0$

$\Rightarrow \quad y = 8 \text{ or } \frac{-1}{8} \Rightarrow x^{3/2n} = 8 \text{ or } \frac{-1}{8}$

$x = (2^3)^{\frac{2n}{3}} \text{ or } \left(\frac{-1}{2^3}\right)^{2n/3}$

$\therefore x = 2^{2n} \text{ or } \frac{1}{2^{2n}}$

56. Let $4 - x = t$

$\therefore \quad 3 - x = t - 1$

$5 - x = t + 1$

$\therefore \quad$ Equation $(t-1)^4 + (t+1)^4 = 16$

$2(t^4 + 6t^2 + 1) = 16$

$t^4 + 6t^2 - 7 = 0$

$(t^2 + 7)(t^2 - 1) = 0$

$t^2 = 1 \quad (t^2 = -7)$

$\therefore \quad t = \pm 1$

$\therefore \quad$ two real root.

57. Let $t = e^{\cos x}$

$\Rightarrow \quad t + \frac{7}{t} = 6 \Rightarrow t^2 - 6t + 7 = 0 \text{ as } t \neq 0$

$t = \frac{6 \pm \sqrt{8}}{2} \Rightarrow t = 3 \pm \sqrt{2}$.

$t = 3 + \sqrt{2}$

$e^{\cos x} = 3 + \sqrt{2} \Rightarrow e^{\cos x} > e$

$\Rightarrow \quad \cos x > 1$ which is not possible.

$t = 3 - \sqrt{2} \Rightarrow e^{\cos x} = 3 - \sqrt{2}$

$\therefore \cos x = \ln\left(3 - \sqrt{2}\right)$

Since $0 < x < \pi/2$, there is only one solution.

58. $x^2 + x - 6 = (x + 3)(x - 2)$

$x < -3$:

Equation is $x^2 + x - 6 - 2x - 4 = 0$

$\Rightarrow \quad x^2 - x - 10 = 0$

$\Rightarrow \quad x = \frac{1 \pm \sqrt{41}}{2}$; there is no root in this case

$-3 \leq x < 0$:

Equation is $-x^2 - x + 6 - 2x - 4 = 0$

$\Rightarrow \quad x^2 + 3x - 2 = 0$

$\Rightarrow \quad x = \frac{-3 \pm \sqrt{17}}{2}$; No roots in this interval.

$0 \leq x < 2$:

Equation is $-x^2 - x + 6 + 2x - 4 = 0$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad (x - 2)(x + 1) = 0$; No roots in this interval.

$x \geq 2$:

Equation is $x^2 + x - 6 + 2x - 4 = 0$

$\Rightarrow \quad x^2 + 3x - 10 = 0$

$\Rightarrow \quad (x + 5)(x - 2) = 0$; $x = 2$ is a root.

Combining, the number of real roots is 1.

59. The equation can be written as

$x^3(2x^2 + 7) + 3(2x^2 + 7) = 0$

$\Rightarrow \quad (x^3 + 3)(2x^2 + 7) = 0 \Rightarrow x^3 + 3 = 0$

or $2x^2 + 7 = 0$

$\Rightarrow \quad x^3 + 3 = 0$, since $2x^2 + 7$ cannot be zero.

$\Rightarrow \quad$ There is only one negative root.

60. $a - \sqrt{b} = \frac{1}{a + \sqrt{b}}$;

Let $\left(a + \sqrt{b}\right)^{x^2 - 15} = y$

Then equation is, $y + \frac{1}{y} = 2a \Rightarrow y^2 - 2ay + 1 = 0$

$\Rightarrow y = a \pm \sqrt{b}$

$\left(a + \sqrt{b}\right)^{x^2 - 15} = \left(a + \sqrt{b}\right) \text{ or } \left(\frac{1}{a + \sqrt{b}}\right)$

$\Rightarrow x^2 - 15 = 1 \text{ or } -1$. So $x = \pm 4, \pm\sqrt{14}$.

61. The equation can be rewritten as

$(6k + 2)x^2 + rx + (3k - 1) = 0$ and $(12k + 4)x^2 + px + (6k - 2) = 0$

$\Rightarrow \quad (6k + 2)x^2 + \frac{p}{2}x + (3k - 1) = 0$

On comparing the coefficients of x in both equations, we get $r = p/2 \Rightarrow p = 2r$.

62. If α is the common root,

$$\frac{\alpha^2}{-4\lambda - 16\lambda} = \frac{\alpha}{-4 - 6} = \frac{1}{-12\lambda + 2\lambda}$$

$$\Rightarrow \frac{\alpha^2}{-20\lambda} = \frac{\alpha}{-10} = \frac{1}{-10\lambda}$$

Since $\lambda \neq 0$, $\alpha^2 = 2, \alpha = \frac{1}{\lambda}$ or $\lambda = \pm\frac{1}{\sqrt{2}}$

63. We have $6x^4 + 3kx^2 + 2 = 0$

$\Rightarrow$ $3x(2x^3 + kx) + 2 = 0$

But $2x^3 + kx = 4$ and the two equations have a common root $\Rightarrow 3x(4) + 2 = 0$

$\Rightarrow$ $x = -\frac{1}{6}$;

Substituting this in the first equation,

$$-2 \times \frac{1}{216} - \frac{k}{6} - 4 = 0$$

$$\Rightarrow \frac{k}{6} = -\frac{1}{108} - 4 = -\frac{433}{108}$$

$$\therefore\ k = \frac{-433}{18}$$

64. The given equations are:

$x^2 + px + qr = 0$ — (1)

$x^2 + qx + rp = 0$ — (2)

and $x^2 + rx + pq = 0$ — (3)

Let α, β be the roots of equation (1), β, γ be the roots of equation (2) and γ, α be the roots of equation (3).

Since β is a common root of (1) and (2).

$\beta^2 + p\beta + qr = 0$

$\beta^2 + q\beta + rp = 0$

$\Rightarrow$ $(p - q)\beta + r(q - p) = 0 \Rightarrow \beta = r$

Now $\alpha\ \ \beta = qr \Rightarrow \alpha = q$

Since β and γ are true roots of (2)

$\beta.\gamma = r.p \Rightarrow r.\gamma \Rightarrow \gamma = p$

$\therefore$ $\alpha + \beta + \gamma = q + r + p = (p + q + r)$

But, sum of the roots is given by

$(\alpha + \beta) + (\beta + \gamma) + (\gamma + \alpha) = -(p + q + r)$

$\Rightarrow$ $2\sum\alpha = -\sum p$

$= -\sum\alpha$

$\Rightarrow$ $\sum\alpha = 0$

65. If α is the common root then

$\alpha^2 + a\alpha + b = 0$ — (1)

$\alpha^2 + b\alpha + a = 0$

$(a - b)\,\alpha = a - b \Rightarrow \alpha = 1$

$\therefore$ From (1) $1 + a + b = 0$

$\Rightarrow$ $a + b = -1$

66. Roots of $x^2 - x - 6 = 0$ are 3 and -2.

$\therefore$ $x = 3$ must satisfy $ax^2 + 3x + 9 = 0$

$\Rightarrow$ $9a + 9 + 9 = 0$

$\Rightarrow$ $a = -2$.

67. $\dfrac{x-2}{x+3} - \dfrac{2x-5}{4x-1} > 0$

$$\Rightarrow \frac{(x-2)(4x-1)-(x+3)(2x-5)}{(x+3)(4x-1)} > 0$$

$$\Rightarrow \frac{2x^2 - 10x + 17}{(x+3)(4x-1)} > 0$$

Since the discriminant of the numerator quadratic is less than zero, numerator is always positive for $x \in R$.

Hence, the inequality will be satisfied for

$x < -3$ or $x > \frac{1}{4}$

68. We must have $[2(a - 3)]^2 + 4a(a + 11) < 0$ and $(a + 11) > 0$

$\Rightarrow$ $(a - 3)^2 + a(a + 11) < 0$ and $a > -11$

$\Rightarrow$ $2a^2 + 5a + 9 < 0$ and $a > -11$

$(2a^2 + 5a + 9)$ is always positive for all values of a

$\Rightarrow$ The expression can never be positive for any real a.

69. $2^{2x^2-10x+3} + 2^{x^2-5x+1}\ 3^{x^2-5x+1}$

$-3^{2x^2-10x+3} \geq 0$

$2^{\{2(x^2-5x+1)+1\}} + 2^{(x^2-5x+1)} \times 3^{(x^2-5x+1)} - 3^{\{2(x^2-5x+1)+1\}} \geq 0$

Let $t = x^2 - 5x + 1$

$2.2^{2t} + 2^t.3^t - 3.3^{2t} \geq 0$

Again, let $2^t = u$ $3^t = v$

We have $2u^2 + uv - 3v^2 \geq 0$

$$2\left(\frac{u}{v}\right)^2 + \left(\frac{u}{v}\right) - 3 \geq 0$$

$$\Rightarrow 2\left(\frac{u}{v}\right)^2 + 3\left(\frac{u}{v}\right) - 2\left(\frac{u}{v}\right) - 3 \geq 0$$

$$\frac{u}{v}\left(2\left(\frac{u}{v}\right)+3\right) - \left(2\left(\frac{u}{v}\right)+3\right) \geq 0$$

$$\Rightarrow \left(\frac{u}{v} - 1\right)\left(2\left(\frac{u}{v}\right) + 3\right) \geq 0$$

$$\Rightarrow \left(\frac{u}{v}\right) \text{ must lie beyond } \frac{-3}{2} \text{ and } 1$$

Since $\frac{u}{v} = \left(\frac{2}{3}\right)^t$ cannot be negative, we conclude

$\frac{u}{v} \geq 1$]

$\Rightarrow \quad \left(\frac{2}{3}\right)^t \geq 1 \Rightarrow t \leq 0$

$x^2 - 5x + 1 \leq 0$

x must lie between the roots of the quadratic

$x^2 - 5x + 1 = 0$

$\Rightarrow \quad x \in \left[\frac{5-\sqrt{21}}{2}, \frac{5+\sqrt{21}}{2}\right]$

70. $x^2 + x - 2 > 0 \Rightarrow (x+2)(x-1) > 0$

$\Rightarrow x < -2$ or $x > 1$ — (1)

Also, $4 - x^2 - 3x > 0 \Rightarrow x^2 + 3x - 4 < 0$

$\Rightarrow -4 < x < 1$ — (2)

Combining (1) and (2), x must lie between – 4 and – 2

71. We must have, $m^2 - 2 < 0$ and

$4(m+3)^2 + 28(m^2 - 2) < 0$

$\Rightarrow \quad -\sqrt{2} < m < \sqrt{2}$ — (i) and

$8m^2 + 6m - 5 < 0 \Rightarrow (4m+5)(2m-1) < 0$

$\Rightarrow \quad m \in \left(-\frac{5}{4}, \frac{1}{2}\right)$ — (ii)

Combining (i) and (ii), $m \in \left(-\frac{5}{4}, \frac{1}{2}\right)$

72. $x^2 + 3x + 10 > 0$, for all real x as the discriminant is < 0.

$x^2 - x - 6 \leq 0 \Rightarrow (x-3)(x+2) \leq 0 \Rightarrow x \in [-2, 3]$

$2x^2 + 7x - 9 \geq 0 \Rightarrow (2x+9)(x-1) \geq 0$

$\Rightarrow \quad x \geq 1$ or $x \leq \frac{-9}{2} \Rightarrow$ Combining $x \in [1, 3]$

73. We have, $2x^2y + xy + 3y = x + 3$

$\Rightarrow \quad 2x^2y + x(y-1) + 3(y-1) = 0$

Since $x \in R$, discriminant of the above quadratic in $x \geq 0$

$(y-1)^2 - 24y(y-1) \geq 0$

or $\quad -(y-1)(23y+1) \geq 0$

or $\quad (y-1)(1+23y) \leq 0$

y lies between $\frac{-1}{23}$ and 1

74. Let $y = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$

$(x^2 + 2x + 4)y = x^2 - 2x + 4$

$(y-1)x^2 + (y+1)x + 4(y-1) = 0$

Roots of the above quadratic are real

$4(y = 1)^2 - 16(y-1)^2 \geq 0$

$(y+1)^2 - 4(y-1)^2 \geq 0$

$-3y^2 + 10y - 3 \geq 0$

$3y^2 - 10y + 3 \leq 0$

$(3y-1)(y-3) \leq 0$

y lies between $\frac{1}{3}$ and 3

Range is $\left[\frac{1}{3}, 3\right]$

75. Let $y = \dfrac{x^2 + 7x - 5}{x - 3}$

$x^2 + (7-y)x + 3y - 5 = 0$

Since x is real $\quad (7-y)^2 - 4(3y-5) \geq 0$

$y^2 - 26y + 69 \geq 0$

$\Rightarrow \quad (y-23)(y-3) \geq 0 \Rightarrow y \geq 23$ or $y \leq 3$

i.e., y takes all values which do not lie between 3 and 23.

76. Let $t = 2^{\sin^2 x}$

Equation reduces to $t + \frac{10}{t} = 7$

$t^2 - 7t + 10 = 0 \Rightarrow t = 5, 2$

$t = 5 \Rightarrow 2^{\sin^2 x} = 5$

As $\sin^2 x \leq 1, \Rightarrow t = 5$ is not admissible

$t = 1 \Rightarrow 2^{\sin^2 x} = 1$

$\sin^2 x = 1$

$\sin x = \pm 1$

$2\sin^2 x - 5\sin x + 4 = 2 \times 1 - 5 \times 1 + 4$ or

$2 \times 1 - 5x - 1 + 4$

$= 1$ or 11

77. If $x = \alpha, y = \beta$ satisfy the equations $2^{x+y} = 6^y$ and $3^{x-1} = 2^{y+1}$, $\alpha + \beta$ equals

$2^{x+y} = 6^y$

Taking logarithms to the base 10,

$(x+y)\log 2 = y\log 6$

$= y(\log 3 + \log 2)$

$\Rightarrow \quad x\log 2 = y\log 3$ — (1)

$3^{x-1} = 2^{y+1}$

Taking logarithms to the base 10,

$(x-1)\log 3 = (y+1)\log 2$

$x\log 3 - y\log 2 = \log 2 + \log 3 = \log 6$ — (2)

From (1) and (2), we get

$$\alpha = x = \frac{\log 6 \log 3}{(\log 3)^2 - (\log 2)^2}$$

$$= \frac{\log 3}{\log\left(\frac{3}{2}\right)}$$

$$\Rightarrow \quad \beta = y = \frac{\log 2}{\log\left(\frac{3}{2}\right)}$$

$$\Rightarrow \quad \alpha + \beta = \frac{\log 3 + \log 2}{\log \frac{3}{2}} = \log_{3/2} 6$$

78. $5\{x\} = x + 2[x]$

$\Rightarrow \quad 5[x - [x]) = x + 2[x]$

$4x = 7[x]$

$\Rightarrow \quad [x] = \frac{4}{7}x$

$x = 0, x = \frac{7}{4}$ satisfy the above equation

sum of the solutions $= \frac{7}{4}$

79. Let $x = \sqrt{15 - 2\sqrt{15 - 2\sqrt{15 \ldots}}}$

Then $x = \sqrt{15 - 2x}$

Squaring, $x^2 = 15 - 2x \Rightarrow x^2 + 2x - 15 = 0$

$\Rightarrow \quad (x + 5)(x - 3) = 0$

Since x cannot be negative, the solution is $x = 3$

80. We have, $(xy)(yz)(zx) = 28 \times 18 \times 14$

$= 7^2 \times 4^2 \times 3^2$

$\Rightarrow \quad xyz = 84$; ($xyz = -84$ is not admissible)

Using the given equations this gives,

$z = 3;\ x = \frac{14}{3};\ y = 6$

$\therefore \quad x + y + z = \frac{41}{3}$

81. $k^4 - 7k^2 - 8 = 0$ or $(k^2 - 8)(k^2 + 1) = 0$

$k^2 = 8, k^2 = -1$ or $k = \pm 2\sqrt{2}, \pm i$.

82. Put $5^x = k$

$k^2 - 30k + 125 = 0 \Rightarrow (k - 25)(k - 5) = 0$

$k = 5, 5^2$

$5^x = 5 \Rightarrow$ or $5^x = 5^2$ $\qquad \Rightarrow x = 1$ or 2.

83. $\sin\alpha + \cos\alpha = \frac{-q}{p};\ \sin\alpha \cdot \cos\alpha = \frac{r}{p}$

$\sin^2\alpha + \cos^2\alpha + 2\sin\alpha\cos\alpha = \frac{q^2}{p^2}$

$1 + \frac{2r}{p} = \frac{q^2}{p^2}$

$p^2 + 2pr = q^2 \Rightarrow p^2 - q^2 + 2pr = 0$

84. $at^2 + bt + c = 0$

$\therefore$ Sum of roots $= \frac{-b}{a}$ — (1)

Put $t = \frac{1}{x} \Rightarrow a.\frac{1}{x^2} + b.\frac{1}{x} + c = 0$

$\Rightarrow a + bx + cx^2 = 0$

Or $cx^2 + bx + a = 0$

$\therefore$ sum of roots $= \frac{-b}{c}$ — (2)

From (1) and (2) $a = c$.

OR

$\alpha, \frac{1}{\alpha}$ are the roots of the equation $\Rightarrow$ Product of the roots $\Rightarrow \frac{c}{a} = 1$ or $c = a$

85. $\text{sum} = \frac{1}{1 + \sqrt{3}} + \frac{1}{1 - \sqrt{3}}$

$= \frac{1 - \sqrt{3} + 1 + \sqrt{3}}{1 - 3} = \frac{-2}{2} = -1$

$\text{Product} = \frac{1}{1 - 3} = -\frac{1}{2}$

$\therefore$ Equations is $k^2 + k - \frac{1}{2} = 0$ and

$2k^2 + 2k - 1 = 0.$

86. Roots of the equation are 3 and $-\frac{1}{2}$

i.e., the polynomial is divisible by $(x - 3)$ and $\left(x + \frac{1}{2}\right)$.

87. By examining the equation, the values 2 and 3 are seen to satisfy the given equation.

88. $\alpha\beta = 7$

$\Rightarrow \quad 2e^{2\log k} - 1 = 7 \Rightarrow 2e^{\log k^2} - 1 = 7$

$\Rightarrow \quad 2k^2 - 1 = 7 \Rightarrow k = \pm 2.$

But $\log k$ is defined only for $k > 0 \Rightarrow k = 2$

89. Put $y = 2x$ or $x = \frac{y}{2}$

$\therefore \quad a\frac{y^2}{4} + b\frac{y}{2} + c = 0$

or $\quad ay^2 + 2by + 4c = 0$

$\equiv 3x^2 + 6x + 4c = 0 \Rightarrow a = 3 = b.$

90. $\alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$

$\alpha + \beta - 2 = \frac{-b}{a} - 2 = \frac{-8}{2}$

$\Rightarrow \frac{-b}{a} = -4 + 2 = -2$

$b = 2a$ not in choice.

$\alpha\beta = \frac{c}{a}, (\alpha - 1)(\beta - 1) = 1$

$(\alpha - 1)(\beta - 1) = \alpha\beta - \alpha - \beta + 1 = 1$

$\alpha\beta = (\alpha + \beta) \Rightarrow c = -b$

91. The roots are got as -2 and -15

Product of the roots must be 30 or $q = 30$ and therefore the correct equation is

$x^2 + 13x + 30 = 0.$

The roots are $-10, -3$.

92. $\alpha\beta = 4\beta^2 = p$

$5\beta = 5 \quad \Rightarrow \beta = 1$

$p = 4.$

93. $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \left(\frac{ac}{a^2}\right)^2 - 4.\frac{1}{a}$

$= \frac{a^2c^2}{a^4} - \frac{4}{a} = \frac{c^2}{a^2} - \frac{4}{a} = \frac{c^2 - 4a}{a^2}$

94. Let $y = \frac{1}{x + 2}$ where, $x = \alpha, \beta$

$\Rightarrow xy + 2y = 1 \Rightarrow x = \frac{1 - 2y}{y}$

Substituting in $3x^2 - x + 8 = 0$,

$\Rightarrow 3\left(\frac{1 - 2y}{y}\right)^2 - \left(\frac{1 - 2y}{y}\right) + 8 = 0$

$\Rightarrow 3(1 - 2y)^2 - y(1 - 2y) + 8y^2 = 0$

$\Rightarrow 22y^2 - 13y + 3 = 0$

The required equation is $22x^2 - 13x + 3 = 0$

95. $\frac{\alpha^3 + \beta^3}{\alpha^2 + \beta^2} = \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{(\alpha + \beta)^2 - 2\alpha\beta}$

$= \frac{-b^3/a^3 + 3\,c/a\,b/a}{b^2/a^2 - 2\,c/a} \Rightarrow \frac{3abc - b^3}{a(b^2 - 2ac)}$

96. $\frac{\alpha}{\alpha - 1} + \frac{\alpha + 1}{\alpha} = \frac{-b}{a}$ — (1)

$\frac{\alpha}{\alpha - 1} \times \frac{\alpha + 1}{\alpha} = \frac{c}{a} \Rightarrow \alpha = \frac{c + a}{c - a}$

$(1) \Rightarrow \frac{c + a}{c + a - (c - a)} + \frac{c + a + (c - a)}{c + a} = \frac{-b}{a}$

$\Rightarrow (c + a)^2 + 4ac = -2b(a + c)$

$c^2 + a^2 + 6ac + 2ab + 2bc = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ac = b^2 - 4ac$

i.e., $(b + a + c)^2 = b^2 - 4ac$

97. $t^2 + 6t + 9 + 9t - 9 + k = 0 \Rightarrow t^2 + 15t + k = 0$

$(\alpha + \beta) = -15, \ (\alpha - \beta) = 5$

$2\alpha = -10 \Rightarrow \alpha = -5, \beta = -10$

$k = \alpha\beta = -5 \times -10 = 50.$

98. $ax^2 + b|x| + c = 0$ represents two quadratic equations

(i) $x \geq 0 \Rightarrow ax^2 + bx + c = 0$

$\Rightarrow$ sum of the roots $= \frac{-b}{a}$

(ii) $x < 0 \Rightarrow ax^2 - bx + c = 0$

$\Rightarrow$ sum of the roots $= \frac{b}{a}$

$\Rightarrow$ sum of all roots $= 0$

99. $(x - 1)^2, (x - 2)^2, (x - 3)^2$ cannot be zero simultaneously for any real value of x.

100. $x^2 - 25 \neq 0 \Rightarrow x \neq \pm 5.$

If $\frac{1}{x^2 - 25}$ is cancelled on both sides the equation

$\Rightarrow x = 5$ which is not possible.

101. Let the roots be pk and qk {since p/q = ratio of roots}

Now, from the equation $px^2 - px + q = 0$

Sum of the roots $= (p + q)k = 1$

$\therefore \quad (p + q)^2 k^2 = 1$ — (1)

Product of roots $pqk^2 = \frac{q}{p}$

$\therefore \quad k^2 = \dfrac{1}{p^2}$ — (2)

Now, from (1), $(p + q)^2 = p^2$

$p^2 + q^2 + 2pq = p^2 \Rightarrow q\,(q + 2p) = 0$

$\Rightarrow \quad q + 2p = 0$

102. $2 + i\sqrt{3}$ and $2 - i\sqrt{3}$ are the roots

$-\,p = \text{sum} = 4 \Rightarrow p = -4$

$q = \text{product} = 4 + 3 = 7 \Rightarrow q = 7.$

103. By the given condition,

$(n - p)\,(n - q) - r = (n - l)\,(n - m)$

$\Rightarrow \quad (n - l)\,(n - m) + r = (n - p)\,(n - q)$

i.e., the roots of $(n - l)\,(n - m) + r = 0$ are p and q.

104. The equation can be written as

$x^2 - 2\,(a + b)\,x + 3ab = 0$

Since sum of the roots is zero, condition, coefficient of $x = 0$

i.e., $a + b = 0.$

105.

	I	II	III
b_1	c_1	a_1	b_1
b_2	c_2	a_2	b_2

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

For common root, $\dfrac{I}{II} = \dfrac{II}{III} \Rightarrow II^2 = (I)(III)$

106. Let α be the common root.

Then $(1 - 2a)\,\alpha^2 - 6a\alpha - 1 = 0$

$a\,\alpha^2 - \alpha + 1 = 0$

$$\frac{\alpha^2}{-(1 + 6a)} = \frac{\alpha}{-a - 1 + 2a} = \frac{1}{-1 + 2a + 6a^2}$$

$\therefore \quad (a - 1)^2 = -(1 + 6a)\,(6a^2 + 2a - 1)$

$\Rightarrow \quad 36a^3 + 19a^2 - 6a = 0$

$\Rightarrow \quad a = 0$ or $36a^2 + 19a - 6 = 0$

or $\quad (4a + 3)\,(9a - 2) = 0$

$a = 0$ or $a = -\dfrac{3}{4}$ or $a = \dfrac{2}{9}$

$\therefore \quad$ a can assume the values $\left(0, -\dfrac{3}{4}, \dfrac{2}{9}\right)$

107. $\dfrac{5}{9x^2 + 6x + 11}$ is maximum when $9x^2 + 6x + 11$ is least.

$9x^2 + 6x + 11 = (3x + 1)^2 + 10 \geq 10$

Equality holds when $3x + 1 = 0 \Rightarrow x = \dfrac{-1}{3}.$

108. $x^2 + 2bx + c = (x + b)^2 - b^2 + c > 0$

$\Rightarrow \quad c - b^2 > 0$

i.e., $c > b^2$ or $b^2 < c$.

109. $f(x) = 0$ is a quadratic equation having roots $x = 2$ and $x = 3$

So $f(x)$ is not always > 0 and $f(0) = 6$.

So $f(x) < 0$ when $2 < x < 3$

Now $\quad f(x) > 6 \quad \Rightarrow x^2 - 5x + 6 > 6$

$x^2 - 5x > 0 \quad \Rightarrow x > 5$ or $x < 0$

$\therefore \quad f(x) < 6 \quad \Rightarrow \quad 0 < x < 5$

110. $b^2 - 4\,ac = 25 - 24 = 1$

$x^2 = \dfrac{5 \pm 1}{2} = 3, 2.$

111. Statement 2 is false

consider statement 1

$a = -1$

$b^2 - 4ac = 16 - 4(-1)(-9) < 0$

Q has the same sign as that of a for all real x

$\Rightarrow \quad Q < 0$

statement 1 is true

112. Statement 2 is true

Consider statement 1

Using statement 2, $x = -1$ is a root of the equation

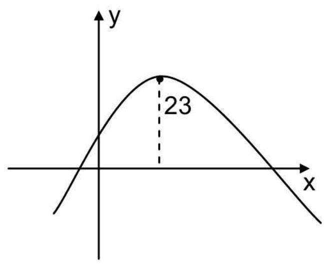
Dividing $(4x^3 + 7x^2 + 7x + 4)$ by $(x + 1)$, we get the quotient as $(4x^2 + 3x + 4)$

$4x^2 + 3x + 4 = 0$

Both roots of the above are complex

$\Rightarrow \quad$ Statement 1 is false

113.



Statement 2 is true

Using statement 2,

If k is chosen a value greater than 23, both roots of $Q - k = 0$ are complex.

Statement is true.

114. $x^2 - 10x + 6 = 0$

$S_{n+2} = \alpha^{n+2} + \beta^{n+2}, S_{n+1} = \alpha^{n+1} + \beta^{n+1}$

$S_n = \alpha^n + \beta^n$

$S_{n+2} - 10S_{n+1} + 6S_n$

$= \alpha^{n+2} + \beta^{n+2} - 10(\alpha^{n+1} + \beta^{n+1}) + 6(\alpha^n + \beta^n)$

$= \alpha^n(\alpha^2 - 10\alpha + 6) + \beta^n(\beta^2 - 10\beta + 6) = 0$

115. We have $S_1 = \alpha + \beta = 10$

$S_{n+2} = 10S_{n+1} - 6S_n$

$n = 1 \rightarrow S_3 = 10S_2 - 6S_1$ — (1)

$S_2 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 100 - 12 = 88$

Substitute in (1)

$S_3 = 10 \times 88 - 6 \times 10 = 880 - 60 = 940$

$n = 2 \rightarrow S_4 = 10S_3 - 6S_2$

$= 10 \times 820 - 6 \times 88 = 8200 - 6 \times 88 = 7672$

116. $S_4 - S_2S_3$

$= 7672 - 88 \times 820 = - 64488$

117. $\alpha + \beta = 3, \alpha\beta = -16$

(a) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \frac{9}{-16} - 2 = \frac{-9}{16} - 2 = -\frac{41}{16}$

$\frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$

Equation is $x^2 - \left(\frac{-41}{16}\right)x + 1 = 0$

$16x^2 + 41x + 16 = 0$

(b) $y = \frac{x+2}{x-2}$

$\Rightarrow \frac{x}{2} = \frac{y+1}{y-1}$

$x = \frac{2y+2}{(y-1)}$

Required equation is

$$\left(\frac{2y+2}{y-1}\right)^2 - 3\left(\frac{2y+2}{y-1}\right) - 16 = 0$$

$4y^2 + 4 + 8y - 6(y + 1)(y - 16(y - 1)^{@} = 0$

$-18y^2 + 40y - 6 = 0$

$9y^2 - 20y + 3 = 0$

(c) $x^2 - 3x - 16$

$$= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 16 = \left(x - \frac{3}{2}\right)^2 - \frac{73}{4}$$

Minimum value $= -\frac{73}{4}$

(d) Note that $3x + 2 = \alpha$ or β

$\Rightarrow x = \frac{\alpha - 2}{3}$ or $\frac{\beta - 2}{3}$

(d) is true

118. $\log_{|\sin x|}(x^2 - 8x + 23) > \frac{3}{\log_2 |\sin x|}$

$$\frac{\log_2(x^2 - 8x + 23)}{\log_2 |\sin x|} > \frac{3}{\log_2 |\sin x|}$$

As $\log_2 |\sin x| < 0$

$\therefore \log_2(x^2 - 8x + 23) < 3$

$\Rightarrow x^2 - 8x + 23 < 8$

$\Rightarrow x^2 - 8x + 15 < 0$

$\Rightarrow 3 < x < 5$

$\log_{|\sin x|}(x^2 - 8x + 23) > \frac{3}{\log_2 |\sin x|}$

$$\frac{\log_2(x^2 - 8x + 23)}{\log_2 |\sin x|} > \frac{3}{\log_2 |\sin x|}$$

As $\log_2 |\sin x| < 0$

$\therefore \log_2(x^2 - 8x + 23) < 3$

$\Rightarrow x^2 - 8x + 23 < 8$

$\Rightarrow x^2 - 8x + 15 < 0$

$\Rightarrow 3 < x < 5$

But the inequality does not hold when

$\sin x = 0, \pm 1$

i.e., when $x = 0, \pi, \pm\pi/2, \pm 3\pi/2,$ …….

119. The equation is $ax^2 + (b - \lambda) x + a - b - \lambda = 0$

The roots are real.

$\Delta \geq 0$

$(b-\lambda)^2 - 4a(a-b-\lambda) \geq 0$

$b^2 + \lambda^2 - 2b\lambda - 4a^2 + 4ab + 4a\lambda \geq 0$

$\lambda^2 + (4a-2b)\lambda - 4a^2 + 4ab + b^2 \geq 0$

This is true for all real values of λ.

$\therefore \quad (4a-2b)^2 - 4(-4a^2 + 4ab + b^2) < 0$

$(2a-b)^2 - (-4a^2 + 4ab + b^2) < 0$

$4a^2 + b^2 - 4ab + 4a^2 - 4ab - b^2 < 0$

$8a^2 - 8ab < 0$

$a^2 - ab < 0.$

$a^2 < ab$

ab is positive.

a and b are of the same sign.

Case I

$a > 0, b > 0$

$a^2 - ab < 0$

$a(a-b) < 0$

$a - b < 0$

$a < b$

$0 < a < b$

(c) is correct.

Case II

$a < 0, b < 0$

$a(a-b) < 0$

$a - b > 0$

$a > b$

$0 > a > b$

(b) is correct

120. (a) For the given condition,

$$k \geq \frac{25^x + 3}{5^x + 1} \quad (5^x + 1 > 0)$$

If $y = \dfrac{25^x + 3}{5^x + 1}$, $X^2 - yX + 3 - y = 0$

$(x = 5^x > 0), y > 0$

$\Rightarrow \quad y^2 - 4(3-y) \geq 0 \qquad \Rightarrow \quad (y+6)(y-2) \geq 0$

$\Rightarrow \quad y \leq -6 \text{ or } \geq 2, y > 0 \qquad \Rightarrow \quad y \geq 2.$

i.e., $y \geq 2 \Rightarrow k \geq 2.$

So (p), (r), (s).

(b) Let α and β denote the roots

$\alpha < \dfrac{1}{3}, \beta > \dfrac{1}{3}$

$\alpha - \dfrac{1}{3} < 0, \beta - \dfrac{1}{3} > 0$

$$\left(\alpha - \frac{1}{3}\right)\left(\beta - \frac{1}{3}\right) < 0$$

$$\alpha\beta - \frac{1}{3}(\alpha + \beta) + \frac{1}{9} < 0$$

$$\frac{1}{4} - \frac{1}{3} \times k + \frac{1}{9} < 0$$

$$\frac{13}{36} - \frac{k}{3} < 0$$

$$\frac{k}{3} > \frac{13}{36} \Rightarrow k > \frac{13}{12}$$

Discriminant of the quadratic > 0

$16k^2 - 16 > 0$

k lies beyond -1 and $+1$

since $x = \dfrac{1}{3}$ has to lie between the roots

$$4\left(\frac{1}{3}\right)^2 - 4k\left(\frac{1}{3}\right) + 1 < 0$$

$$\frac{13}{9} - \frac{4k}{3} < 0$$

$$\frac{4k}{3} > \frac{13}{9}$$

$$k > \frac{13}{12}$$

(c) If α, β, γ denote the roots, given $\alpha = -\beta$

sum of the roots $= -2$

$\Rightarrow \quad \alpha + \beta + \gamma = -2, \gamma = -2$

$-8 + 8 - 8 + k = 0 \Rightarrow k = 8$

(d) $\alpha^2 + \beta^2 = (k-2)^2 + 2(k+1)$

$= k^2 - 2k + 6$

minimum value of Q

$$= \left(\frac{k-2}{2}\right)^2 - (k-2)\frac{(k-2)}{2} - k - 1$$

$$= -\frac{(k-2)^2}{4} - k - 1 = \frac{-\{k^2 + 8\}}{4}$$

$$\alpha^2 + \beta^2 + 4m = (k^2 - 2k + 6) - \frac{4(k^2 + 8)}{4} = 0$$

$-2k + 6 - 8 = 0$

$\Rightarrow \quad k = -1$

(a) $\to$ r

(b) $\to$ p, r, s

(c) $\to$ p

(d) $\to$ q

Additional Practice Exercise

121. We have, Sum of the roots $= \alpha + \beta = \alpha + \alpha^2$

$$= \frac{-2b}{a}$$

Product of the roots $= \alpha\beta = \alpha^3 = \frac{3c}{a}$

Now $\left(\frac{-2b}{a}\right)^3 = \left(\alpha + \alpha^2\right)^3$

$$= \alpha^3 + \alpha^6 + 3\alpha^3\left(\alpha + \alpha^2\right)$$

$$= \frac{3c}{a} + \frac{9c^2}{a^2} + \frac{9c}{a}\left(\frac{-2b}{a}\right)$$

$\Rightarrow \quad -\frac{8b^3}{a^3} = \frac{3c}{a} + \frac{9c^2}{a^2} - \frac{18bc}{a^2}$

$\Rightarrow \quad 8b^3 + 3a^2c + 9ac^2 - 18abc = 0.$

$\Rightarrow \quad 3ac\,(a + 3c) = 18abc - 8b^3.$

122. Let α, β represent the roots of the equation $Ax^2 + Bx + C = 0$

Given: $|\alpha - \beta| =$ a constant $= k$ (say)

or $\quad (\alpha - \beta)^2 = k^2$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = k^2 \Rightarrow \frac{B^2}{A^2} - \frac{4C}{A} = k^2$

$\Rightarrow \quad \frac{B^2 - 4AC}{A^2} = k^2$ (a constant)

$\Rightarrow \quad \frac{B^2 - 4AC}{A^2}$ is an invariant

123. $a(x + b^2 - b) + b(x + a^2 - a)$

$= (x + b^2 - b)(x + a^2 - a)$

$x(a + b) + [a(b^2 - b) + b(a^2 - a)]$

$= x^2 + x(a^2 + b^2 - a - b) + (a^2 - a)(b^2 - b)$

Coefficient of $x = 0 \Rightarrow a^2 + b^2 - a - b - a - b = 0$

$\Rightarrow a^2 + b^2 = 2(a + b)$

124. $ax^2 + bx + c = 0 \quad$ — (1)

$a_1x^2 + b_1x + c_1 = 0 \quad$ — (2)

If α, β are the roots of (1) and γ, δ are the roots of (2), we are given $\frac{\alpha}{\beta} = \frac{\gamma}{\delta} = k$ (say) or $\alpha = k\beta, \ \gamma = k\delta$,

$$\alpha + \beta = \frac{-b}{a} \Rightarrow (k+1)\beta = \frac{-b}{a};$$

$$\alpha\beta = \frac{c}{a} \Rightarrow k\beta^2 = \frac{c}{a}$$

or $\quad \left\{\frac{-b}{a(k+1)}\right\}^2 = \frac{c}{ak} \Rightarrow kb^2 = ac(k+1)^2$

or $\quad \frac{b^2}{ac} = \frac{(k+1)^2}{k}.$

Similarly, $\frac{b_1^2}{a_1c_1} = \frac{(k+1)^2}{k} \Rightarrow \frac{b^2}{ac} = \frac{b_1^2}{a_1c_1}$

125. The roots of the two equations are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $\frac{-b_1 \pm \sqrt{b_1^2 - 4a_1c_1}}{2a_1}$.

Suppose $b^2 - 4ac$ and $b_1^2 - 4a_1c_1$ are not perfect squares. Roots are $\frac{-b \pm \sqrt{q}}{2a}$, $\frac{-b_1 \pm \sqrt{q_1}}{2a_1}$ where, $q = b^2 - 4ac$ and $q_1 = b_1^2 - 4a_1c_1$.

If $\frac{-b + \sqrt{q}}{2a} = \frac{-b_1 + \sqrt{q_1}}{2a_1}$, it immediately follows that $\frac{-b - \sqrt{q}}{2a} = \frac{-b_1 - \sqrt{q_1}}{2a_1}$, which contradicts the hypothesis that only one root is common.

$\therefore \quad b^2 - 4ac$ and $b_1^2 - 4a_1c_1$ are perfect squares.

126. We have $y = \frac{1 - px}{q}$

Substituting in the relation $rx^2 + sy^2 = 1$ and simplifying, we obtain

$$x^2\left(rq^2 + p^2s\right) - 2psx + \left(s - q^2\right) = 0$$

The above quadratic must have equal roots

$\Rightarrow 4p^2s^2 = 4\left(rq^2 + p^2s\right)\left(s - q^2\right)$

$$rs = q^2r + p^2s \quad \text{or} \quad \frac{p^2}{r} + \frac{q^2}{s} = 1$$

Sum of the roots $= \frac{2ps}{rq^2 + p^2s} = \frac{2ps}{rs}$

or $\quad x = \frac{p}{r}$ and $y = \frac{q}{s}$

127. Since the two equations have a common root

$$\frac{\alpha^2}{ab^2 - ac^2} = \frac{\alpha}{ca - ab} = \frac{1}{c - b}$$

or $\alpha = a$ and $\alpha^2 = -a(b+c) \Rightarrow b+c+a = 0$

If α, β are the roots of $x^2 + bx + ca = 0$

$$\alpha + \beta = -b \Rightarrow \beta = -b - a = -(a+b) = c$$

If α, γ are the roots of $x^2 + cx + ab = 0$,

$$\alpha + \gamma = -c \qquad \Rightarrow \gamma = -c - a = -(c+a) = b$$

Equation whose roots are α and γ is

$$x^2 - (b+c)x + bc = 0 \quad \text{or} \quad x^2 + ax + bc = 0$$

128. Let $u = \dfrac{y^2 - 2y + k^2}{y^2 + 2y + k^2}$

$$(y^2 + 2y + k^2)u = y^2 - 2y + k^2$$

$$\Rightarrow \quad y^2(u-1) + 2y(u+1) + k^2(u-1) = 0$$

Since $y \in R, 4(u+1)^2 - 4k^2(u-1)^2 \geq 0$

$$[(u+1) - k(u-1)][(u+1) + k(u-1)] \geq 0$$

$$[u(1-k) + (1+k)][u(1+k) + (1-k)] \geq 0$$

$$\left(u - \frac{k+1}{k-1}\right)\left(u - \frac{k-1}{k+1}\right) \leq 0, \text{since } k^2 - 1 \neq 0$$

OR u must lie between $\left(\dfrac{k-1}{k+1}\right)$ and $\left(\dfrac{k+1}{k-1}\right)$.

129. $\left|\dfrac{12x}{4x^2+9}\right| \leq 1 \Leftrightarrow -1 \leq \dfrac{12x}{4x^2+9} \leq 1$

Let $\dfrac{12x}{4x^2+9} = y$ we have to prove that $-1 \leq y \leq 1$

Now $12x = y(4x^2 + 9 \Rightarrow 4yx^2 - 12x + 9y = 0$

As x is real, discriminant ≥ 0

$\Rightarrow \quad 144 - 4.4y.9y \geq 0$ or $1 - y^2 \geq 0$ or $y^2 \leq 1$

$\therefore \quad -1 \leq y \leq 1 \therefore |y| \leq 1$ so $\left|\dfrac{12x}{4x^2+9}\right| \leq 1$

Here the equality will hold if $\left|\dfrac{12x}{4x^2+9}\right| = 1$,

or $\dfrac{12|x|}{|4x^2+9|} = 1 \quad \left[\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|};\ |ab| = |a|.|b|\right]$

$|4x^2 + 9| = 4x^2 + 9$ because $4x^2 + 9$ is positive

$\therefore \quad 12|x| = 4x^2 + 9$ or $4|x|^2 - 12|x| + 9 = 0$

$\Rightarrow \quad (2|x| - 3)^2 = 0$ or $2|x| = 3$ or $|x| = \dfrac{3}{2}$.

130. Writing the second degree equation as a quadratic in x

$$x^2 + 2x(6y+2) + (4y^2 + 8y + 20) = 0$$

Since x is real, $4(6y+2)^2 - 4(4y^2 + 8y + 20) \geq 0$

$$\Rightarrow \quad (3y+1)^2 - (y^2 + 2y + 5) \geq 0$$

$$8y^2 + 4y - 4 \geq 0 \text{ OR } 2y^2 + y - 1 \geq 0$$

$\Rightarrow \quad$ y lies beyond (-1) and $\dfrac{1}{2}$

Similarly, writing the second degree equation as a quadratic in y

$$4y^2 + 2y(4 + 6x) + (x^2 + 4x + 20) = 0$$

Since y is real, $4(4+6x)^2 - 16(x^2 + 4x + 20) \geq 0$

$$(3x+2)^2 - (x^2 + 4x + 20) \geq 0$$

$$8x^2 + 8x - 16 \geq 0 \text{ OR } x^2 + x - 2 \geq 0$$

$\Rightarrow \quad$ x lies beyond (-2) and 1.

131. The given equation is $\log_5(5^{1/x} + 125) =$

$$\log_5 6 + \log_5 5 + \frac{1}{2x}$$

$$= \log_5 30 + \frac{1}{2x}$$

$$\log_5\left(\frac{5^{1/x} + 125}{30}\right) = \frac{1}{2x}$$

$$5^{1/x} + 125 = 30 \times 5^{1/2x}$$

Let $\quad t = 5^{1/2x} \Rightarrow t^2 - 30t + 125 = 0$

$t = 25$ or 5

$$5^{1/2x} = 5^2 \text{ or } 5^{1/2x} = 5^1 \Rightarrow x = \frac{1}{4}, \frac{1}{2}$$

132. The second equation reduces to $\log_3(y - 2x) = 1$

$\Rightarrow \quad y - 2x = 3 \qquad$ — (1)

Taking logarithm on both sides of the first equation to the base x,

$\log_x y + \log_y x = 2.5 \qquad$ — (2)

Let $\quad \log_x y$ be t

(2) reduces to $t + \dfrac{1}{t} = 2.5 \Rightarrow t = 2, \dfrac{1}{2}$

$\log_x y = 2 \Rightarrow y = x^2$ and using (1)

$\Rightarrow 2x + 3 = x^2 \Rightarrow x = 3, -1$.

Since $x = -1$ is not admissible, $x = 3$ is a solution.

Again, $\log_x y = \frac{1}{2} \Rightarrow y = \sqrt{x} \Rightarrow 2x + 3 = \sqrt{x}$

$\therefore$ No real solution.

$x = 3, y = 9$ is the only solution.

133. For the roots to be rational, discriminant must be perfect square.

$\left[(2a-1)^2 - 4a(a-2)\right]$ is a perfect square or

$[4a + 1]$ is a perfect square.

But $4a + 1$ is odd. So it is an odd perfect square.

So let $4a + 1 = (2n + 1)^2$ $\quad n = 1, 2, 3, \ldots$

$= 4n^2 + 4n + 1$

$\therefore$ $4a = 4n(n + 1)$ $\quad \Rightarrow \therefore a = n(n + 1)$,

$n = 1, 2, 3, \ldots$

134. Let $ax^2 + bx + c = 0$ be an equation satisfying given condition

If α and β are the roots of the above equation

We must have $\alpha^2 + \beta^2 = \alpha + \beta, \alpha^2\beta^2 = \alpha\beta$

or $\quad \frac{b^2 - 2ac}{a^2} = \frac{-b}{a}$ and $\frac{c^2}{a^2} = \frac{c}{a}$

The second condition gives $\frac{c}{a} = 0$ or $\frac{c}{a} = 1$

$\Rightarrow$ $c = 0$ or $c = a$ (Note that a cannot be zero)

Case 1 $\quad c = 0$

The first condition gives $\frac{b^2}{a^2} = \frac{-b}{a} \Rightarrow b = 0$ or $-a$

Case 2 $\quad c = a$

The first condition gives $b^2 - 2a^2 + ab = 0$

$\Rightarrow$ $(b + 2a)(b - a) = 0 \Rightarrow b = a$ or $-2a$

Hence, we get 4 and only 4 solutions

They are $c = 0, b = 0$; $c = 0, b = -a$; $c = a, b = a$ and $c = a, b = -2a$

The corresponding quadratic equations are

$x^2 = 0$; $x^2 - x = 0$, $x^2 + x + 1 = 0$ and

$x^2 - 2x + 1 = 0$

135. It is clear that there is no solution for $|x - 5| < 3$ or there is no solution lying between 2 and 8.

Also $|x - 5| \neq 3$ or $x \neq 2, 8$.

Let $|x - 5| > 3$ i.e., $x < 2$ or $x > 8$.

The inequality becomes $9 \geq \{|x - 5| - 3\}\, |x - 2|$

$\Rightarrow$ $x < 2$

We have $9 \geq \{5 - x - 3\}\{2 - x\}$ or $(2 - x)^2 \leq 9$

$\Rightarrow$ $(2 - x)$ lies between -3 and $+3$

$\Rightarrow$ $(2 - x) \geq -3$ and $(2 - x) \leq +3$

$\Rightarrow$ $x \leq 5$ and $x \geq -1$

Since $x < 2$, solution is $x \in [-1, 2)$

$x > 8$

We have $9 \geq (x - 8)(x - 2)$

$\Rightarrow$ $x^2 - 10x + 7 \leq 0$

Roots of $x^2 - 10x + 7 = 0$ are $5 \pm 3\sqrt{2}$

Since $x > 8$, the solution is $x \in \left(8, 5 + 3\sqrt{2}\right]$

Combining, the solution of the inequality is

$x \in [-1, 2) \cup \left(8, 5 + 3\sqrt{2}\right]$

136. **Case 1**

$0 < \frac{x + 4}{2} < 1$

or $\quad 0 < x + 4 < 2 \quad$ or $\quad -4 < x < -2$ — (1)

Given inequality becomes $\log_2\left(\frac{3x - 1}{3 + x}\right) > 1$

$1 < \frac{3x - 1}{3 + x} < 2$

$\frac{3x - 1}{3 + x} > 2$

$\Rightarrow$ $\frac{3x - 1}{3 + x} - 2 > 0 \Rightarrow \frac{x - 7}{3 + x} > 0$

$x > 7$ and $x > -3$ — (2)

From (1) and (2) we see that there is no solution

Again $x < 7$ and $x < -3$ — (3)

From (1) and (3), x must lie between -4 and -3

Case 2

$\frac{x + 4}{2} > 1$ or $x > -2$

Given inequality becomes

$0 < \log_2\left(\frac{3x - 1}{3 + x}\right) < 1$

$\Rightarrow$ $\frac{3x - 1}{3 + x} - 1 > 0$ and $\frac{3x - 1}{3 + x} - 2 < 0$

$\Rightarrow$ $\frac{2x - 4}{x + 3} > 0$ and $\frac{x - 7}{x + 3} < 0$

$x < 7$ and $x > -3$ — (5)

$(x < -3$ or $x > 2)$ and $-3 < x < 7 \Rightarrow 2 < x < 7$

Combining cases 1 and 2 we get the solution as $(-4 < x < -3) \cup (2 < x < 7)$

137. The equation may be rewritten as $\sqrt{2x-3} + \sqrt{3x-5} = \sqrt{5x-6}$

Squaring both sides,

$$2x-3+3x-5+2\sqrt{(2x-3)(3x-5)} = 5x-6$$

$\Rightarrow\ 2\sqrt{(2x-3)(3x-5)} = -6+8 = 2$

$\Rightarrow\ (2x-3)(3x-5) = 1$

$\Rightarrow\ 6x^2 - 19x + 14 = 0$

$\Rightarrow\ (x-2)(6x-7) = 0 \Rightarrow x = 2, \dfrac{7}{6}$

Clearly, $x = 2$ satisfies the equation and $x = \dfrac{7}{6}$ does not satisfy the equation.

138. It is very clear that $x = 3$ satisfies the given equation. We check whether there is any other solution.

Note that $y = 3^{x-1}$ and $y = 5^{x-1}$ are both increasing functions of x. Therefore their sum $y = 3^{x-1} + 5^{x-1}$ is also an increasing function of x.

This means for $x < 3$ it is $3^{x-1} + 5^{x-1} < 34$ and for $x > 3$, $3^{x-1} + 5^{x-1} > 34$. Thus the equation has no other solution.

139. The given equation can be rewritten as

$\dfrac{3(1-t^2)}{1+t^2} + \dfrac{4\times 2t}{1+t^2} = k$, where t stands for $\tan\theta$.

Simplification gives $(k+3)t^2 - 8t + k - 3 = 0$.

$\tan\theta_1$ and $\tan\theta_2$ are the roots.

$\therefore\ \tan\theta_1 + \tan\theta_2 = \dfrac{8}{k+3}$.

140. We know that $\left(a+\sqrt{a^2-1}\right)\left(a-\sqrt{a^2-1}\right) = a^2 - \left(a^2-1\right) = 1$

$\therefore\ a - \sqrt{a^2-1} = \dfrac{1}{a+\sqrt{a^2-1}}$

The above can be rewritten as $\left(a+\sqrt{a^2-1}\right)^{x^2-2x} + \left(\dfrac{1}{\left(a+\sqrt{a^2-1}\right)}\right)^{x^2-2x} = 2a$

$$\left\{\left(a+\sqrt{a^2-1}\right)^{x^2-2x}\right\}^2 - 2a\left(a+\sqrt{a^2-1}\right)^{x^2-2x} + 1 = 0$$

Solving the above equation,

$$\left(a+\sqrt{a^2-1}\right)^{x^2-2x} = \frac{2a \pm \sqrt{4a^2-4}}{2} = a \pm \sqrt{a^2-1}$$

$$= a+\sqrt{a^2-1}, \frac{1}{a+\sqrt{a^2-1}}$$

$$= a+\sqrt{a^2-1}, \left(a+\sqrt{a^2-1}\right)^{-1}$$

$\therefore\ x^2 - 2x = 1$ or -1

If $x^2 - 2x = 1\ \ \therefore x^2 - 2x - 1 = 0$, $x = 1 \pm \sqrt{2}$

If $x^2 - 2x = -1\ \ \therefore x^2 - 2x + 1 = 0$, $x = 1$

The solutions of the equation are $x = 1, 1 \pm \sqrt{2}$ which are independent of a.

141. The given equation will have equal roots if discriminant = 0

$\Rightarrow (27 \times 3^{1/p} - 15)^2 - 4 \times 4 \times 9 = 0$

$\Rightarrow (27 \times 3^{1/p} - 15)^2 = 144 \Rightarrow 27 \times 3^{1/p} - 15 = \pm 12$

$\Rightarrow 27 \times 3^{1/p} = 27$

$\Rightarrow 3^{1/p} = 1$ or $\Rightarrow 27 \times 3^{1/p} = -12 + 15$

$\Rightarrow 3^{1/p} = \dfrac{1}{9}$

so $\dfrac{1}{p} = 0$ or -2.

As $\dfrac{1}{p}$ cannot be zero, we have $\dfrac{1}{p} = -2$

$\therefore\ p = \dfrac{-1}{2}$

142. Discriminant = 0 $\Rightarrow (2m)^2 - 4(8m - 15) = 0$

$4(m^2 - 8m + 15) = 0 \Rightarrow (m-5)(m-3) = 0$

$\therefore\ m = 5$ or 3

143. The equation is $2ax^2 + (1-a^2)x - (a^3 + a) = 0$

$$x = \frac{\left(a^2-1\right) \pm \sqrt{\left(a^2-1\right)^2 + 8a\left(a^3+a\right)}}{4a}$$

$$= \frac{\left(a^2-1\right) \pm \sqrt{9a^4 + 6a^2 + 1}}{4a}$$

$$= \frac{\left(a^2-1\right) \pm \left(3a^2+1\right)}{4a} = a, \frac{-\left(1+a^2\right)}{2a}$$

144. Since a and x are integers. Therefore, $(x-a)(x-10) + 1 = 0$

$\Rightarrow$ $(x - a)(x - 10) = -1$

$\Rightarrow$ $x - a = 1$ and $x - 10 = -1$ or $x - a = -1$ and $x - 10 = 1$

$\therefore$ $(x = 9$ and $a = 8)$ or $(x = 11$ and $a = 12)$

$\Rightarrow$ $a = 8$ or 12

145. Solving the above equation we get $|x - 1| = 3$ and $|x - 1| = 2$

$x - 1 = \pm 3$ and $x - 1 = \pm 2$

$x = 4, -2 \qquad x = 3, -1.$

So, the equation has four roots.

146. Let us form the equation whose roots are $(\alpha - 3)$ and $(\beta - 3)$.

Setting $y = x - 3$, the roots of the equation

$$2(y + 3)^2 - 6(y + 3) + 1 = 0$$

$$2y^2 + 6y + 1 = 0 \text{ are } \alpha - 3, \beta - 3$$

The equation whose roots are $\frac{1}{\alpha - 3}, \frac{1}{\beta - 3}$ is therefore $x^2 + 6x + 2 = 0$

147. Case I: $x < 1$

The inequality is $1 - x + 2 - x \geq 6$

$\Rightarrow$ $-2x \geq 3 \Rightarrow x \leq \frac{-3}{2}$

Case II: $1 \leq x < 2$

The inequality is $x - 1 + 2 - x \geq 6$

$\Rightarrow$ $1 \geq 6$ which is incorrect.

Case III: $x \geq 2$

The inequality is $2x - 3 \geq 6$

$\Rightarrow$ $x \geq \frac{9}{2}$

$\therefore$ $x \in \left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{9}{2}, \infty\right)$

148. Sum of the roots $= \alpha + \alpha^2 = -b/a$

Product of the roots $= \alpha^3 = \frac{c}{a}$

$\therefore (\alpha + \alpha^2)^3 = \frac{-b^3}{a^3}$ or $\alpha^3 + \alpha^6 + 3\alpha.\alpha^2(\alpha + \alpha^2) = \frac{-b^3}{a^3}$

or $\frac{c}{a} + \frac{c^2}{a^2} + \frac{3c}{a}\left(\frac{-b}{a}\right) = \frac{-b^3}{a^3}$ or

$b^3 + ac(a + c) = 3abc$

149. Conditions for both the roots to be greater than 2 are

(i) Discriminant ≥ 0

(ii) $f(2) > 0$ where $f(x) = x^2 - 2kx + k^2 - 2k + 6$ and

(iii) Sum of the roots > 4

(i) $\Rightarrow 4k^2 - 4(k^2 - 2k + 6) \geq 0 \Rightarrow 2k - 6 \geq 0 \Rightarrow k \geq 3$

(ii) $\Rightarrow 4 - 4k + k^2 - 2k + 6 > 0 \Rightarrow k^2 - 6k + 10 > 0$ which is true for all real k.

(iii) $\Rightarrow 2k > 4 \Rightarrow k > 2.$

Combining the answer is $k \geq 3$.

150. The equation is $(p^2 + p)x^2 + (p^2 + 2)x + (2p + 2) = 0$

$$\alpha + \beta = \frac{-(p^2 + 2)}{p^2 + p}, \alpha\beta = \frac{2p + 2}{(p^2 + p)} = \frac{2}{p} \qquad (p \neq -1)$$

$$\alpha^2\beta^2 + (\alpha\beta + 2)(\alpha + \beta) - 1$$

$$= \frac{4}{p^2} - \left(\frac{2}{p} + 2\right)\left(\frac{p^2 + 2}{p^2 + p}\right) - 1 = -2 - 1 = -3$$

151. Discriminant of the equation $= 4(c + a - 2b)^2 - 4(b + c - 2a)(a + b - 2c)$

$$= 4\left\{c^2 + a^2 + 4b^2 + 2ac - 4bc - 4ab - \begin{bmatrix} ba + b^2 - 2bc + ca + bc - \\ 2c^2 - 2a^2 - 2ab + 4ac \end{bmatrix}\right\}$$

$$= 4\{3a^2 + 3b^2 + 3c^2 - 3bc - 3ca - 3ab\}$$

$$= \frac{12}{2}\left\{(b - c)^2 + (c - a)^2 + (a - b)^2\right\}$$

> 0

We have $a \neq b \neq c$. Otherwise, the equation reduces to an identity.

$\Rightarrow$ Roots are real and distinct.

152. The equation is $x^{x^{3/2}} = \left(x^{3/2}\right)^x = x^{3x/2}$

$x = 1$ is a positive solution.

We have $x^{3/2} = \frac{3x}{2}$ or $x^{1/2} = \frac{3}{2}$ or $x = \frac{9}{4}$

Sum of positive solutions $= 1 + \frac{9}{4} = \frac{13}{4}$

153. $2x^2 + 3ax + 2b = 0 \qquad$ — (i)

$2x^2 + 3bx + 2a = 0 \qquad$ — (ii)

Let α_1 and α_2 be the roots of

(i) and β_1 and β_2 be the roots of

(ii) $(\alpha_1 - \alpha_2)^2 = (\alpha_1 + \alpha_2)^2 - 4\alpha_1\alpha_2$

Similarly, $(\beta_1 - \beta_2)^2 = (\beta_1 + \beta_2)^2 - 4\beta_1\beta_2$

$$\left(\frac{-3a}{2}\right)^2 - \frac{4 \times 2b}{2} = \left(\frac{-3b}{2}\right)^2 - \frac{4 \times 2a}{2}$$

$$\frac{9a^2}{4} - 4b = \frac{9b^2}{4} - 4a \Rightarrow 4(a-b) = \frac{9}{4}(b^2 - a^2)$$

Since $a \neq b$, $\frac{-9}{4}(a+b) = 4 \Rightarrow a + b = \frac{-16}{9}$

154. Let $3^x = y$. Then the inequality becomes

$ay^2 + 4(a-1)y + a > 1$ or $a(y^2 + 4y + 1) > 4y + 1$

or $a\{(y+2)^2 - 3\} > 4y + 1$ — (1)

As $y = 3^x > 0$, $4y + 1 > 1$ and minimum value of $(y+2)^2$ is greater than 4. $\therefore$ From (1)

$a\{(y+2)^2 - 3\} > 4y + 1 > 1$

$\therefore a > \frac{1}{(y+2)^2 - 3}$, since $(y+2)^2 - 3 > 4 - 3 > 1$

$= -\frac{-3}{1}$

If $a \geq 1$, then $a > \frac{1}{(y+2)^2 - 3}$ holds good for all y or for all real x (since $3^x = y$)

So a must be ≥ 1.

155. $a^{\log_a x} = x$

$x^2 - 4x + 5 = x - 1 \quad \Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x = 2, 3.$

156. $\alpha + \beta = 9; \alpha\beta = 14; \Rightarrow (\alpha - \beta) = 5$

157. $\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\alpha^3 + \beta^3}{(\alpha\beta)^3}$; Also $\alpha + \beta = -1; \alpha\beta = 1$

$$= \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3} = \frac{-1+3}{1} = 2.$$

158. $n^2 - n - 6 = n + 2 \Rightarrow n = -2, 4$

$n^2 - n - 6 = -n - 2 \Rightarrow n = \pm 2.$

159. $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3) = (\alpha + \beta)(\alpha - \beta)^2(\alpha^2 + \alpha\beta + \beta^2)$

$= (\alpha + \beta)\left[(\alpha+\beta)^2 - 4\alpha\beta\right][(\alpha+\beta)^2 - \alpha\beta]$

$= 1[1 + 4][1 + 1] = 10$

$\alpha^3\beta^2 + \beta^3\alpha^2 = \alpha^2\beta^2(\alpha + \beta) = 1.1 = 1$

Roots are 10 and 1

Equation is $x^2 - 11x + 10 = 0$.

160. $$\frac{\alpha^{2/3}}{\beta^{1/3}} + \frac{\beta^{2/3}}{\alpha^{1/3}} = \frac{\alpha + \beta}{(\alpha\beta)^{1/3}} = \frac{3/8}{\left(\frac{27}{8}\right)^{1/3}}$$

$$= \frac{3/8}{3/2} = \frac{3}{8} \times \frac{2}{3} = \frac{1}{4}.$$

161. Let α, β be the roots of $ax^2 + 2bx + c = 0$ and γ and δ be the roots of $px^2 + 2qx + r = 0$.

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} = k$$

$$\alpha = k\beta, \gamma = k\delta \text{ and } \beta + k\beta = \frac{-2b}{a}$$

$$\Rightarrow \beta(k+1) = \frac{-2b}{a}, k\beta^2 = \frac{c}{a} \Rightarrow \frac{k}{(k+1)^2} = \frac{ac}{4b^2}$$

It can be proved that $\frac{k}{(k+1)^2} = \frac{pr}{4q^2}$.

162. Quadratic equations of the form $ax^2 + bx + c = 0$ have real roots if $b^2 - 4ac \geq 0$

i.e., $b^2 \geq 4ac$

and a, b, c are taken from $\{2, 3, 6, 7\}$

So if b = 2 or 3, $b^2 < 4ac$ for all possible values of a and c.

If $b = 6$ $b^2 = 36 >$ $4 \times 2 \times 3$,
$4 \times 3 \times 2$

If $b = 7$ $b^2 = 49 >$ $4 \times 2 \times 3$,
$4 \times 3 \times 2$
$4 \times 2 \times 6$
$4 \times 6 \times 2$

So the total number of such equations is 6.

163. Since $a < 1$, $4 - 4a > 0$ and the roots of $x^2 - 2x + a = 0$ are distinct. Since $b^2 + 4$ is positive but not a perfect square the roots of $x^2 + bx - 1 = 0$ are of the form $p \pm \sqrt{q}$. Since the two equations have a common root, the first equation has a root of the form $p + \sqrt{q}$ or $p - \sqrt{q}$. Since irrational roots occur in conjugate pairs both $p + \sqrt{q}$ and $p - \sqrt{q}$ are roots of $x^2 - 2x + a = 0$. So, both the equations have the same pair of roots. Corresponding co-efficients are proportional.

$$\frac{1}{1} = \frac{-2}{b} = \frac{a}{-1}$$

$b = -2 \quad a = -1$

$(a, b) = (-1, -2)$

This point lies on $x + y + 3 = 0$

164. Roots may be assumed as α, β, γ.

Given $\beta^2 = \alpha\gamma$

Product of the roots $= \alpha\beta\gamma = -\dfrac{8}{a^3}$

$\Rightarrow \quad \beta^3 = -\dfrac{8}{a^3} \Rightarrow \beta = \dfrac{-2}{a}$.

β is the root of the equation. Substituting in the equation, we get $ac + 6 = 0$.

165. Sum $= (\sin\alpha - 2)$

Product $= -(1 + \sin\alpha)$

Let the roots of the given equation be p and q.

$p^2 + q^2 = (p + q)^2 - 2pq$

$= (\sin\alpha - 2)^2 + 2(1 + \sin\alpha)$

$= \sin^2\alpha + 4 - 4\sin\alpha + 2 + 2\sin\alpha$

$= \sin^2\alpha - 2\sin\alpha + 6$

$= (\sin\alpha - 1)^2 + 5$ is maximum when

$(\sin\alpha - 1)^2$ is maximum

i.e., $|\sin\alpha - 1|$ is maximum

$\sin\alpha = -1$ when $\alpha = \dfrac{3\pi}{2}$.

166. Maximum value of $3\sin x$ is 3. Since the discriminant of the quadratic expression $7x^2 - 4x + 5$ is negative, and coefficient of $x^2 > 0$, $y = 7x^2 - 4x + 5$ is always positive for all real x. Minimum value of $7x^2 - 4x + 5$ is attained at $x = \dfrac{2}{7}$ and it is equal to $7 \times \dfrac{4}{29} - \dfrac{4 \times 2}{7} + 5 = 5 - \dfrac{4}{7} = \dfrac{31}{7} > 3$ and therefore the two curves do not intersect. The number of points of intersection is zero.

167. $x^2 + x - 2 \geq 0 \Rightarrow (x + 2)(x - 1) \geq 0$

$\Rightarrow \quad x \leq -2$ or $x \geq 1$ — (i)

$2x^2 - 9x - 5 \leq 0 \Rightarrow 2x^2 - 10x + x - 5 \leq 0$

$\Rightarrow \quad 2x(x - 5) + 1(x - 5) \leq 0 \Rightarrow \dfrac{-1}{2} \leq x \leq 5$ — (ii)

$x^2 - 5x + 6 \geq 0 \Rightarrow (x - 2)(x - 3) \geq 0$

$\Rightarrow \quad x \leq 2$ or $x \geq 3$ — (iii)

combining (i), (ii) and (iii), we get

$1 \leq x \leq 2$ or $3 \leq x \leq 5$.

168. A fifth degree polynomial equation with real coefficients can have 1, 3 or 5 real roots ($\because$ complex roots of such an equation occur in conjugate pair)

Let $f(x) = 3x^5 + 10x^3 + 30x + 7$

$f'(x) = 15x^4 + 30x^2 + 30$

$= 15[(x^2 + 1)^2 + 1]$

> 0 for all real x.

$\Rightarrow$ f(x) is monotonically increasing for all real x.

As a m.i. function can have at most 1 real root.

$\Rightarrow$ f(x) has exactly 1 real root

169. Let α and β be the roots one root is less than 2 and the other is greater than 2.

$\alpha < 2 < \beta$.

The roots are real and distinct

$\Delta > 0$

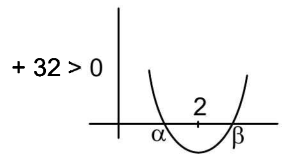
$(\lambda + 1)^2 - 4(\lambda^2 + \lambda - 8) > 0$

$\lambda^2 + 2\lambda + 1 - 4\lambda^2 - 4\lambda + 32 > 0$

$-3\lambda^2 - 2\lambda + 33 > 0$

$3\lambda^2 + 2\lambda - 33 < 0$

$(3\lambda + 11)(\lambda - 3) < 0$



$-\dfrac{11}{3} < \lambda < 3$ ....(1)

$f(2) < 0$

$4 - (\lambda + 1)2 + \lambda^2 + \lambda - 8 < 0$

$\lambda^2 - \lambda - 6 < 0$

$(\lambda - 3)(\lambda + 2) < 0$

$-2 < \lambda < 3$ — (2)

From (1) and (2)

$-2 < \lambda < 3$

170. Put $x - 1 = t^2$

$x = 1 + t^2$

The equation becomes

$\sqrt{1 + t^2 + 3 - 4t} + \sqrt{1 + t^2 + 8 - 6t} = 1$

$$\sqrt{t^2-4t+4}+\sqrt{t^2-6t+9}=1$$

$$\sqrt{(t-2)^2}+\sqrt{(t-3)^2}=1$$

$$|t-2|+|t-3|=1$$

$$2 < t < 3$$

t can take infinite number of values. The given equation has infinite number of solutions.

171. Statement 2 is a well known statement

Statement 1 : $y = f(x)$ is defined if $x^2 - 4x - 5 \geq 0$

$\Rightarrow$ $x \leq -1$ or $x \geq 5$

$\therefore$ 'Statement 1' is true but Statement 2 does not imply Statement 1

172. Statement 2 : is false.

(in $x^2 - 5x + 6 = 0$, $6(1 - 5 + 6) < 0$

but roots are 2, 3).

Statement 1 :

The equation is $(3x - 1)\ 2x - 1) = 0$

roots are $1/3, \frac{1}{2} \in (0, 1)$

'Statement 1' is true.

173. Statement 2 is true

$x^2 - 6x - 7 < 0$

$\Rightarrow$ $(x + 1)(x - 7) < 0$

$\Rightarrow$ $x \in (-1, 7)$

But $x = -1$ and $x = 7$ do not satisfy

$\therefore$ Statement 1 is false

$\therefore$ option (d)

174. Both Statement 2 and Statement 1 are true and Statement 1 follows from Statement 2.

175. Statement 2 is true

Statement 1: Let α be the common root.

$\alpha^2 - 11x + a = 0$

$\alpha^2 - 14\alpha + 2a = 0$

$$\frac{\alpha^2}{-22a+14a}=\frac{\alpha}{a-2a}=\frac{1}{-14+11}$$

$$\frac{\alpha^2}{-8a}=\frac{\alpha}{-a}=\frac{1}{-3}$$

$\therefore$ $a^2 = 24a$

$\Rightarrow$ $a^2 - 24a = 0$

$\Rightarrow$ $a(a - 24) = 0$

$\Rightarrow$ $a = 0, 24$

$\therefore$ Both Statement 2 and Statement 1 are true and Statement 1 follows from Statement 2.

176. Statement 2 : is a standard result.

Statement 1 : Here

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, $h^2 \geq ab$

$\therefore$ Given equation represents two straight lines one with slope 1 and other with slope $\frac{1}{4}$.

Any point on the constituent lines satisfies the given equation.

$\therefore$ no. of (x, y) satisfying the given equation is infinite

$\therefore$ Statement 1 is false.

$\Rightarrow$ choice (d)

177. The roots of $x^2 + x + 1 = 0$ are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}$ i which are complex.

Statement 1 is true.

The roots of $ax^2 + bx + c = 0$ are imaginary if $b^2 - 4ac < 0$

i.e., if $b^2 < 4ac$

i.e., if $-2\sqrt{a}\sqrt{c} < b < 2\sqrt{a}\sqrt{c}$

Statement 2 is true

Statement 2 is the explanation for Statement 1

$\Rightarrow$ choice (a)

178. Statement 2

Discriminant $= (a + c)^2 - 4ac = (a - c)^2$

$\Rightarrow$ Statement 2 is true

Statement 1

Let $y = 2^x$

$\Rightarrow$ $3y^2 - y - 4 = 0$, $\quad y = -1, \frac{4}{3}$

$y = -1$ is not admissible

$\Rightarrow$ $x = \log_2\left(\frac{4}{3}\right)$, which is irrational.

$\Rightarrow$ However, statement 1 does not follow from statement 2

Choice (b)

179. Statement 2

The equation whose roots are the reciprocals of the roots of the equation $ax^3 + bx^2 + bx + a = 0$ is $a + bx + bx^2 + ax^3 = 0$, which is same as the original equation.

$\Rightarrow$ Hence, if α is a root of the given equation $\frac{1}{\alpha}$ is also a root of the equation.

$\Rightarrow$ Statement 2 is true.

Statement 1

Statement 2, using the roots the equation are α, $\frac{1}{\alpha}$, γ (i.e., $\beta = \frac{1}{\alpha}$)

$$\alpha \times \frac{1}{\alpha} \times \gamma = -1$$

$\Rightarrow$ $\gamma = -1$

Dividing the expression $(2x^3 + 5x^2 + 5x + 2)$ by $(x + 1)$, we get $2x^2 + 3x + 2 = 0$

$$\Rightarrow \quad x = \frac{-3 \pm \sqrt{9-16}}{4}$$

$$= \frac{-3}{4} \pm \frac{i\sqrt{7}}{4}$$

Hence, the roots are -1, $\frac{-3}{4} \pm \frac{i\sqrt{7}}{4}$

$\Rightarrow$ choice (a)

180. Statement 2 is true

Statement 1 is true, since, the other root has to be $2-\sqrt{5}$, (by using statement 2)

Polynomial of lowest degree is $(x - 2 - \sqrt{5})(x - 2 + \sqrt{5}) = 0$

$\Rightarrow$ $x^2 - 4x - 1 = 0$

Choice (a)

181. α, β, γ be the roots

$$\sum\alpha = (\alpha + \beta + \gamma) = -\frac{b}{a}$$

$$\sum\alpha\beta = (\alpha\beta + \beta\gamma + \gamma\alpha) = \frac{c}{a}$$

$$\therefore \quad \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$$

182. If α, β, γ are all integers, $\alpha + \beta + \gamma = 4$ — (1)

$\alpha\beta + \beta\gamma + \gamma\alpha = +1$ — (2)

$\alpha\beta\gamma = -9$ — (3)

(3) implies that, α, β, γ are all odd integers contradicting (1). Hence A, B, Γ cannot be all integers

183. $(x - \sqrt{5})(x + \sqrt{5})(2x - 7) = 0$

$(x^2 - 5)(2x - 7) = 0$

$2x^3 - 7x^2 - 10x + 35 = 0$

184. $y = -8[(x - 4)^2 - 1]$, peak value for y is attained when $x = 4$

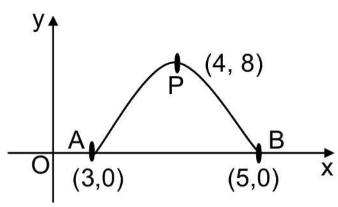
Value of $y = -8[0 - 1] = 8$ units above normal

or 106.4°f

185. Normal means $y = 0$

$\therefore$ Then $x = 3$ or 5

186.



It is increasing, which is deduced from the graph of

$y = -8[x^2 - 8x + 15]$

$y - 8 = -8(x - 4)$

187. $y = 28$

$28 = 3x^2 - 28x + 60$

or $3x^2 - 28x + 32 = 0$

$(3x - 4)(x - 8) = 0$

$x = \frac{4}{3}$ or 8 = 1h 20m after 8 am

(Reasoning forces us to reject the value $x = 8$)

188. When $x = 2$

$y = 3(2)^2 - 28(2) + 60 = 16$ tonnes

189. Balance of oil = 16

Oil added = 44

Total in tank 60

Once again it has the same initial conditions.

Now $y = 0$, when

$3x^2 - 28x + 60 = 0$

$\Rightarrow \quad x = \frac{10}{3}$ or 6

$\therefore$ Another 3 hours 20 minutes it will take

190. Let α, β, γ be the roots of $x^3 - px^2 + qx - r^2 = 0 \rightarrow (1)$ so that $\alpha = -\beta$. Then

$\alpha + \beta + \gamma = p \Rightarrow \gamma = p \rightarrow (2)$

Being a root of (1), x = p satisfies (1)

$\Rightarrow \quad qp - r^2 = 0$

$\Rightarrow$ p, r, q are in GP

$\therefore$ (a) is false whereas (b) is true.

As a consequence $px^2 + 2rx + q = 0$ has equal roots

$\therefore$ (c) is true and (d) is not.

$\therefore$ choices are (b, c)

191. Taking logarithm to the base 2

$$\left[\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right]\log_2 x = \log_2 2^{1/2}$$

Let $y = \log_2 x$

$$y\left(\frac{3}{4}y^2 + y - \frac{5}{4}\right) = \frac{1}{2}$$

$y(3y^2 + 4y - 5) = 2$

$3y^3 + 4y^2 - 5y - 2 = 0$

$(3y + 1)(y + 2)(y - 1) = 0$

$y = 1, -2, -\frac{1}{3}$

$\log_2 x = 1$ or $\log_2 x = -2$ or $\log_2 x = -1/3$

$x = 2$ or $x = 2^{-2}$ or $x = 2^{-1/3}$

$x = 2$ or $x = \frac{1}{4}$ or $x = \frac{1}{\sqrt[3]{2}}$

Choices (b), (c), (d)

192. $\alpha, \beta, \gamma, \delta$ are in HP.

$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ are in AP.

Let $\frac{1}{\alpha} = a - 3d$, $\frac{1}{\beta} = a - d$, $\frac{1}{\gamma} = a + d$ and $\frac{1}{\delta} = a + 3d$

α and γ are the roots of $Ax^2 - 4x + 1 = 0$

$\frac{1}{\alpha}$ and $\frac{1}{\gamma}$ are the roots of $x^2 - 4x + A = 0$

$a - 3d$, $a + d$ are the roots of $x^2 - 4x + A = 0$

$a - 3d + a + d = 4$

i.e., $2a - 2d = 4 \Rightarrow a - d = 2$

$(a - 3d)(a + d) = A$

β and δ are the roots of $Bx^2 - 6x + 1 = 0$

$\frac{1}{\beta}$ and $\frac{1}{\delta}$ are the roots of $x^2 - 6x + B = 0$

$a - d$ and $a + 3d$ are the roots of $x^2 - 6x + B = 0$

$a - d + a + 3d = 6$

$2a + 2d = 6$

$a + d = 3$

and $(a - d)(a + 3d) = B$

solving $a - d = 2$ and

$a + d = 3$

We get $a = \frac{5}{2}$ $d = \frac{1}{2}$

$\therefore$ $A = (a - 3d)(a + d)$

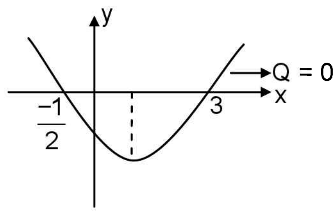
$= (1)(3) = 3$

$B = (a - d)(a + 3d)$

$= (2)(4) = 8$

(a) and (d) are correct.

193.



Zeros of Q are $-1/2$ and 3

$Q + \lambda = 2x^2 - 5x + \lambda - 3.$

Roots of $Q + \lambda = 0$ are equal

if $25 - 8(\lambda - 3) = 0$

i.e. if $\lambda = \frac{49}{8}$; (a) is true

Roots of $Q + \lambda = 0$ are non real

if $25 - 8(\lambda - 3) < 0$

i.e., if $\lambda > \frac{49}{8}$; and $7 > \frac{49}{8}$;

$\Rightarrow$ (b) is true.

Zeros of Q (x – k) are $k \; 1/2$, $k + 3$; both are positive if $k > 1/2$

$\Rightarrow$ (c) is true.

Zeros of Q (x + k) are $\frac{-1}{2} - k, 3 - k$;

Both are negative if $k > 3$;

$\Rightarrow$ (d) is true.

194. $x^2 - (p + q)x + r^2 = 0$ has equal roots.

$\Rightarrow$ D = 0

$\Rightarrow$ $(p + q)^2 - 4r^2 = 0$

$\Rightarrow$ $(p + q + 2r)(p + q - 2r) = 0$

$\Rightarrow$ $p + q + 2r = 0$ or $p + q - 2r = 0$

$p + q - 2r = 0$

$\therefore$ (a) is true and (d) is not

$p + q - 2r = 0 \Rightarrow p^3 + q^3 + (-2r)^3 = 3pq(-2r)$

$\therefore$ (c) is true

$p + q + 2r = 0 \Rightarrow p^3 + q^3 + (2r)^3 = 3pq(2r)$

$\therefore$ (b) is true

195. $\alpha + 1, \beta + 1$

$y = x + 1 \Rightarrow x = y - 1$

$2(y - 1)^2 + 5(y - 1) + 7 = 0$

$2y^2 - 4y + 2 + 5y - 5 + 7 = 0$

$\therefore$ Required equation is $2x^2 + x + 4 = 0$

$2\alpha - 1, 2\beta - 1$

$y = 2x - 1 \Rightarrow x = \frac{y+1}{2}$

$$2\left(\frac{y+1}{2}\right)^2 + 5\left(\frac{y+1}{2}\right) + 7 = 0$$

$$\frac{2y^2 + 4y + 2}{4} + \frac{5y + 5}{2} + 7 = 0$$

$2y^2 + 4y + 2 + 10y + 10 + 28 = 0$

$2y^2 + 14y + 40 = 0 \Rightarrow y^2 + 7y + 20 = 0$

$\therefore$ Required equation is $x^2 + 7x + 20 = 0$

$\frac{1}{\alpha}, \frac{1}{\beta} \Rightarrow 7x^2 + 5x + 2 = 0$

$\alpha^2 + \beta^2$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(\frac{-5}{2}\right)^2 - 2\left(\frac{7}{2}\right) = \frac{25 - 28}{4} = \frac{-3}{4}$$

$$\alpha^2\beta^2 = (\alpha\beta)^2 = \frac{49}{4}$$

$\Rightarrow$ Equation is $x^2 + \frac{3}{4}x + \frac{49}{4} = 0$

$\Rightarrow$ $4x^2 + 3x + 49 = 0$

196. (a) $x^2 + x + 1 > 0$ for all $x \in R$

so f(x) exists for all $x \in R$

(d) Let $\frac{x^2 + 4x + 3}{x^2 + x + 1} = y$

cross multiplying and rearranging $\Rightarrow$

$x^2(1 - y) + x(4 - y) + 3 - y = 0$

But $x \in R$ and so discriminant ≥ 0

$\Rightarrow$ $(4 - y)^2 - 4(3 - y)(1 - y) \geq 0$

$\Rightarrow$ $16 + y^2 - 8y - 12 + 16y - 4y^2 \geq 0$

$\Rightarrow$ $-3y^2 + 8y + 4 \geq 0$

coefficient of the quadratic term is negative and the graph is positive between the roots. Roots are

$$\frac{-8 \pm \sqrt{112}}{-6}$$

$$\therefore \quad y \in \left(\frac{4 - 2\sqrt{7}}{3}, \frac{4 + 2\sqrt{7}}{3}\right)$$

(b) and (c)

since denominator is always positive, sign of f(x) depends as $x^2 + 4x + 3$ only. $x^2 = 4x + 3 \geq 0$ when $x \in (-\infty, -3] \cup [-1, \infty)$

$\therefore$ $f(x) \geq 0$ only when $x \in (-\infty, -3] \cup [-1, \infty)$

(b) false (c) true

197. f(x) being a polynomial has R as its domain

Product of roots = $\frac{-6}{6} = -1$

$b^2 - 4ac = 25 + (4 \times 36) > 0$. There are two distinct real roots. The graph intersects the x-axis at 2 points

Minimum value is obtained at

$$x = -\frac{b}{2a} = \frac{5}{12}$$

198. (a) $(x - a)(x - c) = 0 \Rightarrow$ roots are a, c

$\Rightarrow$ $x - (a + c)x + ac = 0$

$\Rightarrow$ $x - 2bx + ac = 0$

$\therefore$ Sum of the roots = 2b

Roots a, c lie between (1, 5)

$\therefore$ Roots lie between [1, 5]

$\therefore$ $a \to q, r$

(b) $x^2 + (a + c)x + b^2 = 0$

$\Rightarrow$ $x^2 + 2bx + b^2 = 0 \Rightarrow (x + b)^2 = 0$

$\therefore$ $b \to p$

(c) $(x - a)(x - c) + (2b + 1)x = 0$

$\Rightarrow$ $x^2 - (a + c)x + ac + 2bx + x = 0$

$\Rightarrow$ $x^2 + x + ac = 0$

$\Rightarrow$ $x = \dfrac{-1 \pm \sqrt{1 - 4ac}}{2}$

But $1 < 4ac$ ($\because$ $1 < a, c < 5$)

$\therefore$ $c \to s$

(d) $2x^2 - 2bx + 10b = 0$

$\Rightarrow$ $x^2 - bx + 5b = 0$

$\Rightarrow$ $x = \dfrac{b \pm \sqrt{b^2 - 20b}}{2}$

But $b^2 - 20b < 0$

$\therefore$ $d \to s$

199. Since $ax^2 + bx + c < 0 \ \forall \ x > k \in R$, therefore, $a < 0$

(a) Consider $ax^2 + bx + 1 = 0$

Discriminant, $\Delta = b^2 - 4a > 0$ ($\because$ $a < 0$)

$\therefore$ $a \to p, r$

(b) Consider $-ax^2 + bx - 4 = 0$

$\Delta = b^2 - 16a > 0$

$\therefore$ $b \to p, s$

(c) Consider $x^2 + x - a = 0$; $|a| > \dfrac{1}{4}$

$\Delta = 1 + 4a < 0$

$\therefore$ $c \to q$

(d) Consider $ax^2 - b|x| + 5 = 0$

$x < 0 \Rightarrow ax^2 + bx + 5 = 0$

$\Delta = b^2 - 20a > 0$

$x > 0 \Rightarrow ax^2 - bx + 5 = 0$

$\Delta = b^2 - 20a > 0$

$\therefore$ $d \to p, r$

200. (a) Put $3^{\cos^2 x} = t$

$\therefore$ $t + 5.\dfrac{3}{t} = 8$

$\Rightarrow$ $t^2 - 8t + 15 = 0$

$\Rightarrow$ $t = 3, 5$

But $\cos^2 x \le 1 \Rightarrow t = 5$ is not admissible

$\therefore$ $t = 3 \Rightarrow \cos^2 x = 1$

$\therefore$ $a \to s$

(b) $3^{6x} + 7(3^{4x}) = 8(3^{5x})$

$\Rightarrow$ $3^x + \dfrac{7}{3^x} = 8$

Put $3^x = t$

$\Rightarrow$ $t^2 - 8t + 7 = 0$

$t = 1, 7$

$\Rightarrow$ $3^x = 1, 7$

$\therefore$ two solutions

$\therefore$ $b \to r$

(c) $(1024)^{1/x} + 2^{(4x+5)}x = 80$

$\Rightarrow$ $2^{10/x} + 2^4.2^{5/x} = 80$

Put $2^{5/x} = t$

$\Rightarrow$ $t^2 + 16t - 80 = 0$

$\Rightarrow$ $t = -20, 4$

But $2^{5/x} > 0$

$\therefore$ only one solution

$\therefore$ $c \to q$

(d) $y^2 - 3y + 2 > 0 \Rightarrow y < 1$ or >2

$y < 1 \Rightarrow y - 2 < 0$

$\Rightarrow$ $|y - 2| = 2 - y$ and the equation is not satisfied .

$\therefore$ $y > 2 \ \therefore \ \log_4 x - \log_x 16 = 7$

$\Rightarrow$ $X^2 - 7X - 2 = 0$

where, $X = \log_4 x$

$\therefore$ X. and hence x has 2 values.

$\therefore$ $d \to r$

CHAPTER

3

TRIGONOMETRY

CHAPTER OUTLINE

Preview

INTRODUCTION

In this unit, we propose to discuss the following:

(a) Periodic property of circular functions and graphs of circular functions.

(b) If θ is any angle,

$$-\theta,\ \frac{\pi}{2}-\theta,\ \frac{\pi}{2}+\theta,\ \pi-\theta,\ \pi+\theta,\ \frac{3\pi}{2}-\theta,\ \frac{3\pi}{2}+\theta,$$

$2\pi - \theta$ ($-\theta$, $90° \pm \theta$, $180° \pm \theta$, $270° \pm \theta$, $360° - \theta$) are called related angles or allied angles. We show that it is possible to express the circular functions of these related angles in terms of those of θ. This means that if we know the circular functions of θ, those of the related angles can be easily obtained.

(c) If A and B are any two angles and the circular functions of A and B are known, circular functions of (A + B) and (A − B) are obtained in terms of those of A and B. We also derive formulas for the circular functions of multiple angles 2A, 3A in terms of those of A.

(d) Inverse circular functions (inverse trigonometric functions) are defined and the relations between them are developed.

(e) Methods for solving equations involving circular functions called trigonometric equations are developed.

(i) if x lies in the first quadrant, $\left(\text{i.e.,if } 0 < x < \frac{\pi}{2}\right)$, i.e., if the angle x is between 0° and 90°; all the circular functions are positive.

(ii) if x lies in the second quadrant, $\left(\text{i.e.,if } \frac{\pi}{2} < x < \pi\right)$, i.e., if the angle x is between 90° and 180°; sin x and cosec x are positive while all the other circular functions are negative.

(iii) if x lies in the third quadrant, $\left(\text{i.e.,if } \pi < x < \frac{3\pi}{2}\right)$, i.e., if the angle x is between 180° and 270°; tan x and cot x are positive while all the other circular functions are negative.

(iv) if x lies in the fourth quadrant, $\left(\text{i.e.,if } \frac{3\pi}{2} < x < 2\pi\right)$, i.e., if the angle x is between 270° and 360°; cos x and sec x are positive while all the other circular functions are negative.

TRIGONOMETRY FUNDAMENTALS

Definition of an angle

An angle is generated by rotating a ray about a point (called the vertex or pole) from some initial position (called initial side) to some terminal position (called terminal side). The amount of rotation gives the measure of the angle.

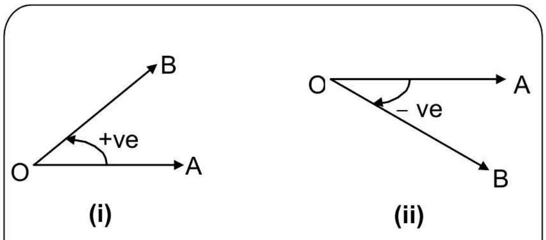


Fig. 3.1

When the rotation is in the counter clockwise sense(or anti-clockwise sense), the measure of the angle is taken as positive (i.e., a positive sign is associated with the angle: refer (i) of Fig. 3.1); and, if the rotation is clockwise, the measure of the angle is taken as negative (i.e., a negative sign is associated with the angle : refer (ii) of Fig. 3.1).

We denote the angles by the letters θ, α, γ, A, B, C,.... (which are the measures of the angles in some units).

It may also be noted that angles having the same initial and terminal sides (known as coterminal angles) may have different measures [Refer Fig. 3.2]

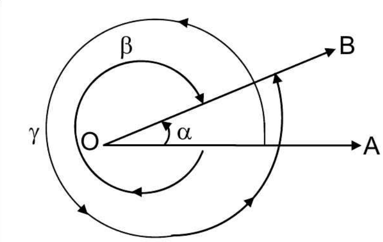


Fig. 3. 2

Consider the following examples:

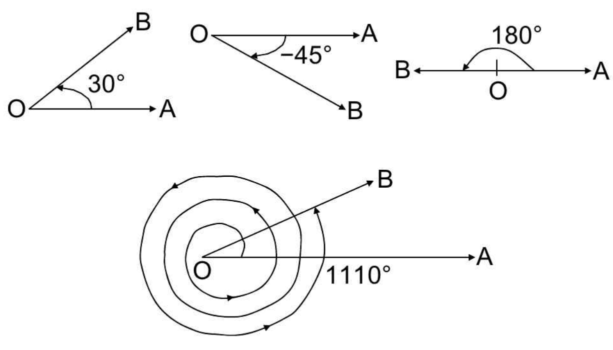


Fig. 3.3

Units of measurement of an angle

One of the units of measurement of an angle is "degree". One degree (denoted by 1°) is defined as the measure of the angle formed by joining the centre of a circle to the extremities of an arc of the circle whose length is $\frac{1}{360}$ of its circumference. A degree is divided into 60 equal parts called minutes of arc and a minute of arc is divided into 60 equal parts called seconds of arc (not to be confused with minutes of time and seconds of time)

1 right angle = 90 degrees = 90°

1° (1 degree) = 60 minutes of arc = 60'

1' (1 minute of arc) = 60 seconds of arc = 60"

There is another unit of measurement of an angle called the "circular measure" or "radian measure". This is the unit of measurement of the angle used for all theoretical purposes.

Definition of radian measure

The angle subtended at the centre of a circle by an arc equal in length to the radius of the circle is called one radian (denoted by 1^c). If r is the radius of the circle, and length of arc AB = r (Refer Fig. 3.4), then $\angle AOB = 1^c$.

AOC is a diameter of the circle with centre O and radius r. The length of the arc varies as the angle subtended by the arc at the centre of the circle. Since the arc AC subtends an angle 180° at the centre and length of arc AC = πr (half the circumference), we have

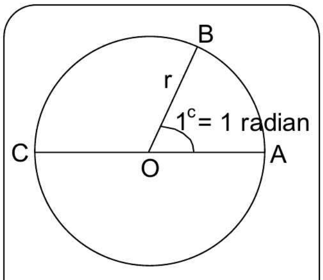


Fig. 3. 4

π radians = 180° or

$$1 \text{ radian} = \frac{180^\circ}{\pi} \approx 57^\circ\ 17' \ (\pi = 3.14159)$$ (≈ symbol means "approximately equal to")

$$1 \text{ degree} \approx 0.0175 \text{ radians}$$

It may be observed from the above that radian measure of an angle is independent of the radius of the circle.

Below given are conversions of a few standard measures of angles.

(i) $90^\circ = \frac{\pi}{2}$ radians

(ii) $45^\circ = \frac{\pi}{4}$ radians

(iii) $30^\circ = \frac{\pi}{6}$ radians

(iv) $\frac{\pi}{5}$ radians = 36°

(v) $\frac{-5\pi}{12}$ radians = −75°

(vi) $\frac{\pi}{3}$ radians = 60°

Area of a sector

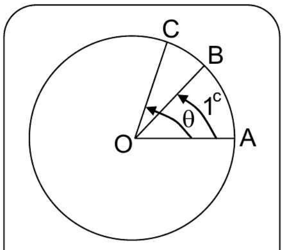


Fig. 3. 5

Let arc AC subtend angle θ at the centre, and arc AB subtend angle 1 radian at the centre.

We have, $\dfrac{\text{length of arc AC}}{\theta} = \dfrac{\text{length of arc AB}}{1} = \dfrac{r}{1}$

$\Rightarrow$ length of arc AC = r θ

Again, $\dfrac{\text{area of sector AOC}}{\theta} = \dfrac{\text{area of the circle}}{2\pi}$

$$= \frac{\pi r^2}{2\pi} = \frac{r^2}{2}$$

$\Rightarrow$ area of the sector AOC = $\frac{1}{2}r^2\theta$ .Where, θ is in radians.

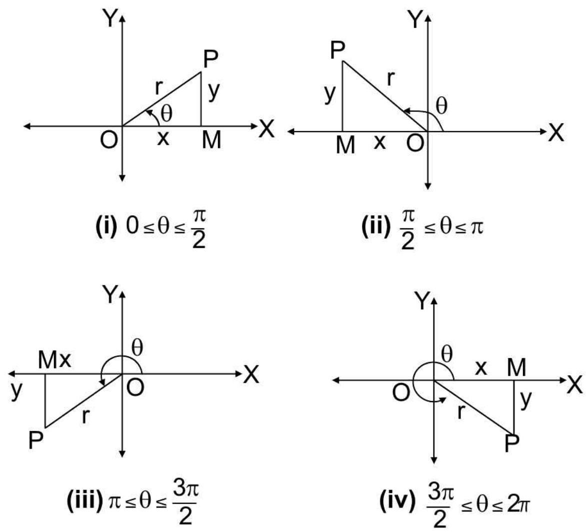


Fig. 3.6

Definitions of trigonometric functions (or circular functions) of an angle

We are already familiar with the definitions for the six trigonometric ratios sine θ (written as sinθ); cosine θ (written as cosθ); tangent θ (written as tanθ); cosecant θ (written as cosecθ); secant θ (written as secθ) and cotangent θ (written as cotθ); for an acute angle θ (i.e., $0 < \theta < 90°$ or $0 < \theta < \frac{\pi}{2}$) and also the relations existing between them.

For the purpose of defining these ratios we used a right-angled triangle in which one of the angles was θ. We are now going to define these ratios for any angle θ (θ need not be restricted to an acute angle, Also, θ can have positive or negative measure). These definitions are such that they automatically hold good for acute angles as well. We also call these as trigonometric functions or circular functions.

We take two mutually perpendicular straight lines XOX' and YOY' intersecting at O. This represents the rectangular Cartesian coordinate system where XOX' is the x-axis and YOY' is the y-axis and O is the origin.

An angle is said to be in standard position if its vertex is at the origin O and its initial side coincides with OX, the positive direction of the x-axis.

Let a ray OP start from OX and trace out ∠XOP (= θ). Then, the terminal side will be in one of the four quadrants. The angle θ can be either in the radian measure or in the degree measure. It is the usual practice to write θ° if the angle is expressed in degree measure. Hence, angle θ means it is in radian measure.

If the terminal side is in the first quadrant and the rotation is in the counter clock wise sense (positive sense), ∠XOP (denoted by θ) will be lying between 0 and $\frac{\pi}{2}$ [Refer (i) of Fig. 3.6].

Similarly, ∠XOP will be lying between $\frac{\pi}{2}$ and π if the terminal side is in the second quadrant; ∠XOP will be lying between π and $\frac{3\pi}{2}$ if the terminal side is in the third quadrant and ∠XOP will be lying between $\frac{3\pi}{2}$ and 2π if the terminal side is in the fourth quadrant [Refer (ii), (iii) and (iv) of Fig. 3.6]

If the rotation of the ray is in the clockwise sense [Refer (i) Fig. 3.7] we will have the angle θ lying between $\frac{-\pi}{2}$ and 0; $-\pi$ and $\frac{-\pi}{2}$; $\frac{-3\pi}{2}$ and $-\pi$; and -2π and $\frac{-3\pi}{2}$ according as the terminal side is in the fourth, third, second or first quadrants respectively.

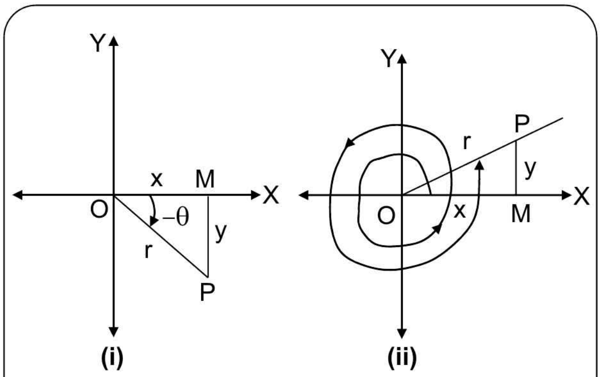


Fig. 3. 7

Suppose the ray OP completes n rotations in the positive sense and occupies a position in one of the quadrants, then the measure of the angle generated is $2n\pi + \angle XOP$ [Refer (ii) of Fig. 3.7].

If (x, y) represents the coordinates of P (referred to X'OX and Y'OY as the axes of coordinates) and if OP is denoted by r,

$$r = \sqrt{OM^2 + PM^2} = \sqrt{x^2 + y^2}.$$

OP is called the radial distance of P and it is always taken as positive.

The six trigonometric functions (or circular functions) of θ are defined in the following manner.

$$\sin\theta = \frac{\text{ordinate of P}}{\text{OP}} = \frac{MP}{OP} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{abscissa of P}}{\text{OP}} = \frac{OM}{OP} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{ordinate of P}}{\text{abscissa of P}} = \frac{MP}{OM} = \frac{y}{x},\ x \neq 0$$

$$\text{cosec}\,\theta = \frac{OP}{MP} = \frac{r}{y},\ y \neq 0$$

$$\sec\theta = \frac{OP}{OM} = \frac{r}{x},\ x \neq 0$$

$$\cot\theta = \frac{OM}{MP} = \frac{x}{y},\ y \neq 0$$

Observation 1

Following are the conventions regarding the signs of the coordinates of a point P in a rectangular Cartesian system.

If P lies in the first quadrant, both x and y are positive;
If P lies in the second quadrant x is negative, y is positive;
If P lies in the third quadrant, both x and y are negative;

and if P lies in the fourth quadrant, x is positive and y is negative.

Recall that OP is always taken as positive.

We therefore see that ,

The signs of the circular functions of any angle θ corresponding to the quadrant in which θ lies can be easily referred from the following diagram.

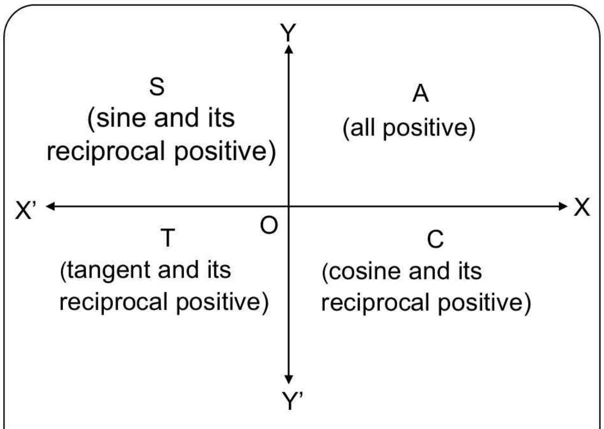


Fig. 3. 8

CONCEPT STRANDS

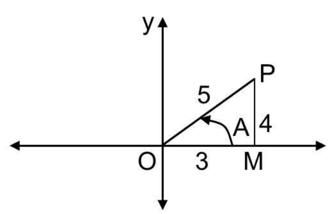
Concept Strand 1

Given $\cos A = \frac{3}{5}$ and A is in the first quadrant find the other trigonometric functions of A.

Solution

A being in the first quadrant, all the trigonometric functions are positive, Referring to the figure

$$PM = \sqrt{5^2 - 3^2} = 4.$$



$\sin A = \frac{4}{5}$, $\tan A = \frac{4}{3}$, $\text{cosec } A = \frac{5}{4}$, $\sec A = \frac{5}{3}$ and $\cot A = \frac{3}{4}$.

Concept Strand 2

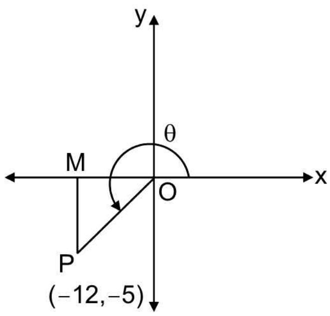
Given θ is in the third quadrant and $\tan\theta = \frac{5}{12}$, find $\frac{2\sin\theta - 3\cos\theta}{5\sin\theta + 7\cos\theta}$.

Solution

Since θ is the third quadrant, $\tan\theta$ and $\cot\theta$ are positive while all the other functions are negative.

$$OP^2 = 5^2 + 12^2 = 13^2$$

Giving OP = 13



$\therefore \quad \sin\theta = \frac{-5}{13}, \cos\theta = \frac{-12}{13}$

$$\frac{2\sin\theta - 3\cos\theta}{5\sin\theta + 7\cos\theta} = \frac{2\times\frac{-5}{13} - 3\times\frac{-12}{13}}{5\times\frac{-5}{13} + 7\times\frac{-12}{13}}$$

$$= \frac{-26}{109}$$

Concept Strand 3

For any angle θ prove that the following relations:

(a) $\cos^2\theta\tan^2\theta + \sin^2\theta\cot^2\theta = 1.$

(b) $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

(c) $(\sin\theta + \cos\theta)(\tan\theta + \cot\theta) = \sec\theta + \operatorname{cosec}\theta$

(d) $\frac{\cos\theta}{1+\sin\theta} + \frac{\cos\theta}{1-\sin\theta} = 2\sec\theta$

(e) $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$

Solution

(a) $\text{L.H.S} = \cos^2\theta \times \left(\frac{\sin\theta}{\cos\theta}\right)^2 + \sin^2\theta \times \left(\frac{\cos\theta}{\sin\theta}\right)^2$

$= \cos^2\theta \times \frac{\sin^2\theta}{\cos^2\theta} + \sin^2\theta \times \frac{\cos^2\theta}{\sin^2\theta}$

$= \sin^2\theta + \cos^2\theta$

$= \text{R.H.S}$

(b) $\text{L.H.S} = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$

$= 1 + 2\sin\theta\cos\theta$

$= \text{R.H.S}$

(c) $\text{L.H.S} = (\sin\theta + \cos\theta)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$

$= (\sin\theta + \cos\theta)\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)$

$= \frac{(\sin\theta + \cos\theta)}{\sin\theta\cos\theta}$

$= \frac{1}{\sin\theta} + \frac{1}{\cos\theta} = \operatorname{cosec}\theta + \sec\theta$

$= \text{R.H.S.}$

(d) $\text{L.H.S} = \frac{\cos\theta(1-\sin\theta) + \cos\theta(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$

$= \frac{2\cos\theta}{(1-\sin^2\theta)} = \frac{2\cos\theta}{\cos^2\theta}$

$= 2\sec\theta$

$= \text{R.H.S}$

(e) $\text{L.H.S} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$

$= \frac{1+\cos\theta}{\sqrt{1-\cos\theta}\sqrt{1+\cos\theta}}$,

(on multiplying numerator and denominator by $\sqrt{1+\cos\theta}$)

$= \frac{1+\cos\theta}{\sqrt{1-\cos^2\theta}} = \frac{1+\cos\theta}{\sqrt{\sin^2\theta}}$

$= \frac{1+\cos\theta}{\sin\theta}$

$= \operatorname{cosec}\theta + \cot\theta$

$= \text{R.H.S}$

Concept Strand 4

Eliminate θ between the relations $x = a \sec\theta$, $y = b \tan\theta$.

Solution

By elimination of θ, we mean, we have to obtain a relation independent of θ by using the two given relations.

We know that $\sec^2\theta - \tan^2\theta = 1$.

$\sec\theta = \frac{x}{a}$ and $\tan\theta = \frac{y}{b}$

$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2\theta - \tan^2\theta = 1$

$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the result of eliminating θ between two given relation.

This relation is called the eliminant.

Concept Strand 5

If $x\sin^2 A + y\cos^2 A = z$, show that $\tan^2 A = \frac{y-z}{z-x}$.

Solution

From the given relation, using $\sin^2 A = 1 - \cos^2 A$, we obtain $x(1-\cos^2 A) + y\cos^2 A = z$

giving $\cos^2 A = \frac{(z-x)}{(y-x)}$.

Since $\tan^2 A = \sec^2 A - 1$, we obtain, by substitution,

$$\tan^2 A = \left(\frac{y-x}{z-x}\right) - 1 = \left(\frac{y-z}{z-x}\right)$$

PERIODIC PROPERTY OF CIRCULAR FUNCTIONS AND GRAPHS OF CIRCULAR FUNCTIONS

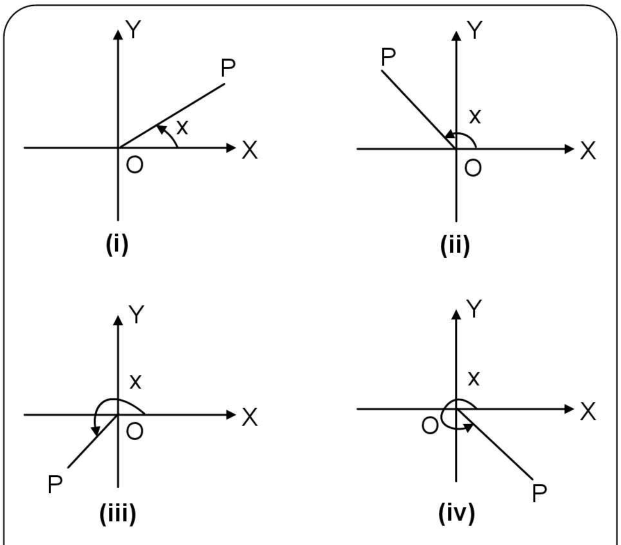


Fig. 3.9

Let OP be the position of the terminal side corresponding to a particular value of the angle x. (P being an arbitrary point on the terminal side, O being the vertex or origin about which the ray rotates to generate the angle x). Then it is clear that the terminal side corresponding to the angle $(x + 2n\pi)$ where n is an integer positive or negative is the same as that for x. (Refer Fig. 1.1)

This means that $\sin(x + 2n\pi) = \sin x$; $\cos(x + 2n\pi) = \cos x$ and so on. In other words, the circular functions of $(x + 2n\pi)$ are those corresponding to x. We call this property as the periodic property of the circular functions.

For example, consider the following:

(i) $\sin 760° = \sin(720°+40°) = \sin 40°$

(ii) $\cos\left(\frac{73\pi}{6}\right) = \cos\left(12\pi + \frac{\pi}{6}\right) = \cos\frac{\pi}{6}$

(iii) $\tan\left(\frac{-31\pi}{4}\right) = \tan\left(-8\pi + \frac{\pi}{4}\right) = \tan\frac{\pi}{4}$

Thus, in the case of circular functions, if their values are known for all x in the interval $0 \le x \le 2\pi$, the values of these functions for any other value of x can be obtained by appealing to the periodic property of these functions. Therefore, for drawing the graphs of these functions, it is enough if we know the values of these functions in $0 \le x \le 2\pi$ only.

f(x) = sin x and f(x) = cos x

Let us now concentrate on the functions f(x) = sin x and f(x) = cos x. In order to observe the variations of the values of sin x and cos x as x varies form 0 to 2π, we proceed as follows:

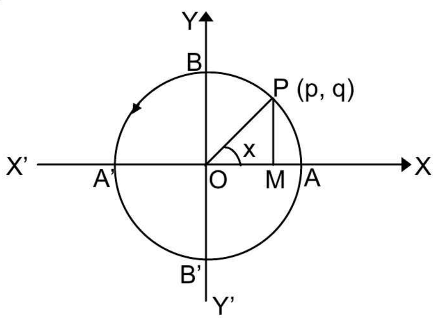


Fig. 3.10

Let $\angle XOP = x$ and OP = 1. Let the coordinates of P referred to the rectangular Cartesian system XOX' and YOY' be (p, q) [p = OM, q = MP].

Then it can be seen that $\sin x = \frac{MP}{OP} = \frac{q}{1}$ and $\cos x = \frac{OM}{OP} = \frac{p}{1}$ or p = cos x, q = sin x.

When x = 0 the terminal side of OP coincides with OA or P coincides with A; when $x = \frac{\pi}{2}$, P coincides with B. when $x = \pi$, P coincides with A'; when $x = \frac{3\pi}{2}$, P coincides with B' and when $x = 2\pi$, P coincides with A.

In other words, as P moves in the counter clockwise sense along the circle whose center is the origin O and whose radius is unity (this circle is called the unit circle), x changes from 0 to 2π.

For any x where $0 \le x < 2\pi$, we have p = cos x and q = sin x. Thus, the x coordinate of P gives cos x, and y coordinate of P gives sin x, where $\angle XOP = x$. Since the coordinates of A, B, A', B' are respectively (1, 0), (0, 1), (–1, 0), (0, –1), we immediately obtain,

$$\sin 0 = 0; \quad \sin \frac{\pi}{2} = 1; \quad \sin \pi = 0; \quad \sin \frac{3\pi}{2} = -1;$$

$$\cos 0 = 1; \quad \cos \frac{\pi}{2} = 0; \quad \cos \pi = -1; \quad \cos \frac{3\pi}{2} = 0.$$

Using the periodic property of sin x and cos x, we have

$$\sin 2\pi = \sin (2\pi + 0) = \sin 0 = 0$$
$$\cos 2\pi = \cos (2\pi + 0) = \cos 0 = 1$$

As P moves from A to B along the circle in the counter clockwise sense, the y coordinate of P increases from 0 to 1. This means that as x increases from 0 to $\frac{\pi}{2}$, sin x increases from 0 to 1.

Again, as P moves from A to B along the circle, the x coordinate of P decreases form 1 to 0. This means that as x increases from 0 to $\frac{\pi}{2}$, cos x decreases from 1 to 0.

We are now in a position to chart the variations of sin x and cos x for $0 \le x \le 2\pi$.

Table 3.1

$0 \le x \le \frac{\pi}{2}$	Corresponds to movement of P from A to B along the circle	sin x increases from 0 to 1 cos x decreases from 1 to 0
$\frac{\pi}{2} \le x \le \pi$	Corresponds to movement of P from B to A' along the circle	sin x decreases from 1 to 0 cos x decreases from 0 to –1
$\pi \le x \le \frac{3\pi}{2}$	Corresponds to movement of P from A' to B' along the circle	sin x decreases from 0 to –1 cos x increases from –1 to 0
$\frac{3\pi}{2} \le x \le 2\pi$	Corresponds to movement of P from B' to A along the circle	sin x increases from –1 to 0 cos x increases from 0 to 1

From the above table, we note that for any x, $-1 \le \sin x \le 1$ and $-1 \le \cos x \le 1$ or the absolute values (or numerical values) of sin x and cos x can never exceed unity.

Since cosec x and sec x are reciprocals of sin x and cos x respectively, for all x, it follows that the absolute values (or numerical values) of cosec x and sec x can never be less than unity.

$$|\sin x| \le 1 \text{ and } |\cos x| \le 1$$
$$|\text{cosec } x| \ge 1 \text{ and } |\sec x| \ge 1$$

The graphs of the functions sin x and cos x are given below:

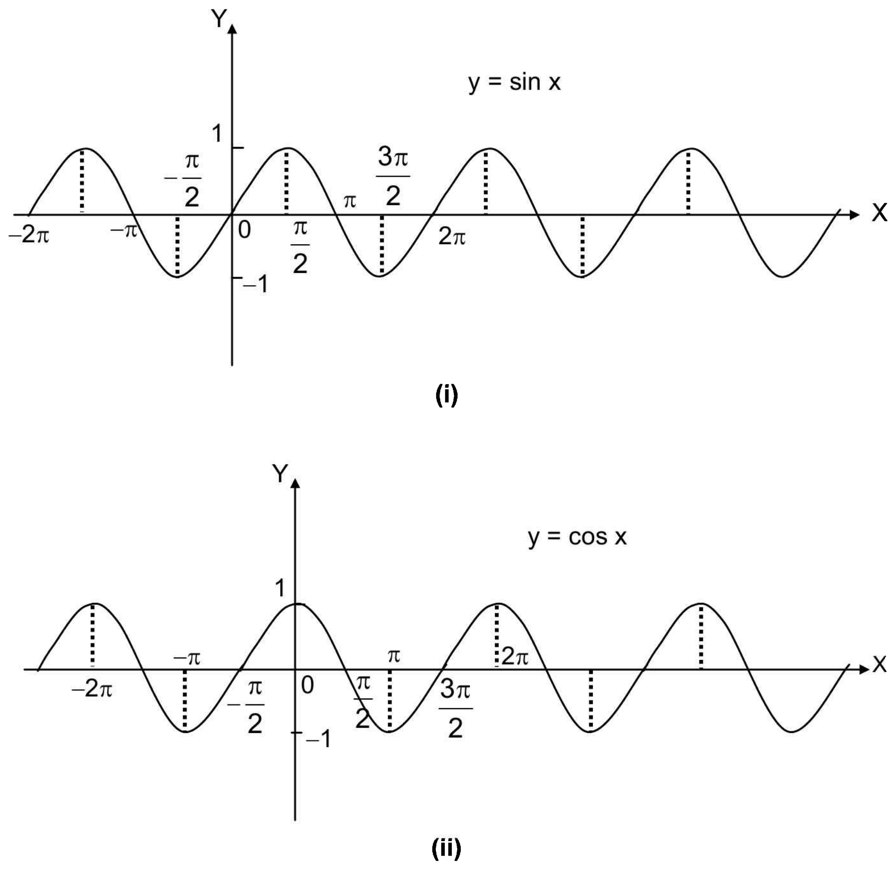


Fig. 3.11

Observations

(i) The domain of both these functions is R and the range is [–1, 1].

(ii) Both the functions sin x and cos x are many one mappings from R onto [–1, 1].

(iii) sin x is an odd function while cos x is an even function.

(iv) Both sin x and cos x are periodic with period 2π.

(v) The above discussion on the values of sin x and cos x as x varies from 0 to 2π leads us to a very interesting observation. Suppose a particle starts from A and moves along the circle in the counterclockwise sense. As the particle moves from A to B along the circle, the foot of perpendicular from P to AA', which is M (Refer Fig. 1.2), (or the projection of P on the x-axis) moves towards O along AA'. Again, as the particle moves from B to A' along the circle, M moves from O to A'. Finally, as the particle moves from A' to B' and then from B' to A along the circle, M moves back from A' to A. That is, as the particle P completes the circular motion, the foot of perpendicular M would have moved from A to A' and then from A' to A back along the x-axis. The motion of the foot of perpendicular is said to be simple harmonic with amplitude unity (i.e., amplitude = 1).

f(x) = tan x

From Fig. 3.10 $\tan x = \frac{MP}{OM}$. When P coincides with A, MP = 0. Consequently, $\tan 0 = 0$.

$$\left(\text{or } \tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0\right)$$

As P moves from A to B along the circle in the counter clockwise sense, OM decreases to zero. This means that $\frac{MP}{OM}$ increases indefinitely (as the denominator OM decreases).

When P coincides with B, OM = 0 and that is, for $x = \frac{\pi}{2}$, $\tan\frac{\pi}{2}$ is undefined. (since division by zero is not defined.)

Therefore, as x increases from 0 to $\frac{\pi}{2}$, tan x increases.

When P coincides with A′, MP = 0 which means that $\tan \pi = 0$. ($\tan\pi = \frac{\sin\pi}{\cos\pi} = \frac{0}{-1} = 0$).

Again, as division by zero is not defined, $\tan\frac{3\pi}{2}$ is also not defined.

Finally, $\tan 2\pi = \tan(2\pi + 0) = \tan 0 = 0$.

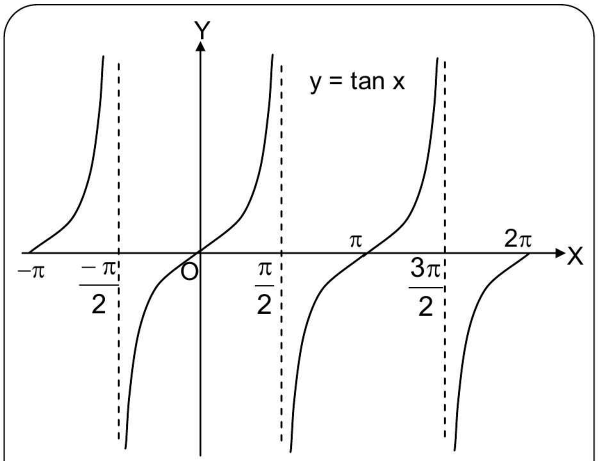


Fig. 3.12

Therefore Domain = $R - (2n+1)\frac{\pi}{2}\ n \in Z$; Range = $(-\infty, +\infty)$ and f(x) = tanx is an odd periodic function. The graph of the function y = tan x is therefore as given above. Its period is π.

f(x) = cosec x

cosec 0 and cosec π are not defined;

$$\operatorname{cosec}\frac{\pi}{2} = 1,\ \operatorname{cosec}\frac{3\pi}{2} = -1$$

Domain = $R - n\pi\ n \in Z$; Range = $(-\infty, -1] \cup [1, \infty)$.

f(x) = cosec x is an odd periodic function and the period of cosec x is the same as that of sin x, i.e., 2π. Graph of y = cosec x is as shown in (i) of Fig. 3.13.

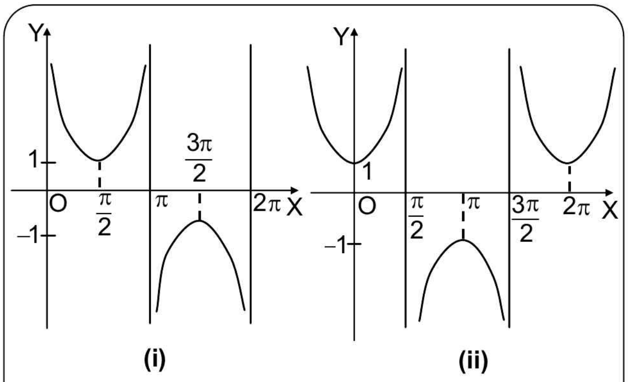


Fig. 3.13

f(x) = sec x

$\sec\frac{\pi}{2}$ and $\sec\frac{3\pi}{2}$ are not defined; sec 0 = 1, sec π = –1.

Domain = $R - (2n+1)\frac{\pi}{2}\ n \in Z$; Range = $(-\infty, -1] \cup [1, \infty)$

f(x) = secx is an even periodic function and the period of sec x is 2π.

Graph of y = sec x is as shown in (ii) of Fig. 3.13

f(x) = cotx

cot 0 and cot π are not defined; $\cot\frac{\pi}{2} = 0$, $\cot\frac{3\pi}{2} = 0$.

Domain = $R - (2n+1)\frac{\pi}{2}\ n \in Z$; Range = $(-\infty, +\infty)$ and f(x) = tanx is an odd periodic function. Period of the function cot x is π.

Graph of f(x) = cot x is shown in Fig. 3.14

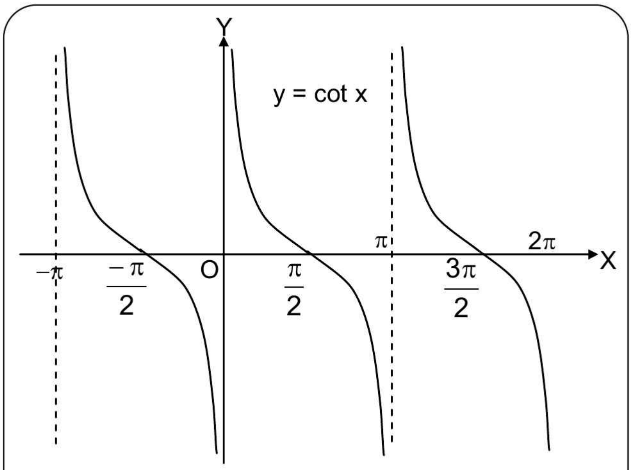


Fig. 3.14

FORMULAS FOR CIRCULAR FUNCTIONS OF RELATED ANGLES

Let θ be an acute angle.

(i) Circular functions of (– θ) in terms of those of θ: (Refer Fig. 3.15)

Let OP = r = OP'
If ∠MOP = θ, ∠MOP' corresponds to – θ. Let the coordinates of P be (x, y).

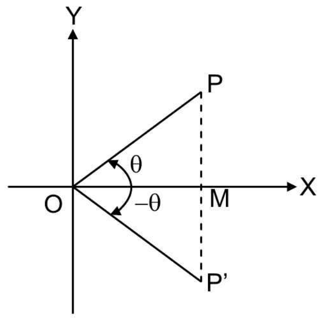


Fig. 3.15

Since triangles OMP and OMP' are congruent, the coordinates of P' will be (x, –y).
We now have, from triangle OMP',

$$\sin(-\theta) = \frac{-y}{r} = -\sin\theta\ ;\ \cos(-\theta) = \frac{x}{r} = \cos\theta$$

It follows that,

tan(–θ) = –tanθ; cosec(–θ) = – cosec θ;

sec(–θ) = sec θ and cot (–θ) = – cot θ.

Since ∠MOP' can also correspond to (2π – θ), circular functions of (2π – θ) are the same as those for –θ.

(ii) Circular functions of $\left(\frac{\pi}{2} - \theta\right)$ or (90° – θ) in terms of those of θ:

(Refer Fig. 3.16)

Let OP = OP' = r

Let ∠MOP = θ and ∠M'OP' $= \frac{\pi}{2} - \theta$

Let P be (x, y).
Since the triangles OMP and P' M'O are congruent, the coordinates of P' will be (y, x).

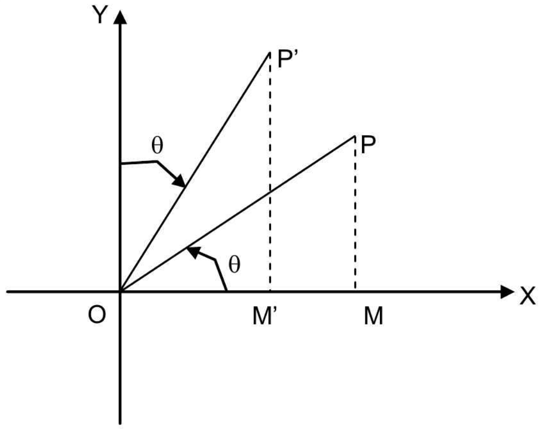


Fig. 3.16

Thus, from triangle OM'P',

$$\sin\left(\frac{\pi}{2} - \theta\right) = \frac{x}{r} = \cos\theta\ ;$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{y}{r} = \sin\theta$$

It follows that

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta, \operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec\theta,$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec}\theta \text{ and } \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta.$$

(iii) Circular functions $\left(\frac{\pi}{2} + \theta\right)$ or (90° + θ) in terms of those of θ. (Refer Fig. 3.17)

Let OP = r = OP'

If ∠MOP = θ, ∠MOP' corresponds to $\left(\frac{\pi}{2} + \theta\right)$

Let the coordinates of P be (x, y). Since triangles OMP and P'M'O are congruent, the coordinates of P' are (– y, x)

We now have

$$\sin\left(\frac{\pi}{2} + \theta\right) = \frac{x}{r} = \cos\theta\ ,$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = \frac{-y}{r} = -\sin\theta$$

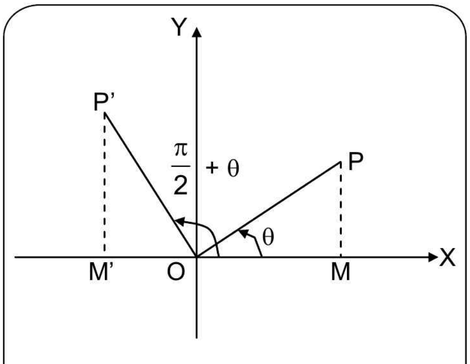


Fig. 3.17

It follows that,

$$\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta\,;\ \operatorname{cosec}\left(\frac{\pi}{2}+\theta\right) = \sec\theta\,;$$

$$\sec\left(\frac{\pi}{2}+\theta\right) = -\operatorname{cosec}\theta \text{ and } \cot\left(\frac{\pi}{2}+\theta\right) = -\tan\theta$$

(iv) Circular functions of $(\pi + \theta)$ or $(180° + \theta)$ in terms of those of θ. (Refer Fig. 3.18)

Let OP = r = OP'
If $\angle MOP = \theta$, $\angle MOP'$ corresponds to $(\pi + \theta)$

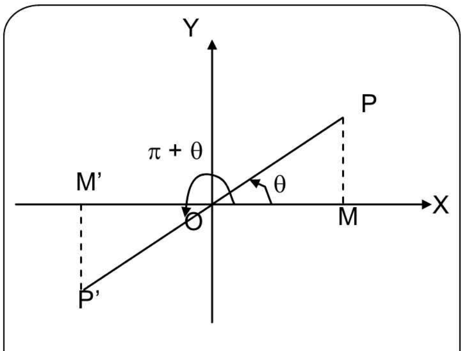


Fig. 3.18

Let the coordinates of P be (x, y). Since triangles OMP and OM'P' are congruent, the coordinates of P' will be (– x, – y)

We now have $\sin(\pi+\theta) = \frac{-y}{r} = -\sin\theta$

$$\cos(\pi+\theta) = -\frac{x}{r} = -\cos\theta$$

It follows that,
$\tan(\pi + \theta) = \tan\theta$, $\operatorname{cosec}(\pi + \theta) = -\operatorname{cosec}\theta$
$\sec(\pi + \theta) = -\sec\theta$ and $\cot(\pi + \theta) = \cot\theta$

Circular functions of other related angles can be similarly obtained. Although the above results were derived by assuming θ to be an acute angle, they hold good for any θ (θ can be in the first, second, third or fourth quadrant).

We give below the formulas for the circular functions of the related angles.

Results

If θ is any angle,

(i) $\sin\left(\frac{\pi}{2}-\theta\right) = \sin(90° - \theta) = \cos\theta$

$\cos\left(\frac{\pi}{2}-\theta\right) = \cos(90° - \theta) = \sin\theta$

$\tan\left(\frac{\pi}{2}-\theta\right) = \tan(90° - \theta) = \cot\theta$

(ii) $\sin\left(\frac{\pi}{2}+\theta\right) = \sin(90° + \theta) = \cos\theta$

$\cos\left(\frac{\pi}{2}+\theta\right) = \cos(90° + \theta) = -\sin\theta$

$\tan\left(\frac{\pi}{2}+\theta\right) = \tan(90° + \theta) = -\cot\theta$

(iii) $\sin(\pi - \theta) = \sin(180° - \theta) = \sin\theta$
$\cos(\pi - \theta) = \cos(180° - \theta) = -\cos\theta$
$\tan(\pi - \theta) = \tan(180° - \theta) = -\tan\theta$

(iv) $\sin(\pi + \theta) = \sin(180° + \theta) = -\sin\theta$
$\cos(\pi + \theta) = \cos(180° + \theta) = -\cos\theta$
$\tan(\pi + \theta) = \tan(180° + \theta) = \tan\theta$

(v) $\sin\left(\frac{3\pi}{2}-\theta\right) = \sin(270° - \theta) = -\cos\theta$

$\cos\left(\frac{3\pi}{2}-\theta\right) = \cos(270° - \theta) = -\sin\theta$

$\tan\left(\frac{3\pi}{2}-\theta\right) = \tan(270° - \theta) = \cot\theta$

(vi) $\sin\left(\frac{3\pi}{2}+\theta\right) = \sin(270° + \theta) = -\cos\theta$

$\cos\left(\frac{3\pi}{2}+\theta\right) = \cos(270° + \theta) = \sin\theta$

$\tan\left(\frac{3\pi}{2}+\theta\right) = \tan(270° + \theta) = -\cot\theta$

(vii) $\sin(2\pi - \theta) = \sin(360° - \theta) = \sin(-\theta) = -\sin\theta$
$\cos(2\pi - \theta) = \cos(360° - \theta) = \cos(-\theta) = \cos\theta$
$\tan(2\pi - \theta) = \tan(360° - \theta) = \tan(-\theta) = -\tan\theta$

Observations

(i) The above formulas enable to us to obtain the circular functions of any angle x between $\frac{\pi}{4}$ and 2π (i.e., between 45° and 360°) in terms of those of θ between 0 and $\frac{\pi}{4}$ (i.e., between 0° and 45°)

For example,

sin 66° = sin (90° – 24°) = cos 24°;

cot 200° = cot (180° + 20°) = cot 20°;

$$\tan\frac{5\pi}{6} = \tan\left(\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6};$$

$$\cos\frac{5\pi}{3} = \cos\left(\frac{3\pi}{2} + \frac{\pi}{6}\right) = \sin\frac{\pi}{6}$$

sin 85° = sin (90° – 5°) = cos 5°

(Note that the angles 24°, $\frac{\pi}{6}$, 20° and 5° are between 0 and $\frac{\pi}{4}$)

(ii) We note from (iv) above that tan (π + θ) = tan θ and therefore cot (π + θ) = cot θ for any angle θ. This means that tan x and cot x are periodic functions with period π. We have already seen that all circular functions are 2π-periodic.

However, since the period of a function is defined as the smallest positive number T for which f(x + T) = f(x) for all x, we infer that tan x and cot x are periodic with period π and the other four circular functions are 2π periodic.

(iii) We here highlight trigonometric ratios of certain standard angles that could be easily derived from periodic nature of these functions.

(a) We have $\sin\frac{\pi}{2} = 1$

Now, $\sin\frac{3\pi}{2} = \sin\left(\pi + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

$\sin\frac{5\pi}{2} = \sin\left(2\pi + \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$ and so on.

Therefore, $\sin\frac{(2n+1)\pi}{2} = (-1)^n$, n any integer.

(b) sin π = sin (π – 0) = sin 0 = 0; sin 2π = sin 0 = 0

sin 3π = sin(2π + π) = sin π = 0

sin (–π) = – sin π = 0 and so on.

Therefore, sin nπ = 0, n any integer.

(c) $\cos\frac{\pi}{2} = 0, \cos\frac{3\pi}{2} = \cos\left(\pi + \frac{\pi}{2}\right) = -\cos\frac{\pi}{2} = 0$

$\cos\frac{5\pi}{2} = \cos\left(2\pi + \frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

$\cos\frac{-\pi}{2} = \cos\frac{\pi}{2} = 0$ and so on.

Therefore, $\cos\frac{(2n+1)\pi}{2} = 0$, n any integer.

or cos (any odd multiple of $\frac{\pi}{2}$) = 0

(d) cos π = cos (π – 0)

= – cos0 = –1

cos 2π = cos 0 = 1

cos 3π = cos (2π + π)

= cos π = –1 and so on.

Therefore, cos nπ = $(-1)^n$ where n is an integer.

From (b) and (d) it follows that tan nπ = 0, where n is any integer.

To sum up, we have the following important results:

For any integer n,

(i) $\sin\frac{(2n+1)\pi}{2} = (-1)^n$

(ii) sin nπ = 0

(iii) $\cos\frac{(2n+1)\pi}{2} = 0$

(iv) cos nπ = $(-1)^n$

(v) tan nπ = 0

(iv) Since division by zero is not defined, $\tan\frac{\pi}{2}$ is not defined. Although $\tan\frac{\pi}{2}$ is not defined, for values of θ differing from $\frac{\pi}{2}$ by very small quantity, it is possible to get tan θ.

Suppose ε (pronounced as epsilon) denotes a very small positive quantity, $\tan\left(\frac{\pi}{2} - \varepsilon\right) = M$ and $\tan\left(\frac{\pi}{2} + \varepsilon\right) = -M$ where M is a very large positive number. (remember that $\frac{\pi}{2} + \varepsilon$ is in the second quadrant and, therefore, $\tan\left(\frac{\pi}{2} + \varepsilon\right)$ is negative).

Similar is the case with $\theta = \frac{3\pi}{2}$. Although $\tan\frac{3\pi}{2}$ is not defined (for the same reason that division by zero is not defined), we have $\tan\left(\frac{3\pi}{2} - \varepsilon\right) = M$ and $\tan\left(\frac{3\pi}{2} + \varepsilon\right) = -M$ (where, M is a very large positive number).

Steps to be followed for writing down the formulas for the related angles:

(a) For $\frac{\pi}{2} \pm \theta, \frac{3\pi}{2} \pm \theta$, sine becomes cosine; cosine becomes sine and tangent becomes cotangent.
For $\pi \pm \theta$, $2\pi - \theta$, $-\theta$, there is no change of the function.

(b) + or – sign to be prefixed in the resulting function depends on in which quadrant the related angle lies. (for which we assume θ to be acute)

Table 3.2 below gives circular functions of a few standard angles.

Table 3.2

Angles	**sin θ**	**cos θ**	**tan θ**
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	Not defined
$\frac{2\pi}{3}$ (120°)	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$\frac{3\pi}{4}$ (135°)	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	–1
$\frac{5\pi}{6}$ (150°)	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
π (180°)	0	–1	0
$\frac{7\pi}{6}$ (210°)	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{5\pi}{4}$ (225°)	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
$\frac{4\pi}{3}$ (240°)	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
$\frac{3\pi}{2}$ (270°)	–1	0	Not defined
$\frac{5\pi}{3}$ (300°)	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
$\frac{7\pi}{4}$ (315°)	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	–1
$\frac{11\pi}{6}$ (330°)	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
2π (360°)	0	1	0

CONCEPT STRANDS

Concept Strand 6

Express the following circular functions in terms of circular functions of angles between 0° and 45°

(i) cos(–300°)
(ii) tan(1640°)
(iii) sin(1170°)
(iv) cosec(1000°)
(v) $\cot\left(\frac{-13\pi}{9}\right)$

Solution

(i) cos(–300°) = cos(300°) = cos(360°–60°) = cos60° = sin30°

(since cos(– θ) = cosθ for any θ)

(ii) tan(1640°) = tan (4 × 360°+200°) = tan 200°
= tan(180° + 20°) = tan 20°

(iii) sin (1170°) = sin(1080°+90°) = sin(3 × 360°+90°)
= sin90° = 1 (OR)

$$\sin(1170°) = \sin\left(\frac{13\pi}{2}\right) = \sin\left\{\frac{(6\times 2+1)\pi}{2}\right\} = (-1)^6 = 1$$

(iv) cosec(1000°) = cosec (720°+280°) =cosec 280°
= cosec (270°+10°) = – sec 10° (OR)
cosec(1000°) = cos(1080° – 80°) = cosec (– 80°)
= –cosec 80° = – sec 10°

(v) $\cot\left(\frac{-13\pi}{9}\right) = \cot\left(\frac{-18\pi}{9}+\frac{5\pi}{9}\right) = \cot\frac{5\pi}{9}$

$= \cot\left(\frac{\pi}{2}+\frac{\pi}{18}\right) = -\tan\frac{\pi}{18}$

(1 rotation in the clockwise sense and then describing angle $\frac{5\pi}{9}$)

Concept Strand 7

Simplify

$$\frac{\sin(2\pi-\theta)+\cos(-\theta)}{\tan(-\theta)+\cot(2\pi+\theta)}-\frac{\sin\left(\frac{\pi}{2}+\theta\right)+\cos\left(\frac{3\pi}{2}-\theta\right)}{\cot(\pi+\theta)+\tan(2\pi-\theta)}$$

Solution

$$\text{Expression}=\frac{-\sin\theta+\cos\theta}{-\tan\theta+\cot\theta}-\frac{\cos\theta+(-\sin\theta)}{\cot\theta+(-\tan\theta)}=0.$$

Concept Strand 5

$$\text{Show that }\frac{\operatorname{cosec}(\pi+\theta)\cot\left(\frac{9\pi}{2}-\theta\right)\operatorname{cosec}^2(2\pi-\theta)}{\cot(2\pi-\theta)\sec^2(\pi-\theta)\sec\left(\frac{3\pi}{2}+\theta\right)}=1$$

Solution

$$\text{L.H.S}=\frac{-\operatorname{cosec}\theta\left[\cot\left(\frac{\pi}{2}-\theta\right)\right](\operatorname{cosec}^2\theta)}{-\cot\theta\sec^2\theta\operatorname{cosec}\theta}$$

$$=\frac{(\tan\theta)(\operatorname{cosec}^2\theta)}{(\cot\theta)(\sec^2\theta)}$$

$$=\frac{\sin\theta}{\cos\theta}\times\frac{1}{\sin^2\theta}\times\frac{\sin\theta}{\cos\theta}\times\cos^2\theta=1$$

Concept Strand 9

Prove that $\sin^2 72° - \sin^2 54° - \sin^2 36° + \sin^2 18° = 0$.

Solution

$$\begin{aligned}\text{Expression} &= \sin^2(90°-18°)-\sin^2(90°-36°)-\sin^2 36° + \sin^2 18°\\ &= \cos^2 18° - \cos^2 36° - \sin^2 36° + \sin^2 18°\\ &= (\cos^2 18° + \sin^2 18°) - (\cos^2 36° + \sin^2 36°)\\ &= 1 - 1 = 0\end{aligned}$$

Concept Strand 10

Show that $\tan\frac{\pi}{20}\tan\frac{3\pi}{20}\tan\frac{5\pi}{20}\tan\frac{7\pi}{20}\tan\frac{9\pi}{20}=1$

Solution

$$\tan\frac{9\pi}{20}=\tan\left(\frac{\pi}{2}-\frac{\pi}{20}\right)=\cot\frac{\pi}{20};$$

$$\tan\frac{7\pi}{20}=\tan\left(\frac{\pi}{2}-\frac{3\pi}{20}\right)=\cot\frac{3\pi}{20};$$

Also, $\tan\frac{5\pi}{20}=\tan\left(\frac{\pi}{4}\right)=1$

Substituting in the given expression, we get

$$=\left(\tan\frac{\pi}{20}\right)\left(\tan\frac{3\pi}{20}\right)(1)\left(\cot\frac{3\pi}{20}\right)\left(\cot\frac{\pi}{20}\right)=1$$

CIRCULAR FUNCTIONS OF COMPOUND ANGLES

Let the circular functions of any two angles say A and B be known. Then, the circular functions of the angles (A + B) and (A – B) can be expressed in terms of those of A and B.

We have the following formulas.

The proofs to the same are described underneath.

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

Proofs

(Refer Fig. 3.19)
Let ∠XOL=A and ∠LOR =B. Let Q be any point on OR. Draw QP perpendicular to OL. From P and Q draw PM and QN perpendicular to OX.

Let PT be perpendicular to QN.

$$\angle PQT = 90° - \angle QPT = \angle TPO = A$$

$$\angle MOQ = A + B$$

From triangle NOQ,

$$\sin(A+B) = \frac{NQ}{OQ} = \frac{QT+TN}{OQ}$$

$$= \frac{MP+TQ}{OQ} \text{(since TN = PM)}$$

$$= \frac{TQ}{OQ} + \frac{MP}{OQ} = \frac{QT}{QP} \times \frac{QP}{OQ} + \frac{MP}{OP} \times \frac{OP}{OQ}$$

$= \cos A \sin B + \sin A \cos B$

$= \sin A \cos B + \cos A \sin B$

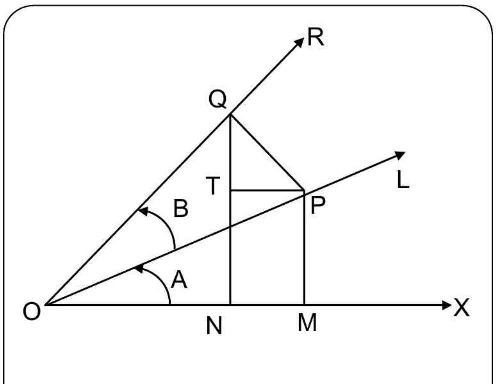


Fig. 3.19

$$\cos(A+B) = \frac{ON}{OQ} = \frac{OM-MN}{OQ}$$

$$= \frac{OM-TP}{OQ} \text{ (since MN = TP)}$$

$$= \frac{OM}{OQ} - \frac{TP}{OQ} = \frac{OM}{OP} \times \frac{OP}{OQ} - \frac{TP}{PQ} \times \frac{PQ}{OQ}$$

$= \cos A \cos B - \sin A \sin B$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)}$$

(on dividing each term of both numerator and denominator by cos A cos B)

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

For deriving the above formulas, A and B were taken such that (A + B) is less than $\frac{\pi}{2}$.

However, the formulas hold good for any values for A and B. Now,

$\sin(A - B) = \sin(A + (-B))$
$= \sin A \cos(-B) + \cos A \sin(-B)$
$= \sin A \cos B - \cos A \sin B$

$\cos(A - B) = \cos(A + (-B))$
$= \cos A \cos(-B) - \sin A \sin(-B)$
$= \cos A \cos B + \sin A \sin B$

$\tan(A-B) = \tan(A+(-B))$

$$= \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

CONCEPT STRANDS

Concept Strand 11

(i) Circular functions of 75° $\left(\text{i.e.,} \frac{5\pi}{12}\right)$

(ii) Circular functions of 15° $\left(\text{i.e.,} \frac{\pi}{12}\right)$

Solution

(i) $\sin 75° = \sin(45° + 30°)$
$= \sin 45° \cos 30° + \cos 45° \sin 30°$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$\cos 75° = \cos(45° + 30°)$
$= \cos 45° \cos 30° - \sin 45° \sin 30°$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\tan 75° = \frac{\sin 75°}{\cos 75°} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3}$$

We may also get tan 75° using the formula for tan (A + B) by taking A = 45° and B = 30°

(ii) $\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

OR

$$\sin 15° = \sin(90° - 75°) = \cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Similarly $\cos 15° = \sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ and

$$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$$

Concept Strand 12

Show that

(i) $\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B$
(ii) $\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B$

Solution

(i) $\sin (A + B) \sin (A - B) = (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$

$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$
$= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B$
$= \sin^2 A - \sin^2 B$

(ii) $\cos (A + B) \cos (A - B) = (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$

$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$
$= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B$
$= \cos^2 A - \sin^2 B$

Concept Strand 13

Prove that if A, B, C are any three angles,

$$\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

Solution

$\tan (A + B + C) = \tan ((A + B) + C)$

$$= \frac{\tan(A + B) + \tan C}{1 - \tan(A + B)\tan C} = \frac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \left(\dfrac{\tan A + \tan B}{1 - \tan A \tan B}\right)\tan C}$$

$$= \frac{\tan A + \tan B + (\tan C)(1 - \tan A \tan B)}{(1 - \tan A \tan B) - (\tan A + \tan B)\tan C}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}$$

Concept Strand 14

Show that

(i) $\tan\left(\dfrac{\pi}{4} + A\right) = \dfrac{1 + \tan A}{1 - \tan A}$

(ii) $\tan\left(\dfrac{\pi}{4} - A\right) = \dfrac{1 - \tan A}{1 + \tan A}$

Solution

(i) $\tan\left(\dfrac{\pi}{4} + A\right) = \dfrac{\tan \pi/4 + \tan A}{1 - \tan \pi/4 \tan A} = \dfrac{1 + \tan A}{1 - \tan A}$

(ii) $\tan\left(\dfrac{\pi}{4} - A\right) = \dfrac{\tan \pi/4 - \tan A}{1 + \tan \pi/4 \tan A} = \dfrac{1 - \tan A}{1 + \tan A}$

Concept Strand 15

If $A + B = \dfrac{\pi}{4}$ (i.e., 45°), prove that

(i) $(1 + \tan A)(1 + \tan B) = 2$
(ii) $\cot A + \cot B = \cot A \cot B - 1$

Solution

(i) Since $A + B = \dfrac{\pi}{4}$, $B = \dfrac{\pi}{4} - A$

$$\tan B = \tan\left(\frac{\pi}{4} - A\right) = \frac{1 - \tan A}{1 + \tan A}$$

$$(1 + \tan A)(1 + \tan B) = (1 + \tan A)\left(1 + \frac{1 - \tan A}{1 + \tan A}\right) = 2$$

(ii) $\cot A + \cot B = \dfrac{1}{\tan A} + \dfrac{1}{\tan B} = \dfrac{\tan A + \tan B}{\tan A \tan B}$

$$\cot A \cot B - 1 = \frac{1}{\tan A \tan B} - 1 = \frac{1 - \tan A \tan B}{\tan A \tan B}$$

We have to prove that,

$$\frac{\tan A + \tan B}{\tan A \tan B} = \frac{1 - \tan A \tan B}{\tan A \tan B}$$

or to prove that $\frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

or to prove that $\tan(A + B) = 1$, which is true, since

$$A + B = \frac{\pi}{4}$$

OR

Since $(1 + \tan A)(1 + \tan B) = 2$

$$\Rightarrow \left(1 + \frac{1}{\cot A}\right)\left(1 + \frac{1}{\cot B}\right) = 2$$

$$\Rightarrow (1 + \cot A)(1 + \cot B) = 2\cot A \cot B$$

$$\Rightarrow 1 + \cot A + \cot B + \cot A \cot B = 2\cot A \cot B$$

$$\Rightarrow \cot A + \cot B = \cot A \cot B - 1$$

PRODUCT FORMULAS

We have,

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \quad \text{— (1)}$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \quad \text{— (2)}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \quad \text{— (3)}$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \quad \text{— (4)}$$

Now,

$(1) + (2) \Rightarrow$

$$\sin(A + B) + \sin(A - B) = 2\sin A \cos B \quad \text{— (5)}$$

$(1) - (2) \Rightarrow$

$$\sin(A + B) - \sin(A - B) = 2\cos A \sin B \quad \text{— (6)}$$

$(3) + (4) \Rightarrow$

$$\cos(A + B) + \cos(A - B) = 2\cos A \cos B \quad \text{— (7)}$$

$(4) - (3) \Rightarrow$

$$\cos(A - B) - \cos(A + B) = 2\sin A \sin B \quad \text{— (8)}$$

OR we have

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

Setting $A + B = C$, $A - B = D$, in (5), (6), (7) and (8), we obtain, what are called product formulas

$$\sin C + \sin D = 2\sin\frac{C + D}{2}\cos\frac{C - D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C + D}{2}\sin\frac{C - D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C + D}{2}\cos\frac{C - D}{2}$$

$$\cos D - \cos C = 2\sin\frac{C + D}{2}\sin\frac{C - D}{2} \quad \text{(or)}$$

$$\cos C - \cos D = -2\sin\frac{C + D}{2}\sin\frac{C - D}{2}$$

CONCEPT STRAND

Concept Strand 16

(i) $\sin 40° \cos 10°$

$$= \frac{1}{2}[\sin(40° + 10°) + \sin(40° - 10°)]$$

$$= \frac{1}{2}[\sin 50° + \sin 30°]$$

(ii) $\cos 70° \sin 35°$

$$= \frac{1}{2}[\sin(70° + 35°) - \sin(70° - 35°)]$$

$$= \frac{1}{2}[\sin 105° - \sin 35°]$$

(iii) $\cos 80° \cos 25°$

$$= \frac{1}{2}[\cos(80° + 25°) + \cos(80° - 25°)]$$

$$= \frac{1}{2}[\cos 105° + \cos 55°]$$

(iv) $\sin 140° \sin 10°$

$$= \frac{1}{2}[\cos(140° - 10°) - \cos(140° + 10°)]$$

$$= \frac{1}{2}[\cos 130° - \cos 150°]$$

(v) $\sin 100° + \sin 70°$

$$= 2\sin\left(\frac{100° + 70°}{2}\right)\cos\left(\frac{100° - 70°}{2}\right)$$

$$= 2\sin 85° \cos 15°$$

(vi) $\sin 20° - \sin 10° = 2\cos\left(\frac{20° + 10°}{2}\right)\sin\left(\frac{20° - 10°}{2}\right)$

$$= 2\cos 15° \sin 5°$$

(vii) $\cos 82° + \cos 46°$

$$= 2\cos\left(\frac{82° + 46°}{2}\right)\cos\left(\frac{82° - 46°}{2}\right)$$

$$= 2\cos 64° \cos 18°$$

(viii) $\cos 41° - \cos 20°$

$$= -2\sin\left(\frac{41° + 20°}{2}\right)\sin\left(\frac{41° - 20°}{2}\right)$$

$$= -2\sin\frac{61°}{2}\sin\frac{21°}{2}$$

CIRCULAR FUNCTIONS OF MULTIPLES OF AN ANGLE A

For any angle A,

$$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1 + \tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$$

$$= 1 - 2\sin^2 A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Proofs

$$\sin 2A = \sin(A + A)$$

$$= \sin A\cos A + \cos A\sin A = 2\sin A\cos A$$

$$= \frac{2\sin A}{\cos A} \times \cos^2 A = \frac{2\tan A}{\sec^2 A} = \frac{2\tan A}{1 + \tan^2 A}$$

$$\cos 2A = \cos(A + A) = \cos A\cos A - \sin A\sin A$$

$$= \cos^2 A - \sin^2 A$$

$$= \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$$

$$= 2(1 - \sin^2 A) - 1 = 1 - 2\sin^2 A$$

Again,

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$= (\cos^2 A)[1 - \tan^2 A]$$

$$= \frac{1 - \tan^2 A}{\sec^2 A} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A\tan A}$$

Again,

$$\sin 3A = \sin(2A + A)$$

$$= \sin 2A\cos A + \cos 2A\sin A$$

$$= (2\sin A\cos A)\cos A + (1 - 2\sin^2 A)\sin A$$

(substituting for sin2A and cos2A)

$$= 2\sin A\cos^2 A + \sin A - 2\sin^3 A$$

$$= (2\sin A)(1 - \sin^2 A) + \sin A - 2\sin^3 A$$

$$= 3\sin A - 4\sin^3 A$$

$$\cos 3A = \cos(2A + A) = \cos 2A\cos A - \sin 2A\sin A$$

$$= (2\cos^2 A - 1)\cos A - 2\sin^2 A\cos A$$

$$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$$

$$= 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{\sin 3A}{\cos 3A} = \frac{3\sin A - 4\sin^3 A}{4\cos^3 A - 3\cos A}$$

$$= \frac{(\sin A)(3 - 4\sin^2 A)}{(\cos A)(4\cos^2 A - 3)}$$

$$= (\tan A)\frac{\left[\frac{3}{\cos^2 A} - 4\frac{\sin^2 A}{\cos^2 A}\right]}{\left(4 - \frac{3}{\cos^2 A}\right)}$$ (on dividing numerator and denominator by $\cos^2 A$)

$$= \frac{(\tan A)\left[3\sec^2 A - 4\tan^2 A\right]}{(4 - 3\sec^2 A)}$$

$$= \frac{(\tan A)\left[3(1 + \tan^2 A) - 4\tan^2 A\right]}{4 - 3(1 + \tan^2 A)}$$

$$= \frac{(\tan A)(3 - \tan^2 A)}{1 - 3\tan^2 A} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

We can also derive the formula for tan3A by writing $\tan 3A = \tan(2A + A)$ and then using the formula for tan2A.

Observations

(i) $1 + \cos 2A = 2\cos^2 A$

$1 - \cos 2A = 2\sin^2 A$

(or) $\sin A = \pm\sqrt{\frac{1 - \cos 2A}{2}}$;

$\cos A = \pm\sqrt{\frac{1 + \cos 2A}{2}}$;

$\tan A = \pm\sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}}$

The proper sign + or − will depend on in which quadrant A lies.

(ii) Replacing 2A by A in the formulas for 2A,

$$\sin A = 2\sin \tfrac{A}{2}\cos\tfrac{A}{2} = \frac{2\tan\frac{A}{2}}{1 + \tan^2\frac{A}{2}}$$

$$\cos A = \cos^2\tfrac{A}{2} - \sin^2\tfrac{A}{2} = 2\cos^2\tfrac{A}{2} - 1$$

$$= 1 - 2\sin^2\tfrac{A}{2} = \frac{1 - \tan^2\frac{A}{2}}{1 + \tan^2\frac{A}{2}}$$

$$\tan A = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}$$

CONCEPT STRANDS

Concept Strand 17

Show that $\tan\theta = \frac{\sin 2\theta}{1 + \cos 2\theta}$ and deduce the value of $\tan 22\frac{1}{2}^\circ$.

Solution

$$\frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \tan\theta$$

Setting $\theta = 22\tfrac{1}{2}^\circ$

$$\tan 22\tfrac{1}{2}^\circ = \frac{\sin 45^\circ}{1 + \cos 45^\circ} = \frac{\frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}} = \frac{1}{(1 + \sqrt{2})}$$

$$= \sqrt{2} - 1$$

Concept Strand 18

Show that $\frac{\sin 3\theta}{1 + 2\cos 2\theta} = \sin\theta$ and deduce the value of $\sin 15^\circ$.

Solution

$$\frac{\sin 3\theta}{1 + 2\cos 2\theta} = \frac{3\sin\theta - 4\sin^3\theta}{1 + 2(1 - 2\sin^2\theta)}$$

$$= \frac{(3 - 4\sin^2\theta)\sin\theta}{(3 - 4\sin^2\theta)} = \sin\theta$$

Setting $\theta = 15^\circ$ in the above result,

$$\sin 15^\circ = \frac{\sin 45^\circ}{1 + 2\cos 30^\circ} = \frac{\frac{1}{\sqrt{2}}}{1 + 2\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{2}\left(\sqrt{3} + 1\right)}$$

$$= \frac{1}{\sqrt{6} + \sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

Concept Strand 19

Show that $\dfrac{1+\sin A}{1-\sin A} = \tan^2\left(\dfrac{\pi}{4}+\dfrac{A}{2}\right)$.

Solution

$$\frac{1+\sin A}{1-\sin A} = \frac{1+2\sin\frac{A}{2}\cos\frac{A}{2}}{1-2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$= \frac{\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2}{\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)^2} = \left(\frac{\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+1}{\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}-1}\right)^2$$

$$= \left(\frac{1+\tan\frac{A}{2}}{\tan\frac{A}{2}-1}\right)^2 = \left(\frac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}}\right)^2$$

$$= \left(\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)\right)^2$$

$$= \tan^2\left(\frac{\pi}{4}+\frac{A}{2}\right)$$

Concept Strand 20

Obtain the values of $\sin 18°$ $\left(\text{i.e.,} \sin\dfrac{\pi}{10}\right)$ and $\cos 36°$ $\left(\text{i.e.,} \cos\dfrac{\pi}{5}\right)$.

Solution

Let $\theta = 18°$. Observe that $3\theta = 54°$, $2\theta = 36°$ and $3\theta = 90° - 2\theta$

$$\cos 3\theta = \cos(90° - 2\theta) = \sin 2\theta$$

Substituting the formulas for $\sin 2\theta$ and $\cos 3\theta$ in the above relation,

$$4\cos^3\theta - 3\cos\theta = 2\sin\theta\cos\theta$$

$\Rightarrow$ $(\cos\theta)[4\cos^2\theta - 3 - 2\sin\theta] = 0$

Since $\cos\theta = \cos 18° \neq 0$, $4\cos^2\theta - 3 - 2\sin\theta = 0$

$\Rightarrow$ $4(1-\sin^2\theta) - 3 - 2\sin\theta = 0$

$\Rightarrow$ $4\sin^2\theta + 2\sin\theta - 1 = 0$

$\Rightarrow$ $\sin\theta = \dfrac{-2\pm 2\sqrt{5}}{8} = \dfrac{\sqrt{5}-1}{4} \text{(or)} \dfrac{-(\sqrt{5}+1)}{4}$

Since $\theta = 18°$ is in the first quadrant, $\sin\theta$ is positive.

Therefore, $\sin 18° = \dfrac{\sqrt{5}-1}{4}$

$\cos 36° = \cos(2 \times 18°) = 1 - 2\sin^2 18°$

$$= 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{\sqrt{5}+1}{4}$$

INVERSE CIRCULAR FUNCTIONS

Circular functions have already been defined. The domain of the functions sin x and cos x is R and their range is [−1, 1]. The domain of the function tan x is R excluding the points $\pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots\ldots\ldots\ldots$ and its range is $(-\infty, \infty)$. These functions are also periodic. Therefore, these functions are not one to one mappings from R onto their ranges. For the inverses of these functions to be defined, the mappings must be bijective.

For the function f(x) = sin x, if we choose $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ as the domain, the mapping is one one and onto its range [−1, 1] (Refer Fig. 1.12 (i)).

For the function f(x) = cos x, if we choose $[0, \pi]$ as the domain, the mapping is one one and onto its range [−1, 1] (Refer Fig. 1.12 (ii)).

For the function f(x) = tan x, if we choose $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ as the domain, the mapping is one one and onto its range $(-\infty, \infty)$ (Refer Fig. 3.20 (iii))

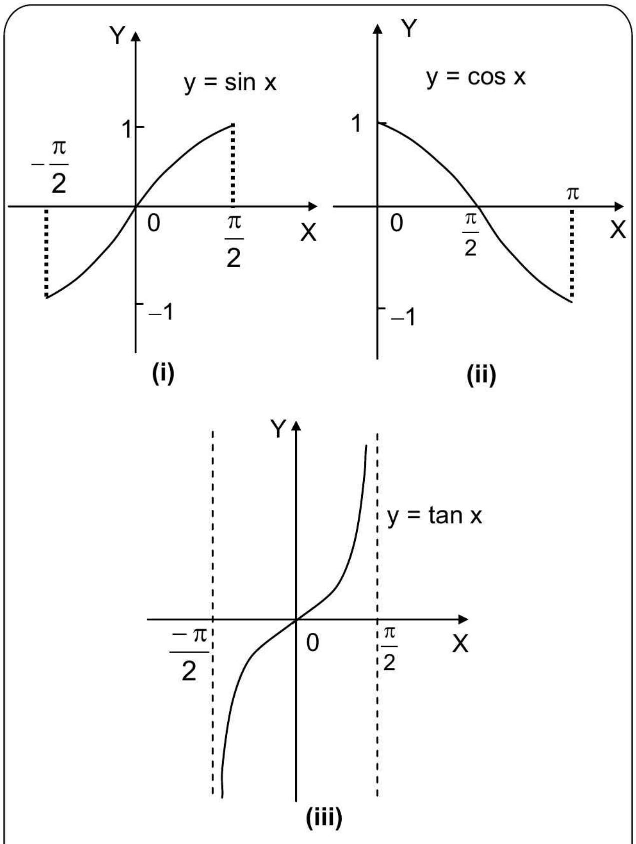


Fig. 3.20

Definition-1

The inverse circular function $\sin^{-1} x$ (pronounced as sine inverse x) is that function of x whose domain is $[-1, 1]$ and whose range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$g(x) = \sin^{-1} x$, $x \in [-1, 1]$ is that angle θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin \theta = x$.

$g(x) = \sin^{-1} x$ is the inverse of $f(x) = \sin x$, i.e., $g \circ f = f \circ g$ = Identity function

Definition-2

The inverse circular function $\cos^{-1} x$ is that function of x whose domain is $[-1, 1]$ and whose range is $[0, \pi]$

$g(x) = \cos^{-1} x$, $x \in [-1, 1]$ is that angle θ in $[0, \pi]$ such that $\cos \theta = x$.

$g(x) = \cos^{-1} x$ is the inverse of $f(x) = \cos x$, i.e., $g \circ f = f \circ g$ = Identity function

Definition-3

The inverse circular function $\tan^{-1} x$ is that function of x whose domain is $(-\infty, \infty)$ and whose range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$g(x) = \tan^{-1} x$, $x \in (-\infty, \infty)$ is that angle θ in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan \theta = x$.

$g(x) = \tan^{-1} x$ is the inverse of $f(x) = \tan x$, i.e., $g \circ f = f \circ g$ = Identity function

The graphs of the above functions are given in Fig. 3.21

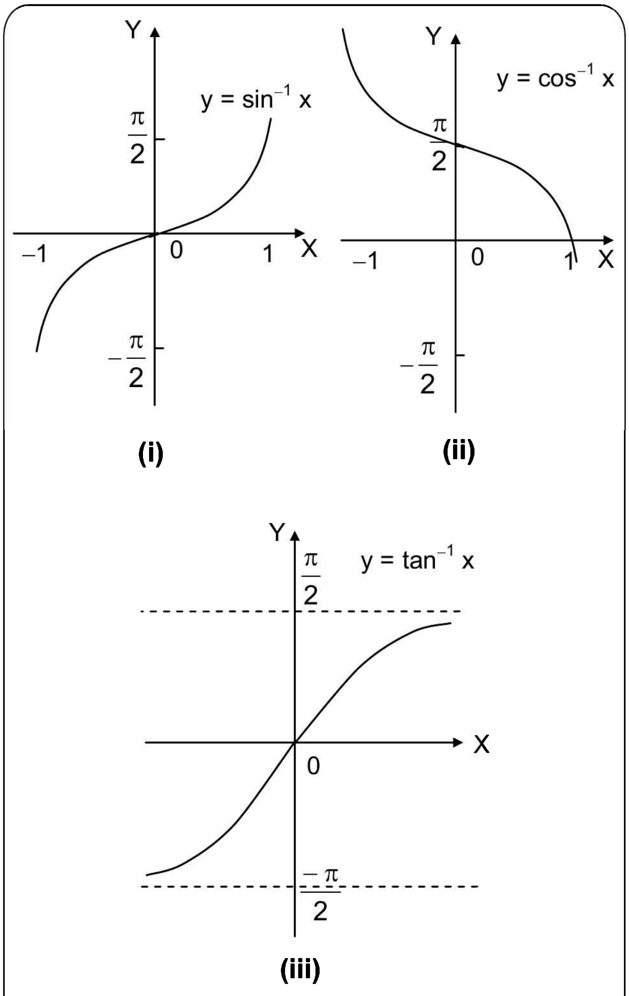


Fig. 3.21

For example,

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4};\quad \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3};\quad \tan^{-1}(1) = \frac{\pi}{4}$$

Remarks

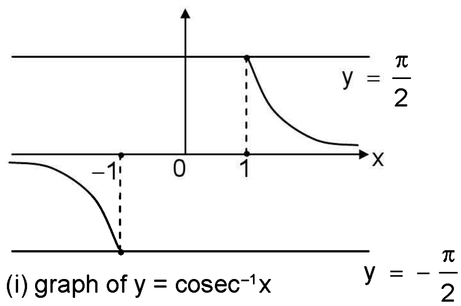
$\text{cosec}^{-1}x$, $\sec^{-1}x$ and $\cot^{-1}x$ can be similarly defined.

g(x) = cosec⁻¹x

Domain: $(-\infty, -1] \cup [1, \infty)$

Range: $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ excluding zero

(i) graph of $y = \text{cosec}^{-1}x$

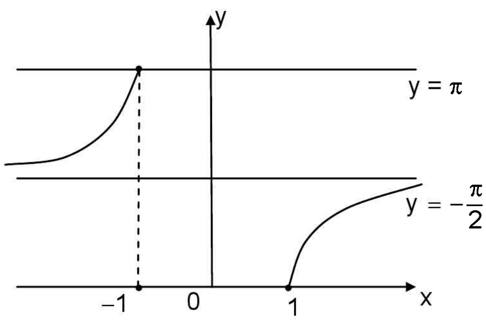


(i) graph of y = cosec⁻¹x

g(x) = sec⁻¹x

Domain: $(-\infty, -1] \cup [1, \infty)$

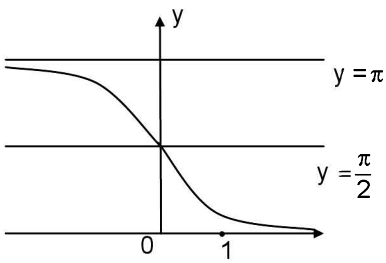
Range: $[0, \pi]$ excluding $\frac{\pi}{2}$



(ii) graph of y = sec⁻¹x

g(x) = cot⁻¹x

Domain: $(-\infty, \infty)$
Range: $(0, \pi)$



(iii) graph of y = cot⁻¹ x

Just as we have proved many relations involving circular functions, we can prove relations involving inverse circular functions also.

Illustration

$\text{cosec}^{-1}(-2) = -\frac{\pi}{6}$, $\text{cosec}^{-1}\left(\sqrt{2}\right) = \frac{\pi}{4}$

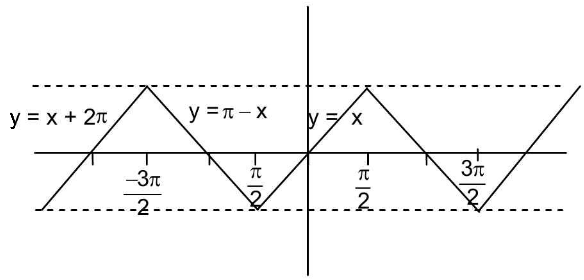
$$\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\sec^{-1}(-2) = \frac{2\pi}{3}$$

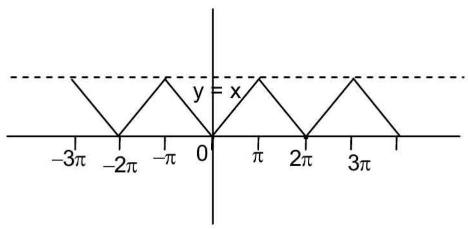
$$\cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}$$

$$\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{2\pi}{3}$$

1. $$y = \sin^{-1}(\sin x) = \begin{cases} x + 2\pi, & -\frac{5\pi}{2} \le x \le \frac{-3\pi}{2} \\ -\pi - x, & \frac{-3\pi}{2} \le x \le -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ x - 2\pi, & \frac{3\pi}{2} \le x \le \frac{5\pi}{2} \end{cases}$$



2. $$y = \cos^{-1}(\cos x) = \begin{cases} x + 2\pi, & -2\pi \le x \le -\pi \\ -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \\ x - 2\pi, & 2\pi \le x \le 3\pi \end{cases}$$



3. $\sin^{-1}x = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}, -1 < x < 1$

$$\sin^{-1}x = \begin{cases} -\pi + \cot^{-1}\dfrac{\sqrt{1-x^2}}{x}, & -1 \le x < 0 \\ \cot^{-1}\dfrac{\sqrt{1-x^2}}{x}, & 0 < x \le 1 \end{cases}$$

4. $$2\sin^{-1}x = \begin{cases} -\sin^{-1}\left(2x\sqrt{1-x^2} - \pi, -1 \le x \le -\dfrac{1}{\sqrt{2}}\right) \\ \sin^{-1}\left(2x\sqrt{1-x^2}\right), & -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), & \dfrac{1}{\sqrt{2}} < x \le 1 \end{cases}$$

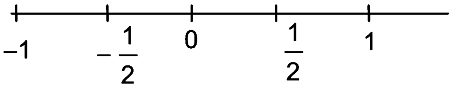
$$2\cos^{-1}x = \begin{cases} 2\pi - \cos^{-1}\left(2x^2 - 1\right), & -1 \le x < 0 \\ \cos^{-1}\left(2x^2 - 1\right), & 0 \le x \le 1 \end{cases}$$

5. $$2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) - \pi, & x < -1 \\ \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & -1 < x < 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & x > 1 \end{cases}$$

$$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) - \pi, & x < -1 \\ \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & x > 1 \end{cases}$$

$$2\tan^{-1}x = \begin{cases} -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & -\infty < x \le 0 \\ \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & 0 \le x < \infty \end{cases}$$

6. $$\sin^{-1}(3x - 4x^3) = \begin{cases} -3\sin^{-1}x - \pi, & -1 \le x \le -\dfrac{1}{2} \\ 3\sin^{-1}x, & -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1}x, & \dfrac{1}{2} \le x \le 1 \end{cases}$$



7. $$\cos^{-1}(3x - 4x^3) = \begin{cases} 3\cos^{-1}x - 2\pi, & -1 \le x \le -\dfrac{1}{2} \\ 2\pi - 3\cos^{-1}x, & -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 3\cos^{-1}x, & \dfrac{1}{2} \le x \le 1 \end{cases}$$

Results

(i) $\sin(\sin^{-1}x) = x$ for all $x \in [-1, 1]$
$\cos(\cos^{-1}x) = x$ for all $x \in [-1, 1]$
$\tan(\tan^{-1}x) = x$ for all $x \in R$

(ii) $\sin^{-1}(\sin x) = x, -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

$\cos^{-1}(\cos x) = x, 0 \le x \le \pi$

$\tan^{-1}(\tan x) = x, -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

(iii) $\sin^{-1}(-x) = -\sin^{-1}x, -1 \le x \le 1$
$\cos^{-1}(-x) = \pi - \cos^{-1}x, -1 \le x \le 1$
$\tan^{-1}(-x) = -\tan^{-1}x, x \in R$
$\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R$

(iv) $\text{cosec}^{-1}x = \sin^{-1}\left(\dfrac{1}{x}\right), |x| \ge 1$

$\sec^{-1}x = \cos^{-1}\left(\dfrac{1}{x}\right), |x| \ge 1$

$$\cot^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{1}{x}\right) & x > 0 \\ \pi + \tan^{-1}\left(\dfrac{1}{x}\right) & x < 0 \end{cases}$$

(v) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}, -1 \le x \le 1$

For, let $\sin^{-1}x = A$, $\cos^{-1}x = B$.
From the definition, we have $\sin A = x = \cos B$, giving

$B = \dfrac{\pi}{2} - A$

$\Rightarrow \sin^{-1}x + \cos^{-1}x = A + B = \dfrac{\pi}{2}$

Similarly,

$\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}, 0 < x \le 1$

(vi) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$, where $x > 0, y > 0$ and $xy < 1$

For, let $\tan^{-1} x = A$, $\tan^{-1} y = B$. Then,

$$\frac{x+y}{1-xy} = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(A+B)$$

or $\quad \tan^{-1}\left(\frac{x+y}{1-xy}\right) = A + B = \tan^{-1} x + \tan^{-1} y$

Also, $\quad \tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

if $\quad x > 0, y > 0, xy > 1$

Similarly,

$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, where, $x > 0, y > 0$

If $xy = 1$, $\tan^{-1}x + \tan^{-1}y = \begin{cases} \dfrac{\pi}{2} & x, y > 0 \\ \dfrac{-\pi}{2} & x, y < 0 \end{cases}$

(vii) $\sin^{-1}x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

where,

$x \geq 0, y \geq 0$ and $x^2 + y^2 \leq 1$

For, let $\sin^{-1}x = \alpha$, $\sin^{-1} y = \beta$

Then, $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$, provided $x^2 + y^2 \leq 1$.

$\Rightarrow \alpha + \beta = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$

If $x \geq 0, y \geq 0$ and $x^2 + y^2 > 1$,

$\sin^{-1}x + \sin^{-1} y =$
$\pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$

CONCEPT STRAND

Concept Strand 21

(i) $\sin^{-1}\left(\sin\frac{4\pi}{3}\right) = \sin^{-1}\left(\sin\left(\pi + \frac{\pi}{3}\right)\right)$

$= \sin^{-1}\left(-\sin\frac{\pi}{3}\right) = \sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$

(ii) $\cos\left(\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right) = \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

(iii) $\tan^{-1}\left(\tan\frac{5\pi}{4}\right) = \tan^{-1}\left(\tan^{-1}\left(\pi + \frac{\pi}{4}\right)\right)$

$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$

(iv) $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right)$

$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right)$

$= \sin\frac{\pi}{2} = 1$

(v) $\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

TRIGONOMETRIC EQUATIONS

A trigonometric equation is an equation involving trigonometric or circular functions. Given below are a few examples:

(i) $\sin x = \frac{1}{2}$

(ii) $\sin^2\theta - 5\sin\theta + 6 = 0$

(iii) $\tan x + 3\cot x = 2$

(iv) $4\cos\theta + 3\sin\theta = 1$

A solution of a trigonometric equation is a value of the unknown angle x (or θ) appearing in the equation which satisfies the equation.

For example, in the case of (i) above, $x = \frac{\pi}{6}$ (or 30°) is a solution of the equation. We also note that $x = \frac{5\pi}{6}$ (or 150°), $x = \frac{13\pi}{6}$ (or 390°), $x = -\frac{11\pi}{6}$ (or −330°)… also satisfy the equation and therefore, these values of x also constitute solutions of (i).

In the case of (ii), we may consider the equation as a quadratic equation in $\sin\theta$, and factoring the expression on the left side, we get $\sin\theta - 2 = 0$ or $\sin\theta - 3 = 0$ or equivalently, we have to solve the equations $\sin\theta = 2$ and $\sin\theta = 3$. As $|\sin\theta|$ cannot assume a value greater than 1, there exists no real value for θ which satisfies the two equations. This means that (ii) has no real solutions.

We therefore see that some trigonometric equations have real solutions while some others have no real solutions.

We consider trigonometric equations which reduce to one of the forms below.

$\sin x = k$ or $\cos x = k$ or $\tan x = k$, where k is a real number.

Since sin x and cos x are 2π -periodic functions, if $x = \alpha$ is a solution of $\sin x = k$ (or $\cos x = k$), $x = 2n\pi + \alpha$ where n is any integer is also a solution of these equations. Again, since tan x is a periodic function with period π, if $x = \alpha$ is a solution of $\tan x = k$, $x = n\pi + \alpha$ where n is any integer is also a solution of this equation. Thus we see that these trigonometric equations have infinitely many solutions. Our task is to obtain general formulas for the solution set of the above equations.

Equations of the form sin x = k — (1)

(Note that $-1 \le k \le 1$ or, otherwise the equation has no real solution)

Let α be a number such that

(i) $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ and

(ii) $\sin\alpha = k$

α is called the 'principal value' of x. It is obvious that $\alpha = \sin^{-1}(k)$.

Since $\sin(\pi - \alpha) = \sin\alpha = k$, $x = (\pi - \alpha)$ is also a solution of (1). Again, since sin x is 2π-periodic,

$\sin(2n\pi + \alpha) = \sin\alpha = k$ and $\sin(2n\pi + \pi - \alpha) = \sin(\pi - \alpha) = \sin\alpha = k$

or $x = 2n\pi + \alpha$ and $x = (2n + 1)\pi - \alpha$ where n is any integer represents the solution of (1).

We may combine the above and state that

$$x = n\pi + (-1)^n\alpha$$

where n is any integer and $\alpha = \sin^{-1} k$ represents the general solution of (1)

[If α is in degree measure, $(-90° \le \alpha \le 90°)$, the solution set may be represented as $x = n \times 180° + (-1)^n\alpha$]

CONCEPT STRAND

Concept Strand 22

(i) Find the general solution of $\sin x = -\frac{1}{2}$.

(ii) Obtain the general solution of the equation $4\sin^2\theta - 3 = 0$

(iii) Write down the solutions of $\sin x = \frac{1}{\sqrt{2}}$ in the interval $[-3\pi, 3\pi]$

Solution

(i) Since $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$, $x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$ where n is any integer

(ii) The given equation can be factored as

$(2\sin\theta - \sqrt{3})(2\sin\theta + \sqrt{3}) = 0 \Rightarrow \sin\theta = \pm\frac{\sqrt{3}}{2}$

For $\sin\theta = \frac{\sqrt{3}}{2}$, general solution is

$\theta = n\pi + (-1)^n\left(\frac{\pi}{3}\right)$

For $\sin\theta = -\frac{\sqrt{3}}{2}$, general solution is

$\theta = n\pi + (-1)^n\left(-\frac{\pi}{3}\right)$

or, the general solution of the equation is

$\theta = n\pi + (-1)^n\left(\pm\frac{\pi}{3}\right)$, where n is any integer.

(iii) We have, $\alpha = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

Therefore, the general solution is $x = n\pi + (-1)^n\left(\frac{\pi}{4}\right)$, where n is any integer.

n = 3 gives $x = \frac{11\pi}{4}$; n = 2 gives $x = \frac{9\pi}{4}$;

n = 1 gives $x = \frac{3\pi}{4}$;

n = 0 gives $x = \frac{\pi}{4}$; n = −1 gives $x = -\frac{5\pi}{4}$;

n = −2 gives $x = -\frac{7\pi}{4}$;

n = −3 gives $x = -\frac{13\pi}{4}$ which is less than -3π

Hence, the solution set of the equation in the interval $[-3\pi, 3\pi]$ is $\left\{-\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}\right\}$

Equations of the form cos x = k — (2)

(Note that $-1 \le k \le 1$)
Let α be a number such that

(i) $0 \le \alpha \le \pi$ and
(ii) $\cos\alpha = k$

α is called the principal value of x. We may note that the principal value is $\cos^{-1} k$.

Since $\cos(-\alpha) = \cos\alpha = k$, $x = -\alpha$ is also a solution of (2). Thus, $x = \alpha$ and $x = -\alpha$ are two particular solutions of (2). By using the periodic property of cos x, we infer that $x = 2n\pi + \alpha$ and $x = 2n\pi - \alpha$ where n is any integer represent the solution sets of (2) or, the general solution of (2) may be represented as

$x = 2n\pi \pm \alpha$ where n is any integer and $\alpha = \cos^{-1} k$

[If α is in degree measure, the solution set is $x = n \times 360° \pm \alpha$ where n is an integer]

CONCEPT STRAND

Concept Strand 23

(i) Obtain the general solution of $\cos\theta = -\frac{1}{2}$
(ii) Obtain the general solution of the equation $2\cos^2 x + \sqrt{3}\cos x = 0$
(iii) Find all solutions of the equation $4\sin^2 x - 4\cos x - 1 = 0$ in the interval $0 \le x \le 2\pi$

Solution

(i) $\theta = 2n\pi \pm \alpha$ where $\alpha = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

$\Rightarrow \theta = 2n\pi \pm \frac{2\pi}{3}$ where n is any integer.

(ii) We have, on factorisation, $(\cos x)(2\cos x + \sqrt{3}) = 0$

giving $\cos x = 0$ or $\cos x = -\frac{\sqrt{3}}{2}$

General solution of $\cos x = 0$ is $x = 2n\pi \pm \frac{\pi}{2}$ and general solution of $\cos x = -\frac{\sqrt{3}}{2}$ is $x = 2n\pi \pm \frac{5\pi}{6}$.

Or, general solution of the given equation is $x = 2n\pi \pm \frac{\pi}{2}$ or $x = 2n\pi \pm \frac{5\pi}{6}$, where n is any integer.

(iii) Given equation may be written as

$4(1 - \cos^2 x) - 4\cos x - 1 = 0$

or $4\cos^2 x + 4\cos x - 3 = 0$

$\Rightarrow \cos x = \frac{-4 \pm \sqrt{16 + 48}}{8} = \frac{-4 \pm 8}{8} = \frac{1}{2}$ or $-\frac{3}{2}$

$\cos x = -\frac{3}{2}$ is not admissible.

Therefore, $\cos x = \frac{1}{2}$

$\Rightarrow$ Principal value $\alpha = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

General solution of the equation is $x = 2n\pi \pm \frac{\pi}{3}$, where n is any integer.

n = 0 gives $x = \pm\frac{\pi}{3}$; n = 1 gives $x = 2\pi \pm \frac{\pi}{3}$

Solutions of the equation in the interval $0 \le x \le 2\pi$ are $\frac{\pi}{3}, \frac{5\pi}{3}$

Equation of the form tan x = k — (3)

(Here, $k \in R$)

Let α be a number such that

(i) $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ and

(ii) $\tan \alpha = k$

α is called the 'principal value' of x. Note that the principal value is $\tan^{-1} k$.

Since tan x is π-periodic, it is clear that $x = n\pi + \alpha$ where n is any integer are solutions of (3).

In other words, the solution set of (3) is represented by $x = n\pi + \alpha$ where n is any integer.

[If α is in degree measure, the solution set is $x = n \times 180° + \alpha$, where n is an integer]

CONCEPT STRAND

Concept Strand 24

(i) Obtain general solution of $\tan \theta = -\sqrt{3}$

(ii) Find the general solution of the equation $28 \tan^2 x - 13\sec^2 x + 8 = 0$

(iii) Write down all the solutions of the equation $\tan \theta = -1$ in the interval $(-2\pi, 2\pi)$

Solution

(i) Since $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$; where n is an integer, general solution is given by $\theta = n\pi + \left(\frac{-\pi}{3}\right)$.

(ii) The equation may be rewritten as $28\tan^2 x - 13(1 + \tan^2 x) + 8 = 0 \Rightarrow 15\tan^2 x - 5 = 0$

$$\tan^2 x = \frac{1}{3}$$

$$\Rightarrow \quad \tan x = \pm\frac{1}{\sqrt{3}}$$

Since $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ and $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$,

General solution of the equation is given by

$x = n\pi \pm \frac{\pi}{6}$, where n is any integer.

(iii) We have, $\tan^{-1}(-1) = -\frac{\pi}{4}$

General solution is $\theta = n\pi - \frac{\pi}{4}$ where n is any integer

$n = 0$ gives $\theta = -\frac{\pi}{4}$; $\quad n = 1$ gives $\theta = \frac{3\pi}{4}$;

$n = -1$ gives $\theta = -\frac{5\pi}{4}$; $\quad n = 2$ gives $\theta = \frac{7\pi}{4}$

The solutions in the interval

$(-2\pi, 2\pi)$ are $\theta = -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

Equations of the form a cos x + b sin x = c — (4)

(a, b, c real numbers)

Our task is to reduce the above equation to either of the forms sin x = k or cos x = k. For this purpose, we divide both sides of the equation (4) by $\sqrt{a^2 + b^2}$ to get

$$\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x = \frac{c}{\sqrt{a^2+b^2}} = k \text{ (say)}$$

We can always find an angle α such that

$$\sin\alpha = \frac{a}{\sqrt{a^2+b^2}} \text{ and } \cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$$

This is possible since $\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$

We may then write (4) as

$$\sin\alpha \cos x + \cos\alpha \sin x = k \text{ or } \sin(x + \alpha) = k \quad \text{— (5)}$$

Alternatively, an angle β can be found such that

$$\cos\beta = \frac{a}{\sqrt{a^2+b^2}} \text{ and } \sin\beta = \frac{b}{\sqrt{a^2+b^2}}$$

(It may be noted that $\beta = \frac{\pi}{2} - \alpha$)

We may then write (4) as

$$\cos\beta \cos x + \sin\beta \sin x = k \text{ or } \cos(x - \beta) = k \text{ — (6)}$$

If $-1 \le k \le 1$, the solution set of (A) can be obtained from (5) or (6)

If $|k| > 1$, (5) and (6) (i.e., (4)) have no real solutions.

CONCEPT STRAND

Concept Strand 25

Find the general solution of the equation

$$\cos x + \sqrt{3}\sin x = \sqrt{2}.$$

Solution

Here $a = 1,\ b = \sqrt{3}$

Dividing the equation by $\sqrt{1+3} = 2$,

we get $\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x = \frac{1}{\sqrt{2}}$

Since $\sin\frac{\pi}{6} = \frac{1}{2}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, we may write the above equation as $\sin\left(x + \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$

Therefore,

$$x + \frac{\pi}{6} = n\pi + (-1)^n\frac{\pi}{4}$$

or the general solution of the given equation is

$x = n\pi - \frac{\pi}{6} + (-1)^n\frac{\pi}{4}$, where n is an integer.

SUMMARY

1. Periodicity property of circular functions and graphs of circular functions

$\sin(x + 2n\pi) = \sin x$

$\cos(x + 2n\pi) = \cos x$

$\tan(x + n\pi) = \tan x$

$\operatorname{cosec}(x + 2n\pi) = \operatorname{cosec} x$

$\sec(x + 2n\pi) = \sec x$

$\cot(x + n\pi) = \cot x$

where n is an integer, positive or negative.

2. Formulas for circular functions of related angles

If θ is an acute angle,

(a) $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$

(b) $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$

(c) $\sin(\pi - \theta) = \sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

$\tan(\pi - \theta) = -\tan\theta$

(d) $\sin(\pi + \theta) = -\sin\theta$

$\cos(\pi + \theta) = -\cos\theta$

$\tan(\pi + \theta) = \tan\theta$

(e) $\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$

$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta$

(f) $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$

$\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$

$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$

(g) $\sin(2\pi - \theta) = -\sin\theta$

$\cos(2\pi - \theta) = \cos\theta$

$\tan(2\pi - \theta) = -\tan\theta$

(h) $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

(i) For any integer n,

(i) $\sin\frac{(2n+1)\pi}{2} = (-1)^n$

(ii) $\sin n\pi = 0$

(iii) $\cos(2n+1)\frac{\pi}{2} = 0$

(iv) $\cos n\pi = (-1)^n$

(v) $\tan n\pi = 0$

3. (a) Circular functions of compound angles. If A and B are any two angles,

(i) $\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$

(ii) $\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$

(iii) $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(iv) $\sin(A + B)\sin(A - B)$
$= \sin^2 A - \sin^2 B$
$= \cos^2 B - \cos^2 A$

(v) $\cos(A + B)\cos(A - B)$
$= \cos^2 A - \sin^2 B$
$= \cos^2 B - \sin^2 A$

(vi) If $A + B = \frac{\pi}{4}, (1+\tan A)(1+\tan B) = 2$

(b) Trigonometric identities

If $A + B + C = 180°$ (or π)

(i) $\sin A + \sin B + \sin C$
$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(ii) $\cos A + \cos B + \cos C$
$= 1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(iii) $\tan A + \tan B + \tan C$
$= \tan A\tan B\tan C.$

4. Product formulas

(a) $\sin A\cos B$

$= \frac{1}{2}(\sin(A+B)+\sin(A-B))$

$\cos A\sin B$

$= \frac{1}{2}(\sin(A+B)-\sin(A-B))$

$\cos A\cos B$

$= \frac{1}{2}(\cos(A+B)+\cos(A-B))$

$\sin A\sin B$

$= \frac{1}{2}(\cos(A-B)-\cos(A+B))$

(b) $\sin C + \sin D$
$= 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

$\sin C - \sin D$
$= 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

$\cos C + \cos D$
$= 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$\cos C - \cos D$
$= -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

5. (a) $\sin\theta\,\sin(60°+\theta)\,\sin(60°-\theta)$
$= \frac{\sin 3\theta}{4}$

(b) $\cos\theta\,\cos(60°+\theta)\cos(60°-\theta)$
$= \frac{\cos 3\theta}{4}$

(c) $\tan\theta\,\tan(60°+\theta)\tan(60°-\theta)$
$= \tan 3\theta.$

6. Circular functions of multiples of an angle A

(i) $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
$= \cos^2 A - \sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$

(iii) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

(iv) $1 + \cos 2A = 2\cos^2 A$

(v) $1 - \cos 2A = 2\sin^2 A$

(vi) $\sin 3A = 3\sin A - 4\sin^3 A$

(vii) $\cos 3A = 4\cos^3 A - 3\cos A$

(viii) $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

(ix) $\sin 18° = \frac{\sqrt{5}-1}{4}, \cos 36° = \frac{\sqrt{5}+1}{4}$

7. Inverse circular functions

(i) $\sin^{-1}x$: Domain $[-1, 1]$

Range : $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

(ii) $\cos^{-1}x$: Domain : $[-1, 1]$
Range $[0, \pi]$

(iii) $\tan^{-1}x$: Domain : $(-\infty, \infty)$ (=R)

Range : $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

(iv) $\mathrm{cosec}^{-1}x$: Domain $(-\infty, -1] \cup [1, \infty)$

Range $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

excluding zero.

(v) $\sec^{-1}x$: Domain : $[-\infty, -1] \cup [1, \infty)$

Range : $[0, \pi]$ excluding $\frac{\pi}{2}$

(vi) $\cot^{-1}x$: Domain : $(-\infty, \infty)$ (= R)

Range : $(0, \pi)$

Important results related to inverse circular functions

(A) 1. $y = \sin^{-1}(\sin x)$

$$= \begin{cases} x + 2\pi, & -\frac{5\pi}{2} \le x \le \frac{-3\pi}{2} \\ -\pi - x, & \frac{-3\pi}{2} \le x \le -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ x - 2\pi, & \frac{3\pi}{2} \le x \le \frac{5\pi}{2} \end{cases}$$

2. $y = \cos^{-1}(\cos x)$

$$= \begin{cases} x + 2\pi, & -2\pi \le x \le -\pi \\ -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le \pi \\ x - 2\pi, & 2\pi \le x \le 3\pi \end{cases}$$

3. $\sin^{-1}(\sin mx) = (-1)^n (mx - n\pi)$ for $(2n-1)\frac{\pi}{2m} <$ $\theta \le (2n+1)\frac{\pi}{2m}$.

4. $\cos^{-1}(\cos mx) = mx - n\pi$ for x even and $= (n+1)\pi - mx$ for n odd,

$\frac{n\pi}{m} \le x \le \frac{(n+1)\pi}{m}$

5. (a) $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$= \sin^{-1}x, -1 < x < 1.$

(b) $\cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sin^{-1}x, 0 < x \le 1$

$= \pi + \sin^{-1}x - 1 \le x < 0.$

6. $\sin^{-1}2x\sqrt{1-x^2} = -\pi - 2\sin^{-1}x, \; -1 \le x < -1/\sqrt{2}$

$= 2\sin^{-1}x, \; -1/\sqrt{2} \le x \le 1/\sqrt{2}$

$= \pi - 2\sin^{-1}, x \; -1/\sqrt{2} < x \le 1$

7. $\cos^{-1}(2x^2 - 1) = 2\pi - 2\cos^{-1}x, \; -1 \le x < 0.$

$= 2\cos^{-1}x, 0 \le x \le 1$

8. $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \pi + 2\tan^{-1}x, x < -1$

$= 2\tan^{-1}x, x < -1.$

$= 2\tan^{-1}x, -1 < x < 1$

$= 2\tan^{-1}x - \pi, x > 1.$

9. $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi + 2\tan^{-1}x, x < -1$

$= 2\tan^{-1}x, -1 \le x \le 1$

$= \pi - 2\tan^{-1}x, x > 1$

10. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = -2\tan^{-1}x, x < 0.$

$= 2\tan^{-1}x - \pi, x \ge 0.$

11. $\sin^{-1}(3x - 4x^3) = -\pi - 3\sin^{-1}x,$

$-1 \le x < -1/2$

$= 3\sin^{-1}x, \; -1/2 \le x \le 1/2$

$= \pi - 3\sin^{-1}x, \; 1/2 < x \le 1.$

12. $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x - 2\pi,$

$-1 \le x \le -1/2$

$= 2\pi - 3\cos^{-1}x, \; -1/2 < x < 1/2$

$= 3\cos^{-1}x, \; 1/2 \le x \le 1.$

(B)

(i) $\sin(\sin^{-1}x) = x$ for all $x \in [-1, 1]$

$\cos(\cos^{-1}x) = x$ for all $x \in [-1, 1]$

$\tan(\tan^{-1}x) = x$ for all $x \in R$

$\sin(\cos^{-1}x) = \cos(\sin^{-1}x) = \sqrt{1-x^2}$ for $x \in [-1,1]$

$\tan(\cot^{-1}x) = \cot(\tan^{-1}x) = \frac{1}{x} \; (x \ne 0)$

$\sec(\mathrm{cosec}^{-1}x) = \mathrm{cosec}(\sec^{-1}x)$

$= \frac{|x|}{\sqrt{x^2-1}} \left(|x| > 1\right)$

(ii) $\sin^{-1}(-x) = -\sin^{-1}x, -1 \le x \le 1$

$\cos^{-1}(-x) = \pi - \cos^{-1}x, -1 \le x \le 1$

$\tan^{-1}(-x) = -\tan^{-1}x, x \in R$

$\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R$

$\sec^{-1}(-x) = \pi - \sec^{-1}x$

$\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1}x$

(iii) $\sin^{-1} = \left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x$

$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x$

$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x, x > 0$

$= \cot^{-1} x - \pi , x < 0.$

(iv) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, -1 \le x \le 1$

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, x \in R$

$\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2},$

$x \in R - (-1, 1)$

(v) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy},$

$x > 0, y > 0$ and $xy < 1$

(vi) If $xy = 1$, $\tan^{-1}x + \tan^{-1}y = \begin{cases} \frac{\pi}{2} & x, y > 0 \\ \frac{-\pi}{2} & x, y < 0 \end{cases}$

(vii) $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right),$

$x > 0, y > 0, xy > 1$

(viii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right),$

$x > 0, y > 0$

(ix) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

where, $x \ge 0, y \ge 0$ and $x^2 + y^2 \le 1$

8. Trigonometric equations

(i) Solution set of
$\sin x = k, -1 \le k \le 1$, is $x = n\pi + (-1)^n \alpha$, n an integer and α is such that $\sin\alpha = k$, where, $\frac{-\pi}{2} \le \alpha \le \frac{\pi}{2}$

(ii) Solution set of
$\cos x = k, -1 \le k \le 1$ is $x = 2n\pi \pm \alpha$, n an integer and α is such that $\cos\alpha = k$ where, $0 \le \alpha \le \pi$

(iii) Solution set of $\tan x = k, -\infty < k < \infty$
$x = n\pi + \alpha$, is $x = n\pi + \alpha$, n an integer and α is such that $\tan\alpha = k$, $\frac{-\pi}{2} \le \alpha \le \frac{\pi}{2}$

(iv) For equations of the form
$a\cos x + b\sin x = c$
where, $\left|\frac{c}{\sqrt{a^2+b^2}}\right| \le 1$, divide throughout by $\sqrt{a^2+b^2}$. The equation reduces to either $\sin\theta = k$ or $\cos\theta = k$

(v) Solution of $\sin^2 x = k^2$ is
$x = n\pi \pm \sin^{-1}k$,
n an integer and $-1 \le k^2 \le 1$

(vi) Solution of $\cos^2 x = k^2$ is
$x = n\pi \pm \cos^{-1}k$.
n an integer and $-1 \le k \le 1$

(vii) Solution of $\tan^2 = k$ is
$x = n\pi \pm \tan^{-1}k$
n an integer and $k \in R$.

CONCEPT CONECTORS

Connector 1: Find the value of $\dfrac{1}{\sin 10^\circ} - \dfrac{\sqrt{3}}{\cos 10^\circ}$.

Solution: Given expression $= \dfrac{\left[\cos 10^\circ - \sqrt{3}\sin 10^\circ\right]}{\sin 10^\circ \cos 10^\circ} = \dfrac{\frac{1}{2}\cos 10^\circ - \frac{\sqrt{3}}{2}\sin 10^\circ}{\left(\frac{1}{2}\sin 10^\circ \cos 10^\circ\right)}$

$$= \frac{\cos 60^\circ \cos 10^\circ - \sin 60^\circ \sin 10^\circ}{\left[\frac{1}{4}\sin 20^\circ\right]}$$

$$= \frac{\cos 70^\circ}{\left[\frac{1}{4}\sin 20^\circ\right]} = 4\text{, since } \cos 70^\circ = \cos(90^\circ - 20^\circ) = \sin 20^\circ$$

Connector 2: (i) Show that $\sin 10^\circ \sin 50^\circ \sin 60^\circ \sin 70^\circ = \dfrac{\sqrt{3}}{16}$.

(ii) Show that $\sin 10^\circ \sin 50^\circ + \sin 50^\circ \sin 250^\circ + \sin 250^\circ \sin 10^\circ = -\dfrac{3}{4}$.

Solution: (i) $\sin 10^\circ \sin 50^\circ \sin 70^\circ = (\sin 10^\circ \sin 70^\circ)(\sin 50^\circ)$

$$= \frac{1}{2}[(\cos 60^\circ - \cos 80^\circ)]\sin 50^\circ = \frac{1}{2}\left[\frac{1}{2} - \cos 80^\circ\right]\sin 50^\circ$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 50^\circ - \cos 80^\circ \sin 50^\circ\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 50^\circ - \frac{1}{2}(\sin 130^\circ - \sin 30^\circ)\right]$$

$$= \frac{1}{4}\left[\sin 50^\circ - \sin 130^\circ + \frac{1}{2}\right]$$

$$= \frac{1}{4}\left[\sin 50^\circ - \sin(180^\circ - 50^\circ) + \frac{1}{2}\right] = \frac{1}{4}\left[\sin 50^\circ - \sin 50^\circ + \frac{1}{2}\right] = \frac{1}{8}$$

Therefore, $\sin 10^\circ \sin 50^\circ \sin 60^\circ \sin 70^\circ = \dfrac{1}{8} \times \dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{3}}{16}$

(ii) L.H.S $= \dfrac{1}{2}\left[\cos 40^\circ - \cos 60^\circ + \cos 200^\circ - \cos 300^\circ + \cos 240^\circ - \cos 260^\circ\right]$

$$= \frac{1}{2}\left[\cos 40^\circ - \frac{1}{2} + \cos 200^\circ - \cos 60^\circ - \cos 60^\circ - \cos 260^\circ\right]$$

$$= \frac{1}{2}\left[\cos 200^\circ + \cos 40^\circ - \cos 260^\circ - \frac{3}{2}\right]$$

$$= \frac{1}{2}\left[2\cos 120^\circ \cos 80^\circ - \cos(180^\circ + 80^\circ) - \frac{3}{2}\right]$$

$$= \frac{1}{2}\left[2\left(-\frac{1}{2}\right)\cos 80^\circ + \cos 80^\circ - \frac{3}{2}\right] = -\frac{3}{4}$$

Connector 3: Show that $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0.$

Solution: $$\text{L.H.S} = \left(\cos\frac{10\pi}{13}+\cos\frac{8\pi}{13}\right)+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}$$

$$=\cos\left(\pi-\frac{3\pi}{13}\right)+\cos\left(\pi-\frac{5\pi}{13}\right)+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13} = -\cos\frac{3\pi}{13}-\cos\frac{5\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$$

Connector 4: If $\cos x+\cos y=a$, $\sin x+\sin y=b$, prove that $\sin(x+y)=\dfrac{2ab}{\left(a^2+b^2\right)}$.

Solution: We have

$$2\cos\frac{x+y}{2}\cos\frac{x-y}{2}=a,\quad 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=b$$

$$\Rightarrow \tan\frac{x+y}{2}=\frac{b}{a}$$

$$\Rightarrow \sin(x+y)=\frac{2\tan\left(\frac{x+y}{2}\right)}{1+\tan^2\left(\frac{x+y}{2}\right)}=\frac{\frac{2b}{a}}{1+\frac{b^2}{a^2}}=\frac{2ab}{(a^2+b^2)}$$

Connector 5: If $\sin\theta=n\sin(\theta+2\alpha)$, show that $\tan(\theta+\alpha)=\left(\dfrac{1+n}{1-n}\right)\tan\alpha$.

Solution: We have $\dfrac{\sin(\theta+2\alpha)}{\sin\theta}=\dfrac{1}{n}$

$$\Rightarrow \frac{\sin(\theta+2\alpha)+\sin\theta}{\sin(\theta+2\alpha)-\sin\theta}=\frac{1+n}{1-n}$$

$$\Rightarrow \frac{2\sin(\theta+\alpha)\cos\alpha}{2\cos(\theta+\alpha)\sin\alpha}=\frac{1+n}{1-n}\Rightarrow\frac{\tan(\theta+\alpha)}{\tan\alpha}=\frac{1+n}{1-n}$$

Connector 6: Prove that $\tan 20^\circ\tan 40^\circ\tan 60^\circ\tan 80^\circ=3$.

Solution: $$\text{L.H.S}=\frac{\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ}{\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ}$$

$$\text{Numerator}=\frac{1}{2}(\cos 20^\circ-\cos 60^\circ)\sin 80^\circ\sin 60^\circ$$

$$=\left[\frac{1}{2}\cos 20^\circ\sin 80^\circ-\frac{1}{4}\sin 80^\circ\right]\sin 60^\circ$$

$$=\left[\frac{1}{4}(\sin 100^\circ+\sin 60^\circ)-\frac{1}{4}\sin 80^\circ\right]\sin 60^\circ$$

$$=\left[\frac{1}{4}\sin(180^\circ-80^\circ)+\frac{\sqrt{3}}{8}-\frac{1}{4}\sin 80^\circ\right]\frac{\sqrt{3}}{2}$$

$$=\left[\frac{1}{4}\sin 80^\circ+\frac{\sqrt{3}}{8}-\frac{1}{4}\sin 80^\circ\right]\frac{\sqrt{3}}{2}=\frac{3}{16}$$

$$\text{Denominator} = \frac{1}{2}(\cos 60^\circ + \cos 20^\circ)\cos 60^\circ \cos 80^\circ$$

$$= \left(\frac{1}{8} + \frac{1}{4}\cos 20^\circ\right)\cos 80^\circ = \frac{1}{8}\cos 80^\circ + \frac{1}{4}\cos 20^\circ \cos 80^\circ$$

$$= \frac{1}{8}\cos 80^\circ + \frac{1}{8}(\cos 100^\circ + \cos 60^\circ) = \frac{1}{8}\cos 80^\circ - \frac{1}{8}\cos 80^\circ + \frac{1}{16} = \frac{1}{16}$$

Result follows.

Connector 7: If A + B + C = 180°, prove the following:

(i) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(ii) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(iii) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Solution: (i) $\sin A + \sin B + \sin C = 2\sin\frac{A+B}{2}\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}$

$$= 2\sin\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} + \sin\frac{C}{2}\right\}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} + \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right\}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right\}$$

$$= 2\cos\frac{C}{2} \times 2\cos\frac{A}{2}\cos\frac{B}{2} = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

(ii) $\cos A + \cos B + \cos C = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \cos C$

$$= 2\cos\left(90^\circ - \frac{C}{2}\right)\cos\frac{A-B}{2} + \cos C$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\frac{C}{2}\right] + 1$$

$$= \left(2\sin\frac{C}{2}\right)\left[\cos\frac{A-B}{2} - \sin\left(90^0 - \frac{A+B}{2}\right)\right] + 1$$

$$= \left(2\sin\frac{C}{2}\right)\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right] + 1$$

$$= \left(2\sin\frac{C}{2}\right)\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right) + 1 = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

(iii) Since A + B + C = 180°, A + B = 180° – C

tan (A + B) = tan (180° – C) = – tan C

$\Rightarrow \quad \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

$\Rightarrow$ tan A + tan B + tan C = tan A tan B tan C

We can also prove this result using the formula for tan (A + B + C)

tan (A + B + C) = tan 180° = 0

But, $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

Result follows.

The above three results are called trigonometric identities.

Connector 8: Obtain the maximum and minimum values of the expression a cosθ + b sinθ + c.

Solution:
$$a\cos\theta + b\sin\theta + c = c + \sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right\}$$

$$= c + \sqrt{a^2+b^2}\left\{\cos\alpha\cos\theta + \sin\alpha\sin\theta\right\}$$

$$\left[\begin{array}{l}\text{Since } \left|\dfrac{a}{\sqrt{a^2+b^2}}\right| \text{ and } \left|\dfrac{b}{\sqrt{a^2+b^2}}\right| \text{ are } < 1, \\ \text{there exists an angle } \alpha \text{ such that } \cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}, \ \sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}\end{array}\right]$$

$$= c + \sqrt{a^2+b^2}\cos(\theta - \alpha)$$

Maximum value of cos(θ – α) is +1 and the minimum value of cos(θ – α) is –1.

Since c is a given number, maximum value of the given expression $= c + \sqrt{a^2+b^2}$ and the minimum value of the given expression $= c - \sqrt{a^2+b^2}$

Connector 9: Given $P = \sin^2 x + \cos^4 x$, show that $\dfrac{3}{4} \le P \le 1$.

Solution:
$$P = \sin^2 x + (1 - \sin^2 x)^2 = \sin^4 x - \sin^2 x + 1 = \left(\sin^2 x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

P is maximum when $\left(\sin^2 x - \dfrac{1}{2}\right)$ is maximum.

i.e., when $\sin^2 x = 1$ and the maximum value $= \left(1 - \dfrac{1}{2}\right)^2 + \dfrac{3}{4} = 1$

P is minimum when $\sin^2 x = \dfrac{1}{2}$ and the minimum value $= \dfrac{3}{4} \Rightarrow \dfrac{3}{4} \le P \le 1$

Connector 10: Prove that tan x sec 4x + tan 4x = tan x + tan 4x sec 2x.

Solution: We have to prove that (tan x) (sec4x – 1) = (tan4x) (sec2x – 1)

or to prove that $\dfrac{(\tan x)\,(1 - \cos 4x)}{\cos 4x} = \dfrac{\sin 4x}{\cos 4x}\left(\dfrac{1 - \cos 2x}{\cos 2x}\right)$

or to prove that $(\tan x)\,(2\sin^2 2x) = \dfrac{(2\sin 2x\cos 2x)\,(2\sin^2 x)}{\cos 2x}$

or to prove that $(\tan x)(\sin 2x) = 2\sin^2 x$

Now, L.H.S $= \dfrac{(\sin x)}{(\cos x)}(2\sin x\cos x) = 2\sin^2 x =$ R.H.S.

Connector 11: Solve the equations:

(i) $4\cos x - 3\sec x = 2\tan x$

(ii) $\sin 5\theta + \sin 3\theta + \sin\theta = 0$

(iii) $\sin x - \cos x = -\sqrt{2}$

Solution: (i) Given equation is $4\cos x - \dfrac{3}{\cos x} = \dfrac{2\sin x}{\cos x}$

Multiplying both sides by cos x (which cannot be zero)

$4\cos^2 x - 3 = 2\sin x$

$\Rightarrow\ 4(1 - \sin^2 x) - 3 = 2\sin x$

$\Rightarrow\ 4\sin^2 x + 2\sin x - 1 = 0$

$$\Rightarrow\ \sin x = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

Taking $\sin x = \dfrac{\sqrt{5}-1}{4}$

principal value $= \sin^{-1}\left(\dfrac{\sqrt{5}-1}{4}\right) = \dfrac{\pi}{10}$

General solution is $x = n\pi + (-1)^n \dfrac{\pi}{10}$, where n is an integer — (1)

Taking $\sin x = -\dfrac{(\sqrt{5}+1)}{4} = -\cos 36^\circ$

$= -\sin 54^\circ = \sin\left(\dfrac{-3\pi}{10}\right)$,

Principle value $= -\dfrac{3\pi}{10}$

General solution is $x = n\pi + \dfrac{(-1)^{n+1}3\pi}{10}$, where, n is an integer — (2)

(1) and (2) constitute the solution.

(ii) Equation may be rewritten as $(\sin 5\theta + \sin\theta) + \sin 3\theta = 0$

$\Rightarrow\ 2\sin 3\theta\cos 2\theta + \sin 3\theta = 0$

$\Rightarrow\ (\sin 3\theta)[2\cos 2\theta + 1] = 0$

$\Rightarrow\ \sin 3\theta = 0$ or $\cos 2\theta = \dfrac{-1}{2}$

Giving $3\theta = n\pi + (-1)^n \times 0$ or $2\theta = 2n\pi \pm \dfrac{2\pi}{3} \Rightarrow \theta = \dfrac{n\pi}{3}$ or $n\pi \pm \dfrac{\pi}{3}$ where, n is any integer.

(iii) $\sin x - \cos x = -\sqrt{2}$

Dividing both sides by $\sqrt{2}$,

$$\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x = -1 \Rightarrow \cos\frac{\pi}{4}\sin x - \sin\frac{\pi}{4}\cos x = -1$$

$$\Rightarrow \quad \sin\left(x - \frac{\pi}{4}\right) = -1$$

$$x - \frac{\pi}{4} = n\pi + (-1)^n\left(\frac{-\pi}{2}\right) = n\pi + (-1)^{n+1}\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad x = \frac{\pi}{4} + n\pi + (-1)^{n+1}\left(\frac{\pi}{2}\right)$$

Connector 12: Solve the equation $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(3x - 2)$.

Solution: If $\sin^{-1}x = A$, then $\cos^{-1}x = \frac{\pi}{2} - A$. The given equation reduces to

$$A - \left(\frac{\pi}{2} - A\right) = \sin^{-1}(3x - 2) \Rightarrow \sin\left(2A - \frac{\pi}{2}\right) = 3x - 2$$

$\Rightarrow \quad -\cos 2A = 3x - 2 \Rightarrow 2\sin^2 A - 1 = 3x - 2$

$\Rightarrow \quad 2x^2 - 1 = 3x - 2 \Rightarrow 2x^2 - 3x + 1 = 0$

$\Rightarrow \quad x = 1, \frac{1}{2}$

Both $x = 1, \frac{1}{2}$ are seen to satisfy the given equation.

$\therefore$ Solution is $x = 1, \frac{1}{2}$

Connector 13: Show that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{77}{85}$.

Solution: Let $\sin^{-1}\frac{3}{5} = A$, $\sin^{-1}\frac{8}{17} = B$

We have $\sin A = \frac{3}{5}$; $\sin B = \frac{8}{17}$

Since both A and B are in the first quadrant, $\cos A = \frac{4}{5}$, $\cos B = \frac{15}{17}$

$\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{77}{85}$, on substitution and simplification.

Connector 14: Show that $\tan^{-1}5 - \tan^{-1}3 = \tan^{-1}\frac{1}{8}$.

Solution: L.H.S $= \tan^{-1}\frac{5 - 3}{1 + 5 \times 3} = \tan^{-1}\frac{1}{8} =$ R.H.S

Connector 15: If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, show that $x^2 + y^2 + z^2 + 2xyz = 1$.

Solution: Let $\cos^{-1}x = A$, $\cos^{-1}y = B$, $\cos^{-1}z = C$

Given $A + B + C = \pi \Rightarrow A + B = \pi - C$

$\Rightarrow \quad \cos(A + B) = \cos(\pi - C) = -\cos C$

i.e., $\cos A \cos B - \sin A \sin B = -\cos C$

i.e., $xy - \left(\sqrt{1 - x^2}\right)\left(\sqrt{1 - y^2}\right) = -z$ or $xy + z = \sqrt{1 - x^2}\sqrt{1 - y^2}$

Squaring both sides and simplifying, we obtain the required result.

Connector 16: Find the periods of

(i) $f(x) = 4\sin^2 x + 7\cos^2 x$

(ii) $f(x) = 2\cos^3 x + 5\cos 4x$

Solution: (i) $f(x) = 4 \times \frac{1}{2}(1 - \cos 2x) + \frac{7}{2}(1 + \cos 2x) = \frac{11}{2} + \frac{3}{2}\cos 2x$

$$f(x+\pi) = \frac{11}{2} + \frac{3}{2}\cos(2(x+\pi)) = \frac{11}{2} + \frac{3}{2}\cos(2x + 2\pi)$$

$$= \frac{11}{2} + \frac{3}{2}\cos 2x = f(x)$$

$\Rightarrow$ Period of f(x) is π.

(ii) Now, $\cos 3x = 4\cos^3 x - 3\cos x$ or $2\cos^3 x = \frac{1}{2}(\cos 3x + 3\cos x)$

$$f(x) = \frac{1}{2}(\cos 3x + 3\cos x) + 5\cos 4x$$

Period of cos3x is $\frac{2\pi}{3}$; Period of cos x is 2π; Period of cos4x is $\frac{2\pi}{4}$ or $\frac{\pi}{2}$

$\therefore$ The period of f(x) is 2π.

Connector 17: What is the integer value of n for which the function $f(x) = \dfrac{\sin x}{\cos\left(\frac{x}{n}\right)}$ has period 4π?

Solution: Period of the numerator function is 2π

Period of the denominator function is $\dfrac{2\pi}{1/n} = 2n\pi$.

Since n is an integer, $2n\pi > 2\pi$.

$\therefore$ The integer value of n for which the period is 4π, is 2. or n = 2.

Connector 18: Show that the functions $f_1(x) = \sin\left(\log\left(x + \sqrt{x^2+1}\right)\right)$ and $f_2(x) = \dfrac{\sin^4 x + 2\cos^4 x}{x + 3x^2 \tan x}$ are odd functions.

Solution: $f_1(-x) = \sin\left(\log\left(-x + \sqrt{x^2+1}\right)\right) = \sin\ \log\left\{\dfrac{1}{\sqrt{x^2+1} + x}\right\}$,

since $\left(\sqrt{x^2+1} + x\right)\left(\sqrt{x^2+1} - x\right) = 1$

$$= \sin\left\{-\log\left(\sqrt{x^2+1} + x\right)\right\} = -\sin\ \log\left(\sqrt{x^2+1} + x\right)$$

$$= -f_1(x)$$

$\Rightarrow$ $f_1(x)$ is an odd function.

$$f_2(-x) = \frac{\sin^4(-x) + 2\cos^4(-x)}{-x + 3x^2\tan(-x)} = \frac{\sin^4 x + 2\cos^4 x}{-x - 3x^2\tan x} = -f_2(x)$$

$\Rightarrow$ $f_2(x)$ is an odd function.

Connector 19: Find the domains of the functions:

(i) $f(x) = \sqrt{\sin^{-1}(\log_4 x)}$

(ii) $f(x) = \log_2 \log_3 \log_{\frac{4}{\pi}}\left(\dfrac{1}{\tan^{-1} x}\right)$

(iii) $f(x) = \cos^{-1}\left(\frac{2-|x|}{4}\right) + \log\left(\frac{1}{3-x}\right)$

(iv) $f(x) = \sin^{-1}\left(\frac{1-2x}{4}\right) + \log(x^2 + x + 4) - \frac{4}{(x+1)(x-2)}$

Solution: (i) $\sin^{-1}(\log_4 x) \geq 0$ and $0 \leq \log_4 x \leq 1 \Rightarrow x \in [1, 4]$

(ii) $\log_3 \log_{\frac{4}{\pi}}\left(\frac{1}{\tan^{-1}x}\right) > 0 \Rightarrow \log_{\frac{4}{\pi}}\left(\frac{1}{\tan^{-1}x}\right) > 1 \Rightarrow \frac{1}{\tan^{-1}x} > \frac{4}{\pi} \Rightarrow \tan^{-1}x < \frac{\pi}{4}$

Also, $\tan^{-1}x$ must be positive. $x \in (0, 1)$

(iii) $-1 \leq \frac{2-|x|}{4} \leq 1$ and $3 - x > 0$

$\Rightarrow -4 \leq 2 - |x| \leq 4$ and $x < 3 \Rightarrow -6 \leq -|x| \leq 2$ and $x < 3$

$\Rightarrow 6 \geq |x| \geq -2$ and $x < 3$. Since $|x|$ is positive, we get $|x| \leq 6$ and $x < 3$

$\Rightarrow x \in [-6, 3)$

(iv) We must have $-1 \leq \frac{1-2x}{4} \leq 1$ and $x \neq -1$ or $x \neq 2$, since $x^2 + x + 4$ is always greater than zero for real x.

$\Rightarrow -4 \leq 1 - 2x \leq 4$ and $x \neq -1$ or $2 \Rightarrow -5 \leq -2x \leq 3$ and $x \neq -1$ or 2.

$\Rightarrow \frac{5}{2} \geq x \geq \frac{-3}{2}$ and $x \neq -1$ or 2.

$\Rightarrow$ Domain of the function is $x \in \left[\frac{-3}{2}, \frac{5}{2}\right]$ excluding the values -1 and 2.

Connector 20: If $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2}$, show that $\cos\varphi = \frac{\cos\theta - e}{1 - e\cos\theta}$

Solution: Let $t = \tan\frac{\phi}{2}$ $\Rightarrow \cos\phi = \frac{1-t^2}{1+t^2} = \frac{\cos\theta - e}{1 - e\cos\theta}$

$1 - e\cos\theta - t^2 + t^2 e\cos\theta = \cos\theta - e + t^2\cos\theta - t^2 e$

$(1 + e)(1 - \cos\theta) = t^2(\cos\theta - e + 1 - e\cos\theta) = t^2(1 - e)(1 + \cos\theta)$

$\therefore \frac{1-\cos\theta}{1+\cos\theta} = \frac{(1-e)t^2}{1+e}$

$\tan^2\frac{\theta}{2} = \frac{1-e}{1+e}t^2 \Rightarrow \tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2}$

TOPIC GRIP

Subjective Questions

1. Find the value of
   (i) $\tan 720° - \cos 270° - \sin 150° \cos 120°$
   (ii) $\sin 600° \cos 330° + \cos 120° \sin 150°$
   (iii) $\cos^2 18° - \cos^2 36° - \cos^2 54° + \cos^2 72°$
   (iv) $\dfrac{\tan(2\pi - \theta)\operatorname{cosec}(\pi - \theta)\sec(\pi + \theta)}{\cot\left(\dfrac{\pi}{2} + \theta\right)\operatorname{cosec}^2\left(\dfrac{3\pi}{2} + \theta\right)\tan\left(\dfrac{3\pi}{2} - \theta\right)}$
   (v) $\cot\dfrac{11\pi}{20}\cot\dfrac{13\pi}{20}\cot\dfrac{15\pi}{20}\cot\dfrac{17\pi}{20}\cot\dfrac{19\pi}{20}$

2. Prove the following:
   (i) $\sin\dfrac{7\pi}{12}\cos\dfrac{\pi}{4} - \cos\dfrac{7\pi}{12}\sin\dfrac{\pi}{4} = \dfrac{\sqrt{3}}{2}$
   (ii) $A + B = \dfrac{\pi}{4}$ if $\tan A = \dfrac{m}{m+1}$, $\tan B = \dfrac{1}{2m+1}$
   (iii) $\tan 70° = \tan 20° + 2\tan 50°$
   (iv) $\dfrac{\cos 8° - \sin 8°}{\cos 8° + \sin 8°} = \tan 37°$
   (v) $\cos^2\dfrac{\pi}{8} + \cos^2\dfrac{3\pi}{8} + \cos^2\dfrac{5\pi}{8} + \cos^2\dfrac{7\pi}{8} = 2$

3. Show that $\tan 5x - \tan 2x - \tan 3x = \tan 2x \tan 3x \tan 5x$.

4. Prove that $\cos\theta \cos 2\theta \cos 2^2\theta \cos 2^3\theta \ldots \cos 2^{k-1}\theta$ (k being a positive integer) $= \dfrac{\sin(2^k\theta)}{2^k \sin\theta}$. If $\theta = \dfrac{\pi}{2^k + 1}$, deduce that the value of the expression is $\dfrac{1}{2^k}$.

5. Prove that $\cos^3\alpha + \cos^3\left(\dfrac{2\pi}{3} + \alpha\right) + \cos^3\left(\dfrac{4\pi}{3} + \alpha\right) = \dfrac{3}{4}\cos 3\alpha$.

6. If $\sqrt{2}\cos\alpha = \cos\beta + \cos^3\beta$ and $\sqrt{2}\sin\alpha = \sin\beta - \sin^3\beta$, prove that $\sin(\alpha - \beta) = \pm\dfrac{1}{3}$.

7. Prove that $\tan^{-1}\left(\dfrac{1}{2}\tan 2\theta\right) + \tan^{-1}(\cot\theta) + \tan^{-1}(\cot^3\theta) = 0$.

8. Solve the equation: $\tan^{-1}(2x) + \tan^{-1}(3x) = \dfrac{\pi}{4}$.

9. Solve the equation $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$.

10. Find the sum of all the solutions of the equation $7\cos 2\theta - \cos 4\theta = 6$ in the interval $(\pi, 45\pi)$.

Straight Objective Type Questions

***Directions:* This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

11. $\dfrac{\sin x}{\sin\left(\dfrac{x}{8}\right)}$ is equal to

(a) $8\sin\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{4}\right)\cos\left(\dfrac{x}{8}\right)$
(b) $8\cos\left(\dfrac{x}{2}\right)\sin\left(\dfrac{x}{4}\right)\cos\left(\dfrac{x}{8}\right)$
(c) $8\cos\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{4}\right)\cos\left(\dfrac{x}{8}\right)$
(d) $4\cos\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{4}\right)\cos\left(\dfrac{x}{8}\right)$

12. If $0 < A < \pi$ and $\sin A + \operatorname{cosec} A = 2$, then $\sin^n A + \operatorname{cosec}^n A$, where n is a positive integer, is
(a) -1 (b) 0 (c) 1 (d) 2

13. $\tan^{-1}\left[\dfrac{a-b}{1+ab}\right] + \tan^{-1}\left[\dfrac{b-c}{1+bc}\right]$ is

(a) $\tan^{-1}\dfrac{b-c}{1-bc}$ (b) $\tan^{-1}\dfrac{a-b}{1-ab}$ (c) $\tan^{-1}\dfrac{a-c}{1+ac}$ (d) $\tan^{-1}\dfrac{a-b}{a+b}$

14. If $3\sin\theta + 4\cos\theta = 5$, then the value of $3\cos\theta - 4\sin\theta$ is equal to
(a) 1 (b) 5 (c) 0 (d) -5

15. The expression $4\cos\left(\theta + \dfrac{\pi}{3}\right) + 3\sin\left(\theta + \dfrac{\pi}{3}\right)$ lies between

(a) -5 and 5 (b) -4 and 4 (c) -3 and 3 (d) -7 and 7

Assertion–Reason Type Questions

***Directions:* Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

16. Consider the following Statements:

Statement 1

The equation $\sin^2 x - 5\sin x + 6 = 0$ has no real solution.

and

Statement 2

The numerical value of $\sin x$ can never exceed 1.

17. **Statement 1**

If $A + B = \dfrac{\pi}{4}$, $(1 + \cot A)(1 + \cot B) = 2\cot A \cot B$.

and

Statement 2

$\tan(A + B) = \dfrac{\cot B + \cot A}{\cot A \cot B - 1}$

18. **Statement 1**

$\sin^2 20° + \sin^2 40° + \sin^2 60° + \sin^2 80° = \dfrac{17}{8}$

and

Statement 2

$1 - 2\sin^2 A = \cos 2A$

19. **Statement 1**

$\tan^{-1}\left(-\sqrt{3}\right) + \cot^{-1}\left(-\sqrt{3}\right) = 0$

and

Statement 2

Range of $\tan^{-1}x$ is $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ and that of $\cot^{-1}x$ is $(0, \pi)$

20. **Statement 1**

$\cos x + \sin x = \dfrac{1}{2}$ has real solutions.

and

Statement 2

$a \cos x + b \sin x = c$ has real solutions if and only if $-1 \le c \le 1$.

Linked Comprehension Type Questions

Directions: **This section contains 2 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

Suppose we are interested in finding the sum to n terms of series of sines of angles which are in A.P. We use the formula $\cos C - \cos D = -2\sin\dfrac{C + D}{2}\sin\dfrac{C - D}{2}$

Or equivalently $-2 \sin A \sin B = \cos(A + B) - \cos(A - B)$

For example, let $S = \sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) \ldots\ldots -$ n terms.

$$-2\sin\frac{\beta}{2}S = -2\sin\alpha\sin\frac{\beta}{2} - 2\sin(\alpha + \beta)\sin\frac{\beta}{2} \ldots\ldots$$

$$= \left[\cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha - \frac{\beta}{2}\right)\right] + \left[\cos\left(\alpha + \frac{3\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right)\right] + \ldots\ldots\ldots$$

$$+\left\{\cos\left[\alpha + \left(n - \frac{1}{2}\right)\beta\right] - \cos\left[\alpha + \left(n - \frac{3}{2}\beta\right)\right]\right\} = \cos\left[\alpha + \left(n - \frac{1}{2}\right)\beta\right] - \cos\left[\left(\alpha - \frac{\beta}{2}\right)\right]$$

$$= -2\sin\frac{2\alpha + (n-1)\beta}{2}\cdot\sin\frac{n\beta}{2} \quad \therefore\ S = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left[\frac{2\alpha + (n-1)\beta}{2}\right] \qquad \text{— (1)}$$

Similarly, we use the formula

2 cos A sin B = sin (A + B) – sin (A – B) when we come across similar series in cosines of angles which are in A.P.

21. $\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots\ldots + \sin n\alpha$ is equal to

(a) $\dfrac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}\sin\left[\dfrac{n+1}{2}\alpha\right]$

(b) $\sin\dfrac{n\alpha}{2}\sin\dfrac{(n+1)\alpha}{2}$

(c) $\cos\dfrac{n\alpha}{2}\sin\dfrac{(n+1)\alpha}{2}$

(d) $\cos\dfrac{n\alpha}{2}\sin\dfrac{(n+1)\alpha}{2}\Big/\sin\dfrac{\alpha}{2}$

22. $\sin 20° + \sin 80° + \sin 140° + \ldots + \sin 560°$ is equal to

(a) $\sqrt{3}\sin 20°$ (b) $\sqrt{3}\cos 20°$ (c) $\sqrt{3}$ (d) $\sin 80°$

23. $\sin^2 5° + \sin^2 10° + \ldots\ldots + \sin^2 90° =$

(a) $\dfrac{19}{2}$ (b) $\dfrac{17}{2}$ (c) $\dfrac{15}{2}$ (d) $\dfrac{13}{2}$

Passage II

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$, where $x > 0$, $y > 0$ and $xy < 1$

24. When $y > x > 0$, $\cot^{-1}x - \cot^{-1}y$ is equal to

(a) $\cot^{-1}\left(\dfrac{1-xy}{y+x}\right)$ (b) $\cot^{-1}\left(\dfrac{y+x}{1+xy}\right)$ (c) $\cot^{-1}\left(\dfrac{1+xy}{y-x}\right)$ (d) $\cot^{-1}\left(\dfrac{1-xy}{y-x}\right)$

25. If $\tan\theta_1$, $\tan\theta_2$ be the roots of $a\tan\theta + b\sec\theta = c$, then $\tan^{-1}\dfrac{2ac}{(a^2-c^2)}$ is equal to

(a) $2(\theta_1 - \theta_2)$ (b) $2(\theta_1 + \theta_2)$ (c) $\theta_1 + \theta_2$ (d) $\theta_1 - \theta_2$

26. If the angles of a triangle are $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ then $\sum x(1-y^2)(1-z^2)$ is equal to

(a) 0 (b) $x^2y^2z^2$ (c) $4xyz$ (d) $8xyz$

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

27. If $\cos(A - B) = \dfrac{3}{5}$ and $\tan A\tan B = 2$, then

(a) $\cos A\cos B = \dfrac{1}{5}$ (b) $\sin A\sin B = \dfrac{-2}{5}$ (c) $\cos(A+B) = \dfrac{-1}{5}$ (d) $\sin A\sin B = \dfrac{4}{5}$

28. If $\sin^{-1}(1-x) = \cos^{-1}x - \sin^{-1}x$, then the value of x is/are

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

29. Given that $4^{\sec^2\alpha+1}x^2 - 8x + 4\tan^2\beta - 8\tan\beta + 5 = 0$ has real solutions, then

(a) $\alpha = n\pi$, $\beta = n\pi + \frac{\pi}{4}$ and $x = \pm 1$

(b) $\alpha = \frac{n\pi}{4}$, $\beta = n\pi$ and $x = \frac{1}{4}$

(c) $\alpha = n\pi$, $\beta = n\pi + \frac{\pi}{4}$ and $x = \frac{1}{4}$

(d) there is no real solution for x if either $\alpha = R - \{n\pi\}$ or $\beta = R - \{n\pi + \pi/4\}$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

30.

Column I	Column II
(a) If $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$, then x equals	(p) 3
(b) If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$, then x equals	(q) $\frac{1}{2}$
(c) If $4\sin^{-1}x + \cos^{-1}x = \pi$, then x equals	(r) 0
(d) If $\cot^{-1}x + \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) = \frac{\pi}{4}$, then x equals	(s) -1

IIT ASSIGNMENT EXERCISE

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

31. $\sin(-420°)\cos(390°) + \cos(-660°)\sin(330°)$ is

(a) 1 (b) −1 (c) 2 (d) −2

32. In any cyclic quadrilateral ABCD, cos A + cos B + cos C + cos D is equal to

(a) 0 (b) 1 (c) −1 (d) 2

33. Value of $\sin\frac{\pi}{6}\sin\frac{2\pi}{6}\sin\frac{4\pi}{6}\sin\frac{5\pi}{6}$ is

(a) $\frac{3}{8}$ (b) $\frac{3}{16}$ (c) $\frac{\sqrt{3}}{8}$ (d) $\frac{3}{4}$

34. $\dfrac{\sin^2 5^\circ + \sin^2 10^\circ + \ldots + \sin^2 85^\circ + \sin^2 90^\circ}{1 + \cos^2 5^\circ + \cos^2 10^\circ + \ldots + \cos^2 85^\circ + \cos^2 90^\circ}$ equals

(a) 1 (b) $\frac{19}{17}$ (c) 2 (d) $\frac{17}{19}$

35. If triangle ABC is equilateral, tan A + tan B + tan C is equal to

(a) $3\sqrt{3}$ (b) $\frac{3}{\sqrt{3}}$ (c) $3+\sqrt{3}$ (d) $3-\sqrt{3}$

36. If $m\tan(\theta - 30°) = n\tan(\theta + 120°)$, then $\cos 2\theta$ is

(a) $\frac{m+n}{m-n}$ (b) $\frac{m-n}{2(m+n)}$ (c) $\frac{m+n}{2(m-n)}$ (d) $\frac{m-n}{m+n}$

37. If $\sin\theta$ and $\cos\theta$ are roots of the equation $ax^2 - bx + c = 0$ then a, b, c satisfy the relation

(a) $a^2 + b^2 + 2ac = 0$ (b) $a^2 - b^2 + 2ac = 0$ (c) $a^2 - b^2 - 2ac = 0$ (d) $b^2 - a^2 + 2ac = 0$

38. $\tan 13\theta - \tan 9\theta - \tan 4\theta$ is equal to

(a) $\tan 13\theta + \tan 9\theta + \tan 4\theta$ (b) $\tan 13\theta \tan 9\theta$

(c) $\tan 9\theta \tan 4\theta$ (d) $\tan 13\theta \tan 9\theta \tan 4\theta$

39. The angles of a triangle are in AP and the largest angle is the sum of the remaining two angles. The triangle is

(a) isosceles (b) equilateral (c) right angled (d) right angled isosceles

40. In a right angled triangle, if the hypotenuse is four times as long as the perpendicular from the opposite vertex, one of the acute angles is

(a) 60° (b) 45° (c) 30° (d) 15°

41. The value of $\sin\frac{\pi}{14}.\sin\frac{3\pi}{14}.\sin\frac{5\pi}{14}$ is equal to

(a) $\frac{1}{8}$ (b) $-\frac{1}{8}$ (c) $\frac{1}{16}$ (d) $-\frac{1}{16}$

42. $2\tan\frac{7\pi}{6}$, $4\tan\frac{9\pi}{4}$ and $8\tan\frac{10\pi}{3}$ are in

(a) AP (b) GP (c) AGP (d) HP

43. If $\sec x = a + \frac{1}{4a}$ then $\sec x + \tan x$ is:

(a) $\frac{2}{a}$ (b) a (c) $\frac{1}{a}$ (d) $\frac{1}{2a}$

44. The equation $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is possible

(a) for any x and y (b) for all x, y > 0 (c) only when x = y (d) None of these

45. If x lies in the 2nd quadrant and $\tan x = \frac{-4}{3}$, then $\sin\frac{x}{2}\cos\frac{x}{2} =$

(a) $\frac{2}{5}$ (b) $\frac{2}{\sqrt{5}}$ (c) $\frac{\sqrt{5}}{2}$ (d) $\frac{1}{2}$

46. $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}}$ is

(a) $2\cos2\theta$ (b) $2\cos\theta$ (c) $2\cos4\theta$ (d) $\cos 2\theta$

47. $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$ is

(a) $4\cos^2\left(\frac{\alpha+\beta}{2}\right)$ (b) $4\cos^2\left(\frac{\alpha-\beta}{2}\right)$ (c) $-4\sin^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}$ (d) $4\sin\frac{\alpha}{2}\cos\frac{\beta}{2}$

48. $2\left[\cos\frac{\pi}{5} - \cos\frac{2\pi}{5}\right]$ is

(a) 1 (b) –1 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

49. Value of $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}$

(a) $\frac{3}{16}$ (b) $\frac{5}{8}$ (c) $\frac{\sqrt{5}}{16}$ (d) $\frac{5}{16}$

50. Value of $\cos^2 48° - \sin^2 12°$

(a) $\frac{\sqrt{5}+1}{4}$ (b) $\frac{\sqrt{5}+1}{8}$ (c) $\frac{\sqrt{5}-1}{8}$ (d) $\frac{\sqrt{5}-1}{4}$

51. Value of $\sin20°\sin40°\sin60°\sin80°$ is

(a) $\frac{3}{8}$ (b) $\frac{3}{16}$ (c) $\frac{3}{9}$ (d) $\frac{5}{16}$

52. If A + B + C = 180°, sin A – sin B + sin C is

(a) $4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$ (b) $4\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$ (c) $4\cos\frac{A}{2}\cos\frac{C}{2}\sin\frac{B}{2}$ (d) $4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

53. $\cos(\tan^{-1}(\sin(\cot^{-1}x)))$ is

(a) $\sqrt{\frac{x^2+1}{x^2+2}}$ (b) $\sqrt{\frac{x^2+2}{x^2+3}}$ (c) $\sqrt{\frac{x^2}{x^2+1}}$ (d) $\sqrt{\frac{x^2-1}{x^2+2}}$

54. If $A = \tan^{-1} x$, then the value of sec 2A is

(a) $\frac{1-x^2}{1+x^2}$ (b) $\frac{2x}{1+x^2}$ (c) $\frac{1-x^2}{2x}$ (d) $\frac{1+x^2}{1-x^2}$

55. $\tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right] =$ is

(a) $\frac{6}{17}$ (b) $\frac{17}{6}$ (c) $\frac{16}{7}$ (d) $\frac{7}{16}$

56. If $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}x$, then the value of x is

(a) $\frac{3}{4}$ (b) $\frac{7}{9}$ (c) $\frac{12}{13}$ (d) $\frac{13}{12}$

57. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2\tan^{-1}x$, then x is

(a) $\frac{ab}{a+b}$ (b) $\frac{a+b}{ab}$ (c) $\frac{a+b}{1-ab}$ (d) $\frac{a-b}{1+ab}$

58. $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$ is

(a) $3+\sqrt{5}$ (b) $3-\sqrt{5}$ (c) $\frac{3+\sqrt{5}}{2}$ (d) $\frac{3-\sqrt{5}}{2}$

59. $\sin\left[\cot^{-1}\left\{\cos\left(\tan^{-1}x\right)\right\}\right]$ is

(a) 1 (b) $\sqrt{\frac{x^2-1}{x^2+2}}$ (c) $\sqrt{\frac{x-2}{x^2+1}}$ (d) $\sqrt{\frac{x^2+1}{x^2+2}}$

60. If $\theta = \cot^{-1}7+\cot^{-1}8+\cot^{-1}18$, then $\cot\theta$ is equal to

(a) 1 (b) 15 (c) 3 (d) 33

61. If $x > 1$, $2\tan^{-1}x$ is equal to

(a) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ (b) $\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ (c) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ (d) $\pi+\tan^{-1}2x$

62. $\sin^{-1}x > \cos^{-1}x$ holds for

(a) All values of x (b) $x\in\left(0,\frac{1}{\sqrt{2}}\right)$ (c) $x\in\left(\frac{1}{\sqrt{2}},1\right)$ (d) $x\in\left(\frac{-1}{\sqrt{2}},0\right)$

63. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then $\cos\left(\theta-\frac{\pi}{4}\right)$ is

(a) $\frac{1}{\sqrt{2}}$ (b) $\pm\frac{1}{2\sqrt{2}}$ (c) 2 (d) $2\sqrt{2}$

64. If tan aθ – tan bθ = 0, then the values of θ form a series in

(a) AP (b) GP (c) HP (d) AGP

65. If tan (cot θ) = cot (tan θ), then sin 2θ = (here k is an integer)

(a) $\frac{1}{2k\pi}$ (b) $\frac{4}{(2k+1)\pi}$ (c) $\frac{2}{(2k+1)\pi}$ (d) $\frac{1}{k\pi}$

66. The solution set of $(2\cos x - 1)(3 + 2\cos x) = 0$ in the interval $0 \le x \le 2\pi$ is

(a) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$ (b) $\left\{\frac{\pi}{3}\right\}$ (c) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}, \cos^{-1}\left(-\frac{3}{2}\right)\right\}$ (d) None of these

67. General solution of $\sin^2 x - 2\cos x + \frac{1}{4} = 0$ is given by x is

(a) $2n\pi \pm \frac{\pi}{3}$ (b) $2n\pi \pm \frac{\pi}{4}$ (c) $2n\pi \pm \frac{\pi}{2}$ (d) $2n\pi \pm \pi$

68. The general solution of sin 6θ + sin 4θ = sin 8θ + sin 2θ is given by θ equals

(a) $\frac{n\pi}{5}$ or $\frac{n\pi}{2}$ or $n\pi$ (b) $n\pi$ or $\frac{n\pi}{2}$ or $\frac{n\pi}{3}$ (c) $n\pi$ or $2n\pi$ or $3n\pi$ (d) $\frac{n\pi}{2}$ or $\frac{2}{5}n\pi$ or $3n\pi$

69. Maximum value of the expression $X = 4\cos\left(\theta + \frac{\pi}{3}\right) - 3\sin\left(\theta - \frac{\pi}{6}\right)$ is

(a) 7 (b) $\frac{7}{2}$ (c) $\frac{-7}{2}$ (d) –7

70. The quadratic equation whose roots are the maximum and minimum values of 3 sin x + 4cos x is

(a) $x^2 + 3x + 4$ (b) $x^2 - 5x - 5$ (c) $x^2 - 25 = 0$ (d) $x^2 - 7x + 12 = 0$

71. A solution of the equation $(1 - \tan\theta)(1 + \tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$ where, θ lies in interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is given by θ is

(a) 0 (b) $\pm\frac{\pi}{3}$ (c) $-\frac{\pi}{6}$ (d) $\frac{\pi}{6}$

72. Period of tan (x + 2x + 3x + … + nx) is

(a) $\frac{2\pi}{n(n+1)}$ (b) $\frac{\pi}{n(n+1)}$ (c) π (d) $\frac{\pi}{2}$

73. If $2\sin^2\theta + 2\sin\theta\cos\theta = 1$, then θ can be

(a) $\frac{\pi}{8} - \frac{n\pi}{2}$, $n \in Z$ (b) $\frac{\pi}{2} - 2n\pi$, $n \in Z$ (c) $(2n+1)\frac{\pi}{2}$, $n \in Z$ (d) $n\pi$, $n \in Z$

74. sin x cos x cos 2x = k has a solution if k lies between

(a) –1 and 1 (b) $-\frac{1}{4}$ and $\frac{1}{4}$ (c) $-\frac{1}{2}$ and $\frac{1}{2}$ (d) $-\frac{1}{3}$ and $\frac{1}{3}$

75. If $1 + \sin\theta + \sin^2\theta + \ldots + \infty = 4 + 2\sqrt{3}$, $0 < \theta < \pi$, $\theta \ne \frac{\pi}{2}$, then

(a) $\theta = \frac{\pi}{3}$ (b) $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ (c) $\theta = \frac{\pi}{4}$ (d) $\theta = \frac{\pi}{4}$ or $\frac{\pi}{6}$

76. Number of solutions of the equation tan x + sec x = 2cos x lying in the interval [0, 2p] is

(a) 0 (b) 1 (c) 2 (d) 3

77. The equation $\sin^6 x + \cos^6 x = a^2$ has real solutions if a belongs to

(a) $\left[\frac{-1}{2}, \frac{1}{2}\right]$ (b) $\left[-1, \frac{-1}{2}\right]$ or $\left[\frac{1}{2}, 1\right]$ (c) $[-1, 1]$ (d) $\left[\frac{-1}{2}, 1\right]$

78. The maximum and minimum values of $\frac{\sec^2 x - \tan x}{\sec^2 x + \tan x}$ where $x \in R$ are

(a) $\left(3, \frac{1}{3}\right)$ (b) $\left(-3, \frac{1}{3}\right)$ (c) $\left(-3, -\frac{1}{3}\right)$ (d) $\left(3, -\frac{1}{3}\right)$

79. If A + B + C = π then cot A cot B + cot B cot C + cot C cot A is equal to

(a) −1 (b) 1 (c) tan A (d) $\frac{1}{2}\tan^2 A$

80. The number of solutions of $\sin^2 x + \sin^2 2x = \sin^2 3x$ if $-\frac{\pi}{2} \le x \le \pi$ are

(a) 4 (b) 7 (c) 6 (d) 5

81. The expression $3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right] - 2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$ is

(a) 0 (b) −1 (c) 1 (d) 3

82. If tan A – tan B = x and cot B – cot A = y, then cot (A – B) is

(a) $\frac{1}{x} + \frac{1}{y}$ (b) $\frac{1}{x} - \frac{1}{y}$ (c) x – y (d) x + y

83. If tan A + cot A = 3 then $\tan^3 A + \cot^3 A =$

(a) 27 (b) 24 (c) 9 (d) 18

84. If sin (θ + α) = a and sin(θ + β) = b, then cos 2(α – β) – 4ab cos(α – β) is equal to

(a) $1 - a^2 - b^2$ (b) $1 - 2a^2 - 2b^2$ (c) $2 + a^2 + b^2$ (d) $2 - a^2 - b^2$

85. $\cos 2\theta \cos 2\phi + \sin^2(\theta - \phi) - \sin^2(\theta + \phi)$ is equal to

(a) sin (2θ + 2ϕ) (b) cos (2θ + 2ϕ) (c) sin (2θ – 2ϕ) (d) cos (2θ – 2ϕ)

86. If tan (α + θ) = n tan(α – θ) then $\frac{\sin 2\theta}{\sin 2\alpha}$ is equal to

(a) $\frac{1+n}{1-n}$ (b) $\frac{n-1}{n+1}$ (c) $\frac{2n}{n+1}$ (d) $\frac{n}{n+1}$

87. If $\sec\theta = \frac{a + b\cos\phi}{a\cos\phi + b}$ then $\tan\left(\frac{\theta}{2}\right)$ is

(a) $\left(\frac{a-b}{a+b}\right)\tan\left(\frac{\phi}{2}\right)$ (b) $\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{\phi}{2}\right)$

(c) $\frac{ab}{a-b}\tan\phi$ (d) $\frac{a+b}{ab}\cot 2\phi$

88. $\frac{\sin(A-C) + 2\sin A + \sin(A+C)}{\sin(B-C) + 2\sin B + \sin(B+C)}$ equals

(a) $\frac{\sin A}{\sin B}$ (b) tan A (c) tan B (d) tan A tan B

89. If $\tan\beta = \dfrac{n\sin\alpha\cos\alpha}{1 - n\cos^2\alpha}$, then $\tan(\alpha + \beta)$ equals

(a) $(n - 1)\tan\alpha$ (b) $(n + 1)\tan\alpha$ (c) $\dfrac{\tan\alpha}{n + 1}$ (d) $\dfrac{\tan\alpha}{1 - n}$

90. $\cos\left((2n + 1)\dfrac{\pi}{2} + \theta\right)$ is given by

(a) $(-1)^n\cos\theta$ (b) $(-1)^{n-1}\cos\theta$ (c) $(-1)^n\sin\theta$ (d) $(-1)^{n+1}\sin\theta$

91. If θ is in the third quadrant, the value of $\sqrt{3 + \cos^2 2\theta - 4\sin^4\theta} - 4\cot\theta\sin^2\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$ is:

(a) $2(\cot\theta + 2\cos\theta)$ (b) $-2(\cot\theta + 2\cos\theta)$ (c) $-2\cot\theta$ (d) $2\cot\theta$

92. $\sin^2\left(\dfrac{\pi}{8} + \dfrac{A}{2}\right) - \sin^2\left(\dfrac{\pi}{8} - \dfrac{A}{2}\right) =$

(a) $\dfrac{1}{2}\sin A$ (b) $\dfrac{1}{\sqrt{2}}\sin A$ (c) $2\sin^2 A$ (d) $\sqrt{2}\sin A$

93. The value of $\tan 9° - \tan 27° - \tan 63° + \tan 81°$ is

(a) 2 (b) 3 (c) 4 (d) 0

94. $\sin^4\dfrac{\pi}{16} + \sin^4\dfrac{3\pi}{16} + \sin^4\dfrac{5\pi}{16} + \sin^4\dfrac{7\pi}{16}$ equals

(a) 2.5 (b) 1.5 (c) $2\sqrt{2}$ (d) $2\sqrt{3}$

95. If $A = \sin^{20}\theta + \cos^{48}\theta$ then

(a) $A \geq 1$ (b) $0 < A \leq 1$ (c) $1 < A < 3$ (d) $A \geq 3$

96. If $\cot B = \dfrac{\sin A - \sin C}{\cos C - \cos A}, -\dfrac{\pi}{2} < A, B, C < \dfrac{\pi}{2}$, then A, B, C are in

(a) AGP (b) GP (c) AP (d) HP

97. If $A + B + C = 180°$, then $\sin^2 A - \sin^2 B + \sin^2 C$ is

(a) $2\sin A\sin B\sin C$ (b) $2\sin A\cos B\sin C$
(c) $2\cos A\sin B\sin C$ (d) $4\sin A\sin B\sin C$

98. The value of $\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right)$ is

(a) $\dfrac{7\pi}{6}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{6}$

99. If $\sin^{-1}\left(\dfrac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\dfrac{5}{4}\right) = \dfrac{\pi}{2}$, x equals

(a) 3 (b) 4 (c) 5 (d) 6

100. If $\sin^{-1}\left(\dfrac{2}{\sqrt{5}}\right)$ and $\sin^{-1}\left(\dfrac{3}{\sqrt{10}}\right)$ are the two angles of a triangle, then the third angle is

(a) 30° (b) 45° (c) 60° (d) 75°

101. If $0 < a < 1$, then, $\tan\left[\dfrac{1}{2}\sin^{-1}\dfrac{2a}{1 + a^2} + \dfrac{1}{2}\cos^{-1}\dfrac{1 - a^2}{1 + a^2}\right]$ is

(a) $\dfrac{a}{1 + a^2}$ (b) $\dfrac{a}{1 - a^2}$ (c) $\dfrac{2a}{1 - a^2}$ (d) $\dfrac{2a}{1 + a^2}$

102. The value of $\cos\left\{\sin^{-1}\frac{1-\sqrt{x}}{1+\sqrt{x}}+\sec^{-1}\frac{1+\sqrt{x}}{1-\sqrt{x}}\right\}, x>0, x\neq 1$ is

(a) 1 (b) 0 (c) –1 (d) ∞

103. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{4}{5}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$ is

(a) $\frac{5}{4}$ (b) $\frac{4}{5}$ (c) $\frac{8}{5}$ (d) $\frac{5}{2}$

104. The number of values of α in the interval $(0, 5\pi)$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is

(a) 0 (b) 5 (c) 6 (d) 10

105. If $\sin 5x + \sin 3x + \sin x = 0$, then the value of x other than zero in the interval $0 < x \leq \frac{\pi}{2}$ is

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{9}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

106. If $\frac{1-\tan x}{1+\tan x}=\tan y$ and $x-y=\frac{\pi}{6}$ then x and y are respectively

(a) $\frac{5\pi}{24}-\frac{n\pi}{2}$ and $\frac{\pi}{24}-\frac{n\pi}{2}$
(b) $\frac{n\pi}{2}+\frac{5\pi}{12}$ and $\frac{n\pi}{2}-\frac{\pi}{24}$
(c) $n\pi-\frac{\pi}{12}$ and $n\pi+\frac{5\pi}{24}$
(d) None of these

107. The equation $a\sin x + b\cos x = c$ where $|c| > \sqrt{a^2+b^2}$ has

(a) one solution
(b) two solutions
(c) no solution
(d) infinite number of solutions

108. The minimum value of $(\sin\alpha + \operatorname{cosec}\alpha)^2 + (\cos\alpha + \sec\alpha)^2$ is

(a) 2 (b) 4 (c) 7 (d) 9

109. The minimum value of the expression $\frac{2}{15+2\sin x-2\sqrt{3}\cos x}$ is

(a) $\frac{19}{2}$ (b) $\frac{2}{19}$ (c) $\frac{2}{11}$ (d) $\frac{2}{15}$

110. The roots of the equation $81^{\sin^2 x}+81^{\cos^2 x}=30, 0\leq x\leq \pi/2$ are

(a) $\frac{\pi}{6},\frac{\pi}{4}$ (b) $\frac{\pi}{4},\frac{\pi}{3}$ (c) $\frac{\pi}{6},\frac{\pi}{3}$ (d) $\frac{\pi}{6},\frac{\pi}{12}$

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

111. Statement 1

Minimum value of $\sin^2 x + \cos^4 x$ is $\frac{3}{4}$.

and

Statement 2

Minimum value of $(ax^2 + bx + c)$ where $a > 0$ is $\frac{-(b^2 - 4ac)}{4a}$.

112. Statement 1

$\tan^{-1}2 + \tan^{-1}\frac{1}{2} = \frac{\pi}{2}$.

and

Statement 2

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$ where $x > 0$, $y > 0$ and $xy < 1$.

113. Statement 1

Number of solutions of the equation

$2\sin^2\theta - 3\sin\theta + 1 = 0$ in the interval $[-2\pi, 2\pi]$ is 6.

and

Statement 2

General solution of $\sin\theta = k$ is given by $\theta = n\pi + (-1)^n\alpha$ where n is an integer and $\sin\alpha = k, 0 \le \alpha < \pi$.

Linked Comprehension Type Questions

Directions: **This section contains 1 paragraph. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

ΔABC is right angled at C.

114. If $\cot\frac{A}{2}$, $\cot\frac{B}{2}$ are the roots of $px^2 + qx + r = 0$ $(p \ne 0)$,

(a) $p + q = r$
(b) $p = q + r$
(c) $p + q + r = 0$
(d) $q = p + r$

115. If $\sin\frac{A}{2}$, $\sin\frac{B}{2}$ are the roots of $px^2 + qx + r = 0$, $(p \ne 0)$ and $A = \frac{\pi}{3}$,

(a) $p + 2q + 4r = 0$
(b) $p + q + r = 0$
(c) $p - 2q + r = 0$
(d) $p + 2q = 4r$

116. If sin A and cos B are the roots of $px^2 + qx + r = 0$ $(p \ne 0)$

(a) $q^2 = 4pr$
(b) $p^2 = 4qr$
(c) $q^2 = p^2 + 2pr$
(d) $q^2 + 4pr = 0$

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

117. $3^{\sin 2x+2\cos^2 x} + 3^{1-\sin 2x+2\sin^2 x} = 28$ is satisfied

(a) for all x (b) if $\cos x = 0$ (c) if $\tan x = -1$ (d) if $\tan x = 1$

118. Value of the expression $\dfrac{\sin^3\theta}{1+\cos\theta} + \dfrac{\cos^3\theta}{1-\sin\theta}$ is

(a) $\sqrt{2}\sin\left(\dfrac{\pi}{4}+\theta\right)$ (b) $\sqrt{2}\cos\left(\dfrac{\pi}{4}-\theta\right)$ (c) $\sqrt{2}\sin\left(\dfrac{\pi}{4}-\theta\right)$ (d) $\sqrt{2}\cos\left(\dfrac{\pi}{4}+\theta\right)$

119. If $\alpha, \beta, \gamma, \delta$ satisfy the equation $\tan\left(x+\dfrac{\pi}{4}\right) = 3\tan 3x$ then

(a) $\tan^2\alpha + \tan^2\beta + \tan^2\gamma + \tan^2\delta = 4$ (b) $\Sigma \tan\alpha\tan\beta = -2$

(c) $\tan\alpha\tan\beta\tan\gamma\tan\delta = -\dfrac{1}{3}$ (d) $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta = 0$

Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

120. $25\cos A - 26\cos B = 5$ and $10\cos A + 13\cos B = 11$

Column I	**Column II**
(a) $\sin(A+B)$	(p) $\dfrac{1}{65}$
(b) $\sin^2\left(\dfrac{A-B}{2}\right)$	(q) $\dfrac{-16}{63}$
(c) $\tan(A-B)$	(r) $\dfrac{65}{64}$
(d) $\sec^2\left(\dfrac{A-B}{2}\right)$	(s) $\dfrac{56}{65}$

ADDITIONAL PRACTICE EXERCISE

Subjective Questions

121. Prove that $(3 + \cos 4x)\cos 2x = 4(\cos^8 x - \sin^8 x)$.

122. If $\cos A = \tan B$, $\cos B = \tan C$, $\cos C = \tan A$ prove that, $\sin A = \sin B = \sin C = \dfrac{\sqrt{5}-1}{2}$.

123. Prove that $\cot 7\dfrac{1^\circ}{2} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$.

124. If $\tan\left(\dfrac{\pi}{4} + \dfrac{A}{2}\right) = \tan^3\left(\dfrac{\pi}{4} + \dfrac{B}{2}\right)$, prove that $\sin A = \left(\dfrac{3 + \sin^2 B}{1 + 3\sin^2 B}\right)\sin B$.

125. If $\cos A = \dfrac{2\cos B - 1}{2 - \cos B}$ prove that $\tan\dfrac{A}{2} = \sqrt{3}\tan\dfrac{B}{2}$. Hence show that $\sin B = \dfrac{\sqrt{3}\sin A}{2 + \cos A}$.

126. If $\sin^{-1}\sqrt{1 - \dfrac{x^2}{4}} + \cos^{-1}\dfrac{y}{5} = \theta$, prove that $25x^2 + 4y^2 - 20\,xy\cos\theta = 100\sin^2\theta$.

127. Prove that $\tan^{-1}x + \tan^{-1}y = \dfrac{1}{2}\sin^{-1}\left(\dfrac{2(x+y)(1-xy)}{(1+x^2)(1+y^2)}\right)$.

128. Find all values of α for which the equation $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$ is valid. Find the general solution.

129. Solve the equation $\cos 3x \sin^3 x + \sin 3x \cos^3 x = 0$.

130. Solve the equation $3\tan 2x - 4\tan 3x = \tan^2 3x \tan 2x$.

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

131. $\sqrt{\sec^2 A + \operatorname{cosec}^2 A}$ is equal to

(a) $\tan A + \cot A$ (b) $\sec A + \operatorname{cosec} A$ (c) $\sqrt{2}\sec A$ (d) $\sec A \tan A$

132. If $a\cos^3\alpha + 3a\cos\alpha.\sin^2\alpha = m$ and $a\sin^3\alpha + 3a\cos^2\alpha.\sin\alpha = n$, then $(m+n)^{2/3} + (m-n)^{2/3}$ is equal to

(a) $2a^2$ (b) $2a^{1/3}$ (c) $2a^{2/3}$ (d) $2a^3$

133. If $\dfrac{\cos(\theta-\alpha)}{\sin(\theta+\alpha)} = \dfrac{m+1}{m-1}$ then m is equal to

(a) $\tan\left(\dfrac{\pi}{4}-\theta\right)\tan\left(\dfrac{\pi}{4}-\alpha\right)$ (b) $\tan\left(\dfrac{\pi}{4}-\theta\right)\tan\left(\dfrac{\pi}{4}+\alpha\right)$

(c) $\tan\left(\dfrac{\pi}{4}+\theta\right)\tan\left(\dfrac{\pi}{4}+\alpha\right)$ (d) $\tan\left(\dfrac{\pi}{4}+\theta\right)\tan\left(\dfrac{\pi}{4}-\alpha\right)$

134. If $0 < \theta < \frac{\pi}{2}$ and $\tan\theta + \cot\theta = 2$, the value of $\tan^{16}\theta + \cot^{16}\theta - 2\tan^2\theta + 3\cot^2\theta$ equals

(a) 3 (b) 2^{16} (c) 3×2^{16} (d) 3×2^{15}

135. If $\sin\alpha + \sin\beta + \sin\gamma = 3$, then $\cos\alpha + \cos\beta + \cos\gamma$ is

(a) 1 (b) 0 (c) -1 (d) 2

136. $(1 + \sec 2\theta)(1+ \sec 2^2\theta)(1 + \sec 2^3\theta) \ldots (1 + \sec 2^n\theta)$ equals

(a) $\tan 2^n\theta \cot\theta$ (b) $\tan 2\theta \cot 2^n\theta$ (c) $\tan 2^n\theta \sec\theta$ (d) $\cot 2^n\theta$

137. If $x\sin\theta = y\sin\left(\theta+\frac{2\pi}{3}\right) = z\sin\left(\theta+\frac{4\pi}{3}\right)$ then $\sum xy$, i.e., $xy + yz + zx$ is equal to

(a) 1 (b) $\frac{1}{2}$ (c) 0 (d) $\frac{1}{4}$

138. The maximum value of $(\cos\alpha_1)(\cos\alpha_2)(\cos\alpha_3)\ldots\ldots(\cos\alpha_n)$ under the restrictions $0 < \alpha_1, \alpha_2, \alpha_3 \ldots, \alpha_n < \pi/2$ and $(\cot\alpha_1)(\cot\alpha_2)(\cot\alpha_3)\ldots(\cot\alpha_n) = 1$ is

(a) $< \frac{1}{2^{n/2}}$ (b) $< \frac{1}{2^n}$ (c) $< \frac{1}{2n}$ (d) $=1$

139. The value of $\tan 3x \cot x$ for all $x \in R$, $x \neq 0, \pm\frac{\pi}{6}$

(a) lies between $\frac{1}{3}$ and 3 (b) lies beyond $\frac{1}{3}$ and 3 (c) lies in $(0, \infty)$ (d) lies in $(-1, 1)$

140. If the equation $\sin^6 x + \cos^6 x = \lambda$ is to have a real solution, then the range of values of λ, is

(a) $\frac{1}{2} \le \lambda \le 1$ (b) $\frac{1}{4} \le \lambda \le 1$ (c) $0 \le \lambda \le 4$ (d) $\frac{1}{4} \le \lambda \le \frac{1}{2}$

141. If $\cos(\alpha + \beta)\sin(\gamma + \delta) = \cos(\alpha - \beta)\sin(\gamma - \delta)$, then which of the following is true?

(a) $\cot\alpha\cot\beta = \cot\gamma\cot\delta$ (b) $\tan\alpha\tan\beta\tan\delta = \tan\gamma$ (c) $\cot\alpha\cot\beta\cot\delta = \cot\gamma$ (d) $\cot\alpha\cot\beta\cot\gamma = \cot\delta$

142. If x and y are acute angles and $\frac{\tan x}{\tan y} = \sqrt{2}$, then $\frac{3\cos 2y - 1}{3 - \cos 2y}$ is

(a) $\cos 2x$ (b) $\tan 2x$ (c) $\sin 2x$ (d) $\frac{1 + \tan^2 x}{1 - \tan^2 x}$

143. If $k_1 = \tan 27\theta - \tan\theta$ and $k_2 = \frac{\sin\theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta}$, then

(a) $2k_1 = k_2$ (b) $k_1 = k_2$ (c) $2k_2 = k_1$ (d) $k_1 + k_2 = 0$

144. Value of $\sec^2\frac{\pi}{16} + \sec^2\frac{3\pi}{16} + \sec^2\frac{5\pi}{16} + \sec^2\frac{7\pi}{16}$ is

(a) 16 (b) 32 (c) 36 (d) 28

145. The value of $\cos\left(\cos^{-1}\left(\frac{12}{15}\right) + \cos^{-1}\left(\frac{4}{5}\right)\right)$ is

(a) $\frac{1}{5}$ (b) $\frac{3}{10}$ (c) $\frac{3}{25}$ (d) $\frac{7}{25}$

146. $\tan\left(\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right)$ is

(a) $\frac{\sqrt{3}}{29}$ (b) $\frac{3}{29}$ (c) $\frac{29}{3}$ (d) $\frac{\sqrt{29}}{3}$

147. If $\cos^{-1}\dfrac{p}{a}+\cos^{-1}\dfrac{q}{b}=\alpha$, then $\dfrac{p^2}{a^2}-\dfrac{2pq}{ab}\cos\alpha+\dfrac{q^2}{b^2}$ is

(a) $\sin^2\alpha$ (b) $\cos^2\alpha$ (c) $\sin\alpha$ (d) $\cos\alpha$

148. $2\tan^{-1}\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{\theta}{2}$ is

(a) $\cos^{-1}\left(\dfrac{a\cos\theta+b}{a}\right)$ (b) $\cos^{-1}\left(\dfrac{a\cos\theta+b}{a+b\cos\theta}\right)$ (c) $\cos^{-1}\left(\dfrac{b\cos\theta+a}{b+a\cos\theta}\right)$ (d) $\cos^{-1}\left(\dfrac{b\cos\theta+a}{b}\right)$

149. $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$ for

(a) All values of x (b) $\dfrac{-1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}}$ (c) $\dfrac{1}{\sqrt{2}}\le x\le 1$ (d) $-1\le x\le\dfrac{-1}{\sqrt{2}}$

150. $\sin\left(2\tan^{-1}\dfrac{1}{3}\right)+\cos(\tan^{-1}2\sqrt{2})+\tan\left(\sin^{-1}\dfrac{4}{5}+\dfrac{3\pi}{2}\right)=$

(a) $\dfrac{11}{30}$ (b) $\dfrac{11}{60}$ (c) $\dfrac{101}{60}$ (d) $-\dfrac{2}{3}$

151. The number of solutions of $\sin x+\sin 2x+\sin 3x=\cos x+\cos 2x+\cos 3x$ in the interval $0\le x\le 2\pi$ is

(a) 6 (b) 4 (c) 0 (d) 10

152. If $2\sin\dfrac{A}{2}=-\sqrt{1+\sin A}+\sqrt{1-\sin A}$ A lies between the limits

(a) $2n\pi-\dfrac{3\pi}{4}$ to $2n\pi-\dfrac{\pi}{4}$ (b) $2n\pi-\dfrac{\pi}{4}$ to $2n\pi+\dfrac{\pi}{4}$

(c) $2n\pi+\dfrac{\pi}{4}$ to $2n\pi+\dfrac{3\pi}{4}$ (d) $2n\pi+\dfrac{3\pi}{4}$ to $2n\pi+\dfrac{5\pi}{4}$

153. General solution of the equation $\sqrt{3}(\cos\theta-\sqrt{3}\sin\theta)=4\sin 2\theta\cos 3\theta$ is,

(a) $r\pi\pm\dfrac{\pi}{4}-\dfrac{\pi}{12}$ or $\dfrac{r\pi}{3}+\dfrac{\pi}{18}$ where, r is an integer (b) $\dfrac{r\pi}{2}-\dfrac{\pi}{3}$ or $\dfrac{r\pi}{3}-\dfrac{\pi}{6}$ where, r is an integer

(c) $\dfrac{r\pi}{3}+\dfrac{\pi}{18}$ or $\dfrac{r\pi}{2}-\dfrac{\pi}{9}$ where, r is an integer (d) None of these

154. Solution of the equation $\cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\dfrac{2x}{x^2-1}\right)=\dfrac{2\pi}{3}$, is

(a) $-\sqrt{3},\sqrt{3}$ (b) $2-\sqrt{3},\sqrt{3}$ (c) $2+\sqrt{3},2-\sqrt{3}$ (d) $\sqrt{2},2+\sqrt{2}$

155. Solution of the equation: $3\tan^2\theta-2\sin\theta=0$

(a) $n\pi$ (b) $n\pi+(-1)^n\dfrac{\pi}{6}$ (c) $n\pi+(-1)^n\dfrac{\pi}{3}$ (d) both (a) and (b)

156. The range of values for $\sin\alpha$ so that the equation

$3\sin x+\sin(\alpha+x)+\sin(\alpha-x)=2\sqrt{3}$ has real solution.

(a) $(-1,0)\cup\left(\dfrac{\sqrt{3}}{2},1\right]$ (b) $\left[-1,\dfrac{-\sqrt{3}}{2}\right)\cup\left(\dfrac{\sqrt{3}}{2},1\right]$

(c) $(-1,1)$ (d) $\left(-1,\dfrac{\sqrt{3}}{2}\right]$

157. The number of solutions of the equation $\tan^{-1}\dfrac{1}{2x+1}+\tan^{-1}\dfrac{1}{4x+1}=\tan^{-1}\dfrac{2}{x^2}$.

(a) 1 (b) 2 (c) 3 (d) 0

158. If $\dfrac{\sin^4\alpha}{a}+\dfrac{\cos^4\alpha}{b}=\dfrac{1}{a+b}$, $\dfrac{\sin^{2n}\alpha}{a^n}+\dfrac{\cos^{2n}\alpha}{b^n}=\dfrac{\lambda}{(a+b)^n}$, then λ equals

(a) 1 (b) 2 (c) -2 (d) 4

159. For real values of θ, $(5\sin^2\theta + 4\sin\theta\cos\theta + 3\cos^2\theta)$ lies between

(a) $\left(4-\sqrt{5}\right)$ and $\left(4+\sqrt{5}\right)$ (b) $2+\sqrt{3}$ and $2-\sqrt{3}$ (c) 1 and -1 (d) 2 and -2

160. If $\dfrac{\cos^3\theta}{\cos(\alpha-3\theta)}=\dfrac{\sin^3\theta}{\sin(\alpha-3\theta)}=k$, then $2k^2-k\cos\alpha-1$ is

(a) 2 (b) -2 (c) 1 (d) 0

161. If $x^2+\tan^2 y-10x-2\tan y+26=0$, then (x, y) is equal to

(a) $\left(5, n\pi-\dfrac{\pi}{4}\right)$ (b) $\left(-5, n\pi+\dfrac{\pi}{4}\right)$ (c) $\left(-5, n\pi-\dfrac{\pi}{4}\right)$ (d) $\left(5, n\pi+\dfrac{\pi}{4}\right)$

162. If $f(x)=2\sin^2\alpha+4\cos(x+\alpha)\sin x\sin\alpha+\cos 2(x+\alpha)$. Then the value of $(f(\alpha))^2+\left[f\left(\dfrac{\pi}{2}-\alpha\right)\right]^2$ is

(a) 1 (b) 2 (c) 0 (d) $\dfrac{1}{2}$

163. If $(\cos p-1)x^2+x\cos p+\sin p=0$ has real roots, the interval of possible values of p is

(a) $(-\pi, 0)$ (b) $(0, \pi)$ (c) $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ (d) $\left(0, \dfrac{3\pi}{2}\right)$

164. If α and β are the solutions of the equation $a\cos 2x + b\sin 2x = c$, $\tan^2\alpha+\tan^2\beta$ is

(a) $\dfrac{2(2b^2+c^2-a^2)}{(a+c)^2}$ (b) $\dfrac{2(2b^2+a^2+c^2)}{(a+c)^2}$ (c) $\dfrac{2(2b^2+a^2-c^2)}{(a+c)^2}$ (d) $\dfrac{2(2b^2-a^2-c^2)}{(a+c)^2}$

165. $\left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\dfrac{5\pi}{8}\right)\left(1+\cos\dfrac{7\pi}{8}\right)=$

(a) $\dfrac{1}{8}$ (b) $\dfrac{1}{4}$ (c) $\dfrac{1}{16}$ (d) $\dfrac{3}{8}$

166. If $\alpha=\dfrac{\pi}{2^5+1}$, $\cos\alpha\cos 2\alpha\cos 4\alpha\cos 8\alpha\cos 16\alpha$ equals

(a) $\dfrac{1}{8}$ (b) $\dfrac{1}{16}$ (c) $\dfrac{1}{32}$ (d) $\dfrac{1}{64}$

167. If $\dfrac{x}{\sin\theta}=\dfrac{y}{\sin\left(\theta-\dfrac{2\pi}{3}\right)}=\dfrac{z}{\sin\left(\theta+\dfrac{2\pi}{3}\right)}$, then $x+y+z$ is

(a) $\cos 2\theta$ (b) $\sin 2\theta$ (c) 0 (d) $\dfrac{1}{3}$

168. If x_1, x_2, x_3 are the roots of the equation, $x^3-x^2\sin 2\beta+x\cos 2\beta-2\sin\beta=0$ then $\tan^{-1}x_1+\tan^{-1}x_2+\tan^{-1}x_3$ is

(a) β (b) -2β (c) $-\beta/2$ (d) $\dfrac{\pi}{2}-\beta$

169. The set of values of a for which the equation $\sin^4 x + \cos^4 x = a$ has real solutions is

(a) $\left(\frac{1}{2},1\right)$ (b) $[0, 1]$ (c) $\left[\frac{1}{2},1\right]$ (d) $\left[\frac{1}{4},\frac{1}{2}\right]$

170. The number of real roots of $\tan^{-1}\sqrt{x^2 + x} + \sin^{-1}\sqrt{1 + x + x^2} = \frac{\pi}{2}$ is

(a) 2 (b) 0 (c) 1 (d) 4

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

171. Statement 1

$2\sin x + 3\cos x = 4$ has no real solutions.

and

Statement 2

$a\cos x + b\sin x = c$ has a real solution if and only if $\frac{c^2}{a^2 + b^2} < 1$.

172. Statement 1

The inequality $\cos(3 + \cos\theta) < 0$ is satisfied for all θ.

and

Statement 2

For all θ, $2 \le 3 + \cos\theta \le 4$.

173. Statement 1

$\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{4}{7}\right)$.

and

Statement 2

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$ provided both x and y are positive.

174. Statement 1

If θ lies in the 4th quadrant and $\tan\theta = \frac{-1}{5}$, $\frac{\cos\theta + 4\sin\theta}{\sin\theta + 4\cos\theta} = \frac{1}{19}$.

and

Statement 2

If θ is in the 4th quadrant $(\sin\theta + \cos\theta)$ is negative.

175. Statement 1

If $A + B = \frac{\pi}{4}$, $(1 + \cot A)(1 + \cot B) = 2$

and

Statement 2

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

176. Statement 1

$\sin^{-1}\left(\dfrac{2x}{1 + x^2}\right) = 2\tan^{-1}x$ for all x

and

Statement 2

$\sin 2\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$

177. Statement 1

In a triangle ABC, if AB = AC then the line joining the orthocentre, centroid and circumcentre passes through A and is perpendicular to BC.

and

Statement 2

In any triangle, the orthocentre, centroid and circumcentre are collinear.

178. Statement 1

In a triangle ABC, $\sum \dfrac{\cot B + \cot C}{\tan B + \tan C} = 2$.

and

Statement 2

In a triangle ABC, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

179. Statement 1

The equation $(\sin x + \cos x)^{1 + \sin 2x} = 2$, has a unique solution in $[-\pi, \pi]$.

and

Statement 2

For all x, $a \sin x + b \cos x + c$ lies in the interval

$\left[c - \sqrt{a^2 + b^2}, c + \sqrt{a^2 + b^2}\right]$

180. Statement 1

In a triangle ABC, if $p = \tan\frac{A}{2}$, $q = \tan\frac{B}{2}$ then $\sin C = \dfrac{2(p + q)(1 - pq)}{(1 + p^2)(1 + q^2)}$

and

Statement 2

$\tan\left(\dfrac{A}{2} + \dfrac{B}{2}\right) = \dfrac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}}$

Linked Comprehension Type Questions

Directions: **This section contains 3 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ defines a curve known as ellipse with center at C(0, 0). Any point on it can be represented parametrically by $x = a\cos\theta$; $y = b\sin\theta$ and is denoted by 'θ'

181. $P(\theta)$ and $Q\left(\frac{\pi}{2}-\theta\right)$ are 2 points on the ellipse. The sum CP + CQ is

(a) $\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}+\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$ (b) $a+b$

(c) $\sqrt{2a^2\cos^2\theta+2b^2\sin^2\theta}$ (d) $2a\cos\theta+2b\sin\theta$

182. If u = CP + CQ, then the maximum value of u^2 is

(a) $2a^2\cos^2\theta+2b^2\sin^2\theta$ (b) a^2+b^2

(c) $2(a^2+b^2)$ (d) $2(a^2\sin^2\theta+b^2\cos^2\theta)$

183. The difference between the maximum and minimum values of u^2 is

(a) $(a-b)^2$ (b) $(a+b)^2$ (c) $2ab$ (d) a^2+b^2

Passage II

$$\cot 2\theta-\cot 4\theta=\frac{1}{\tan 2\theta}-\frac{1}{\tan 2(2\theta)}=\frac{1}{\tan 2\theta}-\frac{1-\tan^2 2\theta}{2\tan 2\theta}$$

$$=\frac{1+\tan^2 2\theta}{2\tan 2\theta}=\frac{\sec^2 2\theta\cos 2\theta}{2\sin 2\theta}=\operatorname{cosec} 4\theta.$$

184. The value of $\cot 2^{n-1}\theta-\cot 2^n\theta$ is

(a) $\operatorname{cosec} 2^n\theta$ (b) 0 (c) $\operatorname{cosec} 2^{n-1}\theta$ (d) $\sin 2^n\theta$

185. $\sum_{r=1}^{n} 2^{r-1}\tan 2^{r-1}\alpha$ is

(a) $\cot\alpha-2^n\cot 2^n\alpha$ (b) $2^n\cot 2^n\alpha-\cot\alpha$ (c) $\cot\alpha+2^n\cot 2^n\alpha$ (d) $2^n\cot 2^n\alpha$

186. The roots of the quadratic equation $(\sin^2 2\theta)x^2-(4\sin\theta\cos^3\theta)x+\cos 2\theta=0$ are

(a) $\operatorname{cosec}\theta, \cot\theta$ (b) $\operatorname{cosec} 2\theta, \cot\theta$ (c) $\operatorname{cosec} 2\theta, \cot 2\theta$ (d) $\operatorname{cosec}\theta, \cot 2\theta$

Passage III

If $|x|\le 1$ and $|y|\le 1$ consider the equations

(i) $\cos^{-1}x+(\sin^{-1}y)^2=\frac{k\pi^2}{4}$

(ii) $(\cos^{-1}x)(\sin^{-1}y)^2=\frac{\pi^4}{16}$

187. The set of value of k for which equation $\cos^{-1}x+(\sin^{-1}y)^2=\frac{k\pi^2}{4}$ hold is

(a) $\left[0,\frac{\pi}{4}\right]$ (b) $[0, 1]$ (c) $\left[0,\frac{4}{\pi}+1\right]$ (d) $\left[0,\frac{4}{\pi}\right]$

188. The value of k if equations (i) and (ii) have solutions

(a) $[-2, 2]$ (b) $(-\infty, 2] \cup [2, \infty)$ (c) $\left[2, \frac{4}{\pi}+1\right]$ (d) $\left[2, \frac{4}{\pi}\right]$

189. The integral value of k for which the system of equation (i) and (ii) have solution is

(a) $\{x : x = n\pi, n \in Z\}$ (b) $\{x : x = 2n + 1\}$ (c) $\{-1, 1\}$ (d) $\{2\}$

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

190. The solutions of $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \cot^{-1}\left(\frac{x^2}{2}\right)$ is

(a) $x = 0$ (b) $x = \frac{2}{3}$ (c) $x = \frac{-2}{3}$ (d) $x = 3$

191. If $\frac{\sec^4 x}{a} + \frac{\tan^4 x}{b} = \frac{1}{a+b}$, then $\frac{\sec^8 x}{a^3} - \frac{\tan^8 x}{b^3}$ equals

(a) $\frac{(a-b)}{(a+b)^4}$ (b) $\frac{1}{(a+b)}$ (c) $\frac{1}{(a-b)}$ (d) $\frac{1}{(a-b)^3}$

192. If $\sin\theta + 2\cos\theta = 2$, then the value of $(\cos\theta - 2\sin\theta)$ is

(a) -1 (b) 0 (c) 1 (d) $\frac{1}{\sqrt{3}}$

193. If $y = \cos^{-1}x - \sin^{-1}x$, then

(a) $x = \frac{1}{2}, y = \frac{\pi}{6}$ (b) $x = \frac{-\sqrt{3}}{2}, y = \frac{7\pi}{6}$ (c) $x = \frac{-1}{2}, y = \frac{5\pi}{6}$ (d) $x = \frac{\sqrt{3}}{2}, y = -\frac{\pi}{6}$

194. If $\tan\frac{\theta}{2} = \operatorname{cosec}\theta - \sin\theta$ then $\tan^2\frac{\theta}{2}$ is equal to

(a) $2 - \sqrt{5}$ (b) $-2 + \sqrt{5}$ (c) $\left(9 - 4\sqrt{5}\right)\left(2 + \sqrt{5}\right)$ (d) $\left(9 + 4\sqrt{5}\right)\left(2 - \sqrt{5}\right)$

195. If $\frac{x}{y} = \frac{\cos A}{\cos B}, A \neq B, 0 \le A, B < \frac{\pi}{2}$ then

(a) $\tan\frac{A+B}{2} = \frac{x\tan A + y\tan B}{x+y}$ (b) $\tan\frac{A-B}{2} = \frac{x\tan A - y\tan B}{x+y}$

(c) $\frac{\sin(A+B)}{\sin(A-B)} = \frac{y\sin A + x\sin B}{y\sin A - x\sin B}$ (d) $x\cos A + y\cos B = 0$

196. Which of the following functions have the period 2π?

(a) $\sin x.\cos x + 2\sin x$ (b) $\frac{\sin 3x + \tan 2x}{\cot 4x + \sec 5x}$

(c) $\tan x \cdot \tan\left(\frac{\pi}{3} - x\right)\tan\left(\frac{\pi}{3} + x\right)$ (d) $3\sin^2 x + \cos^3 x$

197. If $\tan\alpha$ and $\tan\beta$ are the roots of the equation $x^2 + px + q = 0$ where, $p \neq 0$, then

(a) $\sin^2(\alpha+\beta) + p\sin(\alpha+\beta)\cos(\alpha+\beta) + q\cos^2(\alpha+\beta) = q$

(b) $\tan(\alpha+\beta) = \dfrac{p}{(q-1)}$

(c) $\cos(\alpha+\beta) = q - 1$

(d) $\sin(\alpha+\beta) = -p$

Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

198.

	Column I		Column II
(a)	$4\cos 48^\circ \sin 18^\circ \cos 12^\circ$	(p)	$\cos 6^\circ$
(b)	$16\cos 6^\circ \cos 12^\circ \cos 24^\circ \cos 48^\circ \cos 96^\circ$	(q)	$\cos 36^\circ$
(c)	$24\sin 6^\circ \cos^3 6^\circ - 18\sin 6^\circ \cos 6^\circ + 24\sin^3 6^\circ \cos 6^\circ - 32\sin^3 6^\circ \cos^3 6^\circ$	(r)	$\cos 306^\circ$
(d)	$(\cos 6)^{\cos^2 6}.(\cos 6)^{\cos^2 6\sin^2 6}(\cos 6)^{\cos^2 6.\sin^4 6}(\cos 6)^{\cos^2 6\sin^6 6}.....\infty$	(s)	$\cos 906^\circ$

199.

	Column I		Column II
(a)	$\sin\left(\frac{1}{2}\sin^{-1}\left(\frac{-2\sqrt{2}}{3}\right)\right)$	(p)	1
(b)	$\cot\left(\cos^{-1}\left(\frac{-1}{3}\right)\right)$	(q)	$\frac{-1}{2\sqrt{2}}$
(c)	$\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{80}}{9}\right)$	(r)	$\frac{\sqrt{5}-1}{2\sqrt{3}}$
(d)	$\sin\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5}\right)$	(s)	$\frac{-1}{\sqrt{3}}$

200.

	Column I		Column II
(a)	$\frac{1}{3}\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$	(p)	$\frac{5}{16}$
(b)	$\sin 36^\circ \sin 72^\circ \sin 108^\circ \sin 144^\circ$	(q)	$\frac{1}{8}$
(c)	$\sin 12^\circ \sin 48^\circ \sin 54^\circ$	(r)	$\frac{1}{32}$
(d)	$8\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$	(s)	$\frac{1}{16}$

SOLUTIONS

ANSWER KEYS

Topic Grip

1. (i) $\frac{1}{4}$
(ii) -1
(iii) 0
(iv) -1
(v) -1

8. $x = \frac{1}{6}$

9. $2n\pi + \frac{\pi}{2}, n\pi + \frac{\pi}{4}, \frac{n\pi}{2} \pm \frac{\pi}{8}$

10. 989π

11. (c) 12. (d) 13. (c)
14. (c) 15. (a) 16. (a)
17. (a) 18. (d) 19. (d)
20. (c) 21. (a) 22. (b)
23. (a) 24. (c) 25. (c)
26. (c) 27. (a), (c) 28. (a), (c)
29. (c), (d)
30. (a) → (q), (r)
(b) → (s)
(c) → (q)
(d) → (p)

IIT Assignment Exercise

31. (b) 32. (a) 33. (b)
34. (a) 35. (a) 36. (c)
37. (b) 38. (d) 39. (c)
40. (d) 41. (a) 42. (b)
43. (d) 44. (c) 45. (a)
46. (b) 47. (b) 48. (a)
49. (d) 50. (b) 51. (b)
52. (b) 53. (a) 54. (d)
55. (b) 56. (b) 57. (c)
58. (d) 59. (d) 60. (c)
61. (b) 62. (c) 63. (b)
64. (a) 65. (b) 66. (a)
67. (a) 68. (a) 69. (a)
70. (c) 71. (b) 72. (a)
73. (a) 74. (b) 75. (b)
76. (c) 77. (b) 78. (a)
79. (b) 80. (b) 81. (c)
82. (a) 83. (d) 84. (b)
85. (b) 86. (b) 87. (b)
88. (a) 89. (d) 90. (d)
91. (c) 92. (b) 93. (c)
94. (b) 95. (b) 96. (c)
97. (b) 98. (d) 99. (a)
100. (b) 101. (c) 102. (b)
103. (d) 104. (c) 105. (d)
106. (a) 107. (c) 108. (d)
109. (b) 110. (c) 111. (a)
112. (b) 113. (c) 114. (b)
115. (a) 116. (a) 117. (b), (c)
118. (a), (b)
119. (a), (b), (c), (d)
120. (a) → (s)
(b) → (p)
(c) → (q)
(d) → (r)

Additional Practice Exercise

128. $-\frac{3}{2} \le \alpha \le \frac{1}{2}$
$x = \frac{n\pi}{2} + \frac{1}{2}(-1)^n \beta$

129. $x = n\pi, 2n\pi \pm \frac{\pi}{2}, n\pi \pm \frac{\pi}{4}$

130. $x = n\pi, n\pi \pm \tan^{-1}\sqrt{\frac{3}{5}}$

131. (a) 132. (c) 133. (c)
134. (a) 135. (b) 136. (a)
137. (c) 138. (a) 139. (b)
140. (b) 141. (d) 142. (a)
143. (c) 144. (b) 145. (d)
146. (b) 147. (a) 148. (b)
149. (b) 150. (b) 151. (a)
152. (d) 153. (a) 154. (b)
155. (d) 156. (b) 157. (b)
158. (b) 159. (a) 160. (d)
161. (d) 162. (a) 163. (b)
164. (c) 165. (a) 166. (c)
167. (c) 168. (c) 169. (c)
170. (a) 171. (c) 172. (a)
173. (c) 174. (c) 175. (d)
176. (d) 177. (b) 178. (d)
179. (a) 180 (b) 181. (a)
182. (c) 183. (a) 184. (a)
185. (a) 186. (c) 187. (c)
188. (c) 189. (d) 190. (a), (d)
191. (a), (d)
192. (a), (c)
193. (a), (b), (c), (d)
194. (b), (c)
195. (a), (b), (c)
196. (a), (b), (d)
197. (a), (b)
198. (a) → (q)
(b) → (s)
(c) → (r)
(d) → (p)
199. (a) → (s)
(b) → (q)
(c) → (r)
(d) → (p)
200. (a) → (s)
(b) → (p)
(c) → (q)
(d) → (q)

HINTS AND EXPLANATIONS

Topic Grip

1. (i) $\tan 720° - \cos 270° - \sin 150° \cos 120°$

$= \tan(2 \times 360°) - \cos(3 \times 90°)$

$- \sin(180° - 30°)\cos(180° - 60°)$

$= 0 - 0 - \sin 30°(-\cos 60°)$

$= \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{1}{4}$

(ii) $\sin 600° \cos 330° + \cos 120° \sin 150°$

$= \sin(360° + 240°)\cos(360° - 30°)$

$+ \cos(90° + 30°)\sin(180° - 30°)$

$= \sin 240° \cos 30° - \sin 30° \sin 30°$

$= -\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2} = \frac{-3-1}{4} = -1$

(iii) $\cos^2 18° - \cos^2 36° - \cos^2 54° + \cos^2 72°$

$= \cos^2 18° - \cos^2(90° - 54°) - \cos^2 54°$

$+ \cos^2(90° - 18°)$

$= \cos^2 18° - \sin^2 54° - \cos^2 54° + \sin^2 18°$

$= (\sin^2 18° + \cos^2 18°)$

$- (\sin^2 54° + \cos^2 54°) = 0$

(iv) $\frac{(-\tan\theta)(\operatorname{cosec}\theta)(-\sec\theta)}{(-\tan\theta)\sec^2\theta\cot\theta}$

$= -\frac{\sec\theta}{\sec\theta} = -1$

(v) $\cot\frac{11\pi}{20}\cot\frac{13\pi}{20}\cot\frac{15\pi}{20}\cot\frac{17\pi}{20}\cot\frac{19\pi}{20}$

$= \cot\left(\frac{\pi}{2} + \frac{\pi}{20}\right)\cot\left(\frac{\pi}{2} + \frac{3\pi}{20}\right)\cot\left(\frac{\pi}{2} + \frac{5\pi}{20}\right)$

$\cot\left(\pi - \frac{3\pi}{20}\right)\cot\left(\pi - \frac{\pi}{20}\right)$

$= \left(-\tan\frac{\pi}{20}\right)\left(-\tan\frac{3\pi}{20}\right).\left(-\tan\frac{\pi}{4}\right)$

$\left(-\cot\frac{3\pi}{20}\right)\left(-\cot\frac{\pi}{20}\right)$

$= -1$

2. (i) $\sin\frac{7\pi}{12}\cos\frac{\pi}{4} - \cos\frac{7\pi}{12}\sin\frac{\pi}{4} = \sin\left(\frac{7\pi}{12} - \frac{\pi}{4}\right)$

$= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(ii) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{\frac{m}{m+1} + \frac{1}{2m+1}}{1 - \frac{m}{(m+1)(2m+1)}}$

$= \frac{m(2m+1) + m + 1}{(2m+1)(m+1) - m}$

$= \frac{2m^2 + 2m + 1}{2m^2 + 2m + 1} = 1$

$\therefore A + B = \frac{\pi}{4}$

(iii) $\tan 70° = \tan(50° + 20°)$

$= \frac{\tan 50° + \tan 20°}{1 - \tan 50° \tan 20°}$

$\Rightarrow$ $\tan 70° - \tan 70° \tan 50° \tan 20°$

$= \tan 50° + \tan 20°$

$\Rightarrow$ $\tan 70° - \tan(90° - 20°)\tan 50° \tan 20°$

$= \tan 50° + \tan 20°$

$\Rightarrow$ $\tan 70° - \cot 20° \tan 50° \tan 20°$

$= \tan 50° + \tan 20°$

$\Rightarrow$ $\tan 70° - \tan 50° = \tan 50° + \tan 20°$

$\Rightarrow$ $\tan 70° = 2\tan 50° + \tan 20°$

(iv) $\frac{\cos 8° - \sin 8°}{\cos 8° + \sin 8°} = \frac{1 - \tan 8°}{1 + \tan 8°}$ (dividing by $\cos 8°$)

$= \frac{\tan 45° - \tan 8°}{1 + \tan 45° \tan 8°}$

$= \tan(45° - 8°) = \tan 37°$

(v) $\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8}$

$= \frac{1}{2}\left(1 + \cos\frac{\pi}{4} + 1 + \cos\frac{3\pi}{4} + 1\right.$

$\left. + \cos\frac{5\pi}{4} + 1 + \cos\frac{7\pi}{4}\right)$

$= \frac{1}{2}\left(4 + \cos\frac{\pi}{4} - \cos\frac{\pi}{4} - \cos\frac{\pi}{4} + \cos\frac{\pi}{4}\right) = 2$

3. We have $\tan 5x = \tan(2x+3x)$

$$= \frac{\tan 2x + \tan 3x}{1-\tan 2x \tan 3x}$$

$$\Rightarrow \tan 5x(1-\tan 2x\tan 3x) = \tan 2x + \tan 3x$$

$$\tan 5x - \tan 2x \tan 3x \tan 5x = \tan 2x + \tan 3x$$

$$\therefore \tan 5x - \tan 2x - \tan 3x = \tan 2x \tan 3x \tan 5x$$

4. Expression

$$= \frac{1}{2\sin\theta}\left[(2\sin\theta\cos\theta)\cos 2\theta \cos 2^2\theta \ldots \cos 2^{k-1}\theta\right]$$

$$= \frac{1}{2\sin\theta}\left[\sin 2\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{k-1}\theta\right]$$

$$= \frac{1}{2^2\sin\theta}\left[\sin 2^2\theta \cos 2^2\theta \ldots \cos 2^{k-1}\theta\right]$$

$$= \frac{1}{2^{k-1}\sin\theta}\left[\sin 2^{k-1}\theta \cos 2^{k-1}\theta\right] = \frac{1}{2^k\sin\theta}\sin(2^k\theta)$$

When $\theta = \dfrac{\pi}{2^k+1}$,

$$\sin(2^k\theta) = \sin\left(\frac{2^k\pi}{2^k+1}\right)$$

$$= \sin\left(\pi - \frac{\pi}{2^k+1}\right) = \sin\left(\frac{\pi}{2^k+1}\right) = \sin\theta$$

$\therefore$ the given expression $= \dfrac{1}{2^k}$

5. We have $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

$$\Rightarrow \cos^3\alpha = \frac{1}{4}(\cos 3\alpha + 3\cos\alpha)$$

$$\text{L.H.S.} = \frac{1}{4}(\cos 3\alpha + 3\cos\alpha)$$

$$+\frac{1}{4}\left(\cos(2\pi+3\alpha) + 3\cos\left(\frac{2\pi}{3}+\alpha\right)\right)$$

$$+\frac{1}{4}\left(\cos(4\pi+3\alpha) + 3\cos\left(\frac{4\pi}{3}+\alpha\right)\right)$$

$$= \frac{3}{4}\cos 3\alpha +$$

$$\frac{3}{4}\left(\cos\alpha + \cos\left(\frac{2\pi}{3}+\alpha\right) + \cos\left(\frac{2\pi}{3}-\alpha\right)\right)$$

as $\cos\left(\dfrac{4\pi}{3}+\alpha\right) = \cos\left(2\pi - \left(\dfrac{2\pi}{3}-\alpha\right)\right)$

$$= \cos\left(\frac{2\pi}{3}-\alpha\right)$$

$$= \frac{3}{4}\cos 3\alpha + \frac{3}{4}\left(\cos\alpha + 2\cos\frac{2\pi}{3}\cos\alpha\right)$$

$$= \frac{3}{4}\cos 3\alpha + \frac{3}{4}(\cos\alpha - \cos\alpha) = \frac{3}{4}\cos 3\alpha$$

$= \text{R.H.S.}$

6. $\sqrt{2}\cos\alpha = \cos\beta + \cos^3\beta$ — (1)

$\sqrt{2}\sin\alpha = \sin\beta - \sin^3\beta$ — (2)

$(1)^2 + (2)^2$

$$\Rightarrow 2 = 1 + \cos^6\beta + \sin^6\beta + 2(\cos^4\beta - \sin^4\beta)$$

$$= 1 + (\cos^2\beta + \sin^2\beta)^3$$

$$- 3\cos^2\beta\sin^2\beta(\cos^2\beta + \sin^2\beta)$$

$$+ 2(\cos^2\beta + \sin^2\beta)(\cos^2\beta - \sin^2\beta)$$

$$= 1 + 1 - \frac{3}{4}\sin^2 2\beta + 2\cos 2\beta$$

$$= 2 - \frac{3}{4}(1-\cos^2 2\beta) + 2\cos 2\beta$$

$$\Rightarrow 8 = 5 + 3\cos^2 2\beta + 8\cos 2\beta$$

$$\Rightarrow 3\cos^2 2\beta + 8\cos 2\beta - 3 = 0$$

$$\Rightarrow \cos 2\beta = -3 \text{ or } \frac{1}{3}$$

Since $\cos 2\beta \neq -3$, we have $\cos 2\beta = \dfrac{1}{3}$

$$\Rightarrow \sin^2 2\beta = 1 - \frac{1}{9} = \frac{8}{9} \Rightarrow \sin 2\beta = \pm\frac{2\sqrt{2}}{3}$$

Again, (2) $\cos\beta$ – (1) $\sin\beta$

$$\Rightarrow \sqrt{2}\sin(\alpha-\beta) = \cos\beta\left[\sin\beta - \sin^3\beta\right]$$

$$-\sin\beta\left[\cos\beta + \cos^3\beta\right]$$

$$= -\cos\beta\sin^3\beta - \sin\beta\cos^3\beta = -\cos\beta\sin\beta$$

$$= \frac{1}{2}\sin 2\beta$$

$$\sin(\alpha-\beta) = \frac{-1}{2\sqrt{2}}\sin 2\beta$$

$\sin(\alpha-\beta) = \pm\dfrac{1}{3}$, by substituting for $\sin 2\beta$

7. $\tan^{-1}(\cot\theta) + \tan^{-1}(\cot^3\theta) = \tan^{-1}\left(\dfrac{\cot\theta + \cot^3\theta}{1-\cot^4\theta}\right)$

$$= \tan^{-1}\frac{\cot\theta(1+\cot^2\theta)}{(1+\cot^2\theta)(1-\cot^2\theta)}$$

$$= \tan^{-1}\frac{\cos\theta}{\sin\theta}\frac{\sin^2\theta}{\sin^2\theta - \cos^2\theta}$$

$$= \tan^{-1}\left(-\frac{1}{2}\tan 2\theta\right) = -\tan^{-1}\left(\frac{1}{2}\tan 2\theta\right)$$

$$\therefore \tan^{-1}\left(\frac{1}{2}\tan 2\theta\right) + \tan^{-1}(\cot\theta) + \tan^{-1}(\cot^3\theta) = 0$$

8. $\tan^{-1}(2x) + \tan^{-1}(3x) = \tan^{-1}\left(\frac{5x}{1-6x^2}\right)$ provided

$x > 0, 6x^2 < 1$

$$\tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \tan^{-1}(1) \Rightarrow \frac{5x}{1-6x^2} = 1$$

$$\Rightarrow (x+1)(6x-1) = 0 \Rightarrow x = -1 \text{ or } \frac{1}{6}$$

$x = -1$ not admissible as in this case $6x^2 > 1$

$x = \frac{1}{6}$ is the only solution.

Note that if $6x^2 > 1$,

$$\tan^{-1}(2x) + \tan^{-1}(3x) = \pi + \tan^{-1}\left(\frac{5x}{1-6x^2}\right) \neq \frac{\pi}{4}$$

9. Equation is $(\cos x + \cos 7x) + (\cos 3x + \cos 5x) = 0$

$\Rightarrow$ $2\cos 4x \cos 3x + 2\cos 4x \cos x = 0$

$\Rightarrow$ $2\cos 4x (\cos 3x + \cos x) = 0$

$\Rightarrow$ $\cos 4x \cos 2x \cos x = 0,$

giving $\cos x = 0$ or $\cos 2x = 0$ or $\cos 4x = 0$.

The general solution is $x = 2n\pi \pm \frac{\pi}{2}$

or $2x = 2n\pi \pm \frac{\pi}{2}$ or $4x = 2n\pi \pm \frac{\pi}{2}$.

$\Rightarrow$ $x = 2n\pi \pm \frac{\pi}{2}, n\pi \pm \frac{\pi}{4}, \frac{n\pi}{2} \pm \frac{\pi}{8}$, where n is an integer.

10. $7\cos 2\theta - (2\cos^2 2\theta - 1) = 6$

Or $2\cos^2 2\theta - 7\cos 2\theta + 5 = 0$

$\Rightarrow$ $\cos 2\theta = 1, \frac{5}{2}$.

But $\cos 2\theta$ cannot be equal to $\frac{5}{2}$

Therefore $\cos 2\theta = 1$.

$2\theta = 2n\pi \Rightarrow \theta = n\pi$, n is an integer.

Sum of solutions in $(\pi, 45\pi)$

$= 2\pi + 3\pi + 4\pi + \ldots.. + 44\pi$

$= (2 + 3 + 4 + \ldots.. + 44)\pi = 989\pi$

11. $\sin x = 2\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)$

$$= 2\cos\left(\frac{x}{2}\right)2\cos\left(\frac{x}{4}\right)\sin\left(\frac{x}{4}\right)$$

$$= 2\cos\left(\frac{x}{2}\right)2\cos\left(\frac{x}{4}\right)2\cos\left(\frac{x}{8}\right)\sin\left(\frac{x}{8}\right)$$

$$\therefore \quad \frac{\sin x}{\sin\left(x/8\right)} = 8\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)$$

12. $(\sin^2 A + 1 - 2\sin A) = 0$

$\Rightarrow (\sin A - 1)^2 = 0$

$\sin A = 1 = \operatorname{cosec} A$

$\sin^n A + \operatorname{cosec}^n A = (1)^n + (1)^n = 1 + 1 = 2$

13. $\tan^{-1}\frac{a-b}{1+ab} = \tan^{-1} a - \tan^{-1} b$

$\tan^{-1}\frac{b-c}{1+bc} = \tan^{-1} b - \tan^{-1} c$

Required sum $= \tan^{-1} a - \tan^{-1} c = \tan^{-1}\frac{a-c}{1+ac}$.

14. $3\sin\theta + 4\cos\theta = 5$

Let $3\cos\theta - 4\sin\theta = x$

Square and add $9 + 16 = 25 + x^2 \Rightarrow x = 0$.

15. $4\cos\left(\theta + \frac{\pi}{3}\right) + 3\sin\left(\theta + \frac{\pi}{3}\right)$

$= 4\cos\phi + 3\sin\phi\ ;\ \phi = \theta + \frac{\pi}{3}$

This expression lies between $-\sqrt{4^2 + 3^2}$ and $\sqrt{4^2 + 3^2}$ i.e., between -5 and 5

16. Statement 2 is correct

The equation given in statement 1 gives $\sin x = 2$ and 3. As $\sin x$ can never exceed 1, the equation has no real roots. Hence statement 1 is correct.

Choice (a)

17. Statement 2 is true

Consider Statement 1:

$(1 + \cot A)(1 + \cot B)$

$$= \left(1 + \frac{1}{\tan A}\right)\left(1 + \frac{1}{\tan B}\right)$$

$$= \frac{(1+\tan A)(1+\tan B)}{\tan A \tan B} = \frac{2}{\tan A \tan B}, \text{ using statement 2}$$

$\Rightarrow$ statement 1 is true

Choice (a)

18. Statement 2 is true

Consider Statement 1

$\sin^2 20° + \sin^2 40° + \sin^2 60° + \sin^2 80°$

$= \sin^2 20° + \sin^2 40° + \frac{3}{4} + \sin^2 80°$

$$= \frac{3}{4} + \frac{1}{2}(1 - \cos 40° + 1 - \cos 80° + 1 - \cos 160°)$$

$$= \frac{3}{2} + \frac{3}{4} - \frac{1}{2}\left[2\cos 60° \cos 20° + \cos(180° - 20°)\right]$$

$$= \frac{9}{4} - \frac{1}{2}\left(2.\frac{1}{2}.\cos 20° - \cos 20°\right) = \frac{9}{4}$$

$\therefore$ Statement 1 is false

Choice (d)

19. Statement 2 is true

Consider Statement 1

$$\tan^{-1}\left(-\sqrt{3}\right) = \frac{-\pi}{3}$$

$$\cot^{-1}\left(-\sqrt{3}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$\Rightarrow$ Statement 1 is false

Choice (d)

20. Statement 2 is false

Consider Statement 1

Dividing by $\sqrt{2}$,

$$\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \frac{1}{2\sqrt{2}}$$

$$\cos\left(\frac{\pi}{4} - x\right) = \frac{1}{2\sqrt{2}}$$ which has real solutions

$\therefore$ Statement 1 is true

Choice (c)

21. In (1), putting $\beta = \alpha$ we get

$$S = \frac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}} \sin\left[\frac{n+1}{2}\alpha\right]$$

22. Take $\alpha = 20°$; $\beta = 60°$; $n = 10$ in (1)

$$S = \frac{\sin(10 \times 30^\circ)\sin\left[\frac{40 + 540}{2}\right]}{\sin 30°} = \sqrt{3}\cos 20°$$

23. Let $S = \sin^2 5° + \sin^2 10° + \ldots + \sin^2 90°$

$$= \frac{1 - \cos 10°}{2} + \frac{1 - \cos 20°}{2} + \ldots + \frac{1 - \cos 180°}{2}$$

$$= \frac{1}{2}\left[18 - \left(\cos 10° + \cos 20° + \ldots + \cos 180°\right)\right]$$

$\therefore$ $2S = 18 - (\cos 10° + \cos 20° + \ldots + \cos 180°)$

Now consider $C = \cos\alpha + \cos(\alpha + \beta) + \ldots$
$+ \cos[\alpha + (n - 1)\beta)]$

$$2\sin\frac{\beta}{2}C = 2\cos\alpha\sin\frac{\beta}{2} + 2\cos(\alpha + \beta)\sin\frac{\beta}{2} + \ldots\ldots$$
$$+2\cos[\alpha + (n-1)\beta]\sin\frac{\beta}{2}$$

$$= \left[\sin\left(\alpha + \frac{\beta}{2}\right) - \sin\left(\alpha - \frac{\beta}{2}\right)\right]$$
$$+\left[\sin\left(\alpha + \frac{3\beta}{2}\right) - \sin\left(\alpha + \frac{\beta}{2}\right)\right] \ldots\ldots +$$
$$\left[\sin\left(\alpha + \left(n - \frac{1}{2}\right)\beta\right)\right] - \sin\left[\alpha + \left(n - \frac{3}{2}\right)\beta\right]$$

$$= \sin\left(\alpha + \left(n - \frac{1}{2}\right)\beta\right) - \sin\left(\alpha - \frac{\beta}{2}\right)$$

$$= 2\cos\left(\frac{2\alpha + (n-1)\beta}{2}\right)\sin\frac{n\beta}{2}$$

$$C = \frac{\sin\frac{n\beta}{2}\cos\left(\frac{2\alpha + (n-1)\beta}{2}\right)}{\sin\left(\beta/2\right)}$$

Now, using this result we have

$$2S = 18 - \frac{\sin\left(\frac{18 \times 10°}{2}\right)\cos\left(\frac{2 \times 10° + 17 \times 10°}{2}\right)}{\sin\left(\frac{10°}{2}\right)}$$

$$= 18 - \frac{\cos 95°}{\sin 5°}$$

$$= 18 + \frac{\sin 5°}{\sin 5°} = 19$$

$$\therefore\ S = \frac{19}{2}$$

24. Let $\cot^{-1} x = A$ and $\cot^{-1} y = B$

$\Rightarrow$ $\cot A = x$ and $\cot B = y$

$$\cot(A - B) = \frac{1 + \cot A \cot B}{\cot B - \cot A} = \frac{1 + xy}{y - x}$$

$$\therefore\ A - B = \cot^{-1}\left(\frac{1 + xy}{y - x}\right)$$

25. $b^2 \sec^2\theta = c^2 - 2ac\tan\theta + a^2\tan^2\theta$

$\tan^2\theta\,(a^2 - b^2) - 2ac\tan\theta + c^2 - b^2 = 0$

$$\tan\theta_1 + \tan\theta_2 = \frac{2ac}{a^2 - b^2}$$

$$\tan\theta_1 \tan\theta_2 = \frac{c^2 - b^2}{a^2 - b^2}$$

$$\frac{2ac}{a^2 - c^2} = \frac{\frac{2ac}{a^2 - b^2}}{\frac{\left(a^2 - b^2\right) - \left(c^2 - b^2\right)}{a^2 - b^2}}$$

$$= \frac{\frac{2ac}{a^2 - b^2}}{1 - \frac{c^2 - b^2}{a^2 - b^2}}$$

$$= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$= \tan(\theta_1 + \theta_2)$$

26. Let $A = \tan^{-1} x$; $B = \tan^{-1} y$; $C = \tan^{-1} z$

$\therefore \quad A + B + C = 180^0$

$\therefore \quad 2A + 2B + 2C = 2\pi; \qquad \tan(2A + 2B + 2C) = 0$

$\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$

$$\frac{2\tan A}{1 - \tan^2 A} + \frac{2\tan B}{1 - \tan^2 B} + \frac{2\tan C}{1 - \tan^2 C}$$

$$= \frac{2\tan A}{1 - \tan^2 A} \cdot \frac{2\tan B}{1 - \tan^2 B} \cdot \frac{2\tan C}{1 - \tan^2 C}$$

$$\sum 2\tan A\left(1 - \tan^2 B\right)\left(1 - \tan^2 C\right)$$

$$= 8 \tan A \tan B \tan C$$

$$\sum x(1-y^2)(1-z^2) = 4xyz$$

27. $\tan A \tan B = 2$

$$\frac{\sin A \sin B}{\cos A \cos B} = 2$$

$\sin A \sin B = 2 \cos A \cos B$ —(1)

$$\cos(A - B) = \frac{3}{5}$$

$$\cos A \cos B + \sin A \sin B = \frac{3}{5}$$

$$\cos A \cos B + 2\cos A \cos B = \frac{3}{5}$$

$$3\cos A \cos B = \frac{3}{5}$$

$$\cos A \cos B = \frac{1}{5}$$

(a) is correct

From (1), $\sin A \sin B = 2\left(\frac{1}{5}\right) = \frac{2}{5}$

(b) and (d) are not correct

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$= \frac{1}{5} - \frac{2}{5} = -\frac{1}{5}$$

(c) is correct

28. $\sin^{-1}(1 - x) = \frac{\pi}{2} - 2\sin^{-1} x$

$\Rightarrow \quad (1 - x) = \sin\left(\frac{\pi}{2} - 2\sin^{-1} x\right)$

$\Rightarrow \quad 1 - x = \cos(2\sin^{-1}x) = 1 - 2x^2$

$\Rightarrow \quad 2x^2 - x = 0$

$\Rightarrow \quad x = 0, x = \frac{1}{2}$

29. $\Delta = 64 - 4.4^{\sec^2\alpha + 1}\,[(2\tan\beta - 2)^2 + 1) \geq 0$

$64 - 16.4^{\sec^2\alpha}\,[(2\tan\beta - 2)^2 + 1) \geq 0$

$4^{\sec^2\alpha}\,[(2\tan\beta - 2)^2 + 1) \leq 4$

But $4^{\sec^2\alpha} \geq 4$ and $(2\tan\beta - 2)^2 + 1 \geq 1$

$\Rightarrow \quad 4^{\sec^2\alpha} = 4$ and $\tan\beta = 1$

$\Rightarrow \quad \alpha = n\pi, \beta = n\pi + \frac{\pi}{4}$

In each case $x = \frac{1}{4}$

30. (a) Put $\sin^{-1}x = \theta \Rightarrow x = \sin\theta$.

Then $\sin^{-1}x + \sin^{-1}(1 - x) = \cos^{-1}x$ becomes

$\theta + \sin^{-1}(1 - \sin\theta) = \cos^{-1}(\sin\theta)$.

$\Rightarrow \quad \theta + \sin^{-1}(1 - \sin\theta) = \frac{\pi}{2} - \theta$

$\Rightarrow \quad \sin^{-1}(1 - \sin\theta) = \frac{\pi}{2} - 2\theta$

$\Rightarrow \quad 1 - \sin\theta = \sin\left(\frac{\pi}{2} - 2\theta\right)$

$= \cos 2\theta$

$= 1 - 2\sin^2\theta$

$\Rightarrow \quad 2\sin^2\theta - \sin\theta = 0$

$\Rightarrow \quad 2x^2 - x = 0$

$\Rightarrow \quad x = 0$ or $\frac{1}{2}$

$\therefore$ (a) $\to$ (q), (r)

(b) $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$

$\Rightarrow \quad (\tan^{-1}x + \cot^{-1}x)^2$

$- 2\tan^{-1}x\left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{5\pi^2}{8}$

$\Rightarrow \quad \frac{\pi^2}{4} - \pi\tan^{-1} x + 2\left(\tan^{-1} x\right)^2 = \frac{5\pi^2}{8}$

$\Rightarrow \quad 2(\tan^{-1}x)^2 - \pi\tan^{-1}x - \frac{3\pi^2}{8} = 0$

$\Rightarrow \quad \left(\tan^{-1} x + \frac{\pi}{4}\right)\left(2\tan^{-1} x - \frac{3\pi}{2}\right) = 0$

$\Rightarrow \quad \tan^{-1}x = -\frac{\pi}{4}, \frac{3\pi}{4}$

$\Rightarrow \quad x = -1$

(b) $\to$ (s)

(c) $4\sin^{-1}x + \cos^{-1}x = \pi$

$\Rightarrow$ $3\sin^{-1}x + \sin^{-1}x + \cos^{-1}x = \pi$

$\Rightarrow$ $3\sin^{-1}x = \dfrac{\pi}{2}$

$\Rightarrow$ $\sin^{-1}x = \dfrac{\pi}{6}$

$\therefore\ x = \sin\dfrac{\pi}{6} = \dfrac{1}{2}$

(c) $\rightarrow$ (q)

(d) $\cot^{-1}x + \sin^{-1}\left(\dfrac{1}{\sqrt{5}}\right) = \dfrac{\pi}{4}$

$\tan^{-1}\dfrac{1}{x} + \tan^{-1}\dfrac{1}{2} = \dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{\dfrac{1}{x}+\dfrac{1}{2}}{1-\dfrac{1}{2x}}\right) = \dfrac{\pi}{4}$

$\dfrac{2+x}{2x-1} = \tan\dfrac{\pi}{4}$

$2 + x = 2x - 1$

$x = 3$

(d) $\rightarrow$ (p)

IIT Assignment Exericse

31. $\sin(-420°)\cos(390°) + \cos(-660°)\sin(330°)$

$= -\sin(90° \times 4 + 60°) . \cos(90° \times 4 + 30°) +$

$\cos(90° \times 7 + 30°) .\sin(90° \times 3 + 60°)$

$= -(\sin 60°)(\cos 30°) \quad + (\sin 30°)(-\cos 60°)$

$= -\sin 90° = -1$

32. Opposite angles of a cyclic quadrilateral ABCD are supplementary.

$A = \pi - C,\ B = \pi - D$

$\cos A = -\cos C;\ \cos B = -\cos D$

$\therefore$ $\cos A + \cos B = -\cos C + -\cos D$

$\Rightarrow$ $\cos A + \cos B + \cos C + \cos D = 0$

33. $\sin\dfrac{\pi}{6}\sin\dfrac{2\pi}{6}\sin\dfrac{4\pi}{6}\sin\dfrac{5\pi}{6}$

$= \sin\dfrac{\pi}{6}\sin\dfrac{\pi}{3}\sin\dfrac{2\pi}{3}\sin\dfrac{\pi}{6} = \sin^2\dfrac{\pi}{6}\sin\dfrac{\pi}{3}\sin\dfrac{2\pi}{3}$

$= \dfrac{1}{4} \times \dfrac{\sqrt{3}}{2} \times \dfrac{\sqrt{3}}{2} = \dfrac{3}{16}$

34. Numerator

$= \left(\sin^2 5° + \sin^2 85°\right) + \left(\sin^2 10° + \sin^2 80°\right)$

$+ ... + \left(\sin^2 40° + \sin^2 50°\right) + \sin^2 45° + \sin^2 90°$

$= \left(\sin^2 5° + \cos^2 5°\right) + .. + \left(\sin^2 40° + \cos^2 40°\right) + \dfrac{1}{2} + 1$

$= 9.5$

Denominator

$= 1 + \left(\cos^2 5° + \cos^2 85°\right) + ... + \left(\cos^2 40° + \cos^2 50°\right)$

$+ \cos^2 45^0 + \cos^2 90^0 = 9.5$

Answer is 1

35. If ΔABC is equilateral $A = B = C = \dfrac{\pi}{3}$

$\therefore\ \tan A + \tan B + \tan C = 3\sqrt{3}$.

36. $\dfrac{\tan(\theta + 120°)}{\tan(\theta - 30°)} = \dfrac{m}{n}$

$\dfrac{\sin(\theta + 120°)\cos(\theta - 30°)}{\cos(\theta + 120°)\sin(\theta - 30°)} = \dfrac{m}{n}$

Now applying compenendo dividendo, we get

$\dfrac{\sin(\theta + 120° + \theta - 30°)}{\sin(\theta + 120° - \theta + 30°)} = \dfrac{m+n}{m-n}$

$\Rightarrow$ $2\cos 2\theta = \dfrac{m+n}{m-n}$ as $\sin 150° = \cos 60°$

37. Since $\sin\theta$ and $\cos\theta$ are the roots of the given quadratic equation, we have

$\sin\theta + \cos\theta = \dfrac{b}{a}$ and $\sin\theta.\cos\theta = \dfrac{c}{a}$

$\Rightarrow (\sin\theta + \cos\theta)^2 = \dfrac{b^2}{a^2}$

$\Rightarrow \sin^2\theta + \cos^2\theta + 2\sin\theta.\cos\theta = \dfrac{b^2}{a^2}$

$1 + \dfrac{2c}{a} = \dfrac{b^2}{a^2}$

$\Rightarrow$ $a^2 + 2ac - b^2 = 0$

38. $\tan 13\theta = \tan(9\theta + 4\theta)$

$= \dfrac{\tan 9\theta + \tan 4\theta}{1 - \tan 9\theta \tan 4\theta}$

$\Rightarrow$ $\tan 13\theta - \tan 9\theta - \tan 4\theta = \tan 9\theta \tan 4\theta \tan 13\theta$

39. Let $\alpha - \beta, \alpha, \alpha + \beta$ be the angles

Given that the greatest angle = sum of other two angles

$\therefore$ $\alpha + \beta = \alpha - \beta + \alpha$

$\Rightarrow \quad \beta = \alpha/2$

Now, $\dfrac{\alpha}{2} + \alpha + \dfrac{3\alpha}{2} = 180°$

$\Rightarrow \quad \alpha = 60°$

$\Rightarrow \quad 3\alpha/2 = 90°$

$\Rightarrow$ Triangle is right angled.

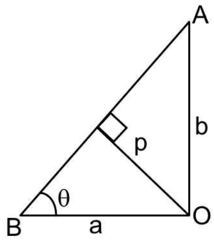
40. Let $\angle B = \theta$ and let a, b be the sides.

$\Rightarrow \quad \angle A = 90° - \theta$

$\therefore \; p/a = \sin\theta$

$p = a \sin\theta$

$p = b \cos\theta$



$$\Rightarrow \frac{p^2}{a^2} + \frac{p^2}{b^2} = 1$$

$$p^2\left[\frac{a^2 + b^2}{a^2 b^2}\right] = 1$$

$$p = \frac{ab}{\sqrt{a^2 + b^2}} \Rightarrow p = \frac{ab}{AB}$$

Now,

$$AB = 4p = \frac{4ab}{AB} \Rightarrow AB^2 = 4ab$$

From ΔAOB, $\sin\theta . \cos\theta = \dfrac{b}{AB}.\dfrac{a}{AB} = \dfrac{ab}{AB^2} = \dfrac{1}{4}$

$$\Rightarrow \quad \frac{1}{2} = \sin 2\theta$$

i.e., $\theta = 15°$.

41. $\sin\left(\dfrac{\pi}{14}\right) = \sin\left(\dfrac{\pi}{2} - 6\dfrac{\pi}{14}\right) = \dfrac{\pi}{14}$

$$= \cos\frac{6\pi}{14} = \cos\left(\pi - \frac{8\pi}{14}\right)$$

$$= -\cos\frac{8\pi}{14}$$

$$\sin\frac{3\pi}{14} = \sin\left(\frac{\pi}{2} - \frac{4\pi}{14}\right) = \cos\frac{4\pi}{14}$$

$$\sin\frac{5\pi}{14} = \sin\left(\frac{\pi}{2} - \frac{2\pi}{14}\right) = \cos\frac{2\pi}{14}$$

$$\therefore \text{L.H.S} = -\cos\frac{2\pi}{14}.\cos\frac{4\pi}{14}.\cos\frac{8\pi}{14}$$

$$= -\frac{1}{2^3 \sin\left(\dfrac{2\pi}{14}\right)}.\sin\left(2^3.\frac{2\pi}{14}\right)$$

$$= -\frac{1}{8\sin\left(\dfrac{\pi}{7}\right)}.\sin\left(\frac{8\pi}{7}\right)$$

$$= \frac{-1}{8\sin\dfrac{\pi}{7}}.\sin\left(\pi + \frac{\pi}{7}\right)$$

$= -\dfrac{1}{8}(-1) = \dfrac{1}{8}$ since $[(\sin \pi + \theta) = -\sin\theta]$

42. $2\tan\dfrac{7\pi}{6} = 2\tan\left(\pi + \dfrac{\pi}{6}\right) = 2\tan\dfrac{\pi}{6} = \dfrac{2}{\sqrt{3}}$

$$4\tan\frac{9\pi}{4} = 4\tan\left(2\pi + \frac{\pi}{4}\right) = 4\tan\frac{\pi}{4} = 4\,;$$

$$8\tan\frac{10\pi}{3} = 8\tan\left(3\pi + \frac{\pi}{3}\right) = 8\tan\frac{\pi}{3} = 8\sqrt{3}$$

They are in G.P.

43. Let $\sec x - \tan x = k \Rightarrow \sec x + \tan x = \dfrac{1}{k}$

Addition gives $2\sec x = k + \dfrac{1}{k} = 2a + \dfrac{1}{2a}$

$\Rightarrow \quad k = 2a$ or $\dfrac{1}{2a}$

44. $\sec^2\theta = \dfrac{4xy}{(x + y)^2} \geq 1$

$\therefore 4xy \geq (x + y)^2 \Rightarrow (x - y)^2 \leq 0$

This is possible only when x = y

45. Since $\tan x = -\dfrac{4}{3}$ and sin x is positive in 2nd quadrant, we get

$$\sin x = \frac{4}{5}$$

$$\Rightarrow \quad 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{4}{5}$$

$$\Rightarrow \quad \sin\frac{x}{2}\cos\frac{x}{2} = \frac{2}{5}$$

46. $\sqrt{2 + \sqrt{2 + \sqrt{2(1 + \cos 8\theta)}}}$

$$= \sqrt{2 + \sqrt{2 + \sqrt{2.2\cos^2 4\theta}}}$$

$$= \sqrt{2 + \sqrt{2 + \sqrt{4\cos^2 4\theta}}}$$

$$= \sqrt{2 + \sqrt{2 + 2\cos 4\theta}}$$

$$= \sqrt{2 + \sqrt{2.2\cos^2 2\theta}}$$

$$= \sqrt{2(1 + \cos 2\theta)} = 2\cos\theta$$

47. $\left[2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right]^2$

$+\left[2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2$

$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left[\cos^2\left(\frac{\alpha+\beta}{2}\right)+\sin^2\left(\frac{\alpha+\beta}{2}\right)\right]$

$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$

48. L.H.S $= 4\sin\frac{3\pi}{10}\sin\frac{\pi}{10}$

$= 4\cos 36\sin 18 = 4\,\frac{\sqrt{5}+1}{4}.\frac{\sqrt{5}-1}{4} = 1$

49. $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}$

$= \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\left(\pi-\frac{2\pi}{5}\right)\sin\left(\pi-\frac{\pi}{5}\right)$

$= \sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5}$

$= \left(\frac{10-2\sqrt{5}}{16}\right)\left(\frac{10+2\sqrt{5}}{16}\right)$

$= \frac{100-20}{16\times 16} = \frac{80}{16\times 16} = \frac{5}{16}$

50. $\cos^2 48° - \sin^2 12°$

$= \frac{1}{2}\left(1+\cos 96° - (1-\cos 24°)\right)$

$= \frac{1}{2}\left(\cos 96° + \cos 24°\right)$

$= \frac{1}{2}\left(2\cos 60°\cos 36°\right)$

$= \frac{1}{2}\times\left(2\times\frac{1}{2}\times\frac{\sqrt{5}+1}{4}\right) = \frac{\sqrt{5}+1}{8}$

51. $\sin 20°\sin 40°\sin 60°\sin 80°$

$= \sin 60°\sin 20°\sin 40°\sin 80°$

$= \frac{\sqrt{3}}{2}\sin 20°\sin(60°-20°)\sin(60°+20°)$

$(\sin\theta\sin(60°-\theta)\sin(60°+\theta) = \frac{\sin 3\theta}{4})$

$= \frac{\sqrt{3}}{2}\frac{\sin(3.20°)}{4} = \frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}\times\frac{1}{4} = \frac{3}{16}$

52. $\sin A - \sin B + \sin C$

$= 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} + \sin C$

$= 2\sin\frac{C}{2}\sin\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$

$= 2\sin\frac{C}{2}\left[\sin\frac{A-B}{2} + \sin\frac{A+B}{2}\right]$

$= 2\left(\sin\frac{C}{2}\right)\left(2\sin\frac{A}{2}\cos\frac{B}{2}\right)$

$= 4\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

53. Let $\cot^{-1} x = \theta \Rightarrow x = \cot\theta$

$\sin\theta = \frac{1}{\sqrt{1+x^2}}$; $\theta = \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

Given expression

$= \cos\left[\tan^{-1}\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right]$

$= \cos\left[\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right] = \cos\left[\cos^{-1}\sqrt{\frac{x^2+1}{x^2+2}}\right]$

$= \sqrt{\frac{x^2+1}{x^2+2}}$

54. $A = \tan^{-1} x$

$\therefore \tan A = x$

$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A} = \frac{1-x^2}{1+x^2}$

$\therefore \sec 2A = \frac{1+x^2}{1-x^2}$

55. $\cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\frac{3}{4}$

$\therefore \tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$

$= \tan\left[\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$

$= \tan\left\{\tan^{-1}\left[\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right]\right\}$

$= \tan\left[\tan^{-1}\left(\frac{17}{6}\right)\right] = \frac{17}{6}.$

56. Let

$\alpha = \tan^{-1}\frac{1}{3}, \beta = \tan^{-1}\frac{1}{5}, r = \tan^{-1}\frac{1}{7}, s = \tan^{-1}x$

$\Rightarrow \quad \tan\alpha = \frac{1}{3}, \tan\beta = \frac{1}{5}, \tan r = \frac{1}{7}, \tan s = x$

$\alpha + \beta + r = s \Rightarrow \alpha + \beta = s - r$

$\tan(\alpha + \beta) = \tan(s - r)$

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\tan s - \tan r}{1 + \tan s\tan r}$$

$$\frac{\frac{1}{3} + \frac{1}{5}}{1 - \frac{1}{3}\times\frac{1}{5}} = \frac{x - \frac{1}{7}}{1 + x \times \frac{1}{7}} \Rightarrow \frac{8}{14} = \frac{7x - 1}{7 + x}$$

$\Rightarrow \quad 4x + 28 = 49x - 7 \Rightarrow 45x = 35 \Rightarrow x = \frac{7}{9}$

Aliter:

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\frac{D_1 - D_3}{1 - D_2}$$

where $D_1 = x + y + z$; $D_2 = xy + yz + xz$;
$D_3 = xyz$

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\left[\frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{3}.\frac{1}{7}.\frac{1}{5}\right]}{1 - \left[\frac{1}{15} + \frac{1}{35} + \frac{1}{21}\right]}$$

$$= \tan^{-1}\left[\frac{\frac{71}{105} - \frac{1}{105}}{1 - \frac{15}{105}}\right] = \tan^{-1}\frac{7}{9} \Rightarrow x = \frac{7}{9}.$$

57. $\tan^{-1}(a) + \tan^{-1}(b) = \tan^{-1}x$

$$\tan^{-1}\left(\frac{a + b}{1 - ab}\right) = \tan^{-1}x \Rightarrow x = \frac{a + b}{1 - ab}.$$

58. Let $\cos^{-1}\frac{\sqrt{5}}{3} = t \Rightarrow \cos t = \frac{\sqrt{5}}{3}$

$$\sin t = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$$

$$\tan\left(\frac{t}{2}\right) = \frac{1 - \cos t}{\sin t} = \frac{1 - \frac{\sqrt{5}}{3}}{2/3} = \frac{3 - \sqrt{5}}{2}.$$

59. Given expression

$$= \sin\left[\cot^{-1}\cos\cos^{-1}\frac{1}{\sqrt{1 + x^2}}\right]$$

$$= \sin\left[\cot^{-1}\frac{1}{\sqrt{1 + x^2}}\right]$$

$$= \sin\left[\tan^{-1}\left(\sqrt{1 + x^2}\right)\right]$$

$$= \sin\left[\sin^{-1}\left(\frac{\sqrt{1 + x^2}}{\sqrt{1 + 1 + x^2}}\right)\right]$$

$$= \sin\sin^{-1}\left(\sqrt{\frac{1 + x^2}{2 + x^2}}\right) = \sqrt{\frac{1 + x^2}{2 + x^2}}.$$

60. $\cot(\cot^{-1}7 + \cot^{-1}8) = \frac{7.8 - 1}{7 + 8} = \frac{55}{15} = \frac{11}{3}$

$$\cot\left(\cot^{-1}\frac{11}{3} + \cot^{-1}18\right) = \frac{\frac{11}{3}.18 - 1}{\frac{11}{3} + 18} = \frac{66 - 1}{\frac{65}{3}} = 3$$

61. Let $x = \tan\theta$

$$\frac{2x}{1 - x^2} = \left(\frac{2\tan\theta}{1 - \tan^2\theta}\right) < 0 \quad \theta = \tan^{-1}x$$

$$\frac{-\pi}{2} < \tan^{-1}x < \frac{\pi}{2} \quad 1 - \tan^2\theta < 0$$

Therefore, we must add π to $\tan^{-1}\frac{2x}{1 - x^2}$ in order to obtain $2\tan^{-1}x$. This is in order to satisfy the range requirements for $2\tan^{-1}x$.

62. The inequality can be written as $\sin^{-1}x > \frac{\pi}{2} - \sin^{-1}x$.

$$\text{or } 2\sin^{-1}x > \frac{\pi}{2} \quad \text{or} \quad \sin^{-1}x > \frac{\pi}{4}.$$

Also, $\sin^{-1}x < \frac{\pi}{2}$ (domain) $\therefore \frac{\pi}{2} > \sin^{-1}x > \frac{\pi}{4}$

$$\therefore 1 > x > \frac{1}{\sqrt{2}}. \text{ i.e., } x \in \left(\frac{1}{\sqrt{2}}, 1\right).$$

63. $\frac{\sin(\pi\cos\theta)}{\cos(\pi\cos\theta)} = \frac{\cos(\pi\sin\theta)}{\sin(\pi\sin\theta)}$

Cross multiplying and shifting everything to left side gives

$\cos[\pi\cos\theta + \pi\sin\theta] = 0$

i.e., $\pi[\cos\theta + \sin\theta] = \pm\frac{\pi}{2}$

$\cos\theta + \sin\theta = \pm\frac{1}{2}$

$$\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = \pm\frac{1}{2\sqrt{2}}$$

i.e., $\cos\left(\theta - \frac{\pi}{4}\right) = \pm\frac{1}{2\sqrt{2}}$

64. $\tan a\theta = \tan b\theta$

$\therefore \quad a\theta = n\pi + b\theta$

$$\theta = \frac{n\pi}{a-b}$$

The values of θ are $0, \frac{\pi}{a-b}, \frac{2\pi}{a-b}, \ldots$which form an A.P.

65. From the given equation, we have

$$\tan(\cot\theta) = \tan\left(\frac{\pi}{2} - \tan\theta\right)$$

$\Rightarrow \quad \cot\theta = k\pi + \left(\frac{\pi}{2} - \tan\theta\right)$, k an integer

$$\Rightarrow \quad \cot\theta + \tan\theta = \frac{(2k+1)\pi}{2}$$

$$\Rightarrow \quad \frac{2}{\sin 2\theta} = \frac{(2k+1)\pi}{2}$$

$$\Rightarrow \quad \sin 2\theta = \frac{4}{(2k+1)\pi}$$

66. $2\cos x - 1 = 0$ or $3 + 2\cos x = 0$

i.e., $\cos x = \frac{1}{2}$

or $\cos x = -\frac{3}{2}$ (inadmissible)

$\therefore \quad \cos x = \frac{1}{2}$

$\therefore \quad$ The values of x in the interval $0 \le x \le 2\pi$ are $\left\{\frac{\pi}{3}, 5\frac{\pi}{3}\right\}$.

67. $\sin^2 x - 2\cos x + \frac{1}{4} = 0$

$1 - \cos^2 x - 2\cos x + \frac{1}{4} = 0$

$\cos^2 x + 2\cos x - \frac{5}{4} = 0$

$4\cos^2 x + 8\cos x - 5 = 0$

i.e., $(2\cos x + 5)(2\cos x - 1) = 0$

$\Rightarrow \quad \cos x = \frac{1}{2}$ or $\cos x = \frac{-5}{2}$ (inadmissible)

$\therefore\ x = 2n\pi \pm \frac{\pi}{3}$, where n is an integer.

68. $\sin 6\theta + \sin 4\theta = \sin 8\theta + \sin 2\theta$

i.e., $2\sin 5\theta \cos\theta = 2\sin 5\theta \cos 3\theta$

$\sin 5\theta(\cos\theta - \cos 3\theta) = 0$

i.e., $\sin 5\theta . 2\sin 2\theta . \sin\theta = 0$

i.e., $\sin 5\theta = 0$ or $\sin 2\theta = 0$ or $\sin\theta = 0$

i.e., $\theta = \frac{n\pi}{5}$ or $\theta = \frac{n\pi}{2}$ or $\theta = n\pi$, $n \in Z$

69. $X = 4\left\{\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right\}$

$$-3\left\{\sin\theta\cos\frac{\pi}{6} - \cos\theta\sin\frac{\pi}{6}\right\}$$

$$= 4\left\{\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right\} - 3\left\{\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right\}$$

$$= \frac{7}{2}\cos\theta - \frac{7\sqrt{3}}{2}\sin\theta = \frac{7}{2}\left[\cos\theta - \sqrt{3}\sin\theta\right]$$

$$\text{Maximum value} = \frac{7}{2} \times \sqrt{1+3} = 7$$

70. $\text{Max}(a\sin x + b\cos x) = \sqrt{a^2 + b^2}$

$\text{Min}(a\sin x + b\cos x) = -\sqrt{a^2 + b^2}$

Here $\sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = 5$

Max = 5 and Min = – 5

The equation is x^2 –(sum) x + product = 0

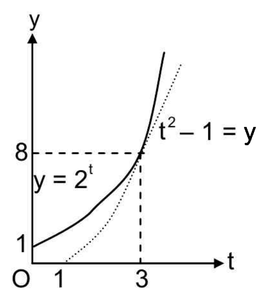
Here sum = 0, product = – 25

$\therefore \quad$ The equation becomes $x^2 - 25 = 0$

71. $(1 - \tan\theta)(1 + \tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$

$(1 - \tan^2\theta)(1 + \tan^2\theta) + 2^{\tan^2\theta} = 0$

$(1 - \tan^4\theta) + 2^{\tan^2\theta} = 0$



Let $t = \tan^2\theta$

$1 - t^2 + 2^t = 0 \Rightarrow 2^t = t^2 - 1$

From the diagram, $y = 2^t$ and $y = t^2 - 1$ intersect at $t = 3$

$t = 3 \Rightarrow \tan^2\theta = 3$

$\Rightarrow \quad \tan\theta = \pm\sqrt{3}$

$\therefore\ \theta = \pm\frac{\pi}{3}$

72. $\tan(x + 2x + \ldots + nx) = \tan\left[\frac{n(n+1)}{2}x\right]$

Its period is $\frac{2\pi}{n(n+1)}$.

73. $2\sin^2\theta + 2\sin\theta\cos\theta = 1$

i.e., $1 - \cos 2\theta + \sin 2\theta = 1$

i.e; $\sin 2\theta = \cos 2\theta$

i.e., $\cos\left(\frac{\pi}{2} - 2\theta\right) = \cos 2\theta$

i.e; $\frac{\pi}{2} - 2\theta = 2n\pi \pm 2\theta$

i.e., $\frac{\pi}{2} - 2n\pi = 2\theta \pm 2\theta$

i.e., $4\theta = \frac{\pi}{2} - 2n\pi$ or $0 = \frac{\pi}{2} - 2n\pi$ (not possible)

$\theta = \frac{\pi}{8} - \frac{n\pi}{2}$

74. $\frac{1}{2}\sin 2x.\cos 2x = k$

$\frac{1}{4}.\sin 4x = k$

$\sin 4x = 4k$

$|\sin 4x| \le 1$

$\therefore$ The solution exists only if k lies between $\frac{-1}{4}$ and $\frac{1}{4}$.

75. ($|\sin\theta| < 1$, sum to infinity of a G.P. $= \frac{a}{1-r}$)

$\frac{1}{1-\sin\theta} = 4 + 2\sqrt{3}$

$\therefore\ 1 - \sin\theta = \frac{1}{4+2\sqrt{3}} = \frac{4-2\sqrt{3}}{4}$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\therefore\ \theta = \frac{\pi}{3}, \frac{2\pi}{3}$.

76. $\frac{1+\sin x}{\cos x} = 2\cos x$

$2 - 2\sin^2 x - \sin x - 1 = 0$

$2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \frac{1}{2}$ or $\sin x = -1$

For $\sin x = -1$, $\cos x = 0$ which does not satisfy the given equation. So $\sin x = 1/2$

Between $[0, 2\pi]$, this has two solutions $\left(\frac{\pi}{6} \text{and} \frac{5\pi}{6}\right)$.

77. $(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x = a^2$

$1 - \frac{3}{4}\sin^2 2x = a^2$ Since $1 \ge \sin^2 2x \ge 0$,

we have $1 \ge 1 - \frac{3}{4}\sin^2 2x \ge \frac{1}{4} \Rightarrow 1 \ge a^2 \ge \frac{1}{4}$

$\Rightarrow\ \frac{1}{4} \le a^2 \le 1$

$a^2 \ge 1/4 \Rightarrow$ a must lie beyond $-1/2$ and $1/2$

$a^2 \le 1 \Rightarrow$ a must lie between -1 and $+1$

Therefore, $a \in \left[-1, \frac{-1}{2}\right]$ or $\left[\frac{1}{2}, 1\right]$

78. Let $t = \tan x$; then $t \in R$

$y = \frac{\sec^2 x - \tan x}{\sec^2 x + \tan x} = \frac{1 + t^2 - t}{1 + t^2 + t}$

$y(1 + t^2 + t) = 1 + t^2 - t$

$t^2(y-1) + t(y+1) + y - 1 = 0$

As $t \in R$, discriminant ≥ 0

$\therefore\ (y+1)^2 - 4(y-1)^2 \ge 0$

$[y + 1 + 2(y-1)]\,[y + 1 - 2(y-1)] \ge 0$

$(3y-1)(y-3) \le 0$

$\Rightarrow$ y lies between 3 and $\frac{1}{3}$.

$\therefore$ Max = 3, min = $\frac{1}{3}$

79. We know that

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Dividing by $\tan A \tan B \tan C$ throughout

$$\frac{1}{\tan B\tan C} + \frac{1}{\tan A\tan C} + \frac{1}{\tan A\tan B} = 1$$

i.e., $\cot B\cot C + \cot A\cot C + \cot A\cot B = 1$

80. $\sin^2 2x = \sin^2 3x - \sin^2 x$

$\sin^2 2x = (\sin 4x)(\sin 2x)$

$\Rightarrow\ \sin 2x = 0$ (or) $\sin 4x - \sin 2x = 0$

$2x = 0, \pm\frac{\pi}{2}, \pi$, $2\cos 3x\sin x = 0$

$\Rightarrow\ x = 0, \frac{\pi}{2}, -\frac{\pi}{2}$ $3x = \pm\frac{\pi}{2}, \frac{5\pi}{2}$, $x = 0, \pi$

$x = \pm\frac{\pi}{6}, \frac{5\pi}{6}$, $x = 0, \pi$

$\therefore\ \frac{-\pi}{2}, \frac{-\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

81. The given expression

$= 3[\cos^4\alpha + \sin^4\alpha] - 2[\cos^6\alpha + \sin^6\alpha]$

$= 3[(\cos^2\alpha + \sin^2\alpha)^2 - 2\sin^2\alpha\cos^2\alpha]$

$\quad - 2[(\cos^2\alpha + \sin^2\alpha)^3 - 3\sin^2\alpha\cos^2\alpha]$

$= 3 - 2 = 1$

82. $\cot B - \cot A = y$

$$\Rightarrow \frac{\tan A - \tan B}{\tan A \tan B} = y$$

$$\therefore \quad \tan A \tan B = \frac{x}{y}, \text{ since } \tan A - \tan B = x$$

$$\cot(A - B) = \frac{1 + \tan A \tan B}{\tan A - \tan B} = \frac{1 + \frac{x}{y}}{x} = \frac{1}{x} + \frac{1}{y}$$

83. $\tan^3 A + \cot^3 A = (\tan A + \cot A)^3$

$\quad - 3\tan A.\cot A(\tan A + \cot A)$

$= 3^3 - 3 \times 1 \times 3 = 27 - 9 = 18$

84. $\cos(\alpha - \beta) = \cos((\theta + \alpha) - (\theta + \beta))$

$= \cos(\theta + \alpha)\cos(\theta + \beta) + \sin(\theta + \alpha)\sin(\theta + \beta)$

$= \sqrt{(1 - a^2)(1 - b^2)} + ab$

$[\cos(\alpha - \beta) - ab]^2 = 1 - a^2 - b^2 + a^2b^2$

$\cos^2(\alpha - \beta) - 2ab\cos(\alpha - \beta) = 1 - a^2 - b^2$

$\therefore$ The given expression is

$2\cos^2(\alpha - \beta) - 1 - 4ab\cos(\alpha - \beta)$

$= 2(1 - a^2 - b^2) - 1 = 1 - 2a^2 - 2b^2$

85. $\cos 2\theta \cos 2\phi$

$+ [\sin(\theta - \phi + \theta + \phi).\sin(\theta - \phi - \theta - \phi)]$

$= \cos 2\theta \cos 2\phi + \sin 2\theta \sin(-2\phi)$

$= \cos 2\theta \cos 2\varphi - \sin 2\theta \sin 2\varphi$

$= \cos(2\theta + 2\varphi)$

86. $$\frac{\sin 2\theta}{\sin 2\alpha} = \frac{\sin(\theta + \alpha + \theta - \alpha)}{\sin(\alpha + \theta + \alpha - \theta)}$$

$$= \frac{\sin(\theta + \alpha)\cos(\theta - \alpha) + \cos(\theta + \alpha)\sin(\theta - \alpha)}{\sin(\alpha + \theta)\cos(\alpha - \theta) + \cos(\alpha + \theta)\sin(\alpha - \theta)}$$

$$= \frac{\tan(\theta + \alpha) - \tan(\alpha - \theta)}{\tan(\theta + \alpha) + \tan(\alpha - \theta)} = \frac{n - 1}{n + 1}$$

87. $$\frac{\cos\theta}{1} = \frac{a\cos\phi + b}{a + b\cos\phi}$$

$$\frac{1 - \cos\theta}{1 + \cos\theta} = \frac{a + b\cos\phi - a\cos\varphi - b}{a + b\cos\phi + a\cos\varphi + b}$$

$$\Rightarrow \frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \left(\frac{a - b}{a + b}\right)\tan^2\left(\frac{\phi}{2}\right)$$

$$\Rightarrow \tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{a - b}{a + b}}\tan\left(\frac{\phi}{2}\right)$$

88. $$\frac{\sin(A - C) + \sin(A + C) + 2\sin A}{\sin(B + C) + \sin(B - C) + 2\sin B}$$

$$= \frac{2\sin A(\cos C + 1)}{2\sin B(\cos C + 1)} = \frac{\sin A}{\sin B}$$

89. Dividing numerator and denominator by $\cos^2\alpha$,

$$\tan\beta = \frac{n\tan\alpha}{\sec^2\alpha - n} = \frac{n\tan\alpha}{1 - n + \tan^2\alpha}$$

Put the value of $\tan\beta$ in $\tan(\alpha + \beta)$, we get

$$\tan(\alpha + \beta) = \frac{\tan\alpha}{1 - n}.$$

90. $$\cos\left((2n + 1)\frac{\pi}{2} + \theta\right) = \pm\sin\theta$$

If n is 0 or even, it is $-\sin\theta$

If n is 1 or odd it is $+\sin\theta$

Therefore, the general value is $(-1)^{n-1}\sin\theta$

or $(-1)^{n+1}\sin\theta$

91. $$3 + \cos^2 2\theta - 4\sin^4\theta = 3 + (1 - 2\sin^2\theta)^2 - 4\sin^4\theta$$

$$= 4 - 4\sin^2\theta = 4\cos^2\theta$$

Square root of the above = + ve value of $2\cos\theta$

$= -2\cos\theta$, since $\cos\theta < 0$

Expression $= -2\cos\theta - 2\cot\theta\,[(1 - \cos(\pi/2 - \theta))]$

$= -2\cos\theta - 2\cot\theta\,[1 - \sin\theta] = -2\cot\theta$

92. $$\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right)$$

$$= \sin\left(\frac{\pi}{8} + \frac{A}{2} + \frac{\pi}{8} - \frac{A}{2}\right).\sin\left(\frac{\pi}{8} + \frac{A}{2} - \frac{\pi}{8} + \frac{A}{2}\right)$$

$$= \sin\left(\frac{\pi}{4}\right)\sin A = \frac{1}{\sqrt{2}}\sin A$$

93. The given expression

$\tan 9° - \tan 27° - \cot 27° + \cot 9°$

$= (\tan 9° + \cot 9°) - (\tan 27° + \cot 27°)$

$$= \frac{1}{\sin 9°\cos 9°} - \frac{1}{\sin 27°\cos 27°}$$

$$= \frac{2}{\sin 18^\circ} - \frac{2}{\sin 54^\circ} = \frac{2}{\sin 18^\circ} - \frac{2}{\cos 36^\circ}$$

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4} \qquad \cos 36^\circ = \frac{\sqrt{5}+1}{4}$$

$\therefore$ The given expression

$$= 2\left[\frac{4}{\sqrt{5}-1} - \frac{4}{\sqrt{5}+1}\right] = 8.\frac{2}{4} = 4$$

94. Expression

$$= \left(\sin^4\frac{\pi}{16} + \sin^4\frac{7\pi}{16}\right) + \left(\sin^4\frac{3\pi}{16} + \sin^4\frac{5\pi}{16}\right)$$

$$= \left(\sin^4\frac{\pi}{16} + \cos^4\frac{\pi}{16}\right) + \left(\sin^4\frac{3\pi}{16} + \cos^4\frac{3\pi}{16}\right)$$

$$= 1 - 2\sin^2\frac{\pi}{16}\cos^2\frac{\pi}{16} + 1 - 2\sin^2\frac{3\pi}{16}\cos^2\frac{3\pi}{16}$$

$$= 2 - \frac{1}{2}\left[\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8}\right]$$

$$= 2 - \frac{1}{2}\left[\sin^2\frac{\pi}{8} + \cos^2\frac{\pi}{8}\right] = 1.5$$

95. $0 \le \sin^{20}\theta \le \sin^2\theta$ and $0 \le \cos^{48}\theta \le \cos^2\theta$

$\therefore 0 \le \sin^{20}\theta + \cos^{48}\theta \le 1$ i.e., $0 \le A \le 1$.

Since $(\sin^{20}\theta + \cos^{48}\theta)$ cannot be zero, answer is $0 < A \le 1$.

96. $$\text{RHS} = \frac{2\sin\left(\frac{A-C}{2}\right)\cos\left(\frac{A+C}{2}\right)}{2\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{A-C}{2}\right)}$$

$$= \cot\left(\frac{A+C}{2}\right)$$

$$\therefore \cot B = \cot\left(\frac{A+C}{2}\right)$$

$$\Rightarrow \quad B = \frac{A+C}{2} \text{ (since } -\frac{\pi}{2} < A,B,C < \frac{\pi}{2})$$

97. $\sin^2 A - \sin^2 B + \sin^2 C$

$$= \frac{1}{2} - \frac{1}{2}(\cos 2A - \cos 2B + \cos 2C) \text{ — (1)}$$

But $\cos 2A - \cos 2B + \cos 2C$

$= -2\sin(A+B)\sin(A-B) + 1 - 2\sin^2 C$

$= -2\sin C\sin(A-B) + 1 - 2\sin^2 C$

(since $A + B = 180^\circ - C$)

$= -2\sin C\,[\sin(A-B) + \sin(A+B)] + 1$

$= -4\sin A\cos B\sin C + 1$

$= 2\sin A\cos B\sin C$

Substituting in (1), result follows.

98. $$\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left(\tan\left(\pi + \frac{\pi}{6}\right)\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$$

99. $$\sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$$

Comparing with $\sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}$, we get

$$\sin^{-1}\frac{x}{5} = \cos^{-1}\frac{4}{5}$$

$$= \sin^{-1}\frac{3}{5}$$

$\therefore x = 3$

100. $$\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) = A = \tan^{-1}(2)$$

$$\sin^{-1}\left(\frac{3}{\sqrt{10}}\right) = B = \tan^{-1}(3)$$

$$A + B = \tan^{-1}(2) + \tan^{-1}(3)$$

$$= \pi + \tan^{-1}(-1) = \pi - \frac{\pi}{4}$$

$$\therefore \text{ 3rd angle } = \pi - \left(\pi - \frac{\pi}{4}\right) = \frac{\pi}{4}.$$

101. Let $a = \tan\theta$

Given expression

$$= \tan\left\{\frac{1}{2}\sin^{-1}\sin 2\theta + \frac{1}{2}\cos^{-1}\cos 2\theta\right\}$$

$$= \tan 2\theta = \frac{2a}{1-a^2}.$$

102. $$\sin^{-1}\frac{1-\sqrt{x}}{1+\sqrt{x}} + \sec^{-1}\frac{1+\sqrt{x}}{1-\sqrt{x}}$$

$$= \sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) + \cos^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \text{Required} = \cos\frac{\pi}{2} = 0.$$

103. Let $\cos^{-1}\frac{4}{5} = 2\alpha$

Given expression $= \tan\left(\frac{\pi}{4}+\alpha\right) + \tan\left(\frac{\pi}{4}-\alpha\right)$

$$= \frac{1+\tan\alpha}{1-\tan\alpha} + \frac{1-\tan\alpha}{1+\tan\alpha} = \frac{2\left(1+\tan^2\alpha\right)}{1-\tan^2\alpha}$$

$$= \frac{2}{\cos 2\alpha} = \frac{2\times 5}{4} = \frac{5}{2}$$

104. Given $3\sin^2 x - 7\sin x + 2 = 0$

$(\sin x - 2)(3\sin x - 1) = 0$

$\therefore \sin x = \frac{1}{3}$ or $\sin x = 2$ (inadmissible)

$\therefore x = n\pi + (-1)^n.\sin^{-1}\left(\frac{1}{3}\right)$

There are 6 values of $x \in (0, 5\pi)$ which satisfy the given equation.

105. $2\sin 3x\cos 2x + \sin 3x = 0$

$2\cos 2x + 1 = 0$ or $\sin 3x = 0$

$\Rightarrow \cos 2x = -\frac{1}{2}$ or $\sin 3x = 0$

$\Rightarrow 2x = \frac{2\pi}{3}$ or $3x = 0$, $\Rightarrow x = 0$ or $\frac{\pi}{3}$

106. $\tan\left(\frac{\pi}{4}-x\right) = \tan y \quad \frac{\pi}{4} - x = n\pi + y$

$x + y = \frac{\pi}{4} - n\pi$ and $x - y = \frac{\pi}{6}$

giving $x = \frac{5\pi}{24} - \frac{n\pi}{2}$ and $y = \frac{\pi}{24} - \frac{n\pi}{2}$

107. The value of $a\sin x + b\cos x$ lies in the interval $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$

Given $|c| > \sqrt{a^2+b^2}$

$\therefore$ No solution.

108. The given expression

$= \sin^2\alpha + \operatorname{cosec}^2\alpha + 2 + \cos^2\alpha + \sec^2\alpha + 2$

$= 5 + 1 + \cot^2\alpha + 1 + \tan^2\alpha$

$= 7 + \cot^2\alpha + \tan^2\alpha \geq 9$

$\left[\because \cot^2\alpha + \tan^2\alpha = (\cot\alpha - \tan\alpha)^2 + 2 \geq 2\right]$

109. Minimum value of the expression

$$= \frac{2}{\left(\text{Maximum value of the denominator of the expression}\right)}$$

$$= \frac{2}{15+\sqrt{4+12}} = \frac{2}{19}$$

110. Let $y = 81^{\sin^2 x}$

Then $81^{\cos^2 x} = 81^{1-\sin^2 x}$

So the given equation is $y + \frac{81}{y} = 30$

$\Rightarrow y^2 - 30y + 81 = 0 \Rightarrow y = 3$ or 27

$\Rightarrow \quad 81^{\sin^2 x} = 3 \quad$ or $\quad 81^{\sin^2 x} = 27$

$3^{4\sin^2 x} = 3$ or $27 \Rightarrow \quad \sin^2 x = 1/4$ or $\frac{3}{4}$

$\Rightarrow \sin x = \pm\frac{1}{2}$ or $\pm\frac{\sqrt{3}}{2}$

Since $0 \leq x \leq \pi/2$, we get $x = \pi/6$ or $\pi/3$

111. Statement 2 is true, since

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2-4ac}{4a}\right)$$

Consider Statement 1

$\sin^2 x + \cos^4 x = \sin^2 x + (1 - \sin^2 x)^2$

$= \sin^4 x - \sin^2 x + 1$

$= \left(\sin^2 x - \frac{1}{2}\right)^2 + \frac{3}{4}$

Minimum value of the expression is $\frac{3}{4}$

Statement 1 is true and it follows from Statement 2.

Choice (a)

112. Statement 2 is true

Considering Statement 1, since $\tan^{-1}2$ and $\tan^{-1}\frac{1}{2}$ both lie in the first quadrant and are complementary angles, the result is true; but it does not follow from statement 2 as $2.\frac{1}{2} = 1$

Choice (b)

113. Statement 2 is true only if $-1 \leq k \leq 1$

Hence, statement 2 is false

Statement 1:

$\sin\theta = 1$ or $\frac{1}{2}$

$\sin\theta = 1$ gives $\theta = n\pi + (-1)^n\frac{\pi}{2}$

$\sin\theta = \frac{1}{2}$ gives $\theta = n\pi + (-1)^n\frac{\pi}{6}$

$\theta = \frac{\pi}{2}, \frac{-3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{-11\pi}{6}, \frac{-7\pi}{6}$

Number of solutions in $[-2\pi, 2\pi]$ is 6

$\Rightarrow$ statement 1 is true

Choice (c)

114. $\cot\left(\frac{A}{2}+\frac{B}{2}\right) = \dfrac{\cot\frac{A}{2}\cot\frac{B}{2}-1}{\cot\frac{B}{2}+\cot\frac{A}{2}}$

i.e., $1 = \dfrac{\frac{r}{p}-1}{-\frac{q}{p}} \Rightarrow -q = r - p$ or $p = q + r$

115. $\sin\frac{A}{2} + \sin\frac{B}{2} = -\frac{q}{p}$; $\quad \sin\frac{A}{2}.\sin\frac{B}{2} = \frac{r}{p}$

$\therefore \sin\frac{A}{2} + \dfrac{\frac{r}{p}}{\sin\frac{A}{2}} = -\frac{q}{p}$;

$p.\sin^2\frac{A}{2} + r = -q\sin\frac{A}{2}$

$p.\sin^2\frac{A}{2} + q\sin\frac{A}{2} + r = o$

But $A = \frac{\pi}{3} \Rightarrow p.\frac{1}{4} + q.\frac{1}{2} + r = 0$

$\Rightarrow$ $p + 2q + 4r = 0$

116. $\cos B = \cos\left(\frac{\pi}{2} - A\right) = \sin A$

$\Rightarrow$ the equation has 2 equal roots

$\therefore q^2 = 4\,pr$

117. The give equation is

$3^{\sin 2x + 2\cos^2 x} + 3^{1-\sin 2x + 2\sin^2 x} = 28$

$3^{\sin 2x + 2\cos^2 x} + 3^{1-\sin 2x + 2(1-\cos^2 x)} = 28$

$3^{\sin 2x + 2\cos^2 x} + 3^3\, 3^{-\sin 2x - 2\cos^2 x} = 28$

$3^{\sin 2x + 2\cos^2 x} + 27\dfrac{1}{3^{\sin 2x + 2\cos^2 x}} = 28$

Put $y = 3^{\sin 2x + 2\cos^2 x}$

$y + \dfrac{27}{y} = 28$

$y^2 - 28y + 27 = 0$

$(y-1)(y-27) = 0$

$y = 1$ or 27

$3^{\sin 2x + 2\cos^2 x} = 1$ or 3^3

$\therefore \sin 2x + 2\cos^2 x = 0$ or $\sin 2x + 2\cos^2 x = 3$

Suppose $\sin 2x + 2\cos^2 x = 0$

$2\sin x\cos x + 2\cos^2 x = 0$

$2\cos x(\sin x + \cos x) = 0$

$2\cos^2 x(\tan x + 1) = 0$

This is satisfied if $\cos x = 0$ or if $\tan x = -1$

Now suppose $\sin 2x + 2\cos^2 x = 3$

$\sin 2x + 1 + \cos 2x = 3$

$\sin 2x + \cos 2x = 2$ which has no solution since the maximum value of $\sin 2x + \cos 2x$ is $\sqrt{2}$.

Choices (b) and (c)

118. Expression

$= \dfrac{\sin^3\theta(1-\cos\theta)}{\sin^2\theta} + \dfrac{\cos^3\theta(1+\sin\theta)}{\cos^2\theta}$

$= \sin\theta - \sin\theta\cos\theta + \cos\theta + \sin\theta\cos\theta$

$= \sin\theta + \cos\theta$

$= \sqrt{2}\left(\dfrac{\sin\theta}{\sqrt{2}} + \dfrac{\cos\theta}{\sqrt{2}}\right)$

$= \sqrt{2}\sin\left(\frac{\pi}{4}+\theta\right)$ or $\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)$

Choices (a) and (b)

119. $\dfrac{1+\tan x}{1-\tan x} = 3\left[\dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\right]$

Let $t = \tan x$.

$1 + t - 3t^2 - 3t^3 = 9t - 3t^3 - 9t^2 + 3t^4$

$\Rightarrow$ $3t^4 - 6t^2 + 8t - 1 = 0$

$t^4 - 2t^2 + \frac{8}{3}t - \frac{1}{3} = 0$

$\tan\alpha + \tan\beta + \tan\gamma + \tan\delta = 0$

$\Sigma \tan\alpha\ \tan\beta = -2$

$\tan\alpha\tan\beta\tan\gamma\ \tan\delta = \frac{-1}{3}$

$\Sigma \tan^2\alpha = (\Sigma\tan\alpha)^2 - 2\,\Sigma\tan\alpha\tan\beta$

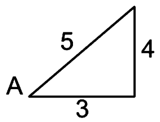
$= 0 - 2(-2) = 4$

Choices (a), (b), (c), (d)

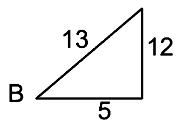
120. $25\cos A - 26\cos B = 5$

$10\cos A + 13\cos B = 11$

Solving $\cos A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$



$\therefore \sin A = \frac{4}{5}, \tan A = \frac{4}{3}$



$\sin B = \frac{12}{13}, \tan B = \frac{12}{5}$

(a) $\sin(A + B) = \sin A \cos B$

$+ \cos A \sin B = \frac{56}{65}$

(b) $\sin^2\left(\frac{A-B}{2}\right) = \frac{1-\cos(A-B)}{2} = \frac{1}{65}$

$(\because \cos(A - B) = \cos A \cos B$

$+ \sin A \sin B = \frac{63}{65})$

(c) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A . \tan B}$

$= \frac{\frac{4}{3} - \frac{12}{5}}{1 + \frac{4}{3} . \frac{12}{15}} = \frac{-16}{63}$

(d) $\sec^2 \frac{A-B}{2} = \frac{2}{1 + \cos(A-B)}$

$= \frac{2}{1 + \frac{63}{65}} = \frac{65}{64}$

$\therefore$ (a) $\to$ (s)
(b) $\to$ (p)
(c) $\to$ (q)
(d) $\to$ (r)

Additional Practice Exercise

121. R.H.S. $= 4\left(\cos^4 x - \sin^4 x\right)\left(\cos^4 x + \sin^4 x\right)$

$= 4\left(\cos^2 x - \sin^2 x\right)\left\{1 - 2\cos^2 x \sin^2 x\right\}$

$= \left(4\cos 2x\right)\left\{1 - \frac{\sin^2 2x}{2}\right\}$

$= \left(2\cos 2x\right)\left\{2 - \left(\frac{1-\cos 4x}{2}\right)\right\}$

$= \left(\cos 2x\right)\left\{3 + \cos 4x\right\}$ = L.H.S.

122. We have $\cos^2 A = \tan^2 B = \sec^2 B - 1$

$= \frac{1}{\cos^2 B} - 1 = \cot^2 C - 1$

or $1 + \cos^2 A = \cot^2 C = \frac{\cos^2 C}{1 - \cos^2 C} = \frac{\tan^2 A}{1 - \tan^2 A}$

(from the third relation)

$\Rightarrow 2 - \sin^2 A = \frac{\sin^2 A}{1 - 2\sin^2 A}$

$\Rightarrow 2\sin^4 A - 6\sin^2 A + 2 = 0$

$\Rightarrow \sin^4 A - 3\sin^2 A + 1 = 0 \Rightarrow \sin^2 A = \frac{3 \pm \sqrt{5}}{2}$

As $\frac{3+\sqrt{5}}{2} > 1, \sin^2 A = \frac{3-\sqrt{5}}{2}$

$= \frac{6 - 2\sqrt{5}}{4} = \left(\frac{\sqrt{5}-1}{2}\right)^2$

$\Rightarrow \sin A = \frac{\sqrt{5}-1}{2}.$

By symmetry, if one had started from $\cos^2 B$ we would get $\sin B = \frac{\sqrt{5}-1}{2}$. Similarly, for sin C

123. $\cot 7\frac{1}{2}° = \frac{\cos 7\frac{1}{2}°}{\sin 7\frac{1}{2}°} = \frac{2\cos^2 7\frac{1}{2}°}{2\cos 7\frac{1}{2}° \sin 7\frac{1}{2}°} = \frac{1 + \cos 15°}{\sin 15°}$

$\sin 15° = \sin\left(45° - 30°\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\cos 15° = \cos\left(45° - 30°\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Hence $\cot 7\frac{1}{2}° = \frac{1 + \frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1}$

$= \frac{\left(2\sqrt{2} + \sqrt{3} + 1\right)\left(\sqrt{3} + 1\right)}{2} = \frac{2\sqrt{6} + 2\sqrt{2} + 2\sqrt{3} + 4}{2}$

$= \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$

124. $\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)=\dfrac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}}=\dfrac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}$

$\Rightarrow\quad \tan^2\left(\frac{\pi}{4}+\frac{A}{2}\right)=\dfrac{1+\sin A}{1-\sin A}$

Given expression

$\Rightarrow\quad \tan^2\left(\frac{\pi}{4}+\frac{A}{2}\right)=\tan^6\left(\frac{\pi}{4}+\frac{B}{2}\right)$

i.e., $\dfrac{1+\sin A}{1-\sin A}=\left(\dfrac{1+\sin B}{1-\sin B}\right)^3$

Taking componendo – dividendo,

$$\sin A=\frac{(1+\sin B)^3-(1-\sin B)^3}{(1+\sin B)^3+(1-\sin B)^3}$$

$$=\frac{2\left(3\sin B+\sin^3 B\right)}{2\left(1+3\sin^2 B\right)}=\left(\frac{3+\sin^2 B}{1+3\sin^2 B}\right)\sin B$$

125. We have $\dfrac{1-\tan^2 A/2}{1+\tan^2 A/2}=\dfrac{2\cos B-1}{2-\cos B}$

$$\Rightarrow \tan^2\frac{A}{2}=\frac{(2-\cos B)-(2\cos B-1)}{(2-\cos B)+(2\cos B-1)}$$

(using componendo dividendo)

$$=\frac{3-3\cos B}{1+\cos B}=\frac{\left(3\sin^2\frac{B}{2}\right)}{\cos^2\frac{B}{2}}=3\tan^2\frac{B}{2}$$

$$\Rightarrow \tan\frac{A}{2}=\sqrt{3}\tan\frac{B}{2}$$

Again, $\sin B=\dfrac{2\tan\frac{B}{2}}{1+\tan^2\frac{B}{2}}=\dfrac{2\times\frac{1}{\sqrt{3}}\tan\frac{A}{2}}{1+\frac{1}{3}\tan^2\frac{A}{2}}$

$$=\frac{3\times\left(2\tan\frac{A}{2}\right)}{\sqrt{3}\left(3+\tan^2\frac{A}{2}\right)}$$

$$=\frac{\sqrt{3}\times 2\sin\frac{A}{2}}{\cos\frac{A}{2}}\times\frac{\cos^2\frac{A}{2}}{\left(3\cos^2\frac{A}{2}+\sin^2\frac{A}{2}\right)}$$

$$=\frac{\sqrt{3}\sin A}{2\cos^2\frac{A}{2}+1}=\frac{\sqrt{3}\sin A}{2+\cos A}$$

126. We have $\sin^{-1}\sqrt{1-\frac{x^2}{4}}=\cos^{-1}\frac{x}{2}$

$\Rightarrow\quad \cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{5}=\theta$

Let $\cos^{-1}\frac{x}{2}=\alpha$; $\quad\cos^{-1}\frac{y}{5}=\beta$

$\cos(\alpha+\beta)=\cos\theta$

$\cos\alpha\cos\beta-\sin\alpha\sin\beta=\cos\theta$

$$\left(\frac{x}{2}\right)\left(\frac{y}{5}\right)-\sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{25}}=\cos\theta$$

Or $\quad\cos\theta=\dfrac{xy}{10}-\dfrac{\sqrt{4-x^2}\sqrt{25-y^2}}{10}$

Now $100\sin^2\theta=100\left(1-\cos^2\theta\right)$

$$=100\left\{1-\frac{1}{100}\left(xy-\sqrt{4-x^2}\sqrt{25-y^2}\right)^2\right\}$$

$$=100-\left(xy-\sqrt{4-x^2}\sqrt{25-y^2}\right)^2$$

$$=100-[x^2y^2+(4-x^2)(25-y^2)$$

$$-2xy\sqrt{4-x^2}\sqrt{25-y^2}$$

$$=25x^2+4y^2-2x^2y^2+2xy\sqrt{4-x^2}\sqrt{25-y^2}$$

$$=25x^2+4y^2-2xy\left(xy-\sqrt{4-x^2}\sqrt{25-y^2}\right)$$

$$=25x^2+4y^2-20xy\cos\theta.$$

127. Let $x=\tan\alpha$, $y=\tan\beta$

We have to prove that

$$2\alpha+2\beta=\sin^{-1}\left[\frac{2(x+y)(1-xy)}{\left(1+x^2\right)\left(1+y^2\right)}\right]$$

or $\quad\dfrac{2(x+y)(1-xy)}{\left(1+x^2\right)\left(1+y^2\right)}=\sin(2\alpha+2\beta)$

RHS = $\sin 2\alpha\cos 2\beta+\cos 2\alpha\sin 2\beta$

$$=\frac{2\tan\alpha}{\left(1+\tan^2\alpha\right)}\times\left(\frac{1-\tan^2\beta}{1+\tan^2\beta}\right)$$

$$+\left(\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right)\left(\frac{2\tan\beta}{1+\tan^2\beta}\right)$$

$$=\frac{2x\left(1-y^2\right)+2y\left(1-x^2\right)}{\left(1+x^2\right)\left(1+y^2\right)}$$

$$=\frac{2(x+y)(1-xy)}{\left(1+x^2\right)\left(1+y^2\right)}$$

128. The given equation can be written as

$(\sin^2x + \cos^2x)^2 - 2\sin^2x\cos^2x + \sin 2x + \alpha = 0$

$1 - \frac{1}{2}\sin^2 2x + \sin 2x + \alpha = 0$

or $\sin^2 2x - 2\sin 2x - 2(1+\alpha) = 0$

If $\sin 2x = S$ then $S^2 - 2S = 2 + 2\alpha$

$S^2 - 2S + 1 = 3 + 2\alpha \Rightarrow (S-1)^2 = 3 + 2\alpha$

Since $\sin\theta$ lies between -1 and 1, $(S-1)^2$ lies between 0 and 4.

$\therefore 0 \le 3 + 2\alpha \le 4$ or $-\frac{3}{2} \le \alpha \le \frac{1}{2}$

and $(S-1) = \pm\sqrt{3+2\alpha}$ or $S = 1 \pm \sqrt{3+2\alpha}$

$\Rightarrow \quad S = 1 - \sqrt{3+2\alpha}$ as $S \le 1$

Under this condition let $1 - \sqrt{3+2\alpha} = \sin\beta$ -- (1)

Thus $\sin 2x = \sin\beta$

The general solution is $2x = n\pi + (-1)^n.\beta$

$x = \frac{n\pi}{2} + \frac{1}{2}(-1)^n\beta$

where, β is given by equation (1).

129. The equation can be rewritten as

$(4c^3 - 3c)s^3 + (3s - 4s^3)c^3 = 0$,

where $c \equiv \cos x$, $s \equiv \sin x$

$\Rightarrow 3sc^3 - 3cs^3 = 0$

$\Rightarrow sc(c^2 - s^2) = 0$

$\Rightarrow s = 0, c = 0, c = \pm s$

$\Rightarrow \sin x = 0, \cos x = 0, \tan x = \pm 1$

$\Rightarrow \quad x = n\pi,\ 2n\pi \pm \frac{\pi}{2},\ n\pi \pm \frac{\pi}{4}$, where n is an integer.

130. The given equation may be written as

$3(\tan 2x - \tan 3x) = \tan 3x + \tan^2 3x \tan 2x$

$= (\tan 3x)(1 + \tan 3x \tan 2x)$

$\Rightarrow \frac{3(\tan 2x - \tan 3x)}{1 + \tan 3x \tan 2x} = \tan 3x$

$\Rightarrow -3\tan(3x - 2x) = \tan 3x$

$\Rightarrow 3\tan x + \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} = 0$ if $\tan^2 x \ne \frac{1}{3}$

$\Rightarrow 6\tan x - 10\tan^3 x = 0$

$\Rightarrow \tan x = 0,\ \tan^2 x = \frac{3}{5} \ne \frac{1}{3}$

$\Rightarrow \quad x = n\pi,\ n\pi + \tan^{-1}\left(\pm\sqrt{\frac{3}{5}}\right)$, n is an integer

$\Rightarrow \quad x = n\pi,\ n\pi \pm \tan^{-1}\left(\sqrt{\frac{3}{5}}\right)$

131. $\sqrt{\sec^2 A + \operatorname{cosec}^2 A} = \sqrt{1 + \tan^2 A + 1 + \cot^2 A}$

$= \sqrt{\tan^2 A + \cot^2 A + 2(\tan A \cot A)}$

$= \tan A + \cot A$

132. $(m+n) = a(\cos\alpha + \sin\alpha)^3$

$(m-n) = a(\cos\alpha - \sin\alpha)^3$

$(m+n)^{2/3} + (m-n)^{2/3}$

$= a^{2/3}\left[(\cos\alpha + \sin\alpha)^2 + (\cos\alpha - \sin\alpha)^2\right]$

$= a^{2/3}\left(2\left(\cos^2\alpha + \sin^2\alpha\right)\right) = 2a^{2/3}$

133. $m = \frac{\cos(\theta-\alpha) + \sin(\theta+\alpha)}{\cos(\theta-\alpha) - \sin(\theta+\alpha)}$

(By componendo-dividendo)

$= \frac{\cos\theta\cos\alpha + \sin\theta\sin\alpha + \sin\theta\cos\alpha + \cos\theta\sin\alpha}{\cos\theta\cos\alpha + \sin\theta\sin\alpha - \sin\theta\cos\alpha - \cos\theta\sin\alpha}$

$= \frac{\cos\theta(\cos\alpha + \sin\alpha) + \sin\theta(\cos\alpha + \sin\alpha)}{\cos\theta(\cos\alpha - \sin\alpha) - \sin\theta(\cos\alpha - \sin\alpha)}$

$= \tan\left(\frac{\pi}{4} + \theta\right)\tan\left(\frac{\pi}{4} + \alpha\right)$

134. Observe that $\tan\theta > 0$

We have, if $x > 0$, $x + \frac{1}{x} \ge 2$ and equality holds when $x = 1$

$\Rightarrow \tan\theta = 1$ and $\cot\theta = 1$

Given expression $= 1 + 1 - 2 + 3 = 3$

135. Maximum value of $\sin x$ is 1

So $\sin\alpha + \sin\beta + \sin\gamma = 3$

$\Rightarrow \sin\alpha = \sin\beta = \sin\gamma = 1$

$\Rightarrow \cos\alpha + \cos\beta + \cos\gamma = 0.$

136. Multiply and divide by $\tan\theta$.

$\frac{\tan\theta(1 + \sec 2\theta)(1 + \sec 2^2\theta)\ldots(1 + \sec 2^n\theta)}{\tan\theta}$

$= \frac{1}{\tan\theta}\left[\frac{\sin\theta}{\cos\theta}\left(\frac{1 + \cos 2\theta}{\cos 2\theta}\right)(1 + \sec 2^2\theta)\ldots(1 + \sec 2^n\theta)\right]$

$$= \frac{1}{\tan\theta}\left[\tan 2\theta.\left(1+\sec 2^2\theta\right)...\left(1+\sec 2^n\theta\right)\right]$$

$$= \frac{1}{(\tan\theta)}.\tan 2^n\theta = (\tan 2^n\theta)(\cot\theta)$$

137. Let each ratio be equal to $1/k$.

$$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=k\left[\sin\theta+\sin\left(\theta+\frac{2\pi}{3}\right)+\sin\left(\theta+\frac{4\pi}{3}\right)\right]$$

$$=k\left[2\sin\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}+\sin\left(\theta+\frac{2\pi}{3}\right)\right]=0$$

$$\left(\text{as }\cos\frac{2\pi}{3}=-\frac{1}{2}\right)$$

$\therefore$ $xy + yz + zx = 0$

138. $(\cos\alpha_1).(\cos\alpha_2).(\cos\alpha_3)\ \ldots\ (\cos\alpha_n) = k$ ---- (1)

and

$(\sin\alpha_1).(\sin\alpha_2).(\sin\alpha_3)\ \ldots\ (\sin\alpha_n) = k$ ---- (2)

(1) x (2) gives

$(\cos\alpha_1).(\cos\alpha_2).(\cos\alpha_3)\ \ldots\ (\cos\alpha_n)$

x $(\sin\alpha_1).(\sin\alpha_2).(\sin\alpha_3)\ \ldots\ (\sin\alpha_n) = k^2$

$$\therefore k^2 = \frac{1}{2\times 2\times \text{....n times}}\times\left(2\sin\alpha_1.\cos\alpha_1\right)\times$$

$$\times\left(2\sin\alpha_2.\cos\alpha_2\right)\times\ \ldots\ldots\ \times\left(2\sin\alpha_n\cos\alpha_n\right)$$

i.e., $k^2 = \frac{1}{2^n}\sin 2\alpha_1\sin 2\alpha_2 \ldots\ldots \sin 2\alpha_n$

Since $0 < \alpha_i < \frac{\pi}{2}$, $i = 1,2,\ldots,n$, we have

$$k^2 < \frac{1}{2^n}.1 \Rightarrow k < \frac{1}{2^{n/2}}$$

139. $y = \tan 3x\cot x = \frac{3t-t^3}{1-3t^2}\times\frac{1}{t} = \frac{3-t^2}{1-3t^2}$

where, $t = \tan x$

$3 - t^2 = y - 3yt^2$

$t^2(3y-1)-(y-3)=0$

Discriminant > 0

$\Rightarrow$ $0 + 4(3y-1)(y-3) > 0$

y should lie beyond $\frac{1}{3}$ and 3

140. $\sin^6 x + \cos^6 x = \lambda$

$\Rightarrow$ $(\sin^2 x + \cos^2 x)^3$

$- 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x) = \lambda$

$\Rightarrow$ $1 - 3\sin^2\cos^2 x = \lambda \Rightarrow 1 - \frac{3}{4}\sin^2 2x = \lambda$

But $0 \le \sin^2 2x \le 1$

$\Rightarrow$ $-\frac{3}{4} \le -\frac{3}{4}\sin^2 2x \le 0$

$\Rightarrow$ $\frac{1}{4} \le 1 - \frac{3}{4}\sin^2 2x \le 1 \Rightarrow \frac{1}{4} \le \lambda \le 1$

141. Given $\Rightarrow \frac{\sin(\gamma+\delta)}{\sin(\gamma-\delta)} = \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}$

Using componendo—dividendo we get

$$\frac{\sin(\gamma+\delta)+\sin(\gamma-\delta)}{\sin(\gamma+\delta)-\sin(\gamma-\delta)} = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)}$$

or $\frac{2\sin\gamma\cos\delta}{2\cos\gamma\sin\delta} = \frac{2\cos\alpha\cos\beta}{2\sin\alpha\sin\beta}$

$\tan\gamma\ \cot\delta = \cot\alpha\ \cot\beta$

$\cot\delta = \cot\alpha\ \cot\beta\ \cot\gamma$

142. $\frac{3\cos 2y - 1}{3 - \cos 2y} = \frac{3(1-\tan^2 y)-(1+\tan^2 y)}{3(1+\tan^2 y)-(1-\tan^2 y)}$

$$= \frac{2-4\tan^2 y}{2+4\tan^2 y} = \frac{2-4\left(\frac{\tan^2 x}{2}\right)}{2+4\left(\frac{\tan^2 x}{2}\right)}$$

$$= \frac{1-\tan^2 x}{1+\tan^2 x} = \cos 2x$$

143. $\frac{\sin\theta}{\cos 3\theta} = \frac{\sin 2\theta}{2\cos 3\theta\cos\theta}$

$$= \frac{\sin(3\theta-\theta)}{2\cos 3\theta\cos\theta}$$

$$= \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{2\cos 3\theta\cos\theta}$$

$$= \frac{1}{2}[\tan 3\theta - \tan\theta]$$

$\therefore$ By symmetry,

$$k_2 = \frac{1}{2}[\tan 3\theta - \tan\theta] + \frac{1}{2}[\tan 9\theta - \tan 3\theta]$$

$$+\frac{1}{2}[\tan 27\theta - \tan 9\theta]$$

$$= \frac{1}{2}[\tan 27\theta - \tan\theta] \quad = \frac{1}{2}k_1$$

144. $\sec^2\dfrac{7\pi}{16} = \sec^2\left(\dfrac{\pi}{2} - \dfrac{\pi}{16}\right) = \operatorname{cosec}^2\dfrac{\pi}{16}$

$$\sec^2\frac{5\pi}{16} = \operatorname{cosec}^2\frac{3\pi}{16}$$

Given expression

$$= \left(\sec^2\frac{\pi}{16} + \operatorname{cosec}^2\frac{\pi}{16}\right) + \left(\sec^2\frac{3\pi}{16} + \operatorname{cosec}^2\frac{3\pi}{16}\right)$$

$$= \frac{1}{\sin^2\frac{\pi}{16}\cos^2\frac{\pi}{16}} + \frac{1}{\sin^2\frac{3\pi}{16}\cos^2\frac{3\pi}{16}}$$

$$= \frac{4}{\sin^2\frac{\pi}{8}} + \frac{4}{\sin^2\frac{3\pi}{8}} = \frac{4}{\sin^2\frac{\pi}{8}} + \frac{4}{\cos^2\frac{\pi}{8}}$$

since $\sin\dfrac{3\pi}{8} = \sin\left(\dfrac{\pi}{2} - \dfrac{\pi}{8}\right)$

$$= \frac{4}{\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}} = \frac{16}{\sin^2\frac{\pi}{4}} = 32$$

145. Let $\theta_1 = \cos^{-1}\dfrac{12}{15}$ $\therefore \cos\theta_1 = \dfrac{12}{15}$

and $\theta_2 = \cos^{-1}\dfrac{4}{5}$ $\therefore \cos\theta_2 = \dfrac{4}{5}$

$\therefore \sin\theta_1 = \dfrac{9}{15}$ and $\sin\theta_2 = \dfrac{3}{5}$, assuming that θ_1 and θ_2 are in the I quadrant

$$\therefore \cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

$$= \frac{12}{15}\times\frac{4}{5} - \frac{9}{15}\times\frac{3}{5} = \frac{7}{25}.$$

146. $\cos^{-1}\left(\dfrac{1}{5\sqrt{2}}\right) = \tan^{-1}(7)$

$$\sin^{-1}\left(\frac{4}{\sqrt{17}}\right) = \tan^{-1}(4)$$

$$\tan\left[\cos^{-1}\left(\frac{1}{5\sqrt{2}}\right) - \sin^{-1}\left(\frac{4}{\sqrt{17}}\right)\right]$$

$$= \ldots\left[\ {}^{-1}(\ ) - {}^{-1}(\ )\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{3}{29}\right)\right] = \frac{3}{29}.$$

147. Let $x = \cos^{-1}\dfrac{p}{a}$; $y = \cos^{-1}\dfrac{q}{b}$

$\cos(x + y) = \cos\alpha$

$$\frac{pq}{ab} - \sqrt{1 - \frac{p^2}{a^2}}\sqrt{1 - \frac{q^2}{b^2}} = \cos\alpha$$

$$\therefore \left(\frac{pq}{ab} - \cos\alpha\right)^2 = 1 - \frac{p^2}{a^2} - \frac{q^2}{b^2} + \frac{p^2q^2}{a^2b^2}$$

$$\frac{p^2q^2}{a^2b^2} + \cos^2\alpha - \frac{2pq}{ab}\cos\alpha$$

$$= 1 - \frac{p^2}{a^2} - \frac{q^2}{b^2} + \frac{p^2q^2}{a^2b^2}$$

$\therefore$ Required sum $= 1 - \cos^2\alpha = \sin^2\alpha$.

148. We know that, $2\tan^{-1}t = \cos^{-1}\dfrac{1 - t^2}{1 + t^2}$

$$2\tan^{-1}\left[\sqrt{\frac{a - b}{a + b}}\cdot\tan\frac{\theta}{2}\right]$$

$$= \cos^{-1}\left[\frac{1 - \left(\frac{a - b}{a + b}\right)\tan^2\frac{\theta}{2}}{1 + \left(\frac{a - b}{a + b}\right)\tan^2\frac{\theta}{2}}\right]$$

$$= \cos^{-1}\left(\frac{a\cos\theta + b}{a + b\cos\theta}\right) \text{(on simplification)}$$

149. $\dfrac{-\pi}{2} \le \sin^{-1}(2x\sqrt{1 - x^2}) \le \dfrac{\pi}{2}$

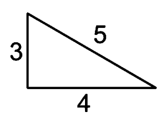
$$-\frac{\pi}{2} \le 2\sin^{-1}x \le \frac{\pi}{2}$$

$$\therefore \frac{-\pi}{4} \le \sin^{-1}x \le \frac{\pi}{4}$$

$$\Rightarrow \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$$

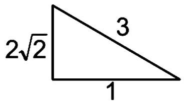
150. $2\tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{3}{4}$

Now $\tan^{-1}\dfrac{3}{4} = \sin^{-1}\dfrac{3}{5}$



$$\therefore \sin\left(2\tan^{-1}\frac{1}{3}\right) = \sin\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{5}$$

$$\tan^{-1}2\sqrt{2} = \cos^{-1}\frac{1}{3}$$



$$\therefore\ \cos\left(\tan^{-1}2\sqrt{2}\right) = \cos\left(\cos^{-1}\frac{1}{3}\right) = \frac{1}{3}$$

$$\tan\left(\sin^{-1}\frac{4}{5} + \frac{3\pi}{2}\right) = -\cot\left(\sin^{-1}\frac{4}{5}\right) = \frac{-3}{4}$$

$$\therefore \text{ Required sum} = \frac{3}{5} + \frac{1}{3} - \frac{3}{4} = \frac{11}{60}$$

151. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x.$

$2\sin 2x\cos x + \sin 2x = 2\cos 2x\cos x + \cos 2x$

$(\sin 2x)(2\cos x + 1) = (\cos 2x)(2\cos x + 1)$

$\Rightarrow\ (\sin 2x - \cos 2x)(2\cos x + 1) = 0$

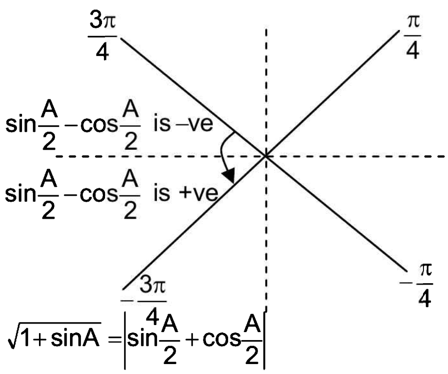
$\tan 2x = 1$ or $\cos x = \dfrac{-1}{2}$

$2x = n\pi + \dfrac{\pi}{4}$ or $x = 2n\pi \pm \dfrac{2\pi}{3}$

$\therefore\ x = \dfrac{n\pi}{2} + \dfrac{\pi}{8}$ or $x = 2n\pi \pm \dfrac{2\pi}{3}$

There are 6 solutions in $[0, 2\pi]$

152.



$$\sqrt{1-\sin A} = \left|\sin\frac{A}{2} - \cos\frac{A}{2}\right|$$

$$\therefore\ \sqrt{1-\sin A} - \sqrt{1+\sin A} = 2\sin\frac{A}{2}$$

$$\Rightarrow \left|\sin\frac{A}{2} - \cos\frac{A}{2}\right| = \sin\frac{A}{2} - \cos\frac{A}{2}$$

$$\text{and } \left|\sin\frac{A}{2} + \cos\frac{A}{2}\right| = -\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)$$

Clearly, two modulli signs are possible in the region shown. So the required region is

$2n\pi + \dfrac{3\pi}{4}$ to $2n\pi + \dfrac{5\pi}{4}$

153. $\sqrt{3}\cos\theta - 3\sin\theta = 2(\sin 5\theta - \sin\theta)$

$$\Rightarrow\ \sqrt{3}\cos\theta - \sin\theta = 2\sin 5\theta$$

$$\Rightarrow\ \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \sin 5\theta$$

$$\Rightarrow\ \sin\left(\frac{\pi}{3} - \theta\right) = \sin 5\theta$$

$$\text{or } \sin 5\theta = \sin\left(\frac{\pi}{3} - \theta\right)$$

$$\Rightarrow\ \sin 5\theta - \sin\left(\frac{\pi}{3} - \theta\right) = 0$$

$$\Rightarrow\ 2\cos\left(2\theta + \frac{\pi}{6}\right).\sin\left(3\theta - \frac{\pi}{6}\right) = 0$$

$$\Rightarrow\ \cos\left(2\theta + \frac{\pi}{6}\right) = 0 \text{ or } \sin\left(3\theta - \frac{\pi}{6}\right) = 0$$

$$2\theta + \frac{\pi}{6} = 2r\pi \pm \frac{\pi}{2} \text{ or } 3\theta - \frac{\pi}{6} = r\pi$$

$$\theta = r\pi \pm \frac{\pi}{4} - \frac{\pi}{12} \text{ or } \theta = \frac{r\pi}{3} + \frac{\pi}{18}$$

154. Case 1: $x < 1$

$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) = \cos^{-1}\left\{\frac{-(1-x^2)}{1+x^2}\right\}$$

$$= \pi - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \pi - 2\tan^{-1}x$$

$$\tan^{-1}\left(\frac{2x}{x^2-1}\right) = -\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$= -2\tan^{-1}x$$

The equation reduces to

$$\pi - 4\tan^{-1}x = \frac{2\pi}{3} \Rightarrow \tan^{-1}x = \frac{\pi}{12}$$

$$\Rightarrow x = \tan\frac{\pi}{12} = 2 - \sqrt{3}$$

Case 2: $x > 1$

$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) = \tan^{-1}\left(\frac{2x}{x^2-1}\right)$$

Equation reduces to

$$2\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}\Rightarrow\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{\pi}{3}$$

$$\Rightarrow\frac{2x}{x^2-1}=\tan\frac{\pi}{3}=\sqrt{3}\Rightarrow x=\sqrt{3},\frac{1}{\sqrt{3}}$$

But here $x>1\Rightarrow x=\sqrt{3}$

Solution is $x=2-\sqrt{3}$ and $x=\sqrt{3}$

155. Equation is $\dfrac{3\sin^2\theta}{\cos^2\theta}-2\sin\theta=0$

$$\sin\theta\left[\frac{3\sin\theta}{\cos^2\theta}-2\right]=0\Rightarrow\sin\theta\left[3\sin\theta-2\cos^2\theta\right]=0,$$

since $\cos\theta$ cannot be zero

$\Rightarrow$ $\sin\theta=0$ or $3\sin\theta-2(1-\sin^2\theta)=0$

$\Rightarrow$ $\theta=n\pi$ or $2\sin^2\theta+3\sin\theta-2=0$

$\Rightarrow$ $\theta=n\pi$ or $\sin\theta=\dfrac{1}{2}$

$\therefore$ General solution is $\theta=n\pi$ or $\theta=n\pi+(-1)^n\dfrac{\pi}{6}$

where, n is an integer.

156. The equation may be rewritten as

$3\sin x+2\sin\alpha\cos x=2\sqrt{3}$

Dividing both sides by $\sqrt{9+4\sin^2\alpha}$, we get

$$\frac{3}{\sqrt{9+4\sin^2\alpha}}\sin x+\frac{2\sin\alpha}{\sqrt{9+4\sin^2\alpha}}\cos x=\frac{2\sqrt{3}}{\sqrt{9+4\sin^2\alpha}}$$

Since both $\dfrac{3}{\sqrt{9+4\sin^2\alpha}}$ and $\dfrac{2\sin\alpha}{\sqrt{9+4\sin^2\alpha}}$ are <1, the above equation will have real solution only if

$$\frac{2\sqrt{3}}{\sqrt{9+4\sin^2\alpha}}\le 1$$

$\Rightarrow$ $12\le 9+4\sin^2\alpha\Rightarrow 4\sin^2\alpha\ge 3\Rightarrow\sin\alpha$ must lie beyond $-\dfrac{\sqrt{3}}{2}$ and $\dfrac{\sqrt{3}}{2}$. However,

$-1\le\sin\alpha\le 1$.

Hence the range of values for $\sin\alpha$ is

$$\left[-1,\frac{-\sqrt{3}}{2}\right]\cup\left[\frac{\sqrt{3}}{2},1\right]$$

157. $$\tan^{-1}\frac{\frac{1}{2x+1}+\frac{1}{4x+1}}{1-\frac{1}{(2x+1)(4x+1)}}=\tan^{-1}\frac{2}{x^2}$$

$$\therefore\frac{6x+2}{8x^2+6x}=\frac{2}{x^2}$$

Assuming $x\ne 0$, we get

$(3x+2)(x-3)=0$

$\Rightarrow$ $x=3$ or $\dfrac{-2}{3}$

$x=3$ satisfies the equation whereas $x=\dfrac{-2}{3}$ does not.

Now we can verify that $x=0$ also satisfies the given equation.

Hence the equation has two solutions.

158. From the given relation, $\dfrac{\sin^4\alpha}{a}+\dfrac{(1-\sin^2\alpha)^2}{b}=\dfrac{1}{a+b}$

$$\Rightarrow\left(\frac{1}{a}+\frac{1}{b}\right)\sin^4\alpha-\frac{2\sin^2\alpha}{b}+\frac{1}{b}=\frac{1}{a+b}$$

$$\Rightarrow(a+b)^2\sin^4\alpha-2a(a+b)\sin^2\alpha+a^2=0$$

$$\Rightarrow[(a+b)\sin^2\alpha-a]^2=0\Rightarrow\sin^2\alpha=\frac{a}{a+b}$$

$$\cos^2\alpha=1-\sin^2\alpha\Rightarrow\cos^2\alpha=\frac{b}{a+b}$$

Hence $\dfrac{\sin^{2n}\alpha}{a^n}=\dfrac{1}{(a+b)^n}$

Also, $\dfrac{\cos^{2n}\alpha}{b^n}=\dfrac{1}{(a+b)^n}$

$$\therefore\frac{\sin^{2n}\alpha}{a^n}+\frac{\cos^{2n}\alpha}{b^n}=\frac{2}{(a+b)^n}\Rightarrow\lambda=2$$

159. Given expression

$$=\frac{5(1-\cos 2\theta)}{2}+2\sin 2\theta+\frac{3(1+\cos 2\theta)}{2}$$

$$=4+2\sin 2\theta-\cos 2\theta$$

Maximum and minimum values of

$4+2\sin 2\theta-\cos 2\theta$ are $4-\sqrt{5}$ and $4+\sqrt{5}$.

Hence for real θ, the given expression lies between $4-\sqrt{5}$ and $4+\sqrt{5}$

160. From the given relations,

$\cos^4\theta - \sin^4\theta$

$= k\left[\cos\theta.\cos(\alpha - 3\theta) - \sin\theta.\sin(\alpha - 3\theta)\right]$

$\Rightarrow \cos 2\theta = k\cos(\alpha - 2\theta)$

$\Rightarrow \cos 2\theta = k[\cos\alpha\cos 2\theta + \sin\alpha\sin 2\theta]$

$\Rightarrow (1 - k\cos\alpha)\cos 2\theta = k\sin\alpha.\sin 2\theta$

$\therefore \tan 2\theta = \dfrac{1 - k\cos\alpha}{k\sin\alpha}$ — (1)

Again,

$\cos^3\theta\sin\theta + \sin^3\theta\cos\theta$

$= k\left[\cos(\alpha - 3\theta)\sin\theta + \sin(\alpha - 3\theta)\cos\theta\right]$

$\Rightarrow \sin\theta\cos\theta = k\sin(\alpha - 2\theta)$

$\Rightarrow \sin 2\theta = 2k[\sin\alpha\cos 2\theta - \cos\alpha\sin 2\theta]$

$\Rightarrow (1 + 2k\cos\alpha)\sin 2\theta = 2k\sin\alpha\cos 2\theta$

$\Rightarrow \tan 2\theta = \dfrac{2k\sin\alpha}{1 + 2k\cos\alpha}$ — (2)

Equating (1) and (2), $\dfrac{1 - k\cos\alpha}{k\sin\alpha} = \dfrac{2k\sin\alpha}{1 + 2k\cos\alpha}$

$\Rightarrow 1 - k\cos\alpha + 2k\cos\alpha - 2k^2\cos^2\alpha$

$= 2k^2\sin^2\alpha$

$\Rightarrow 2k^2 - k\cos\alpha - 1 = 0$

161. $\tan^2 y - 2\tan y + 26 = -x^2 + 10x$

$[\tan y - 1]^2 + (x - 5)^2 = 0 \Rightarrow x = 5, \tan y = 1$

$y = n\pi + \dfrac{\pi}{4}, n \in Z.$

162. $f(x) = 2\sin^2\alpha +$

$2\cos(x + \alpha)\,[2\sin x\sin\alpha + \cos(x + \alpha)] - 1$

$= 2\sin^2\alpha - 1 + 2\cos(x + \alpha)\cos(x - \alpha)$

$= 2\sin^2\alpha - 1 + 2\cos^2 x - 2\sin^2\alpha = \cos 2x$

$f^2(\alpha) = \cos^2 2\alpha$

$f^2\left(\dfrac{\pi}{4} - \alpha\right) = \cos^2 2\left(\dfrac{\pi}{4} - \alpha\right) = \sin^2 2\alpha$

$\therefore f^2(\alpha) + f^2\left(\dfrac{\pi}{4} - \alpha\right) = 1$

163. $(\cos p - 1)x^2 + x\cos p + \sin p = 0$

Discriminant must be ≥ 0

$\cos^2 p - 4(\sin p)(\cos p - 1) \geq 0$

$\cos^2 p - 4\sin p\cos p + 4\sin p \geq 0$

$(\cos p - 2\sin p)^2 - 4\sin^2 p + 4\sin p \geq 0$

$(\cos p - 2\sin p)^2 + 4(\sin p)(1 - \sin p) \geq 0$

$(\sin p)$ must lie between 0 and 1 $\Rightarrow p \in (0, \pi)$

164. $a\cos 2x + b\sin 2x = c$

$\dfrac{a(1 - t^2)}{1 + t^2} + \dfrac{b \times 2t}{1 + t^2} = c$ where, $t = \tan x$

$a - at^2 + 2bt = c + ct^2$

$(a + c)t^2 - 2bt + c - a = 0$

Sum of the roots $= \tan\alpha + \tan\beta = \dfrac{2b}{(a + c)}$

Product of the roots $= \dfrac{c - a}{c + a}$

$\tan^2\alpha + \tan^2\beta = \dfrac{4b^2}{(a + c)^2} - \dfrac{2(c - a)}{(c + a)}$

$= \dfrac{4b^2 - 2(c - a)(c + a)}{(a + c)^2}$

$= \dfrac{4b^2 - 2(c^2 - a^2)}{(a + c)^2} = \dfrac{2(2b^2 + a^2 - c^2)}{(a + c)^2}$

165. $\cos\dfrac{7\pi}{8} = \cos\left(\pi - \dfrac{\pi}{8}\right) = -\cos\dfrac{\pi}{8}$

Also $\cos\dfrac{5\pi}{8} = \cos\left(\dfrac{\pi}{2} + \dfrac{\pi}{8}\right) = -\sin\dfrac{\pi}{8}$

Again $\cos\dfrac{3\pi}{8} = \cos\left(\dfrac{\pi}{2} - \dfrac{\pi}{8}\right) = \sin\dfrac{\pi}{8}$

$\therefore \left(1 + \cos\dfrac{\pi}{8}\right)\left(1 + \cos\dfrac{3\pi}{8}\right)\left(1 + \cos\dfrac{5\pi}{8}\right)\left(1 + \cos\dfrac{7\pi}{8}\right)$

$= \left(1 + \cos\dfrac{\pi}{8}\right)\left(1 + \sin\dfrac{\pi}{8}\right)\left(1 - \sin\dfrac{\pi}{8}\right)\left(1 - \cos\dfrac{\pi}{8}\right)$

$= \sin^2\dfrac{\pi}{8}.\cos^2\dfrac{\pi}{8} = \dfrac{1}{4}\left[2\sin\dfrac{\pi}{8}.\cos\dfrac{\pi}{8}\right]^2$

$= \dfrac{1}{4}\left(\sin\dfrac{\pi}{4}\right)^2 = \dfrac{1}{4}\left(\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{1}{8}$

166. $\cos\alpha\cos 2\alpha\cos 4\alpha\cos 8\alpha\cos 16\alpha$

$= \dfrac{1}{2\sin\alpha}2\sin\alpha\cos\alpha\cos 2\alpha\cos 4\alpha\cos 8\alpha\cos 16\alpha$

$= \dfrac{1}{2\sin\alpha}\sin 2\alpha\cos 2\alpha\cos 4\alpha\cos 8\alpha\cos 16\alpha$

$= \dfrac{1}{4\sin\alpha}\sin 4\alpha\cos 4\alpha\cos 8\alpha\cos 16\alpha$

$$= \frac{1}{8\sin\alpha}\sin 8\alpha\cos 8\alpha\cos 16\alpha$$

$$= \frac{1}{16\sin\alpha}\sin 16\alpha\cos 16\alpha = \frac{1}{32\sin\alpha}\sin 32\alpha$$

$$= \frac{\sin\frac{32\pi}{33}}{32\sin\frac{\pi}{33}} \quad \left[\because \alpha = \frac{\pi}{2^5+1} = \frac{\pi}{33}\right]$$

$$= \frac{\sin\frac{\pi}{33}}{32\sin\frac{\pi}{33}} = \frac{1}{32}$$

167. Let each ratio be equal to k

$$x+y+z = k\left[\sin\theta + \sin\left(\theta - \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{2\pi}{3}\right)\right]$$

$$= k\left[\sin\theta + 2\sin\theta\cos\frac{2\pi}{3}\right] = k\left[\sin\theta - \sin\theta\right] = 0$$

168. We have $x_1 + x_2 + x_3 = \sin 2\beta$

$x_1x_2 + x_1x_3 + x_2x_3 = \cos 2\beta$

$x_1x_2x_3 = 2\sin\beta$

$\tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3$

$$= \tan^{-1}\left[\frac{x_1 + x_2 + x_3 - x_1x_2x_3}{1-(x_1x_2 + x_1x_3 + x_2x_3)}\right]$$

$$= \tan^{-1}\left[\frac{\sin 2\beta - 2\sin\beta}{1-\cos 2\beta}\right]$$

$$= \tan^{-1}\left[\frac{2\sin\beta\cos\beta - 2\sin\beta}{2\sin^2\beta}\right]$$

$$= \tan^{-1}\left(\frac{\cos\beta - 1}{\sin\beta}\right)$$

$$= \tan^{-1}\left(\frac{-2\sin^2\frac{\beta}{2}}{2\sin\frac{\beta}{2}\cos\frac{\beta}{2}}\right)$$

$= \tan^{-1}(-\tan\beta/2) = -\beta/2$

169. We have $\sin^4 x + \cos^4 x \le \sin^2 x + \cos^2 x = 1$

$(\because |\sin x| \le 1 \text{ and } |\cos x| \le 1)$

$\Rightarrow \quad a \le 1$ — (1)

Now,

$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$$= 1 - \frac{2\sin^2 2x}{4} \Rightarrow \frac{\sin^2 2x}{2} = 1 - a$$

$\Rightarrow \quad 1 - a \le 1/2 \Rightarrow a \ge 1/2$ — (2)

From (1) and (2), we get $1/2 \le a \le 1$

170. $\tan^{-1}\sqrt{x^2+x} + \sin^{-1}\sqrt{1+x+x^2} = \dfrac{\pi}{2}$

$$\Rightarrow \quad \cos^{-1}\frac{1}{\sqrt{1+(x+x^2)^2}} = \frac{\pi}{2} - \sin^{-1}\sqrt{x^2+x+1}$$

$$\Rightarrow \quad \cos^{-1}\frac{1}{\sqrt{1+(x+x^2)^2}} = \cos^{-1}\sqrt{1+x+x^2}$$

$x^2 + x + 1 = 1$

$x(x+1) = 0$

$\Rightarrow \quad x = 0, -1$

171. Statement 2 is false.

In fact, the equation has real solutions iff $\left|\dfrac{c^2}{a^2+b^2}\right| \le 1$

Consider Statement 1

$$\frac{c^2}{a^2+b^2} = \frac{16}{13} > 1$$

$\therefore$ the equation has no solutions

$\Rightarrow$ Statement 1 is true

Choice (c)

172. It is clear that the maximum value of $(3+\cos\theta)$ is 4, and its minimum value is 2

So, Statement 2 is true

Consider Statement 1

Using Statement 2, the angle represented by $(3+\cos\theta)$ is in the 2nd and 3rd quadrants

$\Rightarrow \quad \cos(3+\cos\theta) < 0$

$\Rightarrow$ Statement 1 is true

Choice (a)

173. Statement 2 is false

Statement 2 is true if $x, y > 0$ and $xy < 1$

Consider Statement 1.

$$\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{3} = \tan^{-1}\left(\frac{1/5 + 1/3}{1 - 1/15}\right)$$

$$= \tan^{-1}\left(\frac{8}{14}\right) = \tan^{-1}\left(\frac{4}{7}\right)$$

$\Rightarrow$ Statement 1 is true

Choice (c)

174. Statement 2

In the 4th quadrant, sinθ is negative while cosθ is positive

But, we cannot say that (sinθ + cosθ) is negative

⇒ statement 2 is false

Now, $\dfrac{\cos\theta+4\sin\theta}{\sin\theta+4\cos\theta}=\dfrac{1+4\tan\theta}{4+\tan\theta}$

$=\dfrac{1-\frac{4}{5}}{4-\frac{1}{5}}=\dfrac{1}{19}$

⇒ statement 1 is true

Choice (c)

175. Statement 2 is true

Consider statement 1

(1 + cotA)(1 + cotB)

$=\left(1+\dfrac{1}{\tan A}\right)\left(1+\dfrac{1}{\tan B}\right)$

$=\dfrac{(1+\tan A)(1+\tan B)}{\tan A\tan B}=\dfrac{2}{\tan A\tan B}$, using statement 2

⇒ statement 1 is false

Choice (d)

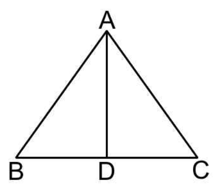
176. Statement 2 is true

Statement 1 is false

Statement 1 is true for $-1\le x\le 1$

⇒ choice (d)

177. Statement 2 is a well known theorem in geometry.



Let ABC be a triangle in which AB = AC. Let D be the mid-point of BC. Then AD is the median through A; it is the altitude through A. It is also the perpendicular bisector of BC.

∴ The orthocentre, centoriod and circumcentre all lie on AD which passes through A and is perpendicular to BC. So Statement 1 is correct.

Statement 2 only asserts the collinearity of orthcentre, centroid and circumcentre. It does not explain why the line of collinearity passes through A and is perpendicular to BC.

So statement 2 is not the explanation for Statement 1.

Choice (b)

178. tan(A+B) = –tan C (∵ A + B + C = 180°)

$\therefore \dfrac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$

⇒ tan A + tan B + tan C = tan A tan B tan C

∴ Statement 2 is true

$\sum\dfrac{\cot A+\cot B}{\tan A+\tan B}=\sum\dfrac{1}{\tan A\tan B}$

$=\dfrac{\tan A+\tan B+\tan C}{\tan A\tan B\tan C}=1$

⇒ Statement 1 is false

Choice (d)

179. Statement 2 is a standard result

Consider statement 1:

Now $-\sqrt{2}\le\sin x+\cos x\le\sqrt{2}$ and

$0\le 1+\sin 2x\le 2$

∴ The equation is satisfied when

$\sin x+\cos x=\pm\sqrt{2}$ and $1+\sin 2x=2$

Now $1+\sin 2x=2\Rightarrow 2x=n\pi+(-2)^n\dfrac{\pi}{2}$

$\Rightarrow x=n\dfrac{\pi}{2}+(-1)^n\dfrac{\pi}{4}$

We observe that $x=\dfrac{\pi}{4}$ and $\dfrac{-3\pi}{4}$ in $[-\pi,\pi]$ satisfy the equation.

∴ statement 1 is false

Choice (d)

180. Statement 2 is a standard result

Consider statement 1

$$\frac{2(p+q)(1-pq)}{(1+p^2)(1+q^2)}=\frac{2\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\left(1-\tan\frac{A}{2}\tan\frac{B}{2}\right)}{\left(1+\tan^2\frac{A}{2}\right)\left(1+\tan^2\frac{B}{2}\right)}$$

$$=\frac{2\left(\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\dfrac{\sin\frac{B}{2}}{\cos\frac{B}{2}}\right)\left(1-\dfrac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}\right)}{\sec^2\frac{A}{2}\sec^2\frac{B}{2}}$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\left(\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}\right)}{\cos^2\frac{A}{2}\cos^2\frac{B}{2}\sec^2\frac{A}{2}\sec^2\frac{B}{2}}$$

$$= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A+B}{2}\right) = \sin(A+B)$$

$= \sin(180° - C) = \sin C$

Statement 1 is true, but it does not follow from statement 2.

Choice (b)

181. P is $(a\cos\theta, b\sin\theta)$, Q is $(a\sin\theta, b\cos\theta)$

$\therefore CP^2 = a^2\cos^2\theta + b^2\sin^2\theta;$

$CQ^2 = a^2\sin^2\theta + b^2\cos^2\theta$

182. $u^2 = a^2\cos^2\theta + b^2\sin^2\theta + a^2\sin^2\theta + b^2\cos^2\theta$

$+ 2\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

$= a^2 + b^2 +$

$2\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

R.H.S is maximum for $\theta = \frac{\pi}{4}$

Max. $u^2 = a^2 + b^2 + 2\sqrt{\frac{a^2}{2} + \frac{b^2}{2}}\sqrt{\frac{a^2}{2} + \frac{b^2}{2}}$

$= a^2 + b^2 + a^2 + b^2 = 2(a^2 + b^2)$

183. Maximum $= 2(a^2 + b^2)$

Minimum $= a^2 + b^2 + 2ab$

Difference $= a^2 + b^2 - 2ab = (a - b)^2$

184. $\cot 2^{n-1}\theta - \cot 2^n\theta = \frac{1}{\tan 2^{n-1}\theta} - \frac{1}{\tan 2\left(2^{n-1}\theta\right)}$

$$= \frac{1}{\tan 2^{n-1}\theta} - \frac{1 - \tan^2\left(2^{n-1}\theta\right)}{2\tan\left(2^{n-1}\theta\right)} = \frac{2 - 1 + \tan^2\left(2^{n-1}\theta\right)}{2\tan\left(2^{n-1}\theta\right)}$$

$$= \frac{\sec^2\left(2^{n-1}\theta\right)\cos\left(2^{n-1}\theta\right)}{2\sin\left(2^{n-1}\theta\right)} = \frac{1}{\sin 2^n\theta} = \operatorname{cosec} 2^n\theta$$

185. $\tan\alpha = \cot\alpha - 2\cot 2\alpha$

$2\tan 2\alpha = 2(\cot 2\alpha - 2\cot 2^2\alpha)$

$2^2\tan 2^2\alpha = 2^2(\cot 2^2\alpha - 2\cot 2^3\alpha)$

....

....

$2^{n-1}\tan 2^{n-1}\alpha = 2^{n-1}[\cot 2^{n-1}\alpha - 2\cot 2^n\alpha]$

Adding we get

$$\sum_{r=1}^{n} 2^{r-1}\tan 2^{r-1}\alpha = \cot\alpha - 2^n\cot 2^n\alpha$$

186. Roots are

$$\frac{4\sin\theta\cos^3\theta \pm \sqrt{16\sin^2\theta\cos^6\theta - 4\sin^2 2\theta\cos 2\theta}}{2\sin^2 2\theta}$$

$= \frac{1}{2}(\cot\theta \pm \tan\theta)$

$= \operatorname{cosec} 2\theta, \cot 2\theta.$

187. $0 \le \cos^{-1}x \le \pi,\ 0 \le (\sin^{-1}y)^2 \le \frac{\pi^2}{4}$

$0 \le \cos^{-1}x + (\sin^{-1}y)^2 \le \pi + \frac{\pi^2}{4}$

$0 \le \frac{k\pi^2}{4} \le \pi + \frac{\pi^2}{4}$

$0 \le k \le \frac{4}{\pi} + 1$ — (1)

188. Eliminating $(\sin^{-1}y)^2$ from (i) and (ii)

$\cos^{-1}x + \frac{\pi^4}{16\cos^{-1}x} = \frac{k\pi^2}{4}$

$\Rightarrow$ $16(\cos^{-1}x)^2 - 4k\pi^2\cos^{-1}x + \pi^4 = 0$

$\cos^{-1}x \in R$

$\Rightarrow$ Discriminant ≥ 0

$\Rightarrow$ $16k^2\pi^4 - 64\pi^4 \ge 0$

$k^2 - 4 \ge 0$

$k \in (-\infty, -2] \cup [2, \infty)$ — (2)

From (1) and (2)

$\left[2, \frac{4}{\pi} + 1\right]$ — (3)

189. From (3), $k \in \left[2, \frac{4}{\pi} + 1\right]$ and the only integer solution in this interval is $k = 2$

190. Let

$x > 0.\ \tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$

$\Rightarrow$ $\tan^{-1}\left[\frac{6x+2}{8x^2+6x}\right] = \tan^{-1}\left(\frac{2}{x^2}\right)$

$\Rightarrow$ $3x^3 + x^2 = 8x^2 + 6x$

$\Rightarrow$ $x(3x^2 - 7x - 6) = 0$

$\Rightarrow$ $x = 3$ as $x > 0$

If $x = 0$, $\tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$, $\cot^{-1}(0) = \frac{\pi}{2}$

$\Rightarrow$ $\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

$\therefore$ $x = 0, 3$

For $x < 0$, there are no solutions

Choices (a), (d)

191. We have $(a + b)\left(\frac{\sec^4 x}{a} + \frac{\tan^4 x}{b}\right) = 1$

$\sec^4 x + \tan^4 x + \frac{b}{a}\sec^4 x + \frac{a}{b}\tan^4 x = 1$

$(\sec^2 x - \tan^2 x)^2 + 2\sec^2 x\tan^2 x$

$+ \frac{b}{a}\sec^4 x + \frac{a}{b}\tan^4 x = 1$

$\Rightarrow$ $\left(\sqrt{\frac{b}{a}}\sec^2 x + \sqrt{\frac{a}{b}}\tan^2 x\right)^2 = 0$

$\Rightarrow$ $\frac{b}{a}\sec^4 x = \frac{a}{b}\tan^4 x$

$\Rightarrow$ $\frac{\sec^4 x}{a^2} = \frac{\tan^4 x}{b^2}$

$\Rightarrow$ $\frac{\sec^2 x}{a} = \frac{\tan^2 x}{b}$

$= \frac{\sec^2 x - \tan^2 x}{a - b} = \frac{1}{(a - b)}$

or

$\frac{\sec^2 x}{a} = \frac{-\tan^2 x}{b} = \frac{\sec^2 x - \tan^2 x}{a + b} = \frac{1}{(a + b)}$

$\Rightarrow$ $\frac{\sec^8 x}{a^3} - \frac{\tan^8 x}{b^3}$

$= \frac{a}{(a - b)^4} - \frac{b}{(a - b)^4}$

or $\frac{a}{(a + b)^4} - \frac{b}{(a + b)^4} = \frac{1}{(a - b)^3}$ or $\frac{(a - b)}{(a + b)^4}$

Choices (a), (d)

192. $\sin\theta + 2\cos\theta = 2$

Let $\cos\theta - 2\sin\theta = k$

Squaring and adding,

$5 = 4 + k^2$

$k^2 = 1 \Rightarrow k = \pm 1$

Choices (a), (c)

193. $y = \cos^{-1}x + \sin^{-1}x - 2\sin^{-1}x = \frac{\pi}{2} - 2\sin^{-1}x$

All the choices are true on verification

Choices (a), (b), (c), (d)

194. $\tan\frac{\theta}{2} = \operatorname{cosec}\theta - \sin\theta$

$= \frac{1 + \tan^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}} - \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

$= \frac{\left(1 + \tan^2\frac{\theta}{2}\right)^2 - 4\tan^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}\left(1 + \tan^2\frac{\theta}{2}\right)}$

$\Rightarrow$ $2\tan^2\frac{\theta}{2}\left(1 + \tan^2\frac{\theta}{2}\right) = \left(1 - \tan^2\frac{\theta}{2}\right)^2$

Put $x = \tan^2\frac{\theta}{2}$

$2x(1 + x) = (1 - x)^2$

$2x + 2x^2 = 1 + x^2 - 2x$

$x^2 + 4x - 1 = 0$

$x = \frac{-4 \pm \sqrt{16 + 4}}{2} = -2 \pm \sqrt{5}$

$x = \tan^2\frac{\theta}{2}$ is positive

$\therefore$ $x = -2 + \sqrt{5}$

Also

$(9 - 4\sqrt{5})(2 + \sqrt{5}) = 18 + 9\sqrt{5} - 8\sqrt{5} - 20$

$= -2 + \sqrt{5}$

Choices (b), (c)

195. $\frac{x}{y} = \frac{\cos A}{\cos B}$

$\Rightarrow$ $\frac{x}{\cos A} = \frac{y}{\cos B} = K$ (say)

$\Rightarrow$ $x = K\cos A$, $y = K\cos B$

$\frac{x\tan A + y\tan B}{x + y} = \frac{K\cos A\tan A + K\cos B\tan B}{K\cos A + K\cos B}$

$= \frac{\sin A + \sin B}{\cos A + \cos B}$

$= \frac{2\sin\frac{A + B}{2}\cos\frac{A - B}{2}}{2\cos\frac{A + B}{2}\cos\frac{A - B}{2}}$

$= \tan\frac{A + B}{2}$

Similarly, we can prove (b)

$$\frac{y\sin A + x\sin B}{y\sin A - x\sin B} = \frac{K\cos B\sin A + K\cos A\sin B}{K\cos B\sin A - K\cos A\sin B}$$

$$= \frac{\sin(A+B)}{\sin(A-B)}$$

x cos A + y cos B

$= K \cos A \cos A + K \cos B \cos B$

$= K(\cos^2 A + \cos^2 B)$

$\neq 0$

Choices (a), (b), (c)

196. Period of $\sin x.\cos x = \frac{1}{2}\sin 2x$ is $\frac{2\pi}{2} = \pi$

$\Rightarrow$ Period of sinx cosx + 2 sin x = 2π

Period of $\frac{\sin 3x + \tan 2x}{\cot 4x + \sec 5x}$ is L.C.M of $\left(\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{2\pi}{5}\right)$

which is 2π.

Period of $\tan x \tan\left(\frac{\pi}{3} - x\right)\tan\left(\frac{\pi}{3} + x\right) = \tan 3x$ is $\frac{\pi}{3}$

Period of $3\sin^2 x + \cos^3 x$ is 2π

Choices (a), (b), (d)

197. $\tan\alpha$ and $\tan\beta$ are the roots of $x^2 + px + q = 0$

$\therefore$ $\tan\alpha + \tan\beta = -p$

$\tan\alpha \tan\beta = q$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{-p}{1-q}$$

$$= \frac{p}{q-1}$$

$\sin^2(\alpha+\beta) + p\sin(\alpha+\beta)\cos(\alpha+\beta) + q\cos^2(\alpha+\beta)$

$= \cos^2(\alpha+\beta)\tan^2(\alpha+\beta) + p\sin(\alpha+\beta)\cos(\alpha+\beta)$

$+ q\cos^2(\alpha+\beta)$

$= \cos^2(\alpha+\beta)\,[\tan^2(\alpha+\beta) + p\tan(\alpha+\beta) + q]$

$$= \cos^2(\alpha+\beta)\left[\frac{p^2}{(q-1)^2} + p.\frac{p}{q-1} + q\right]$$

$$= \frac{1}{1+\tan^2(\alpha+\beta)}\left[\frac{p^2 + p^2(q-1) + q(q-1)^2}{(q-1)^2}\right]$$

$$= \frac{1}{1 + \frac{p^2}{(q-1)^2}}\left[\frac{p^2(1+q-1) + q(q-1)^2}{(q-1)^2}\right]$$

$$= \frac{(q-1)^2}{(q-1)^2 + p^2}.\frac{q[p^2 + (q-1)^2]}{(q-1)^2} = q$$

Also $q - 1 \neq \cos(\alpha+\beta)$ and $\sin(\alpha+\beta) \neq -p$

Choices (a), (b)

198. (a) $4\cos 48° \sin 18° \cos 12°$

$= 4\sin 42° \sin 18° \sin 78°$

$= 4\sin(60° - 18°)\sin 18° \sin(60° + 18°)$

$= 4.\frac{1}{4}\sin 3(18°)$

$= \sin 54°$

$= \cos 36°$

$\therefore$ (a) $\rightarrow$ (q)

(b) Let $C = 16\cos 6° \cos 12° \cos 24° \cos 48°.\cos 96°$

$\therefore C = -16.\sin 6° \cos 6°.\cos 12°.\cos 24°.\cos 48°$;

$(\because \cos 96° = -\sin 6°)$

$= -8\sin 12°.\cos 12° \cos 24° \cos 48°$

$= -4\sin 24° \cos 24° \cos 48°$

$= -2\sin 48° \cos 48°$

$= -\sin 96°$

$= -\cos 6° \quad = \cos(906°)$

$\therefore$ (b) $\rightarrow$ (s)

(c) $24\sin 6° \cos^3 6° + 24\sin^3 6° \cos 6°$

$- 18\sin 6° \cos 6° - 32\sin^3 6° \cos^3 6°$

$= 24\sin 6° \cos 6° [\cos^2 6° + \sin^2 6°]$

$- 18\sin 6° \cos 6° - 4[2\sin 6° \cos 6°]^3$

$= 24\sin 6° \cos 6° - 18\sin 6° \cos 6°$

$- 4\sin^3 12$

$= 6\sin 6° \cos 6° - 4\sin^3 12°$

$= 3\sin 12° - 4\sin^3 12°$

$= \sin 36° = \cos 54° = \cos(306°)$

$\therefore$ (c) $\rightarrow$ (r)

(d) Take $a = \cos^2 6°$ and $b = \sin^2 6°$

$\therefore L.H.S = (\cos 6°)^a.(\cos 6°)^{ab}.(\cos 6°)^{ab^2} \ldots\infty$

$= (\cos 6°)^{a+ab+ab^2 \ldots\ldots\infty}$

$= (\cos 6°)^{\frac{a}{1-b}}$ since $|b| < 1$

$= (\cos 6°)^{\frac{\cos^2 6°}{1-\sin^2 6°}} = \cos 6°$

$\therefore$ (d) $\rightarrow$ (p)

199. (a) Let $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \theta$

Then, $\sin^{-1}\left(\frac{-2\sqrt{2}}{3}\right) = -\theta$

$$\sin\left(\frac{1}{2}\sin^{-1}\left(\frac{-2\sqrt{2}}{3}\right)\right) = \sin\left(\frac{-\theta}{2}\right)$$

$$= -\sin\left(\frac{\theta}{2}\right)$$

Now, since $\sin\theta = \frac{2\sqrt{2}}{3}, \cos\theta = \frac{1}{3}$

$$1 - 2\sin^2\frac{\theta}{2} = \cos\theta = \frac{1}{3}$$

$$2\sin^2\frac{\theta}{2} = \frac{2}{3}$$

$$\Rightarrow \sin\frac{\theta}{2} = \frac{1}{\sqrt{3}}$$

Hence $\sin\left(\frac{1}{2}\sin^{-1}\left(\frac{-2\sqrt{2}}{3}\right)\right) = \frac{-1}{\sqrt{3}}$

$\therefore$ (a) $\rightarrow$ (s)

(b) Let $\cos^{-1}\left(\frac{1}{3}\right) = \theta$

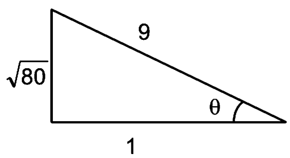
$$\cos^{-1}\left(\frac{-1}{3}\right) = \pi - \theta$$

$$\cot\left(\cos^{-1}\left(\frac{-1}{3}\right)\right) = \cot(\pi - \theta)$$

$$= -\cot\theta = \frac{-1}{2\sqrt{2}}$$

$\therefore$ (b) $\rightarrow$ (q)

(c) Let $\sin^{-1}\left(\frac{\sqrt{80}}{9}\right) = \theta$



$$\Rightarrow \sin\theta = \frac{\sqrt{80}}{9}$$

$$\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{80}}{9}\right) = \sin\frac{\theta}{4}$$

$$\cos\theta = \frac{1}{9}$$

$$1 - 2\sin^2\frac{\theta}{2} = \frac{1}{9}$$

$$\Rightarrow 2\sin^2\frac{\theta}{2} = \frac{8}{9}$$

$$\sin^2\frac{\theta}{2} = \frac{4}{9}$$

$$\Rightarrow \sin\frac{\theta}{2} = \frac{2}{3} \Rightarrow \cos\frac{\theta}{2} = \frac{\sqrt{5}}{3}$$

$$1 - 2\sin^2\frac{\theta}{4} = \cos\frac{\theta}{2} = \frac{\sqrt{5}}{3}$$

$$2\sin^2\frac{\theta}{4} = 1 - \frac{\sqrt{5}}{3} = \frac{3 - \sqrt{5}}{3}$$

$$= \frac{\left(\sqrt{\frac{5}{2}} - \sqrt{\frac{1}{2}}\right)^2}{3}$$

$$\sin^2\frac{\theta}{4} = \frac{\left(\sqrt{\frac{5}{2}} - \sqrt{\frac{1}{2}}\right)^2}{6}$$

$$\Rightarrow \sin\frac{\theta}{4} = \frac{\sqrt{\frac{5}{2}} - \sqrt{\frac{1}{2}}}{\sqrt{6}} = \frac{\sqrt{5} - 1}{2\sqrt{3}}$$

$\therefore$ (c) $\rightarrow$ (r)

(d) Since $3^2 + 4^2 = 5^2$

$$\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5}$$

$$= \sin^{-1}\frac{3}{5} + \cos^{-1}\frac{3}{5} = \frac{\pi}{2}$$

$$\Rightarrow \sin\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5}\right) = 1$$

$\therefore$ (d) $\rightarrow$ (p)

200. (a) Expression

$$= \frac{1}{3} \times \frac{\sqrt{3}}{2}\sin(60^\circ - 20^\circ)\sin(60^\circ + 20^\circ)\sin 20^\circ$$

$$= \frac{1}{2\sqrt{3}}.\frac{1}{4}\sin(3 \times 20^\circ)$$

$$= \frac{1}{8\sqrt{3}}.\frac{\sqrt{3}}{2} = \frac{1}{16}$$

$\therefore$ (a) $\rightarrow$ (s)

(b) $\sin 36°\sin 72°\sin 108°\sin 144°$

$= \sin 36°\sin 72° \times \sin 36°\sin 72°$

$= \sin^2 36° \sin^2 72°$

$= \sin^2 36° \cos^2 18°$

$= (1 - \cos^2 36°)(1 - \sin^2 18°)$

$$= \left[1 - \left(\frac{\sqrt{5}+1}{4}\right)^2\right]\left[1 - \left(\frac{\sqrt{5}-1}{4}\right)^2\right]$$

$$= \left(1 - \frac{6+2\sqrt{5}}{16}\right)\left(1 - \frac{6-2\sqrt{5}}{16}\right)$$

$$= \frac{\left(10-2\sqrt{5}\right)\left(10+2\sqrt{5}\right)}{256} = \frac{80}{256} = \frac{5}{16}$$

$\therefore$ (b) $\to$ (p)

(c) $\sin 12° \sin 48° \sin 54°$

$$= \frac{1}{2}\left[2\sin 12° \sin 48°\right]\sin 54°$$

$$= \frac{1}{2}\left[\cos 36° - \cos 60°\right]\sin 54°$$

$$= \frac{1}{2}\left[\cos^2 36° - \cos 60° \cos 36°\right],$$

$(\because \sin 54° = \cos 36°)$

$$= \frac{1}{2}\left[\left(\frac{\sqrt{5}+1}{4}\right)^2 - \frac{1}{2}\left(\frac{\sqrt{5}+1}{4}\right)\right]$$

$$= \frac{1}{2}\left[\frac{6+2\sqrt{5}}{16} - \frac{\left(\sqrt{5}+1\right)}{8}\right]$$

$$= \frac{1}{32}\left[6 + 2\sqrt{5} - 2\sqrt{5} - 2\right] = \frac{1}{8}$$

$\therefore$ (c) $\to$ (q)

(d) Expression

$$= 8\left[\sin^2\frac{\pi}{14}\sin^2\frac{3\pi}{14}\sin^2\frac{5\pi}{14}\right]$$

$$= 8\left[\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right]^2$$

$$= 8\left\{\frac{1}{2\cos\frac{\pi}{14}} \times \sin\frac{2\pi}{14}\sin\frac{5\pi}{14}\sin\frac{3\pi}{14}\right\}^2$$

$$= 8\left\{\frac{1}{2\cos\frac{\pi}{14}} \times \frac{1}{2}\left(\cos\frac{3\pi}{14} - \cos\frac{7\pi}{14}\right)\sin\frac{3\pi}{14}\right\}^2$$

$$= 8\left\{\frac{1}{4\cos\frac{\pi}{14}} \times \frac{1}{2}\sin\frac{6\pi}{14}\right\}^2 = 8\left(\frac{1}{64}\right) = \frac{1}{8}$$

$\therefore$ (d) $\to$ (q)

CHAPTER

4

PROPERTIES OF TRIANGLES

CHAPTER OUTLINE

INTRODUCTION

In this unit, we shall derive a number of relations connecting the sides and angles of a triangle using trigonometric or circular functions. The important elements of a triangle are its three sides and the three angles (one obvious relation between the angles is that their sum is equal to 180°). There are other elements like the circum radius, in radius, radii of ex-circles etc. connected with a triangle. Formulas for these elements in terms of trigonometric functions of angles or in terms of the sides will also be derived. It may be mentioned that some of these elements can be obtained using geometrical constructions. However, geometrical procedures are cumbersome and, therefore, analytical tools are preferable. The formulas derived in this unit are very important from the point of view of their applications in engineering and technology.

We shall be using the following notations:

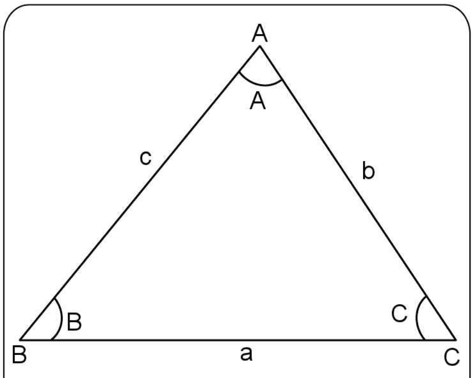


Fig. 4.1

The triangle is taken as ABC, (see Fig. 4.1). Note that the triangle may be acute angled (scalene) or right angled or obtuse angled.

(i) The sides BC, CA, AB are denoted by a, b, c respectively.

(ii) The angles are denoted by A, B, C (angle A is opposite to the side BC or the side a; angle B is opposite to the side CA or the side b and angle C is opposite to the side AB or the side c.

(iii) The semi perimeter of the triangle is denoted by s. i.e., $s = \frac{1}{2}(a + b + c)$

(iv) The area of the triangle ABC, is denoted by Δ.

(v) The circumcentre (denoted by S) is the point of intersection of the perpendicular bisectors of the sides of the triangle. The circle whose centre is S and passing through the vertices of the triangle is called the circumcircle of the triangle. The radius of the circumcircle is denoted by R (called circum radius).

(vi) The incentre (denoted by I) is the point of intersection of the internal bisectors of the angles of the triangle. The circle with centre at I and touching internally the three sides of the triangle is called the incircle of the triangle. The radius of the incircle is denoted by r (called inradius).

(vii) The point of intersection of the internal bisector of the angle A and the external bisectors of angles B and C is denoted by I_1. The circle with centre at I_1 and touching BC internally and AB and AC externally is called the excircle opposite to A and its radius is denoted by r_1. In a similar manner we may define the excircles opposite to B and C and their radii are denoted by r_2 and r_3 respectively.

LAW OF SINES (OR SINE FORMULAS)

In any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \qquad \text{— (I)}$$

Fig. 4.2 (i) refers to acute angled; ΔABC

Fig. 4.2 (ii) refers to right angled (A = 90°); ΔABC, and

Fig. 4.2 (iii) refers to obtuse angled ΔABC, (A > 90°)

Case (i): Δ ABC, acute angled.

Draw SD perpendicular to BC. We have SA = SB = SC = R and ∠BSD = A.

$$\text{From } \Delta SBD, \ \frac{BD}{SB} = \sin A \quad \Rightarrow \frac{\frac{a}{2}}{R} = \sin A$$

$$\Rightarrow \quad \frac{a}{\sin A} = 2R\,.$$

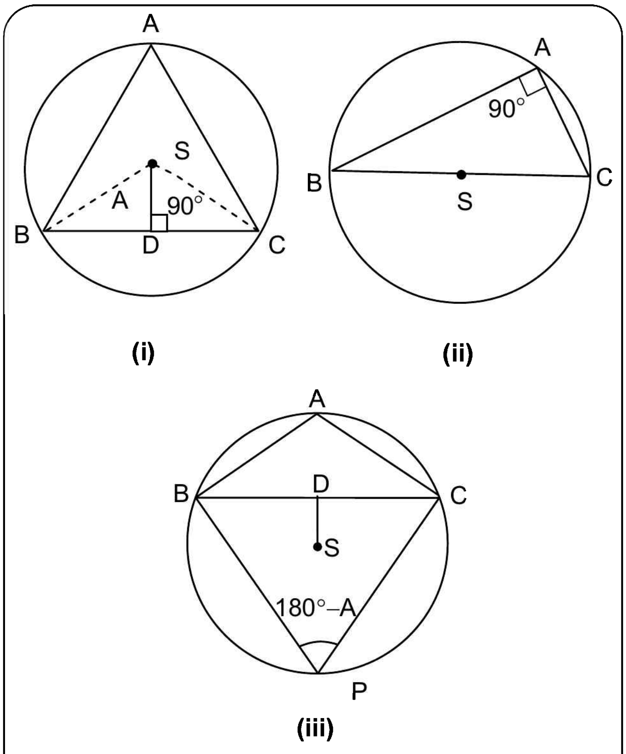


Fig. 4.2

Similarly, by drawing perpendiculars from S to the sides CA and AB and using similar arguments, we obtain $\frac{b}{\sin B} = 2R$ and $\frac{c}{\sin C} = 2R$

Case (ii): ΔABC, is right angled.

It is clear that $a = 2R$ and since $A = 90°$, $\frac{a}{\sin A} = 2R$;

$$\text{Also, } \frac{AC}{BC} = \sin B \text{ i.e., } \frac{b}{2R} = \sin B \text{ or } \frac{b}{\sin B} = 2R.$$

Again, since $\frac{AB}{BC} = \sin C$, we get $\frac{c}{\sin C} = 2R$.

Case (iii): ΔABC, is obtuse angled.

Let P be a point on the arc BC of the circum circle (on the arc not containing A). As the quadrilateral ABPC is cyclic, $\angle BPC = 180° - A$. A being obtuse, $(180° - A)$ will be acute. Draw SD perpendicular to BC. It can be easily seen that $\angle BSD = 180° - A$.

From ΔBSD, $\frac{BD}{SB} = \sin(180^\circ - A) = \sin A$

$$\Rightarrow \frac{\frac{a}{2}}{R} = \sin A \text{ or } \frac{a}{\sin A} = 2R$$

By drawing perpendiculars from S to CA and AB, we can obtain $\frac{b}{\sin B} = 2R, \frac{c}{\sin C} = 2R$

From (I),

$$\left.\begin{aligned} a &= 2R\sin A \\ b &= 2R\sin B \\ c &= 2R\sin C \end{aligned}\right\} \quad \text{— (II)}$$

LAW OF COSINES (OR COSINE FORMULAS)

In any triangle ABC,

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= c^2 + a^2 - 2ca\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}\right\} \quad \text{— (III)}$$

Case (i):

(Ref . Fig. 4.3 (i))
Draw BQ perpendicular to AC.

$$a^2 = BC^2 = BQ^2 + QC^2 = BQ^2 + (AC - AQ)^2 \quad \text{— (1)}$$

Now, from Δ BAQ, $\frac{BQ}{AB} = \sin A, \frac{AQ}{AB} = \cos A$. This means that $BQ = c \sin A$ and $AQ = c \cos A$

Substituting in (1) we get,

$$\begin{aligned} a^2 &= (c \sin A)^2 + (b - c \cos A)^2 \\ &= b^2 + c^2 - 2bc \cos A \end{aligned}$$

Case (ii):

(Refer Fig 4.3 (ii))
Since

$$\begin{aligned} A = 90°, a^2 &= BC^2 = AC^2 + AB^2 = b^2 + c^2 \\ &= b^2 + c^2 - 2bc \cos A \quad (\cos A = \cos 90° = 0) \end{aligned}$$

Case (iii):

(Refer Fig. 4.3 (iii))
The perpendicular from B to AC meets CA produced in Q.

Again, $a^2 = BC^2 = BQ^2 + QC^2 = BQ^2 + (QA + AC)^2$ — (2)

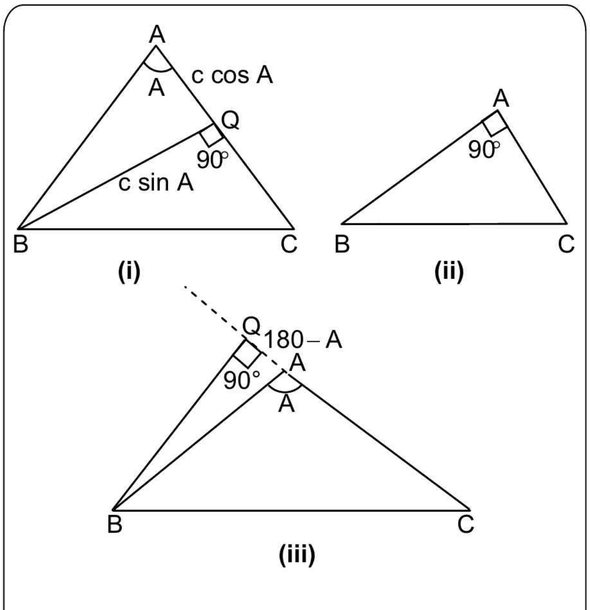


Fig. 4.3

From Δ BAQ , $BQ = c \sin(180° - A) = c \sin A$

and $QA = c \cos(180° - A) = -c \cos A$ (Note that cos A is negative so that QA > 0)

Substituting in (2), we get

$a^2 = (c \sin A)^2 + (-c \cos A + b)^2 = b^2 + c^2 - 2bc \cos A$

In a similar manner, it can be shown that $b^2 = c^2 + a^2 - 2ca \cos B$

$$\text{and } c^2 = a^2 + b^2 - 2ab \cos C$$

From (III), we have

$$\left.\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ \cos B &= \frac{c^2 + a^2 - b^2}{2ca} \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}\right\} \quad \text{— (IV)}$$

$$\begin{aligned} b \cos C + c \cos B &= 2R \sin B \cos C + 2R \sin C \cos B \\ &= 2R \sin (B + C) = 2R \sin A = a \end{aligned}$$

$$\begin{aligned} \text{Using II, } b^2 + c^2 - a^2 &= 4R^2 [\sin^2 B + \sin^2 C - \sin^2 A] \\ &= 4R^2 [\sin^2 B + \sin (C + A) \sin (C - A)] \\ &= 4R^2 [\sin^2 B + \sin B \sin (C - A)] \\ &= 4R^2 \sin B [\sin B + \sin(C - A)] \\ &= 4R^2 \sin B [\sin (C + A) + \sin (C - A)] \\ &= 4R^2 \sin B \, 2\sin C. \cos A \\ &= 2(2R \sin B)\, 2 \sin C . \cos A \\ &= 2\, bc \cos A \end{aligned}$$

$\therefore \quad a^2 = b^2 + c^2 - 2bc \cos A$

PROJECTION FORMULAS

In any triangle ABC,

$$\left.\begin{aligned} a &= b\cos C + c\cos B \\ b &= c\cos A + a\cos C \\ c &= a\cos B + b\cos A \end{aligned}\right\} \quad \text{— (V)}$$

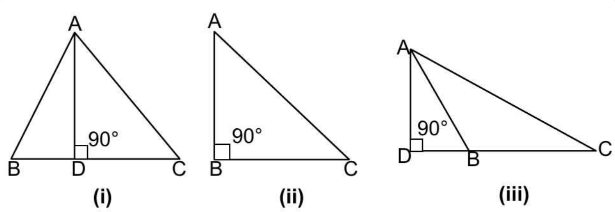


Fig. 4.4

Case (i):

(Refer Fig. 4.4 (i)). Draw AD perpendicular to BC.
a = BC = BD + DC = Projection of AB on BC + projection of AC on BC

$= AB \cos B + AC \cos C$

$= c \cos B + b \cos C$

Case (ii):

(Refer Fig. 4.4 (ii))

$a = BC = AC \cos C = b \cos C$
$= b \cos C + c \cos B$, since
$B = 90°$

Case (iii):

(Refer Fig. 4.4 (iii)). Draw AD perpendicular to BC meeting CB produced in D.

$$\begin{aligned} a = BC &= CD - DB = AC \cos C - AB \cos(180° - B) \\ &= AC \cos C + AB \cos B \quad (\text{from } \Delta ABD) \\ &= b \cos C + c \cos B \end{aligned}$$

The other results in (V) can be proved in a similar manner.

Using the sine formulas and cosine formulas a number of useful results follow. We shall take up these one by one.

Result 1

$$\left.\begin{aligned} \sin A &= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} \\ \sin B &= \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)} \\ \sin C &= \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)} \end{aligned}\right\} \quad \text{—(VI)}$$

$$\text{where } s = \frac{1}{2}(a+b+c)$$

We prove the first of the above set of formulas.

From the cosine formula (IV),

$$\sin^2 A = 1 - \cos^2 A$$

$$= 1 - \left(\frac{b^2+c^2-a^2}{2bc}\right)^2 = \frac{(2bc)^2 - (b^2+c^2-a^2)^2}{4b^2c^2}$$

$$= \frac{(2bc - b^2 - c^2 + a^2)(2bc + b^2 + c^2 - a^2)}{4b^2c^2}$$

$$= \frac{[a^2 - (b-c)^2][(b+c)^2 - a^2]}{4b^2c^2}$$

$$= \frac{(a-b+c)(a+b-c)(b+c-a)(b+c+a)}{4b^2c^2}$$

$$= \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$$

$$= \frac{(2s)(2s-2a)(2s-2b)(2s-2c)}{4b^2c^2}$$

$$= \frac{4s(s-a)(s-b)(s-c)}{b^2c^2}$$

$$\Rightarrow \quad \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

(since $0 < A < 180°$, $\sin A > 0$)

Result 2

$$\left.\begin{aligned} \sin\frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}} \\ \sin\frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{ca}} \\ \sin\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}} \end{aligned}\right\} \quad \text{—(VII)}$$

We shall outline the proof for the first formula of the set.

We have,

$$2\sin^2\frac{A}{2} = 1 - \cos A = 1 - \left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b-c)^2}{2bc} = \frac{(a-b+c)(a+b-c)}{2bc}$$

$$= \frac{(2s-2b)(2s-2c)}{2bc} = \frac{2(s-b)(s-c)}{bc}$$

$$\Rightarrow \quad \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

Result 3

$$\left.\begin{aligned} \cos\frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}} \\ \cos\frac{B}{2} &= \sqrt{\frac{s(s-b)}{ca}} \\ \cos\frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}} \end{aligned}\right\} \quad \text{—(VIII)}$$

For proving the first formula of the set, we write

$$2\cos^2\frac{A}{2} = 1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(b+c+a)(b+c-a)}{2bc} = \frac{2s(2s-2a)}{2bc}$$

$$= \frac{2s(s-a)}{bc}$$

$$\Rightarrow \quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

Result 4

$$\left.\begin{aligned} \tan\frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \\ \tan\frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \\ \tan\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \end{aligned}\right\} \quad \text{— (IX)}$$

The above result follows from (VII) and (VIII)

Result 5

$$\left.\begin{aligned} \tan\frac{B-C}{2} &= \frac{b-c}{b+c}\cot\frac{A}{2} \\ \tan\frac{C-A}{2} &= \frac{c-a}{c+a}\cot\frac{B}{2} \\ \tan\frac{A-B}{2} &= \frac{a-b}{a+b}\cot\frac{C}{2} \end{aligned}\right\} \quad \text{— (X)}$$

Now, $\dfrac{b-c}{b+c}\cot\dfrac{A}{2} = \dfrac{2R(\sin B - \sin C)}{2R(\sin B + \sin C)}\cot\dfrac{A}{2}$, from (II)

$$= \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}\cot\frac{A}{2}$$

$$= \frac{\cos\left(90° - \frac{A}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\sin\left(90° - \frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}\cot\frac{A}{2}$$

$$= \frac{\sin\frac{A}{2}\sin\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}\cos\left(\frac{B-C}{2}\right)}\cot\frac{A}{2} = \tan\left(\frac{B-C}{2}\right)$$

Result 6

Area of the triangle ABC, $= \Delta$

$$= \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C \quad \text{— (XI)}$$

$$= \frac{abc}{4R} \quad \text{— (XII)}$$

$$= 2R^2\sin A\sin B\sin C \quad \text{— (XIII)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)} \quad \text{— (XIV)}$$

Draw AD perpendicular to BC.

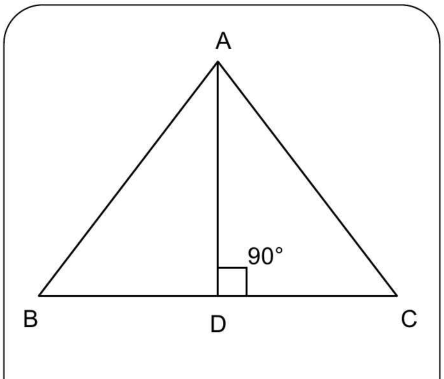


Fig. 4.5

Area of the triangle ABC, $= \dfrac{1}{2}$ base × altitude

i.e., $\Delta = \dfrac{1}{2}BC \times AD = \dfrac{1}{2}a \times AB\sin B$, from ΔABD

$= \dfrac{1}{2}ac\sin B$

or,

$\Delta = \dfrac{1}{2}BC \times AC\sin C$, from $\Delta ADC = \dfrac{1}{2}ab\sin C$

By drawing perpendicular from B to AC, it easily follows that $\Delta = \dfrac{1}{2}bc\sin A$

And thus, $\Delta = \dfrac{1}{2}bc\sin A = \dfrac{1}{2}ca\sin B = \dfrac{1}{2}ab\sin C$

$= \dfrac{1}{2}$ × product of any two sides × (sine of the included angle)

Again,

$\Delta = \dfrac{1}{2}bc\sin A = \dfrac{1}{2}bc \times \dfrac{a}{2R}$, from (I) $= \dfrac{abc}{4R}$

Also,

$\Delta = \dfrac{1}{2}bc \times \sin A$

$= \dfrac{1}{2}bc \times \dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$ from (VI)

$= \sqrt{s(s-a)(s-b)(s-c)}$

Finally,

$\Delta = \dfrac{1}{2}bc\sin A = \dfrac{1}{2} \times (2R\sin B)(2R\sin C)\sin A$

$= 2R^2\sin A\sin B\sin C$

FORMULAS FOR r, r_1, r_2 AND r_3

(i) In any triangle ABC,

$$r = \frac{\Delta}{s} \qquad \text{— (XV)}$$

$$\left.\begin{aligned} r &= (s-a)\tan\frac{A}{2} \\ r &= (s-b)\tan\frac{B}{2} \\ r &= (s-c)\tan\frac{C}{2} \end{aligned}\right\} \qquad \text{— (XVI)}$$

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \qquad \text{— (XVII)}$$

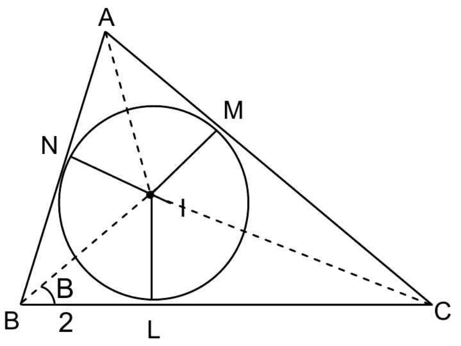


Fig. 4.6

Referring to Fig. 4.6, AI, BI, CI are the internal bisectors of the angles A, B, and C respectively where I is the incentre of the triangle. Draw IL, IM, IN perpendiculars to BC, CA and AB respectively. Then,

In radius of ΔABC, = IL = IM = IN = r

Δ = Area of the triangle ABC

= Area of ΔBIC + Area of ΔCIA + Area of ΔAIB

$= \frac{1}{2}BC \times r + \frac{1}{2}CA \times r + \frac{1}{2}AB \times r$

$= \frac{1}{2}r(a + b + c) = rs$

or $r = \frac{\Delta}{s}$

Again, BN = BL (which represent the lengths of tangents from B to the incircle). Similarly, CL = CM and AM = AN.

$$\begin{aligned} a + b + c &= (BL + LC) + (CM + MA) + (AN + NB) \\ &= 2BL + 2CM + 2AM \\ &= 2BL + 2(CM + AM) \\ &= 2BL + 2CA = 2BL + 2b \end{aligned}$$

$\Rightarrow \quad BL = \frac{1}{2}(a + c - b) = (s - b)$

From ΔBIL, $\frac{IL}{BL} = \tan\angle IBL = \tan\frac{B}{2}$ (since BI is the internal bisector of B)

$\Rightarrow \frac{r}{(s-b)} = \tan\frac{B}{2} \Rightarrow r = (s-b)\tan\frac{B}{2}$

Similarly, we can show that AN = s – a, and CL = s – c and, from triangles AIN and CIM the two results

$r = (s-a)\tan\frac{A}{2}$ and $r = (s-c)\tan\frac{C}{2}$ follow.

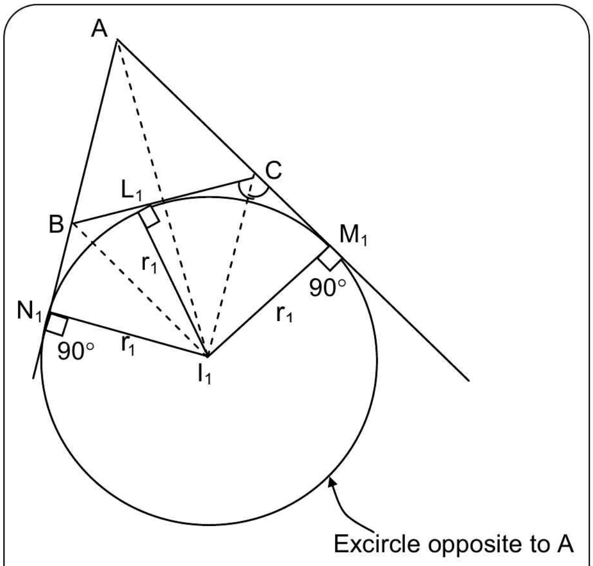


Fig. 4.7

Lastly,

$a = BC = BL + LC = r\cot\frac{B}{2} + r\cot\frac{C}{2}$

$\left(\text{since}\frac{BL}{IL} = \cot\frac{B}{2} \text{and} \frac{CL}{IL} = \cot\frac{C}{2}\right)$

$$= r\left[\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\right] = r\left[\frac{\cos\frac{B}{2}\sin\frac{C}{2} + \cos\frac{C}{2}\sin\frac{B}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}\right]$$

$$= r\frac{\sin\left(\frac{B+C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$= \frac{r\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}} \qquad \left(\text{since } \frac{B+C}{2} = 90° - \frac{A}{2}\right)$$

But, $a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2}$. Hence, we have,

$$4R\sin\frac{A}{2}\cos\frac{A}{2} = \frac{r\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Rightarrow \quad r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

(ii) In any triangle ABC,

$$\left.\begin{aligned} r_1 &= \frac{\Delta}{(s-a)} \\ r_2 &= \frac{\Delta}{(s-b)} \\ r_3 &= \frac{\Delta}{(s-c)} \end{aligned}\right\} \qquad \text{— (XVIII)}$$

$$\left.\begin{aligned} r_1 &= s\tan\frac{A}{2} \\ r_2 &= s\tan\frac{B}{2} \\ r_3 &= s\tan\frac{C}{2} \end{aligned}\right\} \qquad \text{— (XIX)}$$

and

$$\left.\begin{aligned} r_1 &= 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \\ r_2 &= 4R\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{C}{2} \\ r_3 &= 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2} \end{aligned}\right\} \qquad \text{— (XX)}$$

Referring to Fig. 4.7,

AI_1 is the internal bisector of A ; BI_1 and CI_1 are the external bisectors of B and C. I_1L_1, I_1M_1, I_1N_1 are perpendiculars from I_1 to BC, CA, AB.

$\therefore I_1L_1 = I_1M_1 = I_1N_1 = r_1$ = radius of the excircle opposite to A.

We have,

Δ = Area of ΔABC, = Area of ΔI_1AB + Area of ΔI_1AC – Area of ΔI_1BC

$$= \frac{1}{2}AB \times r_1 + \frac{1}{2}AC \times r_1 - \frac{1}{2}BC \times r_1 = \frac{1}{2}r_1$$

$$(AB + AC - BC) = \frac{1}{2}r_1(c + b - a)$$

$$= \frac{1}{2}r_1(2s - 2a) = r_1(s - a)$$

$$\Rightarrow \quad r_1 = \frac{\Delta}{(s-a)}$$

Again, $AM_1 = AN_1$, $BL_1 = BN_1$ and $CL_1 = CM_1$

$2s = a + b + c = BC + CA + AB$

$= (BL_1 + L_1C) + (AM_1 - CM_1) + (AN_1 - BN_1)$

$= 2AN_1$

$\Rightarrow AN_1 = s$

From ΔAI_1N_1, $\frac{r_1}{AN_1} = \tan\frac{A}{2}$, giving $r_1 = s\tan\frac{A}{2}$

Lastly,

$$\begin{aligned} a &= BC = BL_1 + L_1C \\ &= r_1\cot\angle I_1BL_1 + r_1\cot\angle I_1CL_1 \\ &= r_1\cot\left(90° - \frac{B}{2}\right) + r_1\cot\left(90° - \frac{C}{2}\right) \\ &= r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) \end{aligned}$$

$$= r_1\left[\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right] = r_1\left[\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}\right]$$

$$= r_1\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

Since $a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2}$, we obtain

$$r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Formulas for r_2 and r_3 given in (XVIII), (XIX) and (XX) can be easily derived in a similar manner.

HEIGHTS AND DISTANCES

Angle of elevation and Angle of depression

Let O be the position of an observer and P be the position of an object at a higher level than O (refer Fig. 4.8 (i)), (say P is the top of a building or P is a point on a hillock). Let OX be the horizontal line drawn through O to meet the vertical line through P in M. (i.e., the points O, P, M are in the same vertical plane), Let $\angle XOP = \alpha$. Then, α is called the 'angle of elevation' of P viewed from O.

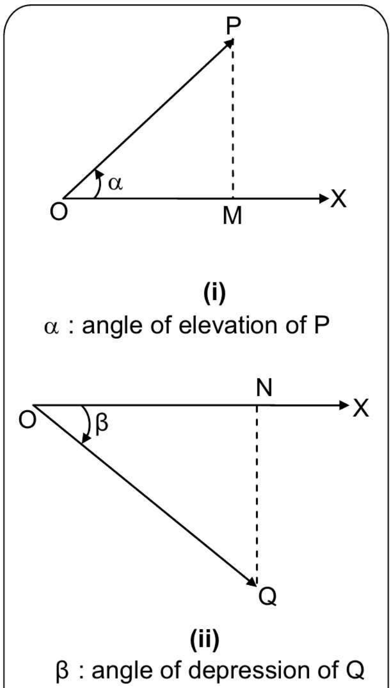


Fig. 4.8

Similarly, if Q is an object at a lower level than O, (say Q is an object on the horizontal ground and O is a point on the terrace of a building) (refer (ii) of Fig. 4.8). The points O, Q, N are in the same vertical plane. Let $\angle XOQ = \beta$. Then, β is called the 'angle of depression' of Q viewed from O.

Bearings of a point

It is expected that the reader is familiar with how the four cardinal directions North, East, South and West are marked. The figure below (Fig. 4.9) is self-explanatory.

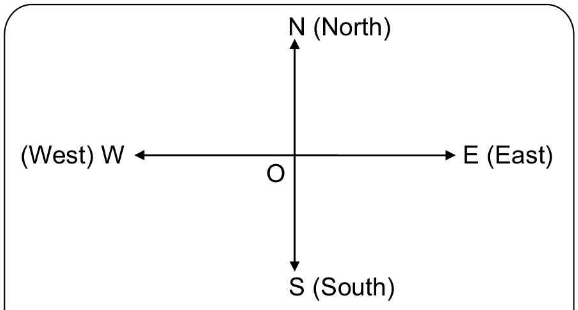


Fig. 4.9

Let A, B, C, D represent 4 objects on the horizontal ground (refer Fig. 4.10).

Suppose $\angle AON = \theta_1°$. Then, bearing of A (viewed from O) is specified as $N\theta_1°E$ (or $\theta_1°$ east of north)

Again, let $\angle WOB = \theta_2°$. Then, bearing of B (viewed from O) is specified as $W\theta_2°N$ (or $\theta_2°$ north of west)

Similarly, bearing of C is specified as $S\theta_3°W$ (or $\theta_3°$ west of south). And bearing of D is specified as $E\theta_4°S$ (or $\theta_4°$south of east)

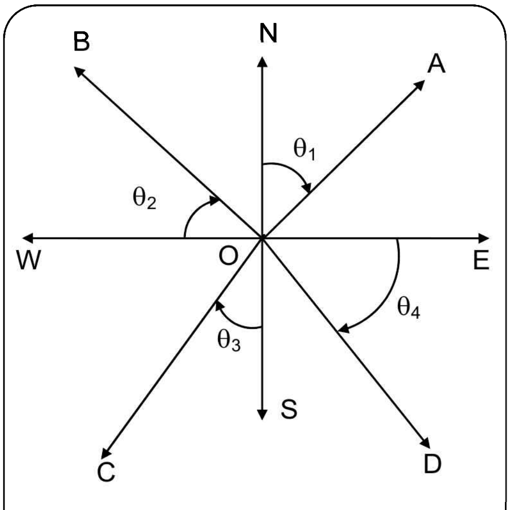


Fig. 4.10

The following examples illustrate the use of the results derived in the earlier part of this unit in solving problems in heights and distances.

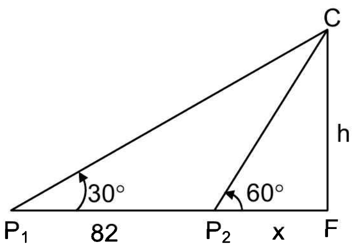
CONCEPT STRANDS

Concept Strand 1

A man walking towards a cliff observes the angle of elevation of its top as 30° and on going 82 metres nearer finds it to be 60°. Find the height of the cliff.

Solution

C represents the top of the cliff. P_1 and P_2 are the two positions from which observations were made. F represents the foot of the cliff.



Let CF = h and P_2F = x. We are given that $P_1P_2 = 82$

From the figure, $\frac{CF}{P_1F} = \tan 30°$ and $\frac{CF}{P_2F} = \tan 60°$

$$\Rightarrow \frac{h}{82+x} = \frac{1}{\sqrt{3}} \text{ and } \frac{h}{x} = \sqrt{3}$$

$$\Rightarrow \frac{82+x}{h} = \sqrt{3}$$

$$\Rightarrow \frac{82}{h} + \frac{1}{\sqrt{3}} = \sqrt{3} \Rightarrow \frac{82}{h} = \frac{2}{\sqrt{3}}$$

$$h = 41\sqrt{3}$$

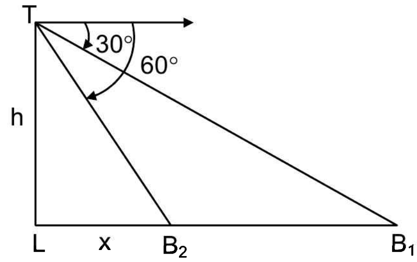
Concept Strand 2

A man on the top of a light house observes a boat coming directly towards it. If it takes 12 minutes for the angle of depression to change from 30° to 60°, how soon will the boat reach the light house? (assume that the boat moves with uniform speed)

Solution

LT : Light house

B_1, B_2 : positions of the boat.



Let the speed of the boat be v units/minute
Then, $B_1B_2 = 12$ v

Let B_2L = x and LT = h

Then, we have

$$\frac{h}{x} = \tan 60^{\circ} = \sqrt{3} \text{ and } \frac{h}{x+12v} = \tan 30^{\circ} = \frac{1}{\sqrt{3}}$$

Dividing, $\frac{x+12v}{x} = 3 \Rightarrow x = 6v$

$\Rightarrow$ Boat will take 6 more minutes to reach the light house.

Concept Strand 3

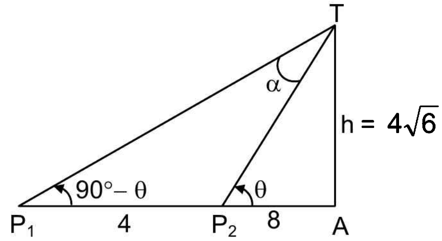
The angles of elevation of the top of a tower from two points distant 12 and 8 units from the base and in the same straight line with it are complementary. Prove that the height of the tower is $4\sqrt{6}$. If θ is the angle subtended at the top of the tower by the line joining these points, then show that $\sin\alpha = \frac{1}{5}$.

Solution

AT : tower

P_1, P_2 : Observation points

$P_1A = 12$, $P_2A = 8$ so that $P_1P_2 = 4$



If h is the height of the tower, $\frac{h}{AP_1} = \tan(90° - \theta) = \cot\theta$ and $\frac{h}{AP_2} = \tan\theta$

Multiplication gives $h^2 = AP_1 \times AP_2 = 96$

or $h = 4\sqrt{6}$

$P_2T = 4\sqrt{10}$

$\therefore \quad \cos\theta = \frac{2}{\sqrt{10}}$

$\alpha = \theta - (90 - \theta) = -(90 - 20)$

$\therefore \quad \sin\alpha = -\cos 2\theta$

$= -2\cos^2\theta + 1 = \frac{1}{5}$

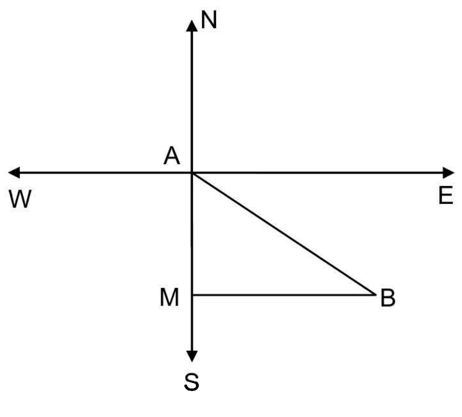
Concept Strand 4

A town B is 12 km south and 18 km east of another town A. Find the distance of B from A.

Solution

Given:

AM = 12, MB = 18



(Refer Figure)

$AB^2 = 12^2 + 18^2$

$= 468$

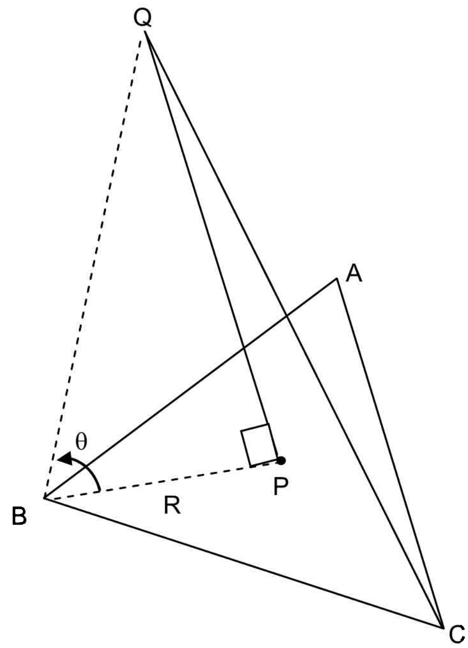
$\Rightarrow \quad AB = \sqrt{468} = 6\sqrt{13}$ km

Concept Strand 5

PQ is a vertical tower. P is the foot and Q is the top of the tower. A, B, C are three points in the horizontal plane through P. The angles of elevation of Q from A, B, C are equal and each is equal to θ. If the sides BC, CA, AB of the ΔABC, are a, b and c respectively and that the area of the triangle ABC, is Δ, show that the height of the tower is $\frac{abc\tan\theta}{4\Delta}$.

Solution

Since the angles of elevation of Q from A, B, C are equal (each equal to θ) , it is clear that the foot of the tower must be at equal distances from the vertices A, B, C of the triangle ABC. This means that P must be the circumcentre of the triangle. We have $\Delta = \frac{abc}{4R}$ where R denotes the circum radius of the triangle ABC.



$PQ = R\tan\theta$

$= \frac{abc}{4\Delta}\tan\theta$

SUMMARY

1. Law of sines (sine formulas)

In any triangle ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ is called sine formula

2. Law of cosines (cosine formulas)

In any triangle ABC

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

3. Projection formulas

In any triangle ABC

$$a = b\cos C + c\cos B$$

$$b = c\cos A + a\cos C$$

$$c = a\cos B + b\cos A$$

4. Half angled formulas

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$

$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

5. Napier Analogies

In any triangle ABC

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$$

$$\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

6. In any triangle ABC

(i) $\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$

(ii) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ where $2s = a + b + c$

(iii) $\Delta = \frac{abc}{4R}$

(iv) $\Delta = 2R^2 \sin A \sin B \sin C$

7. Formulas for r, r_1, r_2 and r_3

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a)\tan A/2$

$= (s-b)\tan B/2$

$= (s-c)\tan C/2$

(iii) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(iv) $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$

(v) $r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$

(vi) $r_1 = s\tan\frac{A}{2}, r_2 = s\tan\frac{B}{2}, r_3 = s\tan\frac{C}{2}$

(vii) $r_1 + r_2 + r_3 - r = 4R$

(viii) $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$

CONECPT CONNECTORS

Connector 1: If A = 30°, B = 60° , find a : b : c

Solution: We have C = 90°

$$\frac{a}{\sin 30^\circ} = \frac{b}{\sin 60^\circ} = \frac{c}{\sin 90^\circ} \qquad \Rightarrow a : b : c = \frac{1}{2} : \frac{\sqrt{3}}{2} : 1 = 1 : \sqrt{3} : 2$$

Connector 2: Prove that (b + c)cos A + (c + a)cos B + (a + b) cos C = 2s

Solution: The expression on the left hand side of the above

= (b cos A + a cos B) + (c cos A + a cos C) + (c cos B + b cos C)

= c + b + a = 2s

Connector 3: Prove the following:

(i) $\sum a\sin(B - C) = 0$

(ii) $\sum a^3 \sin(B - C) = 0$

(iii) $\sum a^3 \cos(B - C) = 3abc$

Solution:

(i) $a \sin(B - C) = 2R \sin A \sin(B - C)$

$= 2R \sin(B + C) \sin(B - C)$

$= 2R (\sin^2 B - \sin^2 C)$

$\Rightarrow \sum a\sin(B - C) = 2R\sum(\sin^2 B - \sin^2 C) = 0$

(ii) $a^3 \sin(B - C) = a^2 \times a \sin(B - C)$

$= (4R^2 \sin^2 A)\,[2R\sin A \sin(B - C)]$

$= 8R^3 \sin^2 A \sin(B + C)\sin(B - C)$

$= 8R^3 \sin^2 A (\sin^2 B - \sin^2 C)$

$\sum a^3 \sin(B - C) = 8R^3 \sum\sin^2 A (\sin^2 B - \sin^2 C) = 0$

(iii) $a^3 \cos(B - C) = a^2 \times a \cos(B - C)$

$= a^2 \times 2R \sin A \cos(B - C) = 2a^2 R \sin(180° - (B + C) \cos(B - C)$

$= 2a^2R\sin(B + C) \cos(B - C) = Ra^2 (\sin 2B + \sin 2C)$

$= Ra^2 (2 \sin B \cos B + 2 \sin C \cos C)$

$= a^2 (b \cos B + c \cos C)$

Similarly, $b^3 \cos(C - A) = b^2 (c \cos C + a \cos A)$

$c^3 \cos(A - B) = c^2 (a \cos A + b \cos B)$

Addition gives

$\sum a^3 \cos(B - C)$

$= bc(b \cos C + c \cos B) + ca(a \cos A + a \cos C) + ab(a \cos B + b \cos A)$

$= bca + cab + ABC,$

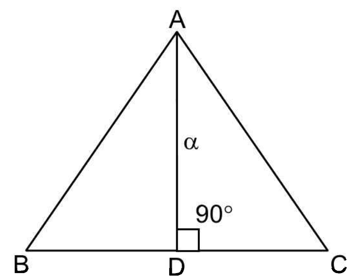
$= 3abc$

Connector 4: If α, β, γ are the lengths of the altitudes of a triangle ABC, prove that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.

Solution: AD is the altitude to the side BC of the triangle ABC.

We have, $\frac{1}{2} BC \times \alpha$ = Area of the ΔABC, = Δ

$$\Rightarrow \alpha = \frac{2\Delta}{a}$$



Similarly, $\beta = \frac{2\Delta}{b}, \gamma = \frac{2\Delta}{c}$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{2\Delta}(a + b + c) = \frac{2s}{2\Delta} = \frac{1}{r},$$

Now, $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$ (using result XVIII)

$$= \frac{3s-2s}{\Delta} = \frac{s}{\Delta}$$

Hence, proved.

Connector 5: If a = 13, b = 14, c = 15, find cos A, cos B, cos C and Δ.

Solution:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{196 + 225 - 169}{2 \times 14 \times 15} = \frac{3}{5}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{225 + 169 - 196}{2 \times 15 \times 13} = \frac{33}{65}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{169 + 196 - 225}{2 \times 13 \times 14} = \frac{5}{13}$$

We have, $s = \frac{13 + 14 + 15}{2} = 21$,

$s - a = 8, s - b = 7$ and $s - c = 6$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \times 8 \times 7 \times 6} = 7 \times 3 \times 4 = 84$$

OR

$$\sin A = \frac{4}{5}$$

$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2} \times 14 \times 15 \times \frac{4}{5} = 84$$

Connector 6: Show that

(i) $\sin\frac{B-C}{2} = \left(\frac{b-c}{a}\right)\cos\frac{A}{2}$

(ii) $\cos\frac{A-B}{2} = \left(\frac{a+b}{c}\right)\sin\frac{C}{2}$

Solution:

(i) $\left(\frac{b-c}{a}\right)\cos\frac{A}{2} = \frac{2R(\sin B - \sin C)}{2R\sin A}\cos\frac{A}{2}$

$$= \frac{2\sin\frac{B-C}{2}\cos\frac{B+C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}\cos\frac{A}{2} = \sin\left(\frac{B-C}{2}\right)$$

$$\left(\text{since } \cos\frac{B+C}{2} = \sin\frac{A}{2}\right)$$

(ii) $\dfrac{a+b}{c}\sin\dfrac{C}{2} = \dfrac{2R(\sin A + \sin B)}{2R\sin C}\sin\dfrac{C}{2}$

$$= \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}\sin\frac{C}{2} = \cos\left(\frac{A-B}{2}\right)$$

Connector 7: Show that $b^2\sin 2C + c^2 \sin 2B = 4\Delta$

Solution: Left hand side expression $= b^2 2\sin C\cos C + c^2 \times 2\sin B\cos B$

$$= 2b^2 \times \frac{c}{2R}\cos C + 2c^2 \times \frac{b}{2R}\cos B$$

$$= \frac{bc}{R}(b\cos C + c\cos B)$$

$$= 4\left(\frac{abc}{4R}\right) = 4\Delta$$

Connector 8: Prove that $4\left(bc\cos^2\dfrac{A}{2} + ca\cos^2\dfrac{B}{2} + ab\cos^2\dfrac{C}{2}\right) = (a+b+c)^2$

Solution: Left hand side expression

$= 2\left(2bc\cos^2\dfrac{A}{2} + 2ca\cos^2\dfrac{B}{2} + 2ab\cos^2\dfrac{C}{2}\right)$

$= 2\{bc(1+\cos A) + ca(1+\cos B) + ab(1+\cos C)\}$

$= 2\{\sum bc + \sum(bc\cos A)\} = 2\sum bc + \sum(2bc\cos A)$

$= 2\sum bc + (b^2 + c^2 - a^2) + c^2 + a^2 - b^2 + a^2 + b^2 - c^2$

$= a^2 + b^2 + c^2 + 2bc + 2ca + 2ab$

$= (a+b+c)^2$

Connector 9: If $\dfrac{1}{c+a} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$ show that $C = 60°$

Solution: The given relation is equivalent to $\dfrac{b+2c+a}{(c+a)(b+c)} = \dfrac{3}{(a+b+c)}$

$\Rightarrow a^2 + b^2 + 2c^2 + 3bc + 3ca + 2ab = 3(bc + c^2 + ab + ac)$

$\Rightarrow a^2 + b^2 = c^2 + ab$ $\qquad \Rightarrow \dfrac{a^2+b^2-c^2}{ab} = 1$

$\Rightarrow \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{1}{2}$ $\qquad \Rightarrow \cos C = \dfrac{1}{2}$

$\Rightarrow C = 60°$

Connector 10: If $3a = b + c$, show that $\cot\dfrac{B}{2}\cot\dfrac{C}{2} = 2$

Solution: $3a = b + c$ gives $3 \times 2R\sin A = 2R(\sin B + \sin C)$

i.e., $3\sin A = \sin B + \sin C$

$\Rightarrow 3 \times 2\sin\dfrac{A}{2}\cos\dfrac{A}{2} = 2\sin\dfrac{B+C}{2}\cos\dfrac{B-C}{2}$

$$\Rightarrow 3\sin\frac{A}{2} = \cos\frac{B-C}{2} \qquad \Rightarrow 3\cos\left(\frac{B+C}{2}\right) = \cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow 3\left\{\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right\} = \cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Rightarrow 2\cos\frac{B}{2}\cos\frac{C}{2} = 4\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Rightarrow \cot\frac{B}{2}\cot\frac{C}{2} = 2$$

Connector 11: Show that $a = (b + c)\sin\theta$ where, $\cos\theta = \dfrac{2\sqrt{bc}}{b+c}\cos\dfrac{A}{2}$.

Solution: Since, $\cos\theta = \dfrac{2\sqrt{bc}}{b+c}\cos\dfrac{A}{2}$,

$$\cos^2\theta = \frac{4bc}{(b+c)^2}\cos^2\frac{A}{2} = \frac{2bc}{(b+c)^2}2\cos^2\frac{A}{2}$$

$$= \frac{2bc}{(b+c)^2}(1+\cos A)$$

$\sin^2\theta = 1 - \cos^2\theta$

$$= 1 - \frac{2bc}{(b+c)^2}(1+\cos A) = \frac{(b+c)^2 - 2bc - 2bc\cos A}{(b+c)^2}$$

$$= \frac{(b^2+c^2) - (b^2+c^2-a^2)}{(b+c)^2} = \frac{a^2}{(b+c)^2}$$ from which the result follows.

Connector 12: The sides of a triangle are $(2x + 3)$, $(x^2 + 3x + 3)$ and $(x^2 + 2x)$ where $x > 0$. Show that the greatest angle is 120°.

Solution: $x^2 + 3x + 3 - (2x + 3) = x^2 + x > 0$

$\Rightarrow (x^2 + 3x + 3) > (2x + 3)$

Again,

$(x^2 + 3x + 3) - (x^2 + 2x) = x + 3 > 0$

$\Rightarrow x^2 + 3x + 3 > (x^2 + 2x)$

$\Rightarrow$ greatest side is $(x^2 + 3x + 3)$

Hence the greatest angle is opposite to the side of length $(x^2 + 3x + 3)$.

Let $a = x^2 + 3x + 3$

$b = x^2 + 2x$

and $c = 2x + 3$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{(x^2+2x)^2 + (2x+3)^2 - (x^2+3x+3)^2}{2(x^2+2x)(2x+3)}$$

$$= \frac{-(2x^3+7x^2+6x)}{2x(x+2)(2x+3)} = \frac{-x(2x^2+7x+6)}{2x(x+2)(2x+3)} = \frac{-1}{2} \Rightarrow A = 120°$$

Connector 13: If a, b, c are in AP prove that $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are also in AP.

Solution: We have to prove that $2\cot\frac{B}{2} = \cot\frac{A}{2} + \cot\frac{C}{2}$

It is clear that conversion has to be made in such a way that the relation reduces to one, which is in terms of sides.

We have to show that $2\sqrt{\frac{s(s-b)}{(s-c)(s-a)}} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

Multiplying both sides by $\sqrt{(s-a)(s-b)(s-c)}$, we have to show that

$$2(s-b) = (s-a) + (s-c)$$

$\Rightarrow$ to show that 2b = a + c, which holds good, since we are given the information that a, b, c are in AP.

Connector 14: The sides of a triangle are in A.P and the greatest angle exceeds the least by 90°. Prove that the sides are proportional to $\sqrt{7}-1, \sqrt{7}$ and $\sqrt{7}+1$.

Solution: We shall assume that a < b < c, so that a is least and C the greatest

$$C - A = 90° \qquad —(1)$$

Given: $2b = a + c \Rightarrow 2 \times 2R\sin B = 2R(\sin A + \sin C)$

$\Rightarrow 2\sin B = \sin A + \sin C$

$2\sin[180 - (A + C)] = \sin A + \sin C \Rightarrow 2\sin(A + C) = \sin A + \sin C$

$$4\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A+C}{2}\right) = 2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)$$

giving $\cos\left(\frac{A+C}{2}\right) = \frac{1}{2}\cos\left(\frac{A-C}{2}\right)$ or $\sin\frac{B}{2} = \frac{1}{2}\cos\left(\frac{A-C}{2}\right)$ —(2)

from (1) substituting, $\sin\frac{B}{2} = \frac{1}{2}\cos 45^\circ = \frac{1}{2\sqrt{2}}$

from which we obtain $\cos\frac{B}{2} = \frac{\sqrt{7}}{2\sqrt{2}}$

Therefore,

$$\sin B = 2\sin\frac{B}{2}\cos\frac{B}{2} = 2 \times \frac{1}{2\sqrt{2}} \times \frac{\sqrt{7}}{2\sqrt{2}} = \frac{\sqrt{7}}{4}$$

We have $\sin A + \sin C = 2\sin B = \frac{\sqrt{7}}{2}$ —(2)

and $\sin A - \sin C = 2\cos\frac{A+C}{2}\sin\frac{A-C}{2}$

$$= 2\sin\frac{B}{2}\sin(-45^\circ) = -\frac{1}{2} \qquad —(3)$$

(2) and (3) yield $\sin A = \frac{\sqrt{7}-1}{4}$ and $\sin C = \frac{\sqrt{7}+1}{4}$

Therefore, $a : b : c = (\sqrt{7}-1) : \sqrt{7} : (\sqrt{7}+1)$

Connector 15: In a ΔABC, if $A = 90°$, show that $r_1 = r + 2R = a$

Solution: Since $A = 90°$, $R = \frac{a}{2} \Rightarrow a = 2R$

Now, $r_1 - r = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$$= \left(4R\sin\frac{A}{2}\right)\left[\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right]$$

$$= \left(4R\sin\frac{A}{2}\right)\cos\frac{B+C}{2} = 4R\sin^2\frac{A}{2} = 4R\left(\frac{1}{\sqrt{2}}\right)^2 = 2R = a$$

Connector 16: If ABC, is an equilateral triangle and each side is of length k units, obtain the values of Δ, R, r, r_1, r_2, r_3. Show that $r = \frac{1}{2}R$.

Solution: $\Delta = \frac{1}{2}k^2\sin 60^\circ = \frac{k^2\sqrt{3}}{4}$

$$R = \frac{k}{2\sin A} = \frac{k}{2\sin 60^\circ} = \frac{k}{\sqrt{3}}$$

$$r = \frac{\Delta}{s} = \frac{\frac{k^2\sqrt{3}}{4}}{\left(\frac{3k}{2}\right)} = \frac{k}{2\sqrt{3}} = \frac{1}{2}R$$

$$r_1 = r_2 = r_3 = \frac{\frac{k^2\sqrt{3}}{4}}{\left(\frac{3k}{2} - k\right)} = \frac{k^2\sqrt{3}}{4} \times \frac{2}{k} = \frac{k\sqrt{3}}{2}$$

Connector 17: In a triangle ABC, prove that $\frac{R}{r} > \left(\frac{a}{c} + \frac{c}{a}\right)$ where, R is the circum radius, r is the inradius and a, b, c represent the sides of the triangle.

Solution: We have $\frac{R}{r} = \frac{R}{\Delta/s} = \frac{Rs}{\Delta} = \frac{\left(abc/4\Delta\right)s}{\Delta}$

$$= \frac{abcs}{4\Delta^2} = \frac{abc}{4(s-a)(s-b)(s-c)}$$

$$= \frac{ac}{4}\left[\frac{b}{(s-a)(s-b)(s-c)}\right] = \frac{ac}{4}\left[\frac{1}{(s-b)(s-c)} + \frac{1}{(s-b)(s-a)}\right] \qquad \text{—(1)}$$

(Since $s - a + s - c = 2s - (a + c) = b$)
Since $(s - a)$ and $(s - b)$ are > 0

$$\Rightarrow \frac{(s-b)+(s-c)}{2} > \sqrt{(s-b)(s-c)} \text{ (AM} \geq \text{GM)}$$

$$\Rightarrow \frac{a}{2} > \sqrt{(s-b)(s-c)} \Rightarrow \frac{2}{a} < \frac{1}{\sqrt{(s-b)(s-c)}}$$

$$\Rightarrow \frac{1}{(s-b)(s-c)} > \frac{4}{a^2}$$

Similarly, $\dfrac{1}{(s-b)(s-a)} > \dfrac{4}{c^2}$

From (1), $\dfrac{R}{r} > \dfrac{ac}{4}\left(\dfrac{4}{a^2} + \dfrac{4}{c^2}\right) > \dfrac{c}{a} + \dfrac{a}{c}$

Connector 18: Prove the following:

(i) $rr_1 + r_2r_3 = bc$

(ii) $\dfrac{1}{bc} + \dfrac{1}{ca} + \dfrac{1}{ab} = \dfrac{1}{2Rr}$

(iii) $\dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} + \dfrac{1}{r^2} = \dfrac{a^2 + b^2 + c^2}{\Delta^2}$

(iv) $\dfrac{r_2 + r_3}{1 + \cos A} = \dfrac{r_3 + r_1}{1 + \cos B} = \dfrac{r_1 + r_2}{1 + \cos C}$

(v) $a \cos A + b \cos B + c \cos C = \dfrac{2rs}{R}$

Solution:

(i) $rr_1 + r_2r_3 = \dfrac{\Delta^2}{s(s-a)} + \dfrac{\Delta^2}{(s-b)(s-c)}$

$$= \frac{\Delta^2}{s(s-a)(s-b)(s-c)}\big[(s-b)(s-c) + s(s-a)\big]$$

$$= 2s^2 - (b+c)s + bc - as = 2s^2 - 2s \times s + as + bc - as = bc$$

(ii) $\dfrac{1}{bc} + \dfrac{1}{ca} + \dfrac{1}{ab} = \dfrac{a+b+c}{abc} = \dfrac{2s}{4\Delta R} = \dfrac{1}{2Rr}$

(iii) $\dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} + \dfrac{1}{r^2} = \dfrac{(s-a)^2 + (s-b)^2 + (s-c)^2 + s^2}{\Delta^2}$

$$= \frac{4s^2 + a^2 + b^2 + c^2 - 2s(a+b+c)}{\Delta^2} = \frac{a^2 + b^2 + c^2}{\Delta^2}$$

(iv) $\dfrac{r_2 + r_3}{1 + \cos A} = \dfrac{4R\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{C}{2} + 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}}{2\cos^2\frac{A}{2}}$

$$= \frac{4R\left[\sin\frac{B}{2}\cos\frac{C}{2} + \cos\frac{B}{2}\sin\frac{C}{2}\right]}{2\cos\frac{A}{2}} = \frac{4R\sin\left(\frac{B+C}{2}\right)}{2\cos\frac{A}{2}} = 2R,$$

since $\sin\dfrac{B+C}{2} = \cos\dfrac{A}{2}$

Similarly, it can be shown that each of the other two ratios reduce to 2R.

(v) $a \cos A + b \cos B + c \cos C$

$= \sum 2R \sin A \cos A = R\sum \sin 2A$

$= R[(\sin 2A + \sin 2B) + \sin 2C]$

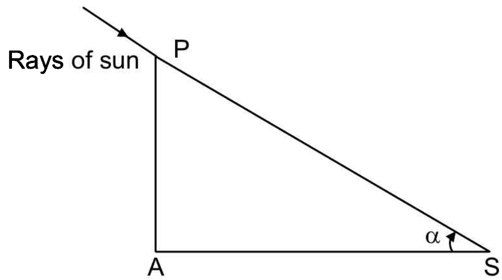
$= R[2\sin(A+B)\cos(A-B) + 2\sin C\cos C]$
$= (2R\sin C)[\cos(A-B) + \cos C]$
$= (2R\sin C)[\cos(A-B) - \cos(A+B)]$
$= 4R\sin A\sin B\sin C$
$= \left(\frac{2}{R}\right)\times 2R^2\sin A\sin B\sin C$
$= \left(\frac{2}{R}\right)\times\Delta = \frac{2}{R}\times(sr) = \frac{2sr}{R}$

Connector 19: What is the elevation of the sun, when the length of the shadow of pole is k times the height of the pole?

Solution: AP is the pole, AS is the shadow of AP

Given, AS = k times AP

If α is the elevation of the sun,



$$\frac{AP}{AS} = \tan\alpha$$

But, $\frac{AP}{AS} = \frac{1}{k}$

$\Rightarrow$ Elevation of the sun $= \tan^{-1}\left(\frac{1}{k}\right)$ or $\cot^{-1}(k)$

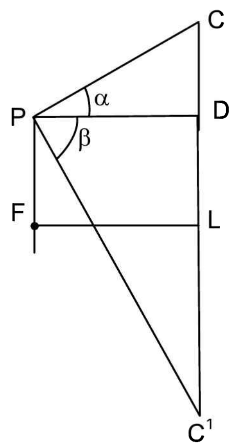
Connector 20: The angle of elevation of a cloud from a point 150 metres above a lake is α and the angle of depression of its reflection in the lake is β. Find the height of the cloud above the lake.

Solution: C : cloud, L : lake, C' : reflection

CL = C'L

P : observer and PF = 150

CL is required. Let CL be denoted by x



$$\frac{CD}{PD} = \tan\alpha, \frac{DC'}{PD} = \tan\beta$$

Division gives $\frac{CD}{DC'} = \frac{\tan\alpha}{\tan\beta}$

$$\Rightarrow \frac{x-150}{x+150} = \frac{\tan\alpha}{\tan\beta}$$

giving $x = 150\left(\frac{\tan\alpha + \tan\beta}{\tan\beta - \tan\alpha}\right) = 150\,\frac{\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$

Connector 21: The elevation of a mountain top as seen at a place A due west of it is α and at a place due north of A and 1 km distant from A, it is β. Find the height of the mountain.

Solution: FT is a mountain

A and B are the two points from which observations were made.

Note that A, B, F are on the horizontal plane and AB = 1 km.

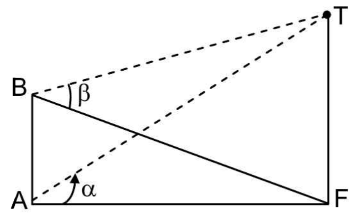
If FT = x, $\frac{x}{AF} = \tan\alpha; \frac{x}{BF} = \tan\beta$

Triangle ABF is right angled at B.

Therefore, $AB^2 + AF^2 = BF^2$

$$1 + \frac{x^2}{\tan^2\alpha} = \frac{x^2}{\tan^2\beta}$$

$$1 = x^2\left[\frac{1}{\tan^2\beta} - \frac{1}{\tan^2\alpha}\right] \Rightarrow x^2 = \frac{\tan^2\alpha . \tan^2\beta}{\left(\tan^2\alpha - \tan^2\beta\right)}$$



Connector 22: An object is observed from three points A, B, C in the same horizontal line passing through the foot of the object. The angle of elevation at B and C are 2θ and 3θ and that at A it is θ. If AB = a, BC = b, prove that the height of the object is $\frac{a}{2b}\sqrt{(a + b)(3b - a)}$.

Solution: T is the top of the object and F is its foot. A, B, C are the points from which observations are made.

We have in Δ BCT,

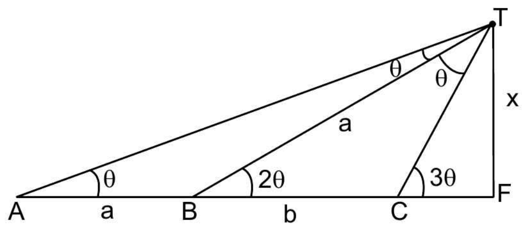
$$\frac{CT}{\sin^2\theta} = \frac{b}{\sin\theta} = \frac{a}{\sin 3\theta}$$

$$\text{Each} = \frac{a+b}{\sin\theta + \sin 3\theta} = \frac{a+b}{2\sin 2\theta \cos\theta}$$

$\therefore$ $2b \sin 2\theta \cos\theta = (a + b)\sin\theta$ (i.e.,) $\cos^2\theta = \frac{a+b}{4b}$

In Δ BTF, $x = a \sin^2\theta = a\, 2 \sin\theta \cos\theta$

$$= 2a\sqrt{1 - \frac{a+b}{4b}} . \sqrt{\frac{a+b}{4b}} = \frac{a}{2b}\sqrt{(3b - a)(a + b)}$$



Connector 23: The elevation of the summit of a hill is α. On walking up 500 metres towards the hill up a slope of θ, the elevation is found to be β. Find the height of the hill.

Solution: HF : hill

P_1 and P_2 are the two points from which observations are made.

$P_1P_2 = 500$

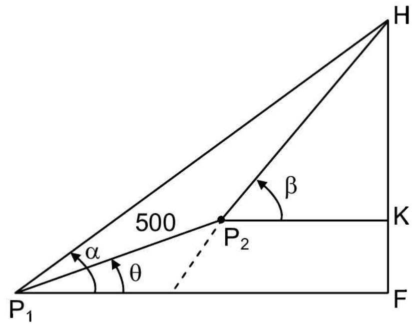
Let HF = h

$\angle P_2P_1H = \alpha - \theta$

$\angle P_1HP_2 = \beta - \alpha$

Therefore, $\angle P_1P_2H = \pi - (\alpha - \theta + \beta - \alpha)$

$= \pi - \beta + \theta$



Using sine formula in ΔP_1P_2H,

$$\frac{500}{\sin\angle P_1HP_2} = \frac{P_1H}{\sin\angle P_1P_2H}$$

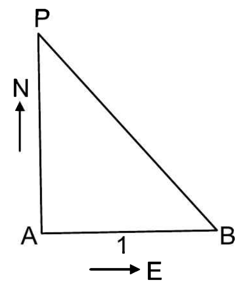
$$\Rightarrow \frac{500}{\sin(\beta - \alpha)} = \frac{P_1H}{\sin(\pi - \beta + \theta)} = \frac{P_1H}{\sin(\beta - \theta)} \qquad \Rightarrow P_1H = \frac{500\sin(\beta - \theta)}{\sin(\beta - \alpha)}$$

From ΔP_1HF, $\frac{h}{P_1H} = \sin\alpha$

$$h = P_1H\sin\alpha = \frac{500\sin(\beta - \theta)\sin\alpha}{\sin(\beta - \alpha)}$$

Connector 24: The angle of elevation of a tower at point A due north of it is 30° and at another point due east of A is 18°. If AB = 1, show that the height of the tower is $\dfrac{1}{\sqrt{2+2\sqrt{5}}}$.

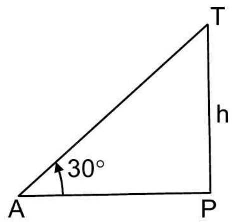
Solution: PT is the tower and the observation points are A and B.

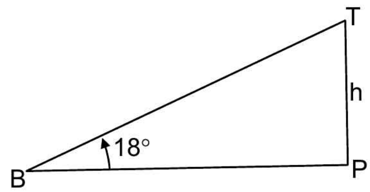


If h is the height of the tower, we have

$$\frac{h}{AP} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$AP = h\sqrt{3} \qquad \text{—(1)}$$

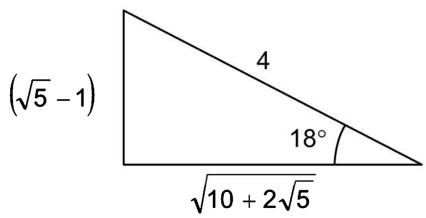




Again,

$$\frac{h}{BP} = \tan 18^\circ = \frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$$

$$BP = \frac{h\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1} \qquad \text{—(2)}$$



A, B, P are in the horizontal plane

$BP^2 = AP^2 + 1^2$

$$\Rightarrow \frac{h^2(10+2\sqrt{5})}{(6-2\sqrt{5})} = 3h^2 + 1 \Rightarrow h^2\left\{\frac{10+2\sqrt{5}}{6-2\sqrt{5}} - 3\right\} = 1$$

$$\Rightarrow h^2 = \frac{(6-2\sqrt{5})}{8(\sqrt{5}-1)} = \frac{\sqrt{5}-1}{8} = \frac{4}{8(\sqrt{5}+1)} = \frac{1}{2(\sqrt{5}+1)}$$

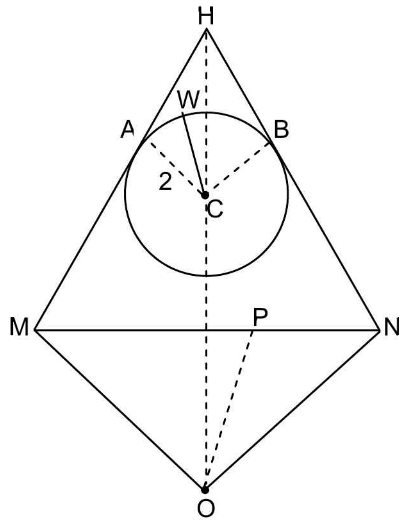
$$\Rightarrow h = \frac{1}{\sqrt{2(\sqrt{5}+1)}} = \frac{1}{\sqrt{2+2\sqrt{5}}}$$

Connector 25: A circular plate of radius 2 units touches a vertical wall. The plate is fixed horizontally at a height x units above the ground. A lighted candle of length y stands vertically, at the centre of the plate. Find an expression for the breadth of the shadow thrown on the wall where it meets the horizontal ground.

Solution: Let C denote the centre of the plate AWB touching the wall at W.

Let CH = y (which denotes the height of the candle); OC = x (height of the plate above the ground).

MN represents the breadth of the shadow on the wall where it meets the ground; OP = 2 is the horizontal distance of MN and MP = PN.



From similar triangles HCB and HON,

$$\frac{CB}{ON} = \frac{HC}{HO} \Rightarrow \frac{2}{ON} = \frac{y}{x+y}$$

$$\Rightarrow \quad ON = \frac{2(x+y)}{y}$$

Therefore, $MN = 2PN = 2\sqrt{ON^2 - OP^2}$

$$= 2\sqrt{\frac{4(x+y)^2}{y^2} - 4} = \frac{4}{y}\sqrt{x^2 + 2xy}$$

TOPIC GRIP

Subjective Questions

1. If in a triangle ABC, a = 13 cm, b = 14 cm, c = 15 cm find
   (i) sinA, cosA, tanA,
   (ii) $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, $\tan\frac{A}{2}$,
   (iii) area of the ΔABC.

2. In ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$; prove that a^2, b^2, c^2 are in AP.

3. In triangle ABC, a = 3 b = 4 and $\sin A = \frac{3}{8}$; find the angle B.

4. If in triangle ABC, $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$, then prove that a, b, c are in AP.

5. In any triangle ABC, prove that
   (i) $R = \frac{abc}{4\Delta}$
   (ii) $2R^2 \sin A \sin B \sin C = \Delta$
   (iii) $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{4R}{\Delta}$
   (iv) $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}$
   (v) $4\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{c}-1\right) = \frac{r}{R}$

6. If A, B, C are the measures of angles of a triangle, show that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$.

7. (i) In a Δ ABC, if cos B + 2 cos C + cos A = 2, find the value of $\tan\frac{A}{2}\tan\frac{B}{2}$.
   (ii) In a Δ ABC, if B = 36° and A = 84° find the value of $\frac{r_2 + r_3}{r_2 + r_1}$.

8. In a Δ ABC, if $a^4 - 2(b^2 + c^2)a^2 + b^4 + b^2c^2 + c^4 = 0$ prove that $\tan A = \pm\sqrt{3}$

9. (i) In a ΔABC, if E is the point on the base CA such that $\frac{CE}{EA} = \frac{3}{5}$ and $\angle AEB = \theta$, show that $8\cot\theta = 5\cot C - 3\cot A$
   (ii) Find the value of $\cot\theta$ if it is given that $\sin C = \frac{5}{13}$ and $\sin A = \frac{3}{5}$.

10. The lengths of the sides BC, CA, AB of a triangle ABC, are 5, 12 and 15 respectively. If x, y, z are lengths of the internal bisectors of angles A, B, C of the triangle, obtain the value of $\frac{1}{x}\cos\frac{A}{2} + \frac{1}{y}\cos\frac{B}{2} + \frac{1}{z}\cos\frac{C}{2}$

Straight Objective Type Questions

***Directions:* This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

11. If a cos A = c cos C, then the triangle ABC, is

(a) isosceles (b) right-angled

(c) equilateral (d) isosceles or right-angled

12. If the angles of a triangle are in the ratio 3 : 4 : 5, then the ratio of the corresponding sides is

(a) $2 : \sqrt{2} : 3$ (b) $2 : \sqrt{6} : \sqrt{3} + 1$ (c) $\sqrt{3} : 2 : \sqrt{5}$ (d) $\sqrt{2} : \sqrt{6} : \sqrt{3} + 1$

13. In a triangle ABC, if a = 10, b = 2 and angle $C = \frac{\pi}{3}$. Then, $\tan \frac{A - B}{2}$ is

(a) $\frac{\sqrt{3}}{2}$ (b) $\sqrt{3}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\sqrt{2}$

14. The sides of a triangle are $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. The greatest angle of the triangle is

(a) 120° (b) 90° (c) 60° (d) 150°

15. If the ex-radii r_1, r_2 and r_3 of a triangle ABC, are in H.P, then a, b, c are in

(a) HP (b) AP (c) GP (d) AGP

Assertion–Reason Type Questions

***Direciton:* Each question contains STATEMENT-1 and STATEMENT-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

16. **Statement 1**

In a triangle ABC, a = 7, b = 5 and ∠C = 60°. Then, $B = \sin^{-1}\left(\frac{5}{2\sqrt{13}}\right)$ and $A = \sin^{-1}\left(\frac{7}{2\sqrt{13}}\right)$.

and

Statement 2

If two sides and the included angle of a triangle are given, the triangle is uniquely fixed so that the remaining side and the other two angles can be found using the tangent formula.

17. **Statement 1**

For an equilateral triangle, R = 2r

and

Statement 2

$r = 4R \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

18. Statement 1

If r_1, r_2, r_3 are in HP, sinA, sinB, sinC are in HP.

and

Statement 2

If a, b, c are in HP $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP.

19. Statement 1

ABC, is a right angled triangle right angled at A. Also, AB = $\sqrt{5}$ and AC = 1. Then, the length of the median of the triangle through A is $\sqrt{\frac{3}{2}}$.

and

Statement 2

Medians of a triangle are concurrent and the point of concurrence is the centroid.

20. The angles A, B, C of a triangle ABC, are in AP.

Statement 1

$\frac{r_1}{r_2} = \sqrt{3}\left(\sqrt{2} - 1\right)$

and

Statement 2

B = 60°

Linked Comprehension Type Questions

Directions: **This section contains 2 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

The angles A, B, C of a triangle ABC, are in the ratio 1: 2 : 7 and the side $a = 3\left(\sqrt{5} - 1\right)$

21. c =

(a) $3\left(\sqrt{5} + 1\right)$ (b) $3\left(5 + \sqrt{5}\right)$ (c) $6\left(\sqrt{5} - 1\right)$ (d) $\frac{3}{2}\left(\sqrt{5} + 1\right)$

22. R =

(a) 12 (b) 18 (c) $\frac{9}{2}$ (d) 6

23. r_2 =

(a) $\frac{3\left(\sqrt{5} + 1\right)}{4} + 3\sqrt{5}$

(b) $\frac{3\left(\sqrt{5} - 1\right)}{4}\left[\left(\sqrt{5} - 1\right) + 2\sqrt{10 - 2\sqrt{5}}\right]$

(c) $2\left(\sqrt{5} + 1\right) + 2\sqrt{10 - 2\sqrt{5}}$

(d) $\frac{3\left(\sqrt{5} - 1\right)}{4}\left[\left(\sqrt{5} + 1\right) + 2\sqrt{10 - 2\sqrt{5}}\right]$

Passage II

In a triangle ABC, D is a point on BC and such that BD : DC = m : n. If given that ∠BAD = α, ∠DAC = β, ∠CDA = θ and AD = x then,

24. b sin β : c sin α

(a) m : n (b) n : m (c) (m + n) : (m – n) (d) (m – n): (m + n)

25. $m \cot \alpha - n \cot \beta$

(a) $(m - n) \cot \theta$ (b) $m \sin \theta - n \cos \theta$ (c) $(m + n) \cot \theta$ (d) $\dfrac{2mn}{m - n}$

26. The area of the above triangle is equal to

(a) $\dfrac{1}{2} a \sin \theta$ (b) $2ax \sin \theta$ (c) $\dfrac{1}{2} ax \sin \theta$ (d) $\dfrac{1}{2} ab \sin \theta$

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

27. The inradius of a right angled triangle with sides a, b, c, where c is the hypotenuse is 5. Then,

(a) $a + b - c = 10$ (b) $ab + 10c = 50$ (c) $ab - 10c = 50$ (d) $r = \dfrac{5 \sin A}{\sqrt{2}}$

28. In a triangle ABC, given that $a = 5, b = 4,\ A = \dfrac{\pi}{2} + B$, the value of angle C

(a) cannot be evaluated (b) $\tan^{-1} \dfrac{9}{40}$ (c) $\tan^{-1}\left(\dfrac{1}{40}\right)$ (d) $2\tan^{-1}\left(\dfrac{1}{9}\right)$

29. In a ΔABC, if $\dfrac{1}{r_1} + \dfrac{1}{r_2} = \dfrac{3}{r_3}$, then,

(a) $c = \dfrac{3s}{4}$ (b) $r = \dfrac{r_3}{4}$ (c) $r = \dfrac{r_3}{2}$ (d) $\sin \dfrac{C}{2} = 3 \sin \dfrac{A}{2} \sin \dfrac{B}{2}$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

30. I is the incentre of ΔABC. $AI = x$, $BI = y$, $CI = z$. R is the circumradius and r is the inradius of ΔABC.

Column I	Column II
(a) xyz	(p) $Rr\left(\sin A/2 + \sin B/2 + \sin C/2\right)$
(b) $a + b + c$	(q) $2Rr\left[\cos A/2 + \cos B/2 + \cos C/2\right]$
(c) $ax + by + cz$	(r) $8R \cos A/2 \cos B/2 \cos C/2$
(d) $xy + yz + xz$	(s) $2R^2 r \sin A/2 \sin B/2 \sin C/2$

IIT ASSIGNMENT EXERCISE

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

31. In a triangle ABC, if $a = \sqrt{7}$, $b = 3$, $c = 4$, cos A is

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) $\frac{1}{4}$ (d) $\frac{1}{5}$

32. In a ΔABC, if $a = 9$, $b = 12$, $c = 15$, then $\tan\frac{A}{2}$ is

(a) $\frac{1}{2}$ (b) 1 (c) 3 (d) $\frac{1}{3}$

33. In a triangle ABC, if $A = 60°$, then

(a) $(a - b)^2 + ab = c^2$ (b) $(c - a)^2 + ca = b^2$

(c) $(b - c)^2 + bc = a^2$ (d) $a^2 + b^2 + c^2 = ab + bc + ca$

34. In a right angled triangle ABC, $\cos^2 A + \cos^2 B + \cos^2 C$ is

(a) 0 (b) 2 (c) 3 (d) 1

35. In a triangle ABC, if b = 2 and c = 5 and area = 5, then $\sin\frac{A}{2}\cos\frac{A}{2}$ is equal to

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{6}$

36. If r is the radius of the incircle of triangle ABC, then $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$ is

(a) $\frac{a+b+c}{2r}$ (b) $\frac{r}{2(a+b+c)}$ (c) $r(a + b + c)$ (d) $\frac{a+b+c}{r}$

37. A tower subtends an angle α at a point A on the ground. B is a point at a height of h meters vertically above A. The angle of depression of the foot of the tower from B is β. The height of the tower is

(a) $h \tan\alpha \tan\beta$ (b) $h \tan\alpha \cot\beta$ (c) $h \cot\alpha \cot\beta$ (d) $h \sin\alpha \sin\beta$

38. If in a triangle ABC, $\angle A = 90°$, then $\frac{b^2 + c^2}{b^2 - c^2}\sin(B - C)$ is

(a) 0 (b) 2 (c) $\frac{1}{2}$ (d) 1

39. In any triangle ABC, the expression $\frac{[(a+b+c)(b+c-a)(c+a-b)(a+b-c)]}{4b^2c^2}$ is equal to

(a) $1 + \cos A$ (b) $1 - \cos A$ (c) $\sin^2 A$ (d) $\cos^2 A$

40. The angles of a triangle are in the ratio 1 : 2 : 7, the ratio of greatest side to least side is

(a) $(\sqrt{2}+1):(\sqrt{2}-1)$ (b) $(\sqrt{5}+1):(\sqrt{5}-1)$ (c) $(\sqrt{3}+1):(\sqrt{3}-1)$ (d) $(2+\sqrt{3}):(2-\sqrt{3})$

41. The perimeter of triangle ABC, is 6 times the arithmetic mean of the sines of its angles. If the side c is 1 unit, the angle C can be

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

42. In a triangle ABC, $a = 5, b = 4$ and $\cos(A - B) = \frac{31}{32}$, then c is

(a) 36 (b) $\sqrt{6}$ (c) 3 (d) 6

43. The angles A, B, C of triangle ABC, are in AP. If $b : c = \sqrt{3} : \sqrt{2}$, then angle A is

(a) 60° (b) 45° (c) 75° (d) 30°

44. In a triangle ABC, If $a = 5, b = 4$ and $A = 60°$, then c is a root of the equation

(a) $x^2 - 4x - 9 = 0$ (b) $x^2 + 4x + 9 = 0$ (c) $x^2 - 4x + 9 = 0$ (d) $x^2 + 4x - 9 = 0$

45. In a triangle ABC, $b^2 \sin 2C + c^2 \sin 2B$ is

(a) ca sin B (b) ab sin B (c) 2bc sin A (d) bc sin A

46. In a triangle ABC, If $\frac{1}{b+c} + \frac{1}{c+a} = \frac{3}{a+b+c}$, then $A + B - C$ is

(a) 90° (b) 120° (c) 45° (d) 60°

47. In any triangle ABC, if $r = r_1 - r_2 - r_3$, then the triangle is

(a) equilateral (b) right-angled (c) isosceles (d) obtuse angled

48. The sides of a triangle are $5x + 12y$, $12x + 5y$ and $13x + 13y$ where x, y are positive numbers. Then the triangle is

(a) right-angled (b) obtuse angled (c) acute angled (d) equilateral

49. If for a triangle ABC, $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then $\sin^3 A + \sin^3 B + \sin^3 C$ is

(a) 1 (b) 0 (c) 3 sin A sin B sin C (d) sin A + sin B + sin C

50. In a triangle ABC, the angle A is greater than angle B. If the measures of angles A and B satisfy the equation $3 \sin x - 4 \sin^3 x - k = 0$, $0 < k < 1$, then the angle C is

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

51. If the tangents of the angles A and B of a triangle ABC, satisfy the equation $abx^2 - c^2 x + ab = 0$, where, a, b, c are the sides of the triangle, then

(a) $\tan A = \frac{a}{b}$ (b) $\tan B = \frac{b}{a}$ (c) $\sin^2 A + \sin^2 B + \sin^2 C = 2$ (d) All the above

52. The angles of elevation of the top of a tower from two points which are at distances of a and b from the base and are collinear with the base, are complementary. Then the height of the tower is

(a) ab (b) $\sqrt{ab}$ (c) a + b (d) $\sqrt{a+b}$

53. The angle of elevation of a cloud from a point 200 m above a lake is 30° and the angle of depression of its reflection in the lake is 60°. The height of the cloud is

(a) 300 m (b) $300\sqrt{3}$ m (c) $200\sqrt{3}$ m (d) 400 m

54. In a triangle ABC, if $(a + b + c)(b + c - a) = \lambda bc$, then

(a) $\lambda < 0$ (b) $\lambda > 6$ (c) $0 < \lambda < 4$ (d) $\lambda > 4$

55. In a triangle ABC, if $a^4 + b^4 + c^4 = 2a^2(b^2 + c^2) + b^2c^2$, then $\angle A$ is

(a) 30° (b) 60° (c) 75° (d) 15°

56. In a triangle ABC, if a = 13, b = 14, c = 15 then $\frac{1}{r_1}:\frac{1}{r_2}:\frac{1}{r_3}$ is

(a) 3 : 4 : 5 (b) 8 : 7 : 6 (c) 6 : 7 : 8 (d) 5 : 12 : 13

57. If A, A_1, A_2, A_3 are respectively the areas of the incircle and the three excircles of a triangle, then $\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}$ is

(a) $\frac{2}{\sqrt{A}}$ (b) $\frac{3}{\sqrt{A}}$ (c) $\frac{1}{2\sqrt{A}}$ (d) $\frac{1}{\sqrt{A}}$

58. In a triangle ABC, if $\frac{a^3+b^3+c^3}{\sin^3 A+\sin^3 B+\sin^3 C}=27$ and $a \ge b \ge c$, then the maximum possible value of a is

(a) 3 (b) 2 (c) $\frac{3}{2}$ (d) $\frac{2}{3}$

59. In an equilateral triangle ABC, $R : r : r_1$ is

(a) 1 : 2 : 3 (b) 3 : 1 : 2 (c) 2 : 1 : 3 (d) 3 : 2 :1

60. The sides of a triangle are three consecutive natural numbers and the greatest angle is twice that of least angle. Then the sides are

(a) 3, 4, 5 (b) 4, 5, 6 (c) 5, 6, 7 (d) 6, 7, 8

61. In any triangle ABC, if $\frac{(a^2+a+1)(b^2+b+1)(c^2+c+1)}{abc}=k$, then

(a) $k \ge 3$ (b) $k \ge 27$ (c) $k \le 3$ (d) $k \le 27$

62. The base angles of a triangle are $22\frac{1}{2}^{\circ}$ and $112\frac{1}{2}^{\circ}$. Then the base of the triangle is equal to

(a) its height (b) half the height (c) twice the height (d) None of these

63. In a triangle ABC, if $a\tan A + b\tan B = (a+b)\tan\frac{A+B}{2}$ then

(a) 2A = B (b) A = 2B (c) A = 3B (d) A = B

64. In a triangle ABC, $a^2\cot A + b^2\cot B + c^2\cot C$ is

(a) 2Δ (b) 3Δ (c) 4Δ (d) Δ

65. In a triangle ABC, $a=(1+\sqrt{3})$ cm, b = 2 cm, $\angle C = 60°$ then the other two angles A and B and the side c are respectively given by

(a) (75°, 45°); $\sqrt{6}$ (b) (45°, 75°); $\sqrt{6}$ (c) (30°, 90°); $\sqrt{6}$ (d) (90°, 30°); $\sqrt{2}$

66. If b + c : c + a : a + b = 11 : 12 : 13, then cos A : cos B : cos C

(a) 7 : 9 : 25 (b) 19 : 7 : 25 (c) 25 : 19 : 7 (d) 7 : 19 : 25

67. If a, b, c, d are the sides of a quadrilateral, then $\frac{a^2+b^2+c^2}{d^2}$ is greater than

(a) $\frac{1}{12}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$. (d) $\frac{1}{4}$

68. The perimeter of a regular polygon of n sides inscribed in a circle of radius r is

(a) $nr\sin\frac{\pi}{n}$ (b) $\frac{2n}{r}\sin n\pi$ (c) $2nr\sin\frac{\pi}{2n}$ (d) $2nr\sin\frac{\pi}{n}$

69. In a triangle ABC, p is the product of sines of the angles and q is the product of their cosines. The tangents of the angles are the roots of the equation

(a) $qx^3 - px^2 + (1 + q)x - p = 0$ (b) $px^3 - qx^2 + (1 + q)x + q = 0$

(c) $qx^3 + px^2 + (1 + q)x + p = 0$ (d) $qx^3 + px^2 - (1 + q)x + p = 0$

70. In a triangle ABC, if $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in HP then the minimum value of $\cot\frac{B}{2}$ is

(a) $\sqrt{2}$ (b) 1 (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

71. In a triangle ABC, if $\frac{a^2 + b^2 + c^2}{4\Delta} = k$, then

(a) $k \geq \sqrt{3}$ (b) $k \leq \sqrt{3}$ (c) $k < \sqrt{3}$ (d) $k > \sqrt{3}$

72. The radius of the circle passing through the vertices B and C and the incentre of triangle ABC, is

(a) $a\sec\frac{A}{2}$ (b) $\frac{a}{2}\sec A$ (c) $a \sec A$ (d) $\frac{a}{2}\sec\frac{A}{2}$

73. If twice the square of the diameter of a circle is equal to sum of the squares of the sides of the inscribed triangle ABC, then $\cos^2 A + \cos^2 B + \cos^2 C$ is equal to

(a) 1 (b) 2 (c) 4 (d) 8

74. In a triangle ABC, if $\angle A = \frac{\pi}{4}$ and $\tan B \tan C = p$, then

(a) $p \notin \left[3 - 2\sqrt{2}, 3 + 2\sqrt{2}\right]$ (b) $3 - 2\sqrt{2} < p < 3 + 2\sqrt{2}$

(c) $p \notin \left(3 - 2\sqrt{2}, 3 + 2\sqrt{2}\right)$ (d) p can take any real value

75. A triangle ABC, is such that b>a>c. Then

(a) $r < r_1 < r_2 < r_3$ (b) $r_2 < r < r_3 < r_1$ (c) $r < r_3 < r_1 < r_2$ (d) $r_1 < r_2 < r_3 < r$

76. If $A_1, A_2, A_3 \ldots A_n$ are the vertices of a regular n-gon and if $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, then the value of n is

(a) 4 (b) 5 (c) 6 (d) 7

77. A ring 10 cm in diameter, is suspended from a point 12 cm above its centre by 6 equal strings attached to its circumference at equal intervals. Then the cosine of angle between two consecutive strings is

(a) $\frac{331}{338}$ (b) $\frac{313}{338}$ (c) $\frac{338}{373}$ (d) $\frac{313}{373}$

78. In a triangle ABC, $\cos\alpha = \frac{a}{b + c}, \cos\beta = \frac{b}{c + a}, \cos\gamma = \frac{c}{a + b}$ and $k = \tan^2\frac{\alpha}{2} \cdot \tan^2\frac{\beta}{2} \cdot \tan^2\frac{\gamma}{2}$. Then,

(a) $27k > 1$ (b) $27k \leq 1$ (c) $9k = 1$ (d) $8k = 1$

79. In a triangle ABC, AD is the altitude from A. $\angle C = 23°$, and $b > c$ and $AD = \frac{abc}{b^2 - c^2}$. Then $\angle B$ is

(a) 113° (b) 67° (c) 90° (d) 45°

80. In a triangle ABC, if A = 45°, C = 75°, then the side a is

(a) $\sqrt{2}b$ (b) $\frac{2}{\sqrt{3}}b$ (c) $\frac{\sqrt{2}}{\sqrt{3}}b$ (d) $\frac{3}{\sqrt{2}}b$

81. For a triangle ABC, $\sum a(\sin B - \sin C)$ is

(a) Δ (b) 1 (c) 0 (d) abc

82. In a triangle ABC, $\frac{\tan B}{\tan C}$ is equal to

(a) $\frac{c^2 + a^2 - b^2}{b^2 + c^2 - a^2}$ (b) $\frac{a^2 + b^2 - c^2}{c^2 + a^2 - b^2}$ (c) $\frac{b^2 + c^2 - a^2}{a^2 + b^2 - c^2}$ (d) $\frac{a^2 + b^2 + c^2}{abc}$

83. In a triangle ABC, if a = 5, b = 6, c = 7, then the value of $\sin\frac{A}{2}$ is

(a) $\sqrt{\frac{1}{7}}$ (b) $\sqrt{\frac{2}{7}}$ (c) $\sqrt{\frac{3}{7}}$ (d) $\sqrt{\frac{3}{8}}$

84. If in a triangle ABC, $a\cos^2\left(\frac{C}{2}\right) + c\cos^2\left(\frac{A}{2}\right) = \frac{3b}{2}$, then the sides a, b, c

(a) are in HP (b) satisfy the relation a + b = c
(c) are in AP (d) are in GP

85. In a triangle ABC, if a, b, c are in AP then $\cot\frac{A}{2}\cot\frac{C}{2}$ is

(a) 1 (b) 2 (c) 3 (d) 4

86. If 2, 5 and 8 units are the radii of three circles, which touch each other, then the area of the triangle formed by joining the centres of these circles is

(a) $10\sqrt{3}$ (b) $20\sqrt{3}$ (c) $5\sqrt{3}$ (d) $12\sqrt{3}$

87. Given that two sides of a parallelogram are 10 cm and 20 cm and one of the angles is 60°, the length of the shorter diagonal is

(a) $8\sqrt{3}$ (b) $6\sqrt{3}$ (c) $4\sqrt{3}$ (d) $10\sqrt{3}$

88. In an equilateral triangle, the ratio of the areas of the incircle and that of the circumcircle is

(a) 1 : 4 (b) 4 : 1 (c) 1 : 2 (d) 1 : 3

89. From the top of a tower, the angles of depression of the top and bottom of a building are 30° and 45° respectively. If the height of the building is 40 m, the height of the tower is

(a) $20\left(3+\sqrt{3}\right)$ (b) $20\left(\sqrt{3}+1\right)$ (c) $40\left(3+\sqrt{3}\right)$ (d) $40\left(\sqrt{3}+1\right)$

90. A circle is inscribed in an equilateral triangle of side a. The area of any square inscribed in this circle is

(a) $\frac{a^2}{3}$ (b) $\frac{a^2}{6}$ (c) $\frac{a^2}{2}$ (d) $\frac{a^2}{\sqrt{16}}$

91. In a triangle ABC, if $\cos A + \cos B = 4\sin^2\frac{C}{2}$, then a, c, b are in

(a) HP (b) AP (c) GP (d) AGP

92. In a triangle ABC, $\tan\frac{A}{2} = \frac{5}{6}$, $\tan\frac{B}{2} = \frac{20}{37}$. Then $\tan\frac{C}{2}$ is

(a) $\frac{2}{5}$ (b) $\frac{5}{2}$ (c) $\frac{3}{5}$ (d) $\frac{5}{3}$

93. In a triangle ABC, $bc\cos^2\frac{A}{2} + ca\cos^2\frac{B}{2} + ab\cos^2\frac{C}{2}$ is equal to

(a) $s^2 - a^2$ (b) $s^2 - b^2$ (c) s^2 (d) r^2

94. In a triangle ABC, it is given that $\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$. Then angle A is

(a) 90° (b) 60° (c) 45° (d) 30°

95. In a triangle ABC, tan A + tan B + tan C = 6 and tan B tan C = 3. Then, the triangle is
(a) equilateral (b) right-angled (c) obtuse angled (d) acute angled

96. If D is the mid-point of side BC of a triangle ABC, and AD is perpendicular to AC, then
(a) $3b^2 = a^2 - c^2$ (b) $3a^2 = b^2 - c^2$ (c) $3c^2 = a^2 - b^2$ (d) $5c^2 = a^2 + b^2$

97. In a right-angled triangle, the hypotenuse is 4-times as long as the perpendicular from the opposite vertex. One of the acute angles of the triangle is
(a) 45° (b) 30° (c) 15° (d) 22.5°

98. An aeroplane flying at a height of 600 m above the ground passes vertically above another plane at an instant when the angles of elevation of the two planes from a point on the ground are 60° and 45° respectively. The height of the second plane above the ground is

(a) $100\sqrt{3}$ metre (b) $\frac{200}{\sqrt{3}}$ metre (c) $200\sqrt{3}$ metre (d) $100\left(\sqrt{3}+1\right)$ metre

99. A regular pentagon and a regular decagon have the same perimeter. Then the ratio of their areas is

(a) $\sqrt{5} : 2$ (b) $3 : \sqrt{5}$ (c) $5 : 2$ (d) $2 : \sqrt{5}$

100. In a triangle, angles A, B, C, are in AP. Then $\lim_{A \to C} \frac{\sqrt{3 - 4\sin A \sin C}}{|A - C|}$ is

(a) 1 (b) 2 (c) 3 (d) 4

101. If s be the semi perimeter of triangle ABC, then $\frac{s}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$ is

(a) $2\sqrt{\frac{abc}{\sin A \sin B \sin C}}$ (b) $\sqrt[3]{\frac{abc}{\sin A \sin B \sin C}}$ (c) $2 \cdot \sqrt[3]{\frac{abc}{\sin A \sin B \sin C}}$ (d) $3 \cdot \sqrt{\frac{abc}{\sin A \sin B \sin C}}$

102. In a triangle ABC, $2abc\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ is equal to

(a) 2s (b) 2sr (c) $2s^2 r$ (d) $2s\,r^2$

103. In a triangle ABC, cos A cos B + sin A sin B sin C = 1. Then triangle ABC, is
(a) only isosceles (b) only right angled (c) isosceles right angled (d) equilateral

104. Three circles, whose radii are a, b, c touch each other externally and the tangents at their points of contact meet at a point. Then the distance of this point from any of the points of contact is

(a) $\sqrt{\frac{a+b+c}{abc}}$ (b) $\sqrt{a+b+c}$ (c) $\sqrt{\frac{abc}{a+b+c}}$ (d) $\sqrt{abc}$

105. If in a triangle ABC, $\sin C + \cos C + \sin(2B + C) - \cos(2B + C) = 2\sqrt{2}$, then the triangle is
(a) equilateral (b) isosceles and right angled
(c) obtuse angled (d) only right angled

106. In triangle ABC, if AC = 3, BC = 4, and medians AD and BE are perpendicular, then cos C is

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{3}{4}$ (d) $\frac{5}{6}$

107. If the sides a, b, c of a triangle ABC, satisfy the equation $x^3 - 11x^2 + 38x - 40 = 0$, then $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ is

(a) $\frac{16}{9}$ (b) 1 (c) $\frac{9}{16}$ (d) $\frac{7}{16}$

108. In a triangle ABC, D is the mid point of AB. Then cot ($\angle CDA$) =

(a) $\frac{a^2 + b^2}{4\Delta}$ (b) $\frac{a^2 - b^2}{4\Delta}$ (c) $\frac{4\Delta}{a^2 + b^2}$ (d) $\frac{4\Delta}{a^2 - b^2}$

109. In a triangle ABC, the tangent of half the difference of two angles equals one third of the tangent of half the sum of the two angles. The ratio of the sides opposite these angles is

(a) 2 : 3 (b) 1 : 3 (c) 2 : 1 (d) 3 : 4

110. In a triangle ABC, if the sides are in A.P and the greatest angle is 90° more than the least angle, then the value of the sine of the third angle is

(a) $\frac{\sqrt{7}}{8}$ (b) $\frac{\sqrt{7}}{4}$ (c) $\frac{\sqrt{7}}{16}$ (d) $\frac{3}{4}$

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

111. Statement 1

In a right angled triangle ABC, if $A = \frac{\pi}{2}, R = \frac{a}{2}$

and

Statement 2

In a right triangle, the circumcenter is at the midpoint of the hypotenuse.

112. Statement 1

In any triangle ABC, minimum value of $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ is $\frac{1}{8}$

and

Statement 2

Arithmetic mean of a set of positive quantities is greater than or equal to their geometric mean and that the equality is attained when the numbers are equal.

113. Statement 1

Triangle ABC, in which $B = \frac{\pi}{6}, C = \frac{\pi}{3}, b = 10, c = 12\sqrt{3}$ does not exist.

and

Statement 2

In any ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Linked Comprehension Type Questions

Directions: **This section contains 1 paragraph. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

In a triangle ABC, $a = 40$, $c = 40\sqrt{3}$ and $B = 30°$

114. $R =$

(a) 20 (b) 40 (c) $20\sqrt{3}$ (d) $40\sqrt{3}$

115. $r =$

(a) $40\left(\sqrt{3}-1\right)$ (b) $20\left(2\sqrt{3}-3\right)$ (c) $20\left(\sqrt{3}+1\right)$ (d) None of the above

116. $r_1 =$

(a) 20 (b) $10\sqrt{3}$ (c) $20\sqrt{3}$ (d) $\frac{20}{\sqrt{3}}$

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

117. In an isosceles Δ ABC, with base BC, $\angle A = 20°$. If P is a point on AC where AP = BC,

(a) $\angle$ ABP = 10° (b) $\angle$ APB = 110°

(c) Circum radius of ΔPBC is BC (d) Circum radius of Δ PBC is AB – PC

118. In a ABC, $\angle C = 90°$. Then

(a) $r + 2R = s$ (b) $r_1 + r_2 + r_3 = r + R$ (c) $r_3 = r + c$ (d) $r_1 + r_2 = r + r_3$

119. In a triangle ABC, with angle $A = \frac{\pi}{3}$, $\tan B + \tan C = p$, if

(a) $p \geq -\frac{2\sqrt{3}}{3}$ (b) $p \leq -\frac{2\sqrt{3}}{3}$ (c) $p \leq 2\sqrt{3}$ (d) $p \geq 2\sqrt{3}$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

120. In ΔABC, AD, BE and CF are bisectors of $\angle A$, $\angle B$ and $\angle C$ respectively

	Column I		**Column II**
(a)	$\frac{2bc}{(b+c)}\cos A/2$	(p)	AD. BD
(b)	$\frac{abc}{R(b+c)}\operatorname{cosec} A/2$	(q)	BD + AD
(c)	$\frac{1}{Rb}\left[\frac{abc}{b+c}\right]^2 \operatorname{cosec} A/2$	(r)	BD/AD
(d)	$\frac{4R\cos A/2 . \sin B \sin C}{\left(\sin B + \sin C\right)}$	(s)	AD

ADDITIONAL PRACTICE EXERCISE

Subjective Questions

121. In a Δ ABC, BE is the altitude through B and O is the orthocenter. If B = 60° and C = 45°, find the value of $\frac{OE}{OB}$.

122. In a Δ ABC, A = 60°, B = 75°. E is a point on BA such that the area of the Δ CBE is $\sqrt{3}$ times the area of ΔCAE. Given that ∠BCE = θ, find sin θ.

123. In a triangle ABC, AC = 9 and AB = 7 and $\cos(B - C) = \frac{4}{5}$. Find the area of the triangle ABC.

124. Find the ratio of the area of the regular polygon of 12 sides circumscribed about a circle to the area of the regular polygon of the same number of sides inscribed in the circle.

125. Three circles whose radii are 4, 7 and 11 touch one another externally and the tangents at their points of contact meet in a point. Find the distance of this point from any of the points of contact.

126. A train is moving at a constant speed in the direction θ east of north. O is a fixed point on the ground and observations are made from O. At a particular instant of time, the train is 45° west of north. 10 minutes later, its bearing was due North of O. After 10 more minutes, its bearing was 30° east of north. Find the value of tan θ.

127. In a triangle ABC,

(i) if $\cos A + \cos B + \cos C = \frac{3}{2}$, prove that the triangle is equilateral.

(ii) find the maximum value of cos A + cos B + cos C

128. In a triangle ABC, if ∠B = 90° then show that

$$\left(1 - \frac{r_2}{r_1}\right)\left(1 - \frac{r_2}{r_3}\right) = 2$$

129. In a triangle ABC, show that

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geq 1$$

130. In triangle ABC, if A = 3B, then prove that

(i) $\cos B = \sqrt{\frac{a+b}{4b}}$

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

131. In a triangle ABC, D is the mid point of BC. If AD is perpendicular AC, then

(a) tan A + tan C = 0 (b) tan A – tan C = 0 (c) tan A + 2 tan C = 0 (d) 2 tan A = tan C

132. In a triangle ABC, maximum value of sin A + sin B + sin C and sin A sin B sin C are respectively equal to

(a) $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{8}$ (b) $\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{8}$ (c) $\frac{3\sqrt{3}}{4}, \frac{3\sqrt{3}}{8}$ (d) $\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}$

133. If ABC, is an acute angled triangle, then tan A tan B tan C is always greater than or equal to

(a) $3\sqrt{3}$ (b) $2\sqrt{2}$ (c) 2 (d) $4\sqrt{2}$

134. The ratio of areas of a regular pentagon to the polygon formed by joining the points of intersection of its diagonals is

(a) 1 : 4 (b) $1 : 16\sin^2 14°$ (c) $1 : 16\sin^4 18°$ (d) $4 : \sin^4 18°$

135. If cotA, cotB, cotC are in H.P then the value of tanB is

(a) 1 (b) $\sqrt{3}$ (c) $\sqrt{2}$ (d) $3\sqrt{2}$

136. If P_1, P_2, P_3 are the perpendiculars drawn from A, B, C to the opposite sides of the triangle ABC, then $P_1P_2P_3$ is

(a) $\frac{2\Delta}{R}$ (b) $\frac{2\Delta}{r}$ (c) $\frac{2\Delta^2}{R}$ (d) $\frac{2\Delta^2}{R^2}$

137. In a triangle ABC, $r_1r_2 + r_2r_3 + r_3r_1$ is

(a) $\frac{s-a}{\Delta}$ (b) $\frac{\Delta}{s}$ (c) s^2 (d) a^2

138. If the sides a, b, c of a triangle ABC, are in AP then $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in

(a) AP (b) GP (c) HP (d) None of these

139. The area of a triangle whose sides are in AP is $\frac{3}{5}$ times the area of an equilateral triangle having the same perimeter then the proportional values of the sides of the triangle.

(a) 2 : 3 : 4 (b) 1 : 3 : 5 (c) 3 : 5 : 7 (d) 3 : 7 : 9

140. An isosceles triangle ABC, has AB = AC = b and $\angle B = \alpha \left(\alpha < \frac{\pi}{4}\right)$. If R, r are the circumradius and inradius, then r =

(a) $\frac{2b\sin\alpha}{1+\cos\alpha}$ (b) $\frac{b\sin\alpha}{2(1+\cos\alpha)}$

(c) $\frac{b\sin 2\alpha}{2(1+\cos\alpha)}$ (d) $\frac{\sin 2\alpha}{1+\cos\alpha}$

141. Three circular coins, of radii 1 cm, 2 cm, 3 cm are placed on a horizontal plane, each touching the other two. The measure of the area (in cm^2) of the region of the open space bounded by the coins is

(a) $2\pi - 6$ (b) $6 - \frac{5}{2}\left(\tan^{-1}\frac{4}{3} + 2\tan^{-1}\frac{3}{4}\right)$

(c) $6 - \frac{5\pi}{4}$ (d) $6 - 5\tan^{-1}\frac{3}{4}$

($\tan^{-1}$ a means the number of radians is $\tan^{-1}$a)

142. In triangle ABC, $a^2 + b^2 = c^2\cos^2 C$; then

(a) $C = \frac{\pi}{2}$ (b) $\frac{\pi}{4} < C < \frac{\pi}{2}$ (c) $C < \frac{\pi}{2}$ (d) $A + B < \frac{\pi}{2}$

143. If, in triangle ABC, $\dfrac{\sin A}{\sin B} = \dfrac{\sin(C - A)}{\sin(B - C)}$, then

(a) a^2, b^2, c^2 are in AP (b) a^2, c^2, b^2 are in AP (c) b^2, a^2, c^2 are in AP (d) a^2, b^2, c^2 are in HP

144. The measures of the sides of a triangle are 3, 4, 5; the distance between the orthocenter and the circumcentre of the triangle is

(a) 4.5 (b) 3.5 (c) 2.5 (d) 2

145. If $\cos B : \cos C = c : b$ in triangle ABC, then

(a) $\sin B = \sin C$ (b) $\sin B = \cos C$ (c) (a) or (b) (d) (a) and (b)

146. The sides of a triangle ABC, are such that $b + c - a = \dfrac{c + a - b}{3} = \dfrac{a + b - c}{5}$; cosA is

(a) $\dfrac{1}{4}$ (b) $-\dfrac{1}{4}$ (c) $\dfrac{1}{2}$ (d) $-\dfrac{1}{2}$

147. If, in triangle ABC, $a^2 + b^2 + c^2 = 8R^2$, then the triangle is

(a) equilateral (b) right-angled
(c) isosceles but not equilateral (d) isosceles but not right-angled

148. Triangle ABC, has area measure Δ and inradius r; the inradius of its medial triangle is

(a) $\dfrac{\Delta}{r}$ (b) $\dfrac{\Delta}{2r}$ (c) r (d) $\dfrac{r}{2}$

149. The triangle ABC, in which $\cot A + \cot B + \cot C = \sqrt{3}$ is

(a) isosceles with vertical angle 120° (b) isosceles with vertical angle 30°
(c) a right triangle with one angle 60° (d) equilateral

150. ℓ, m, n are the lengths of the common chords of the circles, on the sides BC, CA, AB of a triangle ABC, as diameters, taken in pairs. Area measure of ABC, is Δ, circumradius is R and inradius, r. Then, $\dfrac{1}{\ell} + \dfrac{1}{m} + \dfrac{1}{n}$ is equal to

(a) $\dfrac{1}{r}$ (b) $\dfrac{1}{R}$ (c) $\dfrac{2}{R}$ (d) $\dfrac{\Delta}{R}$

151. The measures a, b, c of the sides of triangle ABC, are the roots of

$x^3 - 9x^2 + 26x - 24 = 0$; $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c}$ in equal to

(a) $\dfrac{1}{3}$ (b) $\dfrac{5}{8}$ (c) $\dfrac{29}{48}$ (d) $\dfrac{1}{2}$

152. The harmonic mean of the three ex – radii (in cms) of a triangle with inradius 1 cm and circumradius 3 cm. is

(a) $\dfrac{1}{3}$ (b) 3 (c) $\dfrac{2}{3}$ (d) 1

153. The ratio of the perimeter of triangle ABC, to that of its pedal triangle is, with the usual notation

(a) $\dfrac{R}{r}$ (b) $\dfrac{\Delta}{R}$ (c) $\dfrac{\Delta}{s}$ (d) $\dfrac{\Delta}{r}$

154. In triangle ABC, $\sin\dfrac{B}{2} = \cos\dfrac{B}{2}$ and a, sinA, cosA are rational. Then, in the triangle,

(a) at least one side is irrational (b) b, c are irrational
(c) all sides are rational (d) one of b, c is irrational

155. In a triangle ABC, tanA + tanB + tanC = 2 tanB tanC = 3tanC tanA = 6; (tanA, tanB, tanC) is

(a) (1, 2, 3) (b) (3, 2, 1) (c) (1, 3, 2) (d) (2, 3, 1)

156. The external bisector of angle A of triangle ABC, meets the line BC in D; AD is

(a) $\dfrac{2bc}{|b-c|}\sin\dfrac{A}{2}$ (b) $\dfrac{2bc}{|b-c|}\cos\dfrac{A}{2}$ (c) $\dfrac{2bc}{b+c}\sin\dfrac{A}{2}$ (d) $\dfrac{2bc}{b+c}\cos\dfrac{A}{2}$

157. α, β, γ are respectively the ' length of attitudes' from A, B, C of triangle ABC; then,

$\dfrac{\cos B+\cos C}{\alpha}+\dfrac{\cos C+\cos A}{\beta}+\dfrac{\cos A+\cos B}{\gamma}$ is equal to

(a) $\dfrac{a^2+b^2+c^2}{\Delta}$ (b) $\dfrac{a^2+b^2+c^2}{\Delta^2}$ (c) $\dfrac{2s}{\Delta}$ (d) $\dfrac{s}{\Delta}$

158. a, b, c are the measures of the sides and A, B, C those of the corresponding opposite angles of triangle ABC. Then $\left(\sum a\right)\left(\sum \cos A\right)$ is equal to

(a) $a+b+c$ (b) $a+b+c+\dfrac{abc}{2R^2}$ (c) $a+b+c+\dfrac{abc}{\Delta}$ (d) $\dfrac{abc}{R^2}$

159. If α, β, γ are the length of altitudes of triangle ABC, then, in the usual notation $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$ is equal to

(a) $\dfrac{3}{r}$ (b) $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}$ (c) $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}+\dfrac{1}{r}$ (d) $\dfrac{3}{r}-\left(\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}\right)$

160. If A, B, C are the measures of the angles of triangle ABC, then the minimum value of $\cot^2A+\cot^2B+\cot^2C$ is

(a) 1 (b) $\sqrt{3}$ (c) 2 (d) 3

161. If, in triangle ABC, 2 cosA + cosB + cosC = 2, then

(a) a, b, c are in AP (b) a, b, c are in HP

(c) c, a, b are in AP (d) b, c, a are in AP

162. The areas of a regular polygon of n sides inscribed in a circle, the circle and a regular polygon of n sides circumscribed to the circle are in the proportion

(a) $\sin\dfrac{2\pi}{n} : \dfrac{2\pi}{n} : 2\tan\dfrac{\pi}{n}$ (b) $\sin\dfrac{\pi}{n} : \dfrac{\pi}{n} : \tan\dfrac{\pi}{n}$

(c) $\sin\dfrac{2\pi}{n} : \dfrac{2\pi}{n} : \tan\dfrac{2\pi}{n}$ (d) $\cos\dfrac{2\pi}{n} : \dfrac{2\pi}{n} : \cos\dfrac{2\pi}{n} : \sin\dfrac{2\pi}{n}$

163. P, Q, R are the 'feet of altitudes' of triangle ABC. If circumradius of triangle ABC, is R, then that of triangle PQR is

(a) R (b) $\dfrac{R}{2}$ (c) $\dfrac{R}{4}$ (d) $\dfrac{R}{3}$

164. P, Q, R are the 'feet of altitudes' of triangle ABC; the measures of the sides of triangle PQR are

(a) $\dfrac{a}{2}\cos A, \dfrac{b}{2}\cos B, \dfrac{c}{2}\cos C$ (b) a cosA, b cosB, c cosC

(c) a sinA, b sinB, c sinC (d) $\dfrac{a}{2}\sin A, \dfrac{b}{2}\sin B, \dfrac{c}{2}\sin C$

165. In a triangle ABC, the value of sinA sinB sinC does not exceed

(a) $\dfrac{2}{5}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{1}{3}$ (d) None of these

166. P is foot of perpendicular from A to BC, in triangle ABC, and D is on $\overline{AP}$ such that $\angle PBD = \frac{B}{3}$. Then AD is equal to

(a) $2c \sin \frac{B}{3}$ (b) $c \sin \frac{B}{3}$ (c) $\frac{c}{3} \sin B$ (d) $\frac{2c}{3} \sin B$

167. If, in triangle ABC, a^2, b^2, c^2 are in AP, then

(a) tanA. tanB, tanC are in AP
(b) tanA, tanB, tanC are in HP
(c) cos A, cosB, cosC are in AP
(d) sinA, sinB, sinC are in HP

168. O is a point on AB, OA = OB = OC in triangle ABC; then $\sin^2 A + \sin^2 B + \sin^2 C$ is equal to

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{3}{2}$ (d) 2

169. If in triangle ABC, $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P then

(a) a, b, c are in AP
(b) a, b, c are in HP
(c) a, c, b are in AP
(d) a, c, b are in HP

170. In triangle ABC, a, b, c are in G.P and sinA, sinB, sinC are in AP, then the triangle ABC, is

(a) right-angled but not isosceles (b) isosceles right angled
(c) isosceles but not right-angled or equilateral
(d) equilateral

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

171. Statement 1

ABC, is an isosceles triangle with $\angle A = \angle C = 75°$. Then, $\frac{r_1}{R} = 1 + \frac{\sqrt{3}}{2}$.

and

Statement 2

In any ΔABC, $r_1 < R$

172. ABC, is an isosceles right angled triangle right angled at A such that BC = 8.

Statement 1

The length of the altitude through A is 4.

and

Statement 2

Orthocentre of ΔABC, is at A.

173. Statement 1

If in a triangle ABC, tan A: tan B: tan C = 1: 2: 3, then, tan B = 2.

and

Statement 2

In any triangle ABC, cot A cot B + cot B cot C + cot A cot C = 1.

174. Statement 1

In a triangle ABC, right angled at B, $b^3 < c^3 + a^3$.

and

Statement 2

Circum radius of a triangle ABC, right angled at B is $\frac{1}{2}b$.

175. Statement 1

In any ΔABC, $r \le \frac{R}{2}$

and

Statement 2

In any ΔABC, $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$

176. Statement 1

In a ΔABC, if $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in HP; then, r_1, r_2, r_3 are in AP.

and

Statement 2

If x, y, z are in HP, $y = \frac{2xz}{x+z}$

177. Statement 1

There cannot be a triangle ABC, with a = 7, b = 2, $\sin B = \frac{1}{3}$

and

Statement 2

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

178. Statement 1

In ΔABC, if $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in AP, (b + c – a), (c + a – b), (a + b – c) are in AP

and

Statement 2

In a ΔABC, $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

179. Statement 1

In a ΔABC, the internal bisector of $\angle A$ meets the side BC in D such that BD = 2DC. Then, 4 < AB < 12.

and

Statement 2

Area of a ΔABC, is $\frac{abc}{4R}$

180. Statement 1

In any ΔABC, $\frac{(1+\sin A+\sin^2 A)(1+\sin B+\sin^2 B)(1+\sin C+\sin^2 C)}{\sin A\sin B\sin C}$ is always greater than 27.

and

Statement 2

In any ΔABC, sinA, sinB, sinC are always positive.

Linked Comprehension Type Questions

Directions: **This section contains 2 paragraphs. Based upon the paragraph, multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

The triangle formed by ex-centres of a triangle ABC, is known as ex-central triangle of ABC. If I_1, I_2, I_3 are the ex-centres, I-the incentre, r-the in-radius and R the circum radius of triangle ABC, then

181. Side of the ex-central triangle opposite to I_1 is

(a) $4R\cos\frac{A}{2}$ (b) $4R\cos A$

(c) $4R\sin\frac{A}{2}$ (d) $4R\sin A$

182. Circum radius of the ex-central triangle of $I_1I_2I_3$ is

(a) 2 R (b) 4 R

(c) 8 R (d) 3R

183. Area of the ex-central triangle of ABC, is

(a) $4R^2\cos A\cos B\cos C$ (b) $8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(c) $8R^2\sin A\sin B\sin C$ (d) $4R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

184. $(II_1)\,.\,(II_2)\,.\,(II_3) =$

(a) $16R^2$ (b) $16\frac{r}{R}$ (c) $16R^2r$ (d) $64R^2r$

185. Area of the ex-central triangle of ABC, =

(a) $\frac{abc}{2r}$ (b) $\frac{abc}{2R}$ (c) 4Rr (d) $\frac{4R}{r}$

186. The ortho centre of triangle $I_1I_2I_3$, is

(a) circum centre of triangle ABC (b) ortho centre of triangle ABC

(c) centroid of triangle ABC (d) in centre of triangle ABC

Passage II

If AD, BE, CF are the altitudes of triangle ABC, then the triangle formed by D, E, F is known as the pedal triangle of ABC.'O' is the orthocentre of ABC.

187. The angle of the pedal triangle opposite to vertex A is

(a) $180° + 2A$ (b) $180° - 2A$ (c) $90° - \frac{A}{2}$ (d) $90° + A$

188. If R is the circum radius of ABC, circum radius of its pedal triangle

(a) $\frac{R}{2}$ (b) R (c) $\frac{R}{8}$ (d) $\frac{R}{4}$

189. The ratio of areas of circum circle of ex-central triangle of ABC, to that of pedal triangle of ABC, is

(a) 16 : 1 (b) 8 : 1 (c) 4 : 1 (d) 2 : 1

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

190. In a Δ ABC

(a) $\sum \frac{r}{s-a} = \frac{1}{s}\Sigma r_1$

(b) $\sum \frac{1}{r_1} = 3s$

(c) $\sum \cot\frac{A}{2} = \frac{s}{r}$

(d) $\sum \tan\frac{A}{2} = \frac{r}{s}$

191. ℓ is the length of the median from the vertex B to the side AC of a triangle ABC. Then,

(a) $4\ell^2 = 2c^2 + 2a^2 - b^2$

(b) $4\ell^2 = c^2 + a^2 + 2ca\cos B$

(c) $4\ell^2 = b^2 + 4ca\cos B$

(d) $4\ell^2 = (2s-b)^2 - 4ca\sin^2\frac{B}{2}$

192. In a ΔABC, $\tan^2\frac{C}{2} = 2$. Then,

(a) $\frac{r_1r_2}{r_3} = \frac{\Delta}{2s}$

(b) $\cos C = \frac{1}{3}$

(c) $\cos C = \frac{-1}{3}$

(d) $3s(s-a) = bc$

193. In ΔABC, $C = 2A$; then

(a) $\cos\frac{C}{2} = \frac{a+b}{2c}$

(b) $\cos\frac{C}{2} = \frac{a-b}{2c}$

(c) $\sin A + \sin B = 2\sin C\cos A$

(d) $c^2 = a(a+b)$

194. In a ΔABC, $BC = 2$; the median CF through the vertex C is of length 1 and $\angle CFB = \frac{\pi}{6}$. Then

(a) area of the ΔABC, $= \frac{\sqrt{15}+\sqrt{3}}{4}$

(b) $AB = \sqrt{15}+\sqrt{3}$

(c) $\cos B = \frac{\sqrt{15}}{4}$

(d) $\sin B = \frac{1}{4}$

195. In a ΔABC, if $r_1 < r_2 < r_3$, then

(a) $a < b < c$

(b) $A < B$

(c) $r_2 > r$

(d) $r_2 < r$

196. If in a ΔABC, $b + c = 3a$, then $\cos B + \cos C =$

(a) $6\sin^2\frac{A}{2}$

(b) $3(1 + \cos(B + C))$

(c) $2\cos(B + C)$

(d) $2\cos C$

197. ABC, is a triangle D, E, F are the feet of the perpendiculars form A, B, C in the opposite sides. Then,

(a) $EF = a\cos A$

(b) $AD = c\sin A$

(c) $DE + EF + DF = a\cos A + b\cos B + c\cos C$

(d) $DE + EF + DF = \frac{2\Delta}{R}$ where Δ is the area of the triangle and R, the circumradius.

Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

198.

Column I	Column II
(a) In triangle ABC, of $\cos A + \cos B + \cos C = \frac{7}{4}$, then $\frac{r}{R}$ is equal to	(p) $\frac{9\sqrt{3}}{4}$
(b) In any ΔABC, if I is the incentre, and inradius = 2, then AI + BI + CI has the least value.	(q) 12
(c) If the inradius of the triangle with sides 5k, 6k and 5k is 6 then k is equal to	(r) 4
(d) In a ΔABC, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ and the side a = 3 then the area of the ΔABC, is	(s) $\frac{3}{4}$

199. In ΔABC, $2a^2 + 9b^2 + c^2 = 6ab + 2ac$

Column I	Column II
(a) $\sin\frac{A}{2}$	(p) $\frac{1}{6}$
(b) $\sin 2A$	(q) $\frac{17}{18}$
(c) $\cos B$	(r) $\sqrt{\frac{5}{12}}$
(d) $\cos C$	(s) $\frac{\sqrt{35}}{18}$

200. In ΔABC, altitudes AD, BE and CF intersect at O and AO = x, BO = y, CO = z. Then

Column I	Column II
(a) $x + y + z$	(p) $4R^2\Sigma\sin A.\ \Sigma\cos A$
(b) $(a + b + c)(x + y + z)$	(q) $4R^2\ \Sigma\sin A$
(c) $x(b + c) + y(a + c) + z(a + b)$	(r) $4R^2\Sigma\sin A(\Sigma\cos A - 1)$
(d) $ax + by + cz$	(s) $2R\Sigma\cos A$

SOLUTIONS

ANSWER KEYS

Topic Grip

1. (i) $\sin A = \frac{4}{5}, \cos A = \frac{3}{5}$,

$\tan A = \frac{4}{3}$

(ii) $\sin A = \frac{1}{\sqrt{5}}, \cos A = \frac{2}{\sqrt{5}}$,

$\tan A = \frac{1}{2}$

(iii) 84

3. $B = 30°$ or $150°$

7. (i) $\frac{1}{3}$

(ii) $\frac{1}{12}\left[7 + 4\sqrt{3}\sqrt{10 - 2\sqrt{5}} - \sqrt{5}\right]$

9. (ii) $-\sqrt{3}$ or $\sqrt{3}$

10. $\frac{7}{20}$

11. (d) 12. (b) 13. (c)
14. (a) 15. (b) 16. (b)
17. (a) 18. (d) 19. (b)
20. (d) 21. (a) 22. (d)
23. (b) 24. (b) 25. (c)
26. (c)
27. (a), (c)
28. (b), (d)
29. (a), (b), (d)
30. (a) → (s)
(b) → (r)
(c) → (q)
(d) → (p)

IIT Assignment Exercise

31. (b) 32. (d) 33. (c)
34. (d) 35. (c) 36. (a)
37. (b) 38. (d) 39. (c)
40. (b) 41. (d) 42. (d)
43. (c) 44. (a) 45. (c)
46. (d) 47. (b) 48. (b)
49. (c) 50. (b) 51. (d)
52. (b) 53. (d) 54. (c)
55. (a) 56. (b) 57. (d)
58. (a) 59. (c) 60. (b)
61. (b) 62. (c) 63. (d)
64. (c) 65. (a) 66. (d)
67. (c) 68. (d) 69. (a)
70. (d) 71. (a) 72. (d)
73. (a) 74. (c) 75. (c)
76. (d) 77. (b) 78. (b)
79. (a) 80. (c) 81. (c)
82. (b) 83. (a) 84. (c)
85. (c) 86. (b) 87. (d)
88. (a) 89. (a) 90. (b)
91. (b) 92. (a) 93. (c)
94. (a) 95. (d) 96. (a)
97. (c) 98. (c) 99. (d)
100. (a) 101. (c) 102. (c)
103. (c) 104. (c) 105. (b)
106. (d) 107. (c) 108. (b)
109. (c) 110. (b) 111. (a)
112. (d) 113. (a) 114. (b)
115. (b) 116. (a)
117. (a), (c), (d)
118. (a), (c)
119. (b), (d)
120. (a) → (s)
(b) → (s)
(c) → (p)
(d) → (s)

Additional Practice Exercise

121. $\frac{\sqrt{3} - 1}{2}$

122. $\frac{1}{2}$

123. $\frac{1512}{73}$

124. $4\left(2 - \sqrt{3}\right)$

125. $\sqrt{14}$

126. $\sqrt{3} + 1$

127. (ii) $\frac{3}{2}$

131. (c) 132. (b) 133. (a)
134. (c) 135. (b) 136. (c)
137. (c) 138. (a) 139. (c)
140. (c) 141. (b) 142. (d)
143. (b) 144. (c) 145. (c)
146. (b) 147. (b) 148. (d)
149. (d) 150. (a) 151. (c)
152. (b) 153. (a) 154. (c)
155. (d) 156. (a) 157. (d)
158. (b) 159. (b) 160. (a)
161. (c) 162. (a) 163. (b)
164. (b) 165. (b) 166. (a)
167. (b) 168. (d) 169. (a)
170. (d) 171. (c) 172. (b)
173. (a) 174. (d) 175. (a)
176. (d) 177. (a) 178. (d)
179. (d) 180 (a) 181. (a)
182. (a) 183. (b) 184. (c)
185. (b) 186. (d) 187. (b)
188. (a) 189. (a)
190. (a), (c)
191. (a), (b), (c), (d)
192. (a), (c)
193. (a), (c), (d)
194. (a), (b), (c), (d)
195. (a), (b), (c)
196. (a), (b)
197. (a), (c), (d)
198. (a) → (s)
(b) → (q)
(c) → (r)
(d) → (p)
199. (a) → (r)
(b) → (s)
(c) → (q)
(d) → (p)
200. (a) → (s)
(b) → (p)
(c) → (q)
(d) → (r)

HINTS AND EXPLANATIONS

Topic Grip

1. (i) $2s = 13 + 14 + 15 = 42$

$s = 21 \quad s - a = 8 \quad s - b = 7 \quad s - c = 6$

$$\therefore \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \frac{4}{5}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{3}{5}$$

$$\tan A = \frac{4}{3}$$

(ii) $$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{7\times 6}{14\times 15}} = \frac{1}{\sqrt{5}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{21\times 8}{14\times 15}} = \frac{2}{\sqrt{5}}$$

$$\tan\frac{A}{2} = \frac{1}{2}$$

(iii) Area of ΔABC, $= \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{21\times 8\times 7\times 6}$$
$$= 84$$

2. $$\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$$

$$\frac{\sin(\pi-(B+C))}{\sin(\pi-(A+B))} = \frac{\sin(A-B)}{\sin(B-C)}$$

$$\Rightarrow \frac{\sin(B+C)}{\sin(A+B)} = \frac{\sin(A-B)}{\sin(B-C)}$$

$\Rightarrow \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$

$\Rightarrow 2\sin^2 B = \sin^2 A + \sin^2 C$

$\Rightarrow 2b^2 = a^2 + c^2$

$\therefore$ a^2, b^2, c^2 are in AP $\left(\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}\right)$

3. $$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{3}{\frac{3}{8}} = \frac{4}{\sin B}$$

$$\Rightarrow \sin B = 4\times\frac{3}{8}\times\frac{1}{3} = \frac{1}{2}$$

$\therefore B = 30°$ or $150°$

4. a. $$a.\frac{s(s-c)}{ab} + c.\frac{s(s-a)}{bc} = \frac{3b}{2}$$

$$\Rightarrow \frac{s(s-c)}{b} + \frac{s(s-a)}{b} = \frac{3b}{2}$$

$$\Rightarrow \frac{s}{b}(2s-(a+c)) = \frac{3b}{2}$$

$\Rightarrow 2s = 3b \Rightarrow a + b + c = 3b$

$\Rightarrow 2b = a + c$

$\therefore$ a, b, c are AP

5. (i) $R = \dfrac{a}{2\sin A}$ and $\Delta = \dfrac{1}{2}bc\sin A$

$$\therefore R = \frac{abc}{4\Delta}$$

(ii) $2R^2 \sin A \sin B \sin C$

$$= 2R^2\times\frac{a}{2R}\times\frac{b}{2R}\times\frac{c}{2R} = \frac{abc}{4R} = \Delta$$

(iii) $$\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s}$$

$$= \frac{abc}{s(s-a)(s-b)(s-c)} = \frac{abc}{\Delta^2} = \frac{1}{\Delta}.\frac{abc}{\Delta}$$

$$= \frac{4R}{\Delta}$$

(iv) $$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$$

$$= \frac{a+b+c}{abc} = \frac{2s}{4R\Delta} = \frac{s}{\Delta}.\frac{1}{2R} = \frac{1}{r}.\frac{1}{2R}$$

$$= \frac{1}{2Rr}$$

(v) $$4\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{c}-1\right)$$

$$= 4\left(\frac{s-a}{a}\right)\left(\frac{s-b}{b}\right)\left(\frac{s-c}{c}\right)$$

$$= \frac{4(s-a)(s-b)(s-c)}{abc} = \frac{4\Delta^2}{s(abc)} = \frac{4\Delta}{abc}.\frac{\Delta}{s}$$

$$= \frac{r}{R}$$

6. We have

$\cos A + \cos B + \cos C$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} + \cos C$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2}$$

$$\le 2\sin\frac{C}{2} + 1 - 2\sin^2\frac{C}{2}$$

$$= -2\left\{\left(\sin\frac{C}{2} - \frac{1}{2}\right)^2 - \frac{1}{4}\right\} + 1$$

$$= \frac{3}{2} - 2\left(\sin\frac{C}{2} - \frac{1}{2}\right)^2 \le \frac{3}{2}$$

we prove the inequality $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$ by contradiction.

If possible, let $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} > \frac{1}{8}$

$$\Rightarrow \quad \frac{1}{2}\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right]\sin\frac{C}{2} > \frac{1}{8}$$

$$\Rightarrow \quad \sin\left(\frac{A-B+C}{2}\right) + \sin\left(\frac{-A+B+C}{2}\right)$$

$$-\sin\left(\frac{A+B+C}{2}\right) + \sin\left(\frac{A+B-C}{2}\right) > \frac{1}{2}$$

$$\Rightarrow \quad \cos B + \cos A - 1 + \cos C > \frac{1}{2}$$

$$\Rightarrow \quad \cos A + \cos B + \cos C > \frac{3}{2} \qquad \text{—(2)}$$

(2) contradicts the result obtained in (1)

$\Rightarrow$ Therefore, $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$

OR $\sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2}$ are all > 0

Applying the AM $\ge$ GM inequality,

$$\frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3} \ge \left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \left(\frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3}\right)^3$$

Equality holds good when $\sin\frac{A}{2} = \sin\frac{B}{2} = \sin\frac{C}{2}$

Since A + B + C = 180°, this means that the equality holds good when $\frac{A}{2} = \frac{B}{2} = \frac{C}{2} = 30^\circ$

$$\Rightarrow \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)^3}{27} \le \frac{1}{8}$$

7. (i) We have cos B + cos A = 2(1 − cos C)

$$\Rightarrow \quad 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} = 4\sin^2\frac{C}{2}$$

$$\Rightarrow \cos\left(\frac{A-B}{2}\right) = 2\sin\frac{C}{2} = 2\cos\frac{A+B}{2}$$

$$\Rightarrow \quad \cos\frac{A}{2}\cos\frac{B}{A} + \sin\frac{A}{2}\sin\frac{B}{2}$$

$$= 2\left\{\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}\right\}$$

$$\Rightarrow \quad 3\sin\frac{A}{2}\sin\frac{B}{2} = \cos\frac{A}{2}\cos\frac{B}{2}$$

$$\Rightarrow \quad \tan\frac{A}{2}\tan\frac{B}{2} = \frac{1}{3}$$

(ii) B = 36°, A = 84° $\Rightarrow$ C = 60°

$$r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2};$$

$$r_2 = 4R\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{C}{2};$$

$$r_3 = 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$$

$$r_2 + r_3 = 4R\cos\frac{A}{2}\left[\sin\frac{B}{2}\cos\frac{C}{2} + \cos\frac{B}{2}\sin\frac{C}{2}\right]$$

$$= 4R\cos\frac{A}{2}\left(\cos\frac{A}{2}\right) = 4R\cos^2\frac{A}{2}$$

$$r_2 + r_1 = 4R\cos^2\frac{C}{2};$$

$$\frac{r_2 + r_3}{r_2 + r_1} = \frac{\cos^2\frac{A}{2}}{\cos^2\frac{C}{2}} = \frac{1+\cos A}{1+\cos C} = \frac{1+\cos 84^\circ}{1+\cos 60^\circ}$$

$$= \frac{2}{3}\left[1 + \sin 6^\circ\right]$$

$$= \frac{2}{3}\left[1 + \sin\left(36^\circ - 30^\circ\right)\right]$$

$$= \frac{2}{3}\left[1 + \sin 36^\circ\cos 30^\circ - \cos 36^\circ\sin 30^\circ\right]$$

$$= \frac{2}{3}\left[1 + \frac{\sqrt{3}}{2}\sin 36^\circ - \cos 30^\circ - \frac{1}{2}\left(\frac{\sqrt{5}+1}{4}\right)\right] \quad \text{— (1)}$$

$$\sin 36^\circ = \sqrt{1 - \cos^2 36^\circ}$$

$$= \sqrt{1 - \left(\frac{\sqrt{5}+1^2}{4}\right)} = \frac{1}{4}\sqrt{16 - \left(6 + 2\sqrt{5}\right)}$$

$$= \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$$

substituting in (1),

$$\frac{r_2 + r_3}{r_2 + r_1} = \frac{2}{3}\left[1 + \frac{\sqrt{3}}{2}\frac{\sqrt{10 - 2\sqrt{5}}}{4} - \frac{\left(\sqrt{5}+1\right)}{8}\right]$$

$$= \frac{1}{12}\left[8 + 4\sqrt{3}\sqrt{10 - 2\sqrt{5}} - \sqrt{5} - 1\right]$$

$$= \frac{1}{12}\left[7 + 4\sqrt{3}\sqrt{10 - 2\sqrt{5}} - \sqrt{5}\right].$$

8. $a^4 - 2(b^2 + c^2)a^2 + b^4 + b^2c^2 + c^4 = 0$

$\left[a^2 - \left(b^2 + c^2\right)\right]^2 - b^4 - c^4 - 2b^2c^2 + b^4 + b^2c^2 + c^4 = 0$

$\Rightarrow$ $(a^2 - b^2 - c^2)^2 - b^2c^2 = 0$

$\Rightarrow$ $(a^2 - b^2 - c^2 - bc)(a^2 - b^2 - c^2 + bc) = 0$

$a^2 = b^2 + c^2 + bc$ or $a^2 = b^2 + c^2 - bc$

$\Rightarrow$ $a^2 = b^2 + c^2 - 2bc\cos\frac{2\pi}{3}$

or $a^2 = b^2 + c^2 - 2bc\cos\frac{\pi}{3}$

$\Rightarrow$ $A = \frac{2\pi}{3}$ or $\frac{\pi}{3}$ $\Rightarrow \tan A = -\sqrt{3}$ or $\sqrt{3}$

9. $5CE = 3EA$

$\Rightarrow$ $5(CR - ER) = 3(ER + RA)$

$\Rightarrow$ $8ER = 5CR - 3RA$

$\Rightarrow$ $8\frac{ER}{BR} = 5\frac{CR}{BR} - 3\frac{RA}{BR}$

$\Rightarrow$ $8\cot\theta = 5\cot C - 3\cot A$

Again when $\sin A = \frac{3}{5}$, $\sin C = \frac{5}{13}$

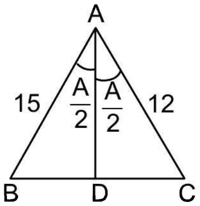
We have $\cot A = \frac{-4}{3}$ and $\cot C = \frac{12}{5}$

or $\cot A = \frac{4}{3}$ and $\cot C = \frac{-12}{5}$

or $\cot A = \frac{4}{3}$ and $\cot C = \frac{12}{5}$

$\therefore$ $\cot\theta = 2$ or -2 or 1

10.



$$\frac{1}{2}.c.x.\sin\frac{A}{2} + \frac{1}{2}.b.x.\sin\frac{A}{2} = \frac{1}{2}.b.c.\sin A$$

$$\Rightarrow \frac{1}{x}\cos\frac{A}{2} = \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right)$$

$$\Rightarrow \sum\frac{1}{x}\cos\frac{A}{2} = \sum\frac{1}{a} = \frac{1}{5} + \frac{1}{12} + \frac{1}{15} = \frac{21}{60} = \frac{7}{20}$$

11. $2R\sin A\cos A = 2R\sin C\cos C$

$\therefore$ $\sin 2A = \sin 2C$

$\therefore$ $2A = 2C = 180^\circ - 2C$

$\therefore$ $A = C$ or $90^\circ - C$

$\therefore$ Triangle is isosceles or right-angled.

12. The angles are 45°, 60°, 75°

$\therefore$ Ratio of sides is

$\sin 45^\circ : \sin 60^\circ : \sin 75^\circ$

$$= \frac{1}{\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{\sqrt{3}+1}{2\sqrt{2}}$$

i.e., $2 : \sqrt{6} : \sqrt{3} + 1$.

13. $$\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \frac{a-b}{a+b}$$

$$\tan\frac{A-B}{2} = \frac{10-2}{10+2}\tan\frac{A+B}{2}$$

$$= \frac{8}{12}\tan\left(90^\circ - \frac{C}{2}\right) = \frac{2}{3}\cot\frac{C}{2}$$

$$= \frac{2}{3}\cot 30^\circ = \frac{2}{3}\sqrt{3} = \frac{2}{\sqrt{3}}.$$

14. $a = x^2 + x + 1$; $b = 2x + 1$; $c = x^2 - 1$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \left(\frac{b}{a/2}\right)$$

$$= \frac{(2x+1)^2 + (2x^2+x)(-x-2)}{2(2x+1)(x^2-1)}$$

$$= \frac{(2x+1) - x(x+2)}{2(x^2-1)} = \frac{-x^2+1}{2(x^2-1)} = -\frac{1}{2}$$

$\therefore$ A = 120°.

15. r_1, r_2, r_3 are in HP.

$\therefore$ $\frac{\Delta}{s-a}, \frac{\Delta}{s-b}, \frac{\Delta}{s-c}$ are in HP.

$\therefore$ s – a, s – b, s – c are in AP.

–a, –b, –c are in AP

$\therefore$ a, b, c are in AP.

16. Statement 2 is true

Statement 1

$c^2 = a^2 + b^2 - 2ab \cos C$

$\Rightarrow$ $c = \sqrt{39}$

$\angle B$ is acute, since b < a

Again $b^2 - c^2 - a^2 = 25 + 39 - 49 > 0$

$\therefore$ $\Rightarrow$ A is acute

$\therefore$ $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow$ $\frac{7}{\sin A} = \frac{5}{\sin B} = \frac{\sqrt{39}}{\sqrt{3/2}}$

$\Rightarrow A = \sin^{-1}\frac{7}{2\sqrt{13}}, B = \sin^{-1}\frac{5}{2\sqrt{13}}$

17. Statement 2 is true

Consider Statement 1

For an equilateral triangle,

A = B = C = 60°

$\Rightarrow$ $r = 4R.\ \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{R}{2}$

Statement 1 is true

Choice (a)

18. Statement 2 is true

Consider Statement 1 : r_1, r_2, r_3 are in HP.

Using Statement 2, $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are in AP.

$\frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta}$ are in AP.

$\Rightarrow$ s – a, s – b, s – c are in AP.

$\Rightarrow$ –a, –b, –c are in AP.

$\Rightarrow$ a, b, c are in AP.

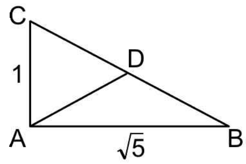
$\Rightarrow$ 2R sinA, 2RsinB, 2RsinC are in AP.

$\Rightarrow$ sinA, sinB, sinC are in AP

Statement 1 is false

Choice (d)

19.



Statement 2 is true

Consider Statement 1

If D is the mid-point of BC, D is the circum centre.

Therefore, $AD = DB = DC = \frac{1}{2}\sqrt{6} = \sqrt{\frac{3}{2}}$

$\Rightarrow$ Statement 1 is true

Choice (b)

20. Statement 2 is true

$$\frac{r_1}{r_2} = \frac{4R \sin A/2 \cos B/2 \cos C/2}{4R \sin B/2 \cos C/2 \cos A/2}$$

$$= \left(\cot\frac{B}{2}\right) \times \tan A/2 = \sqrt{3} \tan A/2$$

$\neq \sqrt{3}\left(\sqrt{2}-1\right)$ as we cannot say that A = 45°

$\Rightarrow$ Statement 1 is false

Choice (d)

21. We have

$A = \frac{1}{10} \times 180^\circ = 18^\circ$; $B = \frac{2}{10} \times 180^\circ = 36^\circ$ and

$C = \frac{7}{10} \times 180^\circ = 126^\circ$

Also, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$$c = \frac{a \sin C}{\sin A} = \frac{3\left(\sqrt{5}-1\right)\sin 126^\circ}{\sin 18^\circ}$$

$$= \frac{3\left(\sqrt{5}-1\right)\cos 36^\circ}{\sin 18^\circ} = \frac{3\left(\sqrt{5}-1\right)\left(\sqrt{5}+1\right)}{4\left(\sqrt{5}-1\right)} \times 4$$

$$= 3\left(\sqrt{5}+1\right)$$

22. $2R = \dfrac{a}{\sin A} = \dfrac{3\left(\sqrt{5}-1\right)}{\sin 18^\circ} = 12$

$R = 6$

23. $r_2 = 4R \sin\dfrac{B}{2}\cos\dfrac{C}{2}\cos\dfrac{A}{2}$

$= 24 \sin 18^\circ \cos 63^\circ \cos 9^\circ$

$= 24 \sin 18^\circ \times \dfrac{1}{2}(\cos 72^\circ + \cos 54^\circ)$

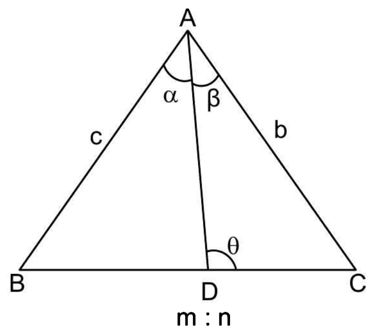
$= 12(\sin 18^\circ)[\sin 18^\circ + \sin 36^\circ]$

$$= \frac{12\left(\sqrt{5}-1\right)}{4}\left[\frac{\left(\sqrt{5}-1\right)}{4} + \frac{\sqrt{10-2\sqrt{5}}}{4}\right]$$

$$= \frac{12\left(\sqrt{5}-1\right)}{16}\left[\left(\sqrt{5}-1\right) + 2\sqrt{10-2\sqrt{5}}\right]$$

$$= \frac{3\left(\sqrt{5}-1\right)}{4}\left[\left(\sqrt{5}-1\right) + 2\sqrt{10-2\sqrt{5}}\right]$$

24.



$\dfrac{ma}{m+n} : \dfrac{na}{m+n}$

From ΔABD

$$\frac{AD}{\sin B} = \frac{BD}{\sin\alpha} = \frac{AB}{\sin\left(180^\circ - \theta\right)} \qquad \text{—(1)}$$

from ΔACD

$$\frac{AD}{\sin C} = \frac{CD}{\sin\beta} = \frac{AC}{\sin\theta} \qquad \text{—(2)}$$

from (1) and (2)

$$\frac{BD \cdot \sin B}{\sin\alpha} = \frac{CD \cdot \sin C}{\sin\beta}$$

$$\frac{ma}{(m+n)} \cdot \frac{\sin\beta}{\sin\alpha} = \frac{na}{(m+n)} \cdot \frac{\sin C}{\sin B} \qquad \text{— (3)}$$

$$\frac{m\sin\beta}{\sin\alpha} = n \cdot \frac{c}{b}$$

$\Rightarrow$ $mb \sin\beta = nc \sin\alpha$ —(1)

25. Substitute $B = \theta - \alpha$,

$C = 180^\circ - \theta - \beta$ in (3)

$$\frac{m}{n} \cdot \frac{\sin\beta}{\sin\alpha} = \frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)}$$

$\Rightarrow$ $m \sin\beta\,[\sin\theta \cos\propto - \cos\theta \sin\propto] = n \sin\alpha \sin\theta$
$\cos\beta + \cos\theta \sin\beta]$

divide with $\sin\theta \sin\alpha \sin r$

$m[\cot\alpha - \cot\theta] = n[\cot\beta + \cot\theta]$

$\Rightarrow$ $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$ —(3)

26. From (1) $AD = \dfrac{AB \cdot \sin B}{\sin\theta}$

$$AD = \frac{C \cdot \sin B}{\sin\theta} = \frac{bc}{2R\sin\theta} = \frac{\frac{1}{2}bc\sin A}{2\sin\theta \cdot \sin\theta}$$

$$= \frac{2\Delta}{a\sin\theta}$$

$\Rightarrow$ $\Delta = \dfrac{1}{2}AD \cdot a \cdot \sin\theta = \dfrac{1}{2}ax \sin\theta$

27. (a) $\tan\dfrac{C}{2} = \dfrac{r}{s-c}$

$\therefore$ $1 = \dfrac{5}{s-c}$ $\because \angle C = 90^\circ$

$s - c = 5$

$\dfrac{a+b+c-2c}{2} = 5 \Rightarrow a + b - c = 10$

$a + b = 10 + c$

$(a+b)^2 = (10+c)^2$

i.e., $a^2 + b^2 + 2ab = 100 + 20c + c^2$

$\therefore$ $2ab = 100 + 20c$

$ab - 10c = 50$

(a) and (c) are true

(b) is false

(d) is obviously not true

Since $\dfrac{5\sin A}{\sqrt{2}} < 5$ and $r = 5$

28. $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

$$\tan\frac{\pi}{4} = \frac{5-4}{5+4}\cot\frac{C}{2}$$

$$1 = \frac{1}{9}\cot\frac{C}{2}$$

$$\cot\frac{C}{2} = 9$$

$$\tan\frac{C}{2} = \frac{1}{9}$$

$$\frac{C}{2} = \tan^{-1}\left(\frac{1}{9}\right)$$

(d) is correct

$$\text{Also } C = \tan^{-1}\frac{2\left(\frac{1}{9}\right)}{1-\frac{1}{81}} = \tan^{-1}\frac{\frac{2}{9}}{\frac{80}{81}}$$

$$= \tan^{-1}\left(\frac{2}{9}\cdot\frac{81}{80}\right) = \tan^{-1}\left(\frac{9}{40}\right)$$

(b) is also correct

29. $\frac{1}{r_1} + \frac{1}{r_2} = \frac{3}{r_3}$

$$r_1 = \frac{\Delta}{s-a},\ r_2 = \frac{\Delta}{s-b},\ r_3 = \frac{\Delta}{s-c}$$

We have

$$\frac{s-a}{\Delta} + \frac{s-b}{\Delta} = \frac{3(s-c)}{\Delta}$$

$$2s - (a+b) = 3(s-c)$$

$$c = 3s - 3c$$

$$3s = 4c \Rightarrow c = \frac{3s}{4}$$

Again, we have the result

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{3s-(a+b+c)}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

Therefore, $\frac{1}{r_1} + \frac{1}{r_2} = \frac{3}{r_3}$

Gives $\frac{1}{r} - \frac{1}{r_3} = \frac{3}{r_3}$

$$\frac{1}{r} = \frac{4}{r_3} \Rightarrow r = \frac{r_3}{4}$$

Consider (d)

Given

$$\frac{1}{4R\sin A/2\cos B/2\cos C/2} + \frac{1}{4R\sin B/2\cos C/2\cos A/2}$$

$$= \frac{3}{4R\sin C/2\cos A/2\cos B/2}$$

$$\Rightarrow \frac{\sin B/2\cos A/2 + \cos B/2\sin A/2}{\sin A/2\cos B/2\cos C/2\sin B/2\cos A/2}$$

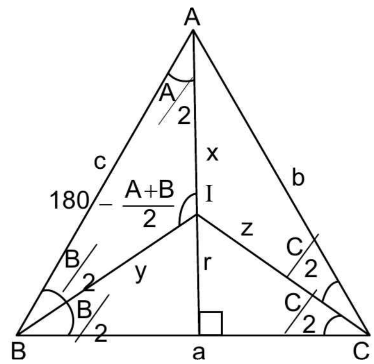
$$= \frac{3}{\sin C/2\cos A/2\cos B/2}$$

$$\Rightarrow \frac{\sin\left(\frac{A+B}{2}\right)}{\sin A/2\sin B/2\cos C/2} = \frac{3}{\sin C/2}$$

$$\frac{\cos C/2}{\sin A/2\sin B/2\cos C/2} = \frac{3}{\sin C/2}$$

$$\sin C/2 = 3\sin A/2\sin B/2$$

30. (a)



$$\frac{y}{\sin A/2} = \frac{x}{\sin B/2} = \frac{c}{\sin\left(180 - \frac{A+B}{2}\right)}$$

$$= \frac{c}{\sin\left(\frac{A+B}{2}\right)} = \frac{c}{\cos C/2} = 2R\sin C/2$$

Similarly, $\frac{y}{\sin C/2} = \frac{z}{\sin B/2} = \frac{a}{\cos A/2} = 2R\sin A/2$

$\therefore\ x = 2R\sin B/2\sin C/2$

$$y = 2R\sin C/2\ \sin A/2$$

$$z = 2R\sin A/2\sin B/2$$

$\therefore \quad xyz = 8R^3 . \sin^2 A/2 \ \sin^2 B/2 \sin^2 C/2$

$= \frac{1}{2}Rr^2, \text{since } r = 4R\sin A/2 \sin B/2 \sin C/2$

$= 2R^2r \sin A/2 \sin B/2 \sin C/2$

(b) $a + b + c = 2s = \frac{2\Delta}{r}$

$= \frac{2.2R^2 \sin A \sin B \sin C}{4R\sin A/2 \sin B/2 \sin C/2}$

$= 8R\cos A/2 \cos B/2 \cos C/2$

(c) $ax + by + cz = 2R\sin A .2R\sin B/2 \ \sin C/2$

$+ 4R^2\sin A/2 \ \sin B . \sin C/2$

$+ 4R^2\sin A/2 \ \sin B/2 \ .\sin C$

$= 8R^2\sin A/2 \ . \sin B/2 \ .\sin C/2$

$\left[\cos A/2 + \cos B/2 + \cos C/2\right]$

$= 2R.r\left[\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right]$

(d) $xy + yz + zx = 4R^2 \sin A/2 \ \sin B/2 \ \sin^2 C/2$

$\left(+4R^2 \sin A/2 \ \sin B/2 \ \sin C/2\right)$

$+4R^2\sin A/2 \ \sin^2 B/2 .\sin C/2$

$= 4R^2 \sin A/2 \ \sin B/2 \ \sin C/2$

$\left[\sin A/2 + \sin B/2 + \sin C/2\right]$

$= R.r\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right)$

IIT Assignment Exercise

31. $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9 + 16 - 7}{2 \times 3 \times 4} = \frac{18}{24} = \frac{3}{4}.$

32. $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$= \sqrt{\frac{(18-12)(18-15)}{18 \times 9}} = \sqrt{\frac{6 \times 3}{18 \times 9}} = \frac{1}{3}.$

33. $a^2 = b^2 + c^2 - 2bc \cos A$

when $A = 60°$,

$a^2 = b^2 + c^2 - bc$

$a^2 = b^2 + c^2 - 2bc + bc$

$a^2 = (b - c)^2 + bc.$

34. Taking $C = 90°$

$A + B = 90°$

$\therefore \quad \cos^2 A + \cos^2 B + \cos^2 C$

$= \cos^2 A + \cos^2 (90° - A) + \cos^2 90°$

$= 1 + 0 = 1.$

35. $\frac{1}{2} bc\sin A = 5$

$\frac{1}{2} \times 2 \times 5\sin A = 5$

$\sin A = 1$

$2\sin\frac{A}{2} \cos\frac{A}{2} = 1$

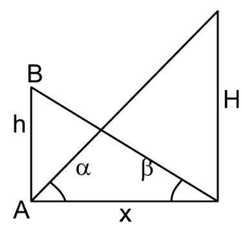
$\sin\frac{A}{2} \cos\frac{A}{2} = \frac{1}{2}.$

36. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$

$= \frac{s-a}{r} + \frac{s-b}{r} + \frac{s-c}{r} = \frac{3s-(a+b+c)}{r}$

$= \frac{3s-2s}{r} = \frac{s}{r} = \frac{2s}{2r} = \frac{a+b+c}{2r}.$

37. $\frac{H}{x} = \tan\alpha$



$\frac{h}{x} = \tan\beta$

$\frac{H}{h} = \frac{\tan\alpha}{\tan\beta}$

$H = h\frac{\tan\alpha}{\tan\beta}.$

38. $\frac{(b^2 + c^2)\sin(B - C)}{b^2 - c^2} = \frac{a^2 \sin(B - C)}{b^2 - c^2}$

$= \frac{4R^2 \sin^2 A \sin(B - C)}{4R^2(\sin^2 B - \sin^2 C)} = \frac{\sin^2 A \sin(B - C)}{\sin(B + C)\sin(B - C)}$

$= \frac{1}{1} = 1.$

39. The given expression is

$$\frac{(2s)(2s-2a)(2s-2b)\times(2s-2c)}{4b^2c^2}$$

$$= \frac{16\,s(s-a)(s-b)(s-c)}{4b^2c^2}$$

$$= \frac{4\Delta^2}{b^2c^2} = \frac{4\times\left(\frac{1}{2}bc\sin A\right)^2}{b^2c^2} = \sin^2 A.$$

40. The greatest angle is 126° and the smallest is 18°.

$$\text{The required ratio} = \frac{\sin 126°}{\sin 18°} = \frac{\cos 36°}{\sin 18°}$$

$$= \frac{\frac{\sqrt{5}+1}{4}}{\frac{\sqrt{5}-1}{4}} = \frac{\sqrt{5}+1}{\sqrt{5}-1}.$$

41. $a + b + c = \frac{6}{3}(\sin A + \sin B + \sin C)$

$\therefore\quad a + b + c = 2(\sin A + \sin B + \sin C)$

$$\therefore\quad \frac{a+b+c}{\sin A + \sin B + \sin C} = 2$$

$$\text{Now, } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$= \frac{a+b+c}{\sin A + \sin B + \sin C} = 2$$

$\therefore\quad c = 2\sin C$

$\therefore\quad \sin C = \frac{1}{2};\ C = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

42. $$\tan^2\frac{A-B}{2} = \frac{1-\cos(A-B)}{1+\cos(A-B)} = \frac{1-\frac{31}{32}}{1+\frac{31}{32}} = \frac{1}{63}$$

$$\therefore\quad \tan\frac{A-B}{2} = \frac{1}{\sqrt{63}}$$

$$\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}} = \frac{a+b}{a-b} = 9$$

$$\therefore\quad \tan\frac{A+B}{2} = 9\times\frac{1}{\sqrt{63}} = \frac{9}{\sqrt{63}}$$

$$\therefore\quad \cot\frac{C}{2} = \frac{9}{\sqrt{63}} \qquad \tan\frac{C}{2} = \frac{\sqrt{63}}{9}$$

$$\therefore\quad \cos C = \frac{1-\tan^2\frac{C}{2}}{1+\tan^2\frac{C}{2}} = \frac{1-\frac{63}{81}}{1+\frac{63}{81}} = \frac{18}{144} = \frac{1}{8}$$

$\therefore\quad c^2 = a^2 + b^2 - 2ab\cos C$

$$= 25 + 16 - 2\times 20\times\frac{1}{8} = 36$$

$c = 6.$

43. A, B, C are in AP.

$\therefore\quad B = 60°$

$$\frac{b}{c} = \frac{\sin B}{\sin C} = \frac{\sqrt{3}}{\sqrt{2}}$$

$$\therefore\quad \frac{\sqrt{3}}{2\sin C} = \frac{\sqrt{3}}{\sqrt{2}}; \qquad \therefore\ \sin C = \frac{1}{\sqrt{2}}$$

$C = 45°;\qquad \therefore\ A = 75°.$

44. $a^2 = b^2 + c^2 - 2bc\cos A$

$$25 = 16 + c^2 - 2\times 4\times c\times\frac{1}{2}$$

$\therefore\ c^2 - 4c - 9 = 0\quad \therefore$ c is a root of $x^2 - 4x - 9 = 0.$

45. $b^2\sin 2C + c^2\sin 2B$

$= 4R^2\sin^2 B \,.\, 2\sin C\cos C$

$\quad + 4R^2\sin^2 C \,.\, 2\sin B\cos B$

$= 8R^2\sin B\sin C\,[\sin B\sin C + \cos B\sin C]$

$= 8R^2\sin B\sin C\sin(B + C)$

$= 8R^2\sin A\sin B\sin C = 4\Delta.$

$= 2bc\sin A$

46. $$\frac{1}{b+c} + \frac{1}{c+a} = \frac{3}{a+b+c}$$

$\therefore\quad (c + a)(a + b + c) + (b + c)(a + b + c)$

$\quad = 3(b + c)(c + a)$

i.e., $ca + cb + c^2 + a^2 + ab + ac + ba + b^2 + bc$

$\quad + ca + cb + c^2 = 3(bc + c^2 + ba + ca)$

$\therefore\quad a^2 + b^2 = c^2 + ab$

$\therefore\quad a^2 + b^2 - c^2 = ab$

$$\therefore\quad \frac{a^2+b^2-c^2}{ab} = 1$$

$$\frac{a^2+b^2-c^2}{2ab} = \frac{1}{2}$$

$$\cos C = \frac{1}{2}$$

$C = 60° \Rightarrow A + B = 120°$

$A + B - C = 60°$

47. $r_1 - r = r_2 + r_3$

$\therefore \quad 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$= 4R\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2} + 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$

$\therefore \quad \sin\frac{A}{2}\cos\frac{B+C}{2} = \cos\frac{A}{2}\sin\frac{B+C}{2}$

i.e., $\sin^2\frac{A}{2} - \cos^2\frac{A}{2} = 0$

$\cos A = 0$

$\therefore \quad A = 90°$

right-angled.

Alternate Method

$$\frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$$

$s(s-a) = (s-b)(s-c)$

$2s(b+c-a) = 2bc$

$b^2 + c^2 = a^2$

48. $a = 5x + 12y$

$b = 12x + 5y$

$c = 13x + 13y$; c is the longest side.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{(5x+12y)^2 + (12x+5y)^2 - (13x+13y)^2}{2ab}$$

$$= \frac{120xy + 120xy - 338xy}{2ab}$$

$$= \frac{-98xy}{2ab} < 0$$

$\therefore$ C is obtuse angle.

49. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

$$\Rightarrow (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} = 0$$

$$(a+b+c)\begin{vmatrix} c-b & a-c \\ a-b & b-c \end{vmatrix} = 0$$

$\Rightarrow \quad (a+b+c)\,[(b-c)^2 + (a-b)(a-c)] = 0$

$\therefore \quad (a+b+c)\,[b^2 + c^2 + a^2 - bc - ca - ab] = 0$

$\therefore \quad a^3 + b^3 + c^3 = 3abc$

$\sin^3 A + \sin^3 B + \sin^3 C = 3 \sin A \sin B \sin C.$

50. $3 \sin x - 4 \sin^3 x = k$

$\sin 3x = k$

$\therefore \quad \sin 3A = \sin 3B$ or $\sin (180° - 3B)$,

since $A > B$

$3A = 180° - 3B$

$A = 60° - B$

$\therefore \quad A + B = 60°$

$C = 120° = \frac{2\pi}{3}$.

51. $\tan A + \tan B = \frac{c^2}{ab}$, $\tan A \tan B = 1$

$\tan A \tan B = 1 \Rightarrow \tan A = \tan (90° - B)$

$\Rightarrow \quad A + B = 90° \Rightarrow C = 90°$

ABC, is a right angled triangle, right angled at C

$\Rightarrow \quad \tan A = \frac{a}{b}$; $\tan B = \frac{b}{a}$

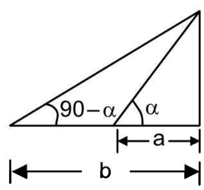
$\tan A + \tan B = \frac{a}{b} + \frac{b}{a} = \frac{c^2}{ab}$

$\sin^2 A + \sin^2 B + \sin^2 C = \frac{a^2}{c^2} + \frac{b^2}{c^2} + 1 = 2.$

52. $\frac{h}{a} = \tan\alpha$, $\frac{h}{b} = \cot\alpha$

$\frac{h^2}{ab} = 1$

$\therefore \quad h = \sqrt{ab}$.



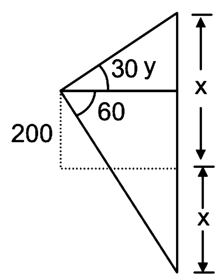
53. $\frac{x-200}{y} = \tan 30°$

$\frac{x+200}{y} = \tan 60°$

$\frac{x+200}{x-200} = \frac{\tan 60}{\tan 30} = 3$

$x + 200 = 3x - 600$

$800 = 2x \Rightarrow x = 400$ m.



54. $2s(2s-2a) = \lambda bc$

$\Rightarrow \dfrac{s(s-a)}{bc} = \dfrac{\lambda}{4} \Rightarrow \cos^2\dfrac{A}{2} = \dfrac{\lambda}{4}$

$\Rightarrow 0 < \dfrac{\lambda}{4} < 1 \quad \Rightarrow 0 < \lambda < 4$

55. $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 = b^2c^2$

$\Rightarrow (b^2 + c^2 - a^2)^2 = 3b^2c^2$

$\Rightarrow \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{\sqrt{3}}{2} \Rightarrow \cos A = \dfrac{\sqrt{3}}{2}$

$\Rightarrow A = 30°$

56. $s = 21$

$\dfrac{1}{r_1} : \dfrac{1}{r_2} : \dfrac{1}{r_3} = \dfrac{s-a}{\Delta} : \dfrac{s-b}{\Delta} : \dfrac{s-c}{\Delta}$

$= 8 : 7 : 6$

57. $A_i = \pi r_i^2 \Rightarrow \dfrac{1}{\sqrt{A_i}} = \dfrac{1}{\sqrt{\pi r_i}}$

$\dfrac{1}{\sqrt{A}} = \dfrac{1}{\sqrt{\pi r}}$

$\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} \Rightarrow \dfrac{1}{\sqrt{A}}$

58. $8R^3 = 27 \Rightarrow 2R = 3$

side of a triangle inscribed in a circle of radius R is maximum when side equals diameter. Maximum possible value of a is 3

59. $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = 4R\cdot\dfrac{1}{2}\cdot\dfrac{1}{2}\cdot\dfrac{1}{2} = \dfrac{R}{2}$

$r_1 = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

$= 4R\cdot\dfrac{1}{2}\cdot\dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{3}}{2} = \dfrac{3R}{2}$

$R : r : r_1 = R : \dfrac{R}{2} : \dfrac{3R}{2} = 2 : 1 : 3$

60. Let the sides be $n-1$, n, $n+1$ and corresponding angles θ, $(\pi - 3\theta)$, 2θ

$\Rightarrow \dfrac{\sin\theta}{n-1} = \dfrac{\sin(\pi-3\theta)}{n} = \dfrac{\sin 2\theta}{n+1}$

$\Rightarrow \dfrac{\sin 3\theta}{\sin\theta} = \dfrac{n}{n-1} \Rightarrow 3 - 4\sin^2\theta = \dfrac{n}{n-1}$

$\Rightarrow \sin^2\theta = \dfrac{2n-3}{4(n-1)}$

$\Rightarrow \dfrac{\sin 2\theta}{\sin\theta} = \dfrac{n+1}{n-1} \Rightarrow \cos\theta = \dfrac{n+1}{2(n-1)}$

$\Rightarrow \cos^2\theta = \dfrac{(n+1)^2}{4(n-1)^2}$

$\dfrac{2n-3}{4(n-1)} + \dfrac{(n+1)^2}{4(n-1)^2} = 1 \quad \Rightarrow n^2 = 5n \Rightarrow n = 5$

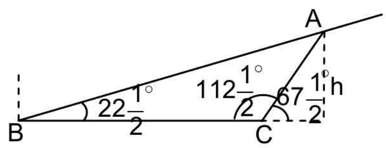
$\therefore$ sides 4, 5, 6

61. $\dfrac{a^2 + a + 1}{a} = a + \dfrac{1}{a} + 1 = \left(\sqrt{a} - \dfrac{1}{\sqrt{a}}\right)^2 + 3 \geq 3$

$\Rightarrow \dfrac{b^2 + b + 1}{b} \geq 3, \dfrac{c^2 + c + 1}{c} \geq 3$

$\therefore$ $k \geq 27$

62.



$BC = h\cot 22\dfrac{1}{2}° - h\cot 67\dfrac{1}{2}°$

$= h\left(\cot 22\dfrac{1}{2}° - \tan 22\dfrac{1}{2}°\right)$

$= \dfrac{2h}{\tan 450} = 2h$

63. $a\left[\tan A - \tan\dfrac{A+B}{2}\right] + b\left[\tan B - \tan\dfrac{A+B}{2}\right] = 0$

$\Rightarrow \dfrac{a\sin\left(\dfrac{A-B}{2}\right)}{\cos A\cos\left(\dfrac{A+B}{2}\right)} + \dfrac{b\sin\left(\dfrac{B-A}{2}\right)}{\cos B\cos\dfrac{A+B}{2}} = 0$

$\Rightarrow$ sinA cosB = cosAsinB

$\Rightarrow A = B$

64. $\Sigma\, 4R^2\sin^2 A\cdot\dfrac{\cos A}{\sin A} = 2R^2\,\Sigma\sin 2A$

$= 8R^2\sin A \text{ Sin } B \sin C = 4\Delta$

65. $c^2 = \left(1+\sqrt{3}\right)^2 + 4 - 2\left(1+\sqrt{3}\right)2\cdot\dfrac{1}{2}$

$= 8 + 2\sqrt{3} - 2 - 2\sqrt{3} \quad c = \sqrt{6}$

$$\sin B = \frac{b\sin C}{c} = \frac{2}{\sqrt{6}} \times \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}$$

$\therefore B = 45° \therefore A = 75°$

66. $b + c = 11k$, $c + a = 12k$, $a + b = 13k$

$\Rightarrow\ 2s = 18k$

$a = 7k, b = 6k, c = 5k \Rightarrow \cos A : \cos B : \cos C$

$= a(b^2 + c^2 - a^2) : b(c^2 + a^2 - b^2) : c(a^2 + b^2 - c^2)$

$= 7 : 19 : 25$

67. $a + b + c > d$

$a^2 + b^2 + c^2 \ge ab + bc + ca$,

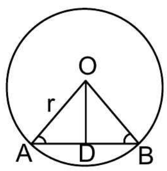
since $(a - b)^2 + (b - c)^2 + (c - a)^2 \ge 0)$

$\Rightarrow\ 2(a^2 + b^2 + c^2) \ge 2(ab + bc + ca)$

$\Rightarrow\ 3(a^2 + b^2 + c^2) \ge (a + b + c)^2 > d^2$

$$\Rightarrow \frac{a^2 + b^2 + c^2}{d^2} > \frac{1}{3}$$

68. $\angle AOB = \frac{2\pi}{n} \Rightarrow \angle AOD = \frac{\pi}{n}$



$AD = r\sin\frac{\pi}{n} \Rightarrow AB = 2r\sin\frac{\pi}{n}$

$\therefore$ Perimeter $= 2nr\sin\frac{\pi}{n}$

69. Given $\sin A \sin B \sin C = p$

$\cos A \cos B \cos C = q$

$\tan A \tan B \tan C = \tan A + \tan B + \tan C = = \frac{p}{q}$

$\tan A \tan B + \tan B \tan C + \tan C \tan A$

$$= \frac{\sin A\sin B\cos C + \sin A\cos B\sin C + \cos A\sin B\sin C}{\cos A\cos B\cos C}$$

$$= \frac{\sin A\sin(B + C) + \cos A\sin B\sin C}{q}$$

$$= \frac{1 - \cos^2 A + \cos A\sin B\sin C}{q}$$

$$= \frac{1 + \cos A(\sin B\sin C + \cos(B + C))}{q} \Rightarrow = \frac{1 + q}{q}$$

$\therefore qx^3 - px^2 + (1 + q)x - p = 0$

70. $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in AP

$$\Rightarrow \cot\frac{A}{2} + \cot\frac{C}{2} = 2\cot\frac{B}{2}$$

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = 3\cot\frac{B}{2}$$

$$\Rightarrow \frac{\Sigma\cot\frac{A}{2}}{3} \ge \sqrt[3]{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}$$

$$\frac{3\cot\frac{B}{2}}{3} \ge \sqrt[3]{3\cot\frac{B}{2}} \Rightarrow \cot^3\frac{B}{2} \ge 3\cot\frac{B}{2}$$

$$\Rightarrow \cot^2\frac{B}{2} \ge 3 \Rightarrow \cot\frac{B}{2} \ge \sqrt{3}$$

71. $\frac{a^2 + b^2 + c^2}{3} \ge \sqrt[3]{a^2b^2c^2}$

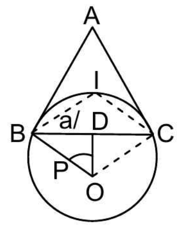
ABC, $= 8R^3 \sin A \sin B \sin C \le 3\sqrt{3}R^3$

$[\because \sin A \sin B \sin C \le \frac{3\sqrt{3}}{8}]$

$$\frac{a^2 + b^2 + c^2}{3} \ge \frac{abc}{(abc)^{\frac{1}{3}}} \ge \frac{abc}{\sqrt{3}R} = \frac{4\Delta}{\sqrt{3}}$$

$$\Rightarrow \frac{a^2 + b^2 + c^2}{4\Delta} \ge \sqrt{3} \Rightarrow k \ge \sqrt{3}$$

72. $\angle BIC = 90° + \frac{A}{2}$



$\Rightarrow \angle BOD = 90^\circ - \frac{A}{2}$

$$\Rightarrow \sin\left(90^\circ - \frac{A}{2}\right) = \frac{\frac{a}{2}}{p}$$

$$\Rightarrow p = \frac{a}{2}\sec\frac{A}{2}$$

73. $8R^2 = (a^2 + b^2 + c^2)$

$= 4R^2(\sin^2 A + \sin^2 B + \sin^2 C)$

$\Rightarrow\ \sin^2 A + \sin^2 B + \sin^2 C = 2$

$\Rightarrow\ 3 - (\cos^2 A + \cos^2 B + \cos^2 C) = 2$

$\Rightarrow \cos^2 A + \cos^2 B + \cos^2 C = 1$

74. $\tan B\tan\left(\frac{3\pi}{4} - B\right) = p$

$$\tan B\left(\frac{-1 - \tan B}{1 - \tan B}\right) = p$$

$\Rightarrow\ \tan^2 B + (1 - p)\tan B + p = 0$

$\tan B$ is real $\Rightarrow (1 - p)^2 - 4p \ge 0$

$\Rightarrow\ p^2 - 6p + 1 \ge 0$

$\Rightarrow\ p \le 3 - 2\sqrt{2}$ or $p \ge 3 + 2\sqrt{2}$

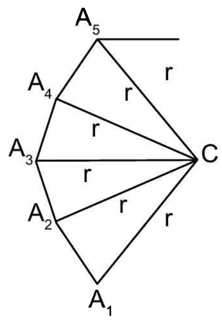
75. $b > a > c \Rightarrow -b < -a < -c$

$\Rightarrow \quad s-b < s-a < s-c < s$

$\Rightarrow \therefore \dfrac{\Delta}{s-b} > \dfrac{\Delta}{s-a} > \dfrac{\Delta}{s-c} > \dfrac{\Delta}{s}$

$r < r_3 < r_1 < r_2$

76.



$\angle A_1CA_2 = \dfrac{2\pi}{n}; \angle A_1CA_3 = \dfrac{4\pi}{n};$

$\angle A_1CA_4 = \dfrac{6\pi}{n}$

$(A_1A_2)^2 = 2r^2 - 2r^2\cos\dfrac{2\pi}{n} = 2r^2\left(2\sin^2\dfrac{\pi}{n}\right)$

$(A_1A_3)^2 = 2r^2 - 2r^2\cos\dfrac{4\pi}{n} = 2r^2\left(2\sin^2\dfrac{2\pi}{n}\right)$

$(A_1A_4)^2 = 2r^2 - 2r^2\cos\dfrac{6\pi}{n} = 2r^2\left(2\sin^2\dfrac{3\pi}{n}\right)$

Given that

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$$

$$\frac{1}{2r\sin\frac{\pi}{n}} - \frac{1}{2r\sin\frac{3\pi}{n}} = \frac{1}{2r\sin\frac{2\pi}{n}}$$

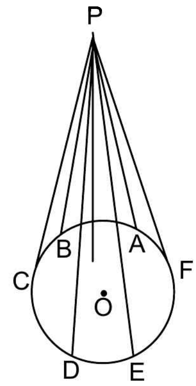
$$\Rightarrow \frac{\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\cdot\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$$

$$\Rightarrow \frac{2\cos\frac{2\pi}{n}\cdot\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\cdot\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$$

$\Rightarrow \sin\dfrac{4\pi}{n} = \sin\dfrac{3\pi}{n} \Rightarrow n = 7$

77. $PA = PB = PC = PD$

$= PE = PF = \sqrt{144 + 25} = 13$



$\cos(\angle APB) = \dfrac{PA^2 + PB^2 - AB^2}{2PA \cdot PB}$

We have OA = OB and

$\angle BOA = \dfrac{2\pi}{6} = 60°$

$\Rightarrow \quad OA = OB = AB = 5$ cm

$\cos\angle APB = \dfrac{169 + 169 - 25}{2\cdot 13\cdot 13}$

$= \dfrac{313}{338}$

78. $\tan^2\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{1+\cos\alpha} = \dfrac{s-a}{s}$

Similarly, $\tan^2\dfrac{\beta}{2} = \dfrac{s-b}{s}, \tan^2\dfrac{\gamma}{2} = \dfrac{s-c}{s}$

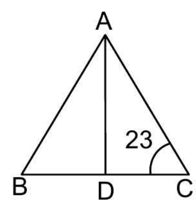
$\tan^2\dfrac{\alpha}{2} + \tan^2\dfrac{\beta}{2} + \tan^2\dfrac{\gamma}{2} = 1$; And by using AM ≥ GM.

$\tan^2 \alpha/2 + \tan^2 \beta/2 + \tan^2 \beta/2 \geq$

$3\cdot\sqrt[3]{\tan^2\dfrac{\alpha}{2}\cdot\tan^2\dfrac{\beta}{2}\cdot\tan^2\dfrac{\gamma}{2}}$

$\Rightarrow \quad \sqrt[3]{k} \leq \dfrac{1}{3} \Rightarrow k \leq \dfrac{1}{27}$

79.



AD = b sin 23°

$\cos B = \dfrac{a^2 - (b^2 - c^2)}{2ac} = \dfrac{a^2 - \dfrac{abc}{AD}}{2ac}$

$= \dfrac{1}{2}\left[\dfrac{a}{c} - \dfrac{b}{AD}\right] = \dfrac{1}{2}\left[\dfrac{a}{c} - \dfrac{1}{\sin 23^\circ}\right]$

We have $\dfrac{a}{\sin(\pi - (B+C))} = \dfrac{c}{\sin C}$

$\Rightarrow \dfrac{a}{c} = \dfrac{\sin(B + 23^\circ)}{\sin 23} \quad \cos B = \dfrac{1}{2}\left[\dfrac{\sin(B+23^\circ) - 1}{\sin 23^\circ}\right]$

$\Rightarrow \quad 2\cos B \sin 23° = \sin(B + 23°) - 1$

$\sin(B + 23°) + \sin(23° - B) = \sin(B + 23°) - 1$

$\sin(23° - B) = -1 \Rightarrow B = 90° + 23° = 113°$

80. $A = 45°;\ B = 60°;\ C = 75°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\therefore \quad a = \frac{b\sin A}{\sin B} = \frac{b\sin 45°}{\sin 60°} = \frac{b \times 2}{\sqrt{2}\,\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}\,b.$$

81. $\sum a(\sin B - \sin C)$

$\sum 2R \sin A(\sin B - \sin C)$

$= 2R[\sin A \sin B - \sin A \sin C + \sin B \sin C - \sin B \sin A + \sin C \sin A - \sin C \sin B] = 0.$

82. $$\frac{\tan B}{\tan C} = \frac{\sin B}{\cos B}\cdot\frac{\cos C}{\sin C} = \frac{b}{c}\,\frac{\cos C}{\cos B}$$

$$= \frac{b\cdot\left(\dfrac{a^2 + b^2 - c^2}{2ab}\right)}{c\cdot\left(\dfrac{a^2 + c^2 - b^2}{2ac}\right)}$$

$$\Rightarrow \quad = \frac{a^2 + b^2 - c^2}{c^2 + a^2 - b^2}.$$

83. $$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{(9-6)(9-7)}{42}}$$

$$\Rightarrow \quad = \sqrt{\frac{1}{7}}.$$

84. $$a\left(\frac{1 + \cos C}{2}\right) + c\left(\frac{1 + \cos A}{2}\right) = \frac{3b}{2}$$

$\therefore \quad a + a\cos C + c + c\cos A = 3b$

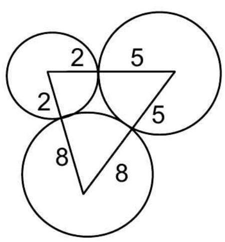
$\therefore \quad a + c = 2b$

$\therefore \quad a, b, c$ are in AP.

85. $$\cot\frac{A}{2}\cdot\cot\frac{C}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{s}{s-b} = \frac{2s}{2s-2b} = \frac{a+b+c}{a+c-b} = \frac{3b}{b} = 3.$$

86.



$a = 7;\ b = 10;\ c = 13;\ s = 15$

$$\Delta = \sqrt{15(8)(5)2}$$

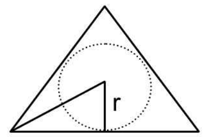
$$= 5 \times 4\sqrt{3} = 20\sqrt{3}.$$

87. $c^2 = 10^2 + 20^2 - 2 \times 10 \times 20 \cos 60°$

$= 100 + 400 - 200 \ = 300$

$c = 10\sqrt{3}.$

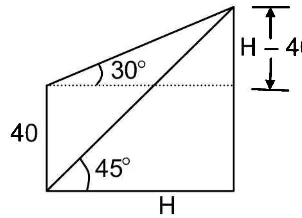
88. $\dfrac{r}{R} = \sin 30°$



$\therefore\ R = 2r.$

Ratio of areas = 1 : 4

89. $$\frac{H - 40}{H} = \tan 30° = \frac{1}{\sqrt{3}}$$



$\sqrt{3}H - 40\sqrt{3} = H$

$\left(\sqrt{3} - 1\right)H = 40\sqrt{3}$

$$H = \frac{40\sqrt{3}}{\sqrt{3} - 1}$$

$$= \frac{40\sqrt{3}\left(\sqrt{3} + 1\right)}{(3 - 1)} = 20\left(3 + \sqrt{3}\right)\ \text{m}.$$

90. Radius of the circle inscribed in an equilateral triangle is $\dfrac{a}{2\sqrt{3}}$

Diameter = diagonal

$$\text{Area} = \frac{d^2}{2} = \frac{a^2}{3 \times 2} = \frac{a^2}{6}.$$

91. $$2\cos\frac{A+B}{2}\cos\frac{A-B}{2} = 4\sin^2\frac{C}{2}$$

$$\cos\frac{A-B}{2} = 2\sin\frac{C}{2}$$

$$\cos\frac{C}{2}\cos\frac{A-B}{2} = 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$\sin\frac{A+B}{2}\cos\frac{A-B}{2} = \sin C$$

$\sin A + \sin B = 2\sin C$

$\therefore \quad a + b = 2c$

$\therefore \quad a, c, b$ are in AP.

92. $$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

$$\Rightarrow \quad \frac{5}{6} \times \frac{20}{37} + \frac{20}{37}\tan\frac{C}{2} + \frac{5}{6}\tan\frac{C}{2} = 1$$

$$\therefore \quad \left(\frac{20}{37}+\frac{5}{6}\right)\tan\frac{C}{2} = 1-\frac{5\times 20}{6\times 37}$$

$$\tan\frac{C}{2} = \frac{1-\frac{5}{6}\times\frac{20}{37}}{\frac{5}{6}+\frac{20}{37}} = \frac{2}{5}.$$

93. $bc\,\dfrac{s(s-a)}{bc} + ca\,\dfrac{s(s-b)}{ca} + ab\,\dfrac{s(s-c)}{ab}$

$= s(s-a) + s(s-b) + s(s-c)$

$= s[3s - (a+b+c)] = s^2.$

94. Given

$$\frac{2bc\cos A + ac\cos B + 2ab\cos C}{abc} = \frac{a^2+b^2}{abc}$$

or $\quad b^2 + c^2 - a^2 + \dfrac{(c^2+a^2-b^2)}{2}$

$+a^2 + b^2 - c^2 = a^2 + b^2$

$\Rightarrow \quad b^2 + c^2 = a^2$

95. $\tan A + \tan B + \tan C = 6 = \tan A \tan B \tan C$

since $A + B + C = 180°$

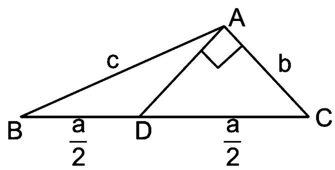
$\therefore \quad \tan A = 2$

$\therefore \quad \tan B + \tan C = 4$ and $\tan B \tan C = 3$

$\therefore \quad \tan B = 3, \tan C = 1$

Thus A, B, C are all acute angles.

96.



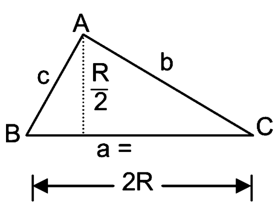
$c^2 = b^2 + a^2 - 2ab\cos C$

$= b^2 + a^2 - 2ab\left(\dfrac{b}{a/2}\right) = b^2 + a^2 - 4b^2$

$c^2 = a^2 - 3b^2$

$3b^2 = a^2 - c^2.$

97.



$$\Delta = \frac{1}{2}\cdot 2R\cdot\frac{R}{2} = \frac{R^2}{2}$$

$$\therefore \quad \frac{R^2}{2} = \tfrac{1}{2}bc.$$

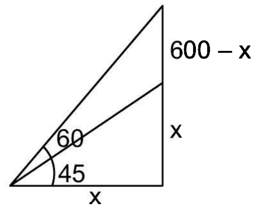
$R^2 = (2R\sin B)(2R\sin C)$

$1 = 4\sin B\sin C = 4\sin B\cos B = 2\,.\,\sin 2B$

$\sin 2B = \dfrac{1}{2}$

$2B = 30° \Rightarrow B = 15°.$

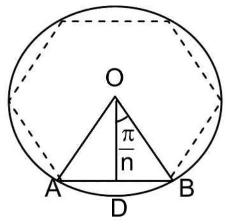
98. $\dfrac{600}{x} = \sqrt{3}$



$x = \dfrac{600}{\sqrt{3}}$

$= \dfrac{600\sqrt{3}}{3} = 200\sqrt{3}\,.$

99.



$AB = a$

Consider a regular polygon of n sides with side length a.

Area of polygon = n Area of ΔOAB

$= n\,\dfrac{1}{2}\,AB.\,OD$

$= n\,\dfrac{1}{2}\,a.\,AD\cot\dfrac{\pi}{n} = n\,\dfrac{1}{2}\,a.\,\dfrac{1}{2}\,a.\cot\dfrac{\pi}{n}$

$= n\,\dfrac{a^2}{4}\cot\dfrac{\pi}{n}$

Let a be the side of the pentagon and b be the side of the decagon.

Their perimeters are equal

$5a = 10b \Rightarrow a = 2b$

$$\frac{\text{Area of pentagon}}{\text{Area of Decagon}} = \frac{5\frac{a^2}{4}\cot\frac{\pi}{5}}{10\frac{b^2}{4}\cot\frac{\pi}{10}} = \frac{5\;4b^2\;\cot 36^\circ}{10\;b^2\;\cot 18^\circ}$$

$$= 2\frac{\cot 36^\circ}{\cot 18^\circ} = 2\frac{\cos 36^\circ}{\sin 36^\circ}.\frac{\sin 18^\circ}{\cos 18^\circ} = \frac{2}{\sqrt{5}}$$

100. A, B, C are in AP $\Rightarrow B = 60°$

$b^2 = a^2 + c^2 - ac \Rightarrow b^2 - ac = (a - c)^2$

$\Rightarrow \quad a - c = \pm\sqrt{b^2 - ac}$

$\Rightarrow \quad \left(2\sin\frac{A-C}{2}\cdot\cos\frac{A+C}{2}\right) = \pm\sqrt{\frac{3}{4} - \sin A\sin C}$

$2\left|\sin\left(\frac{A-C}{2}\right)\right| = \sqrt{3 - 4\sin A\sin C}$

$$\lim_{A\to C}\frac{\sqrt{3-4\sin A\sin C}}{|A-C|} = \lim_{A-C\to 0}\frac{2\left|\sin\left(\frac{A-C}{2}\right)\right|}{\frac{|A-C|}{2}\times 2} = 1$$

101. $s = \frac{1}{2}(a + b + c) = R.(\sin A + \sin B + \sin C)$

$= R\left(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)$

$\Rightarrow \quad \frac{s}{\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}} = 4R = 2.\sqrt[3]{\frac{abc}{\sin A\sin B\sin C}}$

102. $= 2abc\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{s(s-c)}{ab}}$

$= 2s\cdot\Delta = 2s^2 r$

103. $\sin C = \frac{1 - \cos A\cos B}{\sin A\sin B} \le 1$

$\Rightarrow \quad 1 \le \sin A\sin B + \cos A\cos B$

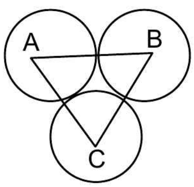
$\Rightarrow \quad \cos(A - B) \ge 1$

$\Rightarrow \quad \cos(A - B) = 1 \Rightarrow A = B$

$\Rightarrow \quad \sin C = 1$

$\Rightarrow \quad \angle C = 90°$

104.



Required distance is in radius of ABC, where the sides are a + b, b + c, c +a

$2s = 2(a + b + c) \Rightarrow s = a + b + c$

$\Delta = \sqrt{(a+b+c)abc}$

$r = \frac{\Delta}{s} = \sqrt{\frac{abc}{a+b+c}}$

105. $\frac{1}{\sqrt{2}}\sin C + \frac{1}{\sqrt{2}}\cos C + \frac{1}{\sqrt{2}}\sin(2B + C)$

$-\frac{1}{\sqrt{2}}\cos(2B + C) = 2$

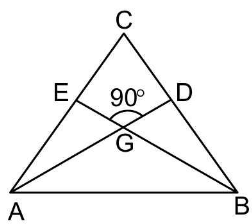
$\sin\left(\frac{\pi}{4} + C\right) + \sin\left(2B + C - \frac{\pi}{4}\right) = 2$

$\Rightarrow \quad \sin\left(\frac{\pi}{4} + C\right) = 1$ and $\sin\left(2B + C - \frac{\pi}{4}\right) = 1$

$\frac{\pi}{4} + C = \frac{\pi}{2}$ and $2B + C - \frac{\pi}{4} = \frac{\pi}{2}$

$C = \frac{\pi}{4} \Rightarrow B = \frac{\pi}{4} \Rightarrow A = \frac{\pi}{2}$

106.



$\cos C = \frac{AC^2 + BC^2 - AB^2}{2AC\cdot BC}$

$\cos C = \frac{25 - AB^2}{24} \Rightarrow AB^2 = 25 - 24\cos C$

Now from BCE

$\cos C = \frac{CE^2 + BC^2 - BE^2}{2CE\cdot BC} = \frac{\left(\frac{3}{2}\right)^2 + 16 - BE^2}{2\cdot\frac{3}{2}\cdot 4}$

$12\cos C = \frac{9}{4} + 16 - BE^2$

$BE^2 = \frac{73}{4} - 12\cos C$

Also $BE = \frac{3}{2}(BG)$

$\frac{9}{4}BG^2 = \frac{73}{4} - 12\cos C$

From ACD

$\cos C = \frac{AC^2 + CD^2 - AD^2}{2AC\cdot CD} = \frac{9 + 4 - AD^2}{2\cdot 3\cdot 2}$

$12\cos C = 13 - AD^2$

$\Rightarrow$ $AD^2 = 13 - 12\cos C$

$\frac{9}{4}AG^2 = 13 - 12\cos C \quad \because AD = \frac{3}{2}AG$

$AB^2 = AG^2 + BG^2$

$25 - 24\cos C$

$= \frac{4}{9}\left(\frac{73}{4} - 12\cos C\right) + \frac{4}{9}(13 - 12\cos C)$

$25 - 24\cos C = \frac{73}{9} - \frac{48}{9}\cos C + \frac{52}{9} - \frac{48}{9}\cos C$

$\Rightarrow \quad 25 - \frac{125}{9} = \left(24 - \frac{96}{9}\right)\cos C$

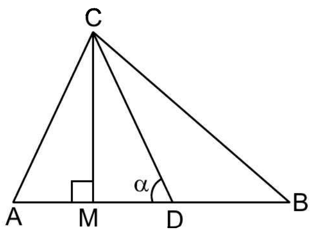
$\therefore \quad \frac{100}{9} = \frac{216 - 96}{9}\cos C \quad \Rightarrow \quad \cos C = \frac{100}{120} = \frac{5}{6}$

107. $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$

$= \frac{(a + b + c)^2 - 2(ab + bc + ca)}{2abc}$

$= \frac{121 - 2(38)}{2 \cdot 40} = \frac{45}{80} = \frac{9}{16}$

108.



Draw CM perpendicular to AB

$\cot(\angle CDA) = \cot\alpha = \frac{DM}{CM}$

$= \frac{c/2 - b\cos A}{b\sin A} = \frac{c/2 - b\frac{(b^2 + c^2 - a^2)}{2bc}}{b\sin A}$

$= \frac{a^2 - b^2}{2bc\sin A} = \frac{a^2 - b^2}{4\Delta}$

109. $\tan\left(\frac{A - B}{2}\right) = \frac{1}{3}\tan\left(\frac{A + B}{2}\right)$

$\Rightarrow \quad \frac{\tan\left(\frac{A + B}{2}\right)}{\tan\left(\frac{A - B}{2}\right)} = \frac{3}{1} = \frac{a + b}{a - b} \Rightarrow \frac{a}{b} = \frac{2}{1}$

110. a, b, c are in AP$\Rightarrow$ $2b = a + c$

Let $A = 90° + C \Rightarrow A - C = 90°$

$2.2R\sin B = 2R(\sin A + \sin C)$

$\Rightarrow \quad 2 \cdot 2\sin\frac{B}{2}\cos\frac{B}{2} = 2 \cdot \sin\frac{A + C}{2} \cdot \cos\frac{A - C}{2}$

$\sin\frac{B}{2} = \frac{1}{2\sqrt{2}} \Rightarrow \cos\frac{B}{2} = \sqrt{\frac{7}{8}}$

$\Rightarrow \quad \sin B = 2 \cdot \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{7}}{2\sqrt{2}} = \frac{\sqrt{7}}{4}$

111. Statement 2 is true

Since $\angle A = 90°$, hypotenuse of the triangle is a diameter of the circum circle.

$\Rightarrow \quad R = \frac{a}{2}$

Statement 1 is true and (1) $\Rightarrow$ (2)

Choice (a)

112. Statement 2 is true

Since $\sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2}$ are > 0,

Using Statement 1,

$$\frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3} \geq \left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^{1/3}$$

$\Rightarrow$ Equality is when $\sin\frac{A}{2} = \sin\frac{B}{2} = \sin\frac{C}{2}$ i.e., when $A = B = C = 60°$

$\Rightarrow$ Therefore,

$$\sin\frac{A}{2} = \sin\frac{B}{2} = \sin\frac{C}{2} \leq \left(\frac{\frac{1}{2} + \frac{1}{2} + \frac{1}{2}}{3}\right)^3 = \frac{1}{8}$$

Or Maximum value of the product

$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ is $\frac{1}{8}$

Statement 1 is false

Choice (d)

113. Statement 2 is true

Using Statement 2, we have

$\frac{b}{\sin B} = \frac{10}{\sin\frac{\pi}{6}} = 20$

$\frac{c}{\sin C} = \frac{12\sqrt{3}}{\sin\frac{\pi}{3}} = 24 \quad \frac{b}{\sin B} \neq \frac{c}{\sin C}$

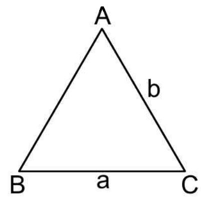
$\Rightarrow$ Statement 1 is true

Choice (a)

114. $b = 40 = a$

$\Rightarrow \angle A = \angle B = 30°$

$$R = \frac{a}{2\sin A} = \frac{40}{2\times\frac{1}{2}} = 40$$



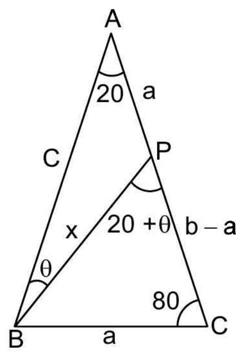
115. $$r = \frac{\Delta}{s} = \frac{400\sqrt{3}}{\left(\frac{40+40+40\sqrt{3}}{2}\right)} = \frac{400\sqrt{3}\times 2}{40\left[2+\sqrt{3}\right]}$$

$$= 20\sqrt{3}\left(2-\sqrt{3}\right) = 20\left(2\sqrt{3}-3\right)$$

116. $$r_1 = \frac{\Delta}{s-a} = \frac{400\sqrt{3}}{\left(\frac{80+40\sqrt{3}}{2}\right)-40}$$

$$= \frac{400\sqrt{3}}{20\sqrt{3}} = 20$$

117.



In triangle PAB $\dfrac{x}{\sin 20°} = \dfrac{a}{\sin\theta}$

$\Delta PBC: \dfrac{x}{\sin 80°} = \dfrac{a}{\sin(20°+\theta)}$

$\therefore \dfrac{\sin 80°}{\sin 20°} = \dfrac{\sin(20°+\theta)}{\sin\theta}$

$\therefore \dfrac{\cos 10°}{2\sin 10°\cos 10°} = \dfrac{\sin(20°+\theta)}{\sin\theta}$

$\therefore \dfrac{1/2}{\sin 10°} = \dfrac{\sin(20°+\theta)}{\sin\theta}$

$\therefore \dfrac{\sin 30°}{\sin 10°} = \dfrac{\sin(20°+\theta)}{\sin\theta} \Rightarrow \theta = 10°$

$\therefore \angle ABP = 10°$

$\therefore \angle APB = 150°$

In ΔPBC, $\dfrac{BC}{\sin(20°+\theta)} = 2R$

$\dfrac{BC}{\sin 30°} = 2R \Rightarrow R = BC$

$\therefore$ Circum radius = BC

Since BC = AP

= AC – PC

= AB – PC

118. Since $\angle C = 90°$, $\tan\dfrac{C}{2} = \dfrac{r}{s-c} = 1$

$\therefore r = s - c$ — (1)

But $\dfrac{c}{\sin 90°} = 2R \Rightarrow c = 2R$

$\therefore r = s - 2R \Rightarrow r + 2R = s$

Again $\tan\dfrac{C}{2} = \dfrac{r_3}{s} = 1 \therefore r_3 = s$

$\therefore r = r_3 - C \Rightarrow r_3 = r + C$

(b) $r_1 + r_2 = 4R\sin\dfrac{A}{2}.\cos\dfrac{B}{2}.\dfrac{1}{\sqrt{2}} + 4R.\sin\dfrac{B}{2}.\cos\dfrac{A}{2}.\dfrac{1}{\sqrt{2}}$

$$= \frac{4R}{\sqrt{2}}\left[\sin\frac{A}{2}.\cos\frac{B}{2} + \cos\frac{A}{2}.\sin\frac{B}{2}\right]$$

$$= \frac{4R}{\sqrt{2}}\sin\left(\frac{A}{2}+\frac{B}{2}\right)$$

$$= \frac{4R}{\sqrt{2}}.\frac{1}{\sqrt{2}} = 2R$$

$r_1 + r_2 = 2R$

$\therefore r_1 + r_2 + r_3 = r_3 + 2R = s + 2R \neq r + R$

Again $r_1 + r_2 = 2R = s - r = r_3 - r$

119.

$$\tan B + \tan\left(\frac{2\pi}{3} - B\right) = p$$

$\Rightarrow \tan B + \dfrac{-\sqrt{3} - \tan B}{1 - \sqrt{3}\tan B} = p$

$\Rightarrow -\sqrt{3}\tan^2 B - \sqrt{3} = p - p\sqrt{3}\tan B$

$\Rightarrow \sqrt{3}\tan^2 B - p\sqrt{3}\tan B + p + \sqrt{3} = 0$

$\Rightarrow$ tan B is real $\Rightarrow \Delta \ge 0$

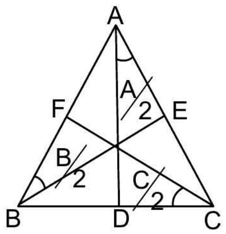
$\Rightarrow 3p^2 - 4\sqrt{3}(p+\sqrt{3}) \ge 0$

$\Rightarrow 3p^2 - 4\sqrt{3}p - 12 \ge 0$

$\Rightarrow \left(p + \frac{2}{\sqrt{3}}\right)(p - 2\sqrt{3}) \geq 0$

$\Rightarrow p \leq \frac{-2}{\sqrt{3}}$ or $p \geq 2\sqrt{3}$

120.



$$\frac{BD}{\sin A/2} = \frac{AD}{\sin B} \text{ and } \frac{BD}{DC} = \frac{c}{b} \Rightarrow \frac{BD}{BD + DC} = \frac{c}{c + b}$$

$\therefore \quad BD = \frac{ac}{b + c}$

$\therefore \quad AD = BD \,.\, \frac{\sin B}{\sin A/2}$

$$= \frac{ac}{(b + c)} \cdot \frac{\sin B}{\sin A/2}$$

$$= \frac{c}{(b + c)} \cdot \frac{a \sin B}{(\sin A/2)}$$

$$= \frac{c}{b + c} \cdot \frac{b \sin A}{\sin A/2}$$

$$= \frac{2bc}{(b + c)} \cdot \cos A/2$$

$$AD = \frac{2abc}{(b + c)} \cdot \frac{\cos A/2}{a}$$

$$= \frac{2abc}{(b + c)} \cdot \frac{\cos A/2}{2R \sin A}$$

$$= \frac{abc}{R(b + c)} \operatorname{cosec} A/2$$

Again $AD.BD = \frac{ac}{(b + c)} \cdot \frac{abc}{R(b + c)} \operatorname{cosec} A/2$

$$= \frac{1}{Rb}\left[\frac{abc}{b + c}\right]^2 . \operatorname{cosec} A/2$$

Again,

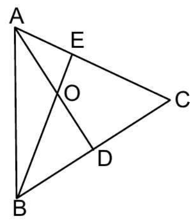
$$\frac{4R \cos A/2 \sin B \sin C}{(\sin B + \sin C)} = \frac{4R \cos A/2 \cdot \frac{b}{2R} \cdot \frac{c}{2R}}{\left(\frac{b}{2R} + \frac{c}{2R}\right)}$$

$$= \frac{1/R \; bc \cos A/2}{1/2R\,(b + c)}$$

$$= \frac{2bc}{(b + c)} \cos A/2 = AD$$

Additional Practice Exercise

121. $BD = AB \cos B = c \cos B$



$$\frac{BD}{OB} = \cos(90° - C) = \sin C$$

$$\Rightarrow OB = \frac{c \cos B}{\sin C}$$

$AE = AB \cos A = c \cos A$

$$\frac{OE}{AE} = \tan(90° - C) = \cot C$$

$$\Rightarrow OE = AE \cot C = \frac{c \cos A}{\tan C}$$

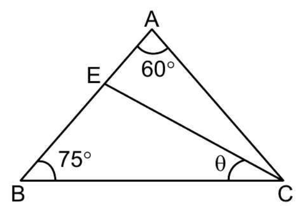
$$\frac{OE}{OB} = \frac{c \cos A}{\tan C} \times \frac{\sin C}{c \cos B} = \frac{\cos A \cos C}{\cos B}$$

Since $B = 60°$, $C = 45° \Rightarrow A = 75°$

$$\cos 75° = \cos(45° + 30°) = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

Substituting, $\frac{OE}{OB} = \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{1}{\sqrt{2}} \times 2 = \frac{\sqrt{3} - 1}{2}$

122.



We have Area of Δ CBE $= \sqrt{3} \times$ Area Δ CAE

$\Rightarrow \frac{1}{2} BC \times CE \sin\theta$

$= \sqrt{3} \times \frac{1}{2}$ CA $\times$ CE $\sin(C - \theta)$

$\Rightarrow\ a\sin\theta = \sqrt{3}\, b\sin(45° - \theta)$

$\Rightarrow\ 2R\sin A\sin\theta = \sqrt{3} \times 2R\sin B\sin(45° - \theta)$

$= \frac{\sqrt{3}}{2}\sin\theta \times \sqrt{3} \times \sin 75^\circ \sin(45^\circ - \theta)$

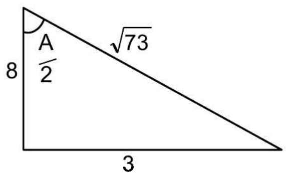
$\Rightarrow\ \frac{\sin\theta}{2} = \frac{(\sqrt{3}+1)}{2\sqrt{2}}\left[\frac{\cos\theta - \sin\theta}{\sqrt{2}}\right]$

$\Rightarrow 2\sin\theta = (\sqrt{3}+1)\ (\cos\theta - \sin\theta)$

$(\sqrt{3}+3)\sin\theta = \frac{(\sqrt{3}+1)}{\sqrt{3}(\sqrt{3}+1)} = \frac{1}{\sqrt{3}}$

$\Rightarrow\ \sin\theta = \frac{1}{2}$

123.



$\cos(B - C) = \frac{4}{5}$

$\Rightarrow\ \frac{1 - \tan^2\left(\frac{B-C}{2}\right)}{1 + \tan^2\left(\frac{B-C}{2}\right)} = \frac{4}{5}$

$\Rightarrow\ \tan^2\left(\frac{B-C}{2}\right) = \frac{1}{9}$

$\Rightarrow\ \tan\left(\frac{B-C}{2}\right) = \frac{1}{3}$, since $B > C$

But, $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{9-7}{9+7}\cot\frac{A}{2}$

$\Rightarrow\ \frac{1}{3} = \frac{1}{8}\cot\frac{A}{2}\quad \Rightarrow\ \cot\frac{A}{2} = \frac{8}{3}$

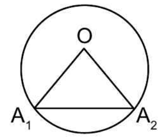
$\Rightarrow\ \sin\frac{A}{2} = \frac{3}{\sqrt{73}},\ \cos\frac{A}{2} = \frac{8}{\sqrt{73}}$

$\Rightarrow \text{s in } A = \frac{2 \times 3 \times 8}{73} = \frac{48}{73}$

Area of the triangle ABC, $= \frac{1}{2}bc\sin A$

$= \frac{1}{2} \times \frac{9 \times 7 \times 48}{73} = \frac{1512}{73}$

124.



Let a be the radius of the circle.

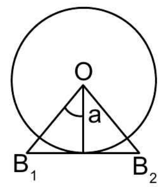
Area of the regular polygon of 12 sides inscribed in the

circle $= 12 \times \frac{1}{2} \times a^2\sin\frac{2\pi}{12} = 6a^2\sin\frac{\pi}{6} = 3a^2$

Let B_1B_2 denote a side of the polygon circumscribed about

the circle. We have $\frac{a}{0B_1} = \cos\frac{2\pi}{2 \times 12} = \cos\frac{\pi}{12}$

$OB_1 = a\sec\frac{\pi}{12}$



Area of the circumscribed polygon

$= 12 \times \frac{1}{2} \times a^2\sec^2\frac{\pi}{12}\sin\frac{2\pi}{12} = 3a^2\sec^2\frac{\pi}{12}$

Ratio of the areas $= \frac{3a^2\sec^2\frac{\pi}{12}}{3a^2}$

$= \sec^2\frac{\pi}{12} = \frac{1}{\cos^2\frac{\pi}{12}} = \frac{2}{1 + \cos\frac{\pi}{6}}$

$= \frac{2}{1 + \frac{\sqrt{3}}{2}} = \frac{4}{2 + \sqrt{3}} = 4(2 - \sqrt{3})$

125. Let C_1, C_2, C_3 be the centres of the three circles respectively and ID, IE, IF be the three common tangents meeting in I. It is clear that I, which is the meeting point of the tangents is the incentre of the triangle $C_1C_2C_3$.

Now, the sides of the triangle $C_1C_2C_3$ are

$(4 + 7), (7 + 11)$ and $(11 + 4)$ ie, 11, 18, 15

$S = 22$

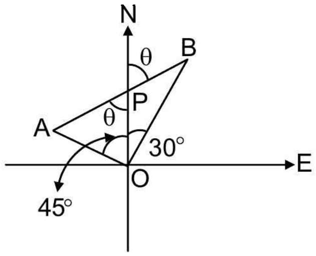
$\Delta = \sqrt{22 \times 11 \times 4 \times 7} = 22\sqrt{14}$.

Distance of I from either of the points of contact of the circles

= In radius of the triangle

$$C_1C_2C_3 = \frac{22\sqrt{14}}{22} = \sqrt{14}.$$

126.



The train is moving along AB.

At the time of observation the train was at A, 10 minutes later, the train was at P and after 10 more minutes, it was at B.

In ΔOPB, $\frac{PB}{\sin 30^\circ} = \frac{OP}{\sin \angle OBP}$

$$\Rightarrow 2PB = \frac{OP}{\sin(\theta - 30^\circ)} \quad \text{— (1)}$$

In ΔOPA,

$$\frac{PA}{\sin 45^\circ} = \frac{OP}{\sin \angle OAP}$$

$$= \frac{OP}{\sin(180^\circ - (45^\circ + \theta))} \quad \text{— (2)}$$

From (1) and (2)

$$2\sin(\theta - 30^\circ) = \frac{\sin(45^\circ + \theta)}{\sin 45^\circ}, \text{ since } PA = PB$$

$$2\left\{\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right\}$$

$$\Rightarrow \sqrt{3}\sin\theta - \cos\theta = \cos\theta + \sin\theta$$

$$\Rightarrow \left(\sqrt{3} - 1\right)\sin\theta = 2\cos\theta$$

$$\Rightarrow \tan\theta = \frac{2}{\sqrt{3} - 1} = \left(\sqrt{3} + 1\right)$$

AP : PB = 1 : 1

So, $(1 + 1)\cot\theta = 1.\cot 30^\circ - 1.\cot 45^\circ$

$$\tan\theta = \frac{2}{\sqrt{3} - 1} = \sqrt{3} + 1$$

127. $\cos A + \cos B + \cos C = \frac{3}{2}$

$$2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2} = \frac{3}{2}$$

$$2\sin^2\frac{C}{2} - 2\sin\frac{C}{2}\cos\frac{A-B}{2} + \frac{1}{2} = 0$$

$$\sin^2\frac{C}{2} - \sin\frac{C}{2}\cdot\cos\frac{A-B}{2} + \frac{1}{4}\left(\cos^2\frac{A-B}{2}\right)$$

$$= \frac{1}{4}\cos^2\left(\frac{A-B}{2}\right) - \frac{1}{4}$$

$$\left[\sin\frac{C}{2} - \frac{1}{2}\cos\left(\frac{A-B}{2}\right)\right]^2 = \frac{1}{4}\left[\cos^2\left(\frac{A-B}{2}\right) - 1\right]$$

LHS ≥ 0 and RHS ≥ 0; RHS cannot be > 0

$$\Rightarrow \cos^2\left(\frac{A-B}{2}\right) - 1 = 0 \Rightarrow \cos\left(\frac{A-B}{2}\right) = 1$$

$\Rightarrow A = B$

$$\sin\frac{C}{2} - \frac{1}{2} = 0 \Rightarrow C = \frac{\pi}{3} \Rightarrow A = B = \frac{\pi}{3}$$

$\therefore$ Triangle is equilateral

(ii) Let $\cos A + \cos B + \cos C = k$

$$\Rightarrow 2\sin\frac{C}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2} = k$$

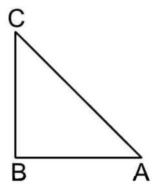
$$2\sin^2\frac{C}{2} - 2\sin\frac{C}{2}\cos\frac{A-B}{2} + k - 1 = 0$$

$\sin\frac{C}{2}$ is real $\Rightarrow \Delta \geq 0$.

$$4\cos^2\left(\frac{A-B}{2}\right) - 8(k-1) \geq 0$$

$$2k - 2 \leq \cos^2\left(\frac{A-B}{2}\right) \leq 1 \Rightarrow k \leq \frac{3}{2}$$

128. $1 - \frac{r_2}{r_1} = \left(1 - \frac{s-a}{s-b}\right) = \frac{a-b}{s-b}$;



$$1 - \frac{r_2}{r_3} = 1 - \frac{s-c}{s-b} = \frac{c-b}{s-b}$$

$$\therefore \left(1 - \frac{r_2}{r_1}\right)\left(1 - \frac{r_2}{r_3}\right) = \frac{(a-b)(c-b)}{(s-b)^2}$$

$$= \frac{ac - bc - ab + b^2}{(a+c-b)^2}.4$$

$$= \frac{2ac - 2bc - 2ab + a^2 + c^2 + b^2}{(a + c - b)^2}.2$$

$$= \frac{(a + c - b)^2}{(a + c - b)^2}.2 = 2$$

129. $\left(\tan\frac{A}{2} - \tan\frac{B}{2}\right)^2 + \left(\tan\frac{B}{2} - \tan\frac{C}{2}\right)^2$

$$+\left(\tan\frac{C}{2} - \tan\frac{A}{2}\right)^2 \geq 0$$

$$\Rightarrow \tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}$$

$$\geq \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2}$$

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geq 1$$

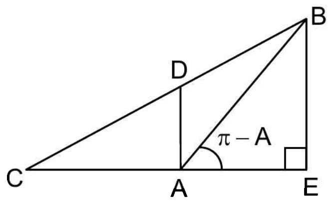
130. $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{b} = \frac{\sin 3B}{\sin B} = 3 - 4\sin^2 B$

$= 4\cos^2 B - 1$

$$\frac{a + b}{4b} = \cos^2 B$$

$$\Rightarrow \cos B = \sqrt{\frac{a + b}{4b}} \text{ (B is acute, being } \frac{A}{3}\text{)}$$

131.



Let the perpendicular from B on CA produced meet it at E. Then triangles CAD and CEB are similar.

$$\Rightarrow \frac{CA}{CE} = \frac{CD}{CB} = \frac{AD}{BE} = \frac{1}{2}$$

From Δ AEB,

$$\tan(\pi - A) = \frac{BE}{AE}$$

$$\tan A = -\frac{BE}{AE};\ \tan C = \frac{BE}{CE}$$

$$= \frac{BE}{2AE}$$

$\therefore \tan A + 2\tan C = 0$

132. (i) Consider three points

P(A, sin A) Q(B, sin B), R(C, sin C) on the curve $y = \sin x$ such that $A + B + C = \pi$

$$\text{centroid of } \Delta PQR = \left(\frac{\pi}{3}, \frac{\sin A + \sin B + \sin C}{3}\right)$$

$\therefore$ centroid lies on $x = \frac{\pi}{3}$ and we know that centroid should lie inside the triangle.

$$\Rightarrow \frac{\sin A + \sin B + \sin C}{3} \leq \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \sqrt[3]{\sin A \sin B \sin C} \leq \frac{\sqrt{3}}{2} \text{ (AM} \geq \text{GM)}$$

$$\Rightarrow \sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}$$

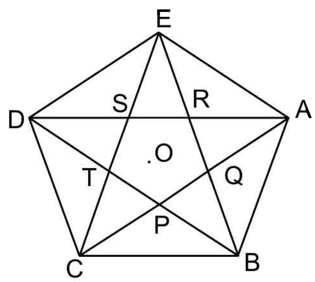
133. In a triangle $\tan A + \tan B + \tan C = \tan A \tan B \tan C = k\ (> 0)$

$$\frac{\tan A + \tan B + \tan C}{3} \geq \sqrt[3]{\tan A \tan B \tan C}$$

$$\frac{k}{3} \geq \sqrt[3]{k} \Rightarrow k^3 \geq 27 \cdot k \Rightarrow \quad k^2 \geq 27$$

$$\Rightarrow k \geq 3\sqrt{3}$$

134.



It may be noted that PQRST is also a regular pentagon.

$$\frac{\text{Area of ABCDE}}{\text{Area of PQRST}} = \frac{x^2}{y^2}$$

x – side of pentagon ABCDE

y – side of pentagon PQRST

From ΔCBP

$$\frac{BC}{\sin 108°} = \frac{CP}{\sin 36°} \quad (\because \angle QPT = 108° = \angle CPB)$$

$$CP = \frac{x \cdot \sin 36°}{\sin 108°} = 2x \sin 18° \qquad \text{— (1)}$$

From ΔCPT

$$\frac{PT}{\sin 36°} = \frac{CP}{\sin 72°}$$

$$y = CP \cdot \frac{\sin 36°}{\cos 18°} = CP \cdot (2\sin 18) = 4x\sin^2 18$$

$$\frac{x}{y} = \frac{1}{4\sin^2 18°}$$

$$x^2 : y^2 = 1 : 16\sin^4 18°$$

135. tan A, tan B, tan C are in AP

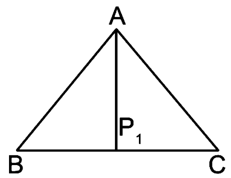
$\Rightarrow$ $\tan A + \tan B + \tan C = 3\tan B = k$

$\Rightarrow$ $k \geq 3\sqrt{3}$

$\Rightarrow$ $3\tan B \geq 3\sqrt{3} \Rightarrow \tan B \geq \sqrt{3}$

Least value is $\sqrt{3}$

136. (i)



$$\Delta = \frac{1}{2}aP_1 \Rightarrow P_1 = \frac{2\Delta}{a}, \frac{1}{P_1} = \frac{a}{2\Delta}$$

Similarly $P_2 = \frac{2\Delta}{b}, \frac{1}{P_2} = \frac{b}{2\Delta}$

$$P_3 = \frac{2\Delta}{c}, \frac{1}{P_3} = \frac{c}{2\Delta}$$

(iii) $P_1P_2P_3 = \frac{8\Delta^3}{abc} = \frac{4\Delta}{abc}(2\Delta^2) = \frac{2\Delta^2}{R}$

137. $r_1r_2 + r_2r_3 + r_3r_1 = \sum r_1r_2 = \sum s(s-a)$

$= s(s - c + s - a + s - b) = s^2$

$$r_1 = s\tan\frac{A}{2} r_2 = (s-a)\cot\frac{A}{2} = s^2$$

138. Given $a + c = 2b \Rightarrow 3b = 2s \Rightarrow s - b = \frac{b}{2}$

(i) $\cot\frac{A}{2} = \frac{s(s-a)}{\Delta}, \cot\frac{C}{2} = \frac{s(s-c)}{\Delta}$

$$\cot\frac{A}{2} + \cot\frac{C}{2} = \frac{s}{\Delta}[2s - (a+c)] = \frac{s}{\Delta}\cdot b$$

$$= 2\cdot\frac{s}{\Delta}(s-b) = 2\cot\frac{B}{2}$$

$\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in AP

139. If 'a' is the side of the equilateral triangle, its area $= \frac{\sqrt{3}}{4}a^2$

perimeter = 3a; thus the sides of the other triangle are $a - d, a, a + d$

Consider area $= \sqrt{\frac{3a}{2}\left(\frac{a}{2}+d\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}-d\right)}$

$$\frac{\sqrt{3}a}{4}\sqrt{a^2 - 4d^2} = \frac{3}{5}\cdot\frac{\sqrt{3}}{4}\cdot a^2$$

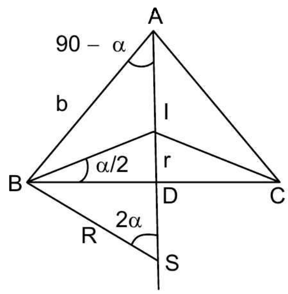
$$a^2 - 4d^2 = \frac{9}{25}a^2 \Rightarrow \frac{16a^2}{25} = 4d^2$$

$$2a = 5d \Rightarrow d = \frac{2a}{5}$$

$$a - d : a : a + d = \frac{3a}{5} : a : \frac{7a}{5} = 3 : 5 : 7$$

140. $B = C = \alpha \Rightarrow B + C < \frac{\pi}{2} \Rightarrow A > \frac{\pi}{2}$

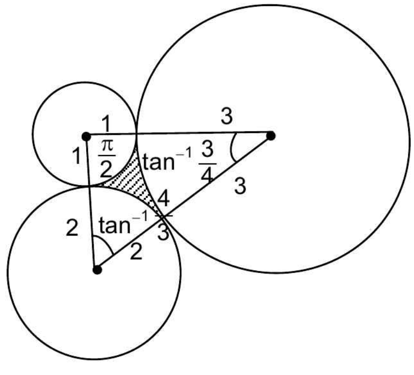
$BD = b\cos\alpha$



$$r = \frac{\Delta}{s} = \frac{\frac{1}{2}b^2\sin 2\alpha}{\frac{1}{2}(b + b + 2b\cos\alpha)}$$

$$= \frac{\frac{1}{2}b^2\sin 2\alpha}{\frac{1}{2}(2b(1+\cos\alpha))} = \frac{b\sin 2\alpha}{2(1+\cos\alpha)}$$

141.



Required area

$$= \frac{1}{2}.3.4 - \frac{1}{2}.1^2.\frac{\pi}{2}$$

$$-\frac{1}{2}.2^2.\tan^{-1}\frac{4}{3} - \frac{1}{2}.3^2.\tan^{-1}\frac{3}{4}$$

$$= 6 - \frac{1}{2}\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{4}{3}\right)$$

$$-2\tan^{-1}\frac{4}{3} - \frac{9}{2}\tan^{-1}\frac{3}{4}$$

$$= 6 - \frac{5}{2}\tan^{-1}\frac{4}{3} - 5\tan^{-1}\frac{3}{4}$$

142. $a^2 + b^2 = c^2\cos^2C$

$\Rightarrow\ a^2 + b^2 - c^2 = -c^2\sin^2C < 0$

$\Rightarrow\ \cos C < 0$

$\Rightarrow\ \cos(A + B) > 0$

$\Rightarrow\ A + B < \frac{\pi}{2}$

143. $\frac{\sin A}{\sin B} = \frac{\sin(C - A))}{\sin(-C)}$

$\Rightarrow\ \sin(B + C)\sin(B - C) = \sin(C + A)\sin(C - A)$

$\Rightarrow\ \sin^2B - \sin^2C = \sin^2C - \sin^2A$

$\Rightarrow\ \sin^2A + \sin^2B = 2\sin^2C$

$\Rightarrow\ a^2 + b^2 = 2c^2$

$\Rightarrow\ a^2, c^2, b^2$ are in AP

144. The triangle is right angled, with hypotenuse 5; orthocentere is right–angle–vertex and circumcentre the midpoint of the hypotenuse. Required distance is circumradius

$$= \frac{1}{2}\text{ hypotenuse} = 2.5$$

145. $b\cos B = c\cos C \Rightarrow \sin 2B = \sin 2C$

$\Rightarrow\ 2B = 2C$ or $2B = \pi - 2C$

$\Rightarrow\ B = C$ or $B = \frac{\pi}{2} - C$

$\Rightarrow\ \sin B = \sin C$ or $\sin B = \cos C$

146. $\frac{b + c - a}{1} = \frac{c + a - b}{3} = \frac{a + b - c}{5} = \frac{a + b + c}{9}$

$$= \frac{2a}{3} = \frac{2b}{6} = \frac{2c}{4}$$

$\Rightarrow\ a : b : c = 4 : 3 : 2$

$$\cos A = \frac{9 + 4 - 16}{2\times3\times2} = \frac{-1}{4}$$

147. $a^2 + b^2 + c^2 = 8R^2 \Rightarrow \sin^2A + \sin^2B + \sin^2C = 2$

$\Rightarrow\ \cos^2B + \cos^2C - \sin^2A = 0$

$\Rightarrow\ \cos^2B + \cos(C + A)\cos(C - A) = 0$

$\Rightarrow\ -\cos B[-\cos B + \cos(C - A)] = 0$

$\Rightarrow\ -2\cos B\cos C\cos A = 0$

$\Rightarrow\ A$ or B or C is $\frac{\pi}{2}$

148. Every linear measure of triangle DEF is half the corresponding measure in ABC, so it is $\frac{r}{2}$

149. $\Sigma\cot B\cot C = 1$ (in any triangle ABC)

$\Sigma(\cot B - \cot C)^2 = 2[\Sigma\cot^2A - \Sigma\cot B\cot C)$

$= 2[(\Sigma\cot A)^2 - 3\Sigma\cot B\cot C] = 2\left[\left(\sqrt{3}\right)^2 - 3\times1\right] = 0$

$\Rightarrow\ \cot A = \cot B = \cot C \Rightarrow A = B = C$

$\Rightarrow$ ABC is equilateral

150. The lengths of the common chords of the 3 circles are the lengths of the three altitudes of the triangle.

$$\therefore\ \sum\frac{1}{\ell} = \sum\frac{a}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

151. $\sum_1^1\frac{\cos A}{a} = \frac{\sum 2bc\cos A}{2abc} = \frac{\sum(b^2 + c^2 - a^2)}{2abc}$

$$= \frac{a^2 + b^2 + c^2}{2abc} = \frac{(a + b + c)^2 - 2\sum bc}{2abc}$$

$$= \frac{9^2 - 2\times26}{2\times24} = \frac{29}{48}$$

152. $r = \frac{\Delta}{s}, r_1 = \frac{\Delta}{s - a}, \ldots\ldots.$

$$\therefore\ \text{H.M of ex – radii} = \frac{3}{\sum\frac{1}{r}} = \frac{3}{\frac{\sum(s - a)}{\Delta}}$$

$$= \frac{3\Delta}{s} = 3r = 3$$

153. The only choice with the correct dimension is

$$\left[\text{It is } \frac{\sum a}{\sum a\cos A} = \frac{2R\sum\sin A}{R\sum\sin 2A}\right.$$

$$= \frac{2.4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{4\sin A\sin B\sin C} = \frac{1}{4}.\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\left.= \frac{R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{R}{r}\right]$$

154. B is a right angle; cosA = sinC

Thus a, sinA, sinC are rational also

$$\frac{a}{\sin A}=\frac{b}{\sin 90^\circ}=\frac{c}{\sin C}$$

$\Rightarrow$ a, b, c are all rational

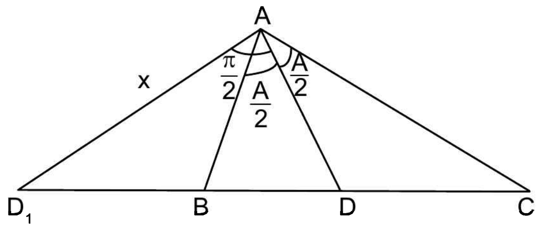
155. In a triangle ABC, $\Sigma\tan A = \tan A \tan B \tan C$

$\therefore$ $\tan A \tan B \tan C = 2 \tan B \tan C$

$= 3 \tan C \tan A = 6$

$\Rightarrow$ $\tan A = 2, \tan B = 3, \tan C = 1$

156.



If x is the required length,

$$\left|\frac{1}{2}x\,b\sin\left(\frac{\pi}{2}+\frac{A}{2}\right)-\frac{1}{2}x\,c\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\right|$$

$$=\Delta=\frac{1}{2}\ b\,c\sin A$$

$$\Rightarrow\ x|b-c|\cos\frac{A}{2}=bc\sin A$$

$$\Rightarrow\ x=\frac{2bc}{|b-c|}.\sin\frac{A}{2}$$

157. $\frac{1}{\alpha}=\frac{a}{2\Delta}$,......

$$\sum\frac{\cos B+\cos C}{\alpha}=\frac{1}{2\Delta}\sum a(\cos B+\cos C)$$

$$=\frac{a+b+c}{2\Delta}=\frac{s}{\Delta}$$

158. $\Sigma a\,.\,\Sigma\cos A=\Sigma a\cos A+\Sigma(b+c)\cos A$

$= R\,\Sigma\sin 2A+\Sigma a$

$= R4\sin A\sin B\sin C+\Sigma a$

$$=\sum a+\frac{1}{2R^2}.abc.$$

159. $\alpha.\ A = 2\Delta$

$$\frac{1}{\alpha}=\frac{a}{2\Delta}$$

$$\sum\frac{1}{\alpha}=\frac{\Sigma\frac{a}{2}}{\Delta}=\frac{s}{\Delta}=\frac{1}{r}$$

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{s-a+s-b+s-c}{\Delta}=\frac{S}{\Delta}=\frac{1}{r}$$

$$\therefore\ \Sigma\frac{1}{\alpha}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$$

160. $\sum\cot^2 A=\frac{1}{2}\sum(\cot B-\cot C)^2+$

$\sum\cot B\cot C$

$$=\frac{1}{2}\ \Sigma(\cot B-\cot C)^2+1\geq 1$$

and $\Sigma\cot^2 A = 1$ if $\cot A = \cot B = \cot C$

$\therefore$ minimum value = 1

161. $2\cos A + \cos B + \cos C = 2$

$$\Rightarrow\ 2\cos\frac{B+C}{2}\cos\frac{B-C}{2}=2.2\sin^2\frac{A}{2}$$

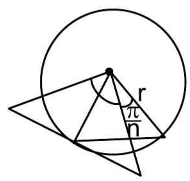
$$\Rightarrow\ \cos\frac{B-C}{2}=2\sin\frac{A}{2}$$

$$\Rightarrow\ 2\sin\frac{B+C}{2}\cos\frac{B-C}{2}=4\sin\frac{A}{2}\cos\frac{A}{2}$$

$\Rightarrow$ $\sin B+\sin C = 2\sin A$

$\Rightarrow$ $b + c = 2a$ $\quad\Rightarrow$ c, a, b are in AP

162.



The 3 areas in order are

$$n.\frac{1}{2}r^2\sin\frac{2\pi}{n},\ \pi r^2,\ n.\frac{1}{2}\frac{r^2}{\cos^2\frac{\pi}{n}}.\sin\frac{2\pi}{n}$$

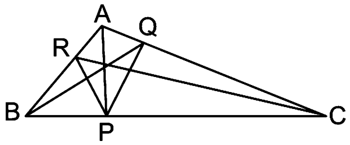
$$\text{i.e., }\frac{nr^2}{2}\sin\frac{2\pi}{n},\frac{nr^2}{2}.\frac{2\pi}{n},\frac{nr^2}{2}.2\tan\frac{\pi}{n}$$

The proportion is $\sin\frac{2\pi}{n}:\frac{2\pi}{n}:2\tan\frac{\pi}{n}$

163. The circumcircle of triangle PQR, the pedal triangle of ABC, is the same as that of the medial triangle of ABC.

so radius $=\frac{R}{2}$

164.



ABPQ is cystic $\Rightarrow \angle CPQ = A$

Similarly, $\angle BPR = A$

$\therefore \angle QPR = \pi - 2A, \ldots\ldots$

$$\frac{QR}{\sin(\pi - 2A} = \frac{RP}{\sin(\pi - 2B)}$$

$$= \frac{PQ}{\sin(\pi - 2C)} = 2.\frac{R}{2} = R$$

$\Rightarrow QR = R . \sin 2A$

$= a\cos A, \ldots\ldots.$

165. $\sin A \sin B \sin C = \frac{1}{4}\Sigma \sin 2A$

$$= \frac{1}{4}\left[3 - \sum(\sin A - \cos A)^2\right]$$

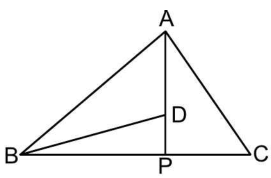
$$\leq \frac{3}{4}$$

But $\Sigma(\sin A - \cos A)^2 \neq 0$

$\therefore \sin A \sin B \sin C < \frac{3}{4}$

$\sin A \sin B \sin C = \frac{1}{2}$ when $A = B = \frac{\pi}{4}, C = \frac{\pi}{2}$

166.



$AD = AP - DP$

$$= c \sin B - c \cos B \tan\frac{B}{3}$$

$$= \frac{c}{\cos\frac{B}{3}}\left[\sin B\cos\frac{B}{3} - \cos B\sin\frac{B}{3}\right]$$

$$= c . \frac{\sin\frac{2B}{3}}{\cos\frac{B}{3}} = 2c\sin\frac{B}{3}$$

167. a^2, b^2, c^2 are in AP

$\Rightarrow b^2 + c^2 - a^2, c^2 + a^2 - b^2, a^2 + b^2 - c^2$ are in AP

$\Rightarrow \frac{\cos A}{a}, \frac{\cos B}{b}, \frac{\cos C}{c}$ are in AP

$\Rightarrow$ cot A, cotB, cotC are in AP

$\Rightarrow$ tanA, tanB, tan C are in HP

168. O is midpoint of AB; ABC, is right – angled at C

$\therefore \sin^2 A + \sin^2 B + \sin^2 C = \sin^2 A + 1 + \cos^2 A = 2$

169. $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in AP

$$\Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

are in H.P

$\Rightarrow \frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c}$ are in HP

$\Rightarrow$ s – a, s – b, s – c are in AP

$\Rightarrow$ a, b, c are in AP

170. $b^2 = ac$, $2 \sin B = \sin A + \sin C$

$\Rightarrow b^2 = ac$ and $2b = a + c$

$\Rightarrow a = b = c$

$\Rightarrow$ triangle is equilateral

171. Let us consider Statement 2

We have $r_1 = 4R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

Suppose Δ ABC, is equilateral

Then, $r_1 = 4R \times \frac{1}{2} \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{3R}{2}$

Therefore, S_2 is false

Consider Statement 1

$$r_1 = 4R \sin\frac{75^\circ}{2}\cos\frac{75^\circ}{2}\cos 15^\circ$$

$$= 2R \sin 75^\circ \cos 15^\circ$$

$$= R \times (\sin 90^\circ + \sin 60^\circ) = R\left(1 + \frac{\sqrt{3}}{2}\right)$$

Statement 1 is true

Choice (c)

172. Statement 2 is true

Since the triangle is isosceles right angled, altitude through A passes through the mid point of BC

Therefore, length of the altitude through A = $\frac{1}{2}BC = 4$. Statement 1 is true

Choice (b)

173. We have

$\tan A = k, \tan B = 2k, \tan C = 3k$

In any Δ ABC, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$6k = 6k^3$

$\Rightarrow \quad k = \pm 1$

k cannot be "–1", since all the angles of a triangle cannot be obtuse.

$\Rightarrow \quad k = 1$

$\Rightarrow \quad \tan B = 2$

Statement 2 is equivalent to

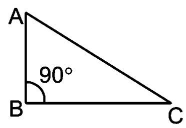
$$\frac{1}{\cot A} + \frac{1}{\cot B} + \frac{1}{\cot C} = \frac{1}{\cot A \cot B \cot C}$$

$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C$

Statement 2 is true and we have used this result in proving Statement 1.

Choice (a)

174. Statement 2 is true



$b^2 = c^2 + a^2$

Multiply both sides by b

$b^3 = c^2b + a^2b$ — (1)

Also, $b > c, \; b > a$

$\Rightarrow \quad c^2b > c^3, a^2b > a^3$

$c^2b + a^2b > c^3 + a^3$

substituting in (1)

$b^3 > c^3 + a^3$

Statement 1 is false

Choice (d)

175. Since $\sin\frac{A}{2}\sin\frac{B}{2}, \sin\frac{C}{2}$ are>0,

we have

$$\frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3} \geq \left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^{\frac{1}{3}}$$

Equality holds for $\sin\frac{A}{2} = \sin\frac{B}{2} = \sin\frac{C}{2}$

Since $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$

$\Rightarrow \quad \frac{A}{2} = \frac{B}{2} = \frac{C}{2}$ implies $\frac{A}{2} = \frac{B}{2} = \frac{C}{2} = \frac{\pi}{6}$

Therefore,

$$\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^{\frac{1}{3}} \leq \frac{\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right)}{3} = \frac{1}{2}$$

$\Rightarrow \quad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}$

Statement 2 is true

Consider Statement 1

$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\leq 4R \times \frac{1}{8}$

$\leq \frac{R}{2}$

Choice (a)

176. Statement 2 is true

Consider Statement 1

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in HP

$\Rightarrow \quad$ a, b, c are in AP.

$\Rightarrow \quad -a, -b, -c$ are in AP

$\Rightarrow \quad s-a, s-b, s-c$, are in AP

$\Rightarrow \quad \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta}$ are in AP

$\Rightarrow \quad \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are in A.P

$\Rightarrow \quad r_1, r_2, r_3$ are in H.P

Statement 1 is false

Choice (d)

177. Statement 2 is true

consider Statement 1

using Statement 2, $\frac{7}{\sin A} = \frac{2}{\sin B} = 6$

$\sin A = \frac{7}{6}$ which is not possible.

Choice (a)

178. Statement 2 is true

Consider Statement 1

Using Statement 2,

$$\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\sqrt{\frac{(s-c)(s-a)}{s(s-b)}},\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

are in AP

$$\Rightarrow \sqrt{\frac{(s-b)(s-c)}{s-a}},\sqrt{\frac{(s-c)(s-a)}{s-b}},\sqrt{\frac{(s-a)(s-b)}{s-c}}$$

$\Rightarrow \frac{1}{s-a},\frac{1}{s-b},\frac{1}{s-c}$ are in A.P

$\Rightarrow$ s – a, s – b, s – c are in H.P

$\Rightarrow \frac{a+b+c}{2}-a, \frac{a+b+c}{2}-b, \frac{a+b+c}{2}-c$, are in H.P $b+c-a, c+a-b, a+b-c$ are in H.P

Statement 1 is false.

Choice (d)

179. Statement 2 is true

Statement 1 is false

AB = 14, AC = 7

then $\frac{BD}{DC} = 2$

Also if AB = 2, AC = 1

then $\frac{BD}{DC} = 2$

Choice (d)

180. Statement 2 is true

Consider Statement 1

$$\frac{\sin^2 A + \sin A + 1}{\sin A} = \sin A + 1 + \frac{1}{\sin A}$$

$$= \sin A + \frac{1}{\sin A} + 1$$

> 3, since sinA > 0

similarly,

$$\frac{\sin^2 B + \sin B + 1}{\sin A} > 3$$

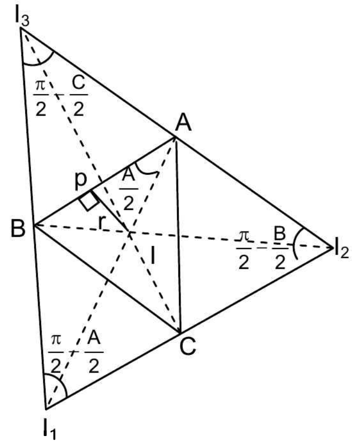
$$\frac{\sin^2 C + \sin C + 1}{\sin C} > 3$$

Hence $(1 + \sin A + \sin^2 A)(1 + \sin B + \sin^2 B)(1 + \sin C + \sin^2 C)$

> 27

Choice (a)

181.



$$\angle BI_1C = \angle BI_1I + \angle CI_1I$$
$$= \angle BCI + \angle CBI = \frac{C}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{A}{2}$$

[$\because$ I, B, I_1, C are concyclic and angles in same segment are equal]

Triangles $I_1 I_2 I_3$ and I_1 BC are similar

$$\frac{I_2I_3}{BC} = \frac{I_3I_1}{I_1C} = \angle I_1 I_2 I_3 = \text{supplement of}$$

$$\angle I_3BC = \angle I_1BC. = \left(90^\circ - \frac{A}{2}\right)$$

$$\Rightarrow I_2I_3 = = a\,\text{cosec}\frac{A}{2} = 4R\cos\frac{A}{2},$$

182. The circumradius of

$$I_1I_2I_3 = \frac{1}{2}.\frac{4R\cos A/2}{\sin\left(\pi/2 - A/2\right)} = 2R$$

183. Sides of $I_1I_2I_3 \to 4R\cos\frac{A}{2}, 4R\cos\frac{B}{2}, 4R\cos\frac{C}{2}$

$$\Rightarrow 2(2R)\sin\left(\frac{\pi}{2}-\frac{A}{2}\right), 2(2R)\sin\left(\frac{\pi}{2}-\frac{B}{2}\right),$$
$$2(2R)\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$\Rightarrow$ circum radius of ex-central triangle is 2R

$$\Rightarrow \Delta = \frac{(I_1I_2)(I_2I_3)(I_3I_1)}{4\cdot(2R)}$$

$$= \frac{64R^3\cdot\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{8R} = 8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

184. $II_1 = 4R\sin\frac{A}{2}, II_2 = 4R\sin\frac{B}{2}, II_3 = 4R\sin\frac{C}{2}$

$\therefore \quad II_1 \cdot II_2 \cdot II_3 = 16R^2\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

$= 16R^2 r$

185. Area of $I_1I_2I_3 = \dfrac{I_1I_2 \cdot I_2I_3 \cdot I_3I_1}{4(2R)}$

$$= \frac{\left(a\operatorname{cosec}\frac{A}{2}\right)\left(b\operatorname{cosec}\frac{B}{2}\right)\left(c\operatorname{cosec}\frac{C}{2}\right)}{8R}$$

$$= \frac{abc}{4R \cdot 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{abc}{2R}$$

186. BI_2 perpendicular to $I_1 I_3$ and

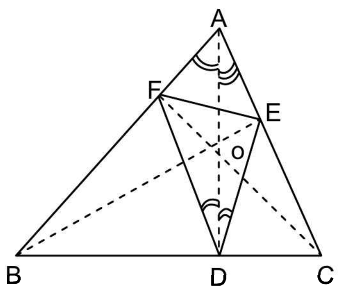
AI_1 perpendicular to $I_2 I_3$

CI_3 perpendicular to $I_1 I_2 \Rightarrow AI_1, BI_2, CI_3$ are altitudes of $I_1 I_2 I_3$

$\therefore$ orthocentre of $I_1 I_2 I_3$ is incentre of ABC

II

187.



B, D, O, F are concyclic. O, D, C, E also concyclic

$\angle ODF = \angle OBF = 90° - A$

$\angle ODE = \angle OCE = 90° - A$

$\Rightarrow \angle FDE = 180° - 2A$

188. Sides of ΔDEF are $R\sin(180° - 2A)$, $R\sin(180° - 2B)$, $R\sin(180° - 2C)$

$\Rightarrow$ circum radius of pedal triangle of ABC, $= \frac{1}{2}R$

189. $\Delta_E = \pi(2R)^2$, $\Delta_P = \pi\left(\frac{R}{2}\right)^2 \Rightarrow \dfrac{\Delta_E}{\Delta_P} = \dfrac{4\pi R^2}{\pi\frac{R^2}{4}} = \dfrac{16}{1}$

190. (a) $r = (s-a)\tan\frac{A}{2} \quad r_1 = s\tan\frac{A}{2}$

$\therefore \quad r = (s-a)\dfrac{r_1}{S}$

$\therefore \quad rs = (s-a)\, r_1$

$\therefore \quad \dfrac{r}{s-a} = \dfrac{r_1}{s}$

$\therefore \quad \sum\dfrac{r}{s-a} = \sum\dfrac{r_1}{s} = \dfrac{1}{s}\sum r_1$

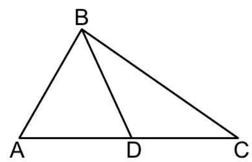
(b) $\sum\dfrac{1}{r_1} = \sum\dfrac{s-a}{\Delta} = \dfrac{1}{\Delta}\sum s-a$

$= \dfrac{1}{\Delta}.s = \dfrac{s}{\Delta} = \dfrac{1}{r}. \neq 3s$

(c) $\sum\cot\frac{A}{2} = \sum\dfrac{s-a}{r} = \dfrac{1}{r}\sum(s-a) = \dfrac{s}{r}$

(d) $\sum\tan\frac{A}{2} \neq \dfrac{r}{s}$

191.



BD is the median through A

we have $AD = DC = \dfrac{b}{2}$

From ΔABD:

$\ell^2 = BD^2 = AB^2 + AD^2 - 2AB.\ AD\cos A$

$= c^2 + \left(\dfrac{b}{2}\right)^2 - 2c\dfrac{b}{2}\cos A$

$= c^2 + \dfrac{b^2}{4} - bc\left(\dfrac{b^2+c^2-a^2}{2bc}\right)$

$= \dfrac{4c^2 + b^2 - 2\left(b^2+c^2-a^2\right)}{4}$

$= \dfrac{2a^2 + 2c^2 - b^2}{4}.$

$\Rightarrow \quad 4\ell^2 = 2a^2 + 2c^2 - b^2$

Again,

$c^2 + a^2 + 2ca\cos B$

$= c^2 + a^2 + \dfrac{2ca\left(c^2 + a^2 - b^2\right)}{2ca}$

$= 2c^2 + 2a^2 - b^2 \;\; = 4\ell^2$

$b^2 + 4ca\cos B = b^2 + 4ca\left(\dfrac{c^2 + a^2 - b^2}{2ca}\right)$

$= b^2 + 2(c^2 + a^2 - b^2)$

$= 2c^2 + 2a^2 - b^2 = 4\ell^2$

$(2s - b)^2 - 4ca\sin^2\dfrac{B}{2}$

$= (a + c)^2 - 2ca(1 - \cos B)$

$= a^2 + c^2 + 2ca\cos B$

$= 4\ell^2$

Choices (a) , (b), (c) and (d)

192. $\dfrac{r_1r_2}{r_3} = \dfrac{\dfrac{\Delta}{(s-a)} \times \dfrac{\Delta}{(s-b)}}{\left(\dfrac{\Delta}{s-c}\right)} = \dfrac{\Delta(s-c)}{(s-a)(s-b)}$

$= \dfrac{\Delta}{s\tan^2\dfrac{c}{2}} = \dfrac{\Delta}{2s}$

We have

$\tan\dfrac{c}{2} = \pm\sqrt{2}$

since $\dfrac{c}{2} < 90^\circ$, $\tan\dfrac{c}{2}$ is > 0

$\Rightarrow \;\; \tan\dfrac{c}{2} = \sqrt{2} \;\; \Rightarrow c > 90^\circ$

$\cos C = \dfrac{1 - \tan^2\dfrac{c}{2}}{1 + \tan^2\dfrac{c}{2}} = \dfrac{1-2}{1+3} = \dfrac{-1}{3}$

Again,

$\tan^2\dfrac{C}{2} = 2 \Rightarrow \cos^2\dfrac{A}{2} = \dfrac{1}{3}$

Choices (a), (c)

193. $B = 180^\circ - (A + C) \;\; = 180^\circ - 3A$

Angles are A, $(180^\circ - 3A)$, 2A

We have $\dfrac{a}{\sin A} = \dfrac{b}{\sin(180^\circ - 3A)} = \dfrac{c}{\sin 2A} = K$

$\dfrac{a+b}{2c} = \dfrac{K\{\sin A + \sin 3A\}}{2K\sin 2A} = \cos A$

$= \cos\dfrac{C}{2}$

$\sin A + \sin B = \sin A + \sin 3A$

$= 2\sin 2A\cos A$

$= 2\sin C\cos A$

$c^2 - a^2 = K^2\sin^2 2A - K^2\sin^2 A$

$= K^2(\sin^2 2A - \sin^2 A)$

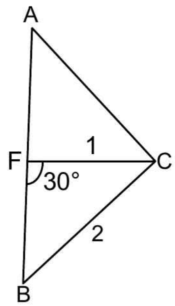
$= K^2\sin 3A\sin A$

$= (K\sin 3A)(K\sin A)$

$= ba$

194. Let BF = x

we have from ΔBFC,



$4 = x^2 + 1 - 2x\dfrac{\sqrt{3}}{2}$

$x^2 - \sqrt{3}x - 3 = 0$

$x = \dfrac{\sqrt{3} \pm \sqrt{3 + 12}}{2}$

Giving $x = \dfrac{(\sqrt{3} + \sqrt{15})}{2}$

(since $x > 0$)

Hence $AB = \sqrt{15} + \sqrt{3}$

Area of ΔABC, = 2 × Area of ΔBFC

$= 2 \times \dfrac{1}{2}FB \times 1 \times \sin 30^\circ$

$= \dfrac{2(\sqrt{15} + \sqrt{3})}{2\times 2} \times \dfrac{1}{2} = \dfrac{\sqrt{15} + \sqrt{3}}{4}$

Again,

from ΔBFC,

$\dfrac{2}{\sin 30^\circ} = \dfrac{1}{\sin B} \Rightarrow \sin B = \dfrac{1}{4}$

and cosB is acute (FB > 1)

$\Rightarrow \;\; \cos B = \sqrt{1 - \dfrac{1}{16}} = \dfrac{\sqrt{15}}{4}$

195. $r_1 < r_2 < r_3$

$\Rightarrow \;\; \dfrac{\Delta}{s-a} < \dfrac{\Delta}{s-b} < \dfrac{\Delta}{s-c}$

$s - a > s - b > s - c$

$-a > -b > -c$

$\Rightarrow \;\; a < b < c$

Since $a < b$, $A < B$

$r_2 - r = \dfrac{\Delta}{s-b} - \dfrac{\Delta}{s}$

$= \dfrac{\Delta}{s(s-b)} \times b > 0$

196. $b + c = 3a$

$\Rightarrow \quad \sin B + \sin C = 3 \times \sin A$

$\Rightarrow \quad 2\sin\frac{B+C}{2}\cos\frac{B-C}{2} = 3 \times 2\sin\frac{A}{2}\cos\frac{A}{2}$

$\Rightarrow \quad 2\cos\frac{A}{2}\cos\frac{B-C}{2} = 3 \times 2\sin\frac{A}{2}\cos\frac{A}{2}$

$\Rightarrow \quad \cos\frac{B-C}{2} = 3\sin\frac{A}{2}$ — (1)

$\cos B + \cos C = 2\cos\frac{B+C}{2}\cos\frac{B-C}{2}$ using (1)

$= 2\sin\frac{A}{2}\cos\frac{B-C}{2}$

$= 2\sin\frac{A}{2}\left(3\sin\frac{A}{2}\right)$, using (1)

$= 6\sin^2\frac{A}{2}$

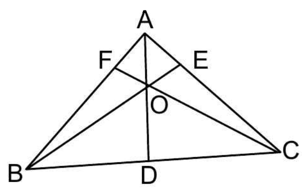
$\cos(B + C) = \cos(180^\circ - A) = -\cos A$

$3[1 + \cos(B + C)] = 3[1 - \cos A]$

$= 6\sin^2\frac{A}{2}$

Clearly, (c) and (d) are false

197.



O is the orthocenter of the triangle

From ΔAEFD

$EF^2 = AF^2 + AE^2 - 2AF.\ AE\cos A$

$= (b\cos A)^2 + (C\cos A)^2 - 2bc\cos^3 A$

$= (\cos^2 A)(b^2 + c^2 - 2bc\cos A)$

$= a^2\cos^2 A$

$\Rightarrow \quad EF = a\cos A$

similarly, $DE = c\cos C$

$DF = b\cos B$

(c) is true

(b) is false

$DE + EF + DF = a\cos A + b\cos B + c\cos C$

$= (2R\sin A\cos A) + 2R\sin B\cos B$
$\quad + 2R\sin C\cos C$

$= R\{\sin 2A + \sin 2B + \sin 2C\}$

$= R\{2\sin(A + B)(\cos A - B) + 2\sin C\cos C\}$

$= R \times 2\sin C\{\cos(A - B) - \cos(A + B)\}$

$= R \times 2\sin C \times 2\sin A\sin B$

$= 4R\sin A\sin B\sin C$

$= 4R \times \left(\frac{\Delta}{2R^2}\right) = \frac{2\Delta}{R}$

198. (a) $\cos A + \cos B + \cos C = \frac{7}{4}$

$1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{7}{4}$

$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{3}{16}$

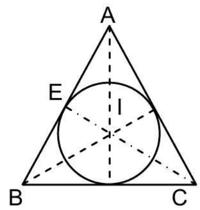
But $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 4R.\frac{3}{16}$

$\therefore \frac{r}{R} = \frac{3}{4}$

(b) In ΔAEI $AI = EI\ \mathrm{cosec}\frac{A}{2}$

$= r\ \mathrm{cosec}\frac{A}{2}$

$CI = r\ \mathrm{cosec}\frac{B}{2}$



$\therefore$ \`AI + BI + CI

$= r\left(\mathrm{cosec}\frac{A}{2} + \mathrm{cosec}\frac{B}{2} + \mathrm{cosec}\frac{C}{2}\right)$

$\geq r\ .\ 3r\left(\mathrm{cosec}\frac{A}{2} + \mathrm{cosec}\frac{B}{2} + \mathrm{cosec}\frac{C}{2}\right)^{1/3}$

(A.M $\geq$ G.M)

$\geq \frac{3r}{\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^{1/3}} \geq 3.(8)^{1/3}$

$\geq 6r$

≥ 12

(c) $S = 8k\ \Delta = 12k^2$

$\therefore r = \frac{\Delta}{S} \Rightarrow 6 = \frac{12k^2}{8k} \Rightarrow k = 4$

(d) $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$

$\Rightarrow \frac{\cos A}{k\sin A}=\frac{\cos B}{k\sin B}=\frac{\cos C}{k\sin C}$

$\cot A=\cot B=\cot C$

$\Rightarrow A=B=C=60°$

$\therefore \Delta ABC$, is equilateral

$\Delta=\frac{\sqrt{3}}{4}\times a^2=\frac{\sqrt{3}}{4}\times 9=\frac{9\sqrt{3}}{4}$

(a) $\rightarrow$ (s)

(b) $\rightarrow$ (q)

(c) $\rightarrow$ (r)

(d) $\rightarrow$ (p)

199. Given $2a^2+9b^2+c^2=6ab+2ac$

i.e., $2a^2+9b^2+c^2-6ab-2ac=0$

i.e., $\left[a^2-6ab+(3b)^2\right]+\left[a^2-2ac+c^2\right]=0$

i.e., $(a-3b)^2+(a-c)^2=0 \Rightarrow a=3b$ and $a=c$

$\therefore$ $A=C; a:b:c=3:1:3$

$\cos C=\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{1+9-9}{2.1.3}=\frac{1}{6}$

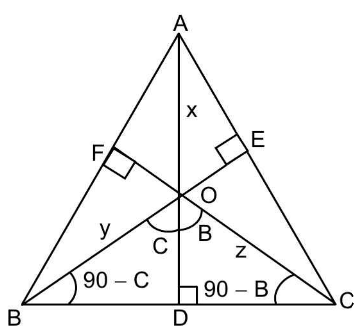
$\therefore \sin\frac{A}{2}=\sqrt{\frac{1-\cos A}{2}}=\sqrt{\frac{5}{12}}$

$\sin A==\sqrt{1-\frac{1}{36}}=\frac{\sqrt{35}}{6}$

$\sin 2A=2.\sin A\cos A=2.\frac{1}{6}.\frac{\sqrt{35}}{6}=\frac{\sqrt{35}}{18}$

$\cos B=\frac{a^2+c^2-b^2}{2ac}=\frac{9+9-1}{2\times3\times3}=\frac{17}{18}$

200.



In ΔBOC

$$\frac{a}{\sin(B+C)}=\frac{y}{\sin(90-A)}=\frac{z}{\sin(90-C)}$$

i.e., $\frac{a}{\sin A}=\frac{y}{\cos A}=\frac{z}{\cos C}$

$\therefore$ $y=a.\frac{\cos A}{\sin A}=a\cot A$

$=2R\cos A$

Similarly, $x=2R\cos B$

$z=2R\cos C$

$\therefore$ $x+y+z=2R(\cos A+\cos B+\cos C)$

$=2R\sum\cos A$

$\therefore$ $(a+b+c)(x+y+z)=4R^2\sum\sin A\sum\cos A$

In ΔBOD, $\cos(90-C)=\frac{BD}{y}$

$\Rightarrow$ $y\sin C=BD$

In ΔCOD, $\cos(90-B)=\frac{CD}{z}$

$\Rightarrow$ $z\sin B=CD$

$\therefore$ $y\sin C+z\sin B=BD+CD=a$

$y\sin C+z\sin B=a$

Similarly,

$x\sin B+y\sin A=c$

and $x\sin C+z\sin A=b$

$\therefore$ $a+b+c=x(\sin B+\sin C)+y(\sin A+\sin C)$

$+z(\sin A+\sin B)$

$2R(a+b+c)=x[b+c]+y[a+c]+z[a+b]$

$\therefore$ $x(b+c)+y(a+c)+z(a+b)=2R(a+b+c)$

$=4R^2(\sin A+\sin B+\sin C)$

$=4R^2\sum\sin A$

$\therefore$ $x(a+b+c)+y(a+b+c)+z(a+b+c)$

$=4R^2(\sin A+\sin B+\sin C)+ax+by+cz$

$(a+b+c)(x+y+z)$

$=2R(a+b+c)+(ax+by+cz)$

$=4Rs+ax+by+cz$

$\therefore$ $ax+by+cz=(a+b+c)(x+y+z)-4RS$

$=2R.\sum\sin A.2R\sum\cos A-4R^2\sum\sin A$

$=4R^2\{\sum\sin A.\sum\cos A-\sum\sin A\}$

$=4R^2\sum\sin A(\sum\cos A-1)$

CHAPTER 5

SEQUENCES AND SERIES

CHAPTER OUTLINE

Totalling or summing up a set of numbers is performed by doing the addition of numbers in the set term by term. Suppose the set of numbers exhibit an order. It is possible to obtain a formula for the sum of the first n terms of this set by appealing to its pattern.

In this unit we, introduce the concept of 'sequences and series'. Arithmetic, geometric and arithmetico-geometric series are defined and formulas for the sum of the first n terms of these series are derived. Sum formulas for a few other series are also developed.

SEQUENCES

Our intuitive concept of the term '*Sequences of numbers*' involves not only a set of numbers but also an order: there is a first number, a second and so on. That is, for each positive integer 1, 2, 3, ……, there is 'associated' a number in the sequence. We may define a sequence as 'A succession of numbers (real or complex) $a_1, a_2, a_3, \ldots, a_n, \ldots$, formed according to some definite law'. We consider sequences of real numbers only. a_n is called the nth term (or the nth number) of the sequence.

Consider the following examples:

(i) $1, 2, 3, 4, \ldots\ldots, n, \ldots$ (sequence of natural numbers)
(ii) $3, 7, 11, 15, \ldots\ldots$
(iii) $2, 6, 18, 54, \ldots\ldots$
(iv) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\ldots$
(v) $2, 3, 5, 7, 11, 13, 17, \ldots$ (sequence of prime numbers)
(vi) $1, 1, 2, 3, 5, 8, 13, 21, \ldots\ldots$
(vii) $1, 6, 20, 56, \ldots\ldots$

In example (ii), it can be observed that any term of the sequence (second term onwards) can be obtained by adding 4 to the immediately preceding term. (ii) is called an 'arithmetic sequence'.

Example (i) also has the same property as that for (ii). Here, any term of the sequence (second term onwards) can be obtained by adding 1 to the immediately preceding term. The sequence of natural numbers is therefore an arithmetic sequence.

In example (iii), any term of the sequence can be obtained by multiplication of the immediately preceding term by 3. (iii) is called a 'geometrical sequence'.

In example (iv), the reciprocals of the numbers i.e., 2, 4, 6, 8, …., is an arithmetic sequence. (iv) is called a 'harmonic sequence'.

In example (vi), any term of the sequence (from third onwards) is obtained by the addition of the two previous terms:

3rd term = 2nd term + 1st term;
4th term = 3rd term + 2nd term;
5th term = 4th term + 3rd term;

This sequence is called the '**Fibonacci sequence**'. The corresponding terms are called Fibonacci numbers. These numbers find a place in many physical and engineering problems.

In example (vii), 2nd term = 3×2
3rd term = 5×2^2
4th term = 7×2^3 and so on

or, example (vii) is a combination of arithmetic and geometric sequence or (vii) is an example of arithmetico–geometric sequence.

For a succession of numbers $a_1, a_2, \ldots\ldots$ to form a sequence, it is not necessary that a_n should be capable of being expressed as a function of n. As a matter of fact, in example, (v), the nth term (or the nth prime number) cannot be expressed as a function of n.

SERIES

An expression of the form $a_1 + a_2 + a_3 + \ldots\ldots + a_n + \ldots$ is called a series, if $a_1, a_2, a_3, \ldots, a_n, \ldots$ form a sequence.

$3 + 7 + 11 + 15 + \ldots\ldots$ is an arithmetic series.

(series is said to be in arithmetic progression written as AP)

$1 + 2 + 3 + 4 + \ldots$ is an arithmetic series.

$2 + 6 + 18 + 54 + \ldots$ is a geometric series.

(series is said to be in geometric progression written as GP)

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\text{.........}$$ is a harmonic series.

(series is said to be in harmonic progression written as HP)

a_n denotes the nth term of the series.

S_n $(= a_1 + a_2 + a_3 + \ldots\ldots + a_n)$ denotes the sum of the first n terms of the series.

If each term of the series is followed by another, it is called an **infinite** series. If the series terminates after a finite number of terms it is called a **finite series**.

(i) $2 + 7 + 12 + \ldots\ldots\ldots\infty$

(ii) $3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\ldots\ldots\ldots\infty$

are examples of **infinite series**. The symbol ∞ is put after some dots to denote that the terms continue indefinitely without end, or there is no last term for an **infinite series.**

(iii) $4 + 8 + 12 + \ldots\ldots\ldots$ 25 terms

(iv) $1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{1000}$

are examples of **finite series.**

ARITHMETIC SERIES (OR SERIES IN AP)

Consider the series $a + (a + d) + (a + 2d) + \ldots\ldots\ldots$

1st term = a

2nd term = a + d = a + (2 – 1)d

3rd term = a + 2d = a + (3 – 1)d

4th term = a + 3d = a + (4 – 1)d and so on.

Any term (second term onwards) of the series is obtained by adding a constant number to the immediately preceding term.
'a' is called the first term of AP; 'd' is called the common difference (CD) of the AP It can be easily seen that the nth term of the above AP = a + (n – 1) d.

To obtain a formula for S_n, the sum of the first n terms of the above AP, we proceed as follows:

$$S_n = a + (a + d) + \ldots\ldots. + (a + (n - 2)d) + (a + (n - 1)d)$$

Also,

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + \ldots\ldots.. + (a + d) + a$$

Addition gives (addition in RHS is done column wise)

$$2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + \ldots\ldots+ [2a + (n - 1)d] + [2a + (n - 1)d]$$

The number $[2a + (n - 1)d]$ occurs n times in the right hand side. Therefore, its sum equals $n[2a + (n - 1)d]$.

$$\therefore\ S_n = \frac{n}{2}\left[2a + (n-1)d\right] \text{ or } \frac{n}{2}\left[a + \left(a + (n-1)d\right)\right]$$

$$= \frac{n}{2}\ [\text{first term} + \text{nth term}]$$

For the AP: $a + (a + d) + (a + 2d) + \ldots..$

nth term = a + (n – 1) d

Sum of the first n terms $= \frac{n}{2}\left[2a + (n-1)d\right]$

$$= \frac{n}{2}\left[\text{ first term} + \text{nth term }\right]$$

For the AP: $4 + 9 + 14 + \ldots\ldots..$: a = first term = 4 and d = common difference = 5

8th term of the AP = a + (8 – 1) d = 4 + 7 × 5 = 39

Sum of the first 25 terms of the series

$$= \frac{25}{2}\left[2 \times 4 + (25-1) \times 5\right] = 1600$$

Remark

The sum of the first n terms of an AP is of the form $An^2 + Bn$ where A and B are numbers depending on the first term and common difference of the AP

CONCEPT STRANDS

Concept Strand 1

4th term of an AP is –15 and its 16th term is –55. Find the 19th term and the sum of the first 40 terms.

Solution

Let a be the first term and d be the common difference of the AP

Given $a + 3d = -15$
$a + 15d = -55$

Solving for a and d, we get $a = -5$,

$$d = -\frac{10}{3}$$

$$19\text{th term} = a + 18d = -5 - \frac{18 \times 10}{3} = -65$$

Sum of the first 40 terms

$$= \frac{40}{2}\left[2 \times -5 + 39 \times \frac{-10}{3}\right] = -2800$$

Concept Strand 2

How many terms of the series 2 + 8 + 14 + ……… are to be taken to obtain the sum 574?

Solution

We have $a = 2$, $d = 6$

Let n denote the number of terms to be taken. Then,

$$\frac{n}{2}\left[2 \times 2 + (n-1) \times 6\right] = 574$$

$\Rightarrow$ $n(3n - 1) = 574$

$\Rightarrow$ $3n^2 - n - 574 = 0$

$$\Rightarrow \quad n = \frac{1 \pm \sqrt{1 + 6888}}{6} = \frac{1 \pm 83}{6} = 14, -\frac{41}{3}$$

Clearly, $n = \frac{-41}{3}$ is not admissible, since n must be a positive integer.

Therefore, n = 14, or 14 terms are to be taken to obtain the sum 574.

Concept Strand 3

Obtain the sum of the first n natural numbers.

Solution

1 + 2 + 3 + 4 + …….. represents the series of natural numbers.

It is an AP with first term a = 1 and common difference d = 1.

$\therefore$ Sum of the first n natural numbers $= 1 + 2 + 3 + \ldots\ldots + n$

$$= \frac{n}{2}\left[2 + (n-1) \times 1\right]$$

$$= \frac{n(n+1)}{2}$$

Concept Strand 4

Obtain the sum of the first n odd numbers.

Solution

1 + 3 + 5 + 7 + ….. represents the series of odd numbers. It is an AP with a = 1, d = 2.

$$n\text{th term} = 1 + (n - 1) \times 2 = (2n - 1)$$

$\therefore$ The sum of the first n odd numbers $= 1 + 3 + 5 + 7 + \ldots + (2n - 1)$

$$= \frac{n}{2}\left[2 \times 1 + (n-1) \times 2\right] = n^2$$

Concept Strand 5

Prove the following:

(i) If the same number k ($\neq 0$) is added to (or is subtracted from) every term of a series in AP, the new series thus obtained is still an AP.

(ii) If one multiplies every term of a series in AP by a number k ($\neq 0$), the new series thus obtained is still an AP.

Solution

Let a + (a + d) + (a + 2d) + …..represent a series in AP

(i) Series obtained by adding k to every term of the above series is

$$[a + k] + [a + d + k] + [a + 2d + k] + \ldots\ldots$$

Clearly, this series is an AP with first term (a + k) and common difference d.

Similar argument for subtraction proves the statement.

(ii) Series obtained by multiplying by k every term of the series is

$$ak + (a + d)\,k + (a + 2d)\,k + \ldots\ldots$$

Clearly, this series is an AP with first term ak and common difference dk.

GEOMETRIC SERIES (OR SERIES IN GP)

Consider the series: $a + ar + ar^2 + ar^3 + \ldots\ldots\ldots$ $(r \neq 1)$

1st term = a

2nd term = $ar = ar^{2-1}$

3rd term = $ar^2 = ar^{3-1}$

4th term = $ar^3 = ar^{4-1}$ and so on.

Any term (second term onwards) of the series is obtained by multiplication of the immediately preceding term by a constant r. a is called the first term of the GP and r is called the common ratio. It can be seen that the nth term of the above GP is $= ar^{n-1}$

We shall now derive the formula for the sum of the first n terms of the above GP

$$S_n = a + ar + ar^2 + \ldots\ldots\ldots\ldots\ldots + ar^{n-2} + ar^{n-1}$$

Multiplying both sides by r,

$$rS_n = ar + ar^2 + ar^3 + \ldots\ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n$$

Subtraction yields $(1 - r)S_n = a - ar^n = a(1 - r^n)$

$$\therefore \quad S_n = a\left(\frac{1-r^n}{1-r}\right) \quad \text{or} \quad a\left(\frac{r^n-1}{r-1}\right)$$

Note

If r = 1, the series reduces to $a + a + a + \ldots\ldots\ldots\ldots$ $S_n = na$, in this case.

For the GP: $a + ar + ar^2 + \ldots\ldots$, where $r \neq 1$,

nth term $= a\,r^{n-1}$

$$\text{Sum of the first n terms} = \frac{a(1-r^n)}{(1-r)} = \frac{a(r^n-1)}{(r-1)}$$

For the GP: $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \ldots\ldots$,

$$a = 1,\ r = \frac{3}{4}$$

$$\text{6th term} = a \times r^5 = 1 \times \left(\frac{3}{4}\right)^5 = \frac{243}{1024}$$

$$\text{Sum of the first 4 terms} = 1\left(\frac{1-\left(\frac{3}{4}\right)^4}{1-\frac{3}{4}}\right) = 4\left(1-\frac{81}{256}\right) = \frac{175}{64}$$

CONCEPT STRANDS

Concept Strand 6

The 7th and 11th terms of a GP whose terms are positive are 1458 and 118098 respectively. Find the series and the sum of the first 8 terms.

Solution

Let a be the first term and r be the common ratio of the GP

Given $ar^6 = 1458$ — (1)

and $ar^{10} = 118098$ — (2)

$\frac{(2)}{(1)}$ gives $r^4 = \frac{118098}{1458} = 81 \Rightarrow r = \pm 3$. Since all the terms are positive, r = 3.

Substituting r = 3 in (1), $a = \frac{1458}{3^6} = 2$

The series is $2 + 6 + 18 + \ldots\ldots$; Sum of the first 8 terms $= \frac{2(3^8-1)}{(3-1)} = 3^8 - 1$

Concept Strand 7

The 5th term of a GP is 32 and the common ratio is 2. Find the first term and the sum of the first 20 terms.

Solution

Let a be the first term.

We are given $a \times 2^4 = 32 \Rightarrow a = 2$

Sum of the first 20 terms

$$= \frac{2(2^{20}-1)}{2-1} = 2(2^{20}-1).$$

Infinite geometric series

As mentioned earlier, an infinite series is one in which each term is followed by another. In other words, there is no last term for an infinite series.

Consider the infinite geometric series:

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots\ldots\ldots\infty$$

If S_n denotes the sum of the first n terms of the above series, $S_n = \dfrac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2\left(1-\left(\frac{1}{2}\right)^n\right)$

We have,

$S_1 = 1$	$S_6 = 1.96875$
$S_2 = 1.5$	$S_7 = 1.984375$
$S_3 = 1.75$	$S_8 = 1.9921875$
$S_4 = 1.875$	$S_9 = 1.99609375$
$S_5 = 1.9375$	$S_{10} = 1.998046875$ and so on.

Observe that as n is increased indefinitely, the sum S_n approaches the finite number 2. We say that the sum of the infinite geometric series above is 2. It may be noted that this does not mean that the terms of the series add up to 2 (the number of terms being infinite, there is no question of adding 'all' the terms). When we say that the sum of the infinite series is 2 we mean that we can make the difference between S_n and 2 as small as we please by taking the value of n sufficiently large.

This is mathematically expressed as 'limit of S_n as n tends to infinity equals 2' and written as $\lim\limits_{n\to\infty} S_n = 2$.

Again, consider the infinite geometric series

$$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+-\ldots\ldots\infty.$$

Let us examine what happens to the sum S_n of the first n terms of the above series as n is increased indefinitely.

Here, $a = 1,\ r = -\dfrac{1}{3}$

$$\Rightarrow \qquad S_n = \frac{1-\left(\frac{-1}{3}\right)^n}{1+\frac{1}{3}} = \frac{3}{4}\left(1-\left(-\frac{1}{3}\right)^n\right)$$

Proceeding as in the first example, we obtain,

$S_1 = 1$
$S_2 = 0.666666667$
$S_5 = 0.753086419$
$S_{10} = 0.749987298$
$S_{25} = 0.750000052$ and so on.

It is clear that the sum S_n tends to 0.75 as n tends to infinity. We express this idea by writing $\lim\limits_{n\to\infty} S_n = 0.75$.

Let us obtain this limit (or the sum of an infinite geometric series) for the infinite geometric series:

$$a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots\infty$$

Here, $S_n = \dfrac{a(1-r^n)}{(1-r)} = \dfrac{a}{(1-r)} - \dfrac{ar^n}{(1-r)}$

Suppose r is numerically less than 1, (i.e., $-1 < r < 1$ or $|r| < 1$).

For large values of n, r^n becomes negligibly small or r^n tends to zero as n tends to infinity. Consequently, as n becomes larger and larger, S_n approaches the number $\dfrac{a}{1-r}$ provided $|r| < 1$.

Or $a + ar + ar^2 + \ldots\ldots\infty = \dfrac{a}{(1-r)}$ provided $|r| < 1$.

For example, the sum of the infinite geometric series $2+\dfrac{2}{5}+\dfrac{2}{25}+\ldots\ldots\ldots\infty$ is given by $\dfrac{2}{1-\frac{1}{5}} = \dfrac{5}{2}$.

Remark

If the common ratio r is numerically greater than 1, r^n becomes larger and larger (numerically) as n becomes large and S_n becomes infinite. We say that, in this case, the geometric series above diverges.

The same is the case with $r = 1$. When $r = -1$, the sum oscillates between 0 and a. It is therefore meaningful to talk about the sum of an infinite geometric series only when $|r| < 1$.

CONCEPT STRANDS

Concept Strand 8

Compute the product $7^{1/2} \times 7^{1/4} \times 7^{1/8} \times 7^{1/16} \times \ldots\ldots\infty$

Solution

Product = $7^{1/2 + 1/4 + 1/8 + 1/16 + \ldots\ldots\infty}$

Index of 7 is an infinite GP with first term $\frac{1}{2}$ and common ratio $\frac{1}{2}$. Its sum equals $\frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$

$\therefore$ Product = $7^1 = 7$.

Concept Strand 9

If S_n and S denote the sum of the first n terms and that to infinity respectively of the series, $1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \ldots\ldots\infty$, how large n should be taken as to have $\left|S - S_n\right| < \frac{1}{10000}$?

Solution

We have, $a = 1$ and $r = \frac{2}{5}$, $S_n = \frac{1 - \left(\frac{2}{5}\right)^n}{1 - \frac{2}{5}} = \frac{5}{3}\left(1 - \left(\frac{2}{5}\right)^n\right)$;

And $S = \frac{1}{1 - \frac{2}{5}} = \frac{5}{3}$

$$|S - S_n| = \left|\frac{5}{3} - \frac{5}{3}\left(1 - \left(\frac{2}{5}\right)^n\right)\right| < \frac{1}{10000}$$

$$\Rightarrow \left(\frac{2}{5}\right)^n < \frac{3}{50000} \Rightarrow \left(\frac{4}{10}\right)^n < \frac{6}{10^5}$$

$\Rightarrow (0.4)^n < 0.00006$

For $n = 9$, $(0.4)^9 = 0.000262 > 0.00006$

For $n = 10$, $(0.4)^{10} = 0.000105 > 0.00006$

For $n = 11$, $(0.4)^{11} = 0.000042 < 0.00006$

We infer that if n is taken as 11 or above, the difference between S_n and S can be made less than $\frac{1}{10000}$.

Concept Strand 10

Express the recurring decimal 0.753753753…. as the ratio of two positive integers.

Solution

$$0.753753\ldots = 0.753 + 0.753 \times 10^{-3} + 0.753 \times 10^{-6} + \ldots\infty$$
$$= 0.753\,[\,1 + 10^{-3} + 10^{-6} + \ldots\ldots\infty]$$

The series inside the square bracket is an infinite geometric series with common ratio $\frac{1}{10^3} < 1$.

Its sum = $\frac{1}{1 - 10^{-3}} = \frac{1000}{999}$

Therefore, $0.753753\ldots = 0.753 \times \frac{1000}{999} = \frac{753}{999}$

OR

Let $S = 0.753753\ldots$

$1000S = 753.753753\ldots$

Subtracting the first from the second,

$999S = 753 \Rightarrow S = \frac{753}{999}$.

ARITHMETICO-GEOMETRIC SERIES

Consider the arithmetic series $a + (a + d) + (a + 2d) + (a + 3d) + \ldots\ldots$ — (1)

and the geometric series $1 + r + r^2 + r^3 + \ldots\ldots\ldots$ — (2)

A series of the form

$a + (a + d)\,r + (a + 2d)\,r^2 + (a + 3d)\,r^3 + \ldots\ldots$ — (3)

in which each term is the product of the corresponding terms in the AP represented by (1) and the GP represented by (2) is called an arithmetico–geometric series (AGP)

Remark

If r =1, the series (3) is an AP
If d = 0, the series (3) is a GP
So, in series (3), we assume $r \neq 1$ and $d \neq 0$.
For example,

(i) $1+\frac{2}{5}+\frac{3}{5^2}+\frac{4}{5^3}+\ldots\ldots\ldots\ldots$ is an AGP The corresponding AP is 1, 2, 3,...... and the corresponding GP is 1, $\frac{1}{5}, \frac{1}{5^2}, \ldots$

(ii) $2 + 4x + 6x^2 + 8x^3 + \ldots\ldots$ is also an AGP The AP in this AGP is 2, 4, 6, 8, ... and the GP is 1, x, x^2, x^3....

(iii) $1 - 3x + 5x^2 - 7x^3 + - \ldots\ldots$ is an AGP formed with the corresponding elements of 1, 3, 5, 7,.... (AP) and 1, $-x$, x^2, $-x^3$ (GP)

To find the nth term of an Arithmetico–geometric series:

Step (1): Find the nth term of the corresponding arithmetic series which is $a + (n-1)d$

Step (2): Find the nth term of the corresponding geometric series which is $r^{(n-1)}$

Step (3): nth term of the AGP = $[a + (n-1)d]\, r^{n-1}$

For example, the nth term of (i) is given by $n \times \frac{1}{5^{n-1}}$.

nth term of (iii) is given by $(2n - 1)(-x)^{n-1} = (-1)^{n-1} x^{n-1} (2n - 1)$

Let S_n represent the sum of the first n terms of an AGP

$S_n = a + (a + d)\, r + (a + 2d)\, r^2 + \ldots\ldots\ldots + [a + (n - 1)\, d]\, r^{n-1}$ — (1)

$(1) \times r \Rightarrow r \times S_n = ar + (a + d)\, r^2 + (a + 2d)\, r^3 + \ldots + [a + (n - 2]\, d]r^{n-1} + [a +(n - 1)d]\, r^n$ — (2)

$(1) - (2) \Rightarrow (1 - r)\, S_n = a + dr + dr^2 + \ldots\ldots + dr^{n-1} - [a + (n - 1)\, d]\, r^n$

$$= a + \frac{dr(1-r^{n-1})}{1-r} - [a+(n-1)d]r^n$$

Or $$S_n = \frac{a}{(1-r)} + \frac{dr}{(1-r)^2} - \frac{dr^n}{(1-r)^2} - \frac{\left[a+(n-1)d\right]r^n}{(1-r)} \quad \text{— (3)}$$

The procedure for obtaining the sum of the first n terms of an arithmetico–geometric series is explained through two examples. As the first example, let S_n represent the sum of the first n terms of the AGP (i).

i.e., $$S_n = 1+\frac{2}{5}+\frac{3}{5^2}+\frac{4}{5^3}+\ldots\ldots\ldots\ldots+\frac{n}{5^{n-1}} \quad \text{— (1)}$$

On multiplying both sides of (1) by $\frac{1}{5}$ (which is the common ratio of the corresponding GP)

$$\frac{1}{5}S_n = \frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}+\ldots\ldots\ldots\frac{(n-1)}{5^{n-1}}+\frac{n}{5^n} \quad \text{— (2)}$$

(1) – (2) gives,

$$\left(1-\frac{1}{5}\right)S_n = 1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\ldots\ldots\ldots+\frac{1}{5^{n-1}}-\frac{n}{5^n}$$

Now, $1+\frac{1}{5}+\frac{1}{5^2}+\ldots+\frac{1}{5^{n-1}}$ is a GP with common ratio $\frac{1}{5}$. Therefore, using the formula for the sum of the first n terms of a GP, we have

$$\frac{4}{5}S_n = \frac{1-\left(\frac{1}{5}\right)^n}{\left(1-\frac{1}{5}\right)} - \frac{n}{5^n}$$

$$= \frac{5}{4}\left(1-\left(\frac{1}{5}\right)^n\right) - \frac{n}{5^n}$$

or, $$S_n = \frac{25}{16}\left[1-\left(\frac{1}{5}\right)^n\right] - \left(\frac{n}{5^n}\right)\times\frac{5}{4}$$

$$= \frac{25}{16}\left[1-\left(\frac{1}{5}\right)^n\right] - \frac{n}{4\times 5^{n-1}}$$

As the second example, consider the AGP

$$3-\frac{5}{2}+\frac{7}{2^2}-\frac{9}{2^3}+-\ldots\ldots$$

nth term of the series $= \frac{(2n+1)(-1)^{n-1}}{2^{n-1}}$

If S_n denotes the sum of the first n terms of the above series,

$$S_n = 3-\frac{5}{2}+\frac{7}{2^2}-\frac{9}{2^3}+-\ldots+\frac{(2n+1)(-1)^{n-1}}{2^{n-1}}$$

On multiplying both sides by $\left(\frac{-1}{2}\right)$ (which is the common ratio of the corresponding GP)

$$\left(-\frac{1}{2}\right)S_n = \frac{-3}{2} + \frac{5}{2^2} - \frac{7}{2^3} + \ldots\ldots$$
$$+\frac{(2n-1)(-1)^{n-1}}{2^{n-1}} + \frac{(2n+1)(-1)^n}{2^n}$$
$$S_n - \left(-\frac{1}{2}S_n\right)$$
$$= \frac{3}{2}S_n = 3 - \frac{2}{2} + \frac{2}{2^2} - \frac{2}{2^3} + \ldots\ldots$$
$$+\frac{2(-1)^{n-1}}{2^{n-1}} - \frac{(2n+1)(-1)^n}{2^n}$$
$$\frac{3}{2}S_n = 3 - \left[1 - \frac{1}{2} + \frac{1}{2^2} - - + \frac{(-1)^{n-2}}{2^{n-2}}\right]$$
$$-\frac{(2n+1)(-1)^n}{2^n}$$
$$= 3 - \frac{1-\left(-\frac{1}{2}\right)^{n-1}}{1-\left(-\frac{1}{2}\right)} - \frac{(2n+1)(-1)^n}{2^n}$$
$$= 3 - \frac{2}{3}\left[1-\left(\frac{-1}{2}\right)^{n-1}\right] - \frac{(2n+1)(-1)^n}{2^n}$$
$$S_n = 2 - \frac{4}{9}\left[1-\left(\frac{-1}{2}\right)^{n-1}\right] - \frac{2}{3}\frac{(-1)^n(2n+1)}{2^n}$$

It is clear from the two examples above, that we can obtain the sum of the first n terms of an arithmetico–geometric series.

Now consider the infinite arithmetico–geometric series

$$a + (a + d)\,r + (a + 2d)\,r^2 + \ldots\ldots\ldots\infty \text{ — (4) with } |r| < 1$$

We can find the sum of this infinite AGP by using the procedure described above.

Since $|r| < 1$, for large n, r^n becomes very small and therefore, r^n tends to zero as n tends to infinity.

Thus, from (3), the sum of the infinite AGP represented by (4) is given by $\frac{a}{(1-r)} + \frac{dr}{(1-r)^2}$

or we have

$$a + (a + d)\,r + (a + 2d)\,r^2 + \ldots\ldots\ldots$$
$$\infty = \frac{a}{(1-r)} + \frac{dr}{(1-r)^2} \text{ provided } |r| < 1$$

As an example for illustration, the sum of the series $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\infty$, where, $|x| < 1$ (here, $a = 1$, $d = 1$, $r = x$) is given by $\frac{1}{(1-x)} + \frac{x}{(1-x)^2} = \frac{1}{(1-x)^2}$.

HARMONIC SERIES (SERIES IN HP)

The series $a_1 + a_2 + a_3 + \ldots\ldots$ is said to be a harmonic series or series is said to be in harmonic progression (written as HP), if the series $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots\ldots$ is in AP (which is called the corresponding AP).

We can find the nth term of an HP by finding the nth term of the corresponding AP

i.e., nth term of the HP = reciprocal of the nth term of the corresponding AP

We do not have a formula for the sum of the first n terms of an HP

Problems involving HP are attempted by converting them into problems in the corresponding AP

For example, consider the series $\frac{1}{2} + \frac{3}{13} + \frac{3}{20} + \ldots\ldots$

This is a series in HP since, the series $2 + \frac{13}{3} + \frac{20}{3} + \ldots\ldots$, whose terms are the reciprocals of the corresponding terms of the given series, is in AP

The corresponding AP has first term 2 and common difference $\frac{7}{3}$.

Suppose we want the 10th term of the HP:

The 10th term of the corresponding AP is given by $a + 9d = 2 + 9 \times \frac{7}{3} = 23$.

Hence, the 10th term of the given HP is $\frac{1}{23}$.

CONCEPT STRANDS

Concept Strand 11

The 4th term of an HP is $\frac{1}{15}$ and its first term is $\frac{1}{3}$. Find the 7th term.

Solution

4th term of the corresponding AP = 15; First term of the corresponding AP = 3

We have, 3 + 3d = 15 or d = 4

7th term of the corresponding AP = 3 + 6d = 27

or 7th term of the HP = $\frac{1}{27}$.

Concept Strand 12

If ℓ, m, n are the pth, qth and rth terms of an HP, prove that $\frac{(q-r)}{\ell} + \frac{(r-p)}{m} + \frac{(p-q)}{n} = 0$

Solution

Note that the pth, qth and rth terms of the corresponding AP are $\frac{1}{\ell}, \frac{1}{m}, \frac{1}{n}$

$$\frac{1}{\ell} = A + (p-1)D;$$

$$\frac{1}{m} = A + (q-1)D;$$

$$\frac{1}{n} = A + (r-1)D,$$

where A denotes the first term and D denotes the common difference of the corresponding AP

$$\frac{(q-r)}{\ell} + \frac{(r-p)}{m} + \frac{(p-q)}{n}$$
$$= [A + (p-1)D](q-r) + [A + (q-1)D](r-p) + [A + (r-1)D](p-q)$$
$$= A\{q - r + r - p + p - q\} + D\{(p-1)(q-r) + (q-1)(r-p) + (r-1)(p-q)\}$$
$$= A \times 0 + D \times 0 = 0$$

ARITHMETIC MEAN, GEOMETRIC MEAN AND HARMONIC MEAN

We introduce the terms 'arithmetic mean', 'geometric mean' and 'harmonic mean' which are closely related to arithmetic, geometric and harmonic sequences respectively. Consider two positive numbers a and b.

Arithmetic Mean

If x is a number such that a, x, b form an arithmetic sequence, x is called the arithmetic mean (AM) of a and b.

Since a, x, b form an arithmetic sequence, x – a = b – x

or $x = \frac{(a+b)}{2}$

Thus, the AM of a and b is $\frac{(a+b)}{2}$.

Geometric Mean

If x is a number such that a, x, b form a geometric sequence, x is called the geometric mean (GM) of a and b.

Since a, x, b form a geometric sequence, $\frac{x}{a} = \frac{b}{x}$ or $x^2 = ab$ or $x = \pm\sqrt{ab}$

It is the usual practice to take the $+\sqrt{ab}$. Thus, the GM of a and b is $+\sqrt{ab}$.

Harmonic Mean

If x is a number such that a, x, b form a harmonic sequence, x is called the harmonic mean (HM) of a and b.

Since a, x, b form a harmonic sequence,

$\frac{1}{a}, \frac{1}{x}, \frac{1}{b}$ form an AP i.e., $\frac{1}{x} - \frac{1}{a} = \frac{1}{b} - \frac{1}{x}$

or $\frac{2}{x} = \frac{1}{a} + \frac{1}{b} \Rightarrow x = \frac{2ab}{(a+b)}$

Thus, the HM of a and b is $\frac{2ab}{(a+b)}$

For example, consider the numbers 3 and 7.

The AM of 3 and 7 is $\frac{3+7}{2} = 5$;

GM of 3 and 7 is $\sqrt{3 \times 7} = \sqrt{21}$ and

HM of 3 and 7 is $\frac{2 \times 3 \times 7}{(3+7)} = \frac{21}{5}$

Remark 1

If A, G, H. represent the AM, GM and HM of two positive numbers a and b,

(i) A > G > H and
(ii) $G^2 = AH$

OR A, G, H form a decreasing GP

We have $A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{(a+b)}$

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{2} > 0;$$

$$G - H = \sqrt{ab} - \frac{2ab}{(a+b)} = \frac{\sqrt{ab}\left(a + b - 2\sqrt{ab}\right)}{(a+b)}$$

$$= \frac{\sqrt{ab}\left(\sqrt{a} - \sqrt{b}\right)^2}{(a+b)} > 0$$

This proves the result A > G > H.

Also, $G^2 = ab$ and $AH = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right) = ab$

or $G^2 = AH$

If a and b are two positive numbers,

Arithmetic Mean (AM) of a and b = $\frac{(a+b)}{2} = A$

Geometric Mean (GM) of a and b = $\sqrt{ab} = G$

Harmonic Mean (HM) of a and b = $\frac{2ab}{(a+b)} = H$

A > G > H and $G^2 = AH$

Remark 2

For two positive numbers, we have just proved that their AM > their GM If the two numbers are equal, it can be easily verified that their AM = their GM This result can be extended to a set of n positive numbers.

Suppose $a_1, a_2, a_3, \ldots\ldots, a_n$ are n positive numbers.

The arithmetic mean (AM) of these numbers is defined as $\frac{(a_1 + a_2 + a_3 + \ldots\ldots + a_n)}{n}$.

The geometric mean (GM) of these numbers is defined as $(a_1 a_2 a_3 \ldots\ldots a_n)^{1/n}$.

The harmonic mean (HM) of these numbers is defined as $\frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)}$

We have the important inequality:

AM of n positive numbers ≥ their GM ≥ their HM

i.e., $$\frac{(a_1 + a_2 + a_3 + \ldots\ldots + a_n)}{n} \geq \left(a_1, a_2 \ldots\ldots a_n\right)^{1/n} \geq \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)}$$

Equality holds good when $a_1 = a_2 = a_3 = \ldots = a_n$ (i.e., when the numbers are equal).

For example, suppose x is a positive number, by considering the two numbers x and $\frac{1}{x}$, their AM= $\frac{x + \frac{1}{x}}{2}$; their GM = 1.

We obtain the inequality, $\frac{x + \frac{1}{x}}{2} \geq 1$ or $x + \frac{1}{x} \geq 2$

for all positive values of x and specifically $x + \frac{1}{x} = 2$ if and only if $x = \frac{1}{x} = 1$

CONCEPT STRANDS

Concept Strand 13

Show that $(a^2 + b^2 + c^2 + d^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}\right) \geq 16$

Solution

Considering the numbers a^2, b^2, c^2, d^2 and applying the AM ≥ GM inequality, $\frac{a^2 + b^2 + c^2 + d^2}{4} \geq \left(a^2 b^2 c^2 d^2\right)^{1/4}$ — (1)

Again, considering the numbers, $\frac{1}{a^2},\frac{1}{b^2},\frac{1}{c^2},\frac{1}{d^2}$ and applying the AM ≥ GM inequality, $\dfrac{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}}{4} \geq \left(\dfrac{1}{a^2b^2c^2d^2}\right)^{1/4}$ — (2)

Multiplying (1) and (2) we obtain the required result.

Concept Strand 14

If the sum of a number of positive numbers is a constant, their product is maximum when the numbers are equal.

Solution

Let $a_1, a_2, a_3, \ldots, a_n$ be n positive numbers such that $a_1 + a_2 + a_3 + \ldots + a_n = k$ (given)

Since $\dfrac{(a_1 + a_2 + \ldots + a_n)}{n} \geq (a_1, a_2 \ldots a_n)^{1/n}$

$$a_1 a_2 \ldots a_n \leq \left(\frac{a_1 + a_2 + \ \ldots + a_n}{n}\right)^n \leq \left(\frac{k}{n}\right)^n$$

This means that the product $a_1\ a_2\ a_3 \ldots a_n$ is always less than $\left(\frac{k}{n}\right)^n$ and is equal to $\left(\frac{k}{n}\right)^n$ only when $a_1 = a_2 = \ldots = a_n$

In other words, the maximum value of $\left(a_1 a_2 \ldots a_n\right) = \left(\frac{k}{n}\right)^n$, when the numbers are equal.

Concept Strand 15

If the product of a number of positive quantities is a constant, their sum is minimum when the numbers are equal.

Solution

If the numbers are assumed as $a_1, a_2, a_3, \ldots a_n$, we are given that $a_1a_2a_3 \ldots a_n = \lambda$ (a constant)

Since $\dfrac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1a_2 \ldots a_n)^{1/n}$

$$a_1 + a_2 + \ldots a_n \geq n\ (a_1a_2 \ldots a_n)^{1/n} \geq n\lambda^{1/n}$$

This means that the sum of the numbers $(a_1 + a_2 + \ldots + a_n)$ is always $\geq n\lambda^{1/n}$ and is equal to $n\lambda^{1/n}$ only when $a_1 = a_2 = \ldots = a_n$.

In other words, the minimum value of $(a_1 + a_2 + \ldots + a_n)$ is equal to $n\lambda^{1/n}$, when the numbers are equal.

Remark 3

Let a and b be any two numbers.

(i) If $a_1, a_2, a_3, \ldots a_n$, are n numbers such that $a, a_1, a_2, a_3, \ldots a_n$, b form an arithmetic sequence, we say that we have inserted n arithmetic means (AMs) $a_1, a_2, a_3, \ldots a_n$ between a and b.

(ii) if $g_1, g_2, g_3 \ldots, g_n$ are n numbers such that $a, g_1, g_2, g_3 \ldots, g_n$, b form a geometric sequence, we say that we have inserted n geometric means (GMs) $g_1, g_2, g_3 \ldots, g_n$ between a and b.

(iii) If $h_1, h_2, h_3, \ldots h_n$ are n numbers such that $a, h_1, h_2, h_3, \ldots h_n$, b form a harmonic sequence, we say that we have inserted n harmonic means (HMs) $h_1, h_2, h_3, \ldots h_n$ between a and b.

PROCEDURE TO FIND THE AMs, GMs, HMs BETWEEN a AND b

Arithmetic Means

$a, a_1, a_2, a_3, \ldots, a_n$, b being an arithmetic sequence, b = (n + 2)th term of the arithmetic series $a + a_1 + a_2 + a_3 + \ldots + a_n + b$

∴ b = a + (n + 1) d (if d represents the common difference). This gives $d = \dfrac{(b-a)}{(n+1)}$

The arithmetic means are given by

$$a_1 = a + \frac{b-a}{n+1} = \frac{an+b}{(n+1)}$$

$$a_2 = a + \frac{2(b-a)}{n+1} = \frac{a(n-1)+2b}{(n+1)}$$

$$a_3 = \frac{a(n-2)+3b}{n+1}$$

……………………..

……………………..

$$a_n = a + \frac{n(b-a)}{n+1} = \frac{a+nb}{n+1}$$

Geometric Means

a, g_1, g_2, g_3, ……., g_n, b being a geometric sequence, b = (n + 2)th term of the geometric series $a + g_1 + g_2 + g_3 + ….. + g_n + b$

$\therefore$ $b = a \times r^{n+1}$ (if r represents the common ratio). This gives $r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$

The geometric means are given by

$$a_1 = a + \frac{b-a}{n+1} = \frac{an+b}{(n+1)}$$

$$a_2 = a + \frac{2(b-a)}{n+1} = \frac{a(n-1)+2b}{(n+1)}$$

$$a_3 = \frac{a(n-2)+3b}{n+1}$$

……………………………………

……………………………………

$$g_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}} = a^{\frac{1}{n+1}} \times b^{\frac{n}{n+1}}$$

Harmonic Means

a, h_1, h_2, h_3, ….. , h_n, b being a harmonic sequence, $\frac{1}{a}, \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, ……, \frac{1}{h_n}, \frac{1}{b}$ will be an arithmetic sequence.

It can be easily observed that $\frac{1}{h_1}, \frac{1}{h_2}, ……, \frac{1}{h_n}$, are the n AMs between $\frac{1}{a}$ and $\frac{1}{b}$ and that we can obtain these by replacing a by $\frac{1}{a}$ and b by $\frac{1}{b}$ in $a_1, a_2, …… a_n$ obtained above.

Therefore,

$$\frac{1}{h_1} = \frac{\frac{n}{a} + \frac{1}{b}}{n+1} = \frac{nb+a}{(n+1)ab};$$

$$\frac{1}{h_2} = \frac{\frac{n-1}{a} + \frac{2}{b}}{(n+1)} = \frac{(n-1)b+2a}{(n+1)ab}$$

……………………………….

……………………………….

$$\frac{1}{h_n} = \frac{\frac{1}{a} + \frac{n}{b}}{(n+1)} = \frac{na+b}{(n+1)ab}$$

OR

The harmonic means are given by $h_1 = \frac{(n+1)ab}{nb+a}$, $h_2 = \frac{(n+1)ab}{(n-1)b+2a}, ……, h_n = \frac{(n+1)ab}{(na+b)}$

From the above,

$$a_1 + a_2 + a_3 + ….+ a_n$$

$$= \frac{a\left(1+2+3+…..+n\right)+b\left(1+2+3+…..+n\right)}{(n+1)}$$

$$= \frac{\frac{an(n+1)}{2} + \frac{bn(n+1)}{2}}{(n+1)}$$

$$= \frac{n(a+b)}{2}$$

and $\quad g_1, g_2, g_3 …. g_n = a^{n/2} \times b^{n/2} = (ab)^{n/2}$

CONCEPT STRANDS

Concept Strand 16

Insert 7 AMs between –5 and 11.

Solution

Let the AMs be $a_1, a_2, a_3, ….., a_7$

11 = 9th term of the AP whose first term is –5

$\therefore$ $11 = -5 + 8d \Rightarrow d = 2$

The AMs are –3, –1, 1, 3, 5, 7, 9

Concept Strand 17

Insert

(i) 4 AMs,
(ii) 4 GMs and
(iii) 4 HMs

between the numbers 3 and 12.

Solution

If a_1, a_2, a_3, a_4 represent the AMs, between 3 and 12.

12 = 6th term of the AP whose first term is 3 = 3 + 5d,

giving $d = \frac{9}{5}$.

$\Rightarrow$ The AMs are $\frac{24}{5}, \frac{33}{5}, \frac{42}{5}, \frac{51}{5}$

If g_1, g_2, g_3, g_4 represent the 4 GMs.

12 = 6th term of the GP whose first term is $3 = 3 \times r^5$

$\Rightarrow$ $r = 2^{2/5}$

$\Rightarrow$ The GMs are $3 \times 2^{2/5}, 3 \times 2^{4/5}, 3 \times 2^{6/5}, 3 \times 2^{8/5}$

If h_1, h_2, h_3, h_4 represent the 4 HMs,

$\frac{1}{3}, \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \frac{1}{h_4}, \frac{1}{12}$ form an arithmetic sequence

$\frac{1}{12} = \frac{1}{3} + 5d$ giving $d = \frac{-1}{20}$

$\Rightarrow$ The HMs are $\frac{60}{17}, \frac{30}{7}, \frac{60}{11}, \frac{15}{2}$

Concept Strand 18

If a_1, a_2, a_3 are the arithmetic means, h_1, h_2, h_3 are the harmonic means between two numbers x and y, show that $a_1h_3 = a_2h_2 = a_3h_1 = xy$

Solution

Since x, a_1, a_2, a_3, y form an arithmetic sequence, y = x + 4d

$\Rightarrow$ $d = \frac{y - x}{4}$

We get $a_1 = \frac{3x + y}{4}$, $a_2 = \frac{2x + 2y}{4}$, $a_3 = \frac{x + 3y}{4}$

Also, $\frac{1}{h_1} = \frac{\frac{3}{x} + \frac{1}{y}}{4}$, $\frac{1}{h_2} = \frac{\frac{2}{x} + \frac{2}{y}}{4}$, $\frac{1}{h_3} = \frac{\frac{1}{x} + \frac{3}{y}}{4}$

Giving $h_1 = \frac{4xy}{3y + x}$, $h_2 = \frac{4xy}{2x + 2y}$, $h_3 = \frac{4xy}{3x + y}$

It is very easy to verify that $a_1h_3 = a_2h_2 = a_3h_1 = xy$.

Concept Strand 19

If between two quantities there be inserted two AMs a_1 and a_2, two GMs g_1 and g_2 and two HM^s h_1 and h_2 show that $\frac{g_1g_2}{a_1 + a_2} = \frac{h_1h_2}{h_1 + h_2}$

Solution

We have, $a_1 = \frac{2x + y}{3}$, $a_2 = \frac{x + 2y}{3}$,

$g_1 = x^{2/3}y^{1/3}$, $g_2 = y^{2/3}x^{1/3}$, $h_1 = \frac{3xy}{x + 2y}$,

$h_2 = \frac{3xy}{2x + y}$

$\frac{g_1g_2}{a_1 + a_2} = \frac{xy}{x + y}$, on substitution and simplification.

$\frac{h_1h_2}{h_1 + h_2} = \frac{1}{\frac{1}{h_1} + \frac{1}{h_2}} = \frac{xy}{x + y}$, on substitution and simplification.

Result follows.

SUMMATION SYMBOL Σ (SIGMA)

Consider the series 1 + 4 + 7 + ….

This is an AP with first term 1 and common difference 3. The nth term of the series is therefore, 1 + (n − 1)3 = 3n − 2.

If S_{50} denotes the sum of the first 50 terms of the above series, S_{50} = 1 + 4 + 7 + ….. + 148 (148 being the 50th term).

Using the summation symbol Σ, we can write the above sum as $S_{50} = \sum_{k=1}^{50}(3k - 2)$

Here, $\sum_{k=1}^{50}(3k - 2)$ means the sum of the series obtained by putting k = 1, 2, 3, ….., 50 in (3k − 2). k is called the summation index. We may as well use λ or n instead of k as the summation index. In other words, S_{50} can be also represented as

$$S_{50} = \sum_{\lambda=1}^{50}(3\lambda - 2) = \sum_{n=1}^{50}(3n - 2)$$

Summation symbol therefore works as a short hand notation for representing the sum of a series. Consider the following examples.

(i) $\sum_{k=1}^{100} k^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2$

(ii) $\sum_{\lambda=1}^{15} (2\lambda + 1)(2\lambda + 5) = 3.7 + 5.9 + 7.11 + \ldots. + 31.35$

(iii) $\sum_{k=1}^{n} (2k - 1)^3 = 1^3 + 3^3 + 5^3 + \ldots. + (2n - 1)^3$

(iv) $\sum_{\lambda=1}^{20} (3\lambda + 2)4^\lambda = 5 \times 4 + 8 \times 4^2 + 11 \times 4^3 + \ldots. + 62 \times 4^{20}$

The sigma symbol is therefore useful in representing the sum of a series whose nth term (or general term) is known.

Remarks

(i) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(ii) $\sum_{r=1}^{n} ka_r = k\sum_{r=1}^{n} a_r$ (k is a constant)

(iii) Suppose we want to represent the sum $S = a^2 (b^2 + c^2) + b^2 (c^2 + a^2) + c^2 (a^2 + b^2)$

Using the summation symbol Σ we can represent S as

$$S = \sum a^2(b^2 + c^2)$$

We get the three terms of S as follows:

First term is taken as $a^2(b^2 + c^2)$.

Second term is obtained from the first term by replacing a by b, b by c, c by a in the first term $a^2(b^2 + c^2)$ and we get $b^2(c^2 + a^2)$.

Third term is obtained from the second term by replacing a by b, b by c, c by a in the second term. We get $c^2(a^2 + b^2)$.

Another example is

$$\sum p(q - r)$$

$$\sum p(q - r)$$

means

$$p(q - r) + q(r - p) + r(p - q).$$

(The letters p, q, r are replaced cyclically (i.e., p to q, q to r and r to p) to get the terms of the series represented by the Σ notation.)

SUMMATION OF SERIES

So far, we have considered the arithmetic series, geometric series, arithmetico-geometric series and we could develop a formula for the sum of the first n terms for each of these series. In general, it may not be possible to develop such a sum formula for every series even though we are able to write the formula for the nth term of such series. However, there are some series for which the sum formulas can be derived provided the nth term of the series is known. We discuss a few such cases below.

(i) Sum of the squares of the first n natural numbers

We have to find the sum $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \sum_{k=1}^{n} k^2$

Let S represent the above sum.

We have,

$$(x + 1)^3 = x^3 + 3x^2 + 3x + 1$$

or $3x^2 + 3x + 1 = (x + 1)^3 - x^3$

Since the above relation holds good for any x, it is an identity in x.

Putting x = 1, 2, 3, ….., n successively in the above,

$$3 \times 1^2 + 3 \times 1 + 1 = 2^3 - 1^3$$

$$3 \times 2^2 + 3 \times 2 + 1 = 3^3 - 2^3$$

$$3 \times 3^2 + 3 \times 3 + 1 = 4^3 - 3^3$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\underline{3 \times n^2 + 3 \times n + 1 = (n + 1)^3 - n^3}$$

On adding, $3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n = (n + 1)^3 - 1$

Since $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots. + n = \frac{n(n + 1)}{2}$, substituting for Σk,

$$\Rightarrow \quad 3S + \frac{3n(n + 1)}{2} + n = (n + 1)^3 - 1$$

$$\Rightarrow \quad 3S = (n + 1)^3 - \frac{3n(n + 1)}{2} - (n + 1)$$

$$= \frac{(n + 1)}{2}\left\{2(n + 1)^2 - 3n - 2\right\}$$

$$= \frac{(n+1)n(2n+1)}{2}$$

or $\quad S = \dfrac{n(n+1)(2n+1)}{6}$

(ii) Sum of the cubes of the first n natural numbers

We have to find the sum $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \sum_{k=1}^{n} k^3$

Let S represent the above sum

We have,

$$(x+1)^4 = (x+1)^2 (x+1)^2$$
$$= (x^2 + 2x + 1)(x^2 + 2x + 1)$$
$$= x^4 + 4x^3 + 6x^2 + 4x + 1$$

or $\quad 4x^3 + 6x^2 + 4x + 1 = (x+1)^4 - x^4$

Since the above relation holds good for any x, it is an identity in x. Putting x = 1, 2, 3, ….., n successively in the above

$$4 \times 1^3 + 6 \times 1^2 + 4 \times 1 + 1 = 2^4 - 1^4$$
$$4 \times 2^3 + 6 \times 2^2 + 4 \times 2 + 1 = 3^4 - 2^4$$
$$4 \times 3^3 + 6 \times 3^2 + 4 \times 3 + 1 = 4^4 - 3^4$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\underline{4 \times n^3 + 6 \times n^2 + 4 \times n + 1 = (n+1)^4 - n^4}$$

On adding,

$$4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + n = (n+1)^4 - 1$$

We have, $\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$ and

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Substituting in the above expression,

$\Rightarrow \quad 4S + n(n+1)(2n+1) + 2n(n+1) + n = (n+1)^4 - 1$

OR

$$4S = (n+1)^4 - n(n+1)(2n+1) - 2n(n+1) - (n+1)$$
$$= (n+1)[(n+1)^3 - n(2n+1) - 2n - 1]$$
$$= (n+1)[(n+1)^3 - (2n+1)(n+1)]$$
$$= (n+1)^2 [(n+1)^2 - 2n - 1] = (n+1)^2 n^2$$

$\Rightarrow \quad S = \left[\dfrac{n(n+1)}{2}\right]^2$ = [sum of the first n natural numbers]^2.

We can consolidate the results as:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2$$
$$= \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3$$
$$= \left[\frac{n(n+1)}{2}\right]^2$$

CONCEPT STRAND

Concept Strand 20

If s and t are respectively the sum and the sum of the squares of n successive positive integers beginning with 'a + 1' show that $(nt - s^2)$ is independent of 'a'.

Solution

$$s = (a+1) + (a+2) + (a+3) + \ldots\ldots\ldots + (a+n)$$
$$= na + \frac{n(n+1)}{2}$$
$$t = (a+1)^2 + (a+2)^2 + (a+3)^2 + \ldots\ldots + (a+n)^2$$
$$= a^2 n + 2a\sum_{r=1}^{n} r + \sum_{r=1}^{n} r^2$$
$$= a^2 n + an(n+1) + \frac{n(n+1)(2n+1)}{6}$$

$\therefore \quad nt = n^2a^2 + an^2(n+1) + \dfrac{n^2(n+1)(2n+1)}{6}$

$$s^2 = n^2a^2 + \frac{n^2(n+1)^2}{4} + an^2(n+1)$$

$\Rightarrow \quad nt - s^2 = \dfrac{n^2(n+1)(2n+1)}{6} - \dfrac{n^2(n+1)^2}{4}$, which is independent of a.

(iii) Sum of the Series using the nth term of the series

(i) Consider $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+$ ......n terms

Let u_n represent the nth term. Observe that the denominator is the product of two numbers. The first factor is the nth term of the arithmetic sequence 1, 2, 3, …… and the second factor is the nth term of the arithmetic sequence 4, 5, 6, …..

It can be easily seen that

$$u_n = \frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+3}\right)$$

First term $= u_1 = \frac{1}{3}\left(\frac{1}{1}-\frac{1}{4}\right)$

Second term $= u_2 = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)$

Third term $= u_3 = \frac{1}{3}\left(\frac{1}{3}-\frac{1}{6}\right)$

....

nth term $= u_n = \frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+3}\right)$,

On addition, Sum of the first n terms

$= S_n = \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right)$, as the other terms cancel.

Observation

As n becomes larger and larger, the numbers $\frac{1}{n+1}$, $\frac{1}{n+2}$, $\frac{1}{n+3}$ become smaller and smaller. Or, as n tends to infinity, $\frac{1}{n+1}$, $\frac{1}{n+2}$ and $\frac{1}{n+3}$ tend to zero.

Therefore, sum of the infinite series

$$\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+....\infty = \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right) = \frac{11}{18}$$

(ii) Consider $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+$ ...n terms

Let u_n represent the nth term of the series

$$u_n = \frac{1}{[2+(n-1)3][5+(n-1)3]}$$

$$= \frac{1}{(3n-1)(3n+2)}$$

$$= \frac{1}{3}\left\{\frac{1}{3n-1}-\frac{1}{3n+2}\right\}$$

$$u_1 = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)$$

$$u_2 = \frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right)$$

$$u_3 = \frac{1}{3}\left(\frac{1}{8}-\frac{1}{11}\right)$$

....

$$u_n = \frac{1}{3}\left(\frac{1}{3n-1}-\frac{1}{3n+2}\right)$$

Addition gives,

Sum of the series $= \frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right) = \frac{n}{2(3n+2)}$

If S_n represents the sum of the first n terms of the above series, $S_n = \frac{1}{6}-\frac{1}{3(3n+2)}$

As n becomes larger and larger, $\frac{1}{(3n+2)}$ becomes smaller and smaller and, therefore, as n tends to infinity, $\frac{1}{(3n+2)}$ tends to zero.

That is, $\lim_{n\to\infty} S_n = \frac{1}{6}$.

Or, sum of the infinite series $\frac{1}{2.5}+\frac{1}{5.8}+....\infty = \frac{1}{6}$

(iii) Method of differences for finding nth term and the sum of the first n terms of a series

Consider the series of real numbers

$$u_1+u_2+u_3+.......+u_n+....... \quad — (1)$$

$(u_2-u_1), (u_3-u_2), (u_4-u_3), ..., (u_n-u_{n-1})$ are called the first order differences of the terms of the series (1). Let these differences be denoted by $v_1, v_2, v_3, ...v_{n-1}$

Then
$v_1 = u_2 - u_1$
$v_2 = u_3 - u_2$
$v_3 = u_4 - u_3$
....
....
$v_{n-1} = u_n - u_{n-1}$

The second order difference of the terms of the series (1) are defined as (v_2-v_1), (v_3-v_2), ... $(v_{n-1}-v_{n-2})$

Suppose we denote the second order differences by $w_1, w_2, w_3, ...w_{n-2}$, the third order differences are defined as $(w_2-w_1), (w_3-w_2), (w_4-w_3)$,............

We may extend the above and define the 4th order, 5th order, ..., nth order differences of the terms of the series (1).

The table of differences is given below:

$$\begin{array}{lllllll}
 & & \text{1st order differences} & & \text{2nd order differences} & & \text{3rd order differences} \\
u_1 & & & & & & \\
 & \longrightarrow & v_1 & & & & \\
u_2 & & & \longrightarrow & w_1 & & \\
 & \longrightarrow & v_2 & & & \longrightarrow & \\
u_3 & & & \longrightarrow & w_2 & & \\
 & \longrightarrow & v_3 & & & & \\
u_4 & & & & & & \\
\cdots & & & & & & \\
\cdots & & & & & & \\
\cdots & & & & & & \\
u_{n-2} & & & & & & \\
 & \longrightarrow & v_{n-2} & & & & \\
u_{n-1} & & & \longrightarrow & w_{n-2} & & \\
 & \longrightarrow & v_{n-1} & & & & \\
u_n & & & & & &
\end{array}$$

Result 1:

If the kth order differences are each equal to a constant λ, the nth term of the series (1) is given by

$$u_n = A_1n^k + A_2n^{k-1} + A_3n^{k-2} + \ldots + A_{n+1} \qquad \text{— (2)}$$

where, $A_1, A_2, \ldots A_{k+1}$ are constants.

$A_1, A_2, \ldots A_{n+1}$ can be determined by setting $n = 1, 2, 3, \ldots, k + 1$ in (2) and solving the linear equations in $A_1, A_2, \ldots A_{k+1}$

For example, consider the series

$$2 + 7 + 16 + 29 + 46 + \ldots\ldots\ldots$$

$$\begin{array}{ccccc}
2 & & & & \\
 & \rightarrow & 5 & & \\
7 & & & \rightarrow & 4 \\
 & \rightarrow & 9 & & \\
16 & & & \rightarrow & 4 \\
 & \rightarrow & 13 & & \\
29 & & & \rightarrow & 4 \\
 & \rightarrow & 17 & & \\
46 & & & & \\
 & & \cdots\cdots\cdots & & \\
 & & \cdots\cdots\cdots & &
\end{array}$$

We note that second order differences of the given series are each equals 4. Therefore, the nth term of the series will be the form

$$u_n = An^2 + Bn + C$$

$$n = 1 \rightarrow 2 = A + B + C$$

$$n = 2 \rightarrow 7 = 4A + 2B + C$$

$$n = 3 \rightarrow 16 = 9A + 3B + C$$

Solving the three equations above,

We get $A = 2, B = -1, C = 1$

$\Rightarrow$ $u_n = 2n^2 - n + 1$

Sum of the first n terms of the series is given by

$$S_n = \sum_{r=1}^{n} u_r$$

$$= 2\Sigma n^2 - \Sigma n + n$$

$$= \frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n$$

$$= \frac{2n(n+1)(2n+1) - 3n(n+1) + 6n}{6}$$

$$= \frac{(4n^2 + 3n + 5)n}{6}$$

Result 2:

If the kth order differences are in GP with common ratio r, the nth term of the given series will be of the form

$$u_n = Ar^n + B_1n^{k-1} + B_2n^{k-2} + \ldots\ldots + B_k \qquad \text{— (3)}$$

where $A, B_1, B_2, \ldots\ldots, B_k$ are constants.

These constants can be determined by setting $n = 1, 2, 3, \ldots, k + 1$ in (3) and solving the linear equations in $A, B_1, B_2, \ldots, B_k$.

For example, consider the series

$$9 + 16 + 29 + 54 + 103 + \ldots\ldots$$

$$\begin{array}{ccccc}
9 & & & & \\
 & \rightarrow & 7 & & \\
16 & & & \rightarrow & 6 \\
 & \rightarrow & 13 & & \\
29 & & & \rightarrow & 12 \\
 & \rightarrow & 25 & & \\
54 & & & \rightarrow & 24 \\
 & \rightarrow & 49 & & \\
103 & & & & \\
 & & \cdots\cdots\cdots & & \\
 & & \cdots\cdots\cdots & &
\end{array}$$

We note that second order differences are in GP with common ratio 2. Therefore, the nth term of the series will be of the form

$$u_n = A \times 2^n + Bn + C$$
$$n = 1 \rightarrow 9 = 2A + B + C$$
$$n = 2 \rightarrow 16 = 4A + 2B + C$$
$$n = 3 \rightarrow 29 = 8A + 3B + C$$

Solving , we get

$$A = 3, B = 1, C = 2$$

Hence, the nth term of the given series is

$$u_n = 3 \times 2^n + n + 2$$

Sum of the first n terms of the series is given by

$$S_n = \sum_{r=1}^{n} u_r = 3\Sigma 2^n + \Sigma n + 2n$$
$$= 3 \times 2(2^n - 1) + \frac{n(n+1)}{2} + 2n$$

CONCEPT STRANDS

Concept Strand 21

Find the nth term and sum of first n terms of the series 6 + 13 + 22 + 33 +……

Solution

Let u_n represent the nth term of the series.
We form table of differences

$$\begin{array}{ccccc} 6 & & & & \\ & \rightarrow & 7 & & \\ 13 & & & \rightarrow & 2 \\ & \rightarrow & 9 & & \\ 22 & & & \rightarrow & 2 \\ & \rightarrow & 11 & & \\ 33 & & & & \end{array}$$

Since 2nd order difference S are each equal to 2.
nth term is of the form $An^2 + Bn + C$
Putting n = 1, 2, 3 in the above, we get

$$A + B + C = 6$$
$$4A + 2B + C = 13$$
$$9A + 3B + C = 22$$

Solving, we get A = 1, B = 4, C = 1

Hence, nth term of the above series is given by $u_n = n^2 + 4n + 1$

If S_n represents the sum of first n terms then,

$$S_n = \sum_{n=1}^{n} u_n = \sum_{n=1}^{n} (n^2 + 4n + 1)$$
$$= \sum_{k=1}^{n} (k^2 + 4k + 1) = \sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + n$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + n$$
$$= \frac{n(2n^2 + 15n + 19)}{6}$$

Concept Strand 22

Find an expression for the nth term of the series 5 + 8 + 17 + 44 +…. and hence find an expression for sum of first n terms

Solution

Let u_n represent the nth term
We form the table of differences

$$\begin{array}{ccc} 5 & & \\ & \rightarrow & 3 \\ 8 & & \\ & \rightarrow & 9 \\ 17 & & \\ & \rightarrow & 27 \\ 44 & & \end{array}$$

1st differences are in GP with common ratio = 3

nth term = $A \times 3^n + B$

By putting n = 1, 2 and solving, we get

$$A = \frac{1}{2}, B = \frac{7}{2}$$

Or, the nth term of the given series = $\frac{3^n + 7}{2}$

If S_n represents the sum of the first n terms,

$$S_n = \sum_{n=1}^{n} u_n = \frac{1}{2}\sum_{n=1}^{n} 3^n + 7$$
$$= \frac{1}{2}\sum_{n=1}^{n} 3^n + \frac{7}{2}\sum_{n=1}^{n} 1$$
$$= \frac{1}{2}(3 + 3^2 + 3^3 + 3^4 + ... + 3^n) + \frac{7n}{2}$$
$$= \frac{1}{2} \times \frac{3(3^n - 1)}{2} + \frac{7}{2}n = \frac{3(3^n - 1)}{4} + \frac{7}{2}n.$$

PARTIAL FRACTIONS

We conclude this unit by a brief discussion on partial fractions decomposition of rational functions.

Expressions of the type $\frac{P(x)}{Q(x)}$ where P(x) and Q(x) are polynomials in x are called rational functions. If the degree of P(x) is less than that of Q(x), $\frac{P(x)}{Q(x)}$ is called a proper fraction.

Otherwise, $\frac{P(x)}{Q(x)}$ is called an improper fraction.

For example, $\frac{3x+5}{(x-2)(2x+1)}$ is a proper fraction while $\frac{4x^3+6x^2+10x-1}{(x^2+2x-6)}$ is an improper fraction.

By division, an improper fraction can be expressed as the sum of a polynomial and a proper fraction. In the above case, we can write $\frac{4x^3+6x^2+10x-1}{(x^2+2x-6)}$ as $4x-2+\frac{38x-13}{(x^2+2x-6)}$

Expressing a given proper fraction as the sum of two or more simple fractions whose denominators are the factors of the denominator of the given fraction is known as decomposition or resolution into partial fractions. We illustrate the procedure by working out the following examples.

CONCEPT STRANDS

Concept Strand 23

Resolve into partial fractions: $\frac{x^2+3x-1}{(x-1)(x+2)(2x+1)}$.

Solution

Choose constants A, B and C such that

$$\frac{x^2+3x-1}{(x-1)(x+2)(2x+1)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{2x+1} \quad \text{— (1)}$$

Multiplying both sides of (1) by (x – 1) (x + 2) (2x + 1),

$$x^2 + 3x - 1 = A(x + 2)(2x + 1) + B(x - 1)(2x + 1) + C(x - 1)(x + 2) \quad \text{— (2)}$$

(2) is an identity, i.e., the coefficients of like powers of x on the left hand side and the right hand side are equal. Equating coefficients of x^2, x and the constant term on both sides we get 3 equations in A, B, C and we then solve these linear equations and obtain the values of A, B and C.

OR, Putting x = 1 on both sides of (2),

$$1 + 3 - 1 = A \times 3 \times 3 \Rightarrow A = \frac{3}{9} = \frac{1}{3}$$

Putting x = –2 on both sides of (2),

$$4 - 6 - 1 = B \times -3 \times -3 \Rightarrow B = -\frac{1}{3}$$

Putting $x = -\frac{1}{2}$ on both sides of (2),

$$\frac{1}{4} - \frac{3}{2} - 1 = C \times \left(-\frac{3}{2}\right)\left(\frac{3}{2}\right) \Rightarrow C = 1$$

Substituting these values of A, B and C in (1), we obtain the partial fractions decomposition as

$$\frac{x^2+3x-1}{(x-1)(x+2)(2x+1)} = \frac{1/3}{x-1} - \frac{1/3}{x+2} + \frac{1}{2x+1}$$

Remark

In general, if the proper fraction to be resolved into partial fractions is $\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)......(a_nx+b_n)}$ where no two factors of the denominator are equal, we write the given proper fraction as $\frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + ... + \frac{A_n}{a_nx+b_n}$.

To find the values $A_1, A_2, ..., A_n$, use the method given in the worked example above.

Concept Strand 24

Resolve into partial fractions, $\frac{x^3-10x^2+29x-14}{(x-2)^3(3x-2)}$

Solution

Note that, in this example, the factor (x – 2) is repeated three times and therefore we assume the given fraction as $\frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{(3x-2)}$

Proceeding similar to example 4.21, we get

$$x^3 - 10x^2 + 29x - 14 = A(x-2)^2(3x-2) + B(x-2)(3x-2) + C(3x-2) + D(x-2)^3$$

Putting $x = 2$ in the above, $12 = 4C \Rightarrow C = 3$.

Putting $x = \frac{2}{3}$, $\frac{8}{27} - \frac{40}{9} + \frac{58}{3} - 14 = D\left(-\frac{64}{27}\right)$

$\Rightarrow \quad D = -\frac{1}{2}$.

Equating the coefficients of x^3 on both sides,

$1 = 3A + D = 3A - \frac{1}{2} \Rightarrow A = \frac{1}{2}$

Putting $x = 0$, $-14 = -8A + 4B - 2C - 8D \Rightarrow B = -2$.

Therefore, the partial fractions decomposition of the given proper fraction is

$$\frac{1}{2(x-2)} - \frac{2}{(x-2)^2} + \frac{3}{(x-2)^3} - \frac{1}{2(3x-2)}$$

Remark

In general, if the given proper fraction contains $(ax + b)^n$ as a factor, corresponding to this factor, take the partial fractions as

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_2x + b_2)^2} + \frac{A_3}{(a_2x + b_2)^3} ..... + \frac{A_n}{(a_nx + b_n)^n}$$

Concept Strand 25

Resolve into partial fractions: $\frac{(2x-1)}{(x-1)(x^2+1)}$.

Solution

In this case, the denominator contains the quadratic factor $(x^2 + 1)$ which cannot be resolved into real linear factors. We proceed as follows for resolution into partial fractions.

Let $\frac{(2x-1)}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx + C}{x^2+1}$

Multiplying both sides by $(x - 1)(x^2 + 1)$.

$2x - 1 = A(x^2 + 1) + (Bx + C)(x - 1)$

Put $x = 1$ in the above, $1 = 2A \Rightarrow A = \frac{1}{2}$

Equating coefficient of x^2 on both side, $0 = A + B \Rightarrow B = -\frac{1}{2}$.

Putting $x = 0$, $-1 = A - C \Rightarrow C = A + 1 = \frac{3}{2}$.

Therefore, the partial fractions decomposition is $\frac{1/2}{(x-1)} + \frac{-1/2\,x + 3/2}{x^2+1}$.

Concept Strand 26

Resolve into partial fractions $\frac{1}{(x^2+1)^2(x-3)}$

Solution

We assume fractions as $\frac{A}{(x-3)} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}$

Proceeding as in earlier examples, we get

$1 = A(x^2 + 1)^2 + (Bx + C)(x - 3)(x^2 + 1) + (Dx + E)(x - 3)$

Putting $x = 3$ $\quad 1 = 100\,A \Rightarrow \frac{1}{100}$

Equating coefficients of x^4 on both sides, $0 = A + B \Rightarrow B = -\frac{1}{100}$

Equating coefficient of x^2, on both sides, $0 = 2A - 3C + B + D$

$\Rightarrow \quad 3C - D = 2A + B = \frac{1}{100}$

Equating coefficient of x^3 on both sides, $0 = C - 3B$

$\Rightarrow \quad C = 3B = -\frac{3}{100}$.

$\therefore \quad D = 3C - \frac{1}{100} = -\frac{9}{100} - \frac{1}{100} = -\frac{1}{10}$

Putting $x = 0$ $\quad 1 = A - 3C - 3E$

$\Rightarrow \quad 3E = A - 3C - 1 = \frac{1}{100} + \frac{9}{100} - 1 = \frac{-90}{100}$

$\therefore \quad E = \frac{-3}{10}$

The partial fractions decomposition is

$$\frac{1/100}{(x-3)} + \frac{-x/100 - 3/100}{(x^2+1)} + \frac{-1/10\,x - 3/10}{(x^2+1)^2}.$$

SUMMARY

1. Arithmetic Series

General form of a series in AP is given by $a + (a + d) + (a + 2d) + \ldots$, where a is the first term and d is the common difference

(i) nth term $t_n = a + (n - 1)d$

(ii) Sum of the first n terms $S_n = \frac{n}{2}[2a + (n-1)d]$

$= \frac{n}{2}[\text{first term} + \text{nth term}]$

(iii) If the same number k ($\neq 0$) is added to (or is subtracted from) every term of a series in AP, the new series thus obtained is still an AP

(iv) If one multiplies every term of a series in AP by a number k ($\neq 0$), the new series thus obtained is still an AP

(v) Any three terms of an AP are generally taken as $(a - d), a, (a + d)$

Any four terms are taken as $(a - 3d), (a - d), (a + d), (a + 3d)$. Here the common difference is 2d.
Likewise, any five terms are written as $(a - 2d), (a - d), a, (a + d), (a + 2d)$

2. Geometric Series

General form of a series in GP is $a + ar + ar^2 + ar^3 + \ldots\ldots\ldots$ ($r \neq 1$), where a is the first term and r is the common ratio

(i) nth term $t_n = a\,r^{n-1}$

(ii) Sum of the first n terms $S_n = \frac{a(1-r^n)}{(1-r)}$

$= \frac{a(r^n-1)}{(r-1)}$

(iii) When $|r| < 1$, sum of the GP $a + ar + ar^2 + ar^3 + \ldots\infty$, i.e., the sum of the infinite GP is given by $S_\infty = \frac{a}{1-r}$

3. Arithmetico-geometric Series

General form of an arithmetico-geometric series is $a + (a + d)r + (a + 2d)r^2 + \ldots$ where a is the first term, r is the common ratio and d is the common difference.

(i) To find the nth term of an Arithmetico-geometric series

(a) Find the nth term of the corresponding arithmetic series which is $a + (n - 1)d$

(b) Find the nth term of the corresponding geometric series which is $r^{(n-1)}$

(c) nth term of the AGP $= [a + (n - 1)d]\,r^{n-1}$

(ii) Sum of an infinite AGP can be found when $|r| < 1$ (here r is the common ratio of the GP). $S_\infty = a + (a + d)r + (a + 2d)r^2 + \ldots\ldots\ldots\infty$

$= \frac{a}{(1-r)} + \frac{dr}{(1-r)^2}$

4. Harmonic Series

General form of a series in HP is $\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$ when $a_1, a_2, a_3 \ldots a_n$ are in AP

5. Means

(i) If a and b are two positive numbers,

Arithmetic Mean (AM) of a and b $= \frac{(a+b)}{2} = A$

Geometric Mean (GM) of a and b $= \sqrt{ab} = G$

Harmonic Mean (HM) of a and b $= \frac{2ab}{(a+b)} = H$

$A > G > H$ and $G^2 = AH$

(ii) Suppose $a_1, a_2, a_3, \ldots\ldots, a_n$ are n positive numbers.
The arithmetic mean (AM) of these numbers is defined as $\frac{(a_1 + a_2 + a_3 + \ldots\ldots + a_n)}{n}$.
The geometric mean (GM) of these numbers is defined as $(a_1 a_2 a_3 \ldots\ldots a_n)^{1/n}$.
The harmonic mean (HM) of these numbers is defined as $\frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)}$

$$\frac{(a_1 + a_2 + a_3 + \ldots\ldots + a_n)}{n} \geq (a_1 a_2 \ldots\ldots a_n)^{1/n} \geq \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)}$$

6. Summation

(i) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(ii) $\sum_{r=1}^{n} ka_r = k\sum_{r=1}^{n} a_r$ (k is a constant)

(iii) $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$

(iv) $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(v) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

CONCEPT CONNECTORS

Connector 1: The sum of three numbers in AP is 15 and the sum of their squares is 83. Find the numbers.

Solution: The three numbers in AP may be assumed as a – d, a, a + d.

Given $(a - d) + a + (a + d) = 15$ and

$(a - d)^2 + a^2 + (a + d)^2 = 83$

The first equation gives $a = 5$, and on substituting for a in the second equation we get $d = \pm 2$.

The numbers are 3, 5, 7.

Note

We may also assume the numbers as **a, a + d, a + 2d**. However, in such a case the solution of the algebraic equation becomes cumbersome.

Any 4 terms of an AP generally taken as a – 3d, a – d, a + d, a + 3d

Any 5 terms a – 2d, a – d, a, a + d, a + 2d.

Connector 2: The pth, qth and rth terms of an AP are x, y, z respectively.

Show that $x(q - r) + y(r - p) + z(p - q) = 0$.

Solution: We have $x = a + (p - 1)d$, $y = a + (q - 1)d$, $z = a + (r - 1)d$

$\Rightarrow\quad x(q - r) + y(r - p) + z(p - q) = [a + (p - 1)d](q - r) + \text{similar terms}$

$\quad = a(q - r + r - p + p - q) + d\{(p - 1)(q - r) + (q - 1)(r - p) + (r - 1)(p - q)\} = 0$

Connector 3: Sum of the first n terms of an AP is given by the formula $(n^2 + 7n)$. Obtain the ratio of its 8th and 17th terms.

Solution: 8th term = Sum of the first 8 terms – Sum of the first 7 terms

$= (8^2 + 7 \times 8) - (7^2 + 7 \times 7) = 22$

Similarly, 17th term = 40, and $\dfrac{\text{8th term}}{\text{17th term}} = \dfrac{20}{40} = \dfrac{11}{20}$. .

Connector 4: Find an AP in which the sum of the first n terms is $3n^2$.

Solution: First term $= 3 \times 1^2 = 3$

Sum of the first two terms $= 3 \times 2^2 = 12$

Therefore, 2nd term $= 12 - 3 = 9 \Rightarrow$ Common difference of the AP = 6.

The AP is 3 + 9 + 15 + …..

Connector 5: If the roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ are equal, show that a, b, c, are in AP

Solution: Observe that x = 1 satisfies the equation. Therefore, x = 1 is a root of the equation.

If β is the other root, $\beta = \alpha$ (given)

Note that the sum of the roots $= -\dfrac{(c - a)}{(b - c)}$

Therefore, $-\dfrac{(c - a)}{(b - c)} = 2$

Or $2b - 2c = a - c \Rightarrow 2b = a + c \Rightarrow$ a, b, c are in AP

Alternate Method

$\Delta = 0$

$\Rightarrow\quad (a + c - 2b)^2 = 0$

$\Rightarrow\quad a + c = 2b.$

Connector 6: If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, prove that bc^2, ca^2, ab^2 are in AP.

Solution: Let α, β represent the roots of the equation $ax^2 + bx + c = 0$

$$\text{Given} = \alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$\Rightarrow \quad -\frac{b}{a} = \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c^2}{a^2}}$$

$$\Rightarrow \quad \frac{-b}{a} \times \frac{c^2}{a^2} = \frac{b^2 - 2ac}{a^2}$$

$\Rightarrow \quad -bc^2 = a\,(b^2 - 2ac)$

$\Rightarrow \quad 2ca^2 = ab^2 + bc^2 \Rightarrow bc^2, ca^2, ab^2$ are in AP

Connector 7: How many 3 digit positive integers are there which leave the remainder 2 when divided by 9?

Solution: Any number leaving remainder 2 when divided by 9 is of the form $9n + 2$ where n is a positive integer. The last 3 digit number of this form is 992.

$9n + 2 = 992$ giving $n = 110$.

The last 2 digit number of this form is 92

$9n + 2 = 92$ gives $n = 10$

Therefore, there are 10 numbers between 1 and 100 which leave remainder 2 when divided by 9. Hence, the total number of 3 digit positive numbers leaving remainder 2 divided by 9 is

$110 - 10 = 100$.

Alternate Method

The numbers are 101, 110, ...992

$992 = 101 + (n - 1)9 \Rightarrow n = 100$.

Connector 8: If $a^2 (b + c)$, $b^2 (c + a)$, $c^2 (a + b)$ are in AP, show that either a, b, c are in AP, or $ab + bc + ca = 0$.

Solution: $2b^2 (c + a) = a^2 (b + c) + c^2 (a + b)$ or $b^2 (c + a) - a^2 (b + c)$

$= c^2 (a + b) - b^2 (c + a)$

$\Rightarrow \quad c(b^2 - a^2) + ab\,(b - a) = a\,(c^2 - b^2) + bc\,(c - b)$

$\Rightarrow \quad (b - a)\,\{c(b + a) + ab\} = (c - b)\,\{a(c + b) + bc\}$

$\Rightarrow \quad (ab + bc + ca)\,(b - a - c + b) = 0 \Rightarrow ab + bc + ca = 0$ or $2b = a + c$

Connector 9: The sum of three numbers in GP is 124 and their product is 8000. Find the numbers.

Solution: We assume the three numbers in GP as: a/r, a, and ar.

Given that $a/r + a + ar = 124$ and $(a/r) \times a \times ar = 8000$

The second equation yields $a = 20$.

Putting this value of 'a' in the first equation, we solve for 'r' and get $r = 5$ or $1/5$.

The numbers are therefore 4, 20, 100.

Note: Any 4 terms of a GP is generally taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

Connector 10: If a, b, c, d are in GP, show that $(ab + bc + cd)^2 = (a^2 + ac + c^2)(b^2 + bd + d^2)$.

Solution: Since the numbers are in GP,

Let $b = ar$, $c = ar^2$ and $d = ar^3$

$(ab + bc + cd)^2 = a^4r^2(1 + r^2 + r^4)^2; (a^2 + ac + c^2)(b^2 + bd + d^2)$

$$= a^2[1 + r^2 + r^4][a^2r^2(1 + r^2 + r^4)].$$

This proves the result.

Connector 11: Prove that the difference between the sum of the first n odd terms and the sum of the first n even terms of the GP of which the first term is 1 and common ratio is 3, is $\frac{1}{4}(3^{2n} - 1)$

Solution: The GP is $1 + 3 + 3^2 + 3^3 + \ldots$

Sum of the first n odd terms $= 1 + 3^2 + 3^4 + 3^6 + \ldots + (3^2)^{n-1}$

$$= +\frac{(3^2)^n - 1}{(3^2 - 1)} = \frac{1}{8}(3^{2n} - 1).$$

Sum of the first n even terms $= 3 + 3^3 + 3^5 + \ldots + 3^{2n-1}$

$$= \frac{3\left[(3^2)^n - 1\right]}{(3^2 - 1)} = \frac{3}{8}(3^{2n} - 1)$$

Difference between the two sums is $\left(\frac{3}{8} - \frac{1}{8}\right)\left(3^{2n} - 1\right) = \frac{1}{4}\left(3^{2n} - 1\right)$

Connector 12: Find the sum of the first n terms of the series: $4 + 44 + 444 + 4444 + \ldots$

Solution: Let S_n represent the sum.

We have $S_n = 4(1+11+111+1111+\ldots$ n terms)

$$= \frac{4}{9}(9 + 99 + 999 + 9999 + \ldots)$$

$$= \frac{4}{9}\{(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots + (10^n - 1)\}$$

$$= \frac{4}{9}[(10 + 10^2 + \ldots + 10^n) - n] = \frac{4}{81}[10^{n+1} - (9n + 10)]$$

Connector 13: x_1, x_2 be the roots of the equation $x^2 - 3x + p = 0$ and x_3, x_4 be the roots of the equation $x^2 - 12x + q = 0$. If x_1, x_2, x_3, x_4 in that order form an increasing GP, show that $\frac{q + p}{q - p} = \frac{17}{15}$.

Solution: We assume $x_1 = a, x_2 = ar, x_3 = ar^2, x_4 = ar^3$.

We have, $x_1 + x_2 = a(1 + r) = 3$

$x_1 x_2 = a^2r = p$

and $x_3 + x_4 = ar^2(1+r) = 12$

$x_3 x_4 = a^2r^5 = q$

The first and third equations give $r = 2$ ($r = -2$ is not admissible)

The second and fourth equations give $\frac{q}{p} = 16$

Therefore, $\frac{q + p}{q - p} = \frac{17}{15}$.

Connector 14: If a, b, c are in GP show that $(a^2 + b^2)(b^2 + c^2) = (ab + bc)^2$.

Solution: a, b, c are in GP $\Rightarrow b^2 = ac$.

LHS $= (a^2 + ac)(ac + c^2) = ac(a + c)^2$

RHS $= b^2(a + c)^2 = ac(a + c)^2$

Thus proved.

Connector 15: Solve the system of equations $2x^4 = y^4 + z^4$, $xyz = 8$, given that $\log_y x$, $\log_z y$, $\log_x z$ form a GP

Solution: Note that $(\log_y x)(\log_z y)(\log_x z) = 1$ — (1)

Since the three numbers are in GP

$$(\log_z y)^2 = (\log_y x)(\log_x z)$$

$$= \frac{1}{\log_z y} \text{ [from (1)]}$$

$\Rightarrow (\log_z y)^3 = 1 \Rightarrow y = z$

We have $2x^4 = y^4 + z^4 = 2y^4 \Rightarrow x = y$

$\therefore$ $x = y = z$ and since $xyz = 8$, the solution is $x = y = z = 2$.

Connector 16: If the positive numbers a, b, c are in HP, prove that $\dfrac{a+b}{2a-b} + \dfrac{c+b}{2c-b} > 4$

Solution: Since a, b, c are in HP, $b = \dfrac{2ac}{a+c}$

Substituting for b in the given expression we get,

$$\frac{a+b}{2a-b} + \frac{c+b}{2c-b} = \frac{a + \dfrac{2ac}{a+c}}{2a - \dfrac{2ac}{a+c}} + \frac{c + \dfrac{2ac}{a+c}}{2c - \dfrac{2ac}{a+c}} = \frac{a^2 + 3ac}{2a^2} + \frac{c^2 + 3ac}{2c^2}$$

$$= \frac{a+3c}{2a} + \frac{c+3a}{2c} = \frac{3(a^2 + c^2) + 2ac}{2ac}$$

$$= \frac{3}{2}\left(\frac{a}{c} + \frac{c}{a}\right) + 1 > \frac{3}{2} \times 2 + 1 = 4$$

(Since $x + \dfrac{1}{x} > 2$ when x is positive)

Connector 17: P,Q, R are defined such that $P = a^2b + ab^2 - a^2c - ac^2$, $Q = b^2c + bc^2 - a^2b - ab^2$, $R = a^2c + ac^2 - b^2c - bc^2$ and $a > b > c > 0$. Show that if $Px^2 + Qx + R = 0$ has equal roots, then a, b, c are in HP

Solution: Since $Px^2 + Qx + R = 0$ has equal roots, $Q^2 - 4PR = 0$

Note that $P + Q + R = 0$. This gives $Q = -(P + R) \Rightarrow Q^2 = (P + R)^2$

$\therefore$ $(P + R)^2 - 4PR = 0$ or $(P - R) = 0$ or $P = R$

$a^2b + ab^2 - a^2c - ac^2 = a^2c + ac^2 - b^2c - bc^2$

$2ac(a + c) = a^2b + ab^2 + b^2c + bc^2 \Rightarrow 2ac(a + c) = ab(a + b) + bc(b + c)$, rearranging

$\Rightarrow ac(a + c) - ab(a + b) = bc(b + c) - ac(a + c)$

$$\Rightarrow \frac{a}{c} = \frac{b^2 + cb - a^2 - ac}{c^2 + ac - b^2 - ab} = \frac{(b-a)(b+a+c)}{(c-b)(c+b+a)} = \frac{b-a}{c-b} \Rightarrow b = \frac{2ac}{(a+c)}$$

Or a, b, c are in HP

Connector 18: If a, b, c form a geometric sequence and x, y represent the arithmetic means between a, b and b, c respectively, show that $\frac{a}{x}+\frac{c}{y}=2$ and $\frac{1}{x}+\frac{1}{y}=\frac{2}{b}$.

Solution: We have, using the given information $x=\frac{1}{2}(a+b)$, $y=\frac{1}{2}(b+c)$ and $b^2 = ac$.

Now, on substituting for x and y and using the third result, $\frac{a}{x}+\frac{c}{y}=2$.

Similarly, $\frac{1}{x}+\frac{1}{y}=\frac{2}{b}$.

Connector 19: The harmonic mean and the geometric mean of two positive numbers are in the ratio 4 : 5. Show that the two numbers are in the ratio 4 : 1.

Solution: Let the numbers be a, b

We have HM h (say) $=\frac{2ab}{a+b}$ and GM g (say) $=\sqrt{ab}$

$$\frac{h}{g}=\frac{4}{5}\text{ (given)} \Rightarrow 5h = 4g \Rightarrow 5\times\frac{2ab}{a+b}=4\times\sqrt{ab} \Rightarrow 5\sqrt{ab}=2(a+b)$$

Squaring, $25ab = 4(a+b)^2 \Rightarrow 4a^2 + 4b^2 - 17ab = 0$

$$4\left(\frac{a}{b}\right)^2 - 17\left(\frac{a}{b}\right)+4=0$$

$$\frac{a}{b}=4 \text{ or } \frac{1}{4} \Rightarrow \text{Ratio of the numbers} = 4:1.$$

Connector 20: If S denotes the sum of the first n terms of a GP, P their product and R the sum of its reciprocals, show that $P^2=\left(\frac{S}{R}\right)^n$.

Solution: Let the series be $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$

$$S=\frac{a(1-r^n)}{(1-r)}$$

$$P = a\times ar\times ar^2\times\ldots\times ar^{n-1} = a^n r^{(1+2+3\ldots+n-1)} = a^n r^{\frac{n(n-1)}{2}}$$

$$R=\frac{1}{a}+\frac{1}{ar}+\frac{1}{ar^2}+\ldots\frac{1}{ar^{n-1}}=\frac{\frac{1}{a}\left[1-\left(\frac{1}{r}\right)^n\right]}{\left(1-\frac{1}{r}\right)}$$

$$=\frac{r^n-1}{r^n(r-1)}\times\left(\frac{r}{a}\right)=\left(\frac{1-r^n}{1-r}\right)\frac{1}{ar^{n-1}}$$

$\therefore$ We have $\frac{S}{R}=a^2r^{n-1}$ and $P^2=a^{2n}r^{n(n-1)}$

Clearly, $P^2=\left(\frac{S}{R}\right)^n$

Connector 21: If a, b, c are in AP; b, c, d are in GP; c, d, e are in HP show that a, c, e are in GP

Solution: We have

$$2b = a + c \qquad — (1)$$

$$c^2 = bd \qquad — (2)$$

$$d = \frac{2ce}{c + e} \qquad — (3)$$

Using (1) and (3) in (2), $c^2 = \dfrac{b \times 2ce}{c + e} = \dfrac{(a + c)ce}{c + e}$

Or$c(c + e) = e(a + c) \Rightarrow c^2 = ae \Rightarrow$ a, c, e are in GP

Connector 22: If a, b, c are real numbers such that $2a^2, (1 - c^2), 2b^2$ are in AP, show that $-\frac{1}{2} < (ab + bc + ca) < 1$.

Solution: Since $2a^2, 1 - c^2, 2b^2$ are in AP, $2(1-c^2) = 2a^2 + 2b^2 \Rightarrow a^2 + b^2 + c^2 = 1$.

We have $(a+ b+ c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 1 + 2 (ab + bc + ca)$

Since $(a + b + c)^2 > 0, \Rightarrow 1 + 2 (ab + bc + ca) > 0 \Rightarrow ab + bc + ca > -\frac{1}{2}$ — (1)

Again,

$$a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$$

$$= \frac{1}{2}\left\{(b - c)^2 + (c - a)^2 + (a - b)^2\right\} > 0$$

But, $a^2 + b^2 + c^2 = 1$.

$\therefore$ We get $1 - (ab + bc + ca) > 0$

$\Rightarrow$ $ab + bc + ca < 1$ — (2)

(1) and (2) give the required result.

Connector 23: If p be the first of n AMs between two positive numbers and q be the first of n HMs between the same two numbers, prove that the value of q can never be between p and $\dfrac{p(n + 1)^2}{(n - 1)^2}$.

Solution: Let x and y be the two positive numbers and $a_1, a_2, \ldots, a_n$ the n AMs between x and y and let $h_1, h_2, \ldots, h_n$ the n HMs between x and y.

Then, $p = a_1 = \dfrac{(nx + y)}{(n + 1)}$ and $q = h_1 = \dfrac{(n + 1)xy}{(ny + x)}$.

$$\frac{p}{q} = 1 + \left[\frac{n}{(n + 1)^2}\right]\left[\frac{(x - y)^2}{xy}\right] > 1, \text{ as x and y} > 0, \text{ giving } q < p. \text{ — (1)}$$

It is clear that $\left[\dfrac{(n + 1)}{(n - 1)}\right] > 1$, n being a natural number or $p\left[\dfrac{(n + 1)}{(n - 1)}\right]^2 > p$, being > 0 — (2)

From (1) and (2), $q < p < \dfrac{p(n + 1)^2}{(n - 1)^2}$ and the result follows.

Connector 24: Find the sum of the first n terms the series $1^2 + 3^2 + 5^2 + \ldots$.

Solution: The nth term of the series is easily obtained as $(2n - 1)^2$. We therefore require $\sum_{k=1}^{n}(2k - 1)^2$

If it is denoted by S_n,

$$S_n = 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= \left[\frac{4n(n+1)(2n+1)}{6}\right] - \left[\frac{4n(n+1)}{2}\right] + n = \left[\frac{n(4n^2-1)}{3}\right], \text{ on simplification.}$$

Connector 25: Find the sum of the first n terms of the series $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots\ldots$

Solution: nth term of the series $= \dfrac{1}{(2n-1)(2n+1)}$

If U_n represents the nth term, $U_n = \left[\dfrac{1}{2(2n-1)}\right] - \left[\dfrac{1}{2(2n+1)}\right]$

Putting n=1, 2, 3,.... successively, we get

$$U_1 = \frac{1}{2} - \frac{1}{(2 \times 3)}$$

$$U_2 = \frac{1}{(2 \times 3)} - \frac{1}{(2 \times 5)} \text{ and so on and finally,}$$

$$U_n = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}$$

Addition gives, Sum $= \dfrac{1}{2} - \left[\dfrac{1}{2(2n+1)}\right] = \dfrac{n}{(2n+1)}$

Connector 26: If x, y, a, b, c are > 0, find the maximum value of xy when $a^2x^4 + b^2y^2 = c^6$.

Solution: We have the result; If the sum of two positive quantities is a constant, their product is maximum when the two numbers are equal.

That is, $(a^2 x^4)$ $(b^2 y^4)$ is maximum when $a^2 x^4 = b^2 y^4 = \dfrac{c^6}{2}$

$\therefore$ Maximum value of $(a^2 x^4)$ $(b^2 y^4)$ is $\left(\dfrac{c^6}{2}\right)\left(\dfrac{c^6}{2}\right) = \dfrac{c^{12}}{4}$

OR Maximum value of $x^4 y^4$ is $= \left(\dfrac{c^{12}}{4a^2b^2}\right)$

OR Maximum value of $xy = \left(\dfrac{c^{12}}{4a^2b^2}\right)^{\frac{1}{4}} = \dfrac{c^3}{\sqrt{2ab}}$

Connector 27: Prove that $a + b + c \geq \dfrac{2bc}{b+c} + \dfrac{2ca}{c+a} + \dfrac{2ab}{a+b}$.

Solution: We use the inequality AM≥HM

$$\frac{b+c}{2} \geq \frac{2}{\frac{1}{b}+\frac{1}{c}} = \frac{2bc}{b+c} \quad \frac{c+a}{2} \geq \frac{2ca}{c+a} \text{ and } \frac{a+b}{2} \geq \frac{2ab}{a+b}$$

Addition gives

$$a + b + c \geq \frac{2bc}{b + c} + \frac{2ca}{c + a} + \frac{2ab}{a + b}$$

Connector 28 If n is a positive integer, show that $(a + b)^n \left(\frac{1}{a^n} + \frac{1}{b^n}\right) \geq 2^{n+1}$.

Solution: Applying the AM≥ GM inequality, $\frac{a + b}{2} \geq \sqrt{ab}$

$$\Rightarrow \quad (a + b)^n \geq 2^n (ab)^{n/2} \qquad \text{— (1)}$$

$$\text{Again, } \frac{\frac{1}{a^n} + \frac{1}{b^n}}{2} \geq \sqrt{\frac{1}{a^n b^n}} = \frac{1}{a^{n/2} b^{n/2}} \qquad \text{— (2)}$$

$$\text{or } \frac{1}{a^n} + \frac{1}{b^n} \geq \frac{2}{a^{n/2} b^{n/2}}$$

Multiplication of (1) and (2) proves the result.

Connector 29: If a, b, c are distinct and positive, show that $(2a + 3b + 5c)^{10} > 10^{10} a^2 b^3 c^5$

Solution: Consider the 10 numbers a, a, b, b, b, c, c, c, c, c

Applying AM > GM inequality to the above 10 numbers

$$\frac{(2a + 3b + 5c)}{10} > \left(a^2 b^3 c^5\right)^{\frac{1}{10}}$$

or $(2a + 3b + 5c)^{10} > 10^{10} a^2 b^3 c^5$

Connector 30: If m and n are distinct positive integers, show that $\left(\frac{mn + 1}{m + 1}\right)^{m+1} > n^m$

Solution: Consider the (m + 1) positive integers n, n, n, …(m numbers) and 1

Applying the AM > GM inequality to the above (m + 1) numbers,

$$\frac{mn + 1}{m + 1} > (n_m \times 1)^{\frac{1}{m+1}} = n^{\frac{m}{m+1}}$$

$$\Rightarrow \quad \left(\frac{mn + 1}{m + 1}\right)^{m+1} > n^m$$

Connector 31: Let x, y, z > 0. Prove that

(i) $(x + y)(y + z)(z + x) \geq 8xyz$

(ii) $\frac{y^2 + z^2}{yz} + \frac{z^2 + x^2}{zx} + \frac{x^2 + y^2}{xy} \geq 6$

Solution: (i) Considering three pairs of numbers (x, y), (y, z) and (z, x) and using the AM ≥ GM inequality

$$\frac{x + y}{2} > \sqrt{xy}, \quad \frac{y + z}{2} > \sqrt{yz}, \text{and } \frac{z + x}{2} > \sqrt{zx}$$

Multiplying the inequalities, $(x + y)(y + z)(z + x) > 8xyz$.

(ii) Considering three pairs of numbers (x^2, y^2), (y^2, z^2) and (z^2, x^2) and again using the AM > GM inequality

$$\frac{y^2 + z^2}{2} > \sqrt{y^2 z^2} = yz \Rightarrow \frac{y^2 + z^2}{yz} \geq 2$$

Similarly, $\dfrac{z^2+x^2}{2} > zx \Rightarrow \dfrac{z^2+x^2}{zx} \geq 2$ and $\dfrac{x^2+y^2}{2} > xy \Rightarrow \dfrac{x^2+y^2}{xy} \geq 2$

Addition gives $\dfrac{y^2+z^2}{yz} + \dfrac{z^2+x^2}{zx} + \dfrac{x^2+y^2}{xy} \geq 6$

Connector 32 If ℓ, m, n are distinct positive integers and $x > 0$ and not equal to 1, prove that $\ell x^{m-n} + mx^{n-\ell} + nx^{\ell-m} > (\ell + m + n)$.

Solution: Consider the set of positive numbers

x^{m-n}, x^{m-n} …….. repeated ℓ times

$x^{n-\ell}, x^{n-\ell}$………... repeated m times

and $x^{\ell-m}, x^{\ell-m}$………. repeated n times

Applying AM > GM for these $(\ell + m + n)$ numbers,

$$\frac{\ell x^{m-n} + mx^{n-\ell} + nx^{\ell-m}}{(\ell+m+n)} > \left[(x^{m-n})^{\ell} \times (x^{n-\ell})^{m} \times (x^{\ell-m})^{n}\right]^{\frac{1}{(\ell+m+n)}} > 1$$

since the quantity inside the square bracket = $x^0 = 1$

or $\ell x^{m-n} + mx^{n-\ell} + nx^{\ell-m} > (\ell + m + n)$.

Connector 33: Show that if a, b, c are real, $(bc + ca + ab)^2 > 3abc\,(a + b + c)$

Solution: $(bc + ca + ab)^2 = \Sigma b^2c^2 + 2abc\,(a + b + c)$

$$= \frac{1}{2}\left\{b^2\left(c^2+a^2\right) + c^2\left(a^2+b^2\right) + a^2\left(b^2+c^2\right)\right\} + 2abc\,(a+b+c) \qquad \text{— (1)}$$

We have, $\dfrac{c^2+a^2}{2} > ac$; $\dfrac{a^2+b^2}{2} > ab$; $\dfrac{b^2+c^2}{2} > bc$

[applying AM > GM for the pairs (c^2, a^2); (a^2, b^2); (b^2, c^2)]

Substituting in (1)

$$(bc + ca + ab)^2 > \frac{1}{2}\,[b^2 \times 2ca + c^2 \times 2ab + a^2 \times 2bc] + 2abc\,(a + b + c)$$

$$> \frac{1}{2} \times 2abc\,(a + b + c) + 2abc\,(a + b + c) > 3abc\,(a + b + c)$$

Connector 34: ABC is a right angled triangle right angled at B. If AB = c, BC = a, CA = b, show that $b^3 > a^3 + c^3$.

Solution: We have $b^2 = a^2 + c^2$ — (1)

Case (i) $a > c$

From (1) $b^3 = a^2b + c^2b > a^2 \times a + c^2 \times a$, (since $b > a$)

$= a^3 + c^2a > a^3 + c^2 \times c$, (since $c < a$)

$= a^3 + c^3$.

Case (ii) $a < c$,

From (1) $b^3 = a^2b + c^2b > a^2 \times c + c^2 \times c$, (since $c < b$)

$> a^3 + c^3$, (since $a < c$).

Thus proved.

Observation

In a right angled triangle, if a, b, c are the sides and b is the hypotenuse then we have $b < a + c$, $b^2 = a^2 + c^2$, and $b^3 > a^3 + c^3$.

Connector 35: Show that $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \geq (a_1b_1 + a_2b_2 + a_3b_3)^2$. The equality holds good, when $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

Solution: Consider the quadratic function

$$f(x) = \sum_{r=1}^{3}(a_r x - b_r)^2 = (a_1x - b_1)^2 + (a_2x - b_2)^2 + (a_3x - b_3)^2$$

Note that $f(x) \geq 0$ for all real x.

Now, $f(x) = (a_1^2 + a_2^2 + a_3^2)\, x^2 - 2x\,(a_1b_1 + a_2b_2 + a_3b_3) + (b_1^2 + b_2^2 + b_3^2)$

Since $f(x) > 0$ for all real x, and $a_1^2 + a_2^2 + a_3^2 > 0$, the discriminant of the quadratic must be ≤ 0

$\Rightarrow$ $4\,(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq 4\,(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$

$\Rightarrow$ $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \geq (a_1b_1 + a_2b_2 + a_3b_3)^2$

Clearly, when $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$, the equality holds good.

TOPIC GRIP

Subjective Questions

1. Find

(i) the 22nd term of the progression 4, 9, 14, 19……..

(ii) the first negative term of the progression 19, $18\frac{1}{5}$, $17\frac{2}{5}$ ……

(iii) the 11th term of the series 5 + 10 + 20 + ……..

(iv) the GP if 4th and 9th terms are 54 and 13122 respectively.

2. Find the sum of the

(i) first 61 terms of the series 2 + 5 + 8 + …..

(ii) first 25 terms of the AP $\frac{1}{9}, \frac{2}{9}, \frac{3}{9}$ ….

(iii) the series 101 + 99 + 97 + …. + 47

(iv) the series 2 + 6 + 18 + …… + 4374

(v) product $6^{\frac{1}{2}}.6^{\frac{1}{4}}.6^{\frac{1}{8}}..........\infty$

(vi) series $\frac{1}{7}+\frac{2}{7^2}+\frac{1}{7^3}+\frac{2}{7^4}+........\infty$

3. The ratio of the sum of the first n terms of two APs is 2n –15: 2n – 1. Find the ratio of the 13th terms of the two APs.

4. For what values of the parameter k are the three values of x such that $5^{1+x} + 5^{1-x}$, $\frac{k}{2}$, $25^x + 25^{-x}$ are three successive terms of an AP

5. If p, q, r are in GP and the equations $px^2 + 2qx + r = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then show that $\frac{d}{p}, \frac{e}{q}, \frac{f}{r}$ are in AP

6. The sum of an infinitely decreasing GP is $\frac{10}{3}$ The sum of the cubes of its terms is $\frac{1000}{63}$. Find the sum of the 4th powers of the terms of the GP.

7. Prove that the three successive terms of a GP will form the sides of a triangle if the common ratio r satisfies the inequality $\frac{\sqrt{5}-1}{2} < r < 1$.

8. The sum of three numbers in a GP is 42. If the first two numbers are increased by 2 and the third is decreased by 4 then the resulting numbers form an AP Find the numbers.

9. If the equations $x^2 - px + q = 0$, $x^2 - rx + s = 0$ have a common root which is the harmonic mean between their other two roots, prove that

(i) $(q - s)^2 = (p - r)(qr - ps)$

(ii) $ps(3q + s) = qr(3s + q)$

10. Prove that $(\underbrace{33...3}_{\text{n digits}})^2 + \underbrace{22...2}_{\text{n digits}} = \underbrace{111...1}_{\text{2n digits}}$. Hence find the value of $9(11111)^2 + 2(11111)$.

Straight Objective Type Questions

***Directions:* This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

11. The three angles of a triangle are in AP If the largest angle is twice the smallest angle, then the largest angle is

(a) 40° (b) 60° (c) 100° (d) 80°

12. If $\frac{3}{14}$, a, $\frac{2}{21}$ are in GP, then a =

(a) $\frac{1}{6}$ (b) $\frac{1}{8}$ (c) $\frac{2}{3}$ (d) $\frac{1}{7}$

13. If $\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$, then a, b, c are in

(a) AP (b) GP (c) HP (d) AGP

14. Sum to n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots\ldots$ is equal to

(a) 2^n-1 (b) $1-2^{-n}$ (c) $n+2^{-n}-1$ (d) $2^{n}-1$

15. If P, Q, R be the AM, GM, HM respectively between numbers a and b, then P – Q is

(a) $\frac{a-b}{a}$ (b) $\frac{a+b}{2}$ (c) $\frac{2ab}{a+b}$ (d) $\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{2}}\right)^2$

Assertion–Reason Type Questions

***Directions:* Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

16. **Statement 1**

If a_1, a_2, a_3, a_4 are four numbers such that $a_2a_3 - a_1a_4$ is positive then a_1, a_2, a_3, a_4 are in AP

and

Statement 2

If a_1, a_2, a_3, a_4 are in AP then $a_2a_3 - a_1a_4$ is positive.

17. **Statement 1**

The sum of any number of terms from the beginning of the series

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots\ldots$, cannot exceed 2.

and

Statement 2

The sum to infinity of a GP whose first term is a and common ratio is r where, $|r| < 1$ is finite and is equal to $\frac{a}{1-r}$.

18. Statement 1

If b + c, c + a and a + b are in HP then $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in HP

and

Statement 2

If each term of a sequence in a GP is squared, the resulting series is a GP

19. Statement 1

If a^2, b^2, c^2 are in GP, then a, b, c are in GP

and

Statement 2

If $y^2 = xz$, then x, y, z are in GP

20. Statement 1

The sum of the first 10 terms of the series

$\frac{1}{2}-\frac{3}{2}+\frac{9}{2}-\frac{27}{2}+....$ is $\frac{-1}{8}\times\left(3^{10}-1\right)$

and

Statement 2

In a GP $a + ar + ar^2 + \ldots$, if r is negative, then $a_{r+1} < a_r, r = 1, 2, \ldots.$

Linked Comprehension Type Questions

Directions: **This section contains 2 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

ABCD is a square. The mid-points of the sides are joined to form a square $A_1 B_1 C_1 D_1$.The same way we form another square $A_2 B_2 C_2 D_2$ from $A_1 B_1 C_1 D_1$. This process of forming squares is continued.

21. The areas X_1, X_2 .... of the squares form a

(a) decreasing GP (b) increasing GP (c) AP (d) HP

22. The diagonals d_1, d_2... of the squares form a

(a) AP (b) GP (c) HP (d) AGP

23. The ratio of the areas of the circum circle and the area of the square is

(a) depends on the radius of the circle.
(b) is independent of the radius of circle
(c) vary inversely with respect to one another
(d) cannot say anything definitely

Passage II

A public maidan of circular shape, of radius 560 metres has a stage at the centre of the maidan. Important functions are held here. Lamp posts have been erected along the circumference of concentric circles with center at the centre of the maidan. The first circle is at a distance of 40 metres from the centre of the maidan and the outer circles are spaced 40 metres apart. The lamp posts are arranged on these concentric circles, such that they are along diameters of the circles, the angle between any two consecutive diameters being 30°.

24. Cabling to connect the lamp posts is to be done to supply power from the stage to all the posts. The cabling is to be done along the circumference of each circular arrangement. The length of the cable required to connect all the lamp posts is

(a) 26960 m (b) 26400 m (c) 53920 m (d) 52800 m

25. If the cost of cable is Rs 40 per metre, the amount to be spent on the purchase of cables, if 5% wastage is incurred

(a) Rs 11, 32, 320 (b) Rs 11, 08, 800 (c) Rs 22, 64, 640 (d) Rs 22, 17, 600

26. The distances between 2 consecutive lampposts on each of these circles along the circumference form a.....when we proceed from the stage to the boundary of the ground

(a) an increasing AP (b) decreasing AP (c) AGP (d) HP

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers of which ONE OR MORE answers will be correct.**

27. If $\cos(\theta - \alpha)$, $\cos\theta$, $\cos(\theta + \alpha)$ are in HP, then $\cos\theta\sec\frac{\alpha}{2}$ is equal to

(a) -1 (b) $-\sqrt{2}$ (c) $\sqrt{2}$ (d) 2

28. If a, b, c, d are in GP then

(a) $a + b, b + c, c + d$ are in GP

(b) $ax^2 + c$ is a factor of $ax^3 + bx^2 + cx + d$

(c) $a^2 + b^2 + c^2, ab + bc + cd, b^2 + c^2 + d^2$ are in GP

(d) $ax + c$ is a factor of $ax^3 + bx^2 + cx + d$

29. Three positive numbers x, y, z are in AP then, x^2, y^2, z^2 are in HP if x, y, z satisfies

(a) $x^2 = y^2 = \frac{z^2}{2}$ (b) $2y^2 + xz = 0$ (c) $x = y = z$ (d) $y^2 = xz$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

30.

Column I	Column II
(a) If $a^x = b^y = c^z$ where a, b, c are in GP, then x, y, z are in	(p) AP
(b) Three distinct numbers a, b, c satisfying $\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$ are in	(q) GP
(c) If x, y, z (all > 1) are in GP, then $\frac{1}{1+\log x}, \frac{1}{1+\log y}, \frac{1}{1+\log z}$ are in	(r) HP
(d) Three consecutive terms $\frac{1}{1+\sqrt{x}}, \frac{1}{1-x}, \frac{1}{1-\sqrt{x}}$ of a sequence are in	(s) AGP

IIT ASSIGNMENT EXERCISE

Straight Objective Type Questions

***Directions:* This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

31. Number of natural numbers between 250 and 800 which are divisible by 7 is

(a) 80 (b) 79 (c) 63 (d) 70

32. If seven times the seventh term of an AP is equal to eleven times its eleventh term, its 18th term is

(a) – 23 (b) –15 (c) 16 (d) 0

33. If a, b, c are in AP, then b + c, c + a, a + b are in

(a) AP (b) HP (c) GP (d) AGP

34. The 9th and 7th terms of a GP are respectively 256 and 64. Then, the common ratio of the GP is

(a) 6 (b) 4 (c) – 3 (d) 2

35. If the fifth term of a GP is 2, then the product of its first 9 terms is

(a) 64 (b) 512 (c) 128 (d) 256

36. If four numbers are in geometric progression, then their logarithms will be in

(a) GP (b) AP (c) HP (d) AGP

37. If $a = 1 + r + r^2 + \ldots\ldots\ldots\ldots \infty$ and $|r| < 1$, then r is

(a) $\frac{a}{a-1}$ (b) $\frac{a-1}{a}$ (c) $\frac{a+1}{a}$ (d) $\frac{2a}{a-1}$

38. $16 \times 16^{1/2} \times 16^{1/4} \times 16^{1/8} \times \ldots\ldots\ldots\ldots\ldots\ldots \infty$ equals

(a) 256 (b) 16 (c) 4 (d) 527

39. Each term of an infinite geometric progression is twice the sum of all the terms which follows it. The common ratio of this GP is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{2}{3}$

40. If A, G, H are the AM, GM and HM between any two distinct positive real numbers, then out of the following 4 Statements

(i) $G^2 = AH$ (ii) $A > G > H$ (iii) $A = G = H$ (iv) $G = A^2H$

(a) Only (i) and (iii) are correct (b) Only (ii) is correct

(c) Only (i) and (ii) are correct (d) All are correct

41. The AM and GM of 2 numbers are 20 and 4 respectively. Then, their HM ÷ GM is equal to

(a) $\frac{1}{5}$ (b) 5 (c) 20 (d) 80

42. The sum to infinity of the arithmeticogeometric series $1 + 3\times\frac{1}{2} + 5\times\frac{1}{4} + 7\times\frac{1}{8} + 9\times\frac{1}{16} + \ldots\ldots\ldots\text{to } \infty$ is

(a) $\frac{1}{4}$ (b) $\frac{5}{2}$ (c) 6 (d) 7

43. The 20th term of the series $2 \times 4 + 4 \times 6 + 6 \times 8 + \ldots\ldots\ldots\ldots$ is

(a) 1600 (b) 1680 (c) 420 (d) 840

44. If the sum of the first n terms of a series is $5n^2 + 2n$, then the second term is

(a) 7 (b) 17 (c) 24 (d) 42

45. If the roots of the equation $x^3 - 12x^2 + 39x - 28 = 0$ are in AP, then the common difference of this AP is

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4

46. If third term of a GP is 8, then the product of first five terms of the GP is

(a) 8^5 (b) 8^{10} (c) 8^2 (d) 8^3

47. If the second number of three numbers in an increasing GP is doubled, we get an AP, then

(a) common ratio of the GP is 1 or –1

(b) common ratio of the GP and common difference of AP are equal.

(c) common difference of AP is equal to both first term and last term.

(d) common ratio of the GP is $2 + \sqrt{3}$.

48. If $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the geometric mean between a and b, then the value of n is

(a) $\dfrac{1}{2}$ (b) $\dfrac{-1}{2}$ (c) $\dfrac{1}{3}$ (d) $\dfrac{-1}{3}$

49. If p, q, r are in GP and p, r, q are in AP then p^2, q^2, pq are in

(a) AP (b) GP (c) AGP (d) HP

50. The recurring decimal $0.55\dot{5}$ is equal to the rational number

(a) $\dfrac{9}{5}$ (b) $\dfrac{5}{55}$ (c) $\dfrac{5}{9}$ (d) $\dfrac{8}{11}$

51. Two AM's A_1 and A_2, two GM's G_1 and G_2 and two HM's H_1 and H_2 are inserted between two positive numbers. Then, $H_1^{-1} + H_2^{-1} =$

(a) $A_1^{-1} + A_2^{-1}$ (b) $G_1^{-1} + G_2^{-1}$ (c) $A_1H_1 + A_2H_2$ (d) $\dfrac{A_1 + A_2}{G_1G_2}$

52. If a, b, c are in AP and a^2, b^2, c^2 are in HP, then

(a) $a = b = c$ (b) $b = 3a + c$ (c) $b^2 = \sqrt{\dfrac{ac}{8}}$ (d) $b^2 = \sqrt{\dfrac{ac}{6}}$

53. If the ratio of the sum of the first m terms and the first n terms of an AP is $m^2 : n^2$, the ratio of its mth and nth term will be

(a) $2m - 1 : 2n - 1$ (b) $m : n$ (c) $2m + 1 : 2n + 1$ (d) None of these

54. If $\dfrac{1}{\log_a x} + \dfrac{1}{\log_c x} = \dfrac{2}{\log_b x}$, then a, b, c are in

(a) AP (b) GP (c) HP (d) None of these

55. $a_1, a_2, \ldots, a_n$ are in AP such that $a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} = 300$. Then, the sum of the first 16 terms of this AP is

(a) 600 (b) 900 (c) 800 (d) 1000

56. The ratio of the sum of the first three terms of a GP to the sum of its first six terms is 125 : 152. The common ratio of the GP is

(a) $\dfrac{1}{5}$ (b) $\dfrac{2}{5}$ (c) $\dfrac{3}{5}$ (d) $\dfrac{4}{5}$

57. If $\sqrt{b}+\sqrt{c},\ \sqrt{c}+\sqrt{a},\ \sqrt{a}+\sqrt{b}$ are in HP, then a, b, c are in

(a) AP (b) HP (c) GP (d) AGP

58. If the harmonic mean of two numbers is 4 and their arithmetic mean A and geometric mean G satisfy the relation $2A + G^2 = 27$, the two numbers are

(a) 6, 3 (b) 5, 4 (c) 5, – 2.5 (d) –3, 1

59. Suppose a, b, c are in AP and a^2, b^2, c^2 are in GP If $a < b < c$ and $a + b + c = \frac{3}{2}$, then the value of 'c' is

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2\sqrt{3}}$ (c) $\frac{1}{2}+\frac{1}{\sqrt{3}}$ (d) $\frac{1}{2}+\frac{1}{\sqrt{2}}$

60. x_1,x_2 ; y_1, y_2 ; z_1, z_2 are the two AM's, two GM's, two HM's respectively between a and b. G_1 is the GM between y_1 and y_2, G_2 is the GM between z_1 and z_2; A_1 is the AM between x_1 and x_2 and A_2 is the AM between z_1 and z_2. Then $A_1 : A_2 =$

(a) $G_1 : G_2$ (b) $G_2 : G_1$ (c) $G_1^2 : G_2^2$ (d) $G_2^2 : G_1^2$

61. Sum of the series $1 + 2.2 + 3.2^2 + 4.\ 2^3 + \ldots + 100.2^{99}$ is

(a) $100.2^{100} + 1$ (b) $99.2^{100} + 1$ (c) $99.2^{99} - 1$ (d) $100.2^{100} - 1$

62. If the harmonic mean between two positive numbers is to their GM as 12 : 13, the numbers are in the ratio

(a) 12 : 13 (b) $\frac{1}{12}:\frac{1}{13}$ (c) 4 : 9 (d)2:3

63. If a, b, c are in AP; p, q, r are in HP and ap, bq, cr are in GP, then $\frac{p}{r}+\frac{r}{p}$ is equal to

(a) $\frac{a}{c}-\frac{c}{a}$ (b) $\frac{a}{c}+\frac{c}{a}$ (c) $\frac{b}{q}-\frac{q}{b}$ (d) $\frac{b}{q}+\frac{q}{p}$

64. If a, b, x, y are positive numbers such that $a + b + x + y = 5$, then the maximum possible value of $(a + b)(x + y)$ is

(a) $\frac{5}{2}$ (b) 5 (c) $\frac{25}{4}$ (d) $\frac{25}{2}$

65. If a, b, c are in AP as well as in GP, then

(a) $a = b \neq c$ (b) $a \neq b = c$ (c) $a \neq b \neq c$ (d) $a = b = c$

66. If $a^2 + 16b^2 + 49c^2 - 4ab - 7ac - 28bc = 0$ then a, b, c are in

(a) AP (b) GP (c) HP (d) None of these

67. The sum of the numbers less than 450 which are divisible by 2, 3 and 5 is

(a) 3210 (b) 1200 (c) 3150 (d) 2350

68. The sum $\underbrace{0.555\ldots5}_{\text{n decimals}}+\underbrace{0.535353\ldots53}_{\text{n decimals}}$ where, n is even, is equal to

(a) $12\left(1-\frac{1}{10^n}\right)$ (b) $\frac{12}{11}\left(1-\frac{1}{10^n}\right)$

(c) $\frac{12}{11}\left(1-\frac{1}{10^{2n}}\right)$ (d) $12\left(1-\left(\frac{1}{10}\right)^{n/2}\right)$

69. In a GP of positive terms, any term other than the first two terms is equal to the sum of the two preceeding terms. Then, the common ratio of the GP is

(a) $\frac{\sqrt{5}+1}{2}$ (b) $\frac{-1+\sqrt{5}}{2}$ (c) $\frac{-1-\sqrt{5}}{2}$ (d) $\frac{1}{2}$

70. If the sides a, b, c of Δ ABC are in GP where log a – log 2b, log 2b – log 3c, log 3c – log a are in AP, then the sides a, b, c are in the ratio

(a) 4 : 6: 9 (b) 9: 4: 6 (c) 9: 6: 4 (d) 3: 6: 8

71. If $x = \sum_{n=0}^{\infty} 3^{\frac{-n}{2}}$ and $y = \sum_{n=0}^{\infty} 2^{\frac{-n}{2}}$, then

(a) $x^2 + y^2 + 2xy + 4x + 2 = 0$ (b) $4x^2 - y^2 - 12x + 4y + 4 = 0$

(c) $(x + y)^2 + 2xy = 0$ (d) $x^2 + y^2 + 6x + 4x + 4 = 0$

72. Let $\sum_{r=1}^{n} r^4 = f(n)$. Then $\sum_{r=1}^{n} (2r-1)^4$ is equal to

(a) $f(2n) - 16 f(n)$ for all $n \in N$ (b) $f(2n) - 16f\left(\frac{n-1}{2}\right)$ when n is odd

(c) $f(n) - 16 f\left(\frac{n}{2}\right)$ when n is even (d) None of these

73. If $<a_n>$ and $<b_n>$ are two sequences given by

$a_1 = 2^{1/2} + 3^{1/2}$
$a_2 = 2^{1/4} + 3^{1/4}$
$a_3 = 2^{1/8} + 3^{1/8}$
....

and

$b_1 = 2^{1/2} - 3^{1/2}$
$b_2 = 2^{1/4} - 3^{1/4}$
$b_3 = 2^{1/8} - 3^{1/8}$
....

Then $a_1 a_2 a_3 \ldots a_n$ equals

(a) $b_1 b_2 \ldots b_n$ (b) $\frac{1}{b_1 b_2 \ldots b_n}$ (c) $\frac{1}{b_n}$ (d) $\frac{-1}{b_n}$

74. If a, $a_1, a_2, \ldots, a_{10}$, b are in AP and a, $h_1, h_2, \ldots, h_{10}$, b are in HP such that $a_1 + a_2 + \ldots + a_{10} = 25$ and $\frac{1}{h_1} + \frac{1}{h_2} + \ldots + \frac{1}{h_{10}} = \frac{25}{6}$, then a and b are

(a) 1, 2 (b) 2, 3 (c) 2.5, 3.5 (d) 3, 4

75. $\frac{4}{19} + \frac{44}{19^2} + \frac{444}{19^3} + \ldots\ldots\infty =$

(a) $\frac{38}{81}$ (b) $\frac{4}{19}$ (c) $\frac{36}{171}$ (d) None of these

76. If $\frac{x^2+3}{(x-3)(x^2+x-2)} = \frac{k_1}{x-3} + \frac{k_2}{x+2} + \frac{k_3}{x-1}$, then k_1, k_2, k_3 are respectively

(a) $\frac{2}{3}, \frac{11}{10}, \frac{6}{5}$ (b) $\frac{-2}{3}, \frac{7}{15}, \frac{6}{5}$

(c) $\frac{3}{2}, \frac{15}{7}, \frac{-10}{11}$ (d) $\frac{2}{3}, \frac{5}{7}, \frac{6}{5}$

77. If x, y, a, b, c are > 0, the maximum value of xy when $a^2x^4 + b^2y^4 = c^6$ is

(a) $\frac{c^3}{2ab}$ (b) $\sqrt{\frac{c^3}{2ab}}$ (c) $\frac{c^3}{\sqrt{2ab}}$ (d) $\frac{c}{2\sqrt{ab}}$

78. If a, b, c are distinct and positive, and $(2a + 3b + 5c)^{10} > \lambda\, a^2 b^3 c^5$, then $\lambda =$

(a) 10 (b) 10^{10} (c) $10^{1/10}$ (d) $\frac{1}{10}$

79. The sum of the first 50 terms of the series: 6 + 9 + 16 + 27 + … is

(a) 82375 (b) 82039 (c) 39450 (d) 83250

80. The sum to infinity of the series $\frac{1}{5.9}+\frac{1}{9.13}+\frac{1}{13.17}+\ldots$

(a) $\frac{1}{5}$ (b) $\frac{1}{20}$ (c) $\frac{1}{10}$ (d) 1

81. The numbers 36, 32, 28, 24, 20 is a

(a) AGP (b) GP (c) HP (d) AP

82. The sum of all 2 digit numbers greater than 19 is

(a) 4760 (b) 4880 (c) 3980 (d) 4580

83. If three consecutive terms in an AP are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, then $\frac{b-c}{a-b} =$

(a) $\frac{a}{c}$ (b) $\frac{b}{a}$ (c) a (d) $\frac{c}{a}$

84. The second term of an infinite GP is $\frac{3}{4}$ and the sum to infinity of the GP is 4. Then, its first term and common ratio are

(a) $\frac{3}{4}$, 1 (b) 3, $\frac{1}{4}$ (c) 3, $\frac{3}{4}$ (d) $\frac{9}{4}, \frac{3}{4}$

85. 1 + 4 + 9 + 16 + 25 + ……… + 400 =

(a) 2780 (b) 2870 (c) 4280 (d) 2650

86. If a and b two AMs between c and d, then a – c =

(a) $\frac{b-c}{2}$ (b) b – c (c) $\frac{d-c}{2}$ (d) $\frac{b-c}{3}$

87. The 20th term of the sequence $\sqrt{5}, \sqrt{20}, \sqrt{45}, \ldots\ldots$ is

(a) $20\sqrt{3}$ (b) $20\sqrt{5}$ (c) $10\sqrt{5}$ (d) $10\sqrt{3}$

88. If m, n, s, t are in GP, then $\frac{1}{m}, \frac{1}{n}, \frac{1}{s}, \frac{1}{t}$ are in

(a) HP (b) AGP (c) AP (d) GP

89. Product 3 numbers in GP is 216 and the sum of their squares is 189. Then, one of the numbers of this set of numbers is

(a) 16 (b) 12 (c) 20 (d) 27

90. If a, b, c are in A. P, then $7^a, 7^b, 7^c$ are in

(a) HP (b) AP (c) AGP (d) GP

91. If the value of 1 + 2 + 3 + ……+ n is 55, then the value of $1^3 + 2^3 + 3^3 + \ldots + n^3$ is

(a) 165 (b) 385 (c) 3025 (d) 555

92. The sum of the first n terms of a series is $\frac{n(n+1)(n+2)}{3}$. The 12th term of the series is

(a) 182 (b) 122 (c) 109 (d) 156

93. If 1 + 6 + 11 +------+ x = 148, then x =

(a) 30 (b) 32 (c) 34 (d) 36

94. If $ax^2 + 2bx + c = 0$ and $a_1x^2 + 2b_1x + c_1 = 0$ have a common root, and if $\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1}$ are in AP then a_1, b_1, c_1 are in

(a) GP (b) AP (c) HP (d) AGP

95. If a, b, c are in GP then $\log_a x, \log_b x, \log_c x$ are in

(a) AP (b) GP (c) HP (d) AGP

96. The sum of an infinite GP with common ratio r (r < 1) is 4. The sum of the infinite GP obtained by squaring the terms of this GP is $\frac{16}{3}$. Then the first term and common ratio of the given GP is

(a) $-2, \frac{1}{2}$ (b) $-2, \frac{1}{3}$ (c) $2, \frac{1}{2}$ (d) $3, \frac{-1}{2}$

97. If A, G, H respectively represent AM, GM and HM of n positive numbers, then

(a) A < G > H always.
(b) A = G = H, if the numbers are equal
(c) A > H always and A = G for equal numbers
(d) A > G always, and G = H for equal numbers

98. If a, b, c are in AP and a, mb, c are in GP, then a, m^2b, c are in

(a) AP (b) GP (c) HP (d) AGP

99. If $t_1 = 2$, $t_2 = 2$ and $t_{n-1} = t_n + 1$ for $n \geq 3$ then $t_5 =$

(a) 1 (b) 0 (c) –1 (d) 5

100. If a, b, c are in AP, then $2^{ax+1}, 2^{bx+1}, 2^{cx+1}$, where $x \neq 0$ are in

(a) AP (b) GP only if x > 0 (c) GP only if x < 0 (d) GP for all $x \neq 0$

101. If the (a + 1)th, 7th and (b +1)th terms of an AP are in GP with a, 6, b being in HP, then 4th term of this AP is

(a) $-\frac{7}{2}$ (b) $\frac{7}{2}$ (c) 0 (d) 3

102. $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+........\infty$ is

(a) $\frac{16}{35}$ (b) $\frac{11}{8}$ (c) $\frac{35}{16}$ (d) $\frac{7}{16}$

103. If $a_r > 0$, $r \in N$ and $a_1 a_2 ....a_{2n}$ are in AP, then, $\frac{a_1+a_{2n}}{\sqrt{a_1}+\sqrt{a_2}}+\frac{a_2+a_{2n-1}}{\sqrt{a_2}+\sqrt{a_3}}+\frac{a_3+a_{2n-2}}{\sqrt{a_3}+\sqrt{a_4}}+.....\ ....+\frac{a_n+a_{n+1}}{\sqrt{a_n}+\sqrt{a_{n+1}}}$ is equal to

(a) n – 1 (b) $\frac{n(a_1+a_{2n})}{\sqrt{a_1}+\sqrt{a_{n+1}}}$ (c) $\frac{n-1}{\sqrt{an}+\sqrt{an+1}}$ (d) None of these

104. If $a_1, a_2, a_3 \ldots$ are in HP and $f(k) = \sum_{r=1}^{n} a_r - a_k$, then $\frac{a_1}{f(1)}, \frac{a_2}{f(2)}, \frac{a_3}{f(3)}, \ldots \frac{a_n}{f(n)}$ are in

(a) AP (b) GP (c) HP (d) AGP

105. a, b, c are 3 consecutive terms of an AP If tan a, tan b, tan c (b ≠ multiple of $\frac{\pi}{2}$) are also in AP, then

(a) tan b = 2 tan a (b) tan a × tan c = tan b (c) tan a = tan b = tan c = 0 (d) tan a = tan b = tan c

106. If α and β are the roots of $x^2 - 3x + a = 0$ and γ, δ that of $x^2 - 12x + b = 0$ and α, β, γ, δ (in that order) form an increasing GP, then

(a) a = 2, b = 32 (b) a =3, b = 12 (c) a = 4, b = 16 (d) a = 12, b = 3

107. If $|x| < 1$, then the sum of the infinite series $\left[x+\frac{1}{2}\right]+\left[x^2+x\frac{1}{2}+\left(\frac{1}{2}\right)^2\right]+\left[x^3+x^2\frac{1}{2}+x\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3\right]+\ldots\infty$ is

(a) $\frac{1}{2}+x$ (b) $\frac{2-x}{1-x}$

(c) $\frac{1+x}{1-x}$ (d) $\frac{2+x}{2-x}$

108. The arithmetic mean between two positive numbers a and b where (a > b) is twice their geometric mean. Then $\frac{a}{b}$

(a) $2+\sqrt{3}$ (b) $7+4\sqrt{3}$ (c) $2-\sqrt{3}$ (d) $7-4\sqrt{3}$

109. If a, b, c are three distinct numbers in AP and b – a, c – b, a are in GP, then a: b: c =

(a) 2: 3: 1 (b) $\frac{1}{2}:\frac{1}{3}:1$ (c) 1: 2: 3 (d) 2: 1: 6

110. If a, b, c are real and $(bc + ca + ab)^2 > k(a + b + c)$, then k equals

(a) abc (b) 3 (c) 1 (d) 3abc

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

111. Statement 1

There is no AP with non-zero common difference whose terms are all prime numbers

and

Statement 2

Any natural number(>1) can be uniquely factored as product of prime numbers.

112. Statement 1

S_r is the sum of an infinite G. P with 1st term r and common ratio $\frac{1}{r+1}$. Then $S_r - r$ depends on r.

and

Statement 2

Sum of an infinite geometric series the common ratio is numerically less than 1 exists only if ÷ r÷ < 1

113. Statement 1

If $a^2 + 2bc$, $b^2 + 2ac$, $c^2 + 2ab$ are in AP then, b – c, c – a, a – b are in HP

and

Statement 2

If $a_1, a_2, \ldots a_n$ are in AP then $\frac{a_1+k}{h}, \frac{a_2+k}{h}, \ldots\ldots$ in AP

Linked Comprehension Type Questions

***Directions:* This section contains 1 paragraph. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Scientists have established that the number of radio-active disintegrations per unit time that occurs in a given sample of a naturally radioactive isotope is directly proportional to that of the isotope present. The more nuclei present, the more will disintegrate per unit of time. The period in which half of the radio-active sample disintegrates is known as its 'half life'. It is the time required for half of the given sample of isotope to disintegrate.

114. Suppose we start with x grams of a sample of radioactive material. y grams of the sample remains after a period of n life cycle. The relationship between x and y is given by

(a) $y = nx + a$ (b) $y = kx\ 0 < k < 1$

(c) $y = \frac{k}{2}$ (d) $y = \frac{x}{2^n}$ where $n = 0, 1, 2.....$

115. The half-life of a certain isotope is 2 years. A sample of 3200 kg of the isotope is considered. If x kg of the sample remains after 12 years x equals

(a) 275 kg (b) 100 kg (c) 50 kg (d) 0

116. 12.5% of an isotope remains after 8 years from the initial stage. After 8 more years, the quantity present will be

(a) $\frac{25}{16}\%$ (b) 6.25% (c) 0% (d) 3.125%

Multiple Correct Objective Type Questions

***Directions:* Each question in this section has four suggested answers of which ONE OR MORE answers will be correct.**

117. In the nth row of the triangle

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 4 & & 9 & & \\ & 16 & & 25 & & 36 & \\ 49 & & 64 & & 81 & & 100 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

(a) last term = sum of the cubes of the first n natural numbers

(b) first term $= \frac{\left(n^2 - n + 2\right)^2}{4}$

(c) positive square root of the first term is one more than the sum of first $n - 1$ natural numbers

(d) sum of the elements in the nth row $= \frac{n\left(n^2+2\right)\left(3n^2+1\right)}{12}$

118. If $\sum_{k=1}^{n}\left(\sum_{m=1}^{k} m^2\right) = an^4 + bn^3 + cn^2 + dn + e$ then

(a) $a = \frac{1}{12}$ (b) $e = 0$ (c) $c = \frac{5}{12}$ (d) $d = \frac{1}{6}$

119. The number of ordered triplets (p, q, r) where $1 \leq p, q, r \leq 10$ and p, q, r are natural numbers such that $2^p + 3^q + 5^r$ is a multiple of 4 is

(a) 50 if $p = 1$ (b) 450 if $p \neq 1$ (c) 500 in all (d) 75 if $p = 1$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

120. Consider $Q = ax^2 + bx + c$ where a, b, c are real and distinct. Also, $Q = 0$ has real roots

Column 1	Column II
(a) If a, b, c are in AP, then the least value of $\left\|\frac{d}{b}\right\|$ (where d is the common difference), is	(p) $-\sec^2 \frac{\pi}{4}$
(b) If a, b, c are in GP, then b cannot be equal to	(q) $\cos\frac{\pi}{3}$
(c) If a, b, c are in AP with common difference 'd', and if zero is a root of $Q = 0$, then $\left\|\frac{d}{a}\right\|$ equals	(r) $\sin\frac{\pi}{3}$
(d) If a, b, c are in AP, and b, c, a are in GP, then the common ratio r equals	(s) $\sin\frac{\pi}{2}$

ADDITIONAL PRACTICE EXERCISE

Subjective Questions

121. If the first three of four given numbers are in AP and the last three are in HP, prove that the four numbers are proportional.

122. If $S_1, S_2, \ldots, S_{2k}$ are the sums of the first n terms of 2k Arithmetic Progressions whose first terms are 1, 2, 3, … , 2k and whose common differences are 1, 3, 5, 7,…, (4k – 1), show that

(i) $S_1 + S_2 + S_3 + \ldots + S_{2k} = kn(1 + 2nk)$

(ii) $S_1 - S_2 + S_3 - \ldots - = -n^2k$

123. Find the sum of the first n terms of the series: 3 + 7 + 23 + 87 + …

124. If $a_1, a_2, a_3, \ldots, a_n$ are in AP where $a_k \neq \frac{(2k-1)\pi}{2}$ and d is the common difference, find

(i) $\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + + \sec a_{n-1} \sec a_n$

(ii) $\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\ldots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)$

Also show that

$$\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\ldots+\frac{1}{a_na_1}=\frac{2}{(a_1+a_n)}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$$

125. Obtain the sum of the first 50 terms of the series $\frac{5}{7}+\frac{19}{7^2}+\frac{69}{7^3}+\frac{263}{7^4}+\ldots$

126. Show that the sum of the cubes of the first n terms of an AP is exactly divisible by the sum of its terms.

127. Prove that the sum to n terms of a geometric series with positive terms is greater than n times the geometric mean of the first and the nth terms of the series.

128. The natural numbers are divided into groups in the following way: (2, 3, 4); (5, 6, 7, 8, 9, 10, 11); (12, 13, 14, 15, …, 22). Obtain a formula for the sum of the numbers in the nth group.

129. If n is a root of the equation $x^2 (1 - ac) - x (a^2 + c^2) - (1 + ac) = 0$ and if n harmonic means are inserted between a and c show that the difference between the first and the last mean is equal to ac(a – c).

130. If a, b, c, d are natural numbers, then prove that

(i) $\left(\frac{a^3+b^3+c^3}{a^2+b^2+c^2}\right)^{a^2+b^2+c^2} \geq a^{a^2}b^{b^2}c^{c^2} \geq \left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a^2+b^2+c^2}$

(ii) $\frac{(a+b+c+d)^2\,abcd}{ab+bc+cd+da} \leq a^2cd+b^2ad+c^2bd+d^2bc$

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

131. Sum to n terms the series: 3.8 + 7.15 + 11.22 + … is

(a) $36n^2 + 15n + 3$ (b) $\frac{n(56n^2+75n+13)}{6}$ (c) $56n^2 + 75n + 13$ (d) $\frac{16n^2+75n+1}{6}$

132. p and q are positive integers. A_1, A_2, A_3 are the three arithmetic means inserted between p and q. H_1, H_2, H_3 are the three harmonic means inserted between p and q. If $A_1A_2A_3 = \frac{15}{2}$ and $H_1H_2H_3 = \frac{18}{5}$, the values of p and q are respectively

(a) 1, 2 (b) 3, 4 (c) 1, 3 (d) 2, 4

133. If $h_1, h_2, h_3, \dots, h_{2n}$ be the 2n harmonic means between a and b, then $\frac{h_1+a}{h_1-a}+\frac{h_{2n}+b}{h_{2n}-b}$

(a) n (b) 2n (c) 4n (d) $\frac{1}{n}$

134. If p, q, r are in AP; ℓ, m, n are in HP; $p\ell$, qm, r n are in GP ($p\,\ell \neq q\,m \neq r\,n$), then $p : q : r =$

(a) $m : n : \ell$ (b) $\frac{1}{n}:\frac{1}{m}:\frac{1}{\ell}$ (c) 1: 2: 3 (d) $n : m : \ell$

135. If $px^2 + 2qx + r = 0$ and $p_1x^2 + 2q_1x + r_1 = 0$ have a common root and $\frac{p_1}{p},\frac{q_1}{q},\frac{r_1}{r}$ are in HP, then p_1, q_1, r_1 are

(a) in AP (b) in GP
(c) in HP (d) not in any progression

136. An AP and a HP have each the first term p and the last term q and the same number of terms n. Then the product of the (r + 1)th term of the first series and the (n – r)th term of the second series

(a) dependent of r (b) independent of r (c) dependent of n (d) dependent of r and n

137. If a, b, c are the sides of a triangle, and $(a + b + c)^3 \geq \lambda\,(a + b - c)(b + c - a)(c + a - b)$, then λ equals

(a) 9 (b) 3 (c) 8 (d) 27

138. Consider a GP with n terms. The product of the AM and HM of all the terms of the GP is equal to

(a) 1 (b) n (GM) (c) GM (d) $(GM)^2$

139. Let a, b, c be the sums of the first n terms, next n terms and next n terms of a GP respectively. Then a, b, c are in

(a) AP (b) GP (c) HP (d) AGP

140. The sum of first n terms of the series 6 + 66 + 666 + 6666 + ……. is

(a) $\frac{27}{20}(10^n-1)-\frac{2}{3}n$ (b) $\frac{20}{27}(10^n-1)-n$ (c) $\frac{20}{27}(10^n-1)-\frac{2}{3}n$ (d) $\frac{1}{27}(10^n-1)$

141. In an AP whose first term is a and the sum of the first p terms is zero, if the sum of the next 'q' terms is $\frac{-a(p+q)}{(p-1)}\lambda$, then the value of λ is

(a) p (b) p + 1 (c) q (d) q + 1

142. If a, b, c, d, are distinct integers in AP such that $d = a^2 + b^2 + c^2$, then the sum a+ b + c + d equals

(a) 0 (b) 1 (c) 2 (d) –2

143. If two arithmetic means A_1, A_2, two geometric means G_1, G_2 and two harmonic means H_1, H_2 are inserted between two numbers then $\frac{A_2 - A_1}{H_2 - H_1}$.

(a) $\frac{A_1A_2}{H_1H_2}$ (b) $\frac{A_1 + A_2}{H_1 + H_2}$ (c) $\frac{A_1 + A_2}{H_1H_2}$ (d) $\frac{H_1 + H_2}{A_1A_2}$

144. If x is the first of the n arithmetic means between two numbers a and b and y is the first of the n harmonic means between a and b, then their product xy is equal to

(a) $\left(\frac{na+b}{nb+a}\right)ab$ (b) $\frac{a+nb}{ab}$ (c) $\left(\frac{a+b}{a-b}\right)ab$ (d) $\left(\frac{nb+a}{na+b}\right)ab$

145. If x_1, x_2, x_3 are in GP as as y_1, y_2, y_3 with the same common ratio, then points A (x_1, y_1), B (x_2, y_2), C (x_3, y_3)

(a) satisfies $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$ (b) form an equilateral triangle
(c) are collinear (d) form an isosceles triangle

146. How many geometric progressions are there containing the terms 48, 27 and 64?

(a) 1 (b) 3 (c) 9 (d) infinite

147. The sum of the series 1 + 3 + 5 + 7 + 9 + ... + 101 is

(a) 2931 (b) 5091 (c) 2601 (d) 2501

148. The nth term of an AP is 164 and the sum of its first n terms is $3n^2 + 5n$. Then, n equals

(a) 53 (b) 36 (c) 27 (d) 28

149. If a, b, c are in AP, then $4(b^2 - ac) =$

(a) $a^2c + 3$ (b) $\frac{a+c}{2}$ (c) $\frac{(a+c)a}{3}$ (d) $(a - c)^2$

150. Which term of the sequence $4, 4\sqrt{3}, 12, 12\sqrt{3}, \ldots\ldots\ldots\ldots$ is 36×3^4?

(a) 13 (b) 12 (c) 11 (d) 14

151. If there are four geometric means between 12 and 384, then the common ratio is

(a) 3 (b) $1/2$ (c) 2 (d) 4

152. Sum of the series $1^3 + 2^3 + 3^3 + 4^3 + \ldots.. + 19^3$ is

(a) 29130 (b) 19400 (c) 36100 (d) 43250

153. If the 4th term of an AP is 4, then the sum of its first 7 terms is

(a) 4 (b) 28 (c) 16 (d) 40

154. Sum of the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \ldots\ldots$ to 9 terms, is

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $\frac{-3}{2}$ (d) $\frac{-2}{3}$

155. If the sum of the first n natural numbers is one seventh of the sum of their squares, n equals

(a) 6 (b) 7 (c) 8 (d) 10

156. If the 10th term of a GP is 9 and its 4th term is 4, then its 7th term is

(a) $4/9$ (b) 36 (c) 6 (d) $9/4$

157. The product of n geometric means between two given positive numbers a and b is

(a) $(ab)^4$ (b) $(ab)^{n/2}$ (c) $(ab)^{2n}$ (d) $(ab)^{-n}$

158. If x, 2x + 2, 3x + 3 are in GP, then the 4th term is

(a) 27 (b) –27 (c) $\frac{27}{2}$ (d) $-\frac{27}{2}$

159. If p is the AM of q and r and q is the GM of r and p, then HM between p and q is

(a) p^2 (b) $pq - r$ (c) r (d) $pq + r$

160. If a^2, b^2, c^2 are in HP, then a^2b^2, a^2c^2, b^2c^2 are in

(a) GP (b) AGP (c) HP (d) AP

161. If a^2, b^2, c^2 are in AP, then $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{b+a}$ are in

(a) AP (b) GP (c) HP (d) None of these

162. If three distinct numbers a, b, c are in HP and a^2, b^2, c^2 are in AP then

(a) $a + c = b$ (b) $a + b + c = 0$ (c) $a + b = c$ (d) $b + c = a$

163. If the three distinct numbers a, b, c are in GP and a + x, b + x, c + x are in HP, then the value of x is

(a) c (b) b (c) a (d) None of these

164. $\log_{18}3, \log_{162}3, \log_{1458}3$ are in

(a) AP (b) GP (c) HP (d) None of these

165. If $a_1, a_2, a_3, \ldots$ are in AP such that $a_i \neq 0$ and $S_n = \frac{1}{a_1a_3} + \frac{1}{a_2a_4} + \frac{1}{a_3a_5} + \ldots + \frac{1}{a_na_{n+2}}$ then $S_{2n} - S_n$ is equal to

(a) $\frac{1}{a_n} + \frac{1}{a_{2n}}$

(b) $\frac{1}{2d}\left[\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_{2n+1}} + \frac{1}{a_{2n+2}}\right]$

(c) $\frac{nd}{\left[\frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_{n+1}} - \frac{1}{a_{n+2}}\right]}$

(d) $\frac{n}{2}\left[\frac{1}{a_{n+1}\,a_{2n+1}} + \frac{1}{a_{n+2}\,a_{2n+2}}\right]$

166. The sum of the first n terms of the series $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots$ is

(a) $\frac{n(n+1)}{2}$ (b) $\frac{(-1)^{n-1}n(n+1)}{2}$ (c) $\frac{-n(n+1)}{2}$ (d) $n(n+1)$

167. If a is any finite negative integer, n is an integer > 2 and $2^a + 2^{2a} + 2^{3a} + \ldots + 2^{na} + \frac{1}{2^a} + \frac{2}{2^{2a}} + \ldots + \frac{1}{2^{na}} \geq \lambda$, then the greastest volue of λ is

(a) n (b) 2n (c) n^2 (d) 0

168. The sum of the products taken two at a time of the n numbers 2, 4, 6, 8,…, 2n is

(a) $\left[\frac{n(n+1)}{2}\right]^2$ (b) $\frac{n(n+1)(2n+1)}{6}$

(c) $\frac{(n-1)n(n+1)}{6}$ (d) $\frac{(n-1)n(n+1)(3n+2)}{6}$

169. When n is even, the sum of the first n terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots\ldots$is $\frac{n(n+1)^2}{2}$. When n is odd, the sum of the series is

(a) $\frac{n(n+1)^2}{4}$ (b) $\frac{n^2(n+1)}{2}$ (c) $\frac{3n(n+1)}{2}$ (d) $\left[\frac{n(n+1)}{4}\right]^2$

170. The infinite geometric series $\sum_{r=0}^{\infty}\frac{a}{\left(y^2-4y+5\right)^{r-1}}$, where $y = x^2 - 6x + 11$ has a finite sum when x takes a value other than

(a) 3 (b) −1 (c) 4 (d) −3

Assertion–Reason Type Questions

***Directions:* Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

171. Statement 1

If a, b, c are in AP then $(b - a)^2 = (c - a)^2$

and

Statement 2

If a, b, c are in AP then $4(b^2 - ac) = (c - a)^2$

172. Statement 1

If a_1, a_2, a_3 are in AP and b_1, b_2, b_3 are in AP then $(a_1 + 2b_1), (a_2 + 2b_2), (a_3 + 2b_3)$ are in AP

and

Statement 2

If $a_1 + a_2 + a_3 + \ldots$ are in AP then $(a_1 + k) + (a_2 + k) + (a_3 + k) + \ldots$ are also in AP

173. Statement 1

If the sum of the first n terms of an AP is zero, then a must be < 0 and d must be positive.

and

Statement 2

Sum of the first n terms of the AP $a + (a + d) + (a + 2d) + \ldots + (a + (n - 1) d)$ is given by $\frac{n}{2}(2a + (n - 1) d)$.

174. Statement 1

If a, b, c are in AP,
$(b + c - a), (c + a - b), (a + b - c)$ are in AP

and

Statement 2

If a_1, a_2, a_3 are in AP,
$a_1 + k, a_2 + k, a_3 + k$ where $k \neq 0$ are also in AP

175. Statement 1

If $3x + 7y + 5z = 45$, maximum value of $x^3 y^7 z^5$ where, x, y, z are > 0 is

and

Statement 2

If A, G, H denote the AM, GM and HM of two positive numbers, $G^2 = AH$

176. Statement 1

For $0 < \theta < \frac{\pi}{2}$, $\cos\theta + \sec\theta > 2$

and

Statement 2

If x is any number not equal to zero, $x + \frac{1}{x}$ is always greater than or equal to 2.

177. Statement 1

If a_1, a_2, a_3 denote the 3 arithmetic means inserted between two numbers x and y then, $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$ are the 3 harmonic means between the numbers x and y.

and

Statement 2

If x_1, x_2, x_3 are in AP, $\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}$ are in HP

178. Statement 1

There cannot be an infinite GP with second term equals –1 and whose sum to infinity equals '–1'.

and

Statement 2

$a + ar + ar^2 + ar^3 + \ldots \infty = \frac{a}{(1-r)}$, provided $-1 < r < 1$.

179. Statement 1

If the terms of a GP are alternately positive and negative, then the common ratio of such a GP has to be a negative number

and

Statement 2

nth term of the GP $a + ar + ar^2 + \ldots$ is ar^{n-1}

180. Statement 1

If the sum of n consecutive terms of a series is of the form $(an^2 + bn + c)$, where $c \neq 0$ then the series is an AP

and

Statement 2

The sum of the first n terms of an AP is of the form $An^2 + Bn$.

Linked Comprehension Type Questions

Directions: **This section contains 3 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

The rearrangement of natural numbers, as per a particular pattern and the formula for finding the sum to n terms of natural numbers, the squares of natural numbers and the cubes of natural numbers are useful in many situations some of which are given below

181. A rehearsal for a parade on an important occasion is arranged as follows:
Each group is occupied by army men of the same rank only. The groups consist of 1, 8, 27, 64…number of persons. There are 3025 army men of different ranks to be arranged starting from the highest rank. The number of groups necessary to accommodate the above, assuming that enough number of persons of the same rank are available?
(a) 8 (b) 9 (c) 10 (d) 12

182. In a parade of 4356 army men, the number of persons having the lowest rank is
(a) 1331 (b) 1000 (c) 729 (d) 512

183. The number of persons in the groups are chosen as 7, 11, 18, 29, 47, 76,123, 199….. the number of persons in the (n–2) th, nth and (n + 1)th groups ($n \geq 3$), form
(a) a GP (b) a HP (c) an AP (d) an AGP

Passage II

The sequence of numbers
1, 5, 9, 13, 17, ……….
is grouped as follows:
{1} : {5, 9, 13} ; {17, 21, 25, 29, 33} ; ……
Then,

184. First number in the 15th group is
(a) 781 (b) 785 (c) 717 (d) 793

185. Last number in the 15th group is
(a) 897 (b) 893 (c) 829 (d) 905

186. Sum of the numbers in the 15th group is
(a) 24273 (b) 23519 (c) 23448 (d) 24389

Passage III

Let

$$S_n = \frac{5}{5} + \frac{10}{65} + \frac{15}{325} + \dots + \frac{5n}{4n^4 + 1}$$

Then,

187. $S_{20} =$

(a) $\frac{1050}{841}$ (b) $\frac{894}{841}$

(c) $\frac{1266}{841}$ (d) $\frac{1066}{841}$

188. $\lim_{n \to \infty} S_n =$

(a) $\frac{5}{6}$ (b) $\frac{5}{4}$

(c) $\frac{5}{2}$ (d) $\frac{1}{4}$

189. $\lim_{n \to \infty} S_n - S_{25}$ is approximately equal to

(a) 7×10^{-3} (b) 9×10^{-3}

(c) 9×10^{-4} (d) 7×10^{-4}

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers of which ONE OR MORE answers will be correct.**

190. If $f(n) = \left[\dfrac{n+50}{100}\right]$, where [ ] stands for greatest integer function then

(a) $\sum_{n=1}^{100} f(n) = 51 = \sum_{n=50}^{100} f(n)$

(b) $\sum_{n=1}^{1000} f(n) = 5010$

(c) $\sum_{n=350}^{549} f(n) = 900$

(d) $\sum_{n=1}^{49} f(n) = 0$

191. Let α and β denote the roots of the quadratic equation $ax^2 + 2bx + c = 0$. Then,

(a) if a, b, c are in AP, one of the roots of the equation is -1.

(b) if a, b, c are in AP one of the roots of the equation is $\dfrac{c}{a}$.

(c) if a, b, c are in GP the roots of the equation are equal.

(d) if a, b, c are in HP, $\beta(1 + \alpha)^2 + \alpha(1+\beta^2) = 0$

192. If the series

$g_1 + g_2 + g_3 + \ldots + g_n$ is in GP with common ratio r, then, for n >1,

(a) $\dfrac{1}{g_1^2 - g_2^2} + \dfrac{1}{g_2^2 - g_3^2} + \dfrac{1}{g_3^2 - g_4^2} + \ldots + \dfrac{1}{g_{n-1}^2 - g_n^2} = \dfrac{r^2}{(1-r^2)^2}\left\{\dfrac{1}{g_n^2} - \dfrac{1}{g_1^2}\right\}$

(b) $\dfrac{1}{g_1 g_2} + \dfrac{1}{g_2 g_3} + \dfrac{1}{g_3 g_4} + \ldots + \dfrac{1}{g_{n-1} g_n} = \dfrac{r}{(1-r^2)}\left\{\dfrac{1}{g_n^2} - \dfrac{1}{g_1^2}\right\}$

(c) $\dfrac{1}{g_1^2 + g_2^2} + \dfrac{1}{g_2^2 + g_3^2} + \dfrac{1}{g_3^2 + g_4^2} + \ldots + \dfrac{1}{g_{n-1}^2 + g_n^2} = \dfrac{r^2}{(1-r^4)}\left\{\dfrac{1}{g_n^2} - \dfrac{1}{g_1^2}\right\}$

(d) $\dfrac{g_1}{g_1 + g_2} + \dfrac{g_2}{g_2 + g_3} + \dfrac{g_3}{g_3 + g_4} + \ldots + \dfrac{g_{n-1}}{g_{n-1} + g_n} = \dfrac{(n-1)}{(1+r)}$

193. Let $0 < \theta < \dfrac{\pi}{4}$.

Then,

(a) $\sum_{k=0}^{\infty} \sin^{2k}\theta = \sec^2\theta$

(b) $\sum_{k=0}^{\infty} \cos^{2k}\theta = \operatorname{cosec}^2\theta$

(c) $\sum_{k=0}^{\infty} \tan^{2k}\theta = \dfrac{\cos^2\theta}{\cos 2\theta}$

(d) $\sum_{k=0}^{\infty} \tan^{2k}\theta + \sum_{k=0}^{\infty} \cot^{2k}\theta = 0$

194. Consider $f(x) = \dfrac{p}{x+p} + \dfrac{q}{x+q} - \dfrac{r}{x+r} - \dfrac{s}{x+s}$ with p, q, r, s ≠ 0. If the equation f(x) = 0 has a pair of equal roots then

(a) p equals r or s

(b) q equals r or s

(c) the HM (p, q) equals HM (r, s)

(d) the distinct root is the HM(p, q) if 0 is the multiple root

195. If a, b, c are positive and not all equal, the expression
$bc(a-b)(a-c) + ca(b-c)(b-a) + ab(c-a)(c-b)$ is always
(a) positive (b) negative
(c) $\Sigma c^2(a-b)^2$ (d) $\Sigma b^2c^2 - abc(a+b+c)$

196. The system of equations
$ax + by + (a\alpha + b)z = 0$
$bx + cy + (b\alpha + c)z = 0$
$(a\alpha + b)x + (b\alpha + c)y = 0$ has trivial solutions only. Then
(a) a, b, c are in GP
(b) α is a root of the equation $ax^2 + bx + c = 0$
(c) $x - \alpha$ is a factor of $ax^2 + 2bx + c = 0$
(d) a, b, c are in HP

197. The expression $p(q-r)x^2 + q(r-p)xy + r(p-q)y^2$ is a perfect square, where p, q, r are non zero reals
(a) Minimum value of p + r is 2q
(b) AM of p, r is q
(c) If p, q, r are the exradii of ΔABC whose smallest side is 4 and largest side is 8 then the third side is 6
(d) The area of ΔABC described in (c) is $3\sqrt{15}$

Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

198.

Column I Series	Column II Sum
(a) $\frac{1}{1.3.5}+\frac{1}{3.5.7}+\ldots+\frac{1}{41.43.45}$ is/are equal to	(p) 68276
(b) $1^3 - 2^3 + 3^3 - 5^3 + \ldots + 51^3$ is/are equal to	(q) $\frac{253}{400}$
(c) $\left[\frac{1+2+\ldots+10}{10^2}+\frac{(1^2+2^2+..+10^2)}{10^3}-\frac{(1^3+2^3+\ldots+10^3)}{10^4}\right]=$	(r) 48048
(d) 1.2.3.4 + 2.3.4.5 +....+ 10.11.12.13 is/are equal to	(s) $\frac{161}{1935}$

199.

Column I	Column II
(a) a, b, c are positive real numbers different from 1. If $\log_a^{100}, 2\log_b^{10}, 2\log_c^{5}+\log_c^{4}$ are in HP, then a, b, c are in	(p) AP
(b) a, b, c are different positive real numbers such that a > b > c. If 2 log(a – c), log $(a^2 - c^2)$, log $(a^2 + 2b^2 + c^2)$ are in AP, then a, b, c are in	(q) GP
(c) If a, b, c are in AP and a^2, b^2, c^2 are in HP then, a^3, b^3, c^3, are in	(r) HP
(d) If a, b, c are in AP, b, c, d are in GP, c, d, e are in HP then a, c, e are in	(s) AGP

200.

Column I	Column II
(a) If x, y, z be respectively AM, GM, HM between two rational numbers a and b then $x - y$ is equal to	(p) $\dfrac{a+b}{ab}$
(b) If A_1, A_2 be two AMs and G_1, G_2 be two GMs; between a and b, then $\dfrac{A_1 + A_2}{G_1G_2}$ is equal to	(q) $\left(\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{2}}\right)^2$
(c) If A_1, A_2 be two AMs; G_1, G_2 be two GMs; H_1, H_2 be two HMs between two positive numbers a and b, then $\dfrac{G_1G_2}{H_1H_2}\,\dfrac{H_1+H_2}{A_1+A_2}$ is	(r) $\dfrac{ab}{a+b}$
(d) If A_1, A_2 be two AMs G_1, G_2 be two GMs H_1, H_2 be two HMs between two numbers a and b, then $\dfrac{1}{H_1}+\dfrac{1}{H_2}$ is equal to	(s) 1

SOLUTIONS

ANSWER KEYS

Topic Grip

1. (i) 109
   (ii) 25th
   (iii) 5120
   (iv) 2, 6, 18.....
2. (i) 5612
   (ii) $\frac{325}{9}$
   (iii) 2072
   (iv) 6560
   (v) 6
   (vi) $\frac{3}{16}$
3. $\frac{5}{7}$
4. Greater than or equal to 12
6. $\frac{2000}{51}$
8. 6, 12, 24 or 24, 12, 6
10. $\underbrace{111...1}_{10\text{ digits}}$

11. (d) **12.** (d) **13.** (c)
14. (c) **15.** (d) **16.** (d)
17. (a) **18.** (b) **19.** (d)
20. (c) **21.** (a) **22.** (b)
23. (b) **24.** (a) **25.** (a)
26. (a)
27. (b), (c)
28. (a), (b), (c)
29. (c), (d)
30. ((a) → (r)
(b) → (r)
(c) → (r)
(d) → (p)

IIT Assignment Exercise

31. (b) **32.** (d) **33.** (a)
34. (d) **35.** (b) **36.** (b)
37. (b) **38.** (a) **39.** (a)
40. (c) **41.** (a) **42.** (c)
43. (b) **44.** (b) **45.** (c)
46. (a) **47.** (d) **48.** (b)
49. (a) **50.** (c) **51.** (d)
52. (a) **53.** (a) **54.** (b)
55. (c) **56.** (c) **57.** (a)
58. (a) **59.** (d) **60.** (c)
61. (b) **62.** (c) **63.** (b)
64. (c) **65.** (d) **66.** (c)
67. (c) **68.** (b) **69.** (a)
70. (c) **71.** (b) **72.** (a)
73. (d) **74.** (b) **75.** (a)
76. (b) **77.** (c) **78.** (b)
79. (a) **80.** (b) **81.** (d)
82. (a) **83.** (d) **84.** (b)
85. (b) **86.** (a) **87.** (b)
88. (d) **89.** (b) **90.** (d)
91. (c) **92.** (d) **93.** (d)
94. (a) **95.** (c) **96.** (c)
97. (b) **98.** (c) **99.** (c)
100. (d) **101.** (c) **102.** (c)
103. (b) **104.** (c) **105.** (d)
106. (a) **107.** (c) **108.** (b)
109. (c) **110.** (d) **111.** (d)
112. (d) **113.** (a) **114.** (d)
115. (c)
116. (a)
117. (a), (b), (c), (d)
118. (a), (b), (c), (d)
119. (a), (b), (c)
120. (a) → (r)
(b) → (p), (q), (r), (s)
(c) → (q)
(d) → (p)

Additional Practice Exercise

123. $\dfrac{4^{n+1}-4+15n}{9}$

125. $\dfrac{4}{3}\left[1-\left(\dfrac{4}{7}\right)^{50}\right]+\dfrac{2}{9}-\dfrac{1}{18\times 7^{49}}-\dfrac{33}{2\times 7^{50}}$

128. (4n – 1) ($2n^2$ – n + 2)
131. (b) **132.** (c) **133.** (c)
134. (b) **135.** (b) **136.** (b)
137. (d) **138.** (d) **139.** (b)
140. (c) **141.** (c) **142.** (c)
143. (b) **144.** (a) **145.** (c)
146. (d) **147.** (c) **148.** (c)
149. (d) **150.** (a) **151.** (c)
152. (c) **153.** (b) **154.** (c)
155. (d) **156.** (c) **157.** (b)
158. (d) **159.** (c) **160.** (d)
161. (a) **162.** (b) **163.** (b)
164. (c) **165.** (d) **166.** (b)
167. (b) **168.** (d) **169.** (b)
170. (a) **171.** (d) **172.** (b)
173. (d) **174.** (a) **175.** (b)
176. (c) **177.** (d) **178.** (a)
179. (a) **180.** (d) **181.** (c)
182. (a) **183.** (c) **184.** (b)
185. (a) **186.** (d) **187.** (a)
188. (b)
189. (c)
190. (a), (b), (c), (d)
191. (a), (c), (d)
192. (a), (b), (c), (d)
193. (a), (b), (c)
194. (a), (b), (c)
195. (a), (d)
196. (a), (c)
197. (a), (c), (d)
198. (a) → (s)
(b) → (p)
(c) → (q)
(d) → (r)
199. (a) → (q)
(b) → (q)
(c) → (p), (q), (r), (s)
(d) → (q)
200. (a) → (q)
(b) → (p)
(c) → (s)
(d) → (p)

HINTS AND EXPLANATIONS

Topic Grip

1. (i) Clearly, progression is an AP first term = 4 Common differences = 5

$\therefore$ $t_n = a + (n-1)d$

$\therefore$ $t_{22} = 4 + (22-1) \times 5 = 4 + 21 \times 5 = 109$

(ii) Let nth of the series be the first negative term 19

$+ (n-1)\left(\frac{-4}{5}\right) < 0$

$\Rightarrow$ $99 - 4n < 0$

$\Rightarrow$ $4n > 99 \Rightarrow n > 24\frac{3}{4}$

$\therefore$ $n = 25$

25th term is the first negative term of the AP

(iii) Given series in a GP

$a = 5$ $r = 2$

$\therefore$ $t_n = ar^{n-1}$

$\therefore$ $t_{11} = 5 \times (2)^{10} = 5120$

(iv) Let a be the first term r be common ratio.

$ar^3 = 54$ —(1) $ar^8 = 13122$

$\therefore$ $\frac{ar^8}{ar^3} = \frac{13122}{54} = 243$

$\Rightarrow$ $r^5 = 3^5 \Rightarrow r = 3$

$ar^3 = 54$

$\Rightarrow$ $a \times 27 = 54 \Rightarrow a = 2$

$\therefore$ GP is a, ar, ar^2, ar^3 …..

$\Rightarrow$ 2, 6, 18, 54 …….

2. (i) $S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{61} = \frac{61}{2}(2 \times 2 + (61-1)3) = \frac{61}{2}(4 + 180)$

$= \frac{61}{2} \times 184 = 61 \times 92 = 5612$

(ii) $S_{25} = \frac{25}{2}\left(\frac{2}{9} + (25-1)\frac{1}{9}\right)$

$= \frac{25}{2}\left(\frac{2}{9} + \frac{24}{9}\right)$

$= \frac{25}{2} \times \frac{26}{9} = \frac{325}{9}$

(iii) $S_n = \frac{n}{2}(a + \ell)$

$t_n = 47 \Rightarrow a + (n-1)d = 47$

$\Rightarrow$ $101 + (n-1)(-2) = 47$

$101 - 2n + 2 = 47$

$2n = 56 \Rightarrow n = 28$

$\therefore$ $S_{28} = \frac{28}{2}(101 + 47)$

$= 14 \times 148$

$= 2072$

(iv) $a = 2$ $r = 3$ $\ell = 4374$

$\therefore$ Required Series $= \frac{\ell r - a}{r - 1}$

$= \frac{4374 \times 3 - a}{3 - 1}$

$= 6560$

(v) $S_\infty = 6^{\frac{1}{2}} . 6^{\frac{1}{4}} . 6^{\frac{1}{8}} ....\infty$

$= 6^{\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...\infty\right)}$

$= 6^{\frac{1/2}{1 - 1/2}} = 6^1 = 6$

(vi) $S_\infty = \left(\frac{1}{7} + \frac{1}{7^3} + ...\infty\right) + \left(\frac{2}{7^2} + \frac{2}{7^4} + ...\infty\right)$

$= \frac{1/7}{1 - 1/7^2} + 2\frac{1/7^2}{1 - 1/7^2} = \frac{7}{48} + \frac{2}{48} = \frac{9}{48} = \frac{3}{16}$

3. Let a_1, a_2 be the first terms and d_1, d_2 be common differences of the two APs

Then ratio of the sums of the first n terms

$= \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{2n - 15}{2n - 1}$

So ratio of the sums of the first 25 terms

$= \frac{2a_1 + 24d_1}{2a_2 + 24d_2} = \frac{50 - 15}{50 - 1}$

$\Rightarrow$ $\frac{a_1 + 12d_1}{a_2 + 12d_2} = \frac{35}{49}$

$\Rightarrow$ $\frac{\text{13th term of Ist AP}}{\text{13th term of IInd AP}} = \frac{5}{7}$

So ratio of 13th terms of the two APs is 5:7

4. We have, $k = 5^{1+x} + 5^{1-x} + 25^x + 25^{-x}$

$= (5^x)^2 + 5.5^x + (5^{-x})^2 + 5.5^{-x}$

$= \left(5^x + 5^{-x}\right)^2 - 2 + 5\left(5^x + 5^{-x}\right)$

$= Y^2 + \frac{1}{Y^2} + 5\left(Y + \frac{1}{Y}\right)$ where $Y = 5^x$

Since Y is always > 0 $k \geq 2 + 5 \times 2 = 12$

So k can take all real numbers greater than or equal to 12.

5. The equation $px^2 + 2qr + r = 0$ has roots given by

$x = \frac{-2q \pm \sqrt{4q^2 - 4rp}}{2p}$

Since p, q, r are in GP Therefore $q^2 = pr$

It follows that $x = \frac{-q}{p}$. But $\frac{-q}{p}$ is also root of $dx^2 + 2ex + f = 0$

Therefore $d\left(\frac{-q}{p}\right)^2 + 2e\left(\frac{-q}{p}\right) + f = 0$

i.e., $dq^2 - 2eqp + fp^2 = 0$

Dividing it by pq^2 and using $q^2 = pr$ we get,

$\frac{d}{p} - \frac{2e}{q} + \frac{f}{r} = 0$

Hence $\frac{d}{p}, \frac{e}{q}, \frac{f}{r}$ are in AP

6. Let the GP be

$a + ar + ar^2 + \ldots\infty$ where $|r| < 1$

Given $\frac{a}{(1-r)} = \frac{10}{3}$ — (1)

$\frac{a^3}{(1-r^3)} = \frac{1000}{63}$ — (2)

From (1), $\frac{a^3}{(1-r)^3} = \frac{1000}{27}$ — (3)

$\frac{(2)}{(3)} \Rightarrow \frac{(1-r)^3}{1-r^3} = \frac{1000}{63} \times \frac{27}{1000} = \frac{3}{7}$

$\Rightarrow \frac{1 + r^2 - 2r}{1 + r^2 + r} = \frac{3}{7}$

$\Rightarrow 4r^2 - 17r + 4 = 0$

$\Rightarrow (r - 4)(4r - 1) = 0$

since $r \neq 4$, $r = 1/4$

Substituting in (1), $a = \frac{10}{3}\left(1 - \frac{1}{4}\right) = \frac{10}{3} \times \frac{3}{4} = \frac{5}{2}$

Sum of the 4th powers of the series in GP

$= \frac{a^4}{(1-r^4)} = \frac{5^4}{16} \times \frac{1}{1 - \left(\frac{1}{4}\right)^4} = \frac{5^4}{16} \times \frac{16 \times 16}{255}$

$= \frac{125 \times 16}{51} = \frac{2000}{51}$

7. Let the sides of the triangle be a, ar ar^2 where a and r are positive. In the case of a triangle, the sum of any two sides is always greater than the third side.

Case 1 : $0 < r < 1$

Side ar^2 is least

We have $a + ar > ar^2$

$\Rightarrow r^2 - r - 1 < 0$

$\Rightarrow$ r must lie between $\frac{-\sqrt{5}+1}{2}$ and $\frac{\sqrt{5}+1}{2}$

Since $0 < r < 1$, we have $0 < r < 1$ — (1)

Case 2 : $r > 1$

Side a is least

We have $ar^2 + ar > a$

$\Rightarrow r^2 + r - 1 > 0$

$\Rightarrow$ r must lie beyond $\frac{-1-\sqrt{5}}{2}$ and $\frac{\sqrt{5}-1}{2}$

Since $r > 0$, this means that $r > \frac{\sqrt{5}-1}{2}$ — (2)

Combining (1) and (2)

$\frac{\sqrt{5}-1}{2} < r < 1$

8. Let the numbers be a, ar, ar^2

Sum = 42 = $a(1 + r + r^2) = 42$ — (1)

and since $a + 2$, $ar + 2$, $ar^2 - 4$ are in AP

$\Rightarrow a(1 - 2r + r^2) = 6$ — (2)

$(1) \div (2) \Rightarrow \frac{r^2 + r + 1}{r^2 - 2r + 1} = 7$

$\Rightarrow 6r^2 - 15r + 6 = 0 \Rightarrow 2r^2 - 5r + 2 = 0$

$\Rightarrow (r - 2)(2r - 1) = 0$

r = 2 or 1/2

$\therefore$ a = 6 if r = 2

a = 24 if r = 1/2

Thus the numbers are 6, 12, 24 or 24, 12, 6.

9. If α is the common root, we have

$$\frac{\alpha^2}{-ps+rq}=\frac{\alpha}{q-s}=\frac{1}{-r+p}$$

$$\Rightarrow \quad \alpha=\frac{q-s}{p-r}, \alpha^2=\frac{qr-ps}{p-r}$$

$$\Rightarrow \quad \frac{qr-ps}{(p-r)}=\left(\frac{q-s}{p-r}\right)^2$$

$$\Rightarrow \quad (p-r)(qr-ps)=(q-s)2 \qquad \text{— (1)}$$

If β and δ are the other roots of the two equations,

$$\alpha\beta=q \Rightarrow \quad \beta=\frac{q}{\alpha}=\frac{q(p-r)}{(q-s)} \text{ and } \alpha\delta=s$$

$$\Rightarrow \quad \delta=\frac{s}{\alpha}=\frac{s(p-r)}{(q-s)}$$

The harmonic mean between β and δ is $\dfrac{2\beta\delta}{(\beta+\delta)}$

$$=\frac{\dfrac{2qs(p-r)^2}{(q-s)^2}}{\dfrac{q(p-r)}{(q-s)}+\dfrac{s(p-r)}{(q-s)}}=\frac{2qs(p-r)}{(q^2-s^2)}$$

Given that $=\dfrac{2qs(p-r)}{(q^2-s^2)}=\alpha=\dfrac{q-s}{(p-r)}$

$$\Rightarrow \quad 2qs(p-r)^2=(q-s)^2(q+s)$$

$$=(q+s)(p-r)(qr-ps), \text{ from (1)}$$

$$\Rightarrow \quad 2qs(p-r)=(q+s)(qr-ps)$$

$$\Rightarrow \quad 3psq+ps2=3qsr+q2r$$

$$\Rightarrow \quad ps(3q+s)=qr(3s+q)$$

10. $\underbrace{33...3}_{\text{n digits}}=3\{1+10+10^2+...+10^{n-1}\}$

$$=\frac{3(10^n-1)}{9}=\frac{10^n-1}{3}$$

$$\underbrace{22...2}_{\text{n digits}}=\frac{2(10^n-1)}{9}$$

$$\underbrace{111...1}_{\text{2n digits}}=1+10+10^2+...+10^{2n-1}$$

$$=\frac{10^{2n}-1}{9}$$

Result follows

$$9(11111)^2=(33333)^2$$

$$2(11111)=(22222)$$

$$9(11111)^2+2(11111)=1111111111$$

$$(\text{or } \underbrace{11...1}_{\text{10 digits}})$$

11. Let the angles be $\alpha-d_1\ \alpha, \alpha+d$

We have $\alpha-d+\alpha+\alpha+d=180°$

Giving $\alpha=60°$

Also, $\alpha+d=2(\alpha-d)$

$$\Rightarrow \quad 3d=\alpha=60° \Rightarrow d=20°$$

Largest angle $=60°+20°=80°$

12. $a^2=\dfrac{3}{14}\times\dfrac{2}{21}=\dfrac{1}{49}$

$$a=\frac{1}{7}$$

13. $\dfrac{1}{b-a}+\dfrac{1}{b-c}=\dfrac{1}{a}+\dfrac{1}{c}$

$$\Rightarrow \quad \frac{1}{b-a}-\frac{1}{c}=\frac{1}{a}-\frac{1}{b-c}$$

$$\Rightarrow \quad \frac{c-b+a}{(b-a)c}=\frac{b-c-a}{a(b-c)}$$

$$\Rightarrow \quad \frac{1}{c(a-b)}=\frac{1}{a(b-c)}$$

$$\Rightarrow \quad ac-bc=ab-ac \text{ or } b=\frac{2ac}{a+c}$$

14. $\dfrac{1}{2}+\dfrac{3}{4}+\dfrac{7}{8}+\dfrac{15}{16}+.........$

$$\text{n th term }=\frac{2^n-1}{2^n}=1-\frac{1}{2^n}=1-2^{-n}$$

Sum to n terms

$$=n-\sum_{n=1}^{n}\left(\frac{1}{2}\right)^n=n-\left\{\frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}}\right\}$$

$$=n-1+2^{-n}.$$

15. $P-Q=\dfrac{a+b}{2}-\sqrt{ab}=\dfrac{a+b-2\sqrt{ab}}{2}$

$$=\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{2}}\right)^2$$

16. Consider Statement 1

Let $a_1 = 1$ $a_2 = 2$ $a_3 = 4$, $a_4 = 5$

$a_2a_3 - a_1a_4 = (2)(4) - (1)(5)$

$= 8 - 5 = 3 > 0$

But 1, 2, 4, 5 are not in AP

So statement 1 is false

Consider statement 2

Let d be the common difference of the AP

$a_2 = a_1 \times d$, $a_3 = a_1 + 2d$, $a_4 = a_1 + 3d$

$a_2a_3 - a_1a_4 = (a_1 + d)(a_1 + 2d) - a_1(a_1 + 3d)$

$= (a_1^2 + 3a_1d + 2d^2) - (a_1^2 + 3a_1d)$

$= 2d^2 > 0$

Statement 2 is true

$\Rightarrow$ Choice (d)

17. Statement 1 is true since $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

For any n, $S_n \le S_\infty$

$S_n \le 2$

Statement 1 is true

Statement 2 is also true

Statement 2 is the explanation for Statement 1

18. Consider Statement 1

$b + c$, $c + a$, $a + b$ are in HP

$\therefore$ $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in AP

$\therefore$ $\frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b}$ are in AP

$\frac{a+b+c}{b+c} - 1, \frac{a+b+c}{c+a} - 1, \frac{a+b+c}{a+b} - 1$ are in AP

$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in AP

$\therefore$ $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in HP

Statement 1 is true

Consider Statement 2

This is true

But statement 2 is not the correct explanation for Statement 1

$\Rightarrow$ Choice (b)

19. Statement 2 is true

statement 1

we have $b^4 = a^2c^2$

$\Rightarrow$ $b^2 = ac$

statement 1 $\rightarrow$ is true

choice (a)

20. Statement 2 is false

For example, consider the GP

$\frac{1}{2} + (-1) + 2 + (-4) + \ldots$

$r = -2 < 1$

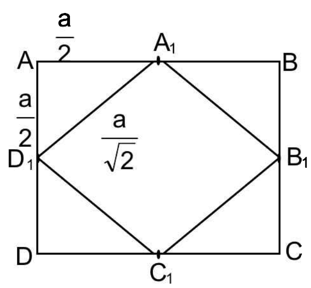
Statement 1 $S_{10} = \frac{\frac{1}{2}\left[1-(-3)^{10}\right]}{4}$

$= \frac{-1}{8}\left(3^{10} - 1\right)$

Statement 1 is true

choice (c)

21.



$X_1 = a^2$ $X_2 = \frac{a}{\sqrt{2}} \times \frac{a}{\sqrt{2}} = \frac{a^2}{2}$

$X_3 = \frac{1}{2}\left(\frac{a^2}{2}\right) = \frac{a^2}{2^2}$; $X_n = \frac{a^2}{2^{n-1}}$

X_1, X_2 ..form a decreasing GP of c. r. $\frac{1}{2}$

22. $d_1 = \sqrt{a^2 + a^2} = \sqrt{2}a$; $d_2 = \sqrt{\frac{a^2}{2} + \frac{a^2}{2}} = a$

$d_3 = \sqrt{\frac{a^2}{4} + \frac{a^2}{4}} = \frac{a}{\sqrt{2}}$ $\quad d_4 = \frac{a}{2}$ and so on

$\therefore$ the diagonals are of length $\sqrt{2}a, a, \frac{a}{\sqrt{2}}, \frac{a}{2}$ .....

A decreasing GP of c.r. $\frac{1}{\sqrt{2}}$

23. The areas $A_1B_1C_1D_1$; A_2, B_2, C_2, D_2, ...are $a^2, \frac{a^2}{2}, \frac{a^2}{4}$ ...

the areas of their circumcircles are $\frac{\pi a^2}{2}, \frac{\pi a^2}{4}, \frac{\pi a^2}{8}$,........

they bear a constant ratio $\frac{\pi}{2}$

24. The circumference of the 1st circle = $2\pi r$

$= 2\pi(40) = 80\pi$

The ....2nd circle.......................= 2π (80) = 160π

The circumference of the outer most circle = 2π x 560 = 1120π

∴ total cable requirement

$= 80\pi + 160\pi + \ldots 1120\pi = 80\pi + 160\pi + \ldots 1120\pi$

$= 80\pi\,[1+2+\ldots 14] = 80\pi \times \frac{14 \times 15}{2}$

$= 80 \times \frac{22}{7} \times 105 = 26400$ metres

To provide connection from stage to 1st circle, 1st circle to 2nd circle....to the last circle

Cable required = 560 meters

∴ total cable = 26960 meters

25. Cable length required = 26960 m

∴ cost including wastage = 26900 x 40 x 1.05

= Rs, 11, 32, 320

26. The distances are $\frac{2\pi r_1}{12}, \frac{2\pi r_2}{12}, \ldots\ldots \frac{2\pi r_{14}}{12}$

$\propto r_1, r_2, \ldots\ldots\ldots r_{14}$

$\propto 40, 80 \ldots\ldots 560$

An increasing AP with C.D = 40

27. $\cos(\theta + \alpha)$, $\cos\theta$, $\cos(\theta + \alpha)$ are in HP

$$\cos\theta = \frac{2\cos(\theta-\alpha)\cos(\theta+\alpha)}{\cos(\theta-\alpha)+\cos(\theta+\alpha)}$$

$$= \frac{\cos^2\theta - \sin^2\alpha}{\cos\theta\cos\alpha}$$

$\cos^2\theta\cos\alpha = \cos^2\theta - \sin^2\alpha$

$\cos^2\theta\cos\alpha - \cos^2\theta = -\sin^2\alpha$

$\cos^2\theta - \cos^2\theta\cos\alpha = \sin^2\alpha$

$\cos^2\theta\,(1 - \cos\alpha) = 1 - \cos^2\alpha$

$\cos^2\theta = 1 + \cos\alpha$

$$= 2\cos^2\frac{\alpha}{2} = \frac{2}{\sec^2\frac{\alpha}{2}}$$

$$\cos^2\theta\sec^2\frac{\alpha}{2} = 2$$

$$\cos\theta\sec\frac{\alpha}{2} = \pm\sqrt{2}$$

Choices (b) and (c) are true.

28. (i) $a + b, b + c, c + d$ in GP

if $(b + c)^2 = (a + b)(c + d)$

if $b^2 + 2bc + c^2 = ac + bc + ad + bd$

if $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$

if a, b, c, d are in GP

∴ (a) is true

(ii) $ax^3 + bx^2 + cx + d = (ax^2 + c)(x + \frac{b}{a}) + (d - \frac{cb}{a})$

a, b, c, d in GP ⇒ ad = bc

⇒ $ax^2 + c$ is a factor of $ax^3 + bx^2 + cx + d$

∴ (b) is true and (d) is not

(iii) Let $b = ar$, $c = ar^2$, $d = ar^3$

We can easily show that

$(ab + bc + cd)^2 = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

∴ (c) is true

⇒ choices (a), (b), (c)

29. Given $2y = x + z$

Clearly, if $y^2 = xz$, $x = y = z$

Also, if $x = y = z$, $y^2 = xz$

30. (a) Let $a^x = b^y = c^z = k$

⇒ $a = k^{1/x}, b = k^{1/y}, c = k^{1/z}$

Since a, b, c are in GP, therefore

$$\frac{k^{1/y}}{k^{1/x}} = \frac{k^{1/z}}{k^{1/y}}$$

⇒ $\frac{1}{y} - \frac{1}{x} = \frac{1}{z} - \frac{1}{y}$

⇒ $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in AP

⇒ x, y, z are in HP

∴ (a) → (r)

(b) $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$

⇒ $\frac{1}{b-a} - \frac{1}{c} = \frac{1}{a} - \frac{1}{b-c}$

$\Rightarrow \quad \dfrac{c-(b-a)}{c(b-a)} = \dfrac{b-c-a}{a(b-c)} \Rightarrow b = \dfrac{2ac}{a+c}$

$\Rightarrow$ a, b, c are in HP

$\therefore$ (b) $\to$ (r)

(c) Let $y = xr$, $z = xr^2$

Then $\dfrac{1}{1+\log y} = \dfrac{1}{1+\log xr} = \dfrac{1}{1+\log r + \log x}$

$\dfrac{1}{1+\log z} = \dfrac{1}{1+\log x + 2\log r}$

$\Rightarrow \dfrac{1}{1+\log x}, \dfrac{1}{1+\log y}, \dfrac{1}{1+\log z}$ are in HP

$\therefore$ (c) $\to$ (r)

(d) $\dfrac{1}{1+\sqrt{x}} + \dfrac{1}{1-\sqrt{x}} = \dfrac{2}{1-x}$

$\Rightarrow \dfrac{1}{1+\sqrt{x}}, \dfrac{1}{1-x}, \dfrac{1}{1-\sqrt{x}}$ are in AP

$\therefore$ (d) $\to$ (p)

IIT Assignment Exercise

31. $a = 252$; $t_n = 798$; $d = 7$

$n = \dfrac{t_n - t_1}{d} + 1 = \dfrac{798-252}{7} + 1 = 78 + 1 = 79$

32. $7 \times t_7 = 11 \times t_{11}$

$7\,[a + 6d] = 11\,[a + 10d]$

$7a + 42d = 11a + 110d$

$4a + 68d = 0$

or $a + 17d = 0$

$\therefore$ $t_{18} = 0$

33. $2b = a + c$

$2\,(c + a) = c + a + c + a = c + a + 2b$

$= c + a + b + b = (a + b) + (b + c)$

34. $ar^8 = 256$

$ar^6 = 64$

$r^2 = 4$

$r = \pm 2$

Since 9th term is > 7th term, $r = +2$

35. $ar^4 = 2$

$a \,.\, ar \,.\, ar^2 \ldots ar^8 = a^9 \,.\, r^{36} = (ar^4)^9 = 2^9 = 512$

36. Let a, b, c, d be in GP

$\therefore$ $b^2 = ac$

$2\log b = \log a + \log c$ — (1)

$\therefore$ $\log a, \log b, \log c$ are in AP

$c^2 = bd$

$2\log c = \log b + \log d$ — (2)

$\therefore$ $\log b, \log c, \log d$ are in AP

37. $a = \dfrac{1}{1-r}$

$1 - r = \dfrac{1}{a}$

$r = 1 - \dfrac{1}{a} = \dfrac{a-1}{a}$

38. $S = 16^{\frac{1}{\left(1-\frac{1}{2}\right)}} = 16^2 = 256$

39. $a = 2\,(ar + ar^2 + \ldots)$

$r + r^2 + \ldots \infty = \dfrac{1}{2}$

$\therefore$ $\dfrac{r}{1-r} = \dfrac{1}{2}$

$r = \dfrac{1}{3}$

40. For any two distinct positive reals AM > GM > HM and $GM^2 = AM\ HM$

41. $A = 20$, $G = 4$

But $G^2 = AH$

$\dfrac{H}{G} = \dfrac{G}{A} = \dfrac{1}{5}$

42. $a = 1$; $d = 2$; $r = \dfrac{1}{2}$

$|r| < 1$

$\therefore$ $S = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

$= \dfrac{1}{\left(\frac{1}{2}\right)} + \dfrac{2 \times \frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}$

$= 2 + 4 = 6$

43. $t_n = 2n(2n + 2)$ and $n = 20$ gives

$t_{20} = 1680$

44. $S_n = 5n^2 + 2n$

$t_2 = S_2 - S_1 = (5 \times 4 + 2 \times 2) - (5 \times 1 + 2 \times 1)$

$= 17.$

45. Let a – d, a, a + d be the roots then sum of roots = 3a = 12 (from equation)

$\Rightarrow$ a = 4

product of the roots = $a(a^2 - d^2) = 28$

$\Rightarrow$ $4(16 - d^2) = 28$

$\Rightarrow$ $d^2 = 9$

$d = \pm 3$

46. $ar^2 = 8$

$a \,.\, ar \,.\, ar^2 \,.\, ar^3 \,.\, ar^4 = a^5 \,.\, r^{10} = (ar^2)^5 = 8^5$

47. Let $\frac{a}{r}$, a, ar be the numbers in GP

Since $\frac{a}{r}$, 2a, ar form an AP (given)

we get $4a = a\left(r + \frac{1}{r}\right) \Rightarrow r^2 - 4r + 1 = 0$

$\Rightarrow$ $r = 2 \pm \sqrt{3}$

Since the given GP is increasing, $r = 2 + \sqrt{3}$.

48. $$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = (ab)^{1/2} \Rightarrow a^{n+1} + b^{n+1}$$

$$= a^{n+1/2}\, b^{1/2} + a^{1/2}\, b^{n+1/2}$$

$$\Rightarrow \quad a^{n+1/2}\left(\sqrt{a} - \sqrt{b}\right) = b^{n+1/2}\left(\sqrt{a} - \sqrt{b}\right)$$

$$\Rightarrow \quad \left(\frac{a}{b}\right)^{n+1/2} = 1 \text{ or } n + 1/2 = 0 \Rightarrow n = \frac{-1}{2}$$

49. $q^2 = pr$

$r = \frac{p+q}{2} \Rightarrow 2q^2 = p^2 + pq$

$\therefore$ p^2, q^2, pq are in AP

50. x = 0.555..............

$\Rightarrow$ $10x - x = 9x = 5 \Rightarrow x = \frac{5}{9}$.

51. $$A_1 + A_2 = 2\left[\frac{a+b}{2}\right] = a + b$$

$$G_1 . G_2 = \left(\sqrt{ab}\right)^2 = ab$$

$$H_1^{-1} + H_2^{-1} = 2\left\{\frac{1/a + 1/b}{2}\right\} = \frac{a+b}{ab} = \frac{A_1 + A_2}{G_1 G_2}$$

52. 2b = a + c .... (1). since a, b, c are in AP

$$b^2 = \frac{2a^2c^2}{a^2 + c^2} \qquad ....(2).$$

since a^2, b^2, c^2 are in HP

$\therefore$ $b^2[(a + c)^2 - 2ac] = 2a^2c^2$

or $b^2[4b^2 - 2ac] = 2a^2c^2$

$\Rightarrow$ $(b^2 - ac)(2b^2 + ac) = 0$

$b^2 - ac = 0$

Then (1) gives $(a + c)^2 - 4ac = 0$

$\Rightarrow$ $(a - c)^2 = 0 \Rightarrow a = c$

$\Rightarrow$ a = b = c.

53. $$\frac{S_m}{S_n} = \frac{m^2}{n^2}$$

$$\therefore \frac{S_m}{m^2} = \frac{S_n}{n^2} = k \text{ (say)}$$

Now $$\frac{T_m}{T_n} = \frac{S_m - S_{m-1}}{S_n - S_{n-1}}$$

$$= \frac{k\left\{m^2 - (m-1)^2\right\}}{k\left\{n^2 - (n-1)^2\right\}} = \frac{2m-1}{2n-1}$$

54. $$\frac{1}{\log_a x} + \frac{1}{\log_c x} = \frac{2}{\log_b x}$$

$\Rightarrow$ $\log_x a + \log_x c = 2\log_x b$

$\Rightarrow$ $\log_x ac = \log_x b^2$

$\therefore$ $ac = b^2$

$\therefore$ a, b, c are in GP

55. Let d be the common difference of the AP

then $a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} = 300$

$\Rightarrow$ $a_1 + a_1 + 3d + a_1 + 6d + a_1 + 9d + a_1 + 12d + a_1 + 15d = 300$

$\Rightarrow$ $6a_1 + d(3 + 6 + 9 + 12 + 15) = 300$

$\Rightarrow$ $6a_1 + 45d = 300$

$\Rightarrow$ $2a_1 + 15d = 100$ — (1)

Now the sum of first 16 terms

$= a_1 + a_2 + \ldots + a_{16}$

$= 16a_1 + d(1 + 2 + 3 + \ldots + 15)$

$= 16a_1 + d \times 8 \times 15 = 8(2a_1 + 15d)$

$= 8 \times 100 = 800$ using (1)

56. $$\frac{S_3}{S_6} = \frac{125}{152}$$

$$\frac{a(r^3 - 1)/(r-1)}{a(r^6 - 1)/(r-1)} = \frac{125}{152}$$

$$\frac{1}{r^3+1}=\frac{125}{152} \text{ or } 125r^3 = 27$$

$$r=\frac{3}{5}$$

57. $\dfrac{1}{\sqrt{a}+\sqrt{b}}+\dfrac{1}{\sqrt{b}+\sqrt{c}}=2\left(\dfrac{1}{\sqrt{c}+\sqrt{a}}\right)$

$$\Rightarrow \frac{\sqrt{b}-\sqrt{a}}{b-a}+\frac{\sqrt{c}-\sqrt{b}}{c-b}=\frac{2\left(\sqrt{c}-\sqrt{a}\right)}{c-a}$$

This holds only if $b - a = c - b = \dfrac{c-a}{2}$

i.e., $2b = a + c$

$\therefore$ a, b, c are in AP

58. $H = 4$ and $G^2 = AH$ gives $G^2 = 4A$

Also $2A + G^2 = 27$ or $2A + 4A = 27$

$6A = 27$ or $2A = 9$. let the numbers be a, b,

or $a + b = 9$ — (1)

Also $G^2 = 4A = 18$ or $ab = 18$ — (2)

From (1) and (2) we get the values a, b as 6 and 3.

59. $a + c = 2b$

$b = 1/2$ [$\because$ $a + b + c = 3/2$ and $a + c = 2b$]

$a^2c^2 = b^4 \Rightarrow ac=\pm\dfrac{1}{4} \Rightarrow c=\pm\dfrac{1}{4a}$

Now $a + c = 1 \Rightarrow a\pm\dfrac{1}{4a}=1$

$\Rightarrow 4a^2 - 4a \pm 1 = 0$

$4a^2 - 4a + 1 = 0 \Rightarrow a = 1/2$

As we should have $a < b < c$, this value of a is neglected.

Consider $4a^2 - 4a - 1 = 0$

$\Rightarrow a=\dfrac{1\pm\sqrt{2}}{2}$

We omit $a=\dfrac{1+\sqrt{2}}{2}$ as this does not satisfy the condition $a < b < c$

$\Rightarrow a=\dfrac{1}{2}-\dfrac{1}{\sqrt{2}},\ b=\dfrac{1}{2},\ c=\dfrac{1}{2}+\dfrac{1}{\sqrt{2}}$

60. a, X_1, X_2, b are in AP

$\therefore X_1 + X_2 = a + b$

a, Y_1, Y_2, b are in GP

$\therefore Y_1Y_2 = ab$

a, z_1, z_2, b are in HP

$\therefore \dfrac{1}{Z_1}+\dfrac{1}{Z_2}=\dfrac{1}{a}+\dfrac{1}{b}$

$\Rightarrow \dfrac{Z_1+Z_2}{Z_1Z_2}=\dfrac{a+b}{ab}=\dfrac{X_1+X_2}{Y_1Y_2}$

$\Rightarrow \dfrac{X_1+X_2}{Z_1+Z_2}=\dfrac{Y_1Y_2}{Z_1Z_2} \Rightarrow \dfrac{A_1}{A_2}=\dfrac{G_1^2}{G_2^2}$

$\left(A_1=\dfrac{X_1+X_2}{2};\ A_2=\dfrac{Z_1+Z_2}{2};\right.$

$\left.G_1=\sqrt{Y_1Y_2};\ G_2=\sqrt{Z_1Z_2};\right)$

61. The given series is an AGP with a = 1, d = 1 and r = 2

Sum of the first n terms

$$=\frac{a}{1-r}+\frac{dr\left(1-r^{n-1}\right)}{\left(1-r\right)^2}+\frac{\left[a+\left(n-1\right)d\right]r^n}{(1-r)}$$

$$=\frac{-1}{2-1}+\frac{2}{\left(2-1\right)^2}-\frac{2^n}{\left(2-1\right)^2}+\frac{n\times2^n}{1}$$

$= 99.\ 2^{100} + 1$ as $n = 100$

62. $\dfrac{H}{G}=\dfrac{12}{13}$

$\therefore \dfrac{2ab}{a+b}=\dfrac{12}{13}\sqrt{ab}$

$\Rightarrow 13\sqrt{ab}=6\left(a+b\right)$ divide by b, we get

$$13\sqrt{\frac{a}{b}}=6\left(\frac{a}{b}+1\right)$$

$\Rightarrow 6x^2 - 13x + 6 = 0;\ x=\sqrt{\dfrac{a}{b}}$

$(2x - 3)(3x - 2) = 0$

$\therefore x=\dfrac{3}{2}$ or $\dfrac{2}{3}$

$\dfrac{a}{b}=\dfrac{9}{4}$ or $\dfrac{4}{9}$

a : b = 4:9

63. $\dfrac{p}{r}+\dfrac{r}{p}=\dfrac{p^2+r^2}{pr}=\dfrac{\left(p+r\right)^2-2pr}{pr}$

$$=\left(\frac{4p^2r^2}{q^2}-2pr\right)\Big/pr$$

$$\left(\because q=\frac{2pr}{p+r}\right)$$

$$= \frac{4pr}{q^2} - 2 = \frac{4b^2}{ac} - 2 \quad (\because b^2q^2 = acpr)$$

$$= \frac{(a+c)^2}{ac} - 2 \text{ (since } 2b = a + c)$$

$$= \frac{a}{c} + \frac{c}{a}$$

64. Let $a + b = A$

$x + y = B$

Then $A + B = 5$

The product of two (+) ve numbers whose sum is constant is maximum when they are equal.

i.e., when $A = B = 5/2$

$\therefore$ Maximum of $AB = 25/4$

$\therefore$ Maximum of $(a + b)(x + y) = 25/4$

65. $2b = a + c, b^2 = ac$

so that $\left\{\frac{a+c}{2}\right\}^2 = ac$

$\therefore (a+c)^2 - 4ac = 0$

or $(a - c)^2 = 0$

or $(a - c)^2 = 0$

or $a = c$ or $2b = 2a$

$\therefore$ $a = b = c$

66. $a^2 + 16b^2 + 49c^2 - 4ab - 7ac - 28bc = 0$

$\Rightarrow$ $2(a^2 + 16b^2 + 49c^2 - 4ab - 7ac - 28bc) = 0$

$\Rightarrow$ $a^2 + 16b^2 - 8ab + a^2 + 49c^2 - 14ac + 16b^2 + 49c^2 - 56bc = 0$

$(a - 4b)^2 + (a - 7c)^2 + (4b - 7c)^2 = 0$

$\Rightarrow$ $a = 4b, a = 7c$ and $4b = 7c$.

$\therefore$ $a = 4b = 7c = k$ (say)

$$\frac{a}{1} = \frac{b}{1/4} = \frac{c}{1/7} = k$$

i.e., $a = \frac{k}{1}, b = \frac{k}{4}, c = \frac{k}{7}$

Since 1, 4, 7 are in AP

$\frac{1}{k}, \frac{4}{k}, \frac{7}{k}$ are in AP

$\therefore$ $\frac{k}{1}, \frac{k}{4}, \frac{k}{7}$ are in HP

67. The numbers divisible by 2, 3 and 5 are the multiples of L.C.M of 2, 3 and 5 i.e., they are multiples of 30. And these numbers are the terms of an AP

First term of the AP = 30 = common difference

Last term = 420

Numbers of such numbers $= \frac{420}{30} = 14$

Required sum

$$= \frac{14}{2}[60 + 13 \times 30] = 7 \times 450 = 3150$$

68. $0.555 \ldots 5 = \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \ldots \frac{5}{10^n}$

$$= 5\left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots + \frac{1}{10^n}\right)$$

$$= \frac{5}{10}\left(\frac{1 - 1/10^n}{1 - 1/10}\right) = \frac{5}{9}\left(1 - \frac{1}{10^n}\right)$$

$$0.535353 \ldots 53 = \frac{53}{10^2} + \frac{53}{10^4} + \frac{53}{10^6} + \ldots + \frac{53}{10^n}$$

$$= 53\left(1/10^2 + 1/10^4 + 1/10^6 + \ldots + 1/10^n\right)$$

$$= 53 \times 1/10^2 \frac{\left(1 - 1/10^n\right)}{\left(1 - 1/10^2\right)} = \frac{53}{99}\left(1 - 1/10^n\right)$$

So required sum $= \left(1 - 1/10^n\right)\left(\frac{5}{9} + \frac{53}{99}\right)$

$$= \left(1 - 1/10^n\right)\frac{108}{99} = 12/11\left(1 - 1/10^n\right)$$

69. Let r be the common ratio and a be the first term of the GP

Then by given condition

$t_n = t_{n-1} + t_{n-2} \qquad n \geq 3$

$ar^{n-1} = ar^{n-2} + ar^{n-3}$

$r^2 = r + 1$

$r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

As t_n's are positive, r cannot be negative

$$\Rightarrow \quad r = \frac{1+\sqrt{5}}{2}$$

70. a, b, c are in GP $\Rightarrow b^2 = ac$ — (1)

$\log\left(\frac{a}{2b}\right), \log\left(\frac{2b}{3c}\right), \log\left(\frac{3c}{a}\right)$ are in AP

$$2\log\left(\frac{2b}{3c}\right) = \log\left(\frac{a}{2b}\right) + \log\left(\frac{3c}{a}\right)$$

$$\Rightarrow \frac{4b^2}{9c^2} = \frac{a}{2b} \times \frac{3c}{a}$$

$8b^3 = 27c^3 \Rightarrow 2b = 3c$ — (2)

From (1) and (2)

$$4ac = 9c^2 \ a = \frac{9c}{4}$$

$$a : b : c = \frac{9}{4} : \frac{3}{2} : 1 = 9 : 6 : 4$$

71. $$x = \sum_{n=0}^{\infty} 3^{-n/2} = \sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{3}}\right)^n = \frac{1}{1-\frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}}{\sqrt{3}-1} = \frac{\sqrt{3}\left(\sqrt{3}+1\right)}{2}$$

$\therefore \quad 2x - 3 = \sqrt{3}$

$$y = \sum_{n=0}^{\infty} 2^{-n/2} = \sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{1-\frac{1}{\sqrt{2}}}$$

$$= \frac{\sqrt{2}\left(\sqrt{2}+1\right)}{1}$$

$y - 2 = \sqrt{2}$

$\therefore \quad (2x-3)^2 - (y-2)^2 = \left(\sqrt{3}\right)^2 - \left(\sqrt{2}\right)^2 = 1$

$4x^2 - y^2 - 12x + 4y + 4 = 0$

72. $$\sum_{r=1}^{n}(2r-1)^4 = 1^4 + 3^4 + 5^4 + \ldots + (2n-1)^4$$

$= 1^4 + 2^4 + 3^4 + \ldots + (2n)^4$

$- \{2^4 + 4^4 + \ldots + (2n)^4\}$

$= f(2n) - 16\{1^4 + 2^4 + \ldots + n^4\}$

$= f(2n) - 16 f(n) \ \forall \ n \in N$

73. $$a_n b_n = \left(2^{1/2^n} + 3^{1/2^n}\right)\left(2^{1/2^n} - 3^{1/2^n}\right)$$

$$= \left(2^{1/2^{n-1}} - 3^{1/2^{n-1}}\right) = b_{n-1}$$

$a_n b_n = b_{n-1}$

$a_{n-1} b_{n-1} = b_{n-2}$

……………………

……………………

……………………

$a_2 b_2 = b_1$

on multiplication, we get

$(a_2 a_3 \ldots\ldots a_n) b_n = b_1$

$$\Rightarrow \ a_1 a_2 a_3 \ldots a_n = \frac{a_1 b_1}{b_n} = \frac{\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}-\sqrt{3}\right)}{b_n}$$

$$= \frac{-1}{b_n}$$

74. We know that the sum of n arithmetic means between two numbers is equal to n times their single mean.

$$\therefore \quad a_1 + a_2 + \ldots + a_{10} = 10\left(\frac{a+b}{2}\right)$$

$25 = 5\,(a+b) \ \therefore \ a + b = 5$

$a, h_1, h_2, \ldots, h_{10}, b$ are in HP

$\Rightarrow$ So $1/a$, $1/h_1 \ldots 1/h_{10}$, $1/b$ are in AP

$$\therefore \frac{1}{h_1} + \frac{1}{h_2} + \ldots + \frac{1}{h_{10}} = 10\left(\frac{\frac{1}{a}+\frac{1}{b}}{2}\right)$$

$$\frac{25}{6} = 5 \times \frac{a+b}{ab} \Rightarrow ab = 6$$

So, a and b are the roots of the equation $t^2 - 5t + 6 = 0$

So, a and b are 2 and 3

75. $$S = \frac{4}{19} + \frac{44}{19^2} + \frac{444}{19^3} + \ldots\ldots + \infty \quad — (1)$$

$$\frac{S}{19} = \frac{4}{19^2} + \frac{44}{19^3} + \frac{444}{19^4} + \ldots + \infty \quad — (2)$$

(1) – (2) $\Rightarrow$

$$S.\frac{18}{19} = \frac{4}{19} + \frac{40}{19^2} + \frac{400}{19^3} + \ldots$$

$$= \frac{4}{19}\left[1 + \frac{10}{19} + \left(\frac{10}{19}\right)^2 + \ldots\right] = \frac{4}{19}\left[\frac{1}{1-\frac{10}{19}}\right] = \frac{4}{9}$$

$$\therefore \quad S = \frac{76}{162} = \frac{38}{81}$$

76. $$\frac{x^2+3}{(x-3)(x^2+x-2)} = \frac{x^2+3}{(x-3)(x+2)(x-1)}$$

$$= \frac{k_1}{x-3} + \frac{k_2}{x+2} + \frac{k_3}{x-1}$$

$x^2 + 3 = k_1(x+2)(x-1) + k_2(x-3)(x-1)$

$+ k_3(x+2)(x-3)$

$$x = 1 \Rightarrow k_3 = \frac{-2}{3}$$

$$x = -2 \Rightarrow k_2 = \frac{7}{15}$$

$$x = 3 \Rightarrow k_1 = \frac{6}{5}$$

77. We have the result; If the sum of two positive quantities is a constant, their product is maximum when the two numbers are equal.

That is, $(a^2 x^4)(b^2 y^4)$ is maximum when $a^2 x^4 = b^2 y^4 = \frac{c^6}{2}$

$\therefore$ Maximum value of $(a^2 x^4)(b^2 y^4)$ is $\left(\frac{c^6}{2}\right)\left(\frac{c^6}{2}\right)$

$$= \frac{c^{12}}{4}$$

OR Maximum value of $x^4 y^4$ is $= \left(\frac{c^{12}}{4a^2b^2}\right)$

OR Maximum value of $xy = \left(\frac{c^{12}}{4a^2b^2}\right)^{\frac{1}{4}}$

$$= \frac{c^3}{\sqrt{2ab}}$$

78. Consider the 10 numbers a, a, b, b, b, c, c, c, c, c

Applying AM > GM inequality to the above 10 numbers

$$\frac{(2a + 3b + 5c)}{10} > \left(a^2b^3c^5\right)^{\frac{1}{10}}$$

or $(2a + 3b + 5c)^{10} > 10^{10} a^2 b^3 c^5$

$\Rightarrow \lambda = 10^{10}$

79. Let the series be $u_1 + u_2 + u_3 + \ldots$

$u_2 - u_1 = 3$

$u_3 - u_2 = 7$

$u_4 - u_3 = 11$

:

:

$u_n - u_{n-1} = 3 + (n - 2)\,4$ (since $(u_n - u_{n-1})$ is the $(n - 1)$th term of the AP $3 + 7 + 11 + \ldots$

$= 4n - 5$

Adding all the above relations

$u_n - u_1 = 3 + 7 + 11 + \ldots + (4n - 5)$

$$= \frac{(n-1)}{2}\{6+(n-2)4\} = (n-1)(2n-1)$$

$= 2n^2 - 3n + 1$

or nth term of the series $= 2n^2 - 3n + 7$

Sum of the first 50 terms $= \sum_{n=1}^{50}\left(2n^2 - 3n + 7\right)$

$$= 2 \times \frac{50 \times 51 \times 101}{6} - \frac{3 \times 50 \times 51}{2} + 7 \times 50 = 82375$$

80. Let un represent the nth term of the series.

$$u_n = \frac{1}{\{5+(n-1)4\}\{9+(n-1)4\}}$$

$$= \frac{1}{(4n+1)(4n+5)}$$

$$= \frac{1}{4}\left(\frac{1}{4n+1} - \frac{1}{4n+5}\right)$$

$$u_1 = \frac{1}{4}\left\{\frac{1}{5} - \frac{1}{9}\right\}$$

$$u_2 = \frac{1}{4}\left\{\frac{1}{9} - \frac{1}{13}\right\}$$

...

$$u_n = \frac{1}{4}\left\{\frac{1}{4n+1} - \frac{1}{4n+5}\right\}$$

Adding the above, Sum of the first n terms

$$= \frac{1}{4}\left(\frac{1}{5} - \frac{1}{4n+5}\right) = \frac{n}{(4n+5)5}$$

Sum to infinity of the series = limit of $\frac{1}{4}\left(\frac{1}{5} - \frac{1}{4n+5}\right)$

as n becomes very large or limit of the sum to n terms as n tends to infinity.

$$= \frac{1}{20}$$

81. The numbers are in AP with common difference -4.

82. $a = 20;\ d = 1;\ t_n = 99$

$$S = \frac{n}{2}(a + t_n)$$

$$n = \frac{99 - 20}{1} + 1 = 80$$

$\therefore \quad S = 40 \times 119 = 4760$

83. $\dfrac{1}{b}-\dfrac{1}{a}=\dfrac{1}{c}-\dfrac{1}{b}$

$\dfrac{a-b}{ab}=\dfrac{b-c}{bc}$

$\dfrac{b-c}{a-b}=\dfrac{bc}{ab}=\dfrac{c}{a}$

84. $ar=\dfrac{3}{4}\Rightarrow r=\dfrac{3}{4a}$

$\dfrac{a}{1-r}=4$

i.e., $\dfrac{a}{1-\dfrac{3}{4a}}=4\ \Rightarrow a=3, 1\Rightarrow\ r=\dfrac{1}{4},\dfrac{3}{4}$

85. S = sum of squares of first 20 numbers

$S_n=\dfrac{n(n+1)(2n+1)}{6}$

$S_{20}=\dfrac{20\times21\times41}{6}=2870$

86. c, a, b, d are in AP

$a-c=b-a=d-b$

$\therefore\ b-c=2(a-c)$

87. $a=\sqrt{5}$, $d=\sqrt{5}$

$t_{20}=a+19d=\sqrt{5}+19\sqrt{5}=20\sqrt{5}$

88. $\dfrac{1/n}{1/m}=\dfrac{m}{n}$ and $\dfrac{1/s}{1/n}=\dfrac{n}{s}$

Since m, n, s, t are in GP, $\dfrac{n}{m}=\dfrac{s}{n}$

So 1/m, 1/n, 1/s, 1/t are in GP with common ratio equal to the reciprocal of common ratio of m, n, s, t.

89. Let the numbers be $\dfrac{a}{r},a,ar$

$\dfrac{a}{r}\times a\times ar=216$

$a^3=216$

$\Rightarrow\ a=6$

$\dfrac{a^2}{r^2}+a^2+a^2r^2=189$

$\dfrac{1}{r^2}+1+r^2=\dfrac{189}{a^2}=\dfrac{21}{4}$

$\dfrac{1}{r^2}+r^2=\dfrac{17}{4}$

$4+4r^4=17r^2$

$4r^4-17r^2+4=0$

$r^2=4$ or $\dfrac{1}{4}$

$r=2$ or $\dfrac{1}{2}$

$\Rightarrow$ one of the numbers of this set = 12

90. $2b=a+c$

$(7^b)^2=7^{2b}=7^{a+c}=7^a\cdot7^c$

91. $1+2+3+....+n=\dfrac{n(n+1)}{2}=55$

$1^3+2^3+3^3+...+n^3=\left[\dfrac{n(n+1)}{2}\right]^2$

$=55\times55=3025$

92. $t_n=S_n-S_{n-1}$

$\therefore\ t_{12}=S_{12}-S_{11}=\dfrac{12\times13\times14}{3}-\dfrac{11\times12\times13}{3}$

$=12\times13=156$

93. $a=1, x=t_n=a+(n-1)d$

where $d=5$

$\Rightarrow\ n=\dfrac{x+4}{5}$

$\therefore\ \dfrac{n}{2}[2a+(n-1)d]=148$ gives

$\left(\dfrac{x+4}{10}\right)(1+x)=148$

$\Rightarrow\ x=-41, 36$

But, the given AP is increasing, and hence $x=36$.

94. Given: $\dfrac{a}{a_1},\dfrac{b}{b_1},\dfrac{c}{c_1}$ are in AP

$\Rightarrow\ \dfrac{b}{b_1}-\dfrac{a}{a_1}=\dfrac{c}{c_1}-\dfrac{b}{b_1}=k$ — (1)

Also given $ax^2+2bx\ c=0$ and

$a_1x^2+2b_1x+c_1=0$ have a common root

$(ac_1-a_1c)^2=2(ab_1-a_1b)(2bc_1-2b_1c)$

$(ac_1-a_1c)^2=4(ab_1-a_1b)(bc_1-b_1c)$

$a_1^2c_1^2\left(\dfrac{a}{a_1}-\dfrac{c}{c_1}\right)^2=4a_1b_1\,b_1c_1\left(\dfrac{a}{a_1}-\dfrac{b}{b_1}\right)\left(\dfrac{b}{b_1}-\dfrac{c}{cl}\right)$

$a_1c_1\left(\dfrac{a}{a_1}-\dfrac{c}{c_1}\right)^2=4b_1\,b_1\left(\dfrac{a}{a_1}-\dfrac{b}{b_1}\right)\left(\dfrac{b}{b_1}-\dfrac{c}{c_1}\right)$

$a_1c_1 = b_1^2 \Rightarrow b_1 = \sqrt{a_1 c_1}$

a_1, b_1, c_1 are in GP.

95. a, b, c are in GP

$\therefore \quad \Rightarrow b^2 = ac$

$\Rightarrow \quad 2\log_x b = \log_x a + \log_x c$

$\therefore \quad \log_x a, \log_x b, \log_x c$ are in AP

$\therefore \quad \log_a x, \log_b x, \log_c x$ are in HP

96. $\dfrac{a}{1-r} = 4$ .... (1)

$\dfrac{a^2}{1-r^2} = \dfrac{16}{3}$ ....(2)

Solving, $a = 2, \quad r = \dfrac{1}{2}$.

97. If numbers are equal,

$$AM = \frac{a + a .... + a(n \text{ terms})}{n} = a$$

$$HM = (a.a.....n \text{ terms})^{\frac{1}{n}} = (a^n)^{\frac{1}{n}} = a$$

$$HM = \frac{1}{\dfrac{\frac{1}{a}+\frac{1}{a}+.....\frac{1}{a}}{n}} = \frac{n}{\frac{n}{a}} = a$$

$\therefore \quad$ AM = GM = HM for equal numbers.

98. Since a, b, c are in A. P, $2b = a + c$ .... (1)

Since a, mb, c are in GP, $m^2 b^2 = ac$ .....(2)

$$m^2 b^2 = m^2 b \,.\, b = m^2 b\left(\frac{a+c}{2}\right) = ac$$

or $m^2 b = \dfrac{2ac}{a+c}$

$\therefore \quad a, m^2 b, c$ are in HP

99. $t_n = t_{n-1} - 1$ for $n \ge 3$

$\therefore \quad t_3 = t_2 - 1 = 1$

$t_4 = t_3 - 1 = 0$

$t_5 = t_4 - 1 = -1$

100. $\dfrac{T_2}{T_1} = \dfrac{T_3}{T_2} \Rightarrow 2^{(b-a)x} = 2^{(c-a)x}$

$\Rightarrow \quad (b - a)x = (c - b)x$

$\Rightarrow \quad (b - a) = (c - b)$ for all x $\quad x \neq 0$

Above is true as a, b, c are in AP

$2^{ax+1}, 2^{bx+1}, 2^{cx+1}$ are in GP for all $x \neq 0$

101. Let the first term of AP = A

Common difference of AP = d

$(a + 1)^h$ term = A + ad

7th term = A + 6d

$(b + 1)^h$ term = A + bd

$\therefore \quad (A + 6d)^2 = (A + ad)(A + bd)$

$36d^2 + 12Ad = abd^2 + (aA + bA)d$

$(36 - ab)d = (a + b - 12)A$

But $\dfrac{2ab}{a+b} = 6 \;\therefore\; ab = 3(a + b)$

$$(36 - ab)d = \left(\frac{ab-36}{3}\right)A$$

$\Rightarrow \quad -A = 3d \Rightarrow A + 3d = 0 \Rightarrow$ 4th term = 0

102. $S = 1 + \dfrac{4}{5} + \dfrac{7}{5^2} + \dfrac{10}{5^3} + ... + \infty$

Then $\dfrac{1}{5}S = \dfrac{1}{5} + \dfrac{4}{5^2} + \dfrac{7}{5^3} + ... + \infty$

$$\therefore S\left(1-\frac{1}{5}\right) = 1 + 3\left[\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + ... + \infty\right]$$

$$\frac{4}{5}S = 1 + 3\left[\frac{\frac{1}{5}}{1-\frac{1}{5}}\right] = 1 + \frac{3}{4} = \frac{7}{4}$$

$$\therefore S = \frac{7}{4} \times \frac{5}{4} = \frac{35}{16}$$

103. $a_1 + a_{2n} = a_2 + a_{2n-1} = a_n + a_{n+1} = k$

$$\text{Expression} = k\left\{\frac{\sqrt{a_1} - \sqrt{a_2}}{a_1 - a_2} + ... + \frac{\sqrt{a_n} - \sqrt{a_{n+1}}}{a_n - a_{n+1}}\right\}$$

$$= \frac{k}{-d}\left(\sqrt{a_1} - \sqrt{a_{n+1}}\right)$$

where, d is common difference

$$= \frac{k}{-d}\frac{a_1 - a_{n+1}}{\sqrt{a_1} + \sqrt{a_{n+1}}}$$

$$= (a_1 + a_{2n}).\frac{-nd}{-d\left(\sqrt{a_1} + \sqrt{a_{n+1}}\right)}$$

$$= \frac{n(a_1 + a_{2n})}{\sqrt{a_1} + \sqrt{a_{n+1}}}$$

104. Given $f(k) = \sum_{r=1}^{n} a_r - a_k$

$\Rightarrow f(k) + a_k = \sum_{r=1}^{n} a_r$

$f(1) + a_1 = f(2) + a_2 = f(3) + a_3 = \ldots$

Since $a_1, a_2, a_3, \ldots$ are in HP

$\therefore \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ are in AP

$\Rightarrow \frac{f(1)+a_1}{a_1}, \frac{f(2)+a_2}{a_2}, \frac{f(3)+a_3}{a_3} \ldots$ are in AP

$\Rightarrow 1+\frac{f(1)}{a_1}, 1+\frac{f(2)}{a_2}, 1+\frac{f(3)}{a_3}, \ldots$ are in AP

$\Rightarrow \frac{f(1)}{a_1}, \frac{f(2)}{a_2}, \frac{f(3)}{a_3}, \ldots$ are in AP

$\therefore \frac{a_1}{f(1)}, \frac{a_2}{f(2)}, \frac{a_3}{f(3)}, \ldots$ are in HP

105. a, b, c are in AP

$\therefore$ $2b = a + c$

tan a, tan b, tan c are in AP

$\therefore$ $2 \tan b = \tan a + \tan c$

Also, $\tan 2b = \tan (a + c)$

$$= \frac{\tan a + \tan c}{(1 - \tan a + \tan c)} = \frac{2 \tan b}{(1 - \tan a \ \tan c)}$$

$$\frac{2 \tan b}{1 - \tan^2 b} = \frac{2 \tan b}{1 - \tan a \tan c}$$

$\Rightarrow 1 - \tan^2 b = 1 - \tan a \tan c, \tan b \neq 0$

$\Rightarrow \tan^2 b = \tan a \tan c$

So, tan a, tan b, tan c are in GP They are in AP(given)

$\Rightarrow \tan a = \tan b = \tan c$

106. Since $\alpha, \beta, \gamma, \delta$ form an increasing GP

$\beta = \alpha r, \gamma = \alpha r^2, \delta = \alpha r^3, r > 1$

Now $\alpha + \beta = 3 \Rightarrow \alpha (1 + r) = 3$ — (1)

$\gamma + \delta = 12 \Rightarrow \alpha r^2 (1 + r) = 12$ — (2)

From (1) and (2) we get $r = 2, \alpha = 1$

$\therefore$ $a = \alpha\beta = \alpha^2 r = 2$

$b = \gamma\delta = \alpha^2 r^5 = 32$

107. The terms of the series

$$\frac{x^2 - \left(\frac{1}{2}\right)^2}{x - \frac{1}{2}}, \frac{x^3 - \left(\frac{1}{2}\right)^3}{x - \frac{1}{2}}, \frac{x^4 - \left(\frac{1}{2}\right)^4}{x - \frac{1}{2}} \ldots\ldots$$

So the sum is

$$= \frac{1}{x - \frac{1}{2}}\left[x^2 + x^3 + x^4 + \ldots - \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots\right]\right]$$

$$= \frac{2}{2x-1}\left[\frac{x^2}{1-x} - \frac{1}{4\left(1-\frac{1}{2}\right)}\right]$$

$$= \frac{2}{2x-1}\left[\frac{2x^2+x-1}{2(1-x)}\right] = \frac{2x^2+x-1}{(2x-1)(1-x)}$$

$$= \frac{(2x-1)(1+x)}{(2x-1)(1-x)} = \frac{1+x}{1-x}$$

108. Given that AM = 2(GM)

We have $\frac{a+b}{2} = 2\sqrt{ab}$

$$\frac{a+b}{2\sqrt{ab}} = \frac{2}{1}$$

Using componendo and dividendo we get

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{3}{1} \Rightarrow \frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{\left(\sqrt{a}-\sqrt{b}\right)^2} = \left(\frac{\sqrt{3}}{1}\right)^2$$

$$\Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{3}}{1}$$

Again using componendo and dividendo we get

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}-\sqrt{a}+\sqrt{b}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \Rightarrow \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\frac{a}{b} = \frac{\left(\sqrt{3}+1\right)^2}{\left(\sqrt{3}-1\right)^2} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$$

109. a, b, c are in AP $\Rightarrow$ $a + c = 2b$ — (1)

$(b - a), (c - b), a$ are in GP

$\therefore$ $(c - b)^2 = (b - a) a$

$(2b - a - b)^2 = (b - a) a$ [from (1)]

$(b - a)^2 = a (b - a)$

$b \neq a \therefore b - a \neq 0 \therefore b - a = a$

$b = 2a$ — (2)

Again from (1) and (2) we get

$a + c = 4a \quad c = 3a$

$a : b : c = a : 2a : 3a = 1 : 2 : 3$

110. $(bc + ca + ab)^2 = \Sigma b^2c^2 + 2abc\,(a + b + c)$

$$= \frac{1}{2}\left\{b^2\left(c^2+a^2\right)+c^2\left(a^2+b^2\right)+a^2\left(b^2+c^2\right)\right\}$$

$$+ 2abc\,(a + b + c) \qquad —(1)$$

We have, $\frac{c^2+a^2}{2} > ac$; $\frac{a^2+b^2}{2} > ab$; $\frac{b^2+c^2}{2} > bc$

[applying AM > GM for the pairs (c^2, a^2); (a^2, b^2); (b^2, c^2)]

Substituting in (1)

$(bc + ca + ab)^2 > \frac{1}{2}\,[b^2 \times 2ca + c^2 \times 2ab + a^2 \times 2bc]$

$+ 2abc\,(a + b + c) > \frac{1}{2} \times 2abc\,(a + b + c) + 2abc$

$(a + b + c) > 3abc\,(a + b + c)$

111. Statement 2 is the prime factorization theorem.

A let $p_1, p_2, \ldots p_n$ be the prime numbers in AP

Their common difference $= p_2 - p_1$

Then the $(p_1 + 1)$th term of this progression is

t_{p_1+1} is a composite number which is a contradiction

∴ Statement 1 is true but Statement 1 does not follow from Statement 2

⇒ Choice (b)

112. Statement 2 is a standard result

Statement 2 : $S_r = \dfrac{r}{1-\dfrac{1}{r+1}} = r + 1$

⇒ $S_r - r = 1$ which does not depend on r

∴ Statement 1 is false

⇒ Choice (d)

113. Statement 2 is a standard result

$a^2 + 2bc, b^2 + 2ac, c^2 + 2ab$ are in AP

⇒ $a^2 + 2bc - k, b^2 + 2ac - k, c^2 + 2ab - k$ in AP

$(k = ab + bc + ca)$

⇒ $-(a - b)(c - a), -(b - c)(a - b), -(c - a)(b - c)$ in AP

⇒ $\frac{(a-b)(c-a)}{m}, \frac{(b-c)(a-b)}{m}, \frac{(c-a)(b-c)}{m}$ in AP

(where $m = -(a - b)(b - c)(c - a)$)

⇒ $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ in AP

⇒ $b - c, c - a, a - b$ in HP

∴ Statement 1 is true and follows from Statement 2

⇒ Choice (a)

114. After the first half life, the remaining sample is $\frac{x}{2}$

After the next half life the remaining sample is $\frac{1}{2}\left(\frac{x}{2}\right) = \frac{x}{2^2}$

Similarly after the third half cycle the remaining sample $= \frac{x}{2^3}$ and so on

$\therefore y = \frac{x}{2^n}$; $n = 0, 1, 2, \ldots\ldots$

115. 12 years = 6 half cycles

∴ quantity left over alter 6 half cycles

$= 3200 \times \frac{1}{2^6} = 50$ kg

116. 12.5% of isotope $= \frac{1}{8}$ of the original

$= \frac{x}{8}$ if we had started with x units initially $= \frac{x}{2^3}$

∴ n = 3

∴ half-life $= \frac{8}{3}$ years

After 8 more years, total period = 8 + 8 = 16 years

∴ quantity remaining $= \frac{100}{2^6} = \frac{100}{64} = \frac{25}{16}$ %

117. We observe that the last term in the nth row

= nth term of the sequence $1^2, 3^2, 6^2, 10^2, \ldots..$

= square of the nth term of the sequence 1, 3, 6, 10,….

$$= \left(\frac{n(n+1)}{2}\right)^2$$

$$= \sum_{k=1}^{n} k^3$$

∴ (a) is true

nth row has n terms of the term

$(x + 1)^2, (x + 2)^2, \ldots(x + n)^2$

We know, $x + n = \frac{n(n+1)}{2} \Rightarrow x = \frac{n(n-1)}{2}$

∴ First term $= \left(\frac{n(n-1)}{2}+1\right)^2$ ⇒ (b, c) are true

Now

S_n = Sum of the elements in the nth row

$$= \sum_{k=1}^{n}(x+k)^2$$

$= nx^2 + xn(n+1) + \dfrac{n(n+1)(2n+1)}{6}$

$= \dfrac{n(n^2+2)(3n^2+1)}{12}$ $\quad$ (as $x = \dfrac{n(n-1)}{2}$)

$\therefore$ (d) is true

118. $\sum\limits_{m=1}^{k} m^2 = 1^2 + 2^2 + \ldots + k^2$

$= \dfrac{k(k+1)(2k+1)}{6} = \dfrac{k(2k^2+3k+1)}{6}$

$= \dfrac{2k^3+3k^2+k}{6} = \dfrac{1}{3}k^3 + \dfrac{1}{2}k^2 + \dfrac{1}{6}k$

$\sum\limits_{k=1}^{n}\left(\sum\limits_{m=1}^{k} m^2\right) = \sum\limits_{k=1}^{n}\left(\dfrac{1}{3}k^3 + \dfrac{1}{2}k^2 + \dfrac{1}{6}k\right)$

$= \dfrac{1}{3}.\dfrac{n^2(n+1)^2}{4} + \dfrac{1}{2}\dfrac{n(n+1(2n+1))}{6} + \dfrac{1}{6}\dfrac{n(n+1)}{2}$

$= \dfrac{1}{12}n(n+1)\left[n(n+1)+2n+1+1\right]$

$= \dfrac{1}{12}(n^2+n)(n^2+3n+2)$

$= \dfrac{1}{12}\left[n^4+4n^3+5n^2+2n\right]$

$= \dfrac{1}{12}n^4 + \dfrac{1}{3}n^3 + \dfrac{5}{12}n^2 + \dfrac{1}{6}n$

This is given as $an^4 + bn^3 + cn^2 + dn + e$

Comparing $a = \dfrac{1}{12}$ $b = \dfrac{1}{3}$, $c = \dfrac{5}{12}$ $d = \dfrac{1}{6}$ $e = 0$

(a), (b), (c), (d) are correct.

119. Given that

$2^p + 3^q + 5^r$ is a multiple of 4

$2^p + (4-1)^q + (4+1)^r$ = multiple of 4

$2^p + 4\lambda_1 + (-1)^q + 4\lambda_2 + (1)^r$ = multiple of 4

Case I

$p = 1$

Then q is even and r is any integer

So $p = 1$, $q = 2, 4, 6, 8, 10$ $r = 1, 2, 3, \ldots\ldots..9, 10$

$\therefore$ Number of ordered triples (p, q, r) is

$1 \times 5 \times 10 = 50$

$\Rightarrow$ (a) is true

Case II

Let $p \neq 1$

Then p can take any of the values 2, 3, ...10

There are 9 values for p

q must be odd and r can be any integer

$q = 1, 3, 5, 7, 9$ $r = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

There are 9 values for p, 5 values for q and 10 values for r

Number of ordered triplets (p, q, r)

$= 9 \times 5 \times 50 = 450$

$\Rightarrow$ (b) is true

Total number of ordered triplets (p, q, r) is

$50 + 450 = 500$

120. (a) Let $b-d, b, b+d$ be the numbers a, b, c respectively

Now $b^2 - 4ac \geq 0$

$\Rightarrow$ $b^2 - 4(b^2 - d^2) \geq 0$

$\Rightarrow$ $4d^2 \geq 3b^2 \Rightarrow \dfrac{d^2}{b^2} \geq \dfrac{3}{4} \Rightarrow \left|\dfrac{d}{b}\right| \geq \dfrac{\sqrt{3}}{2} = \sin\dfrac{\pi}{3}$

$\therefore$ $a \to r$

(b) Let $a = \dfrac{b}{r}$, $c = br$

$b^2 - 4ac \geq 0$

$\Rightarrow$ $b^2 - 4b^2 \geq 0 \Rightarrow -3b^2 \geq 0$

There exists no such real b.

$\therefore$ $b \to p, q, r, s$

(c) Since zero is a root, $\therefore$ $c = 0$

Let $b = a + d$, $c = a + 2d$

$c = 0 \Rightarrow a = -2d$

$\Rightarrow$ $\left|\dfrac{d}{a}\right| = \dfrac{1}{2}$

$= \cos\dfrac{\pi}{3}$

$\therefore$ $c \to q$

(d) Let $a = b - d$, $c = b + d$ and $c = br$, $a = br^2$

Given a, b, c are in AP and b, c, a are in GP

$\therefore$ $b - d = br$

$b + d = br^2 \Rightarrow 2b = br(1+r)$

$\Rightarrow$ $r^2 + r - 2 = 0 \Rightarrow r = -2, 1$

But $r \neq 1$

$\therefore$ $r = -2 = -\sec^2\dfrac{\pi}{4}$

$\therefore$ (d) $\to$ (p)

Additional Practice Exercise

121. Let the numbers be a, b, c, d

a, b, c are in AP $\Rightarrow 2b = a + c$ — (1)

b, c, d are in HP $\Rightarrow c = \dfrac{2bd}{(b+d)}$ — (2)

From (2) $\Rightarrow c(b + d) = 2bd$

$= d(a + c)$, from (1)

$\Rightarrow cb = da$ or $\dfrac{a}{b} = \dfrac{c}{d}$

122. We have

$$S_1 = \frac{n}{2}[2+(n-1)] = \frac{n(n+1)}{2}$$

$$S_2 = \frac{n}{2}[4+(n-1)3] = \frac{(3n+1)n}{2}$$

$$S_3 = \frac{n}{2}[6+(n-1)5] = \frac{(5n+1)n}{2}$$

:

:

$$S_{2k} = \frac{n}{2}[4k+(n-1)(4k-1)] = \frac{n}{2}[(4k-1)n+1]$$

(i) $S_1 + S_2 + S_3 + \ldots + S_{2k}$

$$= \frac{n^2}{2}\{1+3+5+7+\ldots+(4k-1)\}$$

$$+\frac{n}{2}\{1+1+\ldots+2k \text{ terms}\}$$

$$= \frac{n^2}{2}(2k)^2 + \frac{n}{2}\times 2k = 2n^2k^2 + nk$$

$= kn(2nk + 1)$

(ii) $S_1 - S_2 + S_3 - S_4 + - \ldots + S_{2k-1} - S_{2k}$

$= (S_1 - S_2) + (S_3 - S_4) + \ldots + (S_{2k-1} - S_{2k})$ — (1)

$$S_1 - S_2 = \frac{n(n+1)}{2} - \frac{(3n+1)n}{2} = \frac{n}{2}\{-2n\} = -n^2$$

$$S_3 - S_4 = \frac{(3n+1)n}{2} - \frac{(5n+1)n}{2} = \frac{n}{2}(-2n) = -n^2$$

$S_{2k-1} - S_{2k} = -n^2$

Substituting in (1)

$S_1 - S_2 + S_3 - S_4 + \ldots = -n^2k$

123. Let the series be $u_1 + u_2 + u_3 + \ldots$

$u_2 - u_1 = 4 = 4^1$

$u_3 - u_2 = 16 = 4^2$

$u_4 - u_3 = 64 = 4^3$

:

:

$u_n - u_{n-1} = 4^{n-1}$

Addition gives

$u_n - u_1 = 4 + 4^2 + 4^3 + \ldots + 4^{n-1}$

$$= \frac{4(4^{n-1}-1)}{4-1} = \frac{4^n-4}{3}$$

$$u_n = \frac{4^n-4}{3} + u_1 = \frac{4^n-4}{3} + 3 = \frac{4^n+5}{3}$$

We have to find

$$\sum_{r=1}^{n}\left(\frac{4^r+5}{3}\right) = \frac{1}{3}(4+4^2+4^3+\ldots+4^n) + \frac{5n}{3}$$

$$= \frac{4(4^n-1)}{3\times 3} + \frac{5n}{3} = \frac{4^{n+1}-4}{9} + \frac{5n}{3}$$

$$= \frac{4^{n+1}-4+15n}{9}$$

124. We have $d = a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1}$

(i) $\sec a_1 \sec a_2 = \dfrac{\sin(a_2 - a_1)}{\cos a_1 \cos a_2 \sin d}$

$$= \frac{1}{\sin d}\{\tan a_2 - \tan a_1\}$$

Similarly, $\sec a_2 \sec a_3 = \dfrac{\sin(a_3 - a_2)}{\cos a_2 \cos a_3 \sin d}$

$$= \frac{1}{\sin d}\{\tan a_3 - \tan a_2\}$$

Required expression $= \dfrac{1}{\sin d}\{\tan a_2 - \tan a_1\}$

$$+\frac{1}{\sin d}\{\tan a_3 - \tan a_2\} + \ldots$$

$$+\frac{1}{\sin d}\{\tan a_n - \tan a_{n-1}\}$$

$$= \frac{1}{\sin d}\{\tan a_n - \tan a_1\} = \frac{1}{\sin d}\frac{\sin(a_n - a_1)}{\cos a_n \cos a_1}$$

$$= \frac{\sin(n-1)d}{\sin d}\sec a_1 \sec a_n$$

(ii) $\tan^{-1}\left(\dfrac{d}{1+a_1a_2}\right) = \tan^{-1}\left(\dfrac{a_2-a_1}{1+a_1a_2}\right)$

$= \tan^{-1} a_2 - \tan^{-1} a_1$

Similarly,

$$\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)=\tan^{-1}a_3-\tan^{-1}a_2 \;\ldots\ldots\ldots\ldots$$

$$\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)=\tan^{-1}a_n-\tan^{-1}a_{n-1}$$

Addition gives, required expression

$$= \tan^{-1}a_n-\tan^{-1}a_1$$

$$=\tan^{-1}\left(\frac{a_n-a_1}{1+a_1a_n}\right)=\tan^{-1}\left(\frac{(n-1)d}{1+a_1a_n}\right)$$

Again, $\frac{1}{a_1}+\frac{1}{a_n}=\frac{a_n+a_1}{a_1a_n}=\frac{2a_1+(n-1)d}{a_1a_n}$

$$\frac{1}{a_2}+\frac{1}{a_{n-1}}=\frac{a_{n-1}+a_2}{a_2a_{n-1}}=\frac{2a_1+(n-1)d}{a_2a_{n-1}}$$

....

$$\frac{1}{a_n}+\frac{1}{a_1}=\frac{2a_1+(n-1)d}{a_na_1}$$

Addition gives

$$2\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)=\{2a_1+(n-1)d\}\times$$

$$\left[\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\ldots+\frac{1}{a_na_1}\right]$$

$$=(a_1+a_n)\left(\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\ldots+\frac{1}{a_na_1}\right)$$

125. The series can be rewritten as

$$\frac{4+1}{7}+\frac{4^2+3}{7^2}+\frac{4^3+5}{7^3}+\frac{4^4+7}{7^4}+\ldots$$

$$=\left\{\frac{4}{7}+\left(\frac{4}{7}\right)^2+\left(\frac{4}{7}\right)^3+\ldots\right\}+$$

$$\left\{\frac{1}{7}+\frac{3}{7^2}+\frac{5}{7^3}+\frac{7}{7^4}+\ldots\right\} \quad = S1 + S2 \qquad \text{— (1)}$$

where S_1 is a GP

S_1 = sum of the first 50 terms of the series $\frac{4}{7}+\left(\frac{4}{7}\right)^2+\ldots$

$$=\frac{\frac{4}{7}\left(1-\left(\frac{4}{7}\right)^{50}\right)}{1-\left(\frac{4}{7}\right)}=\frac{4}{7}\times\frac{7}{3}\left[1-\left(\frac{4}{7}\right)^{50}\right]=\frac{4}{3}\left[1-\left(\frac{4}{7}\right)^{50}\right]$$

S_2 = sum of the first 50 terms of the series

$$\frac{1}{7}+\frac{3}{7^2}+\frac{5}{7^3}+\ldots=\frac{1}{7}\left\{1+\frac{3}{7}+\frac{5}{7^2}+\ldots\right\}$$

$=\frac{1}{7}$ × sum of 50 terms of an arithmetico–geometric series

where, a = 1, d = 2, r = 1/7

$$=\frac{1}{7}\left\{\frac{1}{1-\frac{1}{7}}+\frac{2\times\frac{1}{7}\left(1-\left(\frac{1}{7}\right)^{49}\right)}{\left(1-\frac{1}{7}\right)^2}-\frac{(1+49\times2)\left(\frac{1}{7}\right)^{50}}{\left(1-\frac{1}{7}\right)}\right\}$$

Using the formula for the sum of the first n terms of an arithmetio-geometric series.

$$=\frac{2}{9}-\frac{1}{18\times7^{49}}-\frac{33}{2\times7^{50}}$$

Required sum = $S_1 + S_2$

126. Let $S_1 = a^3 + (a + d)^3 + (a + 2d)^3 + \ldots + (a + (n - 1)d)^3$

$= na^3 + d^3 [1^3 + 2^3 + 3^3 + \ldots + (n - 1)^3]$

$+ 3a^2d [1 + 2 + 3 + \ldots + (n - 1)]$

$+ 3ad^2 [1^2 + 2^2 + 3^2 + \ldots + (n - 1)^2]$

$$= na^3+\frac{d^3(n-1)^2n^2}{4}+\frac{3a^2d\,n(n-1)}{2}+\frac{3ad^2(n-1)\{2(n-1)+1\}n}{6}$$

$$= na^3+\frac{d^3n^2(n-1)^2}{4}+\frac{3a^2dn(n-1)}{2}+\frac{ad^2n(n-1)(2n-1)}{2} \qquad \text{— (1)}$$

$S = a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d]$

$$=\frac{n}{2}[2a+(n-1)d]$$

$$\frac{S_1}{S}=\frac{2a^3+\frac{d^3n(n-1)^2}{2}+3a^2d(n-1)+ad^2(n-1)(2n-1)}{[2a+(n-1)d]}$$

Set 2a = – (n – 1) d in the numerator

We get $2\times\frac{-(n-1)^3d^3}{8}+\frac{d^3n(n-1)^2}{2}$

$$+\frac{3d(n-1)(n-1)^2d^2}{4}-\frac{d^2(n-1)(2n-1)(n-1)d}{2}$$

$$= d^3\left[\frac{-(n-1)}{4}+\frac{n}{2}+\frac{3(n-1)}{4}-\frac{2n-1}{2}\right](n-1)^2 = 0$$

which means that S is a factor of S_1

127. Let the geometric series be

$a + ar + ar^2 + \ldots + ar^{n-1}$ where a and r are positive

This means that the set of numbers

$a, ar, ar^2, \ldots, ar^{n-1}$ is positive

Applying the result:

Arithmetic mean of a set of positive numbers is greater than their geometric mean, we obtain

$$\frac{a+ar+ar^2+\ldots+ar^{n-1}}{n} > \left[a(ar)(ar^2)\ldots(ar^{n-1})\right]^{1/n}$$

i.e., $> \left[a^n r^{\frac{(n-1)n}{2}}\right]^{1/n}$

i.e., $> ar^{\frac{n-1}{2}}$ = geometric mean of a and ar^{n-1}

128. Number of elements in the first group = 3

Number of elements in the second group = 7

Number of elements in the third group = 11

Number of elements in the nth group

$= 3 + (n - 1)4 = (4n - 1)$

Total number of elements in the first (n – 1) groups

$= 3 + 7 + 11 + \ldots$ to (n – 1) terms

$$= \frac{(n-1)}{2}\left[6+(n-2)4\right]$$

$$= (n-1)(2n-1) = 2n^2 - 3n + 1$$

Since the elements in the group are natural numbers and the first number is 2, the first element in the nth group

$= (2n^2 - 3n + 1) + 2 = 2n^2 - 3n + 3$

Sum of elements (or numbers) in the nth group

$$= \frac{(4n-1)}{2}\left[2(2n^2-3n+3)+(4n-2)\times 1\right]$$

$= (4n - 1)(2n^2 - n + 2)$

129. Since n is a root of the equation $x^2(1 - ac) - x(a^2 + c^2) - (1 + ac) = 0$,

$n^2(1 - ac) - n(a^2 + c^2) - (1 + ac) = 0$ — (1)

Let $h_1, h_2, \ldots, h_n$ be the n harmonic means between a and c, then $\frac{1}{a}, \frac{1}{h_1}, \frac{1}{h_2}, \ldots, \frac{1}{h_n}, \frac{1}{c}$ are in AP and if d is the common difference then $d = \frac{\left(\frac{1}{c}-\frac{1}{a}\right)}{(n+1)} = \frac{a-c}{(n+1)ac}$.

Then,

$$\frac{1}{h_1} = \frac{1}{a} + \frac{a-c}{(n+1)ac} = \frac{nc+a}{(n+1)ac}$$

and $\frac{1}{h_n} = \frac{1}{c} - \frac{a-c}{(n+1)ac}$

$$= \frac{(n+1)a-a+c}{(n+1)ac} = \frac{na+c}{(n+1)ac}$$

$$\Rightarrow h_1 - h_n = (n+1)ac\left[\frac{1}{nc+a}-\frac{1}{na+c}\right]$$

$$= \frac{(n+1)ac(na+c-nc-a)}{(nc+a)(na+c)}$$

$$= \frac{(n^2-1)ac(a-c)}{n^2ac+n(a^2+c^2)+ac}$$

[from (1) $\Rightarrow n(a^2 + c^2) = n^2(1 - ac) - (1 + ac)$]

$$= \frac{(n^2-1)ac(a-c)}{n^2-1} = ac(a-c)$$

130. Consider the numbers

$$\underbrace{a,a,a\ldots a}_{a^2\text{numbers}}\ \underbrace{b,b,b\ldots b}_{b^2\text{numbers}}\ \underbrace{c,c,c..c}_{c^2\text{numbers}}$$

There are $(a^2 + b^2 + c^2)$ numbers

Their $\text{AM} = \frac{a^2\times a + b^2\times b + c^2\times c}{a^2+b^2+c^2} = \frac{a^3+b^3+c^3}{a^2+b^2+c^2}$

Their $\text{GM} = \left(a^{a^2}\times b^{b^2}\times c^{c^2}\right)^{1/(a^2+b^2+c^2)}$

$$\text{HM} = \frac{a^2+b^2+c^2}{a^2\times\frac{1}{a}+b^2\times\frac{1}{b}+c^2\times\frac{1}{c}}$$

$$= \frac{a^2+b^2+c^2}{a+b+c}$$

Now applying the inequality AM ≥ GM ≥ HM

We get $\frac{a^3+b^3+c^3}{a^2+b^2+c^2} \geq \left(a^{a^2}b^{b^2}c^{c^2}\right)^{1/(a^2+b^2+c^2)}$

$$\geq \frac{a^2+b^2+c^2}{a+b+c}$$

$$\left(\frac{a^3+b^3+c^3}{a^2+b^2+c^2}\right)^{a^2+b^2+c^2} \geq a^{a^2}b^{b^2}c^{c^2}$$

$$\geq \left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a^2+b^2+c^2}$$

(ii) Consider

$$\underbrace{b,b,b\ldots b}_{a \text{ nos}}\ \underbrace{c,c,c..c}_{b \text{ nos}}\ \underbrace{d,d,d\ldots d}_{c \text{ nos}}\ \underbrace{a,a,a\ldots a}_{d \text{ nos}}$$

$$AM = \frac{ab+bc+cd+da}{a+b+c+d}$$

$$HM = \frac{a+b+c+d}{a\times\frac{1}{b}+b\times\frac{1}{c}+c\times\frac{1}{d}+d\times\frac{1}{a}}$$

$$= \frac{(a+b+c+d)(abcd)}{a^2cd+b^2ad+c^2ab+d^2bc}$$

Now using the inequality $AM \geq HM$

We get $\frac{ab+bc+cd+ad}{a+b+c+d} \geq \frac{(a+b+c+d)abcd}{a^2cd+b^2ad+c^2ab+d^2bc}$

$$\therefore \frac{(a+b+c+d)^2 abcd}{ab+bc+cd+ad}$$

$$\leq a^2cd+b^2ad+c^2ab+d^2bc$$

131. Let u_n represent the nth term of the series

$u_n = \{3 + (n - 1)4\} \{8 + (n - 1)7\}$

$= (4n - 1) (7n + 1)$

$= 28n^2 - 3n - 1$

Sum of the first n terms $= \sum_{r=1}^{n}(28r^2-3r-1)$

$$= \frac{28n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} - n$$

$$= \frac{28n(n+1)(2n+1)-9n(n+1)-6n}{6}$$

$$= \frac{n(56n^2+75n+13)}{6}$$

132. p, A1, A2, A3 , q are in AP

q = 5th term of the AP = p + 4d, if d is the common difference

$$\Rightarrow \quad d = \frac{q-p}{4}$$

$$A_1 = p+\frac{q-p}{4} = \frac{3p+q}{4};$$

$$A_2 = p+\frac{2(q-p)}{4} = \frac{2p+2q}{4};$$

$$A_3 = p+\frac{3(q-p)}{4} = \frac{p+3q}{4}$$

$$A_1A_2A_3 = \frac{(3p+q)(2p+2q)(p+3q)}{64} = \frac{15}{2} \ (1)$$

We have $\frac{1}{H_1} = \frac{\frac{3}{p}+\frac{1}{q}}{4} = \frac{3q+p}{4pq}$

$$\frac{1}{H_2} = \frac{\frac{2}{p}+\frac{2}{q}}{4} = \frac{2p+2q}{4pq}$$

$$\frac{1}{H_3} = \frac{\frac{1}{p}+\frac{3}{q}}{4} = \frac{3p+q}{4pq}$$

$$H_1H_2H_3 = \frac{64p^3q^3}{(3q+p)(2p+2q)(3p+q)} = \frac{18}{5}$$

(given) (2)

From (1) and (2), pq = 3

Since p and q are positive integers, p = 1, q = 3

133. From the given data,

$\frac{1}{a}, \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \ldots, \frac{1}{h_{2n}}, \frac{1}{b}$ are in AP

If d represents the common difference,

$$\frac{1}{b} = \frac{1}{a}+(2n+1)d$$

$$\Rightarrow \quad d = \frac{(a-b)}{ab(2n+1)}$$

$$\frac{1}{h_1} = \frac{1}{a}+\frac{a-b}{ab(2n+1)} = \frac{2nb+a}{ab(2n+1)}$$

$$\frac{1}{h_{2n}} = \frac{1}{b}-\frac{a-b}{ab(2n+1)} = \frac{2na+b}{ab(2n+1)}$$

$$\frac{h_1-a}{h_1+a} = \frac{(4nab+ab+a^2)}{(ab-a^2)}$$

…………………………………

…………………………………

$$\frac{h_{2n}+b}{h_{2n}-b} = \frac{4nab+ab+b^2}{(ab-b^2)}$$

$$\frac{h_1-a}{h_1+a}+\frac{h_{2n}+b}{h_{2n}-b} = \frac{4nab(b-a)}{ab(b-a)} = 4n$$

134. We have $2q = p + r$; $m = \dfrac{2\ell n}{\ell + n}$; $q^2m^2 = p\,\ell\, rn$

Substituting for q and m from the first and second relations in the third relation,

$$\left(\frac{p+r}{2}\right)^2\left(\frac{2\ell n}{\ell+n}\right)^2 = pr\ell n$$

$$\Rightarrow \frac{(p+r)^2}{pr} = \frac{(\ell+n)^2}{\ell n} \Rightarrow \frac{p}{r}+\frac{r}{p} = \frac{\ell}{n}+\frac{n}{\ell}$$

Multiplying by $\dfrac{p}{r}$

$$\left(\frac{p}{r}\right)^2 - \left(\frac{\ell}{n}+\frac{n}{\ell}\right)\frac{p}{r}+1=0$$

$$\Rightarrow \left(\frac{p}{r}-\frac{\ell}{n}\right)\left(\frac{p}{r}-\frac{n}{\ell}\right)=0$$

$$\Rightarrow \frac{p}{r}=\frac{\ell}{n} \text{ or } \frac{p}{r}=\frac{n}{\ell}$$

The second one is not admissible as $pl \neq rn$

Hence $\dfrac{p}{r}=\dfrac{\ell}{n}$ or $\dfrac{p}{\left(\frac{1}{n}\right)} = \dfrac{r}{\left(\frac{1}{\ell}\right)}$

From the third relation $q^2m^2 = p\,\ell\, rn$

We get $q^2m^2 = pr\lambda n = p^2n^2 \Rightarrow qm = pn$

or $\dfrac{q}{\left(\frac{1}{m}\right)} = \dfrac{p}{\left(\frac{1}{n}\right)}$ which leads to

$$\frac{p}{\left(\frac{1}{n}\right)} = \frac{q}{\left(\frac{1}{m}\right)} = \frac{r}{\left(\frac{1}{\ell}\right)}$$

135. Since the two quadratic equations have a common root,

$$\frac{\alpha^2}{2qr_1-2q_1r} = \frac{\alpha}{(rp_1-r_1p)} = \frac{1}{2pq_1-2p_1q}$$

where α represents the common root.

$$\Rightarrow \alpha^2 = \frac{qr_1-q_1r}{pq_1-p_1q},\ \alpha = \frac{(rp_1-r_1p)}{2(pq_1-p_1q)} \qquad \text{— (1)}$$

Since $\dfrac{p_1}{p},\dfrac{q_1}{q},\dfrac{r_1}{r}$ are in HP

$\dfrac{p}{p_1},\dfrac{q}{q_1},\dfrac{r}{r_1}$ are in AP

$$\Rightarrow \frac{q}{q_1}-\frac{p}{p_1} = \frac{r}{r_1}-\frac{q}{q_1}$$

$$\Rightarrow \frac{qp_1-q_1p}{p_1q_1} = \frac{q_1r-qr_1}{r_1q_1}$$

$$\Rightarrow \frac{pq_1-p_1q}{p_1} = \frac{qr_1-q_1r}{r_1}$$

$$\Rightarrow \frac{qr_1-q_1r}{pq_1-p_1q} = \frac{r_1}{p_1}$$

or $\alpha^2 = \dfrac{r_1}{p_1}$

We have $p_1\alpha^2 + 2q_1\alpha + r_1 = 0$

$$\Rightarrow p_1\left(\frac{r_1}{p_1}\right)+2q_1\left(\pm\sqrt{\frac{r_1}{p_1}}\right)+r_1=0$$

$$\Rightarrow -2r_1 = 2q_1\left(\pm\sqrt{\frac{r_1}{p_1}}\right)$$

Squaring $r_1^2 = \dfrac{q_1^2r_1}{p_1} \Rightarrow r_1p_1 = q_1^2$

$\Rightarrow$ p_1, q_1, r_1 are in GP

136. For the AP

1st term = p, nth term = q

$q = p + (n - 1)d$ where, d is the common difference

$$d = \frac{q-p}{(n-1)}$$

$(r + 1)$th term of the AP = $p+\dfrac{r(q-p)}{(n-1)}$

$$= \frac{p(n-1)+r(q-p)}{(n-1)} \qquad \text{— (1)}$$

For the HP

1st term = p, nth term = q

$\Rightarrow$ the corresponding AP has the first term $1/p$ and the nth term $1/q$

$\dfrac{1}{q} = \dfrac{1}{p}+(n-1)D$, where, D is the common difference.

$$\Rightarrow D = \frac{\frac{1}{q}-\frac{1}{p}}{(n-1)} = \frac{(p-q)}{pq(n-1)}$$

$(n - r)$th term of the corresponding AP

$$= \frac{1}{p}+\frac{(n-r-1)(p-q)}{pq(n-1)}$$

$$= \frac{q(n-1)+(n-r-1)(p-q)}{pq(n-1)}$$

$$= \frac{(n-r-1)p+qr}{pq(n-1)}$$

$$= \frac{p(n-1)+r(q-p)}{pq(n-1)}$$

Therefore, the (n – r)th term of the HP

$$= \frac{pq(n-1)}{p(n-1)+r(q-p)} \qquad \text{— (2)}$$

From (1) and (2) we see that the product of the terms

= pq independent of r

137. Let $\frac{a+b+c}{2} = s$

Then, a + b – c = 2s – 2c, b + c – a = 2s – 2a

c + a – b = 2s – 2b

(2s – 2c), (2s – 2a), (2s – 2b) are positive

(since a + b > c , b + c > a, c + a > b for a triangle)

Applying the result

$AM \geq GM$

To the above set of positive numbers,

$$\frac{(2s-2c)+(2s-2a)+(2s-2b)}{3}$$

$$\geq [(2s-2c)(2s-2a)(2s-2b)]^{1/3}$$

$$\Rightarrow \frac{2s}{3} \geq [(b+c-a)(c+a-b)(a+b-c)]^{1/3}$$

$$\Rightarrow \frac{8s^3}{27} \geq (b+c-a)(c+a-b)(a+b-c)$$

138. Let us consider a GP

$S_n = a + ar + ar^2 + \dots + ar^{n-1}$

$$\text{AM of 'n' terms of GP} = \frac{a+ar+ar^2+..+ar^{n-1}}{n}$$

$$= \frac{a(r^n-1)}{n(r-1)} \text{ If } r > 1.$$

HM of 'n' terms of GP

$$= \frac{n}{\frac{1}{a}+\frac{1}{ar}+\frac{1}{ar^2}+..+\frac{1}{ar^{n-1}}}$$

$$= \frac{n}{\frac{\left(\frac{1}{a}\right)\left(1-\frac{1}{r^n}\right)}{1-\frac{1}{r}}} = \frac{an(r-1)r^{n-1}}{r^n-1}$$

$$\text{Product of AM and HM} = \frac{a(r^n-1)}{n(r-1)} \times \frac{a.\,n(r-1)r^{n-1}}{r^n-1}$$

$$= a^2 r^{n-1}$$

$$= \left\{\left(a^n r^{\frac{n(n-1)}{2}}\right)^{\frac{1}{n}}\right\}^2 = (GM)^2$$

139. Let A be the first term and R is the common ratio of the given GP Then sum of the first n terms of the GP is

$$a = \frac{A(1-R^n)}{1-R} \qquad \text{— (1)}$$

The next n terms form a GP, with the first term as the (n + 1)th term of the given GP which is given by $t_{n+1} = AR^n$.

$\therefore$ The sum of the next n terms of the GP is

$$b = \frac{AR^n(1-R^n)}{1-R} \qquad \text{— (2)}$$

Similarly the sum of the next n terms is

$$c = \frac{AR^{2n}(1-R^n)}{1-R} \qquad \text{— (3)}$$

From (1), (2) and (3) we get $b = aR^n$ and $c = aR^{2n} = a(R^n)^2$

Hence a, b, c are in GP whose common ratio is R^n.

140. Let S = 6 + 66 + 666 + … (n terms)

$$\frac{9s}{6} = 9 + 99 + 999 \,......$$

$= (10 - 1) + (10^2 - 1) \dots. (10^n - 1)$

$= 10 + 10^2 + \dots 10^n - n$

$$\frac{3}{2}S = \frac{10(10^n-1)}{10-1} - n$$

$$S = \frac{20}{27}(10^n-1) - \frac{2}{3}n$$

141. $S_p = \frac{p}{2}[2a+(p-1)d] = 0$

$$\Rightarrow \; d = \frac{-2a}{p-1} \qquad \text{— (1)}$$

Now for the next q terms, t_{p+1} will be the first term

$t_{p+1} = a + pd$

$\therefore$ Sum of next q terms $=\frac{q}{2}\left(2\left(a+pd\right)+\left(q-1\right)d\right)$

$$=\frac{q}{2}\left\{2a-\frac{2a}{p-1}\left(2p+q-1\right)\right\} \text{ from (1)}$$

$$=\frac{-a\left(p+q\right)}{p-1}q$$

142. Let the common difference of the given AP be k

Given $d = a^2 + b^2 + c^2$

$\Rightarrow$ $a + 3k = a^2 + (a + k)^2 + (a + 2k)^2$

$\Rightarrow$ $5k^2 + 6ak - 3k + 3a^2 - a = 0$

$\Rightarrow$ $5k^2 + 3k(2a - 1) + 3a^2 - a = 0$ — (1)

k is real; so $D \geq 0$

$\Rightarrow$ $9(2a - 1)^2 - 4 \times 5(3a^2 - a) \geq 0$

$\Rightarrow$ $36a^2 - 36a + 9 - 60a^2 + 20a \geq 0$

$\Rightarrow$ $24a^2 + 16a - 9 \leq 0 \Rightarrow a^2 + \frac{2}{3}a - \frac{3}{8} \leq 0$

$\Rightarrow$ $\left(a+\frac{1}{3}\right)^2 \leq \frac{3}{8}+\frac{1}{9} \Rightarrow \left(a+\frac{1}{3}\right)^2 \leq \frac{35}{72}=\frac{70}{144}$

$\Rightarrow$ $\frac{-\sqrt{70}}{12} \leq \left(a+\frac{1}{3}\right) \leq \frac{\sqrt{70}}{12}$

$\Rightarrow$ $\frac{-4-\sqrt{70}}{12} \leq a \leq \frac{\sqrt{70}-4}{12}$

$\Rightarrow$ $a = -1, 0$, since a is an integer,

and since $d > 0$, all a, b, c, d are distinct integers, k must be +ve integer.

When $a = 0$ from (1) $k = 0, 3/5$

We reject both these values.

since k must be a positive integer.

When $a = -1$ from (1) $k = 1, 4/5 \Rightarrow k = 1$

$\therefore$ $a + b + c + d = -1 + 0 + 1 + 2 = 2$

143. Suppose a and b are the two numbers

a, A_1, A_2, b are in AP

$\therefore$ $A_1 + A_2 = a + b$ (1)

$3(A_2 - A_1) = b - a$

a, G_1, G_2, b are in GP

$\therefore$ $G_1G_2 = ab$ — (2)

$a, H_1 H_2, b$ are in HP

$\therefore$ $\frac{1}{a},\frac{1}{H_1},\frac{1}{H_2},\frac{1}{b}$ are in AP

$\therefore$ $\frac{1}{H_2}+\frac{1}{H_1}=\frac{1}{a}+\frac{1}{b}$

$$\frac{H_1+H_2}{H_1H_2}=\frac{a+b}{ab}$$

$\therefore$ $\frac{H_1+H_2}{H_1H_2}=\frac{A_1+A_2}{G_1G_2}$ [using (1) and (2)]

$\therefore$ $\frac{A_1+A_2}{H_1+H_2}=\frac{G_1G_2}{H_1H_2}$ — (3)

$$3\left(\frac{1}{H_2}-\frac{1}{H_1}\right)=\frac{1}{b}-\frac{1}{a}$$

$$3\left(\frac{H_2-H_1}{H_1H_2}\right)=\frac{b-a}{ab}$$

$\Rightarrow$ $\frac{A_2-A_1}{H_2-H_1}=\frac{G_1G_2}{H_1H_2}$ — (4) [using (1) and (2)]

From (3) and (4) we get

$$\frac{A_2-A_1}{H_2-H_1}=\frac{A_1+A_2}{H_1+H_2}$$

144. a, x, … b are n +2 terms of an AP

Then $d=\frac{b-a}{n+1}$ $\therefore$ $x = a + d$

$$=a+\frac{b-a}{n+1}=\frac{a\left(n+1\right)+b-a}{n+1}$$

$$=\frac{an+b}{n+1} \quad \text{------ (1)}$$

a, y ….. b are n +2 terms of an HP

$\therefore \frac{1}{a},\frac{1}{y}...\frac{1}{b}$ are in AP

$\therefore$ common difference $=\left(\frac{1}{b}-\frac{1}{a}\right)\Big/_{n+1}$

$$=\frac{a-b}{ab\left(n+1\right)}$$

$$\therefore \frac{1}{y}=\frac{1}{a}+\frac{a-b}{ab\left(n+1\right)}$$

$$=\frac{b\left(n+1\right)+a-b}{ab\left(n+1\right)}=\frac{nb+a}{ab\left(n+1\right)}$$

$$\therefore y=\frac{ab\left(n+1\right)}{nb+a} \quad \text{— (2)}$$

From (1) and (2)

$\therefore$ $xy=\frac{na+b}{nb+a}\times ab$

145. Let $x_1 = a$, $x_2 = ar$ and $x_3 = ar^2$
$y_1 = b$, $y_2 = br$ and $y_3 = br^2$

$\therefore$ Slope of AB $= \dfrac{b(1-r)}{a(1-r)} = \dfrac{b}{a}$

Slope of BC $= \dfrac{br(1-r)}{ar(1-r)} = \dfrac{b}{a}$

Hence AB is parallel to BC

$\therefore$ A, B, C are collinear.

146. Suppose 48, 27 and 64 are respectively the pth, qth and rth terms of the geometric progression where first term is a and common ratio is R.

Then $a R^{p-1} = 48$ — (1)

$a R^{q-1} = 27$ — (2)

$a R^{r-1} = 64$ — (3)

$\dfrac{(3)}{(1)} \Rightarrow R^{r-p} = \dfrac{64}{48} = \dfrac{4}{3}$ — (4)

$\dfrac{(3)}{(2)} \Rightarrow R^{r-q} = \dfrac{64}{27} = \left(\dfrac{4}{3}\right)^3$ — (5)

Comparing (4) and (5) we get

$R^{r-q} = R^{3(r-p)}$

$\therefore$ $R^{3r-3p-r+q} = R^0$

$\therefore$ $2r - 3p + q = 0$

r, p, and q are positive integers. So, we have to find positive integers satisfying $2r - 3p + q = 0$

Three unknowns and one equation. So we can give arbitrary values for two of them.

So we get infinite number of solutions.

So there are infinite number of geometric progressions containing the term 64, 27 and 48.

If $p = 3$ and $q = 1$, then $r = \dfrac{3p-q}{2} = 4$

$\therefore$ from (4) $R^{4-3} = \dfrac{4}{3}$

$R = \dfrac{4}{3}$

From (1) $aR^{3-1} = 48$

$a\left(\dfrac{4}{3}\right)^2 = 48 \Rightarrow a = 27$

So one of such GP containing 27, 48, 64 is

$27, 27\times\dfrac{4}{3}, 27\times\left(\dfrac{4}{3}\right)^2, 27\times\left(\dfrac{4}{3}\right)^3, 27\times\left(\dfrac{4}{3}\right)^4, \ldots\ldots$

i.e., 27, 36, 48, 64, ,……

147. $a = 1; t_n = 101; d = 2; n = 51$

$$S = \frac{51}{2}[1+101] = \frac{51\times102}{2} = 51\times51 = 2601$$

148. $a + (n - 1) d = 164$

$\dfrac{n}{2}.\left[2a+(n-1)d\right] = 3n^2 + 5n$

$\Rightarrow$ $a + 164 = (3n + 5) \times 2$

Now, a = sum of first term

$= 3 \times 1 + 5 = 8$

$\therefore$ $8 + 164 = (3n + 5) \times 2$

$3n = \left(\dfrac{172}{2} - 5\right)$

i.e., $3n = 81 \Rightarrow n = 27$

149. $2b = a + c$

$b^2 - ac = \left(\dfrac{a+c}{2}\right)^2 - ac$

$4(b^2 - ac) = (a - c)^2$

150. $a = 4; r = \sqrt{3}$;

$t_n = 36 \times 3^4$

$ar^{n-1} = 36 \times 3^4$

$4\times\left(\sqrt{3}\right)^{n-1} = 4\times3^6$

$3^{\frac{n-1}{2}} = 3^6$

$\dfrac{n-1}{2} = 6 \Rightarrow n = 13$

151. $a = 12; ar^5 = 384$

$r^5 = \dfrac{384}{12} = 32 \therefore r = 2$

152. We have,

Sum of the cubes of first n natural numbers

$= \left[\dfrac{n(n+1)}{2}\right]^2$

$S = \left(\dfrac{19\times20}{2}\right)^2 = 36100$

153. $a + 3d = 4$

$$S_7 = \frac{7}{2}(2a+6d) = 7\times(a+3d) = 7\times4 = 28.$$

154. $n = 9,\ d = \dfrac{-1}{6};\ a = \dfrac{1}{2}$

$$S_9 = \frac{9}{2}\left[2\times\frac{1}{2}+(9-1)\left(\frac{-1}{6}\right)\right] = \frac{-3}{2}$$

155. $\frac{n(n+1)}{2}=\frac{1}{7}\times\frac{n(n+1)(2n+1)}{6}$

$\Rightarrow$ 2n + 1 = 21 or n = 10

156. $ar^9 = 9$ ……….. (1)

$ar^3 = 4$ …………(2)

Dividing (1) by (2); $r^6=\frac{9}{4}$ or $r^3=\pm\frac{3}{2}$

$\Rightarrow$ $a=\pm\frac{8}{3}\Rightarrow t_7=a\,r^6=\pm\frac{8}{3}\times\frac{9}{4}=\pm 6$.

157. Let $g_1, g_2,\ldots g_n$ be n geometric means between a and b. then a, $g_1, g_2\ldots.g_n$ are in GP

Now if common ratio = r, then $a\,r^{n+1}= b$

$r^{n+1}=\frac{b}{a}$

$\therefore$ Product of geometric means = $a\,r\times a\,r^2\times\ldots a\,r^n$

$= a^n r^{1+2+\ldots n} = a^n r^{\frac{n(n+1)}{2}}$

$= a^n\left(\frac{b}{a}\right)^{\frac{n}{2}} = (ab)^{\frac{n}{2}}$

158. $b^2 = ac$ gives x = – 4 so that numbers are – 4, – 6, – 9

$\therefore$ 4th term = $\frac{-27}{2}$

159. $p=\frac{q+r}{2}$ and $q^2 = pr$ …………….. (1)

$2pq=2q\left(\frac{q+r}{2}\right)=pr+rq=r(p+q)$

$\Rightarrow$ $\frac{2pq}{p+q}=r$

160. $b^2=\frac{2a^2c^2}{a^2+c^2}$

$\Rightarrow$ $a^2b^2+b^2c^2=2a^2c^2$

$\Rightarrow$ a^2b^2, a^2c^2, b^2c^2 are in AP

161. Given that a^2, b^2, c^2 are in AP we have

$b^2 - a^2 = c^2 - b^2$

$\Rightarrow$ (b – a) (b + a) = (c – b) (c + b) — (1)

Consider $\frac{b}{c+a}-\frac{a}{b+c}=\frac{b(b+c)-a(c+a)}{(c+a)(b+c)}$

$=\frac{b^2-a^2+bc-ac}{(c+a)(b+c)}=\frac{(b-a)[a+b+c]}{(c+a)(b+c)}$

$=\frac{(c-b)(a+b+c)}{(c+a)(a+b)}$

$\left[\text{From(1) }\frac{b-a}{b+c}=\frac{c-b}{a+b}\right]$

$=\frac{c^2-b^2+ac-ab}{(c+a)(a+b)}=\frac{c}{a+b}-\frac{b}{c+a}$

$\Rightarrow$ $\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}$ are in AP

162. $\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}\Rightarrow\frac{a-b}{ab}=\frac{b-c}{bc}$

$\Rightarrow$ $\frac{a-b}{a}=\frac{b-c}{c}$ — (1)

$b^2 - a^2 = c^2 - b^2 \Rightarrow (b - a)(b + a) = (c - b)(c + b)$

$\frac{b+a}{b+c}=\frac{c-b}{b-a}=\frac{c}{a}$ from (1)

$\Rightarrow$ $a^2 + ab = c^2 + cb$

$\Rightarrow$ (a – c) (a + c) + b(a – c) = 0

$\Rightarrow$ (a – c) (a + b+ c) = 0

a + b + c = 0 since $a \neq c$

163. a + x, b + x, c + x are in HP

$\Rightarrow$ $b+x=\frac{2(a+x)(c+x)}{(a+x)+(c+x)}$

$\Rightarrow$ (b + x) (a + c + 2x) = 2(a + x) (c + x)

$\Rightarrow$ $(a + c + 2b)\,x + 2x^2 + ab + bc$

$= 2ac + 2x(a + c) + 2x^2$

$\Rightarrow$ x(c + a – 2b) = bc + ab – 2ac

$\Rightarrow$ $x(c + a - 2b) = bc + ab - 2b^2$

[$\because$ a, b, c are in GP]

$\Rightarrow$ x(c + a – 2b) = b(c + a – 2b)

$\Rightarrow$ x = b if $(c + a - 2b) \neq 0$

If c + a – 2b = 0 then a, b, c are in AP as well as in GP and therefore, a = b = c. We have assumed that a, b, c are distinct numbers

164. Consider $\log_3 18, \log_3 162, \log_3 1458$

Factorizing the no: we get

$18 = 3^2 \times 2$

$162 = 3^4 \times 2$

$1458 = 3^6 \times 2$

$\therefore$ $\log_3 18 = \log_3 3^2 + \log_3 2 = 2 + \log_3 2$

$\log_3 162 = \log_3 3^4 + \log_3 2 = 4 + \log_3 2$

$\log_3 1458 = \log_3 3^6 + \log_3 2 = 6 + \log_3 2$

since 2, 4, 6 are in AP

$2 + \log_3 2, 4 + \log_3 2, 6 + \log_3 2$ are in AP

$\therefore$ $\log_3 18, \log_3 162, \log_3 1458$ are in AP

So $\log_{18} 3, \log_{162} 3, \log_{1458} 3$ are in HP

165. Let d be the common difference of the AP

$$\frac{1}{a_1 a_3} = \frac{1}{2d}\left[\frac{a_3 - a_1}{a_1 a_3}\right] = \frac{1}{2d}\left[\frac{1}{a_1} - \frac{1}{a_3}\right]$$

Similarly

$$\frac{1}{a_2 a_4} = \frac{1}{2d}\left[\frac{1}{a_2} - \frac{1}{a_4}\right]$$

$$\frac{1}{a_3 a_5} = \frac{1}{2d}\left[\frac{1}{a_3} - \frac{1}{a_5}\right]$$

....

$$\frac{1}{a_{n-2} a_n} = \frac{1}{2d}\left[\frac{1}{a_{n-2}} - \frac{1}{a_n}\right]$$

$$\frac{1}{a_{n-1} a_{n+1}} = \frac{1}{2d}\left[\frac{1}{a_{n-1}} - \frac{1}{a_{n+1}}\right]$$

$$\frac{1}{a_n a_{n+2}} = \frac{1}{2d}\left[\frac{1}{a_n} - \frac{1}{a_{n+2}}\right]$$

$$\therefore \quad s_n = \frac{1}{2d}\left[\begin{array}{l}\frac{1}{a_1} - \frac{1}{a_3} + \frac{1}{a_2} - \frac{1}{a_4} + \ldots.. \\ \ldots.. + .. \frac{1}{a_{n-1}} - \frac{1}{a_{n+1}} + \frac{1}{a_n} - \frac{1}{a_{n+2}}\end{array}\right]$$

$$= \frac{1}{2d}\left[\frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_{n+1}} - \frac{1}{a_{n+2}}\right]$$

$$s_{2n} = \frac{1}{2d}\left[\frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_{2n+1}} - \frac{1}{a_{2n+2}}\right]$$

$\therefore$ $S_{2n} - S_n$

$$= \frac{1}{2d}\left[\frac{1}{a_{n+1}} + \frac{1}{a_{n+2}} - \frac{1}{a_{2n+1}} - \frac{1}{a_{2n+2}}\right]$$

$$= \frac{1}{2d}\left[\frac{a_{2n+1} - a_{n+1}}{a_{n+1} a_{2n+1}} + \frac{a_{2n+2} - a_{n+2}}{a_{n+2} a_{2n+2}}\right]$$

$$= \frac{n}{2}\left[\frac{1}{a_{n+1} a_{2n+1}} + \frac{1}{a_{n+2} a_{2n+2}}\right]$$

166. The sum depends on n

So if n is even, the sum is

$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (n-1)^2 - n^2$

$= -1 \times (1 + 2) + -1 \times (3 + 4) + \ldots +$

$-1 \times [(n-1) + n]$

$= -1[1 + 2 + 3 + 4 + \ldots + n]$

$$= -1 \times \frac{n(n+1)}{2} \qquad \text{— (1)}$$

If n is odd,

The sum becomes

$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (n-2)^2 - (n-1)^2 + n^2$

$= -1 \times (1 + 2) + -1(3 + 4)$

$+ \ldots + -1 \times [(n-2) + (n-1)] + n^2$

$$= -1 \times \frac{(n-1)n}{2} + n^2$$

$$= n\left[\frac{-n+1+2n}{2}\right] = \frac{n(n+1)}{2} \qquad \text{— (2)}$$

Using (1) and (2) the above sum is $\dfrac{(-1)^{n-1} n(n+1)}{2}$

167. $2^a + 2^{2a} + 2^{3a} + \ldots + 2^{na} + \dfrac{1}{2^a} + \dfrac{1}{2^{2a}} + \ldots + \dfrac{1}{2^{na}}$

$$= \left(2^a + \frac{1}{2^a}\right) + \left(2^{2a} + \frac{1}{2^{2a}}\right) + \ldots + \left(2^{na} + \frac{1}{2^{na}}\right)$$

Let $t = 2^{ka}$ {since $2^{ka} > 0$, $t > 0$ (always)}

Now by using AM $\geq$ GM

$$\frac{t + \frac{1}{t}}{2} \geq t \times \frac{1}{t}$$

$$t + \frac{1}{t} \geq 2$$

$$\therefore 2^{ka} + \frac{1}{2^{ka}} \geq 2 \quad \forall\ k$$

$\therefore$ The required sum is always $\geq 2n$

168. If α_i, I = 1, 2, … n are the given numbers

Then $(\alpha_1 + \alpha_2 \ldots + \alpha_n)^2 = \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2 + 2\sum_{i<j} \alpha_i \alpha_j$

$$\left(\sum \alpha_i\right)^2 = \sum \alpha_i^2 + 2\sum_{i<j} \alpha_i \alpha_j$$

$\therefore$ sum of products taken 2 of a time

$$= \frac{\left(\sum \alpha_i\right)^2 - \sum \alpha_i^2}{2}$$

So here, $\sum \alpha_i = 2 + 4 + \ldots + 2n$

$$= 2\,(1 + 2 + \ldots + n) = 2\frac{n(n+1)}{2}$$

$$\sum \alpha i^2 = 2^2 + 4^2 + 6^2 + \ldots + 4n^2$$

$$= 2^2\,[1 + 2^2 + 3^2 + \ldots + n^2]$$

$$= 4\left[\frac{n(n+1)(2n+1)}{6}\right]$$

So required sum

$$= \frac{1}{2}\left[(n(n+1))^2 - \frac{2n(n+1)(2n+1)}{3}\right]$$

$$= \frac{1}{2}n(n+1)\left[\frac{3n^2 + 3n - 4n - 2}{3}\right]$$

$$= \frac{n(n+1)(n-1)(3n+2)}{6}$$

$$= \frac{(n-1)n(n+1)(3n+2)}{6}$$

169. Let $n = 2m + 1$

Now $S_n = S_{2m+1} = S_{2m} + (2m + 1)$th term

$$= \frac{2m(2m+1)^2}{2} + (2m+1)\text{th term}$$

$$\left[\because S_n = \frac{n(n+1)^2}{2} \text{where n is even}\right]$$

$$= \frac{(n-1)n^2}{2} + n^{th} \text{ term} \quad [\because\ n = 2m + 1]$$

$$= \frac{(n-1)n^2}{2} + n^2 \quad (\therefore\ n \text{ is odd})$$

$$= \frac{n^2}{2}(n-1+2) = \frac{n^2(n+1)}{2}$$

170. $y^2 - 4y + 5 > 1$

$(y - 2)^2 + 1 > 1$ provided $y \neq 2$

i.e., $x^2 - 6x + 11 \neq 2$

$\Rightarrow\ x^2 - 6x + 9 \neq 0 \Rightarrow x \neq 3$

171. Let d be the common difference of the AP

$(b - a)^2 = d^2$

$(c - a)^2 = 4d^2$

$\therefore\ (b - a)^2 \neq (c - a)^2$

Statement 1 is false

$4(b^2 - ac) = 4[(a + d)^2 - a(a + 2d)]$

$= 4[a^2 + 2ad + d^2 - a^2 - 2ad] = 4d^2$

$(c - a)^2 = (2d)^2 = 4d^2$

$\therefore\ 4(b^2 - ac) = (c - a)^2$

Statement 2 is true

Choice (d)

172. Statement 2 is true

Statement 1

Let d_1 be the common difference of the AP a_1, a_2, a_3 and d_2 be the common difference of the AP b_1, b_2, b_3

$a_2 + 2b_2 = (a_1 + 2b_1) + d_1 + 2d_2$ and $a_3 + 2b_3$

$= (a_1 + 2b_1) + (2d_1 + 4d_2)$

$\Rightarrow$ Statement 1 is true

$\Rightarrow$ choice (b)

173. Statement 2 is true

Consider Statement 1

$$\frac{n}{2}(2a + (n-1)d) = 0$$

$2a + (n - 1)\,d = 0$

$$\Rightarrow\ d = \frac{2a}{(1-n)}$$

$1 - n < 0$ since $n > 1$ and if $a < 0$

d is > 0

only if $d = \frac{2a}{(1-n)}$, sum will reduce to zero.

$\Rightarrow$ Statement – 1 is false

Choice (d)

174. Statement 2 is true

Consider Statement 1

To prove that

$b + c - a, c + a - b, a + b - c$ are in AP

$\Rightarrow\ a - c - b, b - c - a, c - a - b$ are in AP

$\Rightarrow$ 2a, 2b, 2c are in AP

[adding $(a + b + c)$]

$\Rightarrow$ a, b, c are in AP

(which is given)

$\Rightarrow$ Statement – 1 is true.

Choice (a)

175. Statement –2 is true

Consider Statement 1

Let us take the AM and GM of the set of 15 numbers.

x, x, x, y, y, y, y, y, y, y, z, z, z, z, z

we have

$$\frac{(x+x+x)+(y+y+y+y+y+y+y)+(z+z+z+z+z)}{15}$$

$$\geq (x^3\ y^7\ z^5)^{\frac{1}{15}}$$

i.e., $(x^3y^7z^5)^{\frac{1}{15}} \leq \frac{45}{15} = 3$

$x^3\ y^7\ z^5 \leq 3^{15}$

Maximum value of $x^3\ y^7\ z^5 = 3^{15}$

$\Rightarrow$ Statement 1 is true.

Choice (b)

176. Statement 2 is false

x has to be positive for Statement 2 to be true

Since $0 < \theta < \frac{\pi}{2}$, both $\cos\theta$ and $\sec\theta$ are > 0

$\Rightarrow$ Statement 1 is true

Choice (c)

177. Statement 2 is true

Since a_1, a_2, a_3 are the AMs between x and y,

$\frac{1}{x}, \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{y}$ are in HP

$\Rightarrow$ Statement 1 is false.

Choice (d)

178. Statement 2 is true

Consider Statement 1

Given : ar = −1

$\frac{a}{1-r} = -1$

$r(1 - r) = 1 \Rightarrow r^2 - r + 1 = 0$

$r = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

$|r| = 1$

$\Rightarrow$ No such infinite GP can exist, since $|r| = 1$

Choice (a)

179. Statement 2 is true

Consider Statement 1

nth term $= ar^{n-1}$

(n + 1)th term $= ar^n$

Since the terms are alternately positive and negative r has to be < 0

Choice (a)

180. Statement 2 is true

However, Statement 1 is false.

$\Rightarrow$ choice (d)

181. If the number of groups is n

$1^3 + 2^3 + \ldots\ldots n^3 = 3025;$

$\frac{n^2(n+1)^2}{4} = 3025 = \frac{100 \times 121}{4}$

n = 10 satisfies the above

$\therefore$ number of groups = 10

182. Let the 'n' be the number of groups

$\frac{n^2(n+1)^2}{4} = 4356 = \frac{11^2.12^2}{4}$

$\therefore$ n = 11

$\therefore$ persons with lowest rank $= 11^3 = 1331$

183. We observe that the series is such that starting from the third term, each item is the sum of the just proceeding two terms. [such a series is known as a Fibonacci series]

$\therefore\ t_n = t_{n-1} + t_{n-2} (n \geq 3)$

$t_{n+1} = t_n + t_{n-1}$

$\therefore\ t_{n+1} - t_n = t_n - t_{n-2}$ or $2\ t_n = t_{n-2} + t_{n+1}$

$\therefore\ t_{n-2}, t_n, t_{n+1}$ form an AP $(n \geq 3)$

II

Number of elements in the 15th group = 2 × 15 − 1 = 29

Number pf numbers of the sequence in the first 14 groups = 1 + 3 + 5 + 7 + … 14 terms

$= 14^2 = 196$

Hence,

the first number in the 15th group

= 197th number of the sequence 1, 5, 9, 13,..

= 1 + 196 × 4 = 785

the last number in the 15th group

= 29th term of the sequence

785, 789, …….

= 785 + 28 × 4 = 897

Sum of the numbers in the 15th group

$= \frac{29}{2}[785 + 897] = 29 \times 841 = 24389$

III

nth term of the series = u_n

$$= \frac{5n}{4n^4+1}$$

$$= \frac{5n}{(2n^2+1)^2-4n^2}$$

$$= \frac{5n}{(2n^2-2n+1)(2n^2+2n+1)}$$

$$= \frac{\frac{5}{4}\left[2n^2+2n+1-(2n^2-2n+1)\right]}{(2n^2-2n+1)(2n^2+2n+1)}$$

$$= \frac{5}{4}\left[\frac{1}{2n^2-2n+1}-\frac{1}{2n^2+2n+1}\right]$$

$$u_1 = \frac{5}{4}\left[\frac{1}{1}-\frac{1}{5}\right]$$

$$u_2 = \frac{5}{4}\left[\frac{1}{5}-\frac{1}{13}\right]$$

$$u_3 = \frac{5}{4}\left[\frac{1}{13}-\frac{1}{25}\right]$$

$$u_4 = \frac{5}{4}\left[\frac{1}{2n^2-2n+1}-\frac{1}{2n^2+2n+1}\right]$$

Addition gives,

$$S_n = \frac{5}{4}\left[1-\frac{1}{2n^2+2n+1}\right] \qquad \text{— (1)}$$

$$S_{20} = \frac{5}{4}\left[1-\frac{1}{841}\right] = \frac{1050}{841}$$

$$S_{25} = \frac{5}{4}\left[1-\frac{1}{1301}\right] = \frac{5}{4}\times\frac{1300}{1301} = \frac{1625}{1301}$$

$$\lim_{n\to\infty} S_n = \frac{5}{4}$$

$$\frac{5}{4}-S_{25} = \frac{5}{4}-\frac{1625}{1301} = \frac{5}{5204}$$

$= 0.0009$ approximately

190. $f(n) = 0$ if $n + 50 < 100 \Rightarrow n < 50$

$$\therefore \quad \sum_{n=1}^{49} f(n) = 0$$

$$50 \le n \le 149 \Rightarrow 1 \le \frac{n+50}{100} < 2$$

$$\Rightarrow \sum_{n=50}^{149} f(n) = 1 + 1 \ldots 100 \text{ terms}$$

$$\Rightarrow \sum_{n=150}^{249} f(n) = 2 + 2 \ldots 100 \text{ terms}$$

....

....

$$\Rightarrow \sum_{n=950}^{1000} f(n) = 10 + 10 + \ldots 51 \text{ terms}$$

$$\sum_{n=1}^{100} f(n) = \sum_{n=1}^{49} f(n) + \sum_{n=50}^{100} f(n) = 0 + 51 = 51$$

$$\sum_{n=1}^{1000} f(n) = \sum_{n=1}^{49} f(n) + \sum_{n=50}^{149} f(n) + \ldots + \sum_{n=950}^{1000} f(n)$$

$$= 0 + 100 + 200 + \ldots + 900 + 510 = 5010$$

$$\sum_{n=350}^{549} f(n) = \sum_{n=350}^{449} f(n) + \sum_{n=450}^{549} f(n) = 400 + 500$$

$$= 900$$

191. (a) $ax^2 + 2bx + c = 0$

a, b, c are in AP

$2b = a + c$ — (1)

substituting $x = -1$ in $ax^2 + 2bx + c_1$

$a - 2b + c$ which is $= 0$, by (1)

$\Rightarrow$ (a) is true

(b) is false.

(c) a, b, c are in GP

$b^2 = ac$

If the roots of the equation are to be equal,

$4b^2 = 4ac$ or $b^2 = ac$

(c) is true

(d) a, b, c are in HP

$$b = \frac{2ac}{a+c}$$

$$\Rightarrow \frac{b}{c} = \frac{2}{1+\frac{c}{a}}$$

$$\Rightarrow \frac{b/a}{(c/a)} = \frac{2}{1+c/a}$$

$$\Rightarrow \frac{-(\alpha+\beta)}{\alpha\beta} = \frac{2}{1+\alpha\beta}$$

$\Rightarrow -(1 + \alpha\beta)(\alpha + \beta) = 2\alpha\beta$

$\Rightarrow 2\alpha\beta + (1 + \alpha\beta)(\alpha + \beta) = 0$

$\Rightarrow 2\alpha\beta + \alpha + \alpha^2\beta + \beta + \alpha\beta^2 = 0$

$\Rightarrow \beta(1+\alpha^2+2\alpha)+\alpha(1+\beta^2)=0$

$\Rightarrow \beta(1+\alpha)^2+\alpha(1+\beta^2)=0$

(d) is true.

192. (a) $\dfrac{1}{g_1^2-g_2^2}=\dfrac{1}{g_1^2-g_1^2r^2}=\dfrac{1}{g_1^2(1-r^2)}$

$$\frac{1}{g_2^2-g_3^2}=\frac{1}{(g_1r)^2-(g_1r^2)^2}=\frac{1}{g_1^2r^2(1-r^2)}$$

$$\frac{1}{g_3^2-g_4^2}=\frac{1}{(g_1r^2)^2-(g_1r^3)^2}=\frac{1}{g_1^2r^4(1-r^2)}$$

$$\frac{1}{g_{n-1}^2-g_n^2}=\frac{1}{(g_1r^{n-2})^2-(g_1r^{n-1})^2}=\frac{1}{g_1^2r^{2n-4}(1-r^2)}$$

Addition gives sum

$$=\frac{1}{g_1^2(1-r^2)}\left\{1+\frac{1}{r^2}+\left(\frac{1}{r^2}\right)^2+\ldots+\left(\frac{1}{r^2}\right)^{n-2}\right\}$$

$$=\frac{1}{g_1^2(1-r^2)}\left\{\frac{1-\left(\frac{1}{r^2}\right)^{n-1}}{1-\frac{1}{r^2}}\right\}$$

$$=\frac{1}{g_1^2(1-r^2)}\frac{r^{2n-2}-1}{(r^2-1)}\times\frac{r^2}{r^{2n-2}}$$

$$=\frac{-r^2}{(1-r^2)^2}\times\frac{1}{g_1^2}\left(\frac{r^{2n-2}-1}{r^{2n-2}}\right)$$

$$=\frac{-r^2}{(1-r^2)^2}\times\frac{1}{g_1^2}\left\{\frac{1}{r^{2n-2}}\right\}$$

$$=\frac{r^2}{(1-r^2)^2}\left\{\frac{1}{r^{2n-2}}-1\right\}\frac{1}{g_1^2}$$

$$=\frac{r^2}{(1-r^2)^2}\left\{\frac{1}{g_n^2}-\frac{1}{g_1^2}\right\}$$

(b) $\dfrac{1}{g_1g_2}=\dfrac{1}{g_1^2r}$

$$\frac{1}{g_2g_3}=\frac{1}{(g_1r)(g_1r^2)}=\frac{1}{g_1^2r^3}$$

$$\frac{1}{g_3g_4}=\frac{1}{(g_1r^2)(g_1r^3)}=\frac{1}{g_1^2r^5}$$

$$\frac{1}{g_{n-1}g_n}=\frac{1}{(g_1r^{n-2})(g_1r^{n-1})}=\frac{1}{g_1^2r^{2n-3}}$$

$$\text{sum}=\frac{1}{g_1^2r}\left\{1+\frac{1}{r^2}+\left(\frac{1}{r^2}\right)^2+\ldots+\left(\frac{1}{r^2}\right)^{n-2}\right\}$$

$$=\frac{1}{g_1^2r}\left\{\frac{1-\left(\frac{1}{r^2}\right)^{n-1}}{\left(1-\frac{1}{r^2}\right)}\right\}$$

$$=\frac{1}{g_1^2r}\times\frac{(r^{2n-2}-1)}{r^{2n-2}}\times\frac{r^2}{(r^2-1)}$$

$$=\frac{-r}{(1-r^2)g_1^2}\left(1-\frac{1}{r^{2n-2}}\right)$$

$$=\frac{-r}{(1-r^2)}\left\{\frac{1}{g_1^2}-\frac{1}{g_n^2}\right\}$$

$$=\frac{r}{(1-r^2)}\left\{\frac{1}{g_n^2}-\frac{1}{g_1^2}\right\}$$

(c) $\dfrac{1}{g_1^2+g_2^2}=\dfrac{1}{g_1^2+g_1^2r^2}=\dfrac{1}{g_1^2(1+r^2)}$

$$\frac{1}{g_2^2+g_3^2}=\frac{1}{g_1^2r^2+g_1^2r^4}=\frac{1}{g_1^2r^2(1+r^2)}$$

....

....

$$\frac{1}{g_{n-1}^2+g_n^2}=\frac{1}{g_1^2r^{2n-4}+g_1^2r^{2n-2}}$$

$$=\frac{1}{g_1^2r^{2n-4}\{1+r^2\}}$$

$$\text{Sum}=\frac{1}{(1+r^2)g_1^2}\left[1+\frac{1}{r^2}+\frac{1}{r^4}+\ldots+\frac{1}{(r^2)^{n-2}}\right]$$

$$=\frac{1}{(1+r^2)g_1^2}\left\{\frac{1-\left(\frac{1}{r^2}\right)^{n-1}}{1-\frac{1}{r^2}}\right\}$$

$$=\frac{r^2}{(1+r^2)(r^2-1)g_1^2}\left[1-\frac{1}{r^{2n-2}}\right]$$

$$=\frac{r^2}{(1-r^4)}\left\{\frac{1}{g_n^2}-\frac{1}{g_1^2}\right\}$$

(d) $\dfrac{g_1}{g_1+g_2}=\dfrac{g_1}{g_1(1+r)}=\dfrac{1}{1+r}$

$\dfrac{g_2}{g_2+g_3}=\dfrac{g_2}{g_2+g_2r}=\dfrac{1}{1+r}$

....

....

$\dfrac{g_{n-1}}{g_{n-1}+g_n}=\dfrac{1}{1+r}$

$\text{sum}=\dfrac{(n-1)}{(1+r)}$

193. Since $0<\theta<\dfrac{\pi}{4}, \sin^2\theta<1,\ \cos^2\theta<1,\ \tan^2\theta<1$

(a) $\sum_{k=0}^{\infty}\sin^{2k}\theta=\dfrac{1}{1-\sin^2\theta}=\sec^2\theta$

(b) $\sum_{k=0}^{\infty}\cos^{2k}\theta=\dfrac{1}{1-\cos^2\theta}=\operatorname{cosec}^2\theta$

(c) $\sum_{k=0}^{\infty}\tan^{2k}\theta=\dfrac{1}{1-\tan^2\theta}=\dfrac{\cos^2\theta}{\cos^2\theta-\sin^2\theta}$

$=\dfrac{\cos^2\theta}{\cos 2\theta}$

(d) $\cot\theta>1$ in $0<\theta<\dfrac{\pi}{4}$. Hence $\sum_{k=0}^{\infty}\cot 2k\theta$ does not exist.

194. $f(x)=0\Rightarrow x^3(p+q-r-s)+2(pq-rs)x^2+(pq(r+s)-rs(p+q))x=0$

Case (i) $x=0$ is a double root

$\Rightarrow$ coefficient of $x=0$

$\Rightarrow$ $pq(r+s)=rs(p+q)$

$\Rightarrow$ $\dfrac{1}{p}+\dfrac{1}{q}=\dfrac{1}{r}+\dfrac{1}{s}\Rightarrow\dfrac{2}{\frac{1}{p}+\frac{1}{q}}=\dfrac{2}{\frac{1}{r}+\frac{1}{s}}$

$\Rightarrow$ HM (p, q) = HM (r, s)

Now, sum of the roots = third root

$=-2\dfrac{(pq-rs)}{p+q-r-s}$

$\because\ \dfrac{2pq}{p+q}=\dfrac{2rs}{r+s}$ we have that each is equal to

$\dfrac{2(pq-rs)}{(p+q)-(r+s)}$

$\therefore$ Third root $=-\dfrac{2pq}{p+q}=-\text{HM}(p, q)$

Case (ii) $x=0$ be a distinct root. Then

$x^2(p+q-r-s)+2(pq-rs)x+pq(r+s)-rs(p+q)=0$

has equal roots. $\Rightarrow$ Discriminant = 0

$(pq-rs)^2=(p+q-r-s)[pq(r+s)-rs(p+q)]$

$\Rightarrow$ $(p-r)(p-s)(q-r)(q-s)=0$

$\Rightarrow$ $p=r$ or s (or) $q=r$ or s

$\therefore$ 'a' and 'b' are correct

195. $bc(a-b)(a-c)=bc(a^2-ac-ab+bc)$

$=abc(a-b-c)+b^2c^2$

By symmetry

$ca(b-c)(b-a)=abc(b-c-a)+c^2a^2$

$ab(c-a)(c-b)=abc(c-a-b)+a^2b^2$

$\therefore$ $\Sigma bc(a-b)(a-c)$

$=abc(-a-b-c)+a^2b^2+b^2c^2+c^2a^2$

$=\Sigma b^2c^2-abc(a+b+c)$

$=\dfrac{1}{2}\Sigma(bc-ca)^2$

which is always positive only

196. $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0\end{vmatrix}=0$

$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha+b & b\alpha+c & -(a\alpha^2+2b\alpha+c)\end{vmatrix}=0$

$(a\alpha^2+2b\alpha+c)(ac-b^2)=0$

$\therefore$ $ac-b^2=0\Rightarrow$ (a) is true

or $a\alpha^2+2b\alpha+c=0\Rightarrow$ (c) is true

197. (a) Given expression is a perfect square

$\Rightarrow$ $q^2(r-p)^2=4pr(p-q)(q-r)$

$\Rightarrow$ $\dfrac{1}{p^2}+\dfrac{1}{r^2}+\dfrac{2}{pr}=\dfrac{4}{q}\left(\dfrac{1}{p}+\dfrac{1}{r}\right)-\dfrac{4}{q^2}$

$\Rightarrow$ $\left(\dfrac{1}{p}+\dfrac{1}{r}\right)^2-\dfrac{4}{q}\left(\dfrac{1}{p}+\dfrac{1}{r}\right)+\dfrac{4}{q^2}=0$

$\Rightarrow$ $\left(\dfrac{1}{p}+\dfrac{1}{r}-\dfrac{2}{q}\right)^2=0\Rightarrow\dfrac{1}{p}+\dfrac{1}{r}=\dfrac{2}{q}$

$\Rightarrow$ p, q, r are in HP

We have HM (p, r) < AM(p, r)

$\Rightarrow \quad q < \dfrac{p+r}{2}$

$\therefore$ Minimum value of p + r is 2q

$\therefore$ 'a' is correct

(b) $\dfrac{1}{p}, \dfrac{1}{q}, \dfrac{1}{r}$ are in AP

(ie) $\dfrac{s-b}{\Delta} - \dfrac{s-a}{\Delta} = \dfrac{s-c}{\Delta} - \dfrac{s-b}{\Delta}$ (p, q, r are the exradii of ΔABC)

$\Rightarrow 2s - b + a = 2s - c + b$

$\Rightarrow a + c = 2b$

$\Rightarrow$ a, b, c in AP

Given a = 4, c = 8 we have b = 6

$\therefore$ 'c' is correct

(c) Area of $\Delta ABC = \sqrt{9 \times 5 \times 3 \times 1} = 3\sqrt{15}$

$\therefore$ 'd' is correct

198. (a) Let $T_n = \dfrac{1}{(2n+1)(2n+3)(2n+5)}$, the nth term of the series

Let $V_n = \dfrac{1}{(2n+3)(2n+5)}$

$V_{n-1} = \dfrac{1}{(2n+1)(2n+3)}$

$V_{n-1} - V_n = \dfrac{1}{(2n+1)(2n+3)} - \dfrac{1}{(2n+3)(2n+5)}$

$= 4T_n$

$\therefore \quad T_n = \dfrac{1}{4}\left(V_{n-1} - V_n\right)$

$\therefore \quad S_n = \sum_{n=1}^{21} \dfrac{1}{4}(V_o - V_{20})$

$= = \dfrac{1}{4}\left(\dfrac{1}{1.3} - \dfrac{1}{43.45}\right)$

$= \dfrac{161}{1935}$

$\therefore \quad a \rightarrow s$

(b) $1^3 - 2^3 + 3^3 - +\ldots+51^3 = 1^3 + 2^3 + 3^3 +\ldots.+51^3 - 2(2^3 + 4^3 +\ldots.+50^3)$

$= 1^3 + 2^3 + 3^3 +\ldots.+51^3 - 16(1^3 + 2^3 +\ldots+25^3)$

$= \left\{\left[\dfrac{(2n+1)(2n+2)}{2}\right]^2 - 16\left[\dfrac{n(n+1)}{2}\right]^2\right\}_{n=25}$

$= \left\{\left[(2n+1)(n+1)\right]^2 - 4\left[n(n+1)\right]^2\right\}_{n=25}$

$= [(n+1)^2(4n+1)]_{n=25}$

$= (26)^2 \times 101 = 68276$

$\therefore \quad b \rightarrow p$

(c) $\dfrac{10\times11}{2\times100} + \dfrac{10\times11\times21}{6\times1000} - \dfrac{10^2\times11^2}{4\times100^2}$

$= \dfrac{10\times11}{2\times100}\left(1 + \dfrac{21}{3\times10} - \dfrac{10\times11}{2\times100}\right)$

$= \dfrac{11}{20}\left(1 + \dfrac{7}{10} - \dfrac{11}{20}\right)$

$= \dfrac{11}{20}\left(\dfrac{20+14-11}{20}\right)$

$= \dfrac{11}{20} \times \dfrac{23}{20} = \dfrac{253}{400}$

$\therefore \quad c \rightarrow q$

(d) The nth term, $t_n = n(n+1)(n+2)(n+3)$

Let $V_n = n(n+1)(n+2)(n+3)(n+4)$

$V_n - V_{n-1} = 5T_n$

$\therefore \quad T_n = \dfrac{1}{5}\left(V_n - V_{n-1}\right)$

$\therefore \quad S_n = \sum_{n=1}^{10} T_n$

$= \sum_{n=1}^{10} \dfrac{1}{5}(V_n - V_{n-1})$

$= \dfrac{1}{5}\left(V_{10} - V_0\right)$

$= \dfrac{1}{5}\left(10.11.12.13.14 - 0\right) = 48048$

$\therefore \quad d \rightarrow r$

199 (a) $\log_a^{100} = \dfrac{\log 100}{\log a}$

$2\log_6^{10} = \dfrac{\log 100}{\log b}$

$2\log_c^{5} + \log_c^{4} = \dfrac{\log 100}{\log c}$

$\therefore \quad \dfrac{\log 100}{\log a}, \dfrac{\log 100}{\log b}, \dfrac{\log 100}{\log c}$ are in HP

$\Rightarrow$ log a, log b, log c are AP

$\Rightarrow$ a, b, c are in GP

(b) $\log(a^2 - c^2) = \dfrac{2\log(a-c) + \log(a^2 + 2b^2 + c^2)}{2}$

$2\log(a^2 - c^2) = 2\log(a-c) + \log(a^2 + 2b^2 + c^2)$

$2\log\left(\dfrac{a^2 - c^2}{a - c}\right) = \log(a^2 + 2b^2 + c^2)$

$(a + c)^2 = a^2 + 2b^2 + c^2$

$\Rightarrow b^2 = ac \therefore$ a, b, c are GP

(c) $b = \dfrac{a+c}{2}\, b^2 = \dfrac{2a^2c^2}{a^2 + c^2}$

$\Rightarrow \dfrac{(a+c)^2}{4} = \dfrac{2a^2c^2}{a^2 + c^2}$

$\Rightarrow (a + c)^2 (a^2 + c^2) - 8a^2c^2 = 0$

$\Rightarrow (a^2 + c^2 + 2ac)(a^2 + c^2) - 8a^2c^2 = 0$

$\Rightarrow (a^2 + c^2)^2 + 2ac(a^2 + c^2) - 8a^2c^2 = 0$

$(a^2 + c^2)^2 - 4a^2c^2 + 2ac(a^2 + c^2) - 4a^2c^2 = 0$

$\Rightarrow (a^2 - c^2)^2 + 2ac(a - c)^2 = 0$

$\Rightarrow (a - c)^2 ((a+c)^2 + 2ac) = 0$

$\Rightarrow (a - c)^2 = 0 \Rightarrow a = c$

$\therefore \quad b = \dfrac{a+c}{2} = \dfrac{2a}{2} = a$

$\therefore \quad a = b = c \therefore \quad a^3, b^3, c^3$, are AP, GP. HP and AGP.

(d) $b = \dfrac{a+c}{2}, c^2 = bd, d = \dfrac{2ce}{c+e}$

$\therefore \quad c^2 = \left(\dfrac{a+c}{2}\right)\dfrac{2ce}{c+e}$

$c = \dfrac{(a+c)e}{c+e}$

$\Rightarrow c^2 + ce = ae + ce$

$\Rightarrow c^2 = ae$

$\therefore$ a, c, e are in GP

200. (a) $x - y = \dfrac{a+b}{2} - \sqrt{ab}$

$= \left(\dfrac{\sqrt{a} - \sqrt{b}}{\sqrt{2}}\right)^2$

(b) a, A_1, A_2, b are AP

$A_1 - a = b - A_2$

$\Rightarrow A_1 + A_2 = a + b$

a, G_1, G_2, b are GP

$\dfrac{G_1}{a} = \dfrac{b}{G_2} \Rightarrow G_1G_2 = ab$

$\therefore \quad \dfrac{A_1 + A_2}{G_1G_2} = \dfrac{a+b}{ab}$

(c) a, A_1, A_2, b are in AP

$\Rightarrow A_1 + A_2 = a + b$

a, G_1, G_2, b are GP

$\Rightarrow G_1 G_2 = ab$

a, H_1, H_2, b are in HP

$\Rightarrow \dfrac{1}{H_1} - \dfrac{1}{a} = \dfrac{1}{b} - \dfrac{1}{H_2}$

$\Rightarrow \dfrac{1}{H_1} + \dfrac{1}{a} = \dfrac{1}{b} - \dfrac{1}{H_2}$

$\Rightarrow \dfrac{1}{H_1} + \dfrac{1}{H_2} = \dfrac{1}{a} + \dfrac{1}{b}$

$\dfrac{H_1 + H_2}{H_1H_2} = \dfrac{a+b}{ab}$

$\therefore \quad \dfrac{G_1G_2}{H_1H_2} \times \dfrac{H_1 + H_2}{A_1 + A_2} = \dfrac{G_1G_2}{A_1 + A_2} \times \dfrac{H_1 + H_2}{H_1H_2}$

$= \dfrac{ab}{a+b} \times \dfrac{a+b}{ab} = 1$

(d) From (c)

$\dfrac{1}{H_1} + \dfrac{1}{H_2} = \dfrac{a+b}{ab} = \dfrac{A_1 + A_2}{G_1G_2}$