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IIT JEE

SUPER COURSE IN

Mathematics

Coordinate Geometry and Vector Algebra

ALWAYS LEARNING

PEARSON

Super Course in Mathematics

COORDINATE GEOMETRY AND VECTOR ALGEBRA

for the IIT-JEE

Volume 4

Trishna Knowledge Systems

A division of
Triumphant Institute of Management Education Pvt. Ltd

PEARSON

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Preface

The IIT-JEE, the most challenging amongst national level engineering entrance examinations, remains on the top of the priority list of several lakhs of students every year. The brand value of the IITs attracts more and more students every year, but the challenge posed by the IIT-JEE ensures that only the *best* of the aspirants get into the IITs. Students require thorough understanding of the fundamental concepts, reasoning skills, ability to comprehend the presented situation and exceptional problem-solving skills to come on top in this highly demanding entrance examination.

The pattern of the IIT-JEE has been changing over the years. Hence an aspiring student requires a step-by-step study plan to master the fundamentals and to get adequate practice in the various types of questions that have appeared in the IIT-JEE over the last several years. Irrespective of the branch of engineering study the student chooses later, it is important to have a sound conceptual grounding in Mathematics, Physics and Chemistry. A lack of proper understanding of these subjects limits the capacity of students to solve complex problems thereby lessening his/her chances of making it to the top-notch institutes which provide quality training.

This series of books serves as a source of learning that goes beyond the school curriculum of Class XI and Class XII and is intended to form the backbone of the preparation of an aspiring student. These books have been designed with the objective of guiding an aspirant to his/her goal in a clearly defined step-by-step approach.

- **Master the Concepts and Concept Strands!**
  This series covers all the concepts in the latest IIT-JEE syllabus by segregating them into appropriate units. The theories are explained in detail and are illustrated using solved examples detailing the different applications of the concepts.
- **Let us First Solve the Examples—Concept Connectors!**
  At the end of the theory content in each unit, a good number of "Solved Examples" are provided and they are designed to give the aspirant a comprehensive exposure to the application of the concepts at the problem-solving level.
- **Do Your Exercise—Daily!**
  Over 200 unsolved problems are presented for practice at the end of every chapter. Hints and solutions for the same are also provided. These problems are designed to sharpen the aspirant's problem-solving skills in a step-by-step manner.
- **Remember, Practice Makes You Perfect!**
  We recommend you work out ALL the problems on your own – both solved and unsolved – to enhance the effectiveness of your preparation.

A distinct feature of this series is that unlike most other reference books in the market, this is not authored by an individual. It is put together by a team of highly qualified faculty members that includes IITians, PhDs etc from some of the best institutes in India and abroad. This team of academic experts has vast experience in teaching the fundamentals and their application and in developing high quality study material for IIT-JEE at T.I.M.E. (Triumphant Institute of Management Education Pvt. Ltd), the number 1 coaching institute in India. The essence of the combined knowledge of such an experienced team is what is presented in this self-preparatory series. While the contents of these books have been organized keeping in mind the specific requirements of IIT-JEE, we are sure that you will find these useful in your preparation for various other engineering entrance exams also.

We wish you the very best!

CHAPTER

1

STRAIGHT LINES

CHAPTER OUTLINE

Preview

STUDY MATERIAL

Coordinate Systems

Slope or Gradient of a Straight Line

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Point of Intersection of Two Lines
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- Straight Objective Type Questions (5)
- Assertion–Reason Type Questions (5)
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- Multiple Correct Objective Type Questions (3)
- Matrix-Match Type Question (1)

IIT ASSIGNMENT EXERCISE
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- Linked Comprehension Type Questions (9)
- Multiple Correct Objective Type Questions (8)
- Matrix-Match Type Questions (3)

COORDINATE SYSTEMS

Coordinate systems or reference systems are essential devices to locate the position of an object in a plane or in space. For this purpose we need to define two dimensional coordinate system (to represent points in a plane) and three dimensional coordinate system (to represent points in space).

Two dimensional coordinate system

We draw two number lines XOX' and YOY' perpendicular to each other and intersecting in O. (refer Fig.1.1) Here O is taken as the zero of both number lines.

O is called the origin, XOX' the x-axis and YOY' the y-axis. OX, OX' are taken as the positive and negative directions of the x-axis respectively and OY, OY' are taken as the positive and negative directions of the y-axis respectively. Therefore the distances measured along OX and OY are taken positive and distances along OX' and OY' are taken as negative.

Let P be a point on the plane. Draw PM perpendicular to the x-axis. Let OM = x, MP = y (measured in the same units). Then, P is identified by the ordered pair of numbers (x, y). We say that the coordinates of P are (x, y). It is clear that if PN is drawn perpendicular to the y- axis, ON = y.

XOX', YOY' represents a rectangular Cartesian system (named after Descartes). x is called the x-coordinate (or abscissa) of P and y is called the y-coordinate (or ordinate) of P.

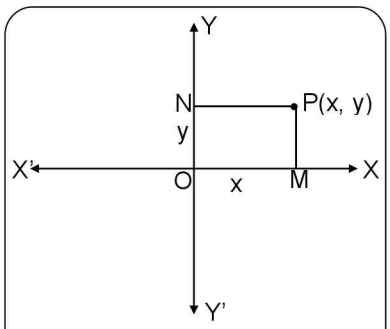


Fig. 1.1

From the above definition, we note that the coordinates of the origin are (0, 0).

The lines XOX' and YOY' divide the plane into 4 parts called quadrants. The point P can be in the first quadrant (XOY part) or in the second quadrant (X'OY part) or in the third quadrant (X'OY' part) or in the fourth quadrant (XOY' part).

If P is in the first quadrant [Refer (i) of Fig. 1.2]; x coordinate and y coordinate of P are positive.

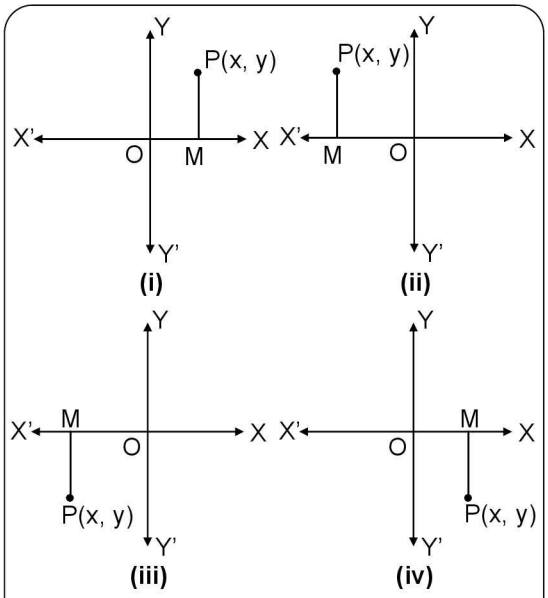


Fig. 1.2

If P is in the second quadrant [Refer (ii) of Fig. 1.2], x coordinate of P is negative, y coordinate of P is positive.

If P is in the third quadrant [Refer (iii) of Fig. 1.2], both x and y coordinates of P are negative.

If P is in the fourth quadrant [Refer (iv) of Fig. 1.2], x coordinate of P is positive, y coordinate of P is negative.

Suppose P, Q, R, S are four points in a plane and their coordinates referred to the rectangular Cartesian coordinate system XOX', YOY' are (1, 3), (–2, 5), (–4, –7) and (3, –6) respectively. Then we infer that P is in the first quadrant, Q is in the second quadrant, R is in the third quadrant and S is in the fourth quadrant.

We observe that points on the x-axis will have their coordinates in the form (x, 0). Points on the y-axis will have their coordinates in the form (0, y).

We also note that there is a one to one correspondence between points in the plane and the ordered pair of numbers (x, y).

We mention that if the coordinate system is changed (i.e., if we choose another rectangular Cartesian system) the representation of P (i.e., the coordinates of P) changes.

Distance between two points in a plane

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the two points, then the distance between P and Q is given by

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \text{ or } PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance of a point (x, y) from the origin is given by $\sqrt{x^2 + y^2}$

Chamber's dictionary defines line as 'that which has length without breadth or thickness'. Mathematically, a straight line can be defined as the curve (or locus) such that the area formed by any three points on the curve is zero.

In this unit, we give different forms of the equation of a straight line (in a plane). Point of intersection of two lines, angle between two lines, perpendicular distance from a point to a line, equations of the bisectors of the angles between two lines are discussed. We also deal with the equation of a pair of lines. Translation and rotation of axes of coordinates are also discussed.

SLOPE OR GRADIENT OF A STRAIGHT LINE

Let α be the angle made by a straight line with the positive direction of the x-axis of a rectangular Cartesian system. Then tan α is called the "slope" or gradient of the straight line. It is usually denoted by m.

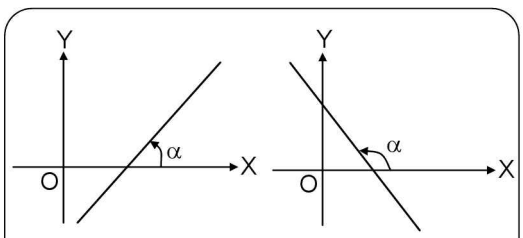


Fig. 1.3

If the straight line is parallel to the x-axis, $\alpha = 0$ and therefore, its slope = tan 0 = 0.

If the straight line is perpendicular to the x-axis (i.e., it is parallel to the y-axis), $\alpha = 90°$. Its slope is tan 90°, which is undefined. However, we write its slope as ∞ (called infinity) (∞ stands for an infinitely large number).

Note that $0° \le \alpha < 180°$.

From the definition of the slope of a line, it is clear that it is dependent on the choice of the coordinate system. For example, consider a line L which makes an angle 30° with the positive direction of the x-axis.

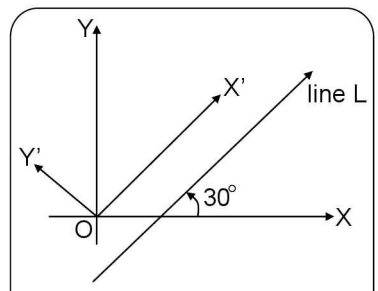


Fig. 1.4

[Refer Fig. 1.4]. The slope of the line L is tan 30° or it is equal to $\frac{1}{\sqrt{3}}$. Suppose we choose another rectangular coordinate system, say OX'Y' where OX' is parallel to the line L, the slope of the same line L in the new system OX'Y' is tan 0 or it is equal to zero.

Also, the slope of a straight line remains a constant with respect to a given coordinate system.

INTERCEPTS OF A STRAIGHT LINE

Let a straight line intersect the X-axis in A and Y-axis in B. Then OA is called the "x intercept" of the straight line and OB is called the "y intercept" of the straight line.

OA is taken as positive or negative according as A is on OX part or OX' part of the X-axis. Similarly, OB is taken as positive or negative according as B is on OY or OY' part of the Y-axis. If a straight line passes through the origin of the coordinate system, it is clear that both its x and y intercepts are zero. [Refer Fig. 1.3 (i) to (v)]

Fig. (i) → both x and y intercepts positive.

Fig. (ii) → x intercept negative, y intercept positive.

Fig. (iii) → both x and y intercepts negative.

Fig. (iv) → x intercept positive, y intercept negative.

Fig. (v) → both x and y intercepts zero.

A line not intersecting the X-axis will be parallel to the X-axis and similarly a line not intersecting the Y-axis will be parallel to the Y-axis.

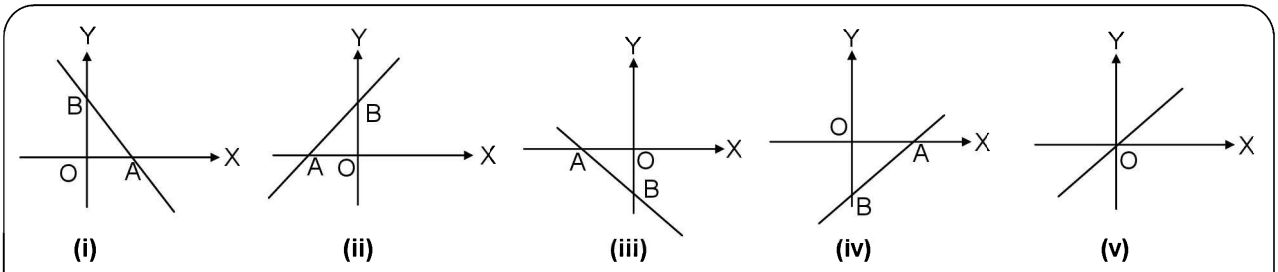


Fig. 1.5

EQUATION OF A STRAIGHT LINE

(i) Gradient form (or slope form)

In this case, we are given the slope and the y intercept of the line. Let the slope of the line be given as m and let the y intercept of the line be given as c.

Referring to Fig. 1.6, let P(x, y) be any point on the line. Draw PM perpendicular to the x-axis and AR perpendicular to PM.

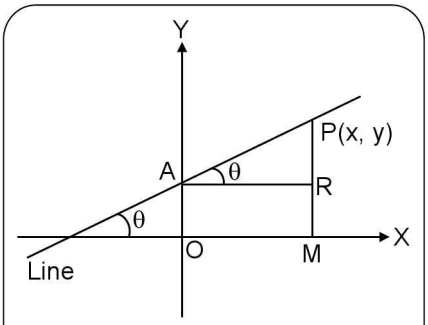


Fig. 1.6

We have

$$m = \tan\theta = \frac{PR}{AR} = \frac{PM - RM}{OM} = \frac{y - OA}{x} = \frac{y - c}{x}$$

Or, we get $y - c = mx \Rightarrow y = mx + c$

Or, the relation satisfied by any point (x, y) on the line is $y = mx + c$.

The equation of the line is $y = mx + c$.

For example,

(i) The equation of the line whose slope is 2 and whose y intercept is 7, is $y = 2x + 7$ or $2x - y + 7 = 0$.

(ii) The equation of the line which makes 60° with the x-axis and whose y intercept is −5 is $y = (\tan 60°)\, x - 5 \Rightarrow y = x\sqrt{3} - 5$

(ii) Slope of the line and a point on it are given

Let the slope of the line be given as m and $A(x_1, y_1)$ be a given point lying on the line [Refer Fig. 1.7].

Let P(x, y) represent a point on the line. Then, we have

$$m = \tan\theta = \text{slope of the line} = \frac{y - y_1}{x - x_1}$$

Or, $y - y_1 = m(x - x_1)$

The equation of the line in this case $y - y_1 = m(x - x_1)$

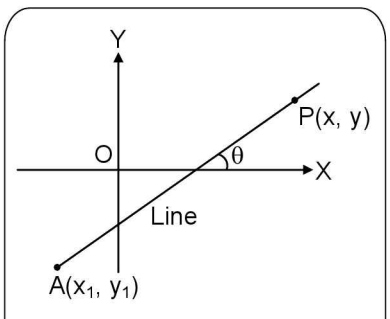


Fig. 1.7

CONCEPT STRANDS

Concept Strand 1

Find the equation of the line passing through (1, –5) and making an angle 135° with the x-axis.

Solution

Slope of the line = tan 135° = –1

The equation of the line is $y - (-5) = -1(x - 1)$

$\Rightarrow \quad x + y + 4 = 0$

Concept Strand 2

Find the equation of the line with slope 2 and passing through the origin.

Solution

Slope m = 2

The equation of the line is

$\therefore \quad y - 0 = 2(x - 0)$

$\Rightarrow \quad y = 2x$

(iii) Two points on the line are given

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points which lie on the line. Let P(x, y) be any point on the line. Refer Fig. 1.8.

Slope of the line joining A and B = $\dfrac{y_2 - y_1}{x_2 - x_1}$

Therefore, using (ii), equation of the line is

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$$

or, $$y - y_2 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_2).$$

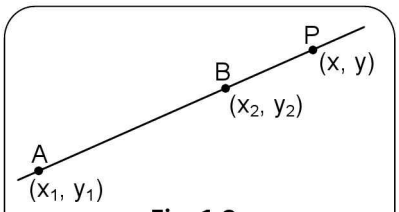


Fig. 1.8

CONCEPT STRAND

Concept Strand 3

Find the equation of the line passing the through the points (1, –6) and (3, 7):

Solution

Slope of the line = $\dfrac{7 + 6}{3 - 1} = \dfrac{13}{2}$

Equation of the line is $y + 6 = \left(\dfrac{13}{2}\right)(x - 1)$

or $y - 7 = \dfrac{13}{2}(x - 3)$

$\Rightarrow \quad 13x - 2y - 25 = 0$

(iv) Intercepts form

In this case, the x and y intercepts of the line are given. Let OA = a, OB = b

$\Rightarrow$ A(a, 0) and B(0, b) are points on the line. (Refer Fig. 1.9).

Its slope is given by $\dfrac{b - 0}{0 - a}$ or $\dfrac{-b}{a}$, (slope using the two points given)

Therefore, the equation of the line is

$$y - 0 = \frac{-b}{a}(x - a)$$

or $y - b$

$$= \frac{-b}{a}(x - 0)$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1.$$

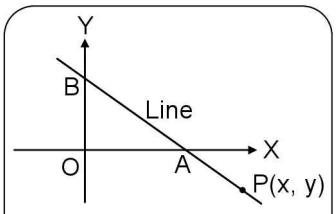


Fig. 1.9

CONCEPT STRAND

Concept Strand 4

Find the equation of the line whose x and y intercepts are

(i) 3 and 8 respectively

(ii) −5 and 7 respectively

Solution

(i) $\frac{x}{3} + \frac{y}{8} = 1 \quad \Rightarrow \quad 8x + 3y - 24 = 0$

(ii) $\frac{x}{-5} + \frac{y}{7} = 1 \quad \Rightarrow \quad 7x - 5y + 35 = 0$

(v) Parametric form (or Symmetric form)

Let a straight line pass through the point $Q(x_1, y_1)$ and make an angle θ with the positive direction of the x-axis.

Let P be a point (x, y) on the line at a distance r from Q (ie QP = r). Draw QN, PM perpendiculars to the x-axis and QR perpendicular to PM. (Refer Fig. 1.10)

Then, QR = PQ cos θ and RP = QP sin θ ; But QR = $x - x_1$ and RP = $y - y_1$

$\Rightarrow \quad x - x_1 = r\cos\theta$ and $y - y_1 = r\sin\theta$.

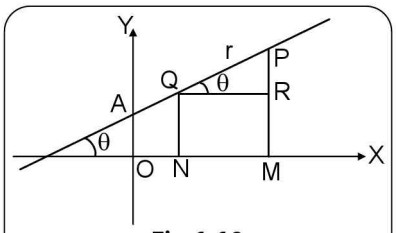


Fig 1.10

Thus we have $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ for any P(x, y) on the line distant r from (x_1, y_1).

This form is called parametric form of the equation of a line, the parameter being r.

This form is very useful when we wish to bring into consideration the distance of a variable point on the line from a fixed point on it.

The coordinates of any point on the line $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ are $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ where, r is the algebraic distance of (x, y) from the point (x, y). The point $Q(x_1, y_1)$ on the line divides it into two portions, r will be positive for points on one portion of the line and r will be negative for points on the other portion of the line.

CONCEPT STRAND

Concept Strand 5

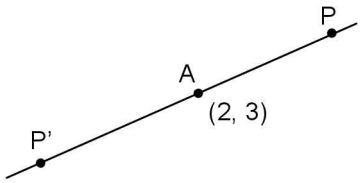
Find the equation of the line passing through the point (2, 3) and having slope $\sqrt{3}$ in its symmetric form (or parametric form).

Solution

$\frac{x - 2}{\cos\theta} = \frac{y - 3}{\sin\theta} = r$ where $\tan\theta = \sqrt{3}$ or $\theta = 60°$

or $\frac{x - 2}{\left(\frac{1}{2}\right)} = \frac{y - 3}{\left(\frac{\sqrt{3}}{2}\right)} = r$

or $\dfrac{x-2}{\left(\frac{1}{2}\right)} \equiv \dfrac{4-3}{\left(\frac{\sqrt{3}}{2}\right)} = r$



Any point on the above line may be represented in the form $(2 + r\cos\theta, 3 + r\sin\theta)$ — (1)

Suppose we want to find the coordinates of P where, AP = 4. We put r = 4 in (1) and get the coordinates of P as $(2 + 4\cos\theta, 3 + 4\sin\theta)$

i.e., $\left(2 + 4 \times \frac{1}{2}, 3 + 4\frac{\sqrt{3}}{2}\right)$

$= \left(4, 3 + 2\sqrt{3}\right)$.

Again, let P' be the point which is at a distance of 3 units from A on the other side (Refer figure). Since P' and P are on opposite sides of A, we have to take r = −3 to get the coordinates of P'. Putting r = −3 in (1),

Coordinates of P' are $(2 - 3\cos 60°, 3 + 3\sin 60°)$

$= \left(\frac{1}{2}, 3 + \frac{3\sqrt{3}}{2}\right)$

On examining the nature of the equation of a straight line in different cases, we infer that its general form is ax + by + c = 0, where a, b, c are constants OR the equation of a straight line will be a linear relation in x and y. (i.e., the relation will contain only first degree terms in x and in y and a constant term). Thus, we are able to identify the straight-line locus by the linear structure of its equation. We can always reduce the general form of the equation of a line to any of the standard forms like slope form or intercepts form.

Observations

(i) A straight line parallel to the x-axis will have its slope equals zero. Therefore, its equation is of the form y = k (k, a constant). In particular, y = 0 represents the x-axis (or, the equation of the x-axis is y = 0).

(ii) Consider a straight line parallel to the y-axis. Every point on the line has the same x-coordinate = k (say). Thus, x = k represents a straight line parallel to the y-axis. In particular, x = 0 represents the y-axis. (or the equation of the y-axis is x = 0)

(iii) We know that the general equation of a line is ax + by + c = 0. Remember that the equation of a line is the relation satisfied by the x and y coordinates of any point on the line. Suppose, the line passes through the origin, i.e., (0, 0) is a point on the line. Then, (0, 0) must satisfy the equation of the line. Thus, a × 0 + b × 0 + c must be = 0 giving c = 0. Therefore, if a straight line passes through the origin, its equation will be of the form ax + by = 0, i.e., the equation will not contain constant term.

CONCEPT STRANDS

Concept Strand 6

Reduce the equation 3x − 4y + 8 = 0 to the slope form.

Solution

We have 4y = 3x + 8 or $y = \frac{3}{4}x + 2$ which is in the slope form.

Slope = $\frac{3}{4}$; y intercept = 2.

Concept Strand 7

Reduce the equation 5x + 12y − 60 = 0 to the intercepts form.

Solution

We have 5x + 12y = 60.

Dividing both sides by 60 we get $\frac{x}{12} + \frac{y}{5} = 1$ which is in the intercepts form.

x intercept = 12; y intercept = 5.

Concept Strand 8

Find the equation of the line cutting off intercepts on the axes of coordinates equal in magnitude but opposite in sign and passing through the point (2, 3).

Solution

If a and b are the x and y intercepts respectively, $a + b = 0$ or $b = -a$

The equation of the line is $\frac{x}{a} - \frac{y}{a} = 1$ or $x - y = a$.

Since the line passes through (2, 3), the coordinates (2, 3) must satisfy the equation.

$\Rightarrow \quad 2 - 3 = a$

or $a = -1$.

So, the equation of the line is $x - y + 1 = 0$.

POINT OF INTERSECTION OF TWO LINES

Let the equations of two lines ℓ_1 and ℓ_2 be given as

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0.$$

Assuming that the two lines are not parallel to each other, let P (x_1, y_1) represent the coordinates of the point of intersection of the lines. Since P lies on both the lines, both equations must be satisfied when x and y are replaced by x_1 and y_1 respectively.

i.e., $a_1x_1 + b_1y_1 + c_1 = 0$

$a_2x_1 + b_2y_1 + c_2 = 0$

Solving the above linear equations in x_1 and y_1, we get the coordinates of the point of intersection as

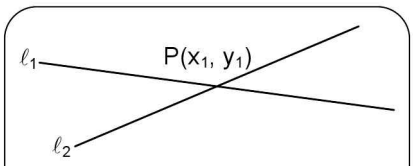


Fig. 1.11

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right).$$

CONCEPT STRANDS

Concept Strand 9

Obtain the coordinates of the point of intersection of the lines $3x + 4y - 18 = 0$ and $4x - 5y + 7 = 0$.

Solution

For obtaining the point of intersection of the two lines, we simply solve the equations $3x + 4y = 18$, $4x - 5y = -7$. This gives (2, 3) as the point of intersection.

Concept Strand 10

Show that the equation $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$ where λ is a constant represents a line passing through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

Solution

Note that the equation is linear in x and y. Therefore, it represents a straight line. Now, if (x_1, y_1) represent the coordinates of the point of intersection of the lines, $a_1x_1 + b_1y_1 + c_1 = 0$ and $a_2x_1 + b_2y_1 + c_2 = 0$. This means that the line $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$ passes through (x_1, y_1) for any value of λ. This completes the proof.

CONCURRENCY OF THREE LINES

Three lines are said to be concurrent if all the three of them pass through the same point.

Consider three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$. Supposing no two of the three lines are parallel to each other, we can find the intersection point of any two lines. If the third line passes through this intersection point, then the three lines are concurrent.

Intersection point of the second and the third lines is found as $\left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}, \frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2}\right)$ and substituting in the first equation we get,

$a_1\left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}\right) + b_1\left(\frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2}\right) + c_1$. If this value equals zero then the three lines are concurrent.

For example, to check if the lines $3x + y - 2 = 0$, $7x - 3y - 10 = 0$, $2y - x + 3 = 0$ are concurrent, let us find the intersection point of the first two lines. This is found to be $= (1, -1)$.

Substituting this point in the third equation $2y - x + 3 = 2 \times -1 - 1 + 3 = 0$, which means that the third line also passes through the point of intersection of the first two lines. Or the three lines are concurrent.

Note that the above condition for concurrency can be expressed in the form of a determinant.

Since, $a_1\left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}\right) + b_1\left(\frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2}\right) + c_1 = 0$

(for concurrency)

$$\Rightarrow \frac{a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)}{a_2b_3 - a_3b_2} = 0$$

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Remark

Concurrency of lines can have a notable algebraic interpretation as 'consistency' of a linear system of equations and the above results are studied in detail in a later Unit 'Matrices and Determinants'.

CONCEPT STRAND

Concept Strand 11

If the lines $ax + 2y + 1 = 0$, $bx + 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent show that a, b, c are in A.P.

Solution

Let us find the point of intersection of the first two lines

$ax + 2y + 1 = 0$ — (1)

$bx + 3y + 1 = 0$ — (2)

$(1) \times 3 - (2) \times 2 \Rightarrow (3a - 2b)x + 1 = 0$ or x

$$= \frac{-1}{(3a - 2b)}$$

Substituting in (1) $y = \frac{-1 - ax}{2} = \frac{-1}{2}\left[1 - \frac{a}{3a - 2b}\right]$

$$= \frac{-(a - b)}{(3a - 2b)}$$

Therefore, the point of intersection of lines (1) and (2) is $\left(\frac{-1}{(3a - 2b)}, \frac{-(a - b)}{3a - 2b}\right)$

Since the three lines are concurrent, the above point must lie on the line $cx + 4y + 1 = 0$

$$\Rightarrow \left(\frac{-1}{3a - 2b}\right) - 4\left(\frac{a - b}{3a - 2b}\right) + 1 = 0$$

$-c - 4(a - b) + 3a - 2b = 0 \Rightarrow -c - a + 2b = 0 \Rightarrow 2b = a + c$ or a, b, c are in A.P.

Alternately,

Given that $\begin{vmatrix} a & 2 & 1 \\ b & 3 & 1 \\ c & 4 & 1 \end{vmatrix} = 0 \Rightarrow a(3 - 4) - b(2 - 4) + c(2 - 3) = -a + 2b - c = 0$

$\Rightarrow 2b = a + c \Rightarrow$ a, b, c are in A.P.

ANGLE BETWEEN TWO LINES

The angle between two lines is defined as the acute angle between the two lines.

Let the slopes of the two lines ℓ_1 and ℓ_2 be m_1 and m_2.

If θ_1 and θ_2 are the angles made by ℓ_1 and ℓ_2 respectively with the x-axis, the acute angle θ between the line is given by $\theta = \theta_1 - \theta_2$ (refer Fig. 1.12)

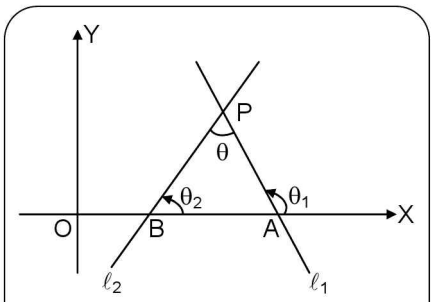


Fig. 1.12

$$\tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

Since m_1 and m_2 are the slopes of the two lines, $\tan\theta_1 = m_1$; $\tan\theta_2 = m_2$.

Therefore, the acute angle θ between the lines ℓ_1 and ℓ_2 is given by $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

If the two lines are parallel to each other, $\theta_1 = \theta_2$. Consequently, $\tan\theta_1 = \tan\theta_2$ or $m_1 = m_2$.

If the two lines are perpendicular to each other, $\theta_1 - \theta_2 = \theta = \dfrac{\pi}{2} \Rightarrow 1 + m_1 m_2 = 0$ or $m_1 m_2 = -1$.

For two lines with slopes m_1 and m_2, the acute angle θ between the two lines is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

The two lines are parallel, if $m_1 = m_2$ i.e., if their slopes are equal.

The two lines are perpendicular to each other if $m_1 m_2 = -1$, i.e., if the product of their slopes $= -1$

For example, the lines $5x + 9y - 8 = 0$ and $10x + 18y + 7 = 0$ are parallel, since, their respective slopes are $\dfrac{-5}{9}$ and $\dfrac{-10}{18}$, which are equal.

Again, the lines $3x - y - 6 = 0$ and $x + 3y + 8 = 0$ are perpendicular to each other, since the product of the slopes of the two lines equals $\left(\dfrac{-1}{3}\right)(3) = -1$

CONCEPT STRAND

Concept Strand 12

Find the acute angle between the lines $3x + y - 7 = 0$ and $x + 2y + 9 = 0$

Solution

Slopes of the lines are $m_1 = -3$, $m_2 = \dfrac{-1}{2}$

If θ is the angle between the lines $\tan\theta = \left|\dfrac{-3 + \frac{1}{2}}{1 + \frac{3}{2}}\right| = 1$

$\Rightarrow \theta = \dfrac{\pi}{4}$

Remark

Suppose the equations of the two lines are $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$. The slopes of the two lines are $\dfrac{-a_1}{b_1}$ and $\dfrac{-a_2}{b_2}$ respectively.

If θ is the angle between the lines,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{-a_1}{b_1} + \frac{a_2}{b_2}}{1 + \frac{a_1 a_2}{b_1 b_2}}\right| = \left|\frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}\right|.$$

Therefore, we can conclude the following:

The two lines are parallel if $\frac{-a_1}{b_1} = \frac{-a_2}{b_2}$

i.e., if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

The two lines are perpendicular to each other if

$\left(\frac{-a_1}{b_1}\right)\left(\frac{-a_2}{b_2}\right) = -1$ i.e., if $a_1a_2 + b_1b_2 = 0$

Results

Let the equation of a line be given as $ax + by + c = 0$ — (1)

The slope of the given line (1) $= \frac{-a}{b}$

Consider the equation $ax + by + k = 0$ — (2) where k is any constant.

The slope of the line represented by (2) is also $\frac{-a}{b}$.

Since the slopes of the lines represented by (1) and (2) are equal, they are parallel.

Again, consider the equation $bx - ay + k = 0$ — (3)

The slope of the line (3) is $\frac{b}{a}$.

Product of the slopes of the lines (1) and (3) is equal to $\left(\frac{-a}{b}\right)\left(\frac{b}{a}\right) = -1$. Hence, the lines (1) and (3) are perpendicular to each other.

Thus, we have

(i) $ax + by + k = 0$, where k is any constant represents the equation of a line parallel to the line $ax + by + c = 0$.

(ii) $bx - ay + k = 0$, where k is any constant represents the equation of a line perpendicular to the line $ax + by + c = 0$

For example, to find the equation of the line passing through (3, –1) and parallel to the line $5x - y + 8 = 0$, we proceed as follows:

The equation of any line parallel to the line $5x - y + 8 = 0$ may be assumed as $5x - y + k = 0$, where k is a constant.

Since this line passes through (3, –1), $x = 3$, $y = -1$ must satisfy $5x - y + k = 0$.

This gives $k = -16$. The required line is $5x - y - 16 = 0$.

CONCEPT STRANDS

Concept Strand 13

Find the value of k for which the lines $kx + y = 6$ and $2x - 5y = 1$ are perpendicular to each other.

Solution

We have $(-k)\left(\frac{2}{5}\right) = -1$, $k = \frac{5}{2}$.

Concept Strand 14

Write down the equation of the line parallel to the line $x + y + 5 = 0$ and passing through the point (6, –8).

Solution

The line may be assumed as $x + y + k = 0$.

Since the line passes through (6, –8), $6 - 8 + k = 0 \Rightarrow k = 2$.

The required line is $x + y + 2 = 0$.

Concept Strand 15

Find the equation of the line passing through (5, 8) and perpendicular to the line $x + 2y - 5 = 0$.

Solution

The equation of any line perpendicular to the line $x + 2y - 5 = 0$ can be assumed as $2x - y + k = 0$.

Since this line passes through (5, 8), we get $k = -2$.

The required line is $2x - y - 2 = 0$.

Concept Strand 16

Write down the equation of the line perpendicular to the line $3x - y + 9 = 0$ and passing through the origin.

Solution

The line may be assumed as $x + 3y + k = 0$. Since the line passes through the origin, $k = 0$.

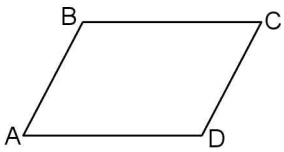
The required line is $x + 3y = 0$.

Concept Strand 17

Three consecutive vertices of a parallelogram are at (–1, 2), (5, 1), (6, 5). Find the coordinates of the remaining vertex.

Solution

Let A, B, C represent the three given vertices and D(x, y), the fourth vertex. AB is parallel to CD and AD is parallel to BC.



These give, $x + 6y = 36$, $4x - y = -6$.

On solving the two equations, we get D as (0, 6).

Concept Strand 18

Find the equation of the line perpendicular to the line $2x + y - 1 = 0$ and passing through the intersection of the lines $x + 2y - 1 = 0$ and $x = y$.

Solution

Any line through the intersection of the lines $x + 2y - 1 = 0$ and $x = y$ may be assumed as

$x + 2y - 1 + \lambda(x - y) = 0$, where λ is a constant.

Slope of the above line $= -\dfrac{(1+\lambda)}{2-\lambda}$

Since the required line is perpendicular to the line $2x + y - 1 = 0$

$$\Rightarrow \left[-\frac{(1+\lambda)}{2-\lambda}\right]\left[\frac{-2}{1}\right] = -1$$

$\Rightarrow 2(1 + \lambda) = -(2 - \lambda)$

$\Rightarrow 2 + 2\lambda = -2 + \lambda \Rightarrow \lambda = -4.$

The required line is $x + 2y - 1 - 4(x - y) = 0 \Rightarrow 3x - 6y + 1 = 0$.

NORMAL FORM OF THE EQUATION OF A LINE

Let a line intersect the axes in A and B. Draw ON perpendicular to the line. ON = length of perpendicular from the origin to the line. This length is usually denoted by p and is always taken as positive.

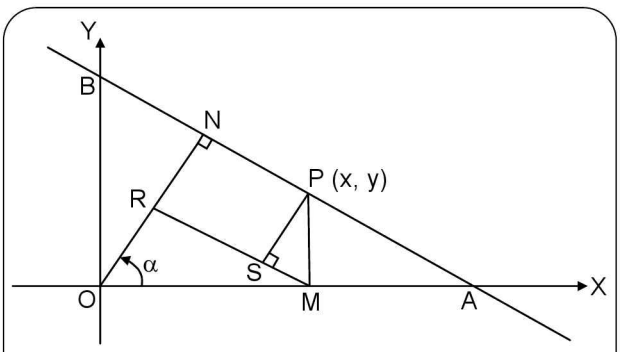


Fig. 1.13

Let ON makes an angle α with the positive direction of the x-axis. Depending on the position of the line in the plane, α may vary from 0° to 360° (Refer Figures 1.13 and 1.14(i) – (iii))

Given p and α, the problem is to find the equation of the line.

Referring to Fig. 1.13, let P(x, y) represent any point on the line. PM is drawn perpendicular to the X-axis. MR is perpendicular to ON, PS is perpendicular to MR.

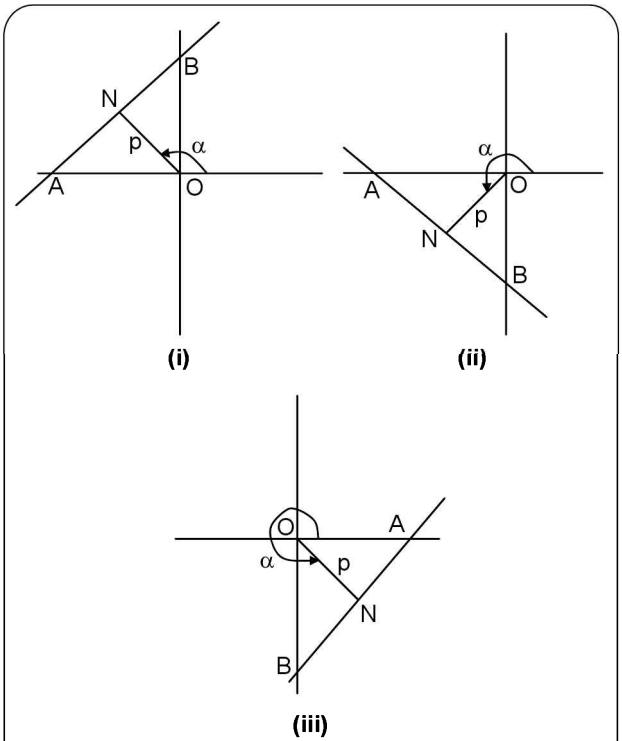


Fig 1.14

We have p = ON = OR + RN

From Δ ORM $= \dfrac{OR}{OM} = \cos\alpha$ or $OR = OM\cos\alpha = x\cos\alpha$.

Also, from Δ PSM, $\dfrac{PS}{PM} = \sin\alpha$, $PS = PM\sin\alpha = y\sin\alpha$ and $RN = PS$.

Therefore, $p = x\cos\alpha + y\sin\alpha$. Or, the equation of the line is $x\cos\alpha + y\sin\alpha = p$. This is called the normal form of the equation of the line.

(i) $p = 3, \alpha = 60°$ The equation is $x + \sqrt{3}\,y = 6$.

(ii) $p = 5, \alpha = 210°$ The equation is $x\sqrt{3} + y + 10 = 0$.

We shall explain how one may write the general equation of a line in the normal form. Let the equation of the line be $ax + by + c = 0$. We assume that the equation of the line is rewritten in such a way that c is negative.

$\therefore$ $ax + by = -c$ (Observe that $-c$ is positive)

We divide both sides by $\sqrt{a^2 + b^2} \Rightarrow$

$$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}} = \frac{-c}{\sqrt{a^2+b^2}}$$

Since $\left(\dfrac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\dfrac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$, $\dfrac{a}{\sqrt{a^2+b^2}}$ may be taken as $\cos\alpha$, and $\dfrac{b}{\sqrt{a^2+b^2}}$ as $\sin\alpha$.

Hence, $p = \dfrac{-c}{\sqrt{a^2+b^2}}$. Or, length of perpendicular from the origin to the line $= \dfrac{-c}{\sqrt{a^2+b^2}}$.

To sum up, in order to write the equation of a line in the normal form, we rewrite the equation with c negative. We then divide the equation throughout by $\sqrt{a^2+b^2}$.

The normal form is

$$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y = \frac{-c}{\sqrt{a^2+b^2}}$$ and therefore the length of perpendicular from the origin to the line is $\dfrac{-c}{\sqrt{a^2+b^2}}$.

CONCEPT STRANDS

Concept Strand 19

Write the equation $3x + 2y - 6 = 0$ in the normal form. Find p and α.

Solution

Since $c = -6$, there is no need to rewrite the equation with c negative. Dividing by $\sqrt{9+4} = \sqrt{13}$, the normal form of the equation is $\dfrac{3}{\sqrt{13}}x + \dfrac{2}{\sqrt{13}}y = \dfrac{6}{\sqrt{13}}$.

$\therefore$ $p = \dfrac{6}{\sqrt{13}}$ and $\alpha = \tan^{-1}\left(\dfrac{2}{3}\right)$

Concept Strand 20

Write the equation $4x - 3y + 7 = 0$ in the normal form.

Solution

The equation is rewritten as $-4x + 3y - 7 = 0$

Dividing by $\sqrt{16+9} = 5$, the normal form is

$\dfrac{-4}{5}x + \dfrac{3}{5}y = \dfrac{7}{5}$.

SIGN OF AX + BY + C

Consider the line whose equation is $ax + by + c = 0$. Let P(x, y) be a point lying in the plane (Refer Fig. 1.15). The ordinate MP is produced to meet the line at P'. It is clear that the abscissa of P' = abscissa of P = x. Since P' lies on the line, $P'M = -\left(\dfrac{ax+c}{b}\right)$ and $PM = y$.

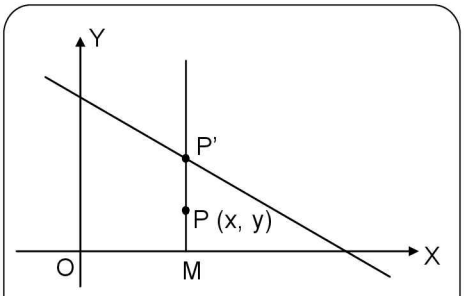


Fig. 1.15

The line divides the plane into two regions; say Region I and Region II.

If P lies in the region I, PM < P'M while if P lies in the region II, PM > P'M.

That is, if P is in region I, $y < \dfrac{-(ax + c)}{b}$

or in region II, $y > \dfrac{-(ax + c)}{b}$.

If b > 0, this in turn gives $ax + by + c \lessgtr 0$, and if b < 0, this gives $ax + by + c \gtrless 0$.

Therefore, we conclude that the line ax + by + c = 0 divides the plane into two regions. The coordinates of the points are such that, in one region, ax + by + c > 0 and in the other region, ax + by + c < 0. This result can be used to test whether two given points lie on the same side or on opposite sides of a line.

Let the equation of the line be ax + by + c = 0 and the two points be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. We compute $(ax_1 + by_1 + c)$ and $(ax_2 + by_2 + c)$. If both these expressions are of the same sign, both P_1 and P_2 lie on the same side of the line ax + by + c = 0. On the other hand, if the two expressions are of opposite signs, P_1 and P_2 lie on opposite sides of the line ax + by + c = 0.

The region in which origin lies is called the 'origin side'. The other region may be called 'non origin side'. By making use of the above arguments, we infer that if $(ax_1 + by_1 + c)$ and c are of the same sign, the point (x_1, y_1) lies on the origin side; while, if $(ax_1 + by_1 + c)$ and c are of opposite signs, the point (x_1, y_1) lies on the non origin side (i.e., origin and (x_1, y_1) are on opposite sides of the line).

For example,

(i) Consider the line 2x – 3y – 6 = 0. The points (1, 2) and (3, –6) are on opposite sides of the line.
(since 2 × 1 – 3 × 2 – 6 < 0 and 2 × 3 – 3 × – 6 – 6 > 0).
Since c = –6, (1, 2) is on the origin side and (3, –6) is on the non origin side of the line.

(ii) Consider the line 5x + 6y + 5 = 0. The points (2, 1), (–3, 2) are both on the same side of the line. Since c = 5, both these points are on the origin side of the line.

LENGTH OF PERPENDICULAR DISTANCE FROM A POINT (X_1, Y_1) TO THE LINE AX + BY + C = 0

Let the equation of the line be ax + by + c = 0. P is the point (x_1, y_1). PR represents the perpendicular distance from P to the line. We want to derive a formula for PR. We first rewrite the equation of the line in the normal form.

The normal form is

$$\frac{a}{\sqrt{a^2 + b^2}}x + \frac{b}{\sqrt{a^2 + b^2}}y = \frac{-c}{\sqrt{a^2 + b^2}}.$$

(Note that the equation is rewritten with c negative).

If ON = p', the equation of the line through $P(x_1, y_1)$ and parallel to the line ax + by + c = 0 may be written as $x \cos\alpha + y \sin\alpha = p'$ where $p' = x_1 \cos\alpha + y_1 \sin\alpha$

$\therefore$ We have $ON = p' = x_1 \cos\alpha + y_1 \sin\alpha = \dfrac{ax_1 + by_1}{\sqrt{a^2 + b^2}}$

Length of perpendicular PR = p' – p or p – p' (Refer figures (i) and (ii) of Fig. 1.16).

$$= \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \text{ or } \frac{-(ax_1 + by_1 + c)}{\sqrt{a^2 + b^2}}.$$

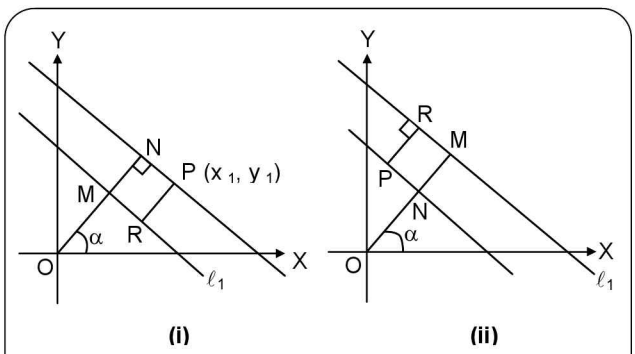


Fig. 1.16

Thus, the length of perpendicular from (x_1, y_1) to the line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.

Note:

Since we are only concerned with the length of perpendicular, which is a positive quantity, we may directly compute the expression $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$. Note that depending on where the point P lies, the value inside the modulus i.e., $\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$ may be greater than or less than zero.

For example, the length of perpendicular from (1, –6) to the line $3x + 4y - 7 = 0$ is $\dfrac{28}{5}$.

(Observe that (1, –6) is on the origin side).

Perpendicular distance between two parallel lines

Let ℓ_1: $ax + by + c = 0$ and ℓ_2: $ax + by + c' = 0$ be two parallel lines.

Case 1

If c and c' are of the same sign, it is clear that both lines lie on the same side of the origin. The perpendicular distance between the lines is the difference between the perpendicular distances of these lines from the origin. (Refer Fig.1.17).

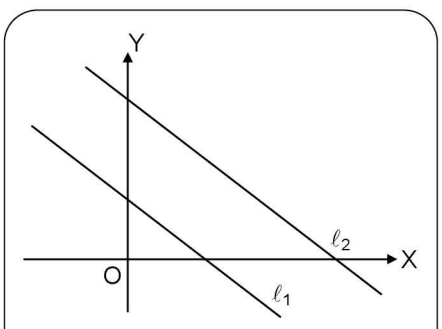


Fig. 1.17

If d is the distance, $d = \left|\dfrac{c - c'}{\sqrt{a^2 + b^2}}\right|$.

Case 2

If c and c' are of opposite signs, the two lines lie on opposite sides of the origin. (Refer Fig.1.18)

In this case, $d = \dfrac{|c| + |c'|}{\sqrt{a^2 + b^2}}$.

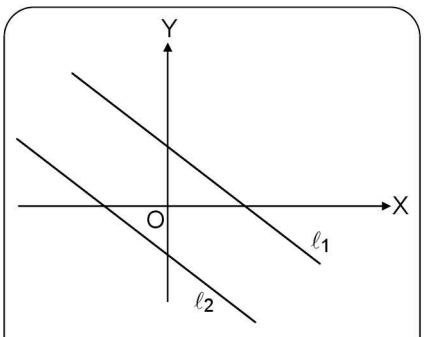


Fig. 1.18

Combining cases 1 and 2, we see that the perpendicular distance between the lines ℓ_1 and ℓ_2 is given by $\left|\dfrac{c - c'}{\sqrt{a^2 + b^2}}\right|$

For example,

(i) The perpendicular distance between the lines $x + 4y + 5 = 0$ and $x + 4y + 8 = 0$ is $\dfrac{8 - 5}{\sqrt{1 + 16}} = \dfrac{3}{\sqrt{17}}$.

(ii) The perpendicular distance between $2x + 5y - 1 = 0$ and $2x + 5y + 9 = 0$ is $\dfrac{1 + 9}{\sqrt{4 + 25}}$ or $\dfrac{10}{\sqrt{29}}$.

CONCEPT STRANDS

Concept Strand 21

Find the perpendicular distance between the lines $3x - 7y + 6 = 0$ and $6x - 14y - 5 = 0$.

Solution

The lines are $3x - 7y + 6 = 0$

and $3x - 7y - \dfrac{5}{2} = 0$.

Since c and c' are of opposite signs (i.e., the constant terms are of opposite signs), the perpendicular distance between the lines $= \dfrac{6 + 5/2}{\sqrt{9 + 49}}$

$$= \frac{17}{2\sqrt{58}}.$$

Concept Strand 22

Find the equation of the line perpendicular to the line $3x - y - 5 = 0$ and at a distance of 3 units from the origin.

Solution

Any line perpendicular to the line $3x - y - 5 = 0$ may be assumed as $x + 3y + k = 0$, where k is a constant.

Since the length of perpendicular to the above line from the origin is of length 3, $\left|\dfrac{-k}{\sqrt{10}}\right| = 3$ or $k = \pm 3\sqrt{10}$

The two lines satisfying the requirements are $x + 3y \pm 3\sqrt{10} = 0$

Concept Strand 23

Find the image of the point (1, 2) in the line $2x + 3y + 7 = 0$.

Solution

Let P be (1, 2). Draw PM perpendicular to the line $2x + 3y + 7 = 0$ and produce it to P' such that PM = MP'. Then, P' is the image or reflection of P in the line.(Refer Figure)

Method 1

Equation of PMP' is $y - 2 = \dfrac{3}{2}(x - 1)$

$2y - 4 = 3x - 3$ or $3x - 2y + 1 = 0$

We solve the two equations

$2x + 3y + 7 = 0$, $3x - 2y + 1 = 0$ and

get $M\left(\dfrac{-17}{13}, \dfrac{-19}{13}\right)$

Let P' be (x, y)

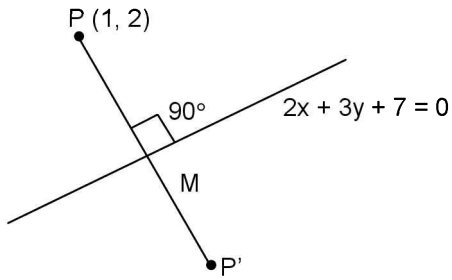
Then $\dfrac{x+1}{2} = \dfrac{-17}{13}$, $\dfrac{y+2}{2} = \dfrac{-19}{13}$

Giving $x = \dfrac{-47}{13}$, $y = \dfrac{-64}{13}$ $\Rightarrow$ Image of (1, 2) is $\left(\dfrac{-47}{13}, \dfrac{-64}{13}\right)$

Method 2

Slope of PMP' $= \dfrac{3}{2}$

i.e., $\tan\theta = 3/2 \Rightarrow \sin\theta = \dfrac{3}{\sqrt{13}}$, $\cos\theta = \dfrac{2}{\sqrt{13}}$



Equation of PMP' may be represented as

$$\frac{x-1}{\frac{2}{\sqrt{13}}} = \frac{y-2}{\frac{3}{\sqrt{13}}} = r$$

Any point on this line may be expressed as

$$\left(1 + \frac{2r}{\sqrt{13}}, 2 + \frac{3r}{\sqrt{13}}\right)$$

If this point is M, $2\left(1 + \dfrac{2r}{\sqrt{13}}\right) + 3\left(2 + \dfrac{3r}{\sqrt{13}}\right) + 7 = 0$

$\Rightarrow \sqrt{13}\,r = -15 \Rightarrow r = \dfrac{-15}{\sqrt{13}}$

Since PP' = 2PM, r corresponding to P' is $\dfrac{-30}{\sqrt{13}}$

Therefore, the coordinates of P' are $\left(1 - \dfrac{60}{13}, 2 - \dfrac{90}{13}\right)$

$\Rightarrow \left(\dfrac{-47}{13}, \dfrac{-64}{13}\right)$

EQUATIONS OF THE BISECTORS OF THE ANGLES BETWEEN TWO STRAIGHT LINES WHOSE EQUATIONS ARE GIVEN

Let the equations of the two lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

Referring to the figure (Fig.1.19), MB_1 and MB_2 are the internal and external bisectors of the angle between the two lines. If $P(x_1, y_1)$ represents a point on any of the bisectors, lengths of perpendiculars from P to the two lines are equal.

i.e., $$\left|\frac{a_1x_1 + b_1y_1 + c_1}{\sqrt{a_1^2 + b_1^2}}\right| = \left|\frac{a_2x_1 + b_1y_2 + c_2}{\sqrt{a_2^2 + b_2^2}}\right|$$

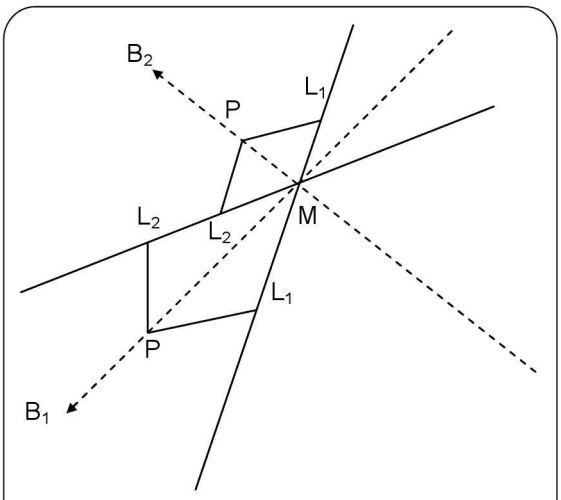


Fig. 1.19

The equations of the bisectors are therefore given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

To find the bisector of the acute angle between the lines we first find the equations of both the bisectors and then find the angle between one of the bisectors and one of the given lines. If this angle is less than 45°, then it means that this bisector represents the internal bisector (i.e., bisector bisecting the acute angle between the lines).

For example, the equations of the bisectors of the angles between the lines $3x + 4y - 5 = 0$ and $12x - 5y - 41 = 0$ are $\frac{3x + 4y - 5}{5} = \pm\frac{12x - 5y - 41}{13}$

or $3x - 11y - 20 = 0$ and $11x + 3y - 30 = 0$.

Note that the two bisectors are at right angles to each other.

The angle θ between $3x + 4y - 5 = 0$ and $3x - 11y - 20 = 0$ is given by $\tan\theta = \left|\frac{\frac{-3}{4} - \frac{3}{11}}{1 - \frac{9}{44}}\right| = \frac{9}{7}$.

Clearly, $\theta > 45°$.

We infer that $3x - 11y - 20 = 0$ is the external bisector of the angle and $11x + 3y - 30 = 0$ is the internal bisector of the angle.

CONCEPT STRANDS

Concept Strand 24

Find the incentre of the triangle formed by the lines $x + y - 7 = 0$, $x - y + 1 = 0$ and $x - 3y + 5 = 0$.

Solution

The two bisectors of $x + y - 7 = 0$ and $x - y + 1 = 0$ are obtained as $x = 3$ and $y = 4$.

Clearly, $x = 3$ is the internal bisector.

The bisectors of the angle between $x - y + 1 = 0$ and $x - 3y + 5 = 0$ are $x + \frac{y}{2}\left(\sqrt{5} - 1\right) - \sqrt{5} = 0$ and

$$x - \frac{y}{2}\left(\sqrt{5} + 1\right) + \sqrt{5} = 0$$

The internal bisector can be seen to be

$$x - \frac{y}{2}\left(\sqrt{5} + 1\right) + \sqrt{5} = 0.$$

Therefore, the incentre is the point of intersection of the lines $x = 3$ and $x - \frac{y}{2}\left(\sqrt{5} + 1\right) + \sqrt{5} = 0$.

Solving, the incentre is at $\left(3, \sqrt{5} + 1\right)$.

PAIR OF STRAIGHT LINES

Consider the two straight lines whose equations are $2x + 3y - 1 = 0$ and $5x - y + 6 = 0$. If we take the product of these equations, we obtain

$$10x^2 + 13xy - 3y^2 + 7x + 19y - 6 = 0 \qquad \text{—(1)}$$

(1) is satisfied by any point (x, y) which lies one any one of the two given straight lines. We may say that (1) represents the combined equation of the two given straight lines. The separate equations of the two lines are obtained by factoring (1) into two linear factors and equating each factor to zero. This leads us to infer that the general equation of the second degree in x and y:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{—(2)}$$

where, a, b, c, f, g, h are constants, represents two straight lines provided the second degree expression on the left side of (2) is factorizable into two real linear factors of the form $(l_1 x + m_1 y + n_1)$ and $(l_2 x + m_2 y + n_2)$. Before going into the condition under which this is possible, let us consider the corresponding homogeneous equation of the second degree in x and y: (A homogenous polynomial is a polynomial in which each term has the same degree.)

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{—(3)}$$

L.H.S of (3) i.e., $ax^2 + 2hxy + by^2$

is a homogeneous polynomial in x and y of degree 2.

Dividing (3) by x^2, $b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$ —(4)

(4) is a quadratic equation in (y/x). Therefore, it gives two roots say m_1 and m_2 (real or complex).

(4) may thus be written as $b\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right) = 0$, meaning thereby that (3) represents two straight lines $y - m_1 x = 0$ and $y - m_2 x = 0$ through the origin. Thus, a homogeneous equation of second degree in x and y always represents a pair of straight lines through the origin.

The roots are real when $4h^2 - 4ab \geq 0$ or $h^2 - ab \geq 0$. We exclude the cases where the two roots are complex, in which case, the two lines are imaginary.

For example,

(i) the equation $2x^2 + xy - 15y^2 = 0$ represents the two lines through the origin given by $x + 3y = 0$ and $2x - 5y = 0$.

(since $2x^2 + xy - 15y^2 \equiv (x + 3y)(2x - 5y)$).

(ii) the equation $4x^2 - 12xy + 9y^2 = 0$ represents the coincident lines $2x - 3y = 0$ and $2x - 3y = 0$.

Although it is possible to study the properties of the two straight lines by considering them independently, in some cases it is more convenient to study these properties by treating the combined equation of the two lines. We propose to undertake this work in the next section.

Let the two lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1 x = 0$ and $y - m_2 x = 0$.

$$\text{Then, } ax^2 + 2hxy + by^2 \equiv b(y - m_1 x)(y - m_2 x)$$
$$\equiv b\,[y^2 - (m_1 + m_2)xy + m_1 m_2 x^2]$$

Equating the coefficients of xy and $x^2 \Rightarrow 2h = -b(m_1 + m_2)$; $a = b\,m_1 m_2$

This gives $m_1 + m_2 = -2h/b$ and $m_1 m_2 = a/b$.

We have just now proved that if m_1 and m_2 are the slopes of the lines represented by (3), then,

$$\left.\begin{aligned} \text{Sum of the slopes} &= \frac{-2h}{b} \\ \text{Product of the slopes} &= \frac{a}{b} \end{aligned}\right\} \qquad \text{—(5)}$$

Again, let θ represent the angle between the lines represented by (3). We have, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

From (5), $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2 = \frac{4}{b^2}(h^2 - ab)$.

$$\text{Therefore, } \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|.$$

We easily deduce that the two lines are coincident (i.e., $m_1 = m_2$) if $h^2 - ab = 0$ and the two lines are perpendicular to each other (i.e., $m_1 m_2 = -1$) if $a + b = 0$.

We may also obtain the combined equation of the pair of bisectors of the angles between the lines represented by (3). If the two lines are $y - m_1 x = 0$ and $y - m_2 x = 0$, the equations of the two bisectors are $\frac{y - m_1 x}{\sqrt{1 + m_1^2}} = \pm \frac{y - m_2 x}{\sqrt{1 + m_2^2}}$

∴ The combined equation of the bisectors is,

$$y^2(m_1 + m_2) - 2xy(m_1 m_2 - 1) - x^2(m_1 + m_2) = 0.$$

Using (5), the equation of the pair of bisectors is,

$$\frac{x^2 - y^2}{(a - b)} = \frac{xy}{h} \qquad \text{— (6)}$$

We note that, coefficient of x^2 + coefficient of y^2 in (6) equals $h + (-h) = 0$

The two bisectors of the pair of lines (internal bisector and external bisector) are perpendicular to each other.

For example, let us consider the pair of lines $3x^2 + 7xy + 2y^2 = 0$.

The angle between the pair of lines can be found using

$$\tan\theta = \left|\frac{2\sqrt{\frac{49}{4} - 6}}{5}\right| = 1 \Rightarrow \theta = \frac{\pi}{4}.$$

Also, the combined equation to the pair of bisectors of the angles between the lines is $\frac{x^2 - y^2}{3 - 2} = \frac{xy}{7/2}$

$\Rightarrow 7x^2 - 2xy - 7y^2 = 0$.

GENERAL EQUATION OF SECOND DEGREE IN X AND Y

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{— (1)}$$

represents the general equation of second degree in x and y.

(1) represents a pair of lines if the left hand side of (1) is factorisable into two real linear factors, i.e., if $[ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ can be written in the form $(\ell x + my + n) \times (\ell' x + m'y + n')$.

Rearranging, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = ax^2 + 2(hy + g)x + by^2 + 2fy + c$. The right hand side can be considered as a quadratic expression in x.

The second-degree expression in x and y is expressible as the product of linear expressions in x and y, only if the discriminant of the above quadratic expression is a perfect square.

i.e., $[4(hy + g)^2 - 4a(by^2 + 2fy + c)]$ or $[(hy + g)^2 - a(by^2 + 2fy + c)]$ must be a perfect square.

i.e.,

$(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)$ must be a perfect square.

The above expression may be considered as a quadratic expression in y and it will be a perfect square only if its discriminant is zero.

This means that $4(gh - af)^2 - 4(h^2 - ab)(g^2 - ac) = 0 \Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

The general equation of second degree in x and y:

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Let (1) represent a pair of straight lines.

Consider $ax^2 + 2hxy + by^2 = 0$ — (2)

Note that (2) is written by taking only the second degree terms from (1), or by taking only the homogenous part of (1).

Let the lines represented by (1) be $y = M_1 x + c_1$ and $y = M_2 x + c_2$

Then, $b(y - M_1 x - c_1)(y - M_2 x - c_2) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

This gives, $M_1 + M_2 = -\frac{2h}{b}$, $M_1 M_2 = \frac{a}{b}$.

We already know that if m_1 and m_2 are the slopes of the lines represented by (2),

$$m_1 + m_2 = -\frac{2h}{b}, \; m_1 m_2 = \frac{a}{b}.$$

It immediately follows that (2) represents a pair of lines through the origin parallel to the lines represented by (1).

Or, in other words, (1) and (2) form a parallelogram.

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, then the corresponding homogeneous equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines through the origin parallel to the original pair.

It also follows that we can find the two equations to the lines constituting (1) by first finding the equations to the lines constituting (2) and then ascertaining the constant terms by comparing coefficients.

Also note that the angle between the pair of lines represented by (1) will be the same as the angle between the pair of lines represented by (2).

CONCEPT STRANDS

Concept Strand 25

Prove that the equation $3x^2 - 10y^2 - xy + 21x + 13y + 30 = 0$ represents a pair of lines.

Solution

We have $a = 3, b = -10, h = \frac{-1}{2}, c = 30, f = \frac{13}{2}, g = \frac{21}{2}$

$$abc + 2fgh - af^2 - bg^2 - ch^2 =$$
$$-900 - \frac{273}{4} - \frac{507}{4} + \frac{4410}{4} - \frac{30}{4}$$
$$= -900 + 900 = 0$$

$\Rightarrow$ the given equation represents a pair of lines.

Concept Strand 26

Find the value of k for which the equation $12x^2 + 7xy + ky^2 + 13x - y + 3 = 0$ represents a pair of lines.

Solution

Given $a = 12, b = k, c = 3, h = \frac{7}{2}, f = \frac{-1}{2}, g = \frac{13}{2}$

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$
$$\Rightarrow 36k + 2 \times \frac{-1}{2} \times \frac{13}{2} \times \frac{7}{2} - 12 \times \frac{1}{4} - k \times \frac{169}{4} - 3 \times \frac{49}{4} = 0$$
$$\Rightarrow 36k - \frac{169}{4}k = \frac{250}{4} \Rightarrow \frac{-25k}{4} = \frac{250}{4}$$
$$\Rightarrow k = -10.$$

CHANGE OF ORIGIN

Consider a straight line $y - 1 = m(x - 2)$ —(1)

(1) is the equation of a line through (2, 1) having slope m (or (1) is the representation of the line with respect to the Ox, Oy coordinate system).

Let us shift the origin of the coordinate system to O'(2, 1). We draw O'X, O'Y parallel to Ox, Oy respectively. If (X, Y) represents the coordinates of a point in the plane referred to O'X, O'Y as the X and Y axes, and (x, y) are its coordinates referred to Ox, Oy as x and y axes, we have $x = X + 2$ and $y = Y + 1$.(refer Fig. 1.20)

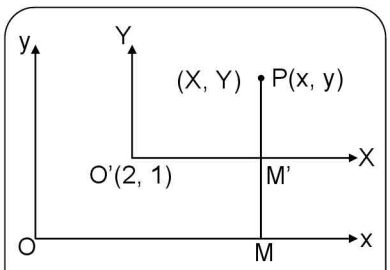


Fig. 1.20

The representation of the line (1) in the new system O'X , O'Y becomes

$$Y + 1 - 1 = m(X + 2 - 2) \text{ or}$$

$$Y = mX \quad \text{—(2)}$$

Observe that (1) and (2) represent the same line in the plane.

(1) is its representation in Ox, Oy system and (2) is its representation in the O'X, O'Y system. Although the equations (or representations) of the curves change when the coordinate system changes, the geometrical properties of the curves do not change with the change in the reference system or these properties are invariant under changes in the coordinate system.

By a proper choice of the coordinate system, the analytical representation (or the equation) of these curves can be made simple, so as to facilitate the study of their geometrical properties.

To sum up, let (h, k) represent the coordinates of a point referred to Ox, Oy as axes of coordinates. We shift the origin to (h, k) and take the lines through (h, k) parallel to Ox and Oy as the new x and y axes. If (x, y) represent the coordinates of any point in the plane referred to Ox and Oy as axes and (X, Y) its coordinates referred to the new axes through (h, k), we have

$$x = X + h \text{ and } y = Y + k$$

or $X = x - h, Y = y - k.$

CONCEPT STRANDS

Concept Strand 27

Origin is shifted to the point (–3, 4) without changing the directions of the axes. Obtain the equation of the line $3x - 4y - 7 = 0$ in the new system.

Solution

We have $x = X - 3$; $y = Y + 4$

The equation of the line in the new system is $3(X - 3) - 4(Y + 4) - 7 = 0$ or

$$3X - 4Y - 32 = 0.$$

Concept Strand 28

Origin is shifted to (1, 1) without changing the directions of the axes. Find the equation of the pair of lines $x^2 + xy - 3x - y + 2 = 0$ in the new system.

Solution

We have $x = X + 1$; $y = Y + 1$

Therefore, the equation of the pair of lines in the new system is

$$(X + 1)^2 + (X + 1)(Y + 1) - 3(X + 1) - (Y + 1) + 2 = 0$$

or $X^2 + XY = 0$

TO OBTAIN THE COORDINATES OF THE POINT OF INTERSECTION OF THE LINES REPRESENTED BY $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{— (1)}$$

Since (1) represents a pair of lines,

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ is satisfied by the coefficients of (1). Let $P(x_1, y_1)$ represent the coordinates of the point of intersection of the lines represented by (1). (Refer Fig. 1.21)

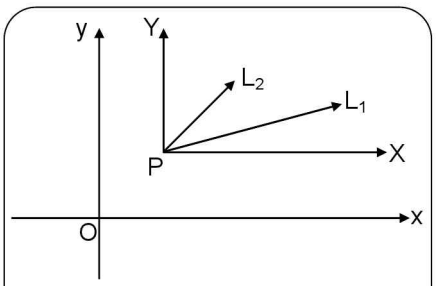


Fig 1.21

Shift the origin to P without changing the directions of the axes. In other words, the new coordinate axes are parallel to the old coordinate axes. Then,

$$x = X + x_1,$$

$$y = Y + y_1,$$

where, (x, y) and (X, Y) represent the coordinates of any point with respect to the old and new system respectively.

By substituting $x = X + x_1$, $y = Y + y_1$ in (1), we get the representation of the pair of lines in the new system.

We have

$$a(X + x_1)^2 + 2h(X + x_1)(Y + y_1) + b(Y + y_1)^2 + 2g(X + x_1) + 2f(Y + y_1) + c = 0$$

i.e.,
$$\begin{aligned} aX^2 + 2hXY + bY^2 &+ 2(ax_1 + hy_1 + g)X \\ &+ 2(hx_1 + by_1 + f)Y \\ &+ (ax_1^2 + 2hx_1y_1 + by_1^2 \\ &+ 2gx_1 + 2fy_1 + c) = 0 \end{aligned}$$

Since (x_1, y_1) is the point of intersection of the lines,

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$$

Also, the equation to the pair of lines in the new system must be a homogenous equation in X and Y of degree 2.

This leads us to $ax_1 + hy_1 + g = 0$, $hx_1 + by_1 + f = 0$ and on solving the two equations,

$$x_1 = \frac{hf - bg}{ab - h^2}, \quad y_1 = \frac{gh - af}{ab - h^2}$$

Therefore, the coordinates of the point of intersection of the pair of lines represented by (1) are obtained by solving the two linear equations $ax + hy + g = 0$, $hx + by + f = 0$ and that the coordinates are $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$.

CONCEPT STRAND

Concept Strand 29

Find the point of intersection of the lines represented by $3x^2 - 3y^2 + 8xy - 46x + 22y - 32 = 0$

Solution

$$a = 3, h = 4, g = -23, f = -11, b = -3$$

The point intersection is obtained by solving the equations is

$$3x + 4y - 23 = 0, 4x - 3y + 11 = 0.$$

The point of intersection is (1, 5).

Rotation of axes

Before concluding this unit, we mention that we can also rotate the axes of coordinates through some specified angle without changing the origin and study the relations between the coordinates of a point in the old and new systems.

Let P(x, y) represent the coordinates of a point referred to the rectangular coordinate system Ox, Oy.

Let the axes of coordinates be rotated through an angle θ in the counterclockwise sense (positive sense) (refer Fig. 1.22). Let the new positions of Ox and Oy be OX, OY. Let the coordinates of P in the new system OX, OY be (X, Y). Draw PM, PN perpendiculars to Ox and OX, NR perpendiculars to Ox and NS perpendicular to PM.

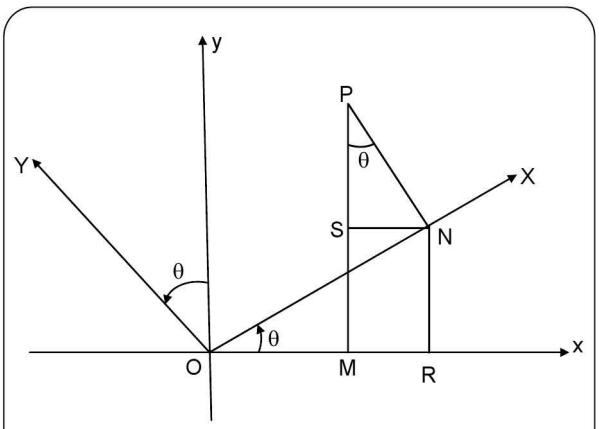


Fig 1.22

$\angle SPN$ = angle between PM and PN
= angle between their perpendiculars Ox and OX.
= θ.

$$x = OM = OR - MR = OR - NS$$
$$= ON \cos\theta - PN \sin\theta$$
$$= X \cos\theta - Y \sin\theta$$

$$y = PM = PS + SM = PS + NR$$
$$= PN \cos\theta + ON \sin\theta$$
$$= X \sin\theta + Y \cos\theta$$

The relations between the old and new coordinates are therefore given by

$$x = X \cos\theta - Y \sin\theta, y = X \sin\theta + Y \cos\theta$$

From the above relations, we can express X and Y in terms of x and y. We have, on multiplying the first of the above equations by cos θ and the second by sin θ and adding, $X = x \cos\theta + y \sin\theta$

Similarly, on multiplying the second equation by cos θ and the first equation by sin θ and subtracting,

$$Y = y \cos\theta - x \sin\theta$$

Thus, we have, if (x, y) and (X, Y) represent the coordinates of a point P in the old and new system respectively,

$$x = X \cos\theta - Y \sin\theta, y = X \sin\theta + Y \cos\theta$$

and

$$X = x \cos\theta + y \sin\theta, Y = -x \sin\theta + y \cos\theta.$$

CONCEPT STRANDS

Concept Strand 30

(1, 2) are the coordinates of a point referred to a rectangular Cartesian coordinate system. The axes are rotated through an angle 45° in the positive sense without changing the origin. Obtain the coordinates of the point in the new system.

Solution

Let (X, Y) represent the coordinates of the point in the new system. Then,

$$X = x\cos\frac{\pi}{4} + y\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$Y = -x\sin\frac{\pi}{4} + y\cos\frac{\pi}{4} = \frac{-1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Therefore, the coordinates of the point in the new system are $\left(\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Concept Strand 31

The coordinates of a point P referred to a rectangular coordinate system where O is the origin are (3, –4). The axes are rotated about O through an angle θ. If the coordinates of P in the new system are (k – 1, 3k + 1), find k.

Solution

Since the distance between O and P is invariant, $(k-3)^2 + (3k+1)^2$ must be = 25

$$10k^2 + 10 = 25 \quad \Rightarrow \quad 10k^2 = 15$$

$$k^2 = \frac{3}{2} \quad \Rightarrow \quad k = \pm\sqrt{\frac{3}{2}}.$$

SUMMARY

1. **Equation of straight line in different forms**
   (i) Slope intercept form $y = mx + c$
   (ii) One-point form $y - y_1 = m(x - x_1)$
   (iii) Two-point form $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ OR $y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$
   (iv) Intercepts form $\frac{x}{a} + \frac{y}{b} = 1$
   (v) Parametric form (symmetric form) $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$
   (vi) Normal form $x\cos\alpha + y\sin\alpha = p$
2. Equation of the line passing through the intersection of two line $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$, where λ is a constant.
3. Condition for three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ to be concurrent (passing through same point) is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$
4. Angle between two lines whose slopes are m_1 and m_2 is $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.
   If the lines are parallel $m_1 = m_2$
   If the lines are perpendicular $m_1m_2 = -1$

Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(ii) perpendicular if $a_1a_2 + b_1b_2 = 0$

(iii) coinciding, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iv) intersecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Equation of line parallel to the given line $ax + by + c = 0$ is $ax + by + k = 0$ (change only the constant term)

Equation of a line perpendicular to the line $ax + by + c = 0$ is $bx - ay + k = 0$

5. **Sign of ax + by + c**

Two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie on the same side of the line $ax + by + c = 0$ if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign; otherwise they lie on the opposite sides of the line.

6. Length of the perpendicular from a point (x_1, y_1) to the line $ax + by + c = 0$ is $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$. Length of the perpendicular from origin is $\frac{|c|}{\sqrt{a^2 + b^2}}$. Perpendicular distance between two parallel line $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

7. Equations of bisectors of the angles between two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

8. **Pair of straight lines**

(i) General second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\text{i.e., if } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(ii) Coordinates of the point of intersection of the pair of lines are $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

(iii) Homogenous second degree equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines through the origin

Sum of the slopes $m_1 + m_2 = \frac{-2h}{b}$

Product of the slopes $m_1m_2 = \frac{a}{b}$

Angle between the lines $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$

Lines are coincident if $h^2 = ab$

Lines are perpendicular if $a + b = 0$

Equation of pair of bisectors is $\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$

9. **Change of origin**

If origin is shifted from O to O' (h, k) axes, then let the coordinates of point P(x, y) with respect to O' be (X, Y) then $x = X + h$ and $y = Y + k$

9. **Change of origin:**

Let the origin O (0, 0) be shifted to O' (h, k) without changing the direction of the axes. If (x, y) and (X, Y) denote the coordinates of a point P in the old and new systems respectively, $x = X + h$ and $y = Y + k$

10. **Rotation of axes**

Let the coordinate axes be rotated through an angle θ in the counter clockwise sense about the origin O. If (x, y) and (X, Y) denote the coordinates of a point P in the old and new systems,

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= X\sin\theta + Y\cos\theta \end{aligned}\right\}; \quad \left.\begin{aligned} X &= x\cos\theta + y\sin\theta \\ Y &= -x\sin\theta + y\cos\theta \end{aligned}\right\}$$

CONCEPT CONNECTORS

Connector 1: Show that the linear equation $ax + by + c = 0$ always represents a straight line.

Solution: Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) denote the coordinates of any three points on the line.

Then,

$ax_1 + by_1 + c = 0$

$ax_2 + by_2 + c = 0$

$ax_3 + by_3 + c = 0$

These give $x_1 = \frac{-1}{a}(c + by_1)$,

$x_2 = \frac{-1}{a}(c + by_2)$ and $x_3 = \frac{-1}{a}(c + by_3)$.

Area of the triangle formed by the three points $= \frac{1}{2}\sum x_1(y_2 - y_3)$.

Substituting for x_1, x_2, x_3 in the above, we get the area as zero.

We have now proved that any three arbitrary points on the locus represented by the equation are collinear. The locus must be only a straight line. The proof is complete.

Connector 2: Show that the equation $Ax + By - 2A + 3B = 0$ for various values of A and B represents a family of lines passing through a fixed point.

Solution: $Ax + By - 2A + 3B = 0$

$\Rightarrow \quad A(x - 2) + B(y + 3) = 0$

Clearly, $(2, -3)$ satisfies the above equation whatever be the values of A and B. Hence the equation represents a family of concurrent lines through the point $(2, -3)$.

Connector 3: A line makes intercepts whose sum is 9 and whose product is 20. Find its equation.

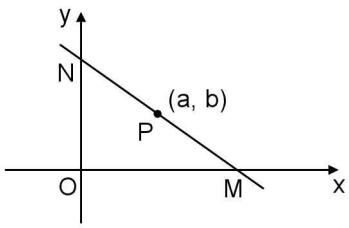
Solution: Given $a + b = 9$, $ab = 20$

It is clear that there are two sets of solutions to the above system of equations.

(i) $a = 5, b = 4$ and (ii) $a = 4, b = 5$.

Accordingly, the two possible lines are $4x + 5y - 20 = 0$ and $5x + 4y - 20 = 0$.

Connector 4: The portion of the straight line intercepted between the axes is bisected at (a, b). Show that the equation of the line is $\frac{x}{a} + \frac{y}{b} = 2$.



Solution: P(a, b) is the mid-point of the line segment MN.

$\Rightarrow$ M is $(2a, 0)$ and N is $(0, 2b)$

The equation of the line is $\frac{x}{2a} + \frac{y}{2b} = 1$

$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 2$

Connector 5: Prove that the points $(3, -2)$, $(7, 6)$, $(-1, 2)$, and $(-5, -6)$ are the vertices of a rhombus.

Solution: If A, B, C, D represent the vertices, Slope of AB = 2, Slope of CD = 2.

$\therefore$ AB is parallel to CD.

Again, slope of AD $= \frac{1}{2}$, slope of BC $= \frac{1}{2}$.

$\therefore$ AD is parallel to BC.

Also, $AB^2 = 80 = AD^2$.

Hence the result.

Connector 6: Obtain the equation of the line passing through the intersection of lines $3x - 2y + 1 = 0$ and $5x + 6y - 17 = 0$ and perpendicular to the line $x + 3y - 6 = 0$.

Solution: Any line through the intersection of $3x - 2y + 1 = 0$ and $5x + 6y - 17 = 0$ can be assumed as

$3x - 2y + 1 + \lambda(5x + 6y - 17) = 0$

i.e., $(3 + 5\lambda)x + (6\lambda - 2)y + 1 - 17\lambda = 0$

Its slope $= -\left(\dfrac{3 + 5\lambda}{6\lambda - 2}\right)$

This line must be perpendicular to the line $x + 3y - 6 = 0$

This gives $-\left(\dfrac{3 + 5\lambda}{6\lambda - 2}\right) \times \left(\dfrac{-1}{3}\right) = -1$

$3 + 5\lambda = -3(6\lambda - 2) = -18\lambda + 6.$

$23\lambda = 3$ or $\lambda = \dfrac{3}{23}$.

The required line is $3x - y - 1 = 0$.

Connector 7: Find the equation of the line through the intersection of the lines $x + 2y - 3 = 0$, $4x + 7y + 9 = 0$ and which has a slope equals -2.

Solution: Any line through the intersection of the lines $x + 2y - 3 = 0$ and $4x + 7y + 9 = 0$ may be written in the form

$x + 2y - 3 + k(4x + 7y + 9) = 0$

Since the slope of the above line is given as (-2), $-\dfrac{1 + 4k}{(2 + 7k)} = -2$

$1 + 4k = +4 + 14k \Rightarrow 10k = -3,\ k = \dfrac{-3}{10}$

Equation of the line is $x + 2y - 3 - \dfrac{3}{10}(4x + 7y + 9) = 0$

$\Rightarrow\ 10x + 20y - 30 - (12x + 21y + 27) = 0$

$\Rightarrow\ -2x - y - 57 = 0$

$\Rightarrow\ 2x + y + 57 = 0$

Connector 8: Find the equations of the medians of the triangle ABC formed by the points A(4, 8), B(3, 2) and C(5, –6).

Solution: Let D, E, F represent the midpoints of the sides BC, CA, AB. Then, D is (4, –2), E is $\left(\frac{9}{2}, 1\right)$ and F is $\left(\frac{7}{2}, 5\right)$.

Median AD: A (4, 8), D (4, –2)

Since the x coordinates of both A and D are same (equals 4), AD must be parallel to the y-axis. Therefore, the equation of AD is $x = 4$.

Median BE: B (3, 2), E $\left(\frac{9}{2}, 1\right)$

Slope of BE $= \dfrac{1 - 2}{\frac{9}{2} - 3} = -\dfrac{2}{3}$.

Equation to BE is $y - 2 = -\frac{2}{3}(x - 3)$

$2x + 3y - 12 = 0.$

Median CF: C (5, –6), F $\left(\frac{7}{2}, 5\right)$

Slope of CF $= -\frac{22}{3}$. So, equation to CF is $22x + 3y - 92 = 0$.

Connector 9: Find the coordinates of the orthocentre of the triangle whose vertices are (1, –2), (3, 1), (–2, 3).

Solution: Let A, B, C represent vertices. Orthocentre is the point of intersection of the altitudes of a triangle. Let AD_1 and BE_1 be the altitudes to BC and AC respectively and O be the orthocentre. We shall find the equations of AD_1 and BE_1. Solving these equations we get the coordinates of O.

Equation to AD_1 is obtained as $5x - 2y - 9 = 0$, and that of BE_1 as $3x - 5y - 4 = 0$.

The two altitudes are therefore $5x - 2y - 9 = 0$ and $3x - 5y - 4 = 0$.

The orthocentre is therefore, $\left(\frac{37}{19}, \frac{7}{19}\right)$.

Connector 10: Find the equations of the lines through the point (2, 3) which make an angle of 45° with the line $3x - y + 5 = 0$.

Solution: Let the slope of the line through (2, 3) be denoted by m. The slope of the line $3x - y + 5 = 0$ is 3. Since the angle between these lines is to be 45°,

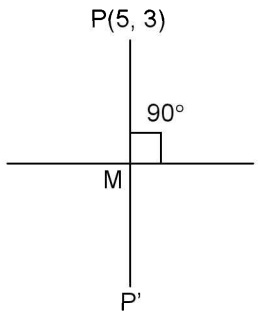
we have $\left|\frac{m - 3}{1 + 3m}\right| = \tan 45° = 1.$

This gives $\frac{m - 3}{1 + 3m} = \pm 1$, or $m = -2, \frac{1}{2}$.

The two lines are $y - 3 = -2(x - 2)$ and $y - 3 = \frac{1}{2}(x - 2)$.

Or $2x + y - 7 = 0$ and $x - 2y + 4 = 0$.

Connector 11: Find the coordinates of the image of the point (5, 3) in the line $3x - 2y + 7 = 0$.



Solution: If P' is the image of P in the line, PM = MP'. We shall find the equation of the line PMP' and obtain the coordinates of M. Since M is the mid-point of PP', the coordinates of P' can be easily obtained.

Since PMP' is perpendicular to the line $3x - 2y + 7 = 0$, its equation may be assumed as $2x + 3y + k = 0$.

(5, 3) lies on it. Therefore, $2 \times 5 + 3 \times 3 + k = 0$, $k = -19$.

Thus, PMP' is $2x + 3y - 19 = 0$.

We solve the two equations $3x - 2y + 7 = 0$ and $2x + 3y - 19 = 0$ to obtain the coordinates of M.

M is found to be $\left(\frac{17}{13}, \frac{71}{13}\right)$.

If P' is (x, y), $\frac{x + 5}{2} = \frac{17}{13}$ and $\frac{y + 3}{2} = \frac{71}{13}$ giving P' as $\left(-\frac{31}{13}, \frac{103}{13}\right)$.

Connector 12: Show that any point on the line $2x + 11y = 5$ is equidistant from the lines $24x + 7y = 20$ and $4x - 3y = 2$.

Solution: To prove that any point on the line $2x + 11y = 5$ is equidistant from the lines $24x + 7y = 20$ and $4x - 3y = 2$ is same as proving that $2x + 11y = 5$ is one of the bisectors of the angles between the lines $24x + 7y = 20$ and $4x - 3y = 2$.

The equations of the bisectors of the angles between the given lines are given by

$$\frac{24x + 7y - 20}{25} = \pm\frac{4x - 3y - 2}{5}$$

$\therefore$ $24x + 7y - 20 = \pm 5(4x - 3y - 2)$

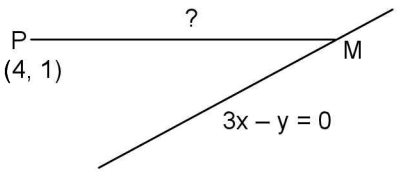
Taking + ve sign, $\Rightarrow 24x + 7y - 20 = 20x - 15y - 10$

$\therefore$ $4x + 22y = 10$

$\therefore$ $2x + 11y = 5$ is one of the bisectors of the angles between the given lines.

Hence the result.

Connector 13: Find the distance of the line $3x - y = 0$ from the point (4, 1) measured along the straight line making an angle of 135° with the x-axis.



Solution: Given that PM makes an angle 135° with the x–axis.

Therefore, slope of PM = –1.

Since it passes through (4, 1), its equation is

$y - 1 = -1(x - 4)$ or $x + y - 5 = 0$.

To get M, we solve $3x - y = 0$ and $x + y = 5$.

$\therefore$ M is $\left(\frac{5}{4}, \frac{15}{4}\right)$.

We require PM. PM^2 is obtained as $\frac{242}{16}$, giving PM = $\frac{11\sqrt{2}}{4}$.

Connector 14: If p is the length of the perpendicular from the origin to the line whose intercepts on the x and y axes are α and β respectively, prove that $p = \frac{\alpha\beta}{\sqrt{(\alpha^2 + \beta^2)}}$.

Solution: The equation of the line is $\frac{x}{\alpha} + \frac{y}{\beta} = 1$

$$p = \frac{1}{\sqrt{\frac{1}{\alpha^2} + \frac{1}{\beta^2}}} = \frac{\alpha\beta}{\sqrt{\alpha^2 + \beta^2}}$$

Connector 15: Find the direction in which a straight line must be drawn through the point (1, 2) so that its point of intersection with the line $x + y = 4$ may be at a distance $\frac{\sqrt{6}}{3}$ from this point.

Solution: A line through (1, 2) may be assumed as $\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = r$ where, θ is the angle made by the line with the x–axis and r is the distance of any point (x, y) from (1, 2).

$(r\cos\theta + 1, r\sin\theta + 2)$ lies on $x + y = 4$.

$r(\cos\theta + \sin\theta) + 3 = 4$

$\Rightarrow$ $r(\cos\theta + \sin\theta) = 1$ — (1)

Also, r must be = $\frac{\sqrt{6}}{3}$

Substituting r in (1), $\cos\theta + \sin\theta = \frac{3}{\sqrt{6}} = \frac{\sqrt{3}}{\sqrt{2}}$

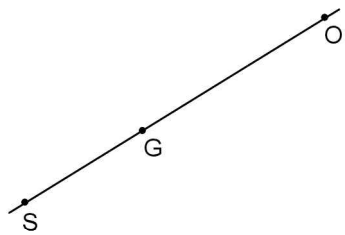
$$\Rightarrow \quad \cos\left(\pi/4 - \theta\right) = \frac{\sqrt{3}}{2}$$

$$\pi/4 - \theta = \pm\frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{12} \text{ or } \frac{5\pi}{12} \ (15^\circ \text{ or } 75^\circ)$$

Connector 16: Prove that the orthocenter of the triangle whose vertices are given as $(a\cos\theta_1, a\sin\theta_1)$, $(a\cos\theta_2, a\sin\theta_2)$, $(a\cos\theta_3, a\sin\theta_3)$ is the point
$(a(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)), a(\sin\theta_1 + \sin\theta_2 + \sin\theta_3))$.

Solution: From the given data on the coordinates of the vertices of the triangle, it is clear that origin is the circum-centre and the circum radius is a. We also know that if S, G, O represent the circum centre, centroid and orthocenter of a triangle, $\frac{SG}{GO} = \frac{1}{2}$

$$G \text{ is } \left(\frac{a(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)}{3}, \frac{a(\sin\theta_1 + \sin\theta_2 + \sin\theta_3)}{3}\right)$$



If O is (x, y),

$$\frac{1 \times x + 2 \times 0}{3} = \frac{a(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)}{3} \text{ and}$$

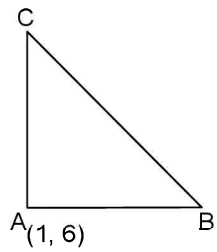
$$\frac{1 \times y + 2 \times 0}{3} = \frac{a(\sin\theta_1 + \sin\theta_2 + \sin\theta_3)}{3}$$

$\Rightarrow$ Orthocentre is at $(\Sigma\, a\cos\theta_1, \Sigma\, a\sin\theta_1)$

Connector 17: Find the equations of the sides AB, AC of a right isosceles triangle if the equation of its hypotenuse BC is $x - 2y - 3 = 0$ and the coordinates of A are (1, 6).

Solution: Equation of AB may be assumed as

$$\frac{x-1}{\cos\alpha} = \frac{y-6}{\sin\alpha} = k \qquad \text{— (1)}$$



and that of AC as

$$\frac{x-1}{-\sin\alpha} = \frac{y-6}{\cos\alpha} = k, \text{ where } AB = AC = k \qquad \text{— (2)}$$

$\Rightarrow$ B is $(1 + k\cos\alpha, 6 + k\sin\alpha)$ and C is $(1 - k\sin\alpha, 6 + k\cos\alpha)$

$$\text{Slope of BC} = \frac{(6 + k\cos)(6 + k\sin\alpha)}{(1 - k\sin\alpha) - (1 + k\cos\alpha)} = \frac{k(\cos\alpha - \sin\alpha)}{-k(\sin\alpha + \cos\alpha)}$$

But, slope of BC $= \frac{1}{2}$. We therefore have

$$-\frac{(\cos\alpha - \sin\alpha)}{(\sin\alpha + \cos\alpha)} = \frac{1}{2}$$

$2\cos\alpha - 2\sin\alpha = -\sin\alpha - \cos\alpha \Rightarrow \quad 3\cos\alpha = \sin\alpha$

$\Rightarrow$ $\tan\alpha = 3$

The sides AB, AC are $y - 6 = 3(x - 1)$ and $y - 6 = \frac{-1}{3}(x - 1)$

or $3x - y + 3 = 0$ and $x + 3y - 19 = 0$.

Connector 18: If m and n are the perpendiculars from the origin to the lines $x \sec\alpha + y \operatorname{cosec}\alpha = a$ and $x\cos\alpha - y\sin\alpha = a\cos 2\alpha$, show that $4m^2 + n^2 = a^2$.

Solution: $x \sec\alpha + y \operatorname{cosec}\alpha = a$ — (1)

$x\cos\alpha - y\sin\alpha = a\cos 2\alpha$ — (2)

m = perpendicular from the origin to (1)

$$= \frac{a}{\sqrt{\sec^2\alpha + \operatorname{cosec}^2\alpha}} = a\cos\alpha\sin\alpha$$

n = perpendicular from the origin to (2)

$$= \frac{a\cos 2\alpha}{\sqrt{\cos^2\alpha + \sin^2\alpha}} = a\cos 2\alpha$$

$4m^2 + n^2 = 4a^2\cos^2\alpha\sin^2\alpha + a^2\cos^2 2\alpha = a^2$

Connector 19: For what value of k the three lines $x - 2y + 1 = 0$, $2x - 5y + 3 = 0$ and $5x - 9y + k = 0$ are concurrent?

Solution: $x - 2y + 1 = 0$ — (1)

$2x - 5y + 3 = 0$ — (2)

$5x - 9y + k = 0$ — (3)

If the three lines are to be concurrent, the point of intersection of (1) and (2) must lie on (3).

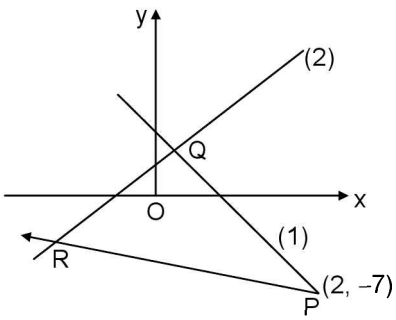
Solving (1) and (2), $x = 1, y = 1$.

$\therefore \quad 5 \times 1 - 9 \times 1 + k = 0, k = 4$

Connector 20: P(2, –7) is a point on the line $4x + y - 1 = 0$. This line meets the line QR whose equation is $3x - 4y + 1 = 0$ at the point Q. Find the equation to line PR such that PQ = PR.

Solution: $4x + y - 1 = 0$ — (1)

$3x - 4y + 1 = 0$ — (2)



Let $\angle PQR = \alpha$

Slope of PQ = –4

Slope of QR = $\frac{3}{4}$

$$\Rightarrow \quad \tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-4 - 3/4}{1 - 3/4 \times 4}\right| = \frac{19}{8}$$

PQ = PR $\quad\Rightarrow\quad \angle PRQ = \alpha$

Now the angle which the straight line PR makes with the x-axis is > $\pi/2$

$\Rightarrow$ Its slope m is negative.

$$\therefore \quad \tan\alpha = \frac{3/4 - m}{1 + 3/4\, m} = \frac{3 - 4m}{4 + 3m} = \frac{19}{8} \Rightarrow m = -\frac{52}{89}$$

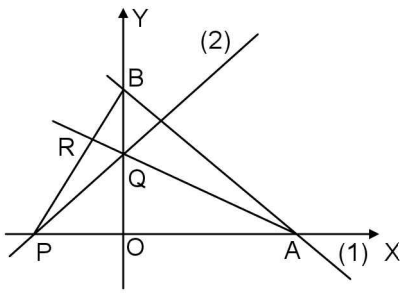
The line PR passes through P(2, –7). Its equation is thus given by

$$y + 7 = -\frac{52}{89}(x - 2)$$

$\Rightarrow \quad 52x + 89y + 519 = 0$

Connector 21: The straight line $\frac{x}{a}+\frac{y}{b}=1$ cuts the x and y axes in A and B respectively. A straight line perpendicular to AB cuts the axes in P and Q respectively. Prove that the lines AQ and BP are perpendicular to each other. Find the locus of the point of intersection of AQ and BP.

Solution: A and B are (a, 0) and (0, b) respectively.



Let a line perpendicular to $\frac{x}{a}+\frac{y}{b}=1$ be $ax-by=k$, where k is a constant

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \text{—(1)}$$

$$ax-by=k \qquad \text{—(2)}$$

P and Q are $\left(\frac{k}{a},0\right)$ and $\left(0,\frac{-k}{b}\right)$ respectively.

Slope of AQ $=\frac{k}{ab}$, Slope of BP $=\frac{-k}{k/a}=\frac{-ab}{k}$

Slope of AQ × Slope of BP = −1

⇒ AQ and BP are perpendicular to each other.

Let the point of intersection of AQ and BP be R. We have to find the locus of R. Since the angle between AQ and BP is 90°, ∠ARB = 90°.

This means that the locus of R is the circle on AB as diameter. Mid-point of AB is $\left(\frac{a}{2},\frac{b}{2}\right)$ which is therefore, the centre of the circle.

Radius of the circle = $=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}$

Therefore, the locus of R is the circle $|z-z_0|=r$, where $z_0=\frac{a}{2}+\frac{ib}{2}$ and $r=|z_0|$

Connector 22: Find the centroid of the triangle, the equations of whose sides are $12x^2-20xy+7y^2=0$ and $2x-3y+4=0$.

Solution: Since $12x^2-20xy+7y^2=0$ represents a pair of lines through the origin, we have to solve the equations $12x^2-20xy+7y^2=0$ and $2x-3y+4=0$ to get the coordinates of the other two vertices.

We have, from the second equation $x=\frac{3y-4}{2}$

Substituting in the first equation and simplifying, we get $y^2-8y+12=0$ giving y = 2, 6. Therefore, x = 1, 7.

The other two vertices are (1, 2) and (7, 6).

The centroid is therefore at $\left(\frac{8}{3},\frac{8}{3}\right)$.

Connector 23: The gradient of one of the lines represented by $ax^2+2hxy+by^2=0$ is twice that of the other. Show that $8h^2=9ab$.

Solution: If m_1 and m_2 are the slopes (or gradients) of the lines represented by the homogeneous equation, $m_1+m_2=-\frac{2h}{b}$, $m_1m_2=\frac{a}{b}$.

We are given the additional information that $m_1=2m_2$.

This means that $3m_2=-\frac{2h}{b}$ and $2m_2^2=\frac{a}{b}$ which gives $8h^2=9ab$.

Connector 24: Find the equation of the pair of lines through the origin perpendicular to the lines represented by $ax^2 + 2hxy + by^2 = 0$.

Solution: If m_1 and m_2 are the slopes of the given lines, the slopes of the perpendicular lines are $\frac{-1}{m_1}$ and $\frac{-1}{m_2}$. Since both these lines pass through the origin, their combined equation is $\left(y + \frac{1}{m_1}x\right)\left(y + \frac{1}{m_2}x\right) = 0$ or

$m_1 m_2 y^2 + x^2 + (m_1 + m_2)\, xy = 0$

Or $bx^2 - 2hxy + ay^2 = 0$.

Connector 25: Prove that the product of the perpendiculars from (α, β) on the lines $ax^2 + 2hxy + by^2 = 0$ is

$$\left|\frac{a\alpha^2 + 2h\alpha\beta + b\beta^2}{\sqrt{4h^2 + (a-b)^2}}\right|.$$

Solution: If the lines are $y - m_1x = 0$ and $y - m_2 x = 0$, the product of the perpendiculars from (α, β) to the lines is given by $\left|\frac{\beta - m_1\alpha}{\sqrt{1 + m_1^2}}\right| \times \left|\frac{\beta - m_2\alpha}{\sqrt{1 + m_2^2}}\right|$

Using the relations $m_1 + m_2 = -\frac{2h}{b}$, $m_1 m_2 = \frac{a}{b}$ and simplifying,

$$\text{Product of perpendiculars} = \left|\frac{a\alpha^2 + 2h\alpha\beta + b\beta^2}{\sqrt{4h^2 + (a-b)^2}}\right|.$$

Connector 26: If the lines $ax^2 + 2hxy + by^2 = 0$ be two sides of a parallelogram and the line $lx + my = 1$ be one of the diagonals, show that the equation of the other diagonal is $y(bl - hm) = x(am - hl)$.

Solution: Let the diagonal $lx + my = 1$ intersect the two lines in A and B. Since diagonals bisect each other, the other diagonal passes through the midpoint M of AB. It is enough if we find the coordinates of M.

Let A be (x_1, y_1) and B be (x_2, y_2). Then, M is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

From $lx + my = 1$ we have $y = \frac{1 - lx}{m}$

On substituting this y in $ax^2 + 2hxy + by^2 = 0$, we get the quadratic equation

$(am^2 - 2hlm + bl^2)x^2 + 2(hm - bl)\, x + b = 0$

This quadratic in x gives two roots which represent the x coordinates of A and B.

Therefore, $x_1 + x_2$ = Sum of the roots

$$= \frac{-2(hm - bl)}{am^2 - 2hlm + bl^2} \quad \text{or}$$

$$\frac{x_1 + x_2}{2} = \frac{bl - hm}{(am^2 - 2hlm + bl^2)}.$$

$$\frac{y_1 + y_2}{2} = \frac{am - hl}{(am^2 - 2hlm + bl^2)}.$$

Since the other diagonal is OM, its equation is $y = \dfrac{am - hl}{bl - hm}x$.

Hence the result.

Connector 27: Show that equation $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$ represents a pair of lines. Find the lines and their point of intersection.

Solution: We have, $a = 3, h = 5, b = 8, g = 7, f = 11$ and $c = 15$.

We see that the condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ is satisfied. Hence it represents two lines.

Taking the second degree terms only, we get

$3x^2 + 10xy + 8y^2 = (3x + 4y)(x + 2y)$.

Let us assume that $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 \equiv (3x + 4y + \ell)(x + 2y + m)$

Equating the x, y and constant terms

$\ell + 3m = 14, 2\ell + 4m = 22, \ell m = 15,$

We easily get $\ell = 5, m = 3$

The two lines are $3x + 4y + 5 = 0$ and $x + 2y + 3 = 0$

We also obtain their point of intersection as $(1, -2)$.

Connector 28: For what value of c will the equation $12x^2 - 10xy + 2y^2 + 14x - 5y + c = 0$ represent two straight lines? Find their point of intersection and the angle between them.

Solution: We have $a = 12, h = -5, b = 2, g = 7, f = -5/2$. Since the equation represents a pair of lines,

$abc + 2fgh - af^2 - bg - ch^2 = 0$ we thus get $c = 2$.

And the lines are $6x - 2y + 1 = 0$ and $2x - y + 2 = 0$.

The point of intersection of the lines is at $\left(\dfrac{3}{2}, 5\right)$ and the angle θ between the lines is given by $\tan\theta = 1/7$.

Connector 29: Show that the equation $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$ represents a pair of parallel lines and find the distance between them.

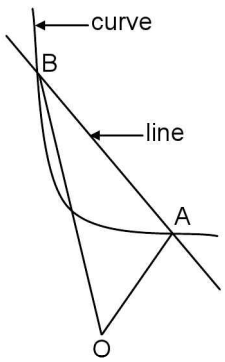
Solution: It is easily seen that the condition for the equation to represent a pair of lines is satisfied.

On factoring, we get the lines as $2x + y - 4 = 0$ and $2x + y + 1 = 0$.

The distance between the lines is therefore, $= \dfrac{4}{\sqrt{5}} + \dfrac{1}{\sqrt{5}} = \sqrt{5}$.

Connector 30: Obtain the equation of the lines joining the origin to the points of intersection of the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and the straight line $\ell x + my = 1$.

Solution: In general, the second degree equation above represents a pair of straight lines or a circle or a parabola or an ellipse or a hyperbola depending on the conditions satisfied by the coefficients in the equation. (This will be treated in detail in the Unit on Circles and Conic Sections). The line $\ell x + my = 1$ therefore intersects this curve in two points say A and B.



Our problem is to find the equation of the pair of lines OA and OB where O is the origin. It is clear that the combined equation of OA and OB is a homogeneous equation in x and y of degree 2.

Let A be (x_1, y_1)

Clearly, $\ell x_1 + my_1 = 1$ — (1)

and $ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$ — (2)

If we make the given second degree equation homogeneous by means of $\ell x + my = 1$, we get

$ax^2 + 2hxy + by^2 + 2(gx + fy)(\ell x + my) + c(\ell x + my)^2 = 0$ — (3)

(3) is a homogeneous equation in x and y of degree 2. Therefore, it represents a pair of lines through the origin. Using (1) and (2) it is clear that (3) is satisfied by (x_1, y_1). Similarly, the coordinates of B can be seen to satisfy (3).Therefore, (3) represents the combined equation of OA and OB.

Connector 31: Show that the lines joining the origin to the points common to $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ and $3x - 2y = 1$ are at right angles.

Solution: Using the technique employed in problem no. 30, the equation to the required pair of lines is obtained as $9x^2 + 10xy - 9y^2 = 0$.

Since coefficient of x^2 + coefficient of y^2 in the above equation is equal to zero, the pair of lines are at right angles.

Connector 32: The curve $x^2 + y^2 = a^2$ cuts off an intercept on the straight line $\ell x + my = 1$, which subtends an angle of $\pi/4$ at the origin.

Show that $4[a^2(\ell^2 + m^2) - 1] = [a^2(\ell^2 + m^2) - 2]^2$

Solution: The equation of the pair of lines through the origin and passing through the points of intersection of the curve $x^2 + y^2 = a^2$ and the line $\ell x + my = 1$ is

$x^2 + y^2 - a^2(\ell x + my)^2 = 0$

$\Rightarrow \quad x^2(1 - a^2\ell^2) + y^2(1 - a^2m^2) - 2\ell ma^2xy = 0$

If θ is the angle between the lines represented by the above equation, $\tan\theta = \tan 45° = 1$(given)

$$1 = \frac{2\sqrt{\ell^2m^2a^4 - (1 - a^2\ell^2))1 - a^2m^2)}}{2 - a^2(\ell^2 + m^2)}$$

$$\Rightarrow \quad [2 - a^2(\ell^2 + m^2)]^2 = 4\{\ell^2m^2a^4 - (1 - a^2\ell^2)(1 - a^2m^2)\}$$

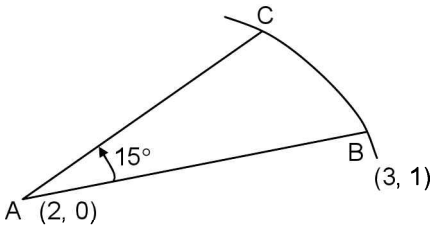
$$= 4\{a^2\ell^2 - 1 + a^2m^2\} = 4\{a^2(\ell^2 + m^2) - 1\}$$

or $4[a^2(\ell^2 + m^2) - 1] = [a^2(\ell^2 + m^2) - 2]^2$

Connector 33: The line joining two points A (2, 0), B (3, 1) is rotated about A in the counter-clockwise sense through an angle 15°. If B goes to C in the new position, what will be the coordinates of C? Find the equation of the line in the new position.

Solution: The complex number representing AB is $(3 - 2) + (1 - 0)i$ or $(1 + i)$

The complex number representing AC is $(1 + i)(\cos 15° + i \sin 15°)$



$$= (1 + i)\left(\frac{\sqrt{3} + 1}{2\sqrt{2}} + i\frac{\sqrt{3} - 1}{2\sqrt{2}}\right)$$

$$= \left(\frac{\sqrt{3} + 1}{2\sqrt{2}} - \frac{\sqrt{3} - 1}{2\sqrt{2}}\right) + i\left(\frac{\sqrt{3} + 1}{2\sqrt{2}} + \frac{\sqrt{3} - 1}{2\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{i\sqrt{3}}{\sqrt{2}}$$

Therefore, C is represented by the complex number $\left(\frac{1}{\sqrt{2}}+2\right)+i\left(\frac{\sqrt{3}}{\sqrt{2}}+0\right)=\frac{4+\sqrt{2}}{2}+i\frac{\sqrt{3}}{\sqrt{2}}$

Or, the new position of B (which is C) has coordinates $\left(\frac{4+\sqrt{2}}{2},\frac{\sqrt{3}}{\sqrt{2}}\right)$

Equation of the line in the new position is

$$y-0=\left(\frac{\sqrt{3}}{\sqrt{2}}\times\frac{\sqrt{2}}{1}\right)(x-2)\Rightarrow y=\sqrt{3}x-2\sqrt{3}$$

Connector 34: Find the point of intersection of lines represented by $3x^2-2y^2-5xy+4x+13y-15=0$. If the origin is shifted to the point of intersection of these lines, without changing the directions of the axes, obtain the equation of the pair of lines in the new coordinate system.

Solution: $3x^2-2y^2-5xy+4x+13y-15=0$

The point of intersection of the lines represented by (1) is given by solving

$ax+hy+g=0$

$hx+by+f=0$

$\Rightarrow\quad 3x-\frac{5}{2}y+2=0;\quad -\frac{5}{2}x-2y+\frac{13}{2}=0$

$\Rightarrow\quad 6x-5y+4=0,\qquad -5x-4y+13=0.\qquad$ or $5x+4y-13=0$

we get $x=1, y=2$.

Thus, the point of intersection of the lines is (1, 2)

Suppose the origin shifted to the point (1, 2). Then, $x=X+1, y=Y+2$, where (x, y) and (X, Y) are respectively the coordinates of a point in the old and new system. The equation of the pair of lines in the new system is

$3(X+1)^2-2(Y+2)^2-5(X+1)(Y+2)+4(X+1)+13(Y+2)-15=0$

$\Rightarrow\quad 3X^2-2Y^2-5XY=0$

Connector 35: One of the bisectors of the angle between the lines

$a(x-1)^2+2h(x-1)(y-2)+b(y-2)^2=0$ is $x+2y-5=0$. Find the equation of the other bisector.

Solution: Shifting the origin to the point (1, 2), the equation of the pair of lines in the new system is

$aX^2+2hXY+bY^2=0$ — (1)

Also, the equation of one of the bisectors of the pair of lines in the new system is

$X+1+2(Y+2)-5=0$

i.e., $X+2Y=0$ — (2)

Since the bisectors are perpendicular to each other the equation of the other bisector in the new system is $2X-Y=0$.

Hence the equation of the other bisector in the old system is

$2(x-1)-(y-2)=0\qquad$ or $2x-y=0$.

OR

It is clear that the point of intersection of the lines represented by the given equation is (1, 2).

The equation of the other bisector can be assumed as $2x-y+k=0$. It passes through (1, 2).

$\Rightarrow 2-2+k=0$ or $k=0$

Hence, equation of the other bisector is $2x-y=0$.

TOPIC GRIP

Subjective Questions

1. Find the equation of the line
   (i) which is equidistant from the lines $x = -4$ and $x = 6$
   (ii) which makes an angle 15° with x-axis and passing through the point (3, 5)
   (iii) passing through the points (3, 5) and (–3, –5)
   (iv) with slope $\frac{-1}{5}$ and y intercept –6
   (v) which cuts off intercepts –3 and 5 on the x-axis and the y-axis respectively.
   (vi) for which $p = 3$ and $\alpha = 30°$ (Normal form)
   (vii) passing through the point $\left(2, \sqrt{3}\right)$ and makes an angle 60° with x-axis in the symmetric form

2. (i) Find the angle between the lines $y - \sqrt{3}\,x + 2 = 0$ and $\sqrt{3}\,y - x + 7 = 0$.
   (ii) Find the equation of the line passing through the point (1, –1) and parallel to the line $3x + 5y = 4$.
   (iii) Find the equation of the perpendicular bisector of the line joining the points (1, 1) and (3, –3).

3. Show that the following lines:
   (i) $7x - 5y + 9 = 0$ and $21x - 15y + 27 = 0$ are coincident
   (ii) $5x - 4y + 7 = 0$, $5x + 4y + 9 = 0$ are intersecting
   (iii) $x + 2y - 5 = 0$, $2x + 3y - 8 = 0$, $3x + y - 5 = 0$ are concurrent.

4. Find the equation of the line(s) passing through a point (2, 2) such that sum of its intercepts on the axes is 9.

5. The coordinates of the midpoints of the sides of a triangle are (2, 1) (5, 3) and (3, 7). Find the equations of the sides of the triangle.

6. Two vertices of a triangle are at (5, –1) and (–2, 3). If the orthocentre of the triangle is at the origin, find the third vertex.

7. Prove that the perpendiculars on the straight line $x \cos\alpha + y \sin\alpha = \frac{-\sin^2\alpha}{\cos\alpha}$ from the points $(k^2, 2k)$, $(km, k + m)$ and $(m^2, 2m)$ form a G.P.

8. Prove that the area of a parallelogram equals $\frac{p_1 p_2}{\sin\alpha}$ where, p_1 and p_2 are the lengths of the altitudes and α is the angle of the parallelogram. Hence deduce that the area of the parallelogram formed by the lines

   $a_1x + b_1y + c_1 = 0$, $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_2x + b_2y + d_2 = 0$ is

   $$\left|\frac{(d_1 - c_1)(d_2 - c_2)}{(a_1b_2 - a_2b_1)}\right|$$

9. Show that if one of the straight lines given by the equation $ax^2 + 2hxy + by^2 = 0$ is common with one of those given by $a_1x^2 + 2h_1xy + b_1y^2 = 0$, then

   $$(ab_1 - a_1b)^2 + 4(ah_1 - a_1h)(bh_1 - b_1h) = 0.$$

10. Show that the general equation of a line may be represented in the form $\bar{A}z + A\bar{z} + B = 0$ where, B is a real constant, A is a given complex number of the form $\alpha + i\beta$ $(\beta \neq 0)$ and z stands for $x + iy$.

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

11. The equation of the line passing through (3, 3) and making an angle of 60° with the positive direction of the x-axis is

(a) $x - \sqrt{3}\,y + 3 - 3\sqrt{3} = 0$ (b) $x + \sqrt{3}\,y + 3 - 3\sqrt{3} = 0$

(c) $\sqrt{3}\,x - y + 3 - 3\sqrt{3} = 0$ (d) $\sqrt{3}\,x - y - 3 + 3\sqrt{3} = 0$

12. The equation of the line passing through the point (2, 3) such that its x intercept equals twice its y intercept, is

(a) $x + 2y = 8$ (b) $2x + y = 7$ (c) $x + y = 5$ (d) $3x + y = 9$

13. The perpendicular distance between the two straight lines $3x + 4y - 20 = 0$ and $3x + 4y + 5 = 0$ is

(a) 4 (b) 3 (c) 5 (d) $\sqrt{5}$

14. If θ is the angle between the straight lines represented by $2x^2 + 5xy + 3y^2 - 6x + 7y + 4 = 0$, then $\tan\theta$ is equal to

(a) $\frac{1}{10}$ (b) $\frac{1}{5}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

15. Origin is shifted to the point (2, 1) without changing the direction of the axes. Then the equation of the line $3x - 4y - 7 = 0$ in the new system is

(a) $3x - 4y + 5 = 0$ (b) $3x + 4y - 5 = 0$ (c) $3x - 4y - 5 = 0$ (d) $4x - 3y + 5 = 0$

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(c) Statement-1 is True, Statement-2 is False

(d) Statement-1 is False, Statement-2 is True

16. **Statement 1**

If the coordinates of the circumcentre and the orthocenter of a triangle are (1, –3) and (3, 5) respectively, the coordinates of the centroid of the triangle are $\left(\frac{5}{3}, \frac{-1}{3}\right)$.

and

Statement 2

Centroid of a triangle divides the line joining circumcentre and orthocentre in the ratio 2 : 1.

17. **Statement 1**

Distance between the parallel lines $3x - y + 6 = 0$ and $9x - 3y - 8 = 0$ is $\frac{26}{3\sqrt{10}}$.

and

Statement 2

Distance between the parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$.

18. Statement 1

The lines $3x + 5y + 7 = 0$, $5x + y - 3 = 0$ and $x + 3y + 4 = 0$ are concurrent.

and

Statement 2

If the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent, then $a_3x + b_3y + c_3 = a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2)$ for a unique value of λ.

19. Statement 1

The pair of lines $3x^2 - 2y^2 - 5xy = 0$ intersect the line $x = 2$ at A and B. Then, AB = 7.

and

Statement 2

$ax^2 + 2hxy + by^2 = 0$ represents a pair of lines through the origin.

20. Consider the pair of lines

$2(x + 1)^2 - 7(x + 1)(y - 2) + 5(y - 2)^2 = 0.$

Statement 1

If the origin is shifted to the point (1, –2) without changing the direction of the axes, the equation of the pair of lines in the new coordinate system is $2X^2 - 7XY + 5Y^2 = 0$.

and

Statement 2

The point of intersection of the, pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is at $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$.

Linked Comprehension Type Questions

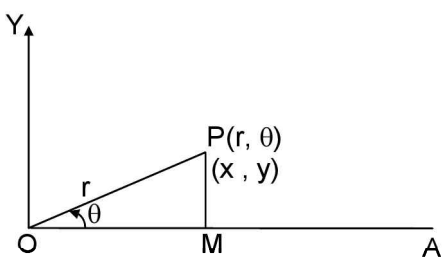
Directions: **This section contains 2 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

Polar coordinate system

A point in a plane can be represented using the polar coordinate system.

Let O be a fixed point and OA be a fixed line in the plane. O is called the pole and OA is called the initial line.



Let P be any point in the plane where OP = r and $\angle AOP = \theta$. Then, P can be represented by the ordered pair (r, θ) where, $0 < r < \infty$ and $0 \le \theta < 2\pi$. $r = 0$ corresponds to the pole. (r, θ) are the polar coordinates of P. Any point on the initial line is of the form $(r, 0)$. Let OY be the perpendicular through O to OA. Then, any point on OY is of the form $\left(r, \frac{\pi}{2}\right)$. Choosing OA as x–axis and OY as y–axis, let the coordinates of P referred to the Cartesian coordinate system OA, OY be (x, y). Then,

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad \text{— (1)}$$

$$\left.\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}\right\} \quad \text{— (2)}$$

Relations (1) and (2) can be used to transform the coordinates of a point in the plane from the polar coordinate system to the Cartesian coordinate system and vice versa.

21. The polar coordinates of the point whose Cartesian coordinates are (3, 4), are

(a) $\left(5, \tan^{-1}\frac{3}{4}\right)$ (b) $\left(5, \tan^{-1}\frac{3}{5}\right)$

(c) $\left(5, \tan^{-1}\frac{4}{3}\right)$ (d) $\left(5, \tan^{-1}\frac{4}{5}\right)$

22. The Cartesian coordinates of the point whose polar coordinates are $\left(4, \frac{\pi}{3}\right)$, are

(a) $\left(2\sqrt{3}, 2\right)$ (b) $\left(2, 2\sqrt{3}\right)$

(c) $\left(2, -2\sqrt{2}\right)$ (d) $\left(2, 2\sqrt{2}\right)$

23. Given the sets of points

(i) $\left\{\left(5, \frac{3\pi}{2}\right), \left(7, \frac{\pi}{2}\right), \left(9, \frac{\pi}{2}\right)\right\}$ (ii) $\left(2, \frac{\pi}{3}\right), \left(2, \frac{\pi}{6}\right), \left(4, \frac{\pi}{3}\right)$

(iii) $\left(10, \frac{\pi}{12}\right), \left(7, \frac{13\pi}{12}\right), \left(7, \frac{\pi}{12}\right)$ (iv) $\left(1, \frac{\pi}{6}\right), \left(2, \frac{\pi}{3}\right), \left(2, \frac{\pi}{6}\right)$

which set/sets of points are collinear?

(a) (i) only (b) (i), (iii) (c) (i), (iii), (iv) (d) (iii) only

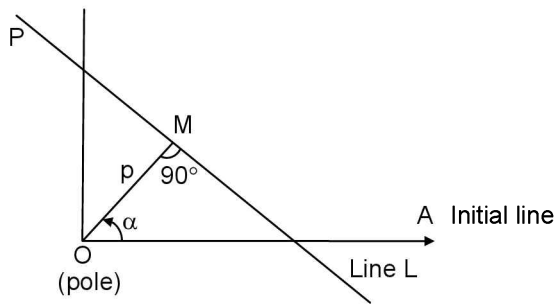
Passage II

(i) Let (r_1, θ_1) and (r_2, θ_2) represent the polar coordinates of two points A and B in a plane. Then,

$AB = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$

(ii) Area of the triangle, the coordinates of whose vertices are (r_1, θ_1), (r_2, θ_2) and (r_3, θ_3) is the numerical value of $\frac{1}{2}\{r_1r_2\sin(\theta_2 - \theta_1) + r_2r_3\sin(\theta_3 - \theta_2) + r_3r_1\sin(\theta_1 - \theta_3)\}$

(iii)



Let the perpendicular distance of a line L in the plane from the pole O be p and let the angle $\angle AOM = \alpha$. Then the equation of the line L is $r\cos(\theta - \alpha) = p$

If $\alpha = 0$, $r\cos\theta = p$ represents a line perpendicular to the initial line.

If $\alpha = \frac{\pi}{2}$, $r\sin\theta = p$ represents a line parallel to the initial line.

24. The distance between the points $\left(3,\frac{\pi}{6}\right)$ and $\left(4,\frac{\pi}{3}\right)$ is

(a) $\sqrt{25+12\sqrt{3}}$ (b) $\sqrt{25-12\sqrt{2}}$

(c) $\sqrt{25-12\sqrt{3}}$ (d) $\sqrt{25+6\sqrt{3}}$

25. Area of the triangle whose vertices are $\left(2,\frac{\pi}{3}\right)$, $\left(3,\frac{\pi}{2}\right)$, $\left(6,\frac{3\pi}{2}\right)$ is

(a) $9\sqrt{2}+2$ (b) $\frac{9}{2}$ (c) 9 (d) $\frac{9\sqrt{3}}{2}$

26. The equation of the initial line is

(a) $\theta = 0$ (b) $\theta = \frac{\pi}{2}$ (c) $\theta = \pi$ (d) $\theta = \frac{3\pi}{2}$

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers of which ONE OR MORE answers will be correct.**

27. Let the abscissa of the point of intersection of the lines $\frac{x}{a}+\frac{y}{b}=2$ and $\frac{x}{b}+\frac{y}{a}=2$ be c. Then

(a) ordinate of the point of intersection of the lines is $\frac{c}{a+b}$

(b) ordinate of the point of intersection of the lines is c.

(c) a, b, c are in A.P.

(d) a, c, b are in H.P.

28. If (2, 3) is one of the vertices and $x - y + 3 = 0$ is one of the sides of an equilateral triangle, then

(a) area of the triangle is cosec 60°

(b) circumcenter of the triangle is at $\left(\frac{4}{3},3\right)$

(c) the slopes of the other two sides of the triangle are $\left(\sqrt{3}-2\right)$ and $-\left(2+\sqrt{3}\right)$

(d) orthocenter of the triangle is at $\left(\frac{4}{3},\frac{11}{3}\right)$

29. The pair of lines $2x^2 - xy - 10y^2 - 3x + 21y - 9 = 0$ intersect at P. The line $x = 3$ intersects this pair of lines in Q and R. If A and B divide QR in the ratio 2 : 1 and 1 : 2 respectively, then

(a) Area of ΔPAB is 0.6 units

(b) $\angle APB$ is $\cot^{-1}\left(\frac{49}{15}\right)$

(c) Equation of pair of lines PA and PB is $x^2 + xy - 4y^2 + 7y - x - 3 = 0$

(d) mid-point of AB is $\left(3,\frac{9}{10}\right)$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

30.

	Column I	**Column II**
(a)	p is the length of the perpendicular from the origin to the line $\frac{x}{a}+\frac{y}{5}=1$. Also a^2, p^2, 25 are in A.P. Then $a^4-2a^2p^2+2p^4+4$ equals	(p) -7
(b)	(p, 3) and (5, q) are the opposite vertices of a rectangle and the other two vertices lie on the line $y = 2x + c$ which passes through (p, q). Then c equals	(q) 2
(c)	The lines $ax + by + p = 0$, $x\cos\alpha + y\sin\alpha - p = 0$, and $x\sin\alpha - y\cos\alpha = 0$ (with $p \neq 0$) are concurrent and the angle between the first two lines is 45°. Then $a^2 + b^2$ equals	(r) 3
(d)	The number of integer values of m for which the x-cooridnate of the point of intersection of the lines $2x + y = 7$ and $y = mx + 2$ is also an integer is	(s) 4

IIT ASSIGNMENT EXERCISE

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

31. The ratio in which the line segment joining (2, –3) and (5, 6) is divided by the x-axis is
(a) 3 : 1 (b) $\sqrt{3}$: 2 (c) 1 : 2 (d) $\sqrt{2}$: 3

32. The angle subtended at the origin by the line segment joining A(5, 2) and B(6, –15) is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

33. The equation of the line passing through the intersection of the lines $5x - 3y + 3 = 0$ and $3x + y - 1 = 0$ and cutting off equal intercepts on the co-ordinate axes is
(a) $x + y + 1 = 0$ (b) $x + y - 1 = 0$ (c) $x - y + 1 = 0$ (d) $x - y - 1 = 0$

34. The y-intercept of the straight line passing through (2, 2) and perpendicular to the line $3x + y - 3 = 0$ is
(a) $\frac{4}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) 1

35. If the acute angle between the lines $x + ky + 3 = 0$ and $2x + y - 7 = 0$ is $\tan^{-1}\frac{3}{4}$, then k is
(a) 1 (b) 2 (c) 3 (d) 4

36. Equation of the line passing the point $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line $x\sec\theta + y\,\text{cosec}\,\theta = a$ is
(a) $x\cos\theta + y\sin\theta = a$ (b) $x\cos\theta - y\sin\theta = \cos\theta$
(c) $x\cos\theta + y\sin\theta = a\sec\theta$ (d) $x\cos\theta - y\sin\theta = a\cos 2\theta$

37. The equation of the line perpendicular to the line $3x - 4y - 5 = 0$ and at a distance of 2 units from the origin is
(a) $4x + 3y \pm 5 = 0$ (b) $3x + 4y \pm 5 = 0$
(c) $4x + 3y \pm 10 = 0$ (d) $4x + 3y \pm 20 = 0$

38. If the area of triangle formed by the line $4x + 3y + c = 0$ and the axes of co-ordinates is 6, then c is equal to
(a) ± 18 (b) ± 16 (c) ± 12 (d) ± 8

39. The quadrilateral formed by the straight lines $x \pm y \pm 1 = 0$ is a
(a) square (b) rhombus but not square
(c) rectangle (d) trapezium

40. P is a point which is at a distance of 7 units from A(1, 1) and B is the point (–2, 5). The maximum value of PB is
(a) 14 (b) 12 (c) 7 (d) 5

41. If the slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is three times the slope of the other line of the pair, then $h^2 : ab =$
(a) 2 : 3 (b) 8 : 9 (c) 3 : 4 (d) 4 : 3

42. The point of intersection of the pair of straight lines represented by $6x^2 + 5xy - 4y^2 + 7x + 13y - 3 = 0$ is
(a) (1, 1) (b) (1, –1) (c) (–1, 1) (d) (–1, –1)

43. The line $2x + y = 5$ divides the line segment joining the points (–1, 1) and (7, 8) in the ratio
(a) 6 : 17 (b) 3 : 2 (c) 4 : 5 (d) 4 : 3

44. The straight line through P(1, 3) such that its intercept between the axes is bisected at P is
(a) $x + y = 6$ (b) $3x + y = 8$ (c) $3x + y = 6$ (d) $x - 3y = 6$

45. If a, b, c are in A.P, then the line $ax + by + c = 0$ will always pass through the point
(a) (2, 1) (b) (1, 2) (c) (1, –2) (d) (–2, –1)

46. P (1, 3) is the image of Q (–3, 1) in the line L. The equation to the line L is
(a) $2x + y = 0$ (b) $2x - y = 0$ (c) $2x + y = 5$ (d) $2x - y + 4 = 0$

47. The area bounded by the curves $x + 2|y| = 1$ and $x = 0$ is
(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

48. The reflection of the point (4, –13) in the straight line $5x + y + 6 = 0$ is the point
(a) (–1, –14) (b) (3, 4) (c) (1, 2) (d) (–4, 13)

49. The orthocentre of the triangle formed by the lines $xy = 0$ and $x + y + 1 = 0$ is
(a) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{3}, \frac{1}{3}\right)$ (c) (0, 0) (d) $\left(\frac{1}{4}, \frac{1}{4}\right)$

50. The value of λ so that $3x^2 + 7xy + \lambda y^2 + 5x + 5y + 2 = 0$ represents a pair of straight lines is
(a) 3 (b) –3 (c) 2 (d) –2

51. Lines represented by $x^2 - 2pxy - y^2 = 0$ bisect the angles between the lines represented by $x^2 - 2qxy - y^2 = 0$. Then
(a) $p + q = 1$ (b) $pq = 1$ (c) $pq + 1 = 0$ (d) $p - q = 1$

52. If $y = mx$ bisects the angle between the pair of straight lines represented by $ax^2 + 2hxy + by^2 = 0$, then
(a) $h(1 - m^2) = m(a - b)$ (b) $h(1 + m^2) = m(a - b)$
(c) $h(1 - m^2) = m(a + b)$ (d) $h(1 + m^2) = m(a + b)$

53. The equations of the diagonals of the square formed by the pairs of straight lines
$3x^2 + 8xy - 3y^2 = 0$ and $3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$ are
(a) $x = 2y, 4x + 2y + 1 = 0$ (b) $2x + y = 0, 2x = 4y + 1$
(c) $x = 2y, 2x = 4y + 1$ (d) $2x + y = 0, 4x + 2y + 1 = 0$

54. The line $3x - 4y + 8 = 0$ is rotated through an angle $\frac{\pi}{4}$ in the clockwise sense about the point (0, 2). The equation of the line in the new position is
(a) $x + 7y - 2 = 0$ (b) $x - 7y + 14 = 0$ (c) $x + 7y - 14 = 0$ (d) $7x - y + 2 = 0$

55. If the sum of the slopes of the lines represented by $x^2 - (5c + 1)xy + 9cy^2 = 0$ is 5 times their product, c equals
(a) $\frac{1}{40}$ (b) $\frac{-1}{40}$ (c) $\frac{-1}{50}$ (d) $\frac{1}{50}$

56. If the points (2a, a) (a, 2a) and (a, a) enclose a triangle of area 72, then the coordinates of the centroid of the triangle is
(a) (16, 16) (b) (4, 4) (c) (12, 12) (d) (–4, –4)

57. Three lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent, if
(a) $a + b + c = 1$ (b) $a^2 + b^2 + c^2 = 2(ab + bc + ca)$
(c) $a^3 + b^3 + c^3 = 3abc$ (d) None of these

58. The circumcentre of the triangle formed by the lines $x = 0$, $y = 0$, $x + y = 1$ is
(a) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (b) (0, 0) (c) (1, 1) (d) (1, 0)

59. If the lines represented by $6x^2 - 6xy + ky^2 + x + 5y - 1 = 0$ are perpendicular to each other, then k equals

(a) 6 (b) −6 (c) $\frac{25}{24}$ (d) 0

60. The equation $\sqrt{(x-2)^2 + y^2} + \sqrt{(x+2)^2 + y^2} = 4$ represents

(a) x axis
(b) the segment $-2 \le x \le 2$ of the x axis
(c) the straight line $2x + 3y = 4$
(d) $\{(x, y)/0 \le x \le 2, 0 \le y \le 2\}$

61. If the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is the square of the other, then

(a) $\frac{a+b}{h} + \frac{8h^2}{ab} = 6$ (b) $\frac{a-b}{h} + \frac{8h^2}{ab} = 0$ (c) $\frac{a+b}{h} - \frac{8h^2}{ab} = 0$ (d) $\frac{a+b}{h} + \frac{8h^2}{ab} = 0$

62. If the distances of any point P from the points A(a + b, a − b) and B (a − b, a + b) are equal, the locus of P is

(a) $ax + by = 0$ (b) $x - y = 0$ (c) $x + y = 0$ (d) $bx + ay = 0$

63. A straight line through P(2, 2) intersects the lines $y + x\sqrt{3} = 0$ and $y - x\sqrt{3} = 0$ at A and B. If OAB (where O is the origin) is an equilateral triangle, then the equation of AB is

(a) $x - 2 = 0$ (b) $y - 2 = 0$ (c) $x - y = 0$ (d) $x + y - 4 = 0$

64. $x + y = 5$, $4x - 7y + 10 = 0$ and $7x + 4y - 15 = 0$ represent the sides of a triangle. The equation of the altitude to the greatest side is

(a) $x - y + 1 = 0$ (b) $xy - x + 1 = 0$ (c) $x - y - 1 = 0$ (d) $y - 2x - 1 = 0$

65. The distance of the point P(4, 0) from the line $x + 2y - 5 = 0$ measured parallel to the line $2x + 3y + 5 = 0$ is

(a) $\sqrt{5}$ (b) $\sqrt{11}$ (c) $\sqrt{13}$ (d) $\sqrt{15}$

66. The line $\frac{x}{a} + \frac{y}{b} = 1$, moves in such a way that $1/a^2 + 1/b^2 = 1/c^2$, where c is a constant. The locus of the foot of the perpendicular from the origin on the line is

(a) $x^2 - y^2 = c^2$ (b) $x^2 + y^2 = c^2$ (c) $1/x^2 + 1/y^2 = 1/c^2$ (d) $x^2 - 2y^2 = c^2$

67. The area of the triangle formed by the lines $18x^2 - 9xy + y^2 = 0$ and the line $y = 9$ is

(a) $\frac{27}{4}$ (b) $\frac{27}{2}$ (c) $\frac{27}{8}$ (d) 27

68. The angle between the lines given by $x^2 - 2xy \operatorname{cosec}\theta + y^2 = 0$ is

(a) $\frac{\pi}{2} + \theta$ (b) $\frac{\pi}{2} - \theta$ (c) $-\frac{\pi}{2} + \theta$ (d) $\frac{\theta}{2}$

69. The pair of straight lines joining the origin to the points of intersection of the straight line $y = mx + c$ and the curve $x^2 + y^2 = a^2$ are at right angles if

(a) $c^2 = 2a^2(1 + m^2)$ (b) $a^2 = 2c^2(1 + m^2)$ (c) $c^2 = a^2(1 + m^2)$ (d) $2c^2 = a^2(1 + m^2)$

70. If 2p represents the length of the perpendicular from the origin to the line $\frac{x}{a} + \frac{y}{b} = 1$, then a^2, $8p^2$, b^2 are in

(a) AP (b) GP (c) HP (d) None of these

71. For different values of p and q, the family of lines defined by
$(p + 2q)x + (3p + q)y - 5(p + q) = 0$ passes through the point

(a) (−2, −1) (b) (1, −2) (c) (2, 1) (d) (−1, −2)

72. The equation of the lines through the origin and which are perpendicular to the pair of straight lines $x^2 - 3xy + 2y^2 = 0$ is

(a) $2x^2 - 3xy + y^2 = 0$ (b) $2x^2 + 3xy - y^2 = 0$ (c) $2x^2 + 3xy + y^2 = 0$ (d) $2x^2 - 3xy - y^2 = 0$

73. Keeping the origin fixed, the coordinate axes are rotated through an angle θ. If the equation of the line $\frac{x}{a}+\frac{y}{b}=1$ with respect to the new axes is $\frac{x}{p}+\frac{y}{q}=1$, then

(a) $a^2p^2 + b^2q^2 = 1$ (b) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$

(c) $a^2 + b^2 = p^2 + q^2$ (d) $\frac{1}{a^2}+\frac{1}{p^2}=\frac{1}{b^2}+\frac{1}{q^2}$

74. The difference between the slopes of the pair of straight lines $(\tan^2\alpha + \cos^2\alpha)\,x^2 - 2xy\tan\alpha + y^2\sin^2\alpha = 0$ is

(a) 1 (b) −1 (c) 3 (d) 2

75. If the pairs of straight lines $px^2 + 2mxy - py^2 = 0$ and $qx^2 + 2nxy - qy^2 = 0$ are equally inclined to each other, then

(a) $pq = mn$ (b) $pm = qn$ (c) $pn = qm$ (d) $pm^2 = qn^2$

76. The equation of the straight line passing through the point of intersection of the straight lines $3x - y + 2 = 0$ and $5x - 2y + 7 = 0$ and which is perpendicular to the x-axis, is

(a) $x = 2$ (b) $x = 3$ (c) $x = 4$ (d) $y = 4$

77. If the equation of the base of a triangle is $2x - y + 3 = 0$ and the vertex opposite to this side is (2, 3), the length of the altitude of the triangle on this base is

(a) $\frac{8}{\sqrt{15}}$ (b) $\frac{4}{\sqrt{5}}$ (c) $\frac{2}{\sqrt{15}}$ (d) $\frac{4}{\sqrt{15}}$

78. The angle made by the perpendicular from the origin to the line $x + \sqrt{3}y - 8 = 0$ with the positive direction of x-axis is

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{5\pi}{6}$

79. The image of the point A(3, 4) in the line mirror $x = 0$ is B and the image of B in the line mirror $x = y$ is (p, q). Then,

(a) $p = 4, q = -3$ (b) $p = 4, q = 3$ (c) $p = -4, q = -3$ (d) $p = -4, q = 3$

80. The lines $2x + y = 3$, $x + 2y = 3$, $2x + y = 5$, $x + 2y = 5$ form the sides of a

(a) square (b) rhombus (c) rectangle (d) trapezium

81. If A(x, y), B(y, z) and C(z, x) are the vertices of a right angled triangle with $\angle A = 90°$, then

(a) $x = z$ (b) $y = z$ (c) $x = y$ (d) $x + y = 0$

82. The equation of a straight line which makes an angle of $\tan^{-1}\sqrt{2}$ with the x-axis and cuts off an intercept of $\frac{-3}{\sqrt{2}}$ with the y-axis is

(a) $y = \sqrt{2}x - 1$ (b) $\sqrt{2}y = 2x - 3$ (c) $\sqrt{2}y - 3 = -2x$ (d) $2x+3+\sqrt{2}y = 0$

83. The quadrilateral formed by the straight lines $5x \pm 2y \pm 5 = 0$ has an area

(a) 2 (b) 4 (c) 5 (d) 6

84. If a, c, b are in GP, then the area of the triangle formed by $ax + by + c = 0$ with the coordinate axes is

(a) a^2b (b) $\frac{1}{2}$ (c) $2a + b$ (d) 25

85. The point of intersection of the two lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$, $a \neq b$ is

(a) $\left(\frac{ab}{a+b},\frac{ab}{a+b}\right)$ (b) $\left(-\frac{ab}{a+b},\frac{ab}{a+b}\right)$ (c) $\left(\frac{a+b}{ab},\frac{a+b}{ab}\right)$ (d) $\left(\frac{a+b}{ab},-\frac{a+b}{ab}\right)$

86. Let $L \equiv 12x + 5y - 13 = 0$. Then, the points $(-3, 1)$ and $(1, -3)$

(a) lie on the same of side of L
(b) lie on the line L
(c) lie on a perpendicular to L
(d) lie on opposite side of L and are equidistant from L

87. One of the bisectors of the angles between the lines $3x - 4y + 12 = 0$ and $4x + 3y + 2 = 0$ is

(a) $x + 7y - 10 = 0$ (b) $x - 7y + 10 = 0$ (c) $7x + y + 14 = 0$ (d) $7x - y - 10 = 0$

88. If the angle between the lines represented by $2x^2 + \lambda y^2 - 7xy + 10x - 20y + c = 0$ where, $\lambda > 0$ is 45°, the values of λ and c are respectively

(a) 3 and –12 (b) 1 and 12 (c) 3 and 12 (d) 1 and –12

89. The origin is shifted to a point P so that the equation $y^2 + 4y + 8x - 2 = 0$ does not contain constant term and the term in y. The coordinates of P are

(a) $(3, -2)$ (b) $\left(\frac{3}{4}, -2\right)$ (c) $(2, -3)$ (d) $(-2, 3)$

90. If the points P (h, k) and Q (k, h) lie on $3x + 2y - 13 = 0$ and $4x - y - 5 = 0$ respectively, the equation of PQ is

(a) $x - y = 6$ (b) $x + y = 5$ (c) $x - y + 5 = 0$ (d) $2x + 2y = 5$

91. The acute angle between the straight lines $3x - 4y + 7 = 0$ and $ax + 4y + 7 = 0$ is $\frac{\pi}{4}$. Then 'a' is

(a) $\frac{4}{7}$ (b) $\frac{-7}{4}$ (c) 1 (d) $\frac{-4}{7}$

92. The foot of the perpendicular from (2, 1) to the line $2x - 4y - 5 = 0$ is at

(a) $\left(\frac{5}{2}, 0\right)$ (b) $\left(\frac{3}{2}, \frac{-1}{2}\right)$ (c) $\left(2, \frac{-1}{4}\right)$ (d) $\left(0, \frac{-5}{4}\right)$

93. Equation of the line which is concurrent with the lines $4x + 3y - 7 = 0$ and $8x + 5y - 1 = 0$ and whose gradient $\frac{-3}{2}$, is

(a) $3x + 2y - 2 = 0$ (b) $2y - 3x - 2 = 0$ (c) $3x + 2y - 63 = 0$ (d) $3x + 2y - 50 = 0$

94. The angle between the lines represented by $x \cos \omega_1 + y \sin \omega_1 = p_1$ and $x \cos \omega_2 + y \sin \omega_2 = p_2$ is

(a) $90° - (\omega_1 + \omega_2)$ (b) $180° - (\omega_1 + \omega_2)$ (c) $\omega_1 - \omega_2$ (d) $\omega_1 + \omega_2$

95. Locus of the midpoint of the portion of the line $x \sin \alpha + y \cos \alpha = \sin 2\alpha$ intercepted between the axes is

(a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = \frac{1}{4}$ (c) $x^2 + y^2 = 1$ (d) $x^2 - y^2 = 4$

96. If the lines $3x + 4y + 1 = 0$, $ax + 2y - 3 = 0$ and $2x - y - 3 = 0$ are concurrent, then a equals

(a) 4 (b) 3 (c) 5 (d) –4

97. The image of the point (3, 8) in the line $x + 3y = 7$ is P. Then, the equation of the line through P with slope $\frac{7}{2}$ is

(a) $7x - 2y + 1 = 0$ (b) $7x - 2y - 1 = 0$ (c) $7x - 2y + 6 = 0$ (d) $2x + 7y + 1 = 0$

98. Area bounded by the straight line $y = 1$ and $\pm 2x + y = 2$ and the x-axis is

(a) $\frac{1}{2}$ (b) 2 (c) 1 (d) $\frac{3}{2}$

99. Equation of the perpendicular bisector of the line segment joining (0, 4) and (4, 8) is

(a) $x + y - 8 = 0$ (b) $x - y + 8 = 0$ (c) $-x + y - 8 = 0$ (d) $x + y + 8 = 0$

100. The product of the perpendiculars from the point (1, 1) to the pair of straight lines represented by $2x^2 + 6xy + 3y^2 = 0$ is

(a) $\frac{11}{\sqrt{37}}$ (b) $\frac{11}{\sqrt{35}}$ (c) $\frac{12}{\sqrt{37}}$ (d) $\frac{13}{\sqrt{37}}$

101. If one of the lines $ax^2 + 2hxy + by^2 = 0$ bisects the angle between the lines $xy = 0$, then

(a) $b^2 - ab = 0$ (b) $(a + b)^2 = h^2$ (c) $(a + b)^2 = 4h^2$ (d) $ab + h^2 = 0$

102. The values of 'h' for which $3x^2 - 2hxy + 4y^2 = 0$ represents a pair of coincident lines are

(a) $\pm 3\sqrt{3}$ (b) $\pm\sqrt{3}$ (c) $\pm 2\sqrt{3}$ (d) $\pm\sqrt{6}$

103. Angle between the lines represented by $x^2 - y^2 = 0$ is

(a) 0° (b) 45° (c) 90° (d) 180°

104. If $a \neq b \neq c$ and the equations $(b - c)x + (c - a)y + (a - b) = 0$ and $(b^3 - c^3)x + (c^3 - a^3)y + (a^3 - b^3) = 0$ represent the same line then the line $ax + by + c = 0$

(a) passes through the fixed point (1, 1) (b) is parallel to the first line

(c) is parallel to the second line (d) passes through the origin.

105. If the pair of straight lines $x^2 - y^2 + 2x + 1 = 0$ meet the y-axis in A and B, AB equals

(a) 1 (b) 0 (c) 2 (d) 4

106. The straight line $3x + y = 9$ divides the line segment joining the points (1, 3) and (2, 7) in the ratio

(a) 4 : 2 (b) 3 : 4 (c) 4 : 5 (d) 5 : 6

107. The equation of straight line which passes through the point (2, –3) and cuts off equal intercepts on the axes is

(a) $x + y = 1$ (b) $2x + y = 1$ (c) $x + y + 1 = 0$ (d) $x - y - 5 = 0$

108. The equation of the line passing through the intersection of the lines $x - 3y + 1 = 0$ and $2x + 5y - 9 = 0$ and at a distance $\sqrt{5}$ from the origin is

(a) $2x - 3y = 5$ (b) $x + 2y = 5$ (c) $2x + y = 5$ (d) $x + 2y = 1$

109. If $x \cos\alpha + y \sin\alpha = a$ and $x \sin\alpha - y \cos\alpha = a$, where α is varied, represent two lines, then the locus of the point of intersection of these two lines is

(a) $x^2 + y^2 = 2a^2$ (b) $x^2 + y^2 = a^2$ (c) $x^2 - y^2 = a^2$ (d) $a^2x^2 + a^2y^2 = 1$

110. If the pair of lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on the y axis, then

(a) $abc - fgh = bg^2 + ch^2$ (b) $abc + 2fgh = bg^2 + ch^2$ (c) $abc + fgh = af^2 + ch^2$ (d) $abc = bg^2 + ch^2$

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(c) Statement-1 is True, Statement-2 is False

(d) Statement-1 is False, Statement-2 is True

111. Statement 1

Orthocentre of the triangle formed by the lines $3y + x = 0$, $x - y - 6 = 0$ and $y - 3x = 0$ is at the origin.

and

Statement 2

Orthocentre of a triangle is the point of concurrence of the attitudes of the triangle.

112. Statement 1

The points $(2, -3)$ and $(1, 9)$ lie on the origin side of the line $5x - y + 9 = 0$.

and

Statement 2

The points (x_1, y_1) and (x_2, y_2) lie on the origin side of the line $ax + by + c = 0$ if $(ax_1 + by_1 + c)$ and $(ax_2 + by_2 + c)$ are both positive.

113. Statement 1

Equation of a pair of lines is given by $3x^2 - 2y^2 + 5xy + 19x - 4y + 6 = 0$.

Let the coordinate axes Ox and Oy be rotated through an angle $\frac{\pi}{4}$ in the positive sense and let the new system of axes be OX and OY. If the equation of the pair of lines in the new coordinate system is $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$, then, $A + B = 1$.

and

Statement 2

Let the equation of a pair of lines be given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

If the axes of coordinate are rotated through an angle θ in the positive sense without changing the origin and if the equation to the pair of lines in the new coordinate system is

$$AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0, \text{ then, } a + b = A + B$$

Linked Comprehension Type Questions

Directions: **This section contains 1 paragraph. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Homogenization.

Let the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (1)

and the line $px + qy = r$ (where $r \neq 0$)

intersect at A and B. Let O be the origin. Then, the combined equation of OA and OB is obtained by homogenizing (1) with the help of (2). We get

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(\frac{px + qy}{r}\right) + c\left(\frac{px + qy}{r}\right)^2 = 0 \text{ as the required equation.}$$

114. If the straight lines joining the origin to the points of intersection of the curve $2x^2 + xy - 9y^2 + 5x + 2y + 7 = 0$ and the line $x + y - \lambda = 0$ are at right angles, the value/values of λ are

(a) 1 and 2 (b) -1 and -2 (c) 1 and -1 (d) -1 and 2

115. If θ is the angle between the pair of lines joining the origin to the points of intersection of the curves $3x^2 + 4xy - y^2 + 2x + y + 6 = 0$ and $x - 6 = 0$, $\tan\theta =$

(a) $\frac{\sqrt{1129}}{30}$ (b) $\frac{\sqrt{1129}}{15}$ (c) $\frac{22}{15}$ (d) $\frac{11}{15}$

116. The area of the triangle formed by the line $x + y - 8 = 0$ and the pair of straight lines through origin, which pass through the intersection of $x^2 + y^2 - 2x - 2y - 2 = 0$ and $x + y - 4 = 0$

(a) 32 (b) 8 (c) 16 (d) 24

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers of which ONE OR MORE answers will be correct.**

117. A_iB_i denote the intercepts made by the lines $x = i$, where $i = 1, 2, 3 \ldots$ on the pair of lines given by $6x^2 + xy - y^2 - 7x + 4y - 3 = 0$. Then

(a) sequence $\{A_iB_i\}$ form an A.P. with common difference 5

(b) sequence $\{x_n\}$ form a G.P. with common ratio 2

(c) $\sum_{i=1}^{n} A_iB_i = \frac{n}{2}(5n + 1)$

(d) $\sum_{i=1}^{n} A_iB_i = \frac{n}{2}(5n + 3)$

118. Let L_1 denote the line $x + y + 3 = 0$

(a) Equation of the line through the point (5, 3) making an angle 45° with L_1 is $y = 3$

(b) Equation of the line through the point (5, 3) making an angle 45° with L_1 is $x = 5$

(c) Projection of the line segment joining the points (5, 3) and (–1, –2) on the line $x + y + 3 = 0$ is $\frac{1}{\sqrt{2}}$

(d) Line joining the origin and the point (5, 3) makes an angle with the line $x + y + 3 = 0$ which is greater than 60°

119. Consider the ΔABC with vertices A(2, 7), B(2, 3) and C(4, 3). Let Q be the image of P(1, 4) in AB; R and S be the images of Q in BC and AC respectively. Then

(a) co-ordinates of Q are (3, 2)

(b) co-ordinates of R are (3, 4)

(c) co-ordinates of S are $\left(\frac{19}{5}, \frac{22}{5}\right)$

(d) centroid of ΔSRP is at $\left(\frac{39}{15}, \frac{52}{15}\right)$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

120.

(a) The points on the line $x + y = 4$, such that distance from these points to the line $4x + 3y = 10$ is unity	(p) (0, 3)
(b) Points on the line $x + y = 3$ such that they are at equal distances from the lines $\lvert x\rvert = \lvert y\rvert$ are	(q) (3, 0)
(c) Image of the point (0, –3) with respect to this line $x + y = 0$ is	(r) (3, 1)
(d) Algebraic sum of distances from the points (–2, 0), (4, 0), and (1, 1) to a variable line $ax + by + c = 0$ is zero. Then the line passes through the point.	(s) (–7, 11)

ADDITIONAL PRACTICE EXERCISE

Subjective Questions

121. The line $\ell x + my + n = 0$ bisects an angle between a pair of straight lines of which one is $px + qy + r = 0$. Show that the equation of the other line is
$(px + qy + r)(\ell^2 + m^2) = 2(\ell p + mq)(\ell x + my + n)$.

122. (i) A triangle is formed by the lines $ax + by + c = 0$, $\ell x + my + n = 0$ and $px + qy + r = 0$. Show that the straight line $\dfrac{ax + by + c}{(ap + bq)} = \dfrac{\ell x + my + n}{(\ell p + mq)}$ passes through the orthocentre of the triangle formed by the three given lines.

(ii) The sides of a triangle are given by the equations $\mu_r = x\cos\alpha_r + y\sin\alpha_r - p_r = 0$, $r = 1, 2, 3$ show that the orthocentre is given by
$\mu_1\cos(\alpha_2 - \alpha_3) = \mu_2\cos(\alpha_3 - \alpha_1) = \mu_3\cos(\alpha_1 - \alpha_2)$.

123. Show that the straight line $\ell x + my + n = 0$ and the straight lines $(\ell x + my)^2 - 3(mx - \ell y)^2 = 0$ form the sides of an equilateral triangle.

124. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines at right angles to each other. Prove that the square of the distance of their point of intersection from the origin is $\left(\dfrac{f^2 + g^2}{b^2 + h^2}\right)$.

125. Show that the condition that one of the two lines $ax^2 + 2hxy + by^2 = 0$ is perpendicular to one of the lines $a_1x^2 + 2h_1xy + b_1y^2 = 0$ is $4(ah_1 + hb_1)(ha_1 + bh_1) + (aa_1 - bb_1)^2 = 0$.

126. Prove that the general equation of the second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel straight lines if $h^2 = ab$ and $bg^2 = af^2$. Prove that the distance between the two straight lines is $2\sqrt{\dfrac{(g^2 - ac)}{a(a + b)}}$.

127. Given the family of lines $\lambda(2x + y + 4) + \mu(x - 2y - 3) = 0$, where λ and μ are some constants, prove that among the members of the family, there exists only one line whose distance from the point $(2, -3)$ is $\sqrt{10}$. Also write the equation of this line.

128. Show that the area of the triangle formed by the lines $y = x\tan\alpha$, $y = x\tan\beta$, $y = x\tan\theta + c$ is
$\dfrac{c^2\sin(\alpha - \beta)\cos^2\theta}{2\sin(\alpha - \theta)\sin(\beta - \theta)}$.

129. A variable straight line drawn through the point of intersection of the straight lines $\dfrac{x}{3} + \dfrac{y}{2} = 1$ and $\dfrac{x}{2} + \dfrac{y}{3} = 1$ meets the coordinate axes at A and B. Show that the locus of midpoint of AB is the curve $10xy = 6(x + y)$.

130. The line $lx + my + n = 0$ bisects the angle between a pair of straight lines of which one is $px + qy + r = 0$. Find the equation of the other line.

Straight Objective Type Questions

***Directions*: This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

131. The distance of the point (x_1, y_1) from each of two straight lines through the origin is λ, and the equation of the pair is $(x_1y - xy_1)^2 = k(x^2 + y^2)$. Then k is
(a) λ^2 (b) $\lambda^2 + 1$ (c) $2\lambda^2$ (d) 1

132. Number of non-zero integer values of k for which the points (k, k^2) lies inside the triangle formed by the lines $11x + 6y + 14 = 0$, $9x + y - 12 = 0$, $2x + 5y - 17 = 0$ is

(a) 0 (b) 2 (c) 3 (d) 4

133. The ends A and B of a straight line segment of constant length 5 units slide on the fixed rectangular axes OX, OY respectively. If the rectangle OAPB is completed, then the locus of the foot of the perpendicular drawn from P upon AB is

(a) $x^{2/3} + y^{2/3} = 5^{2/3}$ (b) $x^{2/3} + y^{2/3} = 2^{2/3}$ (c) $x^2 + y^2 = 2$ (d) $x^{1/3} + y^{1/3} = 5^{1/3}$

134. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is the equation of a conic referred to a rectangular cartesian system Ox, Oy. The axes are rotated through an angle θ in the counter clockwise sense without changing the origin so that the equation of the conic in the new system is independent of xy term then θ equals

(a) $\tan^{-1}\frac{h}{ab}$ (b) $\frac{1}{2}\tan^{-1}\frac{2h}{ab}$ (c) $\frac{1}{2}\tan^{-1}\frac{2h}{a-b}$ (d) $\tan^{-1}\frac{2h}{a-b}$

135. From the point P(–2, 3), a ray of light is sent at an angle $\tan^{-1}(3)$ to the x–axis. Upon reaching the x–axis, the ray is reflected from it. Then, the equation of the straight line which contains the reflected ray is

(a) $3x - y + 3 = 0$ (b) $3x + y + 9 = 0$ (c) $x + 3y + 3 = 0$ (d) $x + y - 3 = 0$

136. The equation of the line passing through the intersection of the lines $x + 2y - 3 = 0$ and $4x - y + 7 = 0$ and parallel to the line $5x + 4y - 20 = 0$

(a) $5x - 4y + 7 = 0$ (b) $5x - 4y - 7 = 0$
(c) $15x + 12y - 7 = 0$ (d) $15x + 12y + 7 = 0$

137. The point A divides the line joining the points (–5, 1) and (3, 5) in the ratio $\lambda : 1$. If the coordinates of the points B and C are respectively (1, 5) and (7, –2), the values of λ for which the area of the triangles 2 sq.units are

(a) $\left\{7, \frac{31}{9}\right\}$ (b) $\left\{7, \frac{-31}{9}\right\}$ (c) $\left\{-7, \frac{31}{9}\right\}$ (d) $\left\{-7, \frac{-31}{9}\right\}$

138. Let a and b be the length of perpendiculars from the origin upon the lines $x \sec\theta + y \operatorname{cosec}\theta = k$ and $x \cos\theta - y \sin\theta = k \cos 2\theta$. Then $4a^2 + b^2$ equals

(a) 0 (b) 1 (c) k^2 (d) $2k^2$

139. The point on the y-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3 is

(a) (0, 1) (b) (0, 3) (c) (0, –9) (d) either (a) or (c)

140. Two equal sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point (1, –10). Then the equation of the third side is

(a) $3x + y + 7 = 0$ (b) $23x + 3y + 7 = 0$
(c) $-x + 3y + 31 = 0$ (d) (a) or (c)

141. The equation of the perpendicular bisectors of the sides AB and AC of triangle ABC are $x - y + 5 = 0$ and $x + 2y = 0$ respectively. If the point A is (1, –2), then the equation of BC.

(a) $14x + 23y - 40 = 0$ (b) $4x - 8y + 12 = 0$ (c) $14x - 23y + 4 = 0$ (d) $x - 4y - 40 = 0$

142. The value of p so that the three lines $3x + y - 2 = 0$, $px + 2y - 3 = 0$ and $2x - y - 3 = 0$ may intersect at one point is

(a) 3 (b) 2 (c) 5 (d) 7

143. If the straight lines given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on the y-axis then b, f, c are

(a) in AP (b) in GP (c) in HP (d) $2(b + c) = 3bfc$

144. Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form the three sides of two squares. Then the equation of fourth side of these squares is

(a) $x + 2y - 14 = 0$ (b) $2x - y - 14 = 0$ (c) $2x - y + 6 = 0$ (d) (b) or (c)

145. The vertex A of triangle ABC is given to be (1, 3) and the medians BE and CF are $x - 2y + 1 = 0$ and $y - 1 = 0$. Then the equation of its one of the sides is

(a) $x + 2y - 7 = 0$ (b) $x - 4y - 1 = 0$ (c) $x - y + 2 = 0$ (d) All the these

146. The coordinates of the point on the line $x + y = 2$ that lies at unit distance from the line $2x + 3y = 5$ are

(a) $\left(1 - \sqrt{13}, 1 + \sqrt{13}\right)$ (b) (1, 1) (c) $\left(1 - \sqrt{3}, 1 + \sqrt{3}\right)$ (d) $\left(1 - \sqrt{5}, 1 + \sqrt{5}\right)$

147. ABC is an isosceles triangle with AB = AC. D is the mid-point of BC; E is the foot of the perpendicular drawn from D to AC and F is he midpoint of D E. Then, the angle between A F and B E is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

148. A line $4x + y = 1$ through a point A(2, –7) meets the line $3x - 4y + 1 = 0$ at the point B. If C is a point on $3x - 4y + 1 = 0$ such that AB = AC, then the equation of the line through A and C.

(a) $52x + 98y + 519 = 0$ (b) $2x + 8y + 9 = 0$ (c) $52x + 89y + 519 = 0$ (d) $89x + 52y + 519 = 0$

149. The range of values of α for which the point (α, α^2) lies inside the triangle formed by lines $2x + 3y - 1 = 0$; $x + 2y - 3 = 0$; $5x - 6y - 1 = 0$ is

(a) $\frac{-3}{2} < \alpha < -1$ (b) $-3 < \alpha < 1$ (c) $\frac{1}{2} < \alpha < 3$ (d) $\frac{-3}{2} < \alpha < 1$

150. One of the equations of the lines represented by $3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$ is

(a) $3x - y - 1 = 0$ (b) $3x + y - 1 = 0$ (c) $x + y + 1 = 0$ (d) $3x + y + 1 = 0$

151. The coordinates of the orthocentre of the triangle formed by the lines $2x + y - 4 = 0$, $x - 2y + 3 = 0$ and $x + y = 5$ are

(a) (1, 2) (b) $\left(\frac{25}{11}, \frac{30}{11}\right)$ (c) $\left(\frac{-5}{3}, \frac{20}{3}\right)$ (d) (0, 0)

152. The image of P(a, b) on $y = -x$ is Q and the image of Q on the line $y = x$ is R. Then, the mid-point of PR is

(a) (a + b, b + a) (b) $\left(\frac{a+b}{2}, \frac{b+a}{2}\right)$ (c) (a –b, b – a) (d) (0, 0)

153. Each side of a square is of length 6. The centre of the square is at (4, 5). If one of its diagonals is parallel to the line $y = x$, then the coordinates of the vertices of the square are

(a) (0, 2), (0, 8), (6, 8), (6, 2) (b) (1, 2), (1, 8), (7, 8), (7, 2)
(c) (2, 3), (2, 9), (8, 9), (8, 3) (d) (3, 4), (3, 10), (9, 10), (9, 4)

154. The equation $2(a + 1)x + 5(a - 1)y - 10(a - 1) = 0$ represents

(a) a line through (0, 2) (b) a line through (0, –2)
(c) a system of lines through (0, 2) (d) a system of lines through (0, –2)

155. One of the lines represented by $x^2 - 2pxy - y^2 = 0$, is rotated about the origin through an angle θ in the clockwise direction and the other line through an angle θ in the anti-clockwise direction. Then the equation of the bisectors of the angles between the line-s in the new position is

(a) $px^2 + 2xy + py^2 = 0$ (b) $x^2 - 2pxy + y^2 = 0$
(c) $px^2 + 2xy - py^2 = 0$ (d) $px^2 - 2pxy + py^2 = 0$

156. The product of the perpendiculars from the points $\left(\pm\sqrt{a^2 - b^2}, 0\right)$ to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is

(a) b^2 (b) a^2 (c) $a^2 - b^2$ (d) $a^2 + b^2$

157. A (1, 2) and B(4, 5) are the end points of a line segment. The area of the trapezium formed by the portion of the line segment AB, the ordinates at the points of trisection of AB and the x-axis is

(a) $\frac{7}{2}$ (b) 6 (c) 7.5 (d) 12

158. The locus of the point that moves so that its distance from the origin is twice its distance from the line $x + y = 1$ is

(a) $x^2 + y^2 + 4xy - 4x - 4y + 2 = 0$
(b) $2xy - 2x - 2y + 1 = 0$
(c) $x^2 + y^2 + 2xy + 2x + 2y - 2 = 0$
(d) $x^2 + y^2 - 2xy - 2x + 2y - 2 = 0$

159. The equation of a straight line passing through the point of intersection of the lines $x - y - 1 = 0$ and $3x + y - 8 = 0$ and perpendicular to one of them is

(a) $x + y - 7 = 0$
(b) $x - 3y - 5 = 0$
(c) $2x - 6y + 3 = 0$
(d) $x - y + 3 = 0$

160. The number of triangles having the points $(3, -1)$ and $(-4, 2)$ as two of the vertices and third vertex on the line $y = x + 1$ and such that its area is 12.5, is

(a) 1
(b) 2
(c) 0
(d) 3

161. $A\left(\sqrt{4 - t^2} + t, \frac{t}{2}\right)$, $B\left(\sqrt{4 - t^2} - 2t, 2t\right)$ are two variable points, t being a parameter. The locus of the point that divides AB internally in the ratio 1 : 2 is

(a) $x^2 + y^2 = 4$
(b) $\frac{x^2}{2} + \frac{y^2}{4} = 1$
(c) $2x + y = 4$
(d) $xy + 1 = 0$

162. The four sides of a quadrilateral are given by $xy - 3x - y + 3 = 0$ and $xy - x - 7y + 7 = 0$. Its area is

(a) 12
(b) 32
(c) 18
(d) 16

163. $x^2 + k_1y^2 + 2k_2y = a^2$ represents a pair of perpendicular straight lines if

(a) $k_1 = 1, k_2 = a$
(b) $k_1 = 1, k_2 = -a$
(c) $k_1 = -1, k_2 = -a$
(d) $k_1 = -1, k_2 = 2a$

164. The equation $(x + y + 2)^2 + k(x^2 + y^2 + 1) = 0$ will represent a pair of straight lines if k equals

(a) 6
(b) -6
(c) 2
(d) -3

165. The origin, the point of intersection of the curve $2x^2 - 21xy + 18y^2 + 15x + 45y - 75 = 0$ and the point at which this curve is cut off by the line $x + 3y - 5 = 0$ form the vertices of a

(a) rectangle
(b) parallelogram
(c) square
(d) trapezium

166. The lines represented by the equation $x^2 - y^2 - 2x + 1 = 0$ are rotated about the point (1, 0). The line making the bigger angle with the positive direction of x-axis is then turned through an angle 45° in the clockwise direction and other line turned through 15° in the anti-clockwise direction. The combined equation of the pair of lines in their new positions is

(a) $\sqrt{3}x^2 - xy + 2\sqrt{3}x - y + \sqrt{3} = 0$
(b) $\sqrt{3}x^2 - xy + 2\sqrt{3}x + \sqrt{3} = 0$
(c) $\sqrt{3}x^2 - xy + 2\sqrt{3}x + y + \sqrt{3} = 0$
(d) $\sqrt{3}x^2 - xy - 2\sqrt{3}x + y + \sqrt{3} = 0$

167. Consider the points A(1, 1), B(2, -3) and the line L whose equation is $7x + 8y - 1 = 0$. Which of the following is true?

(a) AB is parallel to the line L
(b) AB is perpendicular to the line L
(c) The points A and B are on opposite sides of the line L.
(d) The points A and B are on the same side of the line L.

168. If the centroid of the triangle formed by the points (0, 0), (cos, sin) and (sin, −cos) lies on the line $y = 2x$, then is

(a) $\tan^{-1}2$
(b) $\tan^{-1}(-2)$
(c) $\tan^{-1}(-3)$
(d) $\tan^{-1}\left(\frac{1}{3}\right)$

169. If $\tan\theta$ denotes the slope of the straight line through the point $(-5, 4)$ given that its segment intercepted between the lines $3x + y = 1$ and $3x + y + 1 = 0$ is 2 units, $\sin\theta$ equals.

(a) 1
(b) $\frac{-4}{5}$
(c) -1
(d) (a) or (b)

170. If a, b, c are in AP, then the equation $ax - by + c = 0$ represents

(a) a pair of perpendicular lines
(b) a pair of parallel lines
(c) a family of parallel lines
(d) a family of concurrent lines

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

171. Statement 1

AB and AC are two perpendicular lines and AB = 5 and AC = 3.

The circum radius of the triangle ABC is $\sqrt{\frac{17}{2}}$.

and

Statement 2

Circum radius of a triangle is the distance of any vertex from the circum centre of the triangle.

172. Consider the line $L : 4x - 3y - 17 = 0$

Statement 1

The points (2, –3) and (8, 5) lie on L and that (8, 5) is at a distance of 10 units from (2, –3).

and

Statement 2

Equation of L in symmetric form is $\frac{x-2}{3/5} = \frac{y+3}{4/5} = r$ where, r is the distance of any point on L from the point (2, –3).

173. Statement 1

$(2a + b)x + (a + 4b)y - a + 18b = 0$ where a and b any real numbers represent a family of concurrent lines and

and

Statement 2

The lines $3x + y - 1 = 0$ and $x + 4y + 18 = 0$ intersect at the point (2, –5).

174. Statement 1

Equation of the internal bisector of the angles between the lines

$\sqrt{3}\,x - y - 7 = 0$ and

$x - \sqrt{3}\,y - 5\sqrt{3} - 0$ is

$x - y - (\sqrt{3} + 4) = 0$

and

Statement 2

Internal bisector of the angles between the lines $ax + by + c_1 = 0$ and

$a_2x + b_2y + c_2 = 0$ is $\frac{a,x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$.

175. Statement 1

If the numbers a, b, c re in A. P, the line $ax + by + c = 0$ passes through the point $(1, -2)$.

and

Statement 2

If $ax_1 + by_1 + c = 0$, then the point (x_1, y_1) lies on the line $ax + by + c = 0$.

176. Statement 1

Two of the vertices of an equilateral triangle are at the points (3, 1) and (5, 6) . The third vertex can be either

$$\left(4 - 5\frac{\sqrt{3}}{2}\right) + \left(\frac{3}{2} + \sqrt{3}\right)i \text{ OR } \left(4 + \frac{5\sqrt{3}}{2}\right) + \left(\frac{3}{2 - \sqrt{3}}\right)i.$$

and

Statement 2

P is a point represented by the complex number z_0. Let O denote the origin. OP is rotated through an angle x in the positive sense. If the new position of P is denoted by Q, then Q can be represented by the complex number $z_0(\cos\alpha + i\sin\alpha)$.

177. Statement 1

The perpendicular distance between the lines $2x + 3y - 6 = 0$ and $6x + 9y + 8 = 0$ is $\frac{14}{\sqrt{13}}$.

and

Statement 2

The perpendicular distance between the lines $ax + by + d_1 = 0$ and $ax + by + d_2 = 0$ is $\left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2}}\right|$.

178. Statement 1

If the numbers a, b, c are in GP, lines $ax + by + c = 0$ and $cx + by + a = 0$ are perpendicular to each other.

and

Statement 2

If a, b, c are in G . P, $b^2 = ac$.

179. Statement 1

AB and AC are perpendicular lines with AB = 4 and AC = 3. The centroid of the triangle ABC is at a distance $\frac{5}{3}$ from A.

and

Statement 2

Centroid of a triangle divides any median in the ratio 2 : 1.

180. Statement 1

The pair of lines $2x^2 - 3y^2 - x + 19y - xy - 28 = 0$

intersect at P. The equation of the pair of lines through P perpendicular to the given pair of lines is $3x^2 - 2y^2 + xy - 3x + 13y - 18 = 0$.

and

Statement 2

If the pair of lines represented by $ax^2 + 2h\,xy + by^2 + 2gx + 2fy + c = 0$ intersect at P the equation of the pair of lines through P perpendicular to the given pair of lines is

$bx^2 - 2h\,xy + ay^2 + 2gx + 2fy + c = 0$.

Linked Comprehension Type Questions

***Directions*: This section contains 3 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

A research worker is studying the behaviour of rays in different situations. He does various experiments with a powerful source of light. The following result may be used. If (x_2, y_2) is the image (or reflection) of the point (x_1, y_1) in the line $ax + by + c = 0$,

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}.$$

181. A ray of light is sent through the point $\left(0, \frac{5}{2}\right)$ in a direction making an angle of $\tan^{-1}\left(\frac{1}{2}\right)$ with the x-axis. It is reflected in the line $3x - 2y + 7 = 0$. The reflected ray passes through

(a) (0, 16.5) (b) (1, 1) (c) (16.5, 0) (d) $\left(0, \frac{5}{2}\right)$

182. Another ray through $\left(2, -\frac{1}{2}\right)$ is sent in a direction parallel to the line $2y = x$ and it is reflected in the line $3x - 2y - 5 = 0$. The equation of the reflected ray is

(a) $29x - 2y - 31 = 0$ (b) $29x + 2y + 31 = 0$ (c) $29x - 2y + 11 = 0$ (d) None of these

183. Suppose the ray of light through $\left(0, \frac{5}{2}\right)$ is in the direction $4x + 6y - 15 = 0$ and it is reflected in the line $3x - 2y + 7 = 0$, then the reflected ray

(a) passes through the origin (b) passes through $\left(0, \frac{7}{2}\right)$

(c) passes through $\left(1, \frac{5}{2}\right)$ (d) None of the above.

Passage II

For any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in a plane, we can define a real number (say distance d between P and Q) as $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$. Then d satisfies

(i) $d(P, Q) \geq 0$
(ii) $d(P, Q) = d(Q, P)$
(iii) $d(P, Q) \leq d(P, R) + d(R, Q)$

184. The locus of P(x, y) such that $0 < x < 2$, $0 < y < 3$ and $d((0, 0), P) = d((2, 3), P)$ is

(a) $y = \frac{5}{2}$ (b) $2y - 1 = 0$ (c) $2x + 2y = 5$ (d) ϕ

185. If $d((2, 0), P) = d((4, 0), P)$, then the locus of P is

(a) $x + y = 6$ (b) $x = 3$ (c) $y = 0$ (d) ϕ

186. A(3, 4) B(−2, −3) and C (6, −5) are the vertices of _____ triangle.

(a) an isosceles (b) a right angled (c) an equilateral (d) an obtuse angled

Passage III

Let $A_{i+1}B_{i+1}C_{i+1}$ be the triangle formed by joining the mid points of the sides of the triangle $A_iB_iC_i$, i = 1, 2, 3,.... Then for all i = 1, 2, 3,....

(i) Triangles $A_iB_iC_i$ and $A_{i+1}B_{i+1}C_{i+1}$ are similar

(ii) Triangles $A_iB_iC_i$ and $A_{i+1}B_{i+1}C_{i+1}$ have common centroid

(iii) Circum radius of triangle $A_{i+1}B_{i+1}C_{i+1}$ is half of circum radius of triangle $A_iB_iC_i$ and

(iv) Area of triangle $A_{i+1}B_{i+1}C_{i+1}$ is one fourth the area of triangle $A_iB_iC_i$

187. Vertices of triangle $A_3B_3C_3$ are (0, 0) (1, 0) and (0, 1). Then the vertices of triangle $A_1B_1C_1$, are

(a) (−1, 1) (1, 1) (0, −1)
(b) (1, 1) (3, −1) (−1, 3)
(c) (−1, −1) (3, −1) (−1, 3)
(d) (−1, −1) (1, −3) (−3, 1)

188. The area of triangle $A_5B_5C_5$ (in sq. units) is

(a) $\frac{1}{8}$
(b) 8
(c) $\frac{1}{16}$
(d) $\frac{1}{32}$

189. If Δ_{i+1} denotes the area of the triangle formed by the mid points of the sides of the triangle $A_iB_iC_i$ i = 1, 2, 3 and if $\Delta_3 = \frac{1}{2}$, then $\sum_{i=1}^{\infty}\Delta_i$ equals

(a) 16
(b) $\frac{33}{2}$
(c) $\frac{32}{3}$
(d) 32

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers of which ONE OR MORE answers will be correct.**

190. Let 4x + 5y = 4 be a chord of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ intersecting the ellipse at M and N. Then,

(a) mid-point of MN is at $\left(\frac{1}{2},\frac{2}{5}\right)$

(b) $MN = \frac{7}{5}\sqrt{41}$

(c) MN subtends an angle $\tan^{-1}\left(\frac{7}{123}\right)$ at the origin.

(d) Area of the quadrilateral whose diagonals are the major axis and MN is 28 units.

191. Triangle ABC is formed by the lines x − y + 2 = 0, 3x + 2y − 2 = 0 and 2x + y + 1 = 0. Then,

(a) the point (α, 2) lies inside the triangle if $\alpha > \frac{-2}{3}$ or $\alpha < \frac{-3}{2}$.

(b) the point (α, 2) lies inside the triangle for $\frac{-3}{2} < \alpha < \frac{-2}{3}$.

(c) the point (α, 2) lies inside the triangle for $-1 \le \alpha \le \frac{-4}{5}$.

(d) the point (α, 2) lies outside the triangle for α > 0.

192. Two pairs of lines have the equations $10x^2 + 13xy - 3y^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$. The condition that any two lines among them are coincident is

(a) 9a + 12h + 4b = 0
(b) 9a − 12h + 4b = 0
(c) a + 10h + 25b = 0
(d) a + 10h − 56 = 0

193. The point of intersection of the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ lies on the line

(a) $x-y=0$
(b) $(x+y)(a+b)=ab$
(c) $(bx+my)(a+b)=(b+m)ab$
(d) $(bx-my)(a-b)=(b-m)ab$.

194. The lines $(\lambda-2)x+3\lambda y+(\lambda+4)=0$, $x+y-1=0$ and $(2\lambda-5)x+(\lambda+1)y+2\lambda=0$ are concurrent for

(a) 3 values of λ
(b) 2 values of λ
(c) $\lambda=-1$
(d) $\lambda=\frac{-11}{5}$

195. The lines $2x+y=4$ and $3x-y=1$ intersect at M . A variable line is drawn through M meeting the x – w is at A and the y – w is at B

(a) The locus of the mid point of AB is the curve $y=\frac{x}{(x-1)}$
(b) The locus of the foot of perpendicular from the origin on AB is the curve. $x^2+y^2-x-2y=0$
(c) Locus of the centroid of Δ OAB . Where, O is the origin, is the curve $3xy=2x-y$
(d) Locus of the circumcentre of the ΔOAB where, O is the origin, is the curve $y=\frac{x}{(x-1)}$

196. A ray of light is sent along the line $3x+y-9=0$. Upon reaching the line $2x+7y-8=0$, the ray gets reflected from it. Then,

(a) equation of the reflected ray is the form $3x+y-9+\lambda(2x+7y-8)=0$ where λ is a real number.
(b) equation of the reflected ray is $129x-107y-269=0$
(c) equation of the reflected ray is $107x-129y-269=0$
(d) the ray of light hits the line $2x+7y-8=0$ at the point $\left(\frac{55}{19},\frac{6}{19}\right)$

197. The line $x+y-5=0$ intersects the pair of lines $2x^2+3y^2-5xy-5x+5y=0$ at A and B . Then,

(a) coordinates of the point of intersection of the given pair of lines are$(-5,-5)$.
(b) if $\angle AOB=\theta$ where, O is the origin, $\tan\theta=\frac{3}{2}$
(c) $AB=\frac{3}{2}$
(d) angle between the given pair of lines is $\tan^{-1}\left(\frac{1}{5}\right)$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

198.

	Column I	Column II
(a)	The lines joining the origin and the points of intersection of the line $y=mx+2$ and the curve $x^2+y^2=1$ are at right angles. Then m^4 equals	(p) – 15
(b)	The pairs of lines $3x^2-2pxy-3y^2=0$ and $5x^2-2qxy-5y^2=0$ are such that each pair bisects the angle between the other pair. Then pq equals	(q) 12
(c)	Two lines represented by $ax^3+bx^2y+cxy^2+dy^3=0$ are at right angles. Then $a^2+ac+bd+d$ equals	(r) 49
(d)	The slope of one line of the pair of lines represented by $ax^2+8xy+y^2=0$ is three times that of the other. Then 'a' equals	(s) 0

199.

Column I	Column II
(a) The algebraic sum of the distances of a variable line from the points (2, 0), (0, 2) and (−2, −2) is zero. Then the line passes through the point	(p) (−2, −3)
(b) The orthocenter of the triangle formed by the lines $xy = 0$ and $x + y = 4$ is	(q) $\left(\frac{-22}{5}, \frac{6}{5}\right)$
(c) Given $4a^2 + 9b^2 - c^2 + 12ab = 0$. The family of lines $ax + by + c = 0$ is concurrent at	(r) (2, 3)
(d) The image of the point (1, 3) on the line $3x + y + 3 = 0$ is	(s) (0, 0)

200.

Column I	Column II
(a) Equation of the base of an equilateral triangle is $x + y = 2$ and vertex opposite to it is (2, −1). Then the area of the triangle is	(p) 1
(b) Area of the triangle formed by joining the origin to the point of intersection of the line $\sqrt{5}x + 2y = 3\sqrt{5}$ and circle $x^2 + y^2 = 10$ is	(q) 2
(c) Area of triangle formed by the line $xy + 2x + 2y + 4 = 0$ and $x + y + 2 = 0$ is	(r) $\frac{1}{2\sqrt{3}}$
(d) Area of triangle formed by $2x^2 - 5xy + 3y^2 = 0$ and $y = 2$ is	(s) 5

SOLUTIONS

ANSWER KEYS

Topic Grip

1. (i) $x = 1$
   (ii) $\left(2-\sqrt{3}\right)x - y + 3\sqrt{3} - 1 = 0$
   (iii) $5x - 3y = 0$
   (iv) $x + 5y + 30 = 0$
   (v) $5x - 3y + 15 = 0$
   (vi) $\sqrt{3}x + y - 6 = 0$
   (vii) $\dfrac{x-2}{\left(\frac{1}{2}\right)} = \dfrac{y-\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)}$
2. (i) $\dfrac{\pi}{6}$
   (ii) $3x + 5y + 2 = 0$
   (iii) $x - 2y - 4 = 0$
4. $x + 2y - 6 = 0$ or $2x + y - 6 = 0$
5. $2x + y = 5$, $6x - y = 27$ and $2x - 3y + 15 = 0$
6. $(-4, -7)$

11. (c) **12.** (a) **13.** (c)
14. (b) **15.** (c) **16.** (c)
17. (a) **18.** (d) **19.** (b)
20. (d) **21.** (c) **22.** (b)
23. (b) **24.** (c) **25.** (b)
26. (a)
27. (b, d)
28. (a, c, d)
29. (a, b, d)
30. ((a) → (s)
(b) → (p)
(c) → (q)
(d) → (p), (r)

IIT Assignment Exercise

31. (c) **32.** (d) **33.** (b)
34. (a) **35.** (b) **36.** (d)
37. (c) **38.** (c) **39.** (a)
40. (b) **41.** (d) **42.** (c)
43. (a) **44.** (c) **45.** (c)
46. (a) **47.** (b) **48.** (a)
49. (c) **50.** (c) **51.** (c)
52. (a) **53.** (b) **54.** (c)
55. (a) **56.** (a) **57.** (c)
58. (a) **59.** (b) **60.** (b)
61. (a) **62.** (b) **63.** (b)
64. (a) **65.** (c) **66.** (b)
67. (a) **68.** (b) **69.** (d)
70. (c) **71.** (c) **72.** (c)
73. (b) **74.** (d) **75.** (c)
76. (b) **77.** (b) **78.** (a)
79. (a) **80.** (b) **81.** (c)
82. (b) **83.** (c) **84.** (b)
85. (a) **86.** (a) **87.** (a)
88. (c) **89.** (b) **90.** (b)
91. (a) **92.** (a) **93.** (a)
94. (c) **95.** (c) **96.** (c)
97. (b) **98.** (d) **99.** (a)
100. (a) **101.** (c) **102.** (c)
103. (c) **104.** (a) **105.** (c)
106. (b) **107.** (c) **108.** (c)
109. (a) **110.** (c) **111.** (a)
112. (c) **113.** (a) **114.** (d)
115. (b) **116.** (c)
117. (a), (c)
118. (a), (b), (c), (d)
119. (c), (d)
120. (a) → (r), (s)
(b) → (p), (q)
(c) → (q)
(d) → (r)

Additional Practice Exercise

127. $3x - y + 1 = 0$
130. $(\ell^2 + m^2)(px + qy + r) - 2(\ell p + mq)(\ell x + my + n) = 0$

131. (a) **132.** (c) **133.** (a)
134. (c) **135.** (b) **136.** (c)
137. (a) **138.** (c) **139.** (d)
140. (d) **141.** (a) **142.** (c)
143. (b) **144.** (d) **145.** (d)
146. (a) **147.** (c) **148.** (c)
149. (a) **150.** (a) **151.** (a)
152. (d) **153.** (b) **154.** (c)
155. (c) **156.** (a) **157.** (a)
158. (a) **159.** (c) **160.** (b)
161. (a) **162.** (a) **163.** (c)
164. (b) **165.** (b) **166.** (d)
167. (c) **168.** (c) **169.** (d)
170. (d) **171.** (a) **172.** (a)
173. (a) **174.** (c) **175.** (a)
176. (a) **177.** (d) **178.** (d)
179. (a) **180.** (c) **181.** (a)
182. (a) **183.** (b) **184.** (c)

185. (b)
186. (a)
187. (c)
188. (d)
189 (c)
190. (a), (b), (c), (d)
191. (b), (d)
192. (b), (c)
193. (a), (b), (c)
194. (b), (c)
195. (a), (b), (d)
196. (a), (c), (d)
197. (a), (d)
198. (a) → (r)
(b) → (p)
(c) → (s)
(d) → (q)
199. (a) → (s)
(b) → (s)
(c) → (p), (r)
(d) → (q)
200. (a) → (r)
(b) → (s)
(c) → (q)
(d) → (p)

HINTS AND EXPLANATIONS

Topic Grip

1.

(i). Given two lines are parallel to y – axis

Required line I' also parallel to y – axis

Distance of this line from y-axis

$= \frac{1}{2}(-4+6) = 1$

$\therefore$ Its equation is $x = 1$

(ii) Equation of the line passing through (3, 5) and having slope = tan 15° is

$y - 5 = (\tan 15°)(x - 3)$

$y - 5 = (2 - \sqrt{3})(x - 3)$

$(2 - \sqrt{3})x - y + 3\sqrt{3} - 1 = 0$

(iii) Equation of the line is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$y - 5 = \frac{-5-5}{-3-3}(x - 3)$

$y - 5 = \frac{5}{3}(x - 3)$

$3y - 15 = 5x - 15$

$\Rightarrow 5x - 3y = 0$

(iv) $y = mx + c$

$y = -\frac{1}{5}x - 6 \Rightarrow 5y = -x - 30$

$\Rightarrow x + 5y + 30 = 0$

(v) Equation of the line in intercept form is

$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{x}{-3} + \frac{y}{5} = 1 \Rightarrow 5x - 3y = -15$

$\Rightarrow 5x - 3y + 15 = 0$

(vi) Equation of straight line in the normal form is

$x\cos\alpha + y\sin\alpha = p$

$x\cos 30° + y\sin 30° = 3$

$x\frac{\sqrt{3}}{2} + y.\frac{1}{2} = 3 \Rightarrow \sqrt{3}x + y = 6$

$\Rightarrow \sqrt{3}x + y - 6 = 0$

(vii) Equation of straight line in the symmetric form

$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\sin\alpha} \Rightarrow \frac{x-2}{\cos 60^\circ} = \frac{y - \sqrt{3}}{\sin 60^\circ}$

$\Rightarrow \frac{x-2}{\frac{1}{2}} = \frac{y - \sqrt{3}}{\frac{\sqrt{3}}{2}}$

2.

(i) $y = \sqrt{3}x - 2$

$y = \frac{1}{\sqrt{3}}x - \frac{7}{\sqrt{3}}$

$\therefore m_1 = \sqrt{3}$ and $m_2 = \frac{1}{\sqrt{3}}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{1+1}\right| = \frac{1}{\sqrt{3}}$

$\Rightarrow \theta = \frac{\pi}{6}$

(ii). Equation of the line parallel to $3x + 5y = 4$ is of the form $3x + 5y = k$.

Line passes through (1, – 1)

$\Rightarrow 3 - 5 = k \quad \therefore k = -2$

$\therefore$ Equation of the line is $3x + 5y + 2 = 0$

(iii) Equation of the line joining (1, 1)and (3, –3) is

$y - 1 = -2(x - 1) \Rightarrow 2x + y - 3 = 0$

$\therefore$ Equation of the line perpendicular to this line is of the form $x - 2y + k = 0$

Since it is passing through (2, –1), the mid point of (1, 1) and (3, –3), therefore

$2 + 2 + k = 0 \Rightarrow k = -4$

$\therefore$ Equation of the line is $x - 2y - 4 = 0$

3.

(i) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{2}$

$\therefore$ line are coincident .

(ii). $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2} = \frac{1}{2}$

$\therefore$ lines are intersecting.

(iii) $\begin{vmatrix} 1 & 2 & -5 \\ 2 & 3 & -8 \\ 3 & 1 & -5 \end{vmatrix}$

$= 1(-15 + 8) - 2(-10 + 24) - 5(2 - 9)$

$= -7 - 28 + 35 = 0$

$\therefore$ lines are concurrent.

4. Let the intercepts made by line on x-axis and y-axis be a, b respectively.

Therefore the equation of line in intercept form is

$$\frac{x}{a}+\frac{y}{b}=1 \text{ — (1) } a+b=9 \text{ — (2) given}$$

Since this line passes through a point (2, 2) we have

$$\frac{2}{a}+\frac{2}{b}=1 \text{ — (3)}$$

Solving the (3) and (2)

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{2}$$

$ab = 2(a + b) = 2 \times 9 = 18$

$\Rightarrow$ $a = 6,\ b = 3$ or $a = 3,\ b = 6$

Substituting the values in equation (1) the required equations of the lines are $\frac{x}{6}+\frac{y}{3}=1$ or $\frac{x}{3}+\frac{y}{6}=1$

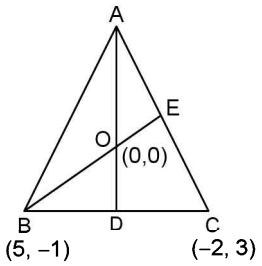
i.e., $x + 2y - 6 = 0$ or $2x + y - 6 = 0$

5. Let D(2,1) E (5, 3) and F(3, 7) be the mid points of the sides BC, CA and AB of the triangle ABC. Since BC passes through D and is parallel to EF its equation is

$$y-1=\frac{7-3}{3-5}(x-2) \qquad \text{or} \qquad 2x+y-5=0$$

Similarly, equations of CA and AB are respectively $6x - y - 27 = 0$ and $2x - 3y + 15 = 0$

6.



Let the third vertex be A(p, q) . O is orthocentre

Now, Slope of BC $= \frac{3-(-1)}{-2-5} = \frac{-4}{7}$

Slope of AD = slope of AO

But $m_1 m_2 = -1$

$$\left(\frac{-4}{7}\right)\cdot\left(\frac{q}{p}\right)=-1$$

$7p = 4q$ — (1)

Slope of CA $= \frac{q-3}{p+2}$

Slope of BE = Slope of BO $= \frac{-1}{5}$

But $m_1 m_2 = -1 \left(\frac{q-3}{p+2}\right)\left(\frac{-1}{5}\right) = -1$

$5p - q + 13 = 0$ — (2)

Solving (1) and (2) we have $p = -4,\ q = -7$

$\therefore$ Third vertex A is the point (–4, –7)

7. p_1 = perpendicular from $(k^2, 2k)$ on the line

$$= k^2\cos\alpha + 2k\sin\alpha + \frac{\sin^2\alpha}{\cos\alpha}$$

p_2 = perpendicular from (km, k + m) on the line

$$= km\cos\alpha + (k+m)\sin\alpha + \frac{\sin^2\alpha}{\cos\alpha}$$

p_3 = perpendicular from $(m^2, 2m)$ on the line

$$= m^2\cos\alpha + 2m\sin\alpha + \frac{\sin^2\alpha}{\cos\alpha}$$

We have $p_2^2 = \left(km\cos\alpha + (k+m)\sin\alpha + \frac{\sin^2\alpha}{\cos\alpha}\right)^2$ — (1)

$$p_1p_3 = \left(k^2\cos\alpha + 2k\sin\alpha + \frac{\sin^2\alpha}{\cos\alpha}\right)$$

$$\times\left(m^2\cos\alpha + 2m\sin\alpha + \frac{\sin^2\alpha}{\cos\alpha}\right)$$

$$= \frac{(k^2\cos^2\alpha + 2k\sin\alpha\cos\alpha + \sin^2\alpha)}{(\cos\alpha)}$$

$$\times\frac{(m^2\cos^2\alpha + 2m\sin\alpha\cos\alpha + \sin^2\alpha)}{(\cos\alpha)}$$

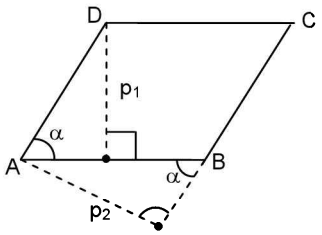
$$= \frac{(k\cos\alpha + \sin\alpha)^2(m\cos\alpha + \sin\alpha)^2}{\cos^2\alpha}$$

$$= \frac{[(k\cos\alpha + \sin\alpha)(m\cos\alpha + \sin\alpha)]^2}{\cos^2\alpha}$$

$$= \frac{[km\cos^2\alpha + (k+m)\sin\alpha\cos\alpha + \sin^2\alpha]^2}{\cos^2\alpha}$$

$$= \left(km\cos\alpha + (k+m)\sin\alpha + \frac{\sin^2\alpha}{\cos\alpha}\right)^2 = p_2^2$$

8.



AB and CD are represented by $a_1x + b_1y + c_1 = 0$ and $a_1x + b_1y + d_1 = 0$

AD and BC are represented by $a_2x + b_2y + c_2 = 0$

$a_2x + b_2y + d_2 = 0$

Area of the parallelogram ABCD = $(AB)p_1$

We have AB $\sin\alpha = p_2$

Therefore, Area of the parallelogram = $\dfrac{p_1p_2}{\sin\alpha}$

p_1 = difference between the lengths of the perpendiculars from the origin to the lines represented by

AB and CD = $\dfrac{|c_1 - d_1|}{\sqrt{a_1^2 + b_2^2}}$

α is the angle between the lines

$a_1x + b_1y + c_1 = 0$

and $a_2x + b_2y + c_2 = 0$

$$\tan\alpha = \pm\frac{\dfrac{a_1}{b_1} - \dfrac{a_2}{b_2}}{1 + \dfrac{a_1a_2}{b_1b_2}} = \pm\frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}$$

$$\sin^2\alpha = \frac{1}{1+\cot^2\alpha} = \frac{1}{1+\left(\dfrac{a_1a_2 + b_1b_2}{a_1b_2 - a_2b_1}\right)^2}$$

$$= \frac{(a_1b_2 - a_2b_1)^2}{(a_1b_2 - a_2b_1)^2 + (a_1a_2 + b_1b_2)^2}$$

$$= \frac{(a_1b_2 - a_2b_1)^2}{(a_1^2b_2^2 + a_2^2b_1^2 + a_1a_2^2 + b_1^2b_2^2}$$

$$= \frac{(a_1b_2 - a_2b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$$

$$\Rightarrow \quad \sin\alpha = \frac{a_1b_2 - a_2b_1}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

Area of the parallelogram

$$= \frac{|c_1 - d_1||c_2 - d_2|}{\sqrt{a_1^2 + b_1^2(a_2^2 + b_2^2)}} \times \frac{1}{\sin\alpha} = \left|\frac{(c_1 - d_1)(c_2 - d_2)}{(a_1b_2 - a_2b_1)}\right|$$

9. Let y = mx be the common line. Then it satisfies both equations.

$\therefore \quad bm^2 + 2hm + a = 0$

$b_1m^2 + 2h_1m + a_1 = 0$

$$\therefore \frac{m^2}{2(ha_1 - h_1a)} = \frac{m}{ab_1 - a_1b} = \frac{1}{2(h_1b - hb_1)}$$

$\therefore \quad (ab_1 - a_1b)^2 = 4(ha_1 - h_1a)(h_1b - hb_1)$

$= -4(ah_1 - ha_1)(bh_1 - hb_1)$

10. Let the line be $lx + my + n = 0$

If $z = x + iy$; $2x = z + \bar{z}$; $2y = \dfrac{z - \bar{z}}{i}$; $= -i(z - \bar{z})$

$$\therefore \quad \ell\frac{(z + \bar{z})}{2} - \frac{mi(z + \bar{z})}{2} + n = 0$$

$z(\ell - mi) + \bar{z}(\ell + mi) + 2n = 0$

Let $\ell + mi = A$ and $2n = B$. Then we have

$\bar{A}z + A\bar{z} + B = 0.$

11. $m = \tan 60° = \sqrt{3}$;

Equation of the line is $\sqrt{3}x - y + 3 - 3\sqrt{3} = 0.$

12. If a and b are the x and y intercepts respectively, then a = 2b

Equation of the line is $\dfrac{x}{2b} + \dfrac{y}{b} = 1$

Since it passes through (2, 3), we get $\dfrac{2}{2b} + \dfrac{3}{b} .= 1$

$\Rightarrow \quad b = 4$ the line $\dfrac{x}{8} + \dfrac{y}{4} = 1$

$\Rightarrow \quad x + 2y = 8$

13. $d = \dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}} = \dfrac{|-20 - 5|}{\sqrt{3^2 + 4^2}} = 5.$

14. $$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \left|\frac{2\sqrt{\dfrac{25}{4} - 2\times 3}}{2 + 3}\right|$$

$$= \frac{2\times\dfrac{1}{2}}{5} = \frac{1}{5}.$$

15. We have $X = x - 2$; $Y = y - 1$

$x = X + 2$ and $y = Y + 1$

$\Rightarrow \quad 3x - 4y - 7 = 3(X + 2) - 4(Y + 1) - 7$

$= 3X - 4Y - 5.$

Equation of the line in the new system is

$3x - 4y - 5 = 0.$

16. Statement 2 is false

Consider Statement 1 we have the result: The centroid divides the line segment joining the circumcentre and the orthocenter in the ratio 1 : 2

Hence, coordinates of the centroid of the triangle are

$$\left(\frac{1\times 3+2\times 1}{2},\frac{1\times 5+2\times -3}{3}\right) \text{ ie., } \left(\frac{5}{3},\frac{-1}{3}\right)$$

Statement 1 is true

Choice (c)

17. Statement 2 is true

Consider Statement 1

The lines are $3x - y + 6 = 0$ and $3x - y - \frac{8}{3} = 0$

Since constant terms in the equation of the lines are of opposite signs, distance between the lines =

$$\frac{6+\frac{8}{3}}{\sqrt{10}} = \frac{26}{3\sqrt{10}}$$

Statement 1 is true

Choice (a)

18. Statement 2 is true

Consider Statement 1 using Statement 2

We must have $x + 3y + 4 = 3x + 5y + 7$

$+ \lambda(5x + y - 3)$, equating the coefficients of x, y and the constant term,

$3 + 5\lambda = 1, \lambda + 5 = 3, -3\lambda + 7 = 4$

There is no value of λ satisfying all the three relations above

Statement 1 is false

Choice (d)

19. Statement 2 is true

Putting $x = 2$ in the equation $3x^2 - 2y^2 - 5xy = 0$

$\Rightarrow \quad y^2 + 5y - 6 = 0$

$\Rightarrow \quad y = -6, 1$

The points A and B are (2, 1) and (2, −6)

Distance AB = 7

Choice (b)

20. Statement 2 is True

Consider Statement 1

The point of intersection of the given pair of lines using Statement 2 is at (−1, 2). Hence when we shift the origin to the point (1, −2) the equation cannot reduce to a homogeneous equation ⇒ Statement 1 is false

Choice (d)

21. $r\cos\theta = 3, r\sin\theta = 4$

$\Rightarrow \quad r^2 = 25 \Rightarrow r = 5$

$\tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\frac{4}{3}$

Polar coordinates are $\left(5, \tan^{-1}\frac{4}{3}\right)$

22. $x = 4\cos\frac{\pi}{3} = 2, \; y = 4\sin\frac{\pi}{3} = 2\sqrt{3}$

Cartesian coordinates are $\left(2, 2\sqrt{3}\right)$

23. Using the definition, sets represented by (i) and (iii) are collinear

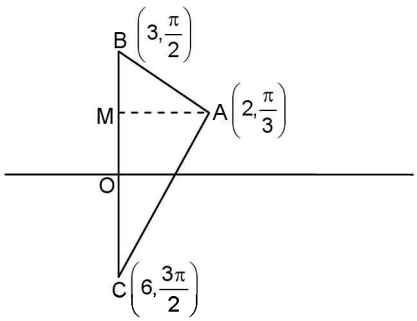
24. $r_1 = 3, r_2 = 4, \theta_1 = \frac{\pi}{6}, \theta_2 = \frac{\pi}{3}$

If d is the distance between the points

$$d^2 = 3^2 + 4^2 - 2\times 3\times 4\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right) 25 - 24\frac{\sqrt{3}}{2}$$

$$= 25 - 12\sqrt{3}$$

25.



Area of $\Delta ABC = \frac{1}{2}(3+6)\times 2\cos\frac{\pi}{3} = \frac{9}{2}$

26. $\theta = 0$, using the definition

27. $\frac{x}{a}+\frac{y}{b} = 2$ — (1)

$\frac{x}{b}+\frac{y}{a} = 2$ — (2)

$(1) - (2) \Rightarrow x\left(\frac{1}{a}-\frac{1}{b}\right)+y\left(\frac{1}{b}-\frac{1}{a}\right) = 0$

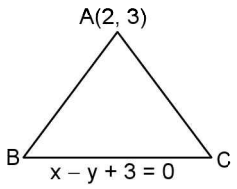
$\Rightarrow \quad x = y$

Given $x = c \Rightarrow y = c$

$(1) \Rightarrow \frac{c}{a}+\frac{c}{b}=2$

$\Rightarrow \frac{1}{a}+\frac{1}{b}=\frac{2}{c}$ a, c, b are in H.P.

28. (2, 3) does not satisfy $x - y + 3 = 0$



$\therefore$ it is opposite vertex of the given side

Since ABC is an equilateral triangle

$\angle B = \angle C = 60°$

$\therefore$ Slopes of AB and AC are given by

$\tan 60° = \left|\frac{m-1}{1+m.1}\right|$

$\therefore \frac{m-1}{m+1} = \pm\sqrt{3} \Rightarrow m = -\left(2+\sqrt{3}\right) = m_1$,

Slope of AC

$m = -2 + \sqrt{3} = m_2$, Slope of AB

(a) Perpendicular distance from (2, 3) to $x - y + 3 = 0$

$= \frac{2-3+3}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

If a is the length of the side of triangle then

$\sqrt{2} = \frac{\sqrt{3}}{2}a \Rightarrow a = \frac{2\sqrt{2}}{\sqrt{3}}$

$\therefore$ area $= \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}.\frac{4.2}{3} = \frac{2}{\sqrt{3}}$ = cosec 60°

(b) Let (α, β) be the foot of the perpendicular from (2, 3) to $x - y + 3 = 0$

i.e., $\alpha - \beta + 3 = 0$ — (1)

$\frac{\beta-3}{\alpha-2}\times 1 = -1 \Rightarrow \beta - 3 = -\alpha + 2$

Solving (1) and (2) $\Rightarrow \alpha = 1$ and $\beta = 4$

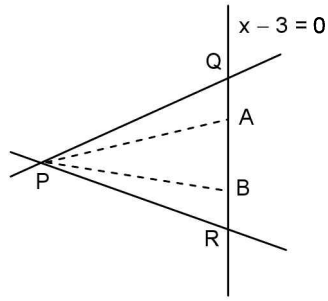
$\therefore$ foot of the perpendicular (1, 4)

In an equilateral triangle, circumcenter, orthocenter and centroid coincide and it divides any median in the ratio 2 : 1

$\therefore$ Circumcentre is $\left(\frac{2.1+1.2}{2+1}, \frac{2.4+1.3}{2+1}\right)$

$= \left(\frac{4}{3}, \frac{11}{3}\right)$

29.



Point of intersection of given pair of lines,

$$x_1 = \frac{\begin{vmatrix} h & g \\ b & f \end{vmatrix}}{\begin{vmatrix} a & h \\ h & b \end{vmatrix}} = \frac{hf - bg}{ab - h^2} = 1$$

$$y_1 = \frac{\begin{vmatrix} g & a \\ f & h \end{vmatrix}}{\begin{vmatrix} a & h \\ h & b \end{vmatrix}} = \frac{gh - af}{ab - h^2} = 1$$

i.e., P is (1, 1)

Co-ordinate of Q and R is obtained by solving

$2x^2 - xy - 10y^2 - 3x + 21y - 9 = 0$ — (1)

and $x = 3$

$18 - 3y - 10y^2 - 9 + 21y - 9 = 0$

$10y^2 - 18y = 0$

$y = 0, 9/5$

$\Rightarrow$ Q is (3, 0) and R is $\left(3, 9/5\right)$

Given that A divides QR in the ratio 2 : 1 and B divides QR in the ratio 1 : 2

$\therefore$ $A = \left(3, 6/5\right)$

$B = \left(3, 3/5\right)$

$\therefore$ Equation of PA is $x - 10y + 9 = 0$

Equation of PB is $x + 5y - 6 = 0$

Equation of the pair of lines PA and PB is $(x - 10y + 9)(x + 5y - 6) = 0$

$\Rightarrow x^2 - 50y^2 - 5xy + 3x + 105y - 54 = 0$

Area of $\Delta PAB = \frac{1}{2} \times AB \times$ perpendicular distance of AB from (1, 1)

$= \frac{1}{2} \times \frac{3}{5} \times 2 = 0.6$

$\angle$ APB = Angle between PA and PB

$$= \tan^{-1}\left\{\frac{\frac{1}{10}+\frac{1}{5}}{1-\frac{1}{50}}\right\} = \tan^{-1}\left\{\frac{15}{49}\right\}$$

Mid-point of AB is $\left(3, \frac{9}{10}\right)$

30. (a) Given $p = \frac{|5a|}{\sqrt{a^2+25}}$

$\Rightarrow \quad p^2 = \frac{25a^2}{a^2+25}$

$\Rightarrow \quad (a^2 + 25)\, p^2 = 25a^2 \qquad$ — (1)

But $a^2 + 25 = 2p^2$ since a^2, p^2, 25 are in A.P.

$\therefore \quad$ (1) $\Rightarrow 2p^4 = 25a^2$

$= (2p^2 - a^2)a^2$

$\Rightarrow \quad a^4 + 2p^4 - 2a^2p^2 = 0$

$\therefore \quad a^4 + 2p^4 - 2a^2p^2 + 4 = 4$

(b) We have $\frac{3+q}{2} = 2.\frac{p+5}{2} + c$

$\Rightarrow \quad 2p + 2c - q + 7 = 0$

Since $y = 2x + c$ passes through (p, q),

$\therefore \quad q = 2p + c$

$\Rightarrow \quad 2p + 2c - 2p - c + 7 = 0$

$\Rightarrow \quad c = -7$

(c) Since the given lines are concurrent,

$$\therefore \quad \begin{vmatrix} a & b & p \\ \cos\alpha & \sin\alpha & -p \\ \sin\alpha & -\cos\alpha & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad a\cos\alpha + b\sin\alpha + 1 = 0 \qquad$ — (1)

Angle between the first two lines is 45°

$$\Rightarrow \quad \tan 45° = \left|\frac{-\frac{a}{b}+\frac{\cos\alpha}{\sin\alpha}}{1+\frac{a}{b}.\frac{\cos\alpha}{\sin\alpha}}\right|$$

$\Rightarrow \quad -a\sin\alpha + b\cos\alpha = \pm 1 \qquad$ — (2)

Squaring and adding the two equations (1) and (2), we get

$a^2 + b^2 = 2$

(d) The x coordinate of the point of intersection of the lines is given by $2x + mx + 2 = 7$

$\Rightarrow \quad x = \frac{5}{m+2}$

x is an integer $\Rightarrow m + 2 = \pm 1$ or ± 5

$\Rightarrow \quad m = -3$ or -1 or 3 or -7

IIT Assingment Exercise

31. $\frac{\lambda(5)+1(2)}{\lambda+1} = x$; $\frac{\lambda(6)+1(-3)}{\lambda+1} = y$

Point is on x–axis. $\therefore y = 0$

$6\lambda = 3 \Rightarrow \lambda = \frac{1}{2}$.

32. Slope of OA $= \frac{2}{5}$; Slope of OB $= \frac{-15}{6} = \frac{-5}{2}$

Slope of OA × Slope of OB = –1

$\therefore \quad$ angle subtended at the origin is $\frac{\pi}{2}$

33. Equation of the line passing through given lines is of the form $5x - 3y + 3 + \lambda(3x + y - 1) = 0$

i.e., $x(5 + 3\lambda) + y(\lambda - 3) + 3 - \lambda = 0$

So x intercept $= \frac{\lambda-3}{5+3\lambda}$; y intercept $= \frac{\lambda-3}{\lambda-3}$

$\Rightarrow \quad 5 + 3\lambda = \lambda - 3$

$2\lambda = -8 \Rightarrow \lambda = -4$

$\therefore \quad$ Required equation is $-7x - 7y + 7 = 0$

$\Rightarrow \quad x + y - 1 = 0$

34. Line perpendicular to $3x + y - 3 = 0$ is of the form $x - 3y + k = 0$

Passes through (2, 2). $\therefore 2 - 6 + k = 0 \Rightarrow k = 4$

$x - 3y + 4 = 0 \Rightarrow y = .\frac{x}{3} + \left(\frac{4}{3}\right)$

35. $m_1 = \frac{-1}{k}$; $m_2 = -2 \quad \Rightarrow \tan\theta = \frac{3}{4} = \frac{\frac{-1}{k}+2}{1+\frac{2}{k}}$

$\Rightarrow \quad \frac{3}{4} = \frac{2k-1}{k+2} \quad \Rightarrow k = 2$

36. Line perpendicular to the given line is of the form x cosec θ – y sec θ = k

x cos θ – y sinθ = k sin θ cos θ — (1)

Since it passes through (a $\cos^3\theta$, a $\sin^3\theta$).

a $\cos^4\theta$ – a $\sin^4\theta$ = k sin θ cos θ.

a ($\cos^2\theta - \sin^2\theta$) ($\cos^2\theta + \sin^2\theta$) = k sin θ cos θ.

$$\therefore \quad k = \frac{a\cos 2\theta}{\sin\theta\cos\theta}$$

So, from (1) we get x cos θ – y sin θ = a cos 2θ as the equation of the line.

37. Equation of the line perpendicular to the given line is of the form 4x + 3y + k = 0

Now, its distance from origin = $\left|\frac{k}{\sqrt{25}}\right| = 2$

$$\Rightarrow \quad \pm\frac{k}{5} = 2$$

$$k = \pm 10$$

Equation of the line is 4x + 3y ± 10 = 0.

38. x-intercept = $\frac{-c}{4}$; y-intercept = $\frac{-c}{3}$

Area of the triangle = $\frac{1}{2}\times\frac{-c}{4}\times\frac{-c}{3} = 6$

$\Rightarrow \quad c^2 = 24 \times 6 = 144 \therefore c = \pm 12.$

39. The given system of lines forms a quadrilateral as shown below.

The above figure is a square.

40. From the given condition, P is a point on the circle with centre A(1, 1) and radius 7. AB = $\sqrt{9+16} = 5$. So, B is a point inside the circle. Now PB is maximum, if A is a point in the segment PB, then PB = PA + AB = 7 + 5 = 12.

41. m + 3m = $\frac{-2h}{b}$ i.e., m = $\frac{-h}{2b}$

$3m^2 = \frac{a}{b}$, i.e., $3 \times \frac{h^2}{4b^2} = \frac{a}{b} \quad \therefore \frac{h^2}{ab} = \frac{4}{3}$.

42. a = 6, b = –4, c = –3, h = $\frac{5}{2}$; g = $\frac{7}{2}$; f = $\frac{13}{2}$

Substituting in $\left(\frac{hf-bg}{ab-h^2}, \frac{hg-af}{ab-h^2}\right)$,

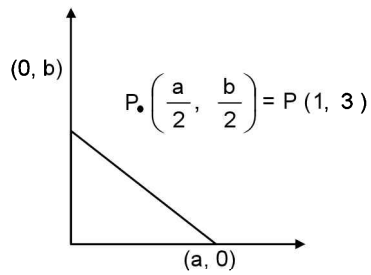
we get (-1, 1).

43. Suppose the line 2x + y = 5 divides the line segment joining A (-1, 1) and B (7, 8) in the ratio k : 1, then

Coordinates of dividing point are $\frac{7k-1}{k+1}, \frac{8k+1}{k+1}$. It lies on 2x + y = 5

$$\Rightarrow \quad 2\frac{(7k-1)}{k+1} + \frac{8k+1}{k+1} = 5 \Rightarrow k = \frac{6}{17}$$

44. $\frac{a}{2} = 1 \Rightarrow a = 2$



$\frac{b}{2} = 3 \Rightarrow b = 6 \quad \frac{x}{2} + \frac{y}{6} = 1 \Rightarrow 3x + y = 6.$

45. a, b, c are in A.P (given)

$\therefore \quad 2b = a + c$

–a + 2b – c = 0 or a – 2b + c = 0.

So ax + by + c will pass through (1, –2)

46. Mid-point of PQ is (–1, 2)

Slope of PQ = $\frac{1}{2}$

$\Rightarrow$ slope of line L = –2

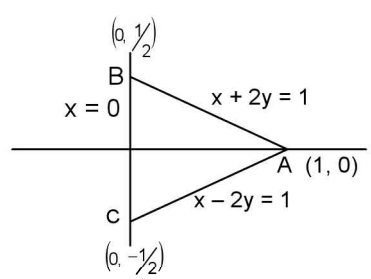
$\Rightarrow$ choice (a)

47. x + 2 |y| = 1 and x = 0 gives |y| = $\frac{1}{2}$

$y = \pm\frac{1}{2}$

Required area = area of ΔABC

where AB $\Rightarrow$ x + 2y = 1 and AC $\Rightarrow$ x – 2y = 1



Required area $= \left(\frac{1}{2}\right)(BC)(OA)$

$= \frac{1}{2}(1)(1) = \frac{1}{2}.$

48. $\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{-2(ax_1+by_1+c)}{a^2+b^2}$

$\Rightarrow \quad \frac{h-4}{5} = \frac{k+13}{1} = \frac{-2[20-13+6]}{25+1}$

$h = -1, k = -14.$

49. $xy = 0 \Rightarrow x = 0$ or $y = 0$

These lines are perpendicular and intersect at origin.

$\therefore$ orthocentre is (0, 0)

50. $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$6\lambda + 2\times\frac{5}{2}\times\frac{5}{2}\times\frac{7}{2} - 3\times\frac{25}{4} - \lambda\times\frac{25}{4} - 2\times\frac{49}{4} = 0$

$24\lambda + 175 - 75 - 25\lambda - 98 = 0$

i.e., $2 - \lambda = 0 \Rightarrow \lambda = 2.$

51. Equation of bisectors of the angles between the lines $x^2 - 2pxy - y^2 = 0$ is

$\frac{x^2-y^2}{1+1} = \frac{xy}{-p}$

$-px^2 + py^2 - 2xy = 0$

Also, $x^2 - 2qxy - y^2 = 0$ represents the bisectors of angle between the above line.

$\Rightarrow \quad \frac{-p}{1} = \frac{p}{-1} = \frac{1}{q} \Rightarrow pq + 1 = 0.$

52. Equation of the bisectors of $ax^2 + 2hxy + by^2 = 0$ is

$\frac{x^2-y^2}{a-b} = \frac{xy}{h}.$

Put $y = mx \Rightarrow h(1 - m^2) = m(a - b).$

53. $3x^2 + 8xy - 3y^2 = (3x - y)(x + 3y)$ and

$3x^2 + 8xy - 3y^2 + 2x - 4y - 1$

$= (3x - y - 1)(x + 3y + 1)$

The first pair intersects at (0, 0) and the second pair intersects as $\left(\frac{1}{5}, -\frac{2}{5}\right)$

$\therefore$ Equation of one diagonal is $y = -2x$

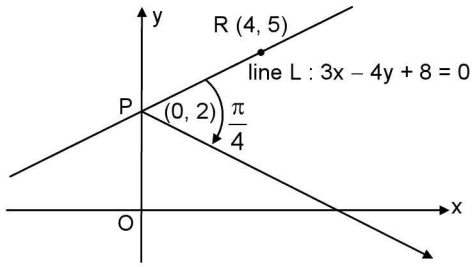
Slope of second diagonal is $\frac{1}{2}$ and it will pass through

$\left(\frac{3}{10}, -\frac{1}{10}\right)$ and $\left(-\frac{1}{10}, -\frac{3}{10}\right)$

So equation of second diagonal is $y + \frac{3}{10} = \frac{1}{2}\left(x + \frac{1}{10}\right)$

$\Rightarrow \quad 2x - 4y = 1$

54. Slope of the line $3x - 4y + 8 = 0$



is $\frac{3}{4}$

Since $\tan^{-1}\left(\frac{3}{4}\right) < \frac{\pi}{4}$, new position of L (which is L') will be making an obtuse angle with the positive direction of the x – axis. If m is the slope of L',

$\frac{m - \frac{3}{4}}{1 + \frac{m\times 3}{4}} = \pm 1$

$\frac{m - \frac{3}{4}}{1 + \frac{3m}{4}} = -1$ gives $m = \frac{-1}{7}$

Slope of L' $= \frac{-1}{7}$

Equation of L' is $y - 2 = \frac{-1}{7}(x)$

$7y - 14 = -x$

$\Rightarrow \quad x + 7y - 14 = 0$

OR

Note that (4, 5) is a point on the line $3x - 4y + 8 = 0$. Let $Q = 4 + 5i$ denote this point.

$\overline{PQ}$ can be represented by the complex number $(4 + 3i)$

Let R denote the new position of Q after PQ is rotated through an angle $\frac{\pi}{4}$ in the clockwise sense.

$\overline{PR} = \frac{(4+3i)(1-i)}{\sqrt{2}} = \frac{1}{\sqrt{2}}(7-i)$

$\Rightarrow$ R is represented by $\frac{1}{2}(7-i)+2i=\frac{7}{\sqrt{2}}+\left(2-\frac{1}{\sqrt{2}}\right)i$

$$\text{Slope of PR} = \frac{2-\frac{1}{\sqrt{2}}-2}{\frac{7}{\sqrt{2}}-0}=\frac{-1}{7}$$

Equation of the line L' is

$$y-2=\frac{-1}{7}x$$

$\Rightarrow x+7y-14=0$

55. Given $5c+1=5\times 9c$

$\Rightarrow \quad 40c=1 \Rightarrow c=\frac{1}{40}$

56. According to the given condition

$$\frac{1}{2}\begin{vmatrix} 2a & a & 1 \\ a & 2a & 1 \\ a & a & 1 \end{vmatrix}=\pm 72$$

$\Rightarrow \quad a^2=144 \Rightarrow a=\pm 12$

The centroid of the triangle is

$$\left(\frac{(2a+a+a)}{3},\frac{(a+2a+a)}{3}\right)$$

$$\Rightarrow \left(\frac{4a}{3},\frac{4a}{3}\right) \Rightarrow (16,16), \text{ or } (-16,-16)$$

57. If these three lines are concurrent then $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$

$\Rightarrow \quad a(bc-a^2)-b(b^2-ac)+c(ac-c^2)=0$

$\Rightarrow \quad a^3+b^3+c^3-3abc=0$

58. The given lines form a right triangle with vertices (0, 0), (1, 0) and (0, 1).

Now, orthocentre of right triangle is the midpoint of hypotenuse

$\therefore$ Here it is $\left(\frac{1}{2},\frac{1}{2}\right)$

59. Coefficient of x^2 + coefficient of $y^2=0$

$k+6=0 \quad \therefore k=-6$

60. Let $A=\sqrt{(x-2)^2+y^2}$, $B=\sqrt{(x+2)^2+y^2}$

$\Rightarrow \quad A+B=4 \qquad$ — (1)

$(A+B)(A-B)=(x-2)^2+y^2-(x+2)^2-y^2$

$=-8x$

$\Rightarrow \quad A-B=-2x \qquad$ — (2)

Solving (1) and (2)

$2A=4-2x$

$A=2-x$

$A^2=(x-2)^2 \Rightarrow y^2=0$

This represents x axis but only the segment $-2\le x\le 2$ of x-axis satisfies the given condition.

61. Let m and m^2 be the slopes of the lines. Then

$m+m^2=\frac{-2h}{b}$, $m^3=\frac{a}{b}$

We have

$(m+m^2)^3=m^3+m^6+3m^3(m+m^2)$

$$\frac{-8h^3}{b^3}=\frac{a}{b}+\frac{a^2}{b^2}-\frac{6ha}{b^2}$$

$-8h^3=ab^2+a^2b-6hab$

$$\frac{8h^3}{ab}=6h-(a+b)$$

$$\frac{8h^2}{ab}=6-\frac{(a+b)}{h}$$

$$\Rightarrow \quad \frac{(a+b)}{h}+\frac{8h^2}{ab}=6$$

62. Locus of P is the perpendicular bisector of AB

$$\text{Slope of AB}=\frac{a+b-a+b}{a-b-a-b}=-1$$

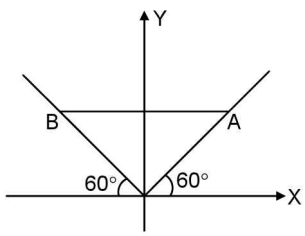
$\therefore$ Slope of locus of P is 1.

Mid-point of AB = (a, a)

$\therefore$ Locus of P is $y-a=x-a$

$y=x$

63. $y=x\sqrt{3}$ has a slope $\sqrt{3}$ and is inclined at 60° with positive direction of x axis. Also $y=-x\sqrt{3}$ makes 60° with negative direction of x axis. So both the lines are equally inclined at 30° with y axis. And since OAB is given as equilateral triangle. We get the triangle as shown in figure. Also we get AB parallel to x axis. Now, since it passes through (2, 2), equation is y = 2



64. Let $L_1 : x + y = 5 = 0$, $L_2 : 4x - 7y + 10 = 0$, $L_3 : 7x + 4y - 15 = 0$ be the equations of three sides.

We observe that $L_2 \perp L_3 \Rightarrow$ given triangle is right angled.

$\Rightarrow$ L_1 represents the greater side, we want the altitude to L_1.

The orthocentre is at the point of intersection of L_2 and L_3

$\Rightarrow$ Altitude to L_1 passes through the intersection of L_2 and L_3.

Any line through the intersection of L_2 and L_3 is of the form

$$(4x - 7y + 10) + k(7x + 4y - 15) = 0$$

$$\Rightarrow (4 + 7k)x + (-7 + 4k)y + 10 - 15k = 0 \quad —(1)$$

$$(1) \perp L_1 \Rightarrow \frac{4+7k}{7-4k} \times (-1) = -1$$

$$\Rightarrow 4 + 7k = 7 - 4k \Rightarrow K = \frac{3}{11}$$

$$\therefore \quad (1) \text{ becomes } \frac{65x}{11} - \frac{65y}{11} + \frac{65}{11} = 0$$

$$\Rightarrow x - y + 1 = 0$$

65. Let the line through P parallel to $2x + 3y + 5 = 0$ be $2x + 3y + k = 0$

$\therefore$ $8 + 0 + k = 0$

$k = -8$

So equation of the line is $2x + 3y - 8 = 0$ — (1)

Solving (1) and $x + 2y - 5 = 0$, we get $x = 1$, $y = 2$

$$\therefore \quad \text{Distance} = \sqrt{(4-1)^2 + 2^2} = \sqrt{13}$$

66. $\frac{x}{a} + \frac{y}{b} = 1$

Line perpendicular to this line and passing through origin is of the form $\frac{x}{b} - \frac{y}{a} = 0$.

So, foot of the perpendicular is the point of intersection of these lines.

Squaring and adding, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} + \frac{x^2}{b^2} + \frac{y^2}{a^2} - \frac{2xy}{ab} = 1$$

$$x^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + y^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = 1$$

$$(x^2 + y^2)\left(\frac{1}{c^2}\right) = 1 \text{(by using the given condition)}$$

$$\Rightarrow x^2 + y^2 = c^2$$

67. $18x^2 - 9xy + y^2 = 0$

$\Rightarrow$ $(6x - y)(3x - y) = 0$

$\Rightarrow$ $y = 6x$ or $y = 3x$

So lines are $y = 6x$, $y = 3x$ and $y = 9$

So the vertices of the triangle formed are $(0, 0)$, $(3, 9)$ and $\left(3/2, 9\right)$

$$\therefore \quad \text{required area} = \frac{1}{2}\left(3 \times 9 - \frac{3}{2} \times 9\right) = \frac{27}{4}$$

68. Angle α between the lines given by $ax^2 + 2hxy + by^2 = 0$ is given by

$$\tan\alpha = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

Here $h = -\text{cosec}\theta$, $a = b = 1$

$$\therefore \tan\alpha = \left|\frac{2\sqrt{\text{cosec}^2\theta - 1}}{2}\right| = |\cot\theta|$$

$$\Rightarrow \alpha = \frac{\pi}{2} - \theta$$

69. $\frac{y - mx}{c} = 1$ — (1)

Homogenizing (1) and $x^2 + y^2 = a^2$,

we have $x^2 + y^2 = a^2\left(\frac{y - mx}{c}\right)^2$

$\Rightarrow$ $c^2x^2 + c^2y^2 = a^2(y^2 + m^2x^2 - 2mxy)$

$\Rightarrow$ $x^2(c^2 - a^2m^2)$

$+ 2mxy\,a^2 + y^2(c^2 - a^2) = 0$ — (2)

(2) will represent perpendicular lines if coefficient of x^2 + coefficient $y^2 = 0$

$\Rightarrow$ $c^2 - a^2m^2 + c^2 - a^2 = 0$

$2c^2 = a^2(1 + m^2)$

70. $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{bx}{\sqrt{a^2+b^2}} + \frac{ay}{\sqrt{a^2+b^2}} = \frac{ab}{\sqrt{a^2+b^2}}$

$$\therefore \quad 2p = \frac{ab}{\sqrt{a^2+b^2}} \Rightarrow 4p^2 = \frac{a^2b^2}{a^2+b^2}$$

$$8p^2 = \frac{2a^2b^2}{a^2+b^2} \Rightarrow a^2, 8p^2, b^2 \text{ are in H.P}$$

71. $p(x + 3y - 5) + q(2x + y - 5) = 0$

i.e., a family of lines passing through the intersection of $x + 3y - 5 = 0$ and $2x + y - 5 = 0$.

Their point of intersection is $(2, 1)$.

72. The pair of lines through the origin perpendicular to the lines given by
$ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.
So here, $2x^2 + 3xy + y^2 = 0$

73. Suppose the axes are rotated through an angle θ. Then equation of $\frac{x}{a}+\frac{y}{b}=1$ with respect to the new axes is obtained by replacing x by $X\cos\theta - Y\sin\theta$ and y by $X\sin\theta + Y\cos\theta$

$$\Rightarrow \quad \frac{X\cos\theta - Y\sin\theta}{a}+\frac{X\sin\theta + Y\cos\theta}{b}=1$$

$$\Rightarrow \quad X\left(\frac{1}{a}\cos\theta+\frac{1}{b}\sin\theta\right)+Y\left(\frac{1}{b}\cos\theta-\frac{1}{a}\sin\theta\right)=1$$

$$\Rightarrow \quad \frac{1}{p}=\frac{1}{a}\cos\theta+\frac{1}{b}\sin\theta \text{ and } \frac{1}{q}=\frac{1}{b}\cos\theta-\frac{1}{a}\sin\theta$$

Squaring and adding $\frac{1}{p^2}+\frac{1}{q^2}=\frac{1}{a^2}+\frac{1}{b^2}$

74. Let lines make θ and ϕ with x-axis
Then $\tan\theta = m_1$, $\tan\phi = m_2$

$$m_1 + m_2 = \frac{-2h}{b}=\frac{2\tan\alpha}{\sin^2\alpha} = \frac{2}{\sin\alpha\cos\alpha}$$

$$m_1 m_2 = \frac{a}{b} = \frac{\tan^2\alpha+\cos^2\alpha}{\sin^2\alpha}$$

$(\tan\theta - \tan\phi)^2 = (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$

$$\Rightarrow \quad \frac{4}{\sin^2\alpha\cos^2\alpha}[1 - \sin^2\alpha - \cos^4\alpha]$$

$$= \frac{4\left(\sin^2\alpha\cos^2\alpha\right)}{\left(\sin^2\alpha\cos^2\alpha\right)}=4$$

$\tan\theta - \tan\phi = \pm 2$.

75. Two pairs of straight lines are equally inclined if they have the same pair of bisectors. $\frac{x^2-y^2}{2p}=\frac{xy}{m}$ — (1)

and $\frac{x^2-y^2}{2q}=\frac{xy}{n}$ — (2)

i.e., $\frac{2p}{m}=\frac{2q}{n}$ $\therefore$ pn = qm.

76. The required line should be
$(3x - y + 2) + \lambda(5x - 2y + 7) = 0$ — (1)
$(3 + 5\lambda)x - (2\lambda + 1)y + (2 + 7\lambda) = 0$

or $y=\frac{3+5\lambda}{2\lambda+1}x+\frac{2+7\lambda}{2\lambda+1}$ — (2)

As (2) is perpendicular to the x- axis,

$2\lambda + 1 = 0 \qquad \therefore \quad \lambda=\frac{-1}{2}$

Putting $\lambda=\frac{-1}{2}$ in (1) we have

$3x - y + 2 + \frac{-1}{2}(5x - 2y + 7) = 0 \Rightarrow x = 3$

77. Altitude = distance from (2, 3) to the base

$$=\left|\frac{2.2-3+3}{\sqrt{2^2+(-1)^2}}\right| = \left|\frac{4}{\sqrt{5}}\right| = \frac{4}{\sqrt{5}}$$

78. Divide by $\sqrt{1^2+\left(\sqrt{3}\right)^2}$ and express in the form of

$x\cos\alpha + y\sin\alpha = p$

$\therefore \cos\alpha=\frac{1}{2}$, $\sin\alpha=\frac{\sqrt{3}}{2}$, $\alpha=\frac{\pi}{3}$

79. If (α, β) is the point, its image in the line x = 0 is $(-\alpha, \beta)$ and its image in the line x = y is $(\beta, -\alpha) \Rightarrow$ B is (–3, 4) and C is (4, –3).

80. Let $L_1 : 2x + y = 3$, $L_2 : x + 2y = 3$, $L_3 : 2x + y = 5$, $L_3 : x + 2y = 5$ be the four lines.
We observe $L_1 \parallel L_3$ and $L_2 \parallel L_4 \Rightarrow$ closed figure could be a parallelogram, rhombus, rectangle or a square.
Let m_i be the slope of the line L_i (i = 1 to 4)
We have $m_1 = m_3 = -2$, $m_2 = m_4 = \frac{-1}{2}$
$m_1.m_2 = m_3.\ m_4 = 1\ (\neq -1)$
$\Rightarrow$ L_1 not perpendicular L_2 and L_3 not perpendicular L_4
$\therefore$ The figure formed cannot be a rectangle or a square

Now, d_1 = distance between L_1 and L_3. $=\left|\frac{5-3}{\sqrt{5}}\right|=\frac{2}{\sqrt{5}}$

d_2 = distance between L_2 and $L_4 = \frac{2}{\sqrt{5}}$

So $d_1 = d_2$, and hence we can say that the given lines form a rhombus.

81. Slope of AB $=\frac{z-y}{y-x}$; Slope of AC $=\frac{x-y}{z-x}$

Slope of AB × slope of AC = –1 $\Rightarrow \frac{z-y}{z-x}=1$
$\Rightarrow \quad z - y = z - x \Rightarrow y = x$

82. $m = \tan\theta = \tan\left(\tan^{-1}\sqrt{2}\right) = \sqrt{2}$

$c = \dfrac{-3}{\sqrt{2}}$

Equation is $y = \sqrt{2}x - \dfrac{3}{\sqrt{2}} \Rightarrow \sqrt{2}y = 2x - 3.$

83. The lines are

$5x + 2y + 5 = 0$ — (1)

$5x + 2y - 5 = 0$ — (2)

$5x - 2y + 5 = 0$ — (3)

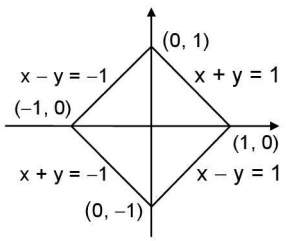
$5x - 2y - 5 = 0$ — (4)

(1) and (3) intersect at (–1, 0), (2) and (4) intersect at (1, 0)

(1) and (4) intersect at $\left(0, \dfrac{-5}{2}\right)$, (2) and (3) intersect at $\left(0, \dfrac{5}{2}\right)$

$\therefore$ Area $= 4 \times \dfrac{1}{2} \times 1 \times \dfrac{5}{2} = 5$

84. $ax + by + c = 0$; x-intercept $= -\dfrac{c}{a}$;



y- intercept $= -\dfrac{c}{b}$

Area of the triangle $= \dfrac{1}{2} \times \dfrac{c^2}{ab}$

But $c^2 = ab$

$\therefore$ Area $= \dfrac{1}{2}$

85. $\dfrac{x}{a} + \dfrac{y}{b} = 1 \Rightarrow bx + ay = ab$ — (1)

$\dfrac{x}{b} + \dfrac{y}{a} = 1 \Rightarrow ax + by = ab$ — (2)

$(1) \times a - (2) \times b \Rightarrow y = \dfrac{ab(a-b)}{a^2 - b^2} = \dfrac{ab}{a+b}$

$\Rightarrow x = \dfrac{ab}{a+b},\ y = \dfrac{ab}{a+b}$

86. $L = 12x + 5y - 13$. Substituting (1, –3) and (–3, 1) in L,

L at $(1, -3) = 12 - 15 - 13 < 0$

L at $(-3, 1) = -36 + 5 - 13 < 0$

So these points lie on same side of L.

87. $\dfrac{3x - 4y + 12}{5} = \pm\dfrac{4x + 3y + 2}{5}$

$3x - 4y + 12 = 4x + 3y + 2$ or $-4x - 3y - 2$

$-x - 7y + 10 = 0$ or $7x - y + 14 = 0$

$x + 7y - 10 = 0$ or $7x - y + 14 = 0.$

88. For the given equation of second degree to represent a pair of lines, we must have

$$\begin{vmatrix} 2 & -7/2 & 5 \\ -7/2 & \lambda & -10 \\ 5 & -10 & c \end{vmatrix} = 0$$

$\Rightarrow 2\lambda c + 150 - \dfrac{49c}{4} - 25\lambda = 0$ — (1)

Since the angle between the lines is 45°,

$\therefore \dfrac{2\sqrt{h^2 - ab}}{a+b} = 1$

$\Rightarrow \dfrac{2\sqrt{\dfrac{49}{4} - 2\lambda}}{2+\lambda} = \pm 1 \Rightarrow \sqrt{49 - 8\lambda} = \pm(2+\lambda)$

$\Rightarrow \lambda^2 + 12\lambda - 45 = 0$

$\lambda = 3$ or -15

Since $\lambda > 0$, $\lambda = 3$

Substituting in (1) we get $c = 12$

89. Suppose that origin is shifted to (h, k). Then $x = X + h$ and $y = Y + k$

Substituting in the given equation, we get

$(Y + k)^2 + 4(Y + k) + 8(X + h) - 2 = 0$

$Y^2 + 2kY + k^2 + 4Y + 4k + 8X + 8h - 2 = 0$

$\Rightarrow Y^2 + Y(2k + 4) + 8X + k^2 + 4k + 8h - 2 = 0$

Now by the given condition, $2k + 4 = 0$ and $k^2 + 4k + 8h - 2 = 0 \Rightarrow k = -2$ and $h = \dfrac{3}{4}$

90. $3h + 2k - 13 = 0$ — (1) × 1

$-h + 4k - 5 = 0$ — (2) × 3

$-3h + 12k - 15 = 0$

$\underline{3h + 2k - 13 = 0}$

$14k = 28 \Rightarrow k = 2\ h = 3$

Points are (3, 2) and (2, 3)

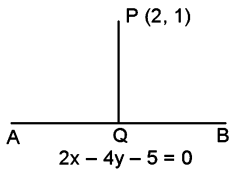
$y - 3 = -1\ (x - 2) \Rightarrow x + y - 5 = 0.$

91. $\tan\theta = \dfrac{|m_1 - m_2|}{1 + m_1 m_2}$

$$1 = \frac{3/4 + a/4}{1 - 3a/16} = \frac{4(3+a)}{16 - 3a} \Rightarrow 16 - 3a = 12 + 4a$$

$4 = 7a \Rightarrow a = \dfrac{4}{7}.$

92. Slope of PQ = –2



Equation of PQ is $y - 1 = -2(x - 2)$

$2x + y - 5 = 0$

Solving the equation we get $\left(\dfrac{5}{2}, 0\right)$

93. Solving the given equations we get $x = -8$, $y = 13$

$\therefore$ Required line is $y - 13 = \dfrac{-3}{2}(x + 8)$

$3x + 2y - 26 + 24 = 0$ or $3x + 2y - 2 = 0.$

94. Angle made by the lines with the x-axis in the positive directions are $(90^0 + \omega_1)$ and $(90° + \omega_2)$ respectively.

Angle between the lines = $(90° + \omega_1) - (90° + \omega_2)$ or $(90° + \omega_2) - (90° + \omega_1)$

$= \omega_1 - \omega_2$ or $\omega_2 - \omega_1$.

95. $y = 0 \Rightarrow x = \dfrac{\sin 2\alpha}{\sin\alpha} = 2\cos\alpha$

$x = 0 \Rightarrow y = \dfrac{\sin 2\alpha}{\cos\alpha} = 2\sin\alpha$,

Points of intersection with the axes of co-ordinates are $(2\cos\alpha, 0)$ and $(0, 2\sin\alpha)$

Mid-point is $(\cos\alpha, \sin\alpha)$. Let it be (x, y)

$\therefore\ x^2 + y^2 = \cos^2\alpha + \sin^2\alpha = 1.$

96. $\begin{vmatrix} 3 & 4 & 1 \\ a & 2 & -3 \\ 2 & -1 & -3 \end{vmatrix} = 0$

$3(-6 - 3) - 4(-3a + 6) + 1(-a - 4) = 0$

$-27 + 12a - 24 - a - 4 = 0 \Rightarrow a = 5.$

97. Line joining required image (x_1, y_1) and (3, 8) must be perpendicular to the given line $\dfrac{y_1 - 8}{x_1 - 3} = 3$

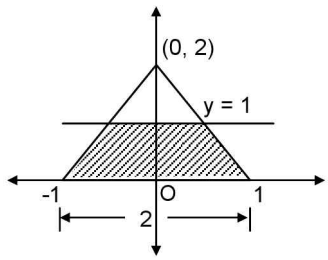
Midpoint is $\left(\dfrac{x_1 + 3}{2}, \dfrac{y_1 + 8}{2}\right)$

The above point should satisfy $x + 3y = 7$

Solving these two conditions we get the point (–1, –4) or p is (–1, –4).

Equation of the required line is $y + 4 = \dfrac{7}{2}(x + 1)$ or $7x - 2y - 1 = 0$

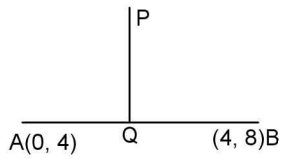
98. $\pm 2x + y = 2$



$\dfrac{x}{\pm 1} + \dfrac{y}{2} = 1$

Area $= \dfrac{1}{2}(2+1)\times 1 = \dfrac{3}{2}$

99. Q = (2, 6)



Slope of AB = 1

Slope of PQ = –1 $y - 6 = -(x - 2)$

$x + y - 8 = 0.$

100. Product of the perpendiculars from the point (x_1, y_1) to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}} = \frac{2+6+3}{\sqrt{(2-3)^2 + 4\times 9}}$$

$$= \frac{11}{\sqrt{37}}$$

101. Since one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ bisects the angle between the lines $xy = 0$, its equation is either $y = x$ or $y = -x$

$\Rightarrow$ $a + b + 2h = 0$ or $a + b - 2h = 0$

$\Rightarrow$ $a + b = -2h$ or $a + b = 2h \Rightarrow (a + b)^2 = 4h^2$.

102. $h^2 = ab \Rightarrow h^2 = (3)(4) \Rightarrow h = \pm 2\sqrt{3}$.

103. Coefficient of x^2 + coefficient of $y^2 = 1 - 1 = 0$

$\Rightarrow$ $\theta = 90°$

104. The given equation represents the same line

$$\Rightarrow \frac{b^3 - c^3}{b - c} = \frac{c^3 - a^3}{c - a} = \frac{a^3 - b^3}{a - b}$$

$\Rightarrow$ $b^2 + bc + c^2 = c^2 + ca + a^2 = a^2 + ab + b^2$

consider $b^2 + c^2 + bc = c^2 + ac + a^2$

$\Rightarrow$ $b^2 + bc = ac + a^2$

$\Rightarrow$ $b^2 - a^2 = ac - bc \Rightarrow (b - a)(b + a) = (a - b)c$

$\Rightarrow$ $a + b + c = 0$ $(\because a \neq b \neq c)$

$\Rightarrow$ line $ax + by + c = 0$ passes through (1, 1)

105. Since the point is on y-axis, x coordinate is zero.

$\Rightarrow$ $y = \pm 1$

$\Rightarrow$ Intercept = 2.

106. Let the required ratio be $k : 1$, then the dividing point is $\left(\frac{2k+1}{k+1}, \frac{7k+3}{k+1}\right)$

This point t lies on the line $3x + y = 9$

$$\Rightarrow 3\left(\frac{2k+1}{k+1}\right) + \left(\frac{7k+3}{k+1}\right) = 9$$

$6k + 3 + 7k + 3 = 9k + 9$

$4k = 3 \Rightarrow k = \frac{3}{4}$

$\Rightarrow$ ratio is 3 : 4

107. Let the line be $\frac{x}{a} + \frac{y}{a} = 1$,

This passes through $(2, -3)$

$\therefore \frac{2}{a} - \frac{3}{a} = 1 \Rightarrow a = -1$

Hence the equation of line is $x + y = -1$ or

$x + y + 1 = 0$

108. The equation of the line passing through the intersection of $x - 3y + 1 = 0$ and $2x + 5y - 9 = 0$ is

$(x - 3y + 1) + \lambda(2x + 5y - 9) = 0$

$(2\lambda + 1)x + (5\lambda - 3)y + 1 - 9\lambda = 0$

This is at distance $\sqrt{5}$ from the origin,

$\therefore \left|\frac{1 - 9\lambda}{\sqrt{(2\lambda+1)^2 + (5\lambda-3)^2}}\right| = \sqrt{5}$ $\therefore \lambda = \frac{7}{8}$ then the required line is $2x + y = 5$

109. $x \cos\alpha + y \sin\alpha = a$

$x \sin\alpha - y \cos\alpha = a$

By squaring and adding (1) and (2) we get

$x^2(\sin^2\alpha + \cos^2\alpha) + y^2(\sin^2\alpha + \cos^2\alpha) + 2xy \sin\alpha \cos\alpha - 2xy \sin\alpha \cos\alpha = a^2 + a^2$

$\therefore$ $x^2 + y^2 = 2a^2$

110. We have $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

Since the lines intersect on the y axis, x coordinate of the point of intersection is zero.

$$\frac{hf - bg}{ab - h^2} = 0 \Rightarrow hf = bg$$

$\therefore$ $fgh = bg^2$

Substituting in the first relation above,

$abc + fgh = af^2 + ch^2$.

111. Statement 2 is true

Consider Statement 1. It can be observed that the triangle is right angled. Using Statement 2, we see that the orthocenter is at the point of intersection of the lines $3y + x = 0$ and $y - 3x = 0$, which is the origin

Statement 1 is true

Choice (a)

112. Statement 2 is false

Consider Statement 1

The constant term in the equation $5x - y + 9 = 0$ is positive. We note that $(5 \times 2 - (-3) + 9)$ and $(5 \times 1 - 9 + 9)$ are both positive ie, they are having the same sign as that of the constant term in the equation of the line. Both points lie on the origin side $\Rightarrow$ Statement 1 is true

Choice (c)

113. Consider Statement 2

Let (x, y) denote the coordinates of a point in the xy system and (X, Y) denote the coordinates of the same point in the new system. Then we have

$x = X\cos\theta - Y\sin\theta$

$y = X\sin\theta + Y\cos\theta$

Substituting in the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, the equation of the pair of lines in the new system is

$a(X\cos\theta - Y\sin\theta)^2 + 2h(X\cos\theta - Y\sin\theta)(X\sin\theta$
$+ Y\cos\theta) + b(X\sin\theta + Y\cos\theta)^2$
$+ 2h(X\cos\theta - Y\sin\theta) + 2f(X\sin\theta + Y\cos\theta) + c = 0$

$\Rightarrow$ $A = a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta$

$B = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta$

$\Rightarrow$ $A + B = a + b$

Statement 2 is true

Consider Statement 1

Using Statement 2, we get

$A + B = 3 + (-2) = 1$

Statement 1 is true

Choice (a)

114. The equation of the pair of lines through the origin is

$$2x^2 + xy - 9y^2 + (5x + 2y)\left(\frac{x+y}{\lambda}\right) + 7\left(\frac{x+y}{\lambda}\right)^2 = 0$$

Since the lines are at right angles,

$$\left(2 + \frac{5}{\lambda} + \frac{7}{\lambda^2}\right) + \left(-9 + \frac{2}{\lambda} + \frac{7}{\lambda^2}\right) = 0 \Rightarrow -7 + \frac{7}{\lambda} + \frac{14}{\lambda^2} = 0$$

$\Rightarrow$ $-\lambda^2 + \lambda + 2 = 0$

$\Rightarrow$ $\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda = -1, 2$

115. The equation of the lines joining the origin and the points of intersection are given by

$$3x^2 + 4xy - y^2 + (2x + y)\left(\frac{x}{6}\right) + 6\left(\frac{x^2}{36}\right) = 0$$

$\Rightarrow$ $18x^2 + 24xy - 6y^2 + 2x^2 + xy + x^2 = 0$

$\Rightarrow$ $21x^2 + 25xy - 6y^2 = 0$

$a = 21, b = -6, \ h = \frac{25}{2}$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{(a+b)}$$

$$= \frac{2\sqrt{\frac{625}{4} + 126}}{15} = \frac{2\sqrt{1129}}{2 \times 15} = \frac{\sqrt{1129}}{15}$$

116. The pair of lines through origin is

$$x^2 + y^2 - (2x + 2y)\frac{(x+y)}{4} - 2\left(\frac{x+y}{4}\right)^2 = 0$$

$3x^2 - 10xy + 3y^2 = 0$

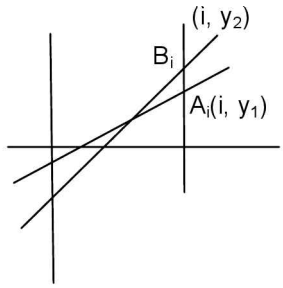
Individual lines are $3x - y = 0$ and $x - 3y = 0$

The vertices of the triangle are (0, 0) ; (6, 2) and (2, 6)

Area of the triangle $= \frac{1}{2}(x_1y_2 - x_2y_1)$

$= \frac{1}{2}(36 - 4) = 16$

117. Let $y = m_1x + c_1$ and $y = m_2x + c_2$ are two equations given by $6x^2 + xy - y^2 - 7x + 4y - 3 = 0$



Equation of the pair of lines may be written as

$(3x - y + \ell)(2x + y + m) = 0$

Identify with the given equation

$\ell m = -3, 2\ell + 3m = -7$

$\ell - m = 4$

$\ell = 1, m = -3$

The lines are $3x - y + 1 = 0$ and $2x + y - 3 = 0$

$|A_iB_i| = |y_2 - y_1| = (3 - 2i) - (3i + 1) = |-5i + 2|$

$A_1B_1 = 3$

$A_2B_2 = 8$

$A_3B_3 = 13$

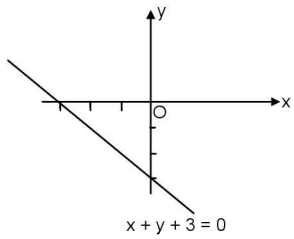
Choice (a) is true

$\therefore$ $\sum_{i=1}^{n} A_iB_i =$ Sum of the first n terms of the A.P.

$$3 + 8 + 13 + .. = \frac{(5n^2 + n)}{2}$$

118. Since L_1 makes an angle 135° with the x-axis, the lines which make angles 45° with L_1 have to be the x-axis and y-axis.

$\Rightarrow$ both (a) and (b) are true

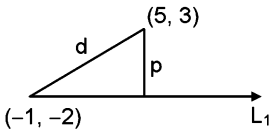


Note that (–1, –2) is a point on L_1

$d^2 = 36 + 25 = 61$

$$p^2 = \left(\frac{5+3+3}{\sqrt{2}}\right)^2 = \frac{121}{2}$$

$$\text{Projection} = \sqrt{61 - \frac{121}{2}} = \frac{1}{\sqrt{2}}$$



Slope of the line joining the origin and the point (5, 3) is $\frac{3}{5}$

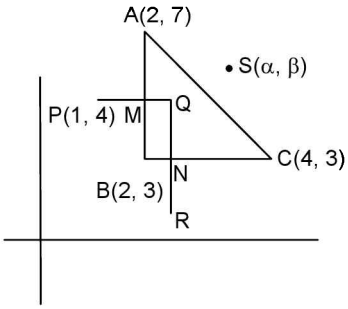
If θ is the angle between the above line and the line $x + y + 3 = 0$,

$$\tan\theta = \frac{3/5 + 1}{1 - 3/5} = \frac{8}{2} = 4$$

$\theta = \tan^{-1}(4)$

$\Rightarrow$ (d) is true

119. Since AB is a line parallel to y-axis, PQ || x-axis and M is mid point of PQ. From the figure coordinates of M is (2, 4)



$\therefore$ Co-ordinates of Q are of M are (3, 4)

Similarly, co-ordinates of R are (3, 2)

Equation of AC is $2x + y - 11 = 0$

Let S(α, β) be the image of Q(3, 4) with respect to AC

$\therefore$ $2(3 + \alpha) + (4 + \beta) - 22 = 0$

$\Rightarrow$ $2\alpha + \beta - 12 = 0$ — (1)

and $(3 - \alpha) - 2(4 - \beta) = 0$ $\alpha - 2\beta + 5 = 0$ — (2)

Solving (1) and (2) $\Rightarrow \alpha = \frac{19}{5}, \beta = \frac{22}{5}$

$\therefore$ S is $\left(\frac{19}{5}, \frac{22}{5}\right)$

$$\text{Centroid of } \Delta\, SRP = \left[\left(\frac{1+3+\frac{19}{5}}{3}\right), \left(\frac{4+2+\frac{22}{5}}{3}\right)\right]$$

$$= \left(\frac{39}{15}, \frac{52}{15}\right)$$

120. (a) (a, b) be this point on this line $x + y = 4$

$\therefore$ $a + b = 4$ — (1)

Distance (a, b) from this line $4x + 3y = 10$ is unity

$$\Rightarrow \frac{4a + 3b - 10}{\sqrt{16+9}} = \pm 1$$

$\Rightarrow$ $4a + 3b = 10 \pm 5$ — (2)

from (1) and (2) $a = -2 \pm 5 \Rightarrow 3, -7$ $b = 1, 11$.

$\therefore$ Required points (3, 1) and (– 7, 11)

(b) $|x| = |y| \Rightarrow x - y = 0$ and $x + y = 0$.

Bisectors of angle between this lines are

$$\frac{x-y}{\sqrt{2}} = \pm\frac{x+y}{\sqrt{2}} \Rightarrow x = 0 \text{ and } y = 0.$$

By solving $x = 0$ and $x + y = 3 \Rightarrow (0, 3)$

$y = 0$ and $x + y = 3 \Rightarrow (3, 0)$

(c) Image of (a, b) with respect to the line $y = -x$ is(– b, – a)

$\therefore$ image of (0, – 3) on the line $x + y = 0$ is (3, 0).

(d) Let variable line is $ax + by + 1 = 0$.

$$\frac{-2a+3}{\sqrt{a^2+b^2}} + \frac{4a+3}{\sqrt{a^2+b^2}} + \frac{a+b+3}{\sqrt{a^2+b^2}} = 0$$

$-2a + 4a + a + b + 9 = 0 \Rightarrow 3a + b + 9 = 0$.

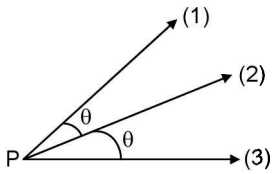
$\therefore$ line $ax + by + 1 = 0$ passes through (3, 1).

Additional Practice Exercise

121. $px + qy + r = 0$ — (1)

$\ell x + my + n = 0$ — (2)

If (3) is the required line, (2) must be a bisector of (1) and (3).



(3) may be assumed as

$px + qy + r + \lambda(\ell x + my + n) = 0$ — (3)

Slope of (1) $= -\dfrac{p}{q}$

Slope of (2) $= -\dfrac{\ell}{m}$

Slope of (3) $= -\dfrac{(p+\lambda\ell)}{(q+\lambda m)}$

We have, angle between (1) and (2) = angle between (2) and (3)

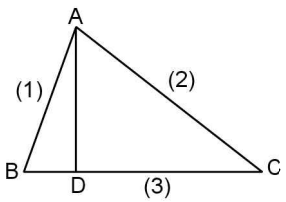
$$\Rightarrow \frac{-\frac{p}{q}+\frac{\ell}{m}}{1+\frac{p\ell}{qm}} = \frac{-\frac{\ell}{m}+\frac{p+\lambda\ell}{q+\lambda m}}{1+\frac{\ell(p+\lambda\ell)}{m(q+\lambda m)}}$$

$$\Rightarrow \frac{(\ell q - pm)}{(p\ell + qm)} = \frac{(pm - \ell q)}{(p\ell + mq) + \lambda(\ell^2 + m^2)}$$

$$\Rightarrow (p\ell + mq) + \lambda(\ell^2 + m^2) = -(p\ell + mq)$$

$$\Rightarrow \lambda = \frac{-2(p\ell + mq)}{(\ell^2 + m^2)}$$ Result follows.

122.



(i) $ax + by + c = 0$ — (1)

$\ell x + my + n = 0$ — (2)

$px + qy + r = 0$ — (3)

Let AD be the altitude through A.

AD may be represented by

$ax + by + c + \lambda(\ell x + my + n) = 0$ — (4)

(3) and (4) are perpendicular to each other.

$$\frac{-(a+\lambda\ell)}{(b+\lambda m)} \times \frac{-p}{q} = -1.$$

$$p(a + \lambda\ell) = -q(b + \lambda m)$$

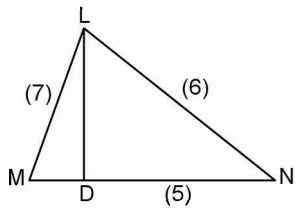
$$\lambda(p\ell + mq) = -(ap + bq)$$

$$\lambda = -\frac{(ap+bq)}{(\ell p + mq)}$$

Since the orthocentre lies on the line AD, orthocentre lies on

$$(ax + by + c) - \frac{(ap+bq)}{(\ell p + mq)}(\ell x + my + n) = 0$$

$$\Rightarrow \frac{ax+by+c}{(ap+bq)} = \frac{\ell x + my + n}{\ell p + mq}$$



(ii) $\mu_1 = x\cos\alpha_1 + y\sin\alpha_1 - p_1 = 0$ — (5)

$\mu_2 = x\cos\alpha_2 + y\sin\alpha_2 - p_2 = 0$ — (6)

$\mu_3 = x\cos\alpha_3 + y\sin\alpha_3 - p_3 = 0$ — (7)

Altitude through L is

$$\frac{x\cos\alpha_2 + y\sin\alpha_2 - p_2}{\cos\alpha_2\cos\alpha_1 + \sin\alpha_2\sin\alpha_1}$$

$$= \frac{x\cos\alpha_3 + y\sin\alpha_3 - p_3}{\cos\alpha_3\cos\alpha_1 + \sin\alpha_3\sin\alpha_1}$$

using result in (i)

$$\Rightarrow \frac{\mu_2}{\cos(\alpha_1 - \alpha_2)} = \frac{\mu_1}{\cos(\alpha_3 - \alpha_1)}$$

Altitude thro M is $\dfrac{\mu_3}{\cos(\alpha_2 - \alpha_3)} = \dfrac{\mu_1}{\cos(\alpha_1 - \alpha_2)}$

Altitude thro N is $\dfrac{\mu_1}{\cos(\alpha_3 - \alpha_1)} = \dfrac{\mu_2}{\cos(\alpha_2 - \alpha_3)}$

The above 3 relations give

$$\mu_1\cos(\alpha_2 - \alpha_3) = \mu_2\cos(\alpha_3 - \alpha_1)$$
$$= \mu_3\cos(\alpha_1 - \alpha_2)$$

123. The lines are

$(\ell x + my)^2 - 3(mx - \ell y)^2 = 0$ — (1)

$\ell x + my + n = 0$ — (2)

Simplifying (1), we get

$$x^2(\ell^2 - 3m^2) + 8\ell mxy + y^2(m^2 - 3\ell^2) = 0$$

If θ is the angle between the above pair,

$$\tan\theta = \left|\frac{2\sqrt{16\ell^2m^2 - (\ell^2 - 3m^2)(m^2 - 3\ell^2)}}{(\ell^2 - 3m^2) + (m^2 - 3\ell^2)}\right|$$

$$= \left|\frac{2\sqrt{3\ell^4 + 3m^4 + 6\ell^2m^2}}{-2(\ell^2 + m^2)}\right| = \sqrt{3}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

(1) can be factored and written in the form

$$\ell x + my + \sqrt{3}\,(mx - \ell y) = 0 \qquad \text{— (3)}$$

and

$$\ell x + my - \sqrt{3}\,(mx - \ell y) = 0 \qquad \text{— (4)}$$

If θ_1 is the angle between (2) and (3),

$$\tan\theta_1 = \pm\frac{\ell(m - \sqrt{3}\ell) - m(\ell + \sqrt{3}m)}{\ell(\sqrt{3}m + \ell) + m(m - \sqrt{3}\ell)}$$

$$= \pm\left(-\sqrt{3}\frac{\ell^2 + m^2}{\ell^2 + m^2}\right) = \pm\sqrt{3}$$

$$\Rightarrow \quad \theta_1 = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

Hence the given lines form the sides of an equilateral triangle.

124. Since the pair of lines are at right angles to each other,

$$a + b = 0 \qquad \text{— (1)}$$

The coordinates of the point of intersection of the pair of lines are $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

Square of the distance of their point of intersection from the origin

$$= \frac{(hf - bg)^2 + (gh - af)^2}{(ab - h^2)^2}$$

$$= \frac{f^2(h^2 + a^2) + g^2(b^2 + h^2)}{(-b^2 - h^2)^2},$$

since $a = -b$

$$= \frac{f^2(h^2 + b^2) + g^2(b^2 + h^2)}{(b^2 + h^2)^2} = \frac{f^2 + g^2}{h^2 + b^2}$$

125. Let $y = mx$ of the 1st pair $\perp$ to a line of second pair

$\Rightarrow \quad y = -\frac{1}{m}x$ is a member of second pair.

$y = mx$ satisfies first equation.

$$\therefore \quad bm^2 + 2hm + a = 0 \qquad \text{— (1)}$$

$y = -\frac{1}{m}x$ satisfies second equation

$$a_1m^2 - 2h_1m + b_1 = 0 \qquad \text{— (2)}$$

From (1) and (2),

$$\frac{m^2}{2(hb_1 + ah_1)} = \frac{m}{aa_1 - bb_1} = \frac{1}{-2(h_1b + ha_1)}$$

$$\therefore \quad (aa_1 - bb_1)^2 = -4(hb_1 + ah_1)(h_1b + ha_1)$$

126. $ax^2 + 2hxy + by^2 = 0$ represents 2 parallel lines if $h^2 = ab$

$h = \sqrt{a}\sqrt{b}$

$$\therefore \quad ax^2 + 2hxy + by^2 = \left(\sqrt{a}\,x + \sqrt{b}\,y\right)^2$$

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$= \left(\sqrt{a}\,x + \sqrt{b}\,y + \lambda_1\right)\left(\sqrt{a}\,x + \sqrt{b}\,y + \lambda_2\right)$$

Comparing like terms

$$2g = (\lambda_1 + \lambda_2)\sqrt{a} \qquad \text{— (1)}$$

$$2f = (\lambda_1 + \lambda_2)\sqrt{b} \qquad \text{— (2)}$$

$$c = \lambda_1\lambda_2 \qquad \text{— (3)}$$

$$(1) \div (2) \Rightarrow \frac{g}{f} = \frac{\sqrt{a}}{\sqrt{b}} \text{ or } bg^2 = af^2$$

$$(\lambda_1 - \lambda_2)^2 = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2$$

$$= \frac{4g^2}{a} - 4c = \frac{4(g^2 - ac)}{a}$$

$\therefore$ Distance between parallel lines

$$= \frac{|\lambda_2 - \lambda_1|}{\sqrt{\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2}} = \frac{2\sqrt{g^2 - ac}}{\sqrt{a}\sqrt{a + b}}$$

127. The distance of $(2, -3)$ from the line

$\lambda(2x + y + 4) + \mu(x - 2y - 3) = 0$ is

$$\frac{5\lambda + 5\mu}{\sqrt{(2\lambda + \mu)^2 + (\lambda - 2\mu)^2}} = \frac{5(\lambda + \mu)}{\sqrt{5}\sqrt{\lambda^2 + \mu^2}}$$

$$= \frac{\sqrt{5}\,(\lambda + \mu)}{\sqrt{\lambda^2 + \mu^2}}$$

Given $\frac{\sqrt{5}(\lambda+\mu)}{\sqrt{\lambda^2+\mu^2}} = \pm\sqrt{10} \Rightarrow \frac{\lambda+\mu}{\sqrt{\lambda^2+\mu^2}} = \pm\sqrt{2}$

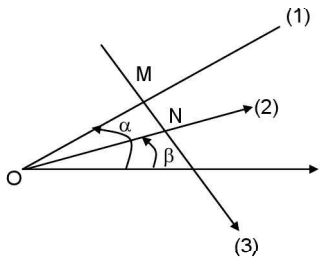
$\Rightarrow \quad (\lambda + \mu)^2 = 2(\lambda^2 + \mu^2)$

$\Rightarrow \quad (\lambda - \mu)^2 = 0, \Rightarrow \lambda = \mu$

The member of the family corresponding to $\lambda = \mu$ is $\lambda(3x - y + 1) = 0$ or $3x - y + 1 = 0$

The condition of the problem gives only one solution or there is only one line.

128. $y = x \tan \alpha$ — (1)



$y = x \tan \beta$ — (2)

$y = x \tan \theta + c$ — (3)

Solving (1) and (3)

$x(\tan\alpha - \tan\theta) = c \Rightarrow x = \frac{c\cos\alpha\cos\theta}{\sin(\alpha-\theta)}$,

$y = \frac{c\sin\alpha\cos\theta}{\sin(\alpha-\theta)}$

M is $\left(\frac{c\cos\alpha\cos\theta}{\sin(\alpha-\theta)}, \frac{c\sin\alpha\cos\theta}{\sin(\alpha-\theta)}\right)$

Similarly, solving (2) an (3),

N is $\left(\frac{c\cos\beta\cos\theta}{\sin(\beta-\theta)}, \frac{c\sin\beta\cos\theta}{\sin(\beta-\theta)}\right)$

Area of Δ ONM =

$$\frac{1}{2}\left[\frac{c^2\cos^2\theta\sin\alpha\cos\beta}{\sin(\alpha-\theta)\sin(\beta-\theta)} - \frac{c^2\cos^2\theta\sin\beta\cos\alpha}{\sin(\alpha-\theta)\sin(\beta-\theta)}\right]$$

$$= \left[\frac{c^2\cos^2\theta\sin(\alpha-\beta)}{2\sin(\alpha-\theta)\sin(\beta-\theta)}\right].$$

129. The equation of line through the point of intersection of $\frac{x}{3}+\frac{y}{2}-1=0$ and $\frac{x}{2}+\frac{y}{3}-1=0$ is of the form

$$\left(\frac{x}{3}+\frac{y}{2}-1\right)+\lambda\left(\frac{x}{2}+\frac{y}{3}-1\right)=0$$

Putting $y = 0$ we get point A as $\left\{\frac{6(1+\lambda)}{2+3\lambda}, 0\right\}$

Putting $y = 0$ we get point B as $\left\{0, \frac{6(1+\lambda)}{3+2\lambda}\right\}$

If (h, k) be the mid-point of AB, then $2h = \frac{6(1+\lambda)}{2+3\lambda}$

$2k = \frac{6(1+\lambda)}{3+2\lambda}$ In order to find the locus of (h, k) we have to eliminate the variable λ.

Now $\frac{1}{2h}+\frac{1}{2k} = \frac{(2+3\lambda)+(3+2\lambda)}{6(1+\lambda)} = \frac{5(1+\lambda)}{6(1+\lambda)}$

$\Rightarrow \frac{h+k}{2hk} = \frac{5}{6} \Rightarrow 6(h + k) = 10hk$

Hence the locus of mid-point (h, k) is obtained by generating (h, k) and is $6(x + y) = 10xy$

130. Let (α, β) be any point on $lx + my + n = 0$

$\Rightarrow \quad l\alpha + m\beta + n = 0$ — (1)

The required line will pass through the intersection of given line and given bisector and hence its equation is

$(px + qy + r) + \lambda(lx + my + n) = 0$ — (2)

Also $px + qy + r = 0$ — (3)

If (α, β) be a point on the bisctor then its perpendicular distances from the lines (2) and (3) are equal.

$$\therefore \frac{(p\alpha+q\beta+r)+\lambda(l\alpha+m\beta+n)}{\sqrt{\left[(p+\ell\lambda)^2+(q+m\lambda)^2\right]}} = \frac{\pm p\alpha+q\beta+r}{\sqrt{p^2+q^2}}$$

$$\frac{1}{\sqrt{(p+l\lambda)^2+(q+m\lambda)^2}} = \frac{1}{\sqrt{p^2+q^2}}$$

$\lambda^2(l^2 + m^2) + 2\lambda(pl + qm) = 0$

$$\Rightarrow \quad \lambda = \frac{-2(pl+qm)}{l^2+m^2}$$

Putting λ value in equation (2) then the required line is

$(l^2 + m^2)(px + qy + r) - 2(lp + mq)(lx + my + n) = 0$

131. Let the straight lines be $y - m_1 x = 0$ and $y - m_2 x = 0$

Then, $\frac{y_1 - m_1x_1}{\sqrt{1+m_1^2}} = \frac{y_1 - m_2x_1}{\sqrt{1+m_2^2}} = \lambda$ (given)

$\Rightarrow \quad (y_1 - m_1x_1)^2 = \lambda^2 (1+m_1^2)$ — (1)

$(y_1 - m_2x_1)^2 = \lambda^2 (1+m_2^2)$ — (2)

(1) – (2) gives

$x_1^2(m_1^2 - m_2^2) + 2x_1y_1(m_2 - m_1) = \lambda^2(m_1^2 - m_2^2)$

$(m_1 - m_2)$ being $\neq 0$, the above reduces to

$x_1^2 (m_1 + m_2) - 2x_1y_1 = \lambda^2 (m_1 + m_2)$

$$m_1 + m_2 = \frac{2x_1y_1}{(x_1^2 - \lambda^2)} \quad \text{— (3)}$$

(1) + (2) gives

$2y_1^2 - 2x_1y_1 (m_1 + m_2) + x_1^2(m_1^2 + m_2^2) = 2\lambda^2 + \lambda^2 (m_1^2 + m_2^2)$

$2(y_1^2 - \lambda^2) - 2x_1y_1(m_1 + m_2) = (\lambda^2 - x_1^2)(m_1^2 + m_2^2)$

Substituting for $(m_1 + m_2)$ from (3),

$$2(y_1^2 - \lambda^2) - 2x_1y_1\left(\frac{2x_1y_1}{x_1^2 - \lambda^2}\right)$$

$$= (\lambda^2 - x_1^2)\left[\frac{4x_1^2y_1^2}{(x_1^2 - \lambda^2)^2} - 2m_1m_2\right]$$

$$\Rightarrow \quad 2(y_1^2 - \lambda^2) = 2m_1m_2 (x_1^2 - \lambda^2)$$

$$\Rightarrow \quad m_1m_2 = \frac{(y_1^2 - \lambda^2)}{(x_1^2 - \lambda^2)} \quad \text{— (4)}$$

Therefore, the combined equation of the lines is $(y - m_1x)(y - m_2x) = 0$

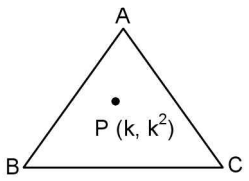
$\Rightarrow y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0$

$$\Rightarrow y^2 - \frac{2x_1y_1}{(x_1^2 - \lambda^2)}xy + \left(\frac{y_1^2 - \lambda^2}{x_1^2 - \lambda^2}\right)x^2 = 0$$

$\Rightarrow x_1^2y^2 - \lambda^2y^2 - 2x_1y_1xy + y_1^2x^2 - \lambda^2x^2 = 0$

$\Rightarrow (x_1y - y_1x)^2 = \lambda^2 (x^2 + y^2)$

132. Let $BC \rightarrow L_1 = 11x + 6y + 14 = 0$



$CA \rightarrow L_2 = 9x + y - 12 = 0$

$AB \rightarrow L_3 = 2x + 5y - 17 = 0$

Solving the equations, the vertices are obtained as

$A(1, 3), B(-4, 5)$ and $C(2, -6)$

$L_1(A) = 11 \times 1 + 6 \times 3 + 14 > 0$

$L_2(B) = 9 \times -4 + 5 - 12 < 0$

$L_3(C) = 2 \times 2 + 5 \times -6 - 17 < 0$

Since P and A are to lie on the same side of BC, $11k + 6k^2 + 14 > 0$, which is true for all real k — (I). (Since discriminant of the quadratic in k is < 0)

Since P and B are to lie on the same of CA, $9k + k^2 - 12 < 0$

$$k = \frac{-9 \pm \sqrt{81+48}}{2} = \frac{-9 \pm \sqrt{129}}{2}$$

k must lie between $\frac{-9-\sqrt{129}}{2}$ and $\frac{-9+\sqrt{129}}{2}$ — (II)

Since P and C are to lie on the same side of AB, $2k + 5k^2 - 17 < 0$

$$k = \frac{-2 \pm \sqrt{4+340}}{10} = \frac{-1 \pm \sqrt{86}}{5}$$

k must lie between $\frac{-1-\sqrt{86}}{5}$ and $\frac{-1+\sqrt{86}}{5}$ — (III)

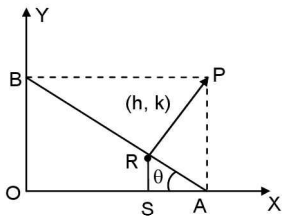
From (I), (II), (III) we see that k should satisfy the inequality

$$\frac{-1-\sqrt{86}}{5} < k < \frac{-9+\sqrt{129}}{2}$$

$\Rightarrow$ k lies between $-(2 + f_1)$ and $(1 + f_2)$ where $0 < f_1, f_2 < 1$

Number of integer values of k = 3

133. Let $\angle OAB = \theta$



θ varies as the position of AB changes.

Let R(h, k) be the foot of the perpendicular from P.

Now,

$h = OS = OA - SA = 5 \cos\theta - RA \cos\theta$

$= 5 \cos\theta - PA \cos(\frac{\pi}{2} - \theta) \cos\theta$

$= 5\cos\theta - 5(\sin\theta)(\sin\theta\cos\theta)$

$= (5\cos\theta)[1-\sin^2\theta] \quad = 5\cos^3\theta$ — (1)

$k = RS = RA\sin\theta = PA\cos(\frac{\pi}{2}-\theta)\sin\theta$

$= (5\sin\theta)\sin^2\theta$

$= 5\sin^3\theta$ — (2)

From (1) and (2)

$\Rightarrow \quad h^{\frac{2}{3}} + k^{\frac{2}{3}} = (5\cos^3\theta)^{\frac{2}{3}} + (5\sin^3\theta)^{\frac{2}{3}} = 5^{\frac{2}{3}}$

or, locus of P is the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5^{\frac{2}{3}}$

134. If the coordinate axes are rotated through an angle θ, the coordinates (X, Y) with respect to the new axes are related with (x, y) with respect to original axes as $x = X\cos\theta - Y\sin\theta$;

$y = X\sin\theta + Y\cos\theta$ Substituting in the given equation,

$a(X\cos\theta - Y\sin\theta)^2 + b(X\sin\theta + Y\cos\theta)^2$

$+ 2h(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + 2g((X\cos\theta - Y\sin\theta) + 2f(X\sin\theta + Y\cos\theta) + c = 0$

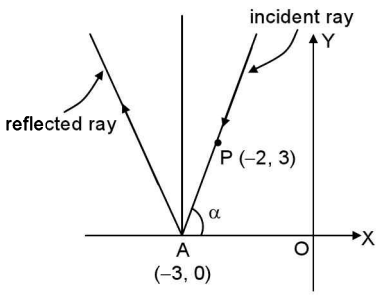
Coefficient of XY is $-2a\cos\theta\sin\theta + 2b\cos\theta\sin\theta + 2h(\cos^2\theta - \sin^2\theta)$

XY term $= 0 \Rightarrow \sin 2\theta(b-a) = -2h\cos 2\theta$

$\Rightarrow \quad \tan 2\theta = \frac{2h}{(a-b)}$

$\therefore \quad \theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$

135. Equation of the incident ray PA is



$y - 3 = 3(x+2)$ i.e., $3x - y + 9 = 0$

It meets X-axis at (–3, 0)

The reflected ray will make $180 - \alpha$ with x-axis and hence slope $= \tan(180-\alpha)$

$= -\tan\alpha = -3.$

$\therefore$ Equation is $y - 0 = -3(x+3)$

136. The equation of line through the point of intersection of given lines is of the form

$x + 2y - 3 + k(4x - y + 7) = 0$

$(1+4k)x + (2-k)y - 3 + 7k = 0$- — (1)

Since this line is parallel to $5x + 4y - 20 = 0$, slopes are equal.

$\Rightarrow \frac{-(1+4k)}{2-k} = \frac{-5}{4} \Rightarrow 4 + 16k = 10 - 5k$

$21k = 6 \quad k = \frac{6}{21}$

Substituting the value of k is (1)

$\left(1+4\times\frac{6}{21}\right)x + \left(2-\frac{6}{21}\right)y - 3 + 7\times\frac{6}{21} = 0$

$\frac{45}{21}x + \frac{36}{21}y - \frac{21}{21} = 0$

$15x + 12y - 7 = 0$

137. A is the point $\left(\frac{3\lambda-5}{\lambda+1}, \frac{5\lambda+1}{\lambda+1}\right)$

Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$\Rightarrow \frac{1}{2}\begin{vmatrix} \frac{3\lambda-5}{\lambda+1} & \frac{5\lambda+1}{\lambda+1} & 1 \\ 1 & 5 & 1 \\ 7 & -2 & 1 \end{vmatrix} = \pm 2$

$\Rightarrow \begin{vmatrix} \frac{2\lambda-6}{\lambda+1} & \frac{-4}{\lambda+1} & 0 \\ 1 & 5 & 1 \\ 6 & -7 & 0 \end{vmatrix} = \pm 4$

$-\left\{\frac{-7(2\lambda-6)}{\lambda+1} + \frac{24}{\lambda+1}\right\} = \pm 4$

$\Rightarrow \left\{\frac{-14\lambda+42+24}{\lambda+1}\right\} = \mp 4$

$-14\lambda + 66 = \mp(4\lambda+4)$

Taking '–' sign $\Rightarrow$

$-14\lambda + 66 = -4\lambda - 4$

$-10\lambda = -70$

$\Rightarrow \lambda = 7$

Taking + sign ⇒

$-14\lambda + 66 = 4\lambda + 4$

$-18\lambda = -62$

$\Rightarrow \lambda = \frac{31}{9}$

138. The equations of given lines are

$x \sec\theta + y \operatorname{cosec}\theta = k$ (1)

$x \cos\theta - y \sin\theta = k\cos 2\theta$ (2)

The distance of origin from the line (1) is

Given by

$$a = \left|\frac{0.\sec\theta + 0.\operatorname{cosec}\theta - k}{\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}}\right|$$

or $a^2 = k^2 \sin^2\theta \,.\, \cos^2\theta$

similarly the distance of the origin form the line (2) is given by

$$b = \left|\frac{0.\cos\theta + 0.\sin\theta - k\cos 2\theta}{\sqrt{\cos^2\theta + \sin^2\theta}}\right|$$

$\therefore\ 4a^2 + b^2 = 4k^2\sin^2\theta \,.\, \cos^2\theta + k^2\cos^2 2\theta$

$= k^2\,[\,(2\sin\theta \,.\, \cos\theta)^2 + \cos^2 2\theta]$

$4a^2 + b^2 = k^2\,[\sin^2 2\theta + \cos^2 2\theta]$

$= k^2 \times 1 = k^2$

$\therefore\ 4a^2 + b^2 = k^2$

139. Let the coordinates of required point on y-axis be $P(0, y_1)$.

The perpendicular distance of the point P from the given line is given by

$$d = \left|\frac{4(0) - 3(y_1) - 12}{\sqrt{4^2 + 3^2}}\right| = 3$$

$$\left|\frac{-3y_1 - 12}{5}\right| = 3$$

$|-3y_1 - 12| = 15$

$\Rightarrow\ -3y_1 - 12 = 15$ or $-3y_1 - 12 = -15$

$\Rightarrow\ y_1 = -9$ or $y_1 = 1$

$\therefore$ The required points are $(0, -9), (0, 1)$

140. Let Δ ABC be isosceles triangle such that AB = AC

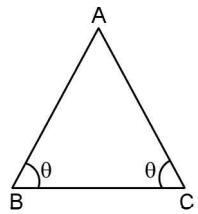
Let AB and BC are given by

$7x - y + 3 = 0$ — (1)

and $x + y - 3 = 0$ — (2)

Slope of (1) line = 7

Slope of (2) line = −1



Let m be the slope of BC, then

$\tan\angle ABC = \tan\theta$

$$\tan\theta = \frac{7-m}{1+7m}$$

Also $\tan\angle ACB = \tan\theta = \frac{m+1}{1-m}$

$\Rightarrow\ \frac{m+1}{1-m} = \frac{7-m}{1+7m} \Rightarrow 3m^2 + 8m - 3 = 0$

$\Rightarrow\ (m + 3)\ (3m - 1) = 0 \Rightarrow m = -3$ or $\frac{1}{3}$

The equation of third side passing through (1, −10) having slope −3 or $\frac{1}{3}$.

$y + 10 = -3\,(x - 1)$

$3x + y + 7 = 0$

or $y + 10 = \frac{1}{3}\,(x - 1)$

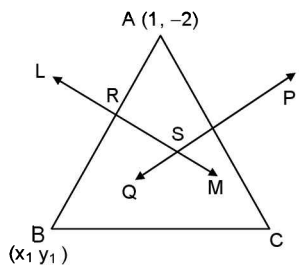
$3y - x + 31 = 0$

Had we taken $\tan\angle ABC = -\tan\theta$

We get $\frac{m+1}{1-m} = \frac{-(7-m)}{1+7m}$

$\Rightarrow m^2 + 1 = 0$ which does not give real values for m.

141. The equation of perpendicular bisector of AB i.e., LM is



$x - y + 5 = 0$ — (1)

Slope of LM = 1

$\Rightarrow$ Slope of AB = −1

Equation of line AB passing through A(1, –2) with a slope of –1.

$y - (-2) = -1\,(x - 1)$

$x + y + 1 = 0$ — (2)

Solving (1) and (2) we have coordinates of R (–3, 2)

Since, R is mid-point of AB we get B is (–7, 6)

Again the equation of PQ is $x + 2y = 0$ — (3)

Slope of PQ = $\frac{-1}{2}$ $\therefore$ slope of AC = 2

$\therefore$ Equation of line AC passing through A (1, –2) with a slope 2

$y + 2 = 2\,(x - 1)$

$y - 2x + 4 = 0$ — (4)

Solving (3) and (4) we have coordinates of S as $\left(\frac{8}{5}, \frac{-4}{5}\right)$

Since S mid point of AC, $\therefore$ C is $\left(\frac{11}{5}, \frac{2}{5}\right)$

$\therefore$ Equation of BC is, $y - 6 = \dfrac{6 - \frac{2}{5}}{-7 - \frac{11}{5}}(x + 7)$

On simplification

$y - 6 = \frac{28}{-46}(x + 7) \Rightarrow 14x + 23y - 40 = 0$

142. The equations of lines are

$3x + y - 2 = 0$ — (1)

$px + 2y - 3 = 0$ — (2)

$2x - y - 3 = 0$ — (3)

Solving the equation (1) and (3) we get

$x = 1, y = -1$

Since the lines are concurrent, the point (1, –1) must satisfy the equation (2).

$\therefore\ p - 2 - 3 = 0 \Rightarrow p = 5$

143. Let the lines intersect at A(0,y). Then we have $by^2 + 2fy + c = 0$ has equal roots

$\Rightarrow$ Discriminant = 0

$\Rightarrow\ 4f^2 - 4bc = 0 \Rightarrow f^2 = bc \Rightarrow$ b, f, c are in G.P.

144. The distance between the parallel lines

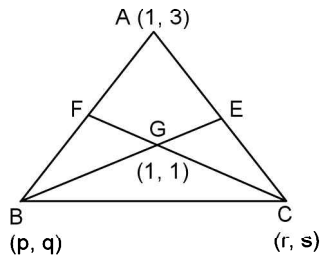
$x + 2y + 3 = 0$ and $x + 2y - 7 = 0$ is $\frac{3+7}{\sqrt{1+4}} = 2\sqrt{5}$

$\Rightarrow$ the length of each side of each square is $2\sqrt{5}$. Let $2x - y + k = 0$ represent the fourth side.

Then, $\frac{k+4}{\sqrt{4+1}} = \pm 2\sqrt{5} \Rightarrow k = -14, 6$

Hence the required equation is $2x - y - 14 = 0$ or $2x - y + 6 = 0$

145. Solving the two medians the centroid G is (1, 1)



Let the other two vertices be B(p, q) and C(r,s)

F lies on y = 1 and it is the midpoint of AB

$\therefore \frac{3+q}{2} = 1\ \therefore\ q = -1$

Again y coordinate of G is

$\Rightarrow\ \frac{3+q+s}{3} = 1\quad \therefore\ q + s = 0$

$\Rightarrow\ s = -q = 1$

E is the mid-point of AC

$\therefore \frac{3+s}{2} = \frac{3+1}{2} = 2 =$ y coordinate of E

which lies on BE $x - 2y + 1 = 0$ (given)

$\therefore\quad x = 3$ as $y = 2$

$\frac{1+r}{2} = 3$ or $r = 5$

$\therefore$ C is (5, 1) B is (p, –1)

$\therefore \frac{1+p+5}{3} = 1\quad \therefore\ p = -3$

B is (–3, –1)

Now the equations of sides of triangle obtained by two point formulae are

$x + 2y - 7 = 0\ x - 4y - 1 = 0$

$x - y + 2 = 0$

146. Let (α, β) be a point on the line x + y = 2 satisfying the condition,

then $\alpha + \beta = 2$.

Now perpendicular distance of (α, β) from

$2x + 3y = 5$ is

$$\left|\frac{2\alpha+3\beta-5}{\sqrt{13}}\right| \text{ which is given as 1}$$

$$\Rightarrow \quad \frac{2\alpha+3\beta-5}{\sqrt{13}}=\pm 1.$$

$$2\alpha + 3\beta = 5 \pm \sqrt{13} \qquad \text{— (1)}$$

$$\alpha + \beta = 2 \qquad \text{— (2)}$$

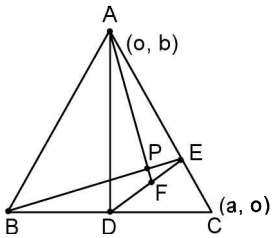
Solving (1) and (2)

$$\alpha = 1 \pm \sqrt{13}$$

$$\beta = 1 \pm \sqrt{13}$$

Points are $\left(1-\sqrt{13}, 1+\sqrt{13}\right)$, $\left(1+\sqrt{13}, 1-\sqrt{13}\right)$

147. Taking D as origin and BC as x – axis, coordinates of B and C are of the form (–a, o), (a, o) since ABC is an isosceles triangle, AD, the altitude through A is the y axis of the system.



Let A be (0, b).

Equation of AC is $\frac{x}{a}+\frac{y}{b}=1$ — (1)

Equation of DE is $y = \frac{a}{b}x$ — (2)

since DE is $\perp^r$ to AC

Solving (1) and (2)

$$E \text{ is } \left(\frac{ab^2}{a^2+b^2}, \frac{a^2b}{a^2+b^2}\right)$$

Since F is the midpoint of DE,

$$F \text{ is } \left(\frac{ab^2}{2\left(a^2+b^2\right)}; \frac{a^2b}{2\left(a^2+b^2\right)}\right)$$

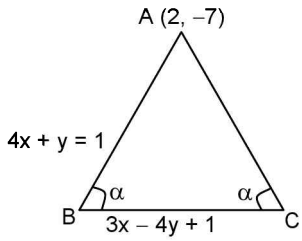
$$\text{Slope of BE} = \frac{\dfrac{a^2b}{a^2+b^2}-0}{\dfrac{ab^2}{a^2+b^2}+a}$$

$$= \frac{a^2b}{ab^2+a\left(a^2+b^2\right)} = \frac{ab}{\left(2b^2+a^2\right)}$$

$$\text{Slope of AF} = \frac{\dfrac{a^2b}{2\left(a^2+b^2\right)}-b}{\dfrac{ab^2}{2\left(a^2+b^2\right)}} = \frac{-\left(2b^2+a^2\right)}{ab}$$

$\Rightarrow$ Angle between AF and BE $= \frac{\pi}{2}$

148. We have



Slope of AB = –4 and slope of BC = $\frac{3}{4}$

If m be the slope of AC then

$$\therefore \frac{\frac{3}{4}-m}{1+\frac{3}{4}m} = \frac{\frac{3}{4}+4}{1-\frac{3}{4}.4} \Rightarrow \frac{4m-3}{3m+4} = \frac{19}{8}$$

$$19\,(3m + 4) = 8\,(4m - 3)$$

$\therefore \quad m = -4$

$\therefore$ Equation of required line is $y + 7 = -4(x - 2)$

$$4x + y + 1 = 0$$

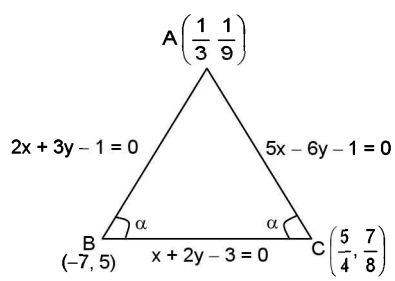
If, on the other hand, we take $\dfrac{\frac{3}{4}-m}{1+\frac{3}{4}m} = -\dfrac{19}{8}$ we get

$$m = -\frac{52}{89}$$

$\Rightarrow$ Equation of AC is $52x + 89y + 519 = 0$

149. The coordinates of the vertices are

$$A\left(\frac{1}{3}, \frac{1}{9}\right), \quad B\left(-7, 5\right), \quad C\left(\frac{5}{4}, \frac{7}{8}\right)$$



If (α, α^2) lies inside the triangle formed by given lines then $\left(\frac{1}{3}, \frac{1}{9}\right)$ and (α, α^2) lie on the same side of the line $x + 2y - 3 = 0$ — (1) Similarly $(-7, 5)$ and (α, α^2) lie on the same side of the line

$5x - 6y - 1 = 0$ — (2)

$\left(\frac{5}{4}, \frac{7}{8}\right)$ and (α, α^2)lie on the same side of the line

$2x + 3y - 1 = 0$ — (3)

In case of (1)

$$\alpha + 2\alpha^2 - 3 < 0 \Rightarrow \frac{-3}{2} < \alpha < 1 \quad \text{— (A)}$$

In case of (2)

$$5\alpha - 6\alpha^2 - 1 < 0 \Rightarrow \alpha > \frac{1}{2} \text{ or } \alpha < \frac{1}{3} \quad \text{— (B)}$$

In case of (3)

$$2\alpha + 3\alpha^2 - 1 > 0 \Rightarrow \text{either } \alpha > \frac{1}{3} \text{ or } \alpha < -1 \quad \text{— (C)}$$

From A, B, C we have either $\frac{-3}{2} < \alpha < -1$ or $\frac{1}{2} < \alpha < 1$

150. The homogenous part of the equation

$3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$ — (1)

can be factored as $(3x - y)(x + 3y)$

So let the lines represented by the given equation be

$3x - y + c_1 = 0$ and $x + 3y + c_2 = 0$

So given equation is same as

$(3x - y + c_1)(x + 3y + c_2) = 0$

$\Rightarrow (3x - y)(x + 3y) + c_1(x + 3y) + c_2(3x - y) + c_1c_2 = 0$

$\Rightarrow (3x - y)(x + 3y) + x(c_1 + 3c_2) + y(3c_1 - c_2) + c_1c_2 = 0$

Comparing the coefficients, we get

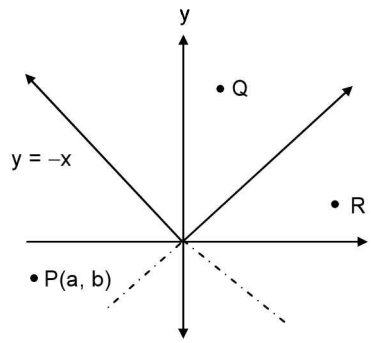
$c_1 + 3c_2 = 2;\ 3c_1 - c_2 = -4$ and $c_1c_2 = -1$

$\Rightarrow c_1 = -1, c_2 = 1.$

So equations of the lines are $3x - y - 1 = 0$ and $x + 3y + 1 = 0$.

151. The sides $2x + y - 4 = 0$ and $x - 2y + 3 = 0$ are perpendicular and so the triangle is right angled at the point of intersection (1, 2) of these two sides . Hence the orthocentre is (1, 2).

152. The image of P (a, b) on $y = -x$ is Q $(-b, -a)$ (Interchanging and change signs) and the image of Q $(-b, -a)$ on $y = x$ is R $(-a, -b)$ (merely interchange).



$\therefore$ The mid-point of PR is (0, 0)

153. Let ABCD be a square whose diagonals are AC and BD equation of AC is

$y = x + k$ — (1)

It passes through (4, 5)

$\therefore\ k = 1 \quad \therefore\ y = x + 1$

Equation of BD is $x + y + k_1 = 0$ — (2)

It passes through (4, 5)

$\therefore\ k_1 = -9$

$\therefore$ The equation of BD is $x + y = 9$.

Only (b) satisfies the equations of AC and BD.

154. The equation is $(2x - 5y + 10) + (2x + 5y - 10)a = 0$

Hence it represents a system of lines through the intersection of $2x - 5y + 10 = 0$ and $2x + 5y - 10 = 0$ i.e., through (0, 2)

155. The equation of the bisectors remain the same

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

$$\Rightarrow \frac{x^2 - y^2}{2} = \frac{xy}{-p}$$

$-px^2 - 2xy + py^2 = 0$

$px^2 + 2xy - py^2 = 0$

156. Given equation is $bx\cos\theta + ay\sin\theta = ab$

So, length perpendicular from $\left(\pm\sqrt{a^2 - b^2}, 0\right)$ is

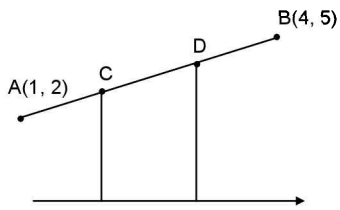
$$p_1 = \frac{\sqrt{a^2 - b^2}\ b\cos\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

Similarly, $p_2 = \dfrac{-\sqrt{a^2 - b^2}\ b\cos\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

$$P_1P_2 = -\left[\frac{(a^2-b^2)(b^2\cos^2\theta)-a^2b^2}{b^2\cos^2\theta+a^2\sin^2\theta}\right]$$

$$= \frac{b^2\left(a^2\sin^2\theta+b^2\cos^2\theta\right)}{b^2\cos^2\theta+a^2\sin^2\theta} = b^2$$

157. Let C and D be the points of trisection of the line joining A and B



$\Rightarrow$ C divides AB, internally in the ratio 1:2 and D in the ratio 2 : 1.

$\therefore$ The coordinates of C and D are respectively (2, 3) and (3, 4)

$\therefore$ Area of the trapezium CDEF $= \frac{1}{2}h(a+b)$

$= \frac{1}{2}\times 1(3+4) = \frac{7}{2}$

158. Let P (α, β) be the variable point

Its distance from the origin is OP $= \sqrt{\alpha^2+\beta^2}$

Its distance from the line $x + y = 1$ is $d = \left|\frac{\alpha+\beta-1}{\sqrt{2}}\right|$

Given OP = 2.d $\Rightarrow$ $OP^2 = 4d^2$

$\Rightarrow \alpha^2 + \beta^2 = 4\frac{(\alpha+\beta-1)^2}{2}$

$\Rightarrow \alpha^2 + \beta^2 = 2(\alpha^2 + \beta^2 + 1 - 2\alpha - 2\beta + 2\alpha\beta)$

$\Rightarrow \alpha^2 + \beta^2 + 4\alpha\beta - 4\alpha - 4\beta + 2 = 0$

$\therefore$ Locus of P is $x^2 + y^2 + 4xy - 4x - 4y + 2 = 0$

159. The equation of any line passing through point of intersection of the given lines is $3x + y - 8 + \lambda(x - y - 1) = 0$.

Since this line is perpendicular to one of the given lines, we have, $\frac{3+\lambda}{\lambda-1} = -1$ or $\frac{1}{3}$

$\therefore \lambda = -1$ or -5

The required line is $3x + y - 8 - 1(x - y - 1) = 0$

$\Rightarrow 2x + 2y - 7 = 0$

or $3x + y - 8 - 5(x - y - 1) = 0$

$\Rightarrow 2x - 6y + 3 = 0$

160. The given line is $y = x + 1$ or $y - x - 1 = 0$

Now $y - x - 1$ give values of opposite signs for the points (3, –1) and (–4, 2) So (3, –1) and (–4, 2) lie on opposite sides of the line $y - x - 1 = 0$

So we can have two triangles of area 12.5 sq. units taking vertices (3, –1), (–4, 2) and a point on $y = x + 1$.

161. Let the point P (x, y) divide line joining A and B in the ratio 1 : 2

Then, $x = \sqrt{4-t^2}$ $y = t$

$\Rightarrow x^2 + y^2 = 4$

162. The equation of the 4 sides are $x = 1$, $y = 3$, $x = 7$, $y = 1$

$\Rightarrow$ Figure formed is a rectangle of length 6 and breadth 2 units.

Its area = 6 x 2 = 12 sq. units

163. We must have

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & k_1 & k_2 \\ 0 & k_2 & -a^2 \end{vmatrix} = 0 \text{ and } 1 + k_1 = 0$$

$\Rightarrow -a^2k_1 - k_2^2 = 0$ and $k_1 = -1$

$\Rightarrow k_2 = \pm a, k_1 = -1$

164. The second degree equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represents a pair of lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Here $a = b = k + 1$, $c = k + 4$, $h = 1$, $g = f = 2$

$\therefore$ We have

$(k + 1)^2 (k + 4) + 8 - 4(k + 1) - (k + 4) = 0$

$\Rightarrow k^3 + 6k^2 = 0 \Rightarrow k = 0, -6$

165. Let L: $2x^2 - 21xy + 18y^2 + 15x + 45y - 75 = 0$

Let $x + 3y = 5$ — (1)

Meet the lines given by L at P and Q. The combined equation of OP and OQ is obtained by homogenising L with (1) as

$2x^2 - 21xy + 18y^2 + 15(x + 3y)\left(\frac{x+3y}{5}\right)$

$- 75\left(\frac{x+3y}{5}\right)^2 = 0$

i.e., $2x^2 - 21xy + 18y^2 = 0 \rightarrow$ (L')

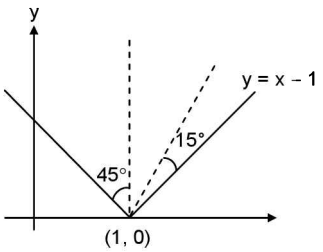
We observe that L and L' are parallel.

Also, lines represented by L (equivalently, by L') are not perpendicular.

$\Rightarrow$ L and L' do not form a rectangle or a square

$\Rightarrow$ The figure formed is a parallelogram

166. $x^2 - y^2 - 2x + 1 = 0$



$(x^2 - 2x + 1) - y^2 = 0$ or $(x - 1)^2 - y^2 = 0$

i.e., $(x - y - 1)(x + y - 1) = 0$ which is the combined equation.

After rotation the lines become $x = 1$ and $y = \sqrt{3}(x - 1)$

So the combined equation after rotation is $(x - 1)\left(\sqrt{3}x - y - \sqrt{3}\right) = 0$

$$\sqrt{3}x^2 - xy - 2\sqrt{3}x + \sqrt{3} + y = 0$$

167. $7x + 8y - 1 = 0$

$7 \times 1 + 8 \times 1 - 1 > 0$

$7 \times 2 - 24 - 1 < 0$

$\therefore$ A and B are on opposite sides of the line.

168. Let (h, k) be the centroid. Then,

$$h = \frac{\cos\theta + \sin\theta}{3}$$

$$k = \frac{\sin\theta - \cos\theta}{3}$$

But $k = 2h$

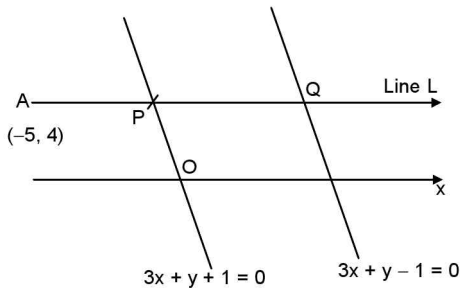
$$\Rightarrow \frac{\sin\theta - \cos\theta}{3} = \frac{2(\cos\theta + \sin\theta)}{3}$$

$\sin\theta - \cos\theta = 2\cos\theta + 2\sin\theta$

$\tan\theta = -3$

$\theta = \tan^{-1}(-3)$.

169. It is clear that the point (–5, 4) is not between the lines and that it is on the non origin side of the lines.



Let L be the required line. If its slope is $\tan\theta$, then equation of L may be written as

$$\frac{x+5}{\cos\theta} = \frac{y-4}{\sin\theta} = r$$

If AP = r, AQ = r + 2

$$\frac{x+5}{\cos\theta} = \frac{y-4}{\sin\theta} = r$$

P is $(r\cos\theta - 5, 4 + r\sin\theta)$

$$\frac{x+5}{\cos\theta} = \frac{y-4}{\sin\theta} = r + 2$$

Q is $\{(r + 2)\cos\theta - 5, (r + 2)\sin\theta + 4\}$

P lies on $3x + y + 1 = 0$

$3(r\cos\theta - 5) + 4 + r\sin\theta + 1 = 0$ — (1)

Q lies on $3x + y - 1 = 0$

$3\{(r + 2)\cos\theta - 5\} + (r + 2)\sin\theta + 4 - 1 = 0$ — (2)

(2) – (1) gives

$b\cos\theta + 2\sin\theta - 2 = 0$

$3\cos\theta + \sin\theta = 1$

$3\cos\theta = 1 - \sin\theta$

Squaring,

$9\cos^2\theta = (1 - \sin\theta)^2$

$= 1 + \sin^2\theta - 2\sin\theta$

$9(1 - \sin^2\theta) = 1 + \sin^2\theta - 2\sin\theta$

$\Rightarrow 10\sin^2\theta - 2\sin\theta - 8 = 0$

$\Rightarrow 5\sin^2\theta - \sin\theta - 4 = 0$

$\Rightarrow \sin\theta = 1, \frac{-4}{5}$

170. $ax - by + c = 0$

$$ax - \left(\frac{a+c}{2}\right)y + c = 0$$

$$a\left(x - \frac{1}{2}y\right) + c\left(1 - \frac{1}{2}y\right) = 0$$

This represents a family of concurrent lines passing through the intersection of $x - \frac{1}{2}y = 0$ and $1 - \frac{1}{2}y = 0$.

171. Statement 2 is true

Since ABC is a right angled triangle,

Circum centre is the midpoint of BC.

$$\text{Circum radius} = \frac{1}{2}BC = \frac{1}{2}\sqrt{34} = \sqrt{\frac{17}{2}}$$

$\Rightarrow$ Statement 1 is true

Choice (a)

172. Statement (2) is true

Consider Statement 2

Equation can be rewritten as

$$\frac{x-2}{3/5}=\frac{y+3}{4/5}=10$$

Gives x = 8, y = 5

⇒ Statement 1 is true

Choice (a)

173. Consider Statement 1

The equation can be rewritten as

$$a(3x+y-1)+b(x+4y+18)=0$$

For any real rules of a and b, the above equation represents a family of lives which passes through the intersection of the lines $3x + y - 1 = 0$ and $x + 4y + 18 = 0$

Statment.1 is true

Choice (a)

174. Statement 2 is not always true as it depends on the signs of C_1 and C_2.

Consider Statment.1 The lines make 60^0 and 30^0 with the x – a, is Hence the internal bisector makes an angle 45^0 with the x – a is Also, the lines intersect at the point $(\sqrt{3},-4)$ Equation of the internal bisector of the lines is $y + 4 = x - \sqrt{3}$ Statment1 is true

Choice (c).

175. Statement 2 is true.

Consider Statement.1 Using Statement.2

(1, –2) lies on the line ax + by + c = 0 if a – 2b + c = 0 which is true since a, b, c are given to be in A.P ⇒ S1 is true.

Choice (a).

176. Statement.2 is true.

Consider Statement.1

If (3, –1) and (5, 6) are the vertices A and B,

$\overline{AB} = (2 + 5i)$

If C is the third vertex,

$\overline{AC} = (2 + 5i)(\cos 60° + I \sin 60°)$

OR

$(2 + 5i)(\cos 60° - I \sin 60°)$

(There are two equilateral triangles possible)

$$\overline{AC} = (2+5i)\left(\frac{1}{2}\right)\pm\frac{i\sqrt{3}}{2}$$

$$=\left(1-\frac{5\sqrt{3}}{2}\right)+\left(\frac{5}{2}+\sqrt{3}\right)$$

$$=\left(1+\frac{5\sqrt{3}}{2}\right)+\frac{5}{2}-\sqrt{3}$$

Hence, the coordinates of the third vertex are

$$\left(1-\frac{5\sqrt{3}}{2}\right)+\left(\frac{5}{2}+\sqrt{3}\right)i+(3-i)$$

and

$$1+\frac{5\sqrt{3}}{2}+\left(\frac{5}{2}-\sqrt{3}\right)i+(3-i)$$

$$\Rightarrow\left(4-\frac{5\sqrt{3}}{2}\right)+\left(\frac{3}{2}+\sqrt{3}\right)i$$

$$\left(4+\frac{5\sqrt{3}}{2}\right)+\left(\frac{3}{2}-\sqrt{3}\right)i$$

Statement 1 is true

Choice (a)

177. Statement 2 is true.

Consider statement 1

Formula given in S_2 cannot be applied in this case.

The lines are

$2x + 3y - 6 = 0$

and $2x + 3y + \frac{8}{3} = 0$

$$\text{perpendicular distance} = \left|\frac{-6-\frac{8}{3}}{\sqrt{2^2+3^2}}\right| = \frac{26}{3\sqrt{13}}$$

Statement 1 is false

Choice (d)

178. Statement 2 is true

Consider Statement 1

Slope of $ax + by + c = 0$ is $\frac{-a}{b}$

Slope of $cx + by + a = 0$ is $\frac{-c}{b}$

If the lines are perpendicular,

$\frac{ac}{b^2}=-1 \Rightarrow b^2=-ac$

Statement 1 is false.

Choice (d)

179. Statement 2 is true.

Consider Statement1.

Choosing A B and AC is the of coordinates, the co-ordinates of the vertices of the Δ ABC are(0, 0), (4, 0) and (0, 3) centroid is at $\left(\frac{4}{3},1\right)$.

If G denotes the centroid,

$AG^2 = \left(\frac{4}{3}\right)^2 + 1^2$ (Since A is the origin)

$= \frac{5^2}{3^2} \Rightarrow AG = \frac{5}{3}$

Statement 1 is true.

Since we can get the coordinates of G by using statement 2 also,

Choice is (a)

180. Statement 2 is false.

Consider statement 1

On factoring, the two lines are

$x + y - 4 = 0$ and $2x - 3y + 7 = 0$

The point of intersection of the lines is at (1, 3).

The lines perpendicular to the above lines are of the form $y - x + k = 0$ and $3x + 2y + k_2 = 0$ where k_1 and k_2 are obtained by using the fact that the lines pass through (1, 3)

We get $k_1 = -2$ and $k_2 = -9$

The lines are $y - x + 2 = 0$ and

$3x + 2y - 9 = 0$

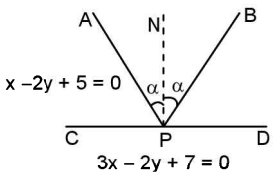
Their combined equation is

$3x^2 - 2y^2 + xy - 3x + 13y - 18 = 0$

$\Rightarrow$ statement 1 is true

Choice (c)

181. Equation of the incident ray is $x - 2y - 5 = 0$



Equation of CD is $3x - 2y + 7 = 0$.

Solving the above two equations we get P as (–1, 2)

Equation to PN is $y - 2 = \frac{-2}{3}(x + 1)$

$\Rightarrow 2x + 3y - 4 = 0$

B is the image of A in the line PN.

If B is (x, y), we have,

$$\frac{x-0}{2} = \frac{y - 5/2}{3} = \frac{-2\left(2\times 0 + 3\times 5/2 - 4\right)}{13} = \frac{-7}{13}$$

$x = \frac{-14}{13}, y = \frac{5}{2} - \frac{21}{13} = \frac{23}{26}$

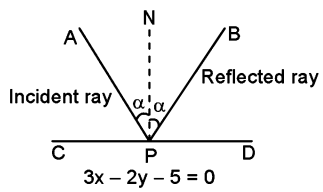
Slope of the reflected ray $= \frac{23/26 - 2}{-14/13 + 1} = \frac{29}{2}$

Equation of the reflected ray is

$y - 2 = \frac{29}{2}(x + 1) \Rightarrow 29x - 2y + 33 = 0$

Clearly, (0, 16.5) lies on the reflected ray.

182.



Equation of the incident ray is

$$y + \frac{1}{2} = \frac{1}{2}(x - 2) \Rightarrow x - 2y - 3 = 0$$

Equation of CD is $3x - 2y - 5 = 0$.

Solving the above two equations we get P as (1, –1)

PN is perpendicular to $3x - 2y - 5 = 0$

$\therefore$ Equation to PN is $y + 1 = \frac{-2}{3}(x - 1)$

$\Rightarrow 2x + 3y + 1 = 0$

B is the image of A in the line PN.

If B is (x, y), we have,

$$\frac{x-2}{2} = \frac{y + 1/2}{3} = \frac{-2\left(2\times 2 + 3\times -1/2 - 1\right)}{13} = \frac{-7}{13}$$

$x = \frac{12}{13}, y = \frac{-55}{26}$

Slope of the reflected ray $= \frac{-55/26 + 1}{12/13 - 1} = \frac{29}{2}$

Equation of the reflected ray is

$y + 1 = \frac{29}{2}(x - 1) \Rightarrow 29x - 2y - 31 = 0$

183. The incident ray is perpendicular to the given line

184. $d((0, 0), P) = d((2, 3), P)$

$\Rightarrow |x| + |y| = |x - 2| + |y - 3)|$

we have, $0 < x < 2, 0 < y < 3$.

$x + y = 2 - x + 3 - y$.

$2x + 2y = 5$

185. $|x-2| + |y| = |x-4| + |y|$

$x < 2 \Rightarrow 2 - x = 4 - x \Rightarrow$ no solution

$2 < x < 4 \Rightarrow x - 2 = 4 - x \Rightarrow x = 3$.

$x > 4 \Rightarrow$ no solution.

186. $d(A, B) = |3 + 2| + |4 + 3| = 12$.

$\Rightarrow |x - 2| = |x - 4|$

$d(B, C) = |-2 - 6| + |-3 + 5| = 8 + 2 = 10$.

$d(C, A) = |6 - 3| + |-5 - 4| = 12$.

$\therefore$ an isosceles triangle.

187. If $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are the mid-points of sides of $A_3B_3C_3$ triangle then the vertices $A_3B_3C_3$ are

$(x_2 + x_3 - x_1, y_2 + y_3 - y_1)$

$(x_3 + x_1 - x_2, y_3 + y_1 - y_2)$ and

$(x_1 + x_2 - x_3, y_1 + y_2 - y_3)$

$\therefore A_2(1, 1)\ B_2(-1, 1)\ C_2(1, -1)$

$\Rightarrow A_1(-1, -1)\ B_1(3, -1)\ C_1(-1, 3)$

188. Area of $A_3B_3C_3 = \frac{1}{2}.1 \times 1 = \frac{1}{2}$ sq.units

Area of $A_4B_4C_4 = \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8}$ sq. units

Area of $A_5B_5C_5 = \frac{1}{4}\left(\frac{1}{8}\right) = \frac{1}{32}$ sq. units

189. $\Delta_3 = \frac{1}{2}$

$\Rightarrow \Delta_2 = 4 \times \frac{1}{2} = 2, \Delta_1 = 4 \times 2 = 8$

$$\sum_{i=1}^{\infty} \Delta_i = 8 + 8.\frac{1}{4} + 8.\left(\frac{1}{4}\right)^2 + \ldots$$

$$= \frac{8}{1 - \frac{1}{4}} = \frac{32}{3} \text{ sq. units}$$

190. (a) $5y = 4 - 4x \Rightarrow y = \frac{4}{5}(1 - x)$

$\therefore \frac{x^2}{25} + \frac{(1-x)^2}{25} = 1 \Rightarrow x^2 - x - 12 = 0$

$\Rightarrow x = -3, 4$

$\therefore y = \frac{4}{5}(4) = \frac{16}{5}$ and $\frac{-12}{5}$

$\therefore$ end point of the chord is $P\left(-3, \frac{16}{5}\right)$ and $Q\left(4, -\frac{12}{5}\right)$

$\therefore$ mid-point is $\left(\frac{1}{2}, \frac{2}{5}\right)$

(b) length of the chord is

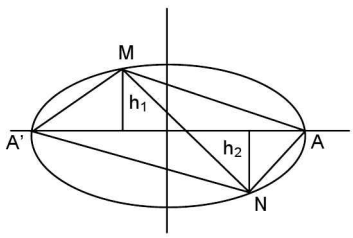
$$= \sqrt{(-3-4)^2 + \left(\frac{16}{5} - \frac{-12}{5}\right)^2} = \sqrt{7^2 + \frac{28^2}{5^2}} = \frac{7}{5}\sqrt{41}$$

(c) If m_1 = slope of OP and m_2 = slope of OQ,

$m_1 = -\frac{16}{15}, m_2 = \frac{-3}{5}$

$$\therefore \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-\frac{3}{5} + \frac{16}{15}}{1 + \frac{16}{15}.\frac{3}{5}} = \frac{7}{123}$$

$$\theta = \tan^{-1}\left(\frac{7}{123}\right)$$



(d)

Area A'PAQ = A'PA + AQA'

$= \frac{1}{2}A'A.h_1 + \frac{1}{2}A'A.h_2$

$= \frac{1}{2}.10 \times \frac{16}{5} + \frac{1}{2}.10 \times \frac{12}{5}$

$= 16 + 12 = 28$

191. Lines are

$x - y + 2 = 0$ — (1) $\Rightarrow$ side BC

$3x + 2y - 2 = 0$ — (2) $\Rightarrow$ side CA

$2x + y + 1 = 0$ — (3) $\Rightarrow$ side AB

Solving the equations we get A is (–4, 7) B as (–1, 1) and C as $\left(\frac{-2}{5}, \frac{8}{5}\right)$

If P (α, 2) is to lie inside the triangle ABC, both A and P must lie on the same side of BC, both B and P must lie on the same side of AB

$x - y + 2 < 0$ for $x = -4, y = 7$.

$\Rightarrow \alpha - 2 + 2 < 0$

$\Rightarrow \alpha < 0$ — (4)

Again, $3x + 2y - 2 < 0$ for $x = -1, y = 1$

$\Rightarrow 3\alpha + 4 - 2 < 0$

$\Rightarrow \alpha < \frac{-2}{3}$ — (5)

Lastly, $2x + y + 1 > 0$ for $x = \frac{-2}{5}, y = \frac{8}{5}$

$\Rightarrow \alpha > \frac{-3}{2}$ — (6)

From (4), (5) and (6) we get

$$\frac{-3}{2} < \alpha < \frac{-2}{3}$$

Note that all the vertices are in the third quadrant. Hence, if $\alpha > 0$, the point (α, 2) lies in the first quadrant.

192. The given equation is $(2x + 3y)(5x - y) = 0$

$\Rightarrow \frac{y}{x} = \frac{-2}{3}$ or $\frac{y}{x} = 5$

$ax^2 + 2hxy + by^2 = 0$

$\Rightarrow a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2 = 0$

$\Rightarrow a + 2h\frac{-2}{3} + b\left(\frac{4}{9}\right) = 0$ or $a + 2h.5 + b.25 = 0$

$\Rightarrow 9a - 12h + 4b = 0$ or $a + 10h + 25b = 0$

193. The lines are $\frac{x}{a} + \frac{y}{b} = 1$

$\frac{x}{b} + \frac{y}{a} = 1$

Solving, point of intersection is $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$

This point satisfies the equations in (a), (b), (c)

194. Lines are concurrent.

$$\begin{vmatrix} 1 & 1 & -1 \\ \lambda - 2 & 3\lambda & \lambda + 4 \\ 2\lambda - 5 & \lambda + 1 & 2\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & -1 \\ (2\lambda + 2) & (4\lambda + 4) & (\lambda + 4) \\ (4\lambda - 5) & (3\lambda + 1) & 2\lambda \end{vmatrix} = 0$$

$\Rightarrow 2(\lambda + 1)(-5\lambda + 11) = 0$

$\lambda = -1, \frac{11}{5}$

195. The point M is (1,2)

Any line through M having slop is

$Y - 2 = m(x - 1)$

$\Rightarrow mx - y + (2 - m) = 0$ (1)

Put $y = 0$ in (1) $\Rightarrow$ A is $\left(\frac{m-2}{m}, 0\right)$

put $x = 0$ in (1) $\Rightarrow$ B is $(0, 2 - m)$

(a) Let P denote the mid-point of AB.

Then, x coordinate of

$P = x = \frac{m-2}{2m} = \frac{1}{2} - \frac{1}{m}$

Y coordinate of $P = y = \frac{2-m}{2}$

$= 1 - \frac{m}{2}$

$\frac{m}{2} = 1 - y$

$\Rightarrow x = \frac{1}{2} - \frac{1}{2(1-y)}$

$\Rightarrow 2x = 1 - \frac{1}{1-y} = \frac{-y}{1-y}$

Locus of P is the curve

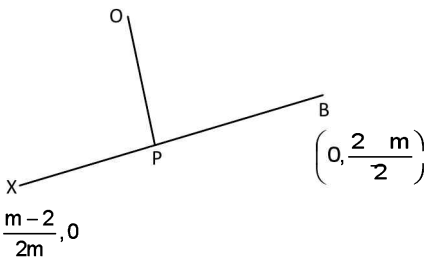
$X - xy + y = 0$ or

$Y = \frac{x}{x-1}$.

(b)

Let P be the foot of the perpendicular from O.

Slope of OP = $\frac{-1}{m}$.



Slove $y = \frac{-1}{m}x$ and — (1)

$mx + \frac{1}{m}x = (m-2)$`

$$\left.\begin{aligned} x &= \frac{m(m-2)}{(m^2+1)} \\ \Rightarrow \quad y &= \frac{-(m-2)}{(m^2+1)} \end{aligned}\right\}$$

Coordinates of P are $\left(\frac{m(m-2)}{m^2+1}, \frac{-(m-2)}{m^2+1}\right)$

$M = \frac{-x}{y}$

Locus of P is the curve

$$y = \frac{2+\frac{x}{y}}{1+\left(\frac{x}{y}\right)^2} = \frac{(2y+x)y}{(x^2+y^2)}$$

$x^2 + y^2 - x - 2y = 0$

(c) Centroid of Δ OAB is the point (x, y) where,

$$x = \frac{m-2}{3m}, \; y = \frac{2-m}{3}$$

$$3mx = m - 2$$

$$3y = 2 - m$$

$$3mx = -3y$$

$$m = -\frac{y}{x}$$

Locus is the curve

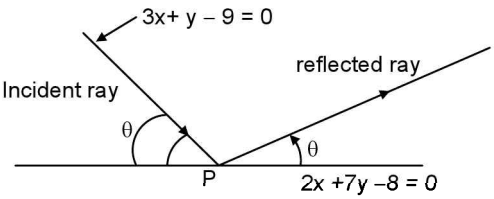
$$X = \frac{\frac{-y}{x} - 2}{\frac{-3y}{x}} = \frac{2x+y}{3y}$$

$$3xy = 2x + y$$

(d) circumcentre is the mid point of AB.

Hence, the locus is the same as in (a)

196. Reflected ray is a line which passes through the intersection of the lines $3x + y - 8 = 0$ and



$2x + 7y - 8 = 0$

(a) is true.

$3x + y - 9 + \lambda(2x + 7y - 8) = 0$

$\Rightarrow \quad (2\lambda + 3)x + (7\lambda + 1)y - (8\lambda + 9) = 0 \rightarrow (1)$

Slope of the line (1) is $\frac{-(2\lambda+3)}{(7\lambda+1)}$.

Slope of $3x + y - 9 = 0$ is $\rightarrow 3$

Slope of $2x + 7y - 8 = 0$ is 0 $\rightarrow \frac{-2}{7}$

If θ is the angle between the above lines

$$\tan\theta = \left|\frac{-3+\frac{2}{7}}{1+\frac{6}{7}}\right| = \left|\frac{-19}{13}\right|$$

$$= \frac{19}{13}$$

For the reflected ray,

$$\frac{-\left(\frac{2\lambda+3}{7\lambda+1)}\right)+\frac{2}{7}}{1+\frac{2(2\lambda+3)}{7(7\lambda+1)}} = \pm\frac{19}{13}$$

$$\Rightarrow \quad \lambda = 0, \frac{-26}{53}.$$

Hence the reflected ray corresponds to $\frac{-26}{53}$

Equation of the reflected ray is

$107x - 129y - 269 = 0$

Solving $3x + y - 9 = 0$ and $2x + 7y - 8 = 0$

We get the point P as $\left(\frac{55}{19}, \frac{6}{19}\right)$

197. If is clear that one of the lines represented by

$2x^2 + 3y^2 - 5xy - 5x + 5y = 0$

Factoring the expression,

$2x^2 - 3xy - 2xy + 3y^2 - 5(x - y) = 0$

$(2x - 3y)(x - y) - 5(x - y) = 0$

$(x - y)(2x - 3y - 5) = 0$

The lines are $x - y = 0$

$2x - 3y - 5 = 0$

The point of intersection of these lines is (–5, –5).

We can find A an B by solving the pairs

$\left.\begin{array}{l} X - y = 0 \\ X + y = 5 \end{array}\right\}$ and $\left.\begin{array}{r} 2x - 3y - 5 = 0 \\ x + y = 5 \end{array}\right\}$

A is $\left(\frac{5}{2}, \frac{5}{2}\right)$ and B is (4, 1)

Slope 0A = 1, Slope of 0B = $\frac{1}{4}$

$$\tan\theta = \frac{1 - \frac{1}{4}}{1 + \frac{1}{4}} = \frac{3}{5}$$

$$AB^2 = (4 - \frac{5}{2})^2 + \left(1 - \frac{5}{2}\right)^2$$

$$= \frac{9}{4} + \frac{9}{4} = \frac{9}{2}$$

$$AB = \frac{3}{\sqrt{2}}$$

If α is the angle between the given pair of lines

$$\tan\alpha = \frac{2\sqrt{\frac{25}{4} - 6}}{(2+3)}$$

$$= 2 \times \frac{1}{2 \times 5} = \frac{1}{5}$$

198. (a) The equation of the line joining the origin and the point of intersection of the line $y = mx + 2$ and the curve

$x^2 + y^2 = 1$ is $x^2 + y^2 = \left(\frac{y - mx}{2}\right)^2$

$\Rightarrow$ $x^2(4 - m^2) + 2mxy + 3y^2 = 0$

Since these lines are at right angles,

$\therefore$ $4 - m^2 + 3 = 0$

$\Rightarrow$ $m^2 = 7$ $\therefore m^4 = 49$

(b) Equation of the bisectors of the angle between the lines $3x^2 + 2pxy - 3y^2 = 0$ is

$$\frac{x^2 - y^2}{3+3} = \frac{xy}{-p} \Rightarrow x^2 + \frac{6}{p}xy - y^2 = 0$$

If this represents the lines

$5x^2 - 2qxy - 5y^2 = 0$

i.e., $x^2 - \frac{2}{5}qxy - y^2 = 0$, then

$$\frac{6}{p} = \frac{-2q}{5} \Rightarrow pq = -15$$

(c) Let the lines represented by

$ax^3 + bx^2y + cxy^2 + dy^3 = 0$ are

$y = m_1x$, $y = m_2x$ and $y = m_3x$. where,

$m_1m_2 = -1$

The given equation can be written as

$$d\left(\frac{y}{x}\right)^3 + c\left(\frac{y}{x}\right)^2 + b\left(\frac{y}{x}\right) + a = 0$$

$\Rightarrow$ $m_1m_2m_3 = \frac{-a}{d}$

$\Rightarrow$ $m_3 = \frac{a}{d}$

m_3 is a root of the above equation

$$\therefore \quad d\left(\frac{a}{d}\right)^3 + c\left(\frac{a}{d}\right)^2 + b\left(\frac{a}{d}\right) + a = 0$$

$$\Rightarrow \quad \frac{a^3}{d^2} + \frac{a^2c}{d^2} + \frac{ab}{d} + a = 0$$

$\Rightarrow$ $a^2 + ac + bd + d = 0$

(d) Let the slopes of the lines be m and 3m

Then $m + 3m = -8$

$\Rightarrow$ $m = -2$

$m.3m = a \Rightarrow a = 12$

199. (a) Let the equation of the variable line be

$ax + by + c = 0$

Given that $\frac{2a + c}{\sqrt{a^2 + b^2}} + \frac{2b + c}{\sqrt{a^2 + b^2}} + \frac{-2a - 2b + c}{\sqrt{a^2 + b^2}} = 0$

$\Rightarrow$ $c = 0$

$\Rightarrow$ the line passes through the origin for all a and b

(b) The given lines are $x = 0$, $y = 0$ and $x + y = 4$. Clearly they form a right triangle, right angled at the origin.

(c) $4a^2 + 9b^2 - c^2 + 12ab = 0$

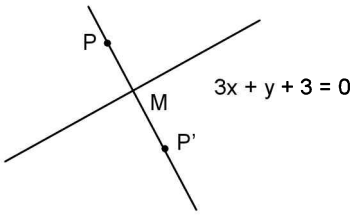
$\Rightarrow$ $(2a + 3b)^2 = c^2$

$\Rightarrow$ $2a + 3b = \pm c$

$\therefore$ either $2a + 3b + c = 0$ or $2a + 3b - c = 0$

The given family of straight lines is

$ax + by + c = 0$ or $-ax - by - c = 0$. They are concurrent at (2, 3) or (–2, –3)



(d) Let P' be the reflection of P(1, 3) on the line

$3x + y + 3 = 0$

Then the equation of PP' is $x - 3y + k = 0$

This passes through (1, 3)

$\Rightarrow$ $1 - 9 + k = 0$

$\Rightarrow$ $k = 8$

The point of intersection of $3x + y + 3 = 0$ and $x - 3y + 8 = 0$ is $M\left(\frac{-17}{10}, \frac{21}{10}\right)$

M is the mid-point of the line segment PP'.

If P' is (h, k) then

$$\frac{1+h}{2} = \frac{-17}{10} \text{ and } \frac{3+k}{2} = \frac{21}{10}$$

$$\Rightarrow \quad h = \frac{-22}{5}, k = \frac{6}{5}$$

200. (a) length of the altitude $p = \frac{|x_1 + y_1 - 2|}{\sqrt{2}}$

$$= \frac{1}{\sqrt{2}}$$

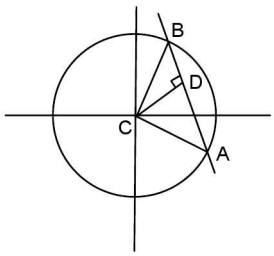
$$\therefore \quad P = a \sin 60 \Rightarrow \frac{1}{\sqrt{2}} = \frac{a\sqrt{3}}{2}$$

$$\Rightarrow \quad a = \sqrt{\frac{2}{3}}$$

$$\therefore \quad \text{Area} = \frac{\sqrt{3}}{4} \times a^2$$

$$= \frac{\sqrt{3}}{4} \times \frac{2}{3} = \frac{1}{2\sqrt{3}}$$

(b) Let C is the centre of the circle $x^2 + y^2 = 10$ its radius $AC = \sqrt{10}$.



line $\sqrt{5}x + 2y = 3\sqrt{5}$ intersect at A and B.

$CD \perp AB$ and D will be the midpoint of AB.

$$\therefore \quad CD = \frac{|-3\sqrt{5}|}{\sqrt{5+4}} = \sqrt{5}$$

Also $AD = \sqrt{AC^2 - CD^2} = \sqrt{10-5} = \sqrt{5}$

$\therefore$ $AB = 2AD = 2\sqrt{5}$

$$\therefore \quad \text{Area} = \frac{1}{2} AB \times CD = \frac{1}{2} \times 2\sqrt{5} \times \sqrt{5}$$

$= 5$

(c) $xy + 2x + 2y + 4 = 0 \Rightarrow (x + 2)(y + 2) = 0$

$\therefore$ Required area $= \frac{1}{2} \times 2 \times 2 = 2$

(d) $$\text{Area} = \left| \frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2} \right| = \frac{4\sqrt{\frac{25}{4} - 6}}{2 + 5.0 + 3.0} = 1$$

CHAPTER

2

CIRCLES AND CONIC SECTIONS

CHAPTER OUTLINE

Preview

STUDY MATERIAL

Circles and Conic Sections

Circles
- Concept Strands (1-8)

Equation of the Tangent to a Circle at a Point on it
- Concept Strands (9-12)

Common Tangents to Two Circles
- Concept Strands (13-21)

Orthogonal Circles
- Concept Strands (22-25)

Conic Sections
- Concept Strands (26-36)

Parabola
- Concept Strand (37)

Ellipse
- Concept Strands (38-40)

Hyperbola
- Concept Strands (41-44)

Conjugate Hyperbola

Rectangular Hyperbola

CONCEPT CONNECTORS
- 40 Connectors

TOPIC GRIP
- Subjective Questions (10)
- Straight Objective Type Questions (5)
- Assertion–Reason Type Questions (5)
- Linked Comprehension Type Questions (6)
- Multiple Correct Objective Type Questions (3)
- Matrix-Match Type Question (1)

IIT ASSIGNMENT EXERCISE
- Straight Objective Type Questions (80)
- Assertion–Reason Type Questions (3)
- Linked Comprehension Type Questions (3)
- Multiple Correct Objective Type Questions (3)
- Matrix-Match Type Question (1)

ADDITIONAL PRACTICE EXERCISE
- Subjective Questions (10)
- Straight Objective Type Questions (40)
- Assertion–Reason Type Questions (10)
- Linked Comprehension Type Questions (9)
- Multiple Correct Objective Type Questions (8)
- Matrix-Match Type Questions (3)

CIRCLES AND CONIC SECTIONS

The curves: circle, parabola, ellipse and hyperbola are obtained as sections of a right circular cone by planes. The curves parabola, ellipse and hyperbola are called conics (or conic sections).

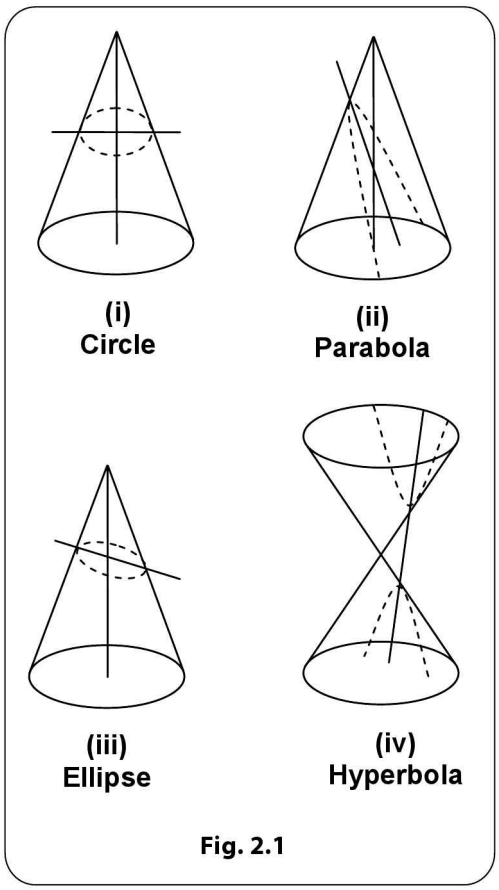


Fig. 2.1

The section of a right circular cone by a plane perpendicular to the axis of the cone is a circle (refer Fig. 2.1 (i));

The section of a right circular cone by a plane parallel to the slant height of the cone but not passing through its vertex is a parabola (refer Fig. 2.1 (ii));

The section of a right circular cone by a plane not perpendicular to the axis of the cone and not parallel to the slant height of the cone is an ellipse (refer Fig. 2.1 (iii)).

If one considers a right circular cone as extending on both sides (refer Fig. 2.1 (iv)), then the section of such a right circular cone by a plane parallel to the axis of the cone but not passing through its vertex is a hyperbola.

We have already introduced these curves in the unit on functions and graphs. These curves find applications in many problems in engineering and technology. The orbits of the planets (including earth) around the sun are ellipses with sun at one of the foci. The orbits of comets are parabolas.

In this sequel, we study some important properties of these curves using analytical methods. For this purpose, these curves have to be looked upon as the loci of points, which move in a plane satisfying some specified conditions. In this way, we will be able to represent these curves by their equations involving the x and y coordinates of a point on these curves. In what follows, the derivations of the equations of these loci will show that they have a general structure, i.e., the equations of the circle, parabola, ellipse or hyperbola are all second degree equations in x and y with some conditions on the coefficients.

It may be recalled that the equation of a pair of lines is also a second degree equation in x and y. This finding is not surprising because the pair of lines can also be obtained as the section of a right circular cone by a plane passing through the vertex of the cone.

CIRCLES

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point in the plane is always a given constant.

The fixed point is called the centre of the circle and the given constant is called the radius of the circle.

Referred to a rectangular Cartesian coordinate system, let (h, k) represent the coordinates of the fixed point C. (Refer Fig. 2.2). Let the constant distance be denoted by r. If P(x, y) is any point on the locus, we have CP = r or

$$CP^2 = r^2$$

i.e., $(x - h)^2 + (y - k)^2 = r^2$

Thus, the equation of the circle whose centre is at (h, k) and whose radius is r is given by

$$(x - h)^2 + (y - k)^2 = r^2$$

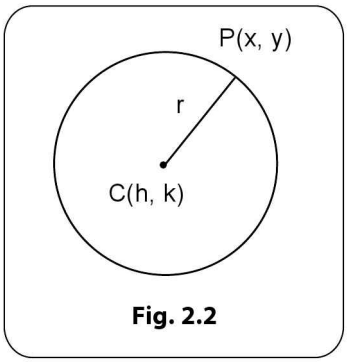


Fig. 2.2

Remarks

(i) Equation of the circle centered at the origin and whose radius is r is given by
$(x - 0)^2 + (y - 0)^2 = r^2$
or $x^2 + y^2 = r^2$
[We note that $x = r \cos \theta$, $y = r \sin \theta$, $0 \le \theta < 2\pi$ satisfies the equation $x^2 + y^2 = r^2$. In other words, any point on the circle $x^2 + y^2 = r^2$ can be represented as $(r \cos \theta, r \sin \theta)$. Or, the parametric form of representation of the circle $x^2 + y^2 = r^2$ is $x = r \cos \theta$., $y = r \sin \theta$, $0 \le \theta < 2\pi$ where θ is the parameter]

(ii) Equation of the circle centered at the origin and whose radius is 1 (i.e., radius is unity) is given by
$x^2 + y^2 = 1$
The above circle is called the unit circle.

(iii) Equation of the circle whose centre is at (h, k) and whose radius is r can also be represented using complex numbers.
Let $Z = x + iy$ and $Z_0 = h + ik$
Since the distance from (h, k) to (x, y) is always the constant r, we have $|Z - Z_0| = r$.
Or, $|Z - Z_0| = r$ represents the equation of the circle with centre at Z_0 and whose radius is r.
If $Z_0 = 0$, the equation of the circle is $|Z| = r$
Equation of the unit circle is $|Z| = 1$.

For example, the equation of the circle whose centre is at (1, –2) and whose radius is 4, is given by $(x - 1)^2 + (y + 2)^2 = 16$ or $x^2 + y^2 - 2x + 4y - 11 = 0$

Similarly, the equation of the circle whose centre is at $\left(\frac{1}{2}, \frac{-3}{2}\right)$ and whose radius is 3, is $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = 9$
$\Rightarrow\ 4(x^2 + y^2) - 4x + 12y - 26 = 0$

Result 1

The general equation of a circle is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$.

For, the equation may be rewritten as $(x - (-g)^2) + (y - (-f))^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$

Therefore, $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle whose centre is at (–g, –f) and whose radius is $\sqrt{g^2 + f^2 - c}$.

The above equation is a second degree equation in x and y.

Recall that the general equation of second degree in x and y is

$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$ — (1)

and (1) represents a pair of straight lines if $ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0$

Now, from the above discussions, we draw the inference that (1) represents a circle if A = B and H = 0. In other words, the general equation of second degree in x and y represents a circle if coefficient of x^2 = coefficient of y^2 and coefficient of xy = 0.

$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$ is the general equation of second degree in x and y. This equation represents a circle if

Coefficient of x^2 = Coefficient of y^2 and Coefficient of xy = 0

Result 2

(i) Observe that there are three independent constants g, f, c in the general equation of a circle
$x^2 + y^2 + 2gx + 2fy + c = 0$.
Hence, there exists one and only one circle satisfying three independent geometrical conditions. For example, there exists only one circle passing through three non-collinear points (which is the circum circle of the triangle formed by the three points)

(ii) If the circle passes through the origin, (0, 0) must satisfy the equation $x^2 + y^2 + 2gx + 2fy + c = 0$. Consequently, c must be zero. Or, the equation of a circle passing through the origin can be assumed as $x^2 + y^2 + 2gx + 2fy = 0$

(iii) Again, equation of a circle whose centre lies on the x-axis will be of the form $x^2 + y^2 + 2gx + c = 0$ and the equation of a circle whose centre lies on the y-axis will be of the form $x^2 + y^2 + 2fy + c = 0$

Result 3

Equation of the circle on the line segment joining the points (x_1, y_1) and (x_2, y_2) as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Let A and B be the points (x_1, y_1) and (x_2, y_2). Since AB is a diameter of the circle, $\angle APB = 90°$ where P is any point on the circle. (refer Fig. 2.3)

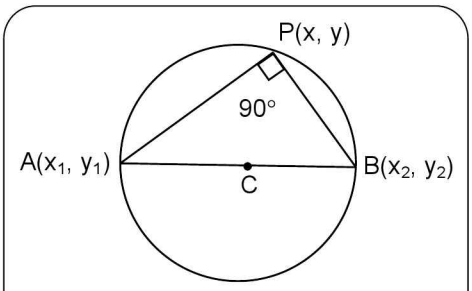


Fig. 2.3

Therefore, for any point P(x, y) on the circle, slope of AP × slope of BP = –1

i.e., $\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right)=-1$.

or $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$, which means that the equation of the circle on AB as diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

Before we take up the next section, let us work out a few examples to illustrate the concepts already developed.

CONCEPT STRANDS

Concept Strand 1

Obtain the coordinates of the centre of the circle $x^2+y^2-4x+7y-9=0$. Also find its radius.

Solution

We have $g=-2,\ f=\frac{7}{2},\ c=-9$

Centre of the circle is at $\left(2,\frac{-7}{2}\right)$ and its radius = $\sqrt{4+\frac{49}{4}+9}=\frac{\sqrt{101}}{2}$

Concept Strand 2

Obtain the coordinates of the centre of the circle $3x^2+3y^2+6x-8y+1=0$. Also find its radius.

Solution

The equation of the circle may be rewritten as

$$x^2+y^2+2x-\frac{8}{3}y+\frac{1}{3}=0$$

We have $g=1,\ f=-\frac{4}{3}$ and $c=\frac{1}{3}$

Centre of the circle is at $\left(-1,\frac{4}{3}\right)$ and radius

$$=\sqrt{1+\frac{16}{9}-\frac{1}{3}}=\frac{\sqrt{22}}{3}$$

Concept Strand 3

Find the equation of the circle on the line segment joining the points (1, –3) and (2, 5) as diameter.

Solution

Equation of the circle is $(x-1)(x-2)+(y+3)(y-5)=0$ or $x^2+y^2-3x-2y-13=0$

Concept Strand 4

Find the equation of the circle whose centre is at the point (5, 2) and which passes through the point (1, –1).

Solution

Since the circle passes through (1, –1), its radius is $\sqrt{(5-1)^2+(2+1)^2}=5$. The equation of the circle is therefore, $(x-5)^2+(y-2)^2=25$

$\Rightarrow\ x^2+y^2-10x-4y+4=0$

Concept Strand 5

Find the equation of the circle passing through the points (1, 2), (2, –1) and (3, 5).

Solution

Let the equation of the circle be $x^2+y^2+2gx+2fx+c=0$

Since it passes through (1, 2), $1+4+2g+4f+c=0$ or $2g+4f+c=-5$ — (1)

Similarly, using the fact that the circle passes through the points (2, −1) and (3, 5), we get the equations

$4g - 2f + c = -5$ — (2)

$6g + 10f + c = -34$ — (3)

Solving (1), (2) and (3) we obtain the values of g, f and c as $\frac{-29}{6}, \frac{-29}{18}$ and $\frac{100}{9}$ respectively.

Thus, the equation of the circle is

$$x^2 + y^2 - \frac{58}{6}x - \frac{58}{18}y + \frac{100}{9} = 0$$

or $9(x^2 + y^2) - 87x - 29y + 100 = 0$

Concept Strand 6

Find the equation of the circle whose centre is on the line $x = 2y$ and which passes through the points (−1, 2) and (3, −2).

Solution

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Since the centre lies on the line $x = 2y$, $-g = -2f$ or $g = 2f$

Since the circle passes through the points (−1, 2) and (3, −2), we have

$1 + 4 - 2g + 4f + c = 0$

and $9 + 4 + 6g - 4f + c = 0$

or $-2g + 4f + c = 5$

and $6g - 4f + c = -13$

Since $g = 2f$, on solving the equations we get $g = -2$, $f = -1$ and $c = -5$.

The equation of the circle is $x^2 + y^2 - 4x - 2y - 5 = 0$

Concept Strand 7

Find the equation of the circle which touches the coordinate axes and passes through the point (2, 1).

Solution

Since the circle has to pass through the point (2, 1) and touch the coordinate axes, the circle or circles must be in the first quadrant.

We may assume the equation of the circle as

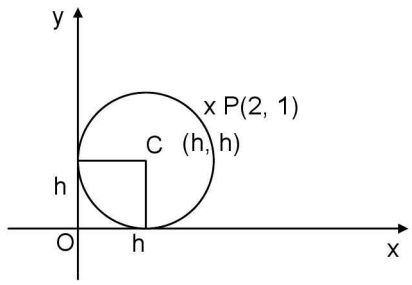
$(x - h)^2 + (y - h)^2 = h^2$

P(2, 1) lies on the circle. $\Rightarrow (2 - h)^2 + (1 - h)^2 = h^2$

$\Rightarrow h^2 - 6h + 5 = 0$

$\Rightarrow h = 1, 5$

The two circles are $(x - 1)^2 + (y - 1)^2 = 1$



and $(x - 5)^2 + (y - 5)^2 = 25$

or $x^2 + y^2 - 2x - 2y + 1 = 0$

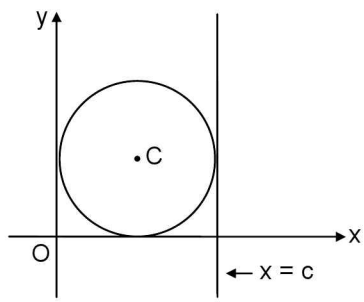
and $x^2 + y^2 - 10x - 10y + 25 = 0$

Concept Strand 8

Find the equation of the circle which touches the lines $x = 0$, $y = 0$ and $x = c$

Solution

The coordinates of the centre of the circle are easily seen as $\left(\frac{c}{2}, \frac{c}{2}\right)$ and its radius is $\frac{c}{2}$.



The equation of the circle is $\left(x - \frac{c}{2}\right)^2 + \left(y - \frac{c}{2}\right)^2 = \left(\frac{c}{2}\right)^2$

$\Rightarrow 4(x^2 + y^2) - 4cx - 4cy + c^2 = 0$

Position of a point in relation to a circle

Consider the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Its centre is at $(-g, -f)$ and its radius is $\sqrt{g^2 + f^2 - c}$. Let $P(x_1, y_1)$ represent a point in the plane of the circle.

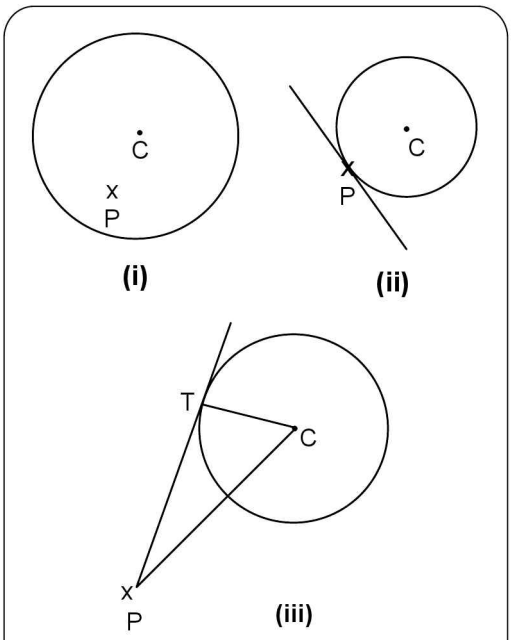


Fig. 2.4

The point P can be either inside the circle, or on the circle or outside the circle. (refer (i), (ii), (iii) of Fig. 2.4).

If C is the centre of the circle, it is clear that P will be inside or on or outside the circle according as $CP \lesseqgtr \sqrt{g^2 + f^2 - c}$

i.e., according as $CP^2 \lesseqgtr (g^2 + f^2 - c)$

$$(x_1 + g)^2 + (y_1 + f)^2 \lesseqgtr (g^2 + f^2 - c)$$

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \lesseqgtr 0$$

Therefore, we infer that the point $P(x_1, y_1)$ will be inside, or on, or outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \lesseqgtr 0$

We know that, only when P is outside the circle (or on the circle) we can draw a line through P touching the circle.

Suppose P is outside the circle. Let PT be a tangent to the circle drawn from P (refer (iii) of Fig. 2.4)

From the right angled triangle PTC,

$PT^2 = CP^2 - CT^2$

$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$\Rightarrow \quad PT = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

We have proved that the length of the tangent from a point $P(x_1, y_1)$ to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Results

(i) In the case of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, if c is positive, origin lies outside the circle; if c is negative, origin lies inside the circle and if c is zero, origin lies on the circle

(ii) The point $P(x_1, y_1)$ lies inside or on outside the circle $x^2 + y^2 = a^2$ according as $x_1^2 + y_1^2 - a^2 \lesseqgtr 0$

(iii) If $P(x_1, y_1)$ is outside the circle $x^2 + y^2 = a^2$, the length of the tangent from P to the circle is given by $\sqrt{x_1^2 + y_1^2 - a^2}$

EQUATION OF THE TANGENT TO A CIRCLE AT A POINT ON IT

Let the point be $P(x_1, y_1)$ and the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre is $C(-g, -f)$; $\therefore$ slope of $CP = \dfrac{y_1 + f}{x_1 + g}$

Since the tangent to the circle is perpendicular to CP, slope of the tangent $= -\dfrac{x_1 + g}{y_1 + f}$

$\therefore$ Equation of the tangent at P to the given circle is $y - y_1 = -\dfrac{x_1 + g}{y_1 + f}(x - x_1)$

$\Rightarrow \quad yy_1 - y_1^2 + fy - fy_1 = -xx_1 + x_1^2 - gx + gx_1$

OR $xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$

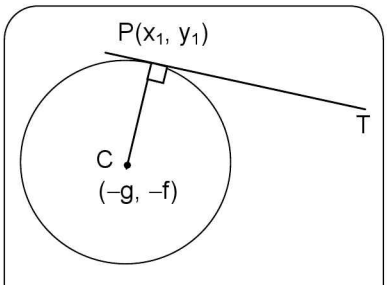


Fig. 2.5

$\Rightarrow \quad xx_1+yy_1+g(x+x_1)+f(y+y_1)+c= x_1^2+y_1^2+2gx_1+2fy_1+c$

$= 0$, as P lies on the circle

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ gives the equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at P (x_1, y_1) on it.

Condition for tangency

Let the line be $y = mx + c$ and the circle be $x^2 + y^2 = a^2$. (Refer Fig. 2.6)

We have to find the condition for the line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$.

We know from geometry that a line will touch the circle if the length of the perpendicular from the centre of the circle to the line equals the radius of the circle.

Therefore, $\dfrac{0-0-c}{\sqrt{1+m^2}} = \pm a$

Squaring both sides, we obtain $c^2 = a^2 (1 + m^2)$.

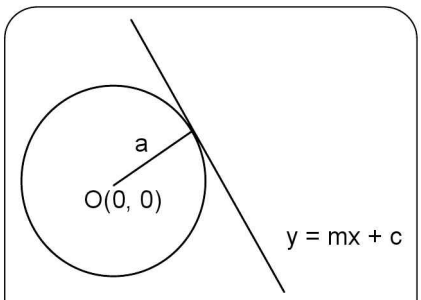


Fig. 2.6

Or the line $y = mx + c$ will touch the circle $x^2 + y^2 = a^2$ (i.e., the line $y = mx + c$ will be a tangent to the circle $x^2 + y^2 = a^2$) if $c^2 = a^2 (1 + m^2)$.

We may also arrive at the above result by an alternative method. A line intersects a circle at two points. The coordinates of the points of intersection of the line and the circle can be obtained by solving the two equations

$$y = mx + c \text{ and } x^2 + y^2 = a^2$$

Substituting $y = mx + c$ in $x^2 + y^2 = a^2$ we get $x^2 + (mx + c)^2 = a^2$

or $(1 + m)^2 x^2 + 2mcx + (c^2 - a^2) = 0$ — (1)

(1) is a quadratic equation in x and gives two values of x (real or complex) which represent the x coordinates of the points of intersection of the line and the circle. If the line is to be a tangent to the circle, the two roots of (1) must be equal, for which, the discriminant of (1) must be zero

i.e., $4m^2c^2 - 4(1 + m^2)(c^2 - a^2) = 0$ giving $c^2 = a^2 (1 + m^2)$.

Condition for the line $y = mx + c$ to touch the circle $x^2 + y^2 = a^2$ (or $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = c^2$) is

$$c^2 = a^2 (1 + m^2).$$

Results

(i) We have $c = \pm a\sqrt{1+m^2}$.

This means that, corresponding to a given value of m, the two parallel lines

$y = mx + a\sqrt{1+m^2}$ and $y = mx - a\sqrt{1+m^2}$ touch the circle $x^2 + y^2 = a^2$

The points at which, the above lines touch the circle are at the extremities of a diameter of the circle (refer Fig. 2.7)

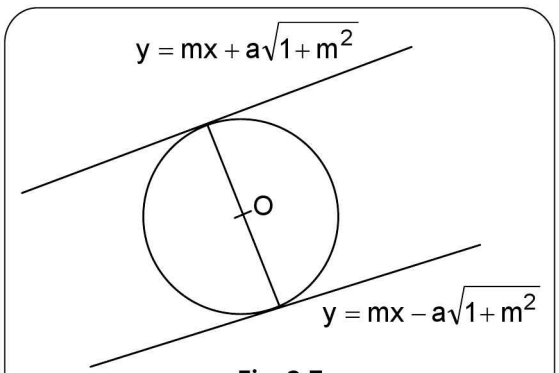


Fig. 2.7

(ii) Consider the equation $y = mx + a\sqrt{1+m^2}$ — (2)

(2) touches the circle $x^2 + y^2 = a^2$ for various values of m or, (2) represents a tangent to the circle $x^2 + y^2 = a^2$ for any value of m. If (2) passes through the point, say $P(x_1, y_1)$, we have

$y_1 = mx_1 + a\sqrt{1+m^2} \quad \Rightarrow \quad y_1 - mx_1 = a\sqrt{1-m^2}$

Squaring, $(y_1 - mx_1)^2 = a^2 (1 + m^2)$

Simplification gives, $(x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 - a^2) = 0$ — (3)

Since x_1, y_1 and a are known, we may treat (3) as a quadratic in m. The roots of (3) (real or complex) give

us the slopes of the tangents that can be drawn from P to the circle.

Discriminant of (3) = $4x_1^2 y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) = 4a^2(x_1^2 + y_1^2 - a^2)$

Clearly, if the discriminant is zero, the two roots are equal which means that only one tangent can be drawn through P to the circle. (Observe that this is the case when P lies on the circle)

Again, if the discriminant is negative, the two roots are complex which means that no tangent can be drawn through P to the circle. (Observe that this is the case when P lies inside the circle)

Let us assume that the discriminant is positive. Then, the roots of (3) are real and distinct. We therefore conclude that two tangents can be drawn from a given point P to the circle and that this is the case when the given point lies outside the circle.

(Note that this is only a confirmation of the well-known result in geometry that two tangents can always be drawn from a given point (lying outside the circle) to a circle and that the lengths of these tangents are equal)

(iii) Suppose the two tangents from $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ are at right angles to each other.

If m_1 and m_2 denote the slopes of the two tangents from P to the circle, m_1 and m_2 are the roots of (3). Since the two tangents are to be perpendicular to each other, $m_1m_2 = -1$

From (3), m_1m_2 = Product of the roots = $\dfrac{(y_1^2 - a^2)}{(x_1^2 - a^2)}$

We must have $\dfrac{(y_1^2 - a^2)}{(x_1^2 - a^2)} = -1 \Rightarrow x_1^2 + y_1^2 = 2a^2$

The conclusion from the above relation is that the coordinates (x_1, y_1) of points such that the tangents from these points to the circle $x^2 + y^2 = a^2$ are at right angles to each other satisfy the equation $x^2 + y^2 = 2a^2$. In other words, such points lie on the circle with radius $a\sqrt{2}$ and concentric with the circle $x^2 + y^2 = a^2$.

The circle $x^2 + y^2 = 2a^2$ is called the director circle of the circle $x^2 + y^2 = a^2$.

We may also say that the locus of a point such that the tangents from this point to the circle $x^2 + y^2 = a^2$ are at right angles to each other is the circle $x^2 + y^2 = 2a^2$. To put it in another way, if one takes any point on the circle $x^2 + y^2 = 2a^2$ and draws tangents to the circle $x^2 + y^2 = a^2$, these two tangents will be perpendicular to each other

CONCEPT STRANDS

Concept Strand 9

Without drawing the figure determine whether the points (0, 0), (−2, 1), (4, −3), (2, −6), (0, −1) lie outside or on or inside the circle $x^2 + y^2 - 5x + 2y - 5 = 0$

Solution

Let the points be denoted by O, P, Q, R, S (O being the origin)

Since c = −5, origin lies inside the circle. Substituting the coordinates of P, Q, R and S in the expression $(x^2 + y^2 - 5x + 2y - 5)$

For P, the value of the expression = 4 + 1 + 10 + 2 − 5 > 0 ⇒ P lies outside the circle,

For Q, the value of the expression = 16 + 9 − 8 − 6 − 5 > 0 ⇒ Q lies outside the circle,

For R, the value of the expression = 4 + 36 − 10 − 12 − 5 > 0 ⇒ R lies outside the circle.

For S, the value of the expression = 0 + 1 + 0 − 2 − 5 < 0 ⇒ S lies inside the circle.

Concept Strand 10

Prove that the tangents from the point (0, 5) to the two circles $x^2 + y^2 + 2x - 4 = 0$ and $x^2 + y^2 - y + 1 = 0$ are of equal length.

Solution

Let P be the point (0, 5) and PT_1 and PT_2 be the tangents to the two circles.

Then, $PT_1^2 = 0 + 25 - 4 = 21$

$PT_2^2 = 0 + 25 - 5 + 1 = 21$

$\Rightarrow PT_1 = PT_2$

Concept Strand 11

Find the points on the straight line x − y + 1 = 0, the tangents from which to the circle $x^2 + y^2 - 3x = 0$ are of length 2.

Solution

If (h, k) is a point on the line such that the tangents from (h, k) to the circle $x^2 + y^2 - 3x = 0$ are of length 2, we have $h - k + 1 = 0$ and $h^2 + k^2 - 3h = 4$

Substituting $k = h + 1$ in the second equation,
$h^2 + (h + 1)^2 - 3h = 4 \Rightarrow 2h^2 - h - 3 = 0$

$\Rightarrow \quad (2h - 3)(h + 1) = 0 \Rightarrow h = -1, \frac{3}{2}$

$\Rightarrow \quad k = 0$ and $\frac{5}{2}$

The two points on the line are $(-1, 0)$ and $\left(\frac{3}{2}, \frac{5}{2}\right)$

Concept Strand 12

Find the equation of the circle which is such that the lengths of the tangents to it from the points (1, 0), (0, 2) and (3, 2) are 1, $\sqrt{7}$ and $\sqrt{2}$ respectively.

Solution

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Given $1 + 2g + c = 1 \quad \Rightarrow 2g + c = 0$

$4 + 4f + c = 7 \quad \Rightarrow 4f + c = 3$

$9 + 4 + 6g + 4f + c = 2 \quad \Rightarrow 6g + 4f + c = -11$

Solving, we get $g = \frac{-7}{3}, f = \frac{-5}{12}, c = \frac{14}{3}$

The equation of the circle is $x^2 + y^2 - \frac{14}{3}x - \frac{5}{6}y + \frac{14}{3} = 0$

$\Rightarrow \quad 6(x^2 + y^2) - 28x - 5y + 28 = 0$

COMMON TANGENTS TO TWO CIRCLES

Consider two circles S_1 and S_2 whose centres are at C_1 and C_2 and whose radii are r_1 and r_2 respectively. Suppose the circles S_1 and S_2 are drawn. Then, we have the following 5 cases. The relevant cases are shown in the figures below. The distance between centres of the circles is C_1C_2.

Case (i):

$C_1C_2 > r_1 + r_2$

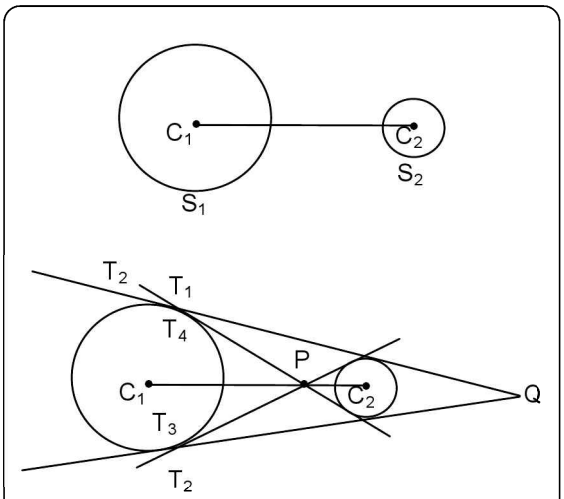


Fig. 2.8

Case (ii):

$C_1C_2 = r_1 + r_2$

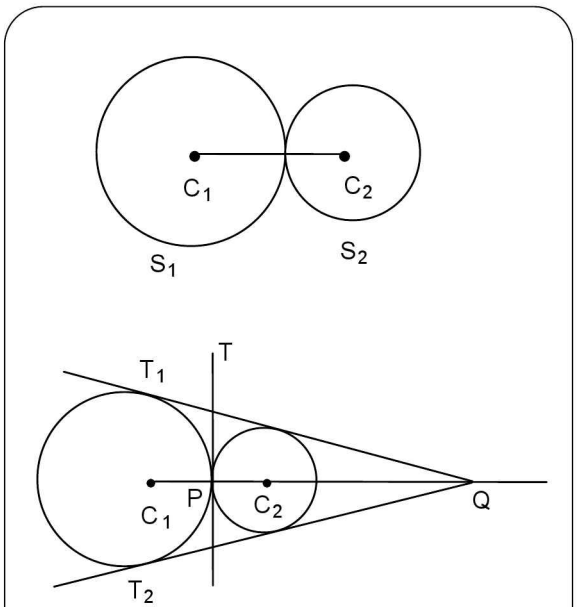


Fig.2.9

Case (iii):

$C_1C_2 < (r_1 + r_2)$ but $> |r_1 - r_2|$

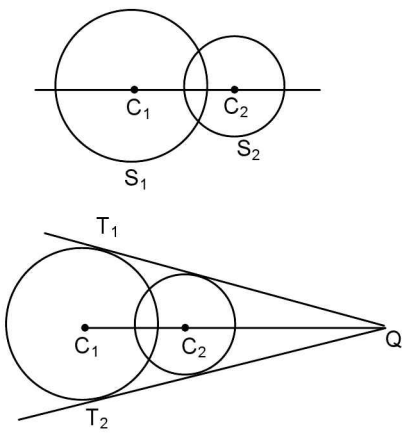


Fig. 2.10

Case (iv):

$C_1 C_2 = |r_1 - r_2|$

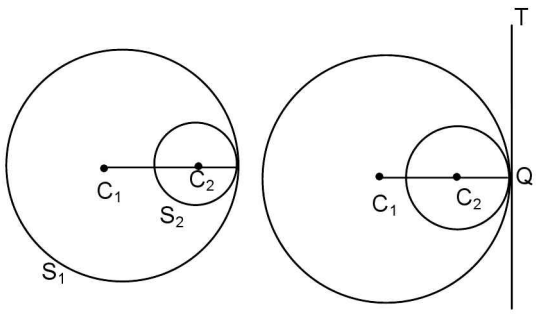


Fig. 2.11

Case (v):

$C_1 C_2 < |r_1 - r_2|$

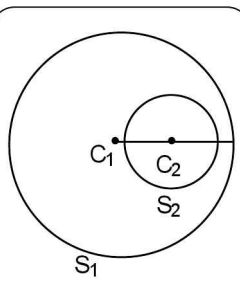


Fig. 2.12

Case (i):

The two circles are such that one lies completely outside the other. In this case, four common tangents can be drawn to the circles. The tangents QT_1 and QT_2 are called the direct common tangents while the tangents PT_3 and PT_4 are called the transverse common tangents to the two circles (refer Fig. 2.8).

Case (ii):

The two circles touch externally. In this case, we can draw three common tangents to the two circles. QT_1 and QT_2 are the direct common tangents while PT is the transverse common tangent (refer Fig. 2.9).

Case (iii):

The two circles intersect. In this case, we can draw only two common tangents to the two circles. QT_1 and QT_2 are the direct common tangents. (refer Fig. 2.10)

Case (iv):

The two circles touch internally. In this case, only one common tangent which is the tangent at the point of contact can be drawn. QT is the common tangent to the two circles (refer Fig. 2.11)

Case (v):

The two circles are such that one is completely inside the other. In this case, no common tangents are possible. (refer Fig. 2.12)

We have the following results in geometry.

Results

(i) The direct common tangents to the circles meet on the line of centres at the point Q, which divides the line of centres C_1C_2 externally in the ratio of the radii.
Q is called the external centre of similitude of the two circles.

(ii) The transverse common tangents to the circles meet on the line of centres at the point P which divides the line of centres C_1C_2 internally in the ratio of the radii.
P is called the internal centre of similitude of the two circles.

We illustrate the above by working out a few examples.

CONCEPT STRANDS

Concept Strand 13

Find the equations of the common tangents to the circles $x^2 + y^2 - 14x + 6y + 33 = 0$ and $x^2 + y^2 + 30x - 2y + 1 = 0$

Solution

The centres of the two circles are $C_1(7, -3)$ and $C_2(-15, 1)$

The radii of the two circles are $r_1 = \sqrt{7^2 + 3^2 - 33} = 5$ and $r_2 = \sqrt{15^2 + 1 - 1} = 15$

$C_1C_2 = \sqrt{22^2 + 4^2} = 10\sqrt{5}$ and $r_1 + r_2 = 20$

Clearly, $C_1C_2 > r_1 + r_2$

(Case (i))

We have two direct common tangents and two transverse common tangents.

The points P and Q dividing C_1C_2 in the ratio 5 : 15 or 1 : 3 internally and externally are respectively $\left(\frac{3}{2}, -2\right)$ and (18, –5).

Direct common tangents to the two circles are the two tangents from the point (18, –5) to any of the circles.

Let the equation of a tangent from (18, –5) to the circle $x^2 + y^2 - 14x + 6y + 33 = 0$ be $y + 5 = m(x - 18)$ — (1)

Then, perpendicular from (7, –3) to the line (1) equals the radius 5.

$$\Rightarrow \frac{-3 + 5 - m(7 - 18)}{\sqrt{1 + m^2}} = \pm 5$$

$$\Rightarrow \frac{2 + 11m}{\sqrt{1 + m^2}} = \pm 5$$

Squaring and simplifying, $96m^2 + 44m - 21 = 0$

$$\Rightarrow m = \frac{7}{24} \text{ or } -\frac{3}{4}$$

Substituting for m in (1), the two direct common tangents are $7x - 24y = 246$ and $3x + 4y = 34$

The transverse common tangents to the two circles are the two tangents from the points $\left(\frac{3}{2}, -2\right)$ to any of the circles.

Let the equation of the tangent from $\left(\frac{3}{2}, -2\right)$ to the circle $x^2 + y^2 - 14x + 6y + 33 = 0$ be $y + 2 = m\left(x - \frac{3}{2}\right)$ — (2)

Then, perpendicular from $\left(\frac{3}{2}, -2\right)$ to the line (2) equals 5.

$$\Rightarrow \frac{-3 + 2 - m\left(7 - \frac{3}{2}\right)}{\sqrt{1 + m^2}} = \pm 5$$

Squaring and simplifying, we get $21m^2 + 44m - 96 = 0$

$\Rightarrow$ giving $m = \frac{4}{3}$ and $\frac{-24}{7}$

Substituting for m in (2), the two transverse common tangents are

$4x - 3y = 12$ and $24x + 7y = 22$

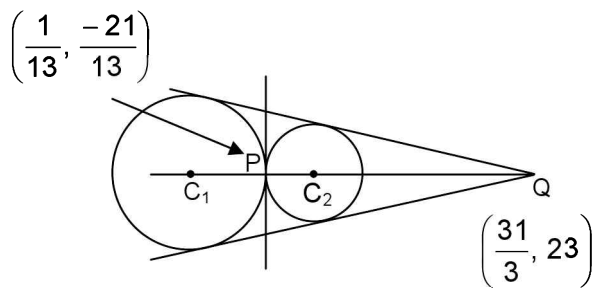
Concept Strand 14

Find the equations of the common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$

and $x^2 + y^2 + 6x + 18y + 26 = 0$

Solution

The centres of the two circles are $C_1(2, 3)$ and $C_2(-3, -9)$.



The radii of the two circles are $r_1 = 5$ and $r_2 = 8$

$C_1C_2 = \sqrt{5^2 + 12^2} = 13$ and $r_1 + r_2 = 13$

(Case (ii)) The two circles touch externally.

We have two direct common tangents and one transverse common tangent.

The points P and Q dividing C_1C_2 in the ratio 5 : 8 internally and externally are respectively $\left(\frac{1}{13}, \frac{-21}{13}\right)$ and $\left(\frac{31}{3}, 23\right)$

Direct common tangents to the two circles are the two tangents from the point $\left(\frac{31}{3}, 23\right)$ to any of the circles.

Transverse common tangent to the two circles is the tangent at the point $\left(\frac{1}{13}, \frac{-21}{13}\right)$ to any of the circles, say $x^2 + y^2 - 4x - 6y - 12 = 0$.

We shall find the slope of the tangent at $\left(\frac{1}{13}, \frac{-21}{13}\right)$ on the circle by using calculus.

Differentiating the relation $x^2 + y^2 - 4x - 6y - 12 = 0$ with respect to x,

we get, $2x + 2yy' - 4 - 6y' = 0$

$\Rightarrow$ $x + yy' - 2 - 3y' = 0$ giving $y' = \frac{(2-x)}{(y-3)}$

$$(y')\text{ at}\left(\frac{1}{13}, \frac{-21}{13}\right) = \frac{2 - \frac{1}{13}}{\frac{-21}{13} - 3} = \frac{-25}{60} = \frac{-5}{12}$$

We may get the slope of the tangent at $\left(\frac{1}{13}, \frac{-21}{13}\right)$ in another way.

Slope of the tangent at $\left(\frac{1}{13}, \frac{-21}{13}\right)$

$$= -\frac{1}{\left[\text{slope of the line joining the points } (2,3) \text{ and } \left(\frac{1}{13}, \frac{-21}{13}\right)\right]}$$

$$= -\frac{2 - \frac{1}{13}}{3 + \frac{21}{13}} = \frac{-5}{12}$$

The equation of the transverse common tangent is

$$y + \frac{21}{13} = \frac{-5}{12}\left(x - \frac{1}{13}\right)$$

$$\Rightarrow \frac{13y + 21}{13} = \frac{-5(13x - 1)}{13 \times 12} \Rightarrow 12\,(13y + 21) = -5\,(13x - 1)$$

$\Rightarrow$ $65x + 156y + 247 = 0 \Rightarrow 5x + 12y + 19 = 0$

Concept Strand 15

Find the equations of the common tangents to the circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 4x - 2y - 4 = 0$

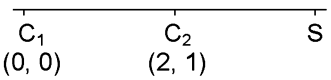
Solution

Centres of the circles are given by $C_1(0, 0)$ and $C_2(2, 1)$

Radii of the circles are given by $r_1 = 4$ and $r_2 = 3$

$C_1C_2 = \sqrt{5} < r_1 + r_2$

But, $C_1C_2 > |r_1 - r_2|$



Therefore, the two circles intersect. (Case (iii))

We have two direct common tangents to the two circles and no transverse common tangent.

The point S is the external point of division of C_1C_2 in the ratio 4 : 3 and it is given by (8, 4)

Direct common tangents to the two circles are the two tangents from (8, 4) to any of the circles, say $x^2 + y^2 = 16$.

Let the tangent be $y - 4 = m(x - 8)$ — (1)

Then, perpendicular from (0, 0) to (1) equals the radius of the circle.

i.e., $\frac{-4 + 8m}{\sqrt{1 + m^2}} = \pm 4 \Rightarrow 3m^2 - 4m = 0$

$\Rightarrow$ $m = 0, \frac{4}{3}$

The tangents are $y - 4 = 0$ and $4x - 3y - 20 = 0$

Concept Strand 16

Find the equations of the common tangents to the circles $x^2 + y^2 = 400$ and $x^2 + y^2 - 10x - 24y + 120 = 0$.

Solution

The centres of the circles are $C_1(0, 0)$ and $C_2(5, 12)$

Radii of the two circles are $r_1 = 20$ and $r_2 = 7$

$C_1C_2 = \sqrt{5^2 + 12^2} = 13$

$C_1C_2 = r_1 - r_2$

The two circles touch internally. (Case (iv))

There is only one common tangent to the two circles. The point of contact Q of the two circles is the external point of division of C_1C_2 in the ratio 20 : 7.

Q is $\left(\frac{100}{13}, \frac{240}{13}\right)$

Slope of the tangent at Q $= \frac{-1}{[\text{slope of the radius } C_1Q]}$

$$= -\frac{100}{240} = \frac{-5}{12}$$

Equation of the common tangent to the two circles is

$$y - \frac{240}{13} = \frac{-5}{12}\left(x - \frac{100}{13}\right)$$

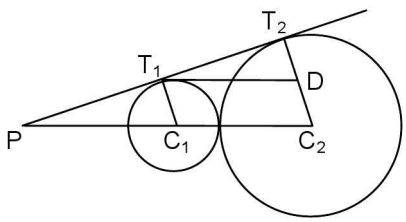
$$\Rightarrow \frac{13y - 240}{13} = \frac{-5(13x - 100)}{12 \times 13}$$

$\Rightarrow$ $12\,(13y - 240) = -5(13x - 100)$

$\Rightarrow$ $65x + 156y - 3380 = 0$

$\Rightarrow$ $5x + 12y - 260 = 0$

Length of the common tangent to two circles which touch externally.



Let the circles $S_1 = 0$ and $S_2 = 0$ touch externally.

Then, if C_1 and C_2 are the centres of the two circles and r_1 and r_2 are the respective radii, we have

$$C_1C_2 = r_1 + r_2$$

The length of the common tangent is T_1T_2. Through T_1 draw T_1D parallel to C_1C_2 to meet C_2T_2 at D. Then, $C_1C_2DT_1$ is a parallelogram.

$T_1D = C_1C_2 = r_1 + r_2$

and $DT_2 = C_2T_2 - C_2D = r_2 - r_1$

Since $\angle T_1T_2D = 90°$, $T_1T_2^2 = T_1D^2 - DT_2^2 = (r_1 + r_2)^2 - (r_2 - r_1)^2 = 4r_1r_2$

$\Rightarrow \quad T_1T_2 = 2\sqrt{r_1r_2}$

Radical axis of two circles

The radical axis of two circles is defined as the locus of a point from which the tangents to two given circles are of equal length.

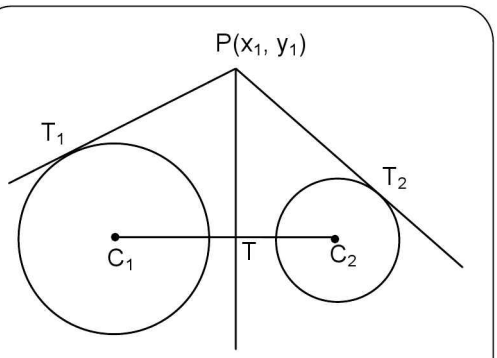


Fig.2.13

Let the equation of the two circles be

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ — (1)

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ — (2)

Let $P(x_1, y_1)$ represent any point on the locus and PT_1 and PT_2 be the tangents from P to the two circles. Then, by definition,

$PT_1 = PT_2$ or $PT_1^2 = PT_2^2$

$x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1 = x_1^2 + y_1^2 + 2g_2x_1 + 2f_2y_1 + c_2$

$\Rightarrow \quad 2(g_1 - g_2)x_1 + 2(f_1 - f_2)y_1 + c_1 - c_2 = 0$ — (3)

(3) means that the coordinates of any point $P(x_1, y_1)$ such that $PT_1 = PT_2$ satisfy the relation

$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$ —(4)

That is, the locus of such a point is the straight line represented by (4). Or, the radical axis of two circles is a straight line.

Observations

(i) The radical axis of two circles whose equations are $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ can be represented by $S_1 - S_2 = 0$

(ii) The radical axis of two circles is perpendicular to the line joining the centres of the two circles.

For, slope of the radical axis is $= -\dfrac{(g_1 - g_2)}{(f_1 - f_2)}$

Slope of the line joining the centres of the two circles = slope of the line joining the point $(-g_1, -f_1)$ and $(-g_2, -f_2)$

$= \dfrac{(f_1 - f_2)}{(g_1 - g_2)}$

Product of the slopes $= \left[-\left(\dfrac{(g_1 - g_2)}{(f_1 - f_2)}\right)\right]\left[\dfrac{(f_1 - f_2)}{(g_1 - g_2)}\right] = -1$

(iii) If two circles touch, the radical axis is the common tangent at their point of contact.
(Refer Fig. 2.14)

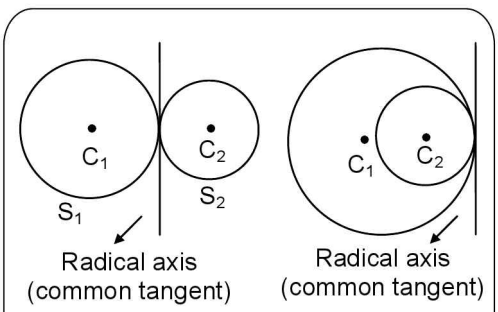


Fig. 2.14

(iv) If two circles intersect, the radical axis is their common chord. (Refer Fig. 2.15)

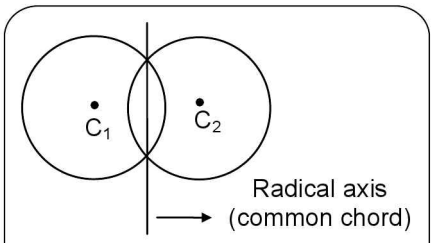


Fig. 2.15

(v) The radical axis of three circles taken in pairs are concurrent. The point of concurrence is called the radical centre of the three circles.

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ — (1)

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ — (2)

$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$ — (3)

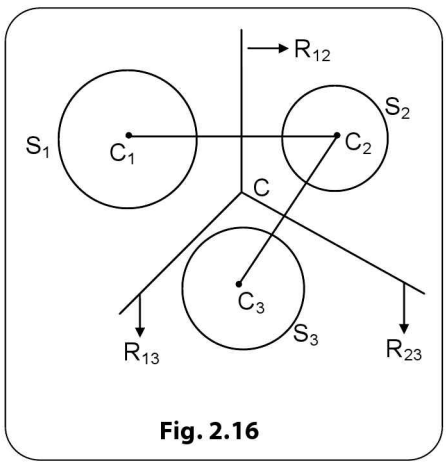


Fig. 2.16

Radical axis of the circles $S_1 = 0$ and $S_2 = 0$ is given by

$R_{12} \equiv 2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$ — (4)

Radical axis of the circles $S_2 = 0$ and $S_3 = 0$ is given by

$R_{23} \equiv 2(g_2 - g_3)x + 2(f_2 - f_3)y + c_2 - c_3 = 0$ — (5)

Radical axis of the circles $S_3 = 0$ and $S_1 = 0$ is given by

$R_{13} \equiv 2(g_3 - g_1)x + 2(f_3 - f_1)y + c_3 - c_1 = 0$ — (6)

Since $\begin{vmatrix} 2(g_1 - g_2) & 2(f_1 - f_2) & (c_1 - c_2) \\ 2(g_2 - g_3) & 2(f_2 - f_3) & (c_2 - c_3) \\ 2(g_3 - g_1) & 2(f_3 - f_1) & (c_3 - c_1) \end{vmatrix}$

$= \begin{vmatrix} 2(g_1 - g_2) & 2(f_1 - f_2) & (c_1 - c_2) \\ 2(g_1 - g_3) & 2(f_1 - f_3) & (c_1 - c_3) \\ 2(g_3 - g_1) & 2(f_3 - f_1) & (c_3 - c_1) \end{vmatrix} (R_2 + R_1)$

$= 0$,

The lines (4), (5) and (6) are concurrent, or radical axes of the three circles taken in pairs are concurrent. The point of concurrence of the lines (4), (5) and (6) gives the radical centre of the three circles (Refer Fig. 2.16)

(vi) Equation of any circle passing through the points of intersection of the two circles

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

can be represented by $S_1 + \lambda S_2 = 0$ or

by $S_1 + \mu(S_1 - S_2) = 0$ or by $S_2 + \alpha S_1 = 0$, where λ, μ, α are constants

For, all the above equations are second degree equations in x and y such that

Coefficient of x^2 = coefficient of y^2, and coefficient of $xy = 0$. This means that all these equations represent circles.

Also, if (x_1, y_1) is a point of intersection of the two circles $S_1 = 0$ and $S_2 = 0$,

$x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1 = 0$

and $x_1^2 + y_1^2 + 2g_2x_1 + 2f_2y_1 + c_2 = 0$

On substituting $x = x_1$ and $y = y_1$ in $S_1 + \lambda S_2$ or $S_1 + \mu(S_1 - S_2)$ or $S_2 + \alpha S_1$,

we can easily see that the expressions $S_1 + \lambda S_2$, $S_1 + \mu(S_1 - S_2)$, $S_2 + \alpha S_1$ reduce to zero. This means that (x_1, y_1) lies on these circles. Similar is the case for the other point of intersection of the two circles $S_1 = 0$ and $S_2 = 0$. Or, the circles $S_1 + \lambda S_2 = 0$ or $S_1 + \mu(S_1 - S_2) = 0$ or $S_2 + \alpha S_1 = 0$ pass through the points of intersection of the circles $S_1 = 0$ and $S_2 = 0$.

We now work out a few problems to illustrate the above results.

CONCEPT STRANDS

Concept Strand 17

Find the equation of the radical axis of the circles $x^2 + y^2 - 4x + 6y - 23 = 0$ and $x^2 + y^2 - 3x + y - 8 = 0$

Solution

Radical axis is $(x^2 + y^2 - 4x + 6y - 23) - (x^2 + y^2 - 3x + y - 8) = 0$

$\Rightarrow \quad -x + 5y - 15 = 0$

$\Rightarrow \quad x - 5y + 15 = 0$

Concept Strand 18

Find the radical centre of the circles $x^2 + y^2 = 9$, $x^2 + y^2 - 2x - 2y - 5 = 0$ and $x^2 + y^2 + 14x + 6y - 19 = 0$

Solution

$x^2 + y^2 - 9 = 0$ — (1)

$x^2 + y^2 - 2x - 2y - 5 = 0$ — (2)

$x^2 + y^2 + 14x + 6y - 19 = 0$ — (3)

Radical axis of circles (1) and (2) is

$2x + 2y - 4 = 0$ or $x + y - 2 = 0$ — (4)

Radical axis of circles (2) and (3) is

$16x + 8y - 14 = 0$ or $8x + 4y - 7 = 0$ — (5)

Radical axis of circles (3) and (1) is

$14x + 6y - 10 = 0$ or $7x + 3y - 5 = 0$ — (6)

Solving (4) and (5) radical centre is at $\left(\frac{-1}{4}, \frac{9}{4}\right)$. It can be seen that $\left(\frac{-1}{4}, \frac{9}{4}\right)$ satisfies (6) also.

Concept Strand 19

Find the radical centre of the circles $3x^2 + 3y^2 - 4x - 6y - 17 = 0$, $x^2 + y^2 - x - y - 9 = 0$ and $2x^2 + 2y^2 + 10x - 6y - 5 = 0$

Solution

The circles are

$x^2 + y^2 - \frac{4}{3}x - 2y - \frac{17}{3} = 0$ — (1)

$x^2 + y^2 - x - y - 9 = 0$ — (2)

$x^2 + y^2 + 5x - 3y - \frac{5}{2} = 0$ — (3)

Radical axis of circles (1) and (2) is $\frac{1}{3}x + y - \frac{10}{3} = 0$

$\Rightarrow$ $x + 3y - 10 = 0$ —(4)

Radical axis of circles (2) and (3) is $6x - 2y + \frac{13}{2} = 0$

$\Rightarrow$ $12x - 4y + 13 = 0$ —(5)

Radical axis of circles (3) and (1) is $\frac{19}{3}x - y + \frac{19}{6} = 0$

$\Rightarrow$ $38x - 6y + 19 = 0$ —(6)

Solving (4) and (5), the radical centre of the three circles is at $\left(\frac{1}{40}, \frac{133}{40}\right)$.

Note that $\left(\frac{1}{40}, \frac{133}{40}\right)$ satisfies (6).

Concept Strand 20

Find the equation of the circle passing through the point $(-1, -1)$ and through the intersection of the circles $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 4x - 2y - 4 = 0$

Solution

Any circle through the common points of the two circles can be assumed as

$x^2 + y^2 - 4x - 6y - 3 + \lambda(x^2 + y^2 + 4x - 2y - 4) = 0$

Since it passes through $(-1, -1)$,

$1 + 1 + 4 + 6 - 3 + \lambda(1 + 1 - 4 + 2 - 4) = 0$

$\Rightarrow$ $9 - 4\lambda = 0 \Rightarrow \lambda = \frac{9}{4}$

The circle is $x^2 + y^2 - 4x - 6y - 3 + \frac{9}{4}(x^2 + y^2 + 4x - 2y - 4) = 0$

$\Rightarrow$ $13(x^2 + y^2) + 20x - 42y - 48 = 0$

OR

The radical axis of the two circles is $8x + 4y - 1 = 0$

The required circle can be assumed as $x^2 + y^2 - 4x - 6y - 3 + k(8x + 4y - 1) = 0$

Since the above circle passes through $(-1, -1)$, substituting for k, $9 - 13k = 0 \Rightarrow k = \frac{9}{13}$

The circle is $13(x^2 + y^2) + 20x - 42y - 48 = 0$

Concept Strand 21

Find the length of the common chord of the circles $x^2 + y^2 - 2x - 4y - 5 = 0$ and $x^2 + y^2 + x - 11 = 0$

Solution

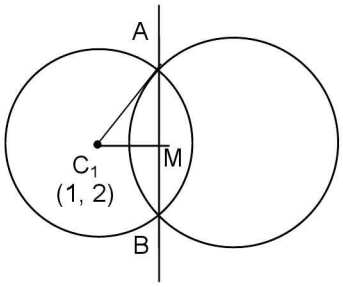
Centres of the circles are $C_1(1, 2)$ and $C_2\left(\frac{-1}{2}, 0\right)$. Radii of the two circles are $r_1 = \sqrt{10}$ and $r_2 = \sqrt{\frac{45}{4}}$

$C_1C_2 = \frac{5}{2}$; Here, $C_1C_2 < (r_1 + r_2)$ but $C_1C_2 > r_2 - r_1$

Therefore, the circles intersect at two points.

Radical axis of the circles is the common chord of the two circles.

Equation of the common chord AB is $3x + 4y - 6 = 0$



Draw C_1M perpendicular to AB

$$C_1M = \frac{3\times1+4\times2-6}{5} = 1$$

$AM^2 = C_1A^2 - CM^2 = 10 - 1 = 9 \Rightarrow AM = 3$

Length of the common chord = 2AM = 6

ORTHOGONAL CIRCLES

The angle between two curves at their point of intersection is defined as the angle between the tangents to the two curves at that point. When two circles intersect at a point so that the tangents at that point are at right angles, the circles are said to intersect (or cut) one another orthogonally. Such circles are often called orthogonal circles.

Let the equations of the two circles be

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ —(1)

and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ —(2)

We will obtain the condition for the circles (1) and (2) to intersect orthogonally.

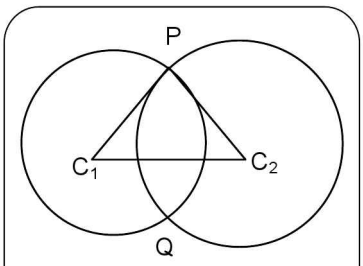


Fig. 2.17

Let the two circles intersect at P and Q (Refer Fig. 2.17). Since the two circles intersect orthogonally, the angle between the tangents to the two circles at P equals $\frac{\pi}{2}$. That is, the radii to the circles drawn at P are the tangents to the two circles at P.

From ΔPC_1C_2, $C_1C_2^2 = C_1P^2 + C_2P^2$

$\Rightarrow (g_2 - g_1)^2 + (f_2 - f_1)^2 = (g_1^2 + f_1^2 - c_1) + (g_2^2 + f_2^2 - c_2)$

$\Rightarrow 2g_1g_2 + 2f_1f_2 = c_1 + c_2$

The condition for the two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ to intersect orthogonally is,

$$2g_1 g_2 + 2f_1 f_2 = c_1 + c_2$$

We may also derive the above condition using calculus. Let P be (x_1, y_1).

Differentiating (1) with respect to x,

$$2x + 2y\frac{dy}{dx} + 2g_1 + 2f_1\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{(x+g_1)}{(y+f_1)}$$

Slope of the tangent at P to the circle (1) is given by

$$m_1 = -\frac{(x_1+g_1)}{(y_1+f_1)}$$

Similarly, the slope of the tangent at P to the circle (2) is given by $m_2 = -\frac{(x_1+g_2)}{(y_1+f_2)}$

Since the two curves (1) and (2) intersect orthogonally, $m_1 m_2 = -1$

$$\Rightarrow \frac{(x_1+g_1)(x_1+g_2)}{(y_1+f_1)(y_1+f_2)} = -1$$

$\Rightarrow x_1^2 + g_1x_1 + g_2x_1 + g_1g_2 + y_1^2 + f_1y_1 + f_2y_1 + f_1f_2 = 0$

Multiplying by 2 and rearranging,

$(x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1) + (x_1^2 + y_1^2 + 2g_2x_1 + 2f_2y_1)$
$+2g_1g_2 + 2f_1f_2 = 0$ — (3)

Since (x_1, y_1) lies on (1) and (2),

$x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1 = 0$

and $x_1^2 + y_1^2 + 2g_2x_1 + 2f_2y_1 + c_2 = 0$

(3) reduces to $-c_1 - c_2 + 2g_1g_2 + 2f_1f_2 = 0$

$\Rightarrow 2g_1g_2 + 2f_1f_2 = c_1 + c_2$

CONCEPT STRANDS

Concept Strand 22

Show that the circles $x^2 + y^2 - 8x - 6y + 21 = 0$ and $x^2 + y^2 - 2y - 15 = 0$ intersect orthogonally. Also find the equation of the common chord.

Solution

Centres of the circles are (4, 3) and (0, 1). The radii are $r_1 = 2$ and 4

$C_1C_2 = \sqrt{16+4} < r_1 + r_2$ but $> (r_2 - r_1)$

Therefore, the circles intersect.

We have, $2g_1g_2 + 2f_1f_2 = 2 \times -4 \times 0 + 2 \times -3 \times -1 = 6$

$c_1 + c_2 = 21 + (-15) = 6$

$\Rightarrow$ circles intersect orthogonally.

The common chord of the two circles is the radical axis of the two circles. Its equation is $8x + 4y - 36 = 0$ or $2x + y - 9 = 0$

Concept Strand 23

Find the equation of the circle which passes through the origin, has its centre on the line $x + y = 4$ and intersects orthogonally the circle $x^2 + y^2 - 4x + 2y + 4 = 0$

Solution

Let the equation of the circle through the origin be $x^2 + y^2 + 2gx + 2fy = 0$ — (1)

Centre of (1) lies on the line $x + y = 4$

$\Rightarrow$ $-g - f = 4$ or $g + f = -4$ — (2)

(1) intersects the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally.

$\Rightarrow$ $2 \times g \times (-2) + 2f \times 1 = 0 + 4$

$\Rightarrow$ $-4g + 2f = 4$

$\Rightarrow$ $-2g + f = +2$ — (3)

Solving (2) and (3), $g = -2$, $f = -2$

The equation of the circle is $x^2 + y^2 - 4x - 4y = 0$

Concept Strand 24

Show that the centre of a circle intersecting two circles orthogonally lies on the radical axis of the two circles.

Solution

For, let the equations of the given circles be

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ — (1)

and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ — (2)

Let the equation of a circle which intersects both (1) and (2) orthogonally be

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ — (3)

Since (3) and (1) intersect orthogonally,

$2gg_1 + 2ff_1 = c + c_1$ — (4)

Again, since (3) and (2) intersect orthogonally,

$2gg_2 + 2ff_2 = c + c_2$ — (5)

(4) – (5) gives

$2g(g_1 - g_2) + 2f(f_1 - f_2) = c_1 - c_2$

$\Rightarrow$ $2(-g)(g_1 - g_2) + 2(-f)(f_1 - f_2) + (c_1 - c_2) = 0$ — (6)

(6) means that the centre of the circle (3) [which is $(-g, -f)$] lies on the line

$2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$

But, the above line represents the radical axis of the two given circles (1) and (2)

$\Rightarrow$ Result follows.

A very interesting byproduct of the result shown above is given below.

Suppose we have three circles

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$

We are interested in finding the equation of the circle which intersects the three circles $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$ orthogonally.

Let the equation of the required circle be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

From the result of Concept Stand 3.24, the centre of S lies on the radical axis of the circles $S_1 = 0$ and $S_2 = 0$; It also lies on the radical axis of $S_2 = 0$ and $S_3 = 0$ and on the radical axis of $S_3 = 0$ and $S_1 = 0$. It immediately follows that the centre of $S = 0$ is the radical centre of the three circles $S_1 = 0$, $S_2 = 0$, $S_3 = 0$.

Thus, we have the result:

If a circle intersects three given circles orthogonally, its centre is the radical centre of the three given circles and its radius is equal to the length of the tangent drawn from the radical centre to any of the circles.

Concept Strand 25

Find the equation of the circle which intersects orthogonally the circles $x^2 + y^2 + 2x + 17y + 4 = 0$, $x^2 + y^2 + 7x + 6y + 11 = 0$ and $x^2 + y^2 - x + 22y + 3 = 0$

Solution

$x^2 + y^2 + 2x + 17y + 4 = 0$ — (1)

$x^2 + y^2 + 7x + 6y + 11 = 0$ — (2)

$x^2 + y^2 - x + 22y + 3 = 0$ — (3)

Radical axis of (1) and (2) is $5x - 11y + 7 = 0$ — (4)

Radical axis of (2) and (3) is $8x - 16y + 8 = 0$

or $x - 2y + 1 = 0$ — (5)

Therefore, the radical centre of the three circles is the point of intersection of (4) and (5), which is (3, 2).

Thus, the centre of the required circle is at (3, 2). The radius of the required circle is equal to the length of the tangent from (3, 2) to (1), (2) or (3).

$$= \sqrt{9+4-3+44+3} = \sqrt{57}$$

The circle is $(x - 3)^2 + (y - 2)^2 = 57$

$\Rightarrow$ $x^2 + y^2 - 6x - 4y - 44 = 0$

CONIC SECTIONS

Focus Directrix property of conics

The locus of a point which moves in a plane such that its distance from a fixed point in the plane bears a constant ratio to its distance from a fixed straight line in the plane is called a conic.

The fixed point is called the **focus** of the conic.

The fixed line is called the **directrix** of the conic.

The constant ratio denoted by e is called the **eccentricity** of the conic.

If e = 1, the conic is a parabola.

If e < 1, the conic is an ellipse.

If e > 1, the conic is a hyperbola.

In order to study the structure of the equation of a conic, let us consider the following three strands.

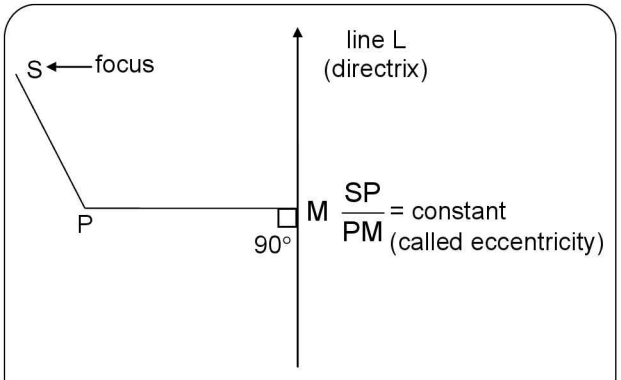


Fig. 2.18

CONCEPT STRANDS

Concept Strand 26

Given: Focus is at (3, 4); Directrix is the line $2x - 3y + 5 = 0$ and eccentricity e = 1. Find the equation of the conic.

Solution

We find the equation of the locus of a point P(x, y) which moves in the plane such that $\frac{SP}{PM} = 1$. (refer Figure. 2.18).

$SP^2 = PM^2$

$SP^2 = (x - 3)^2 + (y - 4)^2$

PM = perpendicular from P(x, y) to the directrix (ie., to the line $2x - 3y + 5 = 0$)

$$= \left|\frac{2x-3y+5}{\sqrt{13}}\right|$$

$SP^2 = PM^2$ gives $(x - 3)^2 + (y - 4)^2 = \left(\frac{2x-3y+5}{\sqrt{13}}\right)^2$

On simplification, we obtain the equation of the conic (here, it is a parabola, since e = 1) as

$9x^2 + 12xy + 4y^2 - 98x - 74y + 300 = 0$ — (1)

Concept Strand 27

If the focus of a conic is at (3, 1), the equation to its directrix is $x - y + 6 = 0$ and its eccentricity $e = \frac{1}{2}$, find the equation of this conic.

Solution

We find the equation of the locus of a point P(x, y) which moves in the plane such that $\frac{SP}{PM} = \frac{1}{2}$ or

$4SP^2 = PM^2$

Proceeding as in example 10.26, $SP^2 = (x - 3)^2 + (y - 1)^2$

$$PM^2 = \left(\frac{x - y + 6}{\sqrt{2}}\right)^2$$

$$4\,SP^2 = PM^2 \text{ gives } 4[(x - 3)^2 + (y - 1)^2] = \left(\frac{x - y + 6}{\sqrt{2}}\right)^2$$

On simplification, we obtain the equation of the conic (here, it is an ellipse, since $e = \frac{1}{2} < 1$) as $7x^2 + 2xy + 7y^2 - 60x - 4y + 44 = 0$ —(2)

Concept Strand 28

Find the equation of the conic with the following data. Focus is at (2, 2), directrix is the line $3x - 4y - 10 = 0$, eccentricity $e = 3$.

Solution

We find the equation of the locus of a point P(x, y) which moves in the plane such that $\frac{SP}{PM} = 3$ or $SP^2 = 9\,P\,M^2$

Proceeding as in Example 10.26, $SP^2 = (x - 2)^2 + (y - 2)^2$

$$PM^2 = \left(\frac{3x - 4y - 10}{5}\right)^2$$

$$SP^2 = 9\,PM^2 \text{ gives } (x - 2)^2 + (y - 2)^2 = \frac{9}{25}(3x - 4y - 10)^2$$

On simplification, we obtain the equation of the conic (here, it is a hyperbola since $e = 3 > 1$) as $56x^2 + 119y^2 - 216xy - 440x + 820y + 700 = 0$ — (3)

On examining the three equations (1), (2), (3) from strands 10.26, 10.27 and 10.28 above, we observe that the equation representing a conic whether it is a parabola, ellipse or hyperbola is of the form.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ — (4)

Again, in strands 10.26, $h^2 - ab = 6^2 - 9 \times 4 = 0$;
in strands 10.27, $h^2 - ab = 1 - 7 \times 7 < 0$; and
in strands 10.28, $h^2 - ab = (108)^2 - 56 \times 119 > 0$

It may be recalled that an equation of the form (4) will represent a pair of straight lines if

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

It can be easily verified that in all the above three examples, Δ is not equal to zero.

Therefore, we infer that a general equation of second degree in x and y of the form (4) represents a conic and it is:

a parabola if	(i) $\Delta \neq 0$ and	(ii) $h^2 = ab$;
an ellipse if	(i) $\Delta \neq 0$ and	(ii) $h^2 < ab$;
and a hyperbola	(i) $\Delta \neq 0$ and	(ii) $h^2 > ab$;

Form (4) of the equation of a conic is not suitable for a study of the geometrical properties of these conics. We therefore obtain simple forms for the equations of these conics (called standard forms) by appropriate choice of the coordinate system.

Standard form of equation of a parabola

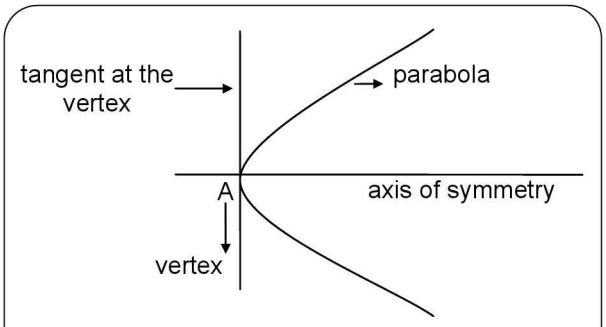


Fig. 2.19

The axis of a parabola is the line about which the curve is symmetrical. A is the vertex of the parabola (see Fig. 2.19). By choosing the axis of the parabola as x-axis, and the tangent at the vertex of the parabola (which is perpendicular to the axis of the parabola) as y-axis, we obtain the standard form of the equation of a parabola.

Let S be the focus and XM be the directrix. Draw SX perpendicular to the directrix and let A be the midpoint of SX. (refer Fig. 2.20)

Let SA = AX = a. Take AS (which is the axis of the parabola) as the x-axis and AY at right angles to AS (which is the tangent at the vertex A of the parabola) as the y-axis. Clearly, the coordinates of the focus S referred to the above coordinate system are (a, 0). Since SA = AX, A is a point on the parabola. (eccentricity e of a parabola = 1).

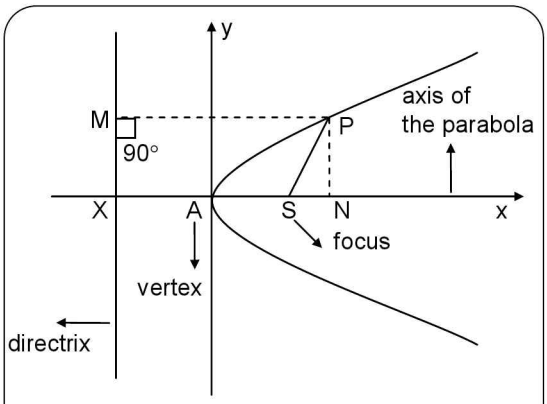


Fig. 2.20

Let P(x, y) be any point on the parabola. Draw PM perpendicular to the directrix. From the definition of the conic, $\frac{SP}{PM}$ = eccentricity = 1

$\Rightarrow$ $SP^2 = PM^2 = NX^2 = (AN + AX)^2$

$\Rightarrow$ $(x - a)^2 + y^2 = (x + a)^2 \Rightarrow y^2 = 4ax$

Since P is any point on the parabola, the relation between x and y satisfied by any point of the parabola is $y^2 = 4ax$, or the standard form of the equation of a parabola is

$$y^2 = 4ax \qquad —(1)$$

Observations

(i) The curve represented by (1) passes through the origin (origin is the vertex of the parabola).

(ii) If y is changed to −y in (1), the equation is unaltered. This means that the curve is symmetrical about the x-axis. (note that the x-axis is the axis of the parabola).

(iii) If x is negative, y^2 is negative. This means that no part of the curve lies to the left of the y-axis.

(iv) As x tends to infinity, y tends to infinity.

Latus rectum of a parabola

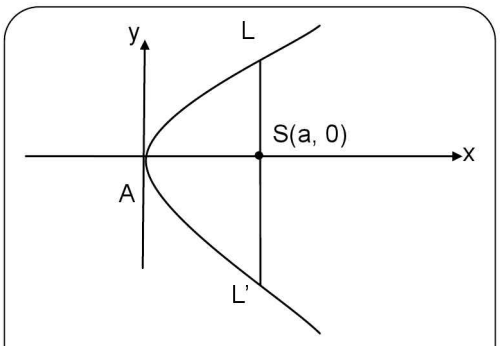


Fig. 2.21

Latus rectum of a conic (parabola, ellipse or hyperbola) is defined as the double ordinate through the focus of the conic. (refer Fig. 2.21)

For the parabola $y^2 = 4ax$, through the focus S draw LSL' perpendicular to the axis. Then, LL' is the latus rectum of the parabola. Since the curve is symmetrical about it axis, SL = SL'. The x–coordinate of L is a and we have

$SL^2 = 4a \times a$ (since L lies on $y^2 = 4ax$)

$= 4a^2$

$\Rightarrow$ SL = 2a

$\Rightarrow$ Latus rectum of the parabola $y^2 = 4ax$ is 4a

Summing up

For the parabola $y^2 = 4ax$, where $a > 0$
Vertex is at the origin
Focus is at (a, 0)
Directrix is the line $x + a = 0$
Axis is the x-axis, i.e., $y = 0$
Tangent at the vertex is the y-axis i.e., $x = 0$
Latus rectum = 4a

Other forms of equations of a parabola

(i) The equation $y^2 = -4ax$, where $a > 0$ represents the parabola whose focus S is at (−a, 0) and whose directrix is $x - a = 0$ (refer Fig. 2.22).

Its axis and the tangent at the vertex are the x and y axes respectively.

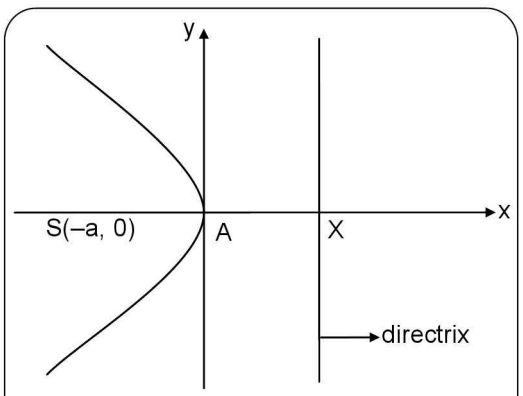


Fig. 2.22

(ii) The equation $x^2 = 4ay$, where $a > 0$ represents the parabola whose focus is at (0, a)

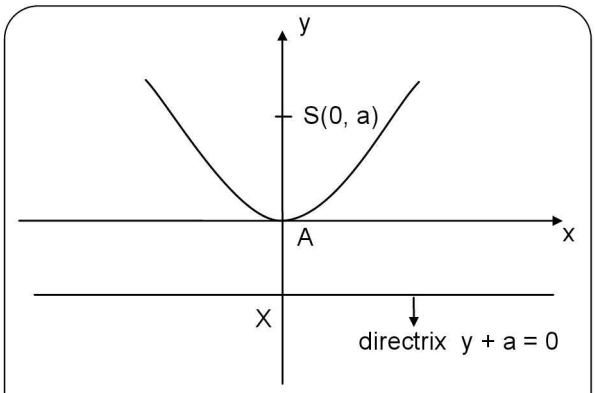


Fig. 2.23

and whose directirx is the line $y + a = 0$. Its axis and the tangent at the vertex are y and x axes respectively. (refer Fig. 2.23)

(iii) The equation $x^2 = -4ay$, where $a > 0$ represents the parabola whose focus S is at $(0, -a)$ and whose directirx is the line $y - a = 0$. Again, its axis and the tangent at the vertex are the y and x axes respectively (see Fig. 2.24).

(iv) In general, an equation of the form $(y + k)^2 = 4a(x + l)$ where $a > 0$ represents a parabola. Its axis and the tangent at the vertex are the lines $y + k = 0$ and $x + l = 0$ respectively. Its vertex (which is the point of intersection of the axis of the parabola and the tangent at the vertex) is at $(-l, -k)$. Its directrix is the line $x + l + a = 0$.

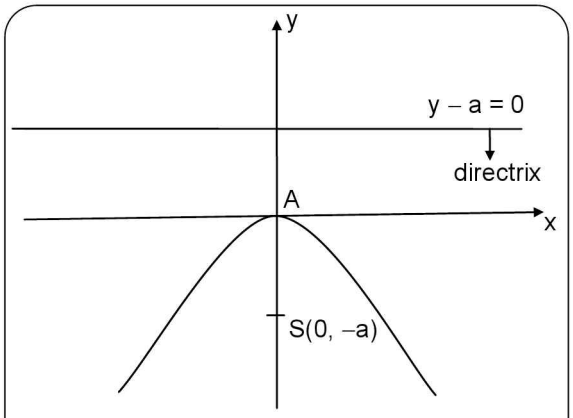


Fig. 2.24

Similarly, the equations
$(y + k)^2 = -4a(x + l)$; $(x + l)^2 = 4a(y + k)$ and $(x + l)^2 = -4a(y + k)$ where $a > 0$ represent parabolas.

Remark

In all the above cases, the length of the latus rectum is equal to 4a. Observe that, in the case of (iv), by shifting the origin to the vertex without changing the directions of the axes, the equation of the parabola in (iv) will reduce to the forms $Y^2 = \pm 4aX$ or $X^2 = \pm 4aY$.

We shall work out a few examples to illustrate the above.

CONCEPT STRANDS

Concept Strand 29

Find the equations of the axis, directrix, the coordinates of the vertex, focus and also the latus rectum of the parabola $y^2 + 4x - 2y + 3 = 0$

Solution

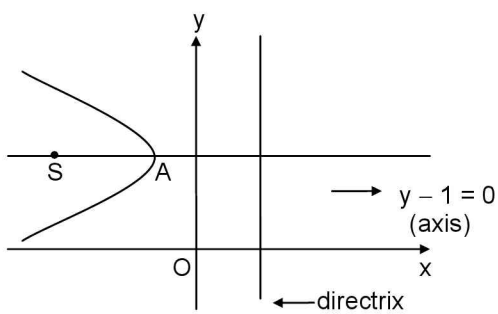
Since $h = 0, a = 0, b = 1, h^2 - ab = 0$

[we compare the given equation with the general equation of second degree in

x and y : $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$]

The equation represents a parabola.

Rewriting the equation, we get



$$y^2 - 2y = -4x - 3$$

$$\Rightarrow \quad (y - 1)^2 = -4x - 2 = -4\left(x + \frac{1}{2}\right)$$

Therefore, for the above parabola,

Vertex is at $\left(\frac{-1}{2},1\right)$; Axis is $y-1=0$;

Tangent at the vertex is $x+\frac{1}{2}=0$;

Focus is at $\left(\frac{-1}{2}-1,0\right)$ i.e. at $\left(\frac{-3}{2},0\right)$:

Directrix is $x-\frac{1}{2}=0$; and latus rectum = 4

Concept Strand 30

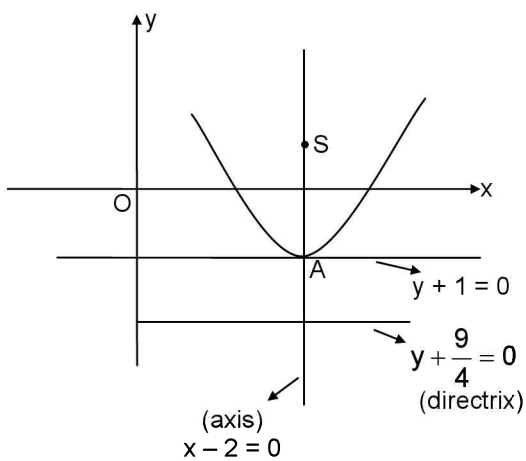
Find the equations of the axis, directrix, the coordinates of the vertex, focus and also the latus rectum of the parabola $x^2-4x-5y-1=0$.

Solution

Since $h^2 = ab$, it represents a parabola.

Rewriting the equation, we get

$(x-2)^2 = 5y+1+4 = 5(y+1)$



For this parabola:

Vertex is at $(2,-1)$;

Axis is $x-2=0$;

Tangent at the vertex is $y+1=0$;

Focus is at $\left(2,-1+\frac{5}{4}\right)$ or at $\left(2,\frac{1}{4}\right)$;

Directrix is $y=-1-\frac{5}{4}$ or $y+\frac{9}{4}=0$; And latus rectum is 5.

Concept Strand 31

Show that the equation $y = ax^2 + bx + c$ represents a parabola. Find its vertex, focus and latus rectum.

Solution

Note that $ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$

The equation $y = ax^2 + bx + c$ may be written as

$$y+\left(\frac{b^2-4ac}{4a}\right)=a\left[\left(x+\frac{b}{2a}\right)^2\right]$$

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{1}{a}\left[y+\left(\frac{b^2-4ac}{4a}\right)\right]$$

The above equation represents a parabola with vertex at $\left(\frac{-b}{2a},\frac{-(b^2-4ac)}{4a}\right)$

Its focus is at $\left(\frac{-b}{2a},\frac{-(b^2-4ac)}{4a}+\frac{1}{4a}\right)$ and its latus rectum is equal to $\frac{1}{a}$.

Standard form of the equation of an ellipse

Let S be the focus, MX the directrix and e (<1) the eccentricity (e is given). Draw SX perpendicular to the directrix. Divide SX internally and externally in the ratio e : 1 at A and A' respectively . (refer Fig. 2.25).

Since $\frac{SA}{AX}=\frac{SA'}{A'X}=\frac{e}{1}$, the points A and A' satisfy the condition of the locus. i.e., A and A' lie on the ellipse.

Let C be the middle point of AA'. Also, let AA' = 2a.(Since the position of S and the directrix are given, A and A' are fixed). Choose CX as x-axis and the line through C perpendicular to CX as y-axis.

We have, SA = e AX and SA' = e A'X

Addition gives,

AA' = e [AX + A'X]

=e [CX – CA + CA' + CX] = e × 2 CX, since CA = CA'

$\Rightarrow$ $2a = 2e \times CX \Rightarrow CX = \frac{a}{e}$

Again, SA' – SA = e(A'X – AX)

$\Rightarrow$ (CA' + CS) – (CA – CS) = e(AA') = 2ae

$\Rightarrow$ 2CS = 2ae $\Rightarrow$ CS = ae

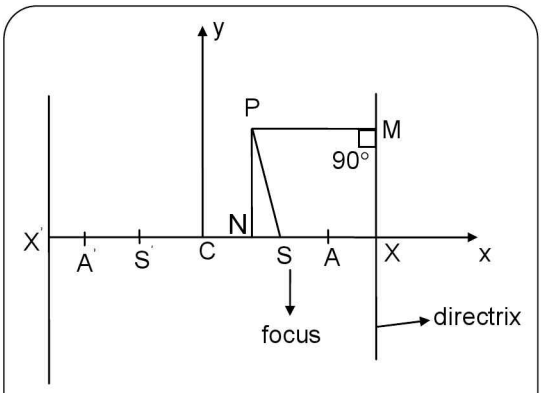


Fig. 2.25

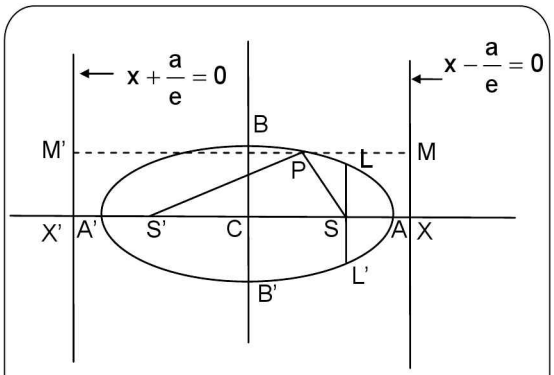


Fig. 2.26

Therefore, referred to the coordinate system chosen, the coordinates of the focus S are (ae, 0) and that of the point X are $\left(\frac{a}{e}, 0\right)$.

Let P(x, y) be any point on the locus. Draw PM perpendicular to the directrix and PN perpendicular to the x-axis.

By definition, $\frac{SP}{PM} = e \Rightarrow SP^2 = e^2 PM^2$

$$\Rightarrow \quad (x - ae)^2 + y^2 = e^2 NX^2 = e^2 (CX - CN)^2 = e^2\left(\frac{a}{e} - x\right)^2$$

Simplification leads to $x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

Let us denote $a^2(1 - e^2)$ by b^2 (observe that $b^2 < a^2$).

Then, the equation of the locus of P is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

or the standard form of the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{— (1),}$$

where $b^2 = a^2(1 - e^2)$.

Observations

(i) If we change x to –x in (1), the equation is not altered. Therefore, the curve is symmetrical about the y-axis. Similarly, it can be seen that the curve is symmetrical about the x-axis.

(ii) The curve passes through A(a, 0) and A'(–a, 0)

(iii) The curve intersects the y-axis at B(0, b) and B'(0, –b)

(iv) x cannot be > a, as, in this case y^2 becomes negative. Similarly, y cannot be > b. This means that no part of the curve lies beyond the lines x = ±a and y = ±b. The ellipse is therefore, a closed curve. (Refer Fig. 2.26)

Results

(i) The lines AA' and BB' (here, the x and y axes) about which the curve is symmetrical are called the major and minor axes of the ellipse. AA' = 2a is the length of the major axis of the ellipse. BB' = 2b is the length of the minor axis of the ellipse. The major and minor axes are perpendicular to each other.
Also, AA' is always > BB'. The point of intersection of the major and minor axes of the ellipse is called the centre of the ellipse. In our case, the centre of the ellipse is C (which is the origin of the coordinate system). Therefore, by choosing the major axis of the ellipse as x-axis and the minor axis of the ellipse as the y-axis, (the centre of the ellipse as the origin), the equation of the ellipse is obtained as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$.

(ii) Latus rectum of the ellipse = LL' = 2 SL, by symmetry (refer Fig. 2.26)
Since the x-coordiante of S is ae, the coordinates of L are (ae, SL).

L lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

We therefore get $\frac{a^2e^2}{a^2} + \frac{SL^2}{b^2} = 1$.

$SL^2 = b^2(1 - e^2) = a^2(1 - e^2)^2$

$SL = a(1 - e^2) = \frac{a^2(1-e^2)}{a} = \frac{b^2}{a}$

$\Rightarrow$ Latus rectum of the ellipse = 2SL $= 2\frac{b^2}{a}$

(iii) Let S' be the reflection of S in the y–axis. Therefore, the coordinates of S' are (–ae, 0). Similarly, if X' is the reflection of X in the y–axis, the coordinates of X' are $\left(\frac{-a}{e}, 0\right)$.

By symmetry, if we take S' as focus and the line through X' perpendicular to AA' as the directrix, we will obtain the same equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as the locus of P. We therefore infer that an ellipse has two foci and two directrices.

$$\text{Set I}: \left.\begin{array}{l}\text{Focus}: S(ae,0)\\ \text{Directrix}: x=\frac{a}{e}\end{array}\right\} \quad \text{Set II}: \left.\begin{array}{l}\text{Focus}: S'(-ae,0)\\ \text{Directrix}: x=\frac{-a}{e}\end{array}\right\}$$

(iv) If P is a point on the ellipse, SP and S'P are called the focal distances of the point P(distances of P from the two foci).

Sum of the focal distances of a point P on the ellipse = SP + S'P = e (PM + PM') (refer Fig. 2.27)

$= e(XX') = e\left(\frac{2a}{e}\right) = 2a$ = length of the major axis of the ellipse.

It is of interest to note that an ellipse can be defined as the locus of a point which moves in a plane such that the sum of its distances from two fixed points S and S' in the plane is a constant k. [Of course, distance between S and S' should be < k]

(v) If a < b, the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ represents an ellipse whose major axis is 2b and the minor axis is 2a. The major axis of the ellipse is the y-axis and the minor axis is the x-axis. The foci are at S(0,be). and S'(0, –be).

The respective directrices are $y=\frac{b}{e}$ and $y=\frac{-b}{e}$.

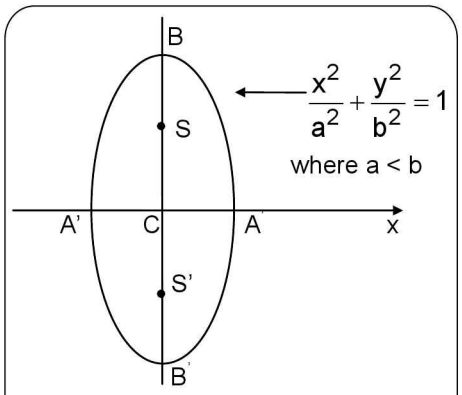


Fig. 2.27

(vi) An equation of the form $\frac{(x-p)^2}{a^2}+\frac{(y-q)^2}{b^2}=1$ represents an ellipse.

If a > b, the major and minor axes of the above ellipse are y – q = 0 and x – p = 0 respectively. The centre of the ellipse is at (p, q). Foci are at (p + ae, q) and (p – ae, q) and the corresponding directrices are

$x=p+\frac{a}{e}, x=p-\frac{a}{e}$.

If a < b, the major and minor axes are x – p = 0 and y – q = 0 respectively. Centre is at (p, q). Foci are at(p, q + be) and (p, q – be). The corresponding directrices are $y=q+\frac{b}{e}, y=q-\frac{b}{e}$.

CONCEPT STRANDS

Concept Strand 32

Find the equation of the ellipse with foci at (±5, 0) and $x=\frac{36}{5}$ as one of the directrices.

Solution

We have ae = 5

Since the directrix corresponding to (5, 0) is $x=\frac{36}{5}, \frac{a}{e}=\frac{36}{5}$

Solving the two equations ae = 5 and $\frac{a}{e}=\frac{36}{5}$, we get a = 6, $e=\frac{5}{6}$ and $b=\sqrt{11}$

Equation of the ellipse is $\frac{x^2}{36}+\frac{y^2}{11}=1$

Concept Strand 33

Find the centre, position of the foci, eccentricity of the ellipse $9x^2 + 25y^2 - 18x - 100y - 116 = 0$.

Solution

Rewriting the equation,

$9(x^2 - 2x) + 25(y^2 - 4y) = 116 \Rightarrow 9(x - 1)^2 + 25(y - 2)^2 = 225$

$\Rightarrow \frac{(x-1)^2}{25}+\frac{(y-2)^2}{9}=1$

We obtain the coordinates of the centre as (1, 2). Here, a = 5, b = 3

$b^2 = a^2(1 - e^2) \Rightarrow 9 = 25(1 - e^2) \Rightarrow e=\frac{4}{5}$

Eccentricity of the ellipse $= \frac{4}{5}$

$\Rightarrow \quad ae = 5 \times \frac{4}{5} = 4$.

Major axis is $y - 2 = 0$; minor axis is $x - 1 = 0$.

Foci are at distances ae on either side of the centre and they are on the major axis. Hence, the foci are at (1 + 4, 2) and (1 − 4, 2) or at (5, 2) and (−3, 2).

Equations of the corresponding directrices are $x = 1 + \frac{25}{4}$ and $x = 1 - \frac{25}{4}$

Or $x = \frac{29}{4}$ and $x = \frac{-21}{4}$.

Concept Strand 34

Find the eccentricity and the coordinates of the foci of the ellipse $9x^2 + 4y^2 = 36$.

Solution

The equation is $\frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow a = 3, b = 2$

$b^2 = a^2(1 - e^2)$ gives $e = \frac{\sqrt{5}}{3}$.

The major and minor axes are $x = 0$ and $y = 0$ respectively. Foci are at $(0, \sqrt{5})$ and $(0, -\sqrt{5})$

Standard form of the equation of a hyperbola

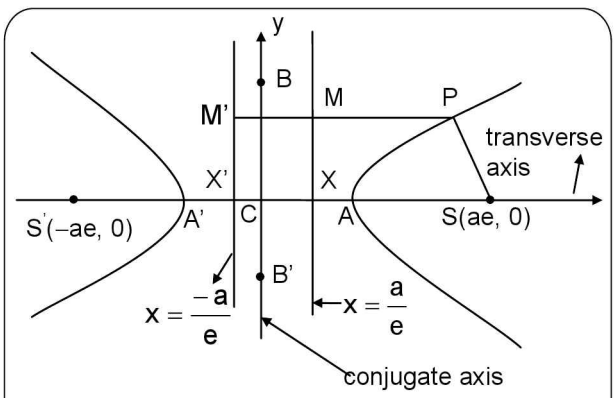


Fig. 2.28

A hyperbola has two axes of symmetry called its transverse and conjugate axes (refer Fig. 2.28). The point of intersection of the transverse and conjugate axes is called the centre of the hyperbola.

Choosing the transverse axis as the x-axis and the conjugate axis as the y-axis, the standard form of the equation of a hyperbola can be derived as $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ — (1)

Here $AA' = 2a$, $BB' = 2b$ and $b^2 = a^2(e^2 - 1)$. Observe that in the case of a hyperbola, e is > 1. [The derivation is similar to that for ellipse]

Observations

(i) The curve is symmetrical about both the x and y-axes.

(ii) Hyperbola intersects the x-axis at the points A(a, 0) and A'(−a, 0).

(iii) From the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we note that if $|x| < a$, y^2 is negative. We therefore infer that no part of the hyperbola lies between $x = -a$ and $x = a$. Again, as x tends to infinity, y tends to infinity. The form of the curve is therefore as shown in Fig. 2.28.

(iv) As in the case of the ellipse, the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by $\frac{2b^2}{a}$, where $b^2 = a^2(e^2 - 1)$.

(v) In the case of a hyperbola a can be greater than or less than or equal to b. If a = b, i.e., the length of the transverse axis equals the length of the conjugate axis, the hyperbola is called a rectangular hyperbola. When a = b, we have $a^2 = a^2(e^2 - 1)$ or $e = \sqrt{2}$. That is, the eccentricity of a rectangular hyperbola is $\sqrt{2}$.

(vi) By symmetry, if S' and X' are the reflections of S and X in the y–axis, S' is (−ae, 0) and X' is $\left(\frac{-a}{e}, 0\right)$. With S' as focus and the line through X' perpendicular to the transverse axis as the directrix, the same hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be generated or, in other words, a hyperbola has two foci and two directrices.

$$\text{Set I}: \left.\begin{array}{l}\text{Focus}: S(ae,0)\\ \text{Directrix}: x = \frac{a}{e}\end{array}\right\} \quad \text{Set II}: \left.\begin{array}{l}\text{Focus}: S'(-ae,0)\\ \text{Directrix}: x = \frac{-a}{e}\end{array}\right\}$$

(vii) If P is any point on the hyperbola, SP and S'P denote the focal distances of P (refer Fig. 2.28). Now,

$|SP - S'P|$ (or $SP \sim S'P$ denote the numerical difference between SP and S'P)

$= e\,|PM - PM'| = e(MM') = e(XX')$

$= e \times \left(\frac{2a}{e}\right) = 2a =$ length of the transverse axis

Hence, a hyperbola may be defined as the locus of a point which moves in a plane such that the difference between its distances from two fixed points in the plane is a constant k. Of course, k must be less than the distance between the two fixed points.

(viii) The equation $\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = 1$ represents a hyperbola whose centre is at (p, q).

The equations of the transverse and conjugate axes are $y - q = 0$ and $x - p = 0$.

The foci are at $(p + ae, 0)$ $(p - ae, 0)$ and the corresponding directrices are $x = p + \frac{a}{e}$ and $x = p - \frac{a}{e}$.

CONCEPT STRANDS

Concept Strand 35

Find the equation of the hyperbola whose centre is at (1, 0), one focus is at (6, 0) and the length of the transverse axis is 6.

Solution

Since the focus is at (6, 0), it follows that the transverse axis is the x–axis and the conjugate axis is $x - 1 = 0$. Also, the distance between the centre and one of the foci is ae. We are given $ae = 5$. Since, $2a = 6$, it immediately follows that $e = \frac{5}{3}$ and $b^2 = a^2(e^2 - 1) = 16$. The equation of the hyperbola is therefore, $\frac{(x-1)^2}{9} - \frac{y^2}{16} = 1$.

Concept Strand 36

Find the centre, eccentricity, foci and directrices of the hyperbola $9x^2 - 16y^2 + 72x - 32y - 16 = 0$.

Solution

The equation may be rewritten as $9(x^2 + 8x) - 16(y^2 + 2y) = 16$

$\Rightarrow 9(x + 4)^2 - 16(y + 1)^2 = 144$

$\Rightarrow \frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1$

The following inferences are made.

transverse axis is $y + 1 = 0$; conjugate axis is $x + 4 = 0$; centre of the hyperbola is at $(-4, -1)$

We have $a = 4$, $b = 3$

$b^2 = a^2(e^2 - 1)$ gives $9 = 16(e^2 - 1) \Rightarrow e = \frac{5}{4}$

Therefore, $ae = 5$ and $\frac{a}{e} = \frac{16}{5}$.

The foci are at $(-4 + 5, -1)$ and $(-4 - 5, -1)$ or at $(1, -1)$ and $(-9, -1)$,

The corresponding directrices are

$x = 4 - \frac{16}{5}$ and $x = -\left(4 + \frac{16}{5}\right)$

$\Rightarrow x = \frac{4}{5}$ and $x = \frac{-36}{5}$.

The length of the latus rectum $= \frac{2 \times 9}{4} = \frac{9}{2}$.

We now take up the three conics (parabola, ellipse and hyperbola) one by one and study a few important geometrical properties of these curves. For this purpose, we use the standard forms of the equations of these conics.

PARABOLA

$y^2 = 4ax$, where $a > 0$ — (1)

Condition for tangency

Let the line be $y = mx + c$. If the line is to be a tangent to the parabola (1), the two points of intersection of the line and the parabola must be coincident.

Solving for the points of intersection, we have

$(mx + c)^2 - 4ax = 0$

Or, $m^2x^2 + 2(cm - 2a)x + c^2 = 0$ — (2)

The above quadratic in x gives two values of x which are the x coordinates of the points of intersection of the line and

the parabola. If the line $y = mx + c$ is to be a tangent to the parabola $y^2 = 4ax$, the two roots of the equation (2) must be equal, for which the discriminant of the quadratic (2) must be zero. This gives

$$4(cm - 2a)^2 - 4c^2m^2 = 0 \Rightarrow c = \frac{a}{m} \qquad \text{— (2)}$$

Hence,

$y = mx + c$ will touch the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$.

We can therefore write the equation of a tangent to the parabola as $y = mx + \frac{a}{m}$ where m is given. Note that if m is given, one can draw only one tangent to the parabola and its equation is $y = mx + \frac{a}{m}$.

Results

(i) Let $P(x_1, y_1)$ be a point.
Draw PM perpendicular to the x-axis and produce MP to meet the parabola at P_1 (refer Fig. 2.29). From the figure,

$PM \lesseqgtr P_1M$, according as P is in the region R_1, or on the curve or in the region R_2

or according as $y_1 \lesseqgtr \sqrt{4ax_1}$ or according as $y_1^2 \lesseqgtr 4ax_1$

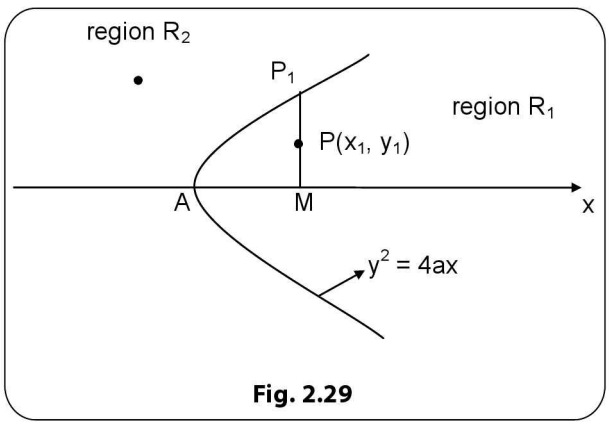


Fig. 2.29

(ii) Let us consider a line through the point $P(x_1, y_1)$. If the line through P is a tangent to the parabola $y^2 = 4ax$, its equation will be of the form $y = mx + \frac{a}{m}$.

Since the tangent passes through $P(x_1, y_1)$, we have

$$y_1 = mx_1 + \frac{a}{m}$$

$$\Rightarrow \quad x_1 m^2 - my_1 + a = 0$$

The above is a quadratic in m and, gives two values for m which are the slopes of the tangents that can be drawn from $P(x_1, y_1)$ to the parabola. We therefore infer that, in general, two tangents can be drawn from a point P to a parabola.

The discriminant of the above quadratic is $y_1^2 - 4ax_1$. It is clear that the two tangents will be real if and only if $y_1^2 - 4ax_1 \geq 0$. i.e., the point P must be in the region R_2 (in this case two tangents) or on the curve(in this case, only one tangent, as the two roots of the quadratic are equal).

Parametric representation of a point on the parabola

It can be seen that $x = at^2$, $y = 2at$ satisfy the equation $y^2 = 4ax$ for all real values of t. The coordinates of any point on the parabola can therefore be expressed in terms of a parameter t. The point 't' on $y^2 = 4ax$ means the point $(at^2, 2at)$.

Remark

$x = at^2$, $y = 2at$ where $t \in R$ represents a point on the parabola $y^2 = 4ax$.

OR the complex number $z = at^2 + i(2at)$ $(i = \sqrt{-1})$ can be said to represent a point on $y^2 = 4ax$.

OR the point P whose position vector relative to the origin of reference O (here it is the vertex) is $\bar{r} = at^2\bar{i} + 2at\bar{j}$, for $t \in R$, lies on the parabola $y^2 = 4ax$.

The first one is parametric representation, second one is complex form of representation and the third one is the vector form of representation where, in all the above cases, the point lies on $y^2 = 4ax$.

Results

(i) Equations of the tangent and normal at any point 't' on the parabola $y^2 = 4ax$.
$x = at^2$, $y = 2at$

$$\frac{dx}{dy} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{1}{t}. \Rightarrow \text{Slope of the tangent at 't' on the}$$

parabola is given by $\frac{1}{t}$.

Therefore, the equation of the tangent at 't' on $y^2 = 4ax$ is given by $y - 2at = \frac{1}{t}(x - at^2)$

$$\Rightarrow \quad y = \frac{x}{t} + at \qquad \text{— (1)}$$

Again, slope of the normal to the curve at 't' is $\frac{-1}{\left(\frac{dy}{dx}\right)} = -t$.

The equation of the normal at 't' on $y^2 = 4ax$ is given by $y - 2at = -t(x - at^2)$ Or

$y + xt = 2at + at^3$ — (2)

Observe that the parameter 't' is the reciprocal of the slope of the tangent at 't'.

(ii) The point of intersection of the tangents at t_1 and t_2 on the parabola $y^2 = 4ax$ is given by $(at_1t_2, a(t_1 + t_2))$.

For, tangent at t_1 is $y = \frac{x}{t_1} + at_1$; Tangent at t_2 is $y = \frac{x}{t_2} + at_2$

On solving the above two equations, the coordinates of the point of intersection of the tangents are obtained as $(at_1t_2, a(t_1 + t_2))$.

(iii) If (x, y) are the coordinates of the point of intersection of the normals to the parabola $y^2 = 4ax$ at t_1 and t_2, then,

$x_1 = 2a + a(t_1^2 + t_2^2 + t_1t_2)$

$y_1 = -at_1t_2 (t_1 + t_2)$

For, equation of the normals at t_1 and t_2 on the parabola are given by $y + xt_1 = 2at_1 + at_1^3$ and $y + xt_2 = 2at_2 + at_2^3$

Subtraction gives $x(t_1 - t_2) = 2a(t_1 - t_2) + a(t_1^3 - t_2^3)$.

Since $t_1 \neq t_2$,

$x = 2a + a(t_1^2 + t_2^2 + t_1t_2)$.

Substituting in one of the equations of the normals,

$y = -t_1[(2a + a(t_1^2 + t_2^2 + t_1 t_2)] + 2at_1 + at_1^3 = -at_1t_2 (t_1 + t_2)$.

(iv) Equation of the chord joining the points t_1 and t_2 on the parabola $y^2 = 4ax$.

The points are $M(at_1^2, 2at_1)$ and $N(at_2^2, 2at_2)$. (refer Fig. 2.30)

Slope of the chord $MN = \frac{2a(t_1 - t_2)}{a(t_1^2 - t_2^2)} = \frac{2}{(t_1 + t_2)}$.

Equation of MN is $y - 2at_1 = \frac{2}{(t_1 + t_2)}(x - at_1^2)$

$\Rightarrow y(t_1 + t_2) = 2x + 2at_1t_2$ — (3)

If a chord of a parabola passes through its focus, it is called a focal chord.

If MN is a focal chord, then, MN must pass through (a, 0). Substituting the coordinates of S in (3), we get,

$0 = 2a + 2at_1 t_2 \Rightarrow t_1 t_2 = -1$

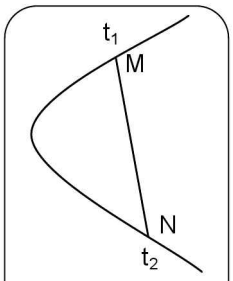


Fig. 2.30

Therefore, the chord joining two points t_1 and t_2 on the parabola $y^2 = 4ax$, is a focal chord if $t_1t_2 = -1$.

$t_1t_2 = -1$ is equivalent to $\left(\frac{1}{t_1}\right)\left(\frac{1}{t_2}\right) = -1$.

But, $\frac{1}{t_1}$ and $\frac{1}{t_2}$ are the slopes of the tangents at t_1 and t_2.

We therefore conclude that these tangents are at right angles. In other words, the tangents at the extremities of a focal chord of a parabola are at right angles.

Again, the coordinates of the point of intersection of tangents at t_1 and t_2 are given by $(at_1 t_2, a(t_1 + t_2))$ (refer (ii) above).

If the chord joining t_1 and t_2 is a focal chord, we have $t_1t_2 = -1$. Consequently, the x coordinate of the point of intersection of the tangents is $-a$. Or, the point of intersection lies on the directrix of the parabola.

We have now shown that the locus of points of intersection of perpendicular tangents to the parabola is the directrix. Or, we may say that the tangents at the extremities of any focal chord are at right angles and that they intersect on the directrix.

(v) If the normal at t_1 on the parabola $y^2 = 4ax$ meets the curve again at the point 't', then $t = -t_1 - \frac{2}{t_1}$.

Equation of the normal at t_1 on $y^2 = 4ax$ is given by

$y + xt_1 = 2at_1 + at_1^3$ — (1)

Since (1) meets the curve at 't' i.e., at $(at^2, 2at)$, the line PQ (which is normal at P) can be considered as the chord joining t_1 and t. (refer Fig. 2.31)

The equation of the chord PQ is $y(t_1 + t) = 2x + 2att_1$ — (2)

(1) and (2) represent the same line.

Therefore, $\frac{t_1 + t}{1} = \frac{-2}{t_1} \Rightarrow t = -t_1 - \frac{2}{t_1}$.

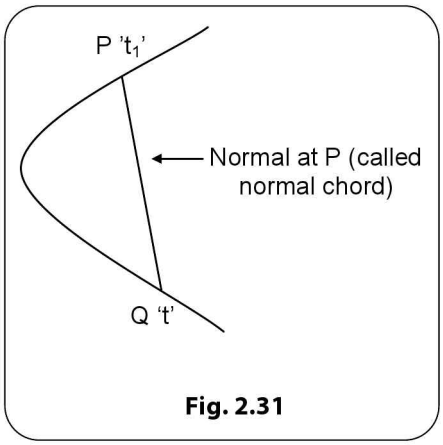


Fig. 2.31

The chord PQ of the parabola which is normal at P is called a normal chord.

(vi) In general, three normals can be drawn from a given point to a parabola.
Let the equation of the parabola be $y^2 = 4ax$
Let the given point be (h, k) and let 't' be the foot of the normal drawn from the given point.
The equation of the normal at 't' is $y + xt = 2at + at^3$ — (1)

(1) passes through the point (h, k).
$\Rightarrow\ k + ht = 2at + at^3 \Rightarrow at^3 + (2a - h)t - k = 0$ — (2)
(2) being a cubic in t, has three roots (either all the roots are real or one root is real). Each of these roots being substituted in (1), gives a normal to the parabola passing through (h, k). So, through any point (h, k) three normals, real or complex can be drawn to the parabola. If t_1, t_2, t_3 are the roots of (2),

$t_1 + t_2 + t_3 = 0$ — (3)

$t_1 t_2 + t_2 t_3 + t_3 t_1 = \dfrac{2a - h}{a}$ — (4)

and $t_1 t_2 t_3 = \dfrac{k}{a}$ — (5)

From (3), $2at_1 + 2at_2 + 2at_3 = 0$
$\Rightarrow$ Sum of the ordinates of the feet of the normals from any point to a parabola is zero.
Suppose the normals at t_1 and t_2 intersect at t_3 on the parabola $\Rightarrow$ Normals at t_1, t_2, t_3 pass through the point $(at_3^2, 2at_3)$

(5) reduces to $t_1 t_2 t_3 = \dfrac{2at_3}{a} \Rightarrow t_1 t_2 = 2.$

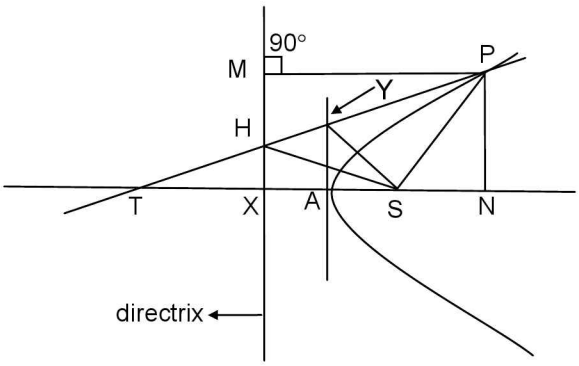
CONCEPT STRANDS

Concept Strand 37

(a) If the tangent at P on the parabola $y^2 = 4ax$ meets the axis in T and PN is the ordinate at P show that
(i) AN = AT
(ii) SP = ST, where A is the vertex and S is the focus.

(b) If the tangent at P meets the tangent at the vertex in Y, then show that SY is perpendicular to TP and $SY^2 = AS \times SP$.

(c) The portion of the tangent intercepted between the point of contact and the directrix subtends a right angle at the focus.



Solution

(a)

(i) Let P be 't'. Then the equation of the tangent at P is $y = \dfrac{x}{t} + at$ — (1)
We put y = 0 to get the x-coordinate of T $\Rightarrow$ T is $(-at^2, 0)$

Therefore, $AT = at^2$
Since AN = x coordinate of P = at^2, we have AT = AN
Observe that NT is the subtangent corresponding to P and we have shown that the subtangent is bisected by the vertex.

(ii) SP = PM, by definition
$= NX = NA + AX = x + a = at^2 + a$

$ST = SA + AT = a + at^2$

$\Rightarrow SP = ST$

(b) Putting $x = 0$ in (1), $y = at$

Therefore, Y is $(0, at)$

$SY^2 = (a - 0)^2 + (0 - at)^2 = a^2(1 + t^2)$

We have $SP = a + at^2$ and $SA = a$.

Therefore, $SP \cdot SA = a^2 (1 + t^2) = SY^2$.

(c) Portion of the tangent intercepted between the point of contact and the directrix is obtained as follows. If H is the point of intersection of the tangent at P and the directrix, we put $x = -a$ in (1)

Then, we get $y = at - \frac{a}{t} = \frac{a(t^2-1)}{t}$; Coordinates of y are $\left(-a, \frac{a(t^2-1)}{t}\right)$

$$\text{Slope of SH} = \frac{\frac{a(t^2-1)}{t} - 0}{-a-a} = \frac{-(t^2-1)}{2t}$$

$$\text{Slope of SP} = \frac{2at-0}{at^2-a} = \frac{2t}{(t^2-1)}$$

Slope of SH × Slope of SP = −1 ⇒ ∠PSH = 90°

ELLIPSE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b) \qquad \text{— (1)}$$

Condition for tangency

Let the line be $y = mx + c$. If the line is to be a tangent to the ellipse (1), the points of intersection of the line and the ellipse must be coincident.

Solving for the points of intersection, we have

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$\Rightarrow \left(\frac{1}{a^2} + \frac{m^2}{b^2}\right)x^2 + \frac{2mc}{b^2}x + \left(\frac{c^2}{b^2} - 1\right) = 0 \qquad \text{— (2)}$$

The quadratic (2) in x gives two values for x which are the x coordinates of the points of intersection of the line and the ellipse. If the line $y = mx + c$ is to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the two roots of (2) must be equal, for which the discriminant of the quadratic (2) must be zero. This gives,

$$\frac{4m^2c^2}{b^2} - 4\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right)\left(\frac{c^2}{b^2} - 1\right) = 0$$

$$\Rightarrow \frac{1}{a^2} - \frac{c^2}{a^2b^2} + \frac{m^2}{b^2} = 0$$

$\Rightarrow c^2 = a^2m^2 + b^2$

Hence,

$y = mx + c$ will touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$

We can therefore write the equation of a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as

$$\left.\begin{array}{c} y = mx + \sqrt{a^2m^2 + b^2} \\ \text{or} \\ y = mx - \sqrt{a^2m^2 + b^2} \end{array}\right\} \qquad \text{— (3)}$$

Note that if m is given, one can draw two tangents to an ellipse and they are given by (3).

Results

(i) Let P be a point in the plane of the ellipse whose coordinates are (x_1, y_1)

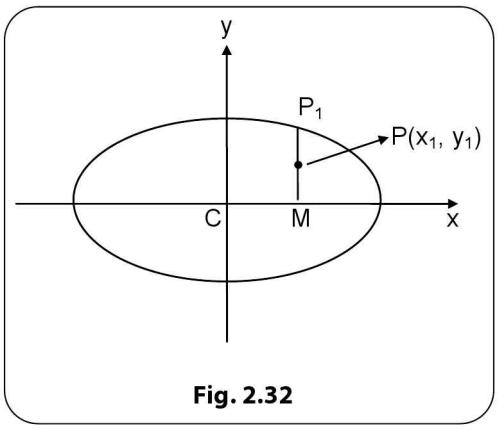


Fig. 2.32

P can be either inside or on or outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Draw PM perpendicular to the x-axis.

Produce MP to meet the ellipse at P_1. If P_1 is(x_1, y), then, P is inside or on or outside the ellipse, according as $y_1 \lesseqgtr y$ (refer Fig. 2.32)

$$\Rightarrow \frac{y_1^2}{b^2} \lesseqgtr \frac{y^2}{b^2}$$

Since $\frac{x_1^2}{a^2}+\frac{y^2}{b^2}=1, \frac{y^2}{b^2}=1-\frac{x_1^2}{a^2}$, substituting in the inequality,

P is inside or on or outside the ellipse according as

$$\frac{y_1^2}{b^2} \lesseqgtr 1-\frac{x_1^2}{a^2}$$

i.e., according as $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 \lesseqgtr 0$ — (4)

(ii) Let us consider a line through the point $P(x_1, y_1)$. If the line through P is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, its equation will be of the form

$$y = mx + \sqrt{a^2m^2+b^2}$$

Or $y = mx - \sqrt{a^2m^2+b^2}$

Since this tangent passes through

$P(x_1, y_1)$, $y_1 = mx_1 + \sqrt{a^2m^2+b^2}$

$$(y_1 - mx_1)^2 = a^2 m^2 + b^2$$

$\Rightarrow (x_1^2 - a^2) m^2 - 2x_1 y_1 m + (y_1^2 - b^2) = 0$ — (5)

The above is a quadratic in m and gives two values for m which are the slopes of the tangents that can be drawn from $P(x_1, y_1)$ to the ellipse. We therefore infer that, in general, two tangents can be drawn from any point to an ellipse.

The discriminant of the above quadratic is $4x_1^2 y_1^2 - 4(x_1^2 - a^2)(y_1^2 - b^2)$

which is equal to $b^2x_1^2 + a^2y_1^2 - a^2b^2$

or $a^2b^2\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$

It is clear that two tangents will be real if and only if $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 \geq 0$ i.e., the point P must be outside the ellipse(in this case, two tangents) or P must be on the ellipse(in this case, only one tangent, as the two roots of the quadratic are equal).

(iii) Let m_1 and m_2 be the slopes of the tangents that can be drawn from $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$.

Then, m_1m_2 = Product of the roots of (5) = $\frac{y_1^2-b^2}{x_1^2-a^2}$.

If the two tangents are perpendicular to each other, $m_1m_2 = -1$

$$\Rightarrow \frac{y_1^2-b^2}{x_1^2-a^2}=-1 \Rightarrow x_1^2 + y_1^2 = a^2 + b^2$$

which means that $P(x_1, y_1)$ must lie on the circle $x^2 + y^2 = a^2 + b^2$. This circle which is concentric with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is called the 'director circle' of the ellipse.

We have just now shown that the locus of points of intersection of perpendicular tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the director circle $x^2 + y^2 = a^2 + b^2$.

(iv) Let us find the locus of the foot of the perpendicular from a focus of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on any tangent to the ellipse.

Any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is of the form

$$y = mx + \sqrt{a^2m^2+b^2} \quad \text{— (1)}$$

Slope of the perpendicular SQ is $\frac{-1}{m}$. Therefore, the equation of SQ is

$$y - 0 = \frac{-1}{m}(x - ae) \Rightarrow my + x = ae \quad \text{— (2)}$$

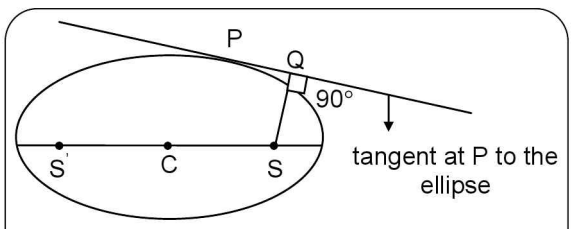


Fig. 2.33

If Q is (x, y) we are required to find the locus of Q. Locus of Q can be obtained by eliminating m (which is the variable parameter here) from (1) and (2).

We have,

$(my + x)^2 + (y - mx)^2 = a^2e^2 + a^2m^2 + b^2 \Rightarrow (x^2 + y^2)(1 + m^2) = a^2(1 + m^2)$,

since $b^2 = a^2(1 - e^2)$

Or, locus of Q is the circle $x^2 + y^2 = a^2$, which is called the auxiliary circle of the ellipse.

Parametric representation of a point on an ellipse

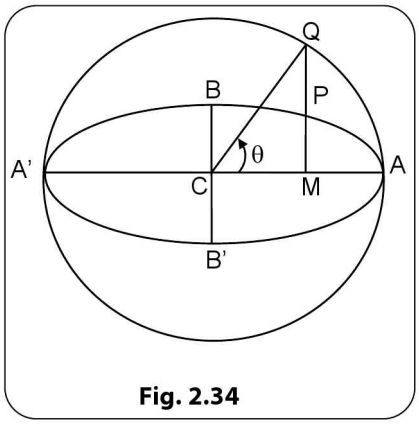


Fig. 2.34

Let P be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Draw PM perpendicular to the major axis (i.e., x-axis here). Produce MP to meet the auxiliary circle of the ellipse at Q. (refer Fig. 2.34).

Let $\angle QCA = \theta$. Since the equation of the auxiliary circle is $x^2 + y^2 = a^2$, the coordinates of Q are $(a\cos\theta, a\sin\theta)$. It is easy to note that the x coordinate of the point P on the ellipse is CM which is equal to $a\cos\theta$. To get the y coordinate of P, we substitute $x = a\cos\theta$ in the equation of the ellipse.

We obtain

$$\frac{a^2\cos^2\theta}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2 = b^2\sin^2\theta$$

$$\Rightarrow \quad y = \pm b\sin\theta$$

Since P is above the x-axis y coordinate of P is $b\sin\theta$. Thus, we can represent any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as $(a\cos\theta, b\sin\theta)$ where θ varies from 0 to 2π. θ is called the eccentric angle of P. A corresponds to $\theta = 0$, B corresponds to $\theta=\frac{\pi}{2}$, A' corresponds to $\theta = \pi$ and B' corresponds to $\theta=\frac{3\pi}{2}$.

Result

Equations of the tangent and normal at any point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

The parametric form of representation of any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\Rightarrow \left.\begin{array}{l} x = a\cos\theta \\ y = b\sin\theta \end{array}\right\} \Rightarrow \frac{dy}{dx}=\frac{b\cos\theta}{-a\sin\theta}$$

Slope of the tangent at 'θ' $= -\frac{b}{a}\times\frac{\cos\theta}{\sin\theta}$; Slope of the normal at 'θ' $= \frac{a\sin\theta}{b\cos\theta}$

Equation of the tangent at 'θ' is given by

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$\Rightarrow \quad ay\sin\theta - ab\sin^2\theta = -xb\cos\theta + ab\cos^2\theta$$

$$\Rightarrow \quad bx\cos\theta + ay\sin\theta = ab$$

$$\Rightarrow \quad \frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

Equation of the normal at 'θ' is given by

$$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

$$yb\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$$

$$\Rightarrow \quad ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$$

$$\Rightarrow \quad \frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$$

CONCEPT STRANDS

Concept Strand 38

P is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If PN is the ordinate of P and the tangent at P meets the major axis produced in T, prove that $CN \cdot CT = CA^2$ where C is the centre of the ellipse.

Solution

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Equation of the tangent at any point 'θ' on the ellipse is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

Putting y = 0 in the above, the coordinates of T are $\left(\frac{a}{\cos\theta}, 0\right)$.

We have CN = a cos θ, $CT = \frac{a}{\cos\theta}$

$\Rightarrow$ CN × CT = $a^2 = CA^2$

Concept Strand 39

P is a point on an ellipse and PR is drawn perpendicular to the minor axis. The tangent at P meets the minor axis produced at V. Show that CR · CV = b^2, where 2b is the length of the minor axis of the ellipse. (Here, C is the centre of the ellipse)

Solution

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let us take P as (a cos θ, b sinθ). The equation of the tangent at P is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

Putting x = 0, $CV = \frac{b}{\sin\theta}$.

$\therefore \quad CR \times CV = (b\sin\theta)\left(\frac{b}{\sin\theta}\right) = b^2$

Concept Strand 40

The normal at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the major axis in G. Prove the following:

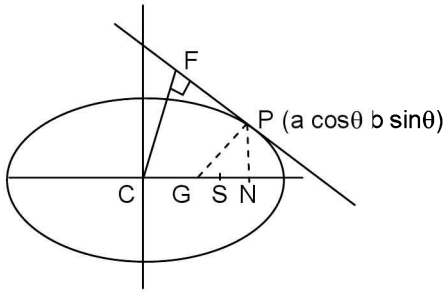
(i) CG = e^2 × CN
(ii) SG = e × SP
(iii) CF × PG = b^2

where C is the centre of the ellipse, S is one of the foci and F is the foot of the perpendicular from C to the tangent at P and N is the foot of the perpendicular from P on major axis.

Solution

Let P be the point (a cos θ, b sin θ).

Normal at P is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$



Putting y = 0,

(i) $CG = \frac{(a^2 - b^2)\cos\theta}{a} = ae^2\cos\theta = e^2 \times CN$

(ii) We have CG = ae^2 cosθ

$\Rightarrow$ SG = CS − CG = ae − ae^2 cos θ = e (a − ae cos θ) = e × SP.

(iii) CF = length of the perpendicular from C to the tangent at P

$$= \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}} = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

We have proved that CG = ae^2 cos θ

$\Rightarrow$ G is (ae^2 cos θ, 0)

$PG^2 = (a\cos\theta - ae^2\cos\theta)^2 + b^2\sin^2\theta$

$= (a^2\cos^2\theta)(1 - e^2)^2 + b^2\sin^2\theta$

$$= (a^2\cos^2\theta)\frac{b^4}{a^4} + b^2\sin^2\theta = \frac{b^2}{a^2}\left(a^2\sin^2\theta + b^2\cos^2\theta\right)$$

$$PG = \frac{b}{a}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$CF \times PG = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \times \frac{b}{a} \times \sqrt{a^2\sin^2\theta + b^2\cos^2\theta} = b^2$$

HYPERBOLA

Standard form of the equation of a hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ — (1),

where a can be less than or equal to or greater than b and $b^2 = a^2 (e^2 - 1)$ (e being > 1).

Condition for tangency

Similar to the cases of parabola and ellipse, we can derive the condition for the line y = mx + c to be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ as $c^2 = a^2 m^2 - b^2$.

Therefore, given the slope m, two tangents can be drawn to the hyperbola and their equations are $y = mx + \sqrt{a^2m^2 - b^2}$ and $y = mx - \sqrt{a^2m^2 - b^2}$

We shall give below a few results for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ which may be proved by the methods employed for ellipse.

Results

(i) Given a point $P(x_1, y_1)$,

P lies in the region I if $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1<0$

P lies on the hyperbola if $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1=0$

and P lies in the region II if $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1>0$

(refer Fig. 2.35)

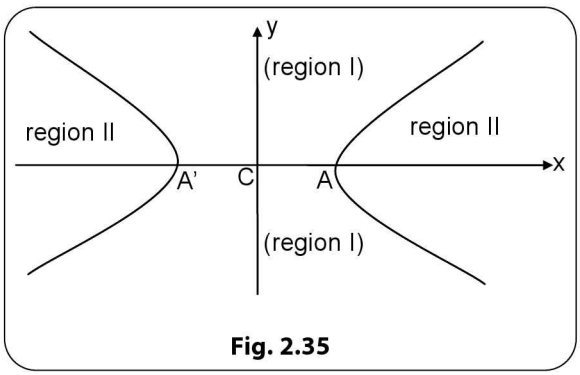


Fig. 2.35

From any point $P(x_1, y_1)$ in region II, two tangents can be drawn to a hyperbola. If P is on the hyperbola, only one tangent can be drawn (which is the tangent at P to the hyperbola)

(ii) The locus of points of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is the circle $x^2 + y^2 = a^2 - b^2$, which is called the director circle of the given hyperbola.

Tangents drawn from any point on the director circle to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ will be at right angles to each other.

It may be mentioned that the above results are valid only for a > b.

(iii) The locus of the foot of perpendicular from a focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to the tangent at any point on the hyperbola is the circle $x^2 + y^2 = a^2$. (may be called the auxiliary circle of the hyperbola)

Parametric form of representation of a point on the hyperbola

It is easily verified that x = a sec θ, y = b tan θ satisfies the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for all θ. Hence, any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ can be represented in the parametric form as x = a sec θ, y = b tan θ, $0 \le \theta < 2\pi$.

The point 'θ' on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ means the point whose coordinates are (a sec θ, b tan θ).

The following results can be established immediately.

Results

(i) Equation of the tangent at 'θ' on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$

(ii) Equation of the normal at 'θ' on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

Asymptotes of the hyperbola

Recall that a straight line L is an asymptote to a curve, if the distance 'd' from a variable point P on the curve to this straight line approaches zero as P recedes to infinity.

OR

A straight line which touches a curve at a point at infinite distance from the origin (i.e., the line is said to touch the curve at infinity) but does not lie altogether at infinity is called an asymptote of the curve.

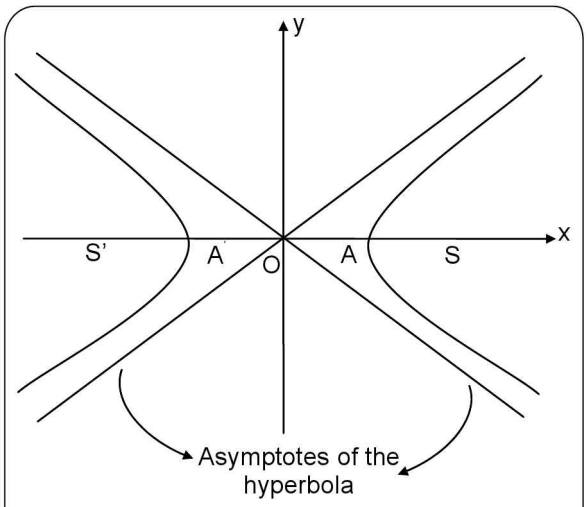


Fig. 2.36

Let $y = mx + c$ — (1)

represent the equation of an asymptote of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{— (2)}$$

The coordinates of the points of intersection of (1) and (2) are given by

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

or $(b^2 - a^2m^2)x^2 - 2ma^2cx - a^2(c^2 + b^2) = 0$ — (3)

If (1) is to be an asymptote of (2) both roots of (3) must be infinite.

The condition for the roots to be infinite are

$b^2 - a^2m^2 = 0$ and $-2ma^2c = 0$ — (4)

[The quadratic equation $Ax^2 + Bx + C = 0$ will have both roots infinite if $A = 0, B = 0$]

(4) leads to $m = \pm\frac{b}{a}$ and $c = 0$

Therefore, the equation of the two asymptotes of the hyperbola are $y = \frac{b}{a}x$ and $y = \frac{-b}{a}x$

Or, the combined equation of the asymptotes is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{— (5)}$$

(refer Fig. 2.36)

Observations

(i) Asymptotes pass through the centre of the hyperbola.

(In the case of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the asymptotes pass through the origin which is the centre of the hyperbola)

(ii) The slopes of the asymptotes are $+\frac{b}{a}$ and $-\frac{b}{a}$.

That is, the asymptotes of a hyperbola are equally inclined to the transverse and conjugate axes of the hyperbola.

[In the case of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the two asymptotes are equally inclined to the axes of coordinates, as the x and y axes happen to be the transverse and conjugate axes]

(iii) The combined equations of the asymptotes of a hyperbola and that of the hyperbola differ only in the constant term.

Suppose $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a hyperbola. Then, its asymptotes will be of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + k = 0$, where k is a constant to be determined by invoking the condition that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + k = 0$ represents a pair of lines.

(iv) If 2θ is the angle between the asymptotes represented by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, we have $\tan\theta = \frac{b}{a}$.

$$\therefore \quad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\frac{b}{a}}{1-\frac{b^2}{a^2}} = \frac{2ab}{(a^2-b^2)}.$$

Again, $\sec^2\theta = 1 + \tan^2\theta$

$$= 1 + \frac{b^2}{a^2} = \frac{a^2 + a^2(e^2-1)}{a^2} = e^2$$

or, $\sec\theta = e \Rightarrow$ The angle between the two asymptotes is given by $2\sec^{-1}(e)$

CONCEPT STRANDS

Concept Strand 41

Prove that the product of the perpendiculars drawn from any point on a hyperbola to its asymptotes is a constant.

Solution

Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

If $P(x_1, y_1)$ is a point on the hyperbola; the length of perpendiculars from P to the two asymptotes $\frac{x}{a}+\frac{y}{b}=0$ and $\frac{x}{a}-\frac{y}{b}=0$ are given by $\dfrac{\frac{x_1}{a}+\frac{y_1}{b}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$ and $\dfrac{\frac{x_1}{a}-\frac{y_1}{b}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$

Their product $\dfrac{\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}}{\frac{1}{a^2}+\frac{1}{b^2}}=\dfrac{1}{\frac{1}{a^2}+\frac{1}{b^2}}$, since (x_1, y_1) is a point on the hyperbola.

= constant

Concept Strand 42

Prove that the foot of the perpendicular from a focus of a hyperbola to one of its asymptotes lies on the corresponding directrix as well as on the auxiliary circle of the hyperbola.

Solution

Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ — (1)

Consider the asymptote $\frac{x}{a}-\frac{y}{b}=0$ — (2)

Equation of the line perpendicular to the above asymptote and passing through the focus S (ae, 0) is

$$y=\frac{-a}{b}(x-ae) \qquad \text{— (3)}$$

Solving the (2) and (3), the coordinates the foot of the perpendicular are $\left(\frac{a}{e}, \frac{b}{e}\right)$. It is clear that the point $\left(\frac{a}{e}, \frac{b}{e}\right)$ lies on the corresponding directrix $x=\frac{a}{e}$ and on the auxiliary circle $x^2 + y^2 = a^2$.

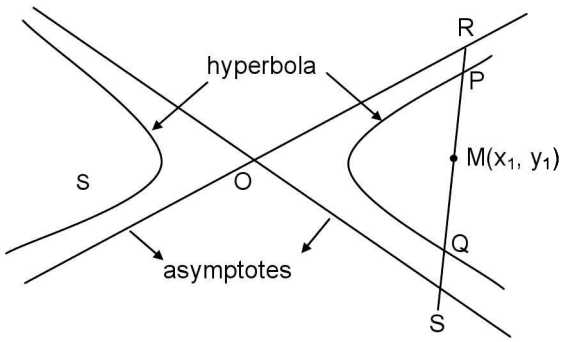
Concept Strand 43

If a straight line intersects a hyperbola P and Q and its asymptotes in R and S, then, PR = QS

Solution

Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ — (1)

Its asymptotes are $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ — (2)



A straight line intersects (1) and (2) in P, Q, R and S. (refer figure)

Let the coordinates of the mid point of PQ be $M(x_1, y_1)$. If the slope of PQ is $\tan\theta$, we may represent the equation of the line as $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$, where r represents the distance of any point (x, y) on the line PQ from the point (x_1, y_1).

Any point on the line can be represented as $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. Substituting these coordinates in (1), we get

$$\frac{(x_1+r\cos\theta)^2}{a^2}-\frac{(y_1+r\sin\theta)^2}{b^2}=1 \text{ giving}$$

$$r^2\left\{\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}\right\}+$$

$$2r\left\{\frac{x_1\cos\theta}{a^2}-\frac{y_1\sin\theta}{b^2}\right\}+\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)= \qquad \text{— (3)}$$

The two values of r obtained by solving (3) represent the distances of P and Q from the point (x_1, y_1). Since these distances are equal in magnitude but opposite in sign, the sum of the roots of (3) must be zero.

$\Rightarrow$ coefficient of r in (3) = 0

$$\Rightarrow \frac{x_1\cos\theta}{a^2} - \frac{y_1\sin\theta}{b^2} = 0$$

$$\Rightarrow \tan\theta = \frac{b^2x_1}{a^2y_1}$$

Therefore, equation of PQ is $y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$

$$\Rightarrow a^2yy_1 - a^2y_1^2 = b^2xx_1 - b^2x_1^2$$

$$\Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} \qquad \text{— (4)}$$

(4) gives the equation of the chord whose mid-point is (x_1, y_1).

Solving (4) and the equations (2), we get the coordinates of R and S as

$$\left(a\left(\frac{x_1}{a}+\frac{y_1}{b}\right), b\left(\frac{x_1}{a}+\frac{y_1}{b}\right)\right) \text{ and}$$

$$\left(a\left(\frac{x_1}{a}-\frac{y_1}{b}\right), -b\left(\frac{x_1}{a}-\frac{y_1}{b}\right)\right) \text{ respectively}$$

The mid-point of RS is easily seen to be (x_1, y_1) which is M, the mid-point of PQ. This means that PR = QS.

Concept Strand 44

Show that the portion of a tangent to a hyperbola intercepted between the asymptotes is bisected at the point of contact.

Solution

Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ — (1)

The two asymptotes are given by

$$\frac{x}{a} - \frac{y}{b} = 0 \qquad \text{— (2)}$$

and $\frac{x}{a} + \frac{y}{b} = 0$ — (3)

Let P (a sec θ, b tan θ) be a point on (1)

The tangent at P is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ — (4)

Let the tangent intersect the two asymptotes in L and M

Solving (4) and (2), we get

$x = \frac{a}{(\sec\theta - \tan\theta)} = a(\sec\theta + \tan\theta)$ and y = b(sec θ + tan θ)

Or L is [a(sec θ + tan θ), b(sec θ + tan θ)]

Similarly, solving (4) and (3), M is [a(sec θ − tan θ), −b(sec θ − tan θ)]

Mid-point of LM is easily seen to be P.

CONJUGATE HYPERBOLA

Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{— (1)}$$

(1) represents the hyperbola whose transverse axis is the x-axis and the conjugate axis is the y-axis. The length of the transverse axis is AA' = 2a and the length of the conjugate axis is BB' = 2b (see Fig. 2.37)

Suppose we have a hyperbola whose transverse axis is the y-axis and whose conjugate axis is the x-axis (i.e., BB' = 2b is the length of the transverse axis and AA' = 2a is the length of the conjugate axis of this hyperbola). (see Fig. 2.37). This hyperbola is called the conjugate hyperbola. Its equation is given by

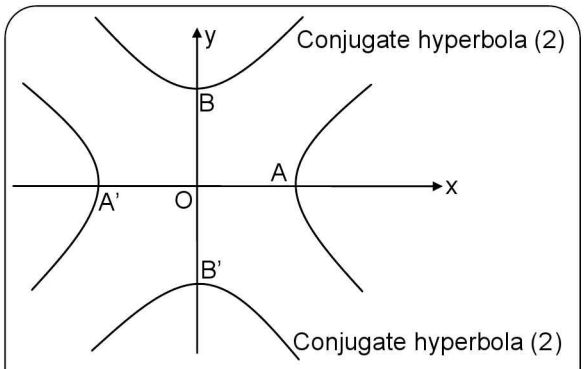


Fig. 2.37

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

or $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1 \qquad \text{— (2)}$$

Let e and e' denote the eccentricities of the hyperbola (1) and the conjugate hyperbola (2) respectively.

Then we have

$b^2 = a^2(e^2 - 1)$

$a^2 = b^2 (e'^2 - 1)$

$\Rightarrow (e^2 - 1)(e'^2 - 1) = 1$

$\Rightarrow \frac{1}{e^2}+\frac{1}{e'^2}=1$

RECTANGULAR HYPERBOLA

Definition

A hyperbola is called a rectangular hyperbola if its transverse and conjugate axes are equal.

The equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ represents a rectangular hyperbola if b = a. Therefore, the equation of a rectangular hyperbola referred to its transverse and conjugate axes as the x and y axes is $x^2 - y^2 = a^2$.

Observations

(i) The eccentricity of a rectangular hyperbola is $\sqrt{2}$. This follows from the relation $b^2 = a^2(e^2 - 1)$ by putting b = a.

(ii) The asymptotes of a rectangular hyperbola are at right angles to each other. This follows from the fact that

$$\tan 2\theta = \frac{2ab}{a^2-b^2}, \text{ as } b^2 = a^2,\ 2\theta = \frac{\pi}{2}$$

Equation of a rectangular hyperbola referred to its asymptotes as axes of coordinates

Observe that the asymptotes of a rectangular hyperbola are at right angles. We choose the asymptotes of the hyperbola as axes of coordinates. The equations of the asymptotes are X = 0, Y = 0. Therefore, the combined equation of the asymptotes is XY = 0. Since the equations of a hyperbola and that of the asymptotes differ only in the constant term, we may assume the equation of the rectangular hyperbola as XY = k. The coordinates of the vertex A of the hyperbola referred to the asymptotes as axes are (OM, ON) [Refer Fig. 2.38].

We have OM = OA $\cos 45^\circ = \frac{a}{\sqrt{2}}$ and ON = OA sin 45°.

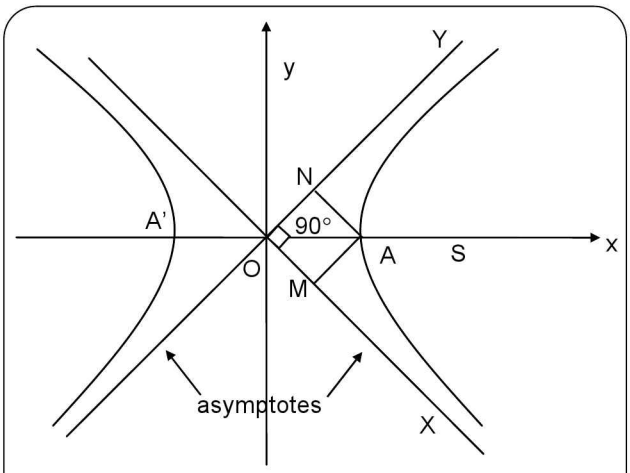


Fig. 2.38

Thus, the point A referred to the asymptotes as axes is $\left(\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}}\right)$. The hyperbola XY = k passes through $\left(\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}}\right)$. This gives k as $\frac{a^2}{2}$. Therefore, the equation of the rectangular hyperbola is $XY = \frac{a^2}{2} = c^2$ (say).

OR

The standard form of the equation of a rectangular hyperbola referred to its asymptotes as axes of coordinates is $xy = c^2$.

Parametric form of representation of a point on the rectangular hyperbola $xy = c^2$

It is easily verified that x = ct, $y = \frac{c}{t}$ where, $t \in R$ satisfy the equation $xy = c^2$. The coordinates of any point on the hyper-

bola can therefore be expressed in terms of a parameter t or, the point 't' on $xy = c^2$ means the point whose coordinates are $\left(ct, \frac{c}{t}\right)$.

Result

The equations of the tangent and normal at 't' on the rectangular hyperbola $xy = c^2$ are

$x + yt^2 = 2ct$ and $t(y - xt^2) = c(1 - t^4)$ respectively.

For, $x = ct$; $y = \frac{c}{t} \Rightarrow \frac{dy}{dx} = \frac{-1}{t^2}$

Equations of the tangent at 't' is $y - \frac{c}{t} = \frac{-1}{t^2}(x - ct)$

$\Rightarrow x + yt^2 = 2ct$

Again, slope of the normal at 't' is t^2.

Therefore, the equation of the normal at 't' is

$y - \frac{c}{t} = t^2(x - ct) \Rightarrow t(y - xt^2) = c(1 - t^4)$

SUMMARY

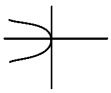
1. **Circles**
   A circle is the locus of a point which moves in a plane such that its distance from a fixed point in the plane is a constant
2. **Equations of a circle in different forms:**
   (i) Centre at (h, k) and radius r:
   $(x - h)^2 + (y - k)^2 = r^2$
   (ii) Centre at (0, 0) and radius r:
   $x^2 + y^2 = r^2$
   (iii) General equation of circle
   $x^2 + y^2 + 2gx + 2fy + c = 0$
   Centre at (–g, –f) and radius = $\sqrt{g^2 + f^2 - c}$ ($g^2 + f^2 \geq c$)
   (iv) End points of a diameter are (x_1, y_1) and (x_2, y_2):
   $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$
   (v) Parametric form of equation of the circle $x^2 + y^2 = a^2$ is $x = a\cos\theta$; $y = a\sin\theta$
3. Condition for general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a circle : coefficient of x^2 = coefficient of $y^2 \neq 0$ and coefficient of $xy = 0$
   i.e., $a = b \neq 0$ and $h = 0$ and $g^2 + f^2 \geq ac$
4. **Circles having same centres are called concentric circles**
   $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2gx + 2fy + k = 0$ are the equations of two concentric circles.
5. **Position of a point in relation to a circle**
   The point (x_1, y_1) lies inside or on or outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ according as
   $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \lesseqgtr 0$
6. Equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point $(x_1\ y_1)$ on it is $xx_1 + yy_1 + y(x + x_1) + f(y + y_1) + c = 0$
7. Length of the tangent from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fg_1 + c}$
   ((x_1, y_1) is outside the circle)
8. Condition that the line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$. Thus, equation of a tangent to the above circle with slope m is $y = mx \pm a\sqrt{1 + m^2}$
9. **Director circle**
   Locus of point of intersection of two perpendicular tangents to a circle is called the director circle of the circle.
   Equation of director circle of $x^2 + y^2 = a^2$ is
   $x^2 + y^2 = 2a^2$
10. **Common tangents to two circles**
    (i) $C_1C_2 > r_1 + r_2$: Four common tangents
    (ii) $C_1C_2 = r_1 + r_2$: Three common tangents
    (iii) $C_1C_2 < (r_1 + r_2)$ but $> |r_1 - r_2|$: Two common tangent.
    (iv) $C_1C_2 = |r_1 - r_2|$: One common tangent
    (v) $C_1C_2 < |r_1 - r_2|$: No common tangent
11. **Radical axis of two circles**
    The radical axis of two circles is defined as the locus of a point such that the tangents from it to the two circles are of equal length.
    Radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is $S_1 - S_2 = 0$
    (Coefficients of x^2 and y^2 in $S_1 = 0$ and $S_2 = 0$ must be 1)
12. **If two circles intersect at right angles to one another they are said to be orthogonal.**
    The two circle $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$
    $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$
    are orthogonal if and only if $2g_1g_2 + 2f_1f_2 = c_1 + c_2$
13. Equation of a circle passing through the points of intersection of the circle $S = 0$ and the line $L = 0$ is $S + \lambda L = 0$
    Equation of a circle passing through intersection of the two circles $S_1 = 0$ and $S_2 = 0$ can be assumed
    $S_1 + \lambda S_2 = 0$
    or $S_1 + \mu(S_1 - S_2) = 0$ (Coefficients of x^2 and y^2 in $S_1 = 0$ and $S_2 = 0$ must be 1)

Conic Sections

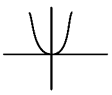
1. Focus directrix property of conics
   The locus of a point which moves in a plane such that its distance from a fixed point in the plane bears a constant ratio to its distance from a fixed line in the plane is called a conic.
   The fixed point is called the focus and the fixed line the directrix.
   The constant ratio is called the eccentricity of the conic, denoted by 'e'
   If $e = 1$, the conic is a parabola, if $e < 1$, it is an ellipse and if $e > 1$, it is a hyperbola
2. The general second degree equation
   $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents
   (i) a parabola if $\Delta = abc + 2fgh - af^2 - by^2 - ch^2 \neq 0$ and $h^2 = ab$
   (ii) an ellipse if $\Delta \neq 0$ and $h^2 < ab$
   (iii) a hyperbola if $\Delta \neq 0$ and $h^2 > ab$
   (iv) a rectangular hyperbola if $\Delta \neq 0$ and $a + b = 0$

Parabola

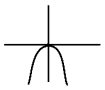
3. **Standard equation of a parabola**
   $y^2 = 4ax$, $a > 0$
   (i) vertex = (0, 0)
   (ii) Focus = (a, 0)
   (iii) Equation of the directrix $x = -a$
   (iv) Equation of axis $y = 0$
   (v) Equation of tangent at the vertex, $x = 0$
   (vi) line of latus rectum : $x = a$
   (vii) Length of the latus rectum = 4a
4. **Other forms of the equations of parabola**
   (i) $y^2 = -4ax$, $a > 0$
   Focus $(-a, 0)$
   Directrix $x = a$



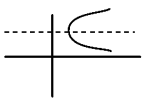
   (ii) $x^2 = 4ay$, $a > 0$
   Focus = (0, a)
   Directrix $y + a = 0$



   (iii) $x^2 = -4ay$, $a > 0$
   Focus $(0, -a)$
   Directrix $y - a = 0$



   (iv) General form
   $(y - k)^2 = 4a(x - \ell)$, $a > 0$
   Vertex(ℓ, k)
   Axis $y - k = 0$
   Directrix $x - \ell + a = 0$



5. The condition that the line $y = mx + c$ is a tangent to the parabola
   $y^2 = 4ax$ is $c = \frac{a}{m}$ $(m \neq 0)$
   Equation of tangent with slope m is $y = mx + \frac{a}{m}$.
6. A point (x_1, y_1) lies inside (focus - side) or on or outside the parabola according as $y_1^2 \lessgtr 4ax_1$
7. Parametric representation of a point on the parabola $y^2 = 4ax$ is $x = at^2, y = 2at$.
8. For the parabola represented by $x = at^2$, $y = 2at$, tangent at 't' is $y = \frac{x}{t} + at$ and normal at 't' is $y + xt = 2at + at^3$

Ellipse

9. Standard form of the equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$
   where, $b^2 = a^2(1 - e^2)$
   (i) Foci are at (ae, 0) and (−ae, 0)
   (ii) The corresponding directrices are $x = \frac{a}{e}$ and $x = \frac{-a}{e}$
   (iii) vertices $(\pm a, 0)$, $(0, \pm b)$
   (iv) length of major axis = 2a, length minor axis = 2b
   (v) eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}}$
   (vi) Length of latus rectum of the ellipse $= \frac{2b^2}{a}$
10. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $a < b$, $a^2 = b^2(1 - e^2)$
    Foci = Foci are at $(0, \pm be)$
    Equation of the directrices are $y = \pm\frac{b}{e}$
11. Condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 + b^2$;
    Equation of tangent of slope m is $y = mx \pm \sqrt{a^2m^2 + b^2}$
12. A point (x_1, y_1) lies outside, on or inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ according as $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \gtreqless 0$
13. Locus of feet of perpendiculars from the foci to a tangent to an ellipse is a circle and is called the auxiliary circle of the ellipse. Auxiliary circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2$

14. Parametric form of representation of a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$x = a\cos\theta$, $y = b\sin\theta$ where $0 \le \theta \le 2\pi$

θ is called the eccentric angle of the point.

15. Equation of the tangent at 'θ' (i.e., at the point (a cosθ, bsinθ) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

Equation of the normal at 'θ' on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$

16. Locus of point of intersection of perpendicular tangents to an ellipse is a circle called the director circle of the ellipse. Director circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$x^2 + y^2 = a^2 + b^2$

Hyperbola

17. Standard form of equation of a hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ $= 1$, where $b^2 = a^2(e^2 - 1)$

$(a \gtreqless b)$

Length of transverse axis = 2a

Length of conjugate axis = 2b

Eccentricity $e = \sqrt{1+\frac{b^2}{a^2}}$

Foci are at $(\pm ae, 0)$

Corresponding directrices are $x = \pm \frac{a}{e}$

18. Condition that the line $y = mx + c$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 - b^2$;

Equation of the tangent of slope m is $y = mx \pm \sqrt{a^2m^2-b^2}$

19. Parametric form representation of a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x = a\sec\theta$, $y = b\tan\theta$

$0 \le \theta < 2\pi$

20. Equation of tangent at 'θ' is

$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta = 1$;

Equation of normal at θ is

$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta} = a^2 + b^2$

21. The locus of point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is the circle $x^2 + y^2 = a^2 - b^2$ called the director circle of the hyperbola. (Circle exists only if a > b)

22. Asymptotes of a hyperbola

Equation of asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$

are $y = \frac{b}{a}x$, $y = \frac{-b}{a}x$

Combined equation of the asymptotes is $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 0$

Combined equation of asymptotes of a hyperbola and that of hyperbola differ only in the constant term,

Angle between the asymptotes

$\tan 2\theta = \frac{2ab}{a^2-b^2}$ or $\theta = 2\sec^{-1}(e)$

23. Conjugate hyperbola

The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = -1$ is the conjugate of $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$. Its transverse axis is y axes and its conjugate axis is x-axes and their lenglts are 2b and 2a respectively.

If e and e' are the eccentricities of a hyperbola and the conjugate hyperbola respectively, then

$\frac{1}{e^2}+\frac{1}{e'^2} = 1$

24. A hyperbola, whose transverse axis and conjugate axis are equal in length is called a rectangular hyperbola.

Equation of a rectangular hyperbola is $x^2 - y^2 = a^2$

Eccentricity $= \sqrt{2}$

Asymptotes of a rectangular hyperbola are at right angles.

Equation of a rectangular hyperbola referred to its asymptotes as axes is $xy = c^2$

Parametric form representation of a point on $xy = c^2$ is $x = ct$, $y = \frac{c}{t}$

CONCEPT CONNECTORS

Connector 1: A circle is drawn with its centre on the line $x + y = 2$ to touch the line $4x - 3y + 4 = 0$ and to pass through the point (0, 1). Find its equation.

Solution: Let the circle be

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{— (1)}$$

We have $h + k = 2$ — (2)

$$(0 - h)^2 + (1 - k)^2 = r^2 \quad \text{— (3)}$$

and $(4h - 3k + 4)^2 = 25r^2$ — (4)

From (3) and (4),

$$(4h - 3k + 4)^2 = 25\,[h^2 + (1 - k)^2] \Rightarrow 9h^2 + 16k^2 + 24hk - 32h - 26k + 9 = 0 \quad \text{— (5)}$$

Using (2), (5) reduces to

$$9h^2 + 16(2 - h)^2 + 24h(2 - h) - 32h - 26(2 - h) + 9 = 0$$

$\Rightarrow \quad h^2 - 22h + 21 = 0 \Rightarrow h = 1, 21$

$\Rightarrow \quad k = 1, -19$

$\Rightarrow \quad r^2 = h^2 + (1 - k)^2 = 1$ or $21^2 + 20^2 = 1$ or 841

The two circles are $(x - 1)^2 + (y - 1)^2 = 1$ and $(x - 21)^2 + (y + 19)^2 = 841$

Connector 2: Find the equations of the circles touching both the axes and passing through the point (4, 2).

Solution: Since the circle touches both the axes, the centre has to be of the form (k, k) and radius equals $|k|$.

The circle may be assumed as $(x - k)^2 + (y - k)^2 = k^2$

Since it passes through (4, 2)

$(4 - k)^2 + (2 - k)^2 = k^2 \Rightarrow k^2 - 12k + 20 = 0$

giving $k = 2, 10$

The circles are $x^2 + y^2 - 4x - 4y + 4 = 0$ and $x^2 + y^2 - 20x - 20y + 100 = 0$

Connector 3: Show that the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ touches the axes of coordinates.

Solution: Putting $y = 0$ in the equation of the circle, we get $x^2 + 4x + 4 = 0$

$\Rightarrow \quad (x + 2)^2 = 0 \Rightarrow$ circle touches the x–axis, at (–2, 0)

Putting $x = 0$ in the equation of the circle, we get $y^2 - 4y + 4 = 0$

$\Rightarrow (y - 2)^2 = 0 \Rightarrow$ circle touches the y – axis at (0, 2)

Connector 4: Find the equation of the circle which touches the x–axis at (a, 0) and cuts off a chord of length p on the positive side of the y–axis. What is the equation of the circle when $a = 12$ and $p = 10$?

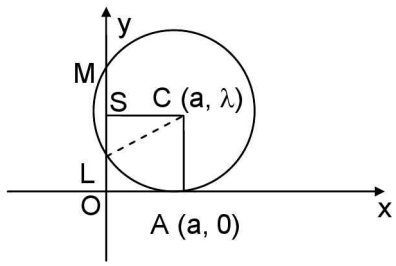
Solution: Let (a, λ) represent the centre of the required circle. The radius of the circle is λ. Now, from ΔCSL,

$$\lambda^2 = CS^2 + SL^2 = a^2 + \frac{p^2}{4}$$

The equation of the circle is

$$(x - a)^2 + \left(y - \sqrt{a^2 + \frac{p^2}{4}}\right)^2 = a^2 + \frac{p^2}{4}$$

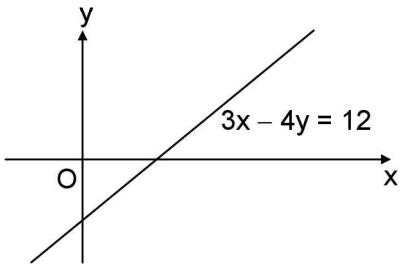
When $a = 12$, $p = 10$, $\sqrt{a^2 + \frac{p^2}{4}} = 13$



The equation of the circle in this case is $(x - 12)^2 + (y - 13)^2 = 169$
or $x^2 + y^2 - 24x - 26y + 144 = 0$

Connector 5: Find the equation of the circle which touches the coordinate axes and the line $3x - 4y = 12$.

Solution: It is clear that the circles touching the coordinate axes and the line $3x - 4y - 12 = 0$ will be either in the first, third or fourth quadrants.



<u>Circle in the first quadrant</u>

Let the circle be $(x - h)^2 + (y - h)^2 = h^2$ where $h > 0$

ie., $x^2 + y^2 - 2hx - 2hy + h^2 = 0$ —(1)

(1) touches the line $3x - 4y - 12 = 0$

$\Rightarrow \quad \dfrac{3h - 4h - 12}{5} = \pm h \Rightarrow (12 + h)^2 = 25h^2$

$24h^2 - 24h - 144 = 0 \Rightarrow h^2 - h - 6 = 0 \Rightarrow h = 3$ or -2

$h = -2$ is not admissible, as we have assured h as positive.

The circle is $x^2 + y^2 - 6x - 6y + 9 = 0$

<u>Circle in the fourth quadrant</u>

Let the circle be $(x - h)^2 + (y + h)^2 = h^2$

$\Rightarrow \quad x^2 + y^2 - 2hx + 2hy + h^2 = 0$ —(2)

(2) touches the line $3x - 4y - 12 = 0$

$\Rightarrow \quad \dfrac{3h + 4h - 12}{5} = \pm h \Rightarrow (7h - 12)^2 = 25h^2$

$\Rightarrow \quad 24h^2 - 168h + 144 = 0 \Rightarrow h^2 - 7h + 6 = 0 \Rightarrow h = 1, 6$

The two circles in the fourth quadrant are $x^2 + y^2 - 2x + 2y + 1 = 0$ and $x^2 + y^2 - 12x + 12y + 36 = 0$

<u>Circle in the third quadrant</u>

Let the circle be $(x + h)^2 + (y + h)^2 = h^2$

$\Rightarrow \quad x^2 + y^2 + 2hx + 2hy + h^2 = 0$ — (3)

(3) touches the line $3x - 4y - 12 = 0$

$\dfrac{-3h + 4h - 12}{5} = \pm h \Rightarrow (h - 12)^2 = 25h^2$

$24h^2 - 24h - 144 = 0 \Rightarrow h^2 - h - 6 = 0 \Rightarrow h = 3, -2$

$h = 3$ is not admissible. Therefore, $h = -2$

The circle is $x^2 + y^2 + 4x + 4y + 4 = 0$

The four circles are

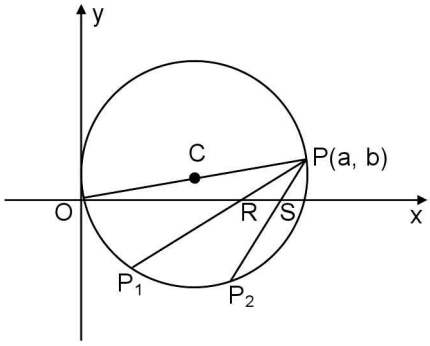
$x^2 + y^2 - 6x - 6y + 9 = 0$; $\quad x^2 + y^2 - 12x + 12y + 36 = 0$;

$x^2 + y^2 - 2x + 2y + 1 = 0$; and $\quad x^2 + y^2 + 4x + 4y + 4 = 0$

Connector 6: Prove that, from the point (a, b) on the circle $x^2 + y^2 - ax - by = 0$ two chords, each bisected by the axis of x can be drawn if $a^2 > 8b^2$.

Solution: It is clear from the equation of the circle that its centre is $C\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$ and OP where, P is (a, b), is a diameter of the circle.

Let PP_1 and PP_2 be the two chords which are bisected by the x-axis. Let these chords intersect the x-axis R and S. Since $PR = RP_1$ and $PS = SP_2$ it is clear that CR is perpendicular to PP_1 and CS is perpendicular to PP_2.



This means that we can draw two chords through P which are bisected by the x-axis only if the semicircle on CP as diameter intersects the x-axis in two real and distinct points.

The equation of the circle on CP as diameter is

$$\left(x-\frac{a}{2}\right)(x-a)+\left(y-\frac{b}{2}\right)(y-b)=0$$

Putting y = 0 in the above we get,

$$\left(x-\frac{a}{2}\right)(x-a)+\frac{a^2+b^2}{2}=0 \text{ or } x^2-\frac{3ax}{2}+\frac{a^2+b^2}{2}=0$$

The above quadratic equation in x must have real and distinct roots.

Discriminant of the above > 0

$$\frac{9a^2}{4}-4\left(\frac{a^2+b^2}{2}\right)>0 \Rightarrow a^2 > 8b^2$$

OR

Centre of the circle is at $\left(\frac{a}{2},\frac{b}{2}\right)$.

The chords through (a, b) to the circle should have their mid points on the x-axis. Let (x, 0) denote the mid point of a chord through (a, b). Then, slope of the chord = $\dfrac{x_1-\frac{a}{2}}{\frac{b}{2}}$ (since the line joining $(x_1, 0)$ and the centre of the circle is perpendicular to the chord). Equation of the chord is

$$y-0=\frac{x_1-\frac{a}{2}}{\frac{b}{2}}(x-x_1)$$

This chord passes through (a, b)

$$\Rightarrow \quad b=\frac{x_1-\frac{a}{2}}{\frac{b}{2}}(a-x_1)$$

$\Rightarrow$ $2x_1^2 - 3ax_1 + a^2 + b^2 = 0$

Since there exists two chords having mid point on the x-axis, the above quadratic equation in x_1 should give 2 distinct real roots.

$\Rightarrow$ Discriminant of the quadratic > 0

$\Rightarrow$ $9a^2 - 8(a^2 + b^2) > 0$

$\Rightarrow$ $a^2 > 8b^2$

Connector 7: A point moves such that the ratio of the lengths of the tangents from it to the circles $x^2 + y^2 + 4x + 3 = 0$ and $x^2 + y^2 - 6x + 5 = 0$ is 1 : 2. Find the locus of the point.

Solution: Let (x_1, y_1) be a point on the locus. Then, we have

$$\frac{\sqrt{x_1^2+y_1^2+4x_1+3}}{\sqrt{x_1^2+y_1^2-6x_1+5}}=\frac{1}{2}$$

$\Rightarrow \quad 4(x_1^2 + y_1^2 + 4x_1 + 3) = x_1^2 + y_1^2 - 6x_1 + 5 \Rightarrow 3(x_1^2 + y_1^2) + 22x_1 + 7 = 0$

The locus is the circle $3(x^2 + y^2) + 22x + 7 = 0$

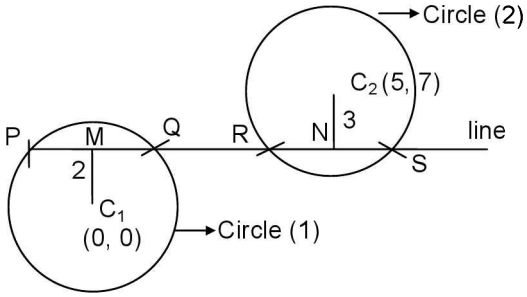
Connector 8:. Find the equation of the line inclined at 45° to the x– axis, such that the two circles $x^2 + y^2 = 4$, $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal lengths on it.

Solution: $x^2 + y^2 = 4$ — (1)

$x^2 + y^2 - 10x - 14y + 65 = 0$ — (2)

Let the equation to the required line be

$y = x + k$ — (1)



Let the intercepts PQ and RS made by these circles on the line be 2l.

Draw C_1M and C_2N perpendiculars to the line $y = x + k$.

Then, $C_1M = \left|\frac{k}{\sqrt{2}}\right|$ and $C_2N = \left|\frac{2-k}{\sqrt{2}}\right|$

$C_1Q^2 = C_1M^2 + MQ^2$ and $C_2R^2 = C_2N^2 + NR^2$

i.e., $4 = \frac{k^2}{2} + l^2$ and $9 = \frac{(2-k)^2}{2} + l^2$

Subtraction gives,

$5 = \frac{(2-k)^2 - k^2}{2} \Rightarrow k = \frac{-3}{2}$

The equation of the line is $y = x - \frac{3}{2}$ or $2x - 2y - 3 = 0$

Connector 9: Prove that the straight line $5x + 12y - 4 = 0$ touches the circle $x^2 + y^2 - 6x + 4y + 12 = 0$. Determine the coordinates of the point of contact.

Solution: Centre of the circle is at (3, –2) and its radius $= \sqrt{9+4-12} = 1$

Now, perpendicular from (3, –2) to the line $5x + 12y - 4 = 0$ is $\left|\frac{15-24-4}{13}\right| = 1$

$\Rightarrow$ Line touches the circle

Let (x_1, y_1) be the coordinates of the point of contact.

Equation of tangent at $P(x_1, y_1)$ to the circle is

$xx_1 + yy_1 - 3(x + x_1) + 2(y + y_1) + 12 = 0$

i.e., $x(x_1 - 3) + y(y_1 + 2) - 3x_1 + 2y_1 + 12 = 0$

$5x + 12y - 4 = 0$

$\therefore \quad \frac{x_1 - 3}{5} = \frac{y_1 + 2}{12} = \frac{-3x_1 + 2y_1 + 12}{-4} = \lambda$

$x_1 = 5\lambda + 3; y = 12\lambda - 2$

$-3x_1 + 2y + 12 = -4\lambda$

$-3(5\lambda + 3) + 2(12\lambda - 2) + 12 + 4\lambda = 0$

$\lambda = \frac{1}{13}$

$\therefore \quad x_1 = \frac{44}{13}; y_1 = -\frac{14}{13}$

OR

The point of contact is the point of intersection of the line $5x + 12y - 4 = 0$ and the circle $x^2 + y^2 - 6x + 4y + 12 = 0$

$$\Rightarrow \quad x^2 + \left(\frac{4-5x}{12}\right)^2 - 6x + 4\left(\frac{4-5x}{12}\right) + 12 = 0$$

$$\Rightarrow \quad 144x^2 + 16 + 25x^2 - 40x - 864x + 192 - 240x + 1728 = 0$$

$$\Rightarrow \quad 169x^2 - 1144x + 1936 = 0$$

$$\Rightarrow \quad (13x - 44)^2 = 0 \Rightarrow x = \frac{44}{13} \Rightarrow y = -\frac{14}{13}$$

Connector 10: Find the equations of the tangents to the circle $x^2 + y^2 - 6x - 4y + 5 = 0$ which make an angle of 45° with the x– axis.

Solution: Since the slope of the tangent to the circle is 1, its equation may be assumed as $y = x + c$. The equation of the circle is $(x-3)^2 + (y-2)^2 = (\sqrt{8})^2$

The length of the perpendicular from the centre (3, 2) to $y = x + c$ equals the radius $\sqrt{8}$.

i.e., $\dfrac{2-3-c}{\sqrt{2}} = \pm\sqrt{8} \Rightarrow c = -5, 3$

The tangents are $y = x - 5$ and $y = x + 3$

Connector 11: Find the area of the quadrilateral formed by the pair of tangents from (4, 5) to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ and the pair of radii corresponding to the points of contact of these tangents.

Solution: The radius of the circle $= \sqrt{4+1+11} = 4$; centre of the circle is at (2, 1)

Length of the tangent from (4, 5) to the circle $= \sqrt{16+25-16-10-11} = 2$

Area of the quadrilateral

$= 2 \times$ area of the right angled triangle whose sides are 2, 4, 2

$= 2 \times \dfrac{1}{2} \times 2 \times 4 = 8$

Connector 12: One of the diameters of the circle circumscribing a rectangle ABCD is $4y = x + 7$. If A and B are the points (–3, 4) and (5, 4) respectively, find the area of ABCD.

Solution: The centre of the circle is the point of intersection of the diameter $4y = x + 7$ and the perpendicular bisector of AB. The equation of the perpendicular bisector of AB is $x = 1$. Hence, the centre is at (1, 2). Also, (1, 2) is the mid-point of BD. We obtain D as (–3, 0).

Area of ABCD = 2 × area of the triangle ABD

$= 2 \times \dfrac{1}{2} \times AB \times AD = 8 \times 4 = 32.$

Connector 13: Show that each of the circles $x^2 + y^2 + 4y - 1 = 0$; $x^2 + y^2 + 6x + y + 8 = 0$; $x^2 + y^2 - 4x - 4y - 37 = 0$ touches the other two.

Solution: Centre and radius of the first circle : (0, –2) and $\sqrt{5}$

Centre and radius of the second circle: $\left(-3, \dfrac{-1}{2}\right)$ and $\dfrac{\sqrt{5}}{2}$

Centre and radius of the third circle: (2, 2) and $3\sqrt{5}$

It can be easily verified that

(i) distance between the centres of the first and second circles = sum of their radii ⇒ first and second circles touch externally.

(ii) Distance between the centres of the second and third circles = difference between their radii ⇒ the second and third circles touch internally.

(iii) Distance between the centres of the the third and first circles = difference between their radii ⇒ the third and the first circles touch internally.

Observe that the third circle envelops the first and second circles.

Connector 14: If the circles $x^2 + y^2 = k^2$ and $x^2 + y^2 - 6x - 8y + 9 = 0$ touch externally, find k.

Solution: Centres of the two circles are at (0, 0) and (3, 4) respectively

Radii of the two circles are k and 4.

Distance between centres = sum of the radii ⇒ $5 = k + 4 \Rightarrow k = 1$

Connector 15: Find the equations of the tangents to the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ parallel to the line $4x - 3y - 1 = 0$

Solution: Let $4x - 3y + k = 0$ represent a tangent to the given circle. Then, perpendicular from the centre (2, 3) to the line $4x - 3y + k = 0$ equals the radius $\sqrt{4+9+12}$

i.e., $\frac{8-9+k}{5} = \pm 5 \Rightarrow k - 1 = \pm 25$ or $k = -24$ or 26

The tangents are $4x - 3y - 24 = 0$ and $4x - 3y + 26 = 0$

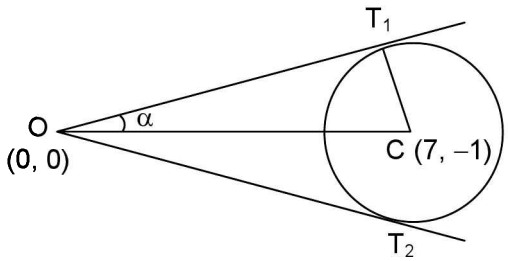
Connector 16: Find the angle between the tangents from the origin to the circle $x^2 + y^2 - 14x + 2y + 25 = 0$

Solution: If 2α is the angle between the two tangents from the origin to the circle,

$$\tan\alpha = \frac{CT_1}{OT_1} = \frac{\sqrt{49+1-25}}{\sqrt{25}} = 1$$

$\alpha = 45°$

or, angle between the tangents $= \frac{\pi}{2}$.



OR

Radius of the given circle is at (7, –1) and its radius is 5. The equation of the director circle is $(x - 7)^2 + (y + 1)^2 = 2 \times 25 = 50$

$\Rightarrow \quad x^2 + y^2 - 14x + 2y = 0$

Since (0, 0) is on this circle, angle between the tangents $= \frac{\pi}{2}$

Connector 17: Find the equation of the circle which passes through (1, 1) and which touches the circle $x^2 + y^2 + 4x - 6y - 3 = 0$ at the point (2, 3).

Solution: On substituting x = 1, y = 1 in the equation of the given circle, we get a negative number. This means that (1, 1) lies inside the given circle. The required circle therefore touches the given circle internally. Since the centre of the circle is at (–2, 3), it is clear that, the common tangents to the circles at (2, 3) is parallel to the y–axis and its equation is therefore x = 2. We may assume the equation of the required circle as $x^2 + y^2 + 4x - 6y - 3 + k(x - 2) = 0$

Since (1, 1) lies on it, $1 + 1 + 4 - 6 - 3 - k = 0$ or $k = -3$.

The circle is, $x^2 + y^2 + x - 6y + 3 = 0$

Connector 18: Find the equations of the two circles that intersect orthogonally the circles $x^2 + y^2 + 2x - 9 = 0$, $x^2 + y^2 - 8x - 9 = 0$ and touch the line $y - x = 4$. Also find the distance between their centres.

Solution: Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ — (1)

We have: $2g \times 1 = c - 9$ and $2g \times -4 = c - 9$, as the circle (1) intersects the given circles orthogonally. The above two equations give $g = 0$, $c = 9$.

Since the circle touches the line $y - x = 4$, perpendicular from the centre $(0, -f)$ to the line = radius $\sqrt{f^2 - c}$

$$\Rightarrow \quad \frac{-f-4}{\sqrt{2}} = \pm\sqrt{f^2-9} \Rightarrow f^2 - 8f - 34 = 0 \text{ or } f = 4 \pm 5\sqrt{2}$$

The two circles are $x^2 + y^2 + \left(8 \pm 10\sqrt{2}\right)y + 9 = 0$

Connector 19: The radical axis of the circles

$x^2 + y^2 + 2gx + 2fy + 10 = 0$ and $2x^2 + 2y^2 + 3x + 8y + 20 = 0$ touches the circle $x^2 + y^2 + 2x - 2y + 1 = 0$.

Show that either $g = \frac{3}{4}$ or $f = 2$.

Solution: Radical axis of the two circles is $\left(2g - \frac{3}{2}\right)x + (2f - 4)y = 0$

Centre of the given circle is $(-1, 1)$ and its radius is 1. Since the radical axis touches the given circle

$$\frac{-\left(2g - \frac{3}{2}\right) + (2f - 4)}{\sqrt{\left(2g - \frac{3}{2}\right)^2 + (2f-4)^2}} = \pm 1$$

$$\text{or} \quad \left(2f - 4 - 2g + \frac{3}{2}\right)^2 = \left(2g - \frac{3}{2}\right)^2 + (2f-4)^2$$

$$\Rightarrow \quad \left(2f - 2g - \frac{5}{2}\right)^2 = \left(2g - \frac{3}{2}\right)^2 + (2f-4)^2$$

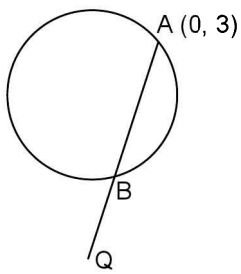
$$\Rightarrow \quad 8g + 3f - 4fg - 6 = 0 \Rightarrow 4g(2 - f) - 3(2 - f) = 0$$

$$\Rightarrow \quad f = 2 \text{ or } g = \frac{3}{4}.$$

Connector 20: From the point $A(0, 3)$ on the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ a chord AB is drawn and extended to a point Q such that $AQ = 2AB$. Find the equation of the locus of Q.

Solution: Equation of AB may be written as

$$\frac{x}{\cos\theta} = \frac{y-3}{\sin\theta} = r,$$



where $\tan\theta$ is the slope of the chord AB and r is the distance of any point on AB from (0, 3). We may represent the coordinates of any point on the line AB as $(r\cos\theta, 3 + r\sin\theta)$

Substituting these coordinates in the equation of the circle,

$$r^2\cos^2\theta + (3 + r\sin\theta)^2 + 4r\cos\theta - 6(3 + r\sin\theta) + 9 = 0$$

$$\Rightarrow \quad r^2 + 4r\cos\theta = 0 \Rightarrow r = 0, -4\cos\theta$$

$r = 0$ corresponds to A and $r = -4\cos\theta$ corresponds to B.

Since $AQ = 2AB$, the value of r corresponding to Q is $2 \times (-4\cos\theta) = -8\cos\theta$.

If (x, y) is the point Q,

$x = (-8\cos\theta)\cos\theta = -8\cos^2\theta$ and $y = 3 - (8\cos\theta)\sin\theta$

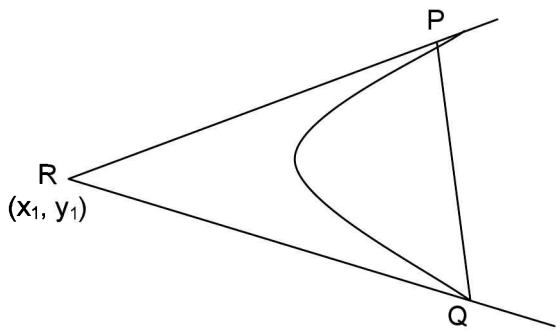
$(y-3)^2 = 64\cos^2\theta\ \sin^2\theta$

$$= 64\left(\frac{-x}{8}\right)\left[1-\left(\frac{-x}{8}\right)\right] = -x\,(8+x)$$

$\Rightarrow$ locus of Q is the circle $x^2 + y^2 + 8x - 6y + 9 = 0$

Connector 21: P and Q are two points on the parabola $y^2 = 4ax$. The tangents at P and Q to the parabola intersect at the point (x_1, y_1). Find the equation of PQ.

Solution: PR and QR are tangents at P and Q to the parabola. We are given the coordinates of R. we have to find the equation of PQ.



Let P be t_1 and Q be t_2.

Then, the tangents at P and Q are

$$y = \frac{x}{t_1} + at_1,\ y = \frac{x}{t_2} + at_2$$

They intersect at the point

$(at_1t_2, a(t_1 + t_2))$

Given: $\left.\begin{aligned} x_1 &= at_1t_2 \\ y_1 &= a(t_1+t_2) \end{aligned}\right\}$ —(1)

The equation of the chord PQ is $y(t_1 + t_2) = 2x + 2at_1t_2$

Substituting for $(t_1 + t_2)$ and t_1t_2, the equation of the chord PQ is

$$\frac{yy_1}{a} = 2x + 2x_1$$

or the equation is $yy_1 = 2a(x + x_1)$

PQ is called the chord of contact of tangents drawn from the point (x_1, y_1) to the parabola $y^2 = 4ax$.

Connector 22: Show that the line $lx + my + n = 0$ is a normal to the parabola $y^2 = 4ax$ if $al^3 + 2alm^2 + m^2n = 0$.

Solution: $lx + my + n = 0$ — (1)

Let (1) be the normal at t.

Then, its equation is $y + xt = 2at + at^3$ — (2)

(1) and (2) represent the same line

$$\Rightarrow \frac{t}{l} = \frac{1}{m} = \frac{2at + at^3}{-n} \Rightarrow t = \frac{l}{m}.$$

Condition is $2a\left(\frac{l}{m}\right) + a\left(\frac{l^3}{m^3}\right) = \frac{-n}{m} \Rightarrow 2alm^2 + al^3 = -nm^2$

$\Rightarrow$ $al^3 + 2alm^2 + m^2n = 0$

Connector 23: Show that the orthocentre of the triangle formed by three tangents to a parabola lies on the directrix.

Solution: Let the equation of the parabola be $y^2 = 4ax$

Let the triangle formed by the tangents at t_1, t_2 and t_3 be DEF. Then D is the point $[at_1t_2, a(t_1 + t_2)]$. The equation of EF is $y = \frac{x}{t_3} + at_3$

Slope of the altitude through D to EF $= -t_3$

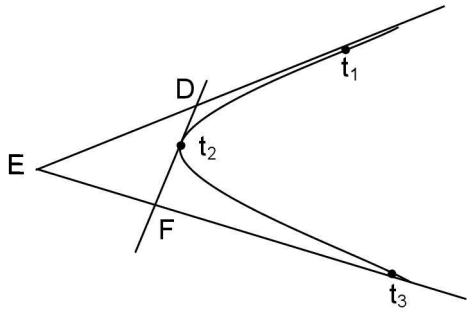
Therefore, the equation of the altitude through D to EF is

$$y - a(t_1 + t_2) = -t_3(x - at_1t_2)$$

$$\Rightarrow \quad y + t_3x = a(t_1 + t_2) + at_1t_2t_3 \qquad —(1)$$

Similarly, the equation of the altitude through F to DE is

$$y + t_2x = a(t_3 + t_1) + at_1t_2t_3 \qquad —(2)$$



The orthocentre of the triangle DEF is the point of intersection of the lines (1) and (2).

Solving (1) and (2) we get,

$$x = -a,\ y = a(t_1 + t_2 + t_3) + at_1t_2t_3$$

$\Rightarrow$ Orthocentre is the point $\{-a, a\,(t_1 + t_2 + t_3 + t_1t_2t_3)\}$

$\Rightarrow$ This point lies on $x + a = 0$ which is the directrix of the parabola.

Connector 24: Find the length of the chord of $4y = 3x - 48$ intercepted by the parabola $y^2 = 64x$. Obtain the angle subtended by the chord at the vertex.

Solution: The focus of the parabola is at (16, 0). Since (16, 0) satisfies the equation of the given line, $14y = 3x - 48$ is a focal chord of the parabola.

Let t_1 and t_2 be the extremities of the chord.

Then, $4y = 3x - 48$ and $y(t_1 + t_2) = 2x + 32t_1t_2$

(Equation of the chord joining t_1 and t_2 on $y^2 = 64x$)are representing the same line

$$\Rightarrow \quad \frac{t_1 + t_2}{4} = \frac{2}{3} = \frac{32t_1t_2}{-48}$$

$$\Rightarrow \quad t_1 + t_2 = \frac{8}{3},\ t_1t_2 = -1$$

$$\Rightarrow \quad t_1 = 3,\ t_2 = \frac{-1}{3}$$

The two extremities of the chord are $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, where $a = 16$

$\Rightarrow$ the coordinates are (144, 96) and $\left(\frac{16}{9}, \frac{-32}{3}\right)$

$$\text{Length of the chord} = \sqrt{\left(144 - \frac{16}{9}\right)^2 + \left(96 + \frac{32}{3}\right)^2}$$

$$= \sqrt{\frac{1280}{81} + \frac{320}{9}} = \frac{1}{9}\sqrt{64 \times 65} = \frac{8\sqrt{65}}{9}$$

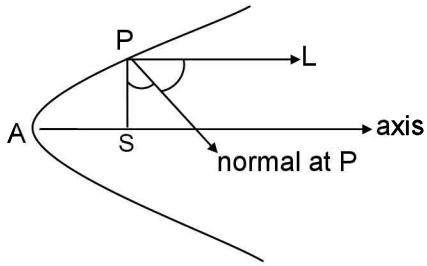
If the points are denoted by P and Q, and A is the vertex,

$$\text{slope of AP} = \frac{2at_1}{at_1^2} = \frac{2}{t_1} = \frac{2}{3};\ \text{slope of AQ} = \frac{2}{t_2} = -6$$

If θ is the angle subtended by the chord at the vertex,

$$\tan\theta = \left|\frac{\frac{2}{3} + 6}{1 - 4}\right| = \frac{20}{9}$$

Connector 25: S is the focus of a parabola and P is a point on it. If PL is the line through P parallel to the axis of the parabola show that SP and PL make the same angle with the normal at P.



Solution: Let the equation of the parabola be $y^2 = 4ax$ —(1)

Let P be $(at^2, 2at)$.

Slope of the normal at P = $-t$

$$\text{Slope of SP} = \frac{2at}{at^2 - a} = \frac{2t}{t^2 - 1} = \frac{2(-t)}{1-(-t)^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$$

where $\tan\theta$ is the slope of the normal. This means that SP makes an angle 2θ with the axis of the parabola (which is the x-axis)

$\Rightarrow$ Normal bisects the angle SPL

Result follows.

Observation

$y^2 - 4ax = 0$ may be considered as a parabolic mirror. The above geometrical property has an interesting physical interpretation. A ray of light emanating from the focus S gets reflected in the mirror in a direction parallel to the axis of the parabola.

Connector 26: Find the centre and eccentricity of the ellipse

$3(3x - 2y + 4)^2 + 2(2x + 3y - 5)^2 = 26$

Solution: The lines $3x - 2y + 4 = 0$ and $2x + 3y - 5 = 0$ are at right angles to each other.

The point of intersection of these lines is $\left(\frac{-2}{13}, \frac{23}{13}\right)$

By shifting the origin to $\left(\frac{-2}{13}, \frac{23}{13}\right)$ and rotating the axes so that the new x-axis coincides with the line $3x - 2y + 4 = 0$ and the new y-axis coincides with the line $2x + 3y - 5 = 0$, the equation of the ellipse can be written as $3Y^2 + 2X^2 = 26$

$$\Rightarrow \quad \frac{X^2}{13} + \frac{Y^2}{\left(\frac{26}{3}\right)} = 1.$$

Centre of the ellipse is at $\left(\frac{-2}{13}, \frac{23}{13}\right)$

Eccentricity e is obtained from the relation, $\frac{26}{3} = 13(1 - e^2)$

$$\Rightarrow \quad 1 - e^2 = \frac{2}{3} \Rightarrow e = \frac{1}{\sqrt{3}}$$

Connector 27: The tangent at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the tangents at A and A' (the extremities of the major axis) in L and M respectively. Prove that $AL \times A'M = b^2$.

Solution: The tangent at 'θ' on the ellipse is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

Putting x = a, and x = −a in the above, we get the coordinates of L and M as

$$\left(a,\frac{b(1-\cos\theta)}{\sin\theta}\right) \text{and} \left(-a,\frac{b(1+\cos\theta)}{\sin\theta}\right)$$

$$AL\times A'M=\frac{b^2(1-\cos^2\theta)}{\sin^2\theta}=b^2$$

Connector 28: If θ_1 and θ_2 be the eccentric angles of the extremities of a focal chord of an ellipse of eccentricity e, prove that $\cos\frac{1}{2}(\theta_1-\theta_2)=e\cos\left(\frac{\theta_1+\theta_2}{2}\right)$

Solution: Let LM be the focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

L is $(a\cos\theta_1, b\sin\theta_1)$ and M is $(a\cos\theta_2, b\sin\theta_2)$

Slope of LM $=\frac{b(\sin\theta_1-\sin\theta_2)}{a(\cos\theta_1-\cos\theta_2)}$

$$=-\frac{b\times 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)}{a\times 2\sin\frac{\theta_1-\theta_2}{2}\sin\frac{\theta_1+\theta_2}{2}}=-\frac{b\cos\frac{\theta_1+\theta_2}{2}}{a\sin\frac{\theta_1+\theta_2}{2}}$$

Equation of the chord LM is $y-b\sin\theta_1=\frac{-b\cos\frac{\theta_1+\theta_2}{2}}{a\sin\frac{\theta_1+\theta_2}{2}}(x-a\cos\theta_1)$ — (1)

Since LM is a focal chord, LM passes through (ae, 0).

Substituting in (1)

$$-b\sin\theta_1=\frac{-b\cos\frac{\theta_1+\theta_2}{2}}{a\sin\frac{\theta_1+\theta_2}{2}}(ae-a\cos\theta_1)$$

$$\Rightarrow \quad \sin\theta_1\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\left[\cos\left(\frac{\theta_1+\theta_2}{2}\right)\right]\left[e-\cos\theta_1\right]$$

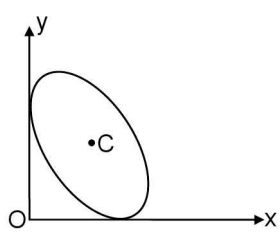
$$\Rightarrow \quad e\cos\frac{\theta_1+\theta_2}{2}=\sin\theta_1\sin\frac{\theta_1+\theta_2}{2}+\cos\theta_1\cos\frac{\theta_1+\theta_2}{2}=\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

Connector 29: An ellipse slides between two lines which are at right angles to each other. Show that the locus of its centre is a circle.

Solution: At any position in the course of the sliding of the ellipse, the tangents at the points of contact of the ellipse with the two lines (which are the x and y axes) meet at O. Since the point of intersection of perpendicular tangents to an ellipse lie on the director circle of the ellipse, O will be on the director circle of the given ellipse and the director circle has C as centre (i.e.,)

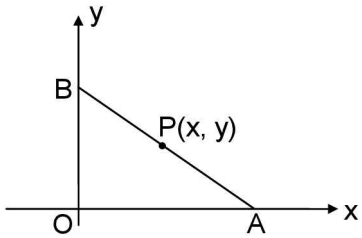
OC = a constant.

Locus of the centre C of the ellipse (C is at the same distance from O at every position of the ellipse in its sliding) is therefore a circle.



Connector 30: A bar of length 20 units moves with its ends in two fixed straight lines at right angles. Prove that a marked point on the bar describes an ellipse. If the point is at distance of 8 units from one end, what is the eccentricity of the ellipse?

Solution: Let the marked point P be dividing AB in the ratio $\lambda : 1$



Let OA = h, OB = k

We have $h^2 + k^2 = 20^2$ — (1)

(here h and k vary)

Now, since $\frac{AP}{PB} = \frac{\lambda}{1}$, $x = \frac{h}{\lambda+1}$, $y = \frac{\lambda k}{\lambda+1}$

$h = (\lambda + 1)x$

$k = \frac{(\lambda+1)y}{\lambda}$

Substituting in (1), locus of (x, y) is

$$(\lambda+1)^2x^2 + \frac{(\lambda+1)^2y^2}{\lambda^2} = 20^2 \text{ or } \frac{x^2}{\left(\frac{20}{\lambda+1}\right)^2} + \frac{y^2}{\left(\frac{20\lambda}{\lambda+1}\right)^2} = 1$$

which is an ellipse. Now, we are given $\lambda = \frac{8}{12} = \frac{2}{3}$

$$\frac{20}{\lambda+1} = \frac{20}{\left(\frac{5}{3}\right)} = 12 \text{ and } \frac{20\lambda}{\lambda+1} = 12 \times \frac{2}{3} = 8$$

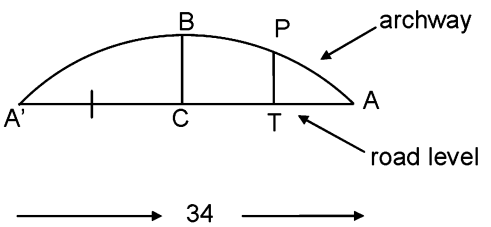
$8^2 = 12^2(1 - e^2) \Rightarrow e = \frac{\sqrt{5}}{3}$

Connector 31: An archway is in the form of a semi ellipse, the major axis of which coincides with the road level. If the breadth of the road is 34 units and a pole of 6 units when in vertical position just touches the top when 2 units from a side of the road, find the greatest height of the arch.

Solution: AA' → road level

PT → pole

Given : AT = 2, TP = 6, CB = b



The equation of the archway (ellipse) is $\frac{x^2}{17^2} + \frac{y^2}{b^2} = 1$

P(CT, TP) lies on the above ellipse

i.e.; P(15, 6) lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{15^2}{17^2} + \frac{6^2}{b^2} = 1 \qquad \Rightarrow \frac{6^2}{b^2} = 1 - \frac{15^2}{17^2} = \frac{32 \times 2}{17^2}$$

$$b^2 = \frac{6^2 \times 17^2}{64} \qquad \Rightarrow b = \frac{6 \times 17}{8} = \frac{51}{4} \text{ units}$$

Connector 32: Find the equation of the hyperbola referred to its transverse and conjugate axes as axes of coordinates and whose distance between the foci is 16 and whose eccentricity is $\sqrt{2}$.

Solution: We have 2ae = 16 and $e = \sqrt{2}$

This yields $a = 4\sqrt{2} = b$ (since the hyperbola is rectangular)

The equation of the hyperbola is $x^2 - y^2 = a^2 = 32$.

Connector 33: Find the equation of the locus of a point such that the difference between its distances from (–4, 0) and (4, 0) is always equal to 6.

Solution: If (x, y) is a point on the locus, $\sqrt{(x+4)^2+y^2}-\sqrt{(x-4)^2+y^2}=6$

We note that $\sqrt{(x+4)^2+y^2}+\sqrt{(x-4)^2+y^2}$

$$\frac{\left(\sqrt{(x+4)^2+y^2}-\sqrt{(x-4)^2+y^2}\right)\left(\sqrt{(x+4)^2+y^2}\right)+\sqrt{(x-4)^2+y^2}}{\sqrt{(x+4)^2+y^2}-\sqrt{(x-4)^2+y^2}}$$

$$=\frac{(x+4)^2+y^2-\left((x-4)^2+y^2\right)}{6}=\frac{16x}{6}=\frac{8x}{3}$$

Addition gives $2\sqrt{(x+4)^2+y^2}=\frac{8x}{3}+6 \qquad \Rightarrow 3\sqrt{(x+4)^2+y^2}=4x+9$

Squaring and simplifying, the equation to the locus is $7x^2 - 9y^2 = 63$, which is a hyperbola.

Connector 34: Show that the normal to the rectangular hyperbola $xy = c^2$ at the point 't' meets the curve again at the point $\left(\frac{-c}{t^3},-ct^3\right)$

Solution: Normal at 't' on $xy = c^2$ is $y-xt^2=\frac{c(1-t^4)}{t}$

Let the normal at t meet the the curve again at 't_1'. This means that $\left(ct_1,\frac{c}{t_1}\right)$ must satisfy the equation of the normal.

We have, $\frac{c}{t_1}-ct^2t_1=\frac{c(1-t^4)}{t}$

$\Rightarrow \frac{1}{t_1}-\frac{1}{t}=t_1t^2-t^3 \qquad \Rightarrow \frac{t-t_1}{tt_1}=t_1t^2-t^3$

$\Rightarrow \frac{t-t_1}{tt_1}=t^2(t_1-t) \qquad \Rightarrow t_1=\frac{-1}{t^3}$

The point is $\left(\frac{-c}{t^3},-ct^3\right)$

Connector 35: If the tangent and normal to a rectangular hyperbola make intercepts a_1 and a_2 on one asymptote and b_1 and b_2 on the other asymptote, prove that $a_1a_2 + b_1b_2 = 0$

Solution: Let the hyperbola be $xy = c^2$

We have $\frac{dy}{dx}=\frac{-y}{x}$

Slope of the tangent at $P(x_1, y_1)$ is $=\frac{-y_1}{x_1}$

Equation of tangent at (x_1, y_1) is $xy_1 + x_1y = 2x_1y_1$ — (1)

Equation of the normal at (x_1, y_1) is $y-y_1=\frac{x_1}{y_1}(x-x_1)$

$\Rightarrow \quad xx_1 - yy_1 = x_1^2 - y_1^2$ — (2)

Since the asymptotes of the hyperbola are the x and y axes, a_1, a_2, b_1, b_2 can be got by putting y = 0 and x = 0 in (1) and (2). We obtain

$$a_1 = 2x_1\ ;\ a_2 = \frac{x_1^2 - y_1^2}{x_1}\ ;$$

$$b_1 = 2y_1\ ;\ b_2 = \frac{-(x_1^2 - y_1^2)}{y_1} \Rightarrow a_1 a_2 + b_1 b_2 = 0$$

Connector 36: Prove that the tangent to a rectangular hyperbola terminated by its asymptotes is bisected at the point of contact and encloses a triangle of constant area.

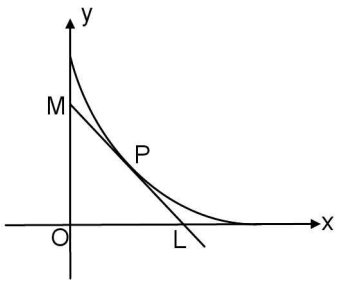
Solution: The equation of the rectangular hyperbola with its asymptotes as axes of coordinates is $xy = c^2$ — (1)

Let $P(x_1, y_1)$ be a point on the hyperbola

We have $x_1y_1 = c^2$

Differentiating (1) with respect to x,

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$



Equation of the tangent at (1) is $y - y_1 = -\frac{y_1}{x_1}(x - x_1)$

$$x_1y - x_1y_1 = -xy_1 + x_1y_1 \Rightarrow xy_1 + x_1y = 2x_1y_1$$

$$\Rightarrow \quad \frac{x}{2x_1} + \frac{y}{2y_1} = 1$$

Intercepts of the tangent on the axes are $2x_1$ and $2y_1$

or L is $(2x_1, 0)$ and $M(0, 2y_1)$

Clearly, the mid-point of LM is P

Area of the triangle OLM $= \frac{1}{2} \times 2x_1 \times 2y_1 = 2x_1y_1 = 2c^2 =$ a constant

Connector 37: Find the asymptotes of the hyperbola $3x^2 - 5xy - 2y^2 + 17x + y + 14 = 0$

Solution: Since the equations of the asymptotes and that of the hyperbola differ only in the constant form, we assume the equation of the asymptotes as

$3x^2 - 5xy - 2y^2 + 17x + y + k = 0$

The above equation must represent a pair of lines. That is, the condition $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ must be satisfied.

Substituting and simplifying, we get $k = 10$.

Asymptotes are given by $3x^2 - 5xy - 2y^2 + 17x + y + 10 = 0$

Connector 38: The asymptotes of a hyperbola are parallel to the lines $2x + 3y = 0$ and $3x - 2y = 0$. The centre of the hyperbola is at (1, 2) and it passes through the point (5, 3). Find the equation of the hyperbola.

Solution: The equations of the asymptotes may be assumed as $2x + 3y + k_1 = 0$ and $3x - 2y + k_2 = 0$.

Since the asymptotes pass through the centre of the hyperbola, (1, 2) must satisfy both the above equations. This gives $k_1 = -8$ and $k_2 = 1$.

The equation of the hyperbola having these lines as its asymptotes may be assumed as

$(2x + 3y - 8)(3x - 2y + 1) = \lambda$

This hyperbola passes through (5, 3).

$\Rightarrow \quad (10 + 9 - 8)(15 - 6 + 1) = \lambda \Rightarrow \lambda = 110$

The equation of the hyperbola is $(2x + 3y - 8)(3x - 2y + 1) = 110$

Connector 39: If P, Q, R are three points on the rectangular hyperbola $xy = c^2$ such that $\angle PRQ = \frac{\pi}{2}$, prove that PQ is parallel to the normal at R.

Solution: Let P, Q, R be t_1, t_2, t_3.

Then, slope of RP $= \dfrac{\frac{c}{t_3} - \frac{c}{t_1}}{ct_3 - ct_1} = -\dfrac{1}{t_1 t_3}$; Similarly, slope of RQ $= \dfrac{-1}{t_3 t_2}$

Since $\angle PRQ = \frac{\pi}{2}$, $\left(\dfrac{-1}{t_1 t_3}\right)\left(\dfrac{-1}{t_3 t_2}\right) = -1 \Rightarrow t_1 t_2 t_3^2 = -1$ — (1)

Slope of PQ $= \dfrac{-1}{t_1 t_2}$; Slope of the normal at R $= t_3^2$

By (1), $\dfrac{-1}{t_1 t_2} = t_3^2$

Result follows.

Connector 40: Two points P and Q on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, are such that $\angle POQ = \frac{\pi}{2}$ where O is the origin. If $a < b$, prove that $\dfrac{1}{OP^2} + \dfrac{1}{OQ^2} = \dfrac{1}{a^2} - \dfrac{1}{b^2}$.

Solution: Let $OP = r_1$ and let the slope of OP be $\tan\theta$. Then,
P may be represented as $(r_1\cos\theta, r_1\sin\theta)$

Since $\angle POQ = \frac{\pi}{2}$, and if we assume that $OQ = r_2$, Q may be represented as

$$\left(r_2\cos\left(\frac{\pi}{2}+\theta\right), r_2\cos\left(\frac{\pi}{2}+\theta\right)\right) \Rightarrow (-r_2\sin\theta, r_2\cos\theta)$$

Given that P and Q lie on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$,

$$r_1^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1 \text{ and } r_2^2\left(\frac{\sin^2\theta}{a^2} - \frac{\cos^2\theta}{b^2}\right) = 1$$

$$\Rightarrow \frac{1}{r_1^2} + \frac{1}{r_2^2} = \left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) + \left(\frac{\sin^2\theta}{a^2} - \frac{\cos^2\theta}{b^2}\right)$$

$$= \frac{1}{a^2} - \frac{1}{b^2}$$

TOPIC GRIP

Subjective Questions

1. (i) Find the equation the circle of radius $\sqrt{73}$ whose centre lies on x – axis and which passes through the point (2, 3).
   (ii) Find the equation of the circle with two diameters along the lines $x + 7y = 23$ and $5x + 2y = 16$ and which passes through the point (5, 7).
   (iii) Find the length of tangent from the point (3, –5) to the circle
   $2x^2 + 2y^2 - 60x + 12y + 210 = 0$
   (iv) Find the equation of the circle with centre at (–4, 2) and touching the line $x - y - 3 = 0$.
   (v) Find the equation of the circle, through the intersection of circles $x^2 + y^2 + 2x = 0$ and $x^2 + y^2 - 3y = 0$ with its centre on the line $x + 2y - 8 = 0$.

2. (i) Find focus, vertex, directrix and axes of the conic represented by the equation $x^2 + 2y - 3x + 5 = 0$.
   (ii) Find the equation of the tangent to the parabola $y^2 = 9x$ which passes through the point (4, 10) Also find the point of contact.
   (iii) Find the eccentricity, focus, vertices, directrices, length of the latus rectum, and equation of a latus rectum of the conic $x^2 + 4y^2 + 2x + 16y + 13 = 0$.
   (iv) Find the equation of the ellipse whose eccentricity is $\frac{2}{3}$, the latus rectum is 5, whose centre is at the origin and whose major and minor axes are x and y axes.
   (v) Find the equation of the hyperbola whose eccentricity is $\sqrt{3}$, focus is at (1, 2) and the corresponding directrix is $2x + y - 1 = 0$.

3. Find the equations of the circles whose centres lie on the line $4x + 3y - 2 = 0$ and which touch the lines $x + y + 4 = 0$ and $7x - y + 4 = 0$.

4. Show that the two circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$.

5. Find the equation of the circle passing through the point (0, 1) and touching the curve $y = x^2$ at the point (2, 4).

6. Let S and S' denote the foci, SL the semi latus rectum of an ellipse. LS' produced cuts the ellipse at P. Show that the length of the ordinate of P is $\frac{(1-e^2)^2}{(1+3e^2)}a$ where, 2a is the length of the major axis and e is the eccentricity of the ellipse.

7. If the normal to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ at the point θ intersects the curve again at the point 2θ, prove that $\sec\theta = -\frac{3}{2}$.

8. If the normal at an end of a latus rectum of an ellipse passes through an extremity of the minor axis, show that the eccentricity e of the ellipse is a solution of the equation $e^4 + e^2 - 1 = 0$

9. Find the equations of the straight lines which are tangents to the parabola $y^2 = 8x$ and the hyperbola $3x^2 - y^2 = 3$.

10. (i) Show that the conic
    $12x^2 + 7xy - 10y^2 + 13x + 45y - 23 = 0$ is a hyperbola
    (ii) find the coordinates of its centre
    (iii) find its asymptotes

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

11. The parametric equation of a point on the circle $x^2 + y^2 - 8x - 6y + 16 = 0$ is

(a) $(4 + 3\cos\theta, 3 + 3\sin\theta)$ (b) $(3\cos\theta - 4, 3\sin\theta - 3)$

(c) $(2 + 3\cos\theta, 3 + 2\sin\theta)$ (d) $(4 + 9\cos\theta, 3 + 9\sin\theta)$

12. Equation of the directrix of the parabola $4y^2 + 12x - 12y + 39 = 0$ is

(a) $x = \frac{-7}{4}$ (b) $y = \frac{3}{4}$ (c) $x - y = 0$ (d) $2x - 3y = 0$

13. If P is any point on the ellipse $4x^2 + 16y^2 = 64$ whose foci are S and S^1, then $SP + S^1P$ is

(a) 4 (b) 8 (c) 12 (d) 16

14. The normal to the hyperbola $4y^2 - 5x^2 = 20$ at $(-4, 5)$ is

(a) $y - x = 9$ (b) $5y - 4x = 41$ (c) $y + x = 1$ (d) $x + y = 5$

15. Which of the following equations does not represent a rectangular hyperbola?

(a) $xy = c^2$ (b) $x^2 - y^2 = a^2$ (c) $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 0$ (d) $x = ct, y = \frac{c}{t}$

Assertion Reason Type Question

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

16. Consider the circles

$C_1 \equiv x^2 + y^2 - 4x + 6y - 2 = 0$

$C_2 \equiv x^2 + y^2 + 9x + 6y - 2 = 0$

Statement 1

The centre of the circle passing through the intersection of the circles $C_1 = 0$ and $C_2 = 0$ and through the point (1, 1) lies on the line $3x + y = 0$

and

Statement 2

Any circle through the intersection of $C_1 = 0$ and $C_2 = 0$ can be written as $C_1 + \lambda C_2 = 0$ where, λ is a constant.

17. **Statement 1**

Let C be a circle intersecting the circles $x^2 + y^2 + 5x - 9y + 3 = 0$ and $x^2 + y^2 - 25 = 0$ orthogonally. Then, the centre of C lies on the line $5x - 9y + 28 = 0$

and

Statement 2

Centre of any circle intersecting two given circles orthogonally lies on the radical axis of the two circles.

18. Statement 1

The line $x + 2y - 6 = 0$ intersects the parabola $x^2 - 4x - 8y + 4 = 0$ at M and N. The, tangents at M and N to the parabola are at right angles.

and

Statement 2

Tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix.

19. Consider the ellipse $E : 16x^2 + 96x + 25y^2 = 256$.

Statement 1

Tangent at any point P on E is drawn. Let M denote the foot of the perpendicular from a focus S of E to the tangent at P. Then, M lies on the circle $x^2 + y^2 + 6x - 16 = 0$.

and

Statement 2

The auxiliary circle of the ellipse is $x^2 + y^2 + 6x = 0$.

20. Statement 1

The asymptotes of the hyperbola $xy - 2x + 4y - 33 = 0$ are the lines $x + 4 = 0$ and $y - 2 = 0$.

and

Statement 2

Asymptotes of the rectangular hyperbola $xy = c^2$ are $y = 0$ and $x = 0$.

Linked Comprehension Type Questions

Directions: **This section contains 2 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

Coaxal system of circles

A system of circles in which every pair of circles in the system has the same radical axis is called a coaxal system of circles.

Let $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

represent two members of the system of circles. Then,

$$\left.\begin{array}{l} S_1 + \lambda S_2 = 0 \\ \text{OR} \\ S_1 + \lambda(S_1 - S_2) = 0 \end{array}\right\} \quad \text{— (1)}$$

where λ is a constant represents a family of circles passing through the intersection of the circles $S_1 = 0$ and $S_2 = 0$. It may be observed that every pair of circles of this family has the same radical axis. (the common radical axis of the system is given by $S_1 - S_2 = 0$).

This means that (1) represents the general equation of the coaxal system of which $S_1 = 0$ and $S_2 = 0$ are members.

Results:

(i) The centres of a system of coaxal circles are collinear.
(ii) If $S_1 = 0$ and $S_2 = 0$ are non-intersecting circles, there will be two circles of the system (1) with zero radius. The centres of these circles are called the limiting points of the coaxal system.
(iii) If $S_1 = 0$ and $S_2 = 0$ are intersecting circles there will be no limiting points.
(iv) By choosing the common line of centres of a coaxal system of circles as the x-axis and the common radical axis as y-axis the equation $x^2 + y^2 + 2\lambda x + c = 0$, where λ is a parameter represents the family of circles of the coaxal system. The different members of the system can be obtained by assigning different values to λ. The limiting points of the coaxal system of circles represented by

$x^2 + y^2 + 2\lambda x + c = 0$, when they exist are given by $\left(-\sqrt{c},0\right)$ and $\left(+\sqrt{c},0\right)$.

If c is negative, the circles of the system are intersecting and hence have no limiting points.

21. Two members of a coaxal system of circles are $x^2 + y^2 - 6 = 0$ and $x^2 + y^2 + 4y - 1 = 0$. The equation of the member of the system passing through the point $(-1, 1)$ is

(a) $9x^2 + 9y^2 + 16y - 34 = 0$
(b) $x^2 + y^2 - 2x - 3y - 1 = 0$
(c) $x^2 + y^2 - 2 = 0$
(d) None of these

22. The limiting points of the coaxal system of circles defined by $x^2 + y^2 - 6x - 6y + 4 = 0$ and $x^2 + y^2 - 2x - 4y + 3 = 0$ are

(a) $(-1, 1), \left(\frac{1}{5}, \frac{-8}{5}\right)$
(b) $(1, -1), \left(\frac{1}{5}, \frac{8}{5}\right)$
(c) $(-1, 1), \left(\frac{1}{5}, \frac{8}{5}\right)$
(d) None of these

23. $(5, -3)$ and $(2, 6)$ are limiting points of a coaxal system of circles. The equation of the radical axis of the system is

(a) $x + 3y - 1 = 0$
(b) $x - 2y - 3 = 0$
(c) $x - 3y + 1 = 0$
(d) $x + y - 5 = 0$

Passage II

The variable hyperbola $\frac{x^2}{a^2} - \frac{y^2}{4} = 1$ with eccentricity e and major axis as x axis intersect the fixed ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ at exactly four points.

24. The range of possible values of the eccentricity e of hyperbola is

(a) $1 < e < \infty$
(b) $\frac{\sqrt{13}}{3} < e < \infty$
(c) $1 < e < \frac{\sqrt{13}}{3}$
(d) $\frac{\sqrt{13}}{3} < e < 2$

25. The value of a for which two vertices of the ellipse coincide with the foci of the hyperbola is

(a) $\sqrt{\frac{3}{2}}$
(b) $\sqrt{5}$
(c) $\sqrt{2}$
(d) 2

26. The value of a for which the vertices of the hyperbola coincide with the foci of the ellipse is

(a) $\sqrt{2}$
(b) 2
(c) $\sqrt{3}$
(d) $\sqrt{5}$

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers of which ONE OR MORE answers will be correct.**

27. For the circles $x^2 + y^2 + 6x - 16 = 0$ and $x^2 + y^2 - 2x + 6y - 6 = 0$

(a) Number of common tangents is 2

(b) Number of common tangents is 3

(c) Distance of the point (1, 2) from the radical axis of the circles is $\frac{7}{5}$

(d) the circles touch internally.

28. Consider the parabola $4x^2 - 4x + 8y + 17 = 0$

(a) latus rectum of the parabola is 2

(b) no part of the curve lies above the line $y + 2 = 0$

(c) focus of the parabola is at $\left(\frac{1}{2}, \frac{-3}{2}\right)$

(d) focus of the parabola is at $\left(\frac{1}{2}, \frac{-5}{2}\right)$

29. For the hyperbola $4x^2 - 25y^2 + 24x - 100y - 164 = 0$, which of the following is/are true?

(a) eccentricity $= \frac{\sqrt{29}}{5}$

(b) length of latus rectum $= \frac{8}{5}$

(c) Equation of the pair of the asymptotes is $4x^2 - 25y^2 + 24x - 100y - 64 = 0$

(d) The asymptotes are inclined to each other at $\frac{1}{2}\tan^{-1}\left(\frac{20}{21}\right)$

Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

30.

	Column I		**Column II**
(a)	$2x^2 + 5xy + 2y^2 - 11x - 7y - 4 = 0$ represents a/an	(p)	Parabola
(b)	The curve $x = 3(\cos t + \sin t)$ $y = 4(\cos t - \sin t)$ represents a/an	(q)	Pair of straight lines
(c)	$x^2 - 2xy + y^2 - 26x - 22y + 25 = 0$ represents a/an	(r)	Ellipse
(d)	$3x^2 - 11xy + 10y^2 - 7x + 13y + 4 = 0$ represents a/an	(s)	Hyperbola

IIT ASSIGNMENT EXERCISE

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

31. The equation of a circle which passes through the origin and cuts off intercepts 'a' and 'b' on the axes is
(a) $x^2 + y^2 = ab$ (b) $x^2 + y^2 - ax - by = 0$
(c) $x^2 + y^2 = a + b$ (d) $x^2 + y^2 - 2ax - 2by = 0$

32. The locus of the centre of the circle which passes through (0, a) and (0, -a) is
(a) $x + y = 0$ (b) $x = y$ (c) $y = 0$ (d) $x = 0$

33. The line $y = mx + c$ and the circle $x^2 + y^2 = a^2$ do not intersect if
(a) $a = \frac{|c|}{\sqrt{1+m^2}}$ (b) $a < \frac{|c|}{\sqrt{1+m^2}}$ (c) $a > \frac{|c|}{\sqrt{1+m^2}}$ (d) $ac = \sqrt{1+m^2}$

34. The length of the tangent from any point on the circle $x^2 + y^2 - 8x + 5y - 4 = 0$ to the circle $x^2 + y^2 - 8x + 5y = 0$ is
(a) 1 (b) 2 (c) 4 (d) 6

35. The angle between the radical axis and the line joining the centres of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0$ is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{6}$

36. The equation of the circle two of whose diameters are along $2x - 3y + 12 = 0$ and $x + 4y - 5 = 0$ and which has area 154 square units $\left(\pi = \frac{22}{7}\right)$ is
(a) $(x + 3)^2 + (y - 2)^2 = 49$ (b) $(x - 2)^2 + (y + 3)^2 = 4$ (c) $x^2 + y^2 = 154$ (d) $x^2 + y^2 = 1$

37. The equations $x = a \cos\theta + b \sin\theta$ and $y = a \sin\theta - b \cos\theta$ represent
(a) a circle (b) a parabola (c) a line (d) an ellipse

38. The distances from the origin to the centres of the circles $x^2 + y^2 + 2k_i x - c^2 = 0$; i = 1, 2, 3 are in G.P. Then the lengths of tangents drawn from any point on the circle $x^2 + y^2 = c^2$ to these circles are in
(a) A.P (b) G.P (c) H.P (d) AGP

39. The length of the common chord of the circles $x^2 + y^2 + 6x = 0$, $x^2 + y^2 + 3y = 0$ is
(a) $\frac{6}{\sqrt{5}}$ (b) $\frac{\sqrt{6}}{10}$ (c) $\frac{6}{10}$ (d) $\frac{3}{\sqrt{5}}$

40. Area of the circle in which a chord of length $2\sqrt{3}$ makes an angle $2\frac{\pi}{3}$ at the centre is
(a) 4π (b) π^2 (c) $\frac{1}{2}$ (d) $\sqrt{2}\pi$

41. If the lengths of the tangents from (3, 5) to the circles $x^2 + y^2 - 4x + 4y - 1 = 0$ and $x^2 + y^2 - 6x + 10y - c = 0$ are in the ratio1 : 3, then the value of c is
(a) −77 (b) −11 (c) − 303 (d) −77

42. Tangents drawn to $x^2 + y^2 = 16$ from P (0, h) meet the x–axis at A and B. The value of h for which the area of triangle PAB is minimum is

(a) 32 (b) $\sqrt{30}$ (c) $\sqrt{31}$ (d) $4\sqrt{2}$

43. The number of points of the form $\left(a+1, \sqrt{3}a\right)$, $a \in Z$ lying inside the region bounded by $x^2 + y^2 - 2x - 3 = 0$, and $x^2 + y^2 - 2x - 15 = 0$ are

(a) 1 (b) 2 (c) 0 (d) 3

44. If the two circles $x^2 + y^2 + 2g_1x + 2f_1y = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y = 0$ touch each other, then

(a) $f_1f_2 = g_1g_2$ (b) $f_1g_1 = f_2g_2$ (c) $f_1g_2 = f_2g_1$ (d) $f_1g_2^2 = f_2g_1^2$

45. For any $\lambda \in R$, $x^2 + y^2 - 2\lambda x - 2\lambda y + \lambda^2 = 0$ touches the line(s)

(a) $x^2 = y^2$ (b) $x = y$ (c) $x + y = 0$ (d) $xy = 0$

46. Four distinct points (0, 0), (0, 1), (1, 0), (2k, 3k) lie on a circle. Then the value of k is

(a) $\frac{13}{5}$ (b) $\frac{5}{13}$ (c) 1 (d) $\frac{1}{2}$

47. The vertex of the parabola $y^2 - 8y - x + 19 = 0$ is at

(a) (3, 3) (b) (4, 4) (c) (3, 4) (d) (4, 3)

48. A double ordinate of the parabola $y^2 = 4ax$ is of length 8a. Then the angle between the lines from the vertex to its ends is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{6}$ (d) π

49. The equation of a parabola is $(y - k)^2 = 4a(x - h)$ where, $a > 0$. The vertex and focus are respectively at

(a) (h, k) and (a, k) (b) (h + a, k + a) and (h, k)
(c) (h, k) and (h – a, k) (d) (h, k) and (h + a, k)

50. If the focus of the parabola $(y - k)^2 = 4(x - h)$ always lies between the lines $x + y = 1$ and $x + y = 3$, then

(a) $h + k < 0$ or $h + k > 2$ (b) $0 < h + k < 2$ (c) $h + k < -2$ (d) $0 < |h - k| < 2$

51. A normal to the parabola $y^2 = 4ax$ with slope 'm' touches the hyperbola $x^2 - y^2 = a^2$ if

(a) $m^6 - 4m^4 + 3m^2 + 1 = 0$ (b) $m^6 - 4m^4 - 3m^2 + 1 = 0$
(c) $m^6 + 4m^4 - 3m^2 + 1 = 0$ (d) $m^6 + 4m^4 + 3m^2 + 1 = 0$

52. The circle $x^2 + y^2 + 2\lambda x = 0$, $\lambda \in R$ touches the parabola $y^2 = 4x$ externally only if

(a) $\lambda > 0$ (b) $\lambda < 0$ (c) $\lambda > 1$ (d) $\lambda = 1$

53. If the line $x - 2 = 0$ is the directrix of the parabola $y^2 - kx - 4 = 0$, then one of the values of k is

(a) 4 (b) – 4 (c) 3 (d) 2

54. The focal distance of a point on the parabola $y^2 = 12x$ is 4. Then the abscissa of this point is

(a) 1 (b) 3 (c) 4 (d) 6

55. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ such that one of the vertices is the vertex of the parabola. Then the length of each side is

(a) $a\sqrt{3}$ (b) $8a\sqrt{3}$ (c) $4a\sqrt{3}$ (d) $2a\sqrt{3}$

56. If y_1, y_2, y_3 are the ordinates of the vertices of a triangle inscribed in the parabola $y^2 = 4ax$ $(a > 0)$, then its area is

(a) $\frac{1}{8a}\left|y_1 y_2 y_3\right|$ (b) $\frac{1}{8a}\left|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)\right|$

(c) $\frac{1}{8a}\left|(y_1 + y_2 + y_3)\right|$ (d) $8a\left|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)\right|$

57. The angle between the tangents at the extremities of any focal chord of a parabola $y^2 = 4ax$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

58. The latus rectum of an ellipse is equal to half the length of its minor axis. Its eccentricity is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

59. The auxiliary circle of the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ is

(a) $x^2 + y^2 = 25$ (b) $x^2 + y^2 = 16$ (c) $x^2 + y^2 = 9$ (d) $x^2 + y^2 = 4$

60. The line $y = 2x + k$ touches the ellipse $4x^2 + y^2 = 8$ if $k =$

(a) ± 2 (b) ± 4 (c) $\pm 2\sqrt{2}$ (d) $\sqrt{2}$

61. Sum of the focal distances of a point on an ellipse whose major and minor axes are of lengths 2a and 2b respectively, is equal to

(a) 2a (b) 2b (c) 2ae (d) 2ab

62. The sum of the squares of the eccentricities of the conics $\frac{x^2}{4}+\frac{y^2}{3}=1$ and $\frac{x^2}{4}-\frac{y^2}{3}=1$ is

(a) 2 (b) $\sqrt{2}$ (c) $\frac{\sqrt{7}}{\sqrt{3}}$ (d) $\sqrt{7}$

63. The distance between the foci of the ellipse $20x^2 + 4y^2 = 5$ is

(a) 1 (b) 2 (c) 4 (d) 3

64. The curve represented by $x = 3 (\cos t + \sin t)$, $y = 4 (\cos t - \sin t)$ is

(a) an ellipse (b) a circle (c) a hyperbola (d) a parabola

65. The equation $\frac{x^2}{r-6}+\frac{y^2}{3-r}+1=0$ represents an ellipse if and only if

(a) $r > 3$ (b) $r > 6$ (c) $3 < r < 6$ (d) $\sqrt{3} > r > \sqrt{6}$

66. The centre of the ellipse $\frac{(x+y-4)^2}{16}+\frac{(x-y)^2}{4}=1$ is

(a) (0, 0) (b) (2, 2) (c) (1, 1) (d) (1, 2)

67. The equation of the hyperbola whose transverse and conjugate axes coincide with the x and y axes respectively and which passes through the point (5, 2) and whose transverse axis is twice the conjugate axis is

(a) $x^2 - 4y^2 = 9$ (b) $4x^2 - y^2 = 96$ (c) $x^2 - 4y^2 = 100$ (d) $x^2 - 4y^2 = 21$

68. The line $x \cos \alpha + y \sin \alpha = P$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ if

(a) $P^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$ (b) $P^2 = b^2 \cos^2 \alpha + a^2 \sin^2 \alpha$

(c) $P^2 = a^2 \cos^2 \alpha - b^2 \sin^2 \alpha$ (d) $P^2 = b^2 \cos^2 \alpha - a^2 \text{ain}^2 \alpha$

69. The equation of the ellipse with eccentricity $\frac{1}{\sqrt{2}}$ circumscribing the rectangle whose sides are given by $x = \pm 2$, $y = \pm 1$, is

(a) $2x^2 + 3y^2 = 11$ (b) $x^2 + 2y^2 = 6$ (c) $3x^2 + 2y^2 = 14$ (d) $2x^2 + y^2 = 9$

70. The eccentricity and the coordinates of a focus of the conic $(5x - 10)^2 + (5y + 15)^2 = (x - 2y + 7)^2$ are given by

(a) $\sqrt{5}$, (2, –3) (b) $\frac{1}{\sqrt{5}}$, (2, –3) (c) $\frac{1}{\sqrt{5}}$, (3, –2) (d) $\frac{1}{\sqrt{5}}$, (3, 2)

71. If the foci of the ellipse $\frac{x^2}{k^2a^2}+\frac{y^2}{a^2}=1$ and hyperbola $\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$ coincide, then a value of k is

(a) $\sqrt{2}$ (b) 1 (c) $\sqrt{3}$ (d) 2

72. If 'θ' is the angle between the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\cos\frac{\theta}{2}$ is

(a) $\sqrt{e}$ (b) $\frac{1}{1+e}$ (c) $\frac{1}{\sqrt{e}}$ (d) $\frac{1}{e}$

73. The number of points from where one can draw mutually perpendicular tangents to the hyperbola $(x - 1)^2 - (y - 2)^2 = 4$ is

(a) 1 (b) 2 (c) 4 (d) infinite

74. If AB is a double ordinate of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that ΔOAB is an equilateral triangle, where O is the centre of the hyperbola, then the eccentricity e of the hyperbola satisfies

(a) $1<e<\frac{2}{\sqrt{3}}$ (b) $e<\frac{1}{\sqrt{3}}$ (c) $e>\frac{2}{\sqrt{3}}$ (d) $e=\frac{2}{\sqrt{3}}$

75. If the tangent to the ellipse $x^2 + 4y^2 = 16$ at the point with eccentric angle ϕ is a normal to the curve $x^2 + y^2 - 8x - 4y = 0$, then ϕ equals

(a) $\frac{\pi}{3}$ (b) $\frac{-\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

76. The minimum distance between the parabolas $y^2 - 4x - 8y + 40 = 0$ and $x^2 - 8x - 4y + 40 = 0$ is

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 1 (d) 2

77. A line with slope m touches both $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $x^2 + y^2 = c^2$ then

(a) $b < c < a$ (b) $c < b < a$ (c) $b < a < c$ (d) $c \in R; a > b$

78. The tangent at a point on $16x^2 + 25y^2 = 400$ meets the tangents at the ends of the major axis at T_1 and T_2. Then the equation of the circle with on T_1T_2 as diameter passes through the point

(a) (1, 0) (b) (3, 0) (c) (0, 3) (d) (0, –3)

79. The eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with latus rectum $\frac{32\sqrt{2}}{5}$ and passing through the point of intersection of $7x + 13y - 87 = 0$ and $5x - 8y + 7 = 0$ is

(a) $\sqrt{\frac{57}{32}}$ (b) $\sqrt{\frac{57}{25}}$ (c) $\sqrt{\frac{27}{21}}$ (d) $\sqrt{2}$

80. If the curves $x^2 + y^2 - 8|x| - 9 = 0$ and $y = |x| + c$ do not intersect, then

(a) $c<-4-5\sqrt{2}$ or $c>-4+5\sqrt{2}$ (b) $-4-5\sqrt{2}<c<-4+5\sqrt{2}$

(c) $-5\sqrt{2}<c<5\sqrt{2}$ (d) $-4 < c < 4$

81. If $4x^2 + 4y^2 + 4hxy + 16x + 32y + 10 = 0$ represents a circle, then h is
(a) 5 (b) 0 (c) −2 (d) −5

82. The length of the diameter of the circle which has the lines $12x + 5y + 52 = 0$ and $12x + 5y = 0$ as its tangents is
(a) 5 (b) 4 (c) 42 (d) 1

83. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches the y-axis, then
(a) $g^2 = f^2$ (b) $f^2 = c$ (c) $f^2 + c = 0$ (d) $f^2 = g^2 c$

84. If the line $3x + 4y + 20 = 0$ touches the circle $x^2 + y^2 = 16$, then the point of contact is
(a) $\left(\frac{-12}{5}, \frac{-16}{5}\right)$ (b) $\left(\frac{12}{5}, \frac{16}{5}\right)$ (c) $\left(0, \frac{-16}{5}\right)$ (d) $\left(\frac{12}{5}, 0\right)$

85. The latus rectum of the parabola $y = x^2 - 2x + 3$ is
(a) 1 (b) 4 (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

86. Two perpendicular tangents to $y^2 = 4ax$ where $a > 0$ always intersect on the line
(a) $x + a = 0$ (b) $x + 4a = 0$ (c) $x = a$ (d) $x - 4a = 0$

87. If one end of a diameter of the ellipse $4x^2 + y^2 = 64$ is at $\left(2\sqrt{3}, 4\right)$, then the other end is at
(a) $\left(-2\sqrt{3}, -4\right)$ (b) $\left(2\sqrt{3}, -4\right)$ (c) $\left(-2\sqrt{3}, 4\right)$ (d) $\left(2\sqrt{3}, 4\right)$

88. The angle between the rectangular hyperbolas $xy = 6$ and $x^2 - y^2 = 5$ at (3, 2) is
(a) 0° (b) 90° (c) 45° (d) 60°

89. The centres of the three circles $x^2 + y^2 - 4x - 6y - 12 = 0$, $x^2 + y^2 + 2x + 4y - 10 = 0$, $x^2 + y^2 - 10x - 16y - 1 = 0$ are
(a) collinear (b) non-collinear
(c) collinear and lie on x-axis (d) collinear and lie on $y = x + 1$

90. The radii of the circles $x^2 + y^2 = 1$, $x^2 + y^2 - 2x - 6y - 6 = 0$ and $x^2 + y^2 - 4x - 12y - 9 = 0$ are in
(a) G.P (b) A.P (c) H.P (d) AGP

91. The parametric equation of the circle $x^2 + y^2 + px + py = 0$ is
(a) $x = \frac{-p}{2}, y = \frac{-p}{2}$ (b) $x = \frac{p}{2}\cos\theta, y = \frac{p}{2}\sin\theta$
(c) $x = \frac{-p}{2} + \frac{p}{\sqrt{2}}\cos\theta, y = \frac{-p}{2} + \frac{p}{\sqrt{2}}\sin\theta$ (d) $x = \frac{p}{\sqrt{2}}\cos\theta, y = \frac{p}{\sqrt{2}}\sin\theta$

92. Diameter of the circle passing through (2, 4) and whose centre is at the point of intersection of lines $x + 2y - 5 = 0$ and $x + 3y - 6 = 0$ is
(a) $\sqrt{10}$ (b) 10 (c) $2\sqrt{10}$ (d) 20

93. The equation of the tangent to the circle $x^2 + y^2 = 25$, at (3, 4) is
(a) $3x + 4y = 25$ (b) $4x + 3y = 16$ (c) $x + y = 10$ (d) $3x - 4y = 20$

94. If the lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$, $(c_1c_2 \neq 0)$ intersect the co-ordinate axes at con-cyclic points, then
(a) $a = b$ (b) $a^2 = b^2$ (c) $ac_1 = bc_2$ (d) $ac_1 + bc_2 = 0$

95. The angle between the tangents to the circle $x^2 + y^2 = 169$ at (5, 12) and at (12, −5) is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

96. The length of the intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$ is

(a) $\sqrt{a^2(1+m^2)-c^2}$ (b) $\dfrac{\sqrt{a^2(1+m^2)-c^2}}{\sqrt{1+m^2}}$ (c) $\dfrac{2\sqrt{a^2(1+m^2)-c^2}}{\sqrt{1+m^2}}$ (d) $a^2(1 + m^2) = c^2$

97. The angle subtended by the common chord of the circles $x^2 + y^2 - 4x - 4y = 0$ and $x^2 + y^2 = 16$ at the origin is

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{6}$

98. The shortest distance from the point (4, 2) to the circle $x^2 + y^2 - 10x - 4y - 7 = 0$ is

(a) 1 (b) 5 (c) 6 (d) 11

99. If (p, q) is the mid-point of a chord of the parabola $y^2 = 4x$, passing through its vertex then

(a) $p^2 = 2q$ (b) $q^2 = 2p$ (c) $p^2 = p + q$ (d) $q^2 = p - q$

100. The angle of intersection of the circles $x^2 + y^2 + 2ax = 0$ and $x^2 + y^2 + 2by = 0$ at (0, 0) is

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{4}$ (c) 0 (d) $\dfrac{\pi}{2}$

101. If the curves $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2 = 0$ intersect at four concyclic points, then

(a) $a_1 - b_1 = a_2 - b_2$ (b) $h_1(a_1 - b_1) = h_2(a_2 - b_2)$ (c) $h_1 = h_2$ and $a_1b_2 = a_2b_1$ (d) $h_2(a_1-b_1)=h_1(a_2-b_2)$

102. One of the common tangents of $x^2 + y^2 = \dfrac{a^2}{2}$ and $y^2 = 4ax$ is

(a) $y = x + a$ (b) $y = -x + a$ (c) $x = y + a$ (d) $x + y = 2a$

103. The centre of the ellipse $\dfrac{(x+y-2)^2}{9}+\dfrac{(x-y)^2}{16}=1$ is at

(a) (1, 1) (b) (2, 0) (c) (0, 2) (d) (0, 0)

104. The point of intersection of the curves whose parametric equations are $x = t^2 + 1$, $y = 2t$ and $x = 2s$, $y = \dfrac{2}{s}$ is given by

(a) (1, –3) (b) (2, 2) (c) (–2, 4) (d) (1, 2)

105. Equation of the circumcircle of the triangle whose vertices are (0,0), (1,0), (–1,–3) is

(a) $x^2 + y^2 - 3x + 11y = 0$ (b) $3(x^2 + y^2 - x) + 11y = 0$

(c) $3(x^2 + y^2) + x - 11y = 0$ (d) $3(x^2 + y^2) - x + 11y = 0$

106. Locus of the foot of the perpendiculars drawn from the centre of the circle $x^2 + y^2 + 2gx + 2fy + 2c = 0$ to its chords which subtend a right angle at the centre is

(a) $(x + g)^2 + (y + f)^2 = f^2 + g^2 - c$ (b) $(x + g)^2 + (y + f)^2 = \dfrac{c}{2}$

(c) $(x + g)^2 + (y + f)^2 = 2(f^2 + g^2 - c)$ (d) $(x + g)^2 + (y + f)^2 = \dfrac{f^2 + g^2 - c}{2}$

107. A line passing through P(3,5) intersects the circle $x^2 + y^2 = 4$ at A and B. A point Q is taken on AB such that 2 PQ = PA + PB. Then, the locus of Q is

(a) $x^2 + y^2 - 3x + 5y = 0$ (b) $x^2 + y^2 + 3x - 5y = 0$

(c) $x^2 + y^2 + 3x + 5y = 0$ (d) $x^2 + y^2 - 3x - 5y = 0$

108. Circum circle of an equilateral triangle ABC is $x^2 + y^2 + 2gx + 2fy = 0$. Then its incircle is

(a) $x^2 + y^2 + 2gx + 2fy = 0$ (b) $4(x^2 + y^2) + 8gx + 8fy + 3g^2 + 3f^2 = 0$

(c) $x^2 + y^2 + 8gx + 8fy + 3g^2 + 3f^2 = 0$ (d) $x^2 + y^2 + 2gx + 2fy + 3g^2 + 3f^2 = 0$

109. The equation of the circle touching the lines given by $|x - 2| + |y - 3| = 4$ is

(a) $x^2 + y^2 + 4x + 6y - 5 = 0$
(b) $x^2 + y^2 - 4x - 6y + 5 = 0$
(c) $x^2 + y^2 - 4x + 6y - 5 = 0$
(d) $x^2 + y^2 - 4x - 6y - 5 = 0$

110. The equation of the circle with latus rectum of $y^2 + 4y + 4x + 2 = 0$ as a diameter is

(a) $4(x^2 + y^2 + x + 4y) + 1 = 0$
(b) $4(x^2 + y^2 + 4y) + 1 = 0$
(c) $x^2 + y^2 + x + 4y + 1 = 0$
(d) $x^2 + y^2 + 3x + 4y + 1 = 0$

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

111. Statement 1

A common tangent to the circles $x^2 + y^2 - 6x = 0$ and $x^2 + y^2 - 10x + 9 = 0$ is given by $2y = x + 3$

and

Statement 2

If two circles touch each other, their radical axis is the common tangent.

112. Statement 1

The equation $xy + 4x + 2y + 8 = 0$ represents a rectangular hyperbola

and

Statement 2

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a rectangular hyperbola if $a + b = 0$ and $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

113. Statement 1

The foci of the ellipse $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{49} = 1$ are at $\left(2, 3+2\sqrt{6}\right)$ and $\left(2, 3-2\sqrt{6}\right)$

and

Statement 2

The foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b$ are at $(ae, 0)$ and $(-ae, 0)$

Linked Comprehension Type Questions

Directions: **This section contains 1 paragraph. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

Let $y = f(x)$ be a curve passing through $(0, 0)$. Set of rays parallel to the x-axis get reflected on the curve. All these reflected rays are concurrent and the point of concurrence is $S(1, 0)$. P is a point on the curve, such that the tangent at P meets the y-axis at A, x-axis at T and the line $x = -1$ at B. Also the normal at P meets the x-axis at N.

114. The equation of the curve is

(a) $y^2 = x, x \in [0, \infty)$ (b) $y^2 = 2x, x \in [0, \infty)$ (c) $y^2 = 4x, x \in [0, \infty)$ (d) $x^2 + y^2 = 1, x \in R$

115. Orthocentre of triangle PAS is

(a) P (b) A (c) S (d) mid-point of PS

116. It is stated that a bisector of the angle between line PS and the line through P parallel to the x-axis is

(i) the line PN (ii) the line PT but not line PN

(iii) the lines PN and PT (iv) the line PN but not line PT

Which of the following is true?

(a) (i) and (iv) (b) (ii) (c) (i) and (iii) (d) None of these

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers of which ONE OR MORE answers will be correct.**

117. If θ is the angle subtended at P(2, 3) by the circle $x^2 + y^2 + 2x - 3y + 1 = 0$, then

(a) $\cot\left(\frac{\theta}{2}\right) = 2$ (b) $\tan\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{2}}$ (c) $\tan\theta = \frac{4}{3}$ (d) $\sec\theta = 3$

118. The equation of a conic is $25x^2 + 16y^2 = 1600$. Then

(a) its latus rectum = 25

(b) focal distances of the point $\left(4\sqrt{3}, 5\right)$ are 7 and 13

(c) line $x + 4y + 4c = 0$ is a tangent to the conic if $c^2 = \frac{281}{4}$

(d) line $2x + 8y + 3 = 0$ is a tangent to the conic

119. The equation $\frac{x^2}{16-k} + \frac{y^2}{6-k} = 1$ represents

(a) an ellipse if $k < 6$

(b) an ellipse if $k > 16$ or $k < 6$

(c) a hyperbola if $6 < k < 16$

(d) if $k = 11$, the equation represents a rectangular hyperbola

Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

120.

	Column I		Column II
(a)	The circles $x^2 + y^2 + 6x + k = 0$ and $x^2 + y^2 + 8y - 20 = 0$ are touching each other, if k equals	(p)	8
(b)	Shortest distance from (2, –7) to the circle $x^2 + y^2 - 14x - 10y - 151 = 0$ is	(q)	2
(c)	If the two circles $x^2 + y^2 + 12x + 10y + 45 = 0$ and $x^2 + y^2 + 2x - 14y + a = 0$ are touching each other externally, then the length of direct common tangent of the circles is	(r)	12
(d)	The circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 4\lambda x + 9 = 0$ have exactly two common tangents if λ equals	(s)	–112

ADDITIONAL PRACTICE EXERCISE

Subjective Questions

121. (i) Find the equation of the circle touching the line $3x + 4y = 25$ at $(3, 4)$ and having radius 5 units.
(ii) If there exists more than one circle find common tangents of these circles.

122. (i) The straight line $\frac{x}{p}+\frac{y}{q}=1$ is a chord of the circle $x^2 + y^2 = r^2$. Find the length of this chord.
(ii) Hence find the condition for the line $\frac{x}{p}+\frac{y}{q}=1$ to touch the circle $x^2 + y^2 = r^2$.

123. (i) From any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, tangents are drawn to the circle $x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$ Find the angle between these tangents.
(ii) If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is the director circle of the circle $x^2 + y^2 + 2gx + 2fy + k = 0$, find the value of k.

124. (i) Find the equations of the common tangents of the circle $(x - 3)^2 + y^2 = 9$ and $y^2 = 4x$.
(ii) Find the area bounded by these tangents.

125. (i) If the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ meets the parabola again at $(at_1^2, 2at_2)$ then prove that $t_1 = -t - \frac{2}{t}$
(ii) Find the minimum length of such chords.
(iii) Find the equation(s) of these shortest normal chord(s).

126. (i) Find the area of the triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, eccentric angles of whose vertices are α, β and γ.
(ii) Find the maximum area of a triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

127. If a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is joined to S_1 and S_2 (foci of the ellipse) then,
(i) show that the incentre of the triangle PS_1S_2 lies on another ellipse.
(ii) find the eccentricity of this ellipse in terms of the eccentricity of the given ellipse.

128. (i) Find the locus of a point of the form $(e^t + e^{-t}, e^t - e^{-t})$
(ii) Find the eccentricity of the locus
(iii) Find the slope of the tangent drawn to the curve at $t = 1$.

129. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the curve $x = pt$, $y = \frac{p}{t}$ (t being a parameter) at A, B, C and D with parameters t_1, t_2, t_3, t_4, prove that
(i) $t_1 t_2 t_3 t_4 = 1$
(ii) $\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4}=\frac{-2f}{p}$

130. The tangents to the rectangular hyperbola $xy = c^2$ and the parabola $y^2 = 4ax$ at their point of intersection are inclined at angles α and β respectively to the x–axis. Show that $\tan\alpha + 2\tan\beta = 0$.

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

131. If one end of a diameter of the circle $x^2 + y^2 - 10x + 6 = 0$ is at (1, 2), then the other end is at

(a) (9, –2) (b) (2, –9) (c) (5, 0) (d) (9, 0)

132. The angle between the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is equal to

(a) $\tan^{-1} a$ (b) $\tan^{-1} b$

(c) $\tan^{-1}\frac{b}{a}$ (d) $2\tan^{-1}\left(\frac{b}{a}\right)$

133. Slope of the tangent at the point $\left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$ on the circle $x^2 + y^2 = a^2$ is

(a) 1 (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{-1}{\sqrt{3}}$ (d) $-\sqrt{3}$

134. Two vertices of an equilateral triangle are at (–1, 0) and at (1, 0) and its third vertex lies above the x-axis. The equation of its circumcircle is

(a) $(x-0)^2 + y^2 = \frac{2}{\sqrt{3}}$ (b) $(x-0)^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \left(\frac{2}{\sqrt{3}}\right)^2$

(c) $\left(x - \frac{1}{\sqrt{3}}\right)^2 + (y-0)^2 = 1$ (d) $\left(x - \frac{1}{\sqrt{3}}\right)^2 + \left(y + \frac{1}{\sqrt{3}}\right)^2 = 1$

135. The circles $x^2 + y^2 - x - y = 0$ and $x^2 + y^2 - x + y = 0$ intersect at an angle

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

136. Locus of point of intersection of perpendicular tangents to the circle $x^2 + y^2 - 4x - 6y - 12 = 0$

(a) $x^2 + y^2 - 4x - 6y + 37 = 0$ (b) $x^2 + y^2 - 4x - 6y - 37 = 0$

(c) $x^2 + y^2 + 4x + 6y - 37 = 0$ (d) $x^2 + y^2 - 4x - 6y - 25 = 0$

137. P and Q are two points on the line through (2, 4) and having slope m. If the line segment AB joining (0,0) and (6,0) subtends right angles at P and Q then

(a) $m \notin \left[\frac{2-3\sqrt{2}}{4}, \frac{2+3\sqrt{2}}{4}\right]$ (b) $\frac{2-3\sqrt{2}}{4} < m < \frac{2+3\sqrt{2}}{4}$

(c) $\frac{2-3\sqrt{2}}{4} \le m \le \frac{2+3\sqrt{2}}{4}$ (d) $m \in R$

138. The triangle formed by the tangent to the parabola $y = x^2$ at the point $x = x_0$ ($x_0 \in [1, 2]$) the y–axis and the straight line $y = x_0^2$ has the greatest area if $x_0 =$

(a) 1 (b) $\frac{4}{3}$ (c) $\frac{3}{2}$ (d) 2

139. If a normal to $y^2 = 4ax$ makes an angle θ with the positive direction of x-axis, perpendicular distance of it from a tangent parallel to it is

(a) $a \operatorname{cosec}^2\theta \sec\theta$ (b) $a \operatorname{cosec}\theta \sec\theta$ (c) $a \operatorname{cosec}\theta \sec^2\theta$ (d) $a \operatorname{cosec}\theta \sec^3\theta$

140. If a normal chord of $y^2 = 4x$ subtends a right angle at the vertex, then the extremities of the chord are

(a) $(3, \pm 2\sqrt{2}), (2, \pm 4\sqrt{2})$ (b) $(2, \pm 2\sqrt{2}), (8, \mp 4\sqrt{2})$

(c) $(3, 4\sqrt{2}), (2, \sqrt{2})$ (d) $(3, \pm 2\sqrt{2}), (8, \mp 4\sqrt{2})$

141. The tangents at P, Q on $y^2 = 4ax$ meet at T, and S is the focus. If SP, ST, SQ are α, β, γ respectively, then the roots of $\alpha x^2 + 2\beta x + \gamma = 0$ are

(a) imaginary (b) real and distinct integers

(c) real and equal (d) real and irrational

142. Two vertices A and B of a triangle ABC are (–3, 0) and (3, 0). The vertex C is varying in such a way that $4\tan\frac{A}{2}\tan\frac{B}{2} = 1$. The locus of C is

(a) $\frac{x^2}{16} + \frac{y^2}{25} = 1$ (b) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ (c) $\frac{x^2}{16} + \frac{y^2}{9} = 1$ (d) $\frac{x^2}{25} + \frac{y^2}{16} = 1$

143. The product of the perpendiculars drawn from the foci on any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (a > b) is equal to

(a) a^2 (b) ab (c) a^2b^2 (d) b^2

144. $P(6, 2\sqrt{3})$ is a point on $\frac{x^2}{9} - \frac{y^2}{4} = 1$. If Q is the projection of P on transverse axis, the point of contact of the tangent drawn from Q to the auxiliary circle of given hyperbola is

(a) $\left(\pm\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ (b) $\left(\pm\frac{3}{2}, \pm\frac{3\sqrt{3}}{2}\right)$ (c) $\left(\frac{3}{2}, \pm\frac{3\sqrt{3}}{2}\right)$ (d) $\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$

145. Length of the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y - b)^2 = b^2$

(a) $\frac{2ab}{a+b}$ (b) $\frac{ab}{a+b}$ (c) $\frac{2ab}{\sqrt{a^2+b^2}}$ (d) $\frac{ab}{\sqrt{a^2+b^2}}$

146. The distances of the origin, from the centres of three circles $x^2 + y^2 - 2\lambda x = c^2$ where c is constant and λ a variable parameter are in geometrical progression. Then the lengths of tangents drawn to them from any point on the circle $x^2 + y^2 = c^2$ are in

(a) A.P (b) G.P.

(c) H.P. (d) not in any progression.

147. Tangents are drawn to the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ at right angles to one another. The locus of their point of intersection is

(a) $x + y = a$ (b) $x^2 + y^2 = a^2 + b^2$ (c) $ax + by = 1$ (d) $x^2 + y^2 + ab = 0$

148. The coordinates of any point on the circle through the points A(2, 2), B(5, 3) and C(3, –1) can be written in the form $(4 + \sqrt{5}\cos\theta, 1 + \sqrt{5}\sin\theta)$. Then, the coordinates of the point P on BC such that AP is perpendicular to BC are

(a) (–1, 4) (b) (4, 1) (c) (1, 4) (d) (2, 3)

149. The normals at two points P and Q on the parabola $y^2 = 4ax$ meet at a point $R(x_1, y_1)$ on the parabola then PQ^2 is
(a) $(x_1 - 8a)^2$ (b) $(x_1 - 8a)(x_1 + 4a)$ (c) $(x_1 - 4a)(x_1 + 8a)$ (d) $x_1^2 - 16a^2$

150. The equation of a tangent to the ellipse $2x^2 + 7y^2 = 14$, drawn from the point (5, 2) is
(a) $x - 9y + 13 = 0$ (b) $x - y - 3 = 0$ (c) $x + y - 3 = 0$ (d) both (a) and (b)

151. P is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and Q is the corresponding point on the auxiliary circle. If the normals at P and Q meet in R, then the locus of R is the circle $x^2 + y^2$ is
(a) $a^2 + b^2$ (b) $(a + b)^2$ (c) a^2b^2 (d) ab

152. If the tangent at a point θ on an ellipse meets the auxiliary circle of the ellipse at two points, which subtend a right angle at the centre, the eccentricity of the ellipse is
(a) $1+\sin^2\theta$ (b) $\sin^2\theta$ (c) $(1+\sin^2\theta)^{-1/2}$ (d) $(1+\sin\theta)^{1/2}$

153. If the chord joining the variable points 'α' and 'β' on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subtends a right angle at the point (a, 0) then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ is
(a) $\frac{-b}{a}$ (b) $\frac{b^2}{a^2}$ (c) $\frac{-a^2}{b^2}$ (d) $\frac{-b^2}{a^2}$

154. The locus of the mid points of chords of the hyperbola $xy = c^2$ which pass through the fixed point (2p, 2q) is the hyperbola $(x - p)(y - q) =$
(a) $p + q$ (b) $2pq$ (c) pq (d) $p^2 + q^2$

155. The equation of a circle touching the lines $x - y = 3$, $x + y = 3$ and passing through $(5, \sqrt{2})$ is
(a) $x^2 + y^2 = 27$ (b) $(x - 10)^2 + y^2 = 27$ (c) $(x - 9)^2 + y^2 = 18$ (d) $(x - 3)^2 + y^2 = 6$

156. The conditions that the two circles passing through (0, a) and (0, –a) and touching the line $y = mx + c$ are orthogonal, is
(a) $c^2 = a^2m^2$ (b) $c^2 = a^2(1 + m^2)$ (c) $c^2 = 2am^2$ (d) $c^2 = a^2(2 + m^2)$

157. The line $2x - y + 4 = 0$ is a diameter of the circle which circumscribes a rectangle ABCD. If the co–ordinates of A and B are A(4, 6), B(1, 9), then the area of rectangle ABCD is
(a) 18 (b) 8 (c) 16 (d) 12

158. If two circles $S = x^2 + y^2 + 2ax + 2by + c = 0$, and $S' = x^2 + y^2 + 2bx + 2ay + c = 0$ are intersecting each other, then
(a) $(a + b)^2 > 2c$ (b) $(a + b)^2 < 2c$ (c) $(a + b)^2 < 2b$ (d) $b + c < c$

159. If the normals at the points P, Q, R on the parabola $y^2 = 4ax$ intersect at the point (α, β), then the orthocentre of triangle PQR is
(a) $\left(\frac{2}{3}a, 0\right)$ (b) $\left(\alpha - 3a, \frac{-\beta}{2}\right)$ (c) $\left(\alpha - 6a, \frac{-\beta}{2}\right)$ (d) $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

160. If the circles $x^2 + y^2 = \lambda^2$ and $x^2 + y^2 - 6x - 8y + 9 = 0$ touch externally, then λ is
(a) –2 (b) 3 (c) 2 (d) 1

161. The equation of the tangent to the circle $x^2 + y^2 = 16$ such that the corresponding normal passes through (2, 4) is
(a) $2x - y \pm 4\sqrt{5} = 0$ (b) $x + 2y \pm 4\sqrt{5} = 0$ (c) $x - 2y \pm 4\sqrt{5} = 0$ (d) $2x + y \pm 4\sqrt{5} = 0$

162. The equation of the parabola whose vertex is at (2, 1) and whose focus is at (1, –1) is
(a) $4x^2 - 4xy + y^2 - 32x - 34y + 89 = 0$ (b) $4x^2 - 4xy + y^2 + 8x + 46y - 71 = 0$
(c) $4x^2 + 4xy + y^2 - 32x - 34y + 89 = 0$ (d) $4x^2 + 4xy + y^2 + 8x + 46y - 71 = 0$

163. The two curves whose equations are given by $7x^2 + 16y^2 = 112$ and $225x^2 - 400y^2 = 1296$

(a) have their latus rectum equal (b) have the same foci

(c) have the same vertices (d) have the same directrices

164. Let P be the mid point of a chord of $x^2 + y^2 = 8$ such that this chord cuts the curve $x^2 - 2x - 2y = 0$ at A and B. Also, AB subtends a right angle at the origin. Then P is

(a) $\left(1, \sqrt{2}\right)$ (b) $(2, 0)$ (c) $(0, -2)$ (d) $(2, 2)$

165. The equation of a common tangent to the parabola $y^2 = 8x$ and the hyperbola $81x^2 - 10y^2 = 40$ is

(a) $3y = 9x + 1$ (b) $3y = -9x - 1$ (c) $3y = 9x + 2$ (d) $3x + 3y = 2$

166. If the line $lx + my + n = 0$ is a normal to the curve $xy = c^2$ (where c is a constant), then

(a) $lm < 0$ (b) $l + m > 0, n > 0$ (c) $l \le 0; m \le 0$ (d) $l > 0, m > 0$

167. AB is a chord of the circle $x^2 + y^2 - 4x - 2y + 1 = 0$ subtending a right angle at the centre. Then the locus of the centroid of the triangle PAB, where P moves on the circle is

(a) a parabola (b) an ellipse (c) a hyperbola (d) a circle

168. Line L is the tangent to the circle $x^2 + y^2 = 1$ at (0, 1). A ray of light incident at the point (3, 1) gets reflected in L. If the reflected ray touches the circle, then the equation of the incident ray is

(a) $3x - 4y = 5$ (b) $3x + 4y = 13$ (c) $4x - 3y = 9$ (d) $4x + 3y = 15$

169. A man is walking along the boundary of the circular path, $x^2 + y^2 - 2x - 2y = 0$. If the speed of the man is $\frac{\pi}{3}$ m/sec and he starts from P(2, 2) and moves in the anti-clockwise direction, then the position of the man after $\frac{5}{\sqrt{2}}$ sec is

(a) $\left(\frac{1-\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}\right)$ (b) $\left(\frac{3-\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2}\right)$ (c) $\left(\frac{-3+\sqrt{3}}{2}, \frac{-3-\sqrt{3}}{2}\right)$ (d) $\left(\frac{-1-\sqrt{3}}{2}, \frac{-3+\sqrt{3}}{2}\right)$

170. The points of contact of the tangents drawn from P(-2, 2) to the parabola $y^2 = 16x$ are A and B. Then AB is

(a) $4\sqrt{17}$ (b) $\sqrt{5}$ (c) $2\sqrt{17}$ (d) $3\sqrt{17}$

Assertion–Reason Type Question

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

171. **Statement 1**

The line $x + 1 = 0$ is a common tangent to the circle $x^2 + y^2 - 2x - 3 = 0$ and the parabola $y^2 + 8x - 4y + 12 = 0$

and

Statement 2

The line $y = mx + c$ is a tangent to the parabola $y^2 + 4ax = 0$ if $c = \frac{a}{m}$

172. Statement 1

Circles

$C_1: x^2 + y^2 - 2x + 4y - 31 = 0$ and

$C_2: x^2 + y^2 - 8x - 4y - 149 = 0$

are such that C_1 is inside C_2

and

Statement 2

If two circles are such that distance between their centres is greater than the sum of their radii, then they do not intersect.

173. Statement 1

$4x - 3y - 5 = 0$ is a chord of the circle $x^2 + y^2 - 6x - 8y + 5 = 0$ through (2, 1).

and

Statement 2

Perpendicular distance from (4, –3) to $4x - 3y - 5 = 0$ is 4.

174. Statement 1

Given two circles $S_1 = 0$ and $S_2 = 0$ having same coefficients in x^2 and y^2, then $S_1 - S_2 = 0$ is the equation of a common tangent of the circles.

and

Statement 2

$4x + 3y - 8 = 0$ is a common tangent of the circles

$x^2 + y^2 + 2x + 2y - 7 = 0$ and $x^2 + y^2 - 6x - 4y + 9 = 0$

175. Statement 1

Distance between the foci of any ellipse of the family $x^2\cos^2\theta + y^2\cot^2\theta = 1$ is a constant.

and

Statement 2

Sum of focal distances of point on an ellipse of the family

$x^2\cos^2\theta + y^2\cot^2\theta = 1$ is not a constant.

176. Statement 1

The centre of the circle $x^2 + y^2 - 6x + 2y + 1 = 0$ lies on the pair of lines represented by the equation $x^2 + xy - 2y^2 + 3y - 1 = 0$, so that the lines are diameters of the circle.

and

Statement 2

If (α, β) be a point on any of the lines represented by $f(x, y) = 0$ where,

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, then $f(\alpha, \beta) = 0$

177. Statement 1

The eccentricity of an ellipse is the ratio of the distance between the foci to the length of its major axis.

and

Statement 2

The eccentricity of an ellipse is independent of the length of minor axis.

178. Statement 1

It is not possible to draw on a plane, a hyperbola and an ellipse having same foci.

and

Statement 2

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a conic with eccentricity e, then for a hyperbola $e > 1$ and for an ellipse $e < 1$.

179. Statement 1

Line perpendicular to x axis through P(3, 1) on the ellipse $x^2 + 3y^2 = 12$ cuts the circle $x^2 + y^2 = 12$ at Q. Then tan $\angle POQ$ is $\frac{3-\sqrt{3}}{3\sqrt{3}+1}$.

and

Statement 2

Eccentric angle θ of any point on an ellipse is the angle made by the ray joining the centre to the point on the auxiliary circle with same abscissa with positive direction of x axis.

180. Statement 1

The line $3x - 2y = 0$ will not meet the curve $9x^2 - 4y^2 = 36$.

and

Statement 2

$y = \frac{b}{a}x$ is an asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Linked Comprehension Type Questions

Directions: **This section contains 3 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

(A) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is a general equation of the second degree in x and y and it represents

(i) a circle, if $a = b \neq 0$, $h = 0$ and $g^2 + f^2 \geq ac$

(ii) a parabola, if $h^2 = ab$ and $\Delta \neq 0$

(iii) an ellipse, if $h^2 < ab$ and $\Delta \neq 0$

(iv) a hyperbola if $h^2 > ab$ and $\Delta \neq 0$

(v) a rectangular hyperbola if $a + b = 0$ and $\Delta \neq 0$

where, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(B) If the equation represents an ellipse or a hyperbola, the coordinates of the centre are obtained by solving the equations

$ax + hy + g = 0$

$hx + by + f = 0$

(C) By rotating the axes in the counter clockwise sense through an angle θ given by $\tan 2\theta = \frac{2h}{(a-b)}$, without changing the origin, the general second-degree equation transforms into a second degree equation without the xy term.

181. The centre of the ellipse $17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$ is at

(a) $(1, 1)$ (b) $(-1, 1)$ (c) $(-1, -1)$ (d) $(1, -1)$

182. If the axes are rotated through an angle $\frac{\pi}{4}$ in the counter clockwise sense without shifting the origin, the equation $x^2 - 3xy + y^2 + 10x - 10y + 21 = 0$ reduces to

(a) $X^2 - 5Y^2 - 20\sqrt{2}X + 20\sqrt{2}Y - 42 = 0$ (b) $X^2 + 5Y^2 - 20\sqrt{2}X + 20\sqrt{2}Y - 42 = 0$

(c) $X^2 - 5Y^2 + 10\sqrt{2}X - 10\sqrt{2}Y - 21 = 0$ (d) None of the above

183. The eccentricity of the ellipse represented by the equation $4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 25$ is

(a) $\frac{5}{9}$ (b) $\frac{\sqrt{5}}{3}$ (c) $\frac{3}{2\sqrt{5}}$ (d) None of these

Passage II

Let (x, y) represent a point in a Cartesian co-ordinate system.

Case (i)

If the axes are rotated (keeping the origin fixed) through an angle θ about origin, in the counter clock wise sense, the co-ordinates of the point (x, y) in the new co-ordinate system are (X, Y) given by

$X = x\cos\theta + y\sin\theta;\quad Y = -x\sin\theta + y\cos\theta$

Or $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

If we rotate the coordinate axes through an angle θ in the counter clockwise sense and then shift the origin to the point (h, k), then (X, Y),

$X = x\cos\theta + y\sin\theta - h$

$Y = -x\sin\theta + y\cos\theta - k$

Case (ii)

If the radius vector corresponding to the point (x, y) is rotated through an angle θ in the counter clockwise sense by keeping the axes fixed, the co-ordinates of the new point are given by

$X + \hat{i}Y = (x + \hat{i}y)e^{i\theta}$

$= (x + \hat{i}y)(\cos\theta + \hat{i}\sin\theta)$

$\Rightarrow X = x\cos\theta - y\sin\theta;$

$Y = x\sin\theta + y\cos\theta$

Or $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Now let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then A is orthogonal.

Hence, $A^{-1} = A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

To sum up, rotation of the radius vector through an angle θ in the clockwise sense has the same effect of rotation of the axes through the same angle in the counter clockwise sense.

184. Let the second-degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent an ellipse. Which of the following is/are invariant under the shifting of origin and the rotation of co-ordinate axes?

(i) $h^2 - ab$ (ii) $a + b$ (iii) eccentricity (iv) coordinates of foci

(a) (i) and (ii) (b) (iii) and (iv) (c) (i), (ii) and (iii) (d) all the four.

185. All the points on the ellipse $\frac{x^2}{2}+\frac{y^2}{1}=1$ are rotated about origin through an angle $\frac{\pi}{4}$ in counter clockwise sense. The equation of ellipse changes to

(a) $3X^2 + 3Y^2 + 2XY = 4$ (b) $3X^2 + 3Y^2 + XY = 2$ (c) $3X^2 + 3Y^2 - 2XY = 4$ (d) $\frac{X^2}{\sqrt{2}}+\frac{Y^2}{\sqrt{2}}-XY=1$

186. All the points on the ellipse of question no.10, after rotation are then shifted two units along positive x-direction and two units along positive y-direction. The equation then becomes

(a) $3X^2 + 3Y^2 + 2X + 2Y + 2XY = 4$ (b) $3X^2 + 3Y^2 + 2X + 2Y + XY = 2$

(c) $\frac{X^2}{\sqrt{2}}+\frac{Y^2}{\sqrt{2}}-2X-2Y-XY=1$ (d) $3X^2 + 3Y^2 - 8X - 8Y - 2XY + 12 = 0$

Passage III

The line $4x - 3y + 4 = 0$ intersects the hyperbola $xy + 4x - 5y + 4 = 0$ at the points A and B. Consider the ellipse with AB as the major axis and $\frac{1}{5}$AB as the minor axis.

187. Eccentricity of the ellipse is

(a) $\frac{\sqrt{6}}{5}$ (b) $\frac{2\sqrt{6}}{5}$ (c) $\frac{3}{5}$ (d) $\frac{\sqrt{6}}{10}$

188. Equation of the minor axis of the ellipse is

(a) $3x + 4y - 1 = 0$ (b) $4x + 3y - 6 = 0$ (c) $6x + 8y - 19 = 0$ (d) $6x - 8y + 19 = 0$

189. If C denotes the circle on AB as diameter, length of tangent from the point (2, 6) to C is given by

(a) $2\sqrt{3}$ (b) $4\sqrt{3}$ (c) 6 (d) point is inside C

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers of which ONE OR MORE answers will be correct.**

190. The tangent at $P(5\cos\theta, 3\sin\theta)$ on the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ meets the auxiliary circle $x^2 + y^2 = 25$ at two points A and B. If AB subtends 90° at the centre of the ellipse, then

(a) $\sin^2\theta = \frac{9}{16}$ (b) $\sin^2\theta = \frac{16}{25}$ (c) $e^2 = \left(1+\sin^2\theta\right)^{-1}$ (d) $e^2 = (1 + \cos^2\theta)^{-1}$

191. A circle of radius a, centre (0, 0) touches the directrix of $y^2 = 4ax$ at P. Tangents are drawn from P to the parabola touches it at Q and R. Then

(a) othocenter of ΔPQR is (a, 0) (b) othocenter of ΔPQR is (–a, 0)

(c) circumcenter of ΔPQR is (a, 0) (d) circumcenter of ΔPQR is (–a, 0)

192. The line $x - y + 2c = 0$ intersects the director circle of $x^2 + y^2 = a^2$, $(a > 0)$ in two real points if

(a) $c + a < 0$ (b) $c + a > 0$ (c) $c - a < 0$ (d) $c - a > 0$

193. A line which touches both $y^2 - 8x = 0$ and $x^2 + y^2 + 8x + 14 = 0$ is

(a) $x + y + 2 = 0$ (b) $x + y - 2 = 0$ (c) $x - y + 2 = 0$ (d) $x - y - 2 = 0$

194. Given $x^2 + y^2 = 8x$ represents the circle C and $y^2 = 4x$ represents the parabola P. Then

(a) (2, 3) is point is region bounded by C and P
(b) (2, –3) is a point inside of both C and P
(c) (2, 4) is a point outside of both C and P
(d) (2, –4) is a point is a region bounded by C and P

195. An ellipse is orthogonal to the hyperbola $x^2 - y^2 = 2$. The eccentricity of the ellipse is reciprocal of that of the hyperbola. Then

(a) equation of the ellipse is $x^2 + 2y^2 = 8$
(b) focus of the ellipse is at $\left(-4\sqrt{2}, 0\right)$
(c) directrix of the ellipse is $x + 4\sqrt{2} = 0$
(d) directrix circle of the ellipse $x^2 + y^2 = 12$

196. Auxiliary circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b) intersects the curve $y = \frac{c^2}{x}$ is $(x_i\ y_i)$, i = 1, 2, 3, 4. Then

(a) A.M of all x_i's is zero
(b) A.M of all y_i's is zero
(c) G.M of all x_i' is $|c|$
(d) G.M of all y_i' is (c)

197. Equations of ellipse E_1 and E_2 are given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (a > b). Then

(a) Director circles of E_1 and E_2 are the same
(b) Auxiliary circles of E_1 and E_2 are the same
(c) Circles, centre origin and passing through foci of E_1 and these of E_2 are the same
(d) Circle, centre origin passing through one of the points of intersection of E_1 and E_2 passes through all the points of intersection.

Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

198. The line y = mx + c is a tangent to

Column I	Column II
(a) the parabola $y^2 = 4x$, if	(p) $m = 2, c = \pm\sqrt{15}$
(b) the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$, if	(q) $m = \frac{1}{\sqrt{5}}, c = \pm 3$
(c) the hyperbola $\frac{x^2}{12} - \frac{y^2}{9} = 1$, if	(r) $m = 2, c = \pm\sqrt{39}$
(d) the circle $x^2 + y^2 = 3$, if	(s) $m = 1, c = 1$

199.

Column I	Column II
(a) Coordinates of the foci of the conic $25x^2 + 16y^2 + 50x - 192y + 201 = 0$	(p) (2, 2)
(b) Coordinates of the focus of the parabola $x^2 - 4x - 12y - 8 = 0$ are	(q) (–1, 3)
(c) Coordinates of the centre of the director circle of the circle $x^2 + y^2 - 4x - 4y - 10 = 0$ are	(r) (–4, 0)
(d) Coordinates of the focus of the parabola $y^2 + 4x + 12 = 0$ are	(s) (–1, 9)

200.

	Column I		**Column II**
(a)	Radical axis of the circles $x^2 + y^2 - 8x - 4y - 10 = 0$ and $x^2 + y^2 - 10x - 4y - 22 = 0$ is	(p)	$4x - 3y + 9 = 0$
(b)	Equation of a common tangent to the circles $x^2 + y^2 + 2x - 10y + 22 = 0$ $x^2 + y^2 - 6x - 4y + 4 = 0$ is	(q)	$2x - y - 4 = 0$
(c)	Equation of the directrix of the parabola $y^2 - 24x = 0$	(r)	$x + y + 1 = 0$
(d)	The parabola $x^2 - 4y = 0$ is touching the line	(s)	$x + 6 = 0$

SOLUTIONS

ANSWERS KEYS

Topic Grip

1. (i) $(x-10)^2+y^2=73$ or $(x+6)^2+y^2=73$
(ii) $(x-2)^2+(y-3)^2=25$
(iii) $\sqrt{19}$
(iv) $2x^2+2y^2+16x-8y-41=0$
(v) $4x^2+4y^2-10x-27y=0$

2. (i) Vertex $\left(\frac{3}{2},\frac{-11}{8}\right)$,
Focus $\left(\frac{3}{2},\frac{-15}{8}\right)$,
Directrix is $y-\frac{7}{8}=0$,
Equation axis is $x=\frac{3}{2}$
(ii) $y=\frac{x}{4}+9$ and $y=\frac{9}{4}x+1$
Points of contact are at (36, 18) and $\left(\frac{4}{9},2\right)$
(iii) $e=\frac{\sqrt{3}}{2}$, Foci $\left(\sqrt{3}-1,-2\right)$ and $\left(-\sqrt{3}-1,-2\right)$
Vertices (1, –2) (–3, –2) (–1, –1) (–1, –3)
Directrices $x=-1\pm\frac{4}{\sqrt{3}}$
Length of latus rectum = 1
Equation of latus rectum
$x=-1\pm\sqrt{3}$
(iv) $\frac{4x^2}{81}+\frac{4x^2}{45}=1$
(v) $7x^2-2y^2+12xy-2x+14y-22=0$

3. $(x-2)^2+(y+2)^2=8$ and $(x+4)^2+(y-6)^2=18$

5. $5(x^2+y^2)+32x-53y+48=0$

9. $y=2x+1$ and $y=-2x-1$

10. (ii) $\left(-\frac{575}{529},\frac{989}{529}\right)$
(iii) $12x^2+7xy-10y^2+13x+45y-35=0$

11. (a) 12. (a) 13. (b)
14. (a) 15. (c) 16. (d)
17. (a) 18. (a) 19. (c)
20. (a) 21. (a) 22. (c)
23. (c) 24. (b) 25. (b)
26. (d)
27. (a,) (c)
28. (a,) (b,) (d)
29. (a,) (b,) (c)
30. (a) → (s)
(b) → (r)
(c) → (p)
(d) → (q)

IIT assingment Exercise

31. (b) 32. (c) 33. (b)
34. (b) 35. (b) 36. (a)
37. (a) 38. (b) 39. (a)
40. (a) 41. (c) 42. (d)
43. (c) 44. (c) 45. (d)
46. (b) 47. (c) 48. (b)
49. (d) 50. (b) 51. (d)
52. (a) 53. (b) 54. (a)
55. (b) 56. (b) 57. (a)
58. (c) 59. (b) 60. (b)
61. (a) 62. (a) 63. (b)
64. (a) 65. (c) 66. (b)
67. (a) 68. (c) 69. (b)
70. (b) 71. (c) 72. (d)
73. (a) 74. (c) 75. (d)
76. (b) 77. (a) 78. (b)
79. (b) 80. (a) 81. (b)
82. (b) 83. (b) 84. (a)
85. (a) 86. (a) 87. (a)
88. (b) 89. (a) 90. (b)
91. (c) 92. (c) 93. (a)
94. (b) 95. (c) 96. (b)
97. (a) 98. (b) 99. (b)
100. (d) 101. (d) 102. (a)
103. (a) 104. (b) 105. (b)
106. (d) 107. (d) 108. (b)
109. (b) 110. (a) 111. (d)
112. (d) 113. (b) 114. (c)
115. (a)
116. (c)
117. (a,) (c)
118. (b)
119. (a), (c,) (d)
120. (a) → (p,) (s)
(b) → (q)
(c) → (r)
(d) → (p,) (q,) (r,) (s)

Additional Practice Exercise

121. (i) $x^2+y^2-12x-16y+75=0$ and $x^2+y^2=25$
(ii) $4x-3y\pm25=0$

122. (i) $2\sqrt{r^2-\frac{p^2q^2}{p^2+q^2}}$
(ii) $\frac{1}{p^2}+\frac{1}{q^2}=\frac{1}{r^2}$

123. (i) 2α or $2(\pi-\alpha)$
(ii) $k=\frac{g^2+f^2+c}{2}$

124. (i) $x+3=\pm\sqrt{3}\,y$ and $x=0$
(ii) $3\sqrt{3}$

125. (i) $t_1=-t-\frac{2}{t}$
(ii) $6\sqrt{3}a$
(iii) $y=\pm\sqrt{2}\left(x-4a\right)$

126. (i)
$2ab\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\beta-\gamma}{2}\right)\sin\left(\frac{\gamma-\alpha}{2}\right)$

(ii) $\frac{3\sqrt{3}}{4}ab$

127. $\sqrt{\frac{2e}{1+e}}$

128. (i) $x^2 - y^2 = 4$

(ii) $\sqrt{2}$

(iii) $\frac{e^2+1}{e^2-1}$

131. (a)

132. (d) **133.** (c) **134.** (b)

135. (a) **136.** (b) **137.** (a)

138. (d) **139.** (c) **140.** (b)

141. (c) **142.** (d) **143.** (d)

144. (c) **145.** (c) **146.** (b)

147. (b) **148.** (b) **149.** (b)

150. (d) **151.** (b) **152.** (c)

153. (d) **154.** (c) **155.** (c)

156. (d) **157.** (a) **158.** (a)

159. (c) **160.** (d) **161.** (b)

162. (b) **163.** (b) **164.** (b)

165. (c) **166.** (a) **167.** (d)

168. (b) **169.** (a) **170.** (d)

171. (c) **172.** (b) **173.** (b)

174. (d) **175.** (b) **176.** (d)

177. (c) **178.** (b) **179.** (a)

180. (a) **181.** (b) **182.** (d)

183. (b) **184.** (c) **185.** (c)

186. (d) **187.** (b) **188.** (c)

189 (a)

190. (a,) (c)

191. (b,) (c)

192. (b,) (c)

193. (a,) (c)

194. (a,) (c)

195. (a,) (d)

196. (a,) (b,) (c,) (d)

197. (a,) (b,) (c,) (d)

198. (a) → (s)
(b) → (q)
(c) → (r)
(d) → (p)

199. (a) → (q,) (s)
(b) → (p)
(c) → (p)
(d) → (r)

200. (a) → (s)
(b) → (p)
(c) → (s)
(d) → (q,) (r)

HINTS AND EXPLANATION

Topic Grip

1. Let centre C(g, o)

Then equation of the circle is

$(x - g)^2 + y^2 = 73$

$x^2 - 2gx + g^2 + y^2 = 73$

Since (2, 3) lies on the circle

$4 - 4g + g^2 + 9 = 73$

$g^2 - 4g - 60 = 0$

$(g - 10)(g + 6) = 0$

$\therefore$ $g = 10, -6$

$\therefore$ Equation of the circle is

$(x - 10)^2 + y^2 = 73$ or $(x + 6)^2 + y^2 = 73$

(ii) Solving $x + 7y - 23 = 0$ and $5x + 2y - 16 = 0$

we get $x = 2$ $y = 3$

$\therefore$ Center of the circle is (2, 3)

Let radius be r

$\therefore$ Equation of the circle is

$(x - 2)^2 + (y - 3)^2 = r^2$

But it passes through (5, 7) $\therefore$ $r^2 = 25$

$\therefore$ Equation of Circle is $(x - 2)^2 + (y - 3)^2 = 25$

(iii) Given equation of the circle is

$x^2 + y^2 - 30x + 6y + 105 = 0$

length of the tangent from $(x_1\ y_1)$ to a circle

$x^2 + y^2 + 2gx + 2fg + c = 0$ is

$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$= \sqrt{9 + 25 - 90 - 30 + 105}$

$= \sqrt{19}$

(iv) Equation of the circle with centre at (–4, 2) is

$x^2 + y^2 + 8x - 4y + c = 0$

given $x - y - 3 = 0$ is tangent a circle. Length of the perpendicular from (–4, 2) is equal to the radius.

$\left|\frac{-4-2-3}{\sqrt{1+1}}\right| = \sqrt{16+4-c} \Rightarrow c = \frac{-41}{2}$

$\therefore$ Equation of the circle is

$x^2 + y^2 + 8x - 4y + \frac{-41}{2} = 0$

$\Rightarrow$ $2x^2 + 2y^2 + 16x - 8y - 41 = 0$

(v) Equation of circle through the intersection of circles is

$S_1 + \lambda S_2 = 0$

i.e $x^2 + y^2 + 2x + \lambda(x^2 + y^2 - 3y) = 0$

i.e., $(1 + \lambda)x^2 + (1 + \lambda)y^2 + 2x - 3\lambda y = 0$

But its centre lies on $x + 2y - 8 = 0$

i.e., $\left(\frac{-1}{1+\lambda}, \frac{3\lambda}{2+2\lambda}\right)$ lies on $x + 2y = 8$

i.e., $\frac{-1}{1+\lambda} + \frac{3\lambda}{1+\lambda} = 8 \Rightarrow \lambda = \frac{-9}{5}$

$\therefore$ equation the circle is

$x^2 + y^2 + 2x - \frac{9}{5}(x^2 + y^2 - 3y) = 0$

$\Rightarrow$ $4x^2 + 4y^2 - 10x - 27y = 0$

2. (i) Clearly $\Delta \neq 0$ and $h^2 = ab$

$\therefore$ Conic is a parabola

$\Rightarrow$ $x^2 - 3x + 2y + 5 = 0$

$\Rightarrow$ $x^2 - 3x + \frac{9}{4} = 2y - \frac{11}{4}$

$\Rightarrow$ $\left(x - \frac{3}{2}\right)^2 = -2\left(y + \frac{11}{8}\right)$

Vertex is at $\left(\frac{3}{2}, \frac{-11}{8}\right)$

$\left(\frac{3}{2}, \frac{-15}{8}\right)$ since the latus rectum is 2.

Equation of the Directrix is $y + \frac{11}{8} = \frac{1}{2}$

$\Rightarrow$ $y = \frac{-7}{8}$

Equation of the axis is $x - \frac{3}{2} = 0$

$\Rightarrow$ $x = \frac{3}{2}$

(ii) $y^2 = 9x$

$\therefore$ $a = \frac{9}{4}$

Equation of tangent to parabola $y^2 = 4ax$ is

$y = mx + \frac{a}{m}$

$\Rightarrow$ $y = mx + \frac{9}{4m}$

Since it passes through (4, 10)

$\Rightarrow$ $10 = 4m + \frac{9}{4m}$

$\Rightarrow$ $16m^2 - 40m + 9 = 0$

$\Rightarrow$ $m = \frac{1}{4}, \frac{9}{4}$

$\therefore$ Equation the tangents are

$y = \frac{1}{4}x + 9$ and $y = \frac{9}{4}x + 1$

$\Rightarrow$ $x - 4y + 36 = 0$ and $9x - 4y + 4 = 0$

$y = mx + \frac{a}{m}$ touches at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$\therefore$ points of contact are (36, 18) and $\left(\frac{4}{9}, 2\right)$

(iii) $x^2 + 4y^2 + 2x + 16y + 13 = 0$

Clearly $\Delta \# 0$ and $h^2 < ab$

$\therefore$ Conic is an ellipse

$(x^2 + 2x + 1) + 4(y^2 + 4y + 16) = 4$

$(x + 1)^2 + 4(y + 2)^2 = 4$

The equation is $\frac{X^2}{2^2} + \frac{Y^2}{1^2} = 1$

where $X = x + 1$, $Y = y + 2$

$e^2 = \frac{2^2 - 1^2}{2^2} \Rightarrow e = \frac{\sqrt{3}}{2}$

foci are given by

i.e., $\left(\sqrt{3} - 1, -2\right)$ and $\left(-\sqrt{3} - 1, -2\right)$

Vertices are at $X = \pm 2$, $Y = 0$ and $X = 0$, $Y = \pm 1$

$(1, -2), (-3, -2), (-1, -1), (-1, -3)$

Equation of the directrices are $x = -1 \pm \frac{4}{\sqrt{3}}$

Length of latus rectum $= \frac{2 \times 1^2}{2} = 1$

Equation of latus rectum are

$x = -1 \pm \sqrt{3}$

(iv) eccentricity $e = \frac{2}{3}$,

latus rectum $\frac{2b^2}{a} = 5$

$\Rightarrow$ $2a = 9 \Rightarrow a = \frac{9}{2}$

$\therefore$ $b^2 = a^2(1 - e^2)$

$= \frac{81}{4}\left(1 - \frac{4}{9}\right) = \frac{45}{4}$

$\therefore$ Equation of the ellipse $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$

(v) Let P(x, y) be a point on the hyperbola focus F (1, 2) and PM be the perpendicular on the directrix. The

$\frac{PF}{PM} = e \Rightarrow PF^2 = e^2 PM^2$

$\Rightarrow$ $(x - 1)^2 + (y - 2)^2 = 3 \frac{(2x + y - 1)^2}{\left(\sqrt{4+1}\right)^2}$

$\Rightarrow$ $x^2 + y^2 - 2x - 4y + 5 = \frac{3}{5}(4x^2 + y^2 + 1 + 4xy - 4x - 2y)$

$\Rightarrow$ $7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

3. Let the equation of the circle be

$(x - h)^2 + (y - k)^2 = r^2$

Centre lies on the line $4x + 3y - 2 = 0$

$\Rightarrow$ $4h + 3k - 2 = 0$ — (1)

Circle touches the lines $x + y + 4 = 0$ and $7x - y + 4 = 0$

$\Rightarrow$ $\left|\frac{h + k + 4}{\sqrt{2}}\right| = \left|\frac{7h - k + 4}{\sqrt{50}}\right|$

or $(h + k + 4) = \frac{7h - k + 4}{5}$ — (2)

or $(h + k + 4) = -\left(\frac{7h - k + 4}{5}\right)$ — (3)

Taking (2), we have

$5h + 5k + 20 = 7h - k + 4$

$\Rightarrow$ $2h - 6k - 16 = 0$

$\Rightarrow$ $h - 3k - 8 = 0$ — (4)

from (1) and (4), $h = 2$, $k = -2$

$r = \left|\frac{h + k + 4}{\sqrt{2}}\right| = 2\sqrt{2}$

equation of one of the required circles is

$(x - 2)^2 + (y + 2)^2 = 8$

Taking (3), $5h + 5k + 20 = -7h + k - 4$

$12h + 4k + 24 = 0$

$\Rightarrow$ $3h + k + 6 = 0$ —(5)

from (1) and (5), $h = -4$, $k = 6$

Also, $r = 3\sqrt{2}$

Equation of the other required circle is

$(x + 4)^2 + (y - 6)^2 = 18$

4. Centres of the circles are (a, 0) and (0, b)

Radii of the circles are $\sqrt{a^2 - c}$ and $\sqrt{b^2 - c}$

The circles touch if

$$a^2 + b^2 = \left(\sqrt{a^2 - c} \pm \sqrt{b^2 - c}\right)^2$$

$$= (a^2 - c) + (b^2 - c) \pm 2\sqrt{(a^2 - c)(b^2 - c)}$$

$\Rightarrow \quad 2c = \pm 2\sqrt{(a^2 - c)(b^2 - c)}$

$\Rightarrow \quad c^2 = (a^2 - c)(b^2 - c)$

$= a^2b^2 - cb^2 - ca^2 + c^2 \Rightarrow ca^2 + cb^2 = a^2b^2$

$\Rightarrow \quad \dfrac{c}{b^2} + \dfrac{c}{a^2} = 1 \quad \Rightarrow \dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c}$

5. $y = x^2$

Differentiating with respect to x, $y' = 2x$

Slope of the tangent at (2, 4) on the curve $y = x^2$ is $2 \times 2 = 4$

Equation of the tangent at (2, 4) on $y = x^2$ is

$y - 4 = 4(x - 2) \Rightarrow 4x - y - 4 = 0$ — (1)

Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ — (2)

(2) passes through (0, 1) and (2, 4)

$\Rightarrow \quad 1 + 2f + c = 0$ or $2f + c = -1$ — (3)

$20 + 4g + 8f + c = 0$ or $4g + 8f + c = -20$ — (4)

Since the circle (2) touches $y = x^2$ at (2, 4), centre of the circle must lie on the line through (2, 4) perpendicular to (1)

$x + 4y - 18 = 0$

Therefore, we have

$-g - 4f - 18 = 0$ — (5)

$\Rightarrow \quad g + 4f = -18$

Solving (5), (3) and (4), we get

$$g = \frac{32}{10}, f = \frac{-53}{10} \quad c = \frac{48}{5}$$

The required circle is $x^2 + y^2 + \dfrac{32}{5}x - \dfrac{53}{5}y + \dfrac{48}{5} = 0$

$\Rightarrow \quad 5(x^2 + y^2) + 32x - 53y + 48 = 0$

6. Let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Semi latus rectum $= \dfrac{b^2}{a} = a(1 - e^2)$

Coordinates of L are $(ae, a(1 - e^2))$

S is (ae, 0) and S' is (–ae, 0)

Slope of LS' $= \dfrac{a(1-e^2)}{2ae} = \dfrac{1-e^2}{2e}$

Equation of LS' is

$$y = \left(\frac{1-e^2}{2e}\right)(x + ae) \quad \text{— (1)}$$

From (1), $x = \dfrac{2ey}{(1-e^2)} - ae$

Substituting in the equation of the ellipse,

$$\frac{1}{a^2}\left[\frac{2ey}{(1-e^2)} - ae\right]^2 + \frac{y^2}{b^2} = 1 \quad \text{— (2)}$$

(2) is a quadratic in y and gives two values which are the y coordinates of the points of intersection of (1) and the ellipse. One value of y (or one root) is the y coordinate of L = y_1 which is = $a(1 - e^2)$

If the y coordinate of P is y_2

$y_1 + y_2$ = sum of the roots of (2)

$$= \frac{\dfrac{4ae^2}{a^2(1-e^2)}}{\dfrac{4e^2}{a^2(1-e^2)^2} + \dfrac{1}{b^2}} = \frac{4ae^2(1-e^2)}{(1+3e^2)}, \text{since } b^2 = a^2(1-e^2)$$

$$y_2 = \frac{4ae^2(1-e^2)}{(1+3e^2)} - y_1 = \frac{4ae^2(1-e^2)}{(1+3e^2)} - a(1-e^2)$$

$$= \frac{a(1-e^2)}{(1+3e^2)}\{4e^2 - 1 - 3e^2\} = \frac{a(1-e^2)(e^2-1)}{(1+3e^2)}$$

$$|y_2| = \frac{a(1-e^2)^2}{(1+3e^2)}$$

7. Normal at θ on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad \text{— (1)}$$

(1) meets the curve again at (a cos 2θ, b sin2θ)

$$\Rightarrow \quad \frac{a \times a\cos 2\theta}{\cos\theta} - \frac{b \times b\sin 2\theta}{\sin\theta} = a^2 - b^2$$

$$\Rightarrow \quad \frac{a^2\cos 2\theta}{\cos\theta} - \frac{b^2\sin 2\theta}{\sin\theta} = a^2 - b^2$$

Substituting for a^2 and b^2,

$$\frac{14(2\cos^2\theta-1)}{\cos\theta}-10\cos\theta=9$$

$$28\cos^2\theta-14-10\cos^2\theta=9\cos\theta$$

$$18\cos^2\theta-9\cos\theta-14=0$$

$$\cos\theta=\frac{9\pm33}{36}=\frac{42}{36}\text{ or }\frac{-2}{3}$$

$\cos\theta$ cannot be $=\frac{42}{36}$

Therefore, $\cos\theta=\frac{-2}{3}$ or $\sec\theta=\frac{-3}{2}$

8. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

L, which is an end of a latus rectum has coordinates $(ae, a(1-e^2))$

$a\cos\theta = ae$

$\cos\theta = e$

$\Rightarrow \sin\theta = \sqrt{1-e^2}$

Normal at "θ" on the ellipse is

$$\frac{ax}{e}-\frac{by}{\sqrt{1-e^2}}=a^2-b^2$$

$$\Rightarrow \frac{ax}{e}-ay=a^2e^2$$

$$\Rightarrow \frac{x}{e}-y=ae^2$$

This line passes through $(0, -b)$

Therefore, $b = ae^2$

$$a\sqrt{1-e^2}=ae^2$$

$\Rightarrow e^4+e^2-1=0$

9. Let $y = mx + c$ be a tangent to the parabola and the hyperbola. Then,

$$c=\frac{2}{m} \qquad \text{— (1)}$$

$c^2 = 1 \times m^2 - 3$ — (2) (since the hyperbola is $\frac{x^2}{1}-\frac{y^2}{3}=1$)

Using (1),

$$\frac{4}{m^2}=m^2-3$$

$$m^4-3m^2-4=0$$

$(m^2-4)(m^2+1)=0 \Rightarrow m=\pm2.$

When $m = +2$, $c = 1$

When $m = -2$, $c = -1$

The two tangents are $y = 2x + 1$ and $y = -2x - 1$

10.

(i) $a = 12, b = -10, h=\frac{7}{2}, g=\frac{13}{2}, f=\frac{45}{2}, c=-23$

$$h^2-ab=\frac{49}{4}+120>0$$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

$\Rightarrow$ It is representing a hyperbola

(ii) centre : $\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)$

$$=\left(\frac{\frac{7\times45}{4}+10\cdot\frac{13}{2}}{-120-\frac{49}{4}},\frac{\frac{13}{2}\cdot\frac{7}{2}-12\cdot\frac{45}{2}}{-120-\frac{49}{4}}\right)$$

$$=\left(\frac{315+260}{-480-49},\frac{91-1080}{-480-49}\right)=\left(\frac{-575}{529},\frac{989}{529}\right)$$

(iii) Asymptotes:

$12x^2 + 7xy - 10y^2 + 13x + 45y + c = 0$

$\Delta = 0$

$$\Rightarrow 12\times-10\times c+2\times\frac{45}{2}\times\frac{13}{2}\times\frac{7}{2}$$

$$-12\cdot\frac{45^2}{4}+10\cdot\frac{13^2}{4}-c\cdot\frac{49}{4}=0$$

on simplification $c = -35$

$\therefore\ 12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$

11. Equation of the circle is $(x-4)^2 + (y-3)^2 = 3^2$

Centre is (4, 3)

Radius is 3.

$x = 4 + 3\cos\theta$

$y = 3 + 3\sin\theta.$

12. Given equation can be written as

$$\left(y-\frac{3}{2}\right)^2=-3\left(x+\frac{5}{2}\right)$$

$4a = 3 \Rightarrow a = \frac{3}{4}$

Directrix is $x = \frac{-5}{2}+\frac{3}{4}=\frac{-7}{4}$ ..

13. $\frac{x^2}{16}+\frac{y^2}{4}=1$

$a = 4, b = 2$

$SP + S^1P = 2a = 8.$

14. $8yy^1 - 10x = 0$

$$y^1 = \frac{5x}{4y} = \frac{-20}{20} = -1$$

Equation of the normal at (–4, 5) is

$\Rightarrow$ $y - 5 = 1\ (x + 4)$

$y - x = 9.$

15. $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 0$, a pair of lines

16. Statement 2 is true

Using Statement 2, let the required circle in Statement 1 be $x^2 + y^2 - 4x + 6y - 2 + \lambda\ (x^2 + y^2 + 9x + 6y - 2) = 0$

Since it passes through (1, 1)

$1 + 1 - 4 + 6 - 2 + \lambda\ (1 + 1 + 9 + 6 - 2) = 0$

$15\lambda + 2 = 0 \Rightarrow \lambda = \frac{-2}{15}$

Circle is

$15(x^2 + y^2 - 4x + 6y - 2)$

$- 2(x^2 + y^2 + 9x + 6y - 2) = 0$

$\Rightarrow$ $13(x^2 + y^2) - 78x + 78y - 26 = 0$

$\Rightarrow$ $x^2 + y^2 - 6x + 6y - 2 = 0$

Centre of the above circle is at (3, –3)

It does not lie on $3x + y = 0$

$\Rightarrow$ Statement 1 is false; Statement 2 is true

Choice (d)

17. Statement 2 is true

Consider Statement 1. Using Statement 2,

Radical axis of the circles is

$5x - 9y + 28 = 0$

Statement 1 is true

Choice (a)

18. Statement 2 is true

Consider Statement 1

$x^2 - 4x - 8y + 4 = 0$

$\Rightarrow$ $(x - 2)^2 = 8(y)$

Focus is at (2, 2)

The line $x + 2y - 6 = 0$ passes through (2, 2)

$\Rightarrow$ Statement 1 is true

Choice (a)

19. Consider Statement 2

E is $16(x + 3)^2 + 25y^2 = 400$

$\Rightarrow$ $\frac{(x+3)^2}{25} + \frac{y^2}{16} = 1$

Centre of E is at (–3, 0)

Auxiliary circle of E is given by

$(x + 3)^2 + y^2 = 25$

$\Rightarrow$ $x^2 + y^2 + 6x - 16 = 0$

Statement 2 is false.

Consider Statement 1

We have the result.

Locus of feet of perpendiculars from the foci to any tangent to an ellipse is the auxiliary circle of the ellipse.

$\Rightarrow$ Statement 1 is true

Choice (c).

20. Statement 2 is true.

Consider Statement 1

$xy - 2x + 4y - 33 = 0$

$x(y - 2) + 4(y - 2) = 25$

$\Rightarrow$ $(x + 4)\ (y - 2) = 25$

Shift the origin to (–4, 2) without changing the directions of the axes. The equation of the hyperbola in the new system is $XY = 25$

Using Statement 2, asymptotes are $Y = 0$ and $X = 0$

$\Rightarrow$ $y - 2 = 0, x + 4 = 0$

$\Rightarrow$ Statement 1 is true

Choice (a)

21. The common radical axis of the system is

$4y + 5 = 0$

The coaxal system can be assumed as

$x^2 + y^2 - 6 + \lambda(4y + 5) = 0$

Since the member passes through (–1, 1),

$1 + 1 - 6 + \lambda(9) = 0 \Rightarrow \lambda = \frac{4}{9}$

The equation of the circle is

$x^2 + y^2 - 6 + \frac{4}{9}(4y + 5) = 0$

$\Rightarrow 9(x^2 + y^2) + 16y - 34 = 0$

22. The common radical axis is $4x + 2y - 1 = 0$

The coaxal system can be assumed as

$x^2 + y^2 - 2x - 4y + 3 + \lambda(4x + 2y - 1) = 0$

$x^2 + y^2 + 2x\ (2\lambda - 1) + 2y(\lambda - 2) + (3 - \lambda) = 0$

Centre of the above circle is $(1 - 2\lambda, 2 - \lambda)$

and radius

$= \sqrt{(1-2\lambda)^2 + (2-\lambda)^2 - 3 + \lambda} = \sqrt{5\lambda^2 - 7\lambda + 2}$

For limiting points, $\sqrt{5\lambda^2 - 7\lambda + 2} = 0$

$\Rightarrow \quad 5\lambda^2 - 7\lambda + 2 = 0$

$\Rightarrow \quad 5\lambda^2 - 5\lambda - 2\lambda + 2 = 0$

$\Rightarrow \quad 5\lambda(\lambda - 1) - 2(\lambda - 1) = 0 \quad \Rightarrow \lambda = 1, \frac{2}{5}$

The limiting points are the centres of the circles with zero radius

Limiting points are $(-1, 1)$ and $\left(\frac{1}{5}, \frac{8}{5}\right)$

23. The members of the system are

$(x-5)^2 + (y+3)^2 = 0$ and $(x-2)^2 + (y-6)^2 = 0$

$\Rightarrow \quad x^2 + y^2 - 10x + 6y + 34 = 0$ and

$x^2 + y^2 - 4x - 12y + 40 = 0$

Common radical axis is $6x - 18y + 6 = 0$

$\Rightarrow \quad x - 3y + 1 = 0$

Passage II

24. The variable a can take values in between 0 and 3 i.e., $0 < a < 3$

$\Rightarrow \quad \frac{13}{9} < 1 + \frac{4}{a^2} < \infty$

$\Rightarrow \quad \frac{\sqrt{13}}{3} < e < \infty$

25. Coordinates of the vertices of the ellipse are $(\pm 3, 0)$

$\Rightarrow \quad \sqrt{a^2 + 4} = 3a^2 + 4 = 9 \quad \Rightarrow a = \sqrt{5}$

26. Coordinates of foci of ellipse are $(\pm\sqrt{5}, 0)$

Coordinates of vertices of hyperbola are $(\pm a, 0)$

$\Rightarrow \quad a = \sqrt{5}$

27. Centres $C_1 = (-3, 0)$ $C_2 = (1, -3)$

Radius $r_1 = 5$ $r_2 = 4$

$C_1C_2 = 5$

$C_1C_2 < r_1 + r_2$

$\therefore$ two circles intersect

$\therefore$ Number of common tangents = 2

(a) is true

(b) and (d) is false.

Equation of the radical axis is $S_1 - S_2 = 0$

$\Rightarrow \quad 4x - 3y - 5 = 0$

Distance from (1, 2) $= \frac{|4-6-5|}{\sqrt{16+9}} = \frac{7}{5}$

$\therefore$ Option (c) is true

28. $4x^2 - 4x + 8y + 17 = 0$

$4\left(x - \frac{1}{2}\right)^2 = -17 - 8y + 1 = -16 - 8y$

$\left(x - \frac{1}{2}\right)^2 = -4 - 2y = -2(y + 2)$

Clearly, the latus rectum = 2

y cannot be > -2

(a) and (b) are true

Vertex is at $\left(\frac{1}{2}, -2\right)$

Focus is at $\left(\frac{1}{2}, -2 - \frac{1}{2}\right)$ i.e., at $\left(\frac{1}{2}, -\frac{5}{2}\right)$

(d) is true

29. $4(x+3)^2 - 25(y+2)^2 = 164 + 36 - 100$

$\frac{(x+3)^2}{25} - \frac{(y+2)^2}{4} = 1$

$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{4}{25}} = \sqrt{\frac{29}{25}} = \frac{\sqrt{29}}{5}$

Length of the latus rectum $= \frac{2b^2}{a} = \frac{8}{5}$

Equation of the asymptotes

$\frac{(x-3)^2}{25} - \frac{(y+2)^2}{4} = 0$

$\Rightarrow \quad 4(x+3)^2 - 25(y+2)^2 = 0$

$\Rightarrow \quad 4x^2 - 25y^2 + 24x - 100y - 64 = 0$

If θ is the angle between the asymptotes,

$\tan\theta = \frac{2ab}{a^2 - b^2} = \frac{20}{21}$

30. (a) $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 > ab$

$\therefore$ It is a hyperbola

(b) $\frac{x}{3} = \cos t + \sin t$ $\frac{y}{4} = \cos t - \sin t$

Squaring and adding

$\frac{x^2}{9} + \frac{y^2}{16} = 2 \Rightarrow \frac{x^2}{18} + \frac{y^2}{32} = 1$ which is an ellipse

(c) $\Delta \neq 0$ and $h^2 = ab$

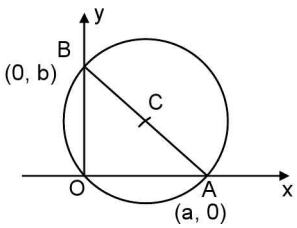
$\therefore$ parabola

(d) $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\therefore$ pair of of lines

IIT Assingment Exercise

31.



$\therefore$ C is centre

$(x - 0)(x - a) + (y - 0)(y - b) = 0$

$x^2 + y^2 - ax - by = 0.$

32. Let (h, k) be a point on the locus. Then (h,k) is equidistant from (0, a) and B (0, –a)

$\Rightarrow$ (h,k) is a point on the perpendicular bisector of line joining (0, a) and (0, –a)

Locus of (h, k) is y = 0.

33. Radius < distance of the centre from the line.

34. The circles are concentric's

Then length of tangent is $\sqrt{c_1 - c} = \sqrt{0+4} = 2.$

35. Radical axis is always perpendicular to the line joining the centres

36. Centre is the point of intersection of both the lines

$x = -3, y = 2$

Area $= \frac{22}{7} \times r^2 = 154$

$r = 7$

$(x + 3)^2 + (y - 2)^2 = 7^2.$

37. $x^2 + y^2 = a^2 + b^2$ represents a circle

38. Centres of 3 circles are

$C_1(-k_1, 0), C_2(-k_2, 0), C_3(-k_3, 0)$

$OC_2^2 = OC_1 . OC_3$

$k_2^2 = k_1 k_3$

Let P (x_1, y_1) be any point on $x^2 + y^2 = c^2$

$x_1^2 + y_1^2 = c^2$

PT_1, PT_2, PT_3 are 3 tangents.

$TT_1^2 = x_1^2 + y_1^2 + 2k_1x_1 - c^2 = 2k_1x_1$, etc

$PT_2^2 = PT_1 . PT_3.$

39. Solving the equations, we get (0,0), $\left(\frac{-6}{5}, \frac{-12}{5}\right)$

$\therefore$ Length of common chord $= \frac{6}{5}\sqrt{5} = \frac{6}{\sqrt{5}}$

40. OA = OB

ΔAOC 30°, 60°, 90°

right angle $1 : \sqrt{3} : 2$

Radius = 2

Area = 4π units.

41. $x_1 = 3, x_2 = 5$

$$\frac{\sqrt{3^2+5^2+2\times-2\times3+2\times2\times5-1}}{\sqrt{3^2+5^2+2\times-3\times3+2\times2\times5-c}} = \frac{1}{3}$$

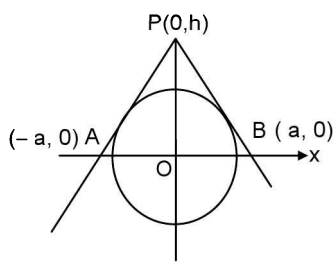
$$\frac{9+25-12+20-1}{9+25-18+50-c} = \frac{1}{9}$$

$$\frac{41}{66-c} = \frac{1}{9}$$

$369 = 66 - c$

$c = -303.$

42.



Let AP and BP be the tangents and B be (a,0) then A is the point (–a,0)

Equation of BP is : $hx + ay = ah;$

BP is a tangent to $x^2 + y^2 = 16$

$\Rightarrow a^2h^2 = 16(h^2 + a^2) \Rightarrow a = \pm\frac{4h}{\sqrt{h^2-16}}$

$$A = \text{Area} = \frac{4h}{\sqrt{h^2-16}} \times h = \frac{4h^2}{\sqrt{h^2-16}}$$

A is minimum $\Rightarrow \frac{dA}{dh} = 0$

$$\Rightarrow \frac{8h\sqrt{h^2-16} - 4h^2\frac{2h}{2\sqrt{h^2-16}}}{\sqrt{h^2-16}} = 0$$

$\Rightarrow \quad 8h(h^2 - 16) - 4h^3 = 0$

$4h^3 - 128h = 0$

$h \neq 0 \Rightarrow 4h^2 = 128$

$h = \sqrt{32} = 4\sqrt{2}$

43. Given circles are concentric with centre (1,0) and radius 2, and 4.

$\therefore \quad (a + 1, \sqrt{3}\, a)$ satisfies the given condition if

$4 < (a + 1 - 1)^2 + (\sqrt{3}\, a - 0)^2 < 16$

$4 < 4a^2 < 16$

$1 < a^2 < 4$

There is no integer a such that $1 < a^2 < 4$

44. If C_1 and C_2 are the centres of the circles, they touch if $C_1C_2 = |r_1 \pm r_2|$

$\Rightarrow \quad C_1C_2^2 = (r_1 \pm r_2)^2$

$\Rightarrow \quad (g_1 - g_2)^2 + (f_1 - f_2)^2$

$= \left(\sqrt{g_1^2 + f_1^2} \pm \sqrt{g_2^2 + f_2^2}\right)^2$

$(g_1g_2 + f_1f_2)^2 = (g_1^2 + f_1^2)(g_2^2 + f_2^2)$

$\Rightarrow \quad (g_1f_2 - g_2f_1)^2 = 0$

$\Rightarrow \quad g_1f_2 = g_2f_1$

Aliter:

The two circles meet at (0, 0)

So, if they touch, they have a common tangent at (0, 0)

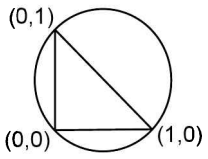
$\therefore \quad g_1x + f_1y = 0$

$g_2x + f_1y = 0$ are identical

$\therefore \quad \dfrac{g_1}{g_2} = \dfrac{f_1}{f_2} \Rightarrow g_1f_2 = g_2f_1$

45. $(g)^2 = (f)^2 = c \Rightarrow$ the given circle touches coordinate axes

46.



The equation of the circle passing through (0,0), (1,0), (0,1) is

$x^2 + y^2 - x - y = 0$

$\Rightarrow \quad 4k^2 + 9k^2 - 2k - 3k = 0$

$\Rightarrow \quad 13k^2 - 5k = 0 \Rightarrow k = 0$ or $\dfrac{5}{13}$; but $k \neq 0$

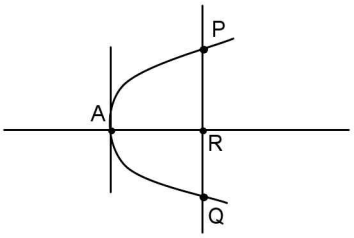
$\therefore \quad k = \dfrac{5}{13}$

47. $y^2 - 8y = x - 19$

$(y - 4)^2 = x - 3$

$\therefore \quad (3, 4).$

48.



Let PQ be the double ordinate of length 8a.

PR = QR = 4a

P (4a, 4a), Q (4a, –4a)

m_1 = Slope of AP = 1

m_2 = Slope of AQ = –1

$m_1m_2 = -1; \theta = \dfrac{\pi}{2}.$

49. Vertex (h, k)

focus (a + h, k).

50. Focus of the parabola is (h + 1, k); (h + 1, k) always lies between x + y = 1 and x+ y = 3, if $1 < h + 1 + k < 3$

$0 < h + k < 2$

51. Normal : $y = mx - 2am - am^3$

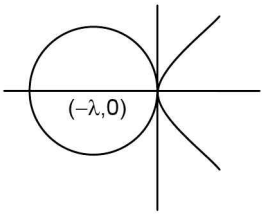
$y = mx - m(2a + am^2)$ is a tangent to $x^2 - y^2 = a^2$

$\Rightarrow \quad c^2 = a^2m^2 - a^2$

$m^2.a^2(2 + m^2)^2 = a^2(m^2 - 1)$

$m^6 + 4m^4 + 3m^2 + 1 = 0$

52.



Since the centre of circle is (–λ,0) and vertex of parabola is (0,0), they touch each other externally only if

$-\lambda < 0 \Rightarrow \lambda > 0$

53. $y^2 = k\left(x + \dfrac{4}{k}\right)$

$y^2 = 4\,ax; 4a = k; Y = y; X = x + \dfrac{4}{k}$

Directrix is X = – a

$x + \frac{4}{k} = \frac{-k}{4}$

$x = \frac{-16-k^2}{4k}$ comparing with x = 2, we get

$2 = \frac{-16-k^2}{4k}$

$k^2 + 8k + 16 = 0$

$(k + 4)^2 = 0 \Rightarrow k = -4.$

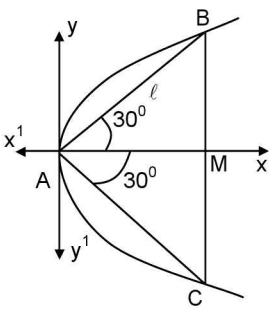
54. $y^2 = 4ax \quad = 12x \qquad a = 3$

Focal distance of any point (x, y) on $y^2 = 4ax$ is x + a i.e., x + 3

$x + 3 = 4$

$\therefore \quad x = 1.$

55.



$AB = \ell$

$AM = \ell \cos 30 = \ell\frac{\sqrt{3}}{2}$

$BM = \ell \sin 30 = \frac{\ell}{2}$

$B = \left(\frac{\sqrt{3}\ell}{2}, \frac{\ell}{2}\right)$

$\frac{\ell^2}{4} = 4a\frac{\ell\sqrt{3}}{2} \Rightarrow \ell = 8a\sqrt{3}.$

56. Let the vertices be $\left(\frac{y_1^2}{4a}, y_1\right), \left(\frac{y_2^2}{4a}, y_2\right)$ and $\left(\frac{y_3^2}{4a}, y_3\right)$

$$\text{Area of triangle} = \frac{1}{2}\begin{vmatrix} \frac{y_1^2}{4a} & y_1 & 1 \\ \frac{y_2^2}{4a} & y_2 & 1 \\ \frac{y_3^2}{4a} & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{8a}\begin{vmatrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$$

57. Let $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola $y^2 = 4ax$ then $t_1 t_2 = -1$

slope of the tangent at $(at_1^2, 2at_1) = \frac{2a}{2at_1} = \frac{1}{t_1}.$

Slope of the tangent at $(at_2^2, 2a\,t_2) = \frac{1}{t_2}$

Product of the slopes $= \frac{1}{t_1 t_2} = -1$

$\therefore$ tangents are perpendicular.

58. Length of the latus rectum

$\frac{2b^2}{a} = \frac{2b}{2} \Rightarrow b = \frac{a}{2}$

$\therefore \quad e = \sqrt{1 - \frac{b^2}{a^2}}$

$e = \frac{\sqrt{3}}{2}.$

59. Major axis is along y-axis

$\therefore$ The auxiliary circle is $x^2 + y^2 = 16.$

60. Ellipse is $\frac{x^2}{2} + \frac{y^2}{8} = 1$

$y = 2x + k$

Substituting y = 2x + k in the equation of the ellipse, we get

$8x^2 + 4kx + k^2 - 8 = 0$

Discriminant = 0

$\Rightarrow \quad k^2 = 16$ or $k = \pm 4$

61. Let P(x, y) be any point on the ellipse with foci S and S'

PS + PS' = e(sum of distances of P from directrices)

$= \frac{ex2a}{e} = 2a.$

62. $e^2 = 1 - \frac{b^2}{a^2}$ for an ellipse

$e^2 = 1 + \frac{b^2}{a^2}$ for hyperbola.

In both the conics, a and b are same.

Their sum of eccentricity is

$$1-\frac{b^2}{a^2}+1+\frac{b^2}{a^2}=2.$$

63. Ellipse is $\frac{x^2}{1/4}+\frac{y^2}{5/4}=1$

$$\frac{1}{4}=\frac{5}{4}\left(1-e^2\right)$$

$$1-e^2=\frac{1}{5}\Rightarrow e=\frac{2}{\sqrt{5}}$$

Distance between the foci = 2ae

$$=\frac{2\times\sqrt{5}}{2}\times\frac{2}{\sqrt{5}}=2$$

64. $\frac{x}{3}=\cos t+\sin t$

$$\frac{y}{4}=\cos t-\sin t$$

$$\left(\frac{x}{3}\right)^2+\left(\frac{y}{4}\right)^2=2\left[\cos^2 t+\sin^2 t\right]$$

$\frac{x^2}{18}+\frac{y^2}{32}=1$ is an ellipse.

65. $\frac{x^2}{r-3}+\frac{y^2}{6-r}=1$

if $r > 3$ and $6 > r$, then only we can say the above equation represents an ellipse

i.e., $3 < r < 6$.

66. $x + y - 4 = 0$

$x - y = 0$

$x = y$

$x + y = 4$

$x = y = 2.$

67. $2a = 2\,(2b)$

$a = 2b$

$$\frac{x^2}{4b^2}-\frac{y^2}{b^2}=1$$

$x^2 - 4y^2 = 4b^2$

Putting $x = 5$; $y = 2$,

$b^2 = \frac{9}{4}$ and $a^2 = 9$. i.e., $x^2 - 4y^2 = 9$.

68. The line $y = mx + c$ is a tangent to the hyperbola if $c^2 = a^2 m^2 - b^2$

$\therefore\quad P^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$

69. Required Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is passing through

$(\pm 2, \pm 1)\quad\Rightarrow\frac{4}{a^2}+\frac{1}{b^2}=1,$

eccentricity $=\frac{1}{\sqrt{2}}\quad\Rightarrow\quad a^2-b^2=\frac{a^2}{2}$

$\Rightarrow a^2 = 2b^2\ \Rightarrow\ \frac{4}{2b^2}+\frac{1}{b^2}=1\ \Rightarrow b^2 = 3 \Rightarrow a^2 = 6$

Ans: $\frac{x^2}{6}+\frac{y^2}{3}=1$

70. $(x-2)^2+(y+3)^2=\frac{(x-2y+7)^2}{25}$

$$\Rightarrow\quad\sqrt{(x-2)^2+(y+3)^2}=\frac{1}{\sqrt{5}}\left|\frac{x-2y+7}{\sqrt{5}}\right|$$

$\therefore$ focus is (2,–3), eccentricity $e=\frac{1}{\sqrt{5}}$, and directrix is $x-2y+7=0$

71. Ellipse

$$e=\sqrt{\frac{k^2a^2-a^2}{k^2a^2}}=\frac{\sqrt{k^2-1}}{k}$$

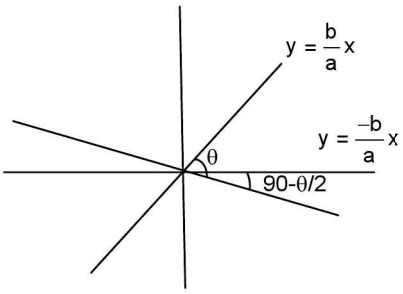
focus : (kae, 0)

Hyperbola

$e=\sqrt{2}$, focus $(a\sqrt{2},0)$

$kae = a\sqrt{2}\Rightarrow\sqrt{k^2-1}=\sqrt{2}\Rightarrow k=\pm\sqrt{3}$

72.



Asymptotes are $y=\pm\frac{b}{a}x$

$\therefore\quad\pm\frac{b}{a}=\text{slope}=\tan\frac{\theta}{2}$ as asymptotes are equally inclined to axes

$\therefore\quad 1+\tan^2\frac{\theta}{2}=1+\frac{b^2}{a^2}=1+e^2-1=e^2$

$$\sec\frac{\theta}{2} = e$$

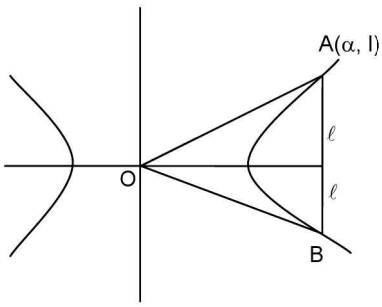
$$\cos\frac{\theta}{2} = \frac{1}{e}$$

73. director circle: $(x-1)^2 + (y-2)^2 = 4 - 4 = 0$

: $(x-1)^2 + (y-2)^2 = 0$ (point circle)

$\Rightarrow$ only one point (1, 2)

74.



Let A be the point (α, ℓ) $\Rightarrow \frac{\alpha^2}{a^2} - \frac{l^2}{b^2} = 1$

$\therefore \quad \alpha = \frac{a\sqrt{b^2+l^2}}{b}$

$OA = OB = 2\ell$

$\alpha^2 + l^2 = 4l^2 \Rightarrow \frac{a^2}{b^2}(b^2+l^2) + l^2 = 4l^2$

$\Rightarrow \quad l^2 = \frac{a^2b^2}{3b^2 - a^2} > 0 \Rightarrow 3b^2 > a^2$

$\Rightarrow 3a^2(e^2-1) > a^2 \Rightarrow e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}$

75. Tangent at $(4\cos\phi, 2\sin\phi)$ is

$$\frac{x}{4}\cos\varphi + \frac{y}{2}\sin\varphi = 1 \qquad \text{— (1)}$$

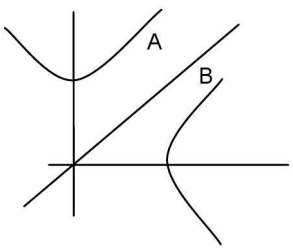
(1) is normal to the circle.

$x^2 + y^2 - 8x - 4y = 0 \Rightarrow$ (1) passes through the centre (4,2)

$\cos\phi + \sin\phi = 1$

$\phi = \frac{\pi}{2}, 0$

76.



$(y-4)^2 = 4(x-6)$ —(1)

$(x-4)^2 = 4(y-6)$ —(2)

these two parabolas are symmetric about $y = x$

$\Rightarrow$ A $(t^2+6, 2t+4)$ is on (1)

$\Rightarrow$ B $[(2t+4), (t^2+6)]$ is a point on (2)

$AB = \sqrt{2}\ (t^2 - 2t + 2)$

Distance between the curves is minimum

When $t^2 - 2t + 2$ is minimum

Ie, at $t = 1$

$\therefore$ minimum distance $= \sqrt{2}$

77. Let $y = mx + \alpha$ be a common tangent

$\alpha^2 = a^2m^2 + b^2$

$\alpha^2 = c^2(1+m^2) \Rightarrow a^2m^2 + b^2 = c^2 + c^2m^2$

$\Rightarrow \quad m^2 = \frac{c^2 - b^2}{a^2 - c^2}$

$(c^2 - b^2)(a^2 - c^2) > 0 \Rightarrow b^2 < c^2 < a^2$

78. Equation of tangent to the ellipse at

$(5\cos\theta, 4\sin\theta)$ is $\frac{x}{5}\cos\theta + \frac{y}{4}\sin\theta = 1$

Tangents at the ends of major axis are $x = \pm 5$

$$T_1 = \left[5, 4\tan\frac{\theta}{2}\right] \quad T_2 = \left[-5, 4\cot\frac{\theta}{2}\right]$$

Required circle is :

$$(x-5)(x+5) + \left(y - 4\tan\frac{\theta}{2}\right)\left(y - 4\cot\frac{\theta}{2}\right) = 0$$

$x^2 + y^2 - 8y\operatorname{cosec}\theta - 9 = 0$

It is passes through $(3, 0)\ \forall\ \theta \in R$

79. $\frac{2b^2}{a} = \frac{32\sqrt{2}}{5} \Rightarrow b^2 = \frac{16\sqrt{2}}{5}a \Rightarrow \frac{5}{a} = \frac{16\sqrt{2}}{b^2}$

point of intersection of given lines (5, 4)

$\frac{25}{a^2} - \frac{16}{b^2} = 1 \Rightarrow \frac{256 \times 2}{b^4} - \frac{16}{b^2} = 1$

$256 \times 2 - 16b^2 = b^4 \Rightarrow b^4 + 16b^2 - 512 = 0$

$\Rightarrow \quad (b^2 - 16)(b^2 + 32) = 0$

$\Rightarrow \quad b^2 = 16 \Rightarrow b = 4 \Rightarrow a = \frac{5}{\sqrt{2}}$

$b^2 = a^2(e^2 - 1) \Rightarrow e^2 = 1 + \frac{32}{25} \Rightarrow e = \sqrt{\frac{57}{25}}$

80. If $x > 0$

$x^2 + y^2 - 8x - 9 = 0$, $y = x + c$

no common point $\Rightarrow$ y = x + c is not a chord/tangent

$$5 < \frac{|4+c|}{\sqrt{2}}$$

if x < 0

$$x^2 + y^2 + 8x - 9 = 0,\ y = -x + c$$

$$\Rightarrow \quad 5 < \frac{|4+c|}{\sqrt{2}}$$

$$\Rightarrow \quad c < -4 - 5\sqrt{2} \text{ (or) } c > -4 + 5\sqrt{2}$$

81. $4x^2 + 4y^2 + 4hxy + 16x + 32y + 4\left(\frac{5}{2}\right) = 0$

$\therefore \quad h = 0.$

82. Distance between parallel line (parallel tangents) is

$$\left|\frac{c_1 - c_2}{\sqrt{a^2+b^2}}\right| = \frac{52-0}{\sqrt{12^2+5^2}} = \frac{52}{13} = 4.$$

83. Putting x = 0 in the equation of the circle,

$$y^2 + 2fy + c = 0$$

Since the circle touches the y-axis, discriminant of the above quadratic = 0

$\Rightarrow \quad f^2 - c = 0$

For y-axis to touch the circle $f^2 = c$.

84. Substitute the value of x or y in the equation of circle. Solve for x and y. By inspection, (a) is the correct choice.

85. $(x - 1)^2 = y - 2$

comparing with $x^2 = 4ay$, $a = \frac{1}{4}$

Latus rectum = 4a = 1.

86. Two perpendicular tangents to $y^2 = 4ax$ always meet on the directrix

i.e., x + a = 0.

87. Since every diameter of an ellipse passes through the centre and is bisected by it

$\therefore$ The co-ordinates of the other end are $\left(-2\sqrt{3}, -4\right)$.

88. $xy = 6 \quad x^2 - y^2 = 5$

$xy' + y = 0 \quad 2x - 2yy' = 0$

$m_1 = \frac{-2}{3} \quad y' = \frac{x}{y}$

$m_1 m_2 = -1 \quad m_2 = \frac{3}{2}$

$\theta = 90°.$

89. The centres of the circles are A(2,3), B(–1,–2), C (5,8)

slope of AB = slope of BC = $\frac{5}{3}$

A, B, C are collinear.

90. Find all the 3 radii we can easily make out that they are in A.P.

91. $\left(x^2 + px + \frac{p^2}{4}\right) + \left(y^2 + py + \frac{p^2}{4}\right) = \frac{p^2}{2}$

$$\left(x + \frac{p}{2}\right)^2 + \left(y + \frac{p}{2}\right)^2 = \left(\frac{p}{\sqrt{2}}\right)^2$$

Centre is $\left(\frac{-p}{2}, \frac{-p}{2}\right)$ and radius is $\frac{p}{\sqrt{2}}$

$\therefore$ parametric equation

$$x = \frac{-p}{2} + \frac{p}{\sqrt{2}}\cos\theta;\ y = \frac{-p}{2} + \frac{p}{\sqrt{2}}\sin\theta$$

92. Centre = (3, 1)

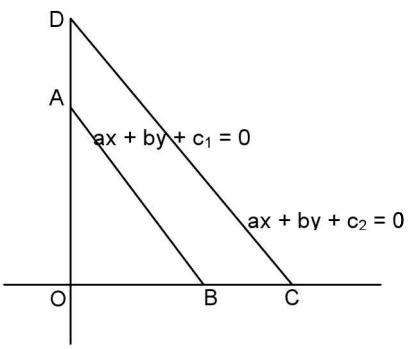
(By solving the two equations).

Radius = $\sqrt{(3-2)^2 + (1-4)^2} = \sqrt{10}$

Diameter = $2\sqrt{10}$.

93. Only option (a) satisfies the condition for tangency as well as the point (3, 4)

94.



$OA = -\frac{c_1}{b}$; $OD = -\frac{c_2}{b}$;

$OB = -\frac{c_1}{a}$; $OC = -\frac{c_2}{a}$

OA.OD = OB.OC

$\therefore \quad a^2 = b^2$

95. At (5, 12), equation of tangent is

$$5x + 12y = 169$$

$$m_1 = \frac{-5}{12}$$

At (12, -5), equation of tangent is

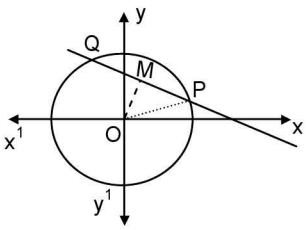
$12x - 5y = 169$

$m_2 = \frac{12}{5}$

$m_1 m_2 = -1$

$\theta = \frac{\pi}{2}$.

96.



$OM = \frac{c}{\sqrt{1+m^2}}$

$OP = a$

$PQ = 2PM = 2\sqrt{OP^2 - OM^2}$

$= 2\sqrt{a^2 - \frac{c^2}{1+m^2}}$.

97. On solving, we get the extremities of chord as A (4, 0), B (0, 4)

AB subtends angle 90° at the origin

98. Radius $= \sqrt{5^2 + 2^2 + 7} = \sqrt{36} = 6$

centre is (5,2)

Since $4^2 + 2^2 - 10 \times 4 - 4 \times 2 - 7 < 0$, the point (4, 2) lies inside the circle

Distance between the centre of the circle and the given point is 1.

$\Rightarrow$ required distance $= 6 - 1 = 5$

99. (p,q) is the mid-point of the chord passing through (0, 0).

$\therefore$ other end of chord is (2p, 2q)

$\therefore$ $4q^2 = 8p$

$\Rightarrow$ $q^2 = 2p$

100. $x^2 + y^2 + 2ax = 0$ —(1)

$x^2 + y^2 + 2by = 0$ —(2); a point of intersection is (0, 0)

at (0, 0) tangent to (1) is $x = 0$

at (0, 0) tangent to (2) is $y = 0$.

The tangent are perpendicular.

$\therefore$ angle is $\frac{\pi}{2}$ (at (0, 0))

101. $(a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1) + \lambda(a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2) = 0$

represents a circle if $a_1 + \lambda a_2 = b_1 + \lambda b_2$ and $2h_1 + \lambda(2h_2) = 0$

$\Rightarrow \frac{b_1 - a_1}{a_2 - b_2} = -\frac{h_1}{h_2} = \lambda \Rightarrow \frac{a_1 - b_1}{h_1} = \frac{a_2 - b_2}{h_2}$

102. $y = mx + \frac{a}{m}$ is a tangent to $y^2 = 4ax$

If this line is to be a tangent to the circle

$x^2 + y^2 = \frac{a^2}{2}$,

$\frac{a^2}{2} = \frac{\frac{a^2}{m^2}}{1+m^2} \Rightarrow m^4 + m^2 - 2 = 0$

$m = \pm 1$

103. solving $x + y - 2 = 0$ and $x - y = 0$, we get (1,1) as the centre of the ellipse

104. $x = t^2 + 1, y = 2t$ and $x = 2s, y = \frac{2}{s}$

$\Rightarrow x = \frac{y^2}{4} + 1, xy = 4$

Ans: (2, 2)

105. Let the circle be $x^2 + y^2 + 2gx + 2fy = 0$

Circle passes through (1, 0) $\rightarrow 1 + 2g = 0$

$\Rightarrow g = \frac{-1}{2}$

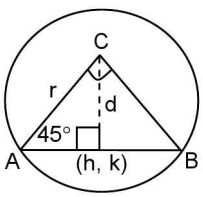
Circle passes through (–1, –3) $\rightarrow 10 - 2g - 6f = 0$

$f = \frac{11}{6}$

Equation of the circle is

$3(x^2 + y^2) - 3x + 11y = 0$

106.



Let (h, k) be a point on the locus.

From figure, $\frac{d}{r} = \sin 45°$

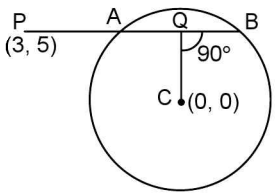
$d^2 = \frac{r^2}{2}$

$$(h+g)^2+(k+f)^2=\frac{g^2+f^2-c}{2}$$

$\therefore$ Locus of (h, k) is

$$x^2+y^2+2gx+2fy+f^2+g^2=\frac{g^2+f^2-c}{2}$$

107. Since 2PQ = PA + PB, we infer that Q is the mid-point of AB.



Let Q be (h, k)

Then, slope of AB $= -\dfrac{1}{\text{slope of CQ}}$

$$=-\frac{h}{k}$$

Equation of PAB is

$$y-5=\frac{h}{k}(x-3)$$

$\Rightarrow$ $ky-5k=-hx+3h$

PAB passes through (h, k)

$\Rightarrow$ $k\times k-5k=-h^2+3h$

$\Rightarrow$ $h^2+k^2-3h-5k=0$

Locus of Q is $x^2+y^2-3x-5y=0$

108. In an equilateral triangle circumcentre coincides with incentre

$$\text{inradius}=\frac{\text{circum radius}}{2}=\frac{\sqrt{g^2+f^2}}{2}$$

Incircle is $(x+g)^2+(y+f)^2=\dfrac{(g^2+f^2)}{4}$

109. The given equations can be transformed to $|x|+|y|=4$, when origin is shifted to the point (2,3), without change in direction of axes.

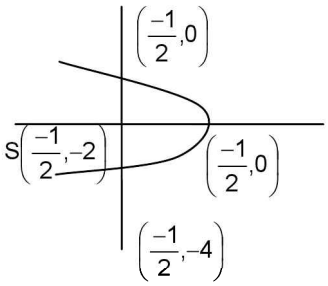
$|x|+|y|=4$ represents a square of side $4\sqrt{2}$

$\therefore$ Radius of the circle touching

$|x-2|+|y-3|=4$ is $2\sqrt{2}$ and centre is (2,3)

$\therefore$ Equation of the circle is $(x-2)^2+(y-3)^2=8$.

110.



$$y^2+4y+4x+2=0\Rightarrow(y+2)^2=-4\left(x-\frac{1}{2}\right)$$

Required circle: $\left(x+\dfrac{1}{2}\right)^2+(y+2)^2=4$

$$x^2+y^2+x+4y+\frac{1}{4}=0$$

111. Centre of the circle $x^2+y^2-6x=0$ is at (3, 0) and its radius equals 3.

Perpendicular from the centre (3, 0) to the line $2y-x-3=0$

Equals $\dfrac{6}{\sqrt{5}}\neq$ its radius

$\Rightarrow$ choice (d)

112. Statement 2 is true

Statement 1

The equation can be written as $(x+2)(y+4)=0$

which represents a pair of lines

$\Rightarrow$ Statement 1 is false

Choice (d)

113. Statement 2 is true

Consider Statement 1

The centre of the given ellipse is at (2, 3)

The foci lie on the major axis $x-2=0$ at distance ae from the centre on either side.

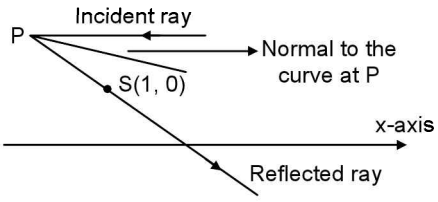
We have

$$e=\frac{2\sqrt{6}}{7}$$

$\Rightarrow$ Statement 1 is true

Choice (b)

114.



Let P be (x, y).

If the slope of the tangent at P(x, y) to the curve is

$m = \left(= \frac{dy}{dx} \text{ at } P\right)$, slope of the normal at P is $\frac{-1}{m}$.

Slope of PS = $\frac{y}{(x-1)}$

If θ is the angle between the reflected ray and the normal;

we have

$$\frac{\frac{y}{x-1} - \left(\frac{-1}{m}\right)}{1 + \left(\frac{y}{x-1}\right)\left(\frac{-1}{m}\right)} = \frac{-1}{m} \Rightarrow \frac{my + x - 1}{m(x-1) - y} = \frac{-1}{m}$$

$m^2y + m(x-1) = -m(x-1) + y$

$\Rightarrow \quad m^2y + 2m(x-1) - y = 0$

is the equation satisfied by the slope of the tangent at any point P on the curve, or it is the differential equation representing this curve.

Taking the option $y^2 = 4x$, $y^1 = \frac{2}{y} = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$

or $m = \frac{1}{\sqrt{x}}$

Substituting $m = \frac{1}{\sqrt{x}}$, $y = 2\sqrt{x}$ in the differential equation representing the curve

$m^2y + 2m(x-1) - y$

$$= \frac{1}{x} \times 2\sqrt{x} + \frac{2}{\sqrt{x}}(x-1) - 2\sqrt{x}$$

$$= \frac{2}{\sqrt{x}} + 2\sqrt{x} + \frac{2}{\sqrt{x}} - 2\sqrt{x} = 0$$

$\Rightarrow \quad y^2 = 4x$ is the curve

or $y^2 = 4x$ is the solution of differential equation.

2,3. Any point P on $y^2 = 4x$ may be represented $(t^2, 2t)$

Slope of tangent at $t = \frac{1}{t}$

Slope of normal a $t = -t$

Equation of the tangent at $P(t^2, 2t)$ is

$$y - 2t = \frac{1}{t}(x - t^2)$$

$$y = \frac{x}{t} + t \qquad \text{— (2)}$$

Equation of the normal at 't' is

$$y + xt = 2t + t^3 \qquad \text{— (3)}$$

We immediately get A as (0, t), T as $(-t^2, 0)$ N as $(2 + t^2, 0)$

ΔPNT is a right angled triangle.

Circumcentre of ΔPNT is the mid-point of TN.

Its coordinates are $\left(\frac{-t^2 + 2 + t^2}{2}, 0\right)$ i.e., (1, 0).

Coordinates of the vertices of ΔPAS are

$P(t^2, 2t)$, A(0, t), S(1, 0)

Slope of PS = $\frac{2t}{t^2 - 1}$

Slope of AP = $\frac{2t - t}{t^2} = \frac{1}{t}$

Slope of AP = $\frac{t}{-1}$

AP is perpendicular to AS ⇒ Orthocentre of the ΔAPS is A.

115. Slope of PS = $\frac{2t}{t^2 - 1}$

Slope of PT = $\frac{1}{t}$

If θ is the angle between PS and PT,

$$\tan\theta = \frac{\frac{2t}{(t^2-1)} - \frac{1}{t}}{1 + \frac{2t}{(t^2-1)}} = \frac{1}{t} = \text{Slope of the tangent PT.}$$

⇒ PT is bisector of the angle between SP and the line through P parallel to the x-axis.

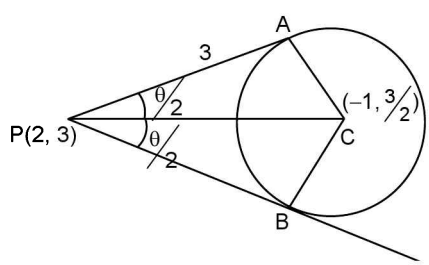
Since PN is perpendicular to PT

(Note that PN is the normal to the curve at P)

PN is the other bisector of the angle between PS and the line through P parallel to the x-axis

116. (c)

117.



Let PA and PB be the tangents from (2, 3) to the circle $x^2 + y^2 + 2x - 3y + 1 = 0$

$\angle APB = \theta$, then $\angle APC = \frac{\theta}{2}$

$\therefore$ From ΔPAC

$$\cot\frac{\theta}{2}=\frac{PA}{AC}$$

$$PA=\sqrt{4+9+4-9+1}=3$$

$$AC=\sqrt{1+\frac{9}{4}-1}=\frac{3}{2}$$

$$\therefore \quad \cot\left(\frac{\theta}{2}\right)=\frac{3}{3/2}=2$$

$$\tan\left(\theta/2\right)=\frac{1}{2}$$

$$\therefore \quad \tan\theta=\frac{2\tan\theta/2}{1-\tan^2\left(\theta/2\right)}=\frac{2.\,1/2}{1-1/4}=\frac{4}{3}$$

$$\sec^2\theta=1+\tan^2\theta=1+\frac{16}{9}$$

$$\therefore \quad \sec\theta=\frac{5}{3}$$

118. The equation of the conic is $\frac{x^2}{64}+\frac{y^2}{100}=1$

$a^2=100 \quad b^2=64$

$64=100(1-e^2)$

$\Rightarrow \quad 100e^2=36$

$e=\frac{3}{5}$

Latus rectum $=\frac{2b^2}{a}=\frac{2\times 64}{10}=\frac{64}{5}$

Foci are at (0, 6) and (0, –6)

Focal distances are $\sqrt{48+1}$ and $\sqrt{48+121}$

i.e., 7 and 13

(b) is true

The ellipse is $\frac{x^2}{64}+\frac{y^2}{100}=1$

Interchange x and y axes

Equation of the ellipse in the new system:

$$\frac{X^2}{100}+\frac{Y^2}{64}=1$$

Equation of the line in the new system.

$Y+4X+4c=0$

$Y=-4X-4c$

Line will be a tangent to the ellipse if

$(-4c)^2=100x\,(-4)^2+64$

$16c^2=100\times 16+64$

$c^2=104$

(c) is false

Again, considering (d), equation of the line is $2y+8X+3=0$

$$Y=-4X-\frac{3}{2}$$

$$\left(\frac{-3}{2}\right)^2=100X(-4)^2+64 \text{ which is not true}$$

(d) is false.

119. If $k<6$, $16-k$ and $6-k$ are positive.

$\therefore$ it is an ellipse

If $k>16$ or $k<6$ equation does not represent an ellipse

If $6<k<16$, $16-k$ is positive and $6-k$ is negative.

It represents a hyperbola

If $k=11$, equation is $\frac{x^2}{5}-\frac{y^2}{5}=1$ which is a rectangular hyperbola

120. (a) Centre and radius of the circles are (–3, 0), (0, –4) and $\sqrt{9-k}$, 6 respectively

Since the circles touches internally

$|r_1-r_2|=C_1C_2$

$\left|\sqrt{9-k}-6\right|=5$

$\Rightarrow \quad k=8,-112$

(b) Shortest distance = difference of the radius and the distance between the centre and the given point

Distance between the given point and the centre = 13

Radius = 15

$\therefore$ shortest distance = 2

(c) Circles touches externally

$C_1C_2=r_1+r_2$

$C_1C_2=\sqrt{25+144}=13$

$r_1+r_2=C_1C_2$

$4+\sqrt{50-a}=13$

$\sqrt{50-a}=9$

$\Rightarrow \quad r_2=9$

$\therefore$ length of the direct common tangent

$=\sqrt{(C_1C_2)^2-(r_1-r_2)^2}$

$=\sqrt{169-(9-4)^2}=12$

(d) The circles have to intersect.

Radical axis is

$4\lambda x - 13 = 0$

Solving $4\lambda x - 13 = 0$ and $x^2 + y^2 = 4$

$$y^2 + \left(\frac{13}{4\lambda}\right)^2 = 4$$

The above equation should have two distinct real roots

$$4 - \left(\frac{13}{4\lambda}\right)^2 > 0$$

$$\lambda > \frac{13}{8} \text{ or } \lambda < -\frac{13}{8}$$

Additional Practice Exercise

121. (i) Let (h, k) be the centre of the circle.

$$\Rightarrow \left|\frac{3h + 4k - 25}{\sqrt{25}}\right| = 5 \Rightarrow 3h + 4k = 0, 50$$

Equation of the line perpendicular to $3x + 4y = 25$ and passing through (3, 4) is

$$(y - 4) = \frac{4}{3}(x - 3) \Rightarrow 3y = 4x$$

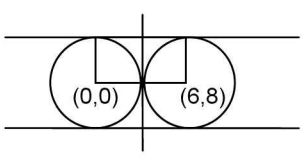
Since (h, k) is a point on this line, $3k = 4h$.

$$\Rightarrow 3h + \frac{16}{3}h = 0, 50$$

$\Rightarrow$ (h, k) = (0, 0), (6, 8)

$\therefore$ There are two circles $x^2 + y^2 = 25$ and $x^2 + y^2 - 12x - 16y + 75 = 0$ satisfying the given conditions.

(ii)



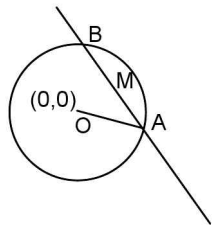
There exists 3 common tangents. One is $3x + 4y = 25$ and remaining two tangents are perpendicular to $3x + 4y = 25$

Let: $4x - 3y + k = 0$ be a tangent to the two circles.

$$\frac{|k|}{5} = 5 \Rightarrow k = \pm 25$$

$\therefore$ $4x - 3y \pm 25 = 0$

122. (i)



$$OM = \frac{1}{\sqrt{\frac{1}{p^2} + \frac{1}{q^2}}}$$

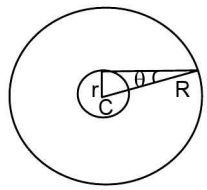
$$AM^2 = OA^2 - OM^2 = r^2 - \frac{p^2q^2}{p^2 + q^2}$$

$$\Rightarrow AM = \sqrt{r^2 - \frac{p^2q^2}{p^2 + q^2}} \Rightarrow AB = 2\sqrt{r^2 - \frac{p^2q^2}{p^2 + q^2}}$$

(ii) If $\overleftrightarrow{AB}$ is a tangent then AB = 0

$$\Rightarrow \frac{1}{\frac{1}{p^2} + \frac{1}{q^2}} = r^2 \Rightarrow \frac{1}{p^2} + \frac{1}{q^2} = \frac{1}{r^2}$$

123. (i)



$$r = \sqrt{g^2 + f^2 - c\sin^2\alpha - g^2\cos^2\alpha - f^2\cos^2\alpha}$$

$$= \sqrt{g^2 + f^2 - c} \cdot \sin\alpha = R\sin\alpha$$

$\sin\theta = \frac{r}{R} = \sin\alpha \Rightarrow \theta = \alpha$ or $\pi - \alpha$ (if α is obtuse)

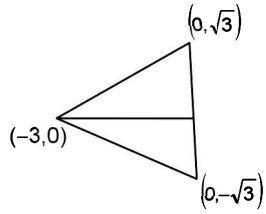
angle between tangents $= 2\theta = 2\alpha$ or $2\pi - 2\alpha$

(ii) We have $\sqrt{g^2 + f^2 - c} = \sqrt{2}\sqrt{g^2 + f^2 - k}$

$$2(g^2 + f^2 - k) = g^2 + f^2 - c$$

$$k = \frac{g^2 + f^2 + c}{2}$$

124. (i)



Any tangent to $y^2 = 4x$ is $y = mx + \frac{1}{m}$. This is a tangent to $(x-3)^2 + y^2 = 9$

$$\Rightarrow \quad 3 = \frac{\left|3m + \frac{1}{m}\right|}{\sqrt{1+m^2}} \Rightarrow 9(1+m^2) = \frac{(3m^2+1)^2}{m^2}$$

$$\Rightarrow \quad 9m^4 + 9m^2 = 9m^4 + 6m^2 + 1$$

$$\Rightarrow \quad 3m^2 = 1 \Rightarrow m = \pm\frac{1}{\sqrt{3}}$$

$\Rightarrow$ equation of common tangents

$$y = \pm\left(\frac{1}{\sqrt{3}}x + \sqrt{3}\right)$$

$$x + 3 = \pm\sqrt{3}\,y$$

Also, both the curves are touching y axis at (0, 0)

$\therefore$ $x = 0$ is another common tangent.

(ii) Area $= \frac{1}{2}\times 3\times 2\sqrt{3} = 3\sqrt{3}$ sq. units

125. (i) Equation of normal at $(at^2, 2at)$ is

$xt + y = at^3 + 2at$ which meets the parabola at $(at_1^2, 2at_1) \Rightarrow \frac{2}{t+t_1} = -t$

$$\Rightarrow \quad t_1 = -t - \frac{2}{t}$$

(ii) Length of the chord joining $(at^2, 2at)$, $(at_1^2, 2at_1)$ is

$$= a(t-t_1)\sqrt{(t+t_1)^2+4} \qquad = a\left(2t+\frac{2}{t}\right)\sqrt{\frac{4}{t^2}+4}$$

$$\ell = \frac{4a(t^2+1)\sqrt{1+t^2}}{t^2} = \frac{4a\left(1+t^2\right)^{\frac{3}{2}}}{t^2}$$

$$\ell \text{ is min} \Rightarrow \frac{d\ell}{dt} = 0$$

$$\frac{d\ell}{dt} = \frac{4a\frac{3}{2}(1+t^2)^{\frac{1}{2}}(2t)t^2 - 2t\cdot 4a(1+t^2)^{\frac{3}{2}}}{t^4} = 0$$

$$= 4a\sqrt{1+t^2}\left[t^3 - 2t\right] = 0 \Rightarrow t = \pm\sqrt{2}$$

[At $t = 0$ normal never meets the parabola again.]

$\Rightarrow$ min. length $= 6\sqrt{3}a$

(iii) Slope of normal at t is $(-t)$

$$\tan\theta = \pm\sqrt{2}$$

$$\theta = \pm\tan^{-1}\left(\sqrt{2}\right) \Rightarrow \text{equation of normal :}$$

$$x\left(\pm\sqrt{2}\right) + y = a\left(\pm 2\sqrt{2}\right) \pm 2a\sqrt{2}$$

$$y = \pm\sqrt{2}\,(x - 4a)$$

126. (i) Let $(a\cos\alpha, b\sin\alpha)$ $(a\cos\beta, b\sin\beta)$ $(a\cos\gamma, b\sin\gamma)$ be the vertices of the triangle. Then its area

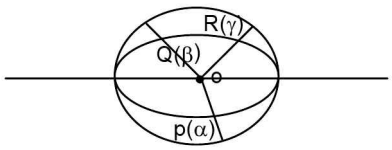
$$= \frac{1}{2}\begin{vmatrix} a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \\ a\cos\gamma & b\sin\gamma & 1 \end{vmatrix}$$

$$= \frac{1}{2}ab[(\cos\beta - \cos\alpha)(\sin\gamma - \sin\alpha) - (\cos\gamma - \cos\alpha)(\sin\beta - \sin\alpha)]$$

$$= \left|\frac{1}{2}\cdot 4ab\cdot\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\beta-\gamma}{2}\right)\sin\left(\frac{\gamma-\alpha}{2}\right)\right|$$

$$= 2ab\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\beta-\gamma}{2}\right)\sin\left(\frac{\gamma-\alpha}{2}\right)$$

(ii)



$(\alpha - \beta) = \angle POQ$

$\beta - \gamma = \angle QOR$ $\gamma - \alpha = \angle ROP$

Let $A = \frac{\alpha-\beta}{2}, B = \frac{\beta-\gamma}{2}, C = \frac{\gamma-\alpha}{2}$

$\Rightarrow$ $A + B + C = \pi$

if $A + B + C = \pi \Rightarrow$ maximum of

$$\sin A \sin B \sin C = \frac{3\sqrt{3}}{8}$$

$$\Rightarrow \quad \text{max. area} = 2ab\cdot\frac{3\sqrt{3}}{8} = \frac{3\sqrt{3}ab}{4}$$

(or)

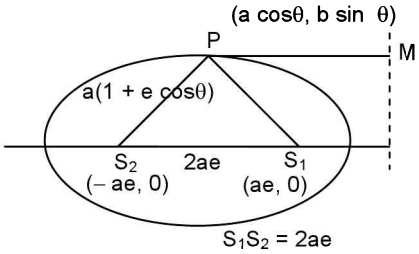
Take the p, q, r corresponding points of P, Q, R on its auxiliary circle then its area will be

$$2a^2\sin A \sin B \sin C \Rightarrow \frac{\Delta PQR}{\Delta pqr} = \frac{b}{a}$$

$\Rightarrow$ Δpqr is max. $\Rightarrow \Delta PQR$ is max.

$$\Rightarrow \quad A = B = C = \frac{\pi}{3} \Rightarrow \frac{\alpha-\beta}{2} = \frac{\beta-\gamma}{2} = \frac{\gamma-\alpha}{2} = \frac{\pi}{3}$$

127. (i)



$S_1(ae, 0)$, $S_2(-ae, 0)$

Let P be (acos θ, b sin θ) and incentre of PS_1S_2 be (x_1, y_1)

S_1P = focal distance of P = exPM

$= a(1 - e\cos\theta)$

$S_2P = a(1 + e\cos\theta)$. Using formula for

in centre, $(x_1, y_1) = \left(ae\cos\theta, \dfrac{be\sin\theta}{1+e}\right)$

$$\cos\theta = \frac{x_1}{ae}, \sin\theta = \frac{(1+e)y_1}{be} \quad \left(\because e^2 = \sqrt{1-\frac{b^2}{a^2}}\right)$$

$$\Rightarrow \frac{x_1^2}{a^2e^2} + \frac{y_1^2}{\frac{b^2e^2}{(1+e)^2}} = 1$$

$$e^1 = \sqrt{1-\frac{b^2}{a^2(1+e)^2}} = \sqrt{1-\frac{1-e^2}{(1+e)^2}}$$

$$= \sqrt{\frac{2e}{1+e}}$$

128. (i) $x = e^t + e^{-t}$, $y = e^t - e^{-t}$

$\Rightarrow$ $x^2 - y^2 = 4$ is a rectangular hyperbola

(ii) eccentricity of above hyperbola = $\sqrt{2}$

(iii) $x = e^t + e^{-t}$

$$\frac{dx}{dt} = e^t - e^{-t} \Rightarrow \frac{dy}{dx} = e^t + e^{-t}$$

$$\therefore \frac{dy}{dx} = \frac{e^t + e^{-t}}{e^t - e^{-t}}$$

$$\frac{dy}{dx} \text{ at } t = 1 = \frac{e^2+1}{e^2-1}$$

129. Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, cuts the hyperbola $xy = p^2$ at the points

$$A\left(pt_1, \frac{p}{t_1}\right) B\left(pt_2, \frac{p}{t_2}\right) C\left(pt_3, \frac{p}{t_3}\right) D\left(pt_4, \frac{p}{t_4}\right)$$

Let $\left(pt, \dfrac{p}{t}\right)$ is a point on $x^2 + y^2 + 2gx + 2fy + c = 0$

$$p^2t^2 + \frac{p^2}{t^2} + 2gpt + 2f\frac{p}{t} + c = 0$$

$$p^2t^4 + 2gpt^3 + ct^2 + 2fpt + p^2 = 0 \qquad \text{— (1)}$$

$\Rightarrow$ $t_1t_2t_3t_4 = 1$,

reciprocal equation of (1)

is $p^2t^4 + 2fpt^3 + 2gpt + p^2 = 0$

$$\Rightarrow \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \frac{-2f}{p}$$

130. $xy = c^2$ —(1)

$y^2 = 4ax$ —(2)

Differentiating (1) with respect to x,

Slope of the tangent to (1) at (x, y) = $\dfrac{-y}{x}$

Similarly, by differentiating (2) with respect to x, slope of the tangent to (2) at

(x, y) = $\dfrac{2a}{y}$

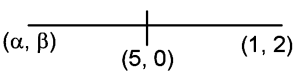
Let (x_1, y_1) be the coordinates of a point of intersection of (1) and (2)

Then, $\dfrac{y_1}{x_1} = \tan\alpha, \dfrac{2a}{y_1} = \tan\beta$ (given)

$$\tan\alpha + 2\tan\beta = \frac{-y_1}{x_1} + \frac{4a}{y_1} = \frac{4ax_1 - y_1^2}{x_1y_1} = 0,$$

since (x_1, y_1) is a point on the parabola $y^2 = 4ax$.

131.



Centre (5, 0)

$$\frac{\alpha+1}{2} = 5, \Rightarrow \alpha = 9$$

$$\frac{\beta+2}{2} = 0, \Rightarrow \beta = -2.$$

132. Asymptotes are $y = \pm\dfrac{b}{a}x$

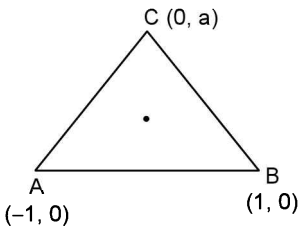
$\therefore$ Angle between them is $2\tan^{-1}\left(\dfrac{b}{a}\right)$.

133. $x^2 + y^2 = a^2 \Rightarrow 2x + 2yy' = 0$

$$y' = \frac{-x}{y}$$

slope of the tangent at $\left(\dfrac{a}{2}, \dfrac{\sqrt{3}}{2}a\right) = \dfrac{-1}{\sqrt{3}}$

134.



$CA^2 = CB^2$

$a = \sqrt{3}$; $C = \left(0, \sqrt{3}\right)$

Since triangle is equilateral,

$G = \left(0, \frac{1}{\sqrt{3}}\right)$ centroid.

Circumradius $GA = \frac{2}{\sqrt{3}}$

Equation is $(x-0)^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \left(\frac{2}{\sqrt{3}}\right)^2$.

135. $2g_1g_2 + 2f_1f_2 = \frac{2}{4} - \frac{2}{4}$

$= C_1 + C_2 = 0 + 0$

Angle $= \frac{\pi}{2}$

136. The locus is the director circle of the given circle. Centre of the given circle is at (2, 3) and its radius equals 5. Locus is,

$(x-2)^2 + (y-3)^2 = 2 \times 25$

$\Rightarrow \quad x^2 + y^2 - 4x - 6y - 37 = 0$

137. AB subtends right angles at P and Q

$\Rightarrow$ P and Q are points on circle with AB as diameter

Equation of PQ is $y - 4 = m(x-2)$

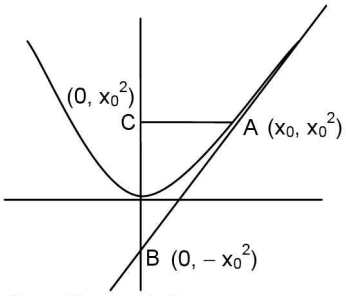
Distance of PQ from centre (3, 0) of the circle < radius.

$\frac{|3m + 4 - 2m|}{\sqrt{1+m^2}} < 3 \Rightarrow (m+4)^2 < 9(1+m^2)$

$8m^2 - 8m - 7 > 0$

$\Rightarrow \quad m < \frac{2-3\sqrt{2}}{4}$ or $m > \frac{2+3\sqrt{2}}{4}$

138.



Equation of tangent at

$x = x_0$ is : $xx_0 = \frac{1}{2}(y + x_0^2)$

$\Rightarrow B(0, -x_0^2)$

Area $= \frac{1}{2} BC.AC = x_0^3$

and its maximum in [1, 2] is when $x_0 = 2$

139. slope of normal = $\tan\theta$

equation of normal

$y = mx + \frac{a}{m} \Rightarrow y = \tan\theta\, x - a\tan^3\theta - 2a\tan\theta$

equation of tangent parallel to above line is

$y = \tan\theta x + a\cot\theta$

distance between these two lines

$= \frac{|a\cot\theta + a\tan^3\theta + 2a\tan\theta|}{\sqrt{1+\tan^2\theta}}$

$= \frac{(a\cot\theta)|1 + \tan^4\theta + 2\tan^2\theta|}{\sec\theta} = \frac{a\cot\theta\sec^4\theta}{\sec\theta}$

$= a\,\text{cosec}\,\theta\,\sec^2\theta$

140. Let $P(at^2, 2at)$ and Q $Q(at_1^2, 2at_1)$ be the ends of normal chord.

$t_1 = -t - \frac{2}{t}$ —(1)

a = 1. Let O be the vertex.

slope of OP $= \frac{2t}{t^2} = \frac{2}{t}$

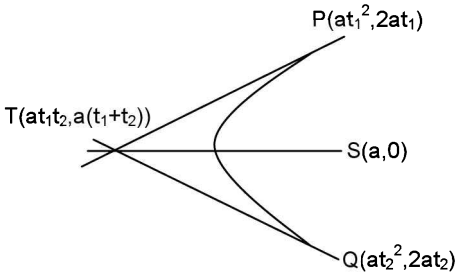
slope of OQ $= \frac{2}{t_1} \Rightarrow \frac{4}{tt_1} = -1 \Rightarrow t_1 = \frac{-4}{t}$ —(2)

From (1) and (2)

$\frac{4}{t} = t + \frac{2}{t} \Rightarrow \frac{2}{t} = t \Rightarrow t = \pm\sqrt{2}$

$\Rightarrow$ points are $\left(2, \pm 2\sqrt{2}\right)$ and $\left(8, \mp 4\sqrt{2}\right)$

141.



$SP = a(t_1^2 + 1)$, $SQ = a(t_2^2 + 1)$

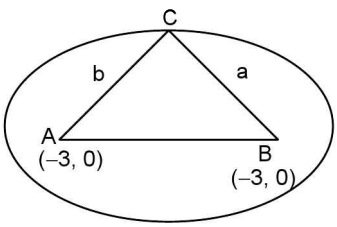
$ST = a\sqrt{t_1^2 t_2^2 + 1 + t_1^2 + t_2^2}$

$4\beta^2 - 4\alpha\gamma = 4a^2(t_1^2 t_2^2 + 1 + t_1^2 + t_2^2)$

$- 4a^2(t_1^2 + 1)(t_2^2 + 1) = 0$

roots are real and equal

142.



$AB = c = 6$

$$4\tan\frac{A}{2}\tan\frac{B}{2} = 1 \Rightarrow 4\cdot\frac{(s-b)(s-c)}{\Delta}\cdot\frac{\Delta}{s(s-b)} = 1$$

$4s - 4c = s \Rightarrow 3s = 24 \Rightarrow s = 8$

$a + b = 2s - c = 10$

Locus is ellipse with focus $(\pm 3, 0)$

$ae = 3 \Rightarrow e = \frac{3}{5}$

$$b^2 = a^2(1 - e^2) = 25\left(1 - \frac{9}{25}\right)$$

$$\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$$

143. Equation of tangent at Q is: $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

Perpendicular from (ae, 0) $= \dfrac{|e\cos\theta - 1|}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}}$

Perpendicular from (–ae, 0) $= \dfrac{|-e\cos\theta - 1|}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}}$

$$\text{Product} = \left|\frac{e^2\cos^2\theta - 1}{\dfrac{\cos^2\theta}{a^2} + \dfrac{1-\cos^2\theta}{b^2}}\right|$$

$$= \left|\frac{b^2\left[(a^2 - b^2)\cos^2\theta - a^2\right]}{a^2 - (a^2 - b^2)\cos^2\theta}\right| = b^2$$

144. Projection of $(6, 2\sqrt{3})$ on transverse axis is (6, 0)

A line with slope m passing through (6,0) is

$y = mx - 6m$

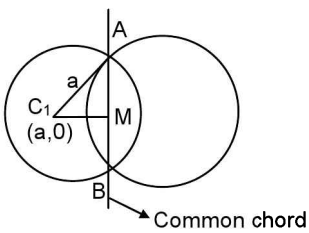
This line is a tangent to the auxillary circle

$x^2 + y^2 = 9$

If $36m^2 = 9(1+m^2) \Rightarrow m = \pm\frac{1}{\sqrt{3}}$

$\therefore$ point of contact of tangent and circle are $\left(\frac{3}{2}, \pm\frac{3\sqrt{3}}{2}\right)$

145.



Centres of the circle are $C_1(a, 0)$ and $C_2(0, b)$

Radii of the circles are $r_1 = a$ and $r_2 = b$

$C_1C_2 = \sqrt{a^2 + b^2}$, $r_1 + r_2 = a + b$

$C_1C_2 < r_1 + r_2$ but $> |r_1 - r_2|$

$\Rightarrow$ circles intersect

The common chord is the radical axis of the circles.

The equation of the common chord is $ax - by = 0$

AM = length of perpendicular from (a, 0) to the line

$ax - by = 0$ equals $\dfrac{a^2}{\sqrt{a^2 + b^2}}$

$$AM^2 = a^2 - \left(\frac{a^2}{\sqrt{a^2 + b^2}}\right)^2$$

$$= \frac{a^2(a^2 + b^2) - a^4}{(a^2 + b^2)} = \frac{a^2b^2}{a^2 + b^2}$$

$$AM = \frac{ab}{\sqrt{a^2+b^2}} \Rightarrow \text{Length of the common chord}$$
$$= \frac{2ab}{\sqrt{a^2+b^2}}$$

146. Let the three circles be

$$x^2 + y^2 - 2\lambda_1 x - c^2 = 0$$
$$x^2 + y^2 - 2\lambda_2 x - c^2 = 0$$
and $x^2 + y^2 - 2\lambda_3 x - c^2 = 0$

Distances of their centres from the origin are $|\lambda_1|$, $|\lambda_2|$, $|\lambda_3|$

since the above distances are in G.P

$$|\lambda_2|^2 = |\lambda_1||\lambda_3| \Rightarrow \lambda_2^2 = |\lambda_1\lambda_2| \quad \text{—(1)}$$

Any point on the circle $x^2 + y^2 = c^2$ may be represented as $(c\cos\theta, c\sin\theta)$

Length of the tangent from $(c\cos\theta, c\sin\theta)$ to a circle $x^2 + y^2 - 2\lambda x - c^2 = 0$ where λ is a parameter, is

$$\sqrt{c^2 - 2\lambda c\cos\theta - c^2} = \sqrt{-2\lambda c\cos\theta}$$

Therefore, the lengths of tangents to the three circles are

$$\sqrt{-2\lambda_1 c\cos\theta}, \sqrt{-2\lambda_2 c\cos\theta}, \sqrt{-2\lambda_3 c\cos\theta}$$

Let these lengths be l_1, l_2 and l_3.

Then, $l_2^2 = |-2\lambda_2 c\cos\theta|$

$$l_1 l_3 = \sqrt{4\lambda_1\lambda_3 c^2\cos^2\theta}$$
$$= \left(\sqrt{|\lambda_1\lambda_3|}\right)|2c\cos\theta|$$

By (1) we see that $l_2^2 = l_1 l_3$

147. Let (x_1, y_1) be a point from which tangents are drawn to the two circles.

Then, the equations of the tangents are

$$y = m_1 x + a\sqrt{1+m_1^2} \quad \text{—(1)}$$
and $$y = m_2 x + b\sqrt{1+m_2^2} \quad \text{—(2)}$$

Given: $m_1 m_2 = -1$ or $m_2 = \frac{-1}{m_1}$

(2) may therefore be written as

$$y = \frac{-1}{m_1}x + b\sqrt{1+\frac{1}{m_1^2}}$$

$$\Rightarrow \quad m_1 y + x = b\sqrt{1+m_1^2} \quad \text{—(3)}$$

(1) may be rewritten as

$$y - m_1 x = a\sqrt{1+m_1^2} \quad \text{—(4)}$$

Since (3) and (4) pass through (x_1, y_1),

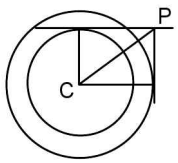
$$m_1 y_1 + x_1 = b\sqrt{1+m_1^2}$$
$$y_1 - m_1 x_1 = a\sqrt{1+m_1^2}$$

squaring and adding

$$(1 + m^2)(x_1^2 + y_1^2) = (b^2 + a^2)(1 + m^2)$$

$\Rightarrow$ Locus of (x_1, y_1) is the circle $x^2 + y^2 = a^2 + b^2$

Aliter:

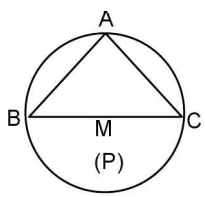


$CP^2 = a^2 + b^2$

So locus is

$x^2 + y^2 = a^2 + b^2$

148.



$$x = 4 + \sqrt{5}\cos\theta;\ y = 1 + \sqrt{5}\sin\theta$$
$$(x-4)^2 + (y-1)^2 = 5 \quad \text{—(1)}$$

We easily see that (2, 2,), (5, 3) and (3, –1) lie on the above circle. Since there can be only one circle passing through three non collinear points, the equations of the circle can be represented in the parametric form

$$x = 4 + \sqrt{5}\cos\theta; \quad y = 1 + \sqrt{5}\sin\theta$$

or by (1)

M is the mid-point of BC

M is (4, 1) which is the centre of the circle. This means that BC is a diameter of the circle. Hence, P coincides with M. or P is (4, 1)

Aliter:

Circle ABC has centre (4, 1), the midpoint of BC and AB = AC; so ABC is right angled, isosceles.

$\therefore$ required point is midpoint of BC viz., (4, 1)

149. As normals at $P(t_1)$ and $Q(t_2)$ meet on the parabola $t_1 t_2 = 2$

Let R be "t".

Then normals at P, Q, R are concurrent at R

$\therefore \quad t_1 + t_2 + t = 0$

or $t_1 + t_2 = -t$ — (2)

$(t_1 - t_2)^2 = (t_1 + t_2)^2 - 4t_1t_2 = t^2 - 8 = \dfrac{x_1 - 8a}{a}$

$\therefore \quad PQ^2 = (at_1^2 - at_2^2)^2 + (2at_1 - 2at_2)^2$

$= a^2(t_1 - t_2)^2 (t_1 + t_2)^2 + 4a^2 (t_1 - t_2)^2$

$= a^2(t_1 - t_2)^2 [(t_1 + t_2)^2 + 4]$

$= a^2 \dfrac{(x_1 - 8a)}{a}\left[t^2 + 4\right]$

$= a^2 \dfrac{x_1 - 8a}{a}\dfrac{x_1 + 4a}{a} = (x_1 - 8a)(x_1 + 4a)$

150. The ellipse is $\dfrac{x^2}{7} + \dfrac{y^2}{2} = 1$

Any tangent from (5, 2) to the ellipse may be assumed as $y - 2 = m(x - 5)$

$\Rightarrow y = mx + (2 - 5m)$ — (1)

Since (1) is a tangent to the given ellipse

$(2 - 5m)^2 = 7m^2 + 2 \Rightarrow 18m^2 - 20m + 2 = 0$

$\Rightarrow 9m^2 - 10m + 1 = 0 \Rightarrow m = \dfrac{1}{9}, 1$

The tangents are $y = \dfrac{1}{9}x + \left(2 - \dfrac{5}{9}\right)$ and

$y = x + (2 - 5)$

i.e., $x - 9y + 13 = 0$ and $x - y - 3 = 0$

151. Let P be $(a\cos\theta, b\sin\theta)$.

Then, Q is $(a\cos\theta, a\sin\theta)$

Equation of the normal at P is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad \text{— (1)}$$

Equation of the normal at Q is

$$\frac{x}{\cos\theta} - \frac{y}{\sin\theta} = 0 \quad \text{— (2)}$$

Using (2), (1) reduces to

$$\frac{ax}{\cos\theta} - \frac{b}{\sin\theta} \times \frac{x\sin\theta}{\cos\theta} = a^2 - b^2$$

$$\Rightarrow \quad \cos\theta = \frac{x}{(a+b)}$$

This gives $\sin\theta = \dfrac{y}{a+b}$

Eliminating θ, locus of R is $x^2 + y^2 = (a + b)^2$

152. Let the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ — (1)

The equation of the tangent at 'θ' on the ellipse is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \quad \text{— (2)}$$

Equation of the auxiliary circle of the ellipse is

$x^2 + y^2 = a^2$ — (3)

The equation of the pair of lines through the origin and through the intersection of (2) and (3) is obtained by homogenizing (3) using (2). The equation of the pair of lines is

$$x^2 + y^2 - a^2\left(\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b}\right)^2 = 0 \quad \text{— (4)}$$

Since the pair of lines represented by (4) are at right angles, coefficient of x^2 + coefficient of y^2 of (4) must be zero

$$(1 - \cos^2\theta) + 1 - \frac{a^2\sin^2\theta}{b^2} = 0$$

$$\left(1 - \frac{a^2}{b^2}\right)(\sin^2\theta) + 1 = 0$$

$(a^2 - b^2)\sin^2\theta = b^2$

$$\sin^2\theta = \frac{b^2}{(a^2 - b^2)} = \frac{a^2(1 - e^2)}{a^2e^2} = \frac{1 - e^2}{e^2}$$

$e^2\sin^2\theta = 1 - e^2$

$e^2(1 + \sin^2\theta) = 1$

$$e^2 = \frac{1}{1 + \sin^2\theta} \quad \Rightarrow e = (1 + \sin^2\theta)^{\frac{-1}{2}}$$

153. Let P be the point α and Q be the point β. Then, the coordinates of P and Q are $(a\cos\alpha, b\sin\alpha)$, $(a\cos\beta, b\sin\beta)$.

Let A be the point (a, 0)

$$\text{Slope of AP} = \frac{b\sin\alpha}{(a\cos\alpha - a)}$$

$$\text{Slope of AQ} = \frac{b\sin\beta}{(a\cos\beta - a)}$$

Since $\angle PAQ = \dfrac{\pi}{2}$,

Slope of AP × slope of AQ = −1

$$\Rightarrow \quad \frac{b^2\sin\alpha\sin\beta}{a^2(1 - \cos\alpha)(1 - \cos\beta)} = -1$$

$$\frac{b^2}{a^2} \times \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \times 2\sin\frac{\beta}{2}\cos\frac{\beta}{2}}{2\sin^2\frac{\alpha}{2} \times 2\sin^2\frac{\beta}{2}} = -1$$

$$\Rightarrow \quad \frac{b^2 \cot\frac{\alpha}{2}\cot\frac{\beta}{2}}{a^2} = -1$$

$$\Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{-b^2}{a^2}$$

154. Let $M(x_1, y_1)$ be the mid-point of chord of $xy = c^2$

Equation of the chord is $T = S_1$

Where, $T \equiv \frac{1}{2}(xy_1 + x_1y) - c^2$

$S_1 \equiv x_1y_1 - c^2$

$\therefore \quad \frac{1}{2}(xy_1 + x_1y) - c^2 = x_1y_1 - c^2$

This passes through (2p, 2q)

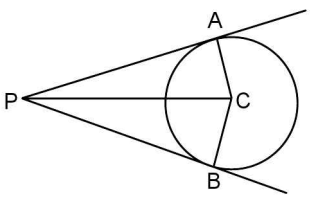
$\therefore \quad \frac{1}{2}(2py_1 + x_1 2q) = x_1y_1$

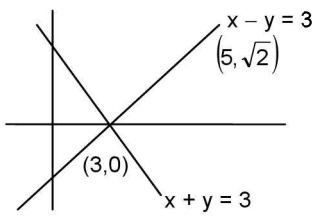
$py_1 + qx_1 - x_1y_1 = 0$

$(x_1 - p)(y_1 - q) = pq$

$\therefore$ Locus of M is $(x - p)(y - q) = pq$

155.





Since the circle touches the lines $x + y = 3$ and $x - y = 3$, the angle bisector of the lines passes through the centre of the circle.

There exists two angular bisectors $x = 3$, $y = 0$

but the circle pass through $(5, \sqrt{2})$

$\therefore \quad y = 0$ passes through the centre.

Let centre of required circle be $(\alpha, 0)$

$$= \left[\frac{|\alpha - 0 - 3|}{\sqrt{2}}\right]^2 = (\alpha - 5)^2 + \left(\sqrt{2}\right)^2$$

$(\alpha - 3)^2 = 2[(\alpha - 5)^2 + 2]$

$\alpha^2 - 6\alpha + 9 = 2\alpha^2 - 20\alpha + 54$

$\alpha^2 - 14\alpha + 45 = 0$

$\alpha = 5$ or $\alpha = 9$

$\therefore \quad$ circles are $(x - 5)^2 + y^2 = 2$

$(x - 9)^2 + y^2 = 18$

156. The circle $x^2 + y^2 = a^2$ and the line $x = 0$ are passing through $(0, \pm a)$.

$\therefore \quad$ The equation of a circle passing through the intersection of $x = 0$ and $x^2 + y^2 = a^2$ is

$x^2 + y^2 - a^2 + \lambda(x) = 0$

Centre $C\left(\frac{-\lambda}{2}, 0\right)$; circle touches $y = mx + c$

$$\Rightarrow \quad r = \sqrt{\frac{\lambda^2}{4} + a^2} = \frac{\left|\frac{-\lambda m}{2} + c\right|}{\sqrt{1 + m^2}}$$

$(\lambda^2 + 4a^2)(1 + m^2) = (\lambda m - 2c)^2$

$\Rightarrow \quad \lambda^2 + 4cm\lambda + 4a^2(1 + m^2) - 4c^2 = 0 \quad$ —(1)

$\therefore \quad$ Required circles $x^2 + y^2 + \lambda_1 x - a^2 = 0 \quad$ —(2)

$x^2 + y^2 + \lambda_2 x - a^2 = 0 \quad$ —(3)

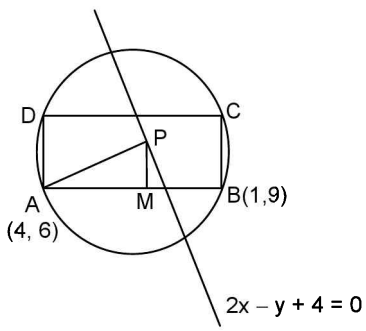
circles are mutually orthogonal, then

$$2\left(\frac{\lambda_1}{2}\right)\left(\frac{\lambda_2}{2}\right) = -2a^2$$

$\lambda_1\lambda_2 = -4a^2 \Rightarrow 4a^2(1 + m^2) - 4c^2 = -4a^2$

$\Rightarrow \quad c^2 = a^2(2 + m^2)$

157.



M is the mid-point of AB $\Rightarrow M\left(\frac{5}{2}, \frac{15}{2}\right)$

(i) Slope of AB $= -1$

$\therefore \quad$ Slope of PM $= 1$

Equation of PM is $y - \frac{15}{2} = \left(x - \frac{5}{2}\right)$

$x - y = -5$

$2x - y = -4$

$-x = -1 \Rightarrow x = 1 \Rightarrow y = 6$

centre (1, 6) ; AB = $3\sqrt{2}$

PM = $\frac{3}{\sqrt{2}} \Rightarrow AD = 3\sqrt{2}$

$\Rightarrow$ Area of ABCD = 18

158. $C_1(-a, -b)$, $r_1 = \sqrt{a^2 + b^2 - c}$

$C_2(-b, -a)$, $r_2 = \sqrt{a^2 + b^2 - c}$

(i) S = 0, S' = 0 intersect if

$|r_1 - r_2| < C_1C_2 < r_1 + r_2$

$\Rightarrow 0 < \sqrt{2(a-b)^2} < 2\sqrt{a^2 + b^2 - c}$

$\Rightarrow (a + b)^2 > 2c$

159. Let the normals at $P(t_1)$, $Q(t_2)$ $2R(t_3)$ intersect.

Let $xt + y = at^3 + 2at$ be a normal passing through (α, β)

$\Rightarrow at^3 + (2a - \alpha)t - \beta = 0$ —(1)

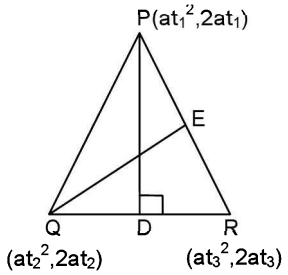
then,

$t_1 + t_2 + t_3 = 0$

$t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a - \alpha}{a}$

$t_1t_2t_3 = +\frac{\beta}{a}$

(i)



Slope of the QR = $\frac{2}{t_2 + t_3}$

$\Rightarrow$ Slope of the PD = $\frac{-(t_2 + t_3)}{2}$

Equation of PD is

$(t_2 + t_3)x + 2y = at_1^2\ (t_2 + t_3) + 4at_1$ — (1)

Similarly,

Equation of QE

$(t_3 + t_1)x + 2y = at_2^2\ (t_3 + t_1) + 4at_2$ — (2)

(1) – (2) $\Rightarrow (t_2 - t_1)x = a(t_1 - t_2)\ [4 + t_1t_2 + t_2t_3 + t_3t_1]$

$x = -a\left(4 + \frac{2a - \alpha}{a}\right) = -4a - 2a + \alpha = \alpha - 6a$

$\Rightarrow 2y = at_1^2t_2 + at_1^2t_3 + 4at_1 - (t_2 + t_3)x$

$2y = at_1^2(-t_1) + 4at_1 - t_1(\alpha - 6a) = -at_1^3 + 4at_1 - t_1\alpha + 6at_1$

$= -\beta$ (from normal (1) $at_1^3 + (2a - \alpha)t_1 = \beta$)

$y = \frac{-\beta}{2}$

$\therefore$ orthocentre $\left(\alpha - 6a, \frac{-\beta}{2}\right)$

160. Distance between the centres = sum of the radii.

$5 = |\lambda| + 4$

$\Rightarrow \lambda = \pm 1$

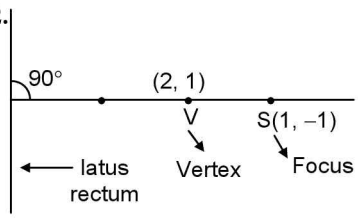
161. Equation of normal is $2x - y = 0$

$\Rightarrow$ tangent equation : $x + 2y + \lambda = 0$

$\Rightarrow 4 = \frac{|\lambda|}{\sqrt{5}} = \lambda = \pm 4\sqrt{5}$

$\therefore$ tangent : $x + 2y \pm 4\sqrt{5} = 0$

162.



We have VS = $\sqrt{5}$

Equation of axis of the parabola is

$y - 1 = 2(x - 2)$

$\Rightarrow \frac{x-2}{\frac{1}{\sqrt{5}}} = \frac{y-1}{\frac{2}{\sqrt{5}}} = R$

Since VS = $\sqrt{5}$

Coordinates of a point on the axis are at a distance $\sqrt{5}$ from (2, 1) is given by

$\frac{x-2}{1/\sqrt{5}} = \frac{y-1}{2/\sqrt{5}} = \pm\sqrt{5}$

Points are (3, 3) or (1, –1)

Hence, X is (3, 3)

Directrix is $y - 3 = \frac{-1}{2}(x - 3)$

$2y - 6 = -x + 3$

$x + 2y - 9 = 0$

Let P be any point on the parabola whose coordinates are (x, y). Then, by definition,

$$(x-1)^2+(y+1)^2=\left(\frac{x+2y-9}{\sqrt{5}}\right)^2$$

$$5\{x^2+y^2-2x+2y+2\}$$

$$=x^2+4y^2+81+4xy-18x-36y$$

$$\Rightarrow\ 4x^2+y^2-4xy+8x+46y-71=0$$

163. $S\equiv\frac{x^2}{16}+\frac{y^2}{7}=1$

$$S'=\frac{25x^2}{144}-\frac{25y^2}{81}=1$$

$$e^2=\frac{16-7}{16}=\frac{9}{16}$$

$$e^2=\frac{\frac{144+81}{25}}{\frac{144}{25}}=\frac{225}{144}$$

$$e=\frac{3}{4}$$

$$e=\frac{15}{12}=\frac{5}{4}$$

$(\pm ae, 0) = (\pm 3, 0)$, for the first

$(\pm ae, 0) = \left(\pm\frac{12}{5}\times\frac{5}{4},0\right) = (\pm 3, 0)$, for the second

Hence, same foci

164. Let $P(x_1, y_1)$ be the mid-point of a chord of $x^2+y^2=8$ then the chord equation $xx_1+yy_1=x_1^2+y_1^2$

$$\Rightarrow\ \frac{xx_1+yy_1}{x_1^2+y_1^2}=1 \qquad —(1)$$

given curve $x^2-2x-2y=0$ —(2)

from (1) and (2) $x^2-2(x+y)\frac{xx_1+yy_1}{x_1^2+y_1^2}=0$ represents pair of perpendicular lines

$$1-\frac{2x_1}{x_1^2+y_1^2}-\frac{2yy_1}{x_1^2+y_1^2}=0$$

$$\Rightarrow\ x^2+y^2-2x-2y=0$$

Clearly, (2, 0) is on the locus.

165. Tangent of $y^2=8x$ in $y=mx+\frac{2}{m}$ —(1)

hyperbola is $81x^2-10y^2=40$ —(2)

$$\Rightarrow\ 81x^2-10\left(m^2x^2+\frac{4}{m^2}+4x\right)=40$$

$$\Rightarrow\ (81-10m^2)x^2-40x-\left(40+\frac{40}{m^2}\right)=0$$

for a tangent the roots are equal

$\Rightarrow\ B^2-4AC=0$

$\Rightarrow\ 10m^4-81m^2-81=0 \quad \Rightarrow m^2=9$ or $m=\pm 3$

$\therefore$ tangent is $y=3x+\frac{2}{3}$ and $y=-3x-\frac{2}{3}$

166. Given normal $lx+my+n=0 \Rightarrow$ slope $=-\frac{l}{m}$;

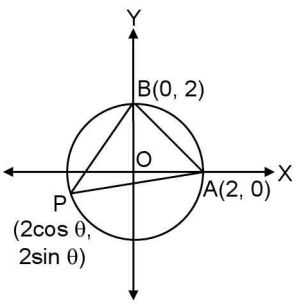
slope of the tangent to $y=\frac{c^2}{x}$ is $\frac{-c^2}{x^2}$

$\therefore$ slope of the normal is $\frac{x^2}{c^2}>0$;

$$\Rightarrow\ \frac{l}{m}<0$$

$$\Rightarrow\ lm<0$$

167.



$x^2+y^2-4x-2y+1=0$

$\Rightarrow\ (x-2)^2+(y-1)^2=4$; shifting the origin to (2, 1), the circle in the new system is $X^2+Y^2=4$

The centroid of ΔPAB is given by

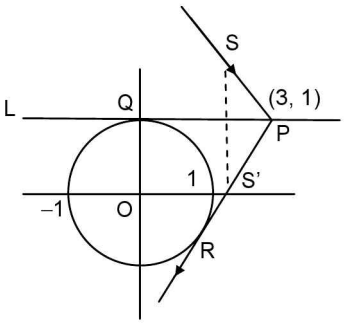
$$(h,k)=\left(\frac{2(1+\cos\theta)}{3};\frac{2(1-\sin\theta)}{3}\right)$$

$\Rightarrow\ (3h-2)^2+(3k-2)^2=4 \quad \therefore$ locus of (h, k) is

$(3(x-2)-2)^2+(3(y-1)-2)^2=4$

$\Rightarrow\ (3x-8)^2+(3y-5)^2=4$, which is a circle

168.



Let P be (3, 1)

Equation of the reflected ray is $y - 1 = m(x - 3)$

$\Rightarrow$ $mx - y + 1 - 3m = 0$ is a tangent to $x^2 + y^2 = 1$

$$1 = \frac{|1-3m|}{\sqrt{1+m^2}}$$

$\Rightarrow$ $m^2 + 1 = 1 + 9m^2 - 6m$

$8m^2 - 6m = 0 \Rightarrow m = 0,\ \ m = \frac{3}{4}$

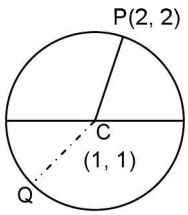
Equation of PR = $3x - 4y - 5 = 0$

Let S' be the reflection of S in L. Then, the equation of the reflected ray is $3x - 4y - 5 = 0$. We have to find the equation of the incident ray. We note that the incident ray and reflected ray are equally inclined to the x-axis. Therefore, the slope of the incident ray = negative of the slope of the reflected ray

$$= \frac{-3}{4}(x-3)$$

$\Rightarrow$ $3x + 4y = 13$

169. Speed of man : $\frac{\pi}{3}$ m/sec



In $\frac{5}{\sqrt{2}}$ sec the man can go $\frac{5\pi}{3\sqrt{2}}$ mts away from P and reach the point Q.

$r = \sqrt{2}$

If $\angle PCQ = \theta$ and $PQ = \frac{5\pi}{3\sqrt{2}}$

$$\Rightarrow \quad \theta = \frac{5\pi}{3\sqrt{2}\times\sqrt{2}} = \frac{5\pi}{6}$$

$$\therefore \quad Q = \left[1+\sqrt{2}\cos\left(\frac{5\pi}{6}+\frac{\pi}{4}\right), 1+\sqrt{2}\sin\left(\frac{5\pi}{6}+\frac{\pi}{4}\right)\right]$$

$$Q\left(1+\sqrt{2}\cos\frac{13\pi}{12}, 1+\sqrt{2}\sin\frac{13\pi}{12}\right)$$

$$\therefore \quad Q = \left[1-\frac{\sqrt{3}+1}{2}, 1-\left(\frac{\sqrt{3}-1}{2}\right)\right]$$

$$Q = \left[\frac{1-\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}\right]$$

170. If $A(at_1^2, 2at_1)$ $B(at_2^2, 2at_2)$ then $P(at_1t_2, a(t_1 + t_2))$

$$AB = a(t_1 - t_2)\sqrt{(t_1+t_2)^2 + 4}$$

$a = 4,\ at_1t_2 = -2$

$a(t_1 + t_2) = 2 \ \Rightarrow\ t_1t_2 = \frac{-1}{2}\ \ t_1 + t_2 = \frac{1}{2}$

$$t_1 - t_2 = \sqrt{\frac{1}{4}+2} = \frac{3}{2}$$

$$AB = 4\cdot\frac{3}{2}\cdot\sqrt{\frac{1}{4}+4} = 3\sqrt{17}$$

171. Consider Statement 2

Substituting $y = mx + c$ in $y^2 + 4ax = 0$, $m^2x^2 + c^2 + 2mcx + 4ax = 0$

$\Rightarrow$ $m^2x^2 + 2(mc + 2a)x + c^2 = 0$

Since $y = mx + c$ is a tangent, the discriminant of the above quadratic in x must be zero

$(mc + 2a)^2 - m^2c^2 = 0$

$4a^2 + 4mca = 0$

$$c = \frac{-a}{m}$$

Statement 2 is false

Consider Statement 1

$x^2 + y^2 - 2x - 3 = 0$ — (1)

$y^2 + 8x - 4y + 12 = 0$ — (2)

Putting $x = -1$ in (1)

$1 + y^2 + 2 - 3 = 0 \Rightarrow y = 0, 0$

Hence $x + 1 = 0$ touches (1)

Putting $x = -1$ in (2)

$y^2 - 8 - 4y + 12 = 0$

$y^2 - 4y + 4 = 0$

$x + 1 = 0$ touches (2)

$\Rightarrow$ Statement 1 is true

Choice (c)

172. Statement 2 is true

Centre of C_1 : (1, –2) and its radius = 6

Centre of C_2 : (4, 2) and its radius = 13

Distance between centres = 5

Difference between their radii = 7

$\Rightarrow$ Statement 1 is true

But, it does not follow from Statement 2

Choice (b)

173. Since (2, 1) is a point inside the given circle and $4x - 3y - 5 = 0$ passes through it. Statement I is true. Statement II is also true but statement I does not follow from statement II.

174. Since $x^2 + y^2 + 2x + 2y - 7 = 0$ and $x^2 + y^2 - 6x - 4y + 9 = 0$ touch externally, the radical axis and common tangent at the point of contact are same and it is $4x + 3y - 8 = 0$. Therefore Statement II is true. Statement I is true only if the two touch circles. So it is false.

175. Distance between the foci of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\sqrt{a^2-b^2}=\sqrt{\frac{1}{\cos^2\theta}-\frac{1}{\cos^2\theta}}=1$ is a constant. Therefore statement I is true. Sum of focal distance of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $2a = 2\cos\theta$ is not a constant. Therefore statement II is also true but I does not follow from II

176. Statement 2 is true

Consider statement 1

The two separate lines are

$x + 2y - 1 = 0$ and $x - y + 1 = 0$

The centre lies on $x + 2y - 1 = 0$, but not on

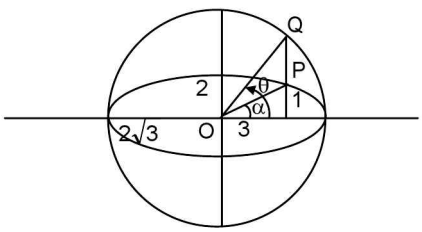
$x - y + 1 = 0$

Therefore, statement 1 is false.

177. Statement I is true but Statement II is false

178. Statement II is true and statement I is also ture however the proof does not follow from II

179.



$P(3, 1) \Rightarrow \sqrt{12}\cos\theta = 3, 2\sin\theta = 1$

$\Rightarrow \quad \theta = \frac{\pi}{6} \Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$

Also $\tan\alpha = \frac{1}{3}$

From fig. $\tan\alpha = \frac{1}{3}$

$\therefore \quad \tan\angle POQ = \tan(\theta - \alpha)$

$$= \frac{\tan\theta - \tan\alpha}{1+\tan\theta\tan\alpha} = \frac{\frac{1}{\sqrt{3}}-\frac{1}{3}}{1+\frac{1}{3\sqrt{3}}}$$

$$= \frac{3-\sqrt{3}}{3\sqrt{3}+1}$$

$\therefore$ Statement I is also true.

180. Statement I and II are true and I $\Rightarrow$ II

181. $a = 17, h = -6, b = 8, g = 23, f = -14, c = 17$

The equations giving the centre are

$17x - 6y + 23 = 0$ and $-6x + 8y - 14 = 0$

Solving the above two equations, we get

$x = -1, y = 1$

The centre of the ellipse is at $(-1, 1)$

182. $x = X\cos\frac{\pi}{4} - Y\sin\frac{\pi}{4}$

$y = X\sin\frac{\pi}{4} + Y\cos\frac{\pi}{4} \Rightarrow x = \frac{X-Y}{\sqrt{2}}; y = \frac{X+Y}{\sqrt{2}}$

$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0$ reduces to

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 - 3\left(\frac{X^2-Y^2}{2}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 + 10\left(\frac{X-Y}{\sqrt{2}}\right)$$

$$-10\left(\frac{X+Y}{\sqrt{2}}\right) + 21 = 0$$

$$\Rightarrow \quad X^2 + Y^2 - \frac{3}{2}\left(X^2 - Y^2\right) - \frac{20Y}{\sqrt{2}} + 21 = 0$$

$$\Rightarrow \quad \frac{-1}{2}X^2 + \frac{5}{2}Y^2 - \frac{20Y}{\sqrt{2}} + 21 = 0$$

$$\Rightarrow \quad X^2 - 5Y^2 + 20\sqrt{2}Y - 42 = 0$$

183. The equation can be rewritten as

$$\frac{(x-2y+1)^2}{\left(\frac{25}{4}\right)} + \frac{(2x+y+2)^2}{\left(\frac{25}{9}\right)} = 1$$

we note that the lines $x - 2y + 1 = 0$ and $2x + y + 2 = 0$ are perpendicular to each other. The two lines intersect at $(-1, 0)$. i.e., the centre of the ellipse is at $(-1, 0)$. Shifting the origin to the point $(-1, 0)$ and rotating the axes through an angle given by $\tan 2\theta = \frac{2h}{(a-b)}$,

$2x + y + 2 = 0$ becomes the new X–axis and $x - 2y + 1 = 0$ becomes the new Y axis .

The equation of the ellipse in the new system will be

$$\frac{X^2}{\left(\frac{25}{4}\right)}+\frac{Y^2}{\left(\frac{25}{9}\right)}=1$$

$$a^2=\frac{25}{4}, b^2=\frac{25}{9}$$

$$\frac{25}{9}=\frac{25}{4}(1-e^2) \Rightarrow 1-e^2=\frac{4}{9}, e=\frac{\sqrt{5}}{3}$$

Eccentricity of the ellipse is $\frac{\sqrt{5}}{3}$

184. $h^2 - ab$, $a + b$ and eccentricity are constants for an ellipse whereas focus is a point. So, any change in the coordinate system affects the co-ordinates of the focus.

185. The equation $\frac{x^2}{2}+\frac{y^2}{1}=1$ represents an ellipse with eccentricity $\frac{1}{\sqrt{2}}$, centre (0, 0) and $y = 0$ as major axis.

Let (X, Y) denote the new coordinates corresponding to (x, y) after rotation. Then,

$x = X\cos\theta + Y\sin\theta$; $y = -X\sin\theta + Y\cos\theta$

The equation becomes

$$\frac{X^2\cos^2\theta+Y^2\sin^2\theta+XY\sin 2\theta}{2}+$$

$$\frac{X^2\sin^2\theta+Y^2\cos^2\theta-XY\sin 2\theta}{1}=1$$

$$X^2\left(\frac{\cos^2\theta}{2}+\frac{\sin^2\theta}{1}\right)+Y^2\left(\frac{\sin^2\theta}{2}+\frac{\cos^2\theta}{1}\right)$$

$$+XY\sin 2\theta\left(\frac{1}{2}-1\right)=1 \qquad \text{— (1)}$$

When $\theta=\frac{\pi}{4}$, we get $3X^2 + 3Y^2 - 2XY = 4$.

186. Let (X, Y) denote the new coordinates corresponding to (x, y) after shifting the centre to (2, 2). Then, $X = x + 2$ and $Y = y + 2$

Equation of ellipse is

$$3(X-2)^2+3(Y-2)^2-2(X-2)(Y-2)=4.$$

$\Rightarrow$ $3X^2 + 3Y^2 - 8X - 8Y - 2XY + 12 = 0.$

187. Solving the two equations

$$x\left(\frac{4x+4}{3}\right)+4x+4-5\left(\frac{4x+4}{3}\right)=0$$

$\Rightarrow$ $x^2 - x - 2 = 0$

$\Rightarrow$ $x = -1, 2$

When $x = 2$, $(1) \to y = 4$

When $y = 0$, $(1) \to x = -1$

$\therefore$ A is the point $(-1, 0)$ and B is the point $(2, 4)$

$|\overline{AB}| = 5 \Rightarrow 2a = 5$

Minor axis $=\frac{1}{5}|AB| = 1$

$\therefore$ $a=\frac{5}{2}$ & $b=\frac{1}{2}$

$$\frac{1}{4}=\frac{25}{4}\left(1-e^2\right)$$

$\Rightarrow$ $e=\frac{2\sqrt{6}}{5}$

188. The centre of the ellipse is the mid-point of AB. It is at $\left(\frac{1}{2},2\right)$

Minor axis is perpendicular to AB.

Slope of the minor axis $=\frac{-3}{4}$

Its equation is

$$y-2=\frac{-3}{4}\left(x-\frac{1}{2}\right)$$

$$4y-8=-3x+\frac{3}{2}$$

$$3x+4y=\frac{19}{2}$$

$\Rightarrow$ $6x + 8y - 19 = 0$

$\Rightarrow$ Choice (c)

189. Equation of the circle on AB as diameter is

$(x + 1)(x - 2) + y(y - 4) = 0$

$x^2 + y^2 - x - 4y - 2 = 0$

Length of tangent from (2, 6) is

$$\sqrt{4+36-2-24-2}=\sqrt{12}=2\sqrt{3}$$

190. Equation of the tangent at $(5\cos\theta, 3\sin\theta)$ is

$$\frac{x}{5}\cos\theta+\frac{y}{3}\sin\theta=1 \qquad \text{— (1)}$$

Homogenizing $x^2 + y^2 = 5^2$ with (1)

$$x^2+y^2=25\left(\frac{x}{5}\cos\theta+\frac{y}{3}\sin\theta\right)^2$$

Since OA and OB are perpendicular

Coefficient x^2 + coefficient $y^2 = 0$

$$\Rightarrow \quad 1 - 25\left(\frac{\cos^2\theta}{25}\right) + 1 - \frac{25\sin^2\theta}{9} = 0$$

$$\Rightarrow \quad \sin^2\theta = \frac{9}{16}$$

Also $e^2 = \dfrac{a^2 - b^2}{a^2} = \dfrac{16}{25}$

$\therefore \quad e^2 = (1 + \sin^2\theta)^{-1}$

191. Equation of the circle is $x^2 + y^2 = a^2$

$\therefore$ co-ordinates of P are $(-a, 0)$

$\therefore$ $\angle QPR = 90°$ and QR is the latus rectum of $y^2 = 4ax$

So orthocentre is $(-a, 0)$ and circumcenter is $(a, 0)$

192. The director circle of $x^2 + y^2 = a^2$ is

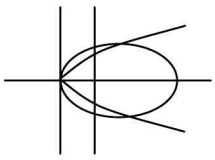
$x^2 + y^2 = 2a^2 \therefore$ or $\left|\dfrac{2c}{\sqrt{2}}\right| < \sqrt{2}a$

$-a < c \Rightarrow 0 < c + a$

$c < a \Rightarrow c - a < 0$

193. Using $C = \dfrac{a}{m}$ for $y^2 = 8x$, we get both $x + y + 2 = 0$ and $x - y + 2 = 0$ are tangents to the parabola.

194. Centre (0, 0) and radius 4 of the circle $x^2 + y^2 = 8x$ Focus of the parabola $y^2 = 4x$ is (1, 0) and Latus rectum 4.



Since abscissa all option are same. Draw a straight line x = 2. the line intersect $(2, 2\sqrt{2})$,. $(2, -2\sqrt{2})$ on P and $(2, 2\sqrt{3})$, $(2, -2\sqrt{3})$ on C. Since $2\sqrt{2} < 3 < 2\sqrt{3}$ and $4 > 2\sqrt{3} > 2\sqrt{2}$

(a) is true, (b) is false, (c) is true and (d) is false

195. Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be the equations of the ellipse, having eccentricity $\dfrac{1}{\sqrt{2}}$

$\therefore \quad e^2 = \dfrac{a^2 - b^2}{a^2} \Rightarrow a^2 = 2b^2$

$\therefore$ ellipse becomes $x^2 + 2y^2 = 2b^2$ — (1)

$\therefore$ $y' = \dfrac{-x}{2y}$ and y' of $x^2 - y^2 = 2$ — (2)

is $\dfrac{x}{y}$

Since (1) and (2) are orthogonal

$\dfrac{-x}{\alpha y} \cdot \dfrac{x}{y} = -1 \Rightarrow x^2 = 2y^2$ — (3)

Solving (2) and (3) we get x = 2 and $y = \sqrt{2}$

$\therefore$ b = 2 from (1)

$\therefore$ required equation of ellipse is

$x^2 + 2y^2 = 8 \Rightarrow \dfrac{x^2}{8} + \dfrac{y^2}{4} = 1$ and $e = \dfrac{1}{\sqrt{2}}$

$\therefore$ foci are $(\pm 2, 0)$ directrices are $x = \pm 4$

$\therefore$ Equation of director circle is

$x^2 + y^2 = a^2 + b^2 \Rightarrow x^2 + y^2 = 12$

$\therefore$ (a) and (d) are true

196. Auxiliary circle of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$x^2 + y^2 = a^2$ —(1) point of intersection

of $y = \dfrac{c^2}{x}$ — (2) with (1) is obtain by

Solving (1) and (2)

Eliminating y we get $x^4 - a^2x^2 + c^4 = 0$

Eliminating x we get $y^4 - a^2y^2 + c^4 = 0$

If $(x_i\ y_i)$ are the roots, (i = 1, 2, 3, 4)

Clearly $\Sigma x_i = 0$, $\Sigma y_i = 0$

and $x_1\ x_2\ x_3\ x_4 = c^4 = y_1\ y_2\ y_3\ y_4$

$\therefore$ AM of $x_i = \dfrac{\sum x_i}{4} = 0$ and

GM of $x_i = (x_1x_2x_3x_4)^{\frac{1}{4}} = C$

$\therefore$ (a) (b) (c) (d) are true

197. Direction circle of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$x^2 + y^2 = a^2 + b^2$

Direction circle of $\dfrac{x^1}{b^2} + \dfrac{y^2}{a^2} = 1$ is

$x^2 + y^2 = b^2 + a^2$

$\therefore$ (a) is true

Auxiliary circle of E_1 and E_2 is $x^2 + y^2 = a^2$

Since $a > b$.

$\therefore$ (b) is true

Foci of E_1 and E_2 are equidistant from the origin.

$\therefore$ (c) true

Points of intersection of E_1 and E_2 are also equidistant from origin

$\therefore$ (d) is true

198. (a) $y = mx + c$ is a tangent to the parabola $y^2 = 4ax$ if

$c = \frac{a}{m}$

$\Rightarrow \quad c = \frac{1}{4m} \Rightarrow m = 1, c = \frac{1}{4}$

(b) $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

if $c^2 = a^2m^2 + b^2$

$c^2 = 25m^2 + 4 \Rightarrow m = \frac{1}{\sqrt{5}} \Rightarrow c = \pm 3$

(c) $y = mx + c$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $c^2 = a^2m^2 - b^2$

$c^2 = 12m^2 - 9 \Rightarrow m = 2, c = \pm\sqrt{39}$

(d) $y = mx + c$ is a tangent to the circle

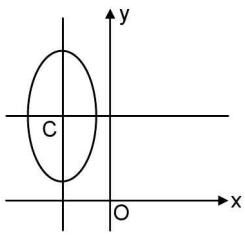
$x^2 + y^2 = r^2$ if $c = r^2(1 + m^2)$

$c^2 = 3(1 + m^2)$

$\Rightarrow m = 2, \quad c^2 = 15$

$\therefore c = \pm\sqrt{15}$

199. (a)



$25(x + 1)^2 - 25 + 16(y - 6)^2 - 576 + 201 = 0$

$25(x + 1)^2 + 16(y - 6)^2 = 400$

$\frac{(x+1)^2}{16}+\frac{(y-6)^2}{25}=1$

Centre of the ellipse is at $(-1, 6)$

$a^2 = 25, \; b^2 = 16$

$\Rightarrow \quad e = \frac{3}{5}$

Foci are at distance $ae = 3$ form the centre

$\Rightarrow$ Foci are at $(-1, 3)$ and $(-1, 9)$

(b) $x^2 - 4x - 12y - 8 = 0$

$(x - 2)^2 = 12y + 8 + 4 = 12(y + 1)$

Vertex of the parabola $\rightarrow (2, -1)$

Axis $\rightarrow x - 2 = 0$

Latus rectum = 12

Focus is at $(2, 2)$

(c) Director circle is concentric with the given circle centre is at $(2, 2)$

(d) $y^2 + 4x + 12 = 0$

$y^2 = -4(x + 3)$

Vertex is at $(-3, 0)$

Latus rectum = 4

Focus is at $(-4, 0)$

200. (a) Radical axis is $2x + 12 = 0$

$\Rightarrow \quad x + 6 = 0$

(b) Centres of the circles are at $(-1, 5)$ and $(3, 2)$ and radii are respectively 2 and 3

Distance between the centres = 5 = sum of the radii

The circles touch externally.

Common tangent is their radical axis

$\Rightarrow \quad 8x - 6y + 18 = 0$

$\Rightarrow \quad 4x - 3y + 9 = 0$

(c) $y^2 = 24x$

Latus rectum = 24

Vertex is at $(0, 0)$

$\Rightarrow$ Directrix is the line $x + 6 = 0$

(d) $x^2 - 4y = 0$

Put $y = 2x - 4$ in $x^2 - 4y = 0$

$x^2 - 8x + 16 = 0$

$\Rightarrow$ equal roots

$y - 2x + 4 = 0$ is a tangent to the parabola.

Put $y = -x - 1$ in $x^2 - 4y = 0$

$x^2 - 4(-x - 1) = 0$

$\Rightarrow$ equal roots

$y + x + 1 = 0$ is a tangent to $x^2 - 4y = 0$

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CHAPTER

3

VECTOR ALGEBRA AND THREE DIMENSIONAL GEOMETRY

CHAPTER OUTLINE

Preview

STUDY MATERIAL

Introduction to Vectors
- Concept Strands (1-15)

Three dimensional Coordinate System
- Concept Strands (16-22)

Components of a Vector Along the Axes of a Rectangular Cartesian Coordinate System
- Concept Strand (23)

Product of Vectors
- Concept Strands (24-32)

Scalar Product or Dot Product or Inner Product of Two Vectors
- Concept Strands (33-39)

Vector Product or Cross Product of Two Vectors
- Concept Strands (40-47)

Scalar and Vector Products Involving More Than Two Vectors
- Concept Strands (48-62)

Products of Four Vectors
- Concept Strands (63-66)

Three Dimensional Geometry
- Concept Strands (67-71)

Plane
- Concept Strands (72-80)

Straight Lines
- Concept Strands (81-87)

Consistency of a Non-homogeneous Linear System of Equations

Vector Algebra

Three Dimensional Geometry

CONCEPT CONNECTORS
- 56 Connectors

TOPIC GRIP
- Subjective Questions (10)
- Straight Objective Type Questions (5)
- Assertion–Reason Type Questions (5)
- Linked Comprehension Type Questions (6)
- Multiple Correct Objective Type Questions (3)
- Matrix-Match Type Question (1)

IIT ASSIGNMENT EXERCISE
- Straight Objective Type Questions (80)
- Assertion–Reason Type Questions (3)
- Linked Comprehension Type Questions (3)
- Multiple Correct Objective Type Questions (3)
- Matrix-Match Type Question (1)

ADDITIONAL PRACTICE EXERCISE
- Subjective Questions (10)
- Straight Objective Type Questions (40)
- Assertion–Reason Type Questions (10)
- Linked Comprehension Type Questions (9)
- Multiple Correct Objective Type Questions (8)
- Matrix-Match Type Questions (3)

INTRODUCTION TO VECTORS

A vector quantity is one which possesses a magnitude and a direction and obeys the triangular law of addition.

Before we venture to introduce the concept of a vector, let us first try to know what we mean by a scalar. We usually hear statements like, 'The price of 1 kg of sugar is Rs 22/-', or 'A particular student has scored 96% in Mathematics', or 'The day temperature today in a city is 32°C".

These statements involve quantifying achievements (example: marks scored in mathematics), physical properties (Example: day temperature in a city) and so on. These quantities or numbers are known as scalars.

We define a scalar as

A scalar is a quantity that is completely characterized by magnitude and sign.

In other words, a scalar is a number positive or negative. Observe that all the examples cited above are examples of scalar. Radius of a circle, distance travelled by a vehicle, age of an individual, capacity of a water tank etc., are other examples of scalars.

Now we introduce the concept of a vector through two examples.

Imagine that a person A walks along a road running from south to north. The person A starts from P on the road and walks a distance of 2 km in the north direction to reach Q. (refer (i) of Fig. 3.1).

Again, suppose another person B walks 2km, but B walks along a road running from East to West. B starts from R and after walking 2 km. along the road reaches S. (refer (ii) of Fig. 3.1).

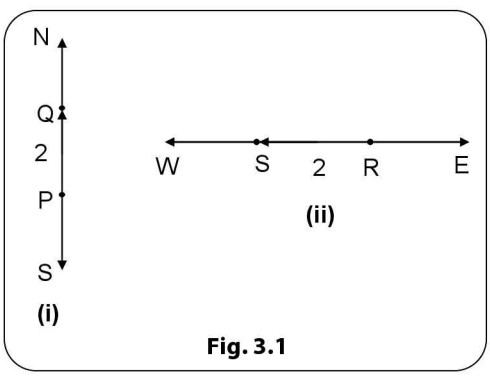


Fig. 3.1

In both cases, the distance walked is 2 km and it is a scalar. We note that the number 2 (equal the distance walked) alone cannot completely characterize the actions of A or B. i.e., a mere scalar representation will not suffice. This is because, in the case of A, walking is in the direction from south to north and in the case of B, walking is in the direction from east to west. So, we need this distinction to be reflected in the representation of their actions. We can represent their actions by a new mathematical entity called 'vector'.

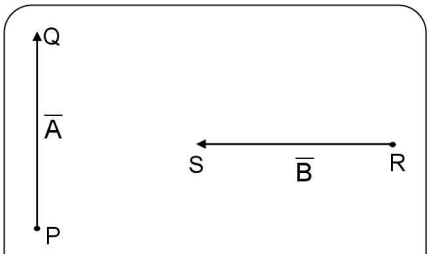


Fig. 3.2

We can say that we can represent the movement of A by the directed line segment PQ (refer Fig. 3.2) or by the vector $\overline{A}$.

P is called the initial point of this vector and Q is its terminal point. Length of PQ = 2 (or proportional to 2 in some unit of measurement. If we take 1 cm as representing 1km, the length of PQ will be equal to 2 cm) and the direction of PQ is towards north.

Similarly, we can represent the movement of B by the directed line segment RS (refer Fig. 3.2) or by the vector $\overline{B}$.

R is called the initial point of this vector and S is called the terminal point of this vector. Length of RS = 2 (or proportional to 2 in some unit of measurement) and the direction of RS will be towards west.

As a second example, consider three forces acting at a point O on a body:

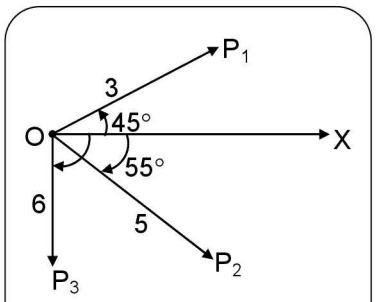


Fig. 3.3

Force F_1 of magnitude 3 kg weight in the direction OP_1 where, $\angle XOP_1 = 45°$;

Force F_2 of magnitude 5 kg weight in the direction of OP_2, where, $\angle XOP_2 = 55°$; and

Force F_3 of magnitude 6 kg weight in the direction of OP_3, where, $\angle XOP_3 = 90°$ (refer Fig. 3.3)

These three forces have different magnitudes and their directions are not same.

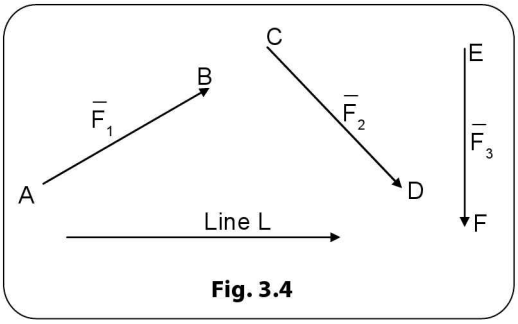


Fig. 3.4

We can represent these forces by the directed line segments $\overline{AB}, \overline{CD}, \overline{EF}$ (or by the vectors $\bar{F}_1, \bar{F}_2, \bar{F}_3$) (refer Fig. 3.4). Here, AB is proportional to 3; CD is proportional to 5 and EF is proportional to 6. If the line L is parallel to OX, AB makes 45° with line L, CD makes 55° with L and EF makes 90° with L, or, in other words, the directions of AB, CD and EF are parallel to OP_1, OP_2 and OP_3 respectively. We are now ready to define a vector.

Definition of a vector

A directed line segment is called a vector. Its length is called the magnitude of the vector (or the length of the vector) and its direction is called the direction of the vector.

We may say that a quantity that is completely characterized (or completely defined) by both magnitude and direction is a vector.

Force is such a quantity. Other examples of vectors are velocity, acceleration, moment of a force about a point etc.

Vectors are usually denoted by $\bar{a}, \bar{b}, \bar{c}, \bar{x}, \bar{y}, \bar{u}, \bar{v}, \ldots$. Graphically, we can represent a vector by an arrow of suitable length and direction.

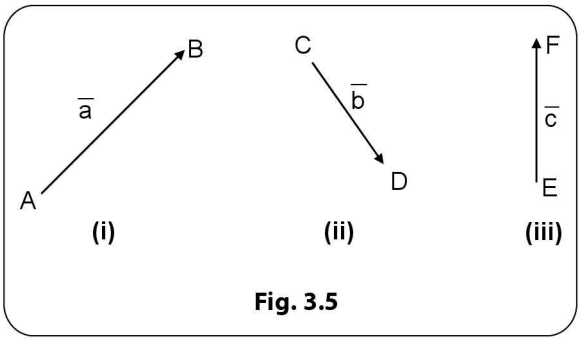


Fig. 3.5

For example, in Fig. 3.5, the vectors $\bar{a}, \bar{b}, \bar{c}$ are represented by the directed line segments AB, CD and EF respectively. A is called the initial point of $\bar{a}$ and B is called the terminal point of $\bar{a}$. Similarly, the initial and terminal points of $\bar{b}$ are C and D respectively; the initial and terminal points of $\bar{c}$ are E and F respectively.

We write

$\bar{a}$ = the vector from A to B = $\overline{AB}$

$\bar{b}$ = the vector from C to D = $\overline{CD}$ and

$\bar{c}$ = the vector from E to F = $\overline{EF}$

The magnitude of a vector, say $\bar{a}$ is denoted by $|\bar{a}|$ and it is proportional to the length AB. The direction of $\bar{a}$ is from A to B. Similarly, $|\bar{b}|$ is proportional to the length CD and the direction of $\bar{b}$ is from C to D. $|\bar{c}|$ is proportional to the length EF and the direction of $\bar{c}$ is from E to F.

A unit vector is a vector having a magnitude of 1 (i.e., the length of a unit vector equals 1 unit). A vector of zero magnitude is called a null vector or a zero vector and is written as $\bar{O}$ (or O). The direction of a null vector is undefined.

Equality of two vectors

$\bar{a}$ and $\bar{b}$ are equal if they have the same magnitude and direction. They need not act at the same location (in other words, $\bar{a}$ and $\bar{b}$ need not have the same initial point).

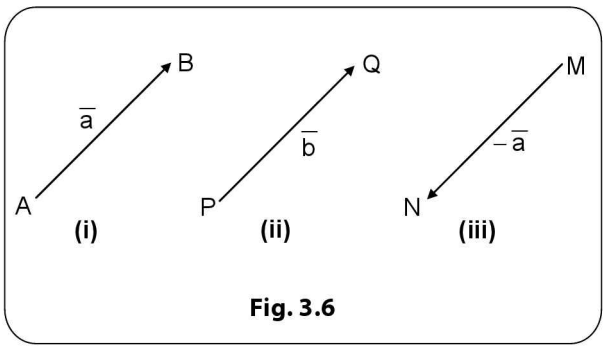


Fig. 3.6

Referring to Fig. 3.6,

(i) Length AB = length PQ and
(ii) AB and PQ have the same direction.

We say that $\bar{a}$ and $\bar{b}$ are equal and write $\bar{a} = \bar{b}$ (which is called a vector equation)

A vector with the same magnitude as $\bar{a}$ but acting in the opposite direction will be denoted by $-\bar{a}$. In Fig. 3.6, length AB = length MN and AB and MN have opposite directions. Therefore, $\overline{MN}$ represents $-\bar{a}$.

Coplanar vectors

Two non-null vectors are said to be collinear if their line segments are parallel. A given set of vectors is said to be coplanar if their line segments are parallel to the same plane. It immediately follows that a set of two vectors is always coplanar.

Addition of vectors

A person walking along $\bar{a}$ and then along $\bar{b}$ would reach the same point as if he walks along $\bar{a}+\bar{b}$.

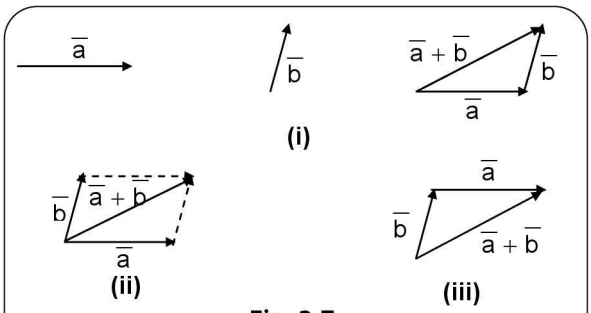


Fig. 3.7

Two vectors $\bar{a}$ and $\bar{b}$ are added by placing the initial point of one at the terminal point of the other as shown in Fig. 3.7. The sum vector (or vector sum) denoted by $\bar{a}+\bar{b}$ is the vector obtained by connecting the initial point of the first vector with the terminal point of the second vector (as shown in (i) and (ii) of Fig. 3.7). If the two vectors to be added are the sides of a parallelogram, their sum, i.e., $\bar{a}+\bar{b}$ is represented by the diagonal of the parallelogram (refer (iii) of Fig. 3.7).

It is clear from the geometry, that vector addition is commutative i.e., $\bar{a}+\bar{b}=\bar{b}+\bar{a}$.

Observations

(i)

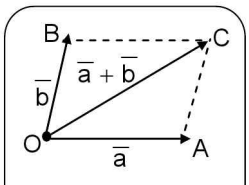


Fig. 3.8

If $\bar{a}$ and $\bar{b}$ are represented by the directed line segments OA and OB, i.e., $\bar{a}=\overline{OA}$ and $\bar{b}=\overline{OB}$, by addition law $\overline{OC}$ represents $\bar{a}+\bar{b}$.

We write $\overline{OA}+\overline{OB}=\overline{OC}$ (refer Fig. 3.8), which means that the sum of the vectors represented by OA and OB is represented by OC.

We hasten to add that $\overline{OA}+\overline{OB}=\overline{OC}$ does not mean that length OA + length OB = length OC. We state that the addition of vectors follows the parallelogram law of addition.

Again, since AC = OB and direction of AC is the same as that of OB, $\overline{AC}=\overline{OB}$ (i.e., vector represented by AC is the same as the vector represented by OB). We thus obtain $\overline{OA}+\overline{AC}=\overline{OC}$

(ii)

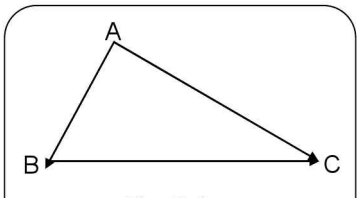


Fig. 3.9

Referring to Fig. 3.9, by the addition law for vectors, $\overline{AB}+\overline{BC}=\overline{AC}$ which is called the triangle law of addition. Since the direction of $\overline{CA}$ is opposite to that of $\overline{AC}$, $\overline{CA}=-\overline{AC}$. We have $\overline{AB}+\overline{BC}=\overline{AC}=-\overline{CA}$

$\Rightarrow \overline{AB}+\overline{BC}+\overline{CA}=0$ (a null vector)

If three vectors can be represented by the sides of a triangle taken in order, their sum is a null vector.

(iii) We show in Fig. 3.10, a graphical demonstration that vector addition is associative; that is,

$\bar{a}+(\bar{b}+\bar{c})=(\bar{a}+\bar{b})+\bar{c}$.

The resultant vector is written as $\bar{a}+\bar{b}+\bar{c}$ since this sum is independent of the grouping.

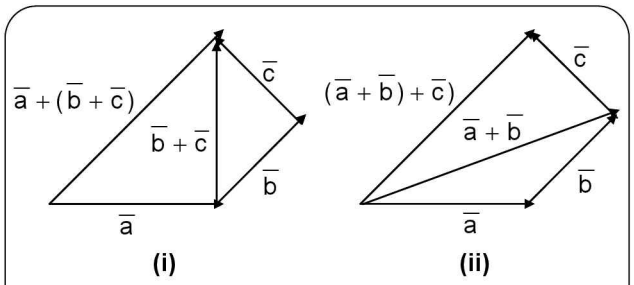


Fig. 3.10

(iv) $\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots, \bar{a}_n$ are n vectors represented by $\overline{OA_1}$, $\overline{A_1A_2}, \ldots \overline{A_{n-1}A_n}$. Then, $\overline{OA_1}+\overline{A_1A_2}+\overline{A_2A_3}+\ldots \overline{A_{n-1}A_n}=\overline{OA_n}$ or sum of the n vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots, \bar{a}_n$ is represented by $\overline{OA_n}$.

The above result is illustrated in Fig. 3. 11.

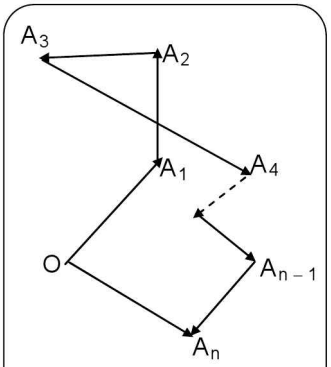


Fig. 3.11

(v) Vector subtraction

Subtraction of vectors may be considered as a special case of addition; that is, $\bar{a} - \bar{b} = \bar{a} + (-1)\bar{b}$

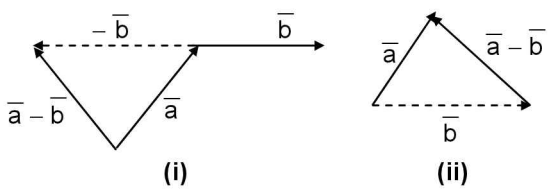


Fig. 3.12

Subtraction is illustrated in Fig. 3.12. Note that $\bar{a} - \bar{b}$ is the other diagonal of the parallelogram whose sides are $\bar{a}$ and $\bar{b}$.

Multiplication of a vector by a scalar

Let $\bar{a}$ be a given vector and k be a scalar ($k \neq 0$). Then $k\bar{a}$ is a vector, which has the magnitude $k\left|\bar{a}\right|$ and direction

(i) same as that of $\bar{a}$ if $k > 0$ and

(ii) opposite to that of $\bar{a}$ if $k < 0$.

Also, we define $0\bar{a}$ = null vector $\bar{O}$. And $-\bar{a}$ is therefore, $(-1)\bar{a}$.

For scalars k_1, k_2 (k_1, $k_2 \neq 0$), we can easily show that $k_1(k_2\bar{a}) = k_2(k_1\bar{a}) = (k_1 k_2)\bar{a}$

If $\bar{a}$ is a non-null vector (i.e, $\left|\bar{a}\right| \neq 0$), $\frac{1}{\left|\bar{a}\right|}\bar{a}$ is the unit vector in the direction of $\bar{a}$ and $\frac{-1}{\left|\bar{a}\right|}\bar{a}$ is the unit vector in the direction opposite to that of $\bar{a}$.

Collinearity of vectors

It may be recalled that two non-null vectors $\bar{a}$ and $\bar{b}$ are collinear if they are parallel. This means that one of the vectors must be a scalar multiple of the other. Therefore, two non-null vectors $\bar{a}$ and $\bar{b}$ are collinear if and only if there exists a scalar $k \neq 0$ such that $\bar{b} = k\bar{a}$.

Results

(i) $\bar{a} + \bar{b} = \bar{b} + \bar{a}$ (commutative law)

(ii) $\bar{a} + \left(\bar{b} + \bar{c}\right) = \left(\bar{a} + \bar{b}\right) + \bar{c}$ (associative law)

(iii) $\bar{a} + \bar{O} = \bar{O} + \bar{a} = \bar{a}$

(iv) $\bar{a} + (-\bar{a}) = \bar{O}$

(v) $k\left(\bar{a}+\bar{b}\right) = k\bar{a} + k\bar{b}$

(vi) $(k_1 + k_2)\bar{a} = k_1\bar{a} + k_2\bar{a}$

(vii) $k_1(k_2\bar{a}) = k_2(k_1\bar{a}) = k_1k_2\bar{a}$

(where k, k_1, k_2 represent scalars $\neq 0$)

Position vector of a point

Although we allow vectors to originate from an arbitrary point in space, as a special case, we will now assume that vectors emanate from an origin O.

The position vector of a point P in space with respect to a fixed point, say O is the vector from O to P.

i.e., position vector of P referred to O as origin = $\overline{OP}$ (refer Fig. 3.13)

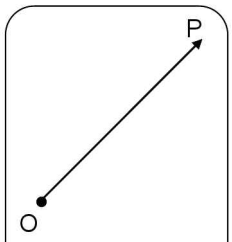


Fig. 3.13

The fixed point is called the origin of reference. Thus, taking O as origin, we can find the position vector of every point in space.

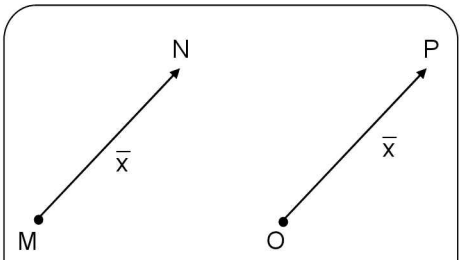


Fig. 3.14

Let $\bar{x}$ be a given vector represented by the directed line segment MN. Through O, the origin (or the fixed point) draw OP parallel to MN such that OP = MN (refer Fig. 3.14). Then, $\overline{OP} = \bar{x}$ or, we can say that

Corresponding to any given vector $\bar{a}$, there is a point P in space such that $\overline{OP} = \bar{a}$.

Remark

A vector drawn parallel to a given vector through a specified point is called a localized vector. But, if the initial point of a vector is not specified, the vector is called a free vector.

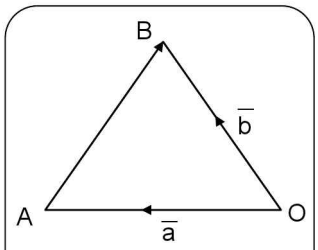


Fig. 3.15

Again, let $\overline{AB}$ be a given vector. Let $\overline{OA}$ and $\overline{OB}$, (which are the position vectors of the initial and terminal points of $\overline{AB}$) be $\bar{a}$ and $\bar{b}$.

Then, by the triangle law of addition, $\overline{OA} + \overline{AB} = \overline{OB}$

$\Rightarrow \overline{AB} = \overline{OB} - \overline{OA}$ = position vector of B − position vector of A = $\bar{b} - \bar{a}$.

CONCEPT STRANDS

Concept Strand 1

If $\bar{A} = 3\bar{i} + 2\bar{j} - 3\bar{k}, \bar{B} = \bar{i} + 5\bar{j} + \bar{k}, \bar{C} = -2\bar{i} + \bar{j}$, find the magnitudes of the vectors

(i) $\bar{A} + 2\bar{B} - 5\bar{C}$

(ii) $\bar{A} + \bar{B} + \bar{C}$

Solution

(i) $\bar{A} + 2\bar{B} - 5\bar{C} = (3\bar{i} + 2\bar{j} - 3\bar{k}) + 2(\bar{i} + 5\bar{j} + \bar{k}) - 5(-2\bar{i} + \bar{j}) = 15\bar{i} + 7\bar{j} - \bar{k}$

$$\left|\bar{A} + 2\bar{B} - 5\bar{C}\right| = \sqrt{15^2 + 7^2 + 1^2} = \sqrt{275} = 5\sqrt{21}$$

(ii) $\bar{A} + \bar{B} + \bar{C} = 2\bar{i} + 8\bar{j} - 2\bar{k}$

$$\left|\bar{A} + \bar{B} + \bar{C}\right| = \sqrt{2^2 + 8^2 + (-2)^2} = \sqrt{72} = 6\sqrt{2}.$$

Concept Strand 2

If $\bar{A} = 2\bar{i} + 4\bar{j} - 5\bar{k}, \bar{B} = \bar{i} + 2\bar{j} + 3\bar{k}$, find the unit vector in the direction of $\bar{A} + \bar{B}$.

$$\bar{A} + \bar{B} = 3\bar{i} + 6\bar{j} - 2\bar{k}$$

$$\left|\bar{A} + \bar{B}\right| = \sqrt{3^2 + 6^2 + 2^2} = \sqrt{49} = 7$$

The unit vector in the direction of $\left(\bar{A} + \bar{B}\right)$ is $\frac{1}{7}\left(\bar{A} + \bar{B}\right)$

i.e., $\frac{1}{7}(3\bar{i} + 6\bar{j} - 2\bar{k}) = \frac{3}{7}\bar{i} + \frac{6}{7}\bar{j} - \frac{2}{7}\bar{k}$.

Concept Strand 3

The following forces act on a particle P:

$$\bar{F_1} = 3\bar{i} + \bar{j}; \bar{F_2} = 4\bar{i} - \bar{j}; \bar{F_3} = -5\bar{j}; \bar{F_4} = 6\bar{i};.$$

Find

(i) resultant of the forces

(ii) magnitude of the resultant

(iii) unit vector in the direction of the resultant.

Solution

Resultant of the forces = vector sum of the forces = $\bar{F_1} + \bar{F_2} + \bar{F_3} + \bar{F_4} = 13\bar{i} - 5\bar{j}$

Magnitude of the resultant = $\sqrt{13^2 + (-5)^2} = \sqrt{194}$.

Unit vector in the direction of the resultant force $= \frac{1}{\sqrt{194}}(13\bar{i} - 5\bar{j}) = \frac{13}{\sqrt{194}}\bar{i} - \frac{5}{\sqrt{194}}\bar{j}$

Note that the resultant of the forces is in the XY plane.

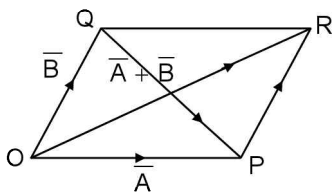
Concept Strand 4

If $\bar{A} = \bar{i} + \bar{j} - 3\bar{k}, \bar{B} = 3\bar{i} + 2\bar{j} + \bar{k}$, verify the result $|\bar{A} + \bar{B}| < |\bar{A}| + |\bar{B}|$.

Solution

By addition law of vectors, $\overline{OR}$ represents the sum vector $\bar{A} + \bar{B}$.

We know that in any triangle, the sum of any two sides is always greater than the third side. Applying this result, we have, from ΔOPR,



$OP + PR > OR$

$\Rightarrow \quad |\bar{A}| + |\bar{B}| > |\bar{A} + \bar{B}|$

$\Rightarrow \quad |\bar{A} + \bar{B}| < |\bar{A}| + |\bar{B}|$

This inequality is called the triangle inequality,

Also, in any triangle, difference between any two sides is less than the third side.

Therefore, for the triangle OPR, $\overline{QP}$ represents $\bar{A} - \bar{B}$ and $|OP - OQ| < QP$

$\Rightarrow \quad \big||\bar{A}| - |\bar{B}|\big| < |\bar{A} - \bar{B}|$

$\Rightarrow \quad |\bar{A} - \bar{B}| > \big||\bar{A}| - |\bar{B}|\big|$

In the given example, $\bar{A} + \bar{B} = 4\bar{i} + 3\bar{j} - 2\bar{k}$;

$|\bar{A}| = \sqrt{1+1+9} = \sqrt{11}$; $|\bar{B}| = \sqrt{9+4+1} = \sqrt{14}$

$|\bar{A} + \bar{B}| = \sqrt{4^2 + 9 + 4} = \sqrt{29}$

Clearly, $\sqrt{29} < \sqrt{11} + \sqrt{14}$

Concept Strand 5

The position vectors of three points A, B, C are $7\bar{i} - 17\bar{j}, \bar{i} + 3\bar{j}$ and $4\bar{i} - 7\bar{j}$ respectively. Show that the three points are collinear.

Solution

$\overline{AB}$ = Position vector of B − Position vector of A $= -6\bar{i} + 20\bar{j} = -2(3\bar{i} - 10\bar{j})$

$\overline{BC}$ = Position vector of C − Position vector of B $= 3\bar{i} - 10\bar{j}$

We see that $\overline{AB} = -2\overline{BC}$ which means that $\overline{AB}$ and $\overline{BC}$ are collinear. Since B is a common point this means that the points A, B, C are collinear.

Note that $\overline{AC} = -(3\bar{i} - 10\bar{j})$, which again proves the above.

Remarks

(i) It is interesting to note here that if A and B are in a plane and that they are represented by the complex numbers z_1 and z_2 respectively, $(z_2 - z_1)$ is represented by the directed line segment AB which means that

(a) length AB gives $|z_2 - z_1|$ and

(b) angle made by AB with the x-axis (with the real axis of the Argand plane) gives $\arg(z_2 - z_1)$.

(ii) **Linear combination of vectors**

Let $\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots, \bar{a}_N$ represent a finite set of vectors and $k_1, k_2, k_3, \ldots, k_N$ scalars. Then the vector $\bar{x} = k_1\bar{a}_1 + k_2\bar{a}_2 + k_3\bar{a}_3 + \ldots + k_N\bar{a}_N$ is called a linear combination of the set of vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots, \bar{a}_N$

For example, if $\bar{a}, \bar{b}, \bar{c}$ are three vectors, $3\bar{a} + 4\bar{b} - 5\bar{c}$, $\bar{a} + 7\bar{c}$, $5\bar{b} + 8\bar{c}$ are linear combinations of the set of vectors $\bar{a}, \bar{b}, \bar{c}$.

Section formula–vector form

Let A and B be two given points and let P be the point dividing AB internally in the ratio m : n $(m \neq n, m, n > 0)$ and let Q be the point dividing AB externally in the ratio m : n (Refer Fig. 3.16).

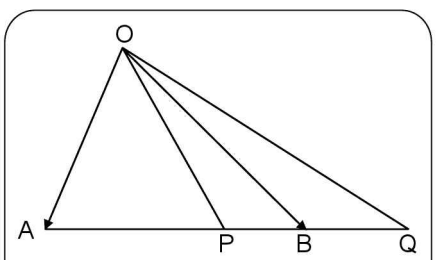


Fig. 3.16

We have $\frac{AP}{PB} = \frac{AQ}{BQ} = \frac{m}{n}$

Let the position vectors of A and B referred to O as origin be $\bar{a}$ and $\bar{b}$, i.e., $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$. We have to find the position vectors of P and Q i.e., $\overline{OP}$ and $\overline{OQ}$ are required.

From Fig. 3.16,

$$\overline{OA} + \overline{AB} = \overline{OB} \Rightarrow \overline{AB} = \overline{OB} - \overline{OA} = \bar{b} - \bar{a}$$

Since $\dfrac{AP}{PB} = \dfrac{m}{n}, \dfrac{AP}{AB} = \dfrac{m}{m+n} \Rightarrow AP = \dfrac{m}{m+n}AB$

Since $\overline{AP}$ and $\overline{AB}$ have the same direction, $\overline{AP}$ is a scalar multiple of $\overline{AB}$

$$\Rightarrow \quad \overline{AP} = \frac{m}{m+n}\overline{AB} = \frac{m}{m+n}(\bar{b} - \bar{a})$$

Again,

$$\overline{OP} = \overline{OA} + \overline{AP} = \bar{a} + \frac{m}{m+n}(\bar{b} - \bar{a})$$

$$= \frac{1}{m+n}(m\bar{b} + n\bar{a}) \qquad \text{— (1)}$$

Since $\dfrac{AQ}{BQ} = \dfrac{m}{n}, \dfrac{AQ}{BQ} - 1 = \dfrac{m-n}{n}$

$$\Rightarrow \quad \frac{AQ - BQ}{BQ} = \frac{m-n}{n} \Rightarrow \frac{AB}{BQ} = \frac{m-n}{n}$$

$$BQ = \frac{n}{m-n}AB$$

Therefore, $\overline{BQ} = \dfrac{n}{m-n}\overline{AB} = \dfrac{n}{m-n}(\bar{b} - \bar{a})$

We have

$$\overline{OQ} = \overline{OB} + \overline{BQ} = \bar{b} + \frac{n}{m-n}(\bar{b} - \bar{a})$$

$$= \frac{1}{(m-n)}(m\bar{b} - n\bar{a}) \qquad \text{— (2)}$$

(1) and (2) give the position vectors of P and Q.

If A and B be two given points with position vectors $\bar{a}$ and $\bar{b}$, P is the point dividing AB internally in the ratio $m : n$ ($m \neq n$, $m, n > 0$) and Q is the point dividing AB externally in the ratio $m : n$, then

$$\text{Position vector of P} = \frac{1}{m+n}(m\bar{b} + n\bar{a})$$

$$\text{Position vector of Q} = \frac{1}{(m-n)}(m\bar{b} - n\bar{a})$$

Note that P.V. of Q is got from that of P by changing (m, n) to $(-m, n)$ or $(m, -n)$

Observation

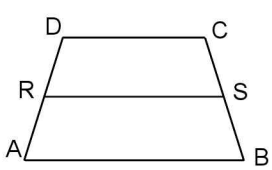
Position vector of the middle point of the line segment AB is obtained by setting $m = n$ in (1); it is $\frac{1}{2}(\bar{a} + \bar{b})$.

We now work out a few examples to reinforce the basic ideas already conveyed.

CONCEPT STRANDS

Concept Strand 6

Show that the segment joining the mid-points of two non-parallel sides of a trapezium is parallel to the parallel sides and is half of their sum.



Solution

Method 1

Let O be the origin of reference. R is mid-point of AD

$$\therefore \quad \overline{OR} = \frac{1}{2}\left(\overline{OA} + \overline{OD}\right); \ \overline{OS} = \frac{1}{2}\left(\overline{OB} + \overline{OC}\right)$$

$$\overline{RS} = \overline{OS} - \overline{OR} = \frac{1}{2}\left(\overline{OB} - \overline{OA} + \overline{OC} - \overline{OD}\right)$$

$$= \frac{1}{2}\left(\overline{AB} + \overline{DC}\right) \qquad \text{— (1)}$$

$$= \frac{1}{2}\left(\overline{AB} + k\overline{AB}\right) \text{ as DC || AB, } k > 0$$

$$= \frac{1}{2}(k+1)\overline{AB} = \overline{RS} \parallel \overline{AB}$$

$$|RS| = \frac{k+1}{2}AB = \frac{1}{2}\left(AB + kAB\right)$$

$$= \frac{1}{2}\left(AB + DC\right)$$

Method 2

We take A as the origin of reference.

Let $\overline{AB} = \bar{b}$ and $\overline{AD} = \bar{d}$

Since DC is parallel to AB, $\overline{DC} = k\overline{AB} = k\bar{b}$, k is a scalar > 0.

R and S are the mid-points of the non-parallel sides.

We have $\overline{AR} = \frac{1}{2}\overline{AD} = \frac{1}{2}\bar{d}$;

$\overline{AC} = \overline{AD} + \overline{DC} = \bar{d} + k\bar{b}$;

$\overline{AS} = \frac{1}{2}\left(\overline{AC} + \overline{AB}\right) = \frac{1}{2}(\bar{d} + k\bar{b} + \bar{b})$;

$\overline{RS} = \overline{AS} - \overline{AR} = \frac{1}{2}(\bar{d} + k\bar{b} + \bar{b}) - \frac{1}{2}\bar{d} = \frac{1}{2}(k+1)\bar{b}$

The above result shows that $\overline{RS}$ is parallel to $\bar{b}$

$\Rightarrow$ RS is parallel to AB

Also, $\left|\overline{RS}\right| = \frac{1}{2}(k+1)\left|\bar{b}\right| = \frac{1}{2}\left(k\left|\bar{b}\right| + \left|\bar{b}\right|\right) = \frac{1}{2}$ (sum of the parallel sides)

Concept Strand 7

In a triangle ABC, D, E, F are the mid-points of the sides BC, CA and AB respectively. Show that $\overline{AD} + \overline{BE} + \overline{CF} = \bar{O}$.

Solution

Taking A as origin of reference, let $\overline{AB} = \bar{x}, \overline{AC} = \bar{y}$. Then, $\overline{BC} = \overline{AC} - \overline{AB} = \bar{y} - \bar{x}$. Since D, E, F are the mid-points of BC, CA, AB,

$$\overline{AD} = \frac{1}{2}\left(\overline{AB} + \overline{AC}\right) = \frac{1}{2}\left(\bar{x} + \bar{y}\right)$$

$$\overline{BE} = \frac{1}{2}\left(\overline{BA} + \overline{BC}\right) = \frac{1}{2}\left(-\bar{x} + \bar{y} - \bar{x}\right) = \frac{1}{2}\left(\bar{y} - 2\bar{x}\right)$$

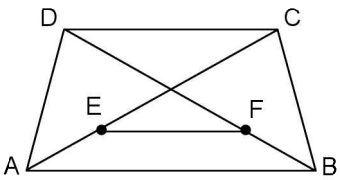
$$\overline{CF} = \frac{1}{2}\left(\overline{CA} + \overline{CB}\right) = \frac{1}{2}\left(-\bar{y} - \bar{y} + \bar{x}\right) = \frac{1}{2}\left(\bar{x} - 2\bar{y}\right)$$

$$\overline{AD} + \overline{BE} + \overline{CF} = \frac{1}{2}\left(\bar{x} + \bar{y}\right) + \frac{1}{2}\left(\bar{y} - 2\bar{x}\right) + \frac{1}{2}\left(\bar{x} - 2\bar{y}\right) = \bar{O}$$

Concept Strand 8

Show that the straight line joining the mid points of the diagonals of a trapezium is parallel to the parallel sides and is half their difference.

Solution



Let E and F be the mid-points of the diagonals of the trapezium ABCD.

Taking A as the origin of reference, let $\overline{AB} = \bar{x}, \overline{AD} = \bar{y}$

$$\overline{BD} = \overline{AD} - \overline{AB} = \bar{y} - \bar{x}$$

Since F is the mid-point of BD,

$$\overline{AF} = \frac{1}{2}\left(\overline{AB} + \overline{AD}\right) = \frac{1}{2}\left(\bar{x} + \bar{y}\right)$$

Since DC is parallel to AB, $\overline{DC} = k\overline{AB} = k\bar{x}$, $k < 1$ (DC < AB)

$$\overline{AC} = \overline{AD} + \overline{DC} = \bar{y} + k\bar{x};$$

$$\overline{AE} = \frac{1}{2}\overline{AC} = \frac{1}{2}\left(\bar{y} + k\bar{x}\right)$$

$$\overline{EF} = \overline{AF} - \overline{AE} = \frac{1}{2}\left(\bar{x} + \bar{y}\right) - \frac{1}{2}\left(\bar{y} + k\bar{x}\right)$$

$$= \frac{1}{2}(1-k)\bar{y}$$

$\Rightarrow$ $\overline{EF}$ is parallel to $\overline{AB}$ and $\left|\overline{EF}\right| = \frac{1}{2}(1-k)\left|\bar{y}\right|$

$$= \frac{1}{2}\left(\left|\bar{y}\right| - k\left|\bar{y}\right|\right) = \frac{1}{2}(AB - DC)$$

Result follows.

Concept Strand 9

If $\bar{a}, \bar{b}$ are the vectors represented by consecutive sides of a regular hexagon ABCDEF, express the vectors $\overline{CD}, \overline{DE}, \overline{EF}, \overline{FA}, \overline{AC}, \overline{AD}, \overline{AE}$, and $\overline{CE}$ in terms of $\bar{a}$ and $\bar{b}$.

Solution

Let $\overline{AB} = \bar{a}, \overline{BC} = \bar{b} \Rightarrow \overline{AC} = \bar{a} + \bar{b}$ — (1)

Since ABCDEF is a regular hexagon, AD is parallel to BC and is twice BC.

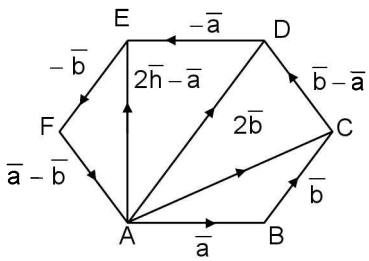
$$\Rightarrow \quad \overline{AD} = 2\overline{BC} = 2\overline{b} \qquad —(2)$$

$$\overline{CD} = \overline{AD} - \overline{AC} = 2\overline{b} - (\overline{a} + \overline{b}) = \overline{b} - \overline{a} \qquad —(3)$$

$$\overline{FA} = -\overline{CD} = \overline{a} - \overline{b} \qquad —(4)$$

$$\overline{DE} = -\overline{AB} = -\overline{a} \qquad —(5)$$

$$\overline{EF} = -\overline{BC} = -\overline{b} \qquad —(6)$$

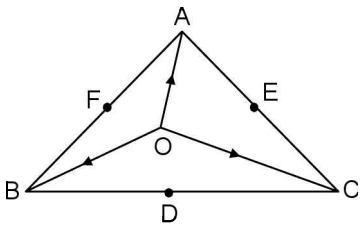


$\overline{AE} = \overline{AD} + \overline{DE} = 2\overline{b} - \overline{a}$ from (2) and (5)

$\overline{CE} = \overline{CD} + \overline{DE} = \overline{b} - \overline{a} - \overline{a} = \overline{b} - 2\overline{a}$, from (3) and (5)

Concept Strand 10

Prove that the system of concurrent forces acting at a point O and represented by $\overline{OA}, \overline{OB}, \overline{OC}$ is equivalent to the system of forces represented by $\overline{OD}, \overline{OE}, \overline{OF}$ (acting at O) if D, E, F are the middle points of the sides BC, CA and AB respectively of triangle ABC. Also prove that



$$\overline{AD} + \frac{2}{3}\overline{BE} + \frac{1}{3}\overline{CF} = \frac{1}{2}\overline{AC} \text{ and } \overline{OE} + \overline{OF} + \overline{DO} = \overline{OA}$$

Solution

We have $\overline{OB} + \overline{OC} = 2\overline{OD}$, since D is the mid-point of BC.

Similarly, $\overline{OC} + \overline{OA} = 2\overline{OE}$ and $\overline{OA} + \overline{OB} = 2\overline{OF}$

Adding, we get

$$2\left(\overline{OA} + \overline{OB} + \overline{OC}\right) = 2\left(\overline{OD} + \overline{OE} + \overline{OF}\right)$$

$$\Rightarrow \quad \overline{OA} + \overline{OB} + \overline{OC} = \overline{OD} + \overline{OE} + \overline{OF}$$

Again, $\overline{AD} = \overline{OD} - \overline{OA} = \frac{1}{2}\left(\overline{OB} + \overline{OC}\right) - \overline{OA}$ — (1)

$$\overline{BE} = \overline{BO} + \overline{OE} = -\overline{OB} + \frac{1}{2}\left(\overline{OA} + \overline{OC}\right)$$

$$\Rightarrow \quad \frac{2}{3}\overline{BE} = -\frac{2}{3}\overline{OB} + \frac{1}{3}\left(\overline{OA} + \overline{OC}\right) \qquad — (2)$$

$$\overline{CF} = \overline{CO} + \overline{OF} = -\overline{OC} + \frac{1}{2}\left(\overline{OA} + \overline{OB}\right)$$

$$\Rightarrow \quad \frac{1}{3}\overline{CF} = -\frac{1}{3}\overline{OC} + \frac{1}{6}\left(\overline{OA} + \overline{OB}\right) \qquad — (3)$$

(1) + (2) + (3) gives,

$$\overline{AD} + \frac{2}{3}\overline{BE} + \frac{1}{3}\overline{CF} = -\frac{1}{2}\overline{OA} + \frac{1}{2}\overline{OC}$$

$$= \frac{1}{2}\left(\overline{OC} - \overline{OA}\right) = \frac{1}{2}\overline{AC}$$

$$\overline{OE} + \overline{OF} + \overline{DO} = \frac{1}{2}\left(\overline{OC} + \overline{OA}\right) +$$

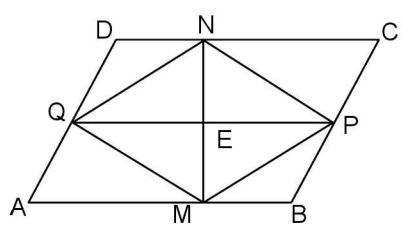
$$\frac{1}{2}\left(\overline{OA} + \overline{OB}\right) - \frac{1}{2}\left(\overline{OB} + \overline{OC}\right) = \overline{OA}$$

Concept Strand 11

ABCD is a quadrilateral and E is the point of intersection of the lines joining the middle points of opposite sides. Show that the resultant of $\overline{OA}, \overline{OB}, \overline{OC}$ and $\overline{OD}$ is equal to $4\overline{OE}$ where O is any point.

Solution

Let M, P, N, Q be the mid-points of the sides of the quadrilateral. Then MPNQ is a parallelogram. Therefore, the diagonals MN and PQ intersect at E where E is the mid point of PQ as well as MN.



Taking O as the origin of reference,

$$\overline{OA} + \overline{OB} = 2\overline{OM}$$

$$\overline{OC} + \overline{OD} = 2\overline{ON}$$

$$\Rightarrow \quad \overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = 2\left(\overline{OM} + \overline{ON}\right)$$

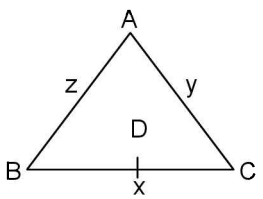
$$= 2 \times 2\overline{OE} = 4\overline{OE}$$

Concept Strand 12

Prove that the internal bisectors of the angles of a triangle are concurrent.

Solution

Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of the vertices A, B, C of the triangle ABC. We know from geometry that the internal bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle.



Let BC = x, CA = y, AB = z. Let the internal bisector of angle A meet the opposite side in D; then, $\frac{BD}{DC} = \frac{z}{y}$.

$\overline{OD} = \frac{z\bar{c} + y\bar{b}}{y + z}$, where O is the origin of reference.

Therefore, the position vector of the point I on the bisector AD which divides it in the ratio $(y + z) : x$ is

$$\frac{x\bar{a} + (y + z)\frac{z\bar{c} + y\bar{b}}{(y + z)}}{(x + y + z)} = \frac{x\bar{a} + y\bar{b} + z\bar{c}}{x + y + z}$$

From the symmetry of the above result, we conclude that this point I lies on the internal bisectors of the angles B and C also. Hence the three bisectors are concurrent at I.

Concept Strand 13

If G is the centroid of a triangle ABC, show that $\overline{GA} + \overline{GB} + \overline{GC} = \bar{0}$, and conversely, if $\overline{GA} + \overline{GB} + \overline{GC} = \bar{0}$, then G is the centroid of the triangle ABC.

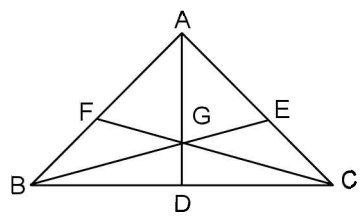
Solution

We know that the medians of a triangle are concurrent and that the point of concurrence is called the centroid of the triangle. Let the position vectors of the vertices A, B, C referred to an origin of reference O be $\bar{a}, \bar{b}, \bar{c}$ respectively.

Then the position vector of G is $\frac{1}{3}\left(\bar{a} + \bar{b} + \bar{c}\right)$

$$\overline{GA} = \text{Position vector of A} - \text{Position vector of G}$$

$$= \bar{a} - \frac{1}{3}\left(\bar{a} + \bar{b} + \bar{c}\right) = \frac{1}{3}\left(2\bar{a} - \bar{b} - \bar{c}\right)$$



Similarly, $\overline{GB} = \frac{1}{3}\left(2\bar{b} - \bar{c} - \bar{a}\right); \overline{GC}$

$$= \frac{1}{3}\left(2\bar{c} - \bar{a} - \bar{b}\right)$$

$$\Rightarrow \quad \overline{GA} + \overline{GB} + \overline{GC}$$

$$= \frac{1}{3}\left(2\bar{a} - \bar{b} - \bar{c} + 2\bar{b} - \bar{c} - \bar{a} + 2\bar{c} - \bar{a} - \bar{b}\right)$$

$$= \bar{O}.$$

Conversely, let $\overline{GA} + \overline{GB} + \overline{GC} = 0$

Then, $\left(\overline{OA} - \overline{OG}\right) + \left(\overline{OB} - \overline{OG}\right) + \left(\overline{OC} - \overline{OG}\right) = 0$

$$\Rightarrow \quad \overline{OG} = \frac{1}{3}\left(\overline{OA} + \overline{OB} + \overline{OC}\right)$$

Hence G is the centroid of the triangle whose vertices are A, B and C.

Concept Strand 14

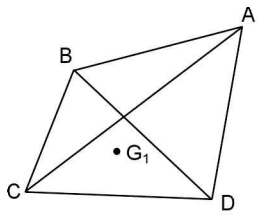
(i) Prove that the lines joining the vertices of a tetrahedron to the centroids of opposite faces are concurrent.

(ii) Prove also that the lines joining the mid points of the opposite edges of a tetrahedron intersect and bisect each other.

Solution

(i) Let ABCD represent the tetrahedron. Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be the position vectors of the vertices A, B, C, D referred to an origin of reference O.

G_1 is the centroid of the ΔBCD and its position vector is $\frac{1}{3}\left(\bar{b} + \bar{c} + \bar{d}\right)$.



The position vector of a point G which divides AG_1 in the ratio 3 : 1 is

$$\frac{3 \times \frac{1}{3}\left(\bar{b} + \bar{c} + \bar{d}\right) + 1 \times \bar{a}}{(3 + 1)} = \frac{1}{4}\left(\bar{a} + \bar{b} + \bar{c} + \bar{d}\right)$$

The symmetry of the result shows that this point will also lie on BG_2, CG_3 and DG_4 where G_2, G_3, G_4 represent the centroids of the triangular faces CAD, ABD and ABC respectively. Hence these four lines (i.e, AG_1, BG_2, CG_3, DG_4) are concurrent and the point of concurrence is G which is called the centroid of the tetrahedron.

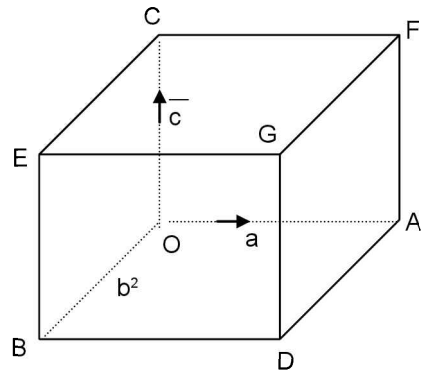
(ii) The position vector of the mid point of AD is $\frac{1}{2}\left(\bar{a} + \bar{d}\right)$ and that of BC is $\frac{1}{2}\left(\bar{b} + \bar{c}\right)$ and the position vector of the mid point of the line segment joining these points is $\frac{1}{4}\left(\bar{a} + \bar{b} + \bar{c} + \bar{d}\right)$ and symmetry of this result proves the statement.

Concept Strand 15

Prove that the four diagonals of a parallelepiped and the joins of the mid-points of opposite edges are concurrent at a common point of bisection.

Solution

Let OA, OB, OC be the coterminous edges of the parallelepiped.



Taking O as the origin of reference, let the position vectors of A, B and C be $\bar{a}, \bar{b}$, and $\bar{c}$ respectively so that those of D, E, F, G are $\bar{a} + \bar{b}, \bar{b} + \bar{c}, \bar{c} + \bar{a}, \bar{a} + \bar{b} + \bar{c}$ respectively. If M_1 is the mid-point of the diagonal OG, then, the position vector of M_1 is $\frac{1}{2}\left(\bar{a} + \bar{b} + \bar{c}\right)$. If M_2 is the mid point of the diagonal AE, position vector of M_2 is $\frac{\bar{a} + \bar{b} + \bar{c}}{2}$ which is the same as that of M_1. Similarly, the position vectors of the mid-points of the other diagonals DC and BF are also $\frac{1}{2}\left(\bar{a} + \bar{b} + \bar{c}\right)$.

Again, midpoint M of BD has position vector $\frac{1}{2}\left(\bar{b} + \bar{a} + \bar{b}\right) = \frac{1}{2}\left(\bar{a} + 2\bar{b}\right)$ and mid point L of CF is having position vector $\frac{1}{2}\left(\bar{c} + \bar{c} + \bar{a}\right) = \frac{1}{2}\left(2\bar{c} + \bar{a}\right)$. $\Rightarrow$ Mid-point of LM is having position vector.

$\frac{1}{4}\left(\bar{a} + 2\bar{b} + 2\bar{c} + \bar{a}\right) = \frac{1}{2}\left(\bar{a} + \bar{b} + \bar{c}\right)$. In a similar manner we can show that the mid-points of the join of other opposite edges are also the same.

THREE DIMENSIONAL COORDINATE SYSTEM

X'OX, Y'OY, Z'OZ are three mutually perpendicular real lines, with their origins coinciding. (In general we shall assume that $\overline{OX}, \overline{OY}, \overline{OZ}$ form a right-handed system).

If P is a point, let the plane through P, parallel to the plane determined by Y'OY and Z'OZ (the yz – plane), meet X'OX at the point representing the real numbers

on it. Let y, z be defined on similar lines. Then the ordered triple(x, y, z) determines and is determined uniquely by the point P. x, y, z are called the Cartesian coordinates of P – the x, y and z coordinates respectively.

X'OX, Y'OY, Z'OZ are the x, y, z-(coordinate) axes and YZ, ZX, XY planes, the coordinate planes. Origin has coordinates (0, 0, 0).

Every point on YZ planes has x-coordinate zero; a point on the x-axis has y- as well as z-coordinate zero.

The three coordinate planes divide space into 8 'octants'. The signs of coordinates of a point depend on the octant in which the point lies.

If $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along $\overline{OX}, \overline{OY}, \overline{OZ}$ then the position vector of P with coordinates x, y, z relative to the origin is $\overline{OP} = xi + yj + zk$, and conversely P can therefore be characterized by its positon vector or coordinates.

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are 2 points,

$$\overline{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}\text{; and hence}$$

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$$OP = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

Section formula

We give below the formula for the coordinates of the point dividing the line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio l : m (l, m positive). (Refer Fig. 3.17)

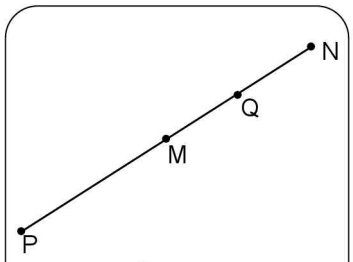


Fig. 3.17

Let M represent the point dividing the line segment PQ in the ratio ℓ : m internally. Then, the coordinates of M are given by $\left(\frac{\ell x_2 + mx_1}{\ell + m}, \frac{\ell y_2 + my_1}{\ell + m}, \frac{\ell z_2 + mz_1}{\ell + m}\right)$.

Let N represent the point dividing the line segment PQ in the ratio l : m externally (for the external point of division, $l \neq m$). Then, the coordinates of N are

$$\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}, \frac{lz_2 - mz_1}{l - m}\right).$$

Note that this is the same as for internal division, with $-\ell$: m or ℓ : –m as ratio.

The coordinates of the midpoint of the line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by

$$\left(\frac{x_2 + x_1}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

The centroid of the triangle with vertices (x_r, y_r, z_r) (r = 1, 2, 3) is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$

We will now work out a few examples.

CONCEPT STRANDS

Concept Strand 16

Find the distance between the points (1, 1, –3) and (3, 2, 6).

Solution

Distance between the points is given by

$$\sqrt{(3-1)^2 + (2-1)^2 + (6+3)^2} = \sqrt{86}$$

Concept Strand 17

Show that the points (2, 3, 5), (–1, 5, –1) and (4, –3, 2) are the vertices of an isoceles right-angled triangle.

Solution

Let the given points be A, B and C.

$$AB^2 = (-1-2)^2 + (5-3)^2 + (-1-5)^2 = 49$$

$$BC^2 = (4+1)^2 + (-3-5)^2 + (2+1)^2 = 98$$

$$CA^2 = (2-4)^2 + (3+3)^2 + (5-2)^2 = 49$$

$\Rightarrow$ $AB^2 = AC^2$ and $AB^2 + AC^2 = BC^2$. Result follows.

Concept Strand 18

Find k such that the distance between the points (7, 1, 3) and (4, 5, k) is 13 units.

Solution

We have $(7-4)^2 + (5-1)^2 + (k-3)^2 = 13^2$

Or $k^2 - 6k - 135 = 0$

$\Rightarrow$ $(k-15)(k+9) = 0$

$\Rightarrow$ $k = -9, 15$

Concept Strand 19

Find the point on the y-axis, which is equidistant from the points (3, 1, 2) and (5, 5, 2).

Solution

Any point on the y-axis is of the form $(0, \lambda, 0)$. We have,

$$(0-3)^2 + (\lambda-1)^2 + (0-2)^2$$
$$= (0-5)^2 + (\lambda-5)^2 + (0-2)^2$$

Or $14 + \lambda^2 - 2\lambda = 54 + \lambda^2 - 10\lambda$

$\Rightarrow$ $8\lambda = 40$

$\Rightarrow$ $\lambda = 5.$

$\therefore$ The point is (0, 5, 0)

Concept Strand 20

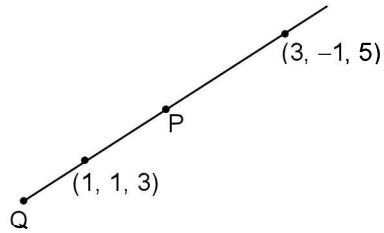
Obtain the coordinates of the points dividing the line joining the points (1, 1, 3) and (3, –1, 5) in the ratio 3 : 4 internally and externally.

Solution

Let P and Q represent the required points of division. Then

P is $\left(\dfrac{3\times3+4\times1}{7}, \dfrac{3\times-1+4\times1}{7}, \dfrac{3\times5+4\times3}{7}\right)$

or $\left(\dfrac{13}{7}, \dfrac{1}{7}, \dfrac{27}{7}\right)$



Q is

$$\left(\frac{-3\times3+4\times1}{4-3}, \frac{-3\times-1+4\times1}{4-3}, \frac{-3\times5+4\times3}{4-3}\right)$$

or (–5, 7, –3)

Concept Strand 21

Find the ratio in which the line joining the points (2, 4, 5) and (3, 5, –4) is divided by the xy-plane.

Solution

Any point on the XY plane will have its z coordinate zero. Therefore, if the ratio is $\ell : m$, $\dfrac{\ell\times-4+m\times5}{\ell+m}$ must be $= 0$. $\Rightarrow \dfrac{\ell}{m} = \dfrac{5}{4} \Rightarrow$ Ratio is 5 : 4

Concept Strand 22

Two vertices of a triangle are (4, –6, 3) and (2, –2, 1) and its centroid is at $\left(\dfrac{8}{3}, -1, 2\right)$. Find the third vertex.

Solution

In the given problem, if (x, y, z) represents the coordinates of the third vertex,

$$\frac{4+2+x}{3} = \frac{8}{3}; \frac{-6-2+y}{3} = -1; \frac{3+1+z}{3} = 2$$

giving x = 2, y = 5, z = 2. Or, the third vertex is at (2, 5, 2)

Direction cosines of a directed line

Let AB be a directed line segment in space. Let AB make angles α, β, γ $(0 \le \alpha, \beta, \gamma \le \pi)$ with the positive directions of the X, Y and Z axes of a rectangular Cartesian coordinate system (Refer Fig. 3.18). Then the numbers $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines (or D.C^s) of the directed line AB. We say that the direction cosines of AB are $\cos\alpha$, $\cos\beta$, $\cos\gamma$. (also written as$[\cos\alpha, \cos\beta, \cos\gamma]$).

It is easy to see that the direction cosines of the directed line segment BA are $[-\cos\alpha, -\cos\beta, -\cos\gamma]$. Also, the directed line OP through the origin O of the coordinate system in the direction

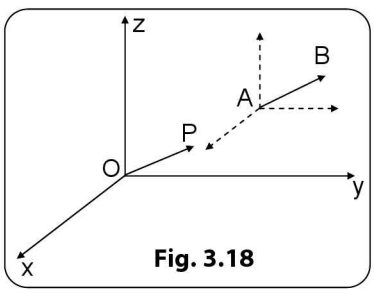


Fig. 3.18

of AB has the same direction cosines as those of AB. In fact, all directed lines in the direction of AB have the same direction cosines as those of AB.

Note that the direction cosine of the directed line determined by the vector xi + yj + zk are

$$\left[\frac{x}{\sqrt{\Sigma x^2}}, \frac{y}{\sqrt{\Sigma x^2}}, \frac{z}{\sqrt{\Sigma x^2}}\right];$$

they are the i–, j–, k–, components of the unit vector in the direction.

Results

(i) From the definition of direction cosines of a directed line, it follows that the direction cosines of the x-axis (OX) are [1, 0, 0] since OX makes angles $0, \frac{\pi}{2}, \frac{\pi}{2}$ with the x, y and z-axes.
Similarly, the direction cosines of the y-axis (OY) are [0, 1, 0] and that of the z-axis (OZ) are [0, 0, 1], direction cosines of $\overline{OX'}$ are (–1, 0, 0) and so on.

(ii) Also, any line in the direction of OX has d.c's [1, 0, 0]; and so on.

Remark

We have defined the direction cosines of a directed line as the cosines of the angles made by the line with the positive directions of the coordinate axes. It is therefore clear that these numbers depend on the choice of the coordinate system.

Projection of a directed line segment AB on a line L

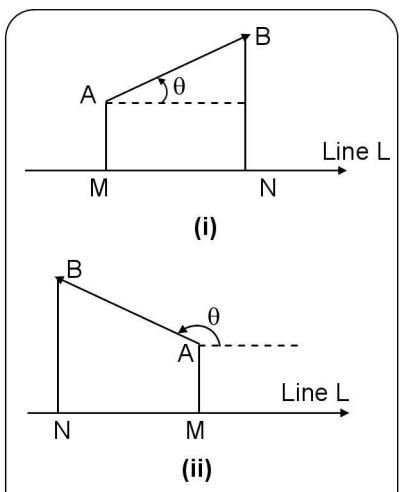


Fig. 3.19

We draw perpendiculars from A and B on the line L. If M and N are the feet of these perpendiculars, then, projection of the directed line segment AB on L is defined as MN. (Refer Fig 3.19)

If θ is the angle between AB and the line L, MN = AB cosθ.

Suppose θ is acute, as in Fig (i) 4.4, the projection is positive and if θ is obtuse, (as in Fig. (ii)) it is negative.

Also note that if $\left(\frac{1}{3}+\frac{11}{3}, \frac{5}{3}+\frac{1}{3}, \frac{8}{3}-\frac{11}{3}\right)$, the projection is zero.

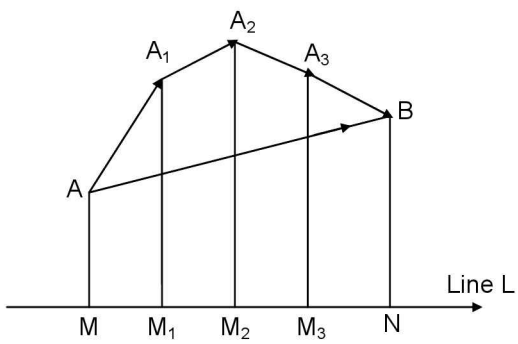


Fig. 3.20

Referring to Fig. 3.20,

Projection of the directed line segment AB on the line L

= MN
= $MM_1 + M_1M_2 + M_2M_3 + M_3N$
= sum of the projections of the segments OA_1, A_1A_2, A_2A_3 and A_3B on L.

It may be mentioned that A and B are any two points in space and that L is any line in space. Also, note that the points A, A_1, A_2, A_3, B need not be in the same plane.

Let P be the point (x, y, z) referred to a rectangular Cartesian coordinate system and let OP make angles α, β, γ with the positive directions of the x, y and z axes. We have $OP = \sqrt{x^2 + y^2 + z^2} = r$ (say). From the definition of the coordinates, it is clear that the x, y and z coordinates of P are the projections of OP on the x, y and z axes respectively. (Refer Fig. 3.21)

This means that

x = OR = Projection of OP on the x-axis
= r cos α.

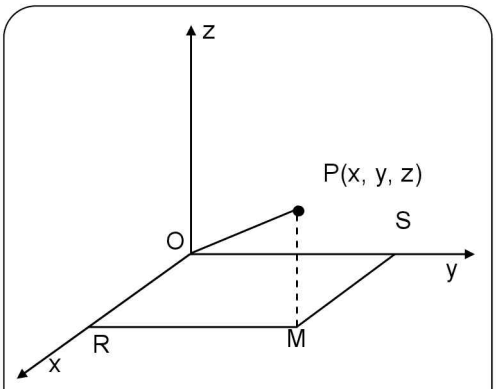


Fig. 3.21

Similarly,

$y = OS =$ Projection of OP on the y-axis $= r \cos \beta$ and

$z = MP =$ Projection of OP on the z-axis $= r \cos \gamma$

Squaring and adding

$$r^2 = x^2 + y^2 + z^2 = r^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)$$

Or, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

Again, $\cos \alpha = \bar{a} = x\bar{i} + 12\bar{j} - \bar{k}, \bar{b} = 2\bar{i} + 2x\bar{j} + \bar{k}$,

$\cos \beta = \dfrac{y}{r}$, $\cos \gamma = \left[\bar{b}, \bar{c}, \bar{a}\right]$

i.e., the direction cosines of OP where, O is the origin and P is the point (x, y, z) are

$$\left[\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right] \text{ or}$$

$$\left[\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}\right]$$

Also, $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = (1 - \cos^2 \alpha)$
$+ (1 - \cos^2 \beta) + (1 - \cos^2 \gamma)$
$= 3 - (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = 2$

Note that since $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, α, β, γ are not independent, they are connected by the relation $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

Also, we see that the direction cosines of OP are proportional to x, y, z.

Direction Ratios of a line

Definition

a, b, c are called the direction ratios (or proportional direction cosines) of a line if a, b, c are proportional to the direction cosines of a directed segment of the line.

Thus direction ratios (d.r's) of the line are a, b, c (also written as [a, b, c]) if $\dfrac{\cos\alpha}{a} = \dfrac{\cos\beta}{b} = \dfrac{\cos\gamma}{c}$, $\cos \alpha$, $\cos \beta$, $\cos \gamma$ being the direction cosines of a directed segment of the line.

From the above,
we have

$$\frac{\cos\alpha}{a} = \frac{\cos\beta}{b} = \frac{\cos\gamma}{c} = \frac{\sqrt{\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma}}{\pm\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{1}{\pm\sqrt{a^2 + b^2 + c^2}}$$

$\Rightarrow$ Direction cosines of the line are

$$\left[\frac{a}{\pm\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\pm\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\pm\sqrt{a^2 + b^2 + c^2}}\right]$$

For example,

If A is (3, 4, 7), the direction ratios of OA, where O is the origin, are given by [3, 4, 7]; and the direction cosines of OA are

$$\left[\frac{3}{\sqrt{3^2 + 4^2 + 7^2}}, \frac{4}{\sqrt{3^2 + 4^2 + 7^2}}, \frac{7}{\sqrt{3^2 + 4^2 + 7^2}}\right]$$

i.e., $\left[\dfrac{3}{\sqrt{74}}, \dfrac{4}{\sqrt{74}}, \dfrac{7}{\sqrt{74}}\right]$

Direction cosines of AO are $\left[\dfrac{-3}{\sqrt{74}}, \dfrac{-4}{\sqrt{74}}, \dfrac{-7}{\sqrt{74}}\right]$.

Note: The direction cosines of a line are not unique. There are two sets of d.c.'s corresponding to a line whereas the direction ratios of a line are unique (apart from the constant of proportion).

Direction cosines of the line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

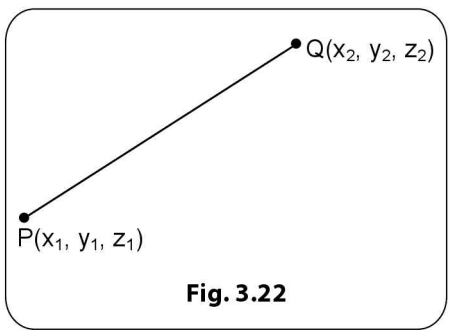


Fig. 3.22

$$\overline{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

$\therefore$ the direction cosines of PQ are

$$\left[\frac{x_2 - x_1}{\lambda}, \frac{y_2 - y_1}{\lambda}, \frac{z_2 - z_1}{\lambda}\right],$$

where $\lambda = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

For example, direction ratios of the line segment PQ where, P is (–1, 3, 7) and Q is (3, –8, 6), are given by $[3 + 1, -8 - 3, 6 - 7]$ or $[4, -11, -1]$.

$\therefore$ The direction cosines of PQ are $\left[\frac{4}{\lambda}, \frac{-11}{\lambda}, \frac{-1}{\lambda}\right]$

where $\lambda = \sqrt{4^2 + (-11)^2 + (-1)^2} = \sqrt{138}$.

The direction cosines of QP are $\left[\frac{-4}{\lambda}, \frac{11}{\lambda}, \frac{1}{\lambda}\right]$

Results

Direction ratios of a line in the XY plane (or on any plane parallel to the XY plane) are of the form $[\ell, m, 0]$ (since any line in this plane makes an angle $\frac{\pi}{2}$ with the z-axis).

Direction ratios of a line in the YZ plane (or on a plane parallel to the YZ plane) are of the form $[0, m, n]$.

Direction ratios of a line in the ZX plane (or on a plane parallel to the ZX plane) are of the form $[\ell, 0, n]$.

COMPONENTS OF A VECTOR ALONG THE AXES OF A RECTANGULAR CARTESIAN COORDINATE SYSTEM

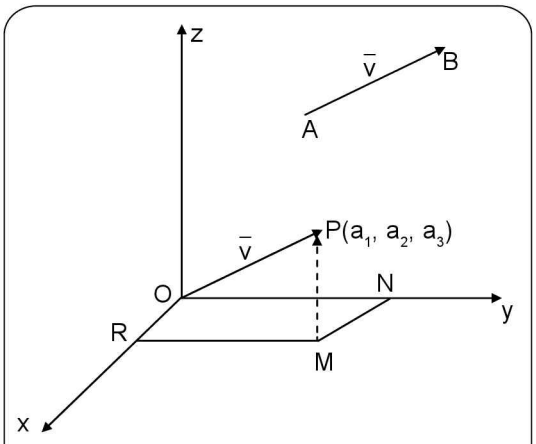


Fig. 3.23

Recall

Let $\vec{i}, \vec{j}, \vec{k}$ denote the unit vectors along the positive directions of the x, y and z axes respectively. Since the vectors $\overline{OR}, \overline{RM}, \overline{MP}$ are along (or parallel to) the axes, we can express these vectors as scalar multiples of the unit vectors $\bar{i}, \bar{j}, \bar{k}$. In other words,

Then, position vector of P,

$$\bar{v} = \overline{OP} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} \qquad \text{— (1)}$$

We have now expressed any vector in terms of its components along the coordinate axes of a rectangular coordinate system. Since the coordinates of P are (a_1, a_2, a_3), the direction cosines of the line OP are

$$\frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}, \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}}, \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

or, the direction cosines of OP give us the direction of the vector $\bar{v}$.

Representation of $\bar{v}$ in the form (1) enables us to know its magnitude (or length) and its direction simultaneously (since the direction cosines of OP tell us what are the angles made by the directed line segment OP with the x, y and z axes of the coordinate system). Of course, the components a_1, a_2, a_3 and the direction cosines depend on the choice of the coordinate system.

We may mention here that the components of $\bar{v}$ need not be a_1, a_2, a_3 if we choose another rectangular coordinate system (It is not even necessary to have a rectangular coordinate system). Although the components may be different from a_1, a_2, a_3, the magnitude of the vector (or the length of the vector) will remain the same irrespective of the coordinate system chosen. However, the numbers representing the direction of the vector (direction cosines) depend on the choice of the coordinate system.

Observations

(i) Since the magnitude of $\bar{v}$ is $\sqrt{a_1^2 + a_2^2 + a_3^2}$, the unit vector in the direction of $\bar{v}$ is

$$\frac{1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\overline{OP} = \frac{1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\left(a_1\bar{i} + a_2\bar{j} + a_3\bar{k}\right)$$

If [cos α, cos β, cos γ] are the direction cosines of OP, we may represent the unit vector in the direction

(ii) If p is the point (x, y, z), $\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$

The direction consists of $\overrightarrow{OP}$ are

$$\left[\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}\right].$$

(iii) Given two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, we know that,

$\overline{PQ} = (x_2 - x_1)\bar{i} + (y_2 - y_1)\bar{j} + (z_2 - z_1)\bar{k}$. The direction cosines of $\overline{PQ}$, are

$$\left(\frac{x_2 - x_1}{\left|\overline{PQ}\right|}, \frac{y_2 - y_1}{\left|\overline{PQ}\right|}, \frac{z_2 - z_1}{\left|\overline{PQ}\right|}\right).$$

A unit vector parallel to PQ may be

$$\frac{1}{\left|\overline{PQ}\right|}\overline{PQ}$$

$$\text{or } \frac{1}{\left|\overline{PQ}\right|}\overline{QP}\left(\text{or } \frac{-1}{\left|\overline{PQ}\right|}\overline{PQ}\right)$$

CONCEPT STRAND

Concept Strand 23

The following forces act on a particle P:

$$\bar{F}_1 = 2\bar{i} + 3\bar{j} - 5\bar{k};\ \bar{F}_2 = -5\bar{i} + \bar{j} + 3\bar{k};\ \bar{F}_3 = \bar{i} - 2\bar{j} + 4\bar{k};\ \bar{F}_4 = 4\bar{i} - 2\bar{j} + 4\bar{k}.$$

Find the resultant of the forces, magnitude of the resultant and its direction.

Solution

Resultant of the forces $= \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$

$= 2\bar{i} + 6\bar{k}$

Magnitude of the resultant $= \sqrt{2^2 + 6^2} = \sqrt{40}$.

$= 2\sqrt{10}$

Direction cosines of the direction of the resultant are

$$\left[\frac{2}{\sqrt{2^2 + 6^2}}, 0, \frac{6}{\sqrt{2^2 + 6^2}}\right]$$

i.e., $$\left[\frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}}\right].$$

Note that the resultant of the forces is parallel to the XZ plane.

PRODUCT OF VECTORS

We had already defined the operations addition and multiplication by a scalar in the case of vectors. We now define the product operations on vectors and discuss their geometrical interpretations.

Let $\bar{a}$ and $\bar{b}$ represent two non-null (or non-wzero) vectors in space. The scalar product or dot product of $\bar{a}$ and $\bar{b}$ written as $\bar{a} \cdot \bar{b}$ is defined as $\bar{a} \cdot \bar{b} = \left|\bar{a}\right|\left|\bar{b}\right|\cos\alpha$, where α is the angle between $\bar{a}$ and $\bar{b}$, (where $0 \le \alpha \le \pi$) computed when the vectors $\bar{a}$ and $\bar{b}$ have their initial point coinciding (refer Fig 3.24)

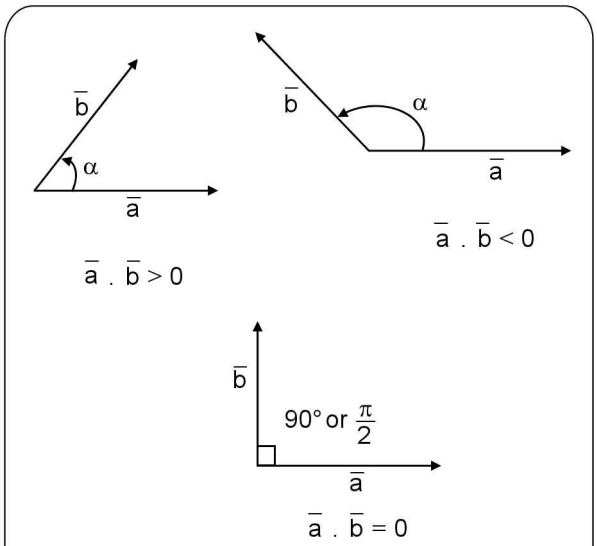


Fig. 3.24

Observe that if α is acute, dot product of $\bar{a}$ and $\bar{b}$ is positive; if α is obtuse, dot product of $\bar{a}$ and $\bar{b}$ is negative and if $\alpha = \frac{\pi}{2}$, dot product of $\bar{a}$ and $\bar{b}$ is zero.

Note that if $\bar{a} = 0$ or $\bar{b} = 0$, $\bar{a} \cdot \bar{b} = 0$.

Results

(i) $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$, since

$\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}|\cos\alpha; \bar{b} \cdot \bar{a} = |\bar{b}||\bar{a}|\cos\alpha = \bar{a} \cdot \bar{b}$

$\Rightarrow$ Dot product operation is commutative.

(ii) $\bar{a} \cdot \bar{a} = |\bar{a}||\bar{a}|\cos 0 = |\bar{a}|^2$

$\bar{a} \cdot \bar{a}$ is usually written as $\bar{a}^2$ and it is equal to $|\bar{a}|^2$.

Also, $\bar{a} \cdot \bar{a} = 0$ if and only if $\bar{a} = 0$. (i.e., if $\bar{a}$ is a null vector)

(iii) $(k_1\bar{a} + k_2\bar{b}) \cdot \bar{c} = k_1\bar{a} \cdot \bar{c} + k_2\bar{b} \cdot \bar{c}$, where k_1 and k_2 are scalars.

(iv) In the case of unit vectors, $\bar{i}, \bar{j}, \bar{k}$, we have the results

$\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1$,

$\bar{i} \cdot \bar{j} = \bar{j} \cdot \bar{i} = 0; \bar{i} \cdot \bar{k} = \bar{k} \cdot \bar{i} = 0; \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{j} = 0$,

since the angles between $\bar{i}$ and $\bar{j}$, $\bar{j}$ and $\bar{k}$, $\bar{k}$ and $\bar{i}$ are each equal to $\frac{\pi}{2}$.

(v) Let $\bar{a}$ and $\bar{b}$ be given in the components form as

$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} \qquad \bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

We have, $\bar{a} \cdot \bar{b} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}) \cdot (b_1\bar{i} + b_2\bar{j} + b_3\bar{k})$

$= a_1 b_1 + a_2 b_2 + a_3 b_3$, using result (iv)

Dot product of two non-null vectors $\bar{a}$ and $\bar{b}$ is defined as

$$\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}|\cos\alpha$$

In the components form, $\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

If the angle between $\bar{a}$ and $\bar{b}$ is 90°, $\bar{a} \cdot \bar{b} = 0$ or $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$. We say that the vectors $\bar{a}$ and $\bar{b}$ are orthogonal if $\bar{a} \cdot \bar{b} = 0$ or if $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$.

Again, if $\bar{a}$ and $\bar{b}$ are collinear (i.e., $\bar{a}$ and $\bar{b}$ are parallel), one of the vectors will be a scalar multiple of the other. This means that $a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \lambda(b_1\bar{i} + b_2\bar{j} + b_3\bar{k})$ where λ is a scalar.

$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

(vi) Again, if α is the angle between $\bar{a}$ and $\bar{b}$ where, $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$,

$\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}|\cos\alpha$

$= \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}\cos\alpha$

Also, $\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Equating the two expressions for $\bar{a} \cdot \bar{b}$, we get the expression for the angle between two given vectors.

$$\cos\alpha = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}} \quad \text{— (1)}$$

(vii) Projection of a directed line segment on a line L

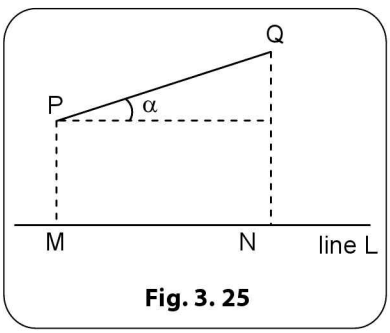


Fig. 3. 25

Let $\overline{PQ} = \bar{a}$. Let $\bar{b}$ be a vector along the line L.

Then, projection of PQ on the line L = MN = PQ cos α, where α is the angle between PQ and the line L.

$\therefore$ Projection of PQ on the line L $= \dfrac{|\overline{PQ}||\bar{b}|\cos\alpha}{|\bar{b}|}$

$$= \frac{|\bar{a}||\bar{b}|\cos\alpha}{|\bar{b}|} = \frac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$$

In general, projection of $\bar{a}$ on $\bar{b}$ is given by $\dfrac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$

(viii) Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

a_1, a_2, a_3 are the components of the vector along the coordinate axes of a rectangular Cartesian coordinate system.

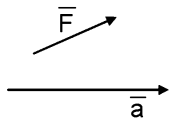
We observe that $a_1 = \bar{a}.\bar{i}$, $a_2 = \bar{a}.\bar{j}$ and $a_3 = \bar{a}.\bar{k}$

This means that the components of a vector along the coordinate axes are the projections of $\bar{a}$ on the coordinate axes.

Again, let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ represent a force whose components along the coordinate axes are F_1, F_2, F_3

Therefore, the vector components of $\bar{F}$ along the x, y, z axes of a rectangular coordinate system are $F_1\bar{i}$, $F_2\bar{j}$ and $F_3\bar{k}$ respectively.

(ix) Consider a force $\bar{F}$ represented by $F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ and $\bar{a}$ is a vector as shown in the figure.



We are interested in obtaining the vector component of $\bar{F}$ in the direction of the vector $\bar{a}$. Using (viii), projection of $\bar{F}$ along $\bar{a}$ is

$$= \frac{\bar{F}.\bar{a}}{|\bar{a}|}$$

A unit vector in the direction of $\bar{a}$ is $\dfrac{1}{|\bar{a}|}\bar{a}$

Hence, the vector component of $\bar{F}$ in the direction of $\bar{a}$ is given by

$$\left(\frac{\bar{F}.\bar{a}}{|\bar{a}|}\right)\frac{1}{|\bar{a}|}\bar{a} = \frac{\bar{F}.\bar{a}}{|\bar{a}|^2}\bar{a}$$

CONCEPT STRANDS

Concept Strand 24

Evaluate

(i) $\bar{k}\cdot(\bar{i}+\bar{j})$

(ii) $(\bar{i} - 3\bar{k})\cdot(\bar{j} + 3\bar{k})$

(iii) $(2\bar{i} - \bar{j} + 3\bar{k})\cdot(3\bar{i} + 2\bar{j} - \bar{k})$

Solution

(i) $\bar{k}\cdot(\bar{i}+\bar{j}) = \bar{k}\cdot\bar{i} + \bar{k}\cdot\bar{j} = 0$.

Recall that the two unit vectors $\bar{i}$ and $\bar{j}$ are perpendicular to each other and the sum $\bar{i}+\bar{j}$ lies on the plane XY. Again, the unit vector $\bar{k}$ is perpendicular to the plane XY and hence the result.

(ii) $(\bar{i} - 3\bar{k})\cdot(\bar{j} + 3\bar{k}) = 0 + 0 - 9 = -9$.

(iii) $(2\bar{i} - \bar{j} + 3\bar{k})\cdot(3\bar{i} + 2\bar{j} - \bar{k}) = 6 - 2 - 3 = 1$.

Concept Strand 25

Find the angle between the vectors

(i) $\bar{A} = 3\bar{i} + 2\bar{j} - 6\bar{k}, \bar{B} = 4\bar{i} - 3\bar{j} + \bar{k}$

(ii) $\bar{C} = 4\bar{i} - 2\bar{j} + 4\bar{k}, \bar{D} = 3\bar{i} - 6\bar{j} - 2\bar{k}$

Solution

(i) $\bar{A}\cdot\bar{B} = 12 - 6 - 6 = 0$

Angle between $\bar{A}$ and $\bar{B}$ is 90°

$\Rightarrow$ The vectors $\bar{A}$ and $\bar{B}$ are orthogonal.

(ii) $\bar{C}\cdot\bar{D} = 12 + 12 - 8 = 16$

If α is the angle between $\bar{C}$ and $\bar{D}$,

$$\bar{C}\cdot\bar{D} = |\bar{C}||\bar{D}|\cos\alpha$$
$$= \sqrt{4^2 + 2^2 + 4^2} \times \sqrt{3^2 + 6^2 + 2^2}\cos\alpha$$
$$= 6 \times 7\cos\alpha$$

We have $42\cos\alpha = 16$ or $\cos\alpha = \dfrac{8}{21}$

Concept Strand 26

For what values of λ, the vectors

(i) $\bar{A} = 5\bar{i} + 2\bar{j} + \lambda\bar{k}, \bar{B} = -2\bar{i} + 3\bar{j} + \bar{k}$ are orthogonal.

(ii) $\bar{A} = \lambda\bar{i} - 2\bar{j} + \bar{k}, \bar{B} = 2\lambda\bar{i} + \lambda\bar{j} - 4\bar{k}$ are orthogonal.

Solution

(i) $\bar{A}$ and $\bar{B}$ are orthogonal if $\bar{A} \cdot \bar{B} = 0$

$\Rightarrow -10 + 6 + \lambda = 0$ giving $\lambda = 4$

(ii) $\bar{A} \cdot \bar{B} = 0$ gives $2\lambda^2 - 2\lambda - 4 = 0$

$\Rightarrow \lambda^2 - \lambda - 2 = 0$

$\Rightarrow \lambda = -1, 2$

Concept Strand 27

Find the projection of the vector $2\bar{i} - 3\bar{j} + 6\bar{k}$ on the vector $\bar{i} + 2\bar{j} + 2\bar{k}$.

Solution

Let $\bar{a} = 2\bar{i} - 3\bar{j} + 6\bar{k}$ and $\bar{b}$
$= \bar{i} + 2\bar{j} + 2\bar{k}$.

Projection of $\bar{a}$ on $\bar{b} = \dfrac{\bar{a} \cdot \bar{b}}{|\bar{b}|}$

$$= \frac{(2\bar{i} - 3\bar{j} + 6\bar{k}) \cdot (\bar{i} + 2\bar{j} + 2\bar{k})}{|\bar{i} + 2\bar{j} + 2\bar{k}|}$$

$$= \frac{8}{3}$$

Concept Strand 28

Find the work done in moving an object along a straight line from the point (4, 6, –9) to (3, 7, –11) in a constant force field given by $\bar{F} = 4\bar{i} - 3\bar{j} + 2\bar{k}$.

Solution

Given force $\bar{F} = 4\bar{i} - 3\bar{j} + 2\bar{k}$

The object is moved from (4, 6, –9) to (3, 7, –11).

$\Rightarrow$ Displacement vector

$\bar{d} = (3\bar{i} + 7\bar{j} - 11\bar{k}) - (4\bar{i} + 6\bar{j} - 9\bar{k}) = -\bar{i} + \bar{j} - 2\bar{k}$

Work done by $\bar{F} = \bar{F} \cdot \bar{d} = (4\bar{i} - 3\bar{j} + 2\bar{k}) \cdot (-\bar{i} + \bar{j} - 2\bar{k})$
$= -4 - 3 - 4 = -11$ units

Vector product or Cross product of two vectors

Let $\bar{a}$ and $\bar{b}$ be two non-null vectors.

The vector product or cross product of $\bar{a}$ and $\bar{b}$ written as $\bar{a} \times \bar{b}$ is defined as the vector $\left(|\bar{a}||\bar{b}|\sin\alpha\right)\hat{c}$, $(0 \le \alpha \le \pi)$ where $\hat{c}$ is the unit vector normal to the plane determined by $\bar{a}$ and $\bar{b}$ such that $\bar{a}, \bar{b}, \bar{a} \times \bar{b}$ form a right handed system. Here, α represents the angle between $\bar{a}$ and $\bar{b}$.

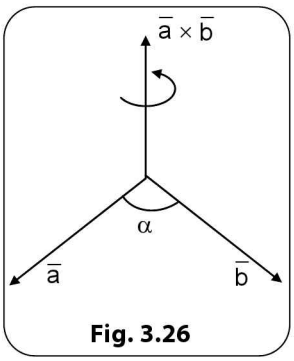


Fig. 3.26

Results

(i) If $\bar{a} \times \bar{b} = \bar{v}$ and $\bar{b} \times \bar{a} = \bar{w}$, then, by definition, $|\bar{v}| = |\bar{w}|$. Since, $\bar{b}, \bar{a}, \bar{w}$ form a right-handed system, we must have $\bar{w} = -\bar{v}$.

Hence, $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$. That is cross product operation is not commutative.

If $\bar{a}$ or $\bar{b}$ is a null vector, $\bar{a} \times \bar{b}$ is defined as null vector.

(ii) A unit vector, which is normal to (or perpendicular to) two given vectors $\bar{a}$ and $\bar{b}$ is $\pm \dfrac{1}{|\bar{a} \times \bar{b}|}(\bar{a} \times \bar{b})$

(iii) $(k\bar{a}) \times (l\bar{b}) = kl(\bar{a} \times \bar{b})$, (where, k and l are scalars)

(iv) $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$ (Distributive law holds good for vector or cross product operation)

(v) If $\bar{a}$ and $\bar{b}$ are two non-null vectors which are collinear (i.e., their directions are parallel), then $\bar{a} \times \bar{b} = 0$.

Conversely, if $\bar{a} \times \bar{b} = 0$, where $\bar{a}$ and $\bar{b}$ are two non-null vectors then, $\bar{a}$ and $\bar{b}$ are collinear or the vectors are parallel.

(vi) For the unit vectors $\bar{i}, \bar{j}, \bar{k}$,

$$\bar{i} \times \bar{i} = 0, \bar{j} \times \bar{j} = 0, \bar{k} \times \bar{k} = 0$$

$$\bar{i} \times \bar{j} = \bar{k} = -\bar{j} \times \bar{i}$$

$$\bar{j} \times \bar{k} = \bar{i} = -\bar{k} \times \bar{j}$$

$$\bar{k} \times \bar{i} = \bar{j} = -\bar{i} \times \bar{k}$$

This is because the rectangular Cartesian coordinate system is a right handed system.

(vii) If the vectors $\bar{a}$ and $\bar{b}$ are expressed in the components form say $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$$\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$$

$$\bar{a} \times \bar{b} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}) \times (b_1\bar{i} + b_2\bar{j} + b_3\bar{k})$$

$$= a_1\bar{i} \times (b_1\bar{i} + b_2\bar{j} + b_3\bar{k}) + a_2\bar{j} \times (b_1\bar{i} + b_2\bar{j} + b_3\bar{k}) + a_3\bar{k} \times (b_1\bar{i} + b_2\bar{j} + b_3\bar{k}),$$

using distributive law

$$= a_1b_2(\bar{i} \times \bar{j}) + a_1b_3(\bar{i} \times \bar{k}) + a_2b_1(\bar{j} \times \bar{i}) + a_2b_3(\bar{j} \times \bar{k}) + a_3b_1(\bar{k} \times \bar{i}) + a_3b_2(\bar{k} \times \bar{j}),$$

since $\bar{i} \times \bar{i} = 0 = \bar{j} \times \bar{j} = \bar{k} \times \bar{k}$

$$= a_1b_2\bar{k} - a_1b_3\bar{j} - a_2b_1\bar{k} + a_2b_3\bar{i} + a_3b_1\bar{j} - a_3b_2\bar{i},$$

using result (vi)

$$= (a_2b_3 - a_3b_2)\bar{i} + (a_3b_1 - a_1b_3)\bar{j} + (a_1b_2 - a_2b_1)\bar{k}.$$

The right hand side vector may be written as a determinant $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

The cross product of two non null vectors $\bar{a}$ and $\bar{b}$ is defined as

$$\bar{a} \times \bar{b} = \left(|\bar{a}||\bar{b}|\sin\alpha\right)\hat{c},$$

where $\hat{c}$ is the unit vector perpendicular to the plane containing $\bar{a}$ and $\bar{b}$, such that $\bar{a}, \bar{b}, \bar{a} \times \bar{b}$ form a right handed system.

In the components form,

$$\bar{a} \times \bar{b} = (a_2b_3 - a_3b_2)\bar{i} + (a_3b_1 - a_1b_3)\bar{j} + (a_1b_2 - a_2b_1)\bar{k}$$

(viii) By definition $|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}|\sin\alpha$

$= OA \times OB \sin\alpha$

= area of the parallelogram whose adjacent sides are OA and OB

(refer Fig. 3.27)

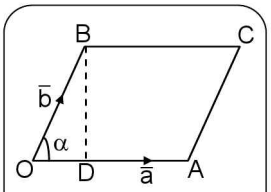


Fig. 3.27

$\bar{a} \times \bar{b}$ may be termed as the vector area of the parallelogram whose adjacent sides are the vectors $\bar{a}$ and $\bar{b}$.

The area of the triangle

$$OAB = \frac{1}{2}OA \times OB\sin\alpha = \frac{1}{2}|\bar{a}||\bar{b}|\sin\alpha = \frac{1}{2}|\bar{a} \times \bar{b}|$$

(viii) Moment of a force about a point P may be expressed as a cross product.

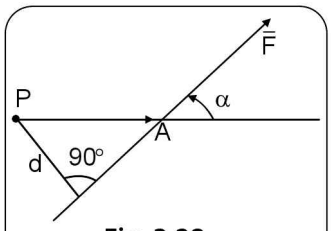


Fig. 3.28

The magnitude of the moment of a force $\bar{F}$ about a point P is defined as the product $|\bar{F}|d$ where d is the perpendicular distance of P from the line of action of $\bar{F}$ (refer Fig. 3.28).

If $\bar{r}$ is the vector from P to any point A on the line of action of $\bar{F}$, $\overline{PA} = \bar{r}$ and $d = |\bar{r}|\sin\alpha$.

Therefore, magnitude of the moment $= |\bar{F}||\bar{r}|\sin\alpha$.

The vector $\bar{r} \times \bar{F}$ is called the moment vector $\bar{M}$ (or the vector moment of $\bar{F}$ about P) and $|\bar{M}| = |\bar{r} \times \bar{F}| = |\bar{F}||\bar{r}|\sin\alpha$ gives the magnitude of the moment of $\bar{F}$ about P.

(The direction of the moment vector due to $\bar{F}$ is along the axis of rotation through P.)

CONCEPT STRANDS

Concept Strand 29

Evaluate following:

(i) $2\bar{j} \times (3\bar{i} - 4\bar{k})$

(ii) $(2\bar{i} - 5\bar{k}) \times (\bar{i} + 2\bar{j})$

(iii) $(2\bar{i} + \bar{j} - \bar{k}) \times (3\bar{i} - 2\bar{j} + 4\bar{k})$

Solution

(i) $2\bar{j} \times (3\bar{i} - 4\bar{k}) = 6(\bar{j} \times \bar{i}) - 8(\bar{j} \times \bar{k})$

$= -6\bar{k} - 8\bar{i} = -8\bar{i} - 6\bar{k}$

(ii) $(2\bar{i} - 5\bar{k}) \times (\bar{i} + 2\bar{j}) = 4(\bar{i} \times \bar{j}) - 5(\bar{k} \times \bar{i}) - 10(\bar{k} \times \bar{j})$

$= 4\bar{k} - 5\bar{j} + 10\bar{i}$

or

$(2\bar{i} - 5\bar{k}) \times (\bar{i} + 2\bar{j}) = (0 + 10)\bar{i} - (0 + 5)\bar{j} + (4 - 0)\bar{k}$

$= 10\bar{i} - 5\bar{j} + 4\bar{k}$

(iii) $(2\bar{i} + \bar{j} - \bar{k}) \times (3\bar{i} - 2\bar{j} + 4\bar{k})$

$= (4 - 2)\bar{i} - (8 + 3)\bar{j} + (-4 - 3)\bar{k} = 2\bar{i} - 11\bar{j} - 7\bar{k}$

Concept Strand 30

Find the area of the parallelogram whose sides are given by $\bar{P} = 3\bar{i} + \bar{j} - 2\bar{k}$ and $\bar{Q} = \bar{i} - 3\bar{j} + 4\bar{k}$.

Area of the parallelogram $= \left|\bar{P} \times \bar{Q}\right|$ — (1)

$\bar{P} \times \bar{Q} = (4 - 6)\bar{i} - (12 + 2)\bar{j} + (-9 - 1)\bar{k}$

$= -2\bar{i} - 14\bar{j} - 10\bar{k}$

$\left|\bar{P} \times \bar{Q}\right| = \sqrt{4 + 196 + 100}$

$= \sqrt{300} = 10\sqrt{3}$

Substituting in (1), area of the parallelogram $= 10\sqrt{3}$.

Concept Strand 31

Find $\left|\bar{a} \times \bar{b}\right|$ if $\left|\bar{a}\right| = 7, \left|\bar{b}\right| = 5$ and $\bar{a} \cdot \bar{b} = 28$.

Solution

$\bar{a} \cdot \bar{b} = \left|\bar{a}\right|\left|\bar{b}\right|\cos\alpha$, where α is the angle between $\bar{a}$ and $\bar{b}$.

Given $\bar{a}.\bar{b} = 28 \Rightarrow 28 = 7 \times 5 \cos\alpha$

$\therefore \quad \cos\alpha = \frac{4}{5}$

$\Rightarrow \quad \sin\alpha = \frac{3}{5}$

$\left|\bar{a} \times \bar{b}\right| = \left|\bar{a}\right|\left|\bar{b}\right|\sin\alpha = 7 \times 5 \times \frac{3}{5} = 21.$

Concept Strand 32

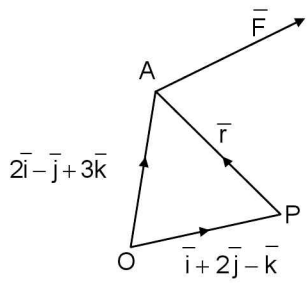
Find the moment of the force $\bar{F} = 3\bar{i} + 2\bar{j} - 4\bar{k}$ acting at the point $2\bar{i} - \bar{j} + 3\bar{k}$ about the point $\bar{i} + 2\bar{j} - \bar{k}$.

Solution

Referring to the figure,

$\bar{r} = \overline{PA} = \bar{i} - 3\bar{j} + 4\bar{k}$

Moment of the force about P is represented by the vector $\bar{r} \times \bar{F}$



$(\bar{i} - 3\bar{j} + 4\bar{k}) \times (3\bar{i} + 2\bar{j} - 4\bar{k})$

$= 4\bar{i} + 16\bar{j} + 11\bar{k}$

SCALAR PRODUCT OR DOT PRODUCT OR INNER PRODUCT OF TWO VECTORS

CONCEPT STRANDS

Concept Strand 33

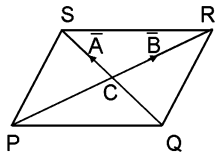
The diagonals of a parallelogram are given by

$$\overline{A} = 3\overline{i} - 4\overline{j} - \overline{k} \text{ and } \overline{B} = 2\overline{i} + 3\overline{j} - 6\overline{k}.$$

Show that the parallelogram is a rhombus and determine the length of its sides and its angles.

Solution

$\overline{A} \cdot \overline{B} = \left(3\overline{i} - 4\overline{j} - \overline{k}\right) \cdot \left(2\overline{i} + 3\overline{j} - 6\overline{k}\right) = 0$; the diagonals of the parallelogram are at right angles; therefore, it is a rhombus.



Now, $\overline{QC} = \frac{1}{2}\overline{A} = \frac{3}{2}\overline{i} - 2\overline{j} - \frac{1}{2}\overline{k}$;

$$\overline{PC} = \frac{1}{2}\overline{B} = \overline{i} + \frac{3}{2}\overline{j} - 3\overline{k}$$

We have, $\overline{PQ} + \overline{QC} = \overline{PC}$

$$\Rightarrow \quad \overline{PQ} = \overline{PC} - \overline{QC} = -\frac{1}{2}\overline{i} + \frac{7}{2}\overline{j} - \frac{5}{2}\overline{k}$$

$$\left|\overline{PQ}\right| = \sqrt{\frac{1}{4} + \frac{49}{4} + \frac{25}{4}} = \sqrt{\frac{75}{4}} = \frac{5\sqrt{3}}{2}.$$

A side of the rhombus is of length $\frac{5\sqrt{3}}{2}$

Again, $\overline{PS} = \overline{PC} + \overline{CS} = \overline{PC} + \overline{QC} = \frac{5}{2}\overline{i} - \frac{1}{2}\overline{j} - \frac{7}{2}\overline{k}$;

$$\overline{PS} = \frac{5\sqrt{3}}{2}$$

If θ is the angle QPS, $\left|\overline{PQ}\right|\left|\overline{PS}\right|\cos\theta = \overline{PQ} \cdot \overline{PS}$

$$\Rightarrow \quad \frac{75}{4}\cos\theta = \frac{-5 - 7 + 35}{4} = \frac{-23}{4}$$

$$\Rightarrow \quad \cos\theta = -\frac{23}{75} \text{ or } \theta = \pi - \cos^{-1}\left(\frac{23}{75}\right)$$

The angles of the rhombus are $\cos^{-1}\left(\frac{23}{75}\right)$ and

$$\pi - \cos^{-1}\left(\frac{23}{75}\right)$$

Concept Strand 34

Find the projection of the vector $\left(3\overline{i} - 4\overline{j} + 2\overline{k}\right)$ on the line segment joining the points (2, 3, –1) and (–2, –4, 3).

Solution

Let $\overline{a} = 3\overline{i} - 4\overline{j} + 2\overline{k}$;

$$\overline{b} = \text{vector from } (2,3,-1) \text{ to } (-2,-4,3) = -4\overline{i} - 7\overline{j} + 4\overline{k}$$

Projection of $\overline{a}$ on $\overline{b}$

$$= \frac{\overline{a} \cdot \overline{b}}{\left|\overline{b}\right|} = \frac{-12 + 28 + 8}{\sqrt{4^2 + 7^2 + 4^2}} = \frac{24}{\sqrt{81}} = \frac{8}{3}$$

Concept Strand 35

Show that

$$\overline{A} = \frac{1}{3}\left(2\overline{i} - 2\overline{j} + \overline{k}\right), \overline{B}$$

$$= \frac{1}{3}\left(\overline{i} + 2\overline{j} + 2\overline{k}\right) \text{and } \overline{C} = \frac{1}{3}\left(2\overline{i} + \overline{j} - 2\overline{k}\right)$$

are mutually orthogonal unit vectors.

Solution

$$\left|\overline{A}\right| = \frac{4 + 4 + 1}{9} = 1; \left|\overline{B}\right| = 1; \left|\overline{C}\right| = 1$$

$\Rightarrow \quad \overline{A}, \overline{B}, \overline{C}$ are unit vectors.

$\overline{A} \cdot \overline{B} = \frac{2 - 4 + 2}{9} = 0 \Rightarrow \overline{A}$ and $\overline{B}$ are orthogonal.

$\overline{B} \cdot \overline{C} = \frac{2 + 2 - 4}{9} = 0 \Rightarrow \overline{B}$ and $\overline{C}$ are orthogonal.

$\overline{C} \cdot \overline{A} = \frac{4 - 2 - 2}{9} = 0 \Rightarrow \overline{C}$ and $\overline{A}$ are orthogonal.

$\Rightarrow \quad \overline{A}, \overline{B}, \overline{C}$ are mutually orthogonal.

Concept Strand 36

If $\overline{a}, \overline{b}, \overline{c}$ are non-null vectors such that $\overline{a} \cdot \overline{b} = \overline{a} \cdot \overline{c}$, show that either $\overline{b} = \overline{c}$ or $\left(\overline{b} - \overline{c}\right)$ is orthogonal to $\overline{a}$.

Solution

We have $\bar{a}\cdot\bar{b}=\bar{a}\cdot\bar{c}$

$\Rightarrow \quad \bar{a}\cdot\left(\bar{b}-\bar{c}\right)=0$

$\Rightarrow$ Either $\bar{b}-\bar{c}$ is a null vector or $\bar{a}$ and $\left(\bar{b}-\bar{c}\right)$ are orthogonal.

Concept Strand 37

If $\bar{X},\bar{Y},\bar{Z}$ be vectors of magnitudes 2, 3 and 4 respectively and $\bar{X}$ and $\bar{Y}+\bar{Z}$ are orthogonal; $\bar{Y}$ and $\bar{Z}+\bar{X}$ are orthogonal; $\bar{Z}$ and $\bar{X}+\bar{Y}$ are orthogonal, find the magnitude of $\left(\bar{X}+\bar{Y}+\bar{Z}\right)$.

Solution

$$\left|\bar{X}+\bar{Y}+\bar{Z}\right|^2=\left(\bar{X}+\bar{Y}+\bar{Z}\right)\cdot\left(\bar{X}+\bar{Y}+\bar{Z}\right)$$

$$=\left|\bar{X}\right|^2+\left|\bar{Y}\right|^2+\left|\bar{Z}\right|^2+2\bar{X}\cdot\bar{Y}+2\bar{Y}\cdot\bar{Z}+2\bar{Z}\cdot\bar{X}$$

$$=4+9+16+2\left(\bar{X}\cdot\bar{Y}+\bar{Y}\cdot\bar{Z}+\bar{Z}\cdot\bar{X}\right) \qquad \text{— (1)}$$

Given $\bar{X}$ and $\left(\bar{Y}+\bar{Z}\right)$ are orthogonal,

$\Rightarrow \quad \bar{X}.\left(\bar{Y}+\bar{Z}\right)=0$

$\Rightarrow \quad \bar{X}\cdot\bar{Y}+\bar{X}\cdot\bar{Z}=0 \qquad$ — (2)

Again, $\bar{Y}$ and $\left(\bar{Z}+\bar{X}\right)$ are orthogonal,

$\Rightarrow \quad \bar{Y}\cdot\bar{Z}+\bar{Y}\cdot\bar{X}=0 \qquad$ — (3)

$\bar{Z}$ and $\left(\bar{X}+\bar{Y}\right)$ are orthogonal,

$\Rightarrow \quad \bar{Z}\cdot\bar{X}+\bar{Z}\cdot\bar{Y}=0 \qquad$ —(4)

(2) + (3) + (4) gives, $2\left(\bar{Y}\cdot\bar{Z}+\bar{Z}\cdot\bar{X}+\bar{X}\cdot\bar{Y}\right)=0$

Substituting in (1),

$\left|\bar{X}+\bar{Y}+\bar{Z}\right|^2=29$ or magnitude of $\left(\bar{X}+\bar{Y}+\bar{Z}\right)=\sqrt{29}$.

Concept Strand 38

If $\bar{X},\bar{Y},\bar{Z}$ are three vectors such that $\bar{X}+\bar{Y}+\bar{Z}=0$ and $|\bar{X}|=3$, $|\bar{Y}|=5$, $|\bar{Z}|=7$, prove that the angle between $\bar{X}$ and $\bar{Y}$ is 60°.

Solution

$\bar{X}+\bar{Y}+\bar{Z}=0$

$\Rightarrow \quad \bar{X}+\bar{Y}=-\bar{Z} \quad \Rightarrow \bar{X}^2+\bar{Y}^2+2\bar{X}.\bar{Y}=\bar{Z}^2$

$\therefore \quad 2.3.5.\cos\theta=49-9-25=15$

$\Rightarrow \quad \cos\theta=\dfrac{1}{2} \quad \Rightarrow \theta=60°$

Concept Strand 39

If $\bar{A}$ is any vector, prove that

$$\bar{A}=(\bar{A}.\hat{i})\bar{i}+(\bar{A}.\bar{j})\bar{j}+(\bar{A}.\bar{k})\bar{k}.$$

Solution

Let $\bar{A}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$

$\bar{A}\cdot\bar{i}=a_1, \bar{A}\cdot\bar{j}=a_2, \bar{A}\cdot\bar{k}=a_3$

$\Rightarrow$ Proved

Observe that $\bar{A}\cdot\bar{i}$ is the projection of $\bar{A}$ on the x-axis. Similarly, for $\bar{B}\cdot\bar{j}, \bar{C}\cdot\bar{k}$. The components of a vector $\bar{A}$ along the axes are therefore the projections of $\bar{A}$ on the axes.

VECTOR PRODUCT OR CROSS PRODUCT OF TWO VECTORS

Let $\bar{a}$ and $\bar{b}$ be two non-null vectors. The vector product or cross product of $\bar{a}$ and $\bar{b}$ written as $\bar{a}\times\bar{b}$ is defined as the vector $\left(\left|\bar{a}\right|\left|\bar{b}\right|\sin\alpha\right)\hat{c}$ $(0\le\alpha\le\pi)$ where $\hat{c}$ is the unit vector normal to the plane determined by $\bar{a}$ and $\bar{b}$ such that $\bar{a},\bar{b},\hat{c}$ form a right handed system. Here, α represents the angle between $\bar{a}$ and $\bar{b}$.

$$\bar{a}\times\bar{b}=\left(\left|\bar{a}\right|\times\left|\bar{b}\right|\sin\alpha\right)\hat{c}$$

If $\bar{a}\times\bar{b}=\bar{v}$ and $\bar{b}\times\bar{a}=\bar{w}$, then, by definition, $\left|\bar{v}\right|=\left|\bar{w}\right|$. Since, $\bar{b},\bar{a},\bar{w}$ form a right-handed system, we must have $\bar{w}=-\bar{v}$.

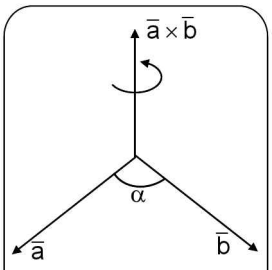


Fig. 3.29

Hence, $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$

That is, cross product operation is not commutative. If $\bar{a}$ or $\bar{b}$ is a null vector, $\bar{a} \times \bar{b}$ is defined as null vector.

Results

(i) A unit vector, which is normal to (or perpendicular to) two given vectors $\bar{a}$ and $\bar{b}$ is $\pm\frac{1}{|\bar{a} \times \bar{b}|}(\bar{a} \times \bar{b})$

(ii) $(k\bar{a}) \times (\ell\bar{b}) = k\ell(\bar{a} \times \bar{b})$, (k, ℓ scalars)

(iii) $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$ (Distributive law holds good for vector or cross product operation)

(iv) If $\bar{a}$ and $\bar{b}$ are two collinear vectors, (i.e., their directions are parallel), then $\bar{a} \times \bar{b} = 0$.

Conversely, if $\bar{a} \times \bar{b} = 0$ where $\bar{a}$ and $\bar{b}$ are two non-null vectors, then, $\bar{a}$ and $\bar{b}$ are collinear.

(v) For the unit vectors $\bar{i}, \bar{j}, \bar{k}$,

$\bar{i} \times \bar{i} = 0, \bar{j} \times \bar{j} = 0, \bar{k} \times \bar{k} = 0$

$\bar{i} \times \bar{j} = \bar{k} = -\bar{j} \times \bar{i}$

$\bar{j} \times \bar{k} = \bar{i} = -\bar{k} \times \bar{j}$

$\bar{k} \times \bar{i} = \bar{j} = -\bar{i} \times \bar{k}$

(since i, j, k form a right handed system)

(vi) If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$, then,

$\bar{a} \times \bar{b} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}) \times (b_1\bar{i} + b_2\bar{j} + b_3\bar{k})$

$= a_1\bar{i} \times (b_1\bar{i} + b_2\bar{j} + b_3\bar{k}) + a_2\bar{j} \times (b_1\bar{i} + b_2\bar{j} + b_3\bar{k})$
$\quad + a_3\bar{k} \times (b_1\bar{i} + b_2\bar{j} + b_3\bar{k})$,

(using distributive law)

$= a_1b_2(\bar{i} \times \bar{j}) + a_1b_3(\bar{i} \times \bar{k}) + a_2b_1(\bar{j} \times \bar{i}) + a_2b_3(\bar{j} \times \bar{k})$
$\quad + a_3b_1(\bar{k} \times \bar{i}) + a_3b_2(\bar{k} \times \bar{j})$,

(since $\bar{i} \times \bar{i} = 0 = \bar{j} \times \bar{j} = \bar{k} \times \bar{k}$)

$= a_1b_2\bar{k} - a_1b_3\bar{j} - a_2b_1\bar{k} + a_2b_3\bar{i} + a_3b_1\bar{j} - a_3b_2\bar{i}$,

using result (v)

$= (a_2b_3 - a_3b_2)\bar{i} + (a_3b_1 - a_1b_3)\bar{j} + (a_1b_2 - a_2b_1)\bar{k}$.

This may be written in the "determinant form"

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Thus, $\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= (a_2b_3 - a_3b_2)\bar{i} + (a_3b_1 - a_1b_3)\bar{j} + (a_1b_2 - a_2b_1)\bar{k}$

(vii)

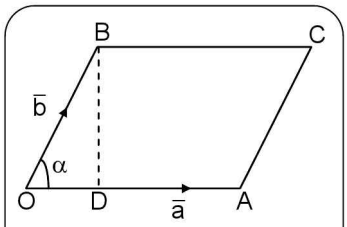


Fig. 3.30

By definition, $|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}|\sin\alpha$

$= (OA) \times (OB) \sin\alpha$

= area of the parallelogram whose adjacent sides are OA and OB

(Refer Fig. 3.30)

$\bar{a} \times \bar{b}$ may be termed as the vector area of the parallelogram whose adjacent sides are the vectors $\bar{a}$ and $\bar{b}$.

The area of the triangle OAB =

$$\frac{1}{2}OA \times OB\sin\alpha = \frac{1}{2}|\bar{a}||\bar{b}|\sin\alpha = \frac{1}{2}|\bar{a} \times \bar{b}|$$

Again, consider the parallelogram PQRS whose diagonals are PR and QS

(Refer Fig. 3.31)

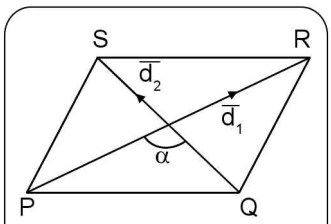


Fig. 3.31

Let $\overline{PR} = \overline{d_1}$ and $\overline{QS} = \overline{d_2}$

Area of the parallelogram =

$$4\left(\frac{1}{2}.\frac{1}{2}PR.\frac{1}{2}QS\sin\alpha\right) = \frac{1}{2}PR.QS.\sin\alpha,$$

α being angle between the diagonals $= \frac{1}{2}\left|\overline{d_1} \times \overline{d_2}\right|$

where, α is the angle between the diagonals.

(viii) Moment of a force about a point P may be expressed as a cross product.

The magnitude of the moment of a force $\overline{F}$ about a point P is defined as the product $\left|\overline{F}\right|d$ where d is the perpendicular distance of P from the line of action of $\overline{F}$ (Refer Fig. 3.14).

If $\overline{r}$ is the vector from P to any point A on the line of action of $\overline{F}$, $\overline{PA} = \overline{r}$ and $d = \left|\overline{r}\right|\sin\alpha$.

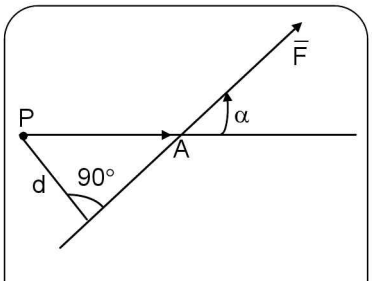


Fig. 3.32

Therefore, magnitude of the moment $= \left|\overline{F}\right|\left|\overline{r}\right|\sin\alpha$.

The vector $\overline{r} \times \overline{F}$ is called the moment vector $\overline{M}$ (or the vector moment of $\overline{F}$ about P) and

$\left|\overline{M}\right| = \left|\overline{r} \times \overline{F}\right| = \left|\overline{F}\right|\left|\overline{r}\right|\sin\alpha$ gives the magnitude of the moment of $\overline{F}$ about P.

CONCEPT STRANDS

Concept Strand 40

If $\overline{A} = 3\overline{i} - \overline{j} - 2\overline{k}$ and $\overline{B} = 2\overline{i} + 3\overline{j} + \overline{k}$, find

(i) $\left|\overline{A} \times \overline{B}\right|$

(ii) $(\overline{A} + 2\overline{B}) \times (2\overline{A} - \overline{B})$

(iii) $\left|(\overline{A} + \overline{B}) \times (\overline{A} - \overline{B})\right|$

Solution

(i) $$\overline{A} \times \overline{B} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = (-1+6)\overline{i} - (3+4)\overline{j}$$

$$+ (9+2)\overline{k} = 5\overline{i} - 7\overline{j} + 11\overline{k}$$

$$|\overline{A} \times \overline{B}| = \sqrt{5^2 + (-7)^2 + 11^2} = \sqrt{195}$$

(ii) $(\overline{A} + 2\overline{B}) \times (2\overline{A} - \overline{B}) = 2\overline{A} \times \overline{A}$

$-\overline{A} \times \overline{B} + 4\overline{B} \times \overline{A} - 2\overline{B} \times \overline{B}$, using distributive law

$= 0 - \overline{A} \times \overline{B} - 4(\overline{A} \times \overline{B}) - 0 = -5(\overline{A} \times \overline{B})$

$= -5(5\overline{i} - 7\overline{j} + 11\overline{k}) = -25\overline{i} + 35\overline{j} - 55\overline{k}$

(iii) $(\overline{A} + \overline{B}) \times (\overline{A} - \overline{B}) = \overline{A} \times \overline{A} - \overline{A} \times \overline{B} + \overline{B} \times \overline{A}$

$-\overline{B} \times \overline{B} = 0 - \overline{A} \times \overline{B} - \overline{A} \times \overline{B} - 0 = -2(\overline{A} \times \overline{B})$

$$\left|(\overline{A} + \overline{B}) \times (\overline{A} - \overline{B})\right| = \left|2(\overline{A} \times \overline{B})\right|$$

$$= 2\left|\overline{A} \times \overline{B}\right| = 2\sqrt{195}$$

Concept Strand 41

Find the area of the parallelogram whose diagonals are given by $\overline{P} = 3\overline{i} + \overline{j} - 2\overline{k}$ and $\overline{Q} = \overline{i} - 3\overline{j} + 4\overline{k}$

Solution

Area of the parallelogram $= \frac{1}{2}\left|\overline{P} \times \overline{Q}\right|$ — (1)

$$\overline{P} \times \overline{Q} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (4-6)\overline{i} - (12+2)\overline{j} + (-9-1)\overline{k}$$

$$= -2\overline{i} - 14\overline{j} - 10\overline{k}$$

$$\left|\overline{P} \times \overline{Q}\right| = \sqrt{4 + 196 + 100}$$

$$= \sqrt{300} = 10\sqrt{3}$$

Substituting in (1), area of the parallelogram $= 5\sqrt{3}$.

Concept Strand 42

Find the area of the triangle with vertices at the points (3, –1, 2), (1, –1, –3) and (4, –3, 1).

Solution

If A(3, –1, 2), B(1, –1, –3), C(4, –3, 1) are the vertices.

$$\overline{BA} = 2\bar{i} + 5\bar{k}$$

$$\overline{BC} = 3\bar{i} - 2\bar{j} + 4\bar{k}$$

$$\overline{BA} \times \overline{BC} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 0 & 5 \\ 3 & -2 & 4 \end{vmatrix}$$

$$= 10\bar{i} - (8-15)\bar{j} + (-4)\bar{k} = 10\bar{i} + 7\bar{j} - 4\bar{k}$$

Area of the triangle ABC

$$= \frac{1}{2}\left|\overline{BA} \times \overline{BC}\right| = \frac{1}{2}\sqrt{100 + 49 + 16} = \frac{\sqrt{165}}{2}$$

Concept Strand 43

If $\bar{A} = 2\bar{i} + 3\bar{j} - 4\bar{k}, \bar{B} = 4\bar{i} + \bar{j} + 7\bar{k}$, find a vector of magnitude 5 perpendicular to both $\bar{A}$ and $\bar{B}$.

Solution

A vector perpendicular to both $\bar{A}$ and $\bar{B}$ is $\bar{A} \times \bar{B}$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{I} & \bar{j} & \bar{k} \\ 2 & 3 & -4 \\ 4 & 1 & 7 \end{vmatrix} = 25\bar{i} - 30\bar{j} - 10\bar{k}$$

A unit vector along $\vec{A} \times \vec{B}$ is

$$\frac{1}{\sqrt{25^2 + 30^2 + 10^2}}\left(25\bar{i} - 30\bar{j} - 10\bar{k}\right)$$

$$= \frac{1}{\sqrt{65}}\left(5\bar{i} - 6\bar{j} - 2\bar{k}\right)$$

A vector of magnitude 5 perpendicular to both $\bar{A}$ and $\bar{B}$ is therefore given by $\frac{5}{\sqrt{65}}\left(5\bar{i} - 6\bar{j} - 2\bar{k}\right)$

Concept Strand 44

Find $\left|\bar{a} \times \bar{b}\right|$ if $\left|\bar{a}\right| = 7, \left|\bar{b}\right| = 5$ and $\bar{a} \cdot \bar{b} = 28$, where α is the angle between $\bar{a}$ and $\bar{b}$.

Solution

$$\bar{a} \cdot \bar{b} = \left|\bar{a}\right|\left|\bar{b}\right|\cos\alpha$$

$$\Rightarrow 28 = 7 \times 5 \cos\alpha \text{ giving } \cos\alpha = \frac{4}{5}$$

$$\Rightarrow \sin\alpha = \frac{3}{5}$$

$$\left|\bar{a} \times \bar{b}\right| = \left|\bar{a}\right|\left|\bar{b}\right|\sin\alpha = 7 \times 5 \times \frac{3}{5} = 21.$$

Concept Strand 45

Prove that $\bar{a} \times \left(\bar{b} + \bar{c}\right) + \bar{b} \times \left(\bar{c} + \bar{a}\right) + \bar{c} \times \left(\bar{a} + \bar{b}\right) = \bar{0}$

Solution

$$\bar{a} \times \left(\bar{b} + \bar{c}\right) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$$

$$\bar{b} \times \left(\bar{c} + \bar{a}\right) = \bar{b} \times \bar{c} + \bar{b} \times \bar{a} = \bar{b} \times \bar{c} - \bar{a} \times \bar{b}$$

$$\bar{c} \times \left(\bar{a} + \bar{b}\right) = \bar{c} \times \bar{a} + \bar{c} \times \bar{b} = -\bar{a} \times c - \bar{b} \times c$$

$$\therefore \sum \bar{a} \times \left(\bar{b} + \bar{c}\right) = 0$$

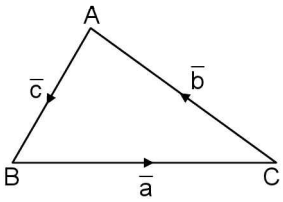
Concept Strand 46

Prove, using vectors that, in any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

where a, b, c denote the lengths of the sides BC, CA, AB respectively.

Solution



Let $\overline{BC} = \bar{a}, \overline{CA} = \bar{b}, \overline{AB} = \bar{c}$

Then,

since $\overline{BC} + \overline{CA} + \overline{AB} = 0$

We have $\bar{a} + \bar{b} + \bar{c} = 0$

$\Rightarrow \bar{a} \times \left(\bar{a} + \bar{b} + \bar{c}\right) = 0$

or $\bar{a} \times \bar{b} + \bar{a} \times \bar{c} = 0$

$\Rightarrow \bar{a} \times \bar{b} = -\bar{a} \times \bar{c} = \bar{c} \times \bar{a}$

Again, $\Rightarrow \bar{b} \times \left(\bar{a} + \bar{b} + \bar{c}\right) = 0$

Or $\bar{b} \times \bar{a} + \bar{b} \times \bar{c} = 0 \Rightarrow \bar{a} \times \bar{b} = \bar{b} \times \bar{c}$

Therefore, $\bar{a} \times \bar{b} = \bar{b} \times \bar{c} = \bar{c} \times \bar{a}$

$$\Rightarrow \left|\bar{a}\times\bar{b}\right| = \left|\bar{b}\times\bar{c}\right| = \left|\bar{c}\times\bar{a}\right|$$

$$\left|\bar{a}\right|\left|\bar{b}\right|\sin(\pi - C) = \left|\bar{b}\right|\left|\bar{c}\right|\sin(\pi - A)$$

$$= \left|\bar{c}\right|\left|\bar{a}\right|\sin(\pi - B)$$

$$\Rightarrow \frac{\sin C}{\left|\bar{c}\right|} = \frac{\sin B}{\left|\bar{b}\right|} = \frac{\sin A}{\left|\bar{a}\right|}$$

$$\Rightarrow \frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$$

Concept Strand 47

What inference can we draw if $\bar{a}$ and $\bar{b}$ are two vectors such that $\bar{a}\times\bar{b} = 0$ and $\bar{a}\cdot\bar{b} = 0$?

Solution

From the given data, vectors $\bar{a}$ and $\bar{b}$ must be parallel (or collinear) and orthogonal simultaneously. This is possible only if either $\bar{a} = 0$ or $\bar{b} = 0$.

SCALAR AND VECTOR PRODUCTS INVOLVING MORE THAN TWO VECTORS

Scalar triple product

Let $\bar{a},\bar{b},\bar{c}$ represent three non-null vectors. The scalar triple product of the three vectors $\bar{a},\bar{b},\bar{c}$ (in that order) is defined as the scalar $\bar{a}\times\bar{b}\cdot\bar{c}$ and is denoted by $\left[\bar{a},\bar{b},\bar{c}\right]$ (or $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$)

$$\left[\bar{a},\bar{b},\bar{c}\right] = \bar{a}\times\bar{b}\cdot\bar{c}$$

(Note that the notation is not ambiguous, since it makes sense only if it is read as $\left(\bar{a}\times\bar{b}\right)\cdot\bar{c}$ or the dot product of the vectors $\bar{a}\times\bar{b}$ and $\bar{c}$)

If

$$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k},\ \bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k},$$

$$\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k},$$

then, $\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(a_2b_3 - a_3b_2\right)\bar{i} + \left(a_3b_1 - a_1b_3\right)\bar{j} + \left(a_1b_2 - a_2b_1\right)\bar{k}$

$$\left[\bar{a},\bar{b},\bar{c}\right] = \bar{a}\times\bar{b}\cdot\bar{c}$$

$$= \bar{a}\times\bar{b}\cdot(c_1\bar{i} + c_2\bar{j} + c_3\bar{k})$$

$$= c_1\left(a_2b_3 - a_3b_2\right) + c_2\left(a_3b_1 - a_1b_3\right) + c_3\left(a_1b_2 - a_2b_1\right)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

CONCEPT STRAND

Concept Strand 48

Find $\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$ if

$\bar{a} = \bar{i} + 2\bar{j} - 3\bar{k};\ \bar{b} = 3\bar{i} + \bar{j} + \bar{k};\ \bar{c} = 5\bar{i} - \bar{j} - 7\bar{k}$,

Solution

$$\left[\bar{a},\bar{b},\bar{c}\right] = \bar{a}\times\bar{b}\cdot\bar{c} = \begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & 1 \\ 5 & -1 & -7 \end{vmatrix}$$

$$= 1(-7+1) - 2(-21-5) - 3(-3-5)$$

$$= -6 + 52 + 24 = 70$$

Results

(i) The scalar triple product is unaltered if the dot and the cross operations are interchanged.

$$\bar{a} \times \bar{b} \cdot \bar{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If the dot and cross operations are interchanged,
$\bar{a} \cdot \bar{b} \times \bar{c} = \bar{b} \times \bar{c} \cdot \bar{a}$

$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

since two interchanges of rows are made
$= \bar{a} \times \bar{b} \cdot \bar{c}$

(ii) The scalar triple product is unaltered by cyclic permutation of the vectors.
i.e., $\bar{a} \times \bar{b} \cdot \bar{c} = \bar{b} \times \bar{c} \cdot \bar{a} = \bar{c} \times \bar{a} \cdot \bar{b}$
This follows immediately from the property of determinants.

We have therefore, $\left[\bar{a}\,\bar{b}\,\bar{c}\right] = \left[\bar{b}\,\bar{c}\,\bar{a}\right] = \left[\bar{c}\,\bar{a}\,\bar{b}\right]$

If any two of the vectors in a scalar triple product are interchanged, the value of the scalar triple product changes in sign. i.e., $\left[\bar{a}\,\bar{b}\,\bar{c}\right] = -\left[\bar{a}\,\bar{c}\,\bar{b}\right]$ (vectors $\bar{c}$ and $\bar{b}$ are interchanged).
Again, this follows from the property of determinants. If any two rows/columns in a determinant are interchanged, the value of the determinant changes in sign.

(iii) The scalar triple product of three vectors is zero if any two of them are collinear.
If $\bar{a}, \bar{b}, \bar{c}$ are the three vectors and $\bar{b} = \lambda\bar{c}$ where, λ is a scalar, $\left[\bar{a}\,\bar{b}\,\bar{c}\right] = 0$, since the third row of the determinant representing the scalar triple product is a multiple of the second row.

(iv) If the vectors $\bar{a}, \bar{b}, \bar{c}$ taken in that order form a right handed system, $\left[\bar{a}\,\bar{b}\,\bar{c}\right]$ will be > 0 otherwise (i.e., if the vectors $\bar{a}, \bar{b}, \bar{c}$ in that order forms a left handed system), $\left[\bar{a}\,\bar{b}\,\bar{c}\right]$ will be < 0

(v) $\bar{a}, \bar{b}, \bar{c}$ are coplanar if $\bar{a} \times \bar{b} \perp \bar{c}$ if $\bar{a} \times \bar{b} . \bar{c} = 0$. Thus $\left[\bar{a} \times \bar{b} . \bar{c}\right] = 0$ if and only if $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

(vi) Geometrical interpretation of the scalar triple product.
Consider the parallelepiped OABCDEFG whose coterminous edges OA, OC, OG are represented by the vectors $\bar{a}, \bar{b}$ and $\bar{c}$ (Refer Fig. 3.33).

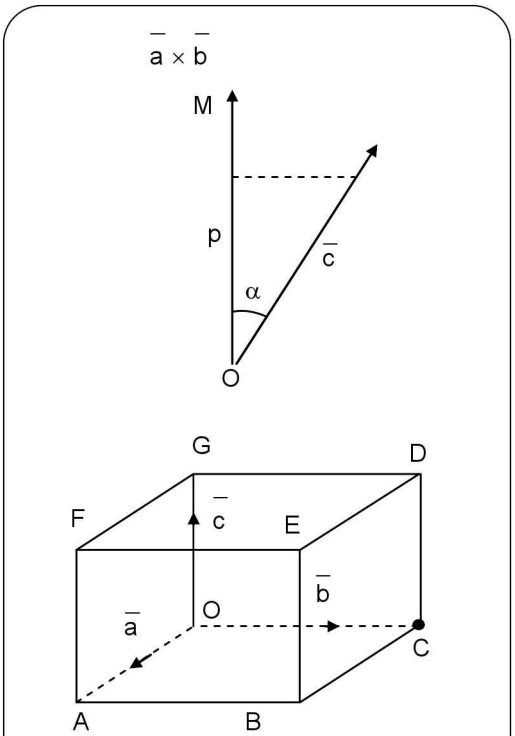


Fig. 3.33

Volume of the parallelepiped

= Area of the base parallelogram multiplied by the perpendicular distance between the parallel faces OABC and DEFG.
= (Area of the base parallelogram) p
= absolute value of $\left|\bar{a} \times \bar{b}\right|\left|\bar{c}\right|\cos\alpha$, where α is the angle between$(a \times b)$ and c
= absolute value of $\bar{a} \times \bar{b} \cdot \bar{c}$
= absolute value of $\left[\bar{a}\,\bar{b}\,\bar{c}\right]$

$\therefore$ Scalar triple product of the vectors $\bar{a}, \bar{b}, \bar{c}$ gives the volume of the parallelepiped whose coterminous edges are $\bar{a}, \bar{b}$ and $\bar{c}$.

CONCEPT STRANDS

Concept Strand 49

Find the volume of the parallelepiped whose edges are represented by the vectors.

$\overline{A} = 2\overline{i} - 3\overline{j} + 4\overline{k}, \overline{B} = \overline{i} + 2\overline{j} - \overline{k}, \overline{C} = 3\overline{i} - \overline{j} + 2\overline{k}$.

Solution

$$\left[\overline{A}\,\overline{B}\,\overline{C}\right] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= 2(4-1) + 3(2+3) + 4(-1-6) = -7$$

$\therefore$ Volume of the parallelopiped = 7

Since $\left[\overline{A}\overline{B}\overline{C}\right] < 0$, we infer that the vectors $\overline{A}, \overline{B}, \overline{C}$, in that order form a left-handed system.

Concept Strand 50

Find the constant λ such that the vectors $2\overline{i} - \overline{j} + \overline{k}, \overline{i} + 2\overline{j} - 3\overline{k}$, and $3\overline{i} + \lambda\overline{j} + 5\overline{k}$ are coplanar.

Solution

We must have $\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0$

$\Rightarrow\ 2(10 + 3\lambda) + 1(5 + 9) + 1(\lambda - 6) = 0$

$\Rightarrow\ 7\lambda + 28 = 0$ giving $\lambda = -4$

Concept Strand 51

Show that $\left(\overline{A} + \overline{B}\right) \cdot \left(\overline{B} + \overline{C}\right) \times \left(\overline{C} + \overline{A}\right) = 2\left[\overline{A}\,\overline{B}\,\overline{C}\right]$.

Solution

$\left(\overline{B} + \overline{C}\right) \times \left(\overline{C} + \overline{A}\right) = \overline{B} \times \overline{C} + \overline{B} \times \overline{A} + \overline{C} \times \overline{C} + \overline{C} \times \overline{A}$, using distributive law

$$= \overline{B} \times \overline{C} + \overline{B} \times \overline{A} + \overline{C} \times \overline{A}$$

$$\left(\overline{A} + \overline{B}\right) \cdot \left(\overline{B} + \overline{C}\right) \times \left(\overline{C} + \overline{A}\right)$$

$$= \left(\overline{A} + \overline{B}\right) \cdot \left(\overline{B} \times \overline{C} + \overline{B} \times \overline{A} + \overline{C} \times \overline{A}\right)$$

$$= \begin{aligned} &\overline{A} \cdot \overline{B} \times \overline{C} + \overline{A} \cdot \overline{B} \times \overline{A} + \overline{A} \cdot \overline{C} \times \overline{A} + \\ &\overline{B} \cdot \overline{B} \times \overline{C} + \overline{B} \cdot \overline{B} \times \overline{A} + \overline{B} \cdot \overline{C} \times \overline{A} \end{aligned}$$

$$= \left[\overline{A}\,\overline{B}\,\overline{C}\right] + 0 + 0 + 0 + \left[\overline{A}\,\overline{B}\,\overline{C}\right] = 2\left[\overline{A}\,\overline{B}\,\overline{C}\right]$$

Concept Strand 52

Show that the 4 points whose coordinates are (6, –7, 0); (16, –19, –4); (0, 3, –6) and (2, –5, 10) lie in the same plane. (or the 4 points are coplanar).

Solution

Let the points be A, B, C, D.

Then,

$$\overline{AB} = 10\overline{i} - 12\overline{j} - 4\overline{k}$$
$$\overline{AC} = -6\overline{i} + 10\overline{j} - 6\overline{k}$$
$$\overline{AD} = -4\overline{i} + 2\overline{j} + 10\overline{k}$$

A, B, C, D are coplanar points iff the vectors $\overline{AB}, \overline{AC}, \overline{AD}$ are coplanar i.e., iff $\left[\overline{AB}, \overline{AC}, \overline{AD}\right] = 0$

$$\left[\overline{AB}, \overline{AC}, \overline{AD}\right] = \begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 2 & 10 \end{vmatrix}$$

= 0, since $R_3 + (R_1 + R_2)$ reduces the third row to zero.

Hence.

Concept Strand 53

If $\overline{X} = 3\overline{i} + 2\overline{j} - 5\overline{k}$; $\overline{Y} = 2\overline{i} + \overline{j}$ and $\overline{Z}$ is a unit vector, find the maximum value of $\left[\overline{X}\,\overline{Y}\,\overline{Z}\right]$.

Solution

$\left[\overline{X}\,\overline{Y}\,\overline{Z}\right] = \overline{X} \times \overline{Y} \cdot \overline{Z}$

$= \left|\overline{X} \times \overline{Y}\right| \left|\overline{Z}\right| \cos\alpha$ where α is the angle between $\left(\overline{X} \times \overline{Y}\right)$ and $\overline{Z}$

$= \left|\overline{X} \times \overline{Y}\right| \cos\alpha$, since $\left|\overline{Z}\right| = 1$

Now, $\left[\overline{X}\ \overline{Y}\ \overline{Z}\right]$ will be maximum when $\alpha = 0$ and the maximum value equals $\left|\overline{X} \times \overline{Y}\right|$

$$\overline{X} \times \overline{Y} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 2 & -5 \\ 2 & 1 & 0 \end{vmatrix} = 5\bar{i} - 10\bar{j} - \bar{k}$$

$$\Rightarrow \left|\overline{X} \times \overline{Y}\right| = \sqrt{25 + 100 + 1} = \sqrt{126}$$

Maximum value of $\left[\overline{X}\ \overline{Y}\ \overline{Z}\right] = \sqrt{126}$

Concept Strand 54

If $\bar{a}, \bar{b}, \bar{c}$ are three mutually orthogonal vectors of magnitudes 4, 5, 6 units, find the value of the scalar triple product $\left[2\bar{b} - \bar{c}, 2\bar{c} - \bar{a}, 2\bar{a} - \bar{b}\right]$.

Solution

Since $\bar{a}, \bar{b}, \bar{c}$ are mutually orthogonal, the parallelopiped with $\bar{a}, \bar{b}, \bar{c}$ as coterminous edges reduces to a rectangular box.

Hence, $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$ = Volume of the rectangular box = $4 \times 5 \times 6 = 120$.

Now, $\left[2\bar{b} - \bar{c}, 2\bar{c} - \bar{a}, 2\bar{a} - \bar{b}\right]$

$$= \left(2\bar{b} - \bar{c}\right) \cdot \left\{4\bar{c} \times \bar{a} - 2\bar{c} \times \bar{b} + \bar{a} \times \bar{b}\right\}$$

$$= 8\bar{b} \cdot \bar{c} \times \bar{a} - \bar{c} \cdot \bar{a} \times \bar{b} = 7\left[\bar{a}\ \bar{b}\ \bar{c}\right]$$

$$= 7 \times 120 = 840$$

Concept Strand 55

Let $\overline{A} = -\bar{i} + \bar{j} + \bar{k}$; $\overline{B} = 2\bar{i} + 3\bar{k}$. The vector $\overline{C}$ is coplanar with $\overline{A}$ and $\overline{B}$. If $\overline{C}$ and $\overline{B}$ are orthogonal and $\overline{C} \cdot \overline{A} = -76$, find $\overline{C}$.

Solution

Let $\overline{C} = x\bar{i} + y\bar{j} + z\bar{k}$

Since $\overline{A}\ \overline{B}, \overline{C}$ are coplanar, $\left[\overline{A}\ \overline{B}\ \overline{C}\right] = 0$

$$\begin{vmatrix} -1 & 1 & 1 \\ 2 & 0 & 3 \\ x & y & z \end{vmatrix} = 0$$

$\Rightarrow$ $x(3) + 5y - 2z = 0$ or $3x + 5y - 2z = 0$ — (1)

$\overline{C}$ and $\overline{B}$ are orthogonal $\Rightarrow \overline{C} \cdot \overline{B} = 0$

$\Rightarrow$ $2x + 3z = 0$ — (2)

$\overline{C} \cdot \overline{A} = -76 \Rightarrow -x + y + z = -76$ — (3)

Solving (1), (2) and (3), we get $x = 30$, $y = -26$, $z = -20$

$\overline{C} = 30\bar{i} - 26\bar{j} - 20\bar{k}$

Remark 1

We have seen that if $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = 0$, then the vectors $\bar{a}, \bar{b}, \bar{c}$ are coplanar. Geometrically, this means that the three vectors lie in the same plane (or that they are parallel to the same plane). We can therefore express one of the vectors in the set $\left\{\bar{a}, \bar{b}, \bar{c}\right\}$ as a linear combination of the other two vectors of the set. If $\bar{a}, \bar{b}, \bar{c}$ are coplanar,

$\bar{a}$ can be expressed as $k_1\bar{b} + k_2\bar{c}$ where at least one of k_1 or k_2 will be different from zero.

OR $\bar{b}$ can be expressed as $k_3\bar{c} + k_4\bar{a}$ where at least one of k_3 or k_4 will be different from zero.

Or

$\bar{c}$ can be expressed as $k_5\bar{a} + k_6\bar{b}$ where at least one of k_5 or k_6 will be different from zero.

In all the above cases, the vectors $\bar{a}, \bar{b}, \bar{c}$ satisfy a relation of the form

$\lambda_1\bar{a} + \lambda_2\bar{b} + \lambda_3\bar{c} = 0$ — (1) where $\lambda_1, \lambda_2, \lambda_3$ are scalars not all zero.

If $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar $\lambda_1\bar{a} + \lambda_2\bar{b} + \lambda_3\bar{c} = 0$

$\Rightarrow$ $\lambda_1 = 0 = \lambda_2 = \lambda_3$.

Remark 2

Let A, B, C denote 3 points in space which are collinear.

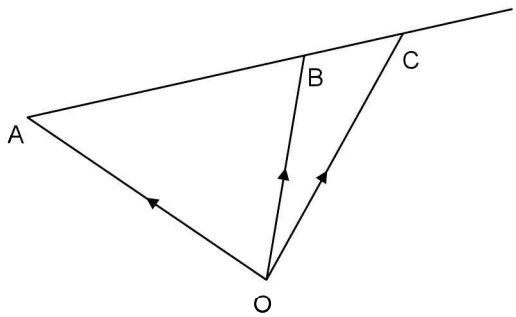
Let the position vectors of A, B, C relative to an origin of reference O be $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively.

We have $\overline{AB} = \overline{OB} - \overline{OA} = \bar{b} - \bar{a}$

and $\overline{BC} = \overline{OC} - \overline{OB} = \bar{c} - \bar{b}$

Since the points A, B, C are collinear,

$\overline{BC} = \lambda\, \overline{AB}$ for some non-zero scalar λ.



$$\Rightarrow \quad \bar{c} - \bar{b} = \lambda\left(\bar{b} - \bar{a}\right)$$

$$\Rightarrow \quad \lambda\bar{a} - (1+\lambda)\bar{b} + \bar{c} = 0$$

Observe that $\lambda + (-1 - \lambda) + 1 = 0$

Hence, if $\bar{a}, \bar{b}, \bar{c}$ denote the position vectors of 3 collinear points in space, then, there exists a relation of the form $\lambda_1\bar{a} + \lambda_2\bar{b} + \lambda_3\bar{c} = 0$
where, $\lambda_1, \lambda_2, \lambda_3$ are non zero scalars satisfying the relation $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

Conversely, if $\lambda_1\bar{a} + \lambda_2\bar{b} + \lambda_3\bar{c} = 0$ where $\lambda_1, \lambda_2, \lambda_3$ are non zero scalars satisfying the relation $\lambda_1 + \lambda_2 + \lambda_3 = 0$, then the points whose positon vectors are $\bar{a}, \bar{b}, \bar{c}$ are collinear.

We now introduce the concepts "linear dependence" and "linear independence" of a set of vectors in space.

Linear dependence and linear independence of a set of vectors

Let $\left\{\overline{a_1}, \overline{a_2}, \overline{a_3}, \ldots \overline{a_n}\right\}$ represent a set of n vectors in space.

The set of vectors $\left\{\overline{a_1}, \overline{a_2}, \ldots \overline{a_n}\right\}$ is said to be linearly dependent if there exists scalars $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ not all zero such that

$$\lambda_1\overline{a_1} + \lambda_2\overline{a_2} + \lambda_3\overline{a_3} + \ldots\ldots + \lambda_n\overline{a_n} = 0\,,$$

where, 0 means null vector — (1)

If relation (1) is satisfied only when $\lambda_1 = 0 = \lambda_2 = \lambda_3 \ldots = \lambda_n$, the set of vectors is said to be linearly independent.

For example, the vectors

$$\bar{a} = 3\bar{i} + 4\bar{j} - \bar{k} \text{ and } \bar{b} = 9\bar{i} + 12\bar{j} - 3\bar{k}$$

are linearly dependent, since $3\bar{a} - \bar{b} = 0$ (i.e., $\lambda_1 = 3$, $\lambda_2 = -1$)

On the other hand, the vectors

$$\bar{a} = \bar{i} + 2\bar{j} + \bar{k}\,, \bar{b} = 3\bar{i} + \bar{j} - \bar{k}$$

are linearly independent, since no relation of the form $\lambda_1\bar{a} + \lambda_2\bar{b} = 0$ is possible unless one chooses the values of λ_1 and λ_2 both as zero.

It follows that

(i) two vectors are linearly dependent iff they are collinear.
(ii) three vectors are linearly independent if they are not coplanar.
(iii) A set of more than 3 vectors is always linearly dependent since any vector can be expressed uniquely as a LC of 3 LI vectors.

The set of vectors $\left\{\bar{i}, \bar{j}, \bar{k}\right\}$ is linearly independent, since $\lambda_1\bar{i} + \lambda_2\bar{j} + \lambda_3\bar{k} = 0$ implies i.e., $(\lambda_1 i + \lambda_2 j + \lambda_3 k) = 0$ $\Rightarrow \lambda_1 = 0$ and similarly, $\lambda_2 = 0 = \lambda_3$.

i, i + j, i + j + k form a linearly independent set where as i, j + k, i + j + k form a linearly dependent system.

We can test the linear dependence of 3 vectors by computing their scalar triple product. If their scalar triple product is 0, the vectors are linearly dependent. If the scalar triple product is non-zero, the vectors are linearly independent.

We now work out a few examples to illustrate the ideas developed above.

CONCEPT STRANDS

Concept Strand 56

Let $\bar{a}, \bar{b}, \bar{c}$ be any three non-coplanar vectors such that $\bar{X} = \lambda_1\bar{a} + \lambda_2\bar{b} + \lambda_3\bar{c} = \mu_1\bar{a} + \mu_2\bar{b} + \mu_3\bar{c}$, where $\bar{X}$ is any vector and $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3$ are scalars. Then, $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$, $\lambda_3 = \mu_3$

Solution

For, we have $\lambda_1\bar{a} + \lambda_2\bar{b} + \lambda_3\bar{c} = \bar{X}$ — (1)

$\mu_1\bar{a} + \mu_2\bar{b} + \mu_3\bar{c} = \bar{X}$ — (2)

$$(1)-(2) \Rightarrow (\lambda_1 - \mu_1)\bar{a} + (\lambda_2 - \mu_2)\bar{b} + (\lambda_3 - \mu_3)\bar{c} = 0$$

Since $\bar{a}, \bar{b}, \bar{c}$ are non coplanar, they are linearly independent. Therefore, the above relation can hold good only if

$$\lambda_1 - \mu_1 = 0, \lambda_2 - \mu_2 = 0, \lambda_3 - \mu_3 = 0$$

$$\Rightarrow \lambda_1 = \mu_1, \lambda_2 = \mu_2, \lambda_3 = \mu_3.$$

Concept Strand 57

Given that the vectors $\bar{X}, \bar{Y}, \bar{Z}$ are linearly independent, check for linear independence the following set of vectors:

(i) $\bar{X} + \bar{Y}, \bar{Y} + \bar{Z}, \bar{Z} + \bar{X}$

(ii) $\bar{X} + 2\bar{Y}, \bar{Y} + 2\bar{Z}, \bar{Z} + 2\bar{X}$

(iii) $\bar{X} + \bar{Y} + \bar{Z}, \bar{X} - \bar{Y} - \bar{Z}, \bar{X} + \bar{Y} - \bar{Z}$

(iv) $\bar{X} - \bar{Y}, \bar{Y} - \bar{Z}, \bar{Z} - \bar{X}$

Solution

(i) $\lambda_1(\bar{X} + \bar{Y}) + \lambda_2(\bar{Y} + \bar{Z}) + \lambda_3(\bar{Z} + \bar{X}) = 0$

$\Rightarrow (\lambda_1 + \lambda_3)\bar{X} + (\lambda_2 + \lambda_1)\bar{Y} + (\lambda_3 + \lambda_2)\bar{Z} = 0$

$\Rightarrow \lambda_1 + \lambda_3 = 0; \lambda_2 + \lambda_1 = 0; \lambda_3 + \lambda_2 = 0$, since $\bar{X}, \bar{Y}, \bar{Z}$ are linearly independent.

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 0, \lambda_2 + \lambda_3 = 0, \lambda_3 + \lambda_1 = 0, \lambda_1 + \lambda_2 = 0$

$\Rightarrow \lambda_1 = 0 = \lambda_2 = \lambda_3$

$\Rightarrow \bar{X} + \bar{Y}, \bar{Y} + \bar{Z}, \bar{Z} + \bar{X}$ are linearly independent.

OR

$$\left[\bar{X} + \bar{Y}, \bar{Y} + \bar{Z}, \bar{Z} + \bar{X}\right]$$
$$= (\bar{X} + \bar{Y}) \cdot \left\{(\bar{Y} + \bar{Z}) \times (\bar{Z} + \bar{X})\right\}$$
$$= (\bar{X} + \bar{Y}) \cdot \left\{\bar{Y} \times \bar{Z} + \bar{Y} \times \bar{X} + \bar{Z} \times \bar{X}\right\}$$
$$= \left[\bar{X}\,\bar{Y}\,\bar{Z}\right] + \left[\bar{Y}\,\bar{Z}\,\bar{X}\right] = 2\left[\bar{X}\,\bar{Y}\,\bar{Z}\right] \neq 0$$

$\Rightarrow$ Vectors are non coplanar.

$\Rightarrow$ They are linearly independent

(ii) $\left[\bar{X} + 2\bar{Y}, \bar{Y} + 2\bar{Z}, \bar{Z} + 2\bar{X}\right]$

$$= (\bar{X} + 2\bar{Y}) \cdot \left\{(\bar{Y} + 2\bar{Z}) \times (\bar{Z} + 2\bar{X})\right\}$$
$$= (\bar{X} + 2\bar{Y}) \cdot \left\{\bar{Y} \times \bar{Z} + 2\bar{Y} \times \bar{X} + 4\bar{Z} \times \bar{X}\right\}$$
$$= \left[\bar{X}\,\bar{Y}\,\bar{Z}\right] + 8\left[\bar{Y}\,\bar{Z}\,\bar{X}\right] = 9\left[\bar{X}\,\bar{Y}\,\bar{Z}\right] \neq 0$$

$\Rightarrow$ Vectors are linearly independent (or the vectors are non-coplanar)

(iii) $(\bar{X} + \bar{Y} + \bar{Z}) \cdot \left\{(\bar{X} - \bar{Y} - \bar{Z}) \times (\bar{X} + \bar{Y} - \bar{Z})\right\}$

$$= (\bar{X} + \bar{Y} - \bar{Z}) \cdot \left\{\begin{matrix}\bar{X} \times \bar{Y} - \bar{X} \times \bar{Z} - \\ \bar{Y} \times \bar{X} + \bar{Y} \times \bar{Z} - \bar{Z} \times \bar{X} - \bar{Z} \times \bar{Y}\end{matrix}\right\}$$
$$= \left[\bar{X}\,\bar{Y}\,\bar{Z}\right] - \left[\bar{X}\,\bar{Z}\,\bar{Y}\right] - \left[\bar{Y}\,\bar{X}\,\bar{Z}\right] - \left[\bar{Y}\,\bar{Z}\,\bar{X}\right] + \left[\bar{Z}\,\bar{X}\,\bar{Y}\right] - \left[\bar{Z}\,\bar{Y}\,\bar{X}\right] = 4\left[\bar{X}\,\bar{Y}\,\bar{Z}\right] \neq 0$$

Vectors are linearly independent (or the vectors are non-coplanar)

(iv) $(\bar{X} - \bar{Y}) \cdot \left\{(\bar{Y} - \bar{Z}), (\bar{Z} - \bar{X})\right\}$

$$= (\bar{X} - \bar{Y}) \cdot (\bar{Y} \times \bar{Z} - \bar{Y} \times \bar{X} + \bar{Z} \times \bar{X})$$
$$= \left[\bar{X}\,\bar{Y}\,\bar{Z}\right] - \left[\bar{Y}\,\bar{Z}\,\bar{X}\right] = 0$$

Vectors are linearly dependent (or the vectors are coplanar)

OR

$$\bar{X} - \bar{Y} = (-1)(\bar{Y} - \bar{Z}) + (-1)(\bar{Z} - \bar{X})$$

$\Rightarrow$ $(\bar{X} - \bar{Y})$ is expressible as a linear combination of the vectors $(\bar{Y} - \bar{Z})$ and $(\bar{Z} - \bar{X})$.

Vector triple product

Let $\bar{a}, \bar{b}, \bar{c}$ represent three non-null vectors. Then, $\bar{a} \times (\bar{b} \times \bar{c})$ is a vector triple product of $\bar{a}, \bar{b}, \bar{c}$. $(\bar{a} \times \bar{b}) \times \bar{c}$ is another vector triple product of $\bar{a}, \bar{b}, \bar{c}$.

It can be shown that

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c} \qquad \text{— (1)}$$

That is, the vector triple product $\bar{a} \times (\bar{b} \times \bar{c})$ lies in the plane containing $\bar{b}$ and $\bar{c}$.

$$(\bar{a} \times \bar{b}) \times \bar{c} = -\bar{c} \times (\bar{a} \times \bar{b}) = -\{(\bar{c} \cdot \bar{b})\bar{a} - (\bar{c} \cdot \bar{a})\bar{b}\}$$

using the relation (1)

$$= (\bar{c} \cdot \bar{a})\bar{b} - (\bar{c} \cdot \bar{b})\bar{a} \qquad \text{— (2)}$$

The vector triple product $(\bar{a} \times \bar{b}) \times \bar{c}$ lies in the plane containing $\bar{a}$ and $\bar{b}$.

Note that the vector triple products $\bar{a} \times (\bar{b} \times \bar{c})$ and $(\bar{a} \times \bar{b}) \times \bar{c}$ are different.

(So, $\bar{a} \times \bar{b} \times \bar{c}$ is misleading)

CONCEPT STRANDS

Concept Strand 58

If $\bar{A} = 3\bar{i} - \bar{j} + 2\bar{k}, \bar{B} = 3\bar{i} + \bar{j} + 4\bar{k}, \bar{C} = \bar{i} + \bar{j} + \bar{k}$, find

(i) $\bar{A} \times (\bar{B} \times \bar{C})$. Verify the formula

$$\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C}$$

(ii) $(\bar{A} \times \bar{B}) \times \bar{C}$

Solution

(i) $\bar{A} \cdot \bar{C} = 3 - 1 + 2 = 4;\ \bar{A} \cdot \bar{B} = 9 - 1 + 8 = 16$

$$(\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C}$$

$$= 4(3\bar{i} + \bar{j} + 4\bar{k}) - 16(\bar{i} + \bar{j} + \bar{k}) = -4\bar{i} - 12\bar{j} \qquad \text{— (1)}$$

$$\bar{B} \times \bar{C} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 1 & 4 \\ 1 & 1 & 1 \end{vmatrix} = -3\bar{i} + \bar{j} + 2\bar{k}$$

$$\bar{A} \times (\bar{B} \times \bar{C}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & -1 & 2 \\ -3 & 1 & 2 \end{vmatrix} = -4\bar{i} - 12\bar{j} \qquad \text{— (2)}$$

From (1) and (2) we see that

$$\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C}$$

(ii) $(\bar{A} \times \bar{B}) \times \bar{C} = -\bar{C} \times (\bar{A} \times \bar{B})$

$$= -\{(\bar{C} \cdot \bar{B})\bar{A} - (\bar{C} \cdot \bar{A})\bar{B}\}$$

$$= (\bar{C} \cdot \bar{A})\bar{B} - (\bar{C} \cdot \bar{B})\bar{A}$$

$\bar{C} \cdot \bar{A} = 3 - 1 + 2 = 4,\ \bar{C} \cdot \bar{B}$

$= 3 + 1 + 4 = 8$

Therefore,

$$(\bar{A} \times \bar{B}) \times \bar{C} = 4(3\bar{i} + \bar{j} + 4\bar{k}) - 8(3\bar{i} - \bar{j} + 2\bar{k})$$

$$= -12\bar{i} + 12\bar{j}$$

Concept Strand 59

Show that $\bar{A} \times (\bar{B} \times \bar{C}) + \bar{B} \times (\bar{C} \times \bar{A}) + \bar{C} \times (\bar{A} \times \bar{B}) = 0$

Solution

$$\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C}$$

$$\bar{B} \times (\bar{C} \times \bar{A}) = (\bar{B} \cdot \bar{A})\bar{C} - (\bar{B} \cdot \bar{C})\bar{A}$$

$$\bar{C} \times (\bar{A} \times \bar{B}) = (\bar{C} \cdot \bar{B})\bar{A} - (\bar{C} \cdot \bar{A})\bar{B}$$

Addition gives

$$\bar{A} \times (\bar{B} \times \bar{C}) + \bar{B} \times (\bar{C} \times \bar{A}) + \bar{C} \times (\bar{A} \times \bar{B}) = 0$$

Since the sum of the vectors

$$\bar{A} \times (\bar{B} \times \bar{C}), \bar{B} \times (\bar{C} \times \bar{A}), \bar{C} \times (\bar{A} \times \bar{B})$$

$= 0$, the 3 vectors are linearly dependent or they are coplanar.

Concept Strand 60

Show that the scalar triple product of $\bar{b} \times \bar{c}, \bar{c} \times \bar{a}, \bar{a} \times \bar{b}$ is $[\bar{a}\ \bar{b}\ \bar{c}]^2$.

Solution

Scalar triple product of $\bar{b}\times\bar{c}, \bar{c}\times\bar{a}, \bar{a}\times\bar{b}$ is

$$= (\bar{b}\times\bar{c})\cdot\{(\bar{c}\times\bar{a})\times(\bar{a}\times\bar{b})\}$$
$$= (\bar{b}\times\bar{c})\cdot\{(\bar{c}\times\bar{a}\cdot\bar{b})\bar{a}-(\bar{c}\times\bar{a}\cdot\bar{a})\bar{b}\}$$
$$= (\bar{b}\times\bar{c})\cdot\{[\overline{abc}]\bar{a}-0\}$$
$$= [\bar{a}\,\bar{b}\,\bar{c}][\bar{a}\,\bar{b}\,\bar{c}] = [\bar{a}\,\bar{b}\,\bar{c}]^2$$

Remark

$\bar{a},\bar{b},\bar{c}$ are non-coplanar, iff $\bar{b}\times\bar{c}, \bar{c}\times\bar{a}, \bar{a}\times\bar{b}$ are non-coplanar.

Concept Strand 61

Prove that if $\bar{a}$ is any vector,

$$\bar{i}\times(\bar{a}\times\bar{i})+\bar{j}\times(\bar{a}\times\bar{j})+\bar{k}\times(\bar{a}\times\bar{k}) = 2\bar{a}$$

Solution

$$\bar{i}\times(\bar{a}\times\bar{i}) = (\bar{i}\cdot\bar{i})\bar{a}-(\bar{i}\cdot\bar{a})\bar{i}$$

$= \bar{a}-a_1\bar{i}$, where we assume $\bar{a}$ as $a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$

$= \bar{a}-(\bar{a}.\bar{i})\bar{i}$

Addition gives $\sum\bar{i}\times(\bar{a}\times\bar{i}) = 3\bar{a}-\sum(\bar{a}.\bar{i})\bar{i}$

$= 3\bar{a}-\bar{a} = 2\bar{a}$

Concept Strand 62

If $\bar{p},\bar{q},\bar{r}$ denote the vectors $\bar{b}\times\bar{c}, \bar{c}\times\bar{a}, \bar{a}\times\bar{b}$ respectively show that $\bar{a}$ is parallel to $\bar{q}\times\bar{r}$; $\bar{b}$ is parallel to $\bar{r}\times\bar{p}$; and $\bar{c}$ is parallel to $\bar{p}\times\bar{q}$.

Solution

$$\bar{q}\times\bar{r} = (\bar{c}\times\bar{a})\times(\bar{a}\times\bar{b}) = (\bar{c}\times\bar{a}\cdot\bar{b})\bar{a}-(\bar{c}\times\bar{a}\cdot\bar{a})\bar{b}$$
$$= [\bar{a}\,\bar{b}\,\bar{c}]\bar{a} - 0 = [\bar{a}\,\bar{b}\,\bar{c}]\bar{a}$$

$\Rightarrow$ $\bar{a}$ is parallel to $\bar{q}\times\bar{r}$

Similarly, the other results follow.

PRODUCTS OF FOUR VECTORS

(i) Scalar product of four vectors

If $\bar{a},\bar{b},\bar{c},\bar{d}$ are four vectors, the scalar products $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})$; $(\bar{a}\times\bar{d})\cdot(\bar{b}\times\bar{c})$; ... are scalar products that can be formed out of 4 vectors.

Result

$$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = \begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{a}\cdot\bar{d}\\ \bar{b}\cdot\bar{c} & \bar{b}\cdot\bar{d}\end{vmatrix}$$
$$= (\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d})-(\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c})$$

For,

$$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})$$

$= (\bar{a}\times\bar{b}).\bar{Y}$, where $\bar{Y}$ stands for $\bar{c}\times\bar{d}$

$$= \bar{a}\cdot\bar{b}\times\bar{Y} = \bar{a}\cdot\{\bar{b}\times(\bar{c}\times\bar{d})\}$$
$$= \bar{a}\cdot\{(\bar{b}\cdot\bar{d})\bar{c}-(\bar{b}\cdot\bar{c})\cdot\bar{d}\}$$
$$= (\bar{b}\cdot\bar{d})(\bar{a}\cdot\bar{c})-(\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c})$$

Note that $(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d}) = (\bar{a}\times\bar{b})\times\bar{c}.\bar{d}$

(ii) Vector product of 4 vectors

as $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})$ and is called a vector product of the four vectors $\bar{a},\bar{b},\bar{c},\bar{d}$.

Result

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\,\bar{b}\,\bar{d}]\bar{c}-[\bar{a}\,\bar{b}\,\bar{c}]\bar{d}$$
$$= [\bar{a}\,\bar{c}\,\bar{d}]\bar{b}-[\bar{b}\,\bar{c}\,\bar{d}]\bar{a}$$

For,

$\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)=\bar{X}\times\left(\bar{c}\times\bar{d}\right)$, where $\bar{X}$ stands for $\bar{a}\times\bar{b}$

$$=\left(\bar{X}\cdot\bar{d}\right)\bar{c}-\left(\bar{X}\cdot\bar{c}\right)\bar{d}=\left[\bar{a}\,\bar{b}\,\bar{d}\right]\bar{c}-\left[\bar{a}\,\bar{b}\,\bar{c}\right]\bar{d}$$

Again,

$\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)=\left(\bar{a}\times\bar{b}\right)\times\bar{Y}$, where $\bar{Y}$ stands for $\bar{c}\times\bar{d}$

$$=-\left\{\bar{Y}\times\left(\bar{a}\times\bar{b}\right)\right\}=-\left\{\left(\bar{Y}\cdot\bar{b}\right)\bar{a}-\left(\bar{Y}\cdot\bar{a}\right)\bar{b}\right\}$$

$$=\left[\bar{c}\,\bar{d}\,\bar{a}\right]\bar{b}-\left[\bar{c}\,\bar{d}\,\bar{b}\right]\bar{a}=\left[\bar{a}\,\bar{c}\,\bar{d}\right]\bar{b}-\left[\bar{b}\,\bar{c}\,\bar{d}\right]\bar{a}$$

From the above results we infer that the vector $\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)$ lies in the plane containing $\bar{c}$ and $\bar{d}$ as well as in the plane containing $\bar{a}$ and $\bar{b}$. This means that $\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)$ is a vector, which is parallel to the line of intersection of the two planes, namely the plane containing $\bar{a}$ and $\bar{b}$ and the plane containing $\bar{c}$ and $\bar{d}$.

Also, since $\left[\bar{a}\ \bar{b}\ \bar{d}\right]\bar{c}-\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}=\left[\bar{a}\ \bar{c}\ \bar{d}\right]\bar{b}-\left[\bar{b}\ \bar{c}\ \bar{d}\right]\bar{a}$, it corroborates on earlier remark that any form vectors are linearly dependent.

CONCEPT STRANDS

Concept Strand 63

Find the values of λ such that there exist scalars p and q not all zero satisfying the relation

$$p(6\bar{i}+3\bar{j})+q(2\bar{i}+7\bar{j})=\lambda(p\bar{i}+q\bar{j}).$$

Solution

We have $(6p+2q-\lambda p)\bar{i}+(3p+7q-\lambda q)\bar{j}=0$

Since $\bar{i}$ and $\bar{j}$ are non collinear (or $\bar{i}$ and $\bar{j}$ are linearly independent),

$$6p+2q-\lambda p=0$$
$$3p+7q-\lambda q=0$$

$\Rightarrow (6-\lambda)p+2q=0$ (i.e.,) $(6-\lambda)\,p=-2q$

$3p+(7-\lambda)q=0$ (i.e.,) $(7-\lambda)\,q=-3p$

Multiplying $\Rightarrow (6-\lambda)(7-\lambda)-6=0$

Or $\lambda^2-13\lambda+36=0$ giving $\lambda=4, 9$.

Concept Strand 64

If $\bar{r}$ is the position vector of the point (x, y, z) and if

$$\left|\bar{r}\times\left(\bar{i}\times\bar{j}\right)+\bar{r}\times\left(\bar{j}\times\bar{k}\right)+\bar{r}\times\left(\bar{k}\times\bar{i}\right)\right|^2$$
$$=2\left\{\left[\bar{r}\,\bar{i}\,\bar{j}\right]+\left[\bar{r}\,\bar{j}\,\bar{k}\right]+\left[\bar{r}\,\bar{k}\,\bar{i}\right]\right\}^2,$$

find the value of xy + yz + zx.

Solution

$$\sum\left\{\bar{r}\times\left(\bar{i}\times\bar{j}\right)\right\}=\bar{r}\times\bar{k}+\bar{r}\times\bar{i}+\bar{r}\times\bar{j}$$
$$=\left(x\bar{i}+y\bar{j}+z\bar{k}\right)\times\left(\bar{k}+\bar{i}+\bar{j}\right)$$

$$=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ x & y & z\\ 1 & 1 & 1\end{vmatrix}=(y-z)\bar{i}+(z-x)\bar{j}+(x-y)\bar{k}$$

$$\left|\sum\bar{r}\times\left(\bar{i}\times\bar{j}\right)\right|=\sqrt{(y-z)^2+(z-x)^2+(x-y)^2}$$
$$=\sqrt{2\left(x^2+y^2+z^2\right)-2(yz+zx+xy)}\qquad\text{— (1)}$$

$$\left[\bar{r}\,\bar{i}\,\bar{j}\right]=\begin{vmatrix}x & y & z\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix}=z$$

Similarly, $\left[\bar{r}\,\bar{j}\,\bar{k}\right]=x, \left[\bar{r}\,\bar{k}\,\bar{i}\right]=y$

$$2\left\{\sum\left[\bar{r}\,\bar{j}\,\bar{k}\right]\right\}^2=2(x+y+z)^2$$
$$=2\,(x^2+y^2+z^2+2xy+2yz+2zx)$$

Given:

$$2(x^2+y^2+z^2)-2(xy+yz+zx)$$
$$=2(x^2+y^2+z^2+2xy+2yz+2zx)$$
$$\Rightarrow\ xy+yz+zx=0$$

Concept Strand 65

If $\bar{x}, \bar{y}, \bar{z}$ are three non coplanar vectors and the points P, Q, R, S with position vectors

$$a_1\bar{x} + b_1\bar{y} + c_1\bar{z}, a_2\bar{x} + b_2\bar{y} + c_2\bar{z}, a_3\bar{x} + b_3\bar{y} + c_3\bar{z}, a_4\bar{x} + b_4\bar{y} + c_4\bar{z}$$

are coplanar, prove that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} = 0$$

Solution

$$\overline{PQ} = (a_2 - a_1)\bar{x} + (b_2 - b_1)\bar{y} + (c_2 - c_1)\bar{z}$$
$$\overline{PR} = (a_3 - a_1)\bar{x} + (b_3 - b_1)\bar{y} + (c_3 - c_1)\bar{z}$$
$$\overline{PS} = (a_4 - a_1)\bar{x} + (b_4 - b_1)\bar{y} + (c_4 - c_1)\bar{z}$$

Since, P, Q, R, S are coplanar, (which means that the 4 points lie on a plane), $\left[\overline{PQ}, \overline{PR}, \overline{PS}\right] = 0$

$$\Rightarrow \begin{vmatrix} a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ a_3 - a_1 & b_3 - b_1 & c_3 - c_1 \\ a_4 - a_1 & b_4 - b_1 & c_4 - c_1 \end{vmatrix} = 0 \qquad \text{— (1)}$$

Now, the given determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ a_1 & a_2 - a_1 & a_3 - a_1 & a_4 - a_1 \\ b_1 & b_2 - b_1 & b_3 - b_1 & b_4 - b_1 \\ c_1 & c_2 - c_1 & c_3 - c_1 & c_4 - c_1 \end{vmatrix}$$

$(C_2 - C_1, C_3 - C_1, C_4 - C_1)$

$$= \begin{vmatrix} a_2 - a_1 & a_3 - a_1 & a_4 - a_1 \\ b_2 - b_1 & b_3 - b_1 & b_4 - b_1 \\ c_2 - c_1 & c_3 - c_1 & c_4 - c_1 \end{vmatrix} = 0, \text{ by (1)}$$

Concept Strand 66

$\bar{A}, \bar{B}, \bar{C}$ are three non-coplanar vectors with magnitudes 2, 3 and 4 respectively and equally inclined to one another at an angle $\frac{\pi}{3}$. If $\bar{B} \times \bar{C} + \bar{C} \times \bar{A} = k_1\bar{A} + k_2\bar{B} + k_3\bar{C}$, where k_1, k_2, k_3 are scalars, obtain the values of k_1, k_2 and k_3.

Solution

Since $\bar{A}, \bar{B}, \bar{C}$ are equally inclined to one another at an angle $\frac{\pi}{3}$,

$$\bar{A}.\bar{B} = 3, \bar{B}.\bar{C} = 6, \bar{C}.\bar{A} = 4$$

Let

$$\bar{A} = 2\bar{i}$$

$$\bar{B} = a_1\bar{i} + b_1\bar{j}$$

and $\bar{C} = a_2\bar{i} + b_2\bar{j} + c_2\bar{k}$

we have $2a_1 = 3 \Rightarrow a_1 = \frac{3}{2}$

$a_1^2 + b_1^2 = 9$

$$b_1^2 = 9 - a_1^2 = 9 - \frac{9}{4} = \frac{27}{4}$$

$$b_1 = \frac{3\sqrt{3}}{2}$$

Again, $2a_2 = \bar{C}.\bar{A} = 4$

$\Rightarrow a_2 = 2$

Also, $a_1a_2 + b_1b_2 = 6$

$$3 + b_1b_2 = 6 \Rightarrow b_1b_2 = 3$$

$$b_2 = \frac{3 \times 2}{3\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$c_2^2 = 16 - a_2^2 - b_2^2$$

$$= 16 - 4 - \frac{4}{3} = \frac{32}{3}$$

$$c_2 = 4\sqrt{\frac{2}{3}}$$

Therefore,

$$\bar{A} = 2\bar{i}$$

$$\bar{B} = \frac{3}{2}\bar{i} + \frac{3\sqrt{3}}{2}\bar{j}$$

$$\bar{C} = 2\bar{i} + \frac{2}{\sqrt{3}}\bar{j} + 4\sqrt{\frac{2}{3}}\bar{k}$$

$$\bar{A}.\bar{B}\times\bar{C} = \begin{vmatrix} 2 & 0 & 0 \\ 3/2 & 3\sqrt{3}/2 & 0 \\ 2 & 2/\sqrt{3} & 4\sqrt{2/3} \end{vmatrix}$$

$$= 2 \times \frac{3\sqrt{3}}{2} \times 4\sqrt{\frac{2}{3}} = 12\sqrt{2}$$

Given: $\bar{B} \times \bar{C} + \bar{C} \times \bar{A} = k_1\bar{A} + k_2\bar{B} + k_3\bar{C}$ — (1)

Dot multiply (1) by $\bar{A}$

$12\sqrt{2} = k_1 \times 4 + k_2 \times 3 + k_3 \times 4$

$\Rightarrow \quad 4k_1 + 3k_2 + 4k_3 = 12\sqrt{2}$ — (2)

Dot multiply (1) by $\bar{B}$

$12\sqrt{2} = 3k_1 + 9k_2 + 6k_3$

$\Rightarrow \quad k_1 + 3k_2 + 2k_3 = 4\sqrt{2}$ — (3)

Dot multiply (1) by $\bar{C}$

$0 = 4k_1 + 6k_2 + 16k_3$

$\Rightarrow \quad 2k_1 + 3k_2 + 8k_3 = 0$ — (4)

Solivng (2), (3) and (4),

$$k_1 = \frac{7\sqrt{2}}{2}, k_2 = \sqrt{2}, k_3 = \frac{-5\sqrt{2}}{4}$$

THREE DIMENSIONAL GEOMETRY

In coordinate geometry of three dimensions or three-dimensional geometry, we study the geometrical properties of surfaces in space like plane, sphere, cone, cylinder, and so on, using algebraic (or analytic) methods.

You may recall that in the unit on Straight Lines in a plane, we found that a linear equation in x and y represents a straight line and that the equation of a plane curve is a relation satisfied by the x and y coordinates of a point on that curve. Similarly, a surface in space is represented by a relation of the form F(x, y, z) = 0 satisfied by every point (x, y, z) on that surface. Here (x, y, z) is a point in space referred to a rectangular Cartesian coordinate system. For example, we will show in the sequel that a linear relation in x, y, z of the form $ax + by + cz + d = 0$ represents a plane.

A vector approach to the study of these surfaces using the algebra of vectors developed earlier in this unit is also possible and in some cases, we will be in a position to present the important results/properties in a more elegant manner using the vector approach. However, we will use algebraic (or analytic) techniques in most of the cases as the analytic presentation of the results bring out the essence of the geometrical properties of the surfaces studied better than that when presented through relations involving vectors.

In what follows, we study in detail planes and straight lines. We have already explained how a point in space can be represented as an ordered triple (x, y, z), where x, y, z are the x, y and z coordinates of the point referred to a rectangular Cartesian coordinate system. We have also defined the direction cosines of a line in space as the cosines of the angles made by the line with the positive directions of the axes of coordinates.

Let P(x, y, z) represent a point in space. Then the position vector of P relative to the origin of reference O (origin of the coordinate system) is $\overline{OP}$ and it can be expressed as $\overline{OP} = x\bar{i} + y\bar{j} + z\bar{k}$. The direction ratios of OP are [x, y, z]

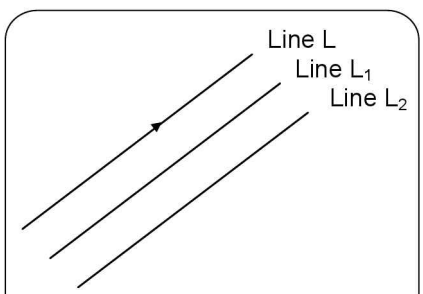


Fig. 3.34

In general, if [ℓ, m, n] are the direction ratios of a line L in space, then, a vector along the line L or along any line parallel to L (like line L_1 or line L_2, refer Fig. 3.34) can be represented by $\ell\bar{i} + m\bar{j} + n\bar{k}$.

The direction cosines of the line L are

$$\left[\frac{\ell}{\sqrt{\ell^2 + m^2 + n^2}}, \frac{m}{\sqrt{\ell^2 + m^2 + n^2}}, \frac{n}{\sqrt{\ell^2 + m^2 + n^2}}\right]$$

We note from the above that the vector $\ell\bar{i} + m\bar{j} + n\bar{k}$ does not represent the line L. However, the components of the vector give us the direction ratios of the line.

For example, consider the vector $5\bar{i} - 6\bar{j} + \bar{k}$. It can represent the position vector of the point (5, –6, 1) relative to the origin of reference O, or it can represent a directed line segment whose length is equal to $\sqrt{5^2 + 6^2 + 1^2}$ and whose direction is parallel to the position vector.

Angle between two lines L_1 and L_2 whose direction ratios are $[\ell_1, m_1, n_1]$ and $[\ell_2, m_2, n_2]$

A vector along the line L_1 is $\ell_1\bar{i} + m_1\bar{j} + n_1\bar{k} = \bar{a}$ (say)

A vector along the line L_2 is $\ell_2\bar{i} + m_2\bar{j} + n_2\bar{k} = \bar{b}$ (say)

Then, if θ is the angle between L_1 and L_2,

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$$

$$= \frac{\ell_1\ell_2 + m_1m_2 + n_1n_2}{\sqrt{\ell_1^2 + m_1^2 + n_1^2}\sqrt{\ell_2^2 + m_2^2 + n_2^2}}$$

The lines L_1 and L_2 are perpendicular (or orthogonal) if $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$ and the lines are parallel if $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

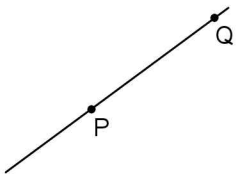
CONCEPT STRANDS

Concept Strand 67

Find the direction cosines of the line joining the points (4, 3, –5) and (–2, 1, –8).

Solution

Let P be (4, 3, –5) and Q (–2, 1, –8)
Direction ratios of the line are $[6, 2, 3]$



Direction cosines of the line are

$$\left[\frac{\pm 6}{\sqrt{6^2 + 2^2 + 3^2}}, \frac{\pm 2}{\sqrt{6^2 + 2^2 + 3^2}}, \frac{\pm 3}{\sqrt{6^2 + 2^2 + 3^2}}\right]$$

$$\Rightarrow \left[\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right] \text{ or } \left[\frac{-6}{7}, \frac{-2}{7}, \frac{-3}{7}\right]$$

Note: d.c's of $\overline{PQ}$ are $\left[\frac{-6}{7}, \frac{-2}{7}, \frac{-3}{7}\right]$,
since $\overline{PQ} = -6\vec{i} - 2\vec{j} - 3\vec{k}$.

Concept Strand 68

Find the direction cosines of lines, which are equally inclined to the axes.

Solution

Let $[\ell, m, n]$ represent the direction cosines of a line, which is equally inclined to the axes.

Then, $\ell = m = n$

$\ell^2 + m^2 + n^2 = 1$

$\Rightarrow 3\ell^2 = 1$

$\Rightarrow \ell = \pm\frac{1}{\sqrt{3}}; m = \pm\frac{1}{\sqrt{3}}, n = \pm\frac{1}{\sqrt{3}};$

Direction cosines of the lines equally inclined to the axes are

$$\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]; \left[\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right];$$

$$\left[\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] \text{ and } \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right]$$

Note:

The other 4 sets of values of ℓ, m, n are $\left[\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right]$,

$$\left[\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right], \left[\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right] \text{ and } \left[\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right],$$

which are only the negatives of the direction cosines of the 4 lines.

Concept Strand 69

Find the angle between the lines whose direction ratios are [1, 2, 1] and [2, –3, 6].

Solution

If θ is the angle between the lines,

$$\cos\theta = \frac{\pm(1\times 2 + 2\times -3 + 1\times 6)}{\sqrt{1^2+2^2+1^2}\sqrt{2^2+3^2+6^2}}$$
$$= \frac{\pm 2}{\sqrt{6}\sqrt{49}} = \frac{\pm 2}{7\sqrt{6}}$$

$\Rightarrow$ The acute angle between the lines

$$= \cos^{-1}\left(\frac{2}{7\sqrt{6}}\right)$$

Concept Strand 70

Find the direction cosines of the line which is perpendicular to the lines with direction ratios [1, –2, –2] and [0, 2, 1].

Solution

We are given the direction ratios of the lines as [1, –2, –2] and [0, 2, 1]

Let the direction ratios of the line perpendicular to the two lines be $[\ell, m, n]$

Then, $\ell \times 1 + m \times -2 + n \times -2 = 0$ and $\ell \times 0 + m \times 2 + n \times 1 = 0$. i.e.,

$$\left.\begin{aligned} \ell - 2m - 2n &= 0 \\ 2m + n &= 0 \end{aligned}\right\}$$

By using the rule of cross multiplication,

$$\frac{\ell}{-2+4} = \frac{m}{0-1} = \frac{n}{2-0}$$

$$\Rightarrow \frac{\ell}{2} = \frac{m}{-1} = \frac{n}{2}$$

Therefore, one set of direction cosines of the line, which is perpendicular to the two given lines, are

$$\left[\frac{2}{\sqrt{2^2+1^2+2^2}}, \frac{-1}{\sqrt{2^2+1^2+2^2}}, \frac{2}{\sqrt{2^2+1^2+2^2}}\right]$$

$$\Rightarrow \left[\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\right]$$

OR

Vectors along the two given lines may be expressed as

$$\bar{a} = \bar{i} - 2\bar{j} - 2\bar{k} \text{ and } \bar{b} = 2\bar{j} + \bar{k}$$

A vector which is perpendicular to both $\bar{a}$ and $\bar{b}$ is along $\bar{a} \times \bar{b}$

Direction cosines of the vector are the components of the unit vector $\frac{1}{|\bar{a}\times\bar{b}|}(\bar{a}\times\bar{b})$.

Concept Strand 71

Find the point in which the join of (–9, 4, 5) and (11, 0, –1) is met by the perpendicular to it from the origin.

Solution

Let A and B represent the points (–9, 4, 5) and (11, 0, –1). Let P be the foot of the perpendicular from O on AB.

Direction ratios of AB are [11 + 9, 0–4, –1–5]

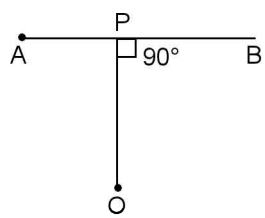
i.e., [20, –4, –6]

or

[10, –2, –3]

Let the coordinates of P be (x, y, z)

Then D.R^s of the line AB can be taken as [x – 11, y – 0, z + 1]



We have $\frac{x-11}{10} = \frac{y-0}{-2} = \frac{z+1}{-3} = k$(say)

$\Rightarrow$ $x = 10k + 11$

$y = -2k$

$z = -3k - 1$

D.R^s of OP are [10k + 11, –2k, –3k – 1]

Since AB and OP are perpendicular to each other,

$$(10k + 11) \times 10 + (-2k)(-2) + (-3k - 1)(-3) = 0$$

$\Rightarrow$ $113k + 113 = 0, k = -1$

Coordinates of P are (10 × –1 + 11, –2 × –1, –3 × –1–1) or (1, 2, 2).

PLANE

A plane may be defined as a surface, such that, for any two points on it, every point on the line joining them also lies on it.

A linear equation in x, y, z of the form $ax + by + cz + d = 0$ — (1), where a, b, c are not all zero represents a plane.

To prove our claim that the above relation represents a plane, we take two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ on the surface represented by (1).

Since A and B lie on (1),

$$\left.\begin{aligned} ax_1 + by_1 + cz_1 + d = 0 \\ ax_2 + by_2 + cz_2 + d = 0 \end{aligned}\right\} \quad \text{— (2)}$$

Any point on AB may be represented in the form $\left(\frac{x_2 + \lambda x_1}{\lambda + 1}, \frac{y_2 + \lambda y_1}{\lambda + 1}, \frac{z_2 + \lambda z_1}{\lambda + 1}\right)$, where λ is a real number not equal to -1.

Substituting the above coordinates in (1)

$$\text{L.H.S} = a\left(\frac{x_2 + \lambda x_1}{\lambda + 1}\right) + b\left(\frac{y_2 + \lambda y_1}{\lambda + 1}\right) + c\left(\frac{z_2 + \lambda z_1}{\lambda + 1}\right) + d$$

$$= ax_2 + by_2 + cz_2 + d + \lambda(ax_1 + by_1 + cz_1 + d)(\lambda + 1)$$

$$= 0 + 0, \text{ by (2)}$$

This means that equation (1) represents a plane.

Remark

Since a, b, c are not simultaneously zero, suppose we assume that $a \neq 0$. Then, dividing (1) by a, we get

$$x + \frac{b}{a}y + \frac{c}{a}z + \frac{d}{a} = 0 \quad \text{— (3)}$$

Thus, from (3) we infer that the equation of a plane involves three constants $\left(\text{here}, \frac{b}{a}, \frac{c}{a} \text{ and } \frac{d}{a}\right)$. Hence, 3 independent conditions to be satisfied by a plane lead us to 3 equations in the three constants and thus the equation of the plane is uniquely determined.

For example, we know that one and only one plane can be drawn passing through 3 non-collinear points.

Results

(i) If a plane passes through the origin, its equation may be assumed as $ax + by + cz = 0$

(ii) Equation of a plane passing through the point (x_1, y_1, z_1) may be assumed as $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

(iii) (a) Equation of the XY plane is $z = 0$. Any plane parallel to the XY plane has the form $z = k$ (k being a constant)

(b) Equation of the YZ plane is $x = 0$. Any plane parallel to the YZ plane has the form $x = k$ (k being a constant).

(c) Equation of the ZX plane is $y = 0$. Any plane parallel to the ZX plane has the form $y = k$ (k being a constant)

(iv) Let a plane pass through the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Let the equation of the plane be

$$ax + by + cz + d = 0 \quad \text{— (1)}$$

We have

$$ax_1 + by_1 + cz_1 + d = 0 \quad \text{— (2)}$$

$$ax_2 + by_2 + cz_2 + d = 0 \quad \text{— (3)}$$

$$ax_3 + by_3 + cz_3 + d = 0 \quad \text{— (4)}$$

Eliminating a, b, c, d from the above 4 equations, we get

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \quad \text{— (5)}$$

(5) gives the equation of the required plane.

OR

A plane through (x_1, y_1, z_1) can be assumed as

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \quad \text{— (6)}$$

Equation (6) passes through (x_2, y_2, z_2) and (x_3, y_3, z_3) also.

$\therefore$ We get

$$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0 \quad \text{— (7)}$$

and

$$A(x_3 - x_1) + B(y_3 - y_1) + C(z_3 - z_1) = 0 \quad \text{— (8)}$$

(6), (7), (8) represent a linear homogeneous system in A, B, C. Since not all A, B, C are zero, the system has non-trivial solutions.

Therefore, $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$ — (9)

(9) gives the equation of the required plane. It can be easily seen that (5) can be reduced to (9) by the operations $R_1 - R_2$, $R_3 - R_2$, $R_4 - R_2$ on (5).

CONCEPT STRAND

Concept Strand 72

Find the equation of the plane passing through the points (2, 2, –1), (3, 4, 2) and (7, 0, 6).

Solution

Equation of a plane passing through the point (2, 2, –1) may be assumed as

$$a(x - 2) + b(y - 2) + c(z + 1) = 0 \quad \text{— (1)}$$

Plane (1) passes through (3, 4,2): $a + 2b + 3c = 0$ — (2)

Plane (1) passes through (7, 0, 6): $5a - 2b + 7c = 0$ — (3)

Eliminating a, b, c from (1), (2), (3), the equation of the phase is

$$\begin{vmatrix} x - 2 & y - 2 & z + 1 \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix} = 0$$

$$20(x - 2) + 8(z - 2) - 12(y + 1) = 0$$

$$\Rightarrow \quad 5x + 2y - 3z - 17 = 0$$

Intercepts form of the equation of a plane

Let a plane intersect the coordinate axes OX, OY, OZ in A, B and C (Refer Fig. 3.35). Then, the x, y and z intercepts of the plane are, respectively the x –, y –, z – coordinates A, B, C.

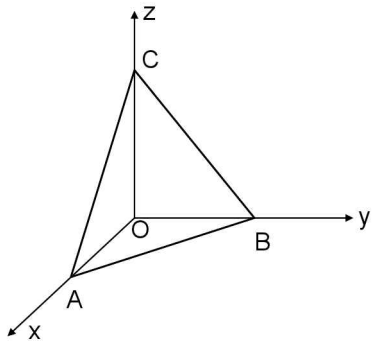


Fig. 3.35

Let a, b, c be respectively the x –, y –, z – intercepts of a plane meeting the x –, y –, z – axes at A, B, C.

The coordinates of A, B, C are easily seen to be (a, 0, 0), (0, b, 0), (0, 0, c) respectively. Therefore, we have to find the equation of the plane passing through the above three points. Since the plane passes through the point (a, 0, 0), its equation may be assumed as

$$A(x - a) + By + Cz = 0 \quad \text{— (1)}$$

(1) passes through (0, b, 0)

$$\Rightarrow -aA + Bb = 0 \quad \text{— (2)}$$

(1) passes through (0, 0, c)

$$\Rightarrow -aA + Cc = 0 \quad \text{—(3)}$$

Eliminate A, B, C

$$\begin{vmatrix} x - a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

$$(x - a)\,bc - y(-ac) + z(-ab) = 0$$

$$\Rightarrow \quad bcx + acy + abz = abc$$

$$\therefore \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad (\text{where } abc \neq 0)$$

For example,
Equation of the plane making intercepts 3, –4, 6 on the axes is:

$$\frac{x}{3}+\frac{y}{-4}+\frac{z}{6}=1$$

$$\Rightarrow\ 4x-3y+2z=12$$

CONCEPT STRAND

Concept Strand 73

A plane meets the coordinate axes in A, B and C such that the centroid of the triangle ABC is the point (5, 9, 17). Find the equation of the plane.

Solution

Let A, B, C be (a, 0, 0), (0, b, 0), (0, 0, c).
Coordinates of the centroid of the triangle ABC are $\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$.

Given: $\frac{a}{3}=5, \frac{b}{3}=9, \frac{c}{3}=17$

$\Rightarrow$ a = 15,
b = 27,
c = 51.

Equation of the plane is $\frac{x}{15}+\frac{y}{27}+\frac{z}{51}=1$

$$\Rightarrow\ \frac{x}{5}+\frac{y}{9}+\frac{z}{17}=3.$$

Normal form of the equation of a plane

Let the length of the perpendicular from the origin (length of the normal from the origin) to a plane be p and let the direction cosines of the normal from the origin to the plane be [cos α, cos β, cos γ]. Then the equation of the plane can be represented in the form

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p \qquad \text{— (1)}$$

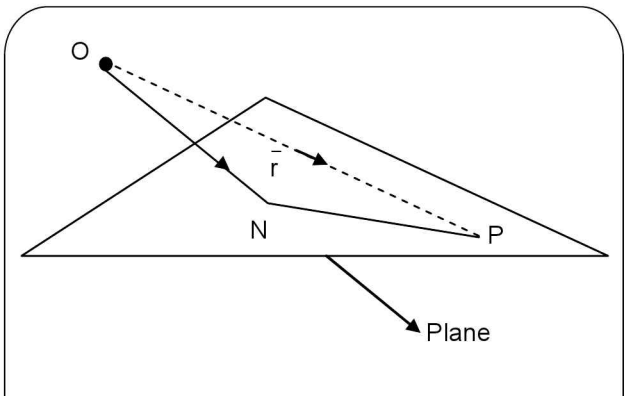


Fig. 3.36

Referring to Fig. 3.36, N is the foot of the perpendicular from the origin O to the plane and the length of the normal from the origin O to the plane = ON = p.

Since the directed line segment ON (or the normal to the plane) makes angles α, β, γ with the axes, a unit vector in the direction of a normal to the plane is $\hat{n}=\cos\alpha\bar{i}+\cos\beta\bar{j}+\cos\gamma\bar{k}$.

Let P(x, y, z) represent a point on the plane. Then, the position vector of P relative to O, is $\overline{OP}=x\bar{i}+y\bar{j}+z\bar{k}$

For any point P on the plane, angle ONP is $\frac{\pi}{2}$.

We have p = ON = Projection of OP on the normal to the plane

$$=\overline{OP}\cdot\hat{n}$$

$$=\left(x\bar{i}+y\bar{j}+z\bar{k}\right)\cdot\left(\cos\alpha\bar{i}+\cos\beta\bar{j}+\cos\gamma\bar{k}\right)$$

$$=x\cos\alpha+y\cos\beta+z\cos\gamma$$

or the equation of the plane is
$x\cos\alpha+y\cos\beta+z\cos\gamma=p$

CONCEPT STRAND

Concept Strand 74

Find the equation of a plane, which is at a distance of 7 units from the origin, the direction ratios of the normal to the plane being [1, 2, −1].

Solution

Direction cosines of the normal to such a plane are $\left[\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right]$

Equation of the plane is

$$x\left(\frac{1}{\sqrt{6}}\right) + y\left(\frac{2}{\sqrt{6}}\right) + z\left(\frac{-1}{\sqrt{6}}\right) = 7$$

Or, $x + 2y - z = 7\sqrt{6}$

Remarks

(i) The general equation of a plane is of the form

$$ax + by + cz + d = 0 \qquad \text{— (1)}$$

Assuming that a, b, c are all different from zero and the plane does not pass through the origin (i.e., $d \neq 0$) we may write (1) as $ax + by + cz = -d$

Dividing both sides by (−d)

$$\frac{x}{\left(\frac{-d}{a}\right)} = \frac{y}{\left(\frac{-d}{b}\right)} = \frac{z}{\left(\frac{-d}{c}\right)} = 1 \qquad \text{— (2)}$$

which gives the intercepts form of the equation to the plane (1). The intercepts are $\frac{-d}{a}, \frac{-d}{b}$ and $\frac{-d}{c}$.

For example, let us find the equation of the plane $5x + 7y - 3z = 9$ in the intercepts form.

Dividing both sides by 9, $\frac{x}{\left(\frac{9}{5}\right)} + \frac{y}{\left(\frac{9}{7}\right)} + \frac{z}{(-3)} = 1$

gives the intercepts form of the equation to the plane.

(ii) To express (1) in the normal form, we proceed as follows:

Observe that in the normal form of the equation of a plane, p is positive (since it represents the length of the perpendicular from the origin to the plane). We now assume that (1) is rewritten so that d is negative.

For example, if (1) is $3x - 4y + 5z - 7 = 0$, there is no need to rewrite the equation as $d = -7$ which is negative. On the other hand, if (1) is $5x + 7y - 8z + 6 = 0$, we rewrite the equation as $-5x - 7y + 8z - 6 = 0$ so that the constant term is negative.

Hence, $ax + by + zx = -d$ (where d is negative)

Divide both sides by $\sqrt{a^2 + b^2 + c^2}$, we get,

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}x + \frac{b}{\sqrt{a^2 + b^2 + c^2}}y + \frac{c}{\sqrt{a^2 + b^2 + c^2}}z = \frac{-d}{\sqrt{a^2 + b^2 + c^2}} \qquad \text{—(3)}$$

(3) represents the normal form of (1) since

$$\left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2 + c^2}}\right)^2 + \left(\frac{c}{\sqrt{a^2 + b^2 + c^2}}\right)^2 = 1,$$

i.e., the coefficients of x, y, z in (3) denote the direction cosines of a line.

Also, $\frac{-d}{\sqrt{a^2 + b^2 + c^2}}$ is positive as d is negative.

Therefore, (3) is of the form

$x \cos\alpha + y \cos\beta + z \cos\gamma = p$.

We draw the following important inferences from the above discussion:

(a) Length of the perpendicular from the origin to the plane (1) i.e., to the plane $ax + by + cz + d = 0$ is given by $p = \left|\frac{-d}{\sqrt{a^2 + b^2 + c^2}}\right|$

(b) Direction ratios of the normal to the plane (1) i.e., to the plane

$ax + by + cz + d = 0$ are [a, b, c]

In other words, the coefficients of x, y, and z in (1) represent the direction ratios of the normal to the plane.

(iii) Equation of a plane in the normal form can be represented as a vector equation,

$$\bar{r} \cdot \hat{n} = p \qquad \text{— (4)}$$

where $\bar{r}$ is the position vector of any point (x, y, z) on the plane, $\hat{n}$ is the unit vector along the normal to the plane (drawn from the origin to the plane) and p is the length of the perpendicular from the origin to the plane.

(iv) Let the direction ratios of the normal to a plane be [a, b, c] and let the plane pass through the point (x_1, y_1, z_1). Then, its equation can be written as

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \qquad \text{— (5)}$$

Or, in the vector form as

$$\bar{N} \cdot (\bar{r} - \bar{r}_1) = 0 \qquad \text{— (6)}$$

In (6), $\bar{N} = a\bar{i} + b\bar{j} + c\bar{k}$ = a vector in the direction of the normal to the plane (5),

$\bar{r}_1 = x_1\bar{i} + y_1\bar{j} + z_1\bar{k}$ = Position vector of the point (x_1, y_1, z_1)

and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ = Position vector of any point (x, y, z) on the plane (5).

We have the following results:

Result 1

Length of the perpendicular from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is equal to $\left|\dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$

Result 2

$P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in space, $ax + by + cz + d = 0$ represents a plane and P, Q do not lie on the plane.

Let $k_1 = ax_1 + by_1 + cz_1 + d$ and $k_2 = ax_2 + by_2 + cz_2 + d$

If k_1 and k_2 are of the same sign, P and Q are on the same side of the plane.

If k_1 and k_2 are of opposite signs, P and Q are on the opposite sides of the plane.

We are also able to say whether a point is on the origin side or non-origin side of the plane.

If k_1 and d are of the same sign, the point P is on the origin side of the plane and if k_1 and d are of opposite signs, P is on the non-origin side of the plane.

The length of perpendicular from the point (1, –3, 5) to the plane $3x+y+7z-8=0$ is $\left|\dfrac{3 \times 1 - 3 + 7 \times 5 - 8}{\sqrt{3^2 + 1^2 + 7^2}}\right| = \dfrac{27}{\sqrt{59}}$

Consider the points (1, 1, 1), (3, 4, –8) and the plane $x + y + z - 6 = 0$

$d = -6 < 0$

$k_1 = 1 + 1 + 1 - 6 < 0$

$k_2 = 3 + 4 - 8 - 6 < 0$

$\Rightarrow$ Both the points (1, 1, 1) and (3, 4, –8) are on the origin side of plane.

Result 3

Angle between the planes

$$a_1x + b_1y + c_1z + d_1 = 0 \qquad \text{— (1)}$$

$$a_2x + b_2y + c_2z + d_2 = 0 \qquad \text{— (2)}$$

The angle between two planes is defined as the acute angle between the normals to the two planes.

D.R.^{s} of the normal to the plane (1) are $[a_1, b_1, c_1]$
D.R.^{s} of the normal to the plane (2) are $[a_2, b_2, c_2]$

If θ is the acute angle between the planes,

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

The two planes (1) and (2) are perpendicular (or are orthogonal) if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

The two planes (1) and (2) are parallel if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

Therefore, equation of a plane parallel to the plane $ax + by + cz + d = 0$ may be assumed as

$$ax + by + cz + k = 0, \text{ where k is a constant.}$$

For example, to find the equation of the plane parallel to the plane $x + 2y + 7z - 1 = 0$ and passing through the origin, we proceed as follows:

Let the plane be $x + 2y + 7z + k = 0$
Since the plane passes through the origin, $k = 0$.
Therefore, the required plane is $x + 2y + 7z = 0$

Result 4

If two planes are not parallel, they intersect along a line.

Equation of a plane (other than the plane 2), passing through the line of intersection of the two planes,

$$a_1x + b_1y + c_1z + d_1 = 0 \text{ — plane 1 and}$$

$$a_2x + b_2y + c_2z + d_2 = 0 \text{ — plane 2}$$

can be represented as

$$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$$

where λ is a constant.

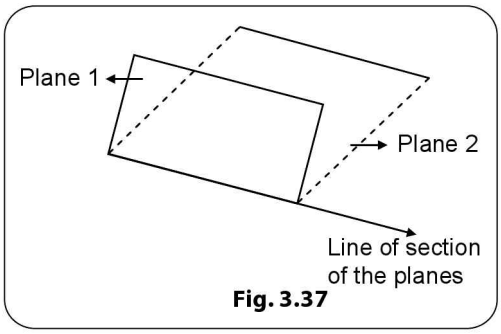


Fig. 3.37

For, the above equation represents a plane since it is linear in x, y and z.

Again, if (x^*, y^*, z^*) is a point on the line of intersection of the planes,

$$a_1 x^* + b_1 y^* + c_1 z^* + d_1 + \lambda(a_2 x^* + b_2 y^* + c_2 z^* + d_2) = 0 + \lambda \times 0 = 0$$

(since (x^*, y^*, z^*) is a point on the line of intersection of the planes, (x^*, y^*, z^*) satisfies the equations of the two planes)

CONCEPT STRAND

Concept Strand 75

Find the equation of the plane through the line intersection of the planes $x + 2y - z + 4 = 0$, $5x + 6y + z - 8 = 0$ and passing through the point $(1, -1, 0)$.

Solution

Let the equation of the plane be $x + 2y - z + 4 + \lambda(5x + 6y + z - 8) = 0$

Since the required plane passes through the point $(1, -1, 0)$,

$$1 + (-2) + 0 + 4 + \lambda(5 - 6 + 0 - 8) = 0$$

$$3 - 9\lambda = 0, \text{ giving } \lambda = \frac{1}{3}$$

The equation of the plane is $x + 2y - z + 4 + \frac{1}{3}(5x + 6y + z - 8) = 0$

$\Rightarrow \quad 8x + 12y - 2z + 4 = 0$ or $4x + 6y - z + 2 = 0$

Result 5

Consider a plane parallel to the z-axis. Let the equation of the plane be $ax + by + cz + d = 0$

D.R^s of the normal to the plane are $[a, b, c]$
D.R^s of the z–axis are $[0, 0, 1]$

Since the plane is parallel to the z-axis, normal to the plane and the z-axis are perpendicular, this gives

$$a \times 0 + b \times 0 + c \times 1 = 0 \Rightarrow c = 0$$

Therefore,

Equation of a plane parallel to the x-axis is of the form $by + cz + d = 0$

Equation of a plane parallel to the y-axis is of the form $ax + cz + d = 0$

Equation of a plane parallel to the z-axis is of the form $ax + by + d = 0$

CONCEPT STRANDS

Concept Strand 76

Find the equation of the plane passing through the point $(-1, 3, 2)$ and perpendicular to the planes $x + 2y + 2z - 5 = 0$ and $3x + 3y + 2z - 8 = 0$.

Solution

Let the equation of the plane be $a(x + 1) + b(y - 3) + c(z - 2) = 0$

Since the required plane is perpendicular to the two given planes,

$$a + 2b + 2c = 0$$

$$3a + 3b + 2c = 0$$

Eliminating a, b, c, equation of the plane is

$$\begin{vmatrix} x+1 & y-3 & z-2 \\ 1 & 2 & 2 \\ 3 & 3 & 2 \end{vmatrix} = 0 \text{ or } 2x - 4y + 3z + 8 = 0$$

Concept Strand 77

Find the equation of the plane passing through the points (–1, 1, 1) and (1, –1, 1) and perpendicular to the plane $x + 2y + 2z = 0$.

Solution

Let the plane be $a(x - 1) + b(y + 1) + c(z - 1) = 0$

Since it passes through (–1, 1, 1), $-2a + 2b = 0$ or $a = b$

Since the required plane is perpendicular to the plane $x + 2y + 2z = 0$

$$a + 2b + 2c = 0 \text{ giving } c = \frac{-3b}{2}$$

The plane is $2x + 2y - 3z + 3 = 0$.

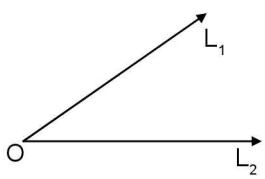
Concept Strand 78

Find the equation of the plane containing the lines through the origin with direction ratios [3, 1, –5] and [2, 3, –1].

Solution

Plane passes through L_1 and L_2 and since both L_1 and L_2 are through the origin, the required plane must pass through the origin.

Let the plane be $ax + by + cz = 0$.



Since the plane passes through L_1, normal to the plane and the line are perpendicular to each other.

This gives $3a + b - 5c = 0$.

Again, since the required plane passes through L_2, $2a + 3b - c = 0$

These relations in a, b, c give $\frac{a}{14} = \frac{b}{-7} = \frac{c}{7}$

Equation of the plane is $14x - 7y + 7z = 0$ Or $2x - y + z = 0$

Concept Strand 79

Find the equation of the plane which passes through the intersection of the planes $2x + y - 4 = 0$, $y + 2z = 0$ and perpendicular to the plane $3x + 2y - 3z = 6$.

Solution

Let the required plane be $2x + y - 4 + k(y + 2z) = 0$ where k is a constant.

D.R^s of normal to the plane are [2, 1 + k, 2k]

Since the required plane is perpendicular to the given plane,

$$2 \times 3 + (1 + k) \times 2 + 2k \times -3 = 0$$
$$\Rightarrow \quad -4k + 8 = 0$$
$$\therefore \quad k = 2$$

Plane is $2x + 3y + 4z - 4 = 0$

Concept Strand 80

Find the equation of the plane which bisects at right angles the join of (1,3, –2) and (3, 1, 6).

Solution

It is clear that the join of the two points is normal to the plane.

Therefore, D.R^s of normal to the plane are [2, –2, 8] or [1, –1, 4]

Since the plane bisects the join of the two given points, it passes through the mid point of the line segment i.e., through (2, 2, 2)

Plane is $1(x - 2) + (-1)(y - 2) + 4(z - 2) = 0$

Or, $x - y + 4z - 8 = 0$

STRAIGHT LINES

A curve in space is usually characterized as the curve of intersection of two surfaces. For example, a circle in space may be characterized as the curve of intersection of a sphere and a plane.

Similarly, a straight line in space can be characterized as the common section of two intersecting planes.

We say that the two equations

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$

together represent a straight line. This means that a curve in space cannot be represented by a single equation.

Equations of a straight line in symmetric form

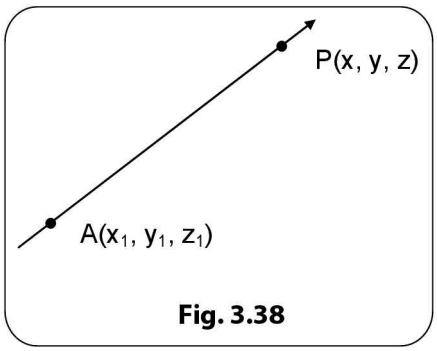


Fig. 3.38

A straight line in space is completely described if we are given

(i) direction ratios of the line and
(ii) a point on the line

Let $[\ell, m, n]$ denote the direction ratios of the line passing through the point $A(x_1, y_1, z_1)$ (refer Fig. 3.38).

Let $P(x, y, z)$ be a point on the line. Then, the direction ratios of the line are $[x - x_1, y - y_1, z - z_1]$

But, the direction ratios of the line are given as $[\ell, m, n]$

Therefore, we have $\dfrac{x - x_1}{\ell} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ — (1)

(1) is known as the symmetric form of the equations to the line.

If we take $\dfrac{x - x_1}{\ell} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n} = r$ — (2), where r is real.

Then,

$$\left.\begin{aligned} x &= x_1 + \ell r \\ y &= y_1 + mr \\ z &= z_1 + nr \end{aligned}\right\} \quad \text{—(3)}$$

We see that any point on the line has the coordinates $(x_1 + \ell r, y_1 + mr, z_1 + nr)$ where r is a real number. By assigning different values to r, we get the coordinates of different points on the line.

(3) may be interpreted as the parametric representation of the equations of the line, r being the parameter.

For example, the equations of the line passing through the point (1, –6, 5) with D.R^s [1, 2, –3] are $\dfrac{x-1}{1} = \dfrac{y+6}{2} = \dfrac{z-5}{-3}$; and any pont on the line could be represented as (r + 1, 2r – 6, 5 – 3r), r being the parameter.

It is clear from the above that the parameter r in (2) is proportional to the distance of the corresponding point (x, y, z) from (x_1, y_1, z_1). If ℓ, m, n represent the direction cosines, then, $\ell^2 + m^2 + n^2 = 1$. The parameter $|r|$ in this case represents the actual distance of any point (x, y, z) from (x_1, y_1, z_1).

Results

(i) Equations of the x-axis are $\dfrac{x}{1} = \dfrac{y}{0} = \dfrac{z}{0}$

Equations of the y-axis are $\dfrac{x}{0} = \dfrac{y}{1} = \dfrac{z}{0}$ and

Equations of the z-axis are $\dfrac{x}{0} = \dfrac{y}{0} = \dfrac{z}{1}$

(ii) Equations of a line in the XOY plane are $\dfrac{x-a}{\ell} = \dfrac{y-b}{m} = \dfrac{z}{0}$;

Equations of a line in the YOZ plane are

$\dfrac{x}{0} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$ and

Equations of a line in the ZOX plane are

$\dfrac{x-a}{\ell} = \dfrac{y}{0} = \dfrac{z-c}{n}$

(iii) Suppose the equations of a line are given as

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0$$

We can represent the equations of the above line in symmetric form:

First, we find the direction ratios of the line.

Let $[\ell, m, n]$ denote the D.R^s of the line. Then, the line is perpendicular to the normals to the two planes. This gives

$$\ell a_1 + mb_1 + nc_1 = 0$$
$$\ell a_2 + mb_2 + nc_2 = 0$$

From which we get,

$$\frac{\ell}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$$

Second, we have to find a point on the given line.

If we put $z = 0$ in the two equations representing the line and solving the resulting equations

$\left.\begin{aligned} a_1x + b_1y + d_1 &= 0 \\ a_2x + b_2y + d_2 &= 0 \end{aligned}\right\}$ for x and y, we are in fact finding the coordinates of the point of intersection of the line with the XY plane. Thus, we get a point on the line. The equations of the line can now be expressed in the symmetric form.

Note:

If the line happens to be parallel to the XOY plane, the two equations in x and y will be inconsistent. Then we proceed to find the point of intersection of the line with the YZ plane by putting $x = 0$ in the equations of the two planes or we find the point of intersection of the line with the ZX plane by putting $y = 0$ in the equations of the two planes.

Illustration

Let the equations of a line be

$$x - 2y + z = 1$$
$$x + 2y - 2z = 5$$

If $[\ell, m, n]$ are the D.R^s of the line, $\dfrac{\ell}{4-2} = \dfrac{m}{1+2} = \dfrac{n}{2+2}$

$$\Rightarrow \quad \frac{\ell}{2} = \frac{m}{3} = \frac{n}{4}$$

Putting $z = 0$ in the two equations representing the line,
We get $x - 2y = 1$

$$x + 2y = 5$$

Giving $x = 3$, $y = 1$.
The line passes through the point (3, 1, 0).
Equations of the line, in symmetric form are

$$\frac{x-3}{2} = \frac{y-1}{3} = \frac{z}{4}.$$

(iv) Equations of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2)
D.R^s of the line are $[x_2 - x_1, y_2 - y_1, z_2 - z_1]$
The equations of the line are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Or $$\frac{x - x_2}{x_2 - x_1} = \frac{y - y_2}{y_2 - y_1} = \frac{z - z_2}{z_2 - z_1}$$

For example, the equations of the line passing through the points (1, 2, –1) and (3, 2, 7) are

$$\frac{x-1}{3-1} = \frac{y-2}{2-2} = \frac{z+1}{7+1} \text{ i.e., } \frac{x-1}{2} = \frac{y-2}{0} = \frac{z+1}{8}$$

or $$\frac{x-1}{1} = \frac{y-2}{0} = \frac{z+1}{4}.$$

Equations of the line may also be written as

$$\frac{x-3}{1} = \frac{y-2}{0} = \frac{z-7}{4}$$

(v) We can also represent a line by a vector equation.
For example, consider the line passing through the point (x_1, y_1, z_1) and having direction ratios $[\ell, m, n]$.

A vector along the line may be written as $\ell\bar{i} + m\bar{j} + n\bar{k}$

Let us denote the position vector of any point P(x, y, z) on the line by r (Refer Fig. 3.38)

$$\bar{r} = \overline{OP} = x\bar{i} + y\bar{j} + z\bar{k}$$

Position vector of $A = x_1\bar{i} + y_1\bar{j} + z_1\bar{k} = \bar{a}$ (say)
From Fig. 3.39,

$$\overline{AP} = \overline{OP} - \overline{OA} = \bar{r} - \bar{a} \qquad \text{— (1)}$$

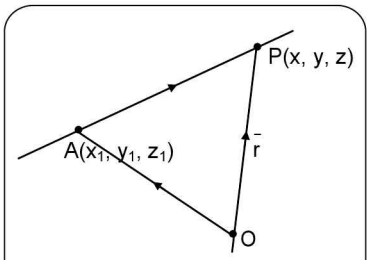


Fig. 3.39

But, $\overline{AP}$ is a vector along the line $\ell\bar{i} + m\bar{j} + n\bar{k}$,

$\overline{AP} = \lambda\left(l\bar{i} + m\bar{j} + n\bar{k}\right)$, where λ is an arbitrary scalar.

Therefore, the vector equation of the line is

$$\bar{r} = \bar{a} + \lambda\left(\ell\bar{i} + m\bar{j} + n\bar{k}\right) \qquad \text{— (2)}$$

Vector form of the equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) may be written in a similar way as

$$\left.\begin{aligned} &\bar{r} = \bar{a} + \lambda \cdot \bar{d} \\ &\text{or} \\ &\bar{r} = \bar{b} + \mu\bar{d} \end{aligned}\right\} \qquad \text{— (3)}$$

where, $\bar{r}$ denotes the position vector of any point on the line;

$$\bar{a} = x_1\bar{i} + y_1\bar{j} + z_1\bar{k}$$

$$\bar{d} = (x_2 - x_1)\bar{i} + (y_2 - y_1)\bar{j} + (z_2 - z_1)\bar{k}$$
$$= \bar{b} - \bar{a}$$

Thus the equation is $\bar{r} = (1-\lambda)\bar{a} + \lambda\bar{b}$, λ being a scalar parameter.

CONCEPT STRANDS

Concept Strand 81

Find the point of intersection of the line $\frac{x+3}{3} = \frac{y-2}{-2} = \frac{z+1}{1}$ and the plane $4x + 5y + 3z - 5 = 0$.

Solution

Any point on the line may be represented as $(3r - 3, -2r + 2, r - 1)$

This point is on the plane if $4(3r - 3) + 5(-2r + 2) + 3(r - 1) - 5 = 0$

$\Rightarrow$ $5r - 10 = 0$ giving $r = 2$

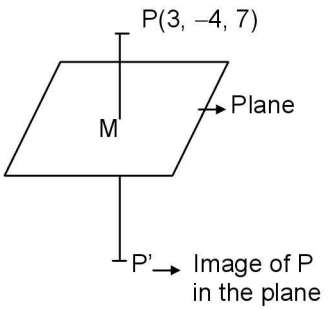
The required point is $(3 \times 2 - 3, -2 \times 2 + 2, 2 - 1)$ or $(3, -2, 1)$

Concept Strand 82

Find the image of the point $(3, -4, 7)$, by reflexion in the plane $x + 2y + 3z - 5 = 0$.

Solution

d.r's of normal to the plane are $[1, 2, 3]$



Therefore, equations to the line PMP^1, which is perpendicular to the plane are

$$\frac{x-3}{1} = \frac{y+4}{2} = \frac{z-7}{3} = r \text{ (say)}$$

Any point on the line can be represented as $(r + 3, 2r - 4, 3r + 7)$

If this point is M, then

i.e., $r + 3 + 2(2r - 4) + 3(3r + 7) - 5 = 0$

$\Rightarrow$ $14r + 11 = 0$

Or $r = \frac{-11}{14}$

P^1 being the image of P, by reflexion in the plane, the value of r (or the value of the parameter r) corresponding to P^1 is $\frac{-11}{14} \times 2 = \frac{-11}{7}$.

Coordinates of the image are

$$\left(\frac{-11}{7} + 3, 2 \times \frac{-11}{7} - 4, 3 \times \frac{-11}{7} + 7\right)$$

i.e., $\left(\frac{10}{7}, \frac{-50}{7}, \frac{16}{7}\right)$

Concept Strand 83

Find the equations of the line passing through the point $(2, -1, -1)$ and perpendicular to the line $2x + y = 0 = x - z + 5$ and parallel to the plane $4x + y + z + 2 = 0$.

Solution

Let $[\ell, m, n]$ be the D.R^s of the required line.

Since the line is parallel to the plane

$$4x + y + z + 2 = 0, 4\ell + m + n = 0 \qquad \text{— (1)}$$

The line whose equations are $2x + y = 0 = x - z + 5$ can be represented in the symmetrical form:

$$\frac{x}{1} = \frac{y}{-2} = \frac{z-5}{1} \qquad \text{— (2)}$$

Since the required line is perpendicular to the line (2)

$$\ell - 2m + n = 0 \qquad \text{— (3)}$$

From (1) and (3), $\frac{\ell}{3} = \frac{m}{-3} = \frac{n}{-9}$

or $\frac{\ell}{1} = \frac{m}{-1} = \frac{n}{-3}$

Equations of the required line are

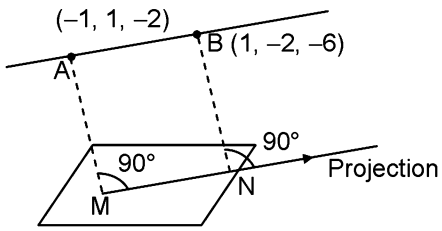
$$\frac{x-2}{1} = \frac{y+1}{-1} = \frac{z+1}{-3}$$

Concept Strand 84

Find the equations of the projection of the line $\frac{x+1}{-2} = \frac{y-1}{3} = \frac{z+2}{4}$ on the plane $2x + y + 4z = 1$.

Solution

To find the projection of the line on the plane, we find the projections of two points of the given line on the plane. The line joining these points is the projection.



Any point on the given line is of the form $(-2r - 1, 3r + 1, 4r - 2)$

Putting $r = -1$, we find that $(1, -2, -6)$ is a point on the line.

Let A be $(-1, 1, -2)$ and B be $(1, -2, -6)$.

D.R^s of normal to the plane are $[2, 1, 4]$

Equations to AM (where M is the foot of the perpendicular from A to the plane) are $\frac{x+1}{2} = \frac{y-1}{1} = \frac{z+2}{4}$

$\Rightarrow (2k - 1, k + 1, 4k - 2)$ is a point on the line.

Substituting in the equation of the plane,

$$2(2k - 1) + (k + 1) + 4(4k - 2) = 1$$

$$21k = 10,\ k = \frac{10}{21}$$

$$M \text{ is } \left(\frac{2 \times 10}{21} - 1, \frac{10}{21} + 1, \frac{4 \times 10}{21} - 2\right)$$

$$\text{i.e., } \left(\frac{-1}{21}, \frac{31}{21}, \frac{-2}{21}\right) \qquad \text{— (1)}$$

Again, equations to BN are $\frac{x-1}{2} = \frac{y+2}{1} = \frac{z+6}{4}$

$\Rightarrow (2\lambda + 1, \lambda - 2, 4\lambda - 6)$ is a point on the above line.

Substituting in the equation of the plane,

$$2(2\lambda + 1) + (\lambda - 2) + 4(4\lambda - 6) = 1$$

$$21\lambda = 25,\ \lambda = \frac{25}{21}.$$

$$N \text{ is } \left(\frac{2 \times 25}{21} + 1, \frac{25}{21} - 2, \frac{4 \times 25}{21} - 6\right)$$

$$\text{i.e, } \left(\frac{71}{21}, \frac{-17}{21}, \frac{-26}{21}\right) \qquad \text{— (2)}$$

From (1) and (2), D.R^s of the projection line are $\left[\frac{24}{7}, \frac{-16}{7}, \frac{-8}{7}\right]$

Equations of the projection line are

$$\frac{x + \frac{1}{21}}{\left(\frac{24}{7}\right)} = \frac{y - \frac{31}{21}}{\left(\frac{-16}{7}\right)} = \frac{z + \frac{2}{21}}{\left(\frac{-8}{7}\right)}$$

$$\Rightarrow \frac{x + \frac{1}{21}}{3} = \frac{y - \frac{31}{21}}{-2} = \frac{z + \frac{2}{21}}{-1}$$

Aliter:

The required line is the intersection of the given plane and the plane through the given line, perpendicular to the given plane.

Plane through given line, perpendicular to the given plane is

$a(x + 1) + b(y - 1) + c(z + 2) = 0$ where

$-2a + 3b + 4c = 0,\ 2a + b + 4c = 0$

$\Rightarrow\ 2a = b = -2c \Rightarrow a = \frac{b}{2} = \frac{c}{-1}$

$\therefore$ plane is $x + 1 + 2y - 2 - z - 2 = 0$

i.e., $x + 2y - z = 3$

$\therefore$ required line is $x + 2y - z = 3,\ 2x + y + 4z = 1$

D.R^s are $9, -6, -3$ i.e., $3, -2, -1$

When $z = 0,\ x + 2y = 3;\ 2x + y = 1$

$$\Rightarrow\ x + y = \frac{4}{3}\ x - y = -2$$

$$\Rightarrow\ x = \frac{-1}{3}, y = \frac{5}{3}$$

$$\therefore \text{ line is } \frac{x + \frac{1}{3}}{3} = \frac{y - \frac{5}{3}}{-2} = \frac{z}{-1}$$

Conditions for a line to lie in a plane

To find the conditions for the line

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad \text{— (1)}$$

to lie in the plane

$$ax + by + cz + d = 0 \qquad \text{— (2)}$$

It is easy to note that if the line (1) is to lie in the plane (2), line(1) must be perpendicular to the normal to the plane; and a point on the line must lie on the plane.
This gives

$$a\ell + bm + cn = 0 \qquad \text{— (3)}$$

and $$ax_1 + by_1 + cz_1 + d = 0 \qquad \text{— (4)}$$

Hence, (3) and (4) are the conditions to be satisfied for line (1) to lie in the plane (2).

CONCEPT STRANDS

Concept Strand 85

Show that the line $\frac{x-2}{3} = \frac{y+2}{-1} = \frac{z-3}{4}$ lies in the plane $2x + 2y - z + 3 = 0$.

Solution

$2 \times 3 + 2 \times -1 + (-1) \times 4 = 0$ and $2 \times 2 + 2 \times -2 - 3 + 3 = 0$
Proved.

Concept Strand 86

Find the equation of the plane which passes through the point $(3,4,-5)$ and contains the line $\frac{x+1}{2} = \frac{y-1}{3} = \frac{z+2}{-1}$.

Solution

Let the plane be

$$a(x-3) + b(y-4) + c(z+5) = 0 \qquad \text{— (1)}$$

We have $2a + 3b - c = 0$ — (2)

and $a(-1-3) + b(1-4) + c(-2+5) = 0$

Or $-4a - 3b + 3c = 0$ — (3)

Eliminate a, b, c between (1), (2), (3). The equation of the plane is $\begin{vmatrix} x-3 & y-4 & z+5 \\ 2 & 3 & -1 \\ -4 & -3 & 3 \end{vmatrix} = 0$

$$3x - y + 3z + 10 = 0$$

Coplanar lines and skew lines

Consider the lines L_1 and L_2 whose equations are

$$L_1 : \frac{x - x_1}{\ell_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \qquad \text{— (1)}$$

$$L_2 : \frac{x - x_2}{\ell_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2} \qquad \text{— (2)}$$

If L_1 and L_2 are such that they lie on a plane we say that the lines are coplanar.

If L_1 and L_2 are such that they do not lie on a plane we say that the lines are skew lines.

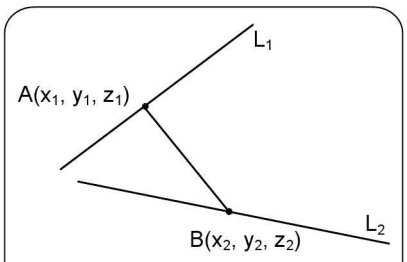


Fig. 3.40

Condition for two lines, L_1 and L_2 to be coplanar

Referring to Fig. 3.40,
A vector along L_1 is $\ell_1\bar{i} + m_1\bar{j} + n_1\bar{k}$ and a vector along L_2 is $\ell_2\bar{i} + m_2\bar{j} + n_2\bar{k}$

Also, $\overline{AB} = (x_2 - x_1)\bar{i} + (y_2 - y_1)\bar{j} + (z_2 - z_1)\bar{k}$

If L_1 and L_2 are coplanar,

$\overline{AB}, \ell_1\bar{i} + m_1\bar{j} + n_1\bar{k}, \ell_2\bar{i} + m_2\bar{j} + n_2\bar{k}$ are coplanar.
Hence, their scalar triple product must vanish.

$$\Rightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad \text{— (3)}$$

(3) gives the condition for the lines L_1 and L_2 to be coplanar.

Equation of the plane containing the lines L_1 and L_2 is given by $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

or $\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

If L_1 and L_2 are not coplanar, the determinant $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} \neq 0$

Suppose L_1 and L_2 are not coplanar, we can find two points M and N on L_1 and L_2 respectively such that MN is perpendicular to both L_1 and L_2. Then, length of MN is the shortest distance between the two skew lines L_1 and L_2.

Let the D.R^s of MN be [a, b, c]

Then,

$$a\ell_1 + bm_1 + cn_1 = 0$$

$$a\ell_2 + bm_2 + cn_2 = 0$$

and we get $\dfrac{a}{m_1n_2 - m_2n_1} = \dfrac{b}{n_1\ell_2 - n_2\ell_1} = \dfrac{c}{\ell_1m_2 - \ell_2m_1}$

Shortest distance between the skew lines L_1 and L_2

= MN

= Projection of the line joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ on the line MN.

$$= \left| \frac{\left[(x_2 - x_1)\bar{i} + (y_2 - y_1)\bar{j} + (z_2 - z_1)\bar{k}\right] \cdot \left[a\bar{i} + b\bar{j} + c\bar{k}\right]}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$$= \left| \frac{a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1)}{\sqrt{a^2 + b^2 + c^2}} \right|$$

where, $a = m_1n_2 - m_2n_1$, $b = n_1\ell_2 - n_2\ell_1$, $c = \ell_1m_2 - \ell_2m_1$.

The shortest distance between the skew lines can be expressed in the determinant form

$$\frac{1}{\sqrt{\sum (m_1n_2 - m_2n_1)^2}} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}$$

Numerical value of the above gives the shortest distance.

Formula for the shortest distance in vector form

Let

$$\overline{p_1} = x_1\bar{i} + y_1\bar{j} + z_1\bar{k}$$

$$\overline{p_2} = x_2\bar{i} + y_2\bar{j} + z_2\bar{k}$$

$$\overline{d_1} = \ell_1\bar{i} + m_1\bar{j} + n_1\bar{k}$$

$$\overline{d_2} = \ell_2\bar{i} + m_2\bar{j} + n_2\bar{k}$$

Shortest distance = numerical value of

$$\frac{1}{\left|\overline{d_1} \times \overline{d_2}\right|}\left(\overline{p_1} - \overline{p_2}\right) \cdot \left(\overline{d_1} \times \overline{d_2}\right)$$

CONCEPT STRAND

Concept Strand 87

Prove that the lines $\dfrac{x-2}{2} = \dfrac{y+2}{1} = \dfrac{z+1}{5}$, $\dfrac{x+1}{1} = \dfrac{y+6}{3} = \dfrac{z}{-6}$ are coplanar. Find the equation of the plane containing them. Find also the coordinates of the point of intersection of the lines.

Solution

Any point on the first line may be represented as $(2r_1 + 2, r_1 - 2, 5r_1 - 1)$

Any point on the second line may be represented as $(r_2 - 1, 3r_2 - 6, -6r_2)$

If the lines are coplanar, they will intersect.

There exists a unique set of values for r_1 and r_2 such that

$$\left.\begin{aligned} 2r_1 + 2 &= r_2 - 1 \\ r_1 - 2 &= 3r_2 - 6 \\ \text{and } 5r_1 - 1 &= -6r_2 \end{aligned}\right\}$$

we find that $r_1 = -1$, $r_2 = 1$ satisfy all the 3 relations above. The coordinates of the point of intersection of the lines are thus $(0, -3, -6)$.

Equation of the plane containing these lines is

$$\begin{vmatrix} x-2 & y+2 & z+1 \\ 2 & 1 & 5 \\ 1 & 3 & -6 \end{vmatrix} = 0$$

$$-21(x-2) - (-12-5)(y+2) + (6-1)(z+1) = 0$$

$$-21(x-2) + 17(y+2) + 5(z+1) = 0$$

$$21x - 17y - 5z - 81 = 0$$

We wind up this unit by highlighting the geometrical interpretation of the consistency of a non-homogeneous linear system of equations.

CONSISTENCY OF A NON-HOMOGENEOUS LINEAR SYSTEM OF EQUATIONS

Consider the non-homogeneous linear system of equations

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

where a_i, b_i, c_i, $i = 1, 2, 3$ are distinct

Let Δ = determinant of the coefficients $= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

$$\Delta_1 = \begin{vmatrix} -d_1 & b_1 & c_1 \\ -d_2 & b_2 & c_2 \\ -d_3 & b_3 & c_3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & -d_1 & c_1 \\ a_2 & -d_2 & c_2 \\ a_3 & -d_3 & c_3 \end{vmatrix}$$

$$\text{and } \Delta_3 = \begin{vmatrix} a_1 & b_1 & -d_1 \\ a_2 & b_2 & -d_2 \\ a_3 & b_3 & -d_3 \end{vmatrix}$$

We have the following results:

(i) Suppose $\Delta \neq 0$.
Then the linear system has a unique solution. That is, the system is consistent and it has a unique solution.

(ii) $\Delta = 0$ and at least one of $\Delta_1, \Delta_2, \Delta_3$ not equal to zero
Then, the linear system has no solution or the system is inconsistent.

(iii) $\Delta = 0 = \Delta_1 = \Delta_2 = \Delta_3$.
Then, the linear system has infinite number of solutions. That is, the system is consistent and possesses infinite number of solutions.

Geometrically, the three equations represent three planes.

In the case of (i), unique solution means, there exists a unique point (x, y, z) satisfying all the three equations. In other words, in this case, the three planes intersect at a point.

In the case of (ii), there is no set of values of x, y, z satisfying all the three equations. Geometrically, this means that the line of intersection of any two planes is parallel to the third plane. In other words, the three planes form a triangular prism.

In the case of (iii), there are infinite number of points (x, y, z) satisfying all the three equations. Geometrically, this means that the three planes pass through a line i.e., they have a common line of intersection.

SUMMARY

VECTOR ALGEBRA

Vectors

A directed line segment is a vector. A vector is completely known once its magnitude and direction are given.

If AB is a directed line segment, $\overline{AB}$ defines a vector. A is its initial point and B is its terminal point. Length of AB represents the magnitude of the vector $\overline{AB}$ and the direction from A to B defines its direction.

Vectors are usually denoted by $\bar{a}$, $\bar{b}$, $\bar{c}$.... Magnitude of $\bar{a}$ is denoted by $|\bar{a}|$.

Unit vector is a vector of magnitude (or length) unity (or 1). A unit vector in the direction of $\bar{a}$ is denoted by $\hat{a}$.

Null vector is a vector of magnitude O and is denoted by $\bar{O}$. The direction of a null vector is not defined.

Equal vectors

Vectors having same magnitude and direction.

Collinear vectors

Two vectors are said to be collinear if they have the same direction. A set of vectors is said to be coplanar if their directions are parallel to the same plane.

Addition and subtraction of vectors

(i) $\bar{a} + \bar{b} = \bar{b} + \bar{a}$ (commutative law)

(ii) $\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$ (associative law)

(iii) $\bar{a} + \bar{O} = \bar{O} + \bar{a} = \bar{a}$

(iv) $\bar{a} + (-\bar{a}) = \bar{O}$

(v) $k(\bar{a} + \bar{b}) = k\bar{a} + k\bar{b}$

(vi) $(k_1 + k_2)\bar{a} = k_1\bar{a} + k_2\bar{a}$

(vii) $k_1(k_2\bar{a}) = k_2(k_1\bar{a}) = k_1k_2\bar{a}$ (where k, k_1, k_2 represent scalars $\neq 0$)

Section formula

If $\bar{a}$ and $\bar{b}$ represent the position vectors of two points A and B, the position vector of the point P which divides the line segment AB in the ratio m : n (m, n > o) internally, is given by

$$\bar{r} = \frac{1}{(m+n)}(m\bar{b} + n\bar{a})$$

The position vector of the point Q which divides the line segment AB in the ratio m : n externally is given by

$$\bar{r} = \frac{1}{(m-n)}(m\bar{b} - n\bar{a}) \qquad (m \neq n)$$

Position vector of the mid point of AB is given by $\frac{1}{2}(\bar{a} + \bar{b})$

Let the position vectors of a 3 points A, B, C be represented by $\bar{a}$, $\bar{b}$, $\bar{c}$ respectively, If ABC form a triangle then, the position vector of the centroid of the ΔABC is given by $\frac{1}{3}(\bar{a} + \bar{b} + \bar{c})$.

Product of vectors

Dot product or scalar product

Let $\bar{a}$ and $\bar{b}$ represent two non-null vectors and let the angle between $\bar{a}$ and $\bar{b}$ be $\theta (0 \le \theta \le \pi)$. Then scalar product of $\bar{a}$ and $\bar{b}$ written as $\bar{a}.\bar{b}$ is defined as

$\bar{a}.\bar{b} = |a|\,|b| \cos\theta$

(i) $\bar{a}.\bar{b} = \bar{b}.\bar{a}$ and $\bar{a}.\bar{a} = |\bar{a}|^2$

(ii) $\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1$ & $\bar{i} \cdot \bar{j} = 0 = \bar{j} \cdot \bar{k} = \bar{i} \cdot \bar{k}$

(iii) $\bar{a} \cdot \bar{b} = 0$ implies $\theta = \frac{\pi}{2}$. Or, we say that $\bar{a}$ and $\bar{b}$ are orthogonal vectors.

(iv) projection of $\bar{b}$ on $\bar{a} = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|}$.

(v) $\bar{a} \cdot (\bar{b} \pm \bar{c}) = \bar{a} \cdot \bar{b} \pm \bar{a} \cdot \bar{c}$.

(vi) $|\bar{a} + \bar{b}|^2 = (\bar{a} + \bar{b}).(\bar{a} + \bar{b}) = \bar{a}^2 + \bar{b}^2 + 2\bar{a} \cdot \bar{b}$

& $|\bar{a} - \bar{b}|^2 = (\bar{a} - \bar{b}).(\bar{a} + \bar{b}) = \bar{a}^2 + \bar{b}^2 - 2\bar{a} \cdot \bar{b}$

and $|\bar{a} + \bar{b}|^2 + |\bar{a} - \bar{b}|^2 = 2(\bar{a}^2 + \bar{b}^2)$

If $|\bar{a} + \bar{b}| = |\bar{a} - \bar{b}|$, $\bar{a}$ and $\bar{b}$ are orthogonal.

(vii) If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$, then

$\bar{a} \cdot \bar{b} = a_1b_1 + a_2b_2 + a_3b_3$ and

$\bar{a} \cdot \bar{a} = a_1^2 + a_2^2 + a_3^2 = |\bar{a}|^2$

$\bar{a}.\bar{b} = ab\cos\theta \qquad \Rightarrow \cos\theta = \dfrac{\bar{a} \cdot \bar{b}}{ab}$

$\Rightarrow \theta = \cos^{-1}\dfrac{\bar{a} \cdot \bar{b}}{ab}$

where

$\bar{a} = |\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

for $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

Hence, angle between the vectors $(a_1\bar{i} + a_2\bar{j} + a_3\bar{k})$ and $b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ is

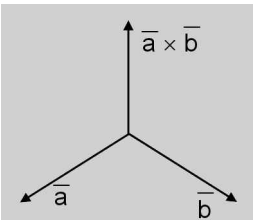
$$\cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}.\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

(viii) Projection of $\bar{a}$ on $\bar{b} = \dfrac{\bar{a}.\bar{b}}{|\bar{b}|}$

(ix) Vector component of $\bar{a}$ in the direction of $\bar{b}$ is $\dfrac{\bar{a}.\bar{b}}{|\bar{b}|}\hat{b}$ where $\hat{b}$ denotes the unit vector along $\bar{b}$ or it is given by $\dfrac{\bar{a}.\bar{b}}{|\bar{b}|^2}\bar{b}$ or by $\dfrac{\bar{a}.\bar{b}}{\bar{b}.\bar{b}}\bar{b}$.

Vector product (cross product)

Let $\bar{a}$ and $\bar{b}$ be two vectors. The vector product or cross product of $\bar{a}$ and $\bar{b}$ written as $a \times b$ is defined as $\bar{a} \times \bar{b} = (|a|\,|b| \sin\theta)\,\bar{c}$ where $\bar{c}$ is the unit vector normal $\bar{a}$ and $\bar{b}$ such that $\bar{a}, \bar{b}, \bar{c}$ form a right handed system.



(i) Unit vector perpendicular to both $\bar{a}$ and $\bar{b}$

$\Rightarrow \pm\dfrac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|}$

(ii) $|\bar{a} \times \bar{b}|^2 = \bar{a}^2\,\bar{b}^2 \sin^2\theta$, $(\bar{a} \cdot \bar{b})^2 = \bar{a}^2\bar{b}^2\cos^2\theta$

(iii) $|\bar{a} \times \bar{b}|^2 + (\bar{a} \cdot \bar{b})^2 = |\bar{a}|^2\,|\bar{b}|^2$

(iv) We write $\bar{a} \times \bar{b} = ab \sin\theta\, \bar{n}$ where $\bar{n}$ is the unit vector perpendicular to both $\bar{a}$ and $\bar{b}$ and $\bar{a}, \bar{b}, \bar{n}$ form a right handed system.

(v) $\bar{b} \times \bar{a} = ba \sin\theta\,(-\bar{n}) = -ba \sin\theta\, \bar{n} = -\bar{a} \times \bar{b}$.

(vi) $\bar{a} \times \bar{b} = \bar{0} \Rightarrow \bar{a}$ is parallel to $\bar{b}$

(vii) $\bar{a} \times (\bar{b} \pm \bar{c}) = \bar{a} \times \bar{b} \pm \bar{a} \times \bar{c}$

(viii) $\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = \bar{0}$

$\bar{i} \times \bar{j} = \bar{k} = -\bar{j} \times \bar{i}$

$\bar{j} \times \bar{k} = i = -\bar{k} \times j$ and $\bar{k} \times i = \bar{j} = -i \times \bar{k}$.

(ix) If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

then $\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \Sigma\,(a_2b_3 - a_3b_2)\,\bar{i}$

(x) Area of the triangles whose adjacent sides represented by $\bar{a}$ and $\bar{b}$ is $= \dfrac{1}{2}|\bar{a} \times \bar{b}|$

Area of the parallelogram whose adjacent sides are $\bar{a}$ and $\bar{b}$ is given by $|\bar{a} \times \bar{b}|$. If the diagonals of a parallelogram are represented by $\overline{d_1}$ and $\overline{d_2}$, area of the parallelogram is given by $\dfrac{1}{2}|\overline{d_1} \times \overline{d_2}|$.

If the vertices of a ΔABC are represented by $\bar{a}$, $\bar{b}$ and $\bar{c}$ area of the ΔABC is $\dfrac{1}{2}(\bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b})$

Scalar triple product

Scalar triple product of the vectors λ_2 is $\bar{a} \times \bar{b}.\bar{c} = [\bar{a}\bar{b}\bar{c}]$ $= [\bar{a}\,\bar{b}\,\bar{c}]$

(i) The scalar triple product is unaltered if the dot and the cross operations are interchanged.

$$\bar{a} \cdot \bar{b} \times \bar{c} = \bar{b} \times \bar{c} \cdot \bar{a}$$

(ii) The scalar triple product is unaltered if the vectors $\bar{a}, \bar{b}, \bar{c}$ are changed cyclically, i.e., if $\bar{a}$ is changed to $\bar{b}$; $\bar{b}$ is changed to $\bar{c}$ and $\bar{c}$ is changed to $\bar{a}$ in $[\bar{a}\,\bar{b}\,\bar{c}]$.

$$\Rightarrow \quad \bar{a} \times \bar{b} \cdot \bar{c} = \bar{b} \times \bar{c} \cdot \bar{a} = \bar{c} \times \bar{a} \cdot \bar{b}$$

$$\left[\bar{a}\,\bar{b}\,\bar{c}\right] = \left[\bar{b}\,\bar{c}\,\bar{a}\right] = \left[\bar{c}\,\bar{a}\,\bar{b}\right]$$

If any two of the vectors in a scalar triple product are interchanged, the value of the scalar triple product changes in sign. i.e., $\left[\bar{a}\,\bar{b}\,\bar{c}\right] = -\left[\bar{a}\,\bar{c}\,\bar{b}\right]$ (vectors $\bar{a}$ and $\bar{b}$ are interchanged).

(iii) The scalar triple product is zero (scalar triple product is said to vanish) if any two of the vectors of the set of three vectors are collinear (or parallel).

If $\bar{a}, \bar{b}, \bar{c}$ are the three vectors and $\bar{b} = \lambda\bar{c}$ where, λ is a scalar, $\left[\bar{a}\,\bar{b}\,\bar{c}\right] = 0$

(iv) If the vectors $\bar{a}, \bar{b}, \bar{c}$ taken in that order form a right handed system, $\left[\bar{a}\,\bar{b}\,\bar{c}\right]$ will be > 0 otherwise (i.e., if the vectors $\bar{a}, \bar{b}, \bar{c}$ in that order forms a left handed system), $\left[\bar{a}\,\bar{b}\,\bar{c}\right]$ will be < 0

If the set of vectors $\{\bar{a}, \bar{b}, \bar{c}\}$ is coplanar, their scalar triple product vanishes (or scalar triple product of $\bar{a}, \bar{b}, \bar{c}$ is zero). Conversely, if $\bar{a}, \bar{b}, \bar{c}$ are three vectors such that $\left[\bar{a}\,\bar{b}\,\bar{c}\right] = 0$, then the set of vectors $\{\bar{a}, \bar{b}, \bar{c}\}$ is coplanar.

Linear dependence and linear independence of a set of vectors

Set of vectors $\{\bar{a}_1, \bar{a}_2, \bar{a}_3 \ldots \bar{a}_n\}$ is said to be linearly dependent if there exist scalars $\lambda_1, \lambda_2, \ldots \lambda_n$ not all zero such that $\lambda_1\bar{a}_1 + \lambda_2\bar{a}_2 + \ldots + \lambda_n\bar{a}_n = 0$ — (1)

If relation (1) is satisfied only when $\lambda_1 = \lambda_2 \ldots = \lambda_n = 0$, then the set of vectors is said to be linearly dependent.

Two vectors are linearly dependent if they are collinear. If $\bar{a}$ and $\bar{b}$ are collinear vectors, $\bar{b} = \lambda\bar{a}$ where, λ is a scalar $\neq 0$.

If the three vectors $\bar{a}$, $\bar{b}$, and $\bar{c}$ are linearly dependent, they are coplanar.

If $\bar{a}$, $\bar{b}$, $\bar{c}$ denote the position vectors of three points A, B, C such that $\lambda_1\bar{a} + \lambda_2\bar{b} + \lambda_3\bar{c} = 0$ where, $\lambda_1 + \lambda_2 + \lambda_3 = 0$, then the points A, B, C are collinear.

The maximum number of linearly independent vectors that can exist in the three dimensional space is 3.

Vector triple product of 3 vectors

If $\bar{a}$, $\bar{b}$, $\bar{c}$ denotes 3 vectors, a vector triple product of $\bar{a}$, $\bar{b}$, $\bar{c}$ is $\bar{a} \times (\bar{b} \times \bar{c})$

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}.\bar{c})\,\bar{b} - (\bar{a}.\bar{b})\,\bar{c}.$$

Another vector triple product of $\bar{a}$, $\bar{b}$, $\bar{c}$ is

$$(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{c}.\bar{a})\bar{b} - (\bar{c}.\bar{b})\bar{a}$$

Results

(i) $(\bar{a} + \bar{b}) \times (\bar{b} + \bar{c}) . (\bar{c} + \bar{a}) = 2\left[\bar{a}, \bar{b}, \bar{c}\right]$

(ii) $\left[(\bar{a} \times \bar{b}), (\bar{b} \times \bar{c}), (\bar{c} \times \bar{a})\right] = \left[\bar{a}, \bar{b}, \bar{c}\right]^2$

(iii) $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) = 0$

THREE DIMENSIONAL GEOMETRY

Distance between the point A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is given by $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

Coordinates of the point P dividing the line segment AB in the ratio m : n (m, n > o) are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

Coordinates of the point Q dividing the line segment AB in the ratio m = n externally, are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right) \quad (m \neq n)$$

Mid-point of AB is at $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

If A (x_1, y_1, z_1), B (x_2, y_2, z_2), C (x_3, y_3, z_3) denote the vertices of a ΔABC, coordinates of the centroid of the ΔABC are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$.

Let a line OP where O is the origin of a rectangular Cartesian system makes angles α, β, γ $(O \leq \alpha, \beta, \gamma \leq \pi)$ with the positive directions of the X, Y and Z axes. Then, $[\cos\alpha, \cos\beta, \cos\gamma]$ are called the direction cosines of the line OP.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

If ℓ, m, n are numbers such that

$$\frac{\ell}{\cos\alpha} = \frac{m}{\cos\beta} = \frac{n}{\cos\gamma},$$

then, [ℓ, m, n] are called the direction ratios of the line OP.

If [ℓ, m, n] are direction ratios of a line, OP the direction cosines of the lines are

$$\left[\frac{\ell}{\sqrt{\ell^2 + m^2 + n^2}}, \frac{m}{\sqrt{\ell^2 + m^2 + n^2}}, \frac{n}{\sqrt{\ell^2 + m^2 + n^2}}\right]$$

Direction ratios of the line AB where A is (x_1, y_1, z_1) and B is (x_2, y_2, z_2) are given by $[x_2 - x_1, y_2 - y_1, z_2 - z_1]$

A unit vector in the direction of AB is

$(\cos\alpha)\,\bar{i} + (\cos\beta)\,\bar{j} + (\cos\gamma)\,\bar{k}$, where

$$\cos\alpha = \frac{x_2 - x_1}{\sqrt{\sum(x_2 - x_1)^2}}, \cos\beta = \frac{y_2 - y_1}{\sqrt{\sum(x_2 - x_1)^2}}, \cos\gamma = \frac{z_2 - z_1}{\sqrt{\sum(x_2 - x_1)^2}}$$

Direction cosines of x axis: [1, 0, 0]

Direction cosines of Y axis: [0, 1, 0]

Direction cosines of Z axis: [0, 0, 1]

Planes

Equation of a plane is of the form $Ax + By + Cz + D = 0$

(i) Intercept form of the equation of a plane:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

(ii) Normal form of the equation of a plane:
$x\cos\alpha + y\cos\beta + z\cos\gamma = p$
Where [$\cos\alpha$, $\cos\beta$, $\cos\gamma$] are the direction cosines of a normal to the plane and p is the length of the perpendicular from the origin to the plane.

(iii) Length of the perpendicular from origin to the plane $Ax + By + Cz + D = 0$ is $\left|\frac{-D}{\sqrt{A^2 + B^2 + C^2}}\right|$

(iv) Length of the perpendicular from the point (x_1, y_1, z_1) to the plane is

$$\left|\frac{Ax_1 + By_1 Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

(v) The direction ratios of a normal to the plane $Ax + By + Cz + D = 0$ are [A, B, C]

(vi) Equation of the plane passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

(vii) $z = k$ represents a plane parallel to the XOY plane
$y = k$ represents a plane parallel to the ZOX plane
and $x = k$ represents a plane parallel to the YOZ plane.

(viii) $Ax + By + D = 0$ represents a plane parallel to the Z axis
$By + Cz + D = 0$ represents a plane parallel to the X axis
and $Ax + Cz + D = 0$ represents a plane parallel to the Y axis

(ix) Acute angle θ between the planes
$A_1x + B_1y + C_1z + D_1 = 0$
$A_2x + B_2y + C_2z + D_2 = 0$ is given by

$$\cos\theta = \left|\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$$

The planes are orthogonal (ie, they are normal to each other) if
$A_1A_2 + B_1B_2 + C_1C_2 = 0$
The phases are parallel if

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

(x) Vector forms of the equation of a plane:
$\bar{r} \rightarrow$ position vector of a point on the plane.
$\hat{n} \rightarrow$ unit vector normal to the plane
$\bar{r}, \hat{n} = p$ where $p > 0$ represents the plane.

Straight planes

(i) Equations of a line passing through the point (x_1, y_1, z_1) and whose direction ratios are [ℓ, m, n] are

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r$$

where r denotes the proportional distance of any point (x, y, z) on the line from the point (x_1, y_1, z_1) (called symmetric form of the equations of a line)

(ii) Equations of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2)

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

OR

$$\frac{x - x_2}{x_2 - x_1} = \frac{y - y_2}{y_2 - y_1} = \frac{z - z_2}{z_2 - z_1}$$

(iii) Equations of the x-axis: $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

y-axis: $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$

z-axis: $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$

(iv) If $\bar{a}$ and $\bar{b}$ are the position vectors of the points A and B, vector equation of the line AB is

$$\bar{r} = \bar{a} + \lambda\left(\bar{b} - \bar{a}\right) = (1 - \lambda)\ \bar{a} + \lambda\,\bar{b}$$

(v) The lines $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ lies in the plane $Ax + By + Cz + D = 0$

if $Ax_1 + By_1 + Cz_1 + D = 0$ and $\ell A + m B + n C = 0$

(vi) The lines $\frac{x - x_1}{\ell_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$

and $\frac{x - x_2}{\ell_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$

are coplanar if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

(vii) Shortest distance between the skew lines (no coplanar lines)

$$\frac{x - x_1}{\ell_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$$

and $\frac{x - x_2}{\ell_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ is given by the numerical value of $\frac{\left(\bar{a}_2 - \bar{a}_1\right).\left(\bar{b}_1 \times \bar{b}_2\right)}{\left|\bar{b}_1 \times \bar{b}_2\right|}$

where $\bar{a}_1 = x_1\,\bar{i} + y_1\,\bar{j} + z_1\,\bar{k}$

$\bar{a}_2 = x_2\,\bar{i} + y_2\,\bar{j} + z_2\,\bar{k}$

$\bar{b}_1 = \ell_1\,\bar{i} + m_1\,\bar{j} + n_1\,\bar{k}$

$\bar{b}_2 = \ell_2\,\bar{i} + m_2\,\bar{j} + n_2\,\bar{k}$

CONCEPT CONNECTORS

Connector 1: P is a point on the side AD of a parallelogram ABCD such that AP:PD = 2 : 1. Find the ratio in which BP is divided by the diagonal AC.

Solution: Let E be the point of intersection of the diagonals of the parallelogram ABCD.
Choose as origin. Let $\bar{a}$ and $\bar{d}$ be the position vectors of A and D respectively.

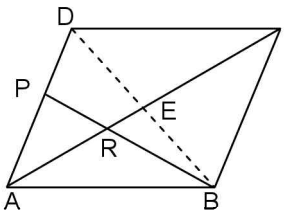
$$\overline{BE} = \frac{d}{2}$$

$\frac{AP}{PD} = \frac{2}{1}$. Then $\overline{BP} = \frac{2\bar{d}+\bar{a}}{2+1} = \frac{\bar{a}+2\bar{d}}{3}$

Equation of BP is $\bar{r} = t\,\frac{\bar{a}+2\bar{d}}{3}$ — (1) where, t is some scalar.

Equation of AE is $\bar{r} = \bar{a} + s\left(\frac{\bar{d}}{2} - \bar{a}\right)$ — (2) for some scalar s.

R lies on (1) and (2)



$\therefore \quad \frac{t}{3} = 1 - s;\ \frac{2t}{3} = \frac{s}{2}$

$\therefore \quad s = \frac{4}{5}$ and $t = \frac{3}{5}$

$\therefore$ R is given by $\bar{r} = \frac{\bar{a}+2\bar{d}}{5}$

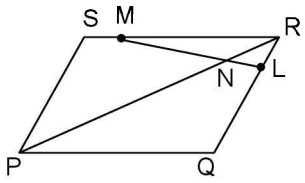
(i.e.) $\overline{BR} = \frac{\bar{a}+2\bar{d}}{5}$

$\overline{BR} = \frac{3}{5}\overline{BP} = \frac{3}{5}\left(\overline{BP} + \overline{RP}\right)$

$\frac{2}{5}\overline{BR} = \frac{3}{5}\overline{RP}$ $\qquad \therefore \frac{BR}{RP} = \frac{3}{2}$

Connector 2: Points L and M are taken on the sides QR and RS respectively of a parallelogram PQRS, so that QL = 4LR and RM = 4MS. The line LM intersects the line PR at N. Find the ratio PN : NR.

Solution: Let us take P as the origin of reference. Let the position vectors of Q, R and S be $\bar{q}$, $\bar{r}$ and $\bar{s}$ respectively, referred to P as origin of reference.
Since PQRS is a parallelogram,



$\overline{PS} = \overline{QR} \Rightarrow \bar{s} = \bar{r} - \bar{q}$

Position vector of L is $\frac{4\bar{r} + \bar{q}}{5}$;

Position vector of M is $\frac{4\bar{s} + \bar{r}}{5} = \frac{4\left(\bar{r} - \bar{q}\right) + \bar{r}}{5} = \bar{r} - \frac{4}{5}\bar{q}$

Let PN : NR = λ : 1 and LN : NM = μ : 1

Position vector of N as a point on LM is $\dfrac{\mu\left(\bar{r}-\frac{4}{5}\bar{q}\right)+\frac{4\bar{r}+\bar{q}}{5}}{(\mu+1)}$

whereas, position vector of N considered as a point on PR is $\dfrac{\lambda\bar{r}}{\lambda+1}$

We must have $\dfrac{1}{(\mu+1)}\left[\left(\mu+\dfrac{4}{5}\right)\bar{r}+\left(\dfrac{-4}{5}\mu+\dfrac{1}{5}\right)\bar{q}\right]=\dfrac{\lambda\bar{r}}{(\lambda+1)}$

Therefore, $\dfrac{5\mu+4}{5(\mu+1)}=\dfrac{\lambda}{\lambda+1}$ and $\dfrac{\frac{-4}{5}\mu+\frac{1}{5}}{(\mu+1)}=0$, since $\bar{r}$ and $\bar{q}$ are non collinear

Solving $\mu=\dfrac{1}{4}\Rightarrow\dfrac{\lambda}{\lambda+1}=\dfrac{21}{25}\Rightarrow\lambda=\dfrac{21}{4}\Rightarrow PN:NR=21:4$

Connector 3: In a quadrilateral ABCD, prove that $AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4PQ^2$ where, P and Q are the middle points of the diagonals AC and BD.

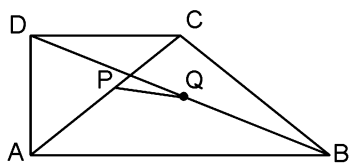
Solution: Let us choose D as our origin of reference.

Let the position vectors of the vertices A, B, C be $\bar{a},\bar{b},\bar{c}$

$\overline{AB}=\bar{b}-\bar{a},\overline{BC}=\left(\bar{c}-\bar{b}\right)$

$\overline{AC}=\bar{c}-\bar{a},\overline{BD}=-\bar{b},\overline{DA}=\bar{a},\overline{CD}=-\bar{c}$

Since P and Q are the mid points of AC and BD,



$\overline{DP}=\dfrac{\bar{a}+\bar{c}}{2},\overline{DQ}=\dfrac{1}{2}\bar{b}$

$\Rightarrow\overline{PQ}=\overline{DQ}-\overline{DP}=\dfrac{1}{2}\left(\bar{b}-\bar{a}-\bar{c}\right)$

$AB^2+BC^2+CD^2+DA^2=\left|\bar{b}-\bar{a}\right|^2+\left|\bar{c}-\bar{b}\right|^2+\left|\bar{c}\right|^2+\left|\bar{a}\right|^2$

$=2\left[\left|\bar{a}\right|^2+\left|\bar{b}\right|^2+\left|\bar{c}\right|^2-\bar{b}\cdot\bar{a}-\bar{c}\cdot\bar{b}\right]$ — (1)

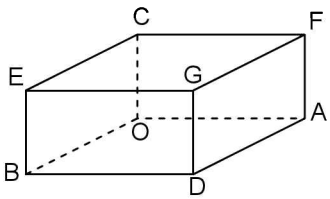
$AC^2+BD^2+4PQ^2=\left|\bar{c}-\bar{a}\right|^2+\left|\bar{b}\right|^2+4\times\dfrac{1}{4}\left|\bar{b}-\bar{a}-\bar{c}\right|^2$

$=2\left[\left|\bar{a}\right|^2+\left|\bar{b}\right|^2+\left|\bar{c}\right|^2-\bar{b}\cdot\bar{c}-\bar{a}\cdot\bar{b}\right]$ —(2)

From (1) and (2) result follows.

Connector 4: Prove by vector methods that the sum of the squares on the diagonals of a parallelepiped is equal to four times the sum of the squares on the three coterminous edges.

Solution: The diagonals of the parallelopiped are OG, CD, AE and BF



Let the position vectors of the vertices A, B, C relative to the origin of reference O be $\bar{a},\bar{b},\bar{c}$ respectively. Then the position vectors of D, E, F, P are $\bar{a}+,\bar{b},\bar{b}+\bar{c},\bar{c}+\bar{a}$ and $\left(\bar{a}+\bar{b}+\bar{c}\right)$ respectively.

$$\Rightarrow \quad \overline{OP} = \left(\bar{a} + \bar{b} + \bar{c}\right)$$

$$\overline{CD} = \overline{OD} - \overline{OC} = \bar{a} + \bar{b} - \bar{c}$$

$$\overline{AE} = \overline{OE} - \overline{OA} = \bar{b} + \bar{c} - \bar{a}$$

$$\overline{BF} = \overline{OF} - \overline{OB} = \bar{c} + \bar{a} - \bar{b}$$

$$OP^2 + CD^2 + AE^2 + BF^2 = \left|\bar{a} + \bar{b} + \bar{c}\right|^2 + \left|\bar{a} + \bar{b} - \bar{c}\right|^2 + \left|\bar{b} + \bar{c} - \bar{a}\right|^2 + \left|\bar{c} + \bar{a} - \bar{b}\right|^2$$

$$= 4\left\{\left|\bar{a}\right|^2 + \left|\bar{b}\right|^2 + \left|\bar{c}\right|^2\right\} = 4\left\{\left|\overline{OA}\right|^2 + \left|\overline{OB}\right|^2 + \left|\overline{OC}\right|^2\right\}$$

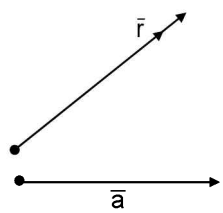
Connector 5: If $\bar{a}$ and $\bar{b}$ are unit vectors and θ is the angle between them, find $\cos\frac{\theta}{2}$.

Solution: $\left|\bar{a} + \bar{b}\right|^2 = \left(\bar{a} + \bar{b}\right).\left(\bar{a} + \bar{b}\right)$

$$\left|\bar{a}\right|^2 + \left|\bar{b}\right|^2 + 2\bar{a}.\bar{b} = 1 + 1 + 2\cos\theta = 2(1 + \cos\theta)$$

$$\Rightarrow 4\cos^2\frac{\theta}{2} = \left|\bar{a} + \bar{b}\right|^2 \Rightarrow \frac{\theta}{2} = \frac{1}{2}\left|\bar{a} + \bar{b}\right|$$

Connector 6: Find the (vector) component of a vector $\bar{r}$ in the direction of a vector $\bar{a}$.



Solution: Projection of $\bar{r}$ on $\bar{a} = \dfrac{\bar{r}\cdot\bar{a}}{\left|\bar{a}\right|}$

unit vector along $\bar{a} = \dfrac{1}{\left|\bar{a}\right|}\bar{a}$

$\therefore$ The vector component of $\bar{r}$ in the direction of $\bar{a}$

$$= \frac{\left(\bar{r}\cdot\bar{a}\right)}{\left|\bar{a}\right|}\frac{1}{\left|\bar{a}\right|}\bar{a} = \frac{\bar{r}\cdot\bar{a}}{\left|\bar{a}\right|^2}\bar{a}$$

Connector 7:. Prove that $\left(\bar{a}\times\bar{b}\right)^2 = \left|\bar{a}\right|^2\left|\bar{b}\right|^2 - \left(\bar{a}\cdot\bar{b}\right)^2$

or

$\left(\bar{a}\times\bar{b}\right)^2 = \begin{vmatrix} a\cdot a & a\cdot b \\ b\cdot a & b\cdot b \end{vmatrix}$ and deduce the identity

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$
$$= (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_2b_3)^2 + (a_3b_{1-}a_1b_3)^2.$$

Solution: $\bar{a}\times\bar{b} = \left(\left|\bar{a}\right|\times\left|\bar{b}\right|\sin\theta\right)\hat{n}$ where, θ is the angle between $\bar{a}$ and $\bar{b}$;and $\hat{n}$ is the unit vector normal to the plane containing $\bar{a}$ and $\bar{b}$ such that $\bar{a}$, $\bar{b}$, $\hat{n}$ form a right handed system

$$\left(\bar{a}\times\bar{b}\right)^2 = \left|\bar{a}\times\bar{b}\right|^2 = \left|\bar{a}\right|^2\left|\bar{b}\right|^2\sin^2\theta$$

$$= \left|\bar{a}\right|^2\left|\bar{b}\right|^2(1 - \cos^2\theta) = \left|\bar{a}\right|^2\left|\bar{b}\right|^2 - \left|\bar{a}\right|^2\left|\bar{b}\right|^2\cos^2\theta$$

$$= \left|\bar{a}\right|^2\left|\bar{b}\right|^2 - \left(\bar{a}\cdot\bar{b}\right)^2$$

$$= \left(\bar{a}\cdot\bar{a}\right)\left(\bar{b}\cdot\bar{b}\right) - \left(\bar{a}\cdot\bar{b}\right)^2 = \begin{vmatrix} \bar{a}\cdot\bar{a} & \bar{a}\cdot\bar{b} \\ \bar{a}\cdot\bar{b} & \bar{b}\cdot\bar{b} \end{vmatrix} = \begin{vmatrix} \bar{a}\cdot\bar{a} & \bar{a}\cdot\bar{b} \\ \bar{b}\cdot\bar{a} & \bar{b}\cdot\bar{b} \end{vmatrix}$$

Deduction:

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$; $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\bar{i} + (a_3b_1 - a_1b_3)\bar{j} + (a_1b_2 - a_2b_1)\bar{k}$$

$$\left|\bar{a}\times\bar{b}\right|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \qquad \text{—(1)}$$

$$\left|\bar{a}\right|^2\left|\bar{b}\right|^2 - \left(\bar{a}.\bar{b}\right)^2 = (a_1^2 + a_3^2 + a_3^2)\ (b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \qquad \text{—(2)}$$

From (1) and (2) , deduction follows.

Connector 8: If $\bar{a},\bar{b},\bar{c}$ are three unit vectors such that $\bar{a}\times\left(\bar{b}\times\bar{c}\right) = \frac{1}{2}\bar{b}$, find the angles which $\bar{a}$ makes with $\bar{b}$ and $\bar{c}$, $\bar{b}$, $\bar{c}$ being non-collinear.

Solution: Given $(\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c} = \frac{1}{2}\bar{b}$ $\quad\Rightarrow \left(\bar{a}\cdot\bar{c} - \frac{1}{2}\right)\bar{b} - (\bar{a}.\bar{b})\bar{c} = 0$

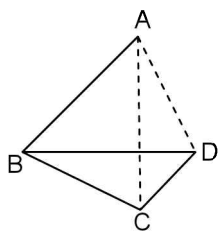
Since $\bar{b}$ and $\bar{c}$ are non-collinear (i.e., $\bar{b}$ and $\bar{c}$ are not parallel) the above equation implies

$\bar{a}\cdot\bar{c} - \frac{1}{2} = 0$; $\bar{a}.\bar{b} = 0$

Angle between $\bar{a}$ and $\bar{b}$ is $\frac{\pi}{2}$; Angle between $\bar{a}$ and $\bar{b}$ is $\frac{\pi}{3}$.

Connector 9: Let F_1, F_2, F_3, F_4 be the areas of the four faces of a tetrahedron. Let $\bar{n}_1,\bar{n}_2,\bar{n}_3,\bar{n}_4$ be the outward drawn normals to the respective faces with magnitudes equal to the corresponding areas. Prove that $\bar{n}_1 + \bar{n}_2 + \bar{n}_3 + \bar{n}_4 = 0$.

Solution: Let A, B, C, D be the vertices of the tetrahedron ABCD
Let $\overline{AB} = \bar{a}$, $\overline{AC} = \bar{b}$, $\overline{AD} = \bar{c}$



Then, $\overline{n_1} = \frac{1}{2}\left(\bar{a}\times\bar{b}\right)$, $\overline{n_2} = \frac{1}{2}\left(\bar{b}\times\bar{c}\right)$, $\overline{n_3} = \frac{1}{2}\left(\bar{c}\times\bar{a}\right)$

$$\overline{n_4} = \frac{1}{2}\left(\overline{CB}\times\overline{CD}\right) = \frac{1}{2}\left\{\left(\bar{a}-\bar{b}\right)\times\left(\bar{c}-\bar{b}\right)\right\}$$

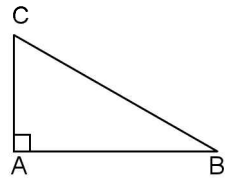
$$\overline{n_1} + \overline{n_2} + \overline{n_3} + \overline{n_4} = \frac{1}{2}\left\{\bar{a}\times\bar{b} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a} + \bar{a}\times\bar{c} - \bar{a}\times\bar{b} - \bar{b}\times\bar{c}\right\}$$

$$= 0$$

Connector 10: Let $\bar{a},\bar{b}$ be non-parallel vectors of which $\bar{a}$ is a unit vector. Find the angles of the triangle whose two sides are represented by the vectors $\sqrt{3}\left(\bar{a}\times\bar{b}\right)$ and $\bar{b} - \left(\bar{a}\cdot\bar{b}\right)\bar{a}$.

Solution: Let $\overline{AB} = \sqrt{3}\left(\bar{a} \times \bar{b}\right)$ and $\overline{AC} = \bar{b} - \left(\bar{a} \cdot \bar{b}\right)\bar{a}$

$\Rightarrow \overrightarrow{AB} \cdot \overrightarrow{AC} = 0$



$\Rightarrow$ angle between the two given sidesof the triangle is $\frac{\pi}{2}$. AB^2

$=3|\vec{a} \times \vec{b}|^2 = 3|\vec{a}|^2|\vec{b}|\sin^2\theta$

where θ is the angle between $\vec{a}$ and $\vec{b}$

$=3k^2\sin^2\theta$ where $k = |\vec{b}|$.

$$AC^2 = \left(\bar{b} - \left(\bar{a} \cdot \bar{b}\right)\bar{a}\right) \cdot \left(\bar{b} - \left(\bar{a} \cdot \bar{b}\right)\bar{a}\right)$$

$$= \left|\bar{b}\right|^2 + \left(\bar{a} \cdot \bar{b}\right)^2\left|\bar{a}\right|^2 - 2\left(\bar{a} \cdot \bar{b}\right)^2 = \left|\bar{b}\right|^2 - \left(\bar{a} \cdot \bar{b}\right)^2, \text{since}\left|\bar{a}\right| = 1$$

$$= \left|\bar{b}\right|^2 - \left|\bar{b}\right|^2\cos^2\theta = \left|\bar{b}\right|^2\sin^2\theta$$

$$= k^2\sin^2\theta$$

Since $\angle BAC = \frac{\pi}{2}$, $BC^2 = 4k^2\sin^2\theta \Rightarrow BC = 2k\sin\theta$

$$\sin B = \frac{AC}{BC} = \frac{k\sin\theta}{2k\sin\theta} = \frac{1}{2} \Rightarrow B = \frac{\pi}{6}$$

Therefore, $C = \pi - \left(\frac{\pi}{2} + \frac{\pi}{6}\right) = \frac{\pi}{3}$

Connector 11: Let $\bar{a} = 3\bar{i} + 4\bar{j}$ and $\bar{b} = 4\bar{j} + 3\bar{k}$. How many vectors $\vec{c}$ are there so that $\vec{a}, \vec{b}, \vec{c}$ are there so that when taken pair wise are equally inclined and that $|\vec{c}| = |\vec{a}| = |\vec{b}|$

Solution: Let $\bar{c} = x\bar{i} + y\bar{j} + z\bar{k}$

Given that $|\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OC} = \bar{c}| = |\bar{a}| = | \Rightarrow \bar{b} \cdot \bar{c} = \bar{a} \cdot \bar{c}|$

Since $|\bar{a} \cdot \bar{c} = \bar{a} \cdot \bar{b}| = |\bar{b}| = 5$, the above condition gives us

$x^2 + y^2 + z^2 = 25$ — (1)

Also, we have

$$\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|} = \frac{\bar{b}.\bar{c}}{|\bar{b}||\bar{c}|} = \frac{\bar{c}.\bar{a}}{|\bar{c}||\bar{a}|}$$

$\Rightarrow 16 = 4y + 3z = 3x + 4y$

$\Rightarrow x = z$ and $3x + 4y = 16$ — (2)

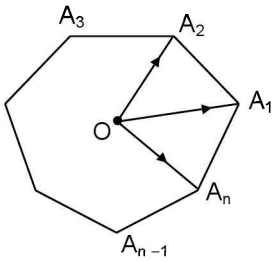
Using (2), (1) reduces to $2\left(\frac{16 - 4y}{3}\right)^2 + y^2 = 25$

$\Rightarrow 41y^2 - 256y + 287 = 0$

Discriminant of the above quadratic in y is $= 256^2 - 4 \times 41 \times 287 > 0$

$\Rightarrow$ there are 2 real y's and hence there are 2 vectors $\bar{c}$ satisfying the given conditions.

Connector 12: If $A_1, A_2, A_3, \ldots A_n$ are the vertices of a regular polygon with n sides and O is its centre, find $\sum_{i=1}^{n-1}\left(\overline{OA_i} \times \overline{OA_{i+1}}\right)$



Solution: $= \frac{1}{|\bar{c}|}\left\{\left(\bar{p}' - \bar{p}\right) \cdot \bar{c}\right\}$ lie in the same plane and $\left|\overline{OA_1}\right| = \left|\overline{OA_2}\right| = \left|\overline{OA_n}\right|$

$\Rightarrow \overline{OA_1} \times \overline{OA_2} = \overline{OA_2} \times \overline{OA_3} = \ldots\ldots\ldots = \overline{OA_{n-1}} \times \overline{OA_n}$

$\Rightarrow \sum_{i=1}^{n-1}\left(\overline{OA_i} \times \overline{OA_{i+1}}\right) = (n-1)\left(\overline{OA_1} \times \overline{OA_2}\right) = (1-n)\left(\overline{OA_2} \times \overline{OA_1}\right)$

Connector 13: Vector $\overline{OP}$, perpendicular to both the z-axis and the vector $8\bar{i} - 15\bar{j} + 3\bar{k}$ makes an acute angle with $\overline{OX}$. Find the coordinates of P if OP = 51

Solution: Let $\overline{OP} = \bar{m} = \lambda\bar{i} + \mu\bar{j} + t\bar{k}$

$\bar{m}$ is perpendicular to the z-axis $\Rightarrow t = 0$

$\bar{m}$ is perpendicular to the vector $\bar{a}$

$\Rightarrow 8\lambda - 15\mu = 0 \Rightarrow \lambda = \frac{15}{8}\mu \quad \bar{m}$

Since $\bar{m}$ makes an acute angle with the x–axis, $\mu > 0$

Again, $\bar{m} = 51 \Rightarrow \frac{15^2}{8^2}\mu^2 + \mu^2 = 51^2$

$\frac{289}{64}\mu^2 = 51^2 \Rightarrow \mu = 24$, since $\mu > 0$.

Therefore, $\bar{m} = 45\bar{i} + 24\bar{j}$

Coordinates of P are (45, 24, 0).

Connector 14: Let $\bar{A}, \bar{B}, \bar{C}$ be unit vectors. Suppose that $\bar{A} \cdot \bar{B} = \bar{A} \cdot \bar{C} = 0$ and that the angle between $\bar{B}$ and $\bar{C}$ is $\frac{\pi}{6}$. Show that $\bar{A} = \pm 2\left(\bar{B} \times \bar{C}\right)$.

Solution: Since $\bar{A}$ is perpendicular to both $\bar{B}$ and $\bar{C}$,

$\bar{A}$ is collinear with $\bar{B} \times \bar{C}$. Let $\bar{A} = \lambda\left(\bar{B} \times \bar{C}\right)$ where, λ is a scalar.

But, $\bar{B} \times \bar{C} = \left\{\left|\bar{B}\right|\left|\bar{C}\right|\sin\frac{\pi}{6}\right\}\bar{j}$ where, $\bar{j}$ is a unit vector normal to $\bar{B}$ and $\bar{C}$

$= \pm\frac{1}{2}\bar{A} \Rightarrow \bar{A} = \pm 2\left(\bar{B} \times \bar{C}\right)$

Connector 15: Given the unit vectors $\bar{a}, \bar{b}, \bar{c}$ satisfying the relation $\bar{a} + \bar{b} + \bar{c} = 0$, find the value of $\bar{a} \cdot \bar{b} + \bar{b} \cdot \bar{c} + \bar{c} \cdot \bar{a}$

Solution: $\left(\bar{a} + \bar{b} + \bar{c}\right)^2 = \bar{a}^2 + \bar{b}^2 + \bar{c}^2 + 2\left(\bar{a}.\bar{b} + \bar{b}.\bar{c} + \bar{c}\,\bar{a}\right)$

$0 = 3 + 2\left(\bar{a}.\bar{b} + \bar{b}.\bar{c} + \bar{c}.\bar{a}\right) \therefore \bar{a}.\bar{b} + \bar{b}.\bar{c} + \bar{c}.\bar{a} = -\frac{3}{2}$

Connector 16: Find the vector $\overline{X}$ perpendicular to the vectors $\overline{a} = 2\overline{i} + 3\overline{j} - \overline{k}$ and $\overline{b} = \overline{i} - 2\overline{j} + 3\overline{k}$ and satisfying the condition $\overline{X} \cdot \left(2\overline{i} - \overline{j} + \overline{k}\right) = -6$.

Solution: A vector perpendicular to both $\overline{a}$ and $\overline{b}$ can be assumed as $\lambda\left(\overline{a} \times \overline{b}\right)$ where λ is a scalar

$\overline{a} \times \overline{b} = 7\left(\overline{i} - \overline{j} - \overline{k}\right)$

Let $\overline{X} = 7\lambda\left(\overline{i} - \overline{j} - \overline{k}\right)$

$\overline{X} \cdot \left(2\overline{i} - \overline{j} + \overline{k}\right) = -6$ gives

$7\lambda(2 + 1 - 1) = -6 \Rightarrow \lambda = \dfrac{-3}{7}$

$\overline{X} = -3\left(\overline{i} - \overline{j} - \overline{k}\right) = 3\left(-\overline{i} + \overline{j} + \overline{k}\right)$

Connector 17: The vector $\overline{OP} = \overline{i} + 2\overline{j} + 2\overline{k}$ turns through a right angle, passing through the positive x–axis on the way. Find the vector in its new position.

Solution: Let the vector in the new position be $a\overline{i} + b\overline{j} + c\overline{k}$

We have

(i) $a^2 + b^2 + c^2 = 9$

(ii) $a > 0$

(iii) $\left(a\overline{i} + b\overline{j} + c\overline{k}\right) \cdot \left(\overline{i} + 2\overline{j} + 2\overline{k}\right) = 0$

(iv) $a\overline{i} + b\overline{j} + c\overline{k}, \overline{i}, \overline{i} + 2\overline{j} + 2\overline{k}$ are coplanar

(v) gives $b = c$; (iii) gives $a + 2b + 2c = 0$

$\Rightarrow \quad a = \dfrac{4}{\sqrt{2}}, b = c = \dfrac{-1}{\sqrt{2}}$

Connector 18: Find values of λ such that for non zero values of a, b, c,

$a\left(\overline{i} + \overline{j} + 3\overline{k}\right) + b\left(3\overline{i} - 3\overline{j} + \overline{k}\right) + c\left(-4\overline{i} + 5\overline{j}\right) = \lambda\left(a\overline{i} + b\overline{j} + c\overline{k}\right)$

Solution: We have $a + 3b - 4c = \lambda a$

$a - 3b + 5c = \lambda b$

$3a + b = \lambda c$

OR

The homogeneous system

$(1 - \lambda)a + 3b - 4c = 0$

$a - (\lambda + 3)b + 5c = 0$

$3a + b - \lambda c = 0$

should have non zero solution

This is possible iff $\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(3+\lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda(\lambda + 1)^2 = 0 \Rightarrow \lambda = 0, -1$

Connector 19: If two pairs of opposite edges of a tetrahedron are perpendicular, then prove that the third pair is also perpendicular. Also show that the sum of the squares on two opposite edges is the same for each pair for such a tetrahedron.

Solution: Let OABC represent the tetrahedron. Taking the origin of reference as O, Let $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OC} = \bar{c}$

Given AC is perpendicular to OB $\Rightarrow (\bar{c} - \bar{a}) \cdot \bar{b} = 0$

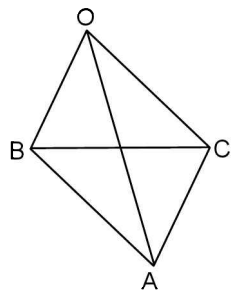
$\Rightarrow \bar{b} \cdot \bar{c} = \bar{a} \cdot \bar{b}$ —(1)

Again AB is perpendicular to OC $\Rightarrow (\bar{b} - \bar{a}) \cdot \bar{c} = 0$

$\Rightarrow \bar{b} \cdot \bar{c} = \bar{a} \cdot \bar{c}$ —(2)

From (1) and (2) $\bar{a} \cdot \bar{c} = \bar{a} \cdot \bar{b}$

$\Rightarrow$ OA is perpendicular to CB



Again $AB^2 + OC^2 = |\bar{b} - \bar{a}|^2 + |\bar{c}|^2 = |\bar{a}|^2 + |\bar{b}|^2 + |\bar{c}|^2 - 2\bar{a} \cdot \bar{b}$

$BC^2 + OA^2 = |\bar{a}|^2 + |\bar{b}|^2 + |\bar{c}|^2 - 2\bar{b} \cdot \bar{c}$

$CA^2 + OB^2 = |\bar{a}|^2 + |\bar{b}|^2 + |\bar{c}|^2 - 2\bar{c} \cdot \bar{a}$

Hence the result, since $\bar{b}.\bar{c} = \bar{c}.\bar{a} = \bar{a}.\bar{b}$

Connector 20: The points P, Q, R have coordinates (1, 1, 1), (1, –1, 2) and (0, 2, –1) respectively. Find a unit vector parallel to the plane determined by P, Q and R and perpendicular to the vector $\bar{i} + \bar{k}$.

Solution: Let $\bar{a} = \overline{PQ} \times \overline{PR}$ and $\bar{b} = \bar{i} + \bar{k}$. Let $\bar{e}$ be the required unit vector. Since $\bar{e}$ is parallel to the plane and perpendicular to $\bar{b}$, $\bar{e}$ is perpendicular to both $\bar{a}$ and $\bar{b}$ or, in other words,

$$\bar{e} = \pm \frac{1}{|\bar{a} \times \bar{b}|}(\bar{a} \times \bar{b})$$

$$\bar{a} = \overline{PQ} \times \overline{PR} = (-2\bar{j} + \bar{k}) \times (-\bar{i} + \bar{j} - 2\bar{k}) \quad = 3\bar{i} - \bar{j} - 2\bar{k}$$

$$\bar{a} \times \bar{b} = -\bar{i} - 5\bar{j} + \bar{k}$$

$$|\bar{a} \times \bar{b}| = \sqrt{27} = 3\sqrt{3}$$

$$\Rightarrow \bar{e} = \pm \frac{1}{3\sqrt{3}}(-\bar{i} - 5\bar{j} + \bar{k})$$

Connector 21: A particle is displaced from the point whose position vector is $5\bar{i} - 5\bar{j} - 7\bar{k}$ to the point whose position vector is $6\bar{i} + 2\bar{j} - 2\bar{k}$ under the action of constant forces $10\bar{i} - \bar{j} + 11\bar{k}, 4\bar{i} + 5\bar{j} - 6\bar{k}$ and $-2\bar{i} + \bar{j} - 9\bar{k}$. Find the total work done by the forces.

Solution: Resultant of the forces

$\bar{F} = 12\bar{i} + 5\bar{j} - 4\bar{k}$

Displacement vector $= \bar{d} = (6\bar{i} + 2\bar{j} - 2\bar{k}) - (5\bar{i} - 5\bar{j} - 7\bar{k}) = \bar{i} + 7\bar{j} + 5\bar{k}$

Work done by the forces $= \bar{F} \cdot \bar{d} = 12 + 35 - 20 = 27$ units

Connector 22: Find the moment of the force $5\bar{i}+10\bar{j}+16\bar{k}$ acting at the point $2\bar{i}-7\bar{j}+10\bar{k}$ about the point $-5\bar{i}+6\bar{j}-10\bar{k}$.

Solution: $\bar{r}$ = Position vector of the point of application of the force

− position vector of the point about which the moment is required

$$= \left(2\bar{i}-7\bar{j}+10\bar{k}\right)-\left(-5\bar{i}+6\bar{i}-10\bar{k}\right)$$

Moment of the force $= \bar{r}\times\bar{F}$

$$= \left(7\bar{i}-13\bar{j}+20\bar{k}\right)\times\left(5\bar{i}+10\bar{j}+16\bar{k}\right)$$

$$= -408\bar{i}-12\bar{j}+135\bar{k}$$

Connector 23: If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the volume of the parallelepiped formed by the vectors $\bar{A}=\bar{i}+a\bar{j}+a^2\bar{k}, \bar{B}=\bar{i}+b\bar{j}+b^2\bar{k}, \bar{C}=\bar{i}+c\bar{j}+c^2\bar{k}$ is 5, find abc.

Solution:

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0 \quad\Rightarrow\quad \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0 \quad\Rightarrow\quad (1+abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0 \qquad \text{— (1)}$$

Volume of the parallelepiped $= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 5$ (given)

From (1), $(1+abc)\times 5 = 0 \Rightarrow abc = -1$

Connector 24: If the vectors $x\bar{i}+\bar{j}+\bar{k}, \bar{i}+y\bar{j}+\bar{k}, \bar{i}+\bar{j}+z\bar{k}$ (where, x, y, z are not equal to 1) are coplanar, compute the value of $\dfrac{1}{(1-x)}+\dfrac{1}{(1-y)}+\dfrac{1}{(1-z)}$

Solution: Since the vectors are coplanar, their scalar triple product is zero $\begin{vmatrix} x & 1 & 1 \\ 1 & y & 1 \\ 1 & 1 & z \end{vmatrix} = 0$

$$\Rightarrow\quad x(yz-1)-1(z-1)+1(1-y)=0$$

$$\Rightarrow\quad xyz-(x+y+z)+2=0 \qquad \text{—(1)}$$

$$\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z} = \frac{(1-y)(1-z)+(1-z)(1-x)+(1-x)(1-y)}{(1-x)(1-y)(1-z)}$$

$$= \frac{3+(xy+yz+zx)-2(x+y+z)}{1-(x+y+z)+(xy+yz+zx)-xyz} = \frac{3+(xy+yz+zx)-2(x+y+z)}{1-(x+y+z)+(xy+yz+zx)+2-(x+y+z)} = 1$$

Connector 25: Show that the vectors $5\bar{a} + 6\bar{b} + 7\bar{c}$, $7\bar{a} - 8\bar{b} + 9\bar{c}$, $\bar{a} - 7\bar{b} + \bar{c}$ are coplanar even for non-coplanar $\bar{a}, \bar{b}, \bar{c}$

Solution: Let k_1, k_2, k_3 be scalar such that $k_1\left(5\bar{a} + 6\bar{b} + 7\bar{c}\right) + k_2\left(7\bar{a} - 8\bar{b} + 9\bar{c}\right) + k_3\left(\bar{a} - 7\bar{b} + \bar{c}\right) = 0$

$$\Rightarrow \left.\begin{aligned} 5k_1 + 7k_2 + k_3 &= 0 \\ 6k_1 - 8k_2 - 7k_3 &= 0 \\ 7k_1 + 9k_2 + k_3 &= 0 \end{aligned}\right\} \qquad \text{— (1)}$$

since the vectors $\bar{a}, \bar{b}, \bar{c}$, are non-coplanar.

Now, $\begin{vmatrix} 5 & 7 & 1 \\ 6 & -8 & -7 \\ 7 & 9 & 1 \end{vmatrix} = 5(-8 + 63) - 7(6 + 49) + 1(54 + 56)$

$= 275 - 385 + 110 = 0$

$\Rightarrow$ Homogeneous linear system (1) has non-trivial solutions, which means that there exists a linear combination of the given vectors, which reduces to the null vector.

$\Rightarrow$ given vectors are coplanar.

Aliter

$\bar{a} - 7\bar{b} + \bar{c} = \frac{1}{2}\left[\left(7\bar{a} - 8\bar{b} + 9\bar{c}\right) - \left(5\bar{a} + 6\bar{b} + 7\bar{c}\right)\right]$

Hence.

Connector 26: If $\bar{a}, \bar{b}, \bar{c}$ are coplanar vectors, prove that

$$\begin{vmatrix} \bar{a} & \bar{b} & \bar{c} \\ \bar{a}\cdot\bar{a} & \bar{a}\cdot\bar{b} & \bar{a}\cdot\bar{c} \\ \bar{b}\cdot\bar{a} & \bar{b}\cdot\bar{b} & \bar{b}\cdot\bar{c} \end{vmatrix} = 0$$

Solution: Since $\bar{a}, \bar{b}, \bar{c}$ are coplanar, these exists scalars k_1, k_2, k_3 not all zero such that $k_1\bar{a} + k_2\bar{b} + k_3\bar{c} = 0$ — (1)

Dot multiplying the vector equation (1) by $\bar{a}, \bar{b}$ we get two equations

$k_1\bar{a}\cdot\bar{a} + k_2\bar{a}\cdot\bar{b} + k_3\bar{a}\cdot\bar{c} = 0$ — (2)

$k_1\bar{b}\cdot\bar{a} + k_2\bar{b}\cdot\bar{b} + k_3\bar{b}\cdot\bar{c} = 0$ — (3)

From (1), (2) and (3)

$$\begin{vmatrix} \bar{a} & \bar{b} & \bar{c} \\ \bar{a}\cdot\bar{a} & \bar{a}\cdot\bar{b} & \bar{a}\cdot\bar{c} \\ \bar{b}\cdot\bar{a} & \bar{b}\cdot\bar{b} & \bar{b}\cdot\bar{c} \end{vmatrix}\cdot = 0$$

Connector 27: (a) Find the volume of a prism on a triangular base where, the three sides of the prism meeting on a vertex are given as $\overline{OA} = 3\bar{i} + 4\bar{j} + 12\bar{k}, \overline{OB} = 12\bar{i} + 3\bar{j} + 4\bar{k}$, $\overline{OC} = 4\bar{i} + 12\bar{j} + 3\bar{k}$

(b) Find a unit vector perpendicular to $\overline{OA}$ and $\overline{OB}$.

(c) Find the angle between $\overline{OA}$ and $\overline{OC}$

(d) Find the area of the triangle OAB.

Solution: (a) volume $= \frac{1}{2} \times$ [volume of the parallelepiped whose coterminous edges are OA, OB, OC]

$$= \frac{1}{2}\begin{vmatrix} 3 & 4 & 12 \\ 12 & 3 & 4 \\ 4 & 12 & 3 \end{vmatrix} = \frac{1387}{2}$$

(b) $\overline{OA} \times \overline{OB} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 4 & 12 \\ 12 & 3 & 4 \end{vmatrix} = -20\bar{i} + 132\bar{j} - 39\bar{k}$

A unit vector normal to $\overline{OA}$ and $\overline{OB}$ is given by

$$\frac{1}{\sqrt{20^2 + 132^2 + 39^2}}(-20\bar{i} + 132\bar{j} - 39\bar{k})$$

$$= \frac{1}{\sqrt{19345}}(-20\bar{i} + 132\bar{j} - 39\bar{k})$$

(c) $\overline{OA} \cdot \overline{OC} = 96$

i.e., $|\overline{OA}| \cdot |\overline{OC}| \cos\theta = 96$

$$\Rightarrow \cos\theta = \frac{96}{\sqrt{3^2 + 4^2 + 12^2}\sqrt{4^2 + 12^2 + 3^2}} = \frac{96}{169} \qquad \Rightarrow \theta = \cos^{-1}\left(\frac{96}{169}\right)$$

(d) Area of $\Delta OAB = \frac{1}{2}\left|\overline{OA} \times \overline{OB}\right| = \frac{\sqrt{19345}}{2}$

Connector 28: Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}, \bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}, \ \bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$ be three non-null vectors such that $\bar{c}$ is a unit vector normal to both the vectors $\bar{a}$ and $\bar{b}$. If the angle between $\bar{a}$ and $\bar{b}$ is $\frac{\pi}{6}$, show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \frac{1}{4}\left(a_1^2 + a_2^2 + a_3^3\right)\left(b_1^2 + b_2^2 + b_3^2\right)$$

Solution: Scalar triple product of $\bar{a}, \bar{b}, \bar{c}$ is given by $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ — (1)

$$\bar{c} = \frac{1}{\left|\bar{a} \times \bar{b}\right|}\left|\bar{a} \times \bar{b}\right|$$

$$= \frac{1}{\left|\bar{a}\right|\left|\bar{b}\right|\sin\frac{\pi}{6}}\left\{(a_2b_3 - a_3b_2)\bar{i} + (a_3b_1 - a_1b_3)\bar{j} + (a_1b_2 - a_2b_1)\bar{k}\right\}$$

$$= \frac{2}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\left\{(a_2b_3 - a_3b_2)\bar{i} + (a_3b_1 - a_1b_3)\bar{j} + (a_1b_2 - a_2b_1)\bar{k}\right\}$$

$$= c_1i + c_2j + c_3k \qquad \text{— (2)}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \bar{c}.\left(\bar{a}\times\bar{b}\right) = \bar{c}.\left[\sum(a_2b_3 - a_3b_2)i\right]$$

$$= c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)$$

$$\bar{c} = \frac{2}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\left\{\sum(a_2b_3 - a_3b_2)^2\right\} \text{ using (2)}$$

$$\Rightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = \frac{4}{\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)}\left\{\sum(a_2b_3 - a_3b_2)^2\right\}^2 \qquad \text{—(3)}$$

We have the result $\left(\bar{a}\times\bar{b}\right)^2 = \left|\bar{a}\right|^2\left|\bar{b}\right|^2 - \left(\bar{a}\cdot\bar{b}\right)^2$

$$= \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) - \left(\frac{\sqrt{3}}{2}\left|\bar{a}\right|\left|\bar{b}\right|\right)^2$$

$$= \frac{1}{4}\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)$$

$$\Rightarrow \sum(a_2b_3 - a_3b_2)^2 = \frac{1}{4}\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)$$

Squaring,

$$\left[\sum(a_2b_3 - a_3b_2)\right]^2 = \frac{1}{16}\left(a_1^2 + a_2^2 + a_3^2\right)^2\left(b_1^2 + b_2^2 + b_3^2\right)^2$$

Substituting in (3), and using (1),

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \frac{1`}{4}\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)$$

Aliter:

$$\frac{\bar{a}\times\bar{b}}{\left|\bar{a}\times\bar{b}\right|} = \pm\bar{c}$$

$$\therefore \quad \left(\bar{a}\times\bar{b}\right).\bar{c} = \pm\bar{c}^2\left|\bar{a}\times\bar{b}\right| = \pm\left|\bar{a}\times\bar{b}\right|$$

$$\Rightarrow \left[\bar{a} \quad \bar{b} \quad \bar{c}\right]^2 = \bar{a}^2\,\bar{b}^2.\sin^2\frac{\pi}{6} = \frac{\bar{a}^2\,\bar{b}^2}{4}$$

Hence.

Connector 29: If $\bar{a},\bar{b},\bar{c}$ are non-coplanar vectors and $\bar{u}$ and $\bar{v}$ are any two vectors, prove that

$$\bar{u}\times\bar{v} = \frac{1}{\bar{a}\,\bar{b}\,\bar{c}}\begin{vmatrix} \bar{u}\cdot\bar{a} & \bar{v}\cdot\bar{a} & \bar{a} \\ \bar{u}\cdot\bar{b} & \bar{v}\cdot\bar{b} & \bar{b} \\ \bar{u}\cdot\bar{c} & \bar{v}\cdot\bar{c} & \bar{c} \end{vmatrix}$$

Solution: Since $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar, we can express $\bar{u} \times \bar{v}$ in the form $k_1\bar{a} + k_2\bar{b} + k_3\bar{c}$ where, k_1, k_2, k_3 are scalars

i.e., $\bar{u} \times \bar{v} = k_1\bar{a} + k_2\bar{b} + k_3\bar{c}$ — (1)

$$\left(\bar{u} \times \bar{v}\right) \cdot \left(\bar{b} \times \bar{c}\right) = k_1\bar{a} \cdot \left(\bar{b} \times \bar{c}\right) + k_2\, b.\left(\bar{b} \times \bar{c}\right) + k_3\, \bar{c}.\left(\bar{b} \times \bar{c}\right)$$

$$= k_1\left[\bar{a}\,\bar{b}\,\bar{c}\right] + 0 + 0 = k_1\left[\bar{a}\,\bar{b}\,\bar{c}\right] \text{ (interchange of dot and cross operations)}$$

$$\Rightarrow \quad \bar{u} \cdot \left[\bar{v} \times \left(\bar{b} \times \bar{c}\right)\right] = k_1\left[\bar{a}\,\bar{b}\,\bar{c}\right]$$

$$\Rightarrow \quad \bar{u} \cdot \left\{\left(\bar{v} \cdot \bar{c}\right)\bar{b} - \left(\bar{v} \cdot \bar{b}\right)\bar{c}\right\} = k_1\left[\bar{a}\,\bar{b}\,\bar{c}\right] \Rightarrow \left(\bar{v} \cdot \bar{c}\right)\left(\bar{u} \cdot \bar{b}\right) - \left(\bar{u}.\bar{c}\right)\left(\bar{v} \cdot \bar{b}\right) = k_1\left[\bar{a}\,\bar{b}\,\bar{c}\right]$$

$$\Rightarrow \quad k_1 = \frac{1}{\left[\bar{a}\,\bar{b}\,\bar{c}\right]}\begin{vmatrix} \bar{u} \cdot \bar{b} & \bar{v} \cdot \bar{b} \\ \bar{u} \cdot \bar{c} & \bar{v} \cdot \bar{c} \end{vmatrix}$$

Similarly, taking dot product of (1) with $\left(\bar{c} \times \bar{a}\right)$ and $\left(\bar{a} \times \bar{b}\right)$ respectively we get

$$k_2 = \frac{1}{\left[\bar{b}\,\bar{c}\,\bar{a}\right]}\begin{vmatrix} \bar{u} \cdot \bar{c} & \bar{v} \cdot \bar{c} \\ \bar{u} \cdot \bar{a} & \bar{v} \cdot \bar{a} \end{vmatrix}$$

$$k_3 = \frac{1}{\left[\bar{c}\,\bar{a}\,\bar{b}\right]}\begin{vmatrix} \bar{u} \cdot \bar{a} & \bar{v} \cdot \bar{a} \\ \bar{u} \cdot \bar{b} & \bar{v} \cdot \bar{b} \end{vmatrix}$$

Since $\left[\bar{a}\,\bar{b}\,\bar{c}\right] = \left[\bar{b}\,\bar{c}\,\bar{a}\right] = \left[\bar{c}\,\bar{a}\,\bar{b}\right]$,

$$\bar{u} \times \bar{v} = \frac{1}{\left[\bar{a}\,\bar{b}\,\bar{c}\right]}\left\{\begin{vmatrix} \bar{u} \cdot \bar{b} & \bar{v} \cdot \bar{b} \\ \bar{u} \cdot \bar{c} & \bar{v} \cdot \bar{c} \end{vmatrix}\bar{a} + \begin{vmatrix} \bar{u} \cdot \bar{c} & \bar{v} \cdot \bar{c} \\ \bar{u} \cdot \bar{a} & \bar{v} \cdot \bar{a} \end{vmatrix}\bar{b} + \begin{vmatrix} \bar{u} \cdot \bar{a} & \bar{v} \cdot \bar{a} \\ \bar{u} \cdot \bar{b} & \bar{v} \cdot \bar{b} \end{vmatrix} + \bar{c}\right\}$$

$$= \frac{1}{\left[\bar{a}\,\bar{b}\,\bar{c}\right]}\begin{vmatrix} \bar{u} \cdot \bar{a} & \bar{v} \cdot \bar{a} & \bar{a} \\ \bar{u} \cdot \bar{b} & \bar{v} \cdot \bar{b} & \bar{b} \\ \bar{u} \cdot \bar{c} & \bar{v} \cdot \bar{c} & \bar{c} \end{vmatrix}$$

Connector 30: If $\bar{a}, \bar{b}, \bar{c}$ are non-null vectors, no two of which are collinear or perpendicular and $\bar{a} \times \left(\bar{b} \times \bar{c}\right) = \left(\bar{a} \times \bar{b}\right) \times \bar{c}$, then prove that

(i) $\left(\bar{c} \times \bar{a}\right) \times \bar{b} = 0$

(ii) $\left(\bar{c} \times \bar{a}\right)$ and $\bar{b}$ are collinear.

Solution: We have $\bar{a} \times \left(\bar{b} \times \bar{c}\right) + \bar{b} \times \left(\bar{c} \times \bar{a}\right) + \bar{c} \times \left(\bar{a} \times \bar{b}\right) = 0$

$$\Rightarrow \quad \bar{a} \times \left(\bar{b} \times \bar{c}\right) - \left(\bar{c} \times \bar{a}\right) \times \bar{b} - \left(\bar{a} \times \bar{b}\right) \times \bar{c} = 0$$

$$\Rightarrow \quad -\left(\bar{c} \times \bar{a}\right) \times \bar{b} = 0, \text{ since } \bar{a} \times \left(\bar{b} \times \bar{c}\right) = \left(\bar{a} \times \bar{b}\right) \times \bar{c}$$

$$\Rightarrow \quad \left(\bar{c} \times \bar{a}\right) \times \bar{b} = 0, \text{ which means } \left(\bar{c} \times \bar{a}\right) \text{ and } \bar{b} \text{ are collinear}$$

Connector 31: A non zero-vector $\bar{v}$ is parallel to the line of intersection of planes P_1 and P_2 where, P_1 is the plane determined by the vectors $\bar{i}+\bar{j}+\bar{k}$ and $\bar{i}-\bar{j}-\bar{k}$ while P_2 is the plane determined by the vectors $\bar{i}-\bar{j}$ and $\bar{i}+\bar{j}+2\bar{k}$. Find the angle of the vectors $\bar{v}$ and the vector $3\bar{i}-5\bar{j}+\bar{k}$.

Solution: Let $\bar{v} = x\bar{i}+y\bar{j}+z\bar{k}$

We have, $\left\{\bar{v}, \bar{i}+\bar{j}+\bar{k}, \bar{i}-\bar{j}-\bar{k}\right\}$ are coplanar

$$\Rightarrow \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 0 \text{ i.e., } \begin{vmatrix} x & y & z \\ 2 & 0 & 0 \\ 1 & -1 & -1 \end{vmatrix} = 0 \Rightarrow y = z$$

Again, $\left\{\bar{v}, \bar{i}-\bar{j}, \bar{i}+\bar{j}+2\bar{k}\right\}$ are coplanar

$$\Rightarrow \begin{vmatrix} x & y & z \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 0 \text{ i.e., } -2x-2y+2z=0 \quad \Rightarrow x = 0, \text{ since } y = z$$

$\bar{v} = \lambda(\bar{j}+\bar{k})$ where, λ is a scalar $\neq 0$

If θ is the angle between $\bar{v}$ and $3\bar{i}-5\bar{j}+\bar{k}$,

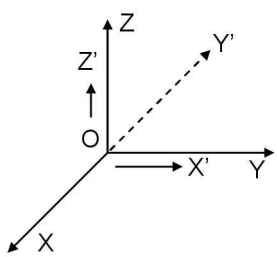
$$\cos\theta = \frac{\bar{v}.(3\bar{i}-5\bar{j}+\bar{k})}{|\lambda|\sqrt{2}\times\sqrt{35}} = \frac{-4\lambda}{|\lambda|\sqrt{70}} = \pm\frac{4}{\sqrt{70}}$$

Connector 32: If $\bar{a}, \bar{b}, \bar{c}$ are any three vectors then, find the scalar triple product of the vectors $\bar{a}-\bar{b}, \bar{c}-\bar{b}, \bar{c}+\bar{a}$.

Solution: Scalar triple product $= \left(\bar{a}-\bar{b}\right)\cdot\left\{\left(\bar{c}-\bar{b}\right)\times\left(\bar{c}+\bar{a}\right)\right\} = \left(\bar{a}-\bar{b}\right)\cdot\left\{\bar{c}\times\bar{a}-\bar{b}\times\bar{c}-\bar{b}\times\bar{a}\right\}$

$$= \bar{a}\cdot\bar{c}\times\bar{a}-\bar{a}\cdot\bar{b}\times\bar{c}-\bar{a}\cdot\bar{b}\times\bar{a}-\bar{b}\cdot\bar{c}\times\bar{a}+\bar{b}\cdot\bar{b}\times\bar{c}+\bar{b}\cdot\bar{b}\times\bar{a} = -2\left[\bar{a}\,\bar{b}\,\bar{c}\right]$$

Connector 33:. A vector $\bar{A}$ has components x, y, z in a right handed rectangular Cartesian coordinate system OXYZ. The xy-plane is rotated about the z-axis through an angle $\frac{\pi}{2}$, so that the new position of OX is aong the old OY. Find the components of $\bar{A}$ in the new coordinate system in terms of x, y, z.



Solution: New X, Y and Z axes are OX^1, OY^1, OZ^1 respectively and they are along OY, along the negative direction of the X axis and along the Z axis respectively. Therefore, the components of $\bar{A}$ in the new system are y, –x and z respectively.

Connector 34: If A, B, C, D are any four points in space prove that $\left|\overline{AB}\times\overline{CD}+\overline{BC}\times\overline{AD}+\overline{CA}\times\overline{BD}\right|$ equals 4 times the area of triangle ABC.

Solution: Let the position vectors of A, B, C, D referred to O as origin be $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ respectively. Then,

$$\overline{AB}\times\overline{CD}+\overline{BC}\times\overline{AD}+\overline{CA}\times\overline{BD}$$
$$= \left(\bar{b}-\bar{a}\right)\times\left(\bar{d}-\bar{c}\right)+\left(\bar{c}-\bar{b}\right)\times\left(\bar{d}-\bar{a}\right)+\left(\bar{a}-\bar{c}\right)\times\left(\bar{d}-\bar{b}\right)$$
$$= \left(\bar{b}\times\bar{d}-\bar{b}\times\bar{c}-\bar{a}\times\bar{d}+\bar{a}\times\bar{c}\right)+\left(\bar{c}\times\bar{d}-\bar{c}\times\bar{a}-\bar{b}\times\bar{d}+\bar{b}\times\bar{a}\right)+\left(\bar{a}\times\bar{d}-\bar{a}\times\bar{b}-\bar{c}\times\bar{d}+\bar{c}\times\bar{b}\right)$$
$$= -2\left(\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}\right)$$

$$\left|\overline{AB}\times\overline{CD}+\overline{BC}\times\overline{AD}+\overline{CA}\times\overline{BD}\right| = 2\left|\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}\right| \qquad —(1)$$

Area of $\Delta ABC = \frac{1}{2}\left|\overline{AB}\times\overline{AC}\right| = \frac{1}{2}\left|\left(\bar{b}-\bar{a}\right)\times\left(\bar{c}-\bar{a}\right)\right| = \frac{1}{2}\left|\bar{b}\times\bar{c}+\bar{a}\times\bar{b}+\bar{c}\times\bar{a}\right|$

From (1) and (2) result follows.

Connector 35: If $\bar{a}=x\bar{i}+12\bar{j}-\bar{k}, \bar{b}=2\bar{i}+2x\bar{j}+\bar{k}$, $\bar{c}=\bar{i}+\bar{k}$ prove that the ordered set $\left[\bar{b},\bar{c},\bar{a}\right]$ is left handed if and only if $-3 < x < 2$. Discuss the case when $c = -3$ or 2.

Solution: $\left[\bar{b},\bar{c},\bar{a}\right]$ will be left-handed if the scalar triple product of $\bar{b},\bar{c},\bar{a}$ in that order is negative.

$$\left[\bar{b}\bar{c}\bar{a}\right]=\left[\bar{a}\bar{b}\bar{c}\right]=\begin{vmatrix} x & 12 & -1 \\ 2 & 2x & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2x^2+2x-12$$

$2x^2+2x-12$ is negative if x lies between -3 and 2. If $x = -3$ or 2, the scalar triple product vanishes which means that the vectors are coplanar.

Connector 36:. Show that the 4 points represented by the position vectors $\bar{p},\bar{q},\bar{r},\bar{s}$ are coplanar if $\left[\bar{q}\bar{r}\bar{s}\right]+\left[\bar{r}\bar{p}\bar{s}\right]+\left[\bar{p}\bar{q}\bar{s}\right]=\left[\bar{p}\bar{q}\bar{r}\right]$

Solution: If the four points are denoted by A, B, C, D

$\overline{AB}=\bar{q}-\bar{p}, \overline{AC}=\bar{r}-\bar{p}, \overline{AD}=\bar{s}-\bar{p}.$

Since the points are coplanar,

$\overline{AB}\cdot\left\{\overline{AC}\times\overline{AD}\right\}=0$

$$\Rightarrow \left(\bar{q}-\bar{p}\right)\cdot\left\{\left(\bar{r}-\bar{p}\right),\times\left(\bar{s}-\bar{p}\right)\right\}. = 0$$

$$\Rightarrow \left(\bar{q}-\bar{p}\right)\cdot\left\{\bar{r}\times\bar{s}-\bar{r}\times\bar{p}-\bar{p}\times\bar{s}\right\}. = 0$$

$$\Rightarrow \left[\bar{q}\bar{r}\bar{s}\right]-\left[\bar{q}\bar{r}\bar{p}\right]-\left[\bar{q}\bar{p}\bar{s}\right]-\left[\bar{p}\bar{r}\bar{s}\right]+\left[\bar{p}\bar{r}\bar{p}\right]+\left[\bar{p}\bar{p}\bar{s}\right] = 0$$

$$\Rightarrow \left[\bar{q}\bar{r}\bar{s}\right]+\left[\bar{r}\bar{p}\bar{s}\right]+\left[\bar{p}\bar{q}\bar{s}\right]=\left[\bar{q}\bar{r}\bar{p}\right]=\left[\bar{p}\bar{q}\bar{r}\right]$$

Connector 37: The points A, B, C have coordinates (1, 0, 4); (0, −11, 3); (2, −3, 1). D is the foot of the perpendicular from A to BC. Find the coordinates of D.

Solution: Direction ratios of BC are [2, 8, −2] or [1, 4, −1]. Therefore, equations of BC are

$$\frac{x}{1}=\frac{y+11}{4}=\frac{z-3}{-1}=r$$

Any point on BC can be represented as $(r, 4r-11, -r+3)$

If it is D, direction ratios of AD are $[r-1, 4r-11, -r-1]$

AD is perpendicular to BC

$\Rightarrow (r-1)+(4r-11)4-(-r-1)=0$

Giving $r=\frac{22}{9} \Rightarrow \text{Dis}\left(\frac{22}{9},\frac{-11}{9},\frac{5}{9}\right)$

Connector 38: Show that the condition for the lines $x = az + b$, $y = cz + d$ and $x = a_1z + b_1$, $y = c_1z + d_1$ to be perpendicular is $aa_1 + cc_1 + 1 = 0$

Solution: $x = az + b, y = cz + d$

$$\Rightarrow \frac{x-b}{a} = \frac{y-d}{c} = \frac{z}{1} \quad \text{—(1)}$$

$x = a_1z + b_1, y = c_1z + d_1$

$$\Rightarrow \frac{x-b_1}{a_1} = \frac{y-d_1}{c_1} = \frac{z}{1} \quad \text{—(2)}$$

The lines (1) and (2) will be perpendicular if $aa_1 + cc_1 + 1 = 0$

Connector 39: Find the equation of the plane containing the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and passing through the point $(0, 7, -7)$.

Solution: Any plane passing through the point $(0, 7, -7)$ may be assumed as

$$A(x-0) + B(y-7) + C(z+7) = 0 \quad \text{—(1)}$$

Since the given line lies in the plane,

$$-3A + 2B + C = 0 \quad \text{—(2)}$$

and $A(-1) + B(3-7) + C(-2+7) = 0$

$$\Rightarrow \quad -A - 4B + 5C = 0 \quad \text{—(3)}$$

From (2) and (3), $\frac{A}{14} = \frac{B}{14} = \frac{C}{14}$

The plane is $x + y - 7 + z + 7 = 0$

$\Rightarrow x + y + z = 0$

Connector 40: Find the equation of the plane parallel to the line $\frac{x-2}{1} = \frac{y-1}{3} = \frac{z-3}{2}$ and passing through the point $(5, 2, -1)$ and the origin.

Solution: Since the required plane passes through the origin, its equation may be assumed as

$$Ax + By + Cz = 0$$

Also, it passes through $(5, 2, -1)$.

$$\Rightarrow \quad 5A + 2B - C = 0 \quad \text{—(1)}$$

Since the plane is parallel to the line with direction ratios $(1, 3, 2)$

$$A + 3B + 2C = 0 \quad \text{—(2)}$$

(we use the fact that the normal to the plane is perpendicular to the line)

From (1) and (2), $\frac{A}{7} = \frac{B}{-11} = \frac{C}{13}$

Plane is, $7x - 11y + 13z = 0$

Connector 41: Find the equation of the plane that passes through the points $A(0, 1, -1)$ and $B(2, -1, 1)$ and is parallel to the line of intersection of the planes $x + 3y - 2z = 6$ and $x - 4y + 3z = 0$

Solution:

1	3	−2
1	−4	3

Direction ratios of the line are $[9 - 8, -2 - 3, -4 - 3] = [1, -5, -7]$

Let the plane be $A(x-0) + B(y-1) + C(z+1) = 0$ —(1)

(1) passes through $(2, -1, 1)$

$\Rightarrow 2A - 2B + 2C = 0$

$\Rightarrow A - B + C = 0$ —(2)

(1) is parallel to the given line with direction ratios $(1, -5, -7)$

$\Rightarrow A - 5B - 7C = 0$ —(3)

From (2) and (3), $\frac{A}{3} = \frac{B}{2} = \frac{C}{-1}$

Required plane is $3x + 2(y - 1) - (z + 1) = 0$

$\Rightarrow 3x + 2y - z - 3 = 0$

Connector 42: A variable plane at a distance of 2 units from the origin cuts the coordinate axes at the points A, B, C. If the centroid G (x_0, y_0, z_0) of Δ ABC satisfies the relation $\frac{1}{x_0^2} + \frac{1}{y_0^2} + \frac{1}{z_0^2} = 2k$, find the value of k.

Solution: Let the plane be given by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ — (1)

Obviously A, B, C are (a, 0, 0), (0, b, 0) and (0, 0, c)

As $G(x_0, y_0, z_0)$ is the centroid,

$$x_0 = \frac{a}{3}; y_0 = \frac{b}{3}; z_0 = \frac{c}{3}$$

$$\therefore \frac{9}{a^2} + \frac{9}{b^2} + \frac{9}{c^2} = 2k \quad \text{— (2)}$$

Further, 2 = length of the perpendicular from (0, 0, 0) on (1)

$$= \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

(i.e.) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{4}$

(2) $\Rightarrow \frac{9}{4} = 2k$ or $k = \frac{9}{8}$

Connector 43: Find the locus of a point, the sum of the squares of whose distances from the planes $x + y + z = 0$, $3x - y + 2z = 0$, $x + 2y + 7z = 0$ is 16.

Solution: Let $P(x_1, y_1, z_1)$ represent any point on the locus.

Then, we have

$$\left(\frac{x_1 + y_1 + z_1}{\sqrt{3}}\right)^2 + \left(\frac{3x_1 - y_1 + 2z_1}{\sqrt{14}}\right)^2 + \left(\frac{x_1 + 2y_1 + 7z_1}{\sqrt{54}}\right)^2 = 16$$

$$\frac{x_1^2 + y_1^2 + z_1^2 + 2x_1y_1 + 2y_1z_1 + 2z_1x_1}{3} + \frac{9x_1^2 + y_1^2 + 4z_1^2 - 6x_1y_1 - 4y_1z_1 + 12z_1x_1}{14}$$

$$+ \frac{x_1^2 + 4y_1^2 + 49z_1^2 + 4x_1y_1 + 28y_1z_1 + 14z_1x_1}{54} = 16$$

$126(x_1^2 + y_1^2 + z_1^2 + 2x_1y_1 + 2y_1z_1 + 2z_1x_1)$

$+ 27(9x_1^2 + y_1^2 + 4z_1^2 - 6x_1y_1 - 4y_1z_1 + 12z_1x_1)$

$+ 7(x_1^2 + 4y_1^2 + 49z_1^2 + 4x_1y_1 + 28y_1z_1 + 14z_1x_1) = 378 \times 16$

$\Rightarrow 376x_1^2 + 181y_1^2 + 577z_1^2 + 118x_1y_1 + 340y_1z_1 + 674z_1x_1 = 6048$

Locus of P is the surface

$376x^2 + 181y^2 + 577z^2 + 118xy + 340yz + 674zx - 6048 = 0$

Connector 44: Find the equations of the line perpendicular to both the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$. Also determine the length of the common perpendicular between these two lines.

Solution: Let $[\ell, m, n]$ represent the direction ratios of the line perpendicular to both the given lines. Then,

$3\ell - 16m + 7n = 0$

$3\ell + 8m - 5n = 0$

$\Rightarrow \frac{\ell}{2} = \frac{m}{3} = \frac{n}{6}$

Any point on the first line is $(3r_1 + 8, -16r_1 - 9, 7r_1 + 10)$

Any point on the second line is $(3r_2 + 15, 8r_2 + 29, -5r_2 + 5)$

The line joining the above two points is perpendicular to both the given lines.

$[3r_1 + 8 - 3r_2 - 15, -16r_1 - 9 - 8r_2 - 29, 7r_1 + 10 + 5r_2 - 5]$

or $[3r_1 - 3r_2 - 7, -16r_1 - 8r_2 - 38, 7r_1 + 5r_2 + 5]$ represent its direction ratios.

We have, $3(3r_1 - 3r_2 - 7) - 16(-16r_1 - 8r_2 - 38) + 7(7r_1 + 5r_2 + 5) = 0$

and $3(3r_1 - 3r_2 - 7) + 8(-16r_1 - 8r_2 - 38) - 5(7r_1 + 5r_2 + 5) = 0$

$\Rightarrow$ $314r_1 + 154r_2 + 622 = 0$ $\Rightarrow 157r_1 + 77r_2 + 311 = 0$

$-154r_1 - 98r_2 - 350 = 0$ $\Rightarrow 77r_1 + 49r_2 + 175 = 0$

Solving the above two equations

$r_1 = \frac{1764}{1764} = -1 \quad r_2 = \frac{3528}{1764} = -2$

The two points are (5, 7, 3) and (9, 13, 15)

Equations of the line perpendicular to both the given lines are $\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$

Shortest distance between the lines or the perpendicular between the given lines

$= \sqrt{(9-5)^2 + (13-7)^2 + (15-3)^2} = 14$

Connector 45: Assuming the line $\frac{x}{4} = \frac{y}{-3} = \frac{z}{7}$ as vertical, find the equations of the line of greatest slope in the plane $2x + y - 5z = 12$ and passing through the point $(2, 3, -1)$.

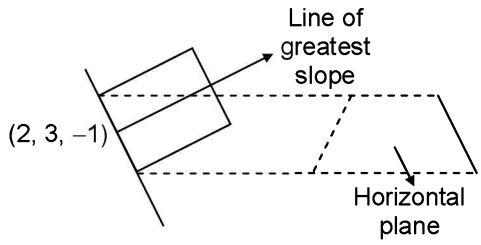
Solution: Since $\frac{x}{4} = \frac{y}{-3} = \frac{z}{7}$ is vertical, the plane $4x - 3y + 7z = 0$ is the horizontal plane.

$$\left.\begin{aligned} 4x - 3y + 7z &= 0 \\ 2x + y - 5z &= 12 \end{aligned}\right\} \qquad \text{— (1)}$$

The line represented by (1) has direction ratios [4, 17, 5] where

$4\ell - 3m + 7n = 0$

$2\ell + m - 5n = 0$



Let $[\ell, m, n]$ be the direction ratios of the line of greatest slope. The line of greatest slope through (2, 3, –1) is perpendicular to line (1). Therefore,

$4\ell + 17m + 5n = 0;$

and $2\ell + m - 5n = 0$ as it should be in $2x + y - 5z = 12$

giving $\frac{\ell}{3} = \frac{m}{-1} = \frac{n}{1}$.

Equations of the line of greatest slope through $(2, 3, -1)$

$$\frac{x-2}{3} = \frac{y-3}{-1} = \frac{z+1}{1}$$

Connector 46: Find the equation of the plane which passes through the points $(0, 17, 5)$, $(4, -4, 5)$ and $(5, 7, 10)$. If the plane $z = 0$ is horizontal, find the equations of the line of greatest slope in the first plane and meeting the z-axis.

Solution: Any plane passing through the point $(0, 17, 5)$ may be assumed as

$A(x-0) + B(y-17) + C(z-5) = 0$ —(1)

(1) passes through the point $(4, -4, 5)$.

$\Rightarrow 4A - 21B = 0$ —(2)

(1) passes through the point $(5, 7, 10)$.

$\Rightarrow 5A - 10B + 5C = 0$

$\Rightarrow A - 2B + C = 0$ —(3)

From (2) and (3), $\frac{A}{21} = \frac{B}{4} = \frac{C}{-13}$

Substituting in (1)

Plane is $21x + 4y - 13z - 3 = 0$

The above plane meets the z–axis at $\left(0,0,\frac{-3}{13}\right)$

$$\left.\begin{aligned} 21x + 4y - 13z - 3 &= 0 \\ z &= 0 \end{aligned}\right\} \quad \text{— (4)}$$

Line represented by (4) has direction ratios $[4, -21, 0]$.

Let the line of greatest slope through $\left(0,0,\frac{-3}{13}\right)$ be $\frac{x}{\ell} = \frac{y}{m} = \frac{z+\frac{3}{13}}{n}$ —(5)

Then, $4\ell - 21m = 0$ —(6)

And $21\ell + 4m - 13n = 0$ — (7)

(6) and (7) give $\frac{\ell}{21 \times 13} = \frac{m}{13 \times 4} = \frac{n}{457}$

Substituting in (5) equations of the line of greatest slope are

$$\frac{x}{21 \times 13} = \frac{y}{13 \times 4} = \frac{z + \frac{3}{13}}{457} \qquad \Rightarrow \frac{x}{21} = \frac{y}{4} = \frac{13z+3}{457}$$

Connector 47: Find the equation of the plane through the line $\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2}$, parallel to the line $\frac{x+1}{3} = \frac{y-1}{-4} = \frac{z+2}{1}$ Hence find the shortest distance between the lines.

Solution: Any plane through the line $\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2}$ may be assumed as

$A(x-1) + B(y-4) + C(z-4) = 0$ — (1)

We have $3A + 2B - 2C = 0$

Also $3A - 4B + C = 0$, as it is parallel to the second line.

Giving $\frac{A}{-6} = \frac{B}{-6-3} = \frac{C}{-18} \Rightarrow \frac{A}{2} = \frac{B}{3} = \frac{C}{6}$

Required plane is $2(x-1) + 3(y-4) + 6(z-4) = 0$

$\Rightarrow 2x + 3y + 6z - 38 = 0$ — (2)

Shortest distance between the lines

= perpendicular distance of the point $(-1, 1, -2)$ which lies in the parallel line from the plane (2)

$$= \left|\frac{-2+3-12-38}{\sqrt{4+9+36}}\right| = \left|\frac{-49}{7}\right| = 7$$

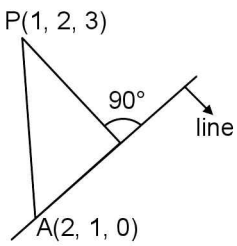
Connector 48: A line is drawn perpendicular to the line $\frac{x-2}{1} = \frac{y-1}{2} = \frac{z}{3}$ from (1, 2, 3); obtain the equations of the line, the length of the perpendicular and the foot of the perpendicular.

Solution: M is the foot of the perpendicular from (1, 2, 3). Let the direction ratios of PM be $[\ell, m, n]$ equations of PM are

$$\frac{x-1}{\ell} = \frac{y-2}{m} = \frac{z-3}{n} = R$$

$(\ell R + 1, mR + 2, nR + 3)$ is point on the line. If it is M,

$\ell + 2m + 3n = 0$ — (1)



$$\frac{\ell R + 1 - 2}{1} = \frac{mR + 2 - 1}{2} = \frac{nR + 3}{3}$$

$2\ell R + 2 - 4 = mR + 1$

$R(2\ell - m) = 3$ — (2)

$3mR + 3 = 2nR + 6$

$R(3m - 2n) = 3$ — (3)

$$\frac{(2)}{(3)} \Rightarrow \frac{2\ell - m}{3m - 2n} = 1$$

$2\ell - m = 3m - 2n$

$2\ell - 4m + 2n = 0$

$\ell - 2m + n = 0$ — (4)

From (1) and (4)

$$\frac{\ell}{4} = \frac{m}{1} = \frac{n}{-2}$$

Equations of PM are $\frac{x-1}{4} = \frac{y-2}{1} = \frac{z-3}{-2}$

$(4R + 1, R + 2, -2R + 3)$ must satisfy

$$\frac{4R+1-2}{1} = \frac{R+2-1}{2} = \frac{-2R+3}{3} \Rightarrow R = \frac{3}{7}$$

Foot of perpendicular M is $\left(\frac{19}{7}, \frac{17}{7}, \frac{15}{7}\right)$

$$PM^2 = \left(\frac{19}{7} - 1\right)^2 + \left(\frac{17}{7} - 2\right)^2 + \left(\frac{15}{7} - 3\right)^2 = \frac{189}{49} \quad \Rightarrow PM = \frac{3\sqrt{21}}{7}$$

Connector 49: Show that the three planes $4x + 3y + 2z + 7 = 0$, $2x + y - 4z + 1 = 0$ and $x - 7z - 2 = 0$ have a common line of intersection.

Solution: Any plane passing through the line of intersection of the planes

$2x + y - 4z + 1 = 0$

$x - 7z - 2 = 0$

may be represented as $2x + y - 4z + 1 + k(x - 7z - 2) = 0$

If the three planes pass through the same line, there exists a unique value of k such that

$$\frac{2 + k}{4} = \frac{1}{3} = \frac{-4 - 7k}{2}$$

$\Rightarrow k = \frac{-2}{3}$ satisfies the above relations.

Result follows.

Connector 50: Show that the planes $x - z - 1 = 0$, $x + y - 2z - 3 = 0$, $x - 2y + z - 3 = 0$ form a triangular prism.

Solution: $x - z - 1 = 0$

$x + y - 2z - 3 = 0$

Let $[\ell, m, n]$ be the direction ratios of the line represented by the first two equations.

Then, $\frac{\ell}{1} = \frac{m}{1} = \frac{n}{1}$

Clearly, (1, 2, 0) is point on the line. Line represented by the first two equations is

$$\frac{x - 1}{1} = \frac{y - 2}{1} = \frac{z}{1} \qquad \text{—(1)}$$

$1 \times 1 - 2 \times 1 + 1 \times 1 = 0$

Also, (1, 2, 0) does not satisfy the equation $x - 2y + z - 3 = 0$ which means that line of intersection of the first two planes is parallel to the third plane.

$\Rightarrow$ The three planes form a triangular prism.

Connector 51: The plane $x - y - z = 2$ is rotated through an angle $\frac{\pi}{2}$ about its line of intersection with the plane $x + 2y + z = 2$. Find its equation in the new position.

Solution: $x - y - z = 2$ —(1)

$x + 2y + z = 2$ —(2)

Any plane through the line of intersection of (1) and (2) may be represented by

$x - y - z - 2 + \lambda(x + 2y + z - 2) = 0$

$\Rightarrow (1 + \lambda)x + (2\lambda - 1)y + (\lambda - 1)z - (2\lambda + 2) = 0$ —(3)

Direction ratios of a normal to (3) are $[1 + \lambda, 2\lambda - 1, \lambda - 1]$

Direction ratios of a normal to (1) are $[1, -1, -1]$

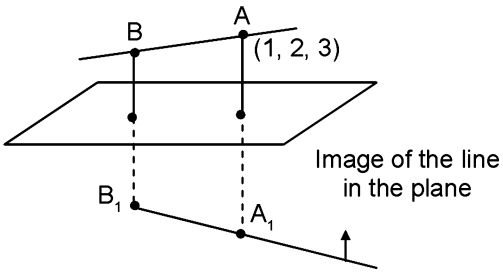
As (1) and (3) are perpendicular,

$1 + \lambda - 2\lambda + 1 - \lambda + 1 = 0 \Rightarrow \lambda = \frac{3}{2}$

Substituting in (3), the required plane is $5x + 4y + z - 10 = 0$

Connector 52: Find the image of the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{4}$ in the plane $2x + y + z = 6$.

Solution: Any point on the line may be represented as $(2r + 1, r + 2, 4r + 3)$



Taking $r = 1$, we get a point on the line as $(3, 3, 7)$

$A(1, 2, 3)$ and $B(3, 3, 7)$ are two points on the line. If A_1 and B_1 are the images of A and B in the plane, line A_1B_1 is the image of the line in the plane.

Direction ratios of a normal to the plane are $[2, 1, 1]$

Line through A normal to the plane has equations

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{1} = R \text{ (say)}$$

If $(2R + 1, R + 2, R + 3)$ lies on the given plane, $2(2R + 1) + R + 2 + R + 3 = 6$

$$6R + 1 = 0 \Rightarrow R = \frac{-1}{6}$$

Therefore, R corresponding to A_1 is $\frac{-2}{6}$ or $\frac{-1}{3}$. Hence the image of $(1, 2, 3)$ in the plane is the point

$$\left(\frac{-2}{3} + 1, \frac{-1}{3} + 2, \frac{-1}{3} + 3\right) \text{ i.e. } A_1\left(\frac{1}{3}, \frac{5}{3}, \frac{8}{3}\right)$$

Again, line through B normal to the plane has equations

$$\frac{x-3}{2} = \frac{y-3}{1} = \frac{z-7}{1} = R_1 \text{ (say)}$$

If $(2R_1 + 3, R_1 + 3, R_1 + 7)$ lies on the given plane,

$2(2R_1 + 3) + R_1 + 3 + R_1 + 7 = 6$

$$6R_1 + 10 = 0 \Rightarrow R_1 = \frac{-5}{3}$$

$\Rightarrow R_1$ corresponding to B_1 is $\frac{-10}{3}$.

Hence, the image of $(3, 3, 7)$ in the plane is the point $\left(\frac{-20}{3} + 3, \frac{-10}{3} + 3, \frac{-10}{3} + 7\right)$

i.e., $B_1\left(\frac{-11}{3}, \frac{-1}{3}, \frac{11}{3}\right)$

Direction ratios of A_1B_1 are $\left(\frac{1}{3} + \frac{11}{3}, \frac{5}{3} + \frac{1}{3}, \frac{8}{3} - \frac{11}{3}\right) \Rightarrow [4, 2, -1]$

Image of the given line is the line $\frac{x - \frac{1}{3}}{4} = \frac{y - \frac{5}{3}}{2} = \frac{z - \frac{8}{3}}{-1}$

OR

Since $2 \times 2 + 1 \times 1 + 1 \times 1 \neq 0$, line is not parallel to the plane. Therefore, the line intersects the plane. Any point on the line may be assumed as $(2R + 1, R + 2, 4R + 3)$

If it lies on the given plane,

$2(2R + 1) + R + 2 + 4R + 3 = 6 \qquad 9R + 1 = 0 \Rightarrow R = \frac{-1}{9}$

Point of intersection of the line and the plane is at $\left(\frac{-2}{9}+1, \frac{-1}{9}+2, \frac{-4}{9}+3\right)$ ie, $\left(\frac{7}{9}, \frac{17}{9}, \frac{23}{9}\right)$

The image of $\left(\frac{7}{9}, \frac{17}{9}, \frac{23}{9}\right)$ in the plane is itself. The image of (1, 2, 3) in the plane was got as $\left(\frac{1}{3}, \frac{5}{3}, \frac{8}{3}\right)$.

Hence the direction ratios of the image line are $\left(\frac{7}{9}-\frac{1}{3}, \frac{17}{9}-\frac{5}{3}, \frac{23}{9}-\frac{8}{3}\right) \Rightarrow \left(\frac{4}{9}, \frac{2}{9}, \frac{-1}{9}\right)$

$\Rightarrow [4, 2, -1]$

Equations of the image line may be written as

$$\frac{x-\frac{7}{9}}{4} = \frac{y-\frac{17}{9}}{2} = \frac{z-\frac{23}{9}}{-1}$$

OR

$$\frac{x-\frac{1}{3}}{4} = \frac{y-\frac{5}{3}}{2} = \frac{z-\frac{8}{3}}{-1}$$

Connector 53: Show that the lines $\frac{x+1}{7} = \frac{y-4}{1} = \frac{z+3}{-4}$

$\frac{x+10}{8} = \frac{y-1}{2} = \frac{z+1}{-3}$ are coplanar and find the equation of the plane containing them. Find also the coordinates of the point of intersection of the two lines.

Solution:

$$\frac{x+1}{7} = \frac{y-4}{1} = \frac{z+3}{-4} = R_1 \qquad \text{— (1)}$$

$$\frac{x+10}{8} = \frac{y-1}{2} = \frac{z+1}{-3} = R_2 \qquad \text{— (2)}$$

Any point on (1) may be represented as $(7R_1 - 1, R_1 + 4, -4R_1 - 3)$ and any point on (2) may be represented as $(8R_2 - 10, 2R_2 + 1, -3R_2 - 1)$

If (1) and (2) are coplanar, there exists a set of values for R_1 and R_2 such that

$7R_1 - 1 = 8R_2 - 10$

$R_1 + 4 = 2R_2 + 1$ and $-4R_1 - 3 = -3R_2 - 1$

$R_1 = 1, R_2 = 2$ satisfy all the three relations above.

Therefore, the two lines intersect, or they are coplanar.

The point of intersection of the lines is (6, 5, –7) {substituting $R_1 = 1$ in $(7R_1 - 1)$ etc.]

Any plane through (1) may be assumed as

$A(x + 1) + B(y - 4) + C(z + 3) = 0$ —(3)

Then, $7A + B - 4C = 0$ —(4)

Since (3) passes through the line (2) also,

$8A + 2B - 3C = 0$ —(5)

Equation of the plane through (1) and (2) is obtained by eliminating A, B, C from (3), (4) an (5)

$$\Rightarrow \begin{vmatrix} x+1 & y-4 & z+3 \\ 7 & 1 & -4 \\ 8 & 2 & -3 \end{vmatrix} = 0$$

$\Rightarrow 5x - 11y + 6z + 67 = 0$

Connector 54: Show that the lines $x - 3y + 2z + 4 = 0 = 2x + y + 4z + 1$ and $3x + 2y + 5z - 1 = 0 = 2y + z$ intersect and find the coordinates of their point of intersection.

Solution: Direction ratios of the line represented by $x - 3y + 2z + 4 = 0 = 2x + y + 4z + 1$ are $[-2, 0, 1]$

Putting $z = 0$ in the above equations, we find that $(-1, 1, 0)$ is a point on the line.

Therefore, the equation of the line are $\dfrac{x+1}{-2} = \dfrac{y-1}{0} = \dfrac{z}{1}$ —(1)

Direction ratios of the line represented by $3x + 2y + 5z - 1 = 0 = 2y + z$ are $[-8, -3, 6]$

Putting $z = 0$, in the above equations, we find that $\left(\dfrac{1}{3}, 0, 0\right)$ is a point on the line.

Therefore, the equation of the line are $\dfrac{x - \frac{1}{3}}{-8} = \dfrac{y}{-3} = \dfrac{z}{6}$ —(2)

Any point on (1) may be represented as $(-2R_1 - 1, 1, R_1)$ and any point on (2) may be represented as $\left(-8R_2 + \dfrac{1}{3}, -3R_2, 6R_2\right)$

It can be easily seen that $R_2 = \dfrac{-1}{3}$, $R_1 = -2$ will lead to the same point $(3, 1, -2)$ or, the point of intersection of the lines is at $(3, 1, -2)$.

Connector 55: A straight line L passes through a point $P(\bar{p})$ and is parallel to $\bar{q}$. A second straight line L' passes through a point $P'(\overline{p^1})$ and is parallel to $\overline{q^1}$. The vector $\bar{s}$ runs from L to L' along the common perpendicular. Show that $\bar{s} = \left\{\dfrac{(\overline{p^1} - \bar{p}) \cdot \bar{c}}{|\bar{c}|^2}\right\}\bar{c}$ where, $\bar{c} = \bar{q} \times \overline{q^1}$

Solution: Since $\bar{s}$ is perpendicular to both L and L', $\bar{s} = \lambda\bar{c}$ where, $\bar{c} = \bar{q} \times \bar{q}'$ and λ is a scalar.

If $\bar{s}$ intersects L and L' in A and B,

AB = shortest distance between the lines L and L'

= Projection of PP' on $\bar{c}$

$$= \frac{1}{|\bar{c}|}\left\{(\bar{p}' - \bar{p}) \cdot \bar{c}\right\}$$

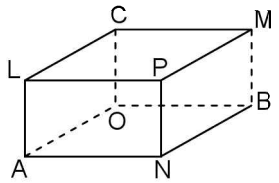
A unit vector along $\bar{c} = \dfrac{1}{|\bar{c}|}\bar{c}$

Therefore, $\bar{s} = \left\{\dfrac{(\bar{p}' - \bar{p}) \cdot \bar{c}}{|\bar{c}|^2}\right\}\bar{c}$

Connector 56: Show that the shortest distances between a diagonal of a rectangular parallelepiped the lengths of whose coterminous edges are a, b, c and the edges not meeting it are $\dfrac{bc}{\sqrt{b^2 + c^2}}, \dfrac{ca}{\sqrt{c^2 + a^2}}, \dfrac{ab}{\sqrt{a^2 + b^2}}$.

Solution: Let a, b, c be the lengths of the sides OA, OB and OC respectively of the rectangular parallelepiped. Taking O as origin and assuming the unit vectors along OA, OB, OC as

$\bar{i}, \bar{j}, \bar{k}, \overline{OA} = a\bar{i}, \overline{OB} = b\bar{j}$ and $\overline{OC} = c\bar{k}$



$\overline{OP} = a\bar{i} + b\bar{j} + c\bar{k}$

The edges which do not meet the diagonal OP are AL, AN, BN and their parallels are BM, CM, CL.

We use the result:

The shortest distance between two non-intersecting lines in space (two skew lines) passing through the points whose position vectors are $\bar{a}$ and $\bar{b}$ and are parallel to the vectors $\bar{c}$ and $\bar{d}$ respectively is given by $\dfrac{\left[\bar{a} - \bar{b}, \bar{c}, \bar{d}\right]}{\left(\bar{c} \times \bar{d}\right)}$

In the present case, the shortest distance between OP and BN is given by

$$p = \frac{\left[o - b\bar{j}, \bar{i}, a\bar{i} + b\bar{j} + c\bar{k}\right]}{\left|\bar{i} \times a(\bar{i} + b\bar{j} + c\bar{k})\right|}$$

$$= \frac{1}{\sqrt{b^2 + c^2}} \begin{vmatrix} 0 & -b & 0 \\ 1 & 0 & 0 \\ a & b & c \end{vmatrix} = \frac{bc}{\sqrt{b^2 + c^2}}$$

Similarly, shortest distance between OP and AN is $\dfrac{ca}{\sqrt{c^2 + a^2}}$ and that between OP and AL is $\dfrac{ab}{\sqrt{a^2 + b^2}}$

TOPIC GRIP

Subjective Questions

1. The lengths a, b, c of the sides of a triangle are related as $a^2 + b^2 = 5c^2$. Prove by vector method that the medians through the vertices A and B are perpendicular.

2. If $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors and $\bar{A} = x_1\bar{a} + y_1\bar{b} + z_1\bar{c}$, $\bar{B} = x_2\bar{a} + y_2\bar{b} + z_2\bar{c}$, $\bar{C} = x_3\bar{a} + y_3\bar{b} + z_3\bar{c}$, prove that

$$\left[\bar{A}\bar{B}\bar{C}\right] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \left[\bar{a}\bar{b}\bar{c}\right]$$

3. (i) If $\bar{a}$ and $\bar{b}$ are vectors such that $\bar{a} \cdot \bar{b} \neq 0$, find the vector $\bar{R}$ which satisfies the equations $\left(\bar{R} - \bar{c}\right) \times \bar{b} = 0$ and $\bar{R} \cdot \bar{a} = 0$

   (ii) It is given that $\left(\bar{R} - \bar{c}\right) \times \bar{b} = 0$ and $\bar{R} \cdot \bar{a} = 0$ Interpret geometrically the, above two vector equations.

4. Prove the following:

   (i) $\bar{a} \times \left\{\bar{b} \times \left(\bar{c} \times \bar{d}\right)\right\} = \left(\bar{b} \cdot \bar{d}\right)\left(\bar{a} \times \bar{c}\right) - \left(\bar{b} \cdot \bar{c}\right)\left(\bar{a} \times \bar{d}\right)$

   (ii) $\bar{d} \cdot \left[\bar{a} \times \left\{\bar{b} \times \left(\bar{c} \times \bar{d}\right)\right\}\right] = \left(\bar{b} \cdot \bar{d}\right)\left[\bar{a}\bar{c}\bar{d}\right]$

   (iii) If $\bar{u} = \bar{a} \times \bar{b}, \bar{v} = \bar{a} \times \bar{c}$ and $\bar{w} = \bar{b} + \bar{c}$, show that $\left(\bar{w} \times \bar{u}\right) + \left(\bar{w} \times \bar{v}\right) = \left|\bar{w}\right|^2 \bar{a} - \left(\bar{w} \cdot \bar{a}\right)\bar{w}$

   (iv) If $\bar{F} \times \bar{A} = \bar{P}$ and $\bar{F} \times \bar{B} = \bar{Q}$, show that $\bar{A} \cdot \bar{Q} + \bar{B} \cdot \bar{P} = 0$. Show also that $\left(\bar{Q} \cdot \bar{A}\right)\left(\bar{P} \times \bar{Q}\right) = \left(\bar{P} \cdot \bar{B}\right)\left(\bar{Q} \times \bar{P}\right)$

   (v) $$\begin{aligned}\left[\bar{a} \times \bar{b}, \bar{c} \times \bar{d}, \bar{e} \times \bar{f}\right] &= \left[\bar{a}\bar{b}\bar{d}\right]\left[\bar{c}\bar{e}\bar{f}\right] - \left[\bar{a}\bar{b}\bar{c}\right]\left[\bar{d}\bar{e}\bar{f}\right] \\ &= \left[\bar{a}\bar{b}\bar{e}\right]\left[\bar{f}\bar{c}\bar{d}\right] - \left[\bar{a}\bar{f}\bar{b}\right]\left[\bar{c}\bar{e}\bar{d}\right] \\ &= \left[\bar{c}\bar{d}\bar{a}\right]\left[\bar{b}\bar{e}\bar{f}\right] - \left[\bar{c}\bar{d}\bar{b}\right]\left[\bar{a}\bar{e}\bar{f}\right]\end{aligned}$$

5. Find the equation of the plane containing the line $\frac{y}{b} + \frac{z}{c} = 1$, $x = 0$ and parallel to the line $\frac{x}{a} - \frac{z}{c} = 1$, $y = 0$. If 2d is the shortest distance between the two given lines, show that $\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

6. $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ are the direction cosines of two lines inclined at an angle θ to each other. Show that the line with direction cosines $\left[\frac{l_1 + l_2}{2\cos\frac{\theta}{2}}, \frac{m_1 + m_2}{2\cos\frac{\theta}{2}}, \frac{n_1 + n_2}{2\cos\frac{\theta}{2}}\right]$ bisects the angle between the two given lines.

7. Prove that the shortest distance between the y-axis and the line represented by $\left.\begin{aligned} ax + by + cz + d &= 0 \\ a'x + b'y + c'z + d &= 0 \end{aligned}\right\}$ is

$$\frac{\left|bd' - b'd\right|}{\sqrt{\left(b'a - b'a\right)^2 + \left(bc' - b'c\right)^2}}$$

8. A plane passes through a fixed point (a, b, c). Show that the locus of the foot of the perpendicular to it from the origin is the surface $x^2 + y^2 + z^2 - ax - by - cz = 0$.

9. (i) Find the equation of the plane through (4, 1, –1), parallel to the plane $2x - y + z - 13 = 0$.

(ii) Find the equation of the plane through (0, 1, 1) and perpendicular to the line joining the points (1, 2, 3) and (2, 3, 5).

(iii) Find angle between the planes got in (i) and (ii).

(iv) If the plane found in (i) is rotated through a right angle about its line of intersection with plane found in (ii), find its equation in the new position.

10. (i) Find the condition on a, b, c for the two lines whose direction cosines l, m, n are given by $al + bm + cn = 0$ and $mn + nl + \ell m = 0$ to be perpendicular.

(ii) For the a, b, c satisfying the above condition show that the vectors $\bar{x} = (bc - a^2)\hat{i} + (ca - b^2)\hat{j} + (ab - c^2)\hat{k}$,
$\bar{y} = (-bc + ca + ab)\hat{i} + (bc - ca + ab)\hat{j} + (bc + ca - ab)\hat{k}$ and
$\bar{z} = (a + b)(a + c)\hat{i} + (b + c)(b + a)\hat{j} + (c + a)(c + b)\hat{k}$ are coplanar

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

11. If $|\bar{a}| = 3, |\bar{b}| = 4$ and $|\bar{a} - \bar{b}| = 5$, then $|\bar{a} + \bar{b}|$ is equal to

(a) 7 (b) 6 (c) 5 (d) 4

12. A particle acted upon by constant forces $4\bar{i} + \bar{j} - 3\bar{k}$ and $3\bar{i} + \bar{j} - \bar{k}$ is displaced from the point $\bar{i} + 2\bar{j} + 3\bar{k}$ to the point $5\bar{i} + 4\bar{j} + \bar{k}$. The total work done by the forces is

(a) 25 units (b) 35 units (c) 45 units (d) 40 units

13. If $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4$ are unit vectors, the maximum value of $\sum_{i \neq j}^{4} |\bar{a}_i - \bar{a}_j|^2$ is

(a) 32 (b) 8 (c) 16 (d) 12

14. If the planes $4x + 3y + 2z + 7 = 0$, $2x + y - 4z + 1 = 0$ and $x + ky - 7z - 2 = 0$ have a common line of intersection then the value of k is

(a) 1 (b) 0 (c) 2 (d) –1

15. If the lines $\frac{x+1}{k+2} = \frac{y-1}{1} = \frac{z-2}{2}$ and $\frac{x-1}{k-2} = \frac{y+2}{k} = \frac{3-z}{3k-2}$ are at right angles, k equals

(a) 0 (b) 5 (c) either (a) or (b) (d) neither (a) nor (b)

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

16. Statement 1

If $\bar{a}, \bar{b}, \bar{c}$, are three non-null vectors such that $\bar{a} \cdot \bar{b} = 0$, $\bar{a} \cdot \bar{c} = 0$, then they are coplanar.

and

Statement 2

If $\bar{a} \cdot (\bar{b} \times \bar{c}) = 0$, then $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

17. Statement 1

If the vectors $\bar{a}, \bar{b}, \bar{c}$, are coplanar, then the vectors $\bar{b} + \bar{c}, \bar{c} + \bar{a}, \bar{a} + \bar{b}$ are coplanar.

and

Statement 2

$\left[\bar{b} + \bar{c}, \bar{c} + \bar{a}, \bar{a} + \bar{b}\right] = 2\left[\bar{a}, \bar{b}, \bar{c}\right]$

18. Statement 1

The line $\dfrac{x-3}{1} = \dfrac{y+1}{-2} = \dfrac{z}{5}$ is parallel to the plane $x - 2y + 5z = 1$

and

Statement 2

Two lines whose direction ratios are $[a_1, b_1, c_1]$ and $[a_2, b_2, c_2]$ are perpendicular to each other if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

19. Statement 1

The plane $x + y + 6z = 7$ passes through the line $\dfrac{x}{2} = \dfrac{y-1}{4} = \dfrac{z-1}{-1}$.

and

Statement 2

If $a\ell + bm + cn = 0$ then the plane $ax + by + cz + d = 0$ passes through the line $\dfrac{x-\alpha}{\ell} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$

20. Statement 1

The point $(2, -2, 0)$ lies on the line $\dfrac{x-1}{\frac{1}{\sqrt{3}}} = \dfrac{y+3}{\frac{1}{\sqrt{3}}} = \dfrac{z-1}{\frac{-1}{\sqrt{3}}}$ and is at a distance $\sqrt{3}$ from the point $(1, -3, 1)$ on the line.

and

Statement 2

If $\dfrac{x-\alpha}{\ell} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n} = R$, then R denotes the distance of any point (x, y, z) on the line from the point (α, β, γ).

Linked Comprehension Type Questions

Directions: **This section contains paragraph. Based upon the paragraph, 6 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

I Reciprocal system of Vectors

Definition

Let $\bar{a}, \bar{b}, \bar{c}$ be any three non coplanar vectors so that $\left[\bar{a}\,\bar{b}\,\bar{c}\right] \neq 0$. Then the three vectors $\overline{a^1}, \overline{b^1}, \overline{c^1}$ defined by the equations $\overline{a^1} = \dfrac{1}{\left[\bar{a}\,\bar{b}\,\bar{c}\right]}\bar{b} \times \bar{c}$; $\overline{b^1} = \dfrac{1}{\left[\bar{a}\,\bar{b}\,\bar{c}\right]}\bar{c} \times \bar{a}$; $\overline{c^1} = \dfrac{1}{\left[\bar{a}\,\bar{b}\,\bar{c}\right]}\bar{a} \times \bar{b}$ are said to form the reciprocal system of vectors of the system $\left[\bar{a}, \bar{b}, \bar{c}\right]$.

Results

(i) $\overline{a^1}, \overline{b^1}, \overline{c^1}$ are non-coplanar.

For, $\left[\overline{a^1 b^1 c^1}\right] = \overline{a^1} \cdot \left(\overline{b^1} \times \overline{c^1}\right)$

$$= \frac{\left(\bar{b} \times \bar{c}\right)}{\left[\bar{a}\bar{b}\bar{c}\right]} \cdot \frac{\left(\bar{c} \times \bar{a}\right)}{\left[\bar{a}\bar{b}\bar{c}\right]} \times \frac{\left(\bar{a} \times \bar{b}\right)}{\left[\bar{a}\bar{b}\bar{c}\right]} = \frac{1}{\left[\bar{a}\bar{b}\bar{c}\right]^3}\left(\bar{b} \times \bar{c}\right) \cdot \left(\bar{c} \times \bar{a}\right) \times \left(\bar{a} \times \bar{b}\right)$$

$$= \frac{1}{\left[\bar{a}\bar{b}\bar{c}\right]^3}\left(\left[\bar{a}\bar{b}\bar{c}\right]^2\right), \text{Since } \left[\bar{b} \times \bar{c}, \bar{c} \times \bar{a}, \bar{a} \times \bar{b}\right] = \left[\bar{a}\bar{b}\bar{c}\right]^2$$

$$= \frac{1}{\left[\bar{a}\bar{b}\bar{c}\right]^3} \Rightarrow \left[\overline{a^1 b^1 c^1}\right] \neq 0$$

$\Rightarrow$ $\overline{a^1}, \overline{b^1}, \overline{c^1}$ are non-coplanar and $\left[\overline{a^1 b^1 c^1}\right]\left[\bar{a}\bar{b}\bar{c}\right] = 1$

(ii) $\bar{a} \cdot \overline{a^1} = \bar{b} \cdot \overline{b^1} = \bar{c} \cdot \overline{c^1} = 1$

For, $\bar{a} \cdot \overline{a^1} = \bar{a} \cdot \frac{1}{\left[\bar{a}\bar{b}\bar{c}\right]}\left(\bar{b} \times \bar{c}\right) = \frac{1}{\left[\bar{a}\bar{b}\bar{c}\right]}\left[\bar{a}\bar{b}\bar{c}\right] = 1$

Similarly, $\bar{b}.\overline{b^1} = 1, \bar{c} \cdot \overline{c^1} = 1$

(iii) $\bar{a} \cdot \overline{a^1} + \bar{b} \cdot \overline{b^1} + \bar{c} \cdot \overline{c^1} = 3$

From (ii), $\bar{a} \cdot \overline{a^1} + \bar{b} \cdot \overline{b^1} + \bar{c} \cdot \overline{c^1} = 1 + 1 + 1 = 3$

(iv) $\bar{a} \cdot \overline{b^1} = \overline{a^1} \cdot \bar{b} = \bar{a} \cdot \overline{c^1} = \overline{a^1} \cdot \bar{c} = \bar{b} \cdot \overline{c^1} = \overline{b^1} \cdot \bar{c} = 0$

For, $\bar{u} \times \bar{v} = \frac{1}{\bar{a}\,\bar{b}\,\bar{c}}\begin{vmatrix} \bar{u} \cdot \bar{a} & \bar{v} \cdot \bar{a} & \bar{a} \\ \bar{u} \cdot \bar{b} & \bar{v} \cdot \bar{b} & \bar{b} \\ \bar{u} \cdot \bar{c} & \bar{v} \cdot \bar{c} & \bar{c} \end{vmatrix}$, since $\left[\bar{a}\bar{c}\bar{a}\right] = 0$

Similarly, the other results can be proved.

(v) If $\bar{i}, \bar{j}, \bar{k}$ have the usual meaning and $\left\{\overline{i^1}, \overline{j^1}, \overline{k^1}\right\}$ represent the reciprocal system to the set of vectors $\left\{\bar{i}, \bar{j}, \bar{k}\right\}, \overline{i^1} = \bar{i}, \overline{j^1} = \bar{j}, \overline{k^1} = \bar{k}$,

In other words, $\left\{\bar{i}, \bar{j}, \bar{k}\right\}$ forms a self-reciprocal system.

(vi) Let $\left\{\overline{a^1}, \overline{b^1}, \overline{c^1}\right\}$ represent the reciprocal system of vectors to the set of non-coplanar vectors $\left\{\bar{a}, \bar{b}, \bar{c}\right\}$. Also, let $\bar{R} = x\bar{a} + y\bar{b} + z\bar{c}$, x, y, z being scalars.

Then, $\overline{R} = \left(\overline{R} \cdot \overline{a^1}\right)\overline{a} + \left(\overline{R} \cdot \overline{b^1}\right)\overline{b} + \left(\overline{R} \cdot \overline{c^1}\right)\overline{c}$

We have $\overline{R} = x\overline{a} + y\overline{b} + z\overline{c} \Rightarrow \overline{R}.\overline{a} = x\left(\overline{a}.\overline{a}'\right) + y\left(\overline{b}.\overline{a}'\right) + z\left(\overline{c}.\overline{a}'\right)$

$$= x.1 + y.0 + z.0 = x$$

Similarly, we can show that $y = \overline{R} \cdot \overline{b^1}, z = \overline{R} \cdot \overline{c^1}$

$\therefore \quad \overline{R} = \left(R.\overline{a}'\right)\overline{a} + \left(R.\overline{b}'\right)\overline{b} + \left(\overline{R}.c'\right)\overline{c}$

21. $\overline{a} \times \overline{a^1} + \overline{b} \times \overline{b^1} + \overline{c} \times \overline{c^1}$ is

(a) $\dfrac{\overline{a} \times \left(\overline{b} \times \overline{c}\right)}{\left[\overline{aca}\right]^2}$ (b) $\dfrac{\overline{a} \times \left(\overline{b} \times \overline{c}\right)}{\left[\overline{aca}\right]}$

(c) 0 (d) None of these

22. $\overline{a^1} \times \overline{b^1} + \overline{b^1} \times \overline{c^1} + \overline{c^1} \times \overline{a^1}$ is

(a) $\dfrac{\left(\overline{a} + \overline{b} + \overline{c}\right)}{\left[\overline{aba}\right]}$ (b) 0

(c) $\dfrac{\overline{a} \times \overline{b} + \overline{b} \times \overline{c} + \overline{c} \times \overline{a}}{\left[\overline{abc}\right]}$ (d) $\dfrac{\overline{a} \times (\overline{b} \times \overline{c}) + \overline{b} \times (\overline{c} \times \overline{a})}{\left[\overline{abc}\right]}$

23. $(\overline{a} + \overline{b}) \cdot \overline{a^1} + (\overline{b} + \overline{c}) \cdot \overline{b^1} + (\overline{c} + \overline{a}) \cdot \overline{c^1}$ is

(a) 0 (b) 1 (c) 2 (d) 3

24. The set of vectors reciprocal to the set $\left\{\overline{j} + \overline{k}, \overline{k} + \overline{i}, \overline{i} + \overline{j}\right\}$ is

(a) $\left\{\overline{j} + \overline{k}, \overline{k} + \overline{r}, \overline{i} + \overline{j}\right\}$ (b) $\left\{\frac{1}{2}(\overline{j} + \overline{k}), \frac{1}{2}(\overline{k} + \overline{i}), \frac{1}{2}(\overline{i} + \overline{j})\right\}$

(c) $\left\{\dfrac{\overline{j} + \overline{k} - \overline{i}}{2}, \dfrac{\overline{k} + \overline{i} - \overline{j}}{2}, \dfrac{\overline{i} + \overline{j} - \overline{k}}{2}\right\}$ (d) $\left\{\dfrac{\overline{i}}{2}, \dfrac{\overline{j}}{2}, \dfrac{\overline{k}}{2}\right\}$

25. The reciprocal system of vectors corresponding to the set of vectors $\left\{2\overline{i} + \overline{j} - 3\overline{k}, 3\overline{i} + 2\overline{j} - \overline{k}, \overline{i} + 3\overline{j} - 2\overline{k}\right\}$ is

(a) $\left\{\dfrac{\overline{i} + 5\overline{j} + 7\overline{k}}{18}, \dfrac{7\overline{i} + \overline{j} - 5\overline{k}}{18}, \dfrac{5\overline{i} + 7\overline{j} - \overline{k}}{18}\right\}$ (b) $\left\{\dfrac{\overline{i} - 5\overline{j} - 7\overline{k}}{18}, \dfrac{7\overline{i} + \overline{j} + 5\overline{k}}{18}, \dfrac{-5\overline{i} + 7\overline{j} - \overline{k}}{18}\right\}$

(c) $\left\{\dfrac{5\overline{j} - 7\overline{k}}{18}, \dfrac{\overline{i} + 7\overline{j} - 5\overline{k}}{18}, \dfrac{3\overline{i} + 2\overline{j} - 5\overline{k}}{18}\right\}$ (d) $\left\{\dfrac{-\overline{i} + 5\overline{j} + 7\overline{k}}{18}, \dfrac{-7\overline{i} - \overline{j} - 5\overline{k}}{18}, \dfrac{5\overline{i} - 7\overline{j} + \overline{k}}{18}\right\}$

26. If $\left(\overline{a}, \overline{b}, \overline{c}\right)$ form the reciprocal system of $\left(\overline{i}, \overline{i} + \overline{j}, \overline{i} + \overline{j} + \overline{k}\right)$, then the reciprocal system of $\left(\overline{a}\ \overline{b}\ \overline{c}\right)$ is

(a) $\left(\overline{i}, + \overline{j} + \overline{k}, \overline{i} + \overline{j}, i\right)$ (b) $\left(\overline{i}, + \overline{j} + \overline{k}, \overline{j} + \overline{k}, \overline{k}\right)$

(c) $\left(\overline{i}, \overline{i} + \overline{j}, \overline{i} + \overline{j} + \overline{k}\right)$ (d) $\left(\overline{i}, \overline{j}, \overline{k}\right)$

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers of which ONE OR MORE answers will be correct.**

27. $\overline{A} = \overline{i} + \overline{j} - \overline{k}$

$\overline{B} = 2\overline{i} + 2\overline{j} + \overline{k}$

$\overline{C} = 8\overline{i} + 8\overline{j} + \overline{k}$

Then,

(a) $\overline{A}, \overline{B}, \overline{C}$ are coplanar

(b) A vector along a bisector of the angle between $\overline{A}$ and $\overline{B}$ is $\frac{1}{2}\left\{\frac{3+2\sqrt{3}}{3\sqrt{3}}(\overline{i}+\overline{j}) - \frac{3-\sqrt{3}}{3\sqrt{3}}\overline{k}\right\}$

(c) A vector along a bisector of the angles between $\overline{A}$ and $\overline{B}$ is $\frac{1}{2}\left\{\left(\frac{3-2\sqrt{3}}{3\sqrt{3}}\right)(\overline{i}+\overline{j}) - \left(\frac{3+\sqrt{3}}{3\sqrt{3}}\right)\overline{k}\right\}$

(d) Projection of $\overline{A}$ on $\overline{B}$ is -1

28. $\overline{a}, \overline{b}, \overline{c}$ are three vectors such that $\overline{a}$ and $\overline{c}$ are orthogonal. Then,

(a) $\overline{a} \times (\overline{b} \times \overline{c})$ is parallel to $\overline{c}$

(b) $\overline{a} \times (\overline{b} \times \overline{c})$ is perpendicular to $\overline{c}$

(c) $(\overline{a} \times \overline{b}) \times \overline{c}$ is perpendicular to $\overline{a}$

(d) If $\overline{a}, \overline{b}, \overline{c}$ are mutually orthogonal, forming a right-handed system, then the scalar triple product of $\overline{b}+\overline{c}, \overline{c}+\overline{a}, \overline{a}+\overline{b}$ is $2|\overline{a}||\overline{b}||\overline{c}|$

29. The equations of two intersecting lines are

$\frac{x-1}{3} = \frac{y+2}{0} = \frac{z-5}{1}$ and $\frac{x-5}{1} = \frac{y+3}{-1} = \frac{z-10}{4}$. Then,

(a) Equation of the plane passing through the lines is $x - 11y - 3z - 8 = 0$

(b) Coordinates of the point of intersection of the lines are $(4, -2, 6)$

(c) An angle between the lines is $\cos^{-1}\left(\frac{7}{6\sqrt{5}}\right)$

(d) Equation of the plane parallel to the plane through the lines and passing through the origin is $x - 11y - 3z = 0$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

30.

Column I	Column II
(a) A vector $\overline{i} + a\overline{j} + 3\overline{k}$ is rotated through an angle 120° and doubled in magnitude. If it now becomes $4\overline{i} - (4a-2)\overline{j} - 2\overline{k}$, the value of a is equal to	(p) -7

(b) The angle between the vectors $\bar{x} = a\bar{i} - 3\bar{j} - \bar{k}$ and $\bar{y} = 2a\bar{i} + a\bar{j} - \bar{k}$ is acute and the angle between the vector $\bar{y}$ and y-axis is obtuse. Then a is equal to	(q) -2
(c) A constant force $\bar{F} = 3\bar{i} + 2j - 4\bar{k}$ is applied at the point $(1, -1, 2)$. The vector moment of $\bar{F}$ about the point $(2, -1, 3)$ is $2\bar{i} + \lambda\bar{j} - 2\bar{k}$. Then λ is equal to	(r) 2
(d) The vectors $\lambda i + j + 2k$, $i + \lambda j - k$ and $2i - j + \lambda k$ are coplanar if λ is equal to	(s) $-2/3$

IIT ASSIGNMENT EXERCISE

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

31. If $\bar{r}$ and $\bar{s}$ are two unit vectors such that the angle between them is 60°, then $\left|\bar{r}-\bar{s}\right|$ is equal to

(a) $\frac{\sqrt{3}}{2}$ (b) $\sqrt{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) 1

32. A unit vector perpendicular to both $\bar{i}+\bar{j}$ and $\bar{j}+\bar{k}$ is

(a) $\frac{1}{3}\left(\bar{i}-\bar{j}+\bar{k}\right)$ (b) $\frac{1}{\sqrt{3}}\left(\bar{i}-\bar{j}+\bar{k}\right)$ (c) $\frac{1}{\sqrt{3}}\left(\bar{i}+\bar{j}-\bar{k}\right)$ (d) $\frac{1}{3}\left(\bar{i}+\bar{j}-\bar{k}\right)$

33. The projection of $\bar{a}$ on $\bar{b}$ is twice the projection of $\bar{b}$ on $\bar{a}$ if

(a) $\bar{a}=\bar{b}$ (b) $\left|\bar{a}\right|=\left|\bar{b}\right|$

(c) $\left|\bar{a}\right|-\left|\bar{b}\right|=\left|\bar{b}\right|$ (d) $\left|\bar{a}\right|-\left|\bar{b}\right|=\left|\bar{a}\right|$

34. A force $\bar{F}=3\bar{i}+\bar{k}$ acts through the point $2\bar{i}-\bar{j}+3\bar{k}$. The moment of this force about the point $\bar{i}+2\bar{j}-\bar{k}$ has the magnitude

(a) 211 (b) 111 (c) $\sqrt{211}$ (d) $\sqrt{111}$

35. The points (10, 7, 0), (6, 6, –1) and (6, 9, –4) form a

(a) right-angled triangle (b) isosceles triangle

(c) both (a) and (b) (d) None of these

36. The ratio in which the plane XOZ divides the join of (1, -1, 5) to (2, 3, 4) is

(a) 1 : 3 (b) 2 : 3 (c) 1 : 1 (d) 2 : 1

37. An angle between the planes 2x – y + 2z = 3 and 6x – 2y + 3z = 5 is

(a) $\cos^{-1}\left(\frac{11}{21}\right)$ (b) $\sin^{-1}\left(\frac{20}{21}\right)$ (c) $\cos^{-1}\left(\frac{20}{21}\right)$ (d) $\tan^{-1}\left(\frac{20}{21}\right)$

38. The distance of the point (2, 1, –1) from the plane x – 2y + 4z = 9 is

(a) $\frac{\sqrt{13}}{\sqrt{7}}$ (b) $\frac{\sqrt{13}}{\sqrt{3}}$ (c) $\frac{13}{\sqrt{7}\sqrt{3}}$ (d) $\frac{\sqrt{7}\sqrt{3}}{13}$

39. The distance between the planes 2x – y + 2z +1=0 and 4x – 2y + 4z + 11 = 0 is

(a) $\frac{9}{2}$ (b) 3 (c) $\frac{7}{2}$ (d) $\frac{3}{2}$

40. The perpendicular distance of the point (2, 4, –1) from the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$ is

(a) $\sqrt{7}$ (b) $\frac{15}{\sqrt{7}}$ (c) $\frac{22}{\sqrt{7}}$ (d) 7

41. The distance between the line $\frac{x-1}{3} = \frac{y+2}{-2} = \frac{z-1}{2}$ and the plane $2x + 2y - z - 6 = 0$ is

(a) 5 (b) 4 (c) 3 (d) 0

42. If θ is an angle between the plane $\bar{r} \cdot \left(2\bar{i} - \bar{j} + \bar{k}\right) = 4$ and the line $\bar{r} = \left(\bar{i} + 2\bar{j} - \bar{k}\right) + \lambda\left(\bar{i} - \bar{j} + \bar{k}\right)$, then

(a) $\cos\theta = \frac{\sqrt{2}}{3}$ (b) $\sin\theta = \frac{\sqrt{2}}{3}$

(c) $\cos\theta = \frac{2\sqrt{2}}{3}$ (d) $\sin\theta = \frac{2\sqrt{2}}{3}$

43. If θ is an angle between the line $\bar{r} = \bar{a} + \lambda\bar{b}$ and the plane $\bar{r} \cdot \bar{n} = d$, then

(a) $\sin\theta = \frac{\bar{b} \cdot \bar{n}}{|\bar{b}||\bar{n}|}$ (b) $\cos\theta = \frac{\bar{b} \cdot \bar{n}}{|\bar{b}||\bar{n}|}$

(c) $\tan\theta = \frac{\bar{b} \cdot \bar{n}}{|\bar{b}||\bar{n}|}$ (d) $\cos\theta = \frac{\bar{a} \cdot \bar{b}}{d}$

44. If $\bar{a}$ is a vector such that $\bar{a} \cdot \left(\bar{i} + \bar{j} + \bar{k}\right) = \bar{a} \cdot \left(\bar{k} + \bar{i}\right) = \bar{a} \cdot \bar{k} = 1$, then $\bar{a}$ is equal to

(a) $\bar{k}$ (b) $\bar{i} + \bar{j}$ (c) $\bar{i} + \bar{j} + \bar{k}$ (d) $\bar{i} + \bar{k}$

45. A set of direction cosines of normal to the plane $6x - 3y - 2z = 10$ is

(a) $\left(\frac{6}{7}, \frac{-3}{7}, \frac{-2}{7}\right)$ (b) $(6, -3, -2)$

(c) $\left(\frac{6}{49}, \frac{-3}{49}, \frac{-2}{49}\right)$ (d) $\left(\frac{-6}{49}, \frac{3}{49}, \frac{2}{49}\right)$

46. The equation of the plane passing through the intersection of the planes $x + 2y + 3z = 4$ and $4x + 3y + 2z = -1$ and the origin is

(a) $17x - 14y + 11z = 0$ (b) $17x + 14y - 11z = 0$

(c) $-17x + 14y + 11z = 0$ (d) $17x + 14y + 11z = 0$

47. If θ is the angle between the vectors $\bar{p}$ and $\bar{q}$ and $2\bar{p} + \bar{q} = \bar{i} + \bar{j}$, $\bar{p} + 2\bar{q} = \bar{i} - \bar{j}$, then

(a) $\cos\theta = \frac{4}{5}$ (b) $\sin\theta = \frac{4}{5}$ (c) $\cos\theta = \frac{3}{5}$ (d) $\sin\theta = \frac{3}{5}$

48. If $\bar{a}$ is any vector, then $\left|\bar{a} \times \bar{i}\right|^2 + \left|\bar{a} \times \bar{j}\right|^2 + \left|\bar{a} \times \bar{k}\right|^2$ is equal to

(a) 2 (b) $2\left|\bar{a}\right|$ (c) $2\left|\bar{a}\right|^2$ (d) $\left|\bar{a}\right|^2$

49. If $\bar{a} = 2\bar{i} + 3\bar{j}$ and $\bar{b} = 3\bar{j} + 4\bar{k}$, then the vector component of $\bar{a}$ along $\bar{b}$ is

(a) $\frac{9}{25}\left(3\bar{j} + 4\bar{k}\right)$ (b) $\frac{9}{25}\left(2\bar{i} + 3\bar{j}\right)$

(c) $\frac{9}{13}\left(2\hat{i} + 3\bar{j}\right)$ (d) $\frac{9}{5}\left(3\bar{j} + 4\bar{k}\right)$

50. Let $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k}$ and $\bar{r} = \bar{i} \times (\bar{a} \times \bar{i}) + \bar{j} \times (\bar{a} \times \bar{j}) + \bar{k} \times (\bar{a} \times \bar{k})$, then $|\bar{r}|$ is

(a) $\sqrt{14}$ (b) $2\sqrt{14}$ (c) 14 (d) $\frac{\sqrt{14}}{2}$

51. If $\bar{a}, \bar{b}, \bar{c}$ are unit vectors such that $\bar{a}$ is perpendicular to the plane of $\bar{b}$ and $\bar{c}$ and the angle between $\bar{b}$ and $\bar{c}$ is $\frac{\pi}{3}$, then the value of $|\bar{a} + \bar{b} + \bar{c}|$ is

(a) 4 (b) 2 (c) 9 (d) 3

52. The vector $\bar{r}$, directed along the internal bisector of the angle between the vectors $\bar{a} = 7\bar{i} - 4\bar{j} - 4\bar{k}$ and $\bar{b} = -2\bar{i} - \bar{j} + 2\bar{k}$ such that $|r| = 3\sqrt{6}$ is

(a) $\bar{i} + 7\bar{j} + 2\bar{k}$ (b) $\bar{i} - 7\bar{j} + 2\bar{k}$ (c) $\bar{i} - 7\bar{j} - 2\bar{k}$ (d) $\bar{i} + 7\bar{j} - 2\bar{k}$

53. The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-\alpha}$ and $\frac{x-1}{\alpha} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar if

(a) $\alpha = 0$ or 3 (b) $\alpha = 0$ or -1 (c) $\alpha = 0$ or -3 (d) $\alpha = \pm 2$

54. A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. The locus of the centroid of the triangle ABC is

(a) $x^2 + y^2 + z^2 = p^2$ (b) $x^2 + y^2 + z^2 = 9p^2$ (c) $x^{-2} + y^{-2} + z^{-2} = p^{-2}$ (d) $x^{-2} + y^{-2} + z^{-2} = 9p^{-2}$

55. Equation of the plane which contains the line $\frac{x-1}{2} = \frac{y-3}{1} = \frac{z-13}{2}$ and perpendicular to the plane $x + y + z = 3$ is

(a) $z - x = 12$ (b) $x + 2y - z + 12 = 0$ (c) $2x + u - z + 12 = 0$ (d) $x + y + z + 6 = 0$

56. If θ is the acute angle between any two diagonals of a cube , then

(a) $\cos\theta = \frac{1}{3}$ (b) $\sin\theta = \frac{1}{3}$ (c) $\cos\theta = \frac{1}{\sqrt{3}}$ (d) $\sin\theta = \frac{1}{\sqrt{3}}$

57. Let $\bar{a}, \bar{b}, \bar{c}$ be three non-coplanar vectors and $\bar{p}, \bar{q}, \bar{r}$ are vectors defined by $\bar{p} = \frac{\bar{b} \times \bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]}$, $\bar{q} = \frac{\bar{c} \times \bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ and $\bar{r} = \frac{\bar{a} \times \bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$. Then the value of $(\bar{b} + \bar{c}) \cdot \bar{p} + (\bar{c} + \bar{a}) \cdot \bar{q} + (\bar{a} + \bar{b}) \cdot \bar{r}$ is equal

(a) 3 (b) 2 (c) 1 (d) 0

58. Shortest distance between the two lines whose vector equations are given by $\bar{r} = (s-1)\bar{i} + (s+1)\bar{j} - (s+1)\bar{k}$ and $\bar{r} = (1-t)\bar{i} + (2t-1)\bar{j} + (t+2)\bar{k}$ is

(a) $5\sqrt{2}$ (b) $15\sqrt{2}$ (c) $\frac{15}{\sqrt{2}}$ (d) $\frac{5}{\sqrt{2}}$

59. $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

$\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$

are three unit vectors such that $\bar{c}$ is perpendicular to both $\bar{a}$ and $\bar{b}$ and $\bar{a}$ and $\bar{b}$ are inclined at an angle of $\frac{\pi}{6}$. Then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to

(a) 0 (b) $\frac{1}{4}$ (c) 1 (d) $4.\frac{\sum a_1^2 \sum b_1^2 \sum c_1^2}{3}$

60. The vectors $2\bar{i} - m\bar{j} + m\bar{k}$ and $(1 + m)\,\bar{i} - 2m\bar{j} - 7\bar{k}$ include an acute angle

(a) for all values of m

(b) if m lies in the interval $\left[\frac{1}{2}, 2\right]$

(c) if m lies outside the interval $\left[\frac{1}{2}, 2\right]$

(d) if m lies outside the interval $\left[-\frac{1}{2}, -2\right]$

61. a, b, c are three distinct non-zero numbers. If the vectors $a\bar{i} - a\bar{j} + b\bar{k}$, $\bar{i} + \bar{k}$ and $c\bar{i} + c\bar{j} + b\bar{k}$ are coplanar, then a, b, c are in

(a) AP (b) HP (c) GP (d) AGP

62. If $\bar{a}, \bar{b}, \bar{c}$ are three vectors $(\bar{a} - \bar{b}) \times (\bar{a} - \bar{b} - \bar{c}) \cdot (\bar{a} + 2\bar{b} - \bar{c})$ is equal to

(a) $-[\bar{a}\ \bar{b}\ \bar{c}]$ (b) $[\bar{a}\ \bar{b}\ \bar{c}]$ (c) $2[\bar{a}\ \bar{b}\ \bar{c}]$ (d) $3[\bar{a}\ \bar{b}\ \bar{c}]$

63. Let L_1 be the line of intersection of the planes $3x + 2y + z = 5$ and $x + y - 2z = 3$ and L_2 be the line of intersection of the planes $2x - y - z = 0$ and $7x + 10y - 8z = 0$. Then the angle between L_1 and L_2 is

(a) 90° (b) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (c) $\cos^{-1}\left(\frac{2}{3}\right)$ (d) $\cos^{-1}\left(\frac{1}{3}\right)$

64. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta =$

(a) $\frac{1}{3}$ (b) $\sqrt{3}$ (c) $\frac{4}{3}$ (d) $\frac{2}{3}$

65. Let ℓ represent the line whose equation is $\bar{r} = 2\bar{i} - 3\bar{j} + 5\bar{k} + \lambda(\bar{i} - \bar{j} + 2\bar{k})$ and p, the plane whose equation is $\bar{r}.(3\bar{i} + \bar{j} - \bar{k}) + 2 = 0$. Then which of following is true?

(a) line ℓ is perpendicular to the plane p

(b) the line ℓ is parallel to the plane p, and is at a distance of 3 units from p

(c) the line ℓ is passing through the point (2, –3, 5) on the plane but include an angle $\frac{\pi}{6}$ with the plane.

(d) the line ℓ lies in the plane

66. $\bar{a}$ and $\bar{b}$ are two vectors of equal magnitude and $|\bar{b}||\bar{c}| = 4$. If $\bar{b} \times (\bar{b} \times \bar{c}) + 4\bar{a} = 0$ and If θ is the angle between $\bar{b}$ and $\bar{c}$, then $\sin 2\theta + \cos 2\theta$ equals

(a) 1 (b) –1 (c) 0 (d) $\frac{1}{2}$

67. If x, y, z are positive real numbers other than unity, $\bar{p} = \bar{i} + (\log_x y)\bar{j} + (\log_x z)\bar{k}$, $\bar{q} = (\log_y x)\bar{i} + \bar{j} + (\log_y z)\bar{k}$, $\bar{r} = (\log_z x)\bar{i} + (\log_z y)\bar{j} + \bar{k}$ then

(a) $\bar{p}, \bar{q}, \bar{r}$ are sides of an equilateral triangle (b) $\bar{p}, \bar{q}, \bar{r}$ are sides of a right angled triangle

(c) $\bar{p}, \bar{q}, \bar{r}$ are collinear (d) $\bar{p}, \bar{q}, \bar{r}$ are coplanar

68. If α, β, γ are the roots of $x^3 + px + q = 0$ where, $pq \neq 0$ then the volume of the parallelepiped with coterminous edges $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$, $\beta\hat{i} + \gamma\hat{j} + \alpha\hat{k}$, $\gamma\hat{i} + \alpha\hat{j} + \beta\hat{k}$ is

(a) 0 (b) $(p + q)^3$ (c) pq^2 (d) $p^3 + q^3$

69. The value of k for which the image of (5, 1, –3) by reflexion in $2x + y - 3z + k = 0$ is the point (1, –1, 3) is

(a) 3 (b) 1 (c) –6 (d) 2

70. The radius of the sphere touching the planes $x + 2y + 2z + 3 = 0$ and $2x + 4y + 4z + 9 = 0$ is

(a) 2 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

71. The set of values of m for which the vectors $\bar{a} = \hat{i} + m\hat{j} + mx^2\hat{k}$, and $\bar{b} = 5\hat{i} + 2x\hat{j} + m\ \bar{k}$ include an obtuse angle between them for all x is

(a) the set of positive real numbers (b) the set of negative real numbers

(c) $R - \{0\}$ (d) ϕ

72. If the sum of two unit vectors is a vector of length $\frac{1+\sqrt{3}}{\sqrt{2}}$, then the angle between the two given vectors is

(a) 0 (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

73. If (i) $\bar{a}$ and $\bar{b}$ are the position vectors of A and B with respect to O; (ii) C is a point on OA dividing it in the ratio 1:3 and (iii) D is a point so that CD is parallel to OB and OB:CD = 4 : 1, then $\overline{AD} \times \overline{AB}$ is equal to

(a) $\frac{\bar{b}+\bar{a}}{2}$ (b) $\frac{\bar{a}\times\bar{b}}{2}$ (c) $\frac{\bar{b}\times\bar{a}}{2}$ (d) $\frac{\bar{b}\times\bar{a}}{4}$

74. D(l, 0, 0) , E(0, m, 0) , F(0, 0, n) are respectively the points that divide the sides BC, CA, AB of a Δ ABC in the ratio 1 : 2. Then $\frac{AB^2 + BC^2 + CA^2}{\ell^2 + m^2 + n^2}$ is

(a) 2 (b) 4 (c) 6 (d) 8

75. If $\bar{a}$ and $\bar{b}$ are unit vectors and θ is the angle between them, then $|\bar{a} - \bar{b}|\cos\frac{\theta}{2} - |\bar{a} + \bar{b}|\sin\frac{\theta}{2}$ is

(a) $\tan\theta$ (b) –1 (c) 0 (d) 1

76. If the vectors $3\hat{i} + 2\hat{j} + 4\hat{k}$ and $p\hat{i} + q\hat{j} + r\hat{k}$ are perpendicular, then the expression $27p^3 + 8q^3 + 64r^3$ is equal to

(a) $p^2 q + q r^2 + r p^2$ (b) 72 p q r (c) 24 p q r (d) 12 p q r

77. If $\bar{i} - \bar{j} + 3\bar{k}$ and $3\hat{i} + \hat{j} - \hat{k}$ represent two of the vertices of an equilateral triangle whose circumcentre is at $\hat{i} + \hat{j} + \hat{k}$, then its third vertex is represented by

(a) $\bar{i} + 3\bar{j} + \bar{k}$ (b) $-\bar{i} + 3\bar{j} + \bar{k}$ (c) $\bar{i} + 3\bar{j} - \bar{k}$ (d) $\bar{i} - 3\bar{j} + \bar{k}$

78. If $p\left(\bar{a}\times\bar{b}\right) + q\left(\bar{b}\times\bar{c}\right) + r\left(\bar{c}\times\bar{a}\right) = 0$ where p, q, r are scalars, not all equal to zero then is

(a) 0 (b) 4 (c) 2 (d) –2

79. If the volume of the parallelepiped having coterminous edges $\hat{i} + x\hat{j} + x^2k$,.$x\hat{i} + x^2\hat{j} + \hat{k}$, $x^2\hat{i} + \hat{j} + x\hat{k}$ is 4 units then the volume of the parallelepiped with coterminous edges $(x^3 - 1)\ \hat{i} + (x - x^4)\ \hat{k}$, $(x - x^4)\ \hat{j} + (x^3 - 1)\ \hat{k}$, $(x - x^4)\ \hat{i} + (x^3 - 1)\ \frac{\sqrt{3}}{2}$ is

(a) 1 (b) 4

(c) 16 (d) cannot be determined

80. $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar, $\bar{a} + \bar{b} + \bar{c}$ is collinear with $\bar{d}$ and $\bar{b} + \bar{c} + \bar{d}$ is collinear with $\bar{a}$. Then $|\bar{a} + \bar{b} + \bar{c} + \bar{d}|$ is

(a) 0 (b) 2 (c) 1 (d) 3

81. The vectors $3\bar{i} + 5\bar{j} + 2\bar{k}$, $2\bar{i} - 3\bar{j} - 5\bar{k}$ and $5\bar{i} + 2\bar{j} - 3\bar{k}$ form the sides of a triangle which is
(a) equilateral
(b) isosceles, but not right-angled
(c) right-angled, but not isosceles
(d) right-angled and isosceles

82. The position vectors of 4 points A, B, C and D are respectively $2\bar{i} + 3\bar{j} + 5\bar{k}$, $\bar{i} + 2\bar{j} + 3\bar{k}$, $-5\bar{i} + 4\bar{j} - 2\bar{k}$ and $\bar{i} + 10\bar{j} + 10\bar{k}$. Then
(a) AC is parallel to BD
(b) AC is perpendicular to BD
(c) AB is parallel to CD
(d) AB is perpendicular to CD

83. If $\left|\bar{a}\right| = 3, \left|\bar{b}\right| = 4, \bar{a} \cdot \bar{b} = 9$, then $\left|\bar{a} \times \bar{b}\right|$ is equal to
(a) $3\sqrt{7}$
(b) $7\sqrt{3}$
(c) 63
(d) 147

84. The number of unit vectors perpendicular to each of the two non collinear vectors $\bar{a} = (a_1, a_2, a_3)$; $\bar{b} = (b_1, b_2, b_3)$ is
(a) 1
(b) 2
(c) 0
(d) infinite

85. A line makes angles 60° and 45° with the x-axis and z-axis respectively. The angle made by it with the y-axis is
(a) 30° or 150°
(b) 60° or 120°
(c) 45° or 135°
(d) 90°

86. Angle between the lines x = 2, y = 3 and y = 3, z = 5 is
(a) $\frac{\pi}{4}$
(b) $\frac{\pi}{3}$
(c) $\frac{\pi}{2}$
(d) $\frac{\pi}{6}$

87. The line $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-7}{7}$ is parallel to the plane
(a) $x - 3y + z + 1 = 0$
(b) $2x + y - z + 2 = 0$
(c) both (a) and (b)
(d) None of these

88. Equation of the line through the point (1, 2, –1) and perpendicular to each of the lines $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$ and $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ is
(a) $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z+1}{1}$
(b) $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z+1}{1}$
(c) $\frac{x}{1} = \frac{y}{-2} = \frac{z}{1}$
(d) $\frac{x+1}{1} = \frac{y+2}{2} = \frac{z+1}{1}$

89. Equations of the line through the point (1, 4, –2) and parallel to the planes $6x + 2y + 2z + 3 = 0$ and $x + 2y - 6z + 4 = 0$ are
(a) $\frac{x-1}{-8} = \frac{y-4}{19} = \frac{z+2}{5}$
(b) $\frac{x-1}{-8} = \frac{y-4}{19} = \frac{z+2}{-5}$
(c) $\frac{x+8}{1} = \frac{y-19}{4} = \frac{z-5}{-2}$
(d) $\frac{x+8}{-1} = \frac{y-19}{-4} = \frac{z-5}{-2}$

90. A set of direction cosines of the line of intersection of the planes $x + y + 2z - 3 = 0$ and $2x + 3y + 7z - 10 = 0$ is
(a) $\left(\frac{-3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$
(b) $\left(\frac{1}{11}, \frac{-3}{11}, \frac{1}{11}\right)$
(c) $\left(\frac{1}{\sqrt{11}}, \frac{-3}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$
(d) (1, -3, 1)

91. The point of intersection of the lines $\bar{r} = \bar{i} + \bar{j} - \bar{k} + \lambda\left(3\bar{i} - \bar{j}\right)$ and $\bar{r} = 4\bar{i} - \bar{k} + \mu\left(2\bar{i} + 3\bar{k}\right)$ is
(a) (4, 0, -1)
(b) (0, 4, 1)
(c) they are parallel lines and hence do not intersect
(d) they are skew lines and hence do not intersect

92. The position vector of a point is $\bar{i} - \bar{j} + 2\bar{k}$. Equation of the line through the point, perpendicular to the plane $\bar{r} \cdot \left(2\bar{i} - \bar{j} + 3\bar{k}\right) = 5$ is

(a) $\frac{x-2}{1} = \frac{y+1}{-1} = \frac{z-3}{2}$ (b) $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-2}{3}$

(c) $\frac{x+1}{-2} = \frac{y-1}{1} = \frac{z+2}{-3}$ (d) $\frac{x+2}{-1} = \frac{y-1}{1} = \frac{z+3}{-2}$

93. If $\left|\bar{a}\right| = 1 = \left|\bar{b}\right|$ and $\left|\bar{a} + \bar{b}\right| = \sqrt{3}$, then $\left(2\bar{a} - 5\bar{b}\right) \cdot \left(3\bar{a} + \bar{b}\right)$ is equal to

(a) $-\frac{11}{2}$ (b) $\frac{11}{2}$ (c) -11 (d) 11

94. Let $\bar{u} = \bar{i} + \bar{j}$, $\bar{v} = \bar{i} - \bar{j}$ and $\bar{w} = \bar{i} + 2\bar{j} + 3\bar{k}$. If $\bar{n}$ is a unit vector such that $\bar{u} \cdot \bar{n} = 0$ and $\bar{v} \cdot \bar{n} = 0$, then $\left|\bar{w} \cdot \bar{n}\right|$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

95. If $\bar{a} = \bar{i} + \bar{j} + \bar{k}$; $\bar{b} = 4\bar{i} + 3\bar{j} + 4\bar{k}$ and $\bar{c} = \bar{i} + 2\alpha\,\bar{j} + 3\beta\bar{k}$ are coplanar and $\left|\bar{c}\right| = \sqrt{6}$, then (α, β) is

(a) $\left(1, \frac{1}{3}\right)$ (b) $\left(-1, \frac{1}{3}\right)$ (c) $\left(-1, -\frac{1}{3}\right)$ (d) both (a) and (b)

96. If $\bar{r} \times \bar{b} = \bar{c} \times \bar{b}$ and $\bar{r} \cdot \bar{a} = 0$ where $\bar{a} = 2\bar{i} + 3\bar{j} - \bar{k}$, $\bar{b} = 3\bar{i} - \bar{j} + \bar{k}$, $\bar{c} = \bar{i} + \bar{j} + 3\bar{k}$, then $\bar{r}$ is

(a) $-2\left(\bar{i} - \bar{j} - \bar{k}\right)$ (b) $2\left(\bar{i} + \bar{j} - \bar{k}\right)$ (c) $2\left(-\bar{i} + \bar{j} - \bar{k}\right)$ (d) $\bar{i} + \bar{j} + \bar{k}$

97. Consider the vector $\bar{r} = \left(\bar{a} \cdot \bar{b}\right)\bar{c} - \left(\bar{a} \cdot \bar{c}\right)\bar{b}$ and the following statements:

(1) $\bar{r}$ is perpendicular to $\bar{a}$ (2) $\bar{r}$ is perpendicular to $\bar{b}$

(3) $\bar{r}$ is perpendicular to $\bar{c}$ (4) $\bar{r}$ is perpendicular to $\bar{b} \times \bar{c}$; then,

(a) (1) and (2) are correct (b) (1) and (3) are correct

(c) (1) and (4) are correct (d) (2), (3) and (4) are correct

98. P, Q, R and S are points in a plane with position vectors $\bar{p}, \bar{q}, \bar{r}$ and $\bar{s}$ respectively such that $\left(\bar{p} - \bar{s}\right) \cdot \left(\bar{q} - \bar{r}\right) = \left(\bar{q} - \bar{s}\right) \cdot \left(\bar{r} - \bar{p}\right) = 0$. Then, for the triangle PQR, S is the

(a) centroid (b) circumcentre (c) orthocentre (d) in-centre

99. The equation of the plane through the points (3, 4, 2) and (7, 0, 6) and perpendicular to the plane $2x - 5y = 15$ is

(a) $5x + 2y - 3z = 17$ (b) $5x + 2y + 3z = 17$

(c) $5x - 2y + 3z = 17$ (d) $5x - 2y - 3z = 17$

100. P is the point (2, 1, 6). The point Q is such that PQ is perpendicular to the plane $x + 1y - 2z = 3$ and such that the midpoint of PQ lies on the plane. Coordinates of Q are

(a) $(2, 1, -6)$ (b) $(6, 5, -2)$ (c) $(8, -1, -4)$ (d) $(6, 1, -4)$

101. Equation of the plane through the origin which contains the line $\frac{x-a}{\ell} = \frac{y-b}{m} = \frac{z-c}{n}$ is

(a) $x(bn - cm) + y(c\ell - an) + z(am - b\ell) = 0$

(b) $(x - a)\ell + (y - b)m + (z - c)n = 0$

(c) $(x - 1)a + (y - m)b + (z - n)c = 0$

(d) $x(a\ell - bm) + y(bm - cn) + z(cn - a\ell) = 0$

102. The three lines $\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}$, $\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}$ and $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$ are coplanar if

(a) $\frac{\ell}{\alpha}(a-b)+\frac{m}{\beta}(b-c)+\frac{n}{\gamma}(c-a)=0$
(b) $\frac{\ell}{a}(\alpha-\beta)+\frac{m}{b}(\beta-\gamma)+\frac{n}{c}(\gamma-\alpha)=0$
(c) $\frac{\ell}{a}(b-c)+\frac{m}{b}(c-a)+\frac{n}{c}(a-b)=0$
(d) $\frac{\ell}{\alpha}(b-c)+\frac{m}{\beta}(c-a)+\frac{n}{\gamma}(a-b)=0$

103. The vector equation of the sphere which has points $\bar{a}$ and $\bar{b}$ as ends of a diameter is

(a) $(\bar{r}+\bar{a}).(\bar{r}+\bar{b})=0$
(b) $(\bar{r}-\bar{a}).(\bar{r}-\bar{b})=0$
(c) $(\bar{r}-\bar{a})\times(\bar{r}-\bar{b})=0$
(d) $(\bar{r}-\bar{a}).(\bar{r}-\bar{b})=1$

104. Let $\bar{a}_i$, i = 1, 2, … n be n unit vectors. If each one is perpendicular to the sum of the remaining n–1 vectors then the magnitude of the sum of all a_i s is

(a) n^2
(b) n+1
(c) n–1
(d) $\sqrt{n}$

105. The position vectors $\bar{a},\bar{b},\bar{c}$ and $\bar{d}$ of four points A, B, C, D on a plane are such that $\frac{|(\bar{d}-\bar{a})\times(\bar{b}-\bar{a})|}{|\bar{b}-\bar{a}|}=\frac{|(\bar{d}-\bar{b})\times(\bar{c}-\bar{b})|}{|\bar{c}-\bar{b}|}=\frac{|(\bar{d}-\bar{c})\times(\bar{a}-\bar{c})|}{|\bar{a}-\bar{c}|}$. Then D is the

(a) centroid of Δ ABC
(b) circumcentre of Δ ABC
(c) incenter of Δ ABC
(d) orthocenter of Δ ABC

106. If the plane containing $\bar{p}$ and $\bar{q}$ is normal to the plane containing $\bar{r}$ and $\bar{s}$ then the projection of $\bar{r}\times\bar{s}$ on $\bar{p}\times\bar{q}$ is

(a) $|\bar{r}\times\bar{s}|$
(b) –1
(c) 0
(d) 1

107. If $\bar{a},\bar{b},\bar{c}$ are 3 non-coplanar unit vectors such that $\bar{a}\times(\bar{b}\times\bar{c})=\frac{1}{2}(\bar{b}-\sqrt{3}\bar{c})$, $\bar{b}$ and $\bar{c}$ are non parallel, then the angles that $\bar{a}$ makes with $\bar{b}$ and $\bar{c}$ are respectively

(a) $\frac{\pi}{6},\frac{\pi}{3}$
(b) $\frac{\pi}{3},\frac{\pi}{6}$
(c) $\frac{5\pi}{6},\frac{\pi}{3}$
(d) $\frac{\pi}{6},\frac{2\pi}{3}$

108. The distance of the point (–1 ,2 ,5) from the line joining (0, 3, 6) and (–2, 1, 4) is

(a) 1
(b) 0
(c) $\sqrt{2}$
(d) 2

109. The number of points common to the line joining A(–4, 0, 2) and B(–1, 2, 1) and the surface $x^2+y^2+z^2=6$ is

(a) 0
(b) 1
(c) 2
(d) 3

110. If, from the point P(α, β, γ), perpendiculars PL, PM are drawn to the planes YOZ and ZOX , and if the plane OLM passes through (γ, 0, β) then

(a) α, β, γ are in GP
(b) α, γ , β are in GP
(c) β, α, γ are in GP
(d) α, γ, β are in AP

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

111. Statement 1

If $\overline{x}, \overline{y}, \overline{z}$ are the position vectors of three points A, B, C which are collinear, then $\ell\overline{x} + m\overline{y} + n\overline{z} = 0$ for scalar ℓ, m, n where $\ell + m + n = 0$.

and

Statement 2

If $\overline{a}$ and $\overline{b}$ are the position vectors of two points A and B, then, the position vector of the mid point of AB is $\frac{1}{2}(\overline{a} + \overline{b})$.

112. Statement 1

If a plane makes intercepts 4, 3, 2 on the x, y and z axes, the direction ratios of the normal to the plane are $\left[\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right]$.

and

The direction ratios of the normal to the plane $Ax + By + Cz + D = 0$ are [A, B, C].

113. Statement 1

The point (1, 0, –5) lies on the line $\frac{x-3}{2} = \frac{y+1}{-1} = \frac{z+3}{2}$

and

Statement 2

The direction ratios of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $[x_2 - x_1, y_2 - y_1, z_2 - z_1]$

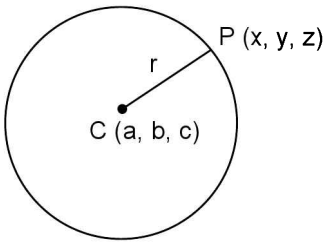
Linked Comprehension Type Questions

Directions: **This section contains 1 paragraph. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage 1

A sphere is the locus of a point, which moves in space such that it is at a constant distance from a fixed point in space.

The fixed point is called the centre of the sphere and the constant distance is called the radius of the sphere.



If P(x, y, z) represents any point of the sphere whose centre is at the point (a, b, c) and whose radius is r, then (x, y, z) satisfies $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$.

The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere whose centre is at(–u, –v, –w) and radius is $\sqrt{u^2 + v^2 + w^2 - d}$ if $u^2 + v^2 + w^2 \nless d$

The equation of the sphere on the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter is given by $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

Example 1

To find the equation of the sphere passing through the points (4, –1, 2); (0, –2, 3); (1, –5, –1) ; (2, 0, 1) we proceed as follows.

Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

Since the sphere passes through the 4 given points we have

$16 + 1 + 4 + 8u - 2v + 4w + d = 0$

$0 + 4 + 9 - 4v + 6w + d = 0$

$1 + 25 + 1 + 2u - 10v - 2w + d = 0$ and

$4 + 0 + 1 + 4u + 2w + d = 0$

Solving the 4 equations above, the equation of the sphere is obtained as

$x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$

Example 2

Consider the sphere $3(x^2 + y^2 + z^2) - 2x - 2y - 2z - 1 = 0$

The equation may be written as $x^2 + y^2 + z^2 - \frac{2}{3}x - \frac{2}{3}y - \frac{2}{3}z - \frac{1}{3} = 0$

Its centre is at $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and its radius is $\sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{3}} = \sqrt{\frac{2}{3}}$

114. The equation of the sphere passing through the three points (3, 0, 2), (–1, 1, 1), (2, –5, 4) and having its centre on the plane $2x + 3y + 4z = 6$ is

(a) $x^2 + y^2 + z^2 + x + 2y - 37 = 0$
(b) $2(x^2 + y^2 + z^2) + x - 2y + 4z - 11 = 0$
(c) $x^2 + y^2 + z^2 + 4y - 6z - 1 = 0$
(d) $x^2 + y^2 + z^2 + 4x - 2y - 25 = 0$

115. Equation of the sphere having its centre on the line $5y + 2z = 0 = 2x - 3y$ and passing through the two points (0, –2, –4) , (2, –1, –1) is

(a) $x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0$
(b) $x^2 + y^2 + z^2 - 8x + 4y + z - 8 = 0$
(c) $x^2 + y^2 + z^2 + x - 3y + 4z + 7 = 0$
(d) None of these

116. A sphere of constant radius 10 passes through the origin and meets the axes in A, B, C. The locus of the centroid of the tetrahedron OABC where O is the origin, is

(a) $x^2 + y^2 + z^2 = 200$
(b) $x^2 + y^2 + z^2 = 25$
(c) $x^2 + y^2 + z^2 = 100$
(d) $x^2 + y^2 + z^2 = 50$

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

117. $\bar{a}$ and $\bar{b}$ are vectors of magnitude 3 each and $|\bar{a} + 2\bar{b}| = 7$. Then

(a) $\bar{a}.\bar{b} = 1$
(b) Angle between and $\bar{a}$ and $\bar{b}$ is obtuse
(c) If θ is the angle between $\bar{a}$ and $\bar{b}$, $\sin\theta = \frac{4\sqrt{5}}{9}$
(d) $|\bar{a} \times \bar{b}| = 2\sqrt{5}$

118. Vectors $\bar{a}, \bar{b}$ and $\bar{c}$ are given to be coplanar. Then

(a) set of vectors $\{\bar{a} + \bar{b} - \bar{c}, \bar{b} + \bar{c} - \bar{a}, \bar{c} + \bar{a} - \bar{b}\}$ is coplanar
(b) set of vectors $\{\bar{a} + 2\bar{b}, \bar{b} + 2\bar{c}, \bar{c} + 2\bar{a}\}$ is coplanar
(c) set of vectors $\{\bar{a} \times \bar{b}, \bar{b} \times \bar{c}, \bar{c} \times \bar{a}\}$ is coplanar
(d) set of vectors $\{3\bar{a} - 7\bar{b} + 16\bar{c}, 3b + \bar{c} - 2\bar{a}, 3\bar{c} + \bar{a} - 2\bar{b}\}$ are coplanar

119. Consider the planes

$3x + y - 5z = 6$

$x + 2y + z + 1 = 0$. Then,

(a) Equation of the plane through the line of intersection of the given planes and perpendicular to the plane $3x + y - 5z = 6$ is $x + 2y + z + 1 = 0$

(b) Angle between the given planes is $\frac{\pi}{2}$

(c) Angle between the given planes is $\cos^{-1}\left(\frac{3}{7}\right)$

(d) Length of perpendicular from (3, 1, 1) on the plane $3x + y - 5z = 6$ is $\frac{1}{\sqrt{35}}$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

120. Let $\bar{a}, \bar{b}, \bar{c}$ and $\bar{x}, \bar{y}, \bar{z}$ be two sets of non coplanar unit vectors such that $\bar{a} \times (\bar{b} \times \bar{c}) = \frac{1}{\sqrt{2}}(\bar{b} + \bar{c})$, $\bar{x} \times (\bar{y} \times \bar{z}) = -\frac{\sqrt{3}}{2}\bar{z}$. Then

Column I	**Column II**
(a) angle between $\bar{a}$ and $\bar{c}$	(p) $\frac{\pi}{4}$
(b) angle between $\bar{a}$ and $\bar{b}$	(q) $\frac{\pi}{6}$
(c) angle between $\bar{x}$ and $\bar{z}$	(r) $\frac{\pi}{2}$
(d) angle between $\bar{x}$ and $\bar{y}$	(s) $\frac{3\pi}{4}$

ADDITIONAL PRACTICE EXERCISE

Subjective Questions

121. In a triangle ABC, the side BC is divided at D in the ratio 5 : 2 and the side AC is divided at E in the ratio 2 : 5. If AD and BE intersect at F, find the ratios $\frac{AF}{FD}$ and $\frac{BF}{FE}$ using vector method.

122. Median AD of a triangle ABC is bisected at E and BE produced meets AC at F. Using vector method, prove that $AF = \frac{1}{3}AC$ and $EF = \frac{1}{4}BF$.

123. $\bar{a}, \bar{b}, \bar{c}$ are three non-coplanar unit vectors equally inclined to one another at an angle of $\frac{\pi}{3}$. If $(\bar{a}\times\bar{b})+(\bar{b}\times\bar{c}) = p\bar{a}+q\bar{b}+r\bar{c}$. Find p, q, r.

124. The direction ratios l, m. n of a line are connected by the relations $l + m + n = 0$, $2lm + 2ln - mn = 0$. Find all possible values that can be assigned to l, m, n.

125. Find the point R in which the line AB cuts the plane CDE, where the position vectors of the points A, B, C, D, E are given by $\bar{i} + 2\bar{j} + \bar{k}$, $2\bar{i} + \bar{j} + 2\bar{k}$, $-4\bar{j} + 4\bar{k}$, $2\bar{i} - 2\bar{j} + 2\bar{k}$, $4\bar{i} + \bar{j} + 2\bar{k}$ respectively.

126. (i) Forces $\bar{P}, \bar{Q}$ act at O and have a resultant $\bar{R}$. If any transversal cuts their lines of action at A, B and C respectively, prove that $\frac{|\bar{P}|}{OA} + \frac{|\bar{Q}|}{OB} = \frac{|\bar{R}|}{OC}$.

(ii) A transversal cuts the sides OL, OM and diagonal ON of a parallelogram at A, B and C respectively. Prove that $\frac{OL}{OA} + \frac{OM}{OB} = \frac{ON}{OC}$.

127. P is any point in the plane $lx + my + nz = k$. O is the origin. A point Q is taken on the line OP such that $OP.OQ = k^2$. Find the locus of Q.

128. A triangle, whose sides are of length a, b, c is placed so that the middle points of its sides are on the axes of coordinates. Find the equation of the plane of the triangle.

129. (i) If ABCD is a trapezium with AB parallel to CD, $\overline{DC} = t\,\overline{AB}$, t is a scalar. If A is the origin and $\bar{b}$ and $\bar{d}$ are the position vectors of B and D respectively, then find the position vector of the point of intersection of the diagonals.

(ii) Find the equation to the line RS where R divides AB in the ratio 3 : 1 and S divides DC in the ratio 1 : 3.

(iii) Show that the point of intersection of diagonals in (i) lies on RS.

130. (i) Q is any point on the plane $x + y + z + 1 = 0$. P is a variable point on OQ such that $OQ \times OP = 4$, where O is the origin. Find the locus of P.

(ii) Find the distance of a point on the locus of P from (–2, –2, –2).

(iii) Hence identify the locus.

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

131. The position vector of the circumcenter of ΔABC where the position vectors of A, B, C are respectively $7\hat{j} + 10\hat{k}$, $-\hat{i} + 6\hat{j} + 6\hat{k}$, and $-4\hat{i} + 9\hat{j} + 6\hat{k}$ is

(a) $-2\hat{i} + 8\hat{j} + 8\hat{k}$ (b) $-2\hat{i} - 8\hat{j} + 8\hat{k}$ (c) $-2\hat{i} + 8\hat{j} - 8\hat{k}$ (d) $2\hat{i} + 8\hat{j} + 8\hat{k}$

132. $(1, 2, -1)$ is the foot of perpendicular from $(0, 1, 2)$ to a plane. The distance of$(2, 2, 1)$ from the plane is

(a) $\frac{5}{\sqrt{11}}$ (b) $\frac{\sqrt{5}}{11}$ (c) $\sqrt{\frac{5}{11}}$ (d) $\frac{5}{11}$

133. If the quadratic equation $ax^2 + (a + b - c)x + b = 0$ where a, b, c are non-negative, has equal roots, then the lines whose direction ratios l, m, n are given by $al + bm + cn = 0$, $mn + nl + lm = 0$ are

(a) perpendicular (b) inclined at 45° to one another

(c) parallel (d) inclined at 60° to one another

134. If $\bar{a}$ and $\bar{b}$ are the position vectors of 2 fixed points A and B respectively and $\bar{r}$ is the position vector of a variable point, then $(\bar{r} - \bar{a}) \times \bar{b} = 0$ represents

(a) line through A and parallel to $\overline{OB}$ (b) line through A and perpendicular to $\overline{OB}$

(c) plane containing A and parallel to $\overline{OB}$ (d) plane containing A and perpendicular to $\overline{OB}$

135. Let $\bar{a} = \bar{i} + 2\bar{j} + \bar{k}$ and $\bar{b} = -2\bar{i} + \bar{j}$ Given that $(\bar{a} \times \bar{c}) + \bar{b} = 0$ and $\bar{a}.\bar{c} = 2$, $|\bar{c}|$ equals

(a) $\sqrt{\frac{2}{3}}$ (b) $\sqrt{\frac{3}{2}}$ (c) $\frac{3}{2}$ (d) $\frac{2}{3}$

136. If $|\ell\bar{x} + m\bar{y}| = 10$ and if $\bar{x}$ and $\bar{y}$ are perpendicular, then $|\ell\bar{x} - m\bar{y}|$ is

(a) ℓ (b) 10 (c) m (d) $\sqrt{\ell^2 + m^2}$

137. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are coplanar vectors and $\bar{d}$ is any vector, $[\bar{a} \times \bar{b}, \bar{a} \times \bar{c}, \bar{d}]$ equals

(a) $|\bar{a}| + |\bar{b}| + |\bar{c}| + |\bar{d}|$ (b) 0

(c) $|\bar{a}||\bar{b}||\bar{c}||\bar{d}|$ (d) 1

138. If $\bar{a}.\bar{b} + \bar{c}.\bar{d} = 0 = \bar{a}.\bar{c} + \bar{b}.\bar{d}$, then the angle between $\bar{a} - \bar{d}$ and $\bar{b} - \bar{c}$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) 0

139. An arc PQ of a circle subtends an angle 120° at the centre O. The point R divides the arc PQ in the ratio 1 : 3 . If $\left(\frac{3}{2}, 1, \frac{3}{2}\right)$ and $\overline{OQ} = \bar{q}$, then $\left(\frac{3}{2}, 1, \frac{3}{2}\right)$ in terms of $\bar{p}$ and $\bar{q}$ is

(a) $\frac{1}{\sqrt{3}}(\bar{p} + \bar{q})$ (b) $\frac{1}{\sqrt{3}}(2\bar{p} + \bar{q})$ (c) $\frac{1}{\sqrt{3}}(\bar{p} + 2\bar{q})$ (d) $\frac{1}{\sqrt{3}}(2\bar{p} - \bar{q})$

140. The set of values of λ for which the three planes $\lambda x + 3y - z - 2 = 0$, $3x + 3y + z - 4 = 0$ and $x - y + \lambda z - 5 = 0$ intersect in a unique point is,

(a) R (b) ϕ (c) Q (c) $R - \{0\}$

141. The sides of a ΔABC are represented by $\bar{p} = (\sin A)\hat{i} + (\sin B)\hat{j} + (\sin C)\hat{k}$; $\bar{q} = (\cos A)\hat{i} + (\cos B)\hat{j} + (\cos C)\hat{k}$; $\bar{r} = (\cos^3 A)\hat{i} + (\cos^3 B)\hat{j} + (\cos^3 C)\hat{k}$. Then ΔABC is

(a) isosceles (b) equilateral (c) either (a) or (b) (d) neither (a) nor (b)

142. If ℓ, m, n are the roots of the cubic equation $x^3 - 2x^2 + x - 5 = 0$, then the acute angle between the lines $\frac{x+1}{\ell} = \frac{y-1}{m} = \frac{z+2}{n}$ and $\frac{x-3}{n} = \frac{y-2}{\ell} = \frac{z+5}{m}$ is

(a) $\frac{\pi}{3}$ (b) 0 (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

143. The acute angle between the planes through the points (0, 4, –3) and (6, –4, 3) but not through the origin which are such that the sum of the intercepts cut off on the coordinate axes is 0, is

(a) $\cos^{-1}\frac{14}{19}$ (b) $\cos^{-1}\frac{15}{49}$ (c) 0 (d) $\frac{\pi}{3}$

144. A(1, –3, 3/2), B(3, –7, 4), C(p, –3, q) are the vertices of a Δ ABC. If the median through A is equally inclined to the positive directions of coordinate axes, then the values of p and q are respectively.

(a) –5, –5 (b) –5, 5 (c) 5, –5 (d) 5, 5

145. The number of possible values of λ for which the line joining the points $(2\lambda + 3, -4, 3)$ and $(4, 7, \lambda + 2)$. also passes through (2,1,3) is

(a) 2 (b) infinitely many (c) 0 (d) –1

146. Through a point $P(\alpha, \beta, \gamma)$ a plane is drawn at right angles to OP where O is the origin to meet the coordinates axes in A, B, C. Then the area of the triangle ABC is

(a) $\frac{\alpha^2 + \beta^2 + \gamma^2}{2\alpha\beta\gamma}$ (b) $\frac{(\alpha^2 + \beta^2 + \gamma^2)^{5/2}}{2\alpha\beta\gamma}$ (c) $\frac{\alpha\beta\gamma}{(\alpha^2 + \beta^2 + \gamma^2)^{5/2}}$ (d) $\frac{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}{2\alpha\beta\gamma}$

147. AB, BC are the diagonals of adjacent edges of a rectangular box with centre at the origin O, its edges being parallel to the axes. If the angles BOC, COA, AOB are α, β, γ respectively, then $\cos\alpha + \cos\beta + \cos\gamma =$

(a) 0 (b) 1 (c) –1 (d) 2

148. The plane $lx + my = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle θ, and the equation of the plane in the new position is $nz = O$. Then, n is

(a) $\pm\sqrt{\ell^2 + m^2}\tan\theta$ (b) $\tan\theta$ (c) $\ell^2 + m^2$ (d) –1

149. The median through the vertex A of Δ ABC is divided by E in the ratio 1 : 3 and BE extended meets the side AC at F. If $\bar{a}$ is the position vector of A and C is the origin then $\left|\overline{EF}\right| =$

(a) 1 (b) $\frac{1}{4}\left|\overline{BF}\right|$ (c) $\frac{1}{2}\left|\overline{BF}\right|$ (d) $\frac{1}{8}\left|\overline{BF}\right|$

150. ABCD is a parallelogram with A at origin. Let E be a point on AB dividing it in the ratio 3 : 1 and DE and AC intersect at a point P which is such that DP : PE is the same as CP : PA, then the common ratio =

(a) 1 : 2 (b) 3 : 4 (c) 4 : 3 (d) 1 : 1

151. L, M and N divide the sides BC, CA, AB respectively of ΔABC in the ratio m : 1. Then the vectors $\overline{AL}, \overline{BM}, \overline{CN}$ form a triangle, of area $\left(\overline{BC} = \bar{a}, \overline{CA} = \bar{b}, \overline{AB} = \bar{c}\right)$

(a) $\frac{m^2 + 1}{2m}\left|\bar{b} \times \bar{c}\right|$ (b) $\frac{m^2 + m + 1}{2(m+1)^2}\left|\bar{b} \times \bar{c}\right|$

(c) $\frac{3(m^2 + m + 1)}{2(m+1)^2}\left|\bar{b} \times \bar{c}\right|$ (d) $\frac{3(m^2 + m + 1)}{2(m+1)}\left|\bar{b} \times \bar{c}\right|$

152. ABCD is a trapezium with A at the origin. P and Q are the points that divide the non-parallel sides BC and AD in the ratio 1 : 3 then the ratio of areas of triangles APD and CQB is

(a) 1 : 1 (b) 2 : 1 (c) 3 : 2 (d) 2 : 3

153. The condition for the vectors $\alpha\hat{i} + \alpha^3\hat{j} + (\alpha^4 - 1)\hat{k}$, $\beta\hat{i} + \beta^3\hat{j} + (\beta^4 - 1)\hat{k}$, $\gamma\hat{i} + \gamma^3\hat{j} + (\gamma^4 - 1)\hat{k}$, where α, β, γ are distinct real numbers, to be coplanar is

(a) $\alpha + \beta + \gamma = \alpha\beta\gamma$ (b) $\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha) = \alpha + \beta + \gamma$

(c) $\alpha\beta + \beta\gamma + \alpha\gamma = 1$ (d) $\alpha\beta\gamma = 1$

154. In question **153** α, β, γ are positive and they are in GP, then β is

(a) 0 (b) 1 (c) 2 (d) 3

155. The angle between the vectors $\overline{AB} = \bar{p} \times (\bar{p} \times \bar{q})$ and $\overline{AC} = \bar{p} \times \bar{q}$ is

(a) π (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

156. In question **155** $\bar{p}$ is a unit vector, the other angles of Δ ABC are

(a) $\frac{\pi}{3}, \frac{\pi}{6}$ (b) $\frac{\pi}{5}, \frac{\pi}{12}$ (c) $\frac{\pi}{4}, \frac{\pi}{4}$ (d) $\frac{\pi}{9}, \frac{7\pi}{18}$

157. The reflection of $x + 2y + 2z = 0$ in the plane $4x - 3y + 12z + 13 = 0$

(a) $7x - 470y + 190z + 572 = 0$ (b) $7x + 470y + 90z - 572 = 0$

(c) $470x - 7y + 190z + 572 = 0$ (d) $90x + 7y + 19z + 72 = 0$

158. The planes $bx - ay = 1$, $cy - bz = 1$, $az - cx = -2$ intersect in a common line where a, b, c are positive numbers; then, the minimum value of $ab + bc + ca$ is

(a) abc (b) 3ac (c) 0 (d) 1

159. The range of values of k for which the planes $x + 2y + 3z = k$, $4x + 8y + 12z = 2k$, $9x - 7y + 16z = 5k$ form a triangular prism

(a) R (b) $R - \{0\}$ (c) $\left(-\frac{1}{4}, \infty\right)$ (d) $\left(-\frac{1}{4}, \frac{1}{4}\right)$

160. The value of m for which the homogeneous second degree equation $2x^2 + 3y^2 - 2z^2 - 5yz + mxy = 0$ represents two planes is

(a) ± 1 (b) -2 (c) ± 7 (d) ± 5

161. The foot of the perpendicular from (7, 5, 2) on $2x + y - 3z + 1 = 0$ is

(a) (–7, –5, –2) (b) (2, 5, 7) (c) (5, 4, 5) (d) (–5, 4, 5)

162. The equation of the projection of the line $\frac{x-7}{2} = \frac{y-5}{1} = \frac{z-2}{2}$ on the plane $2x + y - 3z + 1.= 0$ is

(a) $\frac{x-6}{5} = \frac{y-4}{3} = \frac{z-5}{5}$ (b) $\frac{x-5}{6} = \frac{y-4}{3} = \frac{z-5}{5}$

(c) $\frac{x-5}{5} = \frac{y-3}{9} = \frac{z-5}{2}$ (d) $\frac{x-6}{5} = \frac{y-9}{3} = \frac{z+5}{5}$

163. The 'shortest distance' between the lines $\frac{x-5}{3} = \frac{y-7}{-16} = \frac{z-3}{7}$, $\frac{x-6}{3} = \frac{y-5}{8} = \frac{z-20}{-5}$ is

(a) 10 (b) 12 (c) 14 (d) 24

164. The condition that the line with d.r.'s ℓ, m, n and passing through the origin intersect the straight line $\frac{x-7}{2} = \frac{y-5}{1} = \frac{z-2}{1}$ is

(a) $\ell + m + n = 0$ (b) $\ell - m + n = 0$ (c) $\ell = m + n$ (d) $m = \ell + n$

165. If, in question number 164, it is given that the angle between the two lines mentioned is 90^0, then the equations of the line through the origin is

(a) $\frac{x}{0} = \frac{y}{-1} = \frac{z}{1}$ (b) $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$ (c) $\frac{x}{1} = \frac{y}{0} = \frac{z}{1}$ (d) $\frac{x}{2} = \frac{y}{3} = \frac{z}{1}$

166. The line $\frac{x+1}{5} = \frac{y+7}{3} = \frac{z+6}{8}$ is the hypotenuse of an isosceles right triangle whose opposite vertex is (7, 2, 4) the direction ratios of the remaining sides satisfy the relation is

(a) $6\ell - 3m - 2n = 0$ (b) $3\ell - 6m + 2n = 0$ (c) $6\ell - 2m - 3n = 0$ (d) $6\ell + 2m - 3n = 0$

167. If $(\bar{a} \times \bar{b}).\bar{c} = 5$, then the value of $(2\bar{a} - 3\bar{b} + \bar{c}).(\bar{b} \times \bar{c}) + (\bar{a} - 4\bar{b} + 6\bar{c}).(\bar{a} \times \bar{b}) + (11\bar{a} - 6\bar{b} + 7\bar{c}).(\bar{c} \times \bar{a})$ is

(a) 34 (b) 10 (c) 30 (d) 0

168. Projection of $\bar{a} = 2\bar{i} - 3\bar{j} + \bar{k}$ along a vector perpendicular to $\bar{b} = \bar{i} + 3\bar{j} - \bar{k}$ has the absolute value.

(a) $\sqrt{\frac{10}{17}}$ (b) $3\sqrt{\frac{10}{11}}$ (b) $2\sqrt{\frac{10}{11}}$ (d) $\sqrt{\frac{10}{11}}$

169. The square of the distance between the origin and the image of the point (1, 2, 3) in the line $\frac{x}{-2} = \frac{y+1}{2} = \frac{z-1}{1}$ is

(a) $\frac{46}{3}$ (b) 15 (c) $\frac{23}{3}$ (d) $\frac{35}{3}$

170. The equation of the plane containing the line $\frac{x}{a} - \frac{y}{b} = 1, z = 0$ and parallel to the line $\frac{y}{c} + \frac{z}{a} = -1;\ x = 1$ is

(a) $\frac{x}{bc} + \frac{y}{ca} + \frac{z}{ab} = 1$ (b) $\frac{-x}{ac} + \frac{y}{bc} + \frac{z}{ab} = -1$ (c) $\frac{x}{bc} - \frac{y}{ca} + \frac{z}{ab} = -1$ (d) $\frac{-x}{a} + \frac{y}{b} + \frac{cz}{ab} = -1$

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

171. **Statement 1**

A vector normal to the vector $2\bar{i} + \bar{j} - 3\bar{k}$ lies in the plane $2x + y - 3z = 0$.

and

Statement 2

If $\bar{a}$ and $\bar{b}$ are two non-null vectors they lie in the same plane only if $\bar{a}.\bar{b} = 0$.

172. **Statement 1**

Points (1, –2, –5) and (2, 0, 1) lie on the origin side of the plane

$3x - y - 6z + 9 = 0$

and

Statement 2

If $(ax_1 + by_1 + cz_1 + d)$ and $(ax_2 + by_2 + cz_2 + d)$ are of the same sign, then the points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the origin side of the plane $ax + by + cz + d = 0$

173. Statement 1

Distance between the points (–1, 2, –1) and (0, 5, 2) is equal to the sum of the perpendicular distance of the points form the plane $3x - y + 6z + 5 = 0$.

and

Statement 2

The two points (–1, 2, –1) and (0, 5, 2) are on opposite sides of the plane.

$3x - y + 6z + 5 = 0$

174. Consider the line $\frac{x-1}{5} = \frac{y+2}{6} = \frac{z-1}{-3}$

and the plane $y + 2z = 0$

Statement 1

Line lies in the plane.

and

Statement 2

Line is parallel to the plane.

175. Statement 1

The lines $\frac{x+2}{3} = \frac{y-5}{-6} = \frac{z}{2}$

$\frac{x+3}{4} = \frac{y-2}{-3} = \frac{z}{2}$

and $\frac{x-1}{5} = \frac{y+1}{2} = \frac{z-2}{1}$

are concurrent

and

Statement 2

If three non-parallel lines lie in the same plane, then, the lines must be concurrent.

176. Statement 1

The planes

$2x + 3y - 5z = 0$

$3x - 5y + 2z = 0$

and $5x - 2y - 3z = 0$ meet in a line

and

Statement 2

The planes $ax + by + cz = 0$

$bx + cy + az = 0$

$cx + ay + bz = 0$

meet in a line if $a^2 + b^2 + c^2 \neq 1$ or 0

177. Statement 1

If $[\ell_1, m_1, n_1]$, $[\ell_2, m_2, n_2]$; $[\ell_3, m_3, n_3]$ denote the direction cosines of three mutually perpendicular lines, $\ell^2_1 + \ell_2^2 + \ell_3^2 = 1$.

and

Statement 2

If $[\ell, m, n]$ denote the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$.

178. Statement 1

The position vector of 3 points A, B, C relative to an origin of reference O are $(\bar{i} + 2\bar{j} - 3\bar{k}), (3\bar{i} + 5\bar{j} + 9\bar{k})$ and $(7\bar{i} + 11\bar{y} + 33\bar{k})$ respectively. The points A, B, C are collinear

and

Statement 2

If $x_1\bar{i} + y_1\bar{j} + z_1\bar{k}; x_2\bar{i} + y_2\bar{j} + z_2\bar{k}$ and $x_3\bar{i} + y_3\bar{j} + z_3\bar{k}$ are the position vectors of 3 points relative to an origin of reference O and $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$, then, the points represented by these position vector are collinear.

179. Statement 1

If $\bar{A}$, $\bar{B}$, $\bar{C}$ are any three vectors, then $(\bar{A} - \bar{B}).(\bar{B} - \bar{C}) \times (\bar{C} - \bar{A}) = 0$

and

Statement 2

If $\bar{x}, \bar{y}, \bar{z}$ lie in the same plane, $.\bar{x}.(\bar{y} \times \bar{z}) = 0$

180. Statement 1

Let $\bar{A} = 3\bar{i} + 2\bar{j} + 6\bar{k}, \bar{B} = 2\bar{i} - \bar{j} + 2\bar{k}$ A vector which lies in the plane of $\bar{A}$ and $\bar{B}$ and orthogonal to $\bar{A}$ is of the form $\ell\bar{A}$ and $m\bar{B}$ where $\frac{\ell}{m} = \frac{-16}{49}$.

and

Statement 2

If $\bar{A}, \bar{B}, \bar{C}$ are any three vectors such that $\bar{A} \times (\bar{B} \times \bar{C}) = 0$, then $\bar{A}.\bar{B} \times \bar{C} = 0$.

Linked Comprehension Type Questions

Directions: **This section contains 2 paragraphs. Based upon the paragraph, multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

The section of a sphere by a plane is a circle. Thus the equations

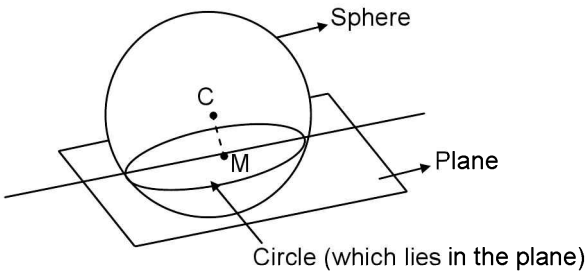
$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{(Sphere) — (1)}$$

$$Ax + By + Cz + D = 0 \qquad \text{(Plane) — (2)}$$

taken together represent a circle, if the plane cuts the sphere.

Let C be the centre of the sphere. The foot of the perpendicular from C to the plane gives the centre of the circle of section. If R is the radius of the sphere and r is the radius of the circle,

$$r^2 = R^2 - CM^2 = R^2 - (\text{length of perpendicular from C to the plane})^2$$



(i) If the plane (2) passes through the centre of the sphere, the circle of section is called a "great circle" (circle of section with the maximum radius) and if (2) does not pass through the centre of the sphere, the circle of section is called a "small circle".

(ii) If (2) touches (1), length of perpendicular from the centre of the sphere to the plane equals its radius.

(iii) Equation of any sphere passing through the circle of section of (1) and (2) may be represented by

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d + \lambda (Ax + By + Cz + D) = 0$$

λ being the parameter.

(iv) Two spheres intersect along a circle

Let $x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$

$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$

represent two spheres. Equation of a sphere through the circle of intersection of the two spheres above can be represented as

$$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 + \lambda(x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2) = 0$$

where, λ is a scalar.

181. The centre and radius of the section of the sphere $x^2 + y^2 + z^2 = 16$ by the plane $2x + y + 2z = 9$ are

(a) $(2,-1,2), \sqrt{5}$ (b) $(2,1,2), \sqrt{7}$ (c) $(2,1,2), \sqrt{3}$ (d) $(1,-2,-1), \sqrt{7}$

182. The equation of the sphere having the circle $x^2 + y^2 + z^2 + 10y - 4z - 8 = 0, x + y + z = 3$ as a great circle is

(a) $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$ (b) $x^2 + y^2 + z^2 + 4x + 6y - 8z + 4 = 0$

(c) $x^2 + y^2 + z^2 + 2x - 3y - 4z + 4 = 0$ (d) $x^2 + y^2 + z^2 - 2x + 3y - 4z + 4 = 0$

183. A sphere of radius 4 and whose centre is at the point (–6, 1, 3) is cut by the plane $x - y + 2z + 5 = 0$. The area of the circle of section is

(a) $\frac{20\pi}{3}$ (b) $\frac{40\pi}{3}$ (c) $\frac{80\pi}{3}$ (d) None of these

Passage II

The vector product of two non-zero, non parallel vectors $\bar{a}$ and $\bar{b}$ is the vector $\bar{a} \times \bar{b}$ whose magnitude is $|\bar{a}||\bar{b}| \sin\theta$ where θ is the angle $(0 \le \theta \le \pi)$ between $\bar{a}$ and $\bar{b}$ and whose direction is perpendicular to both $\bar{a}$ and $\bar{b}$ such that $\bar{a}, \bar{b}$ and $\bar{a} \times \bar{b}$, in that order form a right handed system.

Now, for two non-zero complex numbers $z_1 = x + i\,y$ and $z_2 = u + i\,v$, we define an operation $\otimes$ as $z_1 \otimes z_2 = \overline{z_1} \times \overline{z_2}$ (cross product), where $\overline{z_1}$ and $\overline{z_2}$ are the position vectors $x\hat{i} + y\hat{j}$ and $u\hat{i} + v\hat{j}$ corresponding to the numbers z_1 and z_2 so that $z_1 \otimes z_2$ is a vector perpendicular to both $\overline{z_1}$ and $\overline{z_2}$.

For example, let $z_1 = i$ and $z_2 = 1 + i$, then $\overline{z_1} = \hat{i}$ and $\overline{z_2} = \hat{i} + \hat{j}$

$\therefore \quad z_1 \otimes z_2 = \overline{z_1} \times \overline{z_2} = \hat{i} \times (\hat{i} + \hat{j}) = \hat{k}$

184. $\{z_1 \otimes z_2 / z_1, z_2 \otimes$ the set of complex numbers$\}$ equals

(a) set of all vectors perpendicular to the complex plane.

(b) set of all position vectors of points on z-axis

(c) $\{m\hat{k}/m$ is a non-negative real number$\}$

(d) set of all vectors perpendicular to $\hat{i} + \hat{j}$.

185. If ω is a non-real cube root of unity, then $|\omega \otimes \omega^2|$ equals

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) $\frac{1}{\sqrt{2}}$

186. If z_1 and z_2 are complex numbers having modulus one and if $k = |z_1 \otimes z_2|$, then
(a) $0 \le k \le 1$ (b) $-1 \le k \le 1$ (c) $0 \le k < \infty$ (d) $0 \le k \le 2$

187. If z is a complex number such that $|z| = 1$ then $iz \otimes z$, where, $i = \sqrt{-1}$ equals
(a) $\hat{k}$ (b) 0 (c) $-\hat{k}$ (d) $\sqrt{2}\hat{k}$

188. If $z_1 = 2 + 3i$, $z_2 = 3 - i$, then $(z_1 + z_2) \otimes z_2$ is
(a) $-\hat{k}$ (b) $3\hat{k}$ (c) $-11\hat{k}$ (d) $-10\hat{k}$

189. z_1 and z_2 are two complex numbers such that $z_1 \otimes z_2 = \hat{k}$ and $z_1 = 2i + 3j$ if z_2 is equal to
(a) $i + j$ (b) $-i - j$ (c) $i - j$ (d) $j - i$

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers of which ONE OR MORE answers will be correct.**

190. Consider the plane $2x - y + 7z - 9 = 0$. Then,
(a) Image of the point (4, 1, 8) by reflexion in the given plane is the point (0, 3, –6)
(b) Equations of the line through (4, 1, 8), perpendicular to the plane are $\frac{4-x}{2} = \frac{y-1}{1} = \frac{8-z}{7}$
(c) Image of the point (4, 1, 8), by reflexion in the plane is the point (2, 2, 1)
(d) (3, 1, –5) and (1, 1, 1) are on the same side of the given plane

191. Let $\bar{x}$ and $\bar{y}$ denote two non-collinear vectors where, $|\bar{x}| \ne |\bar{y}|$. Then, $\frac{\bar{x}}{|\bar{x}|} - \frac{\bar{y}}{|\bar{y}|}$ is
(a) a vector equally inclined to $\bar{x}$ and $\bar{y}$
(b) a vector orthogonal to $\bar{x} \times \bar{y}$
(c) a unit vector if the angle between $\bar{x}$ and $\bar{y}$ is $\frac{\pi}{3}$
(d) a vector in the plane of $\bar{x}$ and $\bar{y}$

192. For $\bar{A}, \bar{B}, \bar{C}$, 3 non null vectors $(\bar{A} \times \bar{B}) \times \bar{C} = 0$ if
(a) $\bar{C}$ is orthogonal to both $\bar{B}$ and $\bar{A}$
(b) $\bar{B} = K\bar{A}$ where K is a non-zero scalar
(c) $\bar{A}$ and $\bar{B}$ are noncollinear, but $\bar{A}, \bar{B}, \bar{C}$ are coplanar
(d) $\bar{C}$ is parallel to $(\bar{A} \times \bar{B})$

193. Consider the lines
$L_1 : 3x + y - 2z = 5$
$x + y + z = 3$
$L_2 : \frac{x-3}{5} = \frac{y+4}{-1} = \frac{z-2}{3}$. Then
(a) an angle between L_1 and L_2 is $\cos^{-1}\left(\frac{14}{\sqrt{1330}}\right)$
(b) an angle between L_1 and L_2 is $\cos^{-1}\left(\frac{26}{\sqrt{1330}}\right)$
(c) the two lines are non-coplanar
(d) the two lines are coplanar

194. A point on the line

$$\frac{x-2}{4}=\frac{y+5}{-3}=\frac{z+2}{0}$$

at a distance of 6 units from (2, –5, –2) can be

(a) $\left(\frac{34}{5},\frac{-43}{5},-2\right)$ (b) $\left(\frac{14}{5},\frac{-7}{5},-2\right)$ (c) $\left(\frac{-34}{5},\frac{43}{5},-2\right)$ (d) $\left(\frac{-14}{5},\frac{-7}{5},-2\right)$

195. The plane $x + y + z - 3 = 0$

(a) is parallel to the line $x + 1 = y - 2 = z + 3$

(b) is perpendicular to the line $x + 1 = y - 2 = z + 3$

(c) contains the line $\frac{x+1}{3}=\frac{y+4}{5}=\frac{z-8}{-8}$

(d) makes equal angles with the coordinate planes

196. If a, b, c are distinct non-zero real numbers and $\begin{vmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ x & y & z & 1 \end{vmatrix} = 0$ represent a plane, then

(a) The plane passes through $\left(\frac{1}{3a},\frac{1}{3b},\frac{1}{3c}\right)$

(b) the plane passes through $\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right),\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$

(c) perpendicular distance of (0, 0, 0) from the plane is $\frac{1}{\sqrt{a^2+b^2+c^2}}$

(d) perpendicular distance of (0, 0, 0) from the plane is $= \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$

197. If M is the foot of the perpendicular from P(1, 1, –1) to the line through A(2, –3, 2) parallel to $i - 2\hat{j}+2\hat{k}$ and Q, the image of P by reflexion on the line, then

(a) AM = 5 (b) PM = 1 (c) AP = $\sqrt{26}$ (d) PM = 2

Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

198.

	Column I		Column II
(a)	Equation of the plane containing the line $2x + z - 4 = 0$, $2y + z = 0$ and passing through the point (2, 1, –1) is	(p)	$x - 2y + z + 4 = 0$
(b)	Equation of the plane bisecting angle between the planes $2x + y - 3z + 4 = 0$, $2x - 3y + z + 1 = 0$ is	(q)	$4x - 2y - 2z + 5 = 0$
(c)	Equation of a plane perpendicular to $2x + 3y + z + 3 = 0$ is	(r)	$4y - 4z + 3 = 0$

(d)	Equation of the plane which contains the line $\frac{x}{2}=\frac{y-3}{3}=\frac{z-2}{4}$ and perpendicular to the plane $-3x-4y-5z+3=0$ is	(s)	$x+y+z-2=0$

199. Force $\overline{f}=(2t^2+1)i+(t^2+2t-3)j-5t\,\hat{k}$ and displacement $\overline{s}=(t-1)\,i-tj+(t-4)k$

	Column I		**Column II**
(a)	When $t=2$ work done is	(p)	14
(b)	When $t=4$ work done is	(q)	15
(c)	Maximum work done is	(r)	19
(d)	Minimum work done is	(s)	20

200.

	Column I		**Column II**
(a)	$2i-3j+k$ is linearly dependent on $i+\lambda j$ and $2\lambda i+j+k$, then λ cannot be	(p)	-2
(b)	$\lambda i-k$, $2i+(\lambda+2)j$ and $2i+(\lambda+2)j+(\lambda-2)k$ are coplanar, then λ can be	(q)	-1
(c)	$i+\lambda^2 j+\lambda^2 k$; $i+j+\lambda k$ and $3i+3j+2\lambda$, are non-coplanar; then λ can be	(r)	0
(d)	$-i+2j+k$, $\lambda i+k$, $i+\lambda j+2k$ does not form parallelopiped, then λ cannot be	(s)	1
		(t)	2

SOLUTIONS

ANSWER KEYS

Topic Grip

3. (i) $\bar{R} = \dfrac{(\bar{a}\cdot\bar{b})\bar{c} - (\bar{a}\cdot\bar{c})\bar{b}}{(\bar{a}\cdot\bar{b})}$

5. $\dfrac{x}{a} - \dfrac{y}{b} - \dfrac{z}{c} + 1 = 0$

9. (i) $2x - y + z = 6$
   (ii) $x + y + 2z = 3$
   (iii) $\theta = \dfrac{\pi}{3}$
   (iv) $y + z = 0$

10. (i) $ab + bc + ca = 0$

11. (c)	12. (d)	13. (c)
14. (b)	15. (c)	16. (d)
17. (a)	18. (d)	19. (c)
20. (c)	21. (c)	22. (a)
23. (d)	24. (c)	25. (b)

26. (c)
27. (a), (b), (c)
28. (a), (d)
29. (a), (b), (c), (d)
30. (a) → (r), (s)
    (b) → (p), (q), (s)
    (c) → (p)
    (d) → (q)

IIT Assignment Exercise

31. (d)	32. (b)	33. (c)
34. (c)	35. (c)	36. (a)
37. (c)	38. (c)	39. (d)
40. (d)	41. (c)	42. (d)
43. (a)	44. (a)	45. (a)
46. (d)	47. (d)	48. (c)
49. (a)	50. (b)	51. (b)
52. (b)	53. (c)	54. (d)
55. (a)	56. (a)	57. (d)
58. (d)	59. (b)	60. (c)
61. (b)	62. (d)	63. (a)
64. (c)	65. (d)	66. (b)
67. (d)	68. (a)	69. (c)
70. (d)	71. (d)	72. (d)
73. (c)	74. (c)	75. (c)
76. (b)	77. (b)	78. (a)
79. (c)	80. (a)	81. (a)
82. (c)	83. (a)	84. (b)
85. (b)	86. (c)	87. (c)
88. (b)	89. (a)	90. (c)
91. (a)	92. (b)	93. (a)
94. (d)	95. (d)	96. (a)
97. (c)	98. (c)	99. (a)
100. (b)	101. (a)	102. (d)
103. (b)	104. (d)	105. (c)
106. (c)	107. (a)	108. (b)
109. (b)	110. (b)	111. (b)
112. (a)	113. (a)	114. (c)

115. (a)
116. (b)
117. (a), (c)
118. (a), (b), (c), (d)
119. (a), (b), (d)
120. (a) → (p)
     (b) → (s)
     (c) → (r)
     (d) → (q)

Additional Practice Exercise

121. $\dfrac{AF}{AD} = \dfrac{14}{25}$ $\dfrac{BF}{FE} = \dfrac{35}{4}$

123. $p = \dfrac{1}{\sqrt{2}}; q = -\dfrac{1}{\sqrt{2}}; r = \dfrac{1}{\sqrt{2}}$

124. [1, –2, 1] or [–1, –1, 2]
125. R is (3, 0, 3)
127. $X^2 + Y^2 + Z^2 = k(lX + mY + nZ\})$

128. $$\frac{\sqrt{8}x}{\sqrt{b^2+c^2-a^2}} + \frac{\sqrt{8}y}{\sqrt{c^2+a^2-b^2}} + \frac{\sqrt{8}z}{\sqrt{a^2+b^2-c^2}} = 1$$

129. (i) $T = \dfrac{\bar{d} + t\bar{b}}{1+t}$
     (ii) $\bar{r} = \Delta\left(\dfrac{3\bar{b}}{4}\right) + (1-\Delta)\left(\dfrac{4\bar{d} + t\bar{b}}{4}\right)$
     where 4 is scalar

130. (i) $x^2 + y^2 + z^2 + 4(x + y + z) = 0$
     (ii) $2\sqrt{3}$
     (iii) sphere with centre (–2, –2, –2) radius = $2\sqrt{3}$

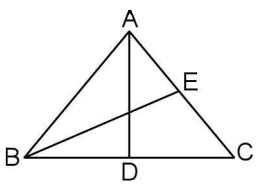
131. (a)	132. (a)	133. (c)
134. (a)	135. (b)	136. (b)
137. (b)	138. (a)	139. (b)
140. (b)	141. (c)	142. (a)
143. (b)	144. (a)	145. (c)
146. (b)	147. (c)	148. (a)
149. (d)	150. (c)	151. (c)
152. (a)	153. (b)	154. (b)
155. (d)	156. (c)	157. (a)
158. (b)	159. (b)	160. (c)
161. (c)	162. (b)	163. (c)
164. (c)	165. (a)	166. (c)
167. (b)	168. (b)	169. (a)
170. (d)	171. (c)	172. (c)
173. (d)	174. (a)	175. (c)
176. (c)	177. (a)	178. (c)
179. (a)	180 (c)	181. (b)
182. (a)	183. (b)	184. (a)
185. (a)	186. (a)	187. (c)
188. (c)	189. (b)	

190. (a), (b), (d)
191. (b), (c), (d)
192. (a), (b), (d)
193. (b), (c)
194. (a), (d)
195. (b), (c), (d)
196. (a), (c)
197. (a), (b), (c)
198. (a) → (s)
     (b) → (q), (r)
     (c) → (q)
     (d) → (p)
199. (a) → (r)
     (b) → (q)
     (c) → (r)
     (d) → (q)
200. (a) → (p), (r), (s)
     (b) → (p), (r), (t)
     (c) → (p), (t)
     (d) → (p), (q), (r)

HINTS AND EXPLANATIONS

Topic Grip

1.



$$\overline{AD}=\frac{\overline{AB}+\overline{AC}}{2}=\frac{2\overline{AB}+\overline{BC}}{2},\ \overline{BE}=\frac{\overline{BC}-\overline{AB}}{2}$$

$$\overline{AD}.\overline{BE}=\frac{BC^2+\overline{AB}.\overline{BC}-2AB^2}{4}$$

$$=\frac{1}{8}\left(2a^2-c^2-a^2+b^2-4c^2\right)$$

$$=\frac{1}{8}\left(a^2+b^2-5c^2\right)=0$$

$$\Rightarrow \quad \overline{AD}\perp\overline{BE}$$

2. Let $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$, $\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$,

$\bar{c}=c_1\bar{i}+c_2\bar{j}+c_3\bar{k}$

Then $\bar{A}=x_1\bar{a}+y_1\bar{b}+z_1\bar{c}$

$$=x_1\left\{a_1\bar{i}+a_2\bar{j}+a_3\bar{k}\right\}+y_1\left\{b_1\bar{i}+b_2\bar{j}+b_3\bar{k}\right\}+z_1\left\{c_1\bar{i}+c_2\bar{j}+c_3\bar{k}\right\}$$

$$=(x_1a_1+y_1b_1+z_1c_1)\bar{i}+(x_1a_2+y_1b_2+z_1c_2)\bar{j}$$

$$B=x_2\,(a_1i+a_2j+a_3k)+\ldots$$

$$=(a_1x_2+b_1y_2+c_1z_2)\,i+\ldots$$

and $\bar{C}=\left(x_3a_1+y_3b_1+z_3c_1\right)\bar{i}+\left(x_3a_2+y_3b_2+z_3c_2\right)\bar{j}$

$$+(x_3a_3+y_3b_3+z_3c_3)\bar{k}\text{ — (3)}$$

$$\lfloor A\ B\ C\rfloor=$$

$$\begin{vmatrix} x_1a_1+y_1b_1+z_1c_1 & x_1a_2+y_1b_2+z_1c_2 & x_1a_3+y_1b_3+z_1c_3 \\ x_2a_1+y_2b_1+z_2c_1 & x_2a_2+y_2b_2+z_2c_2 & x_2a_3+y_2b_3+z_2c_3 \\ x_3a_1+y_3b_1+z_3c_1 & x_3a_2+y_3b_2+z_3c_2 & x_3a_3+y_3b_3+z_3c_3 \end{vmatrix}$$

(from (1), (2) and (3))

$$=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_2 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\left[\bar{a}\,\bar{b}\,\bar{c}\right]$$

3. (i) We have $\left(\bar{R}-\bar{c}\right)\times\bar{b}=0$

$$\Rightarrow \quad \bar{R}\times\bar{b}-\bar{c}\times\bar{b}=0$$

$$\Rightarrow \quad \bar{a}\times\left(\bar{R}\times\bar{b}\right)-\bar{a}\times\left(\bar{c}\times\bar{b}\right)=\bar{a}\times 0=0$$

$$\Rightarrow \quad \left(\bar{a}\cdot\bar{b}\right)\bar{R}-\left(\bar{a}\cdot\bar{R}\right)\bar{b}=\left(\bar{a}\cdot\bar{b}\right)\bar{c}-\left(\bar{a}\cdot\bar{c}\right)\bar{b}$$

$$\Rightarrow \quad \left(\bar{a}\cdot\bar{b}\right)\bar{R}-0=\left(\bar{a}\cdot\bar{b}\right)\bar{c}-\left(\bar{a}\cdot\bar{c}\right)\bar{b}$$

$$\Rightarrow \quad \bar{R}=\frac{\left(\bar{a}\cdot\bar{b}\right)\bar{c}-\left(\bar{a}\cdot\bar{c}\right)\bar{b}}{\left(\bar{a}\cdot\bar{b}\right)}$$

(ii) Let A, B, C and P be the points whose position vectors are $\bar{a},\bar{b},\bar{c}$ and $\bar{R}$ respectively.

Now, since $\left(\bar{R}-\bar{c}\right)\times\bar{b}=0$ either $\bar{R}-\bar{c}=0$ or $\left(\bar{R}-\bar{c}\right)$ is parallel to $\bar{b}$. But $\bar{R}-\bar{c}=0$ means that points P and C coincide. $\left(\bar{R}-\bar{c}\right)$ parallel to $\bar{b}$ implies that $\overline{CP}$ is parallel to $\overline{OB}$, ie, the figure OBPC is a trapezium.

4. (i) $\bar{b}\times\left(\bar{c}\times\bar{d}\right)=\left(\bar{b}\cdot\bar{d}\right)\bar{c}-\left(\bar{b}\cdot\bar{c}\right)\bar{d}$

$$\Rightarrow \quad \bar{a}\times\left\{\bar{b}\times\left(\bar{c}\times\bar{d}\right)\right\}=\bar{a}\times\left\{\left(\bar{b}\cdot\bar{d}\right)\bar{c}-\left(\bar{b}\cdot\bar{c}\right)\bar{d}\right\}$$

$$=\left(\bar{b}\cdot\bar{d}\right)(\bar{a}\times\bar{c})-\left(\bar{b}\cdot\bar{c}\right)(\bar{a}\times\bar{d})$$

(ii) $\bar{d}\cdot\left[\bar{a}\times\left\{\bar{b}\times\left(\bar{c}\times\bar{d}\right)\right\}\right]$

$$=\bar{d}\left\{\left(\bar{b}\cdot\bar{d}\right)\left(\bar{a}\times\bar{c}\right)-\left(\bar{b}\cdot\bar{c}\right)\left(\bar{a}\times\bar{d}\right)\right\}$$

$$=\left(\bar{b}\cdot\bar{d}\right)\left[\bar{d}\,\bar{a}\,\bar{c}\right]-\left(\bar{b}\cdot\bar{c}\right)\left[\bar{d}\,\bar{a}\,\bar{d}\right]$$

$$=\left(\bar{b}\cdot\bar{d}\right)\left[\bar{a}\,\bar{c}\,\bar{d}\right],\ \text{since}\left[\bar{d}\,\bar{a}\,\bar{d}\right]=0$$

(iii) $\bar{w}\times\bar{u}+\bar{w}\times\bar{v}=\bar{w}\times\left(\bar{u}+\bar{v}\right)$

$$=\bar{w}\times(\bar{a}\times\bar{b}+\bar{a}\times\bar{c})=\bar{w}\times\left\{\bar{a}\times(\bar{b}+\bar{c})\right\}$$

$$= \bar{w} \times \{\bar{a} \times \bar{w}\} = (\bar{w} \cdot \bar{w})\bar{a} - (\bar{w} \cdot \bar{a})\bar{w}$$

$$= |\bar{w}|^2 \bar{a} - (\bar{w} \cdot \bar{a})\bar{w}$$

(iv) $\bar{F} \times \bar{A} = \bar{P}$

$\Rightarrow \quad \bar{B} \cdot \bar{P} = \bar{B} \cdot \bar{F} \times \bar{A} = [\overline{BFA}]$

Again, $\bar{F} \times B = \bar{Q}$

$\bar{A} \cdot \bar{Q} = \bar{A} \cdot \bar{F} \times \bar{B} = [\overline{AFB}] = -[\overline{BFA}] = -\bar{B} \cdot \bar{P}$

$\Rightarrow \quad \bar{A} \cdot \bar{Q} + \bar{B} \cdot \bar{P} = 0$

$(\bar{Q} \cdot \bar{A})(\bar{P} \times \bar{Q}) = (-\bar{B} \cdot \bar{P})(\bar{P} \times \bar{Q}) = (\bar{B} \cdot \bar{P})(\bar{Q} \times \bar{P})$

(v) $[\bar{a} \times \bar{b}, \bar{c} \times \bar{d}, \bar{e} \times \bar{f}] = (\bar{a} \times \bar{b}) \cdot \{(\bar{c} \times \bar{d}) \times (\bar{e} \times \bar{f})\}$

$$= (\bar{a} \times \bar{b}) \cdot \{[\overline{cdf}]\bar{e} - [\overline{cde}]\bar{f}\}$$

$$= [\overline{abe}][\overline{cdf}] - [\overline{abf}][\overline{cde}]$$

$$= [\overline{abe}][\overline{fcd}] - [\overline{afb}][\overline{ced}]$$

Again,

$[\bar{a} \times \bar{b}, \bar{c} \times \bar{d}, \bar{e} \times \bar{f}] = (\bar{a} \times \bar{b}) \cdot \{-(\bar{e} \times \bar{f}) \times (\bar{c} \times \bar{d})\}$

$-[(\bar{a} \times \bar{b}) \cdot \{[\overline{efd}]\bar{c} - [\overline{efc}]\bar{d}\}]$

$$= [\overline{abc}][\overline{efc}] - [\overline{abc}][\overline{efd}]$$

$$= [\overline{abd}][\overline{cef}] - [\overline{abc}][\overline{def}]$$

Again,

$[(\bar{a} \times \bar{b}), (\bar{c} \times \bar{d}), (\bar{e} \times \bar{f})] = (\bar{c} \times \bar{d}) \cdot \{(\bar{e} \times \bar{f}) \times (\bar{a} \times \bar{b})\}$

$$= (\bar{c} \times \bar{d}) \cdot \{[\overline{efb}]\bar{a} - [\overline{efa}]\bar{b}\}$$

$$= [\overline{cda}][\overline{efb}] - [\overline{cdb}][\overline{efa}]$$

$$= [\overline{cda}][\overline{bef}] - [\overline{cdb}][\overline{aef}]$$

5. Any line through the line of intersection of the two planes $\frac{y}{b} + \frac{z}{c} = 1, x = 0$ can be assumed as

$$\frac{x}{b} + \frac{y}{c} - 1 + k(x) = 0 \qquad \text{—(1)}$$

where, k is a scalar

(1) is parallel to the line represented by the equations

$$\frac{x}{a} - \frac{z}{c} = 1, \; y = 0$$

Equations of the above line may be written as

$$\frac{x}{a} = \frac{z+c}{c} = \frac{y}{0} \qquad \text{—(2)}$$

(1) is parallel to the line (2)

$\Rightarrow \quad k \times a + \frac{1}{b} \times 0 + \frac{1}{c} \times c = 0$

$k = -\frac{1}{a}$

Equation of the plane is $\frac{y}{b} + \frac{z}{c} - \frac{x}{a} - 1 = 0$

$\Rightarrow \quad \frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$

The two given lines are $\frac{x}{0} = \frac{y}{b} = \frac{z-c}{-c}$ and

$\frac{x}{a} = \frac{y}{0} = \frac{z+c}{c}$

$$2d = \frac{1}{\sqrt{(bc)^2 + (ca)^2 + (ab)^2}} \begin{vmatrix} 0 & 0 & 2c \\ 0 & b & -c \\ a & 0 & c \end{vmatrix}$$

$$= \frac{-2abc}{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}$$

$$d^2 = \frac{a^2b^2c^2}{b^2c^2 + c^2a^2 + a^2b^2}$$

$\Rightarrow \quad \frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

6. We have $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

$$\left[\frac{l_1 + l_2}{2\cos\frac{\theta}{2}}\right]^2 + \left[\frac{m_1 + m_2}{2\cos\frac{\theta}{2}}\right]^2 + \left[\frac{n_1 + n_2}{2\cos\frac{\theta}{2}}\right]^2$$

$$= \frac{1}{4\cos^2\frac{\theta}{2}}\{((l_1 + l_2)^2 + (m_1 + m_2)^2 + (n_1 + n_2)^2)\}$$

$$= \frac{1}{4\cos^2\frac{\theta}{2}} \{(l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) + 2(l_1l_2 + m_1m_2 + n_1n_2)\}$$

$$= \frac{1}{4\cos^2\frac{\theta}{2}}\{1 + 1 + 2\cos\theta\} = 1$$

Let us find the angle between the directions represent by $[l_1, m_1, n_1]$ and $\left[\frac{l_1+l_2}{2\cos\frac{\theta}{2}}, \frac{m_1+m_2}{2\cos\frac{\theta}{2}}, \frac{n_1+n_2}{2\cos\frac{\theta}{2}}\right]$

If α is the angle between the directions,

$$\cos\alpha = \frac{l_1(l_1+l_2)+m_1(m_1+m_2)+n_1(n_1+n_2)}{2\cos\frac{\theta}{2}}$$

$$= \frac{1+\cos\theta}{2\cos\frac{\theta}{2}} = \cos\frac{\theta}{2} \Rightarrow \alpha = \frac{\theta}{2}$$

Therefore, the direction represented by

$\left[\frac{l_1+l_2}{2\cos\frac{\theta}{2}}, \frac{m_1+m_2}{2\cos\frac{\theta}{2}}, \frac{n_1+n_2}{2\cos\frac{\theta}{2}}\right]$ is along the bisector.

7. y–axis has equations $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$ — (1)

If $[\alpha, \beta, \gamma\}$ are the direction ratios of the line represented by the two planes, $\alpha = bc' - b'c$,

$\beta = ca' - c'\ a\ \gamma = ab' - a'b$

Putting $z = 0$ in the equations of the two planes,

$ax + by = -d$

$a'x + b'y = -d'$

Solving, $x = \frac{bd'-b'd}{(ab'-a'b)}, y = \frac{da'-d'a}{(ab'-a'b)}$

Line represented by the two planes is

$$\frac{x - \frac{bd'-b'd}{ab'-a'b}}{\alpha} = \frac{y - \frac{da'-d'a}{ab'-a'b}}{\beta} = \frac{z}{\gamma} \quad \text{— (2)}$$

If D represented the shortest distance between (1) and (2).

D = Numerical value of

$$\frac{1}{\sqrt{\alpha^2+\gamma^2}}\begin{vmatrix} \frac{bd'-b'd}{ab'-a'b} & \frac{da'-d'a}{ab'-a'b} & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & \gamma \end{vmatrix}$$

$$= \frac{1}{\sqrt{\alpha^2+\gamma^2}} \times \left|\frac{bd'-b'd}{ab'-a'b}\right| \times \gamma$$

$$= \frac{|bd'-b'd|}{\sqrt{(ba'-b'a)^2+(bc'-b'c)^2}}$$

8. If (α, β, γ) is foot of perpendicular from the origin to the plane, then the plane is

$\alpha(x-a) + \beta(y-b) + \gamma(z-c) = 0$ and since (α, β, γ) is on the plane

$\alpha(\alpha-a) + \beta(\beta-b) + \gamma(\gamma-c) = 0$

Hence locus of

(α, β, γ) is $x(x-a) + y(y-b) + z(z-c) = 0$

$x^2 + y^2 + z^2 - ax - by - cz = 0.$

9. (i) Any plane parallel to $2x - y + z - 13 = 0$ is of the form $2x - y + z + k = 0$

This passes through $(4, 1, -1)$

$\Rightarrow$ $8 - 1 - 1 + k = 0$

$\Rightarrow$ $k = -6$

$\therefore$ Equation of the required plane is

$2x - y + z = 6$ — (1)

(ii) Direction ratios of the line joining (1, 2, 3) and (2, 3, 5) are 1, 1, 2.

Any plane perpendicular to this line is of the form

$x + y + 2z = k'$

This passes through (0, 1, 1)

$\Rightarrow$ $0 + 1 + 2 = k' \Rightarrow k' = 3$

$\therefore$ Equation of the required plane is

$x + y + 2z = 3$ — (2)

(iii) Angle θ between the planes (1) and (2) is given by

$$\cos\theta = \pm\frac{(2)(1)+(-1)(1)+(1)(2)}{\sqrt{6}\sqrt{6}} = \pm\frac{1}{2}$$

$\Rightarrow$ a value of $\theta = \frac{\pi}{3}$

(iv) Any plane P through the intersection of (1) and (2) is of the form

$(2x - y + z - 6) + \lambda(x + y + 2z - 3) = 0$ — (3)

As (1) is rotated through a right angle about its line of intersection with (2)

We have that angle between the planes (1) and (3) is 90°

$\Rightarrow$ $2(2 + \lambda) - 1(-1 + \lambda) + 1(1 + 2\lambda) = 0$

$\Rightarrow$ $6 + 3\lambda = 0 \Rightarrow \lambda = -2$

$\therefore$ Equation of the plane in the new position is $y + z = 0$

10. (i) Given $al + bm + cn = 0$ — (1)

$mn + nl + lm = 0$ — (2)

Eliminating l from (1) and (2) we have

$$b\left(\frac{m}{n}\right)^2+(b+c-a)\frac{m}{n}+c=0 \qquad \text{— (3)}$$

(3) being a quadratic in $\frac{m}{n}$ gives two values

$\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$, where (l_1, m_1, n_1) and (l_2, m_2, n_2) are the direction cosines of the two lines.

$$\Rightarrow \quad \frac{m_1}{n_1}\cdot\frac{m_2}{n_2}=\frac{c}{b}$$

$$\Rightarrow \quad \frac{m_1m_2}{\frac{1}{b}}=\frac{n_1n_2}{\frac{1}{c}} \qquad \text{— (4)}$$

Similarly we can show by eliminating m, that

$$\frac{\ell_1\ell_2}{\frac{1}{a}}=\frac{n_1n_2}{\frac{1}{c}} \qquad \text{—(5)}$$

(4) and (5) $\Rightarrow \frac{\ell_1\ell_2}{\frac{1}{a}}=\frac{m_1m_2}{\frac{1}{b}}=\frac{n_1n_2}{\frac{1}{c}}$ (= k say)

Lines are perpendicular

$\Rightarrow \quad l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

$\Rightarrow \quad ab+bc+ca=0 \qquad$ — (6)

which is the required condition

(ii) Consider $\left[\overline{X}\,\overline{Y}\,\overline{Z}\right]$

$$=\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ -2bc & -2ca & -2ab \\ a^2 & b^2 & c^2 \end{vmatrix} \text{[using (6)]}$$

$$=-2\begin{vmatrix} 0 & 0 & 0 \\ bc & ca & ab \\ a^2 & b^2 & c^2 \end{vmatrix} \text{[Take (–2) from } R_2 \text{ and then}$$

using $R_1 : R_1 - R_2 + R_3$]

$=0 \Rightarrow \overline{X}, \overline{Y}, \overline{Z}$ are coplanar.

11. $\left(\overline{a}+\overline{b}\right)^2+\left(\overline{a}-\overline{b}\right)^2=2\left(\overline{a}^2+\overline{b}^2\right)$

$\Rightarrow \quad \left(\overline{a}+\overline{b}\right)^2=2\left(3^2+4^2\right)-25 \;=25$

$\Rightarrow \quad |\overline{a}+\overline{b}|=5$

12. $\overline{F}=7\overline{i}+2\overline{j}-4\overline{k}$

$\overline{D}=4\overline{i}+2\overline{j}-2\overline{k}$

$W=\overline{F}\cdot\overline{D} \;=28+4+8\;=40.$

13. $\sum_{\substack{i,j=1\\ i\neq j}}^{4}|\overline{a_i}-\overline{a_j}|^2=3\sum|\overline{a_i}|^2-2\sum_{i\neq j}\overline{a_i}.\overline{a_j}$

$$=12-2\sum_{i\neq j}\overline{a_i}.\overline{a_j} \qquad \text{— (1)}$$

$$\left(\overline{a_1}+\overline{a_2}+\overline{a_3}+\overline{a_4}\right)^2=\sum|a_i|^2+2\sum_{i\neq j}\overline{a_i}.\overline{a_j}$$

$$=4+2\sum_{i\neq j}\overline{a_i}.\overline{a_j}$$

$\therefore \quad \sum\left|\overline{a_i}-\overline{a_j}\right|^2=16-\left(\sum\overline{a_i}\right)^2$

≤ 16

$\therefore \quad 2\sum\overline{a_i}.\overline{a_j}\geq -4$

$\therefore$ Maximum value = 16

14. Using the condition for 3 planes to intersect in a line

we have $\begin{vmatrix} 4 & 3 & 2 \\ 2 & 1 & -4 \\ 1 & k & -7 \end{vmatrix}=0$ giving $k=0$

15. Since the lines are perpendicular, dot product of direction ratios equal to zero.

$k^2-4+k+4-6k=0$

$k^2-5k=0$

$k=0$, or 5

16. Statement 2 is true

$\overline{a}\times\left(\overline{b}\times\overline{c}\right)\;=\left(\overline{a}.\overline{c}\right)\overline{b}-\left(\overline{a}.\overline{b}\right)\overline{c}$

$=0$, since $\overline{a}.\overline{b}=0=\overline{a}.\overline{c}$

$\overline{a}$ is parallel to $\overline{b}\times\overline{c}$

$\Rightarrow$ Statement 1 is not true

choice (d)

17. Statement 2 is true

Using statement 2

$[\overline{b}+\overline{c}, \overline{c}+\overline{a}, \overline{a}+\overline{b}]=2\left[\overline{a}, \overline{b}, \overline{c}\right]$

$=2\times 0$, since $\overline{a}, \overline{b}, \overline{c}$ are coplanar.

$=0$

$\Rightarrow$ Statement 1 is true.

Choice (a)

18. Statement 2 is true

If the line is parallel to plane and the direction ratios of (the line and normal to plane are$[a_1, b_1, c_1]$ and $[a_2, b_2, c_2]$, then

$a_1a_2 + b_1b_2 + c_1c_2 = 0.$

But, $(1)(1) + (-2)(-2) + 5(5) \neq 0$

$\therefore$ Statement 1 is false.

$\Rightarrow$ choice (d)

19. Statement 2 is false, since $a\ell + bm + cn = 0$ ensures that the plane is parallel to the line

Consider statement 1 :

(0, 1, 1) is a point on the line if satisfies the equation of the plane

Also, $2 \times 1 + 4 \times 1 + -1 \times 6 = 0$

$\Rightarrow$ Statement 1 is true

20. Statement 2 is false since $[\ell, m, n]$ need not be the direction cosines of the line.

Consider statement 1

(2, –2, 0) is a point on the line since

$$\frac{2-1}{\frac{1}{\sqrt{3}}} = \frac{-2+3}{\frac{1}{\sqrt{3}}} = \frac{0-1}{\frac{-1}{\sqrt{3}}}$$

$\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right]$ denote the direction cosines of the line.

Hence $\left(1+\frac{R}{\sqrt{3}}, -3+\frac{R}{\sqrt{3}}, 1-\frac{R}{\sqrt{3}}\right)$ denotes the co-ordinates of a point on the line which is at a distance of R from (1, –3, 1)

Put $R = \sqrt{3}$ $\rightarrow$ we get (2, –2, 0)

$\Rightarrow$ Statement 1 is true

choice (c)

21. $$\bar{a} \times \overline{a^1} = \bar{a} \times \frac{\bar{b} \times \bar{c}}{\left[\overline{a\,b\,c}\right]} = \frac{1}{\left[\overline{abc}\right]}\left\{\bar{a} \times \left(\bar{b} \times \bar{c}\right)\right\}$$

$$\bar{b} \times \overline{b^1} = \frac{1}{\left[\overline{abc}\right]}\left\{\bar{b} \times \left(\bar{c} \times \bar{a}\right)\right\}$$

$$\bar{c} \times \overline{c^1} = \frac{1}{\left[\overline{abc}\right]}\left\{\bar{c} \times \left(\bar{a} \times \bar{b}\right)\right\}$$

$$\bar{a} \times \overline{a^1} + \bar{b} \times \overline{b^1} + \bar{c} \times \overline{c^1} = \frac{1}{\left[\overline{abc}\right]}\left\{\bar{a} \times \left(\bar{b} \times \bar{c}\right) + \bar{b} \times \left(\bar{c} \times \bar{a}\right) + \bar{c} \times \left(\bar{a} \times \bar{b}\right)\right\}$$

$$= \frac{0}{\left[\overline{abc}\right]} = 0$$

22. $$\overline{a^1} \times \overline{b^1} = \frac{1}{\left[\overline{abc}\right]^2}\left\{\left(\bar{b} \times \bar{c}\right) \times \left(\bar{c} \times \bar{a}\right)\right\}$$

$$= \frac{1}{\left[\overline{abc}\right]^2}\left\{\left[\overline{bca}\right]\bar{c} - \left[\overline{bcc}\right]\bar{a}\right\}$$

$$= \frac{1}{\left[\overline{abc}\right]^2}\left\{\left[\overline{abc}\right]\bar{c} - 0\right\}$$

$$= \frac{1}{\left[\overline{abc}\right]}\bar{c}$$

Similarly,

$$\overline{b^1} \times \overline{c^1} = \frac{1}{\left[\overline{abc}\right]}\bar{a} \text{ and } \overline{c^1} \times \overline{a^1} = \frac{1}{\left[\overline{abc}\right]}\bar{b}$$

Therefore, $\overline{a^1} \times \overline{b^1} + \overline{b^1} \times \overline{c^1} + \overline{c^1} \times \overline{a^1} = \frac{1}{\left[\overline{abc}\right]}\left\{\bar{a} + \bar{b} + \bar{c}\right\}$

23. $$\left(\bar{a} + \bar{b}\right) \cdot \overline{a^1} = \left(\bar{a} + \bar{b}\right) \cdot \left\{\frac{\bar{b} \times \bar{c}}{\left[\overline{abc}\right]}\right\}$$

$$\left(\bar{a} + \bar{b}\right) \cdot \overline{a^1} = \frac{\bar{a} \cdot \left(\bar{b} \times \bar{c}\right)}{\left[\overline{abc}\right]} + \frac{\bar{b} \cdot \left(\bar{b} \times \bar{c}\right)}{\left[\overline{abc}\right]} = \frac{\left[\overline{abc}\right]}{\left[\overline{abc}\right]} + 0 = 1$$

Similarly, $\left(\bar{b} + \bar{c}\right) \cdot \overline{b^1} = 1$ and $\left(\bar{c} + \bar{a}\right) \cdot \overline{c^1} = 1$

giving the answer to the problem as 3.

24. Scalar triple product of the vectors of the set $\left\{\bar{j} + \bar{k}, \bar{k} + \bar{i}, \bar{i} + \bar{j}\right\}$ is

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$$

Let $\left\{\overline{a^1}, \overline{b^1}, \overline{c^1}\right\}$ represent the reciprocal system of vectors to the given set.

Then, $\overline{a^1} = \dfrac{(\overline{k}+\overline{i})\times(\overline{i}+\overline{j})}{2}$

$$= \frac{1}{2}\left\{\overline{k}\times\overline{i}+\overline{k}\times\overline{j}+\overline{i}\times\overline{i}+\overline{i}\times\overline{j}\right\}$$

$$= \frac{1}{2}\left\{\overline{j}-\overline{i}+0+\overline{k}\right\}$$

$$= \frac{1}{2}\left\{\overline{j}+\overline{k}-\overline{i}\right\}$$

Similarly, we can obtain

$\overline{b^1} = \frac{1}{2}\left\{\overline{k}+\overline{i}-\overline{j}\right\}$ and $\overline{c^1} = \frac{1}{2}\left\{\overline{i}+\overline{j}-\overline{k}\right\}$

25. Scalar triple product of the vectors of the given set

$$\begin{vmatrix} 2 & 1 & -3 \\ 3 & 2 & -1 \\ 1 & 3 & -2 \end{vmatrix}$$

$= 2(-4+3) - 1(-6+1) - 3(9-2)$

$= -2 + 5 - 21 = -18$

Let $\left[\overline{a^1}, \overline{b^1}, \overline{c^1}\right]$ represent the reciprocal system

Then,

$$\overline{a^1} = \frac{1}{-18}\left\{(3\overline{i}+2\overline{j}-\overline{k})\times(\overline{i}+3\overline{j}-2\overline{k})\right\}$$

$$= \frac{1}{-18}\left\{(9\overline{k}+6\overline{j}-2\overline{k}-4\overline{i}-\overline{j}+3\overline{i}\right\}$$

$$= \frac{1}{-18}\left(-\overline{i}+5\overline{j}+7\overline{k}\right)$$

$$\overline{b^1} = \frac{-1}{18}\left\{(\overline{i}+3\overline{j}-2\overline{k})\times(2\overline{i}+\overline{j}-3\overline{k})\right\}$$

$$= \frac{-1}{18}\left\{(\overline{k}+3\overline{j}-6\overline{k}-9\overline{i}-4\overline{j}+2\overline{i})\right\}$$

$$= \frac{-1}{18}\left\{-7\overline{i}-\overline{j}-5\overline{k}\right\}$$

$$\overline{c'} = \frac{-1}{18}\left\{(2\overline{i}+\overline{j}-3\overline{k})\times(3\overline{i}+2\overline{j}-\overline{k})\right\}$$

$$= \frac{-1}{18}\left\{4\overline{k}+2\overline{j}-3\overline{k}-\overline{i}-9\overline{j}+6\overline{i}\right\}$$

$$= \frac{-1}{18}\left\{5\overline{i}-7\overline{j}+\overline{k}\right\}$$

26. If $\left(\overline{a}'\,\overline{b}'\,\overline{c}'\right)$ is reciprocal system of $\left(\overline{a}, \overline{b}, \overline{c}\right)$ then $\left(\overline{a}\ \overline{b}\ \overline{c}\right)$ is the reciprocal system of $\left(\overline{a}'\,\overline{b}'\,\overline{c}'\right)$

Choice (c)

27. $\begin{vmatrix} 1 & 1 & -1 \\ 2 & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 3 & 3 & 0 \\ 9 & 9 & 0 \end{vmatrix} = 0$

$\Rightarrow$ $\overline{A}, \overline{B}, \overline{C}$ are coplanar.

A unit vector along $\overline{A} = \frac{1}{\sqrt{3}}\left(\overline{i}+\overline{j}-\overline{k}\right)$

A unit vector along $\overline{B} = \frac{1}{3}\left(2\overline{i}+2\overline{j}+\overline{k}\right)$

A vector along the bisector of the angle between $\overline{A}$ and $\overline{B}$ can be

$$\frac{\overline{A}+\overline{B}}{2} \text{ or } \frac{\overline{A}-\overline{B}}{2}$$

$$= \frac{1}{2}\left\{\left(\frac{2}{3}+\frac{1}{\sqrt{3}}\right)\overline{i}+\left(\frac{1}{\sqrt{3}}+\frac{2}{3}\right)\overline{j}+\left(\frac{1}{3}-\frac{1}{\sqrt{3}}\right)\overline{k}\right\}$$

or

$$\frac{1}{2}\left\{\left(\frac{1}{\sqrt{3}}-\frac{2}{3}\right)\left(\overline{i}+\overline{j}\right)-\left(\frac{1}{\sqrt{3}}+\frac{1}{3}\right)\overline{k}\right\}$$

Projection of $\overline{A}$ on $\overline{B}$ $= \dfrac{\overline{A}.\overline{B}}{|\overline{B}|} = \dfrac{3}{3} = 1$

28. $\overline{a}\times\left(\overline{b}\times\overline{c}\right) = \left(\overline{a}.\overline{c}\right)\overline{b} - \left(\overline{a}.\overline{b}\right)\overline{c}$

$= -\left(\overline{a}.\overline{b}\right)\overline{c}$, since $\overline{a}.\overline{c} = 0$

$$\left(\overline{a}\times\overline{b}\right)\times\overline{c} = -\left\{\overline{c}\times\left(\overline{a}\times\overline{b}\right)\right\}$$

$$= -\left\{\left(\overline{c}.\overline{b}\right)\overline{a} - \left(\overline{c}.\overline{a}\right)\overline{b}\right\}$$

$$= \left(\overline{c}.\overline{a}\right)\overline{b} - \left(\overline{c}.\overline{b}\right)\overline{a}$$

$$= -\left(\overline{c}.\overline{b}\right)\overline{a}$$

We have

$$\left[\overline{b}+\overline{c}, \overline{c}+\overline{a}, \overline{a}+\overline{b}\right] = 2\left[\overline{a}\overline{b}\overline{c}\right]$$

$= 2|\overline{a}||\overline{b}||\overline{c}|$ (since $\overline{a}, \overline{b}, \overline{c}$ are mutually orthogonal)

29. Equation of the plane containing the lines is

$$\begin{vmatrix} x-1 & y+2 & z-5 \\ 3 & 0 & 1 \\ 1 & -1 & 4 \end{vmatrix} = 0$$

$(x-1) - (y+2)11 + (z-5)(-3) = 0$

$x - 1 - 11y - 22 - 3z + 15 = 0$

$x - 11y - 3z - 8 = 0$

$3r_1 + 1 = r_2 + 5$

$-2 = -r_2 - 3$

$r_1 + 5 = 4r_2 + 10$

$r_2 = -1$

$3r_1 + 1 = 4$

$3r_1 = 3$

$r_1 = 1$

Point of intersection is (4, –2, 6)

If θ is the angle between the lines,

$$\cos\theta = \frac{3+4}{\sqrt{10}.\sqrt{18}} = \frac{7}{6\sqrt{5}}$$

For (d) required plane

$x - 11y - 3z = 0$

30. (a) $2|i + aj + 3k| = |4i - (4a - 2)j - 2k|$

$\Rightarrow \quad 2\sqrt{1+a^2+9} = \sqrt{16+(4a-2)^2+4}$

$\Rightarrow \quad 4(10 + a^2) = 20 + (4a - 2)^2$

$\Rightarrow \quad 4a^2 + 40 = 16a^2 - 16a + 24$

$\Rightarrow \quad 3a^2 - 4a - 4 = 0$

$\Rightarrow \quad a = 2, -\frac{2}{3}$

and $\cos\theta = \frac{-1}{2}$ where, $a = 2$ or $\frac{2}{3}$

(b) Angle between the vectors $\bar{x}$ and $\bar{y}$ is acute and angle between $\bar{b}$ and y-axis is obtuse.

$\Rightarrow \quad \bar{x}.\bar{y} > 0$ and $\bar{y}.\bar{j} < 0$

$\Rightarrow \quad 2a^2 - 3a + 1 > 0$ and $a < 0$

$\Rightarrow \quad a < \frac{1}{2}$ or $a > 1$ and $a < 0$

$\Rightarrow \quad a < 0$

(c) Let P(1, –1, 2) A(2, –1, 3)

Vector moment = AP × F

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix} = 2i - 7j - 2k$$

$\therefore \quad \lambda = -7$

(d) Vectors are coplanar $\Rightarrow \begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$

$\Rightarrow \quad \lambda^3 - 6\lambda - 4 = 0$

$\Rightarrow \quad (\lambda + 2)(\lambda^2 - 2\lambda - 2) = 0$

$\lambda = -2$

IIT Assignment Exercise

31. $|\bar{r} - \bar{s}| = |\bar{r}|^2 + |\bar{s}|^2 - 2|\bar{r}||\bar{s}|\cos 60^\circ = 1.$

32. Required vector $= \dfrac{(\bar{i}+\bar{j})\times(\bar{j}+\bar{k})}{|(\bar{i}+\bar{j})\times(\bar{j}+\bar{k})|}$

$$= \frac{\bar{i}\times\bar{j}+\bar{i}\times\bar{k}+\bar{j}\times\bar{k}}{|\bar{i}\times\bar{j}+\bar{i}\times\bar{k}+\bar{j}\times\bar{k}|}$$

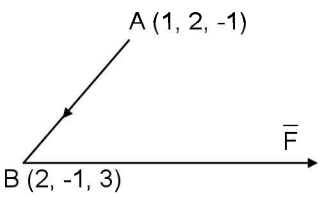
$$= \frac{\bar{k}-\bar{j}+\bar{i}}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}(\bar{i}-\bar{j}+\bar{k}).$$

33. $\dfrac{\bar{a}\cdot\bar{b}}{|\bar{b}|} = \dfrac{2(\bar{b}\cdot\bar{a})}{|\bar{a}|}$

$\therefore \quad 15\hat{i} + \frac{15}{\theta}\hat{j}$

$|\bar{a}| - |\bar{b}| = |\bar{b}|.$

34.



$$\text{Moment} = \overline{AB}\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -3 & 4 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \bar{i}(-3) - \bar{j}(1-12) + \bar{k}(9)$$

$$= -3\bar{i} + 11\bar{j} + 9\bar{k}$$

$\therefore$ Magnitude $= \sqrt{211}$.

35. A (10, 7, 0); B (6, 6, -1); C (6, 9, -4)

$AB^2 = 4^2 + 1^2 + 1^2 = 18$

$BC^2 = 3^2 + 3^2 = 18$

$CA^2 = 4^2 + 2^2 + 4^2 = 36$

$\therefore \quad AB^2 + BC^2 = CA^2$ and $AB = BC$

36. On the XOZ plane $y = 0$

$\therefore \quad ly_2 + my_1 = 0$

i.e., $l \times 3 - m = 0$

$3l = m$

$\frac{\ell}{m} = \frac{1}{3}.$

37. $\cos\theta = \dfrac{12+2+6}{\sqrt{9}\,\sqrt{49}} = \dfrac{20}{21}$

$\therefore \quad \theta = \cos^{-1}\left(\dfrac{20}{21}\right)$

38. Distance $= \left|\dfrac{(2-2-4-9)}{\sqrt{1+4+16}}\right| = \dfrac{13}{\sqrt{21}} = \dfrac{13}{\sqrt{7}\,\sqrt{3}}$.

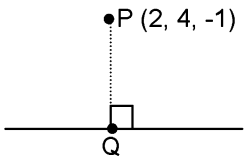
39. $2x - y + 2z + 1 = 0$

$2x - y + 2z + \dfrac{11}{2} = 0$

$$d = \frac{\frac{11}{2}-1}{\sqrt{2^2+1^2+2^2}} = \frac{9/2}{3} = \frac{3}{2}.$$

40. The point Q can be taken as

$r - 5, 4r - 3, -9r + 6$



$\therefore$ Direction ratios of PQ are

$(r - 7, 4r - 7, -9r + 7)$

$\therefore \quad r - 7 + 4(4r - 7) - 9(-9r + 7) = 0$ if PQ $\perp$ line

$\therefore \quad r - 7 + 16r - 28 + 81r - 63 = 0$

$\therefore \quad r = 1$

$\therefore$ Q is $(-4, 1, -3)$

$\therefore \quad PQ = \sqrt{6^2+3^2+2^2} = 7$

41. The line is parallel to the plane since

$3 \times 2 - 2 \times 2 - 2 \times 1 = 0$

$\therefore$ Required distance is the distance of the point $(1, -2, 1)$ from the plane $2x + 2y - z - 6 = 0$ i.e.,

$$\left|\frac{(2-4-1-6)}{\sqrt{4+4+1}}\right| = \frac{9}{3} = 3.$$

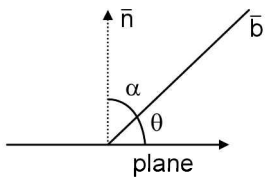
42. Direction ratios of the normal to the plane are $(2, -1, 1)$.

Direction ratios of the line are $(1, -1, 1)$

$\therefore$ Angle between the plane and the line is

$$\sin\theta = \frac{2+1+1}{\sqrt{4+1+1}\,\sqrt{1+1+1}} = \frac{4}{\sqrt{6}\,\sqrt{3}} = \frac{2\sqrt{2}}{3}$$

43.



The line is parallel to the vector $\bar{b}$. The normal to the plane is along the vector $\bar{n}$.

$$\cos\alpha = \frac{\bar{b}\cdot\bar{n}}{|\bar{b}||\bar{n}|}$$

$$\cos(90-\theta) = \frac{\bar{b}\cdot\bar{n}}{|\bar{b}||\bar{n}|}$$

$$\therefore \quad \sin\theta = \frac{\bar{b}\cdot\bar{n}}{|\bar{b}||\bar{n}|}$$

44. Let $\bar{a} = x\bar{i} + y\bar{j} + z\bar{k}$

$\therefore \quad \bar{a}\cdot\bar{k} = 1 \Rightarrow z = 1$

$\bar{a}\cdot(\bar{k}+\bar{i}) = x + z = 1$

$\therefore \quad x = 0$

$\bar{a}\cdot(\bar{i}+\bar{j}+\bar{k}) = x + y + z = 1$

$\therefore \quad y = 0 \therefore \bar{a} = \bar{k}$.

The given condition imply that

$\bar{a}.\bar{j} = 0, \bar{a}.\bar{i} = 0, \bar{a}.\bar{k} = 1$

$\Rightarrow \quad \bar{a} = k$

45. Direction ratios are $(6, -3, -2)$

A set of direction cosines is

$$\left(\frac{6}{\sqrt{36+9+4}}, \frac{-3}{\sqrt{49}}, \frac{-2}{\sqrt{49}}\right)$$

i.e., $\left(\dfrac{6}{7}, \dfrac{-3}{7}, \dfrac{-2}{7}\right)$.

46. The required plane is

$(4x + 3y + 2z + 1) + \lambda(x + 2y + 3z - 4) = 0$

Since it passes through the origin $1 - 4\lambda = 0$

$\lambda = \dfrac{1}{4}$

$\therefore \quad 4x + 3y + 2z + 1 + \left(\dfrac{x+2y+3z-4}{4}\right) = 0$

i.e., $16x + 12y + 8z + 4 + x + 2y + 3z - 4 = 0$

i.e., $17x + 14y + 11z = 0$.

47. Solving, we get

$$\bar{p} = \frac{1}{3}\bar{i} + \bar{j}$$

$$\bar{q} = \frac{1}{3}\bar{i} - \bar{j}$$

$$\bar{p}\cdot\bar{q} = \frac{1}{9} - 1 = -\frac{8}{9}$$

$$|\bar{p}| = \sqrt{\frac{1}{9}+1} = \sqrt{\frac{10}{9}}$$

$$|\bar{q}| = \sqrt{\frac{1}{9}+1} = \sqrt{\frac{10}{9}}$$

$$\therefore \quad \cos\theta = \frac{-8/9}{10/9} = \frac{-4}{5}.$$

$$\Rightarrow \quad \theta \text{ is obtuse and } \cos\theta = \frac{-4}{5}$$

$$\Rightarrow \quad \sin\theta = \frac{3}{5}$$

48. Let $\bar{a} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\therefore \quad \bar{a}\times\bar{i} = z\bar{j} - y\bar{k}$$

$$|\bar{a}\times\bar{i}|^2 = y^2 + z^2$$

$$|\bar{a}\times\bar{j}|^2 = z^2 + x^2$$

$$|\bar{a}\times\bar{k}|^2 = x^2 + y^2$$

Adding, we get

$$\text{RHS} = 2(x^2 + y^2 + z^2) = 2|\bar{a}|^2.$$

49. Vector component of $\bar{a}$ along $\bar{b}$

= (length of projection of $\bar{a}$ on $\bar{b}$) (times unit vector along $\bar{b}$)

$$= \left|\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|}\right|\left(\frac{\bar{b}}{|\bar{b}|}\right)$$

$$= \frac{9\bar{b}}{25} = \frac{9}{25}\left(3\bar{j} + 4\bar{k}\right).$$

50. $\bar{r} = (\bar{i}\cdot\bar{i})\bar{a} - (\bar{i}\cdot\bar{a})\bar{i} + (\bar{j}\cdot\bar{j})\bar{a} - (\bar{j}\cdot\bar{a})\bar{j}$

$$+(\bar{k}\cdot\bar{k})\bar{a} - (\bar{k}\cdot\bar{a})\bar{k} = \bar{a} - \bar{i} + \bar{a} - 2\bar{j} + \bar{a} - 3\bar{k}$$

$$= 3\bar{a} - (\bar{i} + 2\bar{j} + 3\bar{k})$$

$$= 3\bar{a} - \bar{a} = 2\bar{a}$$

$$= 2\bar{i} + 4\bar{j} + 6\bar{k}$$

$$\therefore \quad |\bar{r}| = \sqrt{56} = 2\sqrt{14}.$$

51. $|\bar{a} + \bar{b} + \bar{c}|^2 = (\bar{a} + \bar{b} + \bar{c}).(\bar{a} + \bar{b} + \bar{c})$

$$= |\bar{a}|^2 + |\bar{b}|^2 + |\bar{c}|^2$$

$$+ 2\left[\bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{c} + \bar{c}\cdot\bar{a}\right]$$

$$= 1 + 1 + 1 + 2\left[0 + \cos\frac{\pi}{3} + 0\right] = 4$$

$$\therefore \quad |\bar{a} + \bar{b} + \bar{c}| = 2.$$

52. Unit vector along $\bar{a} = \dfrac{7\bar{i} - 4\bar{j} - 4\bar{k}}{\sqrt{49+16+16}}$

$$\hat{a} = \frac{7}{9}\bar{i} - \frac{4}{9}\bar{j} - \frac{4}{9}\bar{k}$$

$\hat{b}$, unit vector along $\bar{b} = \dfrac{-2\bar{i} - \bar{j} + 2\bar{k}}{3}$

$$\therefore \quad \bar{r} = \left(\frac{\hat{a} + \hat{b}}{|\hat{a} + \hat{b}|}\right)3\sqrt{6} = 3\sqrt{6}\,\frac{(\bar{i} - 7\bar{j} + 2\bar{k})}{\sqrt{54}}$$

$$= \bar{i} - 7\bar{j} + 2\bar{k}.$$

53. For the lines to be coplanar,

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\therefore \quad \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -\alpha \\ \alpha & 2 & 1 \end{vmatrix} = 0$$

$$\therefore \quad \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -\alpha+1 \\ \alpha & \alpha+2 & \alpha+1 \end{vmatrix} = 0$$

$2(\alpha + 1) - (\alpha + 2)(1 - \alpha) = 0$

i.e., $3\alpha + \alpha^2 = 0$

$\alpha(\alpha + 3) = 0$

$\therefore \quad \alpha = 0$ or -3.

54. Equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

where, $p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$

i.e., $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$

Now, if (u, v, w) is the centroid of triangle ABC,

$u=\frac{a}{3}, v=\frac{b}{3}, w=\frac{c}{3}$

$a = 3u, b = 3v, c = 3w$

Substituting, $\frac{1}{9u^2}+\frac{1}{9v^2}+\frac{1}{9w^2}=\frac{1}{p^2}$

$\therefore$ Locus of (u, v, w) is

$x^{-2}+y^{-2}+z^{-2}=9p^{-2}$.

55. Equation of the required plane can be taken as

$A(x-1)+B(y-3)+C(z-13)=0$

where $2A + B + 2C = 0$

Also since this plane is perpendicular to the given plane

$A + B + C = 0$

$\therefore \frac{A}{-1}=\frac{B}{0}=\frac{C}{1}$

i.e., $A : B : C = -1 : 0 : 1$

$\therefore$ The equation of the plane is

$-(x-1)+(z-13)=0$

$-x+z+1-13=0$

$\therefore z - x = 12.$

56. The direction cosines of one such diagonal are

$<\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}>$

Direction cosines of another diagonal are

$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)$

$\therefore \cos\theta=\frac{1}{3}+\frac{1}{3}-\frac{1}{3}=\frac{1}{3}.$

57. $(\bar{b}+\bar{c})\cdot\frac{(\bar{b}\times\bar{c})}{[\bar{a}\ \bar{b}\ \bar{c}]}+$

$(\bar{c}+\bar{a})\cdot\frac{(\bar{c}\times\bar{a})}{[\bar{a}\ \bar{b}\ \bar{c}]}+(\bar{a}+\bar{b})\cdot\frac{(\bar{a}\times\bar{b})}{[\bar{a}\ \bar{b}\ \bar{c}]}$

$=\frac{1}{[\bar{a}\ \bar{b}\ \bar{c}]}\{\bar{b}\cdot\bar{b}\times\bar{c}+\bar{c}\cdot\bar{b}\times\bar{c}+\ldots\}$

$=\frac{1}{[\bar{a}\ \bar{b}\ \bar{c}]}\{0+0\ldots\}=0.$

58. $\bar{r}=(-\bar{i}+\bar{j}-\bar{k})+s(\bar{i}+\bar{j}-\bar{k})$

i.e., $\bar{r}=\bar{a}_1+s\bar{b}_1$

$\bar{r}=(\bar{i}-\bar{j}+2\bar{k})+t(-\bar{i}+2\bar{j}+\bar{k}) \quad =\bar{a}_2+t\bar{b}_2$

$SD=\left|\frac{(\bar{b}_1\times\bar{b}_2)\cdot(\bar{a}_2-\bar{a}_1)}{|\bar{b}_1\times\bar{b}_2|}\right|$

$\bar{b}_1\times\bar{b}_2=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & 1 & -1\\ -1 & 2 & 1\end{vmatrix}=3\bar{i}+3\bar{k}$

$(\bar{b}_1\times\bar{b}_2)\cdot(a_2-a_1)=(3\bar{i}+3\bar{k})\cdot(2\bar{i}-2\bar{j}+3\bar{k})$

$=6+9=15$

$|\bar{b}_1\times\bar{b}_2|=3\sqrt{2}$

$\therefore SD=\frac{15}{3\sqrt{2}}=\frac{5}{\sqrt{2}}.$

59. $\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}^2=[a\ \ b\ \ c]^2$

$=|\bar{a}\times\bar{b}||\bar{c}|\cos 0$ as $\bar{c}$ is $\perp$ to both $\bar{a}$ and $\bar{b}$ and hence parallel to $\bar{a}\times\bar{b}$

$=|\bar{a}\times\bar{b}|^2.1$

$=\left\{|\bar{a}||\bar{b}|\sin\frac{\pi}{6}\right\}^2$

$=\left\{1.1.\frac{1}{2}\right\}^2=\frac{1}{4}$

60. $\bar{a}\cdot\bar{b}>0$

$\therefore 2(1+m)+2m^2-7m>0$

i.e., $2m^2-5m+2>0$

i.e., $(2m-1)(m-2)>0$

$\left(m-\frac{1}{2}\right)(m-2)>0$

$\therefore$ m lies outside the interval $\left[\frac{1}{2},2\right]$.

61. $\begin{vmatrix} a & -a & b \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$

$\therefore \quad \begin{vmatrix} a & -a & b-a \\ 1 & 0 & 0 \\ c & c & b-c \end{vmatrix} = 0$

i.e., $-\{-a(b-c) - c(b-a)\} = 0$

i.e., $ab - ac + bc - ac = 0$

$(a + c)\, b = 2ac$

$b = \dfrac{2ac}{a+c}$

$\therefore$ a, b, c are in HP.

62. Simplifying, we get

$$\left(-\bar{a}\times\bar{b} - \bar{a}\times\bar{c} - \bar{b}\times\bar{a} + \bar{b}\times\bar{c}\right)\cdot\left(\bar{a} + 2\bar{b} - \bar{c}\right)$$

$$= \left(\bar{a} + 2\bar{b} - \bar{c}\right)\cdot\left(\bar{b}\times\bar{c} - \bar{a}\times\bar{c}\right)$$

$$= \bar{a}\cdot\bar{b}\times\bar{c} - 2\bar{b}\cdot\bar{a}\times\bar{c}$$

$$= \bar{a}\cdot\bar{b}\times\bar{c} + 2\bar{a}\cdot\bar{b}\times\bar{c}$$

$$= 3\left[\bar{a}\ \bar{b}\ \bar{c}\right].$$

63. If ℓ_1, m_1, n_1 are the direction ratios of L_1

$3\ell_1 + 2m_1 + n_1 = 0$

$\ell_1 + m_1 - 2n_1 = 0$

$\therefore \quad \dfrac{\ell_1}{-4-1} = \dfrac{m_1}{1+6} = \dfrac{n_1}{3-2}$

$\dfrac{\ell_1}{-5} = \dfrac{m_1}{7} = \dfrac{n_1}{1}$

$\therefore \quad \ell_1 : m_1 : n_1 = -5 : 7 : 1.$

If ℓ_2, m_2, n_2 are the direction ratios of L_2,

$2\ell_2 - m_2 - n_2 = 0$

$7\ell_2 + 10m_2 - 8n_2 = 0$

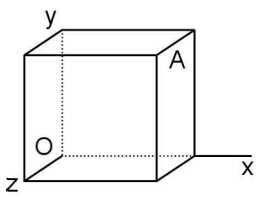
$\dfrac{\ell_2}{8+10} = \dfrac{m_2}{-7+16} = \dfrac{n_2}{20+7}$

$\dfrac{\ell_2}{18} = \dfrac{m_2}{9} = \dfrac{n_2}{27}$

$\therefore \quad \ell_2 : m_2 : n_2 = 2 : 1 : 3$

Since $\ell_1\ell_2 + m_1 m_2 + n_1 n_2 = -10 + 7 + 3 = 0$, angle is 90^0.

64.



Choose one corner as origin and the coterminous edges as the axes of the coordinates.

Let a be the side of the cube.

The direction ratio of one diagonal OA is <a, a, a>

Direction cosines are $< \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}} >$

If ℓ, m, n are the direction cosines of the line,

$$\cos\alpha = \frac{\ell}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}}$$

Similarly,

$$\cos\beta = \frac{\ell}{\sqrt{3}} + \frac{m}{\sqrt{3}} - \frac{n}{\sqrt{3}};$$

$$\cos\gamma = \frac{\ell}{\sqrt{3}} - \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}};$$

$$\cos\delta = -\frac{\ell}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}}$$

Squaring and adding,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= 2\left[\left(\frac{\ell+m}{\sqrt{3}}\right)^2 + \left(\frac{n}{\sqrt{3}}\right)^2 + \left(\frac{\ell-m}{\sqrt{3}}\right)^2 + \left(\frac{n}{\sqrt{3}}\right)^2\right]$$

$$= 4\left(\frac{\ell^2}{3} + \frac{m^2}{3} + \frac{n^2}{3}\right) + \left(\frac{2lm}{3} + \frac{2mn}{3} + \frac{2\ell n}{3}\right)$$

$$= \frac{4}{3}\left(\ell^2 + m^2 + n^2\right)$$

$$= \frac{4}{3}$$

65. The direction ratios of the line are <1, –1, 2>.

Equation of the plane is $3x + y - z + 2 = 0$ whose normal has direction ratios <3, 1, –1>.

The line is parallel to the plane since

$3 \times 1 + 1\,(-1) + 2\,(-1) = 0$

Also the point (2, –3, 5) on the line is also a point on the plane since $3\,(2) - 3 - 5 + 2 = 0$

$\therefore$ The line l lies in the plane p.

66. Given $|\bar{a}| = |\bar{b}|$ — (1)

and $|\bar{b}||\bar{c}| = 4$ — (2)

$\bar{b} \times (\bar{b} \times \bar{c}) + 4\bar{a} = 0$

$\Rightarrow \quad |\bar{b} \times (\bar{b} \times \bar{c})| = |-4\bar{a}|$

$\Rightarrow \quad |\bar{b}|\,|\bar{b} \times \bar{c}| \sin\frac{\pi}{2} = 4|\bar{a}|$

[$\because \bar{b} \times \bar{c}$ is a vector $\perp^r$ to both $\bar{b}$ and $\bar{c}$]

$\Rightarrow \quad |\bar{b} \times \bar{c}| = 4$ [by (1)]

$\Rightarrow \quad |\bar{b}||\bar{c}| \sin\theta = 4$

$\Rightarrow \quad 4(\sin\theta) = 4$ [by (2)]

$\Rightarrow \quad \theta = \frac{\pi}{2}$

$\therefore \quad \sin 2\theta + \cos 2\theta = -1$

67. $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix}$

$= \frac{1}{\log x \log y \log z} \begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0$

$\Rightarrow$ the vectors are coplanar

68. Required Volume

$= \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix}$

[Using $C_1 : C_1 + C_2 + C_3$]

$= \begin{vmatrix} 0 & \beta & \gamma \\ 0 & \gamma & \alpha \\ 0 & \alpha & \beta \end{vmatrix} = 0$ [$\alpha + \beta + \gamma = 0$ as α, β, γ are roots of $x^3 + px + q = 0$]

69. A(1, –1, 3) is the image of B(5, 1, –3) in $2x + y - 3z + k = 0$

$\Rightarrow$ Midpoint of AB lies in the plane and AB $\perp$ the plane

$\Rightarrow$ (3,0,0) is on the plane

$\Rightarrow \quad 6 + k = 0 \Rightarrow k = -6$

70. The given planes are parallel.

$\therefore$ Radius of the sphere touching these 2 planes

$= \frac{1}{2}$ x distance between these 2 planes

$= \frac{1}{2}\left|\frac{\frac{9}{2} - 3}{\sqrt{1+4+4}}\right| = \frac{1}{2} \times \frac{3}{2 \times 3} = \frac{1}{4}$

71. $\bar{a} \cdot \bar{b} = 5 + 2mx + m^2x^2$ (= f(x) say)

We want $m^2x^2 + 2mx + 5 < 0$ for all real x which is not possible, since discriminant of the above quadratic in $x = 4m^2 - 20m^2$ is less than 0 and coefficient of $x^2 = m^2 > 0$. Therefore the expression is always > 0 for any real x and for any real m.

$\Rightarrow$ There is no real value of m

72. Let $|\bar{a}| = |\bar{b}| = 1$; Given $|\bar{a} + \bar{b}| = \frac{1+\sqrt{3}}{\sqrt{2}}$

Then $|\bar{a} + \bar{b}|^2 = 2 + \sqrt{3}$

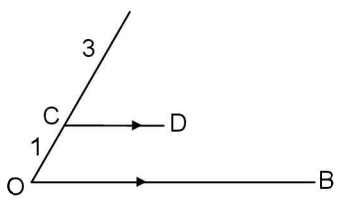
Also $|\bar{a} + \bar{b}|^2 = |\bar{a}|^2 + |\bar{b}|^2 + 2|\bar{a}||\bar{b}|\cos\theta$

i.e., $2 + \sqrt{3} = 2 + 2..1.1\cos\theta$

$\Rightarrow \quad \cos\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{6}$

73.



We have

$\overline{OC} = \frac{1}{4}\overline{OA} = \frac{\bar{a}}{4}$

$\overline{CD} = \frac{1}{4}\overline{OB} = \frac{1}{4}\bar{b}$

Now $\overline{OD} = \overline{OC} + \overline{CD} = \dfrac{\bar{a}+\bar{b}}{4}$

$\therefore \quad \overline{AD} \times \overline{AB} = \left(\overline{OD} - \overline{OA}\right) \times \left(\overline{OB} - \overline{OA}\right)$

$$= \frac{\left(\bar{b} - 3\bar{a}\right)}{4} \times (\bar{b} - \bar{a})$$

$$= \frac{-\bar{b} \times \bar{a} - 3\bar{a} \times \bar{b}}{4} = \frac{-\left(\bar{a} \times \bar{b}\right)}{2}$$

74. Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of the triangle.

Then the x-coordinates of D, E, F are

$$\frac{2x_2 + x_3}{3}, \frac{2x_3 + x_1}{3}, \frac{2x_1 + x_2}{3}$$

$\Rightarrow \quad x_1 = \dfrac{-2\ell}{3}, x_2 = \dfrac{4}{3}\ell, x_3 = \dfrac{\ell}{3}$

In a similar way we get

$y_1 = \dfrac{m}{3}$, $y_2 = \dfrac{-2m}{3}$, $y_3 = \dfrac{4m}{3}$ and

$z_1 = \dfrac{4n}{3}, z_2 = \dfrac{n}{3}, z_3 = \dfrac{-2n}{3}$.

$\therefore \quad A\left(\dfrac{-2\ell}{3}, \dfrac{m}{3}, \dfrac{4n}{3}\right), B\left(\dfrac{4\ell}{3}, \dfrac{-2m}{3}, \dfrac{n}{3}\right),$

$C\left(\dfrac{\ell}{3}, \dfrac{4m}{3}, \dfrac{-2n}{3}\right)$

Now $AB^2 + BC^2 + CA^2 = 6(l^2 + m^2 + n^2)$

$\Rightarrow$ Required ratio = 6

75. $|\bar{a} - \bar{b}|^2 = (\bar{a} - \bar{b}).(\bar{a} - \bar{b})$

$= |\bar{a}|^2 + |\bar{b}|^2 - 2\bar{a}.\bar{b}$

$= 1 + 1 - 2|\bar{a}||\bar{b}|\cos\theta$

$[\,|\bar{a}| = |\bar{b}| = 1]$

$= 2(1 - \cos\theta) = 4\sin^2\theta/2$

Similarly, $|\bar{a} + \bar{b}|^2 = 4\cos^2\dfrac{\theta}{2}$

$\therefore \cos\dfrac{\theta}{2}|\bar{a} - \bar{b}| - \sin\dfrac{\theta}{2}|\bar{a} + \bar{b}| = 0$

76. Given vectors are perpendicular

$\Rightarrow \quad 3p + 2q + 4r = 0$

It $x + y + z = 0$ then $x^3 + y^3 + z^3 = 3xyz$.

$\Rightarrow \quad 27p^3 + 8q^3 + 64r^3 = 3(3p)(2q)(4r)$

$= 72pqr$

77. Let the third vertex be $a\hat{i} + b\hat{j} + c\hat{k}$

ΔABC is equilateral.

$\Rightarrow$ Its circumcenter and centroid coincide

$\Rightarrow \quad \hat{i} + \hat{j} + \hat{k} = \dfrac{(4+a)\hat{i} + b\hat{j} + (c+2)\hat{k}}{3}$

$\Rightarrow \quad a = -1, b = 3, c = 1$

$\Rightarrow$ Third vertex is $-\hat{i} + 3\hat{j} + \hat{k}$

78. Given $p.(\bar{a} \times \bar{b}) + q(\bar{b} \times \bar{c}) + r(\bar{c} \times \bar{a}) = 0$

Take dot product with $\bar{a}$ we have,

$p[\bar{a}\ \bar{b}\ \bar{a}] + q[\bar{b}\ \bar{c}\ \bar{a}] + r[\bar{c}\ \bar{a}\ \bar{a}] = 0$

$\Rightarrow \quad q[\bar{a}\ \bar{b}\ \bar{c}] = 0$

Taking dot product with $\bar{b}$ and $\bar{c}$ we have

$r[\bar{a}\ \bar{b}\ \bar{c}] = 0$ and $p[\bar{a}\ \bar{b}\ \bar{c}] = 0$

$\therefore$ p, q, r are not all zero simultaneously, we must have $[\bar{a}\ \bar{b}\ \bar{c}] = 0$

Now $[\bar{a} + \bar{b}\ \ \bar{b} + \bar{c}\ \ \bar{c} + \bar{a}]$

$= 2[\bar{a}\ \bar{b}\ \bar{c}] = 0$

79. Given volume $= \begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 4$ (= V say)

Required volume

$$= \begin{vmatrix} x^3 - 1 & 0 & x - x^4 \\ 0 & x - x^4 & x^3 - 1 \\ x - x^4 & x^3 - 1 & 0 \end{vmatrix}$$

= Determinant of cofactors = $V^2 = 16$

80. $\bar{a} + \bar{b} + \bar{c}$ is collinear with $\bar{d}$

$\Rightarrow \bar{a} + \bar{b} + \bar{c} = k\bar{d}$ — (1)

Similarly, $\bar{b} + \bar{c} + \bar{d} = \ell\bar{a}$ — (2)

$\Rightarrow \bar{b} + \bar{c} + \dfrac{\bar{a} + \bar{b} + \bar{c}}{k} = \ell\bar{a}$ [using (1)]

$\Rightarrow \dfrac{1}{k}\bar{a} + \left(1 + \dfrac{1}{k}\right)\bar{b} + \left(1 + \dfrac{1}{k}\right)\bar{c} = \ell\bar{a}$

And $\bar{a}, \bar{b}, \bar{c}$, are not coplanar $\Rightarrow \dfrac{1}{k} = \ell$ and

$1 + \dfrac{1}{k} = 0 \Rightarrow k = \ell = -1$

$\Rightarrow \quad \bar{a}+\bar{b}+\bar{c}+\bar{d}=0$ [using(1) or (2)]

$\Rightarrow \quad |\bar{a}+\bar{b}+\bar{c}+\bar{d}|=0$

81. The magnitudes of the vectors given are all equal and the first is sum of the remaining.

82. $\overline{OA}=2\bar{i}+3\bar{j}+5\bar{k}$

$\overline{OB}=\bar{i}+2\bar{j}+3\bar{k}$

$\overline{OC}=-5\bar{i}+4\bar{j}-2\bar{k}$

$\overline{OD}=\bar{i}+10\bar{j}+10\bar{k}$

$\overline{AB}=-\bar{i}-\bar{j}-2\bar{k}$

$\overline{CD}=6\bar{i}+6\bar{j}+12\bar{k} = -6\left(-\bar{i}-\bar{j}-2\bar{k}\right)$

$= -6\,\overline{AB}$

$\therefore \quad \overline{AB}$ is parallel to $\overline{CD}$.

83. $\left|\bar{a}\times\bar{b}\right|=\left|\bar{a}\right|\left|\bar{b}\right|\sin\theta$

$\bar{a}\cdot\bar{b}=\left|\bar{a}\right|\left|\bar{b}\right|\cos\theta$

$\therefore \quad \left|\bar{a}\times\bar{b}\right|^2+\left(\bar{a}.\bar{b}\right)^2=\left|\bar{a}\right|^2\left|\bar{b}\right|^2$

$\therefore \quad \left|\bar{a}\times\bar{b}\right|^2=144-81=63$

$\therefore \quad \left|\bar{a}\times\bar{b}\right|=3\sqrt{7}$.

84. Both $\bar{a}\times\bar{b}$ and $\bar{b}\times\bar{a}$ are perpendicular to $\bar{a}$ and $\bar{b}$ and $\bar{a}\times\bar{b}\neq 0$. So, 2.

85. $\ell^2 + m^2 + n^2 = 1$

$\therefore \quad \cos^2 60 + m^2 + \cos^2 45 = 1$

$\frac{1}{4}+m^2+\frac{1}{2}=1$

$m^2=\frac{1}{4}; m=\pm\frac{1}{2}$

$\cos\beta=\pm\frac{1}{2}$

$\therefore \quad \beta = 60°$ or $120°$.

86. The line $x = 2$, $y = 3$ is parallel to the z-axis. Also $y = 3, z = 5$ is parallel to x-axis. These lines intersect at (2, 3, 5) and since x-axis is perpendicular to the z-axis, angle between the lines is 90°.

87. $2 \times 1 - 3 \times 3 + 7 \times 1 = 0$

$2 \times 2 + 1 \times 3 - 1 \times 7 = 0$

$\therefore$ The given line is parallel to both the given planes.

88. If l, m, n are direction ratios of the line, we have

$l + 0 - n = 0$

$3l + 4m + 5n = 0$

$\therefore \quad l = n$ and $8l + 4m = 0$

$m = -2l$

$\therefore$ Direction ratios are <1, -2, 1>

Equation is

$$\frac{x-1}{1}=\frac{y-2}{-2}=\frac{z+1}{1}.$$

89. Let l, m, n be the direction ratios of the line. Since the line is perpendicular to the normals to the planes

$6l + 2m + 2n = 0$ and $l + 2m - 6n = 0$

$$\therefore \quad \frac{\ell}{-12-4}=\frac{m}{2+36}=\frac{n}{12-2}$$

$\therefore$ Direction ratios are $-16, 38, 10$ i.e., $-8, 19, 5$

$\therefore$ Equation is

$$\frac{x-1}{-8}=\frac{y-4}{19}=\frac{z+2}{5}.$$

90. If l, m, n are the direction ratios of the line of intersection of the given planes,

$l + m + 2n = 0$ and $2l + 3m + 7n = 0$

$$\frac{\ell}{7-6}=\frac{m}{4-7}=\frac{n}{3-2}$$

$\therefore$ Direction ratios are <1, -3, 1>

Direction cosines are

$$\frac{1}{\sqrt{1+9+1}},\frac{-3}{\sqrt{1+9+1}},\frac{1}{\sqrt{1+9+1}}$$

i.e., $\frac{1}{\sqrt{11}},\frac{-3}{\sqrt{11}},\frac{1}{\sqrt{11}}$.

91. The given lines are

$$\frac{x-1}{3}=\frac{y-1}{-1}=\frac{z+1}{0}$$

and $\frac{x-4}{2}=\frac{y-0}{0}=\frac{z+1}{3}$.

A point on the first line is $(3r_1 + 1, -r_1 + 1, -1)$ for some r_1.

A point on the second line is $(2r_2 + 4, r_2, 3r_2 - 1)$ for some r_2.

Common point is when $r_2 = 0$, $r_1 = 1$. The point is $(4, 0, -1)$.

92. The given plane is same as $2x - y + 3z = 5$.

A line perpendicular to this has direction ratios 2, $-1, 3$. Also, the line passes through the point $(1, -1, 2)$.

$\therefore$ The required equation is

$$\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-2}{3}.$$

93. Since $|\bar{a}|=1=|\bar{b}|$ and $|\bar{a}+\bar{b}|=\sqrt{3}$, angle between $\bar{a}$ and $\bar{b}$ is 60°

$\therefore \quad \bar{a}\cdot\bar{b}=\cos 60^\circ=\frac{1}{2}$

Now, $(2\bar{a}-5\bar{b})\cdot(3\bar{a}+\bar{b})$

$$=6|\bar{a}|^2-5|\bar{b}|^2+2\bar{a}\cdot\bar{b}-15\bar{a}\cdot\bar{b}$$

$$=6-5+1-\frac{15}{2}=\frac{-11}{2}.$$

94. $\bar{u}=\hat{i}+\hat{j}$ and $\bar{v}=\hat{i}-\hat{j} \Rightarrow \bar{u}$ and $\bar{v}$ lie on x-y plane.
$\bar{u}.\bar{n}=0$ and $\bar{v}.\bar{n}=0 \Rightarrow \bar{n}$ is a unit vector perpendicular to x-y plane.

$\therefore \quad \bar{n}=\pm\bar{k}$

$\therefore \quad |\bar{w}.\bar{n}|=3$

95. $\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & 2\alpha & 3\beta \end{vmatrix}=0$

(i.e.,) $1-2\alpha(0)+3\beta(-1)=0 \Rightarrow \beta=\frac{1}{3}$

$1+4\alpha^2+9\beta^2=6$

$1+4\alpha^2+1=6$

$\alpha^2=1;\ \alpha=\pm 1$

96. $(\bar{r}-\bar{c})\times\bar{b}=0$

$\therefore \quad \bar{r}-\bar{c}=\lambda\bar{b}$

$\bar{r}=\bar{c}+\lambda\bar{b}=(1+3\lambda)\bar{i}+(1-\lambda)\bar{j}+(3+\lambda)\bar{k}$

Now, $\bar{r}\cdot\bar{a}=0$

$\therefore \quad 2(1+3\lambda)+3(1-\lambda)-(3+\lambda)=0$

$\therefore \quad 2\lambda+2=0$

$\lambda=-1$

$\therefore \quad \bar{r}=\bar{c}-\bar{b}$

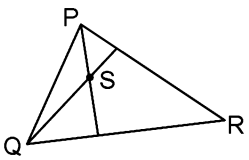
$=-2\bar{i}+2\bar{j}+2\bar{k}=-2(\bar{i}-\bar{j}-\bar{k}).$

97. $(\bar{a}\cdot\bar{b})\bar{c}-(\bar{a}\cdot\bar{c})\bar{b}=\bar{a}\times(\bar{c}\times\bar{b})$

$\therefore \quad \bar{r}=\bar{a}\times(\bar{c}\times\bar{b})$

Hence, $\bar{r}$ is perpendicular to both $\bar{a}$ and $\bar{c}\times\bar{b}$ or $\bar{b}\times\bar{c}$

98.



$(\bar{p}-\bar{s})\cdot(\bar{q}-\bar{r})=0$

$\Rightarrow$ SP is perpendicular to RQ

Similarly, $(\bar{q}-\bar{s})\cdot(\bar{r}-\bar{p})=0$

$\Rightarrow$ SQ is perpendicular to PR

$\therefore$ S is the orthocentre.

99. A plane through (3, 4, 2) is

$A(x-3)+B(y-4)+C(z-2)=0$

Since it passes through (7, 0, 6)

$4A-4B+4C=0$

i.e., $A-B+C=0$

Also, the plane is perpendicular to

$2x-5y=15$

$\therefore \quad 2A-5B=0$

$A=\frac{5}{2}B$

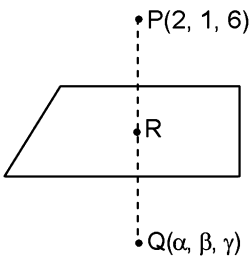
$C=B-A=B-\frac{5}{2}B=-\frac{3}{2}B$

$\therefore \quad A:B:C=\frac{5}{2}:1:-\frac{3}{2}=5:2:-3.$

The plane is $5(x-3)+2(y-4)-3(z-2)=0$

i.e., $5x+2y-3z-17=0$.

100.



Let Q be (α, β, γ)

PQ is normal to the plane.

$\therefore \quad \frac{2-\alpha}{1}=\frac{1-\beta}{1}=\frac{6-\gamma}{-2}=r(\text{say})$

$\alpha=2-r;\ \beta=1-r;\ \gamma=6+2r$

$R\left(\frac{\alpha+2}{2}, \frac{\beta+1}{2}, \frac{\gamma+6}{2}\right)$ lies on plane

(i.e.,) $\left(\frac{4-r}{2}, \frac{2-r}{2}, \frac{12+2r}{2}\right)$ lies on the plane

$\therefore \quad \frac{4-r}{2}, \frac{2-r}{2}, \frac{2(12+2r)}{2} = 3$

$r = -4$

$\therefore$ Q is (6, 5, –2)

101. The plane containing the given line is

$A(x-a) + B(y-b) + C(z-c) = 0$, where, $A\ell + Bm + Cn = 0$

Since the plane passes through the origin

$Aa + Bb + Cc = 0$

$$\therefore \quad \frac{A}{mc-nb} = \frac{B}{na-c\ell} = \frac{C}{b\ell-am}$$

Substituting for A, B, C

$(mc-nb)(x-a) + (na-c\ell)(y-b) + (b\ell-am)(z-c) = 0$

$\therefore \quad x(bn-cm) + y(c\ell-an) + z(am-b\ell) = 0.$

102. The given lines are coplanar if $\begin{vmatrix} \alpha & \beta & \gamma \\ a\alpha & b\beta & c\gamma \\ \ell & m & n \end{vmatrix} = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ \frac{\ell}{\alpha} & \frac{m}{\beta} & \frac{n}{\gamma} \end{vmatrix} = 0$$

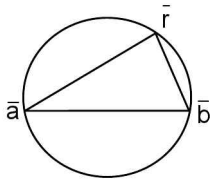
$$\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ \frac{\ell}{\alpha} & \frac{m}{\beta}-\frac{\ell}{\alpha} & \frac{n}{\gamma}-\frac{\ell}{\alpha} \end{vmatrix} = 0$$

$$(b-a)\left(\frac{n}{\gamma}-\frac{\ell}{\alpha}\right) + (c-a)\left(\frac{\ell}{\alpha}-\frac{m}{\beta}\right) = 0$$

i.e., $\frac{\ell}{\alpha}\{c-a-b+a\} - \frac{m}{\beta}(c-a) + \frac{n}{\gamma}(b-a) = 0$

$$\therefore \quad \frac{\ell}{\alpha}(b-c) + \frac{m}{\beta}(c-a) + \frac{n}{\gamma}(a-b) = 0.$$

103.



$\bar{a}, \bar{b}$ are position vectors of the ends of a diameter.

Let $\bar{r}$ be the position vector of any point on the sphere.

Then vector $\bar{r}-\bar{a}$ and $\bar{r}-\bar{b}$ are at right angles

$\therefore \quad (\bar{r}-\bar{a})\cdot(\bar{r}-\bar{b}) = 0$

104. Given $\bar{a}_i \cdot (\bar{a}_1 + \bar{a}_2 + \ldots\ldots + \bar{a}_{i-1} + \bar{a}_{i+1} + \ldots. + \bar{a}_n) = 0$ for $i = 1$ to n

Adding these n equations we have $2 \cdot \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \bar{a}_i \cdot \bar{a}_j = 0$

Now $\left|\bar{a}_1 + \ldots. + \bar{a}_n\right|^2 = \sum_{i=1}^{n} \left|\bar{a}_i\right|^2 + 2\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \bar{a}_i \bar{a}_j$

$= n \left[\because \left|\bar{a}_i\right| = 1 \ \forall i\right]$

$\therefore$ Magnitude of the resultant $= \sqrt{n}$

105. From the given condition, we have

$$\frac{2\text{Area}\Delta DAB}{AB} = \frac{2\text{Area}\Delta DBC}{BC} = \frac{2\text{Area}\Delta DCA}{CA}$$

$$\Rightarrow \quad \frac{P_1 \times (AB)}{AB} = \frac{P_2 \times (BC)}{BC} = \frac{P_3 \times (CA)}{CA}$$

[P_1, P_2, P_3 are respectively, the heights of ΔDAB, ΔDBC and Δ DCA]

$\Rightarrow \quad P_1 = P_2 = P_3$

$\Rightarrow$ The three sides AB, BC, CA are equidistant from D

$\Rightarrow$ D is the incenter of ΔABC

106. Let P_1 and P_2 respectively be the planes containing $\bar{p}, \bar{q}$ and $\bar{r}, \bar{s}$. Then $\bar{p}\times\bar{q}$ is normal to P_1 and $\bar{r}\times\bar{s}$ is normal to P_2

Now $P_1 \perp P_2 \Rightarrow \bar{p}\times\bar{q} \perp \bar{r}\times\bar{s}$

$\Rightarrow$ Projection of one vector on the other is zero

107. We've $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c} = \frac{\bar{b}}{2} - \frac{\sqrt{3}}{2}\bar{c}$

As $\bar{b}$ and $\bar{c}$ are non-parallel we have

$\bar{a}\cdot\bar{c} = \frac{1}{2}$ and $\bar{a}\cdot\bar{b} = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad |\bar{a}||\bar{c}|\cos\theta = \frac{1}{2}$ and $|\bar{a}||\bar{b}|\cos\alpha = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{3}$ and $\alpha = \frac{\pi}{6}$ $\left[\because |\bar{a}| = |\bar{b}| = |\bar{c}| = 1\right]$

108. Equations of the line joining (–1, 2, 5), and (0, 3, 6), are

$$\frac{x+1}{1} = \frac{y-2}{1} = \frac{z-5}{1}$$

Since (–2, 1, 4) satisfies this equation, the given points are collinear.

So distance is 0.

109. Clearly, B lies on the surface.

D.R^s of AB are [3, 2, –1]

D.R^s of the radius joining the point B(–1, 2, 1) are [–1, 2, 1]

Since $3 \times (-1) + 2 \times 2 + (-1) \times 1 = 0$, AB touches the surface, at B.

110. L is the foot of perpendicular from P on YOZ plane

$\Rightarrow$ $L(0, \beta, \gamma)$

Similarly $M(\alpha, 0, \gamma)$

Equation of the plane OLM is of the form

$ax + by + cz = 0$ — (1)

$$\left.\begin{array}{l} L \text{ is on}(1) \Rightarrow b\beta + c\gamma = 0 \\ M \text{ is on}(1) \Rightarrow a\alpha + c\gamma = 0, \end{array}\right\} \Rightarrow a\alpha = b\beta = -c\gamma$$

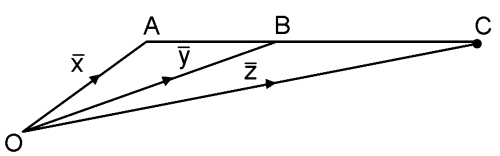
$\Rightarrow \quad a = \frac{k}{\alpha}, b = \frac{k}{\beta}, c = -\frac{k}{\gamma}$

$\therefore$ (1) become, $\frac{x}{\alpha} + \frac{y}{\beta} - \frac{z}{\gamma} = 0$

This passes through $(\gamma, 0, \beta)$. Then

$\frac{\gamma}{\alpha} - \frac{\beta}{\gamma} = 0 \Rightarrow \gamma^2 = \alpha\beta \Rightarrow \alpha, \gamma, \beta$ are in GP.

111.



Statement 2 is true

Consider statement 1 :

Since A, B, C are collinerar,

$\overline{AB} = \lambda\overline{AC}$ where λ is a scalar

$\Rightarrow \quad \bar{y} - \bar{x} = \lambda(\bar{z} - \bar{x})$

$\Rightarrow \quad (\lambda - 1)\bar{x} + \bar{y} - \lambda\bar{z} = 0$

$\ell = \lambda - 1, m = 1, n = -\lambda$

$\Rightarrow \quad \ell + m + n = 0$

$\Rightarrow$ Statement 2 is true

However, it does not follow from statement 2

Choice (b)

112. Statement 2 is true

Consider statement 1 :

Equation of the plane is $\frac{x}{4} + \frac{x}{3} + \frac{x}{2} = 1$ using statement 2, direction ratios of the normal to the plane are $\left[\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right]$

$\Rightarrow$ Statement 1 is true

choice (a)

113. Statement 2 is true

Consider statement 1 : (3, –1, –3) is a point on the line. The direction ratios of the line joining (1, 0, –5) and (3, –1, –3) are [1 – 3, 0 + 1, –5 + 3] i.e., [–2, 1, –2]

or [2, –1, 2]

$\Rightarrow$ true

$\Rightarrow$ choice (a)

114. Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ — (1)

(1) passes through the three given points . This gives

$9 + 4 + 6u + 4w + d = 0$

$1 + 1 + 1 - 2u + 2v + 2w + d = 0$

$4 + 25 + 16 + 4u - 10v + 8w + d = 0$

Centre lies on the plane 2x + 3y + 4z = 6 gives us –2u – 3v – 4w = 6

The 4 equations in u, v, w, d are

$$\left.\begin{array}{l} 6u + 4w + d = -13 \\ -2u + 2v + 2w + d = -3 \\ 4u - 10v + 8w + d = -45 \\ \text{and } 2u + 3v + 4w = -6 \end{array}\right\}$$

Solving these equations we get u = 0, v = 2, w = –3, d = –1

Equation of the sphere is

$x^2 + y^2 + z^2 + 4y - 6z - 1 = 0$

115. The line is $\frac{x}{3}=\frac{y}{2}=\frac{z}{-5}$

Since the centre lies on the above line we may assume its coordinates as (3k, 2k, –5k)

Let the equation of the sphere be

$(x-3k)^2+(y-2k)^2+(z+5k)^2=r^2$ — (1)

(1) passess through (0, –2, –4)

$\Rightarrow$ $9k^2+(2k+2)^2+(5k-4)^2=r^2$

i.e., $38k^2-32k+20=r^2$ — (2)

(1) passess through (2, –1, –1)

$\Rightarrow (2-3k)^2+(1+2k)^2+(5k-1)^2=r^2$

$38k^2-18k+6=r^2$ — (3)

(2) – (3) gives

$-14k+14=0 \Rightarrow k=1 \Rightarrow r^2=38-32+20=26$

Equation of the sphere is

$(x-3)^2+(y-2)^2+(z+5)^2=26$

$x^2+y^2+z^2-6x-4y+10z+12=0$

116. Let the equation of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz=0$ — (1)

Coordinates of A, B, C are given by (–2u, 0, 0); (0, –2v, 0) ; (0, 0, –2w) respectively.

Radius of the sphere (1) is given as 10.

$\Rightarrow$ $u^2+v^2+w^2=100$ — (2)

The centroid of the tetrahedron OABC is at

$$\left(\frac{-2u+0+0+0}{4},\frac{0-2v+0+0}{4},\frac{0+0-2w+0}{4}\right)$$

$$\Rightarrow \left(\frac{-u}{2},\frac{-v}{2},\frac{-w}{2}\right)$$

Let $X=\frac{-u}{2}, Y=\frac{-v}{2}, Z=\frac{-w}{2}$

We have $X^2+Y^2+Z^2=\frac{1}{4}(u^2+v^2+w^2)=25$, from (2)

Locus of the centroid is $x^2+y^2+z^2=25$.

117. $\left|\bar{a}+2\bar{b}\right|=7$

$\left|\bar{a}\right|^2+4\left|\bar{b}\right|^2+4\bar{a}.\bar{b}=49$

$9+4\times 9+4\,\bar{a}.\bar{b}=49$

$\Rightarrow$ $4\bar{a}.\bar{b}=4$

$\bar{a}.\bar{b}=1$

If θ is the angle between $\bar{a}$ and $\bar{b}$

$9\cos\theta=1, \cos\theta=\frac{1}{9}$

$\sin\theta=\frac{4\sqrt{5}}{9}$

$\left|\bar{a}\times\bar{b}\right|=\left|\bar{a}\right|\left|\bar{b}\right|\sin\theta=\frac{9\times 4\sqrt{5}}{9}=4\sqrt{5}$

118. If $\bar{u}=\ell_1\bar{a}+m_1\bar{b}+n_1\bar{c}$

$\bar{v}=\ell_2\bar{a}+m_2\bar{b}+n_2\bar{c}$

$\bar{w}=\bar{u}=\ell_1\bar{a}+m_1\bar{b}+n_1\bar{c}$,

then $\left[\bar{u}\ \ \bar{v}\ \ \bar{w}\right]=\begin{bmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{bmatrix}\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$

$=0$ whereever $\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]=0$

Hence (a), (b), (c), (d)

119. It can be seen that

$3\times 1+1\times 2+-5\times 1=0$

The two given planes are perpendicular to each other

(a) is true

(b) is true, (c) is false.

Length of perpendicular $=\left|\frac{9+1-5-6}{\sqrt{9+1+25}}\right|$

$=\frac{1}{\sqrt{35}}$

120. $\bar{a}\times(\bar{b}\times\bar{c})=\frac{1}{\sqrt{2}}(\bar{b}+\bar{c})$

$(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}=\frac{1}{\sqrt{2}}\bar{b}+\frac{1}{\sqrt{2}}\bar{c}$

$\bar{b}$ and $\bar{c}$ are non-collinear

$\Rightarrow$ $\bar{a}.\bar{c}=\frac{1}{\sqrt{2}}$ $\bar{a}.\bar{b}=-\frac{1}{\sqrt{2}}$

$\Rightarrow$ angle between $\bar{a}$ and $\bar{c}=\frac{\pi}{4}$

angle between $\bar{a}$ and $\bar{b}=\frac{3\pi}{4}$

$\bar{x}\times(\bar{y}\times\bar{z})=-\frac{\sqrt{3}}{2}\bar{z}$

$\Rightarrow$ $(\bar{x}.\bar{z})\bar{y}-(\bar{x}.\bar{y})\bar{z}=-\frac{\sqrt{3}}{2}\bar{z}$

$\Rightarrow$ $(\bar{x}.\bar{z})\bar{y}+\left(\frac{\sqrt{3}}{2}\bar{z}-(\bar{x}.\bar{y})\bar{z}\right)=0$

since $\bar{y}$ and $\bar{z}$ are non-collinear.

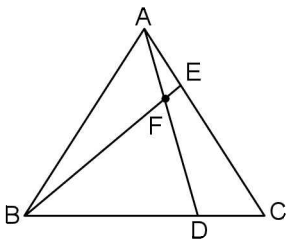
$\bar{x}.\bar{z} = 0$ and $\bar{x}.\bar{y} = \frac{\sqrt{3}}{2}$

$\Rightarrow$ angle between $\bar{x}$ and $\bar{z} = \frac{\pi}{2}$

angle between $\bar{x}$ and $\bar{y} = \frac{\pi}{6}$

Additional Practice Exercise

121.



Given : $\frac{BD}{DC} = \frac{5}{2}; \frac{AE}{EC} = \frac{2}{5}$

$\Rightarrow \quad 7\overline{AD} = 5\overline{AC} + 2\overline{AB}$

$7\overline{BE} = 2\overline{BC} + 5\overline{BA}$

Let $\overline{AB} = \bar{x}$, $\overline{AC} = \bar{y}$

$\Rightarrow \quad \overline{AD} = \frac{5}{7}\bar{y} + \frac{2}{7}\bar{x}$

$\overline{BC} = \overline{AC} - \overline{AB}$

$\overline{BE} = \frac{2}{7}(\bar{y} - \bar{x}) - \frac{5}{7}\bar{x} = \frac{2}{7}\bar{y} - \bar{x}$

Let $\overline{AF} = k\overline{AD}$, $\overline{BF} = \ell\overline{BE}$ where k and ℓ are scalars

We have $\overline{AB} + \overline{BF} = \overline{AF}$

$\bar{x} + \ell\overline{BE} = k\overline{AD}$

$\Rightarrow \quad \bar{x} + \ell\left(\frac{2}{7}\bar{y} - \bar{x}\right) = k\left(\frac{5}{7}\bar{y} + \frac{2}{7}\bar{x}\right)$

$\Rightarrow \quad = \frac{1}{4\cos^2\frac{\theta}{2}} \left\{\left(l_1^2 + m_1^2 + n_1^2\right) + \left(l_2^2 + m_2^2 + n_2^2\right)\right.$

Since $\bar{x}$ and $\bar{y}$ are non-collinear,

$1 - \ell - \frac{2k}{7} = 0$ and $\frac{2\ell}{7} - \frac{5k}{7} = 0$.

Solving the above two equations, we obtain

$\ell = \frac{35}{39}, k = \frac{14}{39}$

Therefore,

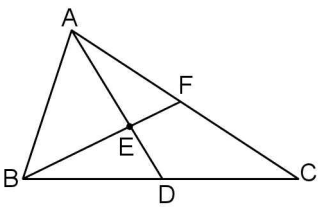
$\overline{AF} = \frac{14}{39}\overline{AD}$ and $\overline{FD} = \frac{25}{39}\overline{AD}$

$\frac{AF}{FD} = \frac{\left|\overline{AF}\right|}{\left|\overline{FD}\right|} = \frac{14}{25}$

Again $\overline{BF} = \frac{35}{39}\overline{BE}$

$\overline{FE} = \frac{4}{39}\overline{BE} \Rightarrow \frac{BF}{FE} = \frac{\left|\overline{BF}\right|}{\left|\overline{BE}\right|} = \frac{35}{4}$

122.



Let $\overline{AB} = \bar{x}$, $\overline{AC} = \bar{y}$

We have $\overline{AD} = \frac{1}{2}(\bar{x} + \bar{y})$

$\Rightarrow \quad \overline{AE} = \frac{1}{4}(\bar{x} + \bar{y})$

$2\overline{BE} = \overline{BD} + \overline{BA} \quad = \frac{1}{2}\overline{BC} - \overline{AB}$

$= \frac{1}{2}(\bar{y} - \bar{x}) - \bar{x} = \frac{1}{2}\bar{y} - \frac{3}{2}\bar{x}$

Also, $\overline{AB} + \overline{BF} = \overline{AF}$

Let $\overline{BF} = k\overline{BE}$ and $\overline{AF} = \ell\overline{AC}$

We have

$\bar{x} + k\left(\frac{\bar{y}}{4} - 3\frac{\bar{x}}{4}\right) = \ell\bar{y}$

$\left(1 - \frac{3k}{4}\right)\bar{x} + \left(\frac{k}{4} - \ell\right)\bar{y} = 0$

Since $\bar{x}$ and $\bar{y}$ are non-collinear, the above vector equation leads us to the two equations

$1 - \frac{3k}{4} = 0$ and $\frac{k}{4} - \ell = 0$

$\Rightarrow \quad k = \frac{4}{3}, \ell = \frac{1}{3}$

Since, $\overline{AF} = \ell\overline{AC}$, we immediately get $AF = \frac{1}{3}AC$ and

$\overline{BF} = k\overline{BE}$, $BF = \frac{4}{3}BE$

$$\overline{EF} = \overline{BF} - \overline{BE} = \frac{4}{3}\overline{BE} - \overline{BE} = \frac{1}{3}\overline{BE}$$

$$\Rightarrow \quad \frac{EF}{BF} = \frac{\frac{1}{3}BE}{\frac{4}{3}BE} = \frac{1}{4}$$

123. $\left[\bar{a}, \bar{b}, \bar{c}\right] \neq 0$ .... (1)

$\bar{a}\times\bar{b} + \bar{b}\times\bar{c} = p\bar{a} + q\bar{b} + r\bar{c}$

Forming dot product with $\bar{a}$,

$\bar{a}.\left(\bar{a}\times\bar{b}\right) + \bar{a}.\left(\bar{b}\times\bar{c}\right) = p\bar{a}.\bar{a} + q\bar{b}.\bar{a} + r\bar{c}.\bar{a} = p + q\cos\frac{\pi}{3} + r\cos\frac{\pi}{3}$

$$\left[\bar{a}\,\bar{b}\,\bar{c}\right] = p + \frac{q}{2} + \frac{r}{2} \qquad \text{— (2)}$$

Similarly, forming dot product with $\bar{b}$ and $\bar{c}$ respectively.

$$0 = \frac{p}{2} + q + \frac{r}{2} \qquad \text{— (3)}$$

$$\left[\bar{a}\,\bar{b}\,\bar{c}\right] = \frac{p}{2} + \frac{q}{2} + r \qquad \text{— (4)}$$

Now $\left[\bar{a}\bar{b}\bar{c}\right]^2 = \begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \\ \bar{c}.\bar{a} & \bar{c}.\bar{b} & \bar{c}.\bar{c} \end{vmatrix} = \begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix}$

$$= 1\left(\frac{3}{4}\right) - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1}{2}\left(-\frac{1}{4}\right) = \frac{1}{2}$$

$$\left[\bar{a}\,\bar{b}\,\bar{c}\right] = \frac{1}{\sqrt{2}} \qquad \text{— (5)}$$

Solving $p = \frac{1}{\sqrt{2}}$; $q = -\frac{1}{\sqrt{2}}$; $r = \frac{1}{\sqrt{2}}$

124. $\ell + m + n = 0 \Rightarrow m = -(\ell + n)$

Substituting for m in the second relation we get

$-2l(\ell + n) + 2ln + n(\ell + n) = 0$

$\Rightarrow \quad 2\ell^2 - n\ell - n^2 = 0$

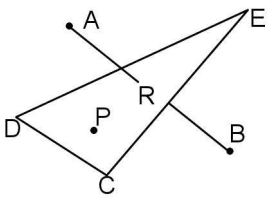
Dividing throughout by n^2 [At least one of ℓ, m, n $\neq 0$ we assume that $n \neq 0$]

$$2\left(\frac{\ell}{n}\right)^2 - \left(\frac{\ell}{n}\right) - 1 = 0$$

$$\Rightarrow \quad \frac{\ell}{n} = 1, -\frac{1}{2}$$

$$\Rightarrow \quad \frac{\ell}{1} = \frac{m}{-2} = \frac{n}{1} \text{ or } \frac{\ell}{1} = \frac{m}{1} = \frac{n}{-2}$$

125.



Let the position vector of a point P on the plane CDE be $\bar{r} = \overline{OP} = x\bar{i} + y\bar{j} + z\bar{k}$ where O is the origin of reference.

Let $\bar{c}$ = Position vector of C

$= -4\bar{j} + 4\bar{k}$

$\bar{d}$ = position vector of D

$= 2\bar{i} - 2\bar{j} + 2\bar{k}$

$\bar{e}$ = position vector of E

$= 4\bar{i} + \bar{j} + 2\bar{k}$

Since P lies on the plane CDE,

$[\bar{r} - \bar{c}, \bar{r} - \bar{d}, \bar{r} - \bar{e}] = 0$

[scalar triple product of $[\bar{r} - \bar{c}, \bar{r} - \bar{d}, \bar{r} - \bar{e}$ is zero]

$$\text{i.e., } \begin{vmatrix} x & y+4 & z-4 \\ x-2 & y+2 & z-2 \\ x-4 & y-1 & z-2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & y+4 & z-4 \\ -2 & -2 & 2 \\ -4 & -5 & 2 \end{vmatrix} = 0 \quad (R_2 - R_1, R_3 - R_1)$$

$$\Rightarrow \begin{vmatrix} x & y+4 & z-4 \\ -1 & -1 & 1 \\ -4 & -5 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x+z-4 & y+z & z-4 \\ 0 & 0 & 1 \\ -2 & -3 & 2 \end{vmatrix} = 0$$

$(C_1 + C_3, C_2 + C_3)$

$\Rightarrow \quad -3(x + z - 4) + 2(y + z) = 0$

$\Rightarrow \quad 3x - 2y + z - 12 = 0$ — (1)

is the equation of the plane CDE

Coordinates of A and B are (1, 2, 1), (2, 1, 2)

$\Rightarrow$ Equations of the line AB are

$$\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-1}{1}=k \text{ (say)}$$

$(k+1, -k+2, k+1)$ is any point on the line joining A and B

If this point is R, then the coordinates

$(k+1, -k+2, k+1)$ must satisfy (1)

$3(k+1)-2(-k+2)+(k+1)-12=0$

$\Rightarrow$ $6k-12=0$ giving $k=2$

R is (3, 0, 3)

OR

$\overline{AB}=\bar{i}-\bar{j}+\bar{k}$

Since R is on AB

$\overline{AR}$ may be assumed as

$\overline{AR}=\lambda(\bar{i}-\bar{j}+\bar{k})$ where λ is a scalar

$\overline{OR}=\overline{OA}+\overline{AR}=\bar{i}+2\bar{j}+\bar{k}+\lambda(\bar{i}-\bar{j}+\bar{k})$

$=(\lambda+1)\bar{i}+(2-\lambda)\bar{j}+(1+\lambda)\bar{k}$

Since R lies in the plane CDE,

$\bar{a},\bar{b}$ are coplanar

$\overline{RC}=\overline{OC}-\overline{OR}$

$=(-4j+4k)-\{(\lambda+1)\bar{i}+(2-\lambda)\bar{j}+(\lambda+1)\bar{k}\}$

$=(-\lambda-1)\bar{i}+(\lambda-6)\bar{j}+(3-\lambda)\bar{k}$

Similarly,

$\overline{RD}=(1-\lambda)\bar{i}+(\lambda-4)\bar{j}+(1-\lambda)\bar{k}$

$\overline{RE}=(3-\lambda)\bar{i}+(\lambda-1)\bar{j}+(1-\lambda)\bar{k}$

We have $\left[\overline{RC},\overline{RD},\overline{RE}\right]=0$ RC

(scalar triple product of $\overline{RC}$, $\overline{RD}$, $\overline{RE}$ equals zero)

$$\begin{vmatrix} -\lambda-1 & \lambda-6 & 3-\lambda \\ 1-\lambda & \lambda-4 & 1-\lambda \\ 3-\lambda & \lambda-1 & 1-\lambda \end{vmatrix}=0$$

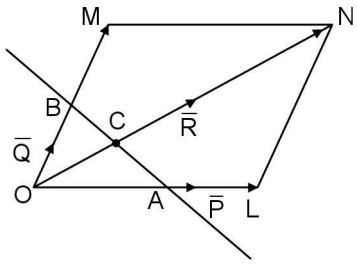
$$\Rightarrow \begin{vmatrix} -7 & \lambda-6 & -3 \\ -3 & \lambda-4 & -3 \\ 2 & \lambda-1 & 0 \end{vmatrix}=0 \quad (C_1-C_2, C_3-C_2)$$

$$\Rightarrow \begin{vmatrix} -7 & \lambda-6 & -3 \\ 4 & 2 & 0 \\ 2 & \lambda-1 & 0 \end{vmatrix}=0 \quad (R_2-R_1)$$

$\Rightarrow$ $4(\lambda-1)-4=0 \Rightarrow \lambda=2$

$\overline{OR}=3\bar{i}+3\bar{k}$ or the point R is (3, 0, 3)

126. (i)



Let the forces $\bar{P}$ and $\bar{Q}$ be represented by $\overline{OL}$ and $\overline{OM}$ so that the diagonal $\overline{ON}$ represents the resultant $\bar{R}$.

$\Rightarrow$ $\bar{P}+\bar{Q}=\bar{R}$ —— (1)

Again, $\bar{P}=\overline{OL}=m\overline{OA}$, $\bar{Q}=\overline{OM}=n\overline{OB}$

And $\bar{R}=\overline{ON}=\lambda\overline{OC}$ — (1)

where, m, n, λ are scalars.

$\Rightarrow$ $m=\frac{|\bar{P}|}{OA}, n=\frac{|\bar{Q}|}{OB}, \lambda=\frac{|\bar{R}|}{OC}$ — (2)

from (1) $m\overline{OA}+n\overline{OB}=\lambda\overline{OC}$

$\Rightarrow$ $m\overline{OA}+n\overline{OB}-\lambda\overline{OC}=0$ — (3)

Now the points A, B, C are collinear.

Let C be the point dividing the line segment AB in the ratio p : q then,

$$\overline{OC}=\frac{q\overline{OA}+p\overline{OB}}{(q+p)}$$

$\Rightarrow$ $q\overline{OA}+p\overline{OB}-(p+q)\overline{OC}=0$

Hence, from (3) we get

$m+n-\lambda=0$

or $\lambda=m+n$

$\Rightarrow$ $\frac{|\bar{R}|}{OC}=\frac{|\bar{P}|}{OA}+\frac{|\bar{Q}|}{OB}$

(ii) we have,

$\bar{P}=\overline{OL}=m\overline{OA}$, $\bar{Q}=\overline{OM}=n\overline{OB}$,

$\bar{R}=\overline{ON}=\lambda\overline{OC}$

$m=\frac{OL}{OA}, n=\frac{OM}{OB}, \lambda=\frac{ON}{OC}$

$m+n=\lambda$ gives

$$\frac{ON}{OC}=\frac{OL}{OA}+\frac{OM}{OB}$$

127. Let P be (x_1, y_1, z_1)

Then, $lx_1 + my_1 + nz_1 = k$ — (1)

Direction cosines of OP are

$$\frac{x_1}{\sqrt{x_1^2+y_1^2+z_1^2}}, \frac{y_1}{\sqrt{x_1^2+y_1^2+z_1^2}},$$

$$\frac{z_1}{\sqrt{x_1^2+y_1^2+z_1^2}}$$

$\Rightarrow \frac{x_1}{R}, \frac{y_1}{R}, \frac{z_1}{R}$ where R denotes $\sqrt{x_1^2+y_1^2+z_1^2}$

Equations of OP are $\frac{x}{\left(\frac{x_1}{R}\right)} = \frac{y}{\left(\frac{y_1}{R}\right)} = \frac{z}{\left(\frac{z_1}{R}\right)} = r$,

where r represents the distance of any point on the line from (0, 0, 0) [ie, the point O].

Since OP. OQ = $k^2 \Rightarrow$ OQ $= \frac{k^2}{OP} = \frac{k^2}{R}$

Therefore, the coordinates of Q are

$$\left(\frac{k^2x_1}{R^2}, \frac{k^2y_1}{R^2}, \frac{k^2z_1}{R^2}\right)$$

Let $X = \frac{k^2x_1}{R^2}$, $Y = \frac{k^2y_1}{R^2}$, $Z = \frac{k^2z_1}{R^2}$

$$X^2 + Y^2 + Z^2 = \frac{k^4}{R^4}\{x_1^2 + y_1^2 + z_1^2\} = \frac{k^4}{R^2} \quad \text{— (1)}$$

Again,

$$lX + mY + nZ = \frac{lk^2x_1}{R^2} + \frac{mk^2y_1}{R^2} + \frac{nk^2z_1}{R^2}$$

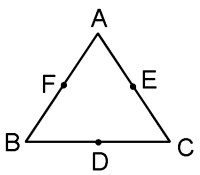
$$= \frac{k^2}{R^2}\{lx_1 + my_1 + nz_1\} = \frac{k^3}{R^2}$$

$$k(lX + mY + nZ) = \frac{k^4}{R^2} \quad \text{— (2)}$$

From (1) and (2) , locus of (X, Y, Z) or the locus of Q is the surface

$X^2 + Y^2 + Z^2 = k(lX + mY + nZ\}$

128.



Let ABC be the triangle where BC = a , CA = b, AB = c. Let D, E, F represent the mid points of the sides BC, CA, AB respectively. It is given that D, E, F are on the X, Y and Z axes. Let the coordinates of D, E, F be (p, 0, 0); (0, q, 0);(0, 0, r) respectively. Then the equation of the plane of the triangle is

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1 \quad \text{—(1)}$$

If B is assumed as (x_2, y_2, z_2), C must be $(2p - x_2, -y_2, -z_2)$

Since the coordinates of D are assumed as (p, 0, 0)

This means that the coordinates of A must be $(x_2 - 2p, 2q + y_2, z_2)$.

Again, since the coordinates of F are (0, 0, r)

$$\frac{x_2 + x_2 - 2p}{2} = 0, \frac{y_2 + 2q + y_2}{2} = 0, z_2 = r$$

$\Rightarrow x_2 = p, y_2 = -q, z_2 = r$

Hence, the coordinates of the vertices A, B, C are (−p, q, r), (p, −q, r); (p, q, −r) respectively.

$a^2 = BC^2 = 4q^2 + 4r^2$

$b^2 = CA^2 = 4p^2 + 4r^2$

$c^2 = AB^2 = 4p^2 + 4q^2$

$b^2 + c^2 - a^2 = 8p^2$

$$\Rightarrow p^2 = \frac{b^2 + c^2 - a^2}{8}; q^2 = \frac{c^2 + a^2 - b^2}{8},$$

$$r^2 = \frac{a^2 + b^2 - c^2}{8} \quad \text{— (2)}$$

Equation of the plane is (1) where p, q, r are given by (2).

129. (i) With A at the origin, let $\bar{b}$ and $\bar{d}$ be the position vectors of B and D respectively.

Now, $\overline{DC} = t\overline{AB}$, t a scalar ($\because$ DC || AB)

$= t\bar{b}$

$\therefore$ $\overline{AC} = \overline{AD} + \overline{DC} = \bar{d} + t\bar{b}$ is the position vector of C.

Equation of $\overline{AC}$ is $\bar{r} = u(\bar{d} + t\bar{b})$, u being a scalar — (1)

and that of $\overline{BD}$ is $\bar{r} = v\bar{b} + (1 - v)\bar{d}$,

v being s scalar — (2)

Let T be the point of intersection of the diagonals.

From (1) and (2) we have

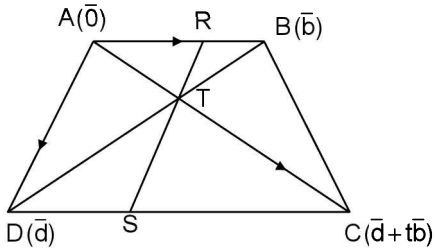
u = 1 − v and v = tu

$\Rightarrow u = \frac{1}{1+t}$ and $v = \frac{t}{1+t}$

Using 'u' in (1) or v in (2) we get the p.v. of

$$T = \frac{\bar{d} + t\bar{b}}{1+t} \quad \text{— (3)}$$

(ii)



R divides AB in the ratio 3 : 1

$\Rightarrow$ Position vector. of R = $\dfrac{3\bar{b}}{4}$

S divides DC in the ratio 1 : 3

$\Rightarrow$ Position vector of S = $\dfrac{3\bar{d}+\left(\bar{d}+t\bar{b}\right)}{4}=\dfrac{\overline{4d}+t\bar{b}}{4}$

$\therefore$ Equation of RS is

$$\bar{r}=\Delta\left(\frac{3\bar{b}}{4}\right)+(1-\Delta)\left(\frac{4\bar{d}+t\bar{b}}{4}\right),$$

'Δ' a scalar — (4)

(iii) If T is a point on RS then we can compare the coefficients of like terms in (3) and (4)

We have

$$1-\Delta=\frac{1}{1+t} \qquad \text{— (5)}$$

and $$\frac{t}{1+t}=\frac{3\Delta}{4}+\frac{t(1-\Delta)}{4} \qquad \text{— (6)}$$

$$(5) \Rightarrow \Delta=\frac{t}{1+t}$$

This value of Δ makes (6) an identity.

$\Rightarrow$ T lies on RS

130. (i) Let Q (a, b, c) be a point on the plane $x + y + z + 1 = 0$

$\Rightarrow$ $a + b + c + 1 = 0$ — (1)

Let P be the point (x_0, y_0, z_0)

Direction cosines of OQ are $\dfrac{a}{OQ}, \dfrac{b}{OQ}, \dfrac{c}{OQ}$

Direction cosines of OP are $\dfrac{x_0}{OP}, \dfrac{y_0}{OP}, \dfrac{z_0}{OP}$

But O, Q and P are collinear

$\Rightarrow$ $\dfrac{a}{OQ}, \dfrac{b}{OQ}, \dfrac{c}{OQ}$ and $\dfrac{x_0}{OP}, \dfrac{y_0}{OP}, \dfrac{z_0}{OP}$ are the direction cosines of the same line.

$\Rightarrow$ $\dfrac{a}{OQ}=\dfrac{x_0}{OP}; \dfrac{b}{OQ}=\dfrac{y_0}{OP}; \dfrac{c}{OQ}=\dfrac{z_0}{OP}$

But OQ . OP = 4

$\Rightarrow$ $a=\dfrac{OQ}{OP}\cdot x_0=\dfrac{4}{OP^2}x_0=\dfrac{4x_0}{x_0^2+y_0^2+z_0^2}$

Similarly, we get

$$b=\frac{4y_0}{x_0^2+y_0^2+z_0^2} \text{ and } c=\frac{4z_0}{x_0^2+y_0^2+z_0^2}$$

Using these values of a, b, c in (1) we've

$x_0^2 + y_0^2 + z_0^2 + 4(x_0 + y_0 + z_0) = 0$

$\therefore$ Locus of $P(x_0, y_0, z_0) = 0$ is

$x^2 + y^2 + z^2 + 4(x + y + z) = 0$

(ii) Let $R(\alpha, \beta, \gamma)$ be a point on the locus of P.

Distance (RC) of R from C(–2, –2, –2) is given by

$RC^2 = (\alpha + 2)^2 + (\beta + 2)^2 + (\gamma + 2)^2$

$= \alpha^2 + \beta^2 + \gamma^2 + 4\alpha + 4\beta + 4\gamma + 12$

$= 12$ [$\because$ R is on the locus of P]

$\therefore$ $RC = 2\sqrt{3}$

(iii) We note that RC does not depend on (α, β, γ)

$\Rightarrow$ Distance of any point R on the locus from C(–2, –2, –2) remains a constant $(2\sqrt{3})$

$\Rightarrow$ Locus of P is a sphere with center at C (–2, –2, –2) and radius equal to $2\sqrt{3}$

131. We have $\overline{AB}=-\hat{i}-\hat{j}-4\hat{k}, \overline{BC}=-3\hat{i}+3\hat{j}$,

$\overline{CA}=4\hat{i}-2\hat{j}+4\hat{k}$ so that

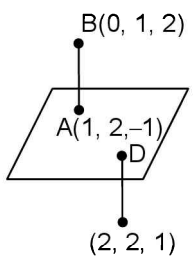
$AB = BC = 3\sqrt{2}$ and $CA = 6$

As $AB^2 + BC^2 = CA^2$, Δ ABC is right angled at B

$\therefore$ Circumcentre is at the mid-point of hypotenuse AC

Position vector of circumcentre $=-2\hat{i}+8\hat{j}+8\hat{k}$

132.



Let A(1, 2, –1) be the foot of perpendicular from B(0, 1, 2).

Direction ratios of AB are 1, 1, –3

As AB is perpendicular to the plane, equation of the plane is of the form.

$x + y - 3z = k$ — (1)

(1) passes through A $\Rightarrow k = 6$

$\therefore$ Distance of C(2, 2, 1) from $x + y - 3z = 6$ is

$$d = \left|\frac{2+2-3-6}{\sqrt{1+1+9}}\right| = \frac{5}{\sqrt{11}}$$

133. We have $al + bm + cn = 0$ — (1)

$mn + nl + lm = 0$ — (2)

Eliminating n from (1) and (2) we have $al^2 + (a+b-c)lm + bm^2 = 0$

i.e., $a\left(\frac{\ell}{m}\right)^2 + (a+b-c)\frac{\ell}{m} + b = 0$ — (3)

[Dividing by m^2]

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction ratios of the lines then $\frac{\ell_1}{m_1}, \frac{\ell_2}{m_2}$ satisfy (3)

[(3) has equal roots

$\Rightarrow \frac{\ell_1}{m_1} = \frac{\ell_2}{m_2}$ or $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2}$

Similarly, we can prove $\frac{\ell_1}{\ell_2} = \frac{n_1}{n_2}$

$\Rightarrow$ The lines are parallel

134. $(\bar{r} - \bar{a}) \times \bar{b} = 0$

$\Rightarrow$ $\bar{r} - \bar{a}$ is parallel to $\bar{b}$

$\Rightarrow$ $\bar{r}$ represents the position vector of a point on the line passing through A and parallel to $\overline{OB}$.

135. Given that $\bar{a} \times \bar{c} = -\bar{b}$ and $\bar{a}.\bar{c} = 2$

$\Rightarrow$ $a^2c^2 \sin^2\theta = b^2$ and $a^2c^2\cos^2\theta = 4$

$\Rightarrow$ $c^2 \sin^2\theta = \frac{5}{6}$ and $c^2\cos^2\theta = \frac{4}{6}$

Adding, $c^2 = \frac{3}{2}$

$\Rightarrow$ $|\bar{c}| = \sqrt{\frac{3}{2}}$

136. $|\ell\bar{x} + m\bar{y}|^2 = |\ell\bar{x} - m\bar{y}|^2 + 4\ell m\,\bar{x}.\bar{y}$

[$\therefore \bar{x} \perp \bar{y}, \bar{x}.\bar{y} = 0$]

$\Rightarrow$ $|\ell\bar{x} - m\bar{y}| = 10$

137. Consider $[\bar{a} \times \bar{b} \quad \bar{a} \times \bar{c} \quad \bar{d}]$

$= \bar{d}.[(\bar{a} \times \bar{b}).\bar{c}].\bar{a} - [(\bar{a} \times \bar{b}).\bar{a}].\bar{c} = 0$

[$\because \bar{a} \quad \bar{b} \quad \bar{c}$ are coplanar and $[\bar{a} \quad \bar{b} \quad \bar{a}] = 0$]

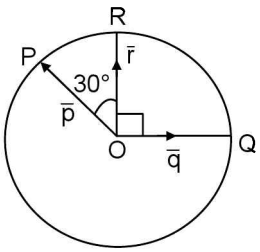
138. Consider

$(\bar{a} - \bar{d}).(\bar{b} - \bar{c}) = (\bar{a}.\bar{b} - \bar{a}.\bar{c}) - (\bar{d}.\bar{b} - \bar{d}.\bar{c})$

$= (\bar{a}.\bar{b} + \bar{c}.\bar{d}) - (\bar{a}.\bar{c} + \bar{c}.\bar{d}) = 0$

$\Rightarrow \bar{a} - \bar{d} \perp \bar{b} - \bar{c}$

139.



Let $\bar{r} = \overline{OR}$

Then $\bar{r} = x\bar{p} + y\bar{q}$ — (1)

[$\therefore \bar{r}$ is coplanar with $\bar{p}, \bar{q}$]

We have $|\bar{p}| = |\bar{q}| = |\bar{r}| = k$ (radius of circle)

Now $\bar{p}.\bar{q} = k^2 \cos 120 = -\frac{k^2}{2}$

$\bar{p}.\bar{r} = k^2 \cos 30° = \frac{k^2\sqrt{3}}{2}$

$\bar{q}.\bar{r} = 0$

Taking dot product with $\bar{p}$ and $\bar{q}$ in (1) we have

$\bar{p}.\bar{r} = x\bar{p}.\bar{p} + y\bar{p}.\bar{q}$ and

$\bar{q}.\bar{r} = x\bar{p}.\bar{q} + y\bar{q}.\bar{q}$

$\Rightarrow \frac{k^2\sqrt{3}}{2} = x k^2 - \frac{k^2}{2}y$ and $0 = -\frac{k^2}{2}x + y k^2$

$\Rightarrow y = \frac{x}{2}$ and $2x - y = \sqrt{3}$

Solving, $x = \frac{2}{\sqrt{3}}, y = \frac{1}{\sqrt{3}}$

$\therefore \bar{r} = \frac{1}{\sqrt{3}}(2\bar{p} + \bar{q})$

140. Using the condition for 3 planes to have a unique common point we have, $\begin{vmatrix} \lambda & 3 & -1 \\ 3 & 3 & 1 \\ 1 & -1 & \lambda \end{vmatrix} \neq 0$

$\Rightarrow \quad 3\lambda^2 - 8\lambda + 9 \neq 0$

Discriminant of the quadratic is < 0

$\Rightarrow$ $3\lambda^2 - 8\lambda + 9$ will never vanish for any real λ.

141. $[\bar{p}\,\bar{q}\,\bar{r}] = \begin{vmatrix} \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \\ \cos^3 A & \cos^3 B & \cos^3 C \end{vmatrix} = 0$, since $\bar{p}, \bar{q}, \bar{r}$ are coplanar.

This is satisfied if A, B, C are not at all different.

i.e., if the triangle is isosceles or equilateral.

142. Angle between the lines θ is given by

$$\cos\theta = \frac{\ell m + mn + n\ell}{\left(\sqrt{\ell^2 + m^2 + n^2}\right)^2}$$

Now ℓ, m, n are the roots of $x^3 - 2x^2 + x - 5 = 0$

$\Rightarrow \quad \ell + m + n = 2$, $\ell m + mn + n\ell = 1$ and $\ell mn = 5$

$\therefore \quad \ell^2 + m^2 + n^2 = (\ell + m + n)^2 - 2(\ell m + mn + n\ell)$
$= 2$

$\therefore \quad \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$.

143. Let the plane be $ax + by + cz + d = 0$ — (1) where

$-\frac{d}{a} - \frac{d}{b} - \frac{d}{c} = 0$ (Sum of intercepts is 0)

$\Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$ — (2) [$\because d \neq 0$]

(1) passes through (0, 4, –3)

$\Rightarrow \quad 4b - 3c + d = 0$ — (3)

(1) passes through (6, –4, 3)

$\Rightarrow \quad 6a - 4b + 3c + d = 0$ —(4)

Solving (2), (3), (4)

we get $a = \frac{-c}{3}$ and $b = \frac{c}{2}$ or

$a = -3c$ and $b = \frac{-3c}{2}$

$\therefore$ The direction ratios of the normal to the two planes obtained are $-\frac{1}{3}, \frac{1}{2}, 1$ and $-3, -\frac{3}{2}, 1$

$\therefore$ Angle between the two planes is given by

$$\cos^{-1}\left(\frac{1 - \frac{3}{4} + 1}{\sqrt{1 + \frac{1}{4} + \frac{1}{9}}\sqrt{9 + \frac{9}{4} + 1}}\right)$$

$$= \cos^{-1}\left(\frac{5/4}{\frac{7}{6} \times \frac{7}{2}}\right)$$

$$= \cos^{-1}\frac{15}{49}$$

144. The mid-point of BC is $\left(\frac{3+p}{2}, -5, \frac{4+q}{2}\right)$

Direction ratios of the median through A are

$\frac{-1-p}{2}, 2, \frac{-1-q}{2}$

As this median is equally inclined to coordinate axes we must have

$$\frac{-1-p}{2} = 2 = \frac{-1-q}{2} \Rightarrow p = -5 = q$$

145. Any point on the line joining $P(2\lambda + 3, -4, 3)$ and $Q(4, 7, \lambda + 2)$ is of the form,

$$\frac{(2\lambda+3) + 4k}{k+1}, \frac{-4 + 7k}{k+1}, \frac{3 + k(\lambda+2)}{k+1}$$

If for some value of λ and k this represents (2, 1, 3) then

$$\frac{2\lambda + 3 + 4.k}{k+1} = 2 \quad \text{— (1)}$$

$$\frac{-4 + 7.k}{k+1} = 1 \quad \text{— (2)}$$

and $$\frac{3 + k.(\lambda+2)}{k+1} = 3 \quad \text{— (3)}$$

$(2) \Rightarrow k = \frac{5}{6}$

Using this value of k in (1) we have $\lambda = -\frac{4}{3}$ But these values do not satisfy (3)

$\Rightarrow$ There is no value of λ satisfying given condition.

Aliter:

For the 3 points to be collinear,

2, 6, $\lambda - 1$ are proportional to $2\lambda + 1, -5, 0$: which is not satisfied.

Hence, the answer.

146. The equation to the plane through $P(\alpha, \beta, \gamma)$ and normal to OP is

$$\alpha(x - \alpha) + \beta(y - \beta) + \gamma(y - \gamma) = 0 \quad \text{—(1)}$$

We easily get A as $\left(\frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha}, 0, 0\right)$ B as

$\left(0, \frac{\alpha^2 + \beta^2 + \gamma^2}{\beta}, 0\right)$ and C as $\left(0, 0, \frac{\alpha^2 + \beta^2 + \gamma^2}{\gamma}\right)$

If θ is the angle between z–axis and the normal to the plane (1)

$$\cos\theta = \frac{0\times\alpha + 0\times\beta + 1\times\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}} = \frac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

Let, Δ be the area of the triangle ABC, Then projection of Δ in the XOY plane is the triangle OAB. We therefore get the relation

$$\Delta\cos\theta = \text{Area of } \Delta\ OAB$$

$$= \frac{1}{2}\left(\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}\right)\left(\frac{\alpha^2+\beta^2+\gamma^2}{\beta}\right)$$

$$= \frac{1}{2\alpha\beta}\left(\alpha^2+\beta^2+\gamma^2\right)$$

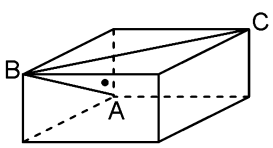
$$\Delta = \frac{\left(\alpha^2+\beta^2+\gamma^2\right)^2}{2\alpha\beta\cos\theta}$$

$$= \frac{\left(\alpha^2+\beta^2+\gamma^2\right)^2}{2\alpha\beta\gamma}\sqrt{\alpha^2+\beta^2+\gamma^2},$$

on substituting for $\cos\theta$.

$$= \frac{\left(\alpha^2+\beta^2+\gamma^2\right)^{\frac{5}{2}}}{2\alpha\beta\gamma}$$

147.



Let the edges of the box be of lengths 2a, 2b, 2c. Since origin is the centre of the box, the coordinates of A, B, C are easily obtained as (a, –b, –c); (–a, –b, c); (a, b, c) respectively.

We have

$$\cos\alpha = \frac{-a^2-b^2+c^2}{a^2+b^2+c^2},$$

$$\cos\beta = \frac{a^2-b^2-c^2}{a^2+b^2+c^2}$$

$$\cos\gamma = \frac{-a^2+b^2-c^2}{a^2+b^2+c^2}$$

$$\cos\alpha + \cos\beta + \cos\gamma = -1$$

148. Any plane passing through the line of intersection of $lx + my = 0$, $z = 0$ can be assumed as

$lx + my + \lambda z = 0$ where, λ is a scalar. — (1)

The direction cosines of a normal to (1) are

$$\left[\frac{l}{\sqrt{l^2+m^2+\lambda^2}}, \frac{m}{\sqrt{l^2+m^2+\lambda^2}}, \frac{\lambda}{\sqrt{l^2+m^2+\lambda^2}}\right]$$

The direction cosines of a normal to the plane $lx + my = 0$ are $\left[\frac{l}{\sqrt{l^2+m^2}}, \frac{m}{\sqrt{l^2+m^2}}, o\right]$

Since θ is the angle between (1) and the plane $lx + my = 0$,

$$\cos\theta = \frac{l\times l + m\times m + \lambda\times 0}{\sqrt{l^2+m^2}\sqrt{l^2+m^2+\lambda^2}} = \frac{\sqrt{l^2+m^2}}{\sqrt{l^2+m^2+\lambda^2}}$$

$\Rightarrow$ $(l^2 + m^2 + \lambda^2)\cos^2\theta = (l^2 + m^2)$

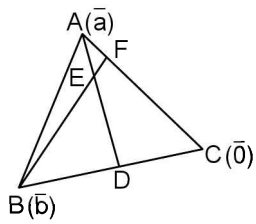
$\Rightarrow$ $\lambda^2\cos^2\theta = (l^2 + m^2)\sin^2\theta$

$\Rightarrow$ $\lambda^2 = (l^2 + m^2)\tan^2\theta$

$\Rightarrow$ $\lambda = \pm\sqrt{l^2+m^2}\tan\theta$

Substituting in (1), the equations of the planes are $\pm\left(\sqrt{\ell^2+m^2}\tan\theta\right)z$

149.



With C as the origin, let the position vector of A and B be $\bar{a}$ and $\bar{b}$ respectively.

Then the position vector of D, the midpoint of BC is $\frac{\bar{b}}{2}$. The point E that divides AD in the ratio 1 : 3 has the position vector $\frac{3\bar{a}+\frac{\bar{b}}{2}}{4}$ i.e., $\frac{6\bar{a}+\bar{b}}{8}$

Equation of CA is $\bar{r} = t\bar{a}$, t - a scalar — (1)

Equation of BE is $\bar{r} = (1-\Delta)\bar{b} + \Delta\left(\frac{6\bar{a}+\bar{b}}{8}\right)$

Δ a scalar — (2).

To find the position vector of F, the point of intersection of BE and CA

From (1) and(2)

$$t = \frac{6\Delta}{8}, 1 - \Delta + \frac{\Delta}{8} = 0$$

$$\Rightarrow \quad \Delta = \frac{8}{7}, t = \frac{6}{7}.$$

Substituting the value of t in (1) we get the position vector of $F = \frac{6}{7}\bar{a}$.

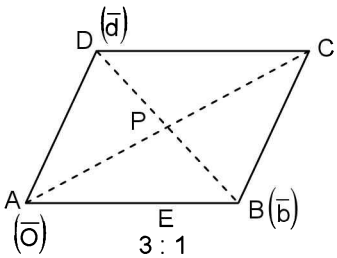
$\overline{EF}$ = position vector of F – position vector of E

$$=\frac{6\bar{a}-7\bar{b}}{56}$$

$$\overline{BF}=\frac{6\bar{a}-7\bar{b}}{7} \quad \overline{EF}=\frac{1}{8}\overline{BF}$$

$$\Rightarrow \quad \left|\overline{EF}\right|=\frac{1}{8}\left|\overline{BF}\right|$$

150.



With A at origin, let $\bar{b}$ and $\bar{d}$ be the position vectors of B and D respectively.

Now, $\overline{BC}=\overline{AD}=\bar{d}$ (ABCD is a parallelogram)

$\therefore \quad \overline{AC}=\overline{AB}+\overline{BC}=\bar{b}+\bar{d}$

E is a point on AB dividing it in the ratio 3 : 1

$\therefore$ Position vector of E = $\overline{AE}=\frac{3\bar{b}}{4}$

Let P divide CA in the ratio m : 1

Then position vector of

$$P=\frac{m(\bar{O})+1(\bar{b}+\bar{d})}{m+1}=\frac{\bar{b}+\bar{d}}{m+1}$$

Let Q divide DE in the ratio n : 1. Then

$$\text{Position vector of Q} = \frac{n\cdot\left(\frac{3\bar{b}}{4}\right)+1\cdot\bar{d}}{n+1}$$

If DE and AC intersect then for some value of n, position vectors of P and Q are equal.

$$\Rightarrow \quad \frac{\bar{b}+\bar{d}}{m+1}=\frac{\frac{3}{4}n\bar{b}+\bar{d}}{n+1}$$

$$\Rightarrow \quad \frac{1}{m+1}=\frac{3n}{4(n+1)} \qquad \text{—(1)}$$

$$\text{and} \quad \frac{1}{m+1}=\frac{1}{n+1} \qquad \text{—(2)}$$

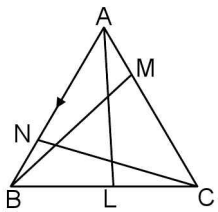
(2) $\Rightarrow$ m = n

$\Rightarrow$ DE and AC intersect at P which is such that DP : PE = CP : PA

Taking m = n in (1) we have $n=\frac{4}{3}$

$\therefore$ Common ratio = 4 : 3

151. (i)



$\bar{a}+\bar{b}+\bar{c}=0$

L divides the side BC in the ratio m : 1. Then

$$\overline{AL}=-\frac{m\bar{b}+\bar{c}}{m+1}$$

Similarly,

$$\overline{BM}=\frac{-m\bar{c}+\bar{a}}{m+1}=\frac{-(m+1)\bar{c}-\bar{b}}{m+1}$$

$$\overline{CN}=\frac{(m+1)\bar{b}+m\bar{c}}{m+1}$$

$$\overline{AL}+\overline{BM}+\overline{CN}=0$$

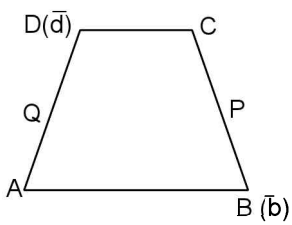
$\therefore$ they form a triangle.

Area of the triangle

$$=\frac{1}{2}\left|\sum\left(\overline{BM}\times\overline{CN}\right)\right|$$

$$=\frac{3}{2}\frac{m^2+m+1}{(m+1)^2}\left|\bar{b}\times\bar{c}\right|$$

152.



With A as the origin let $\bar{b}$ and $\bar{d}$ be the position vectors of B and D respectively.

Now $\overline{DC}=t\bar{b}$, t - a scalar ($\because$ CD || AB)

$\therefore$ Position vector of C $=\overline{AC}=\overline{AD}+\overline{DC}=\bar{d}+t\bar{b}$

Q and P respectively divide AD and BC in the ratio 1 : 3

$\therefore$ Position vector of P $= \overline{AP} = \dfrac{1(\bar{d}+t\bar{b})+3\bar{b}}{4}$

$$= \frac{\bar{d}+(t+3)\bar{b}}{4}$$

and position vector of Q $= \overline{AQ} = \dfrac{1\bar{d}+3(\bar{0})}{4} = \dfrac{\bar{d}}{4}$

$\therefore \quad \overline{CQ} = \overline{AQ} - \overline{AC} = -\dfrac{3\bar{d}}{4} - t\bar{b}$

$\overline{CB} = \overline{AB} - \overline{AC} = (1-t)\bar{b} - \bar{d}$

$\therefore \quad \Delta_1 = \text{Area } \Delta APD$

$$\left(\frac{1}{3a},\frac{1}{3b},\frac{1}{3c}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{|t+3|}{8}\left|\bar{b}\times\bar{d}\right|$$

Similarly, $\Delta_2 = \text{Area } \Delta CQB$

$$= \frac{1}{2}\left|\overline{CQ}\times\overline{CB}\right| = \frac{|t+3|}{8}\left|\bar{b}\times\bar{d}\right|$$

$\Delta APD : \Delta CQB = 1 : 1$

153. Given vectors are coplanar

$$\Rightarrow \quad 0 = \begin{vmatrix} \alpha & \alpha^3 & \alpha^4-1 \\ \beta & \beta^3 & \beta^4-1 \\ \gamma & \gamma^3 & \gamma^4-1 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha & \alpha^3 & \alpha^4 \\ \beta & \beta^3 & \beta^4 \\ \gamma & \gamma^3 & \gamma^4 \end{vmatrix} + \begin{vmatrix} \alpha & \alpha^3 & -1 \\ \beta & \beta^3 & -1 \\ \gamma & \gamma^3 & -1 \end{vmatrix}$$

$$= \alpha\beta\gamma\begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 1 & \beta^2 & \beta^3 \\ 1 & \gamma^2 & \gamma^3 \end{vmatrix} - \begin{vmatrix} 1 & \alpha & \alpha^3 \\ 1 & \beta & \beta^3 \\ 1 & \gamma & \gamma^3 \end{vmatrix}$$

$$= \alpha\beta\gamma\begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 0 & \beta^2-\alpha^2 & \beta^3-\alpha^3 \\ 0 & \gamma^2-\beta^2 & \gamma^3-\beta^3 \end{vmatrix} - \begin{vmatrix} 1 & \alpha & \alpha^3 \\ 0 & \beta-\alpha & \beta^3-\alpha^3 \\ 0 & \gamma-\beta & \gamma^3-\beta^3 \end{vmatrix}$$

[using $R_2 : R_2 - R_1$,

$R_3 : R_3 - R_2$ for both determinants]

$$= \alpha\beta\gamma(\beta-\alpha)(\gamma-\beta)\begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 0 & \beta+\alpha & \beta^2+\alpha\beta+\alpha^2 \\ 0 & \gamma+\beta & \gamma^2+\gamma\beta+\beta^2 \end{vmatrix}$$

$$-(\beta-\alpha)(\gamma-\beta)\begin{vmatrix} 1 & \alpha & \alpha^3 \\ 0 & 1 & \beta^2+\alpha\beta+\alpha^2 \\ 0 & 1 & \gamma^2+\gamma\beta+\beta^2 \end{vmatrix}$$

[Taking out $\beta-\gamma$, $\gamma-\beta$ from R_2 and R_3 of both determinants]

$= \alpha\beta\gamma\,(\beta-\alpha)\,(\gamma-\beta)\,[(\alpha+\beta)\,(\beta^2+\beta\gamma+\gamma^2) - (\beta+\gamma)(\alpha^2+\alpha\beta+\beta^2)]$

$\quad -(\beta-\alpha)\,(\gamma-\beta)\,[(\beta^2+\beta\gamma+\gamma^2) - (\alpha^2+\alpha\beta+\beta^2)]$

$= \alpha\beta\gamma\,(\beta-\alpha)\,(\gamma-\beta)\,[\alpha\gamma^2+\beta\gamma^2-\alpha^2\beta-\alpha^2\gamma] - (\beta-\alpha)\,(\gamma-\beta)[\gamma^2+\beta\gamma-\alpha\beta-\alpha^2]$

$= \alpha\beta\gamma\,(\beta-\alpha)\,(\gamma-\beta)\,(\gamma-\alpha)\,(\alpha\beta+\beta\gamma+\gamma\alpha) - (\beta-\alpha)\,(\gamma-\beta)\,(\gamma-\alpha)\,(\alpha+\beta+\gamma)$

$= (\beta-\alpha)\,(\gamma-\beta)\,(\gamma-\alpha)\,[\alpha\beta\gamma\,[\alpha\beta+\beta\gamma+\gamma\alpha) - (\alpha+\beta+\gamma)]$

$= \alpha\beta\gamma\,(\alpha\beta+\beta\gamma+\gamma\alpha) - (\alpha+\beta+\gamma)$

[α, β, γ are distinct]

$\Rightarrow \quad \alpha\beta\gamma\,(\alpha\beta+\beta\gamma+\gamma\alpha) = \alpha+\beta+\gamma \qquad$ — (1)

This is the required condition for the given vectors to be coplanar.

154. α, β, γ are in GP $\Rightarrow \beta^2 = \alpha\gamma$

Now (1) becomes

$\beta \,.\, \beta^2\,(\alpha\beta+\beta\gamma+\beta^2) = \alpha+\beta+\gamma$

$\Rightarrow \quad (\beta^4-1)\,(\alpha+\beta+\gamma) = 0$

$\Rightarrow \quad \beta^4 = 1$ [$\because \alpha, \beta, \gamma$ are positive. $\Rightarrow \alpha+\beta+\gamma \neq 0$]

$\Rightarrow \quad \beta = 1$ [$\because$ β is a positive real number]

155. (i) $\bar{p}\times(\bar{p}\times\bar{q}) \perp \bar{p}\times\bar{q}$ by definition

So $\theta = \dfrac{\pi}{2}$

156. We have

$$\left|\overline{AB}\right| = \left|\bar{p}\right|\left|\bar{p}\times\bar{q}\right|\sin\frac{\pi}{2} \qquad \left[\because \bar{p}\times\bar{q} \text{ is} \perp^r \text{ to } \bar{p}\right]$$

$$= \left|\bar{p}\times\bar{q}\right| \qquad \left[\because \left|\bar{p}\right| = 1\right]$$

and, $\left|\overline{AC}\right| = \left|\bar{p}\times\bar{q}\right|$

$\Rightarrow$ Δ ABC which is right angled at A is isosceles.

$\Rightarrow \quad \angle ABC = \angle ACB = \dfrac{\pi}{4}$

157. Reflection P of the plane P_1 in the plane P_2 is such that it is through the intersection of P_1 and P_2. Perpendicular distance of a point on P_2 to P_1 and P are equal.

Here $\quad P_1 : x + 2y + 2z = 0 \qquad$ — (1)

$\qquad P_2 : 4x - 3y + 12z + 13 = 0 \qquad$ — (2)

Plane passing through the intersection of P_1 and P_2 is of the form

$(x + 2y + 2z) + \lambda(4x - 3y + 12z + 13) = 0 \quad$ — (3)

i.e., $x(1 + 4\lambda) + y(2 - 3\lambda) + z(2 + 12\lambda) + 13\lambda = 0$

We note that Q(–1, –1, –1) is a point on P_2. Let M and N be the feet of the perpendiculars from Q on P and P_1 respectively. Then we have

QM = QN, if (3) is the reflection of (1) on (2)

$$\Rightarrow \frac{(1+4\lambda)(-1)+(2-3\lambda)(-1)+(2+12\lambda)(-1)+13\lambda}{\sqrt{(1+4\lambda)^2+(2-3\lambda)^2+(2+12\lambda)^2}}$$

$$\frac{1}{\sqrt{2}}$$

$$\Rightarrow \lambda(169\lambda + 44) = 0 \Rightarrow \lambda = \frac{-44}{169} \qquad [\because \lambda \neq 0]$$

$\therefore$ Equation of the plane of reflection of (1) in (2) is

$7x - 470y + 190z + 572 = 0$

158. (i) Equation of the plane through the intersection of $bx - ay = 1$ and $cy - bz = 1$ is of the form

$(bx - ay - 1) + \lambda(cy - bz - 1) = 0$

$bx + (-a + c\lambda)y - b\lambda z + (-1 - \lambda) = 0$

Comparing with $az - cx = -2$, we get

$$\frac{b}{-c} = \frac{-a+c\lambda}{0} = \frac{-b\lambda}{a} = \frac{+1+\lambda}{-2}$$

$$\Rightarrow \lambda = \frac{a}{c} \text{ and } \frac{b}{-c} = \frac{1+\frac{a}{c}}{-2} \text{ or } a + c = 2b$$

We know that for any two positive number $AM \geq HM$

$\Rightarrow AM(a, c) \geq HM(a, c)$

$$\Rightarrow b \geq \frac{2ac}{a+c}$$

$\Rightarrow ab + bc + ca \geq 3ac$

Minimum value of $ab + bc + ca = 3ac$

159. Equation of a plane passing though

$x + 2y + 3z = k$

and $4x + 8y + 12z = 2k$ is of the form

$x + 2y + 3z - k + \lambda(4x + 8y + 12z - 2k) = 0$ —(1)

Comparing the coefficients with

$9x - 7y + 16z = 5k$ — (2)

$$\frac{1+4\lambda}{9} = \frac{2+8\lambda}{-7} = \frac{3+12\lambda}{16}$$

$\Rightarrow \lambda = \overline{p}.\overline{c}$ and the ratio equals 0.

So line of intersection of the first two planes is parallel to the third plane. And $\lambda = \frac{-1}{4}$ in the ratio of constants of (1) and (2) does not give 0 for any value of k.

So, the third plane is not passing through the intersection of the first two for any k except for $k = 0$

$\therefore$ Three planes form a triangular prism for any $k \neq 0$.

OR

We have determinant of the coefficients

$$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 9 & -7 & 6 \end{vmatrix} = 0$$

as the second row is multiple of the first row

It is clear that for $k \neq 0$, Δ_x, Δ_y, Δ_z do not vanish.

$\therefore$ the three planes form a triangular prism for $k \neq 0$.

Aliter:

The first two equation are inconsistent unless $k = 0$. Hence $k \in R - \{0\}$

160. (i) Homogeneous second degree equation represents two planes

$$\Rightarrow \begin{vmatrix} 2 & m/2 & 0 \\ m/2 & 3 & -5/2 \\ 0 & -5/2 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \frac{m^2}{2} - \frac{49}{2} = 0 \Rightarrow m = \pm 7$$

161. (i) Line through (7, 5, 2) and perpendicular to $2x + y - 3z = -1$

Is $\frac{x-7}{2} = \frac{y-5}{1} = \frac{z-2}{-3}$ — (1)

A point on (1) is (7 + 2r, 5 + r, 2 – 3r)

To get the foot of the perpendicular

$2(7 + 2r) + (5 + r) - 3(2 - 3r) = -1$

$13 + 14r = -1$

$r = -1$

$\therefore$ Foot of perpendicular is Q (5, 4, 5)

162. Given $\frac{x-7}{2} = \frac{y-5}{1} = \frac{z-2}{2}$ — (1)

Any point R on (2) is (7 + 2r, 5 + r, 2 + 2r)

If R is on the given plane then

$2(7 + 2r) + (5 + r) - 3(2 + 2r) = -1$

$13 - r = -1 \Rightarrow r = 14$

$\therefore$ R(35, 19 , 30) (i.e.,) (1) meets the plane at R.

Equations of the required projection is the line joining Q and R

$\therefore \quad \dfrac{x-5}{30}=\dfrac{y-4}{15}=\dfrac{z-5}{25}$ i.e., $\dfrac{x-5}{6}=\dfrac{y-4}{3}=\dfrac{z-5}{5}$

$\therefore$ the reflexion is (3, 3, 8)

163. $P(5+3r_1, 7-16r_1, 3+7r_1)$, $Q(6+3r_2, 5+8r_2, 20-5r_2)$ are coordinates of the general points on the two lines respectively.

Direction ratios of PQ are

$3r_1-3r_2-1, -16r_1-8r_2+2, 7r_1+5r_2-17$

PQ is the line of shortest distance if it is perpendicular to both the given lines.

$\Rightarrow \quad 3(3r_1-3r_2-1)-16(-16r_1-8r_2+2)+7(7r_1+5r_2-17)=0$

and $3(3r_1-3r_2-1)+8(-16r_1-8r_2+2)-5(7r_1+5r_2-17)=0$

$\Rightarrow \quad 314r_1+154r_2-154=0$

and $-154r_1-98r_2+98=0$

$\Rightarrow \quad r_1=0\ r_2=1$

$\therefore \quad P(5, 7, 3)$ and $Q(9, 13, 15)$

$\therefore$ Shortest distance = PQ = 14

164. Given $\ell_1: \dfrac{x-7}{2}=\dfrac{y-5}{1}=\dfrac{z-2}{1}$

Let $l_2: \dfrac{x}{\ell}=\dfrac{y}{m}=\dfrac{z}{n}$ be the line thro' origin

The two lines intersect if $\begin{vmatrix} 7 & 5 & 2 \\ 2 & 1 & 1 \\ \ell & m & n \end{vmatrix}=0$

$\Rightarrow \quad \ell=m+n$ — (1) is the required condition

165. In addition it is given that angle between ℓ_1 and ℓ_2 is 90°.

Then $\dfrac{2\ell+m+n}{\sqrt{6}\sqrt{\ell^2+m^2+n^2}}=0$

$\Rightarrow \quad 2\ell+m+n=0$ and we have $-\ell+m+n=0$ [from (1)]

Solving we get $\ell=0, m=-n$

The two sets of direction ratios are 0, 1, −1 and 0, −1, 1.

$\therefore$ There are two such lines given by

$\dfrac{x}{0}=\dfrac{y}{1}=\dfrac{z}{-1}$ and $\dfrac{x}{0}=\dfrac{y}{-1}=\dfrac{z}{1}$.

166. Equation of hypotenuse is $\dfrac{x+1}{5}=\dfrac{y+7}{3}=\dfrac{z+6}{8}$ (1)

and opposite vertex P(7, 2, 4)

Then the equations of the other two sides are of the form $\dfrac{x-7}{\ell}=\dfrac{y-2}{m}=\dfrac{z-4}{n}$ — (2)

(1) and (2) intersect $\Rightarrow \begin{vmatrix} 8 & 9 & 10 \\ 5 & 3 & 8 \\ \ell & m & n \end{vmatrix}=0$

$\Rightarrow \quad 6l-2m-3n=0$ — (3)

167. $(\bar{a}\times\bar{b}).\bar{c}=5$

$\Rightarrow \quad [\bar{a}\ \bar{b}\ \bar{c}]=5$

$(2\bar{a}-3\bar{b}+\bar{c}).(\bar{b}\times\bar{c})+(\bar{a}-4\bar{b}+6\bar{c}).(\bar{a}\times\bar{b})$

$+(11\bar{a}-6\bar{b}+7\bar{c}).(\bar{c}\times\bar{a})$

$=2[\bar{a}\ \bar{b}\ \bar{c}]-3[\bar{b}\ \bar{b}\ \bar{c}]+[\bar{c}\ \bar{b}\ \bar{c}]+[\bar{a}\ \bar{a}\ \bar{b}]$

$-4[\bar{b}\ \bar{a}\ \bar{b}]+6[\bar{c}\ \bar{a}\ \bar{b}]+11[\bar{a}\ \bar{c}\ \bar{a}]$

$-6[\bar{b}\ \bar{c}\ \bar{a}]$

$+7[\bar{c}\ \bar{c}\ \bar{a}]$

$=2(5)-0+0+0-0+6(2)+0-6(2)+0$

$=10+12-12=10$

168. Projection of $\bar{a}$ along a vector perpendicular to $\bar{b}$ is given by $\dfrac{|\bar{a}\times\bar{b}|}{|\bar{b}|}$

$\bar{a}\times\bar{b}=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & -3 & 1 \\ 1 & 3 & -1 \end{vmatrix}=0\bar{i}+3\bar{j}+9\bar{k}$

Required projection $=\dfrac{|\bar{a}\times\bar{b}|}{|\bar{b}|}$

$=\dfrac{\sqrt{0+9+81}}{\sqrt{1+9+1}}=\dfrac{\sqrt{90}}{\sqrt{11}}=3\sqrt{\dfrac{10}{11}}$

169. Any point on the line is $(-2r, 2r+1, r+1)$

This is foot of perpendicular from (1, 2, 3) if

$-2(-2r-1)+2(2r-3)+r-2=0$

$\Rightarrow \quad r=\dfrac{2}{3}$

$\therefore$ the foot of perpendicular from (1, 2, 3) is $\left(\dfrac{-4}{3}, \dfrac{1}{3}, \dfrac{5}{3}\right)$

$\therefore$ reflexion is (x, y, z)

Where $x + 1 = \frac{-8}{3} \Rightarrow x = \frac{-11}{3}$

$y + 2 = \frac{2}{3} \Rightarrow y = \frac{-4}{3}$

$z + 3 = \frac{10}{3} \Rightarrow z = \frac{1}{3}$

$\therefore \quad Q\left(\frac{-11}{3}, \frac{-4}{3}, \frac{1}{3}\right)$

$\therefore \quad OQ^2 = \frac{46}{3}$

170. The first line is the line of intersection of the planes

$$\frac{x}{a} - \frac{y}{b} - 1 = 0$$

and $z = 0$

We may assume the required plane as

$$\frac{x}{a} - \frac{y}{b} - 1 + \lambda z = 0 \qquad \text{— (1)}$$

D.R^s of normal to the plane (1) are

$$\left[\frac{1}{a}, \frac{-1}{b}, \lambda\right]$$

The second line is

$$\frac{x-1}{0} = \frac{y+c}{c} = \frac{z}{-a}$$

Since (1) is parallel to the second line,

$$\frac{1}{a} \times 0 + \frac{-1}{b} \times c - \lambda a = 0$$

$$\lambda = \frac{-c}{ab}$$

Plane is $\frac{x}{a} - \frac{y}{b} - \frac{cz}{ab} = 1$

171. Statement 2 is false

Consdier statement 1:

Direction ratios of normal to the plane $2x + y - 3z = 0$ are $[2, 1, -3]$ i.e., normal is in the direction of the vector $2\bar{i} + \bar{j} - 3\bar{k}$. All vectors in the plane are perpendicular to $\left(2\bar{i} + \bar{j} - 3\bar{k}\right) \Rightarrow$ statement 1 is true

$\Rightarrow$ choice (c)

172. Statement 2 is false

Consider statement 1:

"d" in the equation of the plane is positive.

Also, $(3 \times 1 - 1 \times -2 - 6 \times -5 + 9) > 0$ and $3 \times 2 - 1 \times 0 - 6 \times 1 + 9 > 0$

$\Rightarrow$ Statement 1 is true

choice (c)

173. Since $3 \times 9 - 1_ -2 + 6(-1) + 5$ is < 0 and $3 \times 0 - 5 + 12 + 5$ is > 0

Statement 2 is true

Consider statement 1: The distance between the points is not equal to the sum of the perpendicular distances.

174. Direction ratio of the line $\rightarrow [5, 6, -3]$

Direction ratio of normal to the plane $\rightarrow [0, 1, 2]$

$\Rightarrow$ Line is parallel to the plane

Statement 2 is true

Consider Statement 1

The point $(1, -2, 1)$ lies in the plane. Using Statement 2, we conclude that the line lies in the plane

175. Statement 2 is false

Consider Statement 1

Any point on line I is of the form

$3r, -2, -6r_1 + 5, 2r_1)$

Any point on line II is of the form

$4r_2 - 3, -3r_2 + 2, 2r_2)$

If the two lines are coplanar,

$3r_1 - 2 = 4v_2 - 3,$

$-6r_1 + 5 = -3r_2 + 2,$

$2r_1 = 2r_2$

$\Rightarrow \quad r_1 = r_2$

Solving, we get $r_1 = r_2 = 1$

The two lines are coplanar and their point of intersection is $(4 - 3, -3 + 2, 2)$

$\Rightarrow \quad (1, -1, 2)$

Consider the 3rd line in the set

Substituting $x = 1, y = -1, z = 2$

$$\frac{0}{5} = \frac{0}{2} = \frac{0}{1}$$

$\Rightarrow$ the point $(1, -1, 2)$ lies on 3rd line.

Statement 1 is true

176. Statement 2 is false

$$\text{Consider Statement 1} = \begin{vmatrix} 2 & 3 & -5 \\ 3 & -5 & 2 \\ 5 & -2 & -3 \end{vmatrix} = 0$$

$\Rightarrow$ Planes meet in a line

177. Statement 2 is true

We have $\ell_1^2 + m_1^2 + n_1^2 = 1$

$\ell_2^2 + m_2^2 + n_2^2 = 1$

$\ell_3^2 + m_3^2 + n_3^2 = 1$

$\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$ — (1)

$\ell_1\ell_3 + m_1m_3 + n_1n_3 = 0$ — (2)

$\ell_2\ell_3 + m_2m_3 + n_2n_3 = 0$ — (3)

Consider the vector

$\overline{A} = \ell_1\overline{\ell} + m_1\overline{j} + n_1\overline{k}$

$\overline{B} = \ell_2\overline{i} + m_2\overline{j} + n_2\overline{k}$

$\overline{C} = \ell_3\overline{i} + m_3\overline{j} + n_3\overline{k}$

$\overline{A}$, $\overline{B}$, $\overline{C}$ are three mutually orthogonal unit vector

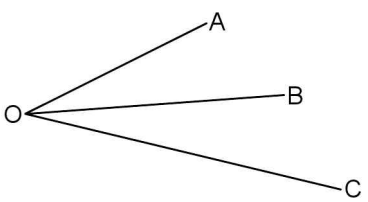
$$\begin{vmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{vmatrix} = \pm 1$$

$$\Rightarrow \quad X = \begin{pmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{pmatrix}$$

is an orthogonal matrix

$\Rightarrow \quad \ell_1^2 + \ell_2^2 + \ell_3^2 = 1$

178.



Statement 2 is false

(see figure)

→ all though

$\overline{OA}, \overline{OB}, \overline{OC}$ are coplanar.

A, B, C need not be collinear

Statement 1 $\begin{vmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 7 & 11 & 33 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -1 & 18 \\ 0 & -3 & 54 \end{vmatrix} = 0$

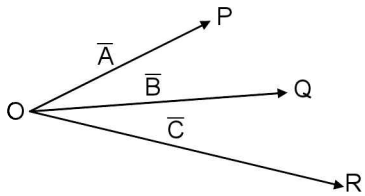
Also, $\overline{AB} = 2\overline{i} + 3\overline{j} + 12\overline{k}$

$\overline{BC}\ \ 4\overline{i} + 6\overline{j} + 24\overline{k}$

$2\,\overline{AB} = \overline{BC}$

$\Rightarrow$ A, B, C are collinear $\Rightarrow$ Statement 1 is true

179.



Let the terminal points of $\overline{A}, \overline{B}, \overline{C}$ be P,Q, R

$\overline{A} - \overline{B} = \overline{QP}$

$\overline{B} - \overline{C} = \overline{RQ}$

$\overline{C} - \overline{A} = \overline{PR}$

$\overline{QP}, \overline{PR}, \overline{PR}$ lie in the same plane

Using Statement 2, we prove that statement 1 is true.

180. Consider Statement 2

$\overline{A} \times (\overline{B} \times \overline{C}) = 0, \ \not\Rightarrow \ \overline{A}.(\overline{B} \times \overline{C}) = 0$

If $\overline{A} = i, \overline{B} = j, \overline{C} = k$, then $\overline{A} \times (\overline{B} \times \overline{C}) = 0$

But $A.(B \times C) = 0$

Statement 2 is false

Consider Statement 1

Let $\overline{C}$ be the required vector

$\overline{C} = \ell\overline{A} + m\overline{B}$ (ℓ, m scalars)

$= \ell\left(3\overline{i} + 2\overline{j} + 6\overline{k}\right) + m\left(2\overline{i} - \overline{j} + 2\overline{k}\right)$

$= (3\ell + 2m)\ \overline{i} + (2\ell - m)\overline{j} + (6\ell + 2m)\overline{k}$

$\overline{C}.\overline{A} = 0$

$3(3\ell + 2m) + 2(2\ell - m) + 6(6\ell + 2m) = 0$

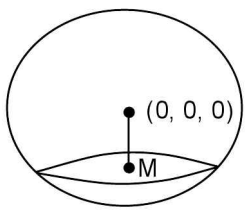
$\Rightarrow \quad 49\ell + 16m = 0$

$\Rightarrow \quad \dfrac{\ell}{m} = \dfrac{-16}{49}$

Statement 1 is true

Choice (c)

181.



Centre of the sphere is at (0, 0, 0). Length of perpendicular from the centre of the sphere to the plane

$$= \frac{9}{\sqrt{2^2 + 1^2 + 2^2}} = 3$$

Radius of the circle $=\sqrt{4^2-3^2}=\sqrt{7}$

Equations of the normal through the centre of the sphere to the plane are

$$\frac{x}{2}=\frac{y}{1}=\frac{z}{2}=k$$

Any point on the line is (2k, k, 2k). If this is the centre of the circle, it lies on the plane.

$\Rightarrow$ $4k+k+4k=9 \Rightarrow k=1$

Therefore, centre of the circle of section is (2, 1, 2)

182. Let the equation of the sphere be

$$x^2+y^2+z^2+10y-4z-8+\lambda(x+y+z-3)=0 \qquad —(1)$$

Centre of the sphere (1) is at $\left[\frac{-\lambda}{2},-\left(5+\frac{\lambda}{2}\right),2-\frac{\lambda}{2}\right]$

Centre lies on the plane

$$\frac{-\lambda}{2}-5-\frac{\lambda}{2}+2-\frac{\lambda}{2}=3$$

$$\frac{3\lambda}{2}=-6\ ;\ \lambda=-4$$

Sphere is

$$x^2+y^2+z^2+10y-4z-8-4(x+y+z-3)=0$$

$\Rightarrow$ $x^2+y^2+z^2-4x+6y-8z+4=0$

183. Length of perpendicular from (–6, 1, 3) to the plane

$$=\frac{-6-1+6+5}{\sqrt{1+1+4}}=\frac{4}{\sqrt{6}}$$

Radius of the circle of section

$$=\sqrt{16-\frac{16}{6}}=4\sqrt{\frac{5}{6}}$$

Area of the circle of section $=\frac{\pi\times16\times5}{6}=\frac{40\pi}{3}$

184. Since $\overline{z_1}$ and $\overline{z_2}$ are position vectors, they intersect at origin.

$\therefore$ $\overline{z_1}\times\overline{z_2}$ is a vector perpendicular to x-y plane and passing through origin.

So $\overline{z_1}\times\overline{z_2}$ is along positive or negative z-axis

185. $\overline{\omega}\times\overline{\omega^2}=|\overline{\omega}|\times|\omega^2|\sin\frac{2\pi}{3}\hat{k}=\frac{\sqrt{3}}{2}\hat{k}$

$$\therefore |\omega\otimes\omega^2|=\frac{\sqrt{3}}{2}.$$

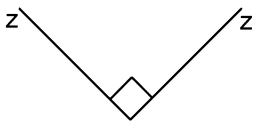
186. Given that $|z_1|=|z_2|=1$

$\Rightarrow$ $|\overline{z_1}|=|\overline{z_2}|=1$

$$k=|z_1\otimes z_2|=|\overline{z_1}\times\overline{z_2}|$$

$$=|\overline{z_1}||\overline{z_2}||\sin\theta||\hat{k}| \quad =|\sin\theta|$$

187.



iz can be obtained from z by rotating z through an angle $\frac{\pi}{2}$ about origin in anti-clockwise sense.

$\therefore$ $iz\otimes z=|iz|\,|z|\sin\frac{\pi}{2}\left(-\hat{k}\right)$

$=-\hat{k}$ (since $i\overline{z},\overline{z}$ and $-\hat{k}$ form a right handed system.

188. $z_1+z_2=5i+2j$

$(z_1+z_2)\times z_2=(5i+2j)\times(3i-j)$

$=-5k-6k=-11k$

189. Let $z_2=\lambda i+\mu j$

$z_1\times z_2=(2i+3j)(\lambda i+\mu j)$

$=(2\mu-3\lambda)\overline{k}$

$z_2\times z_2=k$

$2\mu-3\lambda=1$

$\therefore$ which is satisfied if

$\lambda=-1, \mu=-1$

$z_2=-i-j$

190. Line through (4, 1, 8) perpendicular to the plane is

$$\frac{x-4}{2}=\frac{y-1}{-1}=\frac{z-8}{7}=r$$

(2r + 4, 1 – r, 7r + 8) lies on $2x-y+7z-9=0$

$4r+8-1+r+49r+56-9=0$

$\Rightarrow$ $r=-1$

The foot of perpendicular is (2, 2, 1)

If the required point is (x, y, z),

$x+4=4, y+1=4, z+8=2$

$\Rightarrow$ $x=0, y=3, z=-6$

The image is (0, 3, –6)

So, (a)

For (b) we see that the direction ratios of the given line are the d.r.'s of the normal to the plane viz., [2, –1, 7]

$\therefore$ Equations are

$$\frac{x-4}{2}=\frac{y-1}{-1}=\frac{z-8}{7}$$

i.e., $\frac{4-x}{2}=\frac{y-1}{1}=\frac{8-z}{7}$

$2\times 3-1+7\times -5-9<0$

$2\times 1-1+7-9<0$

$\Rightarrow$ both points are on the same side of the plane

191. $\left(\frac{\bar{x}}{|\bar{x}|}-\frac{\bar{y}}{|\bar{y}|}\right)^2$

$$=1+1-\frac{2\bar{x}.\bar{y}}{|\bar{x}||\bar{y}|}=2-2\cos\theta$$

where θ is the angle between $\bar{x}$ and $\bar{y}$

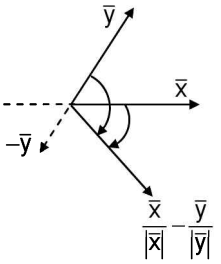
$=1$ if $\theta=\frac{\pi}{3}$ (c) is true,

It is clear that (d) is true

Since the given vector is a linear combination of $\bar{x}$ and `$\bar{y}$

$\left(\frac{\bar{x}}{|\bar{x}|}-\frac{\bar{y}}{|\bar{y}|}\right).(\bar{x}\times\bar{y})=0$ since $(\bar{x}\times\bar{y})$ is normal to the plane of $\bar{x}$ and `$\bar{y}$

(b) is true



$\frac{\bar{x}}{|\bar{x}|}-\frac{\bar{y}}{|\bar{y}|}$ is along the external bisector of angle between $\bar{x}$ and $\bar{y}$ and hence is not equally inclined to $\bar{x}$ and $\bar{y}$.

(a) is not true.

192. Clearly, (d) is true

$$(\bar{A}\times\bar{B})\times\bar{C}=(\bar{A}.\bar{C})\bar{B}-(\bar{B}.\bar{C})A=0$$

If $\bar{A}.\bar{C}=0=\bar{B}.\bar{C}$

(a) is true

$\bar{B}=k\bar{A}\Rightarrow\bar{A}\times\bar{B}=0$

$\Rightarrow (\bar{A}\times\bar{B})\times\bar{C}=0$

$\Rightarrow$ (b) is true

(c) is not true (e.g., i, j, i + j)

(d) is true for $C\,||\,\bar{A}\times\bar{B}\Rightarrow(\bar{A}\times\bar{B})\times\bar{C}=0$

193. Direction ratio of $L_1\rightarrow[3,-5,2]$

Direction ratio of $L_2\rightarrow[5,-1,3]$

If θ is the angle between L_1 and L_2,

$$\cos\theta=\frac{15+5+6}{\sqrt{38}\sqrt{35}}=\frac{26}{\sqrt{1330}}$$

Putting $z=0$ in L_1,

$3x+y=5\quad\Rightarrow x=1$

$x+y=3\qquad y=2$

(1, 2, 0) is a point on the line L_1,

Line L_1 may be represented as

$$\frac{x-1}{3}=\frac{y-2}{-5}=\frac{z}{2}$$

$$\begin{vmatrix}3-1 & -4-2 & 2-0\\ 3 & -5 & 2\\ 5 & -1 & 3\end{vmatrix}=\begin{vmatrix}2 & -6 & 2\\ 3 & -5 & 2\\ 5 & -1 & 3\end{vmatrix}$$

$=2(-15+2)+6(9-10)+2(-3+25)$

$=-26-6+44\neq 0$

Lines are not coplanar

194. Line can be written as

$$\frac{x-2}{\frac{4}{5}}=\frac{y+5}{\frac{-3}{5}}=\frac{z+2}{0}$$

Any point on the line at a distance of R units from (2, –5, –2) may be represented as

$$\left(\frac{4R}{5}+2,\frac{-3R}{5}-5,-2\right)$$

$R=+6$ or -6

$R=6\rightarrow\left(\frac{24}{5}+2,\frac{-18}{5}-5,-2\right)$

$R=-6\ \left(\frac{-24}{5}+2,\frac{18}{5}-5,-2\right)$

i.e., $\left(\frac{34}{5},\frac{-43}{5},-2\right)$ or $\left(\frac{-14}{5},\frac{-7}{5},-2\right)$

195. Direction ratio of normal to the plane $\rightarrow [1, 1, 1]$

(a) is false (b) is true

(c) $1 \times 3 + 1 \times 5 + 1 \times -8 = 0$

line is parallel to the plane

Also, $-1 - 4 + 8 - 3 = 0$

$\Rightarrow$ $(-1, -4, 8)$ lies on the plane

(c) is true

Angles made by the plane with the coordinate axes are each equal to $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(d) is true

196. $C_4 \rightarrow C_4 - (ac_1 + bc_2 + cc_3)$ we get

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & z & 1-(ax+by+cz) \end{vmatrix} = 0$$

i.e., $1 - (ax + by + cz) = 0 \Rightarrow ax + by + cz = 1$ clearly, the plane passes through $\left(\frac{1}{3a}, \frac{1}{3b}, \frac{1}{3c}\right)$ and does not passes through $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$. Also the perpendicular distance from the origin is $\frac{1}{\sqrt{a^2+b^2+c^2}}$

197. Equation of line through $A(2, -3, 2)$ and parallel to $i - 2j + 2k$ is

$$\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z-2}{2}$$

AM = Length of the projection of $\overline{AP}$ on $i - 2j + 2k$

$$= (-1i + 4j - 3k) \cdot \frac{(i - 2j + 2k)}{\sqrt{9}}$$

$$= \left|\frac{-1-8-6}{3}\right| = \left|\frac{-15}{3}\right| = 5$$

$$AP = \sqrt{(2-1)^2 + (-3-1)^2 + (2--1)^2}$$

$$= \sqrt{1+16+9} = \sqrt{26}$$

$$PM = \sqrt{AP^2 - AM^2} = \sqrt{26-25} = 1$$

198. (a) Plane through the given line $2x + z - 4 + \lambda(2y + z) = 0$ passes through $(2, 1, -1)$ if

$2(2) + (-1) + (-4) + \lambda(2 - 1) = 0 \Rightarrow \lambda = 1$

$\therefore$ Equation of the plane is $x + y + z - 2 = 0$

(b) Equation of the planes bisecting the angles between the given planes is

$$\frac{2x+y-3z+4}{\sqrt{4+1+9}} = \pm\frac{2x-3y+z+1}{\sqrt{4+9+1}}$$

$\Rightarrow$ $4y - 4z + 3 = 0$ and $4x - 2y - 2z + 5 = 0$

(c) $4(2) - 2(3) - 2(1) = 0$

$\therefore$ $4x - 2y - 2z + 5 = 0$

(d) Plane through the given line is

$ax + b(y - 3) + c(z - 2) = 0$ — (1)

and $2a + 3b + 4c = 0$ — (2)

(1) is perpendicular to $-3x - 4y - 5z + 3 = 0$

$\Rightarrow$ $-3a - 4b - 5c = 0$ — (3)

Solving (2) and (3) $\frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$

$\therefore$ $x - 2y + z + 4 = 0$

199. When $t = 2$, $\bar{f} = 9i + 5j - 10k$

$\bar{s} = i - 2j - 2k$. $\therefore$ $\bar{f} \cdot \bar{s} = 19$

when $t = 4$ $\bar{f} \cdot = 33i + 21j - 20k$

$\bar{s} = 3i - 4j + 0k$ $\bar{f} \cdot \bar{s} = 15$

Again, work done $w = \bar{f}(t) \cdot \bar{s}(t)$

$= (2t^2 + 1)(t - 1) + (t^2 + 2t - 3)(-t) + (-5t)(t - 4)$

$= t^3 - 9t^2 + 24t - 1$

$$\frac{dw}{dt} = 3t^2 - 18t - 24 \text{ and } \frac{d^2w}{dt^2} = 6t - 18$$

$$\frac{d\omega}{dt} = 0 \Rightarrow 3t^2 - 18t - 24 = 0 \text{ is } t = 2, 4$$

when $t = 2$, $\frac{d^2w}{dt^2} < 0$ work done is maxim

when $t = 4$ $\frac{d^2w}{dt^2} > 0$ work done is minimum

200. (a) For linearly dependent vectors,

$$\begin{vmatrix} 2 & -3 & 1 \\ 1 & \lambda & 0 \\ 2\lambda & 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $\lambda^2 - \lambda - 2 = 0$

i.e., $\lambda = -1$ or 2

i.e., when $\lambda = -1$ or 2, given vectors are linearly dependent.

(b) Using condition of co planarity

$$\begin{vmatrix} \lambda & 0 & -1 \\ 2 & (\lambda+2) & 0 \\ 2 & (\lambda+2) & (\lambda-2) \end{vmatrix} = 0$$

$\lambda(\lambda^2 - 4) = 0 \Rightarrow \lambda = 0, 2, -2$

(c) If $\bar{a}$ is orthogonal to both $\bar{b}$ and $\bar{c}$ the $\bar{a}.\bar{b}\times\bar{c} = 0$

$\Rightarrow \begin{bmatrix} a & b & c \end{bmatrix} = 0$

i.e., $\begin{vmatrix} 1 & +\lambda^2 & \lambda^2 \\ 1 & 1 & \lambda \\ 3 & 3 & 2\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda(\lambda^2 - 1) = 0$

$\Rightarrow \lambda = 0, 1, -1$

(d) Since the vectors not form any parallelopiped

$\begin{bmatrix} a & b & c \end{bmatrix} = 0$

i.e., $\begin{vmatrix} -1 & 2 & 1 \\ \lambda & 0 & +1 \\ 1 & \lambda & 2 \end{vmatrix} = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = 0$

i.e., $\lambda = 1, 2$

i.e., when $\lambda = 1, 2$ they will not form any parallelopiped.