Mathematics for IIT-JEE

Differential Calculus
Algebra
Trigonometry

Volume-1

Second Edition 2015

This book contains theory and a large collection of about 7500 questions and is useful for students and learners of IIT-JEE, Higher and Technical Mathematics; and also for the students who are preparing for Standardized Tests, Achievement Tests, Aptitude Tests and other competitive examinations all over the world

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First Edition 2014

Second Edition 2015

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ISBN-13: 978-1507743584
ISBN-10: 1507743580

ABOUT THE AUTHOR

Indian Institute of Technology Joint Entrance Examination (IIT-JEE) is considered to be one of the toughest competitive entrance examinations in the world with success ratio well below 1%. The author of this book Sanjiva Dayal was selected in IIT-JEE in his first attempt. After completing B.Tech. from IIT Kanpur, since the last more than 25 years, he has been teaching IIT-JEE Mathematics to the students aspiring for success in IIT-JEE and other engineering entrance examinations with an objective to provide training of the highest order by way of specialized and personalized oral coaching, teaching, training, guidance, educational facilities including tutorials, tests, assignments, reading materials etc. and all such facilities which are helpful to the students seeking admission in the IIT and other Engineering Institutes of India.

His excellent teaching and result-oriented methods have combined to produce outstanding performance by his students in the past years. A large number of his students were selected in their First Attempt with top ranks. Most of his students have secured admission in the IIT and other Engineering Institutes of India and they are doing well in their engineering education and professional career also.

Conceptual understanding of Mathematics provided by him not only helps his students to do well in the IIT-JEE and other engineering entrance examinations but also helps them to do well in their engineering studies where they are pitted against the best students of the country and competition is much more intense.

ACKNOWLEDGEMENTS

I thank the Almighty God for my very existence and for providing me the opportunity and resources to write this book.

Mother is the first teacher and father is the second teacher. My respects and deepest gratitude to my mother Late Pushpa Dayal, my father Late Chandra Mauleshwar Dayal, my grandfather Late Lakshmeshwar Dayal, my grandmother Late Sarju Dayal, my maternal grandfather Late Ranbir Jang Bahadur and my maternal grandmother Late Chandra Kishori Bahadur who are no longer in this world but their blessings are always with me.

I am extremely thankful to all my school teachers and all my IIT Kanpur Professors who have taught and trained me.

I am also extremely thankful to all my batch mates in IIT Kanpur because studying and training with such brilliant persons was an amazing experience for me which will be cherished by me throughout my life.

I am also extremely thankful to all my students because teaching and interacting with such brilliant minds has been an amazing learning experience for me which will be cherished by me throughout my life.

I am also extremely thankful to my wife Dr. Vidya Dayal, my sister Ms. Smita Prakash and my niece Ms. Somya Prakash for their love, cooperation and moral support.

I am also extremely thankful to my childhood friends Mr. Navneet Butan, Mr. Pankaj Agarwal, Mr. Akhil Srivastava and Mr. Subir Nigam for their love, cooperation and moral support.

I am also thankful to all other persons who have ever helped me directly or indirectly.

I am also thankful to my student Mr. Ashutosh Verma for his help and efforts in publishing this book.

Place: Kanpur, India.
Date: 27th January, 2015.

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PREFACE TO THE SECOND EDITION 2015

"If there is a God, he's a great mathematician."~Paul Dirac

"How is an error possible in mathematics?"~Henri Poincare

"Go down deep enough into anything and you will find mathematics."~Dean Schlicter

- **About Indian Institute of Technology Joint Entrance Examination (IIT-JEE)**

  Indian Institute of Technology (IIT) are the premier Engineering Institutes of India. Career in engineering calls for high level of confidence, motivation and capacity to do hard work besides intelligence and analytical & deductive thinking. Therefore, in order to select the brightest students for admission in the Indian Institute of Technology (IIT), National Institute of Technology (NIT) and other Engineering Institutes of India, the competitive entrance examination Indian Institute of Technology Joint Entrance Examination (IIT-JEE) is conducted on all India basis every year, along with other state level engineering entrance examinations. IIT-JEE is considered to be one of the toughest competitive entrance examinations in the world with success ratio well below 1%. The number of candidates appearing in IIT-JEE as well as the level of competition has grown tremendously in the past years thus creating a need for highly specialized course content and result-oriented coaching and training.

- **About the student**

  An engineering aspirant should possess sharp mental reflexes, acumen and skill to understand and master the fundamental concepts of Science and Mathematics; should have the basic qualities of an ideal student; should be capable of going through the rigorous and tough training schedule to achieve the standards set by the IIT-JEE and other engineering entrance examinations; and must be hungry for achievement. With specialized coaching and training, such talented students can achieve success in IIT-JEE and other engineering entrance examinations.

- **<u>Purpose of the book</u>**

This book is useful for students and learners of IIT-JEE, Higher and Technical Mathematics; and also for the students who are preparing for Standardized Tests, Achievement Tests, Aptitude Tests and other competitive examinations all over the world.

- **<u>Organization of the book</u>**

This book is a part of book series which is divided into two volumes, seven parts and twenty eight chapters as under:-

<u>Volume-1</u>

<u>Part-I: Differential Calculus</u>
Chapter-1: Real Functions, Domain, Range
Chapter-2: Limit
Chapter-3: Continuity
Chapter-4: Derivatives
Chapter-5: Applications Of Derivatives
Chapter-6: Investigation Of Functions And Their Graphs

<u>Part-II: Algebra</u>
Chapter-7: Equations And Inequalities
Chapter-8: Quadratic Expressions
Chapter-9: Progressions
Chapter-10: Determinants And Matrices

<u>Part-III: Trigonometry</u>
Chapter-11: Trigonometric Identities And Expressions
Chapter-12: Trigonometric And Inverse Trigonometric Functions, Equations And Inequalities
Chapter-13: Properties Of Triangles

<u>Volume-2</u>

<u>Part-IV: Two Dimensional Coordinate Geometry</u>
Chapter-14: Point And Straight Line
Chapter-15: Circle
Chapter-16: Parabola
Chapter-17: Ellipse
Chapter-18: Hyperbola

<u>Part-V: Vector And Three Dimensional Geometry</u>
Chapter-19: Vector
Chapter-20: Three Dimensional Coordinate Geometry

<u>Part-VI: Integral Calculus</u>
Chapter-21: Indefinite Integrals
Chapter-22: Definite Integral
Chapter-23: Differential Equations
Chapter-24: Applications Of Calculus

<u>Part-VII: Algebra</u>
Chapter-25: Complex Numbers
Chapter-26: Binomial Theorem

This book contains theory and a large collection of about 7500 questions. In each chapter, theory is divided into Sections and questions are divided into Question Categories. For each Section there are one or more corresponding Question Category/ Categories in order to make this book more readable and more useful for the readers and students.

- **Using the book**

Each Section and its corresponding Question Category/ Categories is the prerequisite of its following Sections and their corresponding Question Category/ Categories. It is recommended that readers and students should read a Section and then solve the questions of its corresponding Question Category/ Categories; and thereafter go to the next Section and so on. This approach is necessary for proper conceptual development and problem solving skills.

- **Reader's feedback**

All readers and students are requested and welcome to send their feedback, comments, corrections, suggestions, improvements, mathematical theory and questions on my email ID sanjivadayal@yahoo.com. This will help in continuous improvement and updation of this book. All readers and students are invited to join me as friends on Facebook at URL www.facebook.com/sanjiva.dayal and they are also invited to join my Facebook Group "Mathematics By Sanjiva Dayal".

Place: Kanpur, India.
Date: 27th January, 2015.

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Mathematics for IIT-JEE

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DIFFERENTIAL CALCULUS

CHAPTER-1
REAL FUNCTIONS, DOMAIN, RANGE

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CHAPTER-1
REAL FUNCTIONS, DOMAIN, RANGE

LIST OF THEORY SECTIONS

LIST OF QUESTION CATEGORIES

CHAPTER-1
REAL FUNCTIONS, DOMAIN, RANGE

SECTION-1.1. INTRODUCTION TO MATHEMATICS

1. **The subject of Mathematics**
   Mathematics is the study of relationships among quantities, magnitudes and properties and of logical operations by which unknown quantities, magnitudes and properties may be deduced.
2. **The language of Mathematics**
   Mathematics can be regarded as a language which uses symbols to provide an exact theory of logical deduction and inference based on definitions, axioms, postulates and rules for combining and transforming primitive elements into more complex relations and theorems.
3. **Definition**
   Definition is a brief precise statement describing or stating clear and unambiguous meaning of a word or an expression. A word or an expression which has a definition is said to be *defined* otherwise is said to be *not defined*.
4. **Symbol**
   Symbols are various signs and abbreviations used in mathematics to indicate entities, relations or operations.
5. **Notation**
   Notation is any system of marks or symbols used to represent entities, processes, facts or relationships in an abbreviated or nonverbal form.
6. **Proof**
   Proof is an argument that is used to show the truth of a mathematical assertion.
7. **Axiom and postulate**
   Axiom is a basic principle that is assumed to be true without proof. The terms *axiom* and *postulate* are often used synonymously.
8. **Rule and law**
   Rule is a statement or relationship that is assumed or proved to hold under given conditions. The terms *rule* and *law* are often used synonymously.
9. **Formula**
   Formula is a set of symbols and numbers that expresses a fact or rule.
10. **Theorem**
    Theorem is a proposition or formula that is derivable or provable from a set of axioms and basic assumptions.
11. **Standard result**
    Standard result is a result that can be used without giving reason or proof.
12. **Method**
    Method is a mathematical procedure that can be used to solve problems of a particular type.
13. **Quantity**
    Quantity is anything that is completely characterized by its numerical value.
14. **Constant and variable quantities**
    A *constant quantity* (or constant) is a quantity which has one and only one value within the framework of a given problem. A *variable quantity* (or variable) is a quantity which can take different values within the framework of a given problem. A quantity can be a constant in one problem and a variable in the other.
15. **Parameter**

A *parameter* is a quantity which can take different constant values within the framework of a given problem.

16. Expression

An *expression* is a mathematical phrase constructed with numbers, variables and mathematical operations (addition, substraction, multiplication, division, power) formed according to the rules.

SECTION-1.2. REAL NUMBER SYSTEM

1. Natural numbers and Integers

i. Set of natural numbers, denoted by *N*, is
$N = \{1, 2, 3, 4, 5, \cdots\cdots\cdots\}$.

ii. Set of integers, denoted by *Z* or *I*, is
$I = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots\cdots\cdots\}$.

iii. $N \subset I$.

iv. The three basic rules i.e. rules of comparison, addition and multiplication are defined for numbers. Subtraction and division are defined as inverse operations to addition and multiplication respectively.

v. Division by zero is not defined.

vi. In decimal notation symbol 0 denotes zero, symbols 1, 2, 3, 4, 5, 6, 7, 8, 9 denotes the first nine natural numbers, ten is designated as 10 and each integer is represented as $\pm\left(a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots\cdots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0\right)$ which is written as $\pm a_n a_{n-1} \cdots\cdots a_2 a_1 a_0$ where a_n is one of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and each of a_0, a_1, a_2, ..., a_{n-1} is one of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

2. Rational numbers

i. Concept of fraction and reciprocal.

ii. Set of rational numbers, denoted by *Q*, is
$$Q = \left\{ \frac{p}{q} \middle| \, p, q \in I, \, q \neq 0 \right\}.$$

iii. $N \subset I \subset Q$.

iv. Any rational number $\frac{p}{q}$, where the natural number *q* does not have any prime divisors other than 2 and 5 can be uniquely represented as a terminating decimal fraction which is written as $\pm a \cdot a_1 a_2 a_3 \ldots\ldots.. a_n$ where *a* is a non-negative integer and each of a_1, a_2, ..., a_n is one of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

v. Any rational number $\frac{p}{q}$, where the natural number *q* contains at least one prime factor different from 2 and 5, can be uniquely represented as a non-terminating repeating decimal fraction which is written as $\pm a \cdot a_1 a_2 a_3 \ldots\ldots..$ where *a* is a non-negative integer and each of a_1, a_2, ... is one of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and one or several digits (a block) are repeated in an unchanged order.

vi. Any rational number can be represented as a terminating decimal fraction or as a non-terminating repeating decimal fraction and conversely, any terminating decimal fraction or non-terminating repeating decimal fraction represents a definite rational number.

vii. Sum, difference, product and ratio of two rational numbers is always a rational number.

3. Irrational numbers

i. Irrational numbers are those numbers which are defined such that they cannot be expressed in $\frac{p}{q}$ form and can be contained between two rational numbers *a* and *b* $(b > a)$, whose difference $b - a$ can be made smaller than any positive number.

ii. Any irrational number *x* is contained between two rational numbers $a \cdot a_1 a_2 a_3 .... a_n$ and $a \cdot a_1 a_2 a_3 .... a_n + \frac{1}{10^n}$ such that $a \cdot a_1 a_2 a_3 .... a_n < x < a \cdot a_1 a_2 a_3 .... a_n + \frac{1}{10^n}$ for any number *n* whose difference $\frac{1}{10^n}$ can be made smaller than any positive number. The rational number $a \cdot a_1 a_2 a_3 .... a_n$ is called its *approximate value by defect* and the rational number $a \cdot a_1 a_2 a_3 .... a_n + \frac{1}{10^n}$ is called its *approximate value by excess.* Therefore any irrational number can be approximated by rational numbers with any degree of accuracy.

iii. Therefore for every irrational number *x*, there exists rational numbers d_i as its approximate values by defect and there exists rational numbers e_i as its approximate values by excess such that $d_0 < d_1 < d_2 < d_3 < .... < x < .... < e_3 < e_2 < e_1 < e_0$.

iv. An irrational number has non-terminating non-repeating decimal fraction form.

v. Irrational numbers cannot be expressed as fraction or decimal fraction and so they are denoted by symbols.

vi. Types of irrational numbers:- roots, π , *e*.

4. Real numbers

i. Set of real numbers, denoted by *R*, is the set of all rational numbers and irrational numbers.
ii. Sum of a rational number and an irrational number is a irrational number.
iii. Product of a non-zero rational number and an irrational number is an irrational number.
iv. Between any two real numbers, there are innumerable rational as well as irrational numbers.

5. Geometric representation of Real numbers on number line

i. On a given a horizontal straight line, an arbitrary chosen point corresponds to number 0. The portion of the line to the right of 0 is taken the positive side and to the left of 0 is taken the negative side. This line is known as number line.

ii. There is a one-to-one correspondence between the set of all points of the number line and the set of all real numbers. Thus,
- every point of the number line is associated with one and only one real number,
- different points of the number line are associated with different numbers,
- there is not a single number which would not be associated with some point of the number line.

iii. "A real number" is also called "a point".

6. Definition of power

i. Powers with 0 or 1 as base or exponent

a. 0^0 is not defined

b. $0^x = 0, x > 0$

c. 0^x is not defined for $x < 0$

d. $a^0 = 1, a \neq 0$

e. $1^x = 1$

f. $a^1 = a$

ii. Negative exponent is defined as

$a^{-x} = \frac{1}{a^x}$, $x > 0$.

iii. Positive integer exponent is defined as

$a^n = aaa\cdots\cdots a\ (n\ times)$.

iv. Positive non-integer rational exponent

a. $a^{\frac{1}{n}}\ (n \geq 2)$ is defined as the number such that $\left(a^{\frac{1}{n}}\right)^n = a$. $a^{\frac{1}{n}}$ is also written as $\sqrt[n]{a}$ and is called the *n*th root of *a*.

b. By definition, $a^{\frac{p}{q}} = \left(a^p\right)^{\frac{1}{q}}$.

c. Even roots of negative numbers are not defined but odd roots of negative numbers are defined

v. Positive irrational exponent

a. If *x* is a positive irrational number and rational numbers d_i approximate the number *x* by defect and rational numbers e_i approximate the number *x* by excess, i.e.

$d_0 < d_1 < d_2 < d_3 < .... < x < .... < e_3 < e_2 < e_1 < e_0$; then a^x is the number which is contained between any number a^{d_i} and a^{e_i} and such a number exists and is unique.

b. Irrational exponent power of negative numbers are not defined

vi. Binomial theorem, Binomial series, Exponential series

7. Logarithm

i. Definition of natural logarithm

If $x > 0$, then the real number denoted by $\log_e x$ is the logarithm of the number x to the base e if $e^{\log_e x} = x$. $\log_e x$ is also called the natural logarithm of the number x.

a. $\log_e x$ is also denoted as $\ln x$.

b. $\ln 1 = 0$ and $\ln e = 1$.

c. Log of negative numbers and 0 are not defined.

d. Logarithmic series.

e. $a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$.

f. Series of a^x.

ii. Definition of $\log_a x$

If $x > 0$, $a > 0$ and $a \neq 1$, then the real number denoted by $\log_a x$ is the logarithm of the number x to the base *a* if $a^{\log_a x} = x$.

a. Base of log must be positive and not 1, i.e. $a > 0$, $a \neq 1$

b. $\log_a 1 = 0$ and $\log_a a = 1$.

c. $\log_a x = \frac{\ln x}{\ln a}$

d. $\log_{10} x$ is also denoted as $\log x$.

8. Concept of 'Plus infinity (+∞)' and 'Minus infinity (-∞)'

i. If a quantity 'increases without bound' then it is said that it 'approaches $+\infty$'. If a quantity 'decreases without bound' then it is said that it 'approaches $-\infty$'. $+\infty$ is also denoted as ∞.

ii. $+\infty$ and $-\infty$ are not numbers and hence mathematical operations cannot be used with them.

9. Set notation for representing sub-sets of real numbers

i. $[a,b]$ denotes the set of all real number x satisfying the condition $a \le x \le b$.

ii. $[a,b)$ denotes the set of all real number x satisfying the condition $a \le x < b$.

iii. $(a,b]$ denotes the set of all real number x satisfying the condition $a < x \le b$.

iv. (a,b) denotes the set of all real number x satisfying the condition $a < x < b$.

v. $[a,+\infty)$ denotes the set of all real number x satisfying the condition $x \ge a$.

vi. $(a,+\infty)$ denotes the set of all real number x satisfying the condition $x > a$.

vii. $(-\infty,b]$ denotes the set of all real number x satisfying the condition $x \le b$.

viii. $(-\infty,b)$ denotes the set of all real number x satisfying the condition $x < b$.

ix. $(-\infty,+\infty)$ or R denotes the set of all real numbers.

x. $[a]$ denotes the set of real number a.

xi. Any sub-set of real number can be written using above mentioned sets alongwith set union and subtraction

10. Interval

i. An *interval* is any of the set of real numbers $[a,b]$, $[a,b)$, $(a,b]$, (a,b), $[a,+\infty)$, $(a,+\infty)$, $(-\infty,b]$, $(-\infty,b)$ and $(-\infty,+\infty)$.

ii. Intervals $[a,b]$ are called closed intervals.

iii. Intervals $[a,b)$, $(a,b]$, $[a,+\infty)$ and $(-\infty,b]$ are called semi-open or semi-closed intervals.

iv. Intervals (a,b), $(a,+\infty)$, $(-\infty,b)$ and $(-\infty,+\infty)$ are called open intervals.

SECTION-1.3. REAL FUNCTIONS

1. Calculus of real quantities

i. For the study of calculus, all quantities are considered within the set of real numbers.

2. Concept and Definition of real function

i. A real function, denoted by *f*, is a rule which relates a real number *x* to another real number $f(x)$ such that:-

a. for at least one value of *x*, $f(x)$ should have a real value;

b. for each value of *x* for which $f(x)$ has a value, $f(x)$ should have only one value.

3. One-one (injective) and many-one functions

i. A function is said to be a One-one (injective) function if for any two different values of *x*, $f(x)$ has different value; i.e. $\forall x_1, x_2, x_1 \ne x_2 \Rightarrow f(x_1) \ne f(x_2)$.

ii. A function is said to be a Many-one function if there exists two different values of *x* for which $f(x)$ has same value; i.e. $\exists x_1, x_2, x_1 \ne x_2 \Rightarrow f(x_1) = f(x_2)$.

SECTION-1.4. DEFINING AND REPRESENTING REAL FUNCTIONS

1. Methods to define and represent real functions

i. Graphical method

a. The graph of a function $f(x)$ is the set of points on a rectangular coordinate axes having coordinates $(x, f(x))$.

b. Conventions in graph drawing
   - Graph should contain all the properties of a function
   - End-point marking
   - Single point exclusion
   - Solid and dotted lines

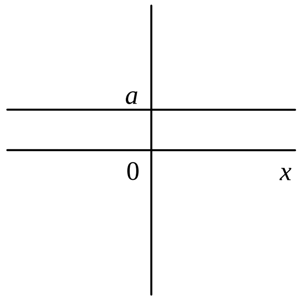
ii. Tabular method
   a. Values of x and corresponding values of $f(x)$ are tabulated in horizontal or vertical table.

iii. Analytical method
   a. $f(x)$ is expressed as an expression containing x.

2. Basic functions and their graphs

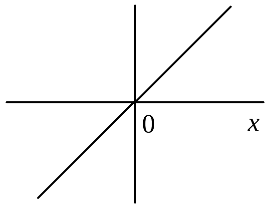
i. Constant functions

$f(x) = a$



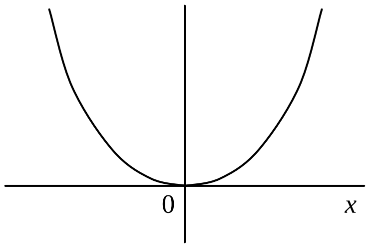
ii. Identity function

$f(x) = x$

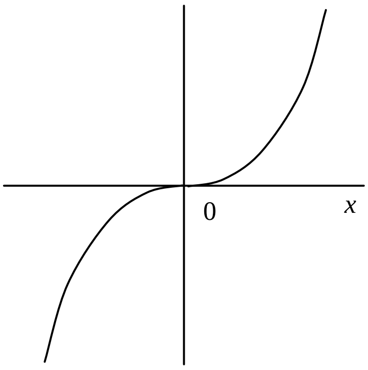


iii. Power functions

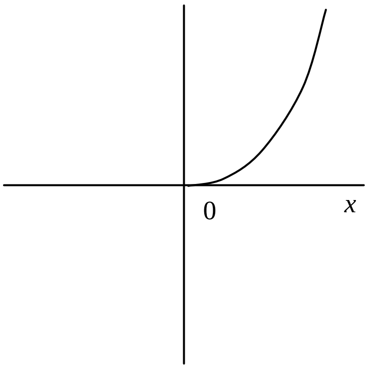
a. $f(x) = x^a$, $a > 1$, $a = \dfrac{p}{q}$ (p even, q odd); eg. x^2, x^4, $x^{\frac{4}{3}}$, etc.



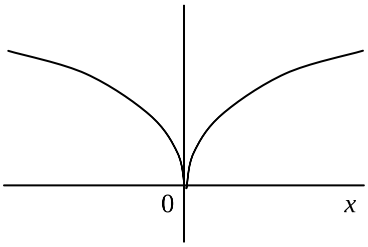
b. $f(x) = x^a$, $a > 1$, $a = \dfrac{p}{q}$ (p odd, q odd) ; eg. x^3, x^5, $x^{\frac{5}{3}}$, etc.



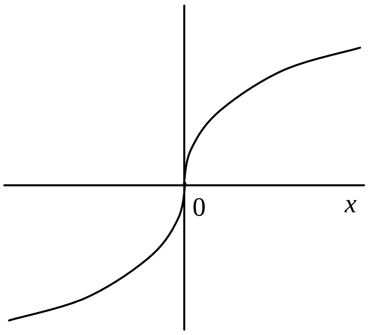
c. $f(x) = x^a$, $a > 1$, $a = \frac{p}{q}$ (p odd, q even) or a is irrational; eg. $x^{\frac{3}{2}}$, $x^{\sqrt{2}}$, etc.



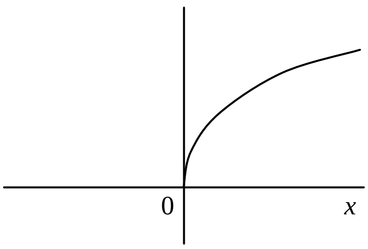
d. $f(x) = x^a$, $0 < a < 1$, $a = \frac{p}{q}$ (p even, q odd) ; eg. $x^{\frac{2}{3}}$, etc.



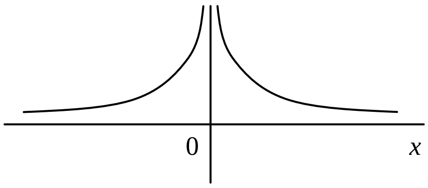
e. $f(x) = x^a$, $0 < a < 1$, $a = \frac{p}{q}$ (p odd, q odd) ; eg. $\sqrt[3]{x}$, $x^{\frac{3}{5}}$, etc.



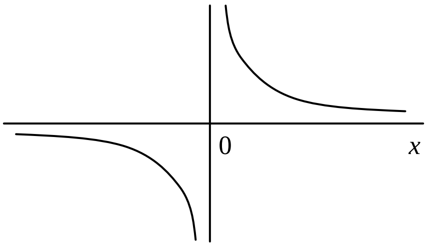
f. $f(x) = x^a$, $0 < a < 1$, $a = \frac{p}{q}$ (p odd, q even) or a is irrational; eg. $\sqrt{x}$, $x^{\frac{1}{\sqrt{2}}}$, etc.



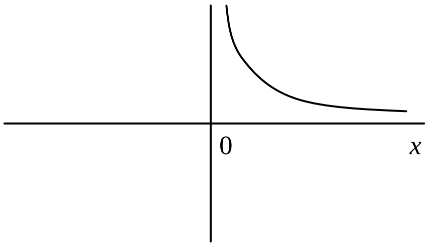
g. $f(x) = x^a$, $a < 0$, $a = \dfrac{p}{q}$ (p even, q odd) ; eg. $\dfrac{1}{x^2}$, etc.



h. $f(x) = x^a$, $a < 0$, $a = \dfrac{p}{q}$ (p odd, q odd) ; eg. $\dfrac{1}{x}$, $\dfrac{1}{x^3}$, etc.

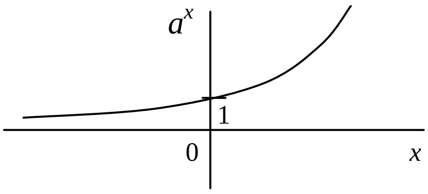


i. $f(x) = x^a$, $a < 0$, $a = \dfrac{p}{q}$ (p odd, q even) or a is irrational; eg. $x^{-\frac{1}{2}}$, $x^{-\sqrt{2}}$, etc.

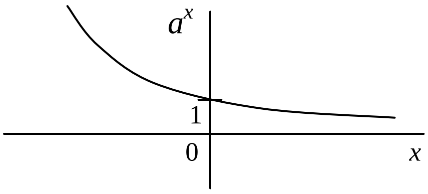


iv. Exponential functions

a. $f(x) = a^x$, $a > 1$; eg. e^x, 2^x, 10^x, etc.

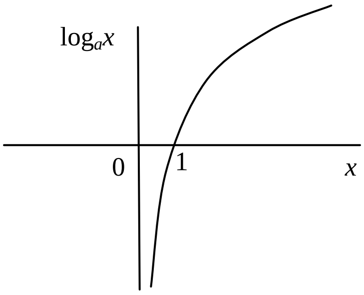


b. $f(x) = a^x$, $0 < a < 1$; eg. 0.5^x, etc.

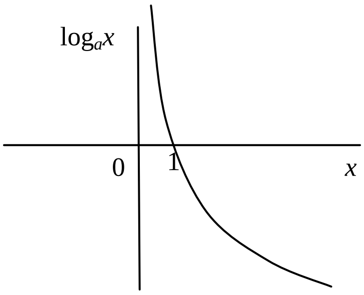


v. Logarithmic functions

a. $f(x) = \log_a x$, $a > 1$; eg. $\ln x$, $\log_2 x$, $\log x$, etc.

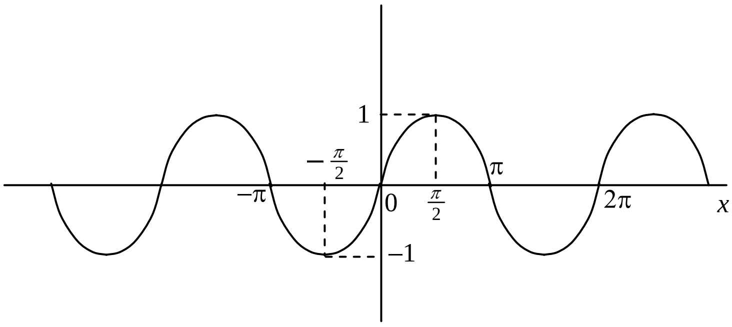


b. $f(x) = \log_a x$, $0 < a < 1$; eg. $\log_{0.5} x$, etc.

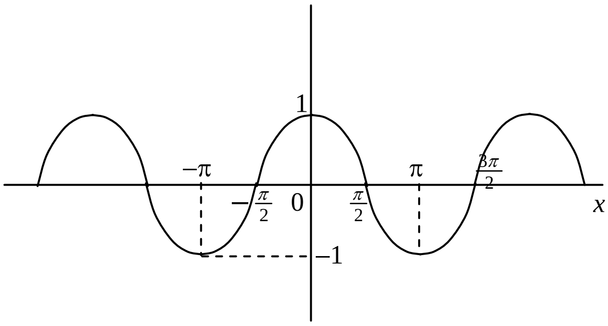


vi. Trigonometric functions

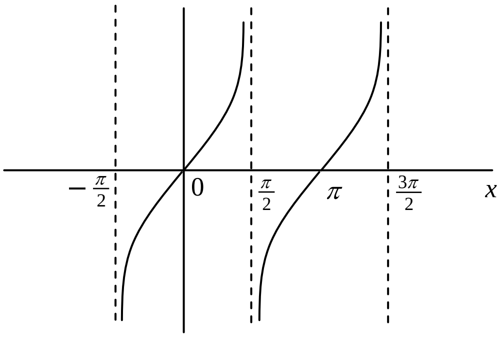
a. $f(x) = \sin x$



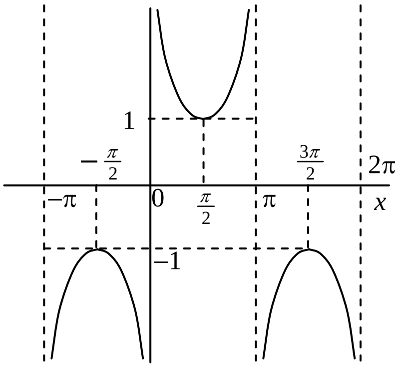
b. $f(x) = \cos x$



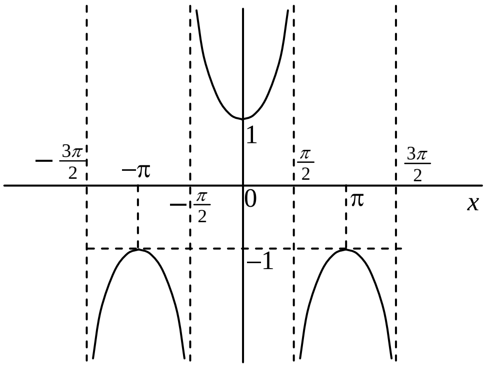
c. $f(x) = \tan x$



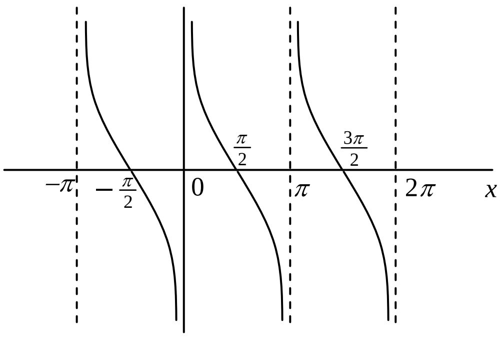
d. $f(x) = \operatorname{cosec} x$



e. $f(x) = \sec x$

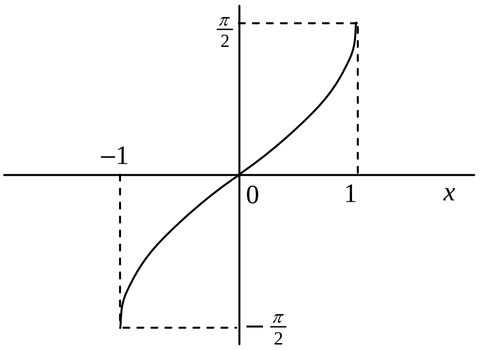


f. $f(x) = \cot x$

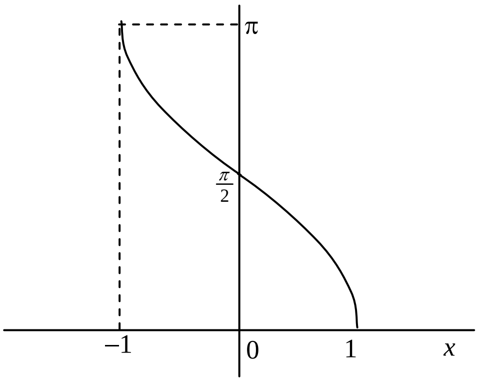


vii. Inverse trigonometric functions

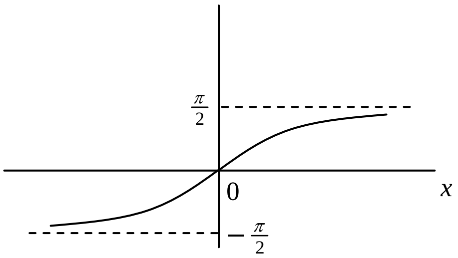
a. $f(x) = \sin^{-1} x$



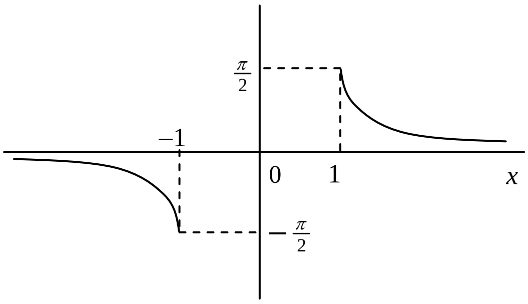
b. $f(x) = \cos^{-1} x$



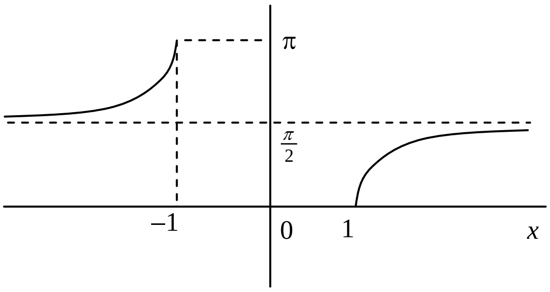
c. $f(x) = \tan^{-1} x$



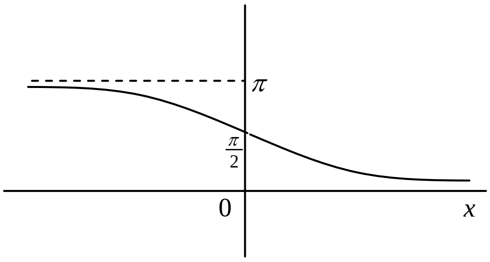
d. $f(x) = \cos ec^{-1} x$



e. $f(x) = \sec^{-1} x$

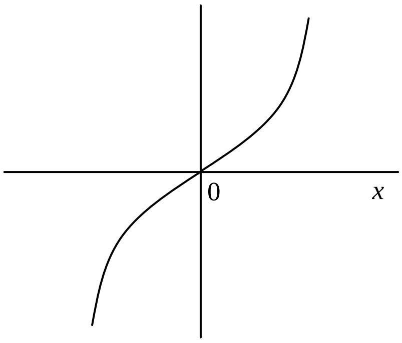


f. $f(x) = \cot^{-1} x$

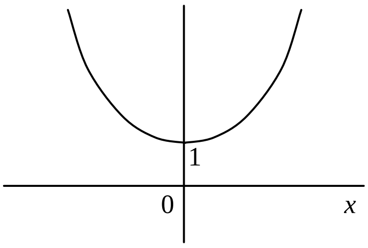


viii. Hyperbolic functions

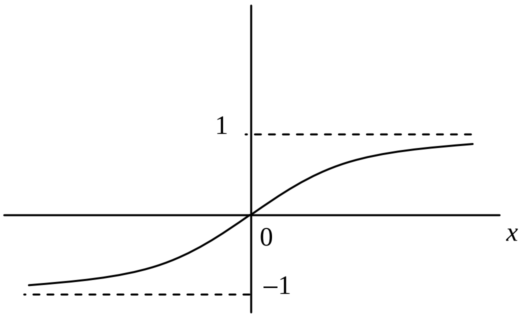
a. $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$



b. $f(x) = \cosh x = \dfrac{e^x + e^{-x}}{2}$



c. $f(x) = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$



3. Methods to define functions from basic/predefined functions

i. Defining analytical functions by taking part (or parts) of basic/ predefined functions (Piecewise functions)

a. If $f_1(x)$ is a basic/predefined function and a set $X_1 \subset R$ then a function $f(x)$ may be defined as $f(x) = f_1(x), \quad x \in X_1$.

b. If $f_1(x),\ f_2(x),\ f_3(x),\ \ldots\ldots\ldots..,\ f_n(x)$ are basic/predefined functions and sets $X_1, X_2, X_3, \ldots\ldots\ldots\ldots, X_n \subset R$ then a function $f(x)$ may be defined as

$$\begin{aligned} f(x) &= f_1(x), \quad x \in X_1 \\ &= f_2(x), \quad x \in X_2 \\ &= f_3(x), \quad x \in X_3 \\ &= \cdots\cdots\cdots\cdots \\ &= \cdots\cdots\cdots\cdots \\ &= \cdots\cdots\cdots\cdots \\ &= f_n(x), \quad x \in X_n. \end{aligned}$$

ii. Defining analytical functions by adding/ multiplying two basic/ predefined functions

a. If $f_1(x)$ and $f_2(x)$ are basic/predefined functions then a function $f(x)$ may be defined as $f(x) = f_1(x) + f_2(x)$.

b. If $f_1(x)$ and $f_2(x)$ are basic/predefined functions then a function $f(x)$ may be defined as $f(x) = f_1(x) \times f_2(x)$.

c. Negative of a function

d. Subtraction of two functions

iii. Defining analytical functions as Composite function of two basic/ predefined functions

a. If $f_1(x)$ and $f_2(x)$ are basic/predefined functions then a function $f(x)$ may be defined as $f(x) = f_1(f_2(x))$.

b. $f_1(f_2(x))$ is also written as $f_1 o f_2(x)$.

c. Defining $\dfrac{1}{f(x)}$ type of functions

d. Division of two functions

e. Defining $(f(x))^a$ and $a^{f(x)}$ type of functions

f. Defining $f(x)^{g(x)}$ type of expo-logarithmic functions

4. Explicit functions

i. If x and y are related as $y=\phi(x)$, where $\phi(x)$ denotes an expression containing x, then y is said to be an explicit function of x.

5. Implicit functions

i. If x and y are related by an equation $\phi(x,y)=0$, where $\phi(x,y)$ denotes an expression containing x and y, then y is said to be an implicit function of x.

6. Parametric functions

i. If x and y are related as $y=\phi(t),\ x=\varphi(t)$ where $\phi(t)$ and $\varphi(t)$ denote expressions containing t, then y is said to be a parametric function of x and t is called parameter.

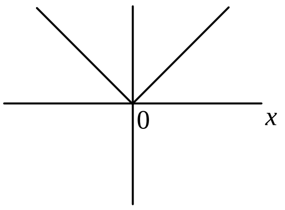
7. Other ways of defining functions

i. Height of falling body as a function of time

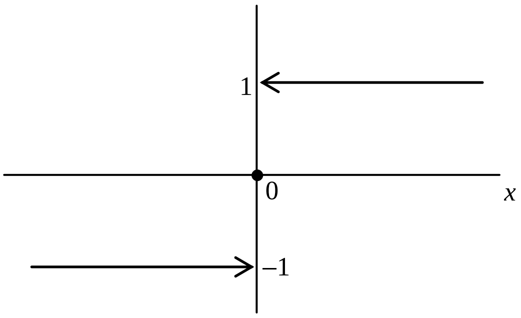
ii. Rotation of fan as a function of voltage

8. Standard functions $|x|$ and $sgnx$ and their graphs

i. $|x|=x,\ x\geq 0$
$\quad =-x,\ x<0$

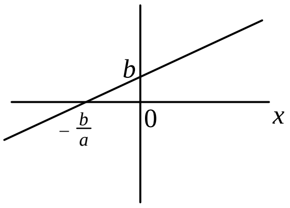


ii. $sgnx=1,\ x>0$
$\quad =0,\ x=0$
$\quad =-1,\ x<0$

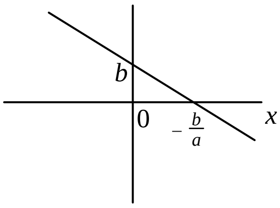


9. Graphs of linear functions

i. $f(x)=ax+b,\ a>0$



ii. $f(x) = ax + b, a < 0$



10. How function is different from it's ways of representations

i. Every graph is not a function
ii. A function may not be represented graphically
iii. Every tabular data is not a function
iv. A function may not be represented completely by tables
v. Every expression is not a function
vi. A function may not have an expression
vii. Two (or more) different expressions may represent the same function
viii. A function may involve more than one expression

11. Graphical representation shows all the values and properties of a function

i. Checking whether a graph is of a function or not- Vertical line test
   a. If a vertical line cuts the curve at two (or more) points then the graph is not a function, otherwise the graph is a function.
ii. Checking whether a graph is of one-one (injective) or many-one function- Horizontal line test
   a. If a horizontal line cuts the curve at two (or more) points then the function is a many-one function, otherwise the function is a one-one (injective) function.
iii. Checking basic functions and standard functions for being one-one (injective) or many-one functions

- $f(x) = a$; many-one function.
- $f(x) = x$; one-one (injective) function.
- $f(x) = x^a$, $a > 1$, $a = \frac{p}{q}$ (p even, q odd) ; many-one function.
- $f(x) = x^a$, $a > 1$, $a = \frac{p}{q}$ (p odd, q odd) ; one-one (injective) function.
- $f(x) = x^a$, $a > 1$, $a = \frac{p}{q}$ (p odd, q even) or a is irrational; one-one (injective) function.
- $f(x) = x^a$, $0 < a < 1$, $a = \frac{p}{q}$ (p even, q odd) ; many-one function.
- $f(x) = x^a$, $0 < a < 1$, $a = \frac{p}{q}$ (p odd, q odd) ; one-one (injective) function.
- $f(x) = x^a$, $0 < a < 1$, $a = \frac{p}{q}$ (p odd, q even) or a is irrational; one-one (injective) function.
- $f(x) = x^a$, $a < 0$, $a = \frac{p}{q}$ (p even, q odd) ; many-one function.
- $f(x) = x^a$, $a < 0$, $a = \frac{p}{q}$ (p odd, q odd) ; one-one (injective) function.

- $f(x) = x^a$, $a < 0$, $a = \frac{p}{q}$ (p odd, q even) or a is irrational; one-one (injective) function.
- $f(x) = a^x$; one-one (injective) function.
- $f(x) = \log_a x$; one-one (injective) function.
- $f(x) = \sin x$; many-one function.
- $f(x) = \cos x$; many-one function.
- $f(x) = \tan x$; many-one function.
- $f(x) = \operatorname{cosec} x$; many-one function.
- $f(x) = \sec x$; many-one function.
- $f(x) = \cot x$; many-one function.
- $f(x) = \sin^{-1} x$; one-one (injective) function.
- $f(x) = \cos^{-1} x$; one-one (injective) function.
- $f(x) = \tan^{-1} x$; one-one (injective) function.
- $f(x) = \operatorname{cosec}^{-1} x$; one-one (injective) function.
- $f(x) = \sec^{-1} x$; one-one (injective) function.
- $f(x) = \cot^{-1} x$; one-one (injective) function.
- $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$; one-one (injective) function.
- $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$; many-one function.
- $f(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$; one-one (injective) function.
- $f(x) = |x|$; many-one function.
- $f(x) = sgn x$; many-one function.

SECTION-1.5. DOMAIN AND RANGE

1. **Properties of functions: Domain, Range, Limit, Continuity, Derivatives**
2. **Definition of Domain of a function**
   Domain is the set of all real values of x for which $f(x)$ attains real value. Domain $\neq \phi$ and Domain $\subseteq R$.
3. **Definition of Range of a function**
   Range is the set of all real values attained by $f(x)$. Range $\neq \phi$ and Range $\subseteq R$.
4. **Reading Domain and Range of a function from it's graph**
5. **Reading Domain and Range of a function from it's table**
6. **Domain and Range of Basic functions and standard functions**
   i. $f(x) = a$; Domain: R; Range: $[a]$.
   ii. $f(x) = x$; Domain: R; Range: R.
   iii. $f(x) = x^a$, $a > 1$, $a = \frac{p}{q}$ (p even, q odd) ; Domain: R; Range: $[0, \infty)$.

iv. $f(x)=x^a$, $a>1$, $a=\frac{p}{q}$ (p odd, q odd) ; Domain: R; Range: R.

v. $f(x)=x^a$, $a>1$, $a=\frac{p}{q}$ (p odd, q even) or a is irrational; Domain: $[0,\infty)$; Range: $[0,\infty)$.

vi. $f(x)=x^a$, $0<a<1$, $a=\frac{p}{q}$ (p even, q odd) ; Domain: R; Range: $[0,\infty)$.

vii. $f(x)=x^a$, $0<a<1$, $a=\frac{p}{q}$ (p odd, q odd) ; Domain: R; Range: R.

viii. $f(x)=x^a$, $0<a<1$, $a=\frac{p}{q}$ (p odd, q even) or a is irrational; Domain: $[0,\infty)$; Range: $[0,\infty)$.

ix. $f(x)=x^a$, $a<0$, $a=\frac{p}{q}$ (p even, q odd) ; Domain: $R-[0]$; Range: $(0,\infty)$.

x. $f(x)=x^a$, $a<0$, $a=\frac{p}{q}$ (p odd, q odd) ; Domain: $R-[0]$; Range: $R-[0]$.

xi. $f(x)=x^a$, $a<0$, $a=\frac{p}{q}$ (p odd, q even) or a is irrational; Domain: $(0,\infty)$; Range: $(0,\infty)$.

xii. $f(x)=a^x$; Domain: R; Range: $(0,\infty)$.

xiii. $f(x)=\log_a x$; Domain: $(0,\infty)$; Range: R.

xiv. $f(x)=\sin x$; Domain: R; Range: $[-1,1]$.

xv. $f(x)=\cos x$; Domain: R; Range: $[-1,1]$.

xvi. $f(x)=\tan x$; Domain: $R-\left[(2n+1)\frac{\pi}{2}\right], n\in I$; Range: R.

xvii. $f(x)=\operatorname{cosec} x$; Domain: $R-[n\pi], n\in I$; Range: $(-\infty,-1]\cup[1,\infty)$.

xviii. $f(x)=\sec x$; Domain: $R-\left[(2n+1)\frac{\pi}{2}\right], n\in I$; Range: $(-\infty,-1]\cup[1,\infty)$.

xix. $f(x)=\cot x$; Domain: $R-[n\pi], n\in I$; Range: R.

xx. $f(x)=\sin^{-1}x$; Domain: $[-1,1]$; Range: $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

xxi. $f(x)=\cos^{-1}x$; Domain: $[-1,1]$; Range: $[0,\pi]$.

xxii. $f(x)=\tan^{-1}x$; Domain: R; Range: $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

xxiii. $f(x)=\operatorname{cosec}^{-1}x$; Domain: $(-\infty,-1]\cup[1,\infty)$; Range: $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$.

xxiv. $f(x)=\sec^{-1}x$; Domain: $(-\infty,-1]\cup[1,\infty)$; Range: $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$.

xxv. $f(x)=\cot^{-1}x$; Domain: R; Range: $(0,\pi)$.

xxvi. $f(x)=\sinh x=\frac{e^x-e^{-x}}{2}$; Domain: R; Range: R.

xxvii. $f(x)=\cosh x=\dfrac{e^x+e^{-x}}{2}$; Domain: R; Range: $[1,\infty)$.

xxviii. $f(x)=\tanh x=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$; Domain: R; Range: $(-1,1)$.

xxix. $f(x)=|x|$; Domain: R; Range: $[0,\infty)$.

xxx. $f(x)=sgnx$; Domain: R; Range: $[-1]\cup[0]\cup[1]$.

7. Domain of defined analytical functions

i. <u>Theorem:</u> If D_1 is the domain of the function f_1 and D_2 is the domain of the function f_2 and the function $f(x)=f_1(x)+f_2(x)$, then the domain of the function $f=\{x|\ x\in D_1,\ x\in D_2\}$, i.e. domain of $f(x)$ is $D_1\cap D_2$.

ii. <u>Theorem:</u> If D_1 is the domain of the function f_1 and D_2 is the domain of the function f_2 and the function $f(x)=f_1(x)\times f_2(x)$, then the domain of the function $f=\{x|\ x\in D_1,\ x\in D_2\}$, i.e. domain of $f(x)$ is $D_1\cap D_2$.

iii. <u>Theorem:</u> If D_1 is the domain of the function f_1 and D_2 is the domain of the function f_2 and the function $f(x)=f_1(f_2(x))$, then the domain of the function $f=\{x\mid x\in D_2,\ f_2(x)\in D_1\}$.

iv. Domain conditions

8. Domain of piecewise functions

If D_1 is the domain of the function f_1 and D_2 is the domain of the function f_2 and the function

$$\begin{aligned} f(x) &= f_1(x), \quad x\in X_1 \\ &= f_2(x), \quad x\in X_2, \end{aligned}$$

then the Domain of $f=(D_1\cap X_1)\cup(D_2\cap X_2)$.

9. Applications of Domain

10. Range of defined analytical functions

11. Applications of Range

12. Equivalent functions

Two functions $f(x)$ and $g(x)$ are said to be equivalent functions, written as $f(x)\equiv g(x)$, if they are defined in different ways but they have the same domain and $f(x)=g(x)$ for all values of *x* in their domain; otherwise $f(x)$ and $g(x)$ are said to be non-equivalent functions, written as $f(x)\not\equiv g(x)$.

SECTION-1.6. MATHEMATICAL OPERATIONS ON FUNCTIONS

1. Properties of modulus

i. $$\begin{aligned} |x| &= x, \quad x\geq 0 \\ &= -x, \quad x<0 \end{aligned}$$

ii. $|-x|\equiv|x|$

iii. $|x|^2\equiv x^2$

iv. $|x|\equiv\sqrt{x^2}$

v. $|x|\geq 0$

vi. $|xy| = |x||y|$

vii. $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$

viii. $|x+y| = |x| + |y|$, if $x \ge 0,\ y \ge 0$ or $x < 0,\ y < 0$

ix. $|x+y| \le |x| + |y|$

x. $|x-y| \ge |x| \sim |y|\ \left(\left||x|-|y|\right|\right)$

2. Properties of power

i. $a^0 = 1, \quad a \ne 0$

ii. $0^x = 0, \quad x > 0$

iii. $a^1 = a$

iv. $1^a = 1$

v. $a^{-x} = \dfrac{1}{a^x}$

vi. $(ab)^x = a^x b^x$

vii. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

viii. $a^x a^y = a^{x+y}$

ix. $\dfrac{a^x}{a^y} = a^{x-y}$

x. $(a^x)^y = a^{xy}$

3. Properties of logarithm

i. $a^{\log_a x} = x, \quad a > 0, a \ne 1, x > 0$

ii. $\log_a 1 = 0, \quad a > 0, a \ne 1$

iii. $\log_a a = 1, \quad a > 0, a \ne 1$

iv. $\log_a x + \log_a y = \log_a xy, \quad a > 0, a \ne 1, x > 0, y > 0$

v. $\log_a xy = \log_a |x| + \log_a |y|, \quad a > 0, a \ne 1, xy > 0$

vi. $\log_a x - \log_a y = \log_a \dfrac{x}{y}, \quad a > 0, a \ne 1, x > 0, y > 0$

vii. $\log_a \dfrac{x}{y} = \log_a |x| - \log_a |y|, \quad a > 0, a \ne 1, \dfrac{x}{y} > 0$

viii. $\log_a x^b = b \log_a x, \quad a > 0, a \ne 1, x > 0$

ix. $\log_a x^2 \equiv 2 \log_a |x|, \quad a > 0, a \ne 1, x \ne 0$

x. $\log_a b = \dfrac{\log_c b}{\log_c a}, \quad a > 0, a \ne 1, b > 0, c > 0, c \ne 1$

xi. $\log_a b = \dfrac{1}{\log_b a}, \quad a > 0, a \ne 1, b > 0, b \ne 1$

xii. $\log_a x = \log_{a^b} x^b, \quad a > 0, a \neq 1, x > 0, b \neq 0$

4. Mathematical operations on functions

i. To find the value of a given function $f(x)$ at a given point a.

ii. Given two functions $f(x)$ and $g(x)$, to determine the functions $f(x)+g(x)$, $f(x)-g(x)$, $f(x)\times g(x)$, $\dfrac{f(x)}{g(x)}$ and $f(g(x))$.

5. Mathematical operations on graphical functions

6. Mathematical operations on tabular functions

7. Mathematical operations on single expression analytical functions

8. Mathematical Operations On Piecewise Functions

i. To find the value of a given piecewise function $f(x)$ at a given point a.

ii. Given a single expression function $f(x)$ and a piecewise function $g(x)$, to determine the functions $f(x)+g(x)$, $f(x)-g(x)$, $f(x)\times g(x)$, $\dfrac{f(x)}{g(x)}$ and $f(g(x))$.

iii. Given two piecewise functions $f(x)$ and $g(x)$, to determine the functions $f(x)+g(x)$, $f(x)-g(x)$, $f(x)\times g(x)$, $\dfrac{f(x)}{g(x)}$ and $f(g(x))$.

iv. Given a single expression function $f(x)$ and a piecewise function $g(x)$, to determine the function $f(g(x))$.

v. Given a single expression function $f(x)$ and a piecewise function $g(x)$, to determine the function $g(f(x))$.

vi. Given two piecewise functions $f(x)$ and $g(x)$, to determine the function $f(g(x))$.

EXERCISE-1

CATEGORY-1.1. DEFINING ANALYTICAL FUNCTIONS

1. Define the following analytical functions using basic functions:-

i. $f(x) = \sin x + \ln x,\ x > 1$

ii. $f(x) = x^2 - \cos x,\ x \ge 0$

iii. $|x|$

iv. *sgnx*

v. $f(x) = e^x - \sin x,\ x < 0$

$= 3\sqrt{x},\ \ x \ge 0$

vi. $f(x) = \dfrac{\sin x}{\sqrt{\ln x} + \cosh x}$

vii. $f(x) = \dfrac{\sqrt{e^x}}{1 + x^2},\ x \le 0$

$= \dfrac{\frac{1}{x} + \ln x}{e^x + \ln^2 x},\ x > 0$

viii. $f(x) = \sqrt{\ln(\sin x) + \sin\sqrt{\ln x}}$

ix. $f(x) = \left(\dfrac{x}{1 + \sin x}\right)^3$

x. $f(x) = 2^{\cos x + \sqrt{x}}$

xi. $f(x) = x^x$

xii. $f(x) = (\sin x)^{\cos x}$

xiii. $f(x) = \left(\dfrac{1 + x}{1 - x}\right)^x$

xiv. $f(x) = \sin^{-1}(\ln x + \sin^{-1} x)$

xv. $f(x) = \cos^{-1}\left(x \ln x + \sqrt{\tan^{-1}\sqrt{x}}\right)$

xvi. $f(x) = \log_{\sin x} \cos x$

CATEGORY-1.2. DOMAIN AND RANGE

2. Find domain (write conditions only):-

i. $f(x) = \dfrac{1}{\ln(1 - x)} + \sqrt{x + 2}$

ii. $f(x) = \sqrt{\sin^{-1}(\log_3 x)}$

iii. $f(x) = \sqrt{\ln(\sin x)} + \sin^{-1}\left(\sqrt{\ln x}\right)$

iv. $f(x)=\cos^{-1}\left(\dfrac{3}{4+2\sin x}\right)$

v. $f(x)=\log_2(\log_3 x)$

vi. $f(x)=\dfrac{x}{\ln(1+\sec^{-1}(\ln x))}$

vii. $f(x)=\sqrt{4+x}-\sqrt{x+2}+\sqrt{15-x}$

viii. $f(x)=\dfrac{1}{\sqrt{\ln\{\cosh(\sin x)\}}}$

ix. $f(x)=\sqrt{-x}+\dfrac{1}{\sqrt{2+cosec^{-1}(\sin x)}}$

x. $f(x)=2^{\frac{1}{\cos^{-1}x}}+\cos^{-1}(2^x)$

xi. $f(x)=\tan\left(\dfrac{1}{1-\tan^{-1}(e^x)}\right)$

xii. $f(x)=\sqrt[3]{\sin x}+\sqrt[4]{\cos x}$

xiii. $f(x)=x^x$

xiv. $f(x)=(\sin x)^{\cos^{-1}x}$

xv. $f(x)=\left(\dfrac{1+x}{1-x}\right)^x$

xvi. $f(x)=\log_{\sin x}\cos x$

3. Find domain of the function

$$f(x)=\frac{1}{x-3},\quad x>2$$
$$=\frac{1}{x},\quad -1<x<1$$
$$=\frac{1}{x-1},\quad x\le -2.$$ {Ans. $(-\infty,-2]\cup(-1,0)\cup(0,1)\cup(2,3)\cup(3,\infty)$}

4. If $f(x)=\dfrac{x^2+1}{x-1},\quad x<3$

$$=\frac{\sin x}{x-3},\quad x>3,$$

for what values of x is the function not defined? {Ans. 1, 3}

CATEGORY-1.3. EQUIVALENT FUNCTIONS

5. Are the following functions equivalent?

i. $f(x)=\dfrac{x}{x}$ and $\phi(x)=1$ {Ans. No}

ii. $f(x)=\frac{1}{\frac{1}{x}}$ and $\phi(x)=x$ {Ans. No}

iii. $f(x)=\frac{1}{x}-\frac{1}{x}$ and $\phi(x)=0$ {Ans. No}

iv. $f(x)=\left(\sqrt{x}\right)^2$ and $\phi(x)=x$ {Ans. No}

v. $f(x)=\sqrt{x^2}$ and $\phi(x)=x$ {Ans. No}

vi. $f(x)=\sqrt{x^2}$ and $\phi(x)=|x|$ {Ans. Yes}

vii. $f(x)=\log x^2$ and $\phi(x)=2\log x$ {Ans. No}

viii. $f(x)=\log(x-2)+\log(x-3)$ and $\phi(x)=\log(x-2)(x-3)$ {Ans. No}

ix. $f(x)=\cot x$ and $\phi(x)=\frac{1}{\tan x}$ {Ans. No}

x. $f(x)=\sin^2 x+\cos^2 x$ and $\phi(x)=1$ {Ans. Yes}

xi. $f(x)=\frac{1}{\cos ecx}$ and $\phi(x)=\sin x$ {Ans. No}

CATEGORY-1.4. SIMPLIFYING EXPRESSIONS CONTAINING POWERS AND LOGARITHMS

6. $3^{-\frac{1}{2}\log_3 9}$. {Ans. $\frac{1}{3}$}

7. $2^{2-\log_2 5}$. {Ans. $\frac{4}{5}$}

8. $10^{\log m+\log n}$. {Ans. mn}

9. $2^{\log_{2\sqrt{2}} 15}$. {Ans. $\sqrt[3]{225}$}

10. $(5.8)^{\log_{5.8} 10+1}$. {Ans. 58}

11. $8^{\log_2 \sqrt[3]{121}+\frac{1}{3}}$. {Ans. 242}

12. $\sqrt{\log_{0.5}^2 4}$. {Ans. 2}

13. $\log_\pi \tan(0.25\pi)$. {Ans. 0}

14. $\log_2(\log_3 81)$. {Ans. 2}

15. $81^{\left(\frac{1}{\log_5 3}\right)}+27^{\log_9 36}+3^{\frac{4}{\log_7 9}}$. {Ans. 890}

16. $2^{\log_3 5}-5^{\log_3 2}$. {Ans. 0}

17. $\log_3 5\cdot\log_{25} 27$. {Ans. $\frac{3}{2}$}

18. $\log_9 27-\log_{27} 9$. {Ans. $\frac{5}{6}$}

19. $\log_3 4\cdot\log_4 5\cdot\log_5 6\cdot\log_6 7\cdot\log_7 8\cdot\log_8 9$. {Ans. 2}

20. $\log_3 2\cdot\log_4 3\cdot\log_5 4\cdots\cdots\log_{15}14\cdot\log_{16}15$. {Ans. $\frac{1}{4}$}

21. $\log_6\left(216\sqrt{6}\right)$. {Ans. $\frac{7}{2}$}

22. $\log_2 \log_2 \log_4 256 + 2\log_{\sqrt{2}} 2$. {Ans. 5}

23. $\sqrt{\left(\frac{1}{\sqrt{(27)}}\right)^{2-\frac{(\log_5 13)}{2\log_5 9}}}$. {Ans. $\frac{13^{\frac{3}{16}}}{3^{\frac{3}{2}}}$}

24. $7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80}$. {Ans. $\log 2$}

25. $\log_2 \sqrt[3]{16} + \log_8 \sqrt[4]{2} - \log_3\left(27\sqrt{3}\right) - \log_5 \sqrt{5\sqrt{5}}$. {Ans. $-\frac{17}{6}$}

26. $\log_2\left(\frac{1}{4\sqrt{4}}\right) + \log_3\left(\frac{\sqrt[3]{3\sqrt{3}}}{27}\right) + \log_4\left(\frac{\sqrt[3]{8}}{128\sqrt{2}}\right) - \log_7\left(\frac{\sqrt{7}}{\sqrt[3]{49}}\right)$. {Ans. $-\frac{103}{12}$}

27. $\log_{\frac{1}{3}} \sqrt{9} + \log_{\sqrt[3]{\frac{1}{3}}} 9 - \log_{\frac{1}{8}} \sqrt[4]{32} + \log_{\frac{1}{\sqrt{2}}} \sqrt[3]{128\sqrt{2}}$. {Ans. $-\frac{139}{12}$}

28. $\log_3 27 - \log_{\sqrt{3}} 27 - \log_{\frac{1}{3}} 27 - \log_{\frac{\sqrt{3}}{2}}\left(\frac{64}{27}\right)$. {Ans. 6}

29. $\log_2\left(\frac{\sqrt[3]{4}\sqrt{2\sqrt[5]{16}}}{\sqrt{2}}\right) - \log_{\frac{1}{2}} \sqrt[3]{\frac{4}{\sqrt{2}}} + \log_{\frac{1}{\sqrt{3}}}\left(9\sqrt[3]{3}\right)$. {Ans. $-\frac{31}{10}$}

30. $\log_{0.4}\left(\frac{1}{5}\cdot\sqrt[3]{50}\right) + \log_{0.6}\left(\frac{\sqrt{15}}{5}\right) + \log_{0.32}\left(\frac{2\sqrt{2}}{5}\right)$. {Ans. $\frac{4}{3}$}

31. $\log^2_{\sqrt[5]{5}} \sqrt{5} - \log_{\sqrt[3]{5}}\left(5\sqrt{5}\right) + \log_{\left(\sqrt{3}+1\right)}\left(4+2\sqrt{3}\right)$. {Ans. $\frac{15}{4}$}

32. $\sqrt{\log_{\sqrt{2}} \sqrt{\sqrt{2}\sqrt{\sqrt{2}}} + \log_{\sqrt{\sqrt{2}}} \sqrt[4]{\sqrt{2\sqrt{2}}}}$. {Ans. $\sqrt{\frac{3}{2}}$}

33. $\sqrt{\log_{\sqrt{3}} \sqrt[4]{\frac{\left(\sqrt{3}\right)^{\frac{1}{2}}}{\sqrt{3}}} + \log_{\sqrt[4]{2}} \sqrt[4]{\sqrt{\frac{2}{\sqrt{2}}}}}$. {Ans. $\frac{1}{2\sqrt{2}}$}

34. $\left(\log_{\sqrt{5}} \frac{1}{5}\right)\sqrt{\log_{\frac{1}{5}}\left(5\sqrt{5}\right) + \log_{\sqrt{5}}\left(5\sqrt{5}\right)}$. {Ans. $-\sqrt{6}$}

35. $2\log_5 \sqrt[4]{5} + \frac{1}{2}\cdot\log_{\sqrt{5}} 25 - \log^2_5 \sqrt{5} - 2$. {Ans. $\frac{1}{4}$}

36. $\frac{1}{2}\left(9^{\log_{25} 5+1} - 3^{2\left(\log_{16} 2+\frac{1}{4}\right)}\right) - \log_{\sqrt{2}}\left(2\sqrt{2}\right)$. {Ans. 9}

37. $\log_3 \log_8 \log_2 16$. {Ans. $\log_3 2 - 1$}

38. $\log_8 \log_4 \log_2 64$. {Ans. $\frac{\log_2\left(\log_2 3+1\right)-1}{3}$}

39. $\log_4 \log_2 \log_3 81$. {Ans. $\frac{1}{2}$ }

40. $\log_3\left[\log_2^2\left(\frac{1}{2}\right)+6\log_2\sqrt{2}+5\right]$. {Ans. 2}

41. $\left(\log_{\sqrt{5}}125\div\log_5^2 25\right)\cdot\left(\log_{\frac{1}{5}}\sqrt{5}\div\log_{0.2}\sqrt[3]{25}\right)$. {Ans. $\frac{9}{8}$ }

42. $\left[\log_{\frac{1}{2}}\sqrt{\frac{1}{4}}+6\log_{\frac{1}{4}}\left(\frac{1}{2}\right)-2\log_{\frac{1}{16}}\left(\frac{1}{4}\right)\right]\div\log_{\sqrt{2}}\sqrt[5]{8}$. {Ans. $\frac{5}{2}$ }

43. $3^{1+\log_3 4}+2^{\log_2 3-2}$. {Ans. $\frac{51}{4}$ }

44. $4^{3+\log_4 2}-(1.5)^{\log_{\frac{3}{2}}3-1}$. {Ans. 126}

45. $2^{3-\log_4 3}+7^{2\log_7 2+1}$. {Ans. $28+\frac{8}{\sqrt{3}}$ }

46. $16^{1-\log_8 5}+4^{\frac{1}{2}\log_2 3+3\log_8 5}$. {Ans. $75+\frac{16}{5^{\frac{4}{3}}}$ }

47. $9^{2\log_3 2+4\log_{81}2}\cdot\sqrt{3^{2+\frac{1}{2}\log_3 16}}$. {Ans. 384}

48. $(0.1)^{2\log 0.1-1.5\log 0.1}\cdot(0.1)^{\left(\log\frac{8}{3}+2-\log 20\right)}$. {Ans. $\frac{3}{4\sqrt{10}}$ }

49. $72\cdot\left(49^{\frac{1}{2}\log_7 9-\log_7 6}+5^{-\log_{\sqrt{5}}4}\right)$. {Ans. $\frac{45}{2}$ }

50. $\frac{\log_3 81}{\log_3 9}\left(36^{1-\log_6 2}+49^{-\log_7 6}\right)$. {Ans. $\frac{325}{18}$ }

51. $\frac{\log_{\sqrt{2}}16}{\log_4\sqrt{2}}\left[\log_{\sqrt{2}}\left(2\cdot\sqrt[4]{2}\right)+100^{\frac{1}{2}\log 8-2\log 2}\right]$. {Ans. 96}

52. $10^{\frac{1}{2}\log 9-\log 5+\log 2}\cdot 7^{\log_{3\sqrt{3}}27}$. {Ans. $\frac{294}{5}$ }

53. $\log_{\sqrt{6}}3\cdot\log_3 36+\log_{\sqrt{3}}8\cdot\log_4 81$. {Ans. 16}

54. $72\log_2\left(\sqrt{\frac{1}{5}}\right)\cdot\log_{25}\sqrt[3]{2}+10\log_2\left(\frac{\sqrt[5]{8}}{2}\right)$. {Ans. -10 }

55. $3^{\frac{2}{5}\log_3 32-\frac{1}{3}\log_3 64+\log_3 10}$. {Ans. 10}

56. $(0.2)^{\frac{1}{2}(9\log_{0.2}2-3\log_{0.2}4)}$. {Ans. $2\sqrt{2}$ }

57. $\left(\sqrt{2}\right)^{3\log_{\sqrt{2}}5-2\log_{\sqrt{2}}25-\log_{\sqrt{2}}10+2\log_{\sqrt{2}}\sqrt{5}}$. {Ans. $\frac{1}{10}$ }

58. $(\log 2+\log 5+\log 300-\log 3)\cdot 3^{\frac{1}{5\log_5 3}}$. {Ans. $3\cdot 5^{\frac{1}{5}}$}

59. $\left(\log_8 27-\log_{0.5}\frac{1}{3}\right)\cdot\left(\frac{\log_3 12}{\log_{36}3}-\frac{\log_3 4}{\log_{108}3}\right)$. {Ans. 0}

60. $\frac{\log_2\sqrt[3]{\frac{2}{3}}}{\log_2^2\sqrt{7}}-\frac{2\log_{\sqrt{7}}\sqrt[3]{\frac{2}{3}}}{\log_{\sqrt{3}}\sqrt{7}}-\log_{\sqrt[6]{7}}\sqrt[3]{\frac{2}{3}}\cdot\log_{\sqrt{7}}\sqrt[3]{\frac{2}{3}}$. {Ans. 0}

61. $2^{\log_5 3}\cdot\left(\frac{1}{3}\right)^{1-\log_5 2.5}\cdot\log_9 2\cdot\log_4 81$. {Ans. 1}

62. $\log_3 2\cdot\log_4 3\cdot\log_5 4\cdot\log_6 5\cdot\log_7 6\cdot\log_8 7$. {Ans. $\frac{1}{3}$}

CATEGORY-1.5. MISCELLANEOUS QUESTIONS ON POWER AND LOGARITHM

63. Given that $\log_6 2=a$, find $\log_{24}72$ in terms of a. {Ans. $\frac{a+2}{2a+1}$}

64. Given that $\log_{36}8=a$, find $\log_{36}9$ in terms of a. {Ans. $\frac{3-2a}{3}$}

65. Given that $\log_4 125=a$, find $\log 64$ in terms of a. {Ans. $\frac{18}{3+2a}$}

66. Given that $\log_{100}3=a$ and $\log_{100}2=b$, find $\log_5 6$ in terms of a and b. {Ans. $\frac{2(a+b)}{1-2b}$}

67. Given that $\log_6 15=a$ and $\log_{12}18=b$, find $\log_{25}24$ in terms of a and b. {Ans. $\frac{5-b}{2(a+ab+1-2b)}$}

68. If $\ln 2\cdot\log_a 625=\log 16\cdot\ln 10$, then find the value of a. {Ans. 5}

69. Prove that $\log_b a\log_c b\log_a c=1$.

70. Prove that $\log_b a\log_c b\log_d c\log_a d=1$.

71. If $\log_a(ab)=x$, then evaluate $\log_b(ab)$ in terms of x. {Ans. $\frac{x}{x-1}$}

72. Prove that $\frac{\log_a n}{\log_{ab}n}=1+\log_a b$.

73. Prove that $\log_{ab}x=\frac{\log_a x\log_b x}{\log_a x+\log_b x}$.

74. If $a^2+b^2=7ab$, prove that $\log\frac{1}{3}(a+b)=\frac{1}{2}[\log a+\log b]$.

75. Show that $\frac{1}{\log_2 n}+\frac{1}{\log_3 n}+\cdots\cdots+\frac{1}{\log_{43}n}=\frac{1}{\log_{43!}n}$.

76. If $n=1983!$, compute the sum $\frac{1}{\log_2 n}+\frac{1}{\log_3 n}+\frac{1}{\log_4 n}+\cdots\cdots+\frac{1}{\log_{1983}n}$. {Ans. 1}

77. If $y = a^{\frac{1}{(1-\log_a x)}}$ and $z = a^{\frac{1}{(1-\log_a y)}}$, prove that $x = a^{\frac{1}{(1-\log_a z)}}$.

78. If $\dfrac{\log a}{b-c} = \dfrac{\log b}{c-a} = \dfrac{\log c}{a-b}$, prove that $a^a \cdot b^b \cdot c^c = 1$.

79. If $\dfrac{\log x}{q-r} = \dfrac{\log y}{r-p} = \dfrac{\log z}{p-q}$, prove that $x^{q+r} \cdot y^{r+p} \cdot z^{p+q} = x^p \cdot y^q \cdot z^r$.

80. Prove that $\log_a n \log_b n + \log_b n \log_c n + \log_c n \log_a n = \dfrac{\log_a n \log_b n \log_c n}{\log_{abc} n}$.

81. If $a > 0$, $c > 0$, $b = \sqrt{ac}$, $a \neq 1$, $c \neq 1$, $ac \neq 1$ and $n > 0$, prove that $\dfrac{\log_a n}{\log_c n} = \dfrac{\log_a n - \log_b n}{\log_b n - \log_c n}$.

82. Prove that if $x = \log_c b + \log_b c$, $y = \log_a c + \log_c a$, $z = \log_b a + \log_a b$ then $xyz = x^2 + y^2 + z^2 - 4$.

CATEGORY-1.6. MATHEMATICAL OPERATIONS ON FUNCTIONS

83. If $f(x) = ax^2 + bx + c$, find $f(0)$, $f(1)$, $f(-1)$, $f(a)$ and $f(b)$. {Ans. c, $a+b+c$, $a-b+c$, a^3+ab+c, ab^2+b^2+c }

84. If $f(x) = \dfrac{x-1}{x}$ and $g(x) = x^2+1$, find $g\left(f(1)\right)$ and $f\left(g(-1)\right)$. {Ans. $1, \dfrac{1}{2}$ }

85. If $f(x) = x^2+3x+1$ and $g(x) = 2x-3$. Find *fog, gof, fof* and *gog*. {Ans. $(fog)(x) = 4x^2-6x+1$; $(gof)(x) = 2x^2+6x-1$; $(fof)(x) = x^4+6x^3+14x^2+15x+5$; $(gog)(x) = 4x-9$ }

86. Let $f(x) = x^2$ and $g(x) = 2x+1$. Find *fog* and *gof*. Also show that $fog \neq gof$. {Ans. $(fog)(x) = 4x^2+4x+1$; $(gof)(x) = 2x^2+1$}

87. If $f(x) = x^2+2$ and $g(x) = \dfrac{x}{x-1}$. Find *fog* and *gof*. {Ans. $(fog)(x) = \dfrac{3x^2-4x+2}{(x-1)^2}$; $(gof)(x) = \dfrac{x^2+2}{x^2+1}$ }

88. If $f(x) = x^2-3x+2$, find $f\left(f(x)\right)$. {Ans. $f\left(f(x)\right) = x^4-6x^3+10x^2-3x$}

89. Find $\phi(\psi(x))$ and $\psi(\phi(x))$ if $\phi(x) = x^2+1$ and $\psi(x) = 3^x$. {Ans. $3^{2x}+1, 3^{x^2+1}$}

90. If $f(x) = \sin x$ and $g(x) = x^2$, then find $fog(x)$ and $gof(x)$. {Ans. $fog(x) = \sin x^2$, $gof(x) = (\sin x)^2$}

91. If $f(x) = x\cos x$ and $g(x) = \dfrac{x}{1+x^2}$, then find $fog(x)$ and $gof(x)$. {Ans. $fog(x) = \dfrac{x}{1+x^2}\cos\dfrac{x}{1+x^2}$; $gof(x) = \dfrac{x\cos x}{1+x^2\cos^2 x}$ }

92. If $f(x) = x^2 - \dfrac{1}{x^2}$, prove that $f(x) = -f\left(\dfrac{1}{x}\right)$.

93. If $f(x) = x + \dfrac{1}{x}$, prove that $\left(f(x)\right)^3 = f\left(x^3\right) + 3f\left(\dfrac{1}{x}\right)$.

94. Given the function $f(x) = \dfrac{a^x + a^{-x}}{2}, (a > 0)$, show that $f(x+y) + f(x-y) = 2f(x)f(y)$.

95. Given the functions $f(x)$ and $g(x)$.

x	$f(x)$
1	3
2	1
3	4
4	2

x	$g(x)$
1	2
2	4
3	1
4	3

i. Determine the function $h(x) = f^2(x) + g^2(x) - f(x)g(x)$;

ii. Determine the function $\phi(x) = f(g(x))$;

iii. Determine the function $\varphi(x) = g(f(x))$.

iv. Which of the above functions are injective?

{Ans.

i.

x	$h(x)$
1	7
2	13
3	13
4	7

ii.

x	$\phi(x)$
1	1
2	2
3	3
4	4

iii.

x	$\varphi(x)$
1	1
2	2
3	3
4	4

iv. $\phi(x)$ and $\varphi(x)$ are injective}

CATEGORY-1.7. MATHEMATICAL OPERATIONS ON PIECEWISE FUNCTIONS

96. Given the function

$$
\begin{aligned}
f(x) &= 3^{-x} - 1, \quad -1 \le x < 0 \\
&= \tan\frac{x}{2}, \quad 0 \le x < \pi \\
&= \frac{x}{x^2 - 2}, \quad \pi \le x \le 6.
\end{aligned}
$$

Find:-

i. $f(-1)$; {Ans. 2 }

ii. $f\left(\frac{\pi}{2}\right)$; {Ans. 1}

iii. $f\left(\frac{2\pi}{3}\right)$; {Ans. $\sqrt{3}$ }

iv. $f(4)$; {Ans. $\frac{2}{7}$ }

v. $f(6)$. {Ans. $\frac{3}{17}$ }

97. Let $f(x)=1$, if x is a rational number

= 0, if x is a irrational number.

Find:-

i. $f\left(\frac{1}{3}\right)$; {Ans. 1}

ii. $f(\sqrt{7})$; {Ans. 0}

iii. $f\left(\frac{22}{7}\right)$; {Ans. 1}

iv. $f(\pi)$; {Ans. 0}

v. $f(e)$; {Ans. 0}

vi. $f(f(1.4327))$; {Ans. 1}

vii. $f(f(\sqrt{3}))$. {Ans. 1}

98. Given $f(x)=x^2, \quad x \geq 0$

$= x, \quad x<0$

and $g(x)=\frac{1}{x}, \quad x \geq 1$

$=1, \quad x<1$.

Determine the following functions:-

i. $h(x)=x \times g(x)$;

ii. $\phi(x)=f(x)+g(x)$;

iii. $\psi(x)=f(x) \times g(x)$.

{Ans.

i. $h(x)=1, \quad x \geq 1$

$= x, \quad x<1$;

ii. $\phi(x)=x^2+\frac{1}{x}, \quad x \geq 1$

$= x^2+1, \quad 0 \leq x<1$

$= x+1, \quad x<0$;

iii. $\psi(x)=x, \quad x \geq 1$

$= x^2, \quad 0 \leq x<1$

$= x, \quad x<0$ }

99. Given $f(x)=x+1, \quad -2 \leq x<1$

$=1-x, \quad x>1$

and $g(x) = -x, \quad x < 0$

$= -x - 1, \quad 0 \le x \le 2$.

i. Determine the function $\phi(x) = f(x) + g(x)$;

ii. Determine the function $\varphi(x) = \phi(x) + g(x)$;

iii. Plot graph of $f(x)$ and write its domain and range;

iv. Plot graph of $g(x)$ and write its domain and range;

v. Plot graph of $\phi(x)$ and write its domain and range;

vi. Plot graph of $\varphi(x)$ and write its domain and range;

vii. Which of the above functions are injective?

{Ans.

i. $\phi(x) = 1, \quad -2 \le x < 0$

$= 0, \quad 0 \le x < 1$

$= -2x, \quad 1 < x \le 2$

ii. $\varphi(x) = 1 - x, \quad -2 \le x < 0$

$= -x - 1, \quad 0 \le x < 1$

$= -3x - 1, \quad 1 < x \le 2$

vii. $g(x)$ and $\varphi(x)$ are injective}

100. Determine the function $f(x) = |x| \times sgn x$. {Ans. $f(x) = x$ }

101. Determine the following functions:-

i. $f_1(x) = \sin|x|$;

ii. $f_2(x) = \sin^{-1}(\operatorname{sgn} x)$;

iii. $f_3(x) = |x|^2$;

iv. $f_4(x) = (\operatorname{sgn} x)^2$;

v. $f_5(x) = e^{\operatorname{sgn} x}$.

{Ans.

i. $f_1(x) = \sin x, \quad x \ge 0$

$= -\sin x, \quad x < 0$

ii. $f_2(x) = \frac{\pi}{2}, \quad x > 0$

$= 0, \quad x = 0$

$= -\frac{\pi}{2}, \quad x < 0$

iii. $f_3(x) = x^2$

iv. $f_4(x) = 1, \quad x \ne 0$

$= 0, \quad x = 0$

v. $f_5(x) = e, \quad x > 0$

$= 1, \quad x = 0$

$= \frac{1}{e}, \quad x < 0$ }

102. Determine the following functions:-

i. $f_1(x) = |x-1|$;

ii. $f_2(x) = |2x+1|$;

iii. $f_3(x) = |\ln x|$;

iv. $f_4(x) = |e^x|$;

v. $f_5(x) = \operatorname{sgn}(x+3)$;

vi. $f_6(x) = \operatorname{sgn}(3x+2)$;

vii. $f_7(x) = \operatorname{sgn}(\ln x)$;

viii. $f_8(x) = \operatorname{sgn}(e^x)$.

{Ans.

i. $f_1(x) = x-1, \quad x \ge 1$

$= 1-x, \quad x < 1$

ii. $f_2(x) = 2x+1, \quad x \ge -\frac{1}{2}$

$= -2x-1, \quad x < -\frac{1}{2}$

iii. $f_3(x) = \ln x, \quad x \ge 1$

$= -\ln x, \quad 0 < x < 1$

iv. $f_4(x) = e^x$

v. $f_5(x) = 1, \quad x > -3$

$= 0, \quad x = -3$

$= -1, \quad x < -3$

vi. $f_6(x) = 1, \quad x > -\frac{2}{3}$

$= 0, \quad x = -\frac{2}{3}$

$= -1, \quad x < -\frac{2}{3}$

vii. $f_7(x) = 1, \quad x > 1$

$= 0, \quad x = 1$

$= -1, \quad 0 < x < 1$

viii. $f_8(x) = 1$ }

103. Given $f(x) = x+2, \quad -2 \le x \le 0$

$= x-1, \quad 0 < x \le 4$.

i. Determine the function $\phi(x) = f\left(\frac{x}{2}\right)$ and plot its graph and write its domain and range.

ii. Determine the function $\varphi(x) = f(2x)$ and plot its graph and write its domain and range.

iii. Determine the function $h(x) = f(-x)$ and plot its graph and write its domain and range.

iv. Determine the function $f_1(x) = f(x+1)$ and plot its graph and write its domain and range.

v. Determine the function $f_2(x) = f(2x+1)$ and plot its graph and write its domain and range.

vi. Which of the above functions are one-one functions?

{Ans.

i. $\phi(x) = \dfrac{x}{2} + 2, \quad -4 \le x \le 0$

$\qquad = \dfrac{x}{2} - 1, \quad 0 < x \le 8$

ii. $\varphi(x) = 2x + 2, \quad -1 \le x \le 0$

$\qquad = 2x - 1, \quad 0 < x \le 2$

iii. $h(x) = -x - 1, \quad -4 \le x < 0$

$\qquad = -x + 2, \quad 0 \le x \le 2$

iv. $f_1(x) = x + 3, \quad -3 \le x \le -1$

$\qquad = x, \quad -1 < x \le 3$

v. $f_2(x) = 2x + 3, \quad -\dfrac{3}{2} \le x \le -\dfrac{1}{2}$

$\qquad = 2x, \quad -\dfrac{1}{2} < x \le \dfrac{3}{2}$

vi. None of these functions are one-one function}

104. Given $f(x) = x^2, \quad x < 1$

$\qquad = \dfrac{1}{x}, \quad 1 \le x \le 2.$

Determine the functions $g(x) = f(x^2)$.

{Ans.

$g(x) = \dfrac{1}{x^2}, \quad -\sqrt{2} \le x \le -1$

$\qquad = x^4, \quad -1 < x < 1$

$\qquad = \dfrac{1}{x^2}, \quad 1 \le x \le \sqrt{2}$ }

105. Given

$f(x) = 1 + x, \quad -1 \le x \le 0$

$\qquad = -x, \quad 0 < x \le 1$

$g(x) = -1 - x, \quad -1 \le x < 0$

$\qquad = 1 - x, \quad 0 \le x \le 1.$

i. Determine the function $f_1(x) = f(g(x))$ and plot its graph and write its domain and range.

ii. Determine the function $f_2(x) = g(f(x))$ and plot its graph and write its domain and range.

iii. Determine the function $f_3(x) = f_1(f_2(x))$ and plot its graph and write its domain and range.

iv. Determine the function $f_4(x) = f_2(f_1(x))$ and plot its graph and write its domain and range.

v. Determine the function $f_5(x) = f(|x|)$ and plot its graph and write its domain and range.

vi. Determine the function $f_6(x) = |f(x)|$ and plot its graph and write its domain and range.

vii. Determine the function $f_7(x) = sgn(f(x))$ and plot its graph and write its domain and range.

viii. Determine the function $f_8(x) = f(sgnx)$ and plot its graph and write its domain and range.

ix. Which of the above functions are injective functions?

{Ans.

i. $f_1(x) = -x, \quad -1 \le x < 0$
$= x-1, \quad 0 \le x < 1$
$= 1, \quad x = 1$

ii. $f_2(x) = -x, \quad -1 \le x \le 0$
$= x-1, \quad 0 < x \le 1$

iii. $f_3(x) = 1, \quad x = -1$
$= -x-1, \quad -1 < x \le 0$
$= 1-x, \quad 0 < x < 1$
$= -1, \quad x = 1$

iv. $f_4(x) = -x-1, \quad -1 \le x < 0$
$= 1-x, \quad 0 \le x \le 1$

v. $f_5(x) = x, \quad -1 \le x < 0$
$= 1, \quad x = 0$
$= -x, \quad 0 < x \le 1$

vi. $f_6(x) = 1+x, \quad -1 \le x \le 0$
$= x, \quad 0 < x \le 1$

vii. $f_7(x) = 0, \quad x = -1$
$= 1, \quad -1 < x \le 0$
$= -1, \quad 0 < x \le 1$

viii. $f_8(x) = 0, \quad x < 0$
$= 1, \quad x = 0$
$= -1, \quad x > 0$

ix. $f(x)$ is injective function.}

106. Given $f(x) = 1+x, \quad 0 \le x \le 2$
$= 3-x, \quad 2 < x \le 3$.

i. Determine the function $g(x) = f(f(x))$ and plot its graph and write its domain and range.

ii. Determine the function $h(x) = f(g(x))$ and plot its graph and write its domain and range.

iii. Determine the function $\phi(x) = f(x) + h(x)$ and plot its graph and write its domain and range.

iv. Determine the function $\varphi(x) = \phi(f(x))$ and plot its graph and write its domain and range.

{Ans.

i. $g(x) = 2+x, \quad 0 \le x \le 1$
$= 2-x, \quad 1 < x \le 2$

$= 4 - x, \quad 2 < x \le 3$

ii. $h(x) = 3, \quad x = 0$

$= 1 - x, \quad 0 < x \le 1$

$= 3 - x, \quad 1 < x \le 2$

$= 5 - x, \quad 2 < x \le 3$

iii. $\phi(x) = 4, \quad x = 0$

$= 2, \quad 0 < x \le 1$

$= 4, \quad 1 < x \le 2$

$= 8 - 2x, \quad 2 < x \le 3$

iv. $\varphi(x) = 2, \quad x = 0$

$= 4, \quad 0 < x \le 1$

$= 6 - 2x, \quad 1 < x \le 2$

$= 2, \quad 2 < x < 3$

$= 4, \quad x = 3$ }

107. Given $f(x) = -1, \quad -2 \le x \le 0$

$= x - 1, \quad 0 < x \le 2$.

i. Determine the function $g(x) = f(|x|) + |f(x)|$ and plot its graph and write its domain and range.

ii. Determine the function $\phi(x) = g(|x|) + |g(x)|$ and plot its graph and write its domain and range.

{Ans.

i. $g(x) = -x, \quad -2 \le x < 0$

$= 0, \quad 0 \le x < 1$

$= 2x - 2, \quad 1 \le x \le 2$

ii. $\phi(x) = -3x - 2, \quad -2 \le x \le -1$

$= -x, \quad -1 < x < 0$

$= 0, \quad 0 \le x < 1$

$= 4x - 4, \quad 1 \le x \le 2$ }

108. Given

$f(x) = 1 + x, \quad -2 \le x < 0$

$= 0, \quad x = 0$

$= x - 1, \quad 0 < x \le 2$,

$\phi(x) = f(f(x))$ and $F(x) = 2\phi\left(\dfrac{x}{2}\right)$. Determine the function $g(x) = \phi(F(x))$ and plot its graph and write its domain and range.

{Ans. $g(x) = x + 4, \quad -4 \le x < -3$

$= 0, \quad x = -3$

$= x + 2, \quad -3 < x < -1$

$= 0, \quad x = -1$

$= x, \quad -1 < x < 1$

$= 0, \quad x = 1$

$= x-2, \quad 1 < x < 3$

$= 0, \quad x = 3$

$= x-4, \quad 3 < x \le 4$ }

109. Determine the function $f(x) = \left||x+1| - |x|\right|$ and plot its graph and write its domain and range.

{Ans. $f(x) = 1, \quad x < -1$

$= -2x-1, \quad -1 \le x < -\frac{1}{2}$

$= 2x+1, \quad -\frac{1}{2} \le x < 0$

$= 1, \quad x \ge 0$ }

CATEGORY-1.8. ADDITIONAL QUESTIONS

Mathematics for IIT-JEE

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PART-I
DIFFERENTIAL CALCULUS

CHAPTER-2
LIMIT

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CHAPTER-2
LIMIT

LIST OF THEORY SECTIONS

LIST OF QUESTION CATEGORIES

CHAPTER-2
LIMIT

SECTION-2.1. DEFINITION OF LIMIT

1. Definition of Neighbourhood (nbd) of a point, Deleted neighbourhood, δ-neighbourhood, Right hand and Left hand neighbourhood

i. Any interval containing a point *a* as its interior point is called the *neighbourhood* (nbd) of point *a*.

ii. Any neighbourhood of point *a* which does not contain the point *a* is called the *deleted neighbourhood* (nbd) of point *a*.

iii. The interval $(a-\delta, a+\delta)$ is called the δ-neighbourhood of point *a*.

iv. Any interval whose left hand end-point is *a* is called *right hand neighbourhood* of point *a*.

v. Any interval whose right hand end-point is *a* is called *left hand neighbourhood* of point *a*.

2. Meaning of tends to ($\rightarrow$) and its use

i. *Tends to*, denoted by symbol $\rightarrow$, means "approaching without bound but not equal to".

ii. $x \rightarrow a$ means "*x* approaches *a* without bound but $x \neq a$".

iii. $x \rightarrow a^+$ means $x > a$ and $x \rightarrow a$.

iv. $x \rightarrow a^-$ means $x < a$ and $x \rightarrow a$.

v. $x \rightarrow \infty$ means '*x* approaches ∞ without bound' or '*x* is increasing without bound'.

vi. $x \rightarrow -\infty$ means '*x* approaches $-\infty$ without bound' or '*x* is decreasing without bound'.

3. Definition of Right hand limit of a function at a finite point

i. A function $f(x)$ is said to have a right hand limit at a finite point *a* if $f(x)$ is either tending to or equal to *b* (a real number or $+\infty$ or $-\infty$) as $x \rightarrow a^+$ and *b* is said to be the right hand limit of $f(x)$ at the point *a*, denoted by symbol $\lim\limits_{x \rightarrow a^+} f(x) = b$; otherwise $f(x)$ has no right hand limit at the point *a*.

4. Definition of Left hand limit of a function at a finite point

i. A function $f(x)$ is said to have a left hand limit at a finite point *a* if $f(x)$ is either tending to or equal to *b* (a real number or $+\infty$ or $-\infty$) as $x \rightarrow a^-$ and *b* is said to be the left hand limit of $f(x)$ at the point *a*, denoted by symbol $\lim\limits_{x \rightarrow a^-} f(x) = b$; otherwise $f(x)$ has no left hand limit at the point *a*.

5. Definition of limit of a function at a finite point

i. A function $f(x)$ is said to have a limit at a finite point *a* if $f(x)$ is either tending to or equal to *b* (a real number or $+\infty$ or $-\infty$) as $x \rightarrow a$ and *b* is said to be the limit of $f(x)$ at the point *a*, denoted by symbol $\lim\limits_{x \rightarrow a} f(x) = b$; otherwise $f(x)$ has no limit at the point *a*.

6. Definition of limit of a function at $\pm\infty$

i. A function $f(x)$ is said to have a limit at $+\infty$ $(-\infty)$ if $f(x)$ is either tending to or equal to *b* (a real number or $+\infty$ or $-\infty$) as $x \rightarrow +\infty$ $(x \rightarrow -\infty)$ and *b* is said to be the limit of $f(x)$ at $+\infty$ $(-\infty)$, denoted by symbol $\lim\limits_{x \rightarrow +\infty} f(x) = b$ $\left(\lim\limits_{x \rightarrow -\infty} f(x) = b\right)$; otherwise $f(x)$ has no limit at $+\infty$ $(-\infty)$.

7. A function may not have a limit at a point or $\pm\infty$

SECTION-2.2. EXISTENCE OF LIMIT

1. Existence of limit of a function at a finite point

i. The limit of a function $f(x)$ is said to exist at a finite point *a* iff:-
   a. $\lim\limits_{x\to a^+} f(x)$ is finite,
   b. $\lim\limits_{x\to a^-} f(x)$ is finite,
   c. $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ are both equal,

   otherwise the limit of the function $f(x)$ does not exist at *a*.

2. Existence of limit at the end points of the Domain

i. If $f(x)$ is not defined in a left hand nbd of the point *a* and $\lim\limits_{x\to a^+} f(x)$ is finite, then the limit of the function $f(x)$ is said to exist at the point *a*, otherwise the limit of the function $f(x)$ does not exist at *a*.

ii. If $f(x)$ is not defined in a right hand nbd of the point *a* and $\lim\limits_{x\to a^-} f(x)$ is finite, then the limit of the function $f(x)$ is said to exist at the point *a*, otherwise the limit of the function $f(x)$ does not exist at *a*.

3. Existence of limit at $\pm\infty$

i. If $\lim\limits_{x\to\infty} f(x)$ is finite, then the limit of the function $f(x)$ is said to exist at $+\infty$, otherwise the limit of the function $f(x)$ does not exist at $+\infty$.

ii. If $\lim\limits_{x\to-\infty} f(x)$ is finite, then the limit of the function $f(x)$ is said to exist at $-\infty$, otherwise the limit of the function $f(x)$ does not exist at $-\infty$.

4. Cases when limit does not exist

SECTION-2.3. FINDING LIMIT OF A FUNCTION BY THEOREMS

1. Reading the limit of a function from it's graph

2. Limit of analytical functions

i. Limit of Basic functions
   a. From values/graphs
   b. Theorem of limit for Basic functions

      If $f(x)$ is a basic function and a finite point *a* belongs to its domain, then $\lim\limits_{x\to a} f(x)$ exists and is equal to $f(a)$.

ii. Limit of defined analytical functions
   a. Meaningless (Indeterminate) forms

      Certain forms of limits are said to be meaningless when merely knowing the limiting behavior of individual parts of the expression is not sufficient to actually determine the overall limit since the value of the overall limit actually depends on the limiting behavior of the combination of the two expressions. There are seven meaningless forms $\frac{0}{0}$ form, $\frac{\infty}{\infty}$ form, $0\times\infty$ form, $\infty-\infty$ form, 0^0 form, ∞^0 form, 1^∞ form.

   b. Non-meaningless forms and Properties of mathematical operations

      All other forms of limits except the seven meaningless forms are non-meaningless forms and merely knowing the limiting behavior of individual parts of the expression is sufficient to actually determine the overall limit by Properties of mathematical operations.

   c. Theorems of limit & its applications

      If *a* is a finite point or $+\infty$ or $-\infty$ and $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist and $\lim\limits_{x\to a} f(x)=l$ and

$\lim_{x\to a} g(x) = m$ then:-

- $\lim_{x\to a}[f(x) \pm g(x)]$ must exist and is equal to $l \pm m$.
- $\lim_{x\to a}[f(x) \times g(x)]$ must exist and is equal to $l \times m$.
- $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]$ must exist and is equal to $\frac{l}{m}$ provided that $m \neq 0$.
- $\lim_{x\to a}\left[f(x)^{g(x)}\right]$ must exist and is equal to l^m provided that l^m is defined.

d. Finding limit by values, theorems and properties in cases of non-meaningless forms
e. Limit of piecewise functions

SECTION-2.4. FINDING LIMIT OF A FUNCTION BY METHODS

1. Methods to calculate limit

i. There are methods to calculate limits in cases of meaningless forms and in other cases where theorems/properties are not applicable.
ii. Limit of rational functions at a finite point
iii. Limit of irrational functions at a finite point
iv. Limit involving ∞
v. Substitution
vi. Use of Standard limits

a. $\lim_{x\to 0}\frac{\sin x}{x} = 1$

b. $\lim_{x\to 0}\frac{\tan x}{x} = 1$

c. $\lim_{x\to 0}\frac{\sin^{-1} x}{x} = 1$

d. $\lim_{x\to 0}\frac{\tan^{-1} x}{x} = 1$

e. $\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$ (a, n are constants)

f. $\lim_{x\to 0}(1+kx)^{\frac{1}{x}} = e^k$ (k is a non-zero constant); $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

g. $\lim_{x\to \pm\infty}\left(1+\frac{k}{x}\right)^x = e^k$ (k is a non-zero constant); $\lim_{x\to \pm\infty}\left(1+\frac{1}{x}\right)^x = e$

h. $\lim_{x\to 0}\frac{a^x - 1}{x} = \ln a$ (a is a positive constant); $\lim_{x\to 0}\frac{e^x - 1}{x} = 1$

vii. Use of Binomial theorem & Series.

a. Binomial Theorem

If $n \in N$ and a, b and x are numbers, then

- $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \ldots\ldots\ldots\ldots\ldots + {}^nC_{n-1} a b^{n-1} + {}^nC_n b^n$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + ... + nab^{n-1} + b^n;$$

- $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_{n-1}x^{n-1} + {}^nC_n x^n$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + nx^{n-1} + x^n;$$

where ${}^nC_r = \frac{n!}{(n-r)!r!}$; ${}^nC_0 = {}^nC_n = 1$.

b. Sine series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

c. Cosine series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

d. Exponential series

- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$
- $a^x = 1 + \frac{(\ln a)x}{1!} + \frac{(\ln a)^2 x^2}{2!} + \frac{(\ln a)^3 x^3}{3!} + ...$

e. Logarithmic series

If $-1 < x \le 1$, then

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

f. Binomial series

If $a \notin N$ and x is a number such that $-1 < x < 1$, then

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \frac{a(a-1)(a-2)(a-3)}{4!}x^4 + ...$$

g. Proving standard limits

viii. Application of methods

ix. L' Hospital's rule & it's restricted use

If a is a finite point or $+\infty$ or $-\infty$; and if as $x \to a$ $f(x)$ and $g(x)$ either both tends to 0 or both tends to $\pm\infty$; and if $\lim_{x\to a}\frac{f(x)}{g(x)}$ and $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ both exist or both are infinity, then they must be equal.

x. Sandwich theorem

a. Let $h(x) \le f(x) \le g(x)$ in a neighbourhood of a except probably at a and $h(x)$, $f(x)$ and $g(x)$ have limits at a then $\lim_{x\to a} h(x) \le \lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

b. Let $h(x) \le f(x) \le g(x)$ in a neighbourhood of a except probably at a and let $h(x)$ and $g(x)$ have the same limit L at a, then $\lim_{x\to a} f(x) = L$.

SECTION-2.5. LIMITS WITH PARAMETERS

1. Finding limit of a function containing parameters

2. Finding values of the parameters given the value of limit
3. Finding the limit as function of parameter

EXERCISE-2

CATEGORY-2.1. LIMIT OF DEFINED ANALYTICAL FUNCTIONS BY THEOREMS

1. $\lim_{x\to\infty}\frac{\tan^{-1}x}{\ln x}$ {Ans. 0}
2. $\lim_{x\to\infty}\frac{\tanh x}{x^3}$ {Ans. 0}
3. $\lim_{x\to 0}\operatorname{sgn}(x^2)$ {Ans. 1}
4. $\lim_{x\to 0}\operatorname{sgn}(x^3)$ {Ans. 1, –1}
5. $\lim_{x\to 0}(\operatorname{sgn} x)^2$ {Ans. 1}
6. $\lim_{x\to 0}(\operatorname{sgn} x)^{\operatorname{sgn} x}$. {Ans. 1, –1}
7. $\lim_{x\to 0}\operatorname{sgn}(\ln x)$ {Ans. -1}
8. $\lim_{x\to 0}e^{\operatorname{sgn} x}$ {Ans. e, $\frac{1}{e}$}
9. $\lim_{x\to 0}e^{|\operatorname{sgn} x|}$ {Ans. e}
10. $\lim_{x\to 0}(\operatorname{sgn} x)^{e^x}$ {Ans. 1}
11. $\lim_{x\to 0}(\tan x)^{\ln(\operatorname{sgn} x)}$ {Ans. 1}
12. $\lim_{x\to 0}\frac{\ln x-1}{x-1}$ {Ans. ∞}
13. $\lim_{x\to 1}3x^2+4x+5$. {Ans. 12}
14. $\lim_{x\to 1}\frac{4x^5+9x+7}{3x^6+x^3+1}$ {Ans. 4}
15. $\lim_{x\to 0}\frac{\cos x}{1+\sin x}$. {Ans. 1}
16. $\lim_{x\to 0}\frac{2^x+\sin x}{\cos^{-1}x+\tan^{-1}x}$ {Ans. $\frac{2}{\pi}$}
17. $\lim_{x\to e}x^2\ln x$ {Ans. e^2}
18. $\lim_{x\to 1}\cos^{-1}x\cdot\ln x+\frac{1}{x}+\frac{1}{x^2}$ {Ans. 2}
19. $\lim_{x\to 1}\ln(\sin\pi x-\cos\pi x)$ {Ans. 0}
20. $\lim_{x\to 0}\sin^{-1}\{\ln(\cos x)\}$ {Ans. 0}
21. $\lim_{x\to 0}\operatorname{sgn}\{\ln(\cos x)\}$ {Ans. –1}
22. $\lim_{x\to 0}\operatorname{sgn}\{\ln(1+x)\}$ {Ans. 1, –1}
23. $\lim_{x\to 0}\operatorname{sgn}\{\ln(\operatorname{sgn} x)\}$ {Ans. 0}

24. $\lim\limits_{x\to 0}\frac{\ln(\operatorname{sgn} x)}{x}$ {Ans. 0}

25. $\lim\limits_{x\to \infty} x\ln(\operatorname{sgn} x)$ {Ans. 0}

26. $\lim\limits_{x\to 0}\{\ln(\operatorname{sgn} x)\}^{x}$ {Ans. 0}

27. $\lim\limits_{x\to \infty} x^{\ln(\operatorname{sgn} x)}$ {Ans. 1}

28. $\lim\limits_{x\to \infty}(\operatorname{sgn} x)^{x}$ {Ans. 1}

29. $\lim\limits_{x\to 0}(1+x)^{\sin x}$ {Ans. 1}

30. $\lim\limits_{x\to 0}(\cos^{-1} x)^{\cos x}$ {Ans. $\frac{\pi}{2}$}

31. $\lim\limits_{x\to 1}\frac{x}{x-1}$ {Ans. +∞, -∞}

32. $\lim\limits_{x\to 0}\ln x+e^{x}$ {Ans. -∞}

33. $\lim\limits_{x\to \frac{\pi}{2}}\tan x+\tan 2x$ {Ans. -∞, +∞}

34. $\lim\limits_{x\to 0}\cos^{-1} x\cdot\cot x+\tan^{-1} x$ {Ans. +∞, -∞}

35. $\lim\limits_{x\to 0}\frac{e^{x}}{x^{2}}$ {Ans. ∞}

36. $\lim\limits_{x\to \infty} x\ln x$ {Ans. ∞}

37. $\lim\limits_{x\to 1}\frac{\sqrt{x+3}}{\sqrt[3]{x+7}}$ {Ans. 1}

38. $\lim\limits_{x\to \infty} x+e^{x}$ {Ans. ∞}

39. $\lim\limits_{x\to 0}\frac{\ln x}{x}$ {Ans. -∞}

40. $\lim\limits_{x\to \frac{\pi}{2}}\frac{1}{\ln(\sin x)}$ {Ans. -∞}

41. $\lim\limits_{x\to \frac{\pi}{2}}\frac{x-\frac{\pi}{2}}{\tan x}$ {Ans. 0}

42. $\lim\limits_{x\to 0} x^{\frac{1}{x}}$ {Ans. 0}

43. $\lim\limits_{x\to \infty} x^{x}$ {Ans. ∞}

44. $\lim\limits_{x\to 0}(\sin x)^{\cot x}$ {Ans. 0}

45. $\lim\limits_{x\to \infty} x^{\ln x}$ {Ans. ∞}

46. $\lim\limits_{x\to -\infty}\frac{x}{e^{x}}$ {Ans. -∞}

47. $\lim\limits_{x\to 0} 2^{-2^{\frac{1}{x}}}$ {Ans. 0, 1}

48. $\lim\limits_{x\to\frac{\pi}{2}} \frac{1}{1+e^{\tan x}}$ {Ans. 1, 0}

49. $\lim\limits_{x\to 0} \frac{\ln|x|}{\sin x}$ {Ans. -∞, ∞}

50. $\lim\limits_{x\to 0} \log_{\cos x} \sin x$ {Ans. ∞}

CATEGORY-2.2. LIMIT OF PIECEWISE FUNCTIONS

51. Given

$f(x) = -2x+3, \quad x \le 1$

$\quad = 3x-5, \quad x > 1$

Find $\lim\limits_{x\to 1} f(x)$. {Ans. –2, 1}

52. Given

$f(x) = x+2, \quad x < 1$

$\quad = 4x-1, \quad 1 \le x < 3$

$\quad = x^2+5, \quad x > 3$.

Find:-

i. $\lim\limits_{x\to 1} f(x)$. {Ans. 3}

ii. $\lim\limits_{x\to 3} f(x)$. {Ans. 14, 11}

53. A function is defined as

$f(x) = 1, \quad x \ne 0$

$\quad = 2, \quad x = 0$

Does the limit $\lim\limits_{x\to 0} f(x)$ exists? {Ans. Yes}

54. Draw the graph of function $f(x) = \frac{|x|}{x}$. Is $f(0)$ defined? Does $\lim\limits_{x\to 0} f(x)$ exist? {Ans. No, No}

55. Given

$f(x) = x, \quad x < 0$

$\quad = 1, \quad x = 0$

$\quad = x^2, \quad x > 0$

Does $\lim\limits_{x\to 0} f(x)$ exist? {Ans. Yes}

56. Evaluate the limit of the function

$f(x) = \frac{|x-4|}{x-4}, \quad x \ne 4$

$\quad = 0, \quad x = 4$

at $x = 4$. Whether the limit exists or not. {Ans. 1, -1, does not exist}

57. Evaluate the limit of the function

$f(x) = 1 + x^2, \quad 0 \le x \le 1$

$\quad = 2 - x, \quad x > 1$

at $x = 1$. Whether the limit exists or not. {Ans. 1, 2, does not exist}

58. If

$$f(x) = \frac{x - |x|}{x}, \quad x \neq 0$$
$$= 2, \quad x = 0,$$

show that $\lim_{x \to 0} f(x)$ does not exist.

59. If

$$f(x) = 5x - 4, \quad 0 < x \leq 1$$
$$= 4x^3 - 3x, \quad 1 < x < 2,$$

show that $\lim_{x \to 1} f(x)$ exists.

60. Let

$$f(x) = \cos x, \quad x \geq 0$$
$$= x + k, \quad x < 0.$$

Find the value of constant *k*, given that $\lim_{x \to 0} f(x)$ exists. {Ans. $k = 1$}

61. Given

$$f(x) = x, \quad 0 \leq x < 1$$
$$= 4 - 2x, \quad 1 \leq x \leq 2$$
$$g(x) = x + 1, \quad 0 \leq x \leq 1$$
$$= 2 - x, \quad 1 < x \leq 2.$$

Find:-

i. $\lim_{x \to 0} f(x) + g(x)$; {Ans. 1}
ii. $\lim_{x \to 1} f(x) + g(x)$; {Ans. 3}
iii. $\lim_{x \to 2} f(x) + g(x)$; {Ans. 0}
iv. $\lim_{x \to 0} f(g(x))$; {Ans. 2}
v. $\lim_{x \to 1} f(g(x))$; {Ans. 1, 0}
vi. $\lim_{x \to 2} f(g(x))$; {Ans. 0}
vii. $\lim_{x \to 0} g(f(x))$; {Ans. 1}
viii. $\lim_{x \to 1} g(f(x))$; {Ans. 0, 2}
ix. $\lim_{x \to 2} g(f(x))$; {Ans. 1}
x. $\lim_{x \to 1} f(f(f(x)))$; {Ans. 0, 1}
xi. $\lim_{x \to 1} g(g(g(x)))$. {Ans. 0, 1}

CATEGORY-2.3. LIMIT OF RATIONAL FUNCTIONS AT A FINITE POINT

62. $\lim_{x \to -2} \frac{x^2 + 5x + 6}{x + 2}$ {Ans. 1}

63. $\lim_{x \to 5} \frac{2x^2 - 11x + 5}{4x^2 - 16x - 20}$ {Ans. $\frac{3}{8}$}

64. $\lim\limits_{x\to 3}\dfrac{x^2-6x+9}{2x^2+x-21}$ {Ans. 0}

65. $\lim\limits_{x\to -3}\dfrac{x^3+27}{2x^2-3x-27}$ {Ans. $-\frac{9}{5}$}

66. $\lim\limits_{x\to \frac{1}{2}}\dfrac{8x^3-1}{6x^2-5x+1}$ {Ans. 6}

67. $\lim\limits_{x\to 2}\dfrac{x^3-6x^2+11x-6}{x^2-6x+8}$. {Ans. $\frac{1}{2}$}

68. $\lim\limits_{x\to 2}\dfrac{x^3-5x^2+8x-4}{x^3-3x^2+4}$ {Ans. $\frac{1}{3}$}

69. $\lim\limits_{x\to 3}\dfrac{1}{x-3}-\dfrac{6}{x^2-9}$ {Ans. $\frac{1}{6}$}

70. $\lim\limits_{x\to 1}\dfrac{x^3+x^2-x-1}{x^3-x^2-x+1}$ {Ans. $+\infty, -\infty$}

71. $\lim\limits_{x\to 4}\dfrac{2x^2-4x-24}{x^2-16}-\dfrac{1}{4-x}$ {Ans. $\frac{13}{8}$}

72. $\lim\limits_{x\to 2}\dfrac{x^3+3x^2-9x-2}{x^3-x-6}$ {Ans. $\frac{15}{11}$}

73. $\lim\limits_{x\to 1}\dfrac{x^3-3x+2}{x^4-4x+3}$ {Ans. $\frac{1}{2}$}

74. $\lim\limits_{x\to 1}\dfrac{1}{1-x}-\dfrac{3}{1-x^3}$ {Ans. -1}

75. $\lim\limits_{x\to -2}\dfrac{x^5+2x^4+x^2+3x+2}{x^4+2x^3+3x^2-5x-22}$ {Ans. $-\frac{3}{5}$}

76. $\lim\limits_{x\to 2}\dfrac{1}{x(x-2)^2}-\dfrac{1}{x^2-3x+2}$ {Ans. ∞}

77. $\lim\limits_{x\to -2}\dfrac{2}{x+2}+\dfrac{1}{x^2-2x+4}-\dfrac{24}{x^3+8}$ {Ans. $-\frac{11}{12}$}

CATEGORY-2.4. LIMIT OF IRRATIONAL FUNCTIONS AT A FINITE POINT

78. $\lim\limits_{x\to 1}\dfrac{1-\sqrt{x}}{1-x^2}$ {Ans. $\frac{1}{4}$}

79. $\lim\limits_{x\to 4}\dfrac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$ {Ans. $-\frac{1}{3}$}

80. $\lim\limits_{x\to 1}\dfrac{\sqrt{x+8}-\sqrt{8x+1}}{\sqrt{5-x}-\sqrt{7x-3}}$ {Ans. $\frac{7}{12}$}

81. $\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}-\sqrt{4-x}}{4x}$ {Ans. $\frac{1}{8}$}

82. $\lim\limits_{x\to 7}\dfrac{2-\sqrt{x-3}}{x^2-49}$ {Ans. $-\frac{1}{56}$}

83. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$ {Ans. 1}

84. $\lim\limits_{x\to 3}\dfrac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$ {Ans. $-\frac{1}{3}$}

85. $\lim\limits_{x\to -1}\dfrac{x+1}{2-\sqrt{4+x+x^2}}$ {Ans. 4}

86. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$ {Ans. 4}

87. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+9}-3}$. {Ans. 3}

88. $\lim\limits_{x\to 2}\dfrac{\sqrt[3]{10-x}-2}{x-2}$ {Ans. $-\frac{1}{12}$}

89. $\lim\limits_{x\to 1}\dfrac{x-1}{\sqrt[3]{26+x}-3}$ {Ans. 27}

90. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x^2}-1}{x^2}$ {Ans. $\frac{1}{3}$}

91. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$ {Ans. $\frac{2}{3}$}

CATEGORY-2.5. LIMITS INVOLVING ∞

92. $\lim\limits_{x\to \infty}\dfrac{2x^2-4x+1}{4x^2+6x-5}$ {Ans. $\frac{1}{2}$}

93. $\lim\limits_{x\to \infty}\dfrac{(x-1)^3}{x^3+1}$ {Ans. 1}

94. $\lim\limits_{x\to \infty}\dfrac{x^3}{3x^2-4}-\dfrac{x^2}{3x+2}$ {Ans. $\frac{2}{9}$}

95. $\lim\limits_{x\to \infty}\dfrac{x^3+x}{x^4-3x^2+1}$ {Ans. 0}

96. $\lim\limits_{x\to \infty}\dfrac{3x^2+2}{x^4+1}$ {Ans. 0}

97. $\lim\limits_{x\to \infty}\dfrac{x^3}{x^2+1}-x$ {Ans. 0}

98. $\lim\limits_{x\to \infty}\dfrac{x^4-5x}{3x-x^2+1}$ {Ans. -∞}

99. $\lim\limits_{x\to \infty}\dfrac{\sqrt{x^2+2}}{x-2}$ {Ans. 1}

100. $\lim\limits_{x\to \infty}\dfrac{\sqrt{1+x^2}-\sqrt{4+x^2}}{x}$ {Ans. 0}

101. $\lim\limits_{x\to\infty} \dfrac{\sqrt{1+x^4}-1-x^2}{x^2}$. {Ans. 0}

102. $\lim\limits_{x\to\infty} x^{\frac{3}{2}}\left(\sqrt{x^3+1}-\sqrt{x^3-1}\right)$ {Ans. 1}

103. $\lim\limits_{x\to\infty} \sqrt{x^2+1}-\sqrt{x^2-1}$ {Ans. 0}

104. $\lim\limits_{x\to\infty} \dfrac{\sqrt{1+9x^2}+\sqrt{x^2-1}}{\sqrt{1+9x^2}-\sqrt{x^2-1}}$ {Ans. $\frac{4}{3}$}

105. $\lim\limits_{x\to 0} \dfrac{\ln x+1}{2-\ln x}$ {Ans. –1}

106. $\lim\limits_{x\to\infty} \dfrac{e^x-1}{e^x+1}$ {Ans. 1}

107. $\lim\limits_{x\to\infty} \dfrac{\ln^2 x-\ln x+1}{\ln x+3}$ {Ans. ∞}

108. $\lim\limits_{x\to\infty} \dfrac{e^{2x}-e^x+1}{e^{3x}-e^{2x}+2e^x+3}$ {Ans. 0}

CATEGORY-2.6. SUBSTITUTION

109. $\lim\limits_{x\to 1} \dfrac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$ {Ans. $\frac{1}{9}$}

110. $\lim\limits_{x\to 1} \dfrac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$ {Ans. $\frac{4}{3}$}

111. $\lim\limits_{x\to -1} \dfrac{1+\sqrt[3]{x}}{1+\sqrt[5]{x}}$ {Ans. $\frac{5}{3}$}

112. $\lim\limits_{x\to -1} \dfrac{x+1}{\sqrt[4]{x+17}-2}$ {Ans. 32}

113. $\lim\limits_{x\to \frac{\pi}{6}} \dfrac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$ {Ans. -3}

CATEGORY-2.7. USE OF STANDARD LIMITS $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$ *AND* $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$

114. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x}$ {Ans. 3}

115. $\lim\limits_{x\to 0} \dfrac{\tan kx}{x}$ {Ans. k}

116. $\lim\limits_{x\to 0} \dfrac{\sin\alpha x}{\sin\beta x}$ {Ans. $\frac{\alpha}{\beta}$}

117. $\lim\limits_{x\to 0} \dfrac{\tan 2x}{\sin 5x}$ {Ans. $\frac{2}{5}$}

118. $\lim_{x\to 1}\frac{\sin(\ln x)}{\ln x}$ {Ans. 1}

119. $\lim_{x\to -\infty}\frac{\tan(e^x)}{e^x}$ {Ans. 1}

120. $\lim_{x\to 0}\frac{\sin(x^2)}{x}$ {Ans. 0}

121. $\lim_{x\to 0}\frac{\sin^2 2x}{x^2}$ {Ans. 4}

122. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ {Ans. $\frac{1}{2}$}

123. $\lim_{x\to 1}\frac{\cos\left(\frac{\pi x}{2}\right)}{1-x}$ {Ans. $\frac{\pi}{2}$}

124. $\lim_{x\to \frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\tan x$ {Ans. 1}

125. $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$ {Ans. $\frac{2}{\pi}$}

CATEGORY-2.8. USE OF STANDARD LIMITS $\lim_{x\to 0}\frac{\sin^{-1} x}{x}=1$ *AND* $\lim_{x\to 0}\frac{\tan^{-1} x}{x}=1$

126. $\lim_{x\to 0}\frac{\sin^{-1} 2x}{3x}$ {Ans. $\frac{2}{3}$}

127. $\lim_{x\to 0}\frac{x}{\tan^{-1} 2x}$. {Ans. $\frac{1}{2}$}

128. $\lim_{x\to 0}\frac{\sin^{-1}(x^2)}{(\tan^{-1} x)^2}$ {Ans. 1}

129. $\lim_{x\to -3}\frac{x^2-9}{\tan^{-1}(x+3)}$ {Ans. -6}

130. $\lim_{x\to 0}\frac{2x-\sin^{-1} x}{2x+\tan^{-1} x}$ {Ans. $\frac{1}{3}$}

CATEGORY-2.9. USE OF STANDARD LIMIT $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$

131. $\lim_{x\to 1}\frac{x^{10}-1}{x-1}$ {Ans. 10}

132. $\lim_{x\to -1}\frac{x^{13}+1}{x^{17}+1}$ {Ans. $\frac{13}{17}$}

133. $\lim_{x\to 1}\frac{x^m-1}{x^n-1}$ {Ans. $\frac{m}{n}$}

CATEGORY-2.10. USE OF STANDARD LIMITS $\lim_{x\to 0}(1+kx)^{\frac{1}{x}}=e^k$ *AND* $\lim_{x\to\pm\infty}\left(1+\frac{k}{x}\right)^x=e^k$

134. $\lim_{x\to 0}(1-2x)^{\frac{1}{x}}$ {Ans. e^{-2}}

135. $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$. {Ans. e^2}

136. $\lim_{x\to 0}(1+x^2)^{\frac{1}{x}}$ {Ans. 1}

137. $\lim_{x\to 0}(1+x)^{\frac{1}{3x}}$ {Ans. $e^{\frac{1}{3}}$}

138. $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x$ {Ans. $\frac{1}{e}$}

139. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{7x}$ {Ans. e^7}

140. $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{mx}$ {Ans. e^{mk}}

141. $\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)^x$ {Ans. 1}

142. $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^x$ {Ans. e^{-1}}

143. $\lim_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{x+3}$. {Ans. e^4}

144. $\lim_{x\to 0}(1+\sin x)^{\operatorname{cosec} x}$ {Ans. e}

145. $\lim_{x\to 0}(1+3\tan^2 x)^{\cot^2 x}$ {Ans. e^3}

146. $\lim_{x\to 0}\frac{\ln(1+kx)}{x}$ {Ans. k}

147. $\lim_{x\to 0}\frac{1}{\sin x}\ln(1+a\sin x)$ {Ans. a}

148. $\lim_{x\to e}\frac{\ln x-1}{x-e}$ {Ans. $\frac{1}{e}$}

CATEGORY-2.11. USE OF STANDARD LIMIT $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$

149. $\lim_{x\to 0}\frac{e^{-x}-1}{x}$ {Ans. -1}

150. $\lim_{x\to 0}\frac{e^{2x}-1}{3x}$ {Ans. $\frac{2}{3}$}

151. $\lim\limits_{x\to 0}\dfrac{a^{2x}-1}{x}$ {Ans. $2\ln a$ }

152. $\lim\limits_{x\to 0}\dfrac{2^{x^2}-1}{x}$ {Ans. 0}

153. $\lim\limits_{x\to \infty} x(e^{\frac{1}{x}}-1)$ {Ans. 1}

154. $\lim\limits_{x\to \infty}(a^{\frac{1}{x}}-1)x\ (a>0)$ {Ans. ln$a$}

155. $\lim\limits_{x\to 0}\dfrac{e^{ax}-e^{bx}}{x}$ {Ans. a - b}

156. $\lim\limits_{x\to 0}\dfrac{e^{x}-1}{\sin x}$ {Ans. 1}

157. $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{3^{x}-1}$ {Ans. $\frac{1}{\ln 3}$ }

158. $\lim\limits_{x\to 0}\dfrac{e^{4x}-1}{\tan x}$ {Ans. 4}

159. $\lim\limits_{x\to 1}\dfrac{e^{x}-e}{x-1}$ {Ans. e }

160. $\lim\limits_{x\to 0}\dfrac{10^{x}-1}{x^2}$ {Ans. +∞, -∞}

CATEGORY-2.12. USE OF SERIES

161. $\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^3}$ {Ans. $-\frac{1}{6}$ }

162. $\lim\limits_{x\to 0}\dfrac{e^{x}-1-x}{x^2}$ {Ans. $\dfrac{1}{2}$ }

163. $\lim\limits_{x\to 0}\dfrac{\sin x-x+\dfrac{x^3}{6}}{x^5}$. {Ans. $\dfrac{1}{120}$ }

164. $\lim\limits_{x\to 0}\dfrac{e^{x}+e^{-x}-2}{x^2}$ {Ans. 1}

165. $\lim\limits_{x\to 0}\dfrac{e^{x}-e^{-x}}{\sin x}$ {Ans. 2}

166. $\lim\limits_{x\to 0}\dfrac{e^{x}-e^{-x}+2\sin x-4x}{x^5}$ {Ans. $\frac{1}{30}$ }

167. $\lim\limits_{x\to 0}\dfrac{x\cos x-\sin x}{x^3}$ {Ans. $-\frac{1}{3}$ }

168. $\lim\limits_{x\to 0}\dfrac{e^{x}-\frac{x^3}{6}-\frac{x^2}{2}-x-1}{\cos x+\frac{x^2}{2}-1}$ {Ans. 1}

169. $\lim\limits_{x\to 0}\dfrac{\ln(1+x)^4-4x+2x^2-\frac{4}{3}x^3+x^4}{6\sin x-6x+x^3}$ {Ans. 16}

170. $\lim\limits_{x\to 0}\dfrac{(1+2x)^7-(1+3x)^5}{x}$ {Ans. -1}

171. $\lim\limits_{x\to 0}\dfrac{(1+x)^8-(1+2x)^4}{x^2}$ {Ans. 4}

172. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x}-\sqrt[4]{1+2x}}{x}$ {Ans. $-\frac{1}{6}$}

173. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+4x}-\sqrt[3]{1+6x}}{x^2}$ {Ans. 2}

CATEGORY-2.13. APPLICATION OF METHODS

174. $\lim\limits_{x\to a}\dfrac{\sqrt[3]{x}-\sqrt[3]{a}}{\sqrt{x}-\sqrt{a}}$ {Ans. $\frac{2}{3\sqrt[6]{a}}$}

175. $\lim\limits_{x\to 2}\dfrac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}}$ {Ans. $\frac{34}{23}$}

176. $\lim\limits_{x\to 3}\log_a\dfrac{x-3}{\sqrt{x+6}-3}$ {Ans. $\log_a 6$}

177. $\lim\limits_{x\to \pm\infty}\sqrt{9x^2+1}-3x$ {Ans. 0, $+\infty$}

178. $\lim\limits_{x\to +\infty}\dfrac{2\sqrt{x}+3\sqrt[3]{x}+5\sqrt[5]{x}}{\sqrt{3x-2}+\sqrt[3]{2x-3}}$ {Ans. $\frac{2}{\sqrt{3}}$}

179. $\lim\limits_{x\to -\infty}\sqrt{2x^2-3}-5x$ {Ans. $+\infty$}

180. $\lim\limits_{x\to \pm\infty}x\left(\sqrt{x^2+1}-x\right)$ {Ans. $\frac{1}{2}, -\infty$}

181. $\lim\limits_{x\to \pm\infty}\dfrac{\sqrt{2x^2+3}}{4x+2}$ {Ans. $\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}$}

182. $\lim\limits_{x\to \infty}5^{\frac{2x}{x+3}}$ {Ans. 25}

183. $\lim\limits_{x\to 0}\dfrac{\sqrt[k]{1+x}-1}{x}$ (k is positive integer) {Ans. $\frac{1}{k}$}

184. $\lim\limits_{x\to \frac{\pi}{6}}\dfrac{\sin(x-\frac{\pi}{6})}{\sqrt{3}-2\cos x}$ {Ans. 1}

185. $\lim\limits_{x\to 0}\dfrac{\ln(a+x)-\ln a}{x}$ {Ans. $\frac{1}{a}$}

186. $\lim\limits_{x\to \infty}\dfrac{ax^2+bx+c}{Ax^2+Bx+C}$. {Ans. $\dfrac{a}{A}$}

187. $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$. {Ans. $\dfrac{2}{3\sqrt{3}}$}

188. $\lim\limits_{x\to \infty}\dfrac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$. {Ans. $\dfrac{\sqrt{3}-\sqrt{2}}{4}$}

189. $\lim\limits_{x\to\infty}\sqrt{x^2+x+1}-\sqrt{x^2+1}$. {Ans. $\frac{1}{2}$}

190. $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)}$. {Ans. 1}

191. $\lim\limits_{x\to a}\dfrac{\cos\sqrt{x}-\cos\sqrt{a}}{x-a}$. {Ans. $-\dfrac{\sin\sqrt{a}}{2\sqrt{a}}$}

192. $\lim\limits_{x\to 0}\dfrac{\ln(5+x)-\ln(5-x)}{x}$. {Ans. $\frac{2}{5}$}

193. $\lim\limits_{x\to 0}(1+\sin x)^{2\cot x}$. {Ans. e^2}

194. $\lim\limits_{x\to 1}\left(\dfrac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}$ {Ans. $\sqrt{\frac{2}{3}}$}

195. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+4x-3}{x^2-2x+5}\right)^x$. {Ans. e^6}

196. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{x-1}}$ {Ans. $\frac{1}{4}$}

197. $\lim\limits_{x\to\infty}\left(\dfrac{2x^2+3}{2x^2+5}\right)^{8x^2+3}$ {Ans. e^{-8}}

198. $\lim\limits_{x\to 0}\left(\dfrac{1+5x^2}{1+3x^2}\right)^{\frac{1}{x^2}}$. {Ans. e^2}

199. $\lim\limits_{x\to 0}\dfrac{x(5^x-1)}{1-\cos x}$. {Ans. $2\ln 5$}

200. $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)\sin 5x}{x^2\sin 3x}$. {Ans. $\frac{10}{3}$}

201. $\lim\limits_{x\to 0}\dfrac{\sin 4x}{1-\sqrt{1-x}}$. {Ans. 8}

202. $\lim\limits_{x\to\infty}\left(\dfrac{x-1}{x+1}\right)^x$. {Ans. e^{-2}}

203. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+6}{x^2-6}\right)^x$. {Ans. 1}

204. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+2x+3}{2x^2+x+5}\right)^{\frac{3x-2}{3x+2}}$. {Ans. $\frac{1}{2}$}

205. $\lim\limits_{x\to 0}\dfrac{\sin x}{\sqrt{x^2}}$. {Ans. 1, -1}

206. $\lim_{x\to 0}(1+ax)^{\frac{b}{x}}$. {Ans. e^{ab}}

207. $\lim_{x\to 3}\dfrac{x-3}{|x-3|}$. {Ans. 1, -1}

208. $\lim_{x\to 0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{\frac{1}{\sin x}}$ {Ans. 1}

209. $\lim_{x\to 1}(1+\sin\pi x)^{\cot\pi x}$ {Ans. $\frac{1}{e}$}

210. $\lim_{x\to 0}\dfrac{\cos x+4\tan x}{2-x-2x^4}$ {Ans. $\frac{1}{2}$}

211. $\lim_{x\to\infty}\dfrac{1-2x}{\sqrt[3]{1+8x^3}}+2^{-x^2}$ {Ans. -1}

212. $\lim_{x\to 0}\dfrac{e^{ax}-e^{-2ax}}{\ln(1+x)}$ {Ans. $3a$}

213. $\lim_{x\to -1}\dfrac{\sqrt[3]{1+2x}+1}{\sqrt{2+x}+x}$ {Ans. $\frac{4}{9}$}

214. $\lim_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$ {Ans. 2}

215. $\lim_{x\to 0}\dfrac{\sqrt{x+4}-2}{\sin 5x}$ {Ans. $\frac{1}{20}$}

216. $\lim_{x\to\infty}\left(\dfrac{x^2+5x+3}{x^2+x+2}\right)^{x}$. {Ans. e^4}

217. $\lim_{x\to 1}\dfrac{\sin(1-x)}{\sqrt{x}-1}$ {Ans. -2}

218. $\lim_{\alpha\to\pi}\dfrac{\sin\alpha}{1-\frac{\alpha^2}{\pi^2}}$ {Ans. $\frac{\pi}{2}$}

219. $\lim_{x\to\frac{\pi}{4}}\tan 2x\cdot\tan(\frac{\pi}{4}-x)$ {Ans. $\frac{1}{2}$}

220. $\lim_{x\to\frac{\pi}{3}}\dfrac{\tan^3 x-3\tan x}{\cos(x+\frac{\pi}{6})}$ {Ans. -24}

221. $\lim_{x\to\infty}\left(\dfrac{x+4}{x}\right)^{x+3}$ {Ans. e^4}

222. $\lim_{x\to\infty}\left(\dfrac{2x-1}{2x+1}\right)^{x}$ {Ans. $\frac{1}{e}$}

223. $\lim_{x\to\infty}\left(\dfrac{3x^2+2x+1}{x^2+x+2}\right)^{\frac{6x+1}{3x+2}}$ {Ans. 9}

224. $\lim_{x\to\infty}\left(\dfrac{1+3x}{2+3x}\right)^{\frac{1-\sqrt{x}}{1-x}}$ {Ans. 1}

225. $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\ln(\tan x)}{1-\cot x}$ {Ans. 1}

226. $\lim\limits_{x\to\sqrt{3}} \dfrac{x^2-3}{x^4+x^2+1}$ {Ans. 0}

227. $\lim\limits_{x\to 1} \dfrac{x^3+x-2}{x^3-x^2-x+1}$ {Ans. $+\infty$, $-\infty$}

228. $\lim\limits_{x\to 1} \dfrac{x+2}{x^2-5x+4}+\dfrac{x-4}{3(x^2-3x+2)}$ {Ans. 0}

229. $\lim\limits_{x\to\infty} \dfrac{x^3}{2x^2-1}-\dfrac{x^2}{2x+1}$ {Ans. $\frac{1}{4}$}

230. $\lim\limits_{x\to\infty} \dfrac{3x^2}{2x+1}-\dfrac{(2x-1)(3x^2+x+2)}{4x^2}$ {Ans. $-\frac{1}{2}$}

231. $\lim\limits_{x\to\infty} \dfrac{(x+1)^{10}+(x+2)^{10}+\ldots+(x+100)^{10}}{x^{10}+10^{10}}$ {Ans. 100}

232. $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x^2+1}+\sqrt{x}}{\sqrt[4]{x^3+x}-x}$ {Ans. -1}

233. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$ {Ans. 1}

234. $\lim\limits_{x\to+\infty} \dfrac{\sqrt[5]{x^7+3}+\sqrt[4]{2x^3-1}}{\sqrt[6]{x^8+x^7+1}-x}$ {Ans. ∞}

235. $\lim\limits_{x\to\infty} \dfrac{\sqrt[3]{x^4+3}-\sqrt[5]{x^3+4}}{\sqrt[3]{x^7+1}}$ {Ans. 0}

236. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1}{x^2}$ {Ans. $+\infty$, $-\infty$}

237. $\lim\limits_{x\to a} \dfrac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}\ (a>b)$ {Ans. $\frac{1}{4a\sqrt{a-b}}$}

238. $\lim\limits_{x\to 1} \dfrac{\sqrt[n]{x}-1}{\sqrt[m]{x}-1}$ {Ans. $\frac{m}{n}$}

239. $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+x^2}-\sqrt[4]{1-2x}}{x+x^2}$ {Ans. $\frac{1}{2}$}

240. $\lim\limits_{x\to\infty} \sqrt{x+a}-\sqrt{x}$ {Ans. 0}

241. $\lim\limits_{x\to\pm\infty} \sqrt{x^2+1}-x$ {Ans. 0, $+\infty$}

242. $\lim\limits_{x\to\pm\infty} \sqrt{(x+a)(x+b)}-x$ {Ans. $\frac{a+b}{2},+\infty$}

243. $\lim\limits_{x\to\pm\infty} \sqrt{x^2-2x-1}-\sqrt{x^2-7x+3}$ {Ans. $\frac{5}{2},-\frac{5}{2}$}

244. $\lim\limits_{x\to\infty} \sqrt[3]{(x+1)^2}-\sqrt[3]{(x-1)^2}$ {Ans. 0}

245. $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{x\sin 2x}$ {Ans. $\frac{3}{4}$ }

246. $\lim\limits_{\alpha\to 0}\dfrac{\tan\alpha}{\sqrt[3]{(1-\cos\alpha)^2}}$ {Ans. +∞, -∞}

247. $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x}{1-\sin x-\cos x}$ {Ans. -1}

248. $\lim\limits_{\alpha\to 0}\dfrac{\tan\alpha-\sin\alpha}{\alpha^3}$ {Ans. $\frac{1}{2}$ }

249. $\lim\limits_{\alpha\to 0}\dfrac{(1-\cos\alpha)^2}{\tan^3\alpha-\sin^3\alpha}$ {Ans. +∞, -∞}

250. $\lim\limits_{x\to 0}\dfrac{1}{\sin x}-\dfrac{1}{\tan x}$ {Ans. 0}

251. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-\sin x}{\left(\dfrac{\pi}{2}-x\right)^2}$ {Ans. $\frac{1}{2}$ }

252. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos x}{\sqrt[3]{(1-\sin x)^2}}$ {Ans. -∞, +∞}

253. $\lim\limits_{x\to\pi}\dfrac{\sin 3x}{\sin 2x}$ {Ans. $-\frac{3}{2}$ }

254. $\lim\limits_{y\to a}\sin\dfrac{y-a}{2}\cdot\tan\dfrac{\pi y}{2a}$ {Ans. $-\frac{a}{\pi}$ }

255. $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\cos x-\sin x}{\cos 2x}$ {Ans. $\frac{1}{\sqrt{2}}$ }

256. $\lim\limits_{x\to\pi}\dfrac{1-\sin\dfrac{x}{2}}{\cos\dfrac{x}{2}\cdot\left(\cos\dfrac{x}{4}-\sin\dfrac{x}{4}\right)}$ {Ans. $\frac{1}{\sqrt{2}}$ }

257. $\lim\limits_{x\to\frac{\pi}{2}}2x\tan x-\dfrac{\pi}{\cos x}$ {Ans. -2}

258. $\lim\limits_{x\to 0}\dfrac{\cos(a+x)-\cos(a-x)}{x}$ {Ans. $-2\sin a$ }

259. $\lim\limits_{x\to 0}\dfrac{\cos\alpha x-\cos\beta x}{x^2}$ {Ans. $\frac{\beta^2-\alpha^2}{2}$ }

260. $\lim\limits_{x\to 0}\dfrac{\sin(a+x)-\sin(a-x)}{\tan(a+x)-\tan(a-x)}$ {Ans. $\cos^3 a$ }

261. $\lim\limits_{\alpha\to\beta}\dfrac{\sin^2\alpha-\sin^2\beta}{\alpha^2-\beta^2}$ {Ans. $\frac{\sin 2\beta}{2\beta}$ }

262. $\lim\limits_{h\to 0}\dfrac{\sin(a+2h)-2\sin(a+h)+\sin a}{h^2}$ {Ans. $-\sin a$ }

263. $\lim\limits_{h\to 0}\dfrac{\tan(a+2h)-2\tan(a+h)+\tan a}{h^2}$ {Ans. $\frac{2\sin a}{\cos^3 a}$ }

264. $\lim\limits_{x\to 0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$ {Ans. $\frac{1}{4\sqrt{2}}$}

265. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}$ {Ans. 1}

266. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x\sin x}-\sqrt{\cos 2x}}{\tan^2\frac{x}{2}}$ {Ans. 6}

267. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{\cos 2x}\cdot\cos x}{x^2}$ {Ans. $\frac{3}{2}$}

268. $\lim\limits_{x\to a}\dfrac{x^m-a^m}{x^n-a^n}$ {Ans. $\frac{m}{n}a^{m-n}$}

269. $\lim\limits_{x\to 0}\dfrac{e^{x^2}-1}{\cos x-1}$ {Ans. -2}

270. $\lim\limits_{x\to e}(\ln x)^{\frac{1}{x-e}}$. {Ans. $e^{\frac{1}{e}}$}

271. $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}}{\sin x\cos x}$ {Ans. 2}

272. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{\frac{x+1}{x}}$ {Ans. 1}

273. $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x-2}\right)^{2x-1}$ {Ans. e^6}

274. $\lim\limits_{x\to\infty}\left(\dfrac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}$ {Ans. $e^{-\frac{2}{3}}$}

275. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x^2-1}\right)^{x^2}$ {Ans. e^2}

276. $\lim\limits_{x\to\pm\infty}\left(\dfrac{x+1}{2x-1}\right)^{x}$ {Ans. 0, $+\infty$}

277. $\lim\limits_{x\to\pm\infty}\left(\dfrac{2x+1}{x-1}\right)^{x}$ {Ans. $+\infty$, 0}

278. $\lim\limits_{x\to\pm\infty}\left(1+\dfrac{1}{x}\right)^{x^2}$ {Ans. $+\infty$, 0}

279. $\lim\limits_{x\to\infty}\left(\dfrac{x^2-2x+1}{x^2-4x+2}\right)^{x}$ {Ans. e^2}

280. $\lim\limits_{x\to 0}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ {Ans. $\sqrt{e}$}

281. $\lim\limits_{x\to\infty} x(\ln(x+a)-\ln x)$ {Ans. a}

282. $\lim\limits_{x\to 0}\dfrac{e^{x^2}-\cos x}{x^2}$ {Ans. $\frac{3}{2}$}

283. $\lim\limits_{x\to 0}\dfrac{e^{\sin 2x}-e^{\sin x}}{x}$ {Ans. 1}

284. $\lim\limits_{x\to 0}\dfrac{\cos x-e^{-\frac{x^2}{2}}}{x^4}$ {Ans. $-\frac{1}{12}$}

285. $\lim\limits_{x\to\pm\infty}\cosh x-\sinh x$ {Ans. 0, +∞}

286. $\lim\limits_{x\to\pm\infty}\tanh x$ {Ans. 1, -1}

287. $\lim\limits_{x\to\pm\infty}x\left(\sqrt{x^2+\sqrt{x^4+1}}-x\sqrt{2}\right)$ {Ans. 0, -∞}

288. $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}$ {Ans. 0}

289. $\lim\limits_{x\to\infty}\dfrac{\tan^{-1}x}{x}$ {Ans. 0}

290. $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x+\cos x}$ {Ans. 1}

291. $\lim\limits_{x\to 1}\dfrac{\sin^{-1}x}{\tan\frac{\pi x}{2}}$ {Ans. 0}

292. $\lim\limits_{x\to\frac{\pi}{2}}\tan^2 x\cdot\left(\sqrt{2\sin^2 x+3\sin x+4}-\sqrt{\sin^2 x+6\sin x+2}\right)$ {Ans. $\frac{1}{12}$}

293. $\lim\limits_{x\to 1}\dfrac{x^x-x}{1-x+\ln x}$ {Ans. −2}

294. $\lim\limits_{x\to 0}\dfrac{1-\cos(1-\cos x)}{x^4}$ {Ans. $\frac{1}{8}$}

295. $\lim\limits_{x\to\infty}x^2\left(1-\cos\dfrac{1}{x}\right)$ {Ans. $\frac{1}{2}$}

296. $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin x}}$ {Ans. 1}

297. $\lim\limits_{x\to 0}\dfrac{x-\sin^{-1}x}{x^3}$ {Ans. $-\dfrac{1}{6}$}

298. $\lim\limits_{x\to 0}\dfrac{\ln\cos x}{x^2}$ {Ans. $-\frac{1}{2}$}

299. $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$ {Ans. $\frac{1}{e}$}

300. $\lim\limits_{x\to 0}(\cos x+\sin x)^{\frac{1}{x}}$ {Ans. e}

301. $\lim\limits_{x\to 0}(\cos x+a\sin bx)^{\frac{1}{x}}$ {Ans. e^{ab}}

302. $\lim\limits_{x\to\infty}\cos\sqrt{x+1}-\cos\sqrt{x}$ {Ans. 0}

303. $\lim\limits_{x\to 0}\dfrac{a^x-1-x\ln a}{x^2}$ {Ans. $\dfrac{\ln^2 a}{2}$}

304. $\lim\limits_{x\to\infty}\dfrac{3^x+4^x}{5^x+6^x}$ {Ans. 0}

305. $\lim\limits_{x\to 0}\dfrac{\cos^{-1}(1-x)}{\sqrt{x}}$ {Ans. $\sqrt{2}$ }

306. $\lim\limits_{x\to -1}\dfrac{\sqrt{\pi}-\sqrt{\cos^{-1}x}}{\sqrt{x+1}}$ {Ans. $\frac{1}{\sqrt{2\pi}}$ }

307. $\lim\limits_{x\to\infty}\dfrac{e^{\frac{1}{x^2}}-1}{2\tan^{-1}x^2-\pi}$ {Ans. $-\frac{1}{2}$ }

308. $\lim\limits_{x\to 0}\dfrac{\sqrt{1-\cos 2x}}{x}$ {Ans. $\sqrt{2},-\sqrt{2}$ }

309. $\lim\limits_{x\to 0}\dfrac{1}{2-2^{\frac{1}{x}}}$ {Ans. 0, $\frac{1}{2}$ }

310. $\lim\limits_{x\to 1}3+\dfrac{1}{1+7^{\frac{1}{1-x}}}$ {Ans. 4, 3}

311. $\lim\limits_{x\to 0}\dfrac{|x|}{x^2}$ {Ans. ∞}

312. If

$$f(x)=x\sin\frac{1}{x},\quad x\neq 0$$
$$=1,\quad x=0$$

then find $\lim\limits_{x\to 0}f(x)$. {Ans. 0}

313. $\lim\limits_{x\to 2}\dfrac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}$. {Ans. $\dfrac{1}{8\sqrt{3}}$ }

314. $\lim\limits_{x\to 0}\dfrac{\sqrt{1-\cos x^2}}{1-\cos x}$. {Ans. $\sqrt{2}$ }

315. $\lim\limits_{x\to 0}\dfrac{\tan x-x}{x^3}$. {Ans. $\dfrac{1}{3}$ }

316. $\lim\limits_{x\to\infty}\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}$. {Ans. $\dfrac{1}{2}$ }

317. $\lim\limits_{x\to 0}\dfrac{3x+|x|}{7x-5|x|}$. {Ans. 2, $\dfrac{1}{6}$ }

318. $\lim\limits_{x\to -\infty}\dfrac{\left(3x^4+2x^2\right)\sin\frac{1}{x}+|x|^3+5}{|x|^3+|x|^2+|x|+1}$. {Ans. -2 }

319. $\lim\limits_{x\to a}\dfrac{a\sin x-x\sin a}{x^2-ax}$ $(a\neq 0)$. {Ans. $\cos a-\dfrac{\sin a}{a}$ }

CATEGORY-2.14. L' HOSPITAL'S RULE

320. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}$ {Ans. 0}

321. $\lim\limits_{x\to 0} x\ln x$ {Ans. 0}

322. $\lim\limits_{x\to 0} x^x$ {Ans. 1}

323. $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$ {Ans. 1}

324. Show that the following limits cannot be evaluated by L' Hospital's rule:-

i. $\lim\limits_{x\to 0}\dfrac{x^2\sin\frac{1}{x}}{\sin x}$ {Ans. 0}

ii. $\lim\limits_{x\to\infty}\dfrac{2+2x+\sin 2x}{(2x+\sin 2x)e^{\sin x}}$ {Ans. no limit}

iii. $\lim\limits_{x\to\infty}\dfrac{x-\sin x}{x+\sin x}$ {Ans. 1}

iv. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\tan x}{\sec x}$ {Ans. 1}

CATEGORY-2.15. APPLICATION OF METHODS AND L' HOSPITAL'S RULE

325. $\lim\limits_{x\to 0}\dfrac{\ln(1+x^2)}{\cos 3x-e^{-x}}$ {Ans. 0}

326. $\lim\limits_{x\to 0}\dfrac{\sin 3x^2}{\ln\cos(2x^2-x)}$ {Ans. -6}

327. $\lim\limits_{x\to+\infty}\dfrac{\log_a x}{x^k}$ $(k>0)$ {Ans. 0}

328. $\lim\limits_{x\to 1}\dfrac{1}{\ln x}-\dfrac{1}{x-1}$ {Ans. $\frac{1}{2}$ }

329. $\lim\limits_{x\to 0}\cot x-\dfrac{1}{x}$ {Ans. 0}

330. $\lim\limits_{x\to 0}\dfrac{1}{x}-\dfrac{1}{e^x-1}$ {Ans. $\frac{1}{2}$ }

331. $\lim\limits_{x\to 0} x^n\ln x$ $(n>0)$ {Ans. 0}

332. $\lim\limits_{x\to 0}\{\ln(1+\sin^2 x)\}\{\cot\ln^2(1+x)\}$ {Ans. 1}

333. $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{\sin x}$ {Ans. 1}

334. $\lim\limits_{x\to 0}(\sin x)^{\frac{1}{x}}$ {Ans. 0}

335. $\lim\limits_{x\to 0} x^{\frac{1}{\ln(e^x-1)}}$ {Ans. e}

336. $\lim\limits_{x\to\frac{\pi}{2}}(\tan x)^{\cot x}$ {Ans. 1}

337. $\lim\limits_{x\to 2}\dfrac{\ln(x^2-3)}{x^2+3x-10}$ {Ans. $\frac{4}{7}$ }

338. $\lim\limits_{x\to 1}\dfrac{a^{\ln x}-x}{\ln x}$ {Ans. $\ln a-1$}

339. $\lim\limits_{x\to 1}\dfrac{1-4\sin^2(\frac{\pi x}{6})}{1-x^2}$ {Ans. $\frac{\pi\sqrt{3}}{6}$ }

340. $\lim\limits_{x\to a}\sin^{-1}\left(\dfrac{x-a}{a}\right)\cot(x-a)$ {Ans. $\frac{1}{a}$ }

341. $\lim\limits_{x\to +\infty}(\pi-2\tan^{-1}x)\ln x$ {Ans. 0}

342. $\lim\limits_{x\to 0}(\cos mx)^{\frac{n}{x^2}}$ {Ans. $e^{-\frac{m^2n}{2}}$ }

343. $\lim\limits_{x\to 0}\dfrac{1}{x^2}-\cot^2 x$ {Ans. $\frac{2}{3}$ }

344. $\lim\limits_{x\to\infty}x-x^2\ln\left(1+\dfrac{1}{x}\right)$ {Ans. $\frac{1}{2}$ }

345. $\lim\limits_{x\to\infty}x^2\left[\cosh\dfrac{a}{x}-1\right]$ {Ans. $\frac{a^2}{2}$ }

346. $\lim\limits_{x\to 0}\left(\dfrac{5}{2+\sqrt{9+x}}\right)^{\frac{1}{\sin x}}$ {Ans. $e^{-\frac{1}{30}}$ }

347. $\lim\limits_{x\to 0}(\ln\cot x)^{\tan x}$ {Ans. 1}

348. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{1+x^2}\,\cos x}{\tan^4 x}$ {Ans. $\frac{1}{3}$ }

349. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+3x}-\sqrt{1+2x}}{x^2}$ {Ans. $-\frac{1}{2}$ }

350. $\lim\limits_{x\to 0}\dfrac{e^x\sin x-x(1+x)}{x^3}$ {Ans. $\frac{1}{3}$ }

351. $\lim\limits_{x\to 0}\dfrac{e^{\alpha x}-\cos\alpha x}{e^{\beta x}-\cos\beta x}$ {Ans. $\frac{\alpha}{\beta}$ }

352. $\lim\limits_{x\to 0}\dfrac{x-\tan^{-1}x}{x^3}$ {Ans. $\frac{1}{3}$ }

353. $\lim\limits_{x\to 0}\dfrac{e^{a\sqrt{x}}-1}{\sqrt{\sin bx}}$ {Ans. $\frac{a}{\sqrt{b}}$ }

354. $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x-\tan x}$ {Ans. $-\frac{1}{2}$ }

355. $\lim\limits_{x\to\infty}\dfrac{\pi-2\tan^{-1}x}{\ln\left(1+\dfrac{1}{x}\right)}$ {Ans. 2}

356. $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{c^x-d^x}$ {Ans. $\frac{\ln\frac{a}{b}}{\ln\frac{c}{d}}$ }

357. $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{x\sqrt{1-x^2}}$ {Ans. $\ln\frac{a}{b}$ }

358. $\lim\limits_{x\to a}\dfrac{\cos x\ln(x-a)}{\ln(e^x-e^a)}$ {Ans. $\cos a$}

359. $\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^x}{\tan x-x}$ {Ans. 1}

360. $\lim\limits_{x\to 0}\dfrac{e^{x^3}-1-x^3}{\sin^6 2x}$ {Ans. $\frac{1}{128}$}

361. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$ {Ans. -1}

362. $\lim\limits_{x\to 0}\dfrac{2x^2-2e^{x^2}+2\cos x^{\frac{3}{2}}+\sin^3 x}{x^4}$ {Ans. -1}

363. $\lim\limits_{x\to 0}\dfrac{\ln\sin 2x}{\ln\sin x}$ {Ans. 1}

364. $\lim\limits_{x\to 0}\dfrac{\ln x}{\ln\sin x}$ {Ans. 1}

365. $\lim\limits_{x\to 1}\dfrac{\ln(1-x)+\tan\dfrac{\pi x}{2}}{\cot\pi x}$ {Ans. -2}

366. $\lim\limits_{x\to +\infty} x^n e^{-x}$ $(n>0)$ {Ans. 0}

367. $\lim\limits_{x\to\infty} x\sin\dfrac{a}{x}$ {Ans. a}

368. $\lim\limits_{\varphi\to a}\left(a^2-\varphi^2\right)\tan\dfrac{\pi\varphi}{2a}$ {Ans. $\frac{4a^2}{\pi}$}

369. $\lim\limits_{x\to 1}\dfrac{1}{\cos\dfrac{\pi x}{2}\ln(1-x)}$ {Ans. $-\infty$}

370. $\lim\limits_{x\to\infty}\sqrt[3]{(a+x)(b+x)(c+x)}-x$ {Ans. $\frac{a+b+c}{3}$}

371. $\lim\limits_{x\to 0} x^2 e^{\frac{1}{x^2}}$ {Ans. ∞}

372. $\lim\limits_{x\to\frac{\pi}{2}}(\tan x)^{2x-\pi}$ {Ans. 1}

373. $\lim\limits_{x\to 0}\left(e^x+x\right)^{\frac{1}{x}}$ {Ans. e^2}

374. $\lim\limits_{x\to\infty}(1+e^x)^{\frac{1}{x}}$ {Ans. e}

375. $\lim\limits_{x\to 0}\left(\dfrac{1+a^x}{2}\right)^{\frac{1}{x}}$ {Ans. $\sqrt{a}$}

376. $\lim\limits_{x\to a}\left(2-\dfrac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$ {Ans. $e^{\frac{2}{\pi}}$}

377. $\lim\limits_{x\to 0}\dfrac{\ln(1+x)^{1+x}}{x^2}-\dfrac{1}{x}$ {Ans. $\frac{1}{2}$}

378. $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\ln \sin x}$ {Ans. 2}

379. $\lim\limits_{x\to 1} \dfrac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1}$ {Ans. $-\frac{1}{4}$ }

380. $\lim\limits_{x\to 0} \dfrac{(1+x)^{\frac{1}{x}}-e}{x}$ {Ans. $-\dfrac{e}{2}$ }

381. $\lim\limits_{x\to\frac{\pi}{4}} (\sin 2x)^{\tan^2 2x}$ {Ans. $e^{-\frac{1}{2}}$ }

382. $\lim\limits_{x\to\frac{\pi}{2}} (\sin x)^{\tan x}$ {Ans. 1}

383. $\lim\limits_{x\to 0} \left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}$ {Ans. $e^{\frac{1}{3}}$ }

384. $\lim\limits_{x\to 0} \left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}}$ {Ans. e^2 }

385. $\lim\limits_{x\to\infty} \left(\dfrac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)^{x}$ $(a,b>0)$ {Ans. $\sqrt{ab}$ }

386. $\lim\limits_{x\to 0} \left(\dfrac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$. {Ans. $\sqrt[3]{abc}$ }

387. $\lim\limits_{x\to a} \left(\dfrac{\sin x}{\sin a}\right)^{\frac{1}{x-a}}$ ($a\neq k\pi$, k is an integer) {Ans. $e^{\cot a}$ }

388. $\lim\limits_{x\to 0} \dfrac{(1+x)^{\frac{1}{x}}+\dfrac{ex}{2}-e}{x^2}$ {Ans. $\dfrac{11e}{24}$ }

389. $\lim\limits_{x\to 0} \dfrac{\log_{\sin x}\cos x}{\log_{\sin\frac{x}{2}}\cos\dfrac{x}{2}}$ {Ans. 4}

390. $\lim\limits_{x\to\infty} \left(\dfrac{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+..........+a_n^{\frac{1}{x}}}{n}\right)^{nx}$

{Ans. $a_1a_2......a_n$ }

391. $\lim\limits_{x\to 0} (1+a_1x+a_2x^2+\ldots+a_nx^n)^{\frac{1}{x}}$ {Ans. e^{a_1} }

392. $\lim\limits_{x\to 1} \dfrac{x^x-x}{(x-1)^2}$ {Ans. 1}

393. $\lim\limits_{x\to 0} \dfrac{\cos(\frac{\pi}{2}\cos x)}{x^2}$ {Ans. $\dfrac{\pi}{4}$ }

394. $\lim\limits_{x\to 0} \left(\dfrac{\ln(1+x)}{x}\right)^{\frac{1}{x}}$ {Ans. $e^{-\frac{1}{2}}$ }

395. $\lim\limits_{x\to 2}\dfrac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}$ {Ans. 8}

396. $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{2\sqrt{2}-(\cos x+\sin x)^3}{1-\sin 2x}$ {Ans. $\dfrac{3}{\sqrt{2}}$ }

397. $\lim\limits_{x\to a}\dfrac{x^a-a^x}{x^x-a^a}$ $(a>0)$ {Ans. $\dfrac{1-\ln a}{1+\ln a}$ }

398. $\lim\limits_{x\to 2}\dfrac{(\sin\theta)^x+(\cos\theta)^x-1}{x-2}$

{Ans. $\sin^2\theta\ln(\sin\theta)+\cos^2\theta\ln(\cos\theta)$}

399. $\lim\limits_{x\to\infty}x^2\left(1+\dfrac{1}{x}\right)^x-ex^3\ln\left(1+\dfrac{1}{x}\right)$ {Ans. $\dfrac{e}{8}$ }

400. $\lim\limits_{x\to 0}\dfrac{\sqrt{a^2+ax+x^2}-\sqrt{a^2+ax}}{\ln\left(\cos\dfrac{x}{a}\right)}$ {Ans. $-|a|$ }

401. $\lim\limits_{x\to 0}\dfrac{x^2-\sin^2 x-\dfrac{x^4}{3}}{x^6}$ {Ans. $-\dfrac{2}{45}$ }

402. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+2x}-e^x+x^2}{x^3}$ {Ans. $\dfrac{1}{3}$ }

403. $\lim\limits_{x\to 0}\dfrac{\sin(\tan x)-x}{x^3}$ {Ans. $\dfrac{1}{6}$ }

404. $\lim\limits_{x\to a}\sqrt{a^2-x^2}\cot\left(\dfrac{\pi}{2}\sqrt{\dfrac{a-x}{a+x}}\right)$ {Ans. $\dfrac{4a}{\pi}$ }

405. $\lim\limits_{x\to 0}\dfrac{(a^x-1)}{x^n\sin x}\left(\dfrac{b\sin x-\sin bx}{\cos x-\cos bx}\right)^n$ {Ans. $\left(\dfrac{b}{3}\right)^n\ln a$ }

406. $\lim\limits_{x\to 0}\dfrac{a^x\sin bx-b^x\sin ax}{\tan bx-\tan ax}$ {Ans. 1}

407. $\lim\limits_{h\to 0}\dfrac{\sin(a+3h)-3\sin(a+2h)+3\sin(a+h)-\sin a}{h^3}$ {Ans. $-\cos a$ }

408. $\lim\limits_{x\to 0}\dfrac{x\sin(\sin x)-\sin^2 x}{x^6}$ {Ans. $\dfrac{1}{18}$ }

409. $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{1-(\sin x)^{\sin x}}{\cos^2 x}$ {Ans. $\dfrac{1}{2}$ }

410. $\lim\limits_{n\to\infty}\sin\pi\sqrt{n^2+1}$ (n is an integer) {Ans. 0}

411. $\lim\limits_{x\to\infty}(x^n+a_1x^{n-1}+\ldots+a_n)^{\frac{1}{n}}-(x^n+b_1x^{n-1}+\ldots+b_n)^{\frac{1}{n}}$ {Ans. $\dfrac{a_1-b_1}{n}$ }

412. $\lim_{x\to 0}\frac{e^{-\frac{1}{x}}}{x^a}$ ($a > 0$) {Ans. 0}

413. $\lim_{x\to 0}[\ln(\sec x)]^x$ {Ans. 1}

414. $\lim_{n\to\infty}\tan^n\left[\frac{\pi-4}{4}+\left(1+\frac{1}{n}\right)^\alpha\right]$ {Ans. $e^{2\alpha}$ }

415. $\lim_{x\to 0}\frac{\log_{\sec\frac{x}{2}}\cos x}{\log_{\sec x}\cos\frac{x}{2}}$ {Ans. 16}

416. $\lim_{x\to 0}\frac{e^{\frac{a}{x}}-e^{-\frac{a}{x}}}{e^{\frac{a}{x}}+e^{-\frac{a}{x}}}$ ($a > 0$). {Ans. 1, –1}

417. $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}$ (a, b, c, d are positive). {Ans. $e^{\frac{d}{b}}$ }

418. $\lim_{n\to\infty}\frac{n^K\cos n!}{n+1}\ (0<K<1)$. {Ans. 0}

419. $\lim_{n\to\infty}(2^n+3^n)^{\frac{1}{n}}$. {Ans. 3}

420. Find $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m$ without using L' Hospital's rule. {Ans. 1}

421. Find $\lim_{x\to\infty}\frac{x^2\sin\left(\frac{1}{x}\right)-x}{1-|x|}$. {Ans. 0}

422. $\lim_{x\to-\infty}\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{1+|x|^3}$ {Ans. –1}

423. If

$$f(x)=x^4,\quad x^2<1$$
$$=x,\quad x^2\ge 1.$$

Discuss the existence of limit at x = 1 and x = −1. {Ans. exists, does not exist}

424. If $f(x)=\sin x,\quad x\ne n\pi,\quad n=0,\pm1,\pm2,\ldots$

$$=2,\quad x=n\pi$$

and $g(x)=x^2+1,\quad x\ne 0, 2$

$$=4,\quad x=0$$
$$=5,\quad x=2$$

then find $\lim_{x\to 0}g[f(x)]$. {Ans. 1}

425. $\lim_{x\to 0}\left[\frac{\cos x+\cos 2x+\ldots+\cos nx}{n}\right]^{\frac{1}{x^2}}$ {Ans. $e^{-\frac{(n+1)(2n+1)}{12}}$ }

CATEGORY-2.16. SANDWICH THEOREM

426. If $\sin x < f(x) < \operatorname{cosec} x$ in the neighbourhood of $\frac{\pi}{2}$, find $\lim\limits_{x \to \frac{\pi}{2}} f(x)$. {Ans. 1}

427. If $x^2 \le f(x) \le |x|$ in the neighbourhood of 0, find $\lim\limits_{x \to 0} f(x)$. {Ans. 0}

428. If $|f(x)| \le |x| \ \forall \ x$, find $\lim\limits_{x \to 0} f(x)$. {Ans. 0}

429. If $f(x) > \ln x \ \forall x > 0$, find $\lim\limits_{x \to \infty} f(x)$. {Ans. ∞}

CATEGORY-2.17. FINDING LIMIT OF A FUNCTION CONTAINING PARAMETERS

430. $\lim\limits_{x \to \pm\infty} \frac{a^x}{a^x + 1} (a > 0)$

$$\left\{\begin{matrix} Ans. \\ \left\{\begin{matrix} 1, & a > 1 \\ 0, & a < 1 \\ \frac{1}{2}, & a = 1 \end{matrix}\right\}, \left\{\begin{matrix} 0, & a > 1 \\ 1, & a < 1 \\ \frac{1}{2}, & a = 1 \end{matrix}\right\} \end{matrix}\right\}$$

431. $\lim\limits_{x \to \pm\infty} \frac{a^x - a^{-x}}{a^x + a^{-x}} (a > 0)$

$$\left\{\begin{matrix} Ans. \\ \left\{\begin{matrix} 1, & a > 1 \\ -1, & a < 1 \\ 0, & a = 1 \end{matrix}\right\}, \left\{\begin{matrix} -1, & a > 1 \\ 1, & a < 1 \\ 0, & a = 1 \end{matrix}\right\} \end{matrix}\right\}$$

432. $\lim\limits_{x \to \infty} \sqrt{a^2x^2 + ax + 1} - \sqrt{a^2x^2 + 1}$.

$$\left\{\begin{matrix} Ans. \\ \frac{1}{2}, & a > 0 \\ 0, & a = 0 \\ -\frac{1}{2}, & a < 0 \end{matrix}\right\}$$

433. $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x^a}\right)^x$.

$$\left\{\begin{matrix} Ans. \\ 1, & a > 1 \\ e, & a = 1 \\ \infty, & a < 1 \end{matrix}\right\}$$

434. $\lim\limits_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

$$\begin{Bmatrix} Ans. & \\ \infty, & x=0 \\ \dfrac{1}{2\sqrt{x}}, & x>0 \end{Bmatrix}$$

435. $\lim_{x\to 0}\dfrac{\sin(x^n)}{(\sin x)^m}$ (m and n are positive integers)

$$\begin{Bmatrix} Ans. & \\ 0, & n-m>0 \\ 1, & n=m \\ \infty, & n-m<0\ \&\ (n-m)\ is\ even \\ +\infty, -\infty & n-m<0\ \&\ (n-m)\ is\ odd \end{Bmatrix}$$

436. How do the roots of the equation $ax^2+bx+c=0$ change when b and c retain constant values $(b\neq 0)$ and a tends to zero. {Ans. One root tends to $-\dfrac{c}{b}$ and the other root tends to $\pm\infty$}

437. $\lim_{x\to+\infty}\dfrac{x^b}{a^x}$

$$\begin{Bmatrix} Ans. & \\ 0, & b\geq 0\ \&\ a>1 \\ \infty, & b>0\ \&\ 0<a\leq 1 \\ 1, & b=0\ \&\ a=1 \\ 0, & b<0\ \&\ a\geq 1 \\ \infty, & b\leq 0\ \&\ 0<a<1 \end{Bmatrix}$$

438. $\lim_{x\to\infty}\dfrac{(ax+1)^n}{x^n+A}$. Consider separately the cases when n is (1) positive integer, (2) negative integer, (3) zero.

$$\begin{Bmatrix} Ans. & \\ (1)\ a^n, & n>0 \\ (2)\ 0, & A\neq 0, a\neq 0, n<0 \\ \quad a^n, & A=0, a\neq 0, n<0 \\ \quad \infty, & A=a=0, n<0 \\ \quad \frac{1}{A}, & A\neq 0, a=0, n<0 \\ (3)\ \frac{1}{1+A}, & A\neq -1, n=0 \\ \quad \text{not a function}, & A=-1, n=0 \end{Bmatrix}$$

CATEGORY-2.18. FINDING VALUES OF THE PARAMETERS GIVEN THE VALUE OF LIMIT

439. Find the constants a and b such that $\lim_{x\to\infty}\dfrac{x^2+1}{x+1}-ax-b=0$ {Ans. a = 1, b = -1}

440. If $\lim_{x\to\infty}\frac{x^2+1}{x+1}-ax-b=\infty$, find the value of a and b. {Ans. $a<1, b\in R$}

441. If $\lim_{x\to\infty}\frac{x^2-1}{x+1}-ax-b=2$, find the values of a and b. {Ans. $a=1, b=-3$}

442. If $\lim_{x\to\infty}\frac{x^3+1}{x^2+1}-ax-b=2$, then find a and b. {Ans. $a=1, b=-2$}

443. Find the constants a and b such that $\lim_{x\to-\infty}\sqrt{x^2-x+1}-ax-b=0$ {Ans. a = -1, $b=\frac{1}{2}$}

444. If $\lim_{x\to0}\frac{ae^x+be^{-x}+cx}{x^3}=1$, find the values of a, b, c. {Ans. $a=3, b=-3, c=-6$}

445. Find the values of a, b, c so that $\lim_{x\to0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}=2$. {Ans. $a=1, b=2, c=1$}

446. If $\lim_{x\to0}\frac{\sin x+ae^x+be^{-x}+c\ln(1+x)}{x^3}$ is finite, find a, b, c and the value of the limit. {Ans. $a=-\frac{1}{2}, b=\frac{1}{2}, c=0, -\frac{1}{3}$}

447. If $\lim_{x\to0}\frac{\cos 4x+a\cos 2x+b}{x^4}$ is finite, find a, b and the value of the limit. {Ans. a = -4, b = 3, 8}

448. If $\lim_{x\to\infty}\frac{e^{-x}+x\ln x+x^2\sin ax}{1+x^2}$ is finite, find a and the value of the limit. {Ans. a = 0, 0}

449. If $\lim_{x\to0}\frac{x^2+\ln(1-ax)+b\sin x}{x^3}$ is finite, find a, b and the value of the limit. {Ans. $a=b=\sqrt{2}, -\frac{5\sqrt{2}}{6}$, $a=b=-\sqrt{2}, \frac{5\sqrt{2}}{6}$}

450. If $\lim_{x\to0}\frac{\sinh 3x+a\sinh 2x+b\sinh x}{x^5}$ is finite, find a, b and the value of the limit. {Ans. a = -4, b = 5, 1}

451. Given

$$f(x)=\frac{\sin x}{x},\quad x<0$$
$$=ax+b,\quad x>0.$$

If $\lim_{x\to0}f(x)$ exists, find a, b and the value of the limit. {Ans. $a\in R$, b = 1, 1}

452. If $\lim_{x\to0}\frac{\{(a-n)nx-\tan x\}\sin x}{x^2}=0$, where n is non-zero real no., then find the value of a. {Ans. $n+\frac{1}{n}$}

CATEGORY-2.19. FINDING THE LIMIT AS FUNCTION OF PARAMETER

453. Sketch the graph of the function

$$f(x)=\lim_{n\to\infty}\sqrt[n]{1+x^n},\quad x\ge0.$$

$$\left\{\begin{array}{l} Ans. \\ f(x)=1,\quad 0\le x\le1 \\ \quad\quad =x,\quad x>1 \end{array}\right\}$$

454. Sketch the graph of the function $f(x)=\lim_{n\to\infty}\sin^{2n}x$.

$$\left\{\begin{array}{l} Ans. \\ f(x)=1, \quad x=(2m+1)\dfrac{\pi}{2} \\ \qquad =0, \quad x\neq(2m+1)\dfrac{\pi}{2} \end{array}\right\}$$

455. Determine the function $f(x)=\lim\limits_{n\to\infty}\left(\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdots\cdots\cos\frac{x}{2^n}\right)$.

$$\left\{\begin{array}{l} \text{Ans. } f(x)=\dfrac{\sin x}{x}, \quad x\neq 0 \\ \qquad\qquad =1, \quad x=0 \end{array}\right\}$$

456. Determine the function $f(x)=\lim\limits_{n\to\infty}\lim\limits_{m\to\infty}(\cos m!\,\pi x)^n$.

$$\left\{\begin{array}{l} \text{Ans.} \\ f(x)=1, \quad x \text{ is rational} \\ \qquad =0, \quad x \text{ is irrational} \end{array}\right\}$$

CATEGORY-2.20. ADDITIONAL QUESTIONS

457. If $\lim\limits_{x\to a} f(x)g(x)$ exists, then both $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exists. {Ans. False}

458. Find the value of k if $\lim\limits_{x\to 1}\frac{x^4-1}{x-1}=\lim\limits_{x\to k}\frac{x^3-k^3}{x^2-k^2}$. {Ans. $\frac{8}{3}$}

459. If $\lim\limits_{x\to -a}\frac{x^9+a^9}{x+a}=9$, find the value of a. {Ans. ± 1}

460. Let $f(x)=\frac{1}{\sqrt{18-x^2}}$. Find the value of $\lim\limits_{x\to 3}\frac{f(x)-f(3)}{x-3}$. {Ans. $\frac{1}{9}$}

461. Find the polynomial function $f(x)$ of least degree satisfying $\lim\limits_{x\to 0}\left(1+\frac{f(x)}{x^3}\right)^{\frac{1}{x}}=e^2$. {Ans. $2x^4$}

462. Find a polynomial $f(x)$ of least degree such that $\lim\limits_{x\to 0}\left(2+\frac{f(x)}{x^2}\right)^{\frac{1}{x}}=e^2$. {Ans. $f(x)=2x^3-x^2$}

463. Evaluate $\lim\limits_{x\to 0}\frac{x-\sin x}{x^3}$ without using series and L' Hospital's rule. {Ans. $\frac{1}{6}$}

464. The function $f(x)$ satisfies the condition $f(x)=\frac{1}{3}\left[f(x+1)+\frac{5}{f(x+2)}\right]$. Find $\lim\limits_{x\to\infty} f(x)$. {Ans. $\pm\sqrt{\frac{5}{2}}$}

465. Find $\lim\limits_{n\to\infty} n^2\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\cdots\cdots\infty}}}$. {Ans. $\frac{1}{2}$}

466. Find $\lim\limits_{x\to 0}\dfrac{\cos^2\left(1-\cos^2\left(1-\cos^2\left(1-\cos^2\cdots\cdots-\cos^2\left(1-\cos^2 x\right)\right)\right)\right)}{\sin\left(\pi\dfrac{\sqrt{x+4}-2}{x}\right)}$. {Ans. $\sqrt{2}$ }

467. $\lim\limits_{n\to\infty}\dfrac{1-2+3-4+\cdots\cdots\cdots+(2n-1)-2n}{\sqrt{n^2+1}}$. {Ans. -1}

468. $\lim\limits_{n\to\infty}(1+x)(1+x^2)(1+x^4)\cdots\cdots\cdots(1+x^{2^n})$, where $|x|<1$. {Ans. $\dfrac{1}{1-x}$ }

469. Given $f(x)=\lim\limits_{n\to\infty}\tan^{-1}nx$; $g(x)=\lim\limits_{n\to\infty}\sin^{2n}x$ and $h(x)=\dfrac{1}{2}\left[\cos(\pi g(x))+\cos(2f(x))\right]$. Find the function $h(x)$.

{Ans.

$$
\begin{aligned}
h(x)&=-1, \quad x=(2n+1)\frac{\pi}{2}\\
&=0, \quad x\neq(2n+1)\frac{\pi}{2}, 0\\
&=1, \quad x=0
\end{aligned}
$$
}

Mathematics for IIT-JEE

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PART-I
DIFFERENTIAL CALCULUS

CHAPTER-3
CONTINUITY

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CHAPTER-3
CONTINUITY

LIST OF THEORY SECTIONS

LIST OF QUESTION CATEGORIES

CHAPTER-3
CONTINUITY

SECTION-3.1. CONTINUITY AT A POINT

1. Continuity of a function at a finite point

i. A function $f(x)$ is said to be continuous at a finite point *a* iff:-

a. $\lim_{x \to a} f(x)$ exists,

b. $f(a)$ is defined,

c. $\lim_{x \to a} f(x)$ and $f(a)$ are both equal, i.e. $\lim_{x \to a} f(x) = f(a)$,

otherwise $f(x)$ is said to be discontinuous at the point *a* and the point *a* is called a point of discontinuity.

2. Right hand and Left hand continuity at a finite point

i. A function $f(x)$ is said to be continuous from the right at a finite point *a* iff:-

a. $\lim_{x \to a^+} f(x)$ is finite,

b. $f(a)$ is defined,

c. $\lim_{x \to a^+} f(x)$ and $f(a)$ are both equal, i.e. $\lim_{x \to a^+} f(x) = f(a)$,

otherwise $f(x)$ is said to be discontinuous from the right at the point *a*.

ii. A function $f(x)$ is said to be continuous from the left at a finite point *a* iff:-

a. $\lim_{x \to a^-} f(x)$ is finite,

b. $f(a)$ is defined,

c. $\lim_{x \to a^-} f(x)$ and $f(a)$ are both equal, i.e. $\lim_{x \to a^-} f(x) = f(a)$,

otherwise $f(x)$ is said to be discontinuous from the left at the point *a*.

3. Continuity of a function at the end points of the Domain

i. If a function $f(x)$ is not defined in a left (right) hand neighbourhood of a finite point *a* and is continuous from the right (left) at the point *a* then the function $f(x)$ is said to be continuous at the point *a*, otherwise $f(x)$ is said to be discontinuous at the point *a*.

4. Checking continuity of a function at a point by first principles

i. Graphical

a. If the curve of the function $f(x)$ is unbroken at the point *a* then $f(x)$ is continuous at *a* and vice-versa.

b. If the curve of the function $f(x)$ is broken at the point *a* then $f(x)$ is discontinuous at *a* and vice-versa.

ii. Analytical

a. To check continuity at the point *a*, check whether $f(a)$ is defined or not, $\lim_{x \to a} f(x)$ exists or not and $\lim_{x \to a} f(x) = f(a)$ or not.

5. Removable and non-removable discontinuities

i. A point a is called *removable discontinuity* of the function $f(x)$ if $\lim_{x\to a} f(x)$ exists but either $f(a)$ is not defined or $f(a)$ is different from $\lim_{x\to a} f(x)$. This removable discontinuity can be removed and the function can be made continuous at a by defining the value of $f(a)$ equal to the value of $\lim_{x\to a} f(x)$.

ii. A point a is called *non-removable discontinuity* of the function $f(x)$ if $\lim_{x\to a} f(x)$ does not exist.

SECTION-3.2. CONTINUITY IN A SET/ DOMAIN

1. Continuity of a function in a set

i. A function $f(x)$ is said to be continuous on a set A $(A \subseteq R)$ if $f(x)$ is continuous at every point of set A, otherwise $f(x)$ is said to be discontinuous on the set A.

ii. If $\lim_{h\to 0} f(x+h) - f(x) = 0 \ \forall x \in A$ then $f(x)$ is continuous on set A.

2. Continuity of a function in it's Domain

i. A function $f(x)$ is said to be a continuous function if $f(x)$ is continuous at every point in its domain, otherwise $f(x)$ is said to be a discontinuous function.

ii. If $\lim_{h\to 0} f(x+h) - f(x) = 0 \ \forall x \in D_f$ then $f(x)$ is a continuous function.

3. Checking continuity of a function in a set/Domain by first principles

i. Graphical

a. If the curve of the function $f(x)$ is unbroken in a set/domain then $f(x)$ is continuous in the set/domain and vice-versa.

b. If the curve of the function $f(x)$ is broken in a set/domain then $f(x)$ is discontinuous in the set/domain and vice-versa.

ii. Analytical

a. Check whether $\lim_{h\to 0} f(x+h) - f(x) = 0 \ \forall x \in A$ or not without using Theorem of limit of basic functions and L' Hospital's rule.

4. Theorem of continuity of Basic functions

i. All Basic functions are continuous functions, i.e. all basic functions are continuous at every point in their domain..

5. Theorems of continuity and its converse

i. If the functions $f(x)$ and $g(x)$ are continuous at a point/set then:-

a. the function $f(x)+g(x)$ must be continuous at the point/set;

b. the function $f(x)-g(x)$ must be continuous at the point/set;

c. the function $f(x)\times g(x)$ must be continuous at the point/set;

d. the function $\frac{f(x)}{g(x)}$ must be continuous at the point/set provided $g(x)\neq 0$ at the point/set.

ii. If $f(x)$ is continuous at the point a and $g(x)$ is continuous at the point $x = f(a)$ then the composite function $g(f(x))$ is continuous at the point a.

iii. If $f(x)$ is continuous in the set A and the values of $f(x)$ is the set B and $g(x)$ is continuous in the set B then the composite function $g(f(x))$ is continuous in the set A.

6. Applications of Theorems of continuity

7. Cases where theorems of continuity are not applicable

8. Checking continuity by theorems

i. For checking the continuity of a function in a set, identify the points at which theorems of continuity are applicable and the points at which theorems of continuity are not applicable. At the points where theorems of continuity are applicable, the function is continuous. At the points where theorems of continuity are not applicable, check each point for continuity by first principles.

9. Intermediate value theorem

i. If $f(x)$ is continuous on a closed interval $[a,b]$ then it assumes all intermediate values between $f(a)$ and $f(b)$ within the interval, i.e. if $f(x)$ be continuous on a closed interval $[a,b]$ and let $f(a)=A$ and $f(b)=B$ and if C is any value between A and B then there exists at least one point $\alpha \in (a,b)$ such that $f(\alpha)=C$.

ii. If $f(x)$ be continuous on a closed interval $[a,b]$ and if $f(a)$ and $f(b)$ be of opposite signs, then there exists at least one point in the open interval (a,b) at which the value of $f(x)$ is zero.

iii. If the value of a function is zero at a point then that point is called a *zero* of the function.

EXERCISE-3

CATEGORY-3.1. CONTINUITY AT A POINT BY FIRST PRINCIPLES

1. Given

$$f(x) = (1+x)^{\frac{1}{x}}, \quad x \neq 0$$
$$= e, \quad x = 0.$$

Check continuity at $x = 0$. {Ans. continuous}

2. Given

$$f(x) = x\sin\left(\frac{1}{x}\right), \quad x \neq 0$$
$$= 0, \quad x = 0.$$

Check continuity at $x = 0$. {Ans. continuous}

3. Given

$$f(x) = \frac{e^x - 1}{x}, \quad x \neq 0$$
$$= e, \quad x = 0.$$

Check continuity at $x = 0$. {Ans. discontinuous}

4. Given

$$f(x) = \frac{\sin 3x}{x}, \quad x \neq 0$$
$$= 1, \quad x = 0.$$

Check continuity at $x = 0$. {Ans. discontinuous}

5. Given

$$f(x) = \frac{1}{5}(2x^2 + 3), \quad x \leq 1$$
$$= 6 - 5x, \quad 1 < x < 3$$
$$= x - 3, \quad x \geq 3.$$

Check continuity at $x = 1$ and $x = 3$. {Ans. continuous, discontinuous}

6. Given

$$f(x) = \frac{\cos x - \sin x}{\cos 2x}, \quad x \neq \frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}}, \quad x = \frac{\pi}{4}.$$

Check continuity at $x = \frac{\pi}{4}$. {Ans. continuous}

7. Check the function

$$f(x) = \frac{\cos x}{\frac{\pi}{2} - x}, \quad x \neq \frac{\pi}{2}$$
$$= 1, \quad x = \frac{\pi}{2}$$

for continuity at $x = \frac{\pi}{2}$. {Ans. Continuous}

8. Given

$$f(x) = \frac{\sqrt{1+x} - \sqrt[3]{1+x}}{x}, \quad x \neq 0$$
$$= \frac{1}{6}, \quad x = 0.$$

Check continuity at $x = 0$. {Ans. continuous}

9. Given

$$f(x) = 0, \quad x < 0$$
$$= x, \quad 0 \le x < 1$$
$$= -x^2 + 4x - 2, \quad 1 \le x < 3$$
$$= 4 - x, \quad x \ge 3.$$

Check continuity at $x = 0$, 1 and 3. {Ans. continuous}

10. A function $f(x)$ is defined as

$$f(x) = \frac{x^2 - 4x + 3}{x^2 - 1}, \quad x \neq 1$$
$$= 2, \quad x = 1.$$

Test the continuity of the function at $x = 1$. {Ans. discontinuous}

11. Construct the graph of the function given below

$$f(x) = x - 1, \quad x < 0$$
$$= \frac{1}{4}, \quad x = 0$$
$$= x^2, \quad x > 0.$$

Find $\lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to 0^-} f(x)$ discuss the continuity of $f(x)$ at $x = 0$. {Ans. 0, −1, discontinuous}

12. Let $f(x) = \dfrac{x^2 - 1}{x^2 - 2|x-1| - 1}, \quad x \neq 1$

$$= \frac{1}{2}, \quad x = 1.$$

Discuss the continuity of function at $x = 1$. {Ans. discontinuous}

13. For what value of a will the function $f(x)$ be continuous at $x = 1$

$$f(x) = x + 1, \quad x \le 1$$
$$= 3 - ax^2, \quad x > 1.$$ {Ans. $a = 1$}

14. Let $f(x) = \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}, \quad x \neq 2$

$$= k, \quad x = 2.$$

If $f(x)$ is continuous for all x, then find k. {Ans. 7}

15. Let $f(x) = \left[\sin\left(x + \dfrac{\pi}{4}\right)\right]^{\frac{1}{x^2}}, \quad x \neq 0$

$= K, \quad x = 0$.

If $f(x)$ is continuous at $x = 0$, then find *K*. {Ans. 0}

16. $f(x) = \dfrac{1-\cos 4x}{x^2}, \quad x < 0$

$= a, \quad x = 0$

$= \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, \quad x > 0.$

If possible find the values of *a* so that the function may be continuous at *x* = 0. {Ans. 8}

17. For what value of *a*, the function

$f(x) = x^a \sin\dfrac{1}{x}, \quad x \neq 0$

$= 0, \quad x = 0$

is continuous at $x = 0$. {Ans. $a > 0$ }

18. Choose *A* and *B* so as to make the function *f* (*x*) continuous at $x = \pm\dfrac{\pi}{2}$

$f(x) = -2\sin x, \quad x \le -\dfrac{\pi}{2}$

$= A\sin x + B, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$= \cos x, \quad x \ge \dfrac{\pi}{2}$. {Ans. *A* = -1, *B* = 1}

19. Let $f(x) = \left(1+|\sin x|\right)^{\frac{a}{|\sin x|}}, \quad -\dfrac{\pi}{6} < x < 0$

$= b, \quad x = 0$

$= e^{\frac{\tan 2x}{\tan 3x}}, \quad 0 < x < \dfrac{\pi}{6}$.

Determine *a* and *b* such that $f(x)$ is continuous at *x* = 0. {Ans. $\dfrac{2}{3}, e^{\frac{2}{3}}$ }

20. Given the function

$f(x) = \left(\dfrac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}}, \quad 0 < x < \dfrac{\pi}{2}$

$= b + 2, \quad x = \dfrac{\pi}{2}$

$= \left(1+|\cos x|\right)^{\frac{a|\tan x|}{b}}, \quad \dfrac{\pi}{2} < x < \pi$.

Determine the values of *a* and *b*, if $f(x)$ is continuous at $x = \dfrac{\pi}{2}$. {Ans. $a = 0$, $b = -1$}

21. Determine *a*, *b* and *c* for which the function

$f(x) = \dfrac{\sin(a+1)x + \sin x}{x}, \quad x < 0$

$= c, \quad x = 0$

$= \dfrac{(x+bx^2)^{\frac{1}{2}} - x^{\frac{1}{2}}}{bx^{\frac{3}{2}}}, \quad x > 0$

is continuous at $x = 0$. {Ans. $a = -\frac{3}{2}, b \in R - [0], c = \frac{1}{2}$ }

22. Given the function

$f(x) = \dfrac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x}, \quad x > 0$

$= \dfrac{\ln(1 + x + x^2) + \ln(1 - x + x^2)}{\sec x - \cos x}, \quad x < 0.$

If $f(x)$ is continuous at $x = 0$, find the value of a. Now, $g(x) = \ln\left(2 - \dfrac{x}{a}\right) \cdot \cot(x - a), \quad x \neq a$. If $g(x)$ is continuous at $x = a$, show that $g\left(\dfrac{1}{e}\right) = -e$. {Ans. $a = \dfrac{1}{e}$ }

23. Given

$f(x) = 1, \quad x = 0$

$= x, \quad 0 < x < 1$

$= 0, \quad x = 1$

$= 3 - x, \quad 1 < x < 2$

$= 2, \quad x = 2$

$g(x) = 0, \quad x = 0$

$= 1 + x, \quad 0 < x < 1$

$= 0, \quad x = 1$

$= 2 - x, \quad 1 < x < 2$

$= 1, \quad x = 2.$

Check the continuity of the following functions at the indicated points:-

i. $f(x) + g(x)$ at $x = 0$ {Ans. Continuous}

ii. $f(x) + g(x)$ at $x = 1$ {Ans. Discontinuous}

iii. $f(x) + g(x)$ at $x = 2$ {Ans. Discontinuous}

iv. $f(x)g(x)$ at $x = 0$ {Ans. Continuous}

v. $f(x)g(x)$ at $x = 1$ {Ans. Discontinuous}

vi. $f(x)g(x)$ at $x = 2$ {Ans. Discontinuous}

vii. $f(g(x))$ at $x = 0$ {Ans. Discontinuous}

viii. $f(g(x))$ at $x = 1$ {Ans. Continuous}

ix. $f(g(x))$ at $x = 2$ {Ans. Continuous}

x. $g(f(x))$ at $x = 0$ {Ans. Discontinuous}

xi. $g(f(x))$ at $x = 1$ {Ans. Discontinuous}

xii. $g(f(x))$ at $x = 2$ {Ans. Continuous}

xiii. $f(f(x))$ at $x = 0$ {Ans. Continuous}

xiv. $f(f(x))$ at $x=1$ {Ans. Continuous}
xv. $f(f(x))$ at $x=2$ {Ans. Continuous}
xvi. $g(g(x))$ at $x=0$ {Ans. Discontinuous}
xvii. $g(g(x))$ at $x=1$ {Ans. Discontinuous}
xviii. $g(g(x))$ at $x=2$ {Ans. Discontinuous}

CATEGORY-3.2. RIGHT HAND AND LEFT HAND CONTINUITY

24. Given

$$f(x)=\left(\frac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}}, \quad x<0$$
$$=\frac{1}{\sqrt{e}}, \quad x=0$$
$$=(1+\ln(\cos(\sin x)))^{\frac{1}{x}}, \quad x>0.$$

Check the left hand and right hand continuity at $x = 0$. {Ans. Continuous from the left but discontinuous from the right}

25. Find the constant A such that the function

$$f(x)=\frac{e^{4x}-1}{\tan x}, \quad x<0$$
$$=A^2-7A+16, \quad x=0$$
$$=A, \quad x>0$$

is continuous from the left but discontinuous from the right at $x = 0$. {Ans. $A = 3$}

CATEGORY-3.3. REMOVABLE AND NON-REMOVABLE DISCONTINUITY

26. Define the following functions at $x = 0$ so as to make them continuous:-

i. $f(x)=\dfrac{5x^2-3x}{2x}$ {Ans. $-\frac{3}{2}$}

ii. $f(x)=\dfrac{\sqrt{1-\cos x}}{x}$ {Ans. Non-removable discontinuity}

iii. $f(x)=\dfrac{\ln(1+x)-\ln(1-x)}{x}$ {Ans. 2}

iv. $f(x)=\dfrac{2-\sqrt{x+4}}{\sin 2x}$ {Ans. $-\frac{1}{8}$}

v. $f(x)=\dfrac{\cos^2 x-\sin^2 x-1}{\sqrt{x^2+1}-1}$ {Ans. -4}

vi. $f(x)=\dfrac{1-\cos x}{\sin^2 x}$ {Ans. $\frac{1}{2}$}

vii. $f(x)=(x+1)^{\cot x}$. {Ans. e}

CATEGORY-3.4. CONTINUITY IN A SET/ DOMAIN BY FIRST PRINCIPLES

27. Prove the continuity of the following functions by first principles:-
    i. $f(x) = x^n$
    ii. $f(x) = \dfrac{1}{x}$
    iii. $f(x) = e^x$
    iv. $f(x) = \ln x$
    v. $f(x) = \sin x$
    vi. $f(x) = x \sin x$
    vii. $f(x) = \cos x \cdot \ln x$
28. If $f(x+y) = f(x) + f(y) \forall x, y$ and $f(x)$ is continuous at $x = 0$, then show that $f(x)$ is continuous $\forall x$.
29. If $f(x+y) = f(x) \cdot f(y) \forall x, y$ and $f(x)$ is continuous at $x = 0$, then show that $f(x)$ is continuous $\forall x$.
30. If $f(xy) = f(x) + f(y) \forall x, y \neq 0$ and $f(x)$ is continuous at $x = 1$, then check the continuity of $f(x)$. {Ans. Continuous for all x except $x = 0$}
31. If $f(xy) = f(x) \cdot f(y) \forall x, y$ and $f(x)$ is continuous at $x = 1$, then show that $f(x)$ is continuous for all x except $x = 0$.
32. If $f(x+y) = f(x) + f(y) \forall x, y$ and $f(x)$ is continuous at a point $x = a$, then show that $f(x)$ is continuous $\forall x$.
33. If $f\left(\dfrac{x+y}{2}\right) = \dfrac{f(x) + f(y)}{2} \ \forall\ x, y$ and $f(x)$ is continuous at $x = 0$, then check the continuity of $f(x)$. {Ans. Continuous for all x}
34. If $f(x+2y) = f(x) + 2f(y) - 2f(0) \ \forall\ x, y$ and $f(x)$ is continuous at $x = 0$, then check the continuity of $f(x)$. {Ans. Continuous for all x}
35. If $f\left(\dfrac{x+y}{3}\right) = \dfrac{f(x) + f(y) + f(0)}{3} \ \forall\ x, y$ and $f(x)$ is continuous at $x = 0$, then check the continuity of $f(x)$. {Ans. Continuous for all x}
36. If $f\left(\dfrac{x+2y}{3}\right) = \dfrac{f(x) + 2f(y)}{3} \ \forall\ x, y$ and $f(x)$ is continuous at $x = 0$, then check the continuity of $f(x)$. {Ans. Continuous for all x}

CATEGORY-3.5. CONTINUITY BY THEOREMS

37. Show that a polynomial function $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ is a continuous function.
38. Show that a rational function is a continuous function.
39. Test the following functions for continuity $f(x) = \dfrac{\cos(\ln x) + \sin^3 x \cdot \tan^{-1} x}{e^x + \cosh x}$. {Ans. continuous}
40. Check the continuity of $|x|$ and $\operatorname{sgn} x$. {Ans. Continuous, Discontinuous at $x = 0$}
41. Test the following functions for continuity

$$\phi(x) = 0, \quad x = 0$$
$$= \frac{1}{2} - x, \quad 0 < x < \frac{1}{2}$$

$= \frac{1}{2}, \quad x = \frac{1}{2}$

$= \frac{3}{2} - x, \quad \frac{1}{2} < x < 1$

$= 1, \quad x = 1.$

Also draw the graph of the function. {Ans. Discontinuous at $0, \frac{1}{2}, 1$}

42. Find the values of *a* and *b* so that the function

$f(x) = x + a\sqrt{2}\sin x, \quad 0 \le x < \frac{\pi}{4}$

$= 2x\cot x + b, \quad \frac{\pi}{4} \le x \le \frac{\pi}{2}$

$= a\cos 2x - b\sin x, \quad \frac{\pi}{2} < x \le \pi$

is continuous for $0 \le x \le \pi$. {Ans. $a = \frac{\pi}{6}, b = -\frac{\pi}{12}$}

43. Given the function $f(x) = \frac{1}{1-x}$. Find the points of discontinuity of the function $f(x)$, $f(f(x))$ & $f(f(f(x)))$. {Ans. 1; 0 & 1; 0 & 1}

44. Let $f(x) = \frac{1}{x-1}$ & $g(x) = \frac{1}{x^2 + x - 2}$. Find the points where $g(f(x))$ is discontinuous. {Ans. $\frac{1}{2}$, 1, 2}

45. Given

$f(x) = x - 1, \quad x \ge 0$

$= x + 1, \quad x < 0.$

Test the function $\phi(x) = [f(x)]^2$ for continuity. {Ans. continuous}

46. Investigate the functions $f(g(x))$ and $g(f(x))$ for continuity if $f(x) = \text{sgn}x$ and $g(x) = x(1 - x^2)$. {Ans. discontinuous at $x = 0, \pm 1$, continuous}

47. Given

$f(x) = 1 + x, \quad 0 \le x \le 2$

$= 3 - x, \quad 2 < x \le 3.$

Test the function $f(f(x))$ for continuity. {Ans. Continuous $\forall x \ne 1,2$ & discontinuous at $x = 1,2$ }

48. Show that

$f(x) = 1, \quad x$ is rational

$= -1, \quad x$ is irrational

is discontinuous for all *x*.

49. Given

$f(x) = x, \quad x$ is rational

$= -x, \quad x$ is irrational.

Show that $f(x)$ is continuous at *x* = 0 only.

50. Test the function for continuity

$f(x)=\frac{1}{2^n},\ \frac{1}{2^{n+1}}<x\le\frac{1}{2^n}\quad n=0,1,2\ldots$

$=0,\quad x=0$. {Ans. Discontinuous at $x=\frac{1}{2^n}$, $n=1,2,3,.......$ }

51. Discuss the continuity of the function $f(x)=\lim\limits_{n\to\infty}\frac{(1+\sin\pi x)^n-1}{(1+\sin\pi x)^n+1}$ at the point x = 1. {Ans. Discontinuous}

52. Check the continuity of the function $f(x)=\lim\limits_{n\to\infty}\frac{\log(2+x)-x^{2n}\sin x}{1+x^{2n}}$. {Ans. Discontinuous at $x=\pm1$ }

53. If $f(x)$ and $g(x)$ are two functions continuous everywhere and $h(x)=\lim\limits_{n\to\infty}\frac{f(x)+x^{2n}g(x)}{1+x^{2n}}$, then prove that $h(x)$ is continuous everywhere except at $x=1,-1$. Find the condition on $f(x)$ and $g(x)$ which makes $h(x)$ continuous everywhere. {Ans. $f(1)=g(1)$ & $f(-1)=g(-1)$}

54. If $f(x+2y)=f(x)[f(y)]^2\ \forall\ x,y$ and $f(x)$ is continuous at x = 0, then check the continuity of $f(x)$. {Ans. Continuous for all x}

55. Can one assert that the square of a discontinuous function is also a discontinuous function? Give an example of a function discontinuous everywhere whose square is a continuous function. {Ans. No}

56. Let $f(x)$ be a continuous and $g(x)$ be a discontinuous function. Prove that $f(x)+g(x)$ is a discontinuous function.

CATEGORY-3.6. INTERMEDIATE VALUE THEOREM

57. Does the function $f(x)=\frac{x^3}{4}-\sin\pi x+3$ take the value $2\frac{1}{3}$ within the interval $[-2,2]$? {Ans. Yes}

58. Given $f(x)=x^3+x+1$, show that $f(x)$ has a zero in the interval $[-1,0]$.

59. Show that the equation $x^5-3x-1=0$ has at least one root lying between 1 and 2.

60. Show that the equation $x2^x=1$ has at least one positive root not exceeding 1.

61. Show that $f(x)=x^3-3x+1$ has a zero in the interval $[1,2]$.

62. Does the equation $x^5-18x+2=0$ has a root in the interval $[-1,1]$? {Ans. Yes}

63. Does the equation $\sin x-x+1=0$ has a root? {Ans. Yes}

64. Show that the equation $x=a\sin x+b$, where $0<a<1,\ b>0$, has at least one positive root which does not exceed $a+b$.

65. Show that $x+\ln x=0$ has a solution in the interval $(0,1)$.

66. Show that the equation $\frac{a_1}{x-\lambda_1}+\frac{a_2}{x-\lambda_2}+\frac{a_3}{x-\lambda_3}=0$, where $a_1>0$, $a_2>0$, $a_3>0$ and $\lambda_1<\lambda_2<\lambda_3$, has two real roots lying in the intervals (λ_1,λ_2) and (λ_2,λ_3).

67. Given

$f(x)=x^2+1,\quad -2\le x<0$

$=-x^2-1,\quad 0\le x\le 2$.

Is there a point in the interval $[-2,2]$ at which $f(x)=0$? {Ans. No}

68. The function $f(x)$ is continuous in the interval $[a,b]$ and has values of the same sign on its end-points.

Can one assert that there is no point in $[a,b]$ at which the function becomes zero? {Ans. No}

69. Show that the function $f(x)=x^5-4x+1$ has at least two zeros in the interval $(0,2)$.

70. If $f(x)$ be continuous in $[a,b]$ and the equation $f(x)=0$ has only two roots α and β ($\alpha<\beta$) in the interval $[a,b]$, then prove that $f(x)$ retains the same sign in the interval $[\alpha,\beta]$.

71. Let $f(x)$ be a continuous function defined for $1\le x\le 3$. If $f(x)$ takes rational values for all x and $f(2)=10$, then find $f(1.5)$. {Ans. 10}

72. Let the function $f(x)$ be continuous in the interval $[a,b]$ and $a\le f(x)\le b \forall x\in[a,b]$. Prove that in this close interval there exists at least one point at which $f(x)=x$. Explain this geometrically.

73. If $f(x)$ is continuous and $f(0)=f(1)$ then prove that there exists $c\in\left[0,\frac{1}{2}\right]$ such that $f(c)=f\left(c+\frac{1}{2}\right)$.

74. Let the function $f(x)$ be continuous in the interval $[a,b]$. Prove that in this close interval there exists at least one point at which $f(x)=\frac{f(a)+f(b)}{2}$.

75. Prove that if the function $f(x)$ is continuous in the interval (a,b) and x_1, x_2, ………, x_n are any values in this open interval, then we can always find a real number c in this open interval such that $f(c)=\frac{f(x_1)+f(x_2)+\dots\dots\dots+f(x_n)}{n}$.

CATEGORY-3.7. ADDITIONAL QUESTIONS

76. If $f(x)$ is continuous and $f\left(\frac{9}{2}\right)=\frac{2}{9}$ then find $\lim\limits_{x\to 0} f\left(\frac{1-\cos 3x}{x^2}\right)$. {Ans. $\frac{2}{9}$}

77. If $f(x)$ is continuous in $[0,1]$ and $f\left(\frac{1}{3}\right)=1$ then find $\lim\limits_{n\to\infty} f\left(\frac{n}{\sqrt{9n^2+1}}\right)$. {Ans. 1}

78. If $f(x)$ and $h(x)$ are continuous functions and $g(x)=\lim\limits_{m\to\infty}\frac{x^m f(x)+h(x)+1}{2x^m+3x+3}$, $x\ne 1$ and $g(1)=\lim\limits_{x\to 1}(\ln ex)^{\frac{2}{\ln x}}$ and $g(x)$ is continuous at $x=1$, then find the value of $2g(1)+2f(1)-h(1)$. {Ans. 1}

Mathematics for IIT-JEE

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PART-I
DIFFERENTIAL CALCULUS

CHAPTER-4
DERIVATIVES

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CHAPTER-4
DERIVATIVES

LIST OF THEORY SECTIONS

LIST OF QUESTION CATEGORIES

CHAPTER-4
DERIVATIVES

SECTION-4.1. DIFFERENTIABILITY AND FIRST DERIVATIVE AT A POINT

1. Differentiability and First derivative of a function at a point

i. The first derivative of the function $f(x)$ at a finite point *a*, denoted by $f'(a)$, is defined as $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ if this limit exists. If this limit exists then it is said that the function $f(x)$ is differentiable at the point *a* and the value of this limit is the value of $f'(a)$. If this limit does not exist then it is said that the function is not differentiable at the point *a* and $f'(a)$ does not exist.

2. Right hand & left hand derivative at a point

i. The right hand derivative of the function $f(x)$ at a finite point *a*, denoted by $f'(a^+)$, is defined as $f'(a^+)=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$ if this limit is finite. If this limit is not finite then $f'(a^+)$ has no value.

ii. The left hand derivative of the function $f(x)$ at a finite point *a*, denoted by $f'(a^-)$, is defined as $f'(a^-)=\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0^+}\frac{f(a)-f(a-h)}{h}$ if this limit is finite. If this limit is not finite then $f'(a^-)$ has no value.

3. Differentiability and First derivative at the end points of the domain

i. If a function $f(x)$ is not defined in a left hand neighbourhood of a finite point *a* and $f'(a^+)$ is finite, then it is said that the function $f(x)$ is differentiable at the point *a* and the value of $f'(a^+)$ is the value of $f'(a)$, otherwise it is said that the function is not differentiable at the point *a* and $f'(a)$ does not exist.

ii. If a function $f(x)$ is not defined in a right hand neighbourhood of a finite point *a* and $f'(a^-)$ is finite, then it is said that the function $f(x)$ is differentiable at the point *a* and the value of $f'(a^-)$ is the value of $f'(a)$, otherwise it is said that the function is not differentiable at the point *a* and $f'(a)$ does not exist.

4. Continuity is a necessary condition for differentiability

i. Continuity is a necessary but not sufficient condition for differentiability.
ii. If a function is discontinuous at a point then it cannot be differentiable at that point.
iii. If a function is differentiable at a point then it must be continuous at that point.

5. Checking differentiability and finding first derivative at a point by first principles

i. Graphical
   a. If the curve of the function $f(x)$ is unbroken and smooth at the point *a* and the tangent at the point *a* is not vertical (not perpendicular to *x*-axis) then the function $f(x)$ is differentiable at *a* and $f'(a)$ = slope of tangent at *a*.
   b. If the curve of the function $f(x)$ is unbroken and smooth at the point *a* and the tangent at the point *a* is vertical (perpendicular to *x*-axis) then the function $f(x)$ is continuous but not differentiable at *a*.

c. If the curve of the function $f(x)$ is unbroken but not smooth at the point *a* then the function $f(x)$ is continuous but not differentiable at *a*.
d. If the curve of the function $f(x)$ is broken at the point *a* then the function $f(x)$ is discontinuous and not differentiable at *a*.

ii. Analytical
a. Find $f'(a)$ by finding the limit given in the definition without using L' Hospital's rule.

SECTION-4.2. DIFFERENTIABILITY IN A SET/DOMAIN AND FIRST DERIVATIVE FUNCTION

1. Differentiability in a set

i. A function $f(x)$ is said to be differentiable on a set A $(A \subseteq R)$ if $f(x)$ is differentiable at every point of set *A*, otherwise it is said that $f(x)$ is not differentiable on the set *A*.

2. Differentiability in the domain

i. A function $f(x)$ is said to be a differentiable function if $f(x)$ is differentiable at every point in its domain, otherwise it is said that $f(x)$ is not a differentiable function.

3. First derivative function

i. The First derivative function of a function $f(x)$, denoted by $f'(x)$, is defined as
$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $\forall x$ at which this limit exists.

ii. The value of $f'(x)$ at a point is the value of first derivative of $f(x)$ at that point.

iii. Domain of First derivative function is the set of all the points at which the function is differentiable.

iv. If a function $f(x)$ is a differentiable function then the domain of $f'(x)$ is same as the domain of the function $f(x)$, otherwise domain of $f'(x)$ is a subset of the domain of the function $f(x)$.

4. Symbolic notations of first derivative

i. First derivative function is usually denoted by $f'(x)$ or $\frac{d}{dx} f(x)$.

ii. First derivative at a point *a* is usually denoted by $f'(a)$ or $\left(\frac{d}{dx} f(x)\right)_{x=a}$.

5. Checking differentiability in a set/domain and finding first derivative function by first principles

i. Graphical
a. If the curve of the function $f(x)$ is unbroken and smooth and tangent is not vertical in a set/domain then $f(x)$ is differentiable in the set/domain, otherwise $f(x)$ is not differentiable in the set/domain.
b. Plot the graph of $f'(x)$ by plotting slope of tangent at the points on the curve of the function $f(x)$.

ii. Analytical
a. Find $f'(x)$ by finding the limit given in the definition without using L' Hospital's rule.

6. Process of differentiation

i. Differentiability of Basic functions
a. All basic functions are differentiable functions except the following basic functions at the indicated points:-
- x^a $(0 < a < 1)$ at $x = 0$;
- $\sin^{-1} x$ at $x = \pm 1$;
- $\cos^{-1} x$ at $x = \pm 1$;

- $\operatorname{cosec}^{-1}x$ at $x = \pm 1$;
- $\sec^{-1}x$ at $x = \pm 1$.

ii. First derivative function of basic functions

a. $f(x) = c$, $\quad f'(x) = 0$

b. $f(x) = x^a$, $\quad f'(x) = ax^{a-1}$

c. $f(x) = e^x$, $\quad f'(x) = e^x$

d. $f(x) = a^x$, $\quad f'(x) = a^x \ln a$

e. $f(x) = \ln x$, $\quad f'(x) = \dfrac{1}{x}, \quad x > 0$

f. $f(x) = \log_a x$, $\quad f'(x) = \dfrac{1}{x \ln a}, \quad x > 0$

g. $f(x) = \sin x$, $\quad f'(x) = \cos x$

h. $f(x) = \cos x$, $\quad f'(x) = -\sin x$

i. $f(x) = \tan x$, $\quad f'(x) = \sec^2 x$

j. $f(x) = \operatorname{cosec} x$, $\quad f'(x) = -\operatorname{cosec} x \cot x$

k. $f(x) = \sec x$, $\quad f'(x) = \sec x \tan x$

l. $f(x) = \cot x$, $\quad f'(x) = -\operatorname{cosec}^2 x$

m. $f(x) = \sin^{-1} x$, $\quad f'(x) = \dfrac{1}{\sqrt{1-x^2}}$

n. $f(x) = \cos^{-1} x$, $\quad f'(x) = -\dfrac{1}{\sqrt{1-x^2}}$

o. $f(x) = \tan^{-1} x$, $\quad f'(x) = \dfrac{1}{1+x^2}$

p. $f(x) = \operatorname{cosec}^{-1} x$, $\quad f'(x) = -\dfrac{1}{|x|\sqrt{x^2-1}}$

q. $f(x) = \sec^{-1} x$, $\quad f'(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$

r. $f(x) = \cot^{-1} x$, $\quad f'(x) = -\dfrac{1}{1+x^2}$

s. $f(x) = \sinh x$, $\quad f'(x) = \cosh x$

t. $f(x) = \cosh x$, $\quad f'(x) = \sinh x$

u. $f(x) = \tanh x$, $\quad f'(x) = \operatorname{sech}^2 x$

v. $f(x) = \operatorname{cosech} x$, $\quad f'(x) = -\operatorname{cosech} x \coth x$

w. $f(x) = \operatorname{sech} x$, $\quad f'(x) = -\operatorname{sech} x \tanh x$

x. $f(x) = \coth x$, $\quad f'(x) = -\operatorname{cosech}^2 x$

iii. Theorems (Rules) of differentiation

If the functions $f(x)$ and $g(x)$ are differentiable at a point/set then:-

a. the function $\phi(x) = f(x) \pm g(x)$ must be differentiable at the point/set and $\phi'(x) = f'(x) \pm g'(x)$;

b. the function $\phi(x) = f(x)\cdot g(x)$ must be differentiable at the point/set and
$\phi'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$;

c. the function $\phi(x) = \dfrac{f(x)}{g(x)}$ must be differentiable at the point/set and $\phi'(x) = \dfrac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$
if $g(x) \neq 0$.

iv. Theorems (Rules) of differentiation for composite functions (Chain rule)

a. If $f(x)$ is differentiable at the point *a* and $g(x)$ is differentiable at the point $x = f(a)$ then the composite function $\phi(x) = g(f(x))$ must be differentiable at the point *a* and $\phi'(a) = g'(f(a))\cdot f'(a)$.

b. If $f(x)$ is differentiable in the set *A* and the values of $f(x)$ is the set *B* and $g(x)$ is differentiable in the set *B* then the composite function $\phi(x) = g(f(x))$ must be differentiable in the set *A* and
$\phi'(x) = g'(f(x))\cdot f'(x)$ or $\dfrac{d}{dx}\phi(x) = \dfrac{d}{du}\phi(u)\cdot\dfrac{du}{dx}$, where $u = f(x)$.

7. Differentiating functions by Process of differentiation

i. Explicit functions
ii. Logarithmic differentiation
iii. Implicit function
iv. Parametric function
v. Differentiation of one function w.r.t. another function

8. Checking differentiability and finding first derivative by differentiation and by first principles

i. For checking the differentiability of a function in a set, identify the points at which theorems of differentiation are applicable and the points at which theorems of differentiation are not applicable. At the points where theorems of differentiation are applicable, find first derivative by differentiation. At the points where theorems of differentiation are not applicable, find first derivative by first principles.

SECTION-4.3. HIGHER ORDER DERIVATIVES

1. Definition of higher order derivatives of a function

i. For $n \geq 2$, the *n*th order derivative of $f(x)$, denoted by $f^{(n)}(x)$, is the derivative of the $(n-1)$th order derivative of $f(x)$, i.e. $f^{(n)}(x) = \dfrac{d}{dx} f^{(n-1)}(x)$.

2. Symbolic notations of higher order derivatives

i. Higher order derivative functions of $f(x)$ are usually denoted by
$f'(x),\ f''(x),\ f'''(x),\ f^{IV}(x)$, ...
or $f^{(2)}(x),\ f^{(3)}(x),\ f^{(4)}(x)$, ...
or $\dfrac{d}{dx}f(x),\ \dfrac{d^2}{dx^2}f(x),\ \dfrac{d^3}{dx^3}f(x)$, ...

ii. Higher order derivative functions of $y(x)$ are usually denoted by
$y_1(x),\ y_2(x),\ y_3(x)$, ...

iii. Higher order derivatives of $f(x)$ at a point *a* are usually denoted by
$f'(a),\ f''(a),\ f'''(a),\ f^{IV}(a)$, ...
or $f^{(2)}(a),\ f^{(3)}(a),\ f^{(4)}(a)$, ...

or $\left(\frac{d}{dx}f(x)\right)_{x=a}$, $\left(\frac{d^2}{dx^2}f(x)\right)_{x=a}$, $\left(\frac{d^3}{dx^3}f(x)\right)_{x=a}$, ...

iv. Higher order derivatives of $y(x)$ at a point a are usually denoted by $y_1(a),\ y_2(a),\ y_3(a)$, ...

3. Finding higher order derivatives by differentiation

i. Explicit functions
ii. Implicit functions
iii. Parametric functions

4. Leibniz theorem

i. $$\frac{d^n}{dx^n}(uv) = {}^nC_0\left(\frac{d^n u}{dx^n}\right)v + {}^nC_1\left(\frac{d^{n-1}u}{dx^{n-1}}\right)\left(\frac{dv}{dx}\right) + {}^nC_2\left(\frac{d^{n-2}u}{dx^{n-2}}\right)\left(\frac{d^2 v}{dx^2}\right) + \cdots\cdots + {}^nC_{n-1}\left(\frac{du}{dx}\right)\left(\frac{d^{n-1}v}{dx^{n-1}}\right) + {}^nC_n u\left(\frac{d^n v}{dx^n}\right)$$

$$= \left(\frac{d^n u}{dx^n}\right)v + n\left(\frac{d^{n-1}u}{dx^{n-1}}\right)\left(\frac{dv}{dx}\right) + \frac{n(n-1)}{2!}\left(\frac{d^{n-2}u}{dx^{n-2}}\right)\left(\frac{d^2 v}{dx^2}\right) + \cdots\cdots + n\left(\frac{du}{dx}\right)\left(\frac{d^{n-1}v}{dx^{n-1}}\right) + u\left(\frac{d^n v}{dx^n}\right)$$

where ${}^nC_r = \frac{n!}{(n-r)!r!}$; ${}^nC_0 = {}^nC_n = 1$.

ii. $\frac{d^n}{dx^n}c = 0$.

iii. $\frac{d^n}{dx^n}x^m = \frac{m!}{(m-n)!}x^{m-n}, \quad n \le m$
$= 0, \quad n > m$.

iv. $\frac{d^n}{dx^n}e^x = e^x$.

v. $\frac{d^n}{dx^n}a^x = (\ln a)^n a^x$.

vi. $\frac{d^n}{dx^n}\ln x = (-1)^{n+1}(n-1)!\,x^{-n}$.

vii. $\frac{d^n}{dx^n}\sin x = \sin\left(x + \frac{n\pi}{2}\right)$.

viii. $\frac{d^n}{dx^n}\cos x = \cos\left(x + \frac{n\pi}{2}\right)$.

5. Finding higher order derivatives of a function by differentiation and by first principles

i. Graphical
ii. Analytical

EXERCISE-4

CATEGORY-4.1. FIRST DERIVATIVE OF A FUNCTION AT A POINT BY FIRST PRINCIPLES

1. Given $f(x)=x^2$, find $f'(0)$, $f'(1)$ & $f'(-1)$ by first principles. {Ans. $0, 2, -2$}
2. Given $f(x)=\sqrt{x}$, find $f'(0)$ & $f'(1)$ by first principles. {Ans. does not exist, $\frac{1}{2}$}
3. Given $f(x)=\sqrt[3]{x}$, find $f'(0)$ by first principles. {Ans. does not exist}
4. Given $f(x)=x^a$, show that $f'(0)$ does not exist if $0 < a < 1$.
5. Given $f(x)=e^x$, find $f'(0)$, $f'(1)$ & $f'(-1)$ by first principles. {Ans. $1, e, \frac{1}{e}$}
6. Given $f(x)=\ln x$, find $f'(1)$ & $f'(e)$ by first principles. {Ans. $1, \frac{1}{e}$}
7. Given $f(x)=\sin x$, find $f'(0)$, $f'(\frac{\pi}{2})$ & $f'(\pi)$ by first principles. {Ans. $1, 0, -1$}
8. Given $f(x)=\tan x$, find $f'(0)$ & $f'(\frac{\pi}{4})$ by first principles. {Ans. $1, 2$}
9. Given $f(x)=\sin^{-1} x$, find $f'(0)$, $f'(-1)$ & $f'(1)$ by first principles. {Ans. 1, does not exist, does not exist}
10. Given $f(x)=\cos^{-1} x$, find $f'(0)$, $f'(-1)$ & $f'(1)$ by first principles. {Ans. -1, does not exist, does not exist}
11. Given $f(x)=x^2e^x$, find $f'(0)$ & $f'(1)$ by first principles. {Ans. $0, 3e$}
12. Given $f(x)=x\ln x$, find $f'(1)$ by first principles. {Ans. 1}
13. Given $f(x)=e^x \sin x$, find $f'(0)$ & $f'(\pi)$ by first principles. {Ans. $1, -e^{\pi}$}
14. Given $f(x)=\sin(\ln x)$, find $f'(1)$ by first principles. {Ans. 1}
15. Given $f(x)=|x|$, find $f'(0)$ by first principles. {Ans. does not exist}
16. Prove that $f(x)=|\ln x|$ is continuous but not differentiable at $x = 1$.
17. Given

    $$\begin{aligned} f(x) &= x^2, \quad x \ge 0 \\ &= x, \quad x < 0. \end{aligned}$$

    Find $f'(0)$. {Ans. does not exist}
18. Given

    $$\begin{aligned} f(x) &= x^2, \quad x \ge 0 \\ &= 0, \quad x < 0 \end{aligned}$$

    Find $f'(0)$. {Ans. 0}
19. Given

    $$\begin{aligned} f(x) &= x^3, \quad x \ge 0 \\ &= x^2, \quad x < 0. \end{aligned}$$

    Find $f'(0)$. {Ans. 0}
20. Given

    $$\begin{aligned} f(x) &= x^3-1, \quad x > 1 \\ &= x-1, \quad x \le 1, \end{aligned}$$

    find $f'(1)$. {Ans. does not exist}

21. Given
$f(x)=e^x,\quad x\ge 0$
$\quad = x+1,\quad x<0.$
Find $f'(0)$. {Ans. **1** }

22. Given
$f(x)=\dfrac{x}{\sqrt{x^2}},\quad x\ne 0$
$\quad =0,\quad x=0,$
find $f'(0)$. {Ans. does not exist}

23. If $f(x)=1,\quad x<0$
$\quad =1+\sin x,\quad 0\le x<\dfrac{\pi}{2},$
find $f'(0)$. {Ans. does not exist}

24. Given
$f(x)=\ln x,\quad x\ge 1$
$\quad =\sin(x-1),\quad x<1.$
Find $f'(1)$. {Ans. **1** }

25. Given
$f(x)=x,\quad x<1$
$\quad =2-x,\quad 1\le x\le 2$
$\quad =-2+3x-x^2,\quad x>2$
Find $f'(1)$ & $f'(2)$. {Ans. does not exist, -**1** }

26. Show that the function
$f(x)=x\sin\dfrac{1}{x},\ x\ne 0$
$\quad =0,\quad x=0$
is continuous but not differentiable at $x = 0$.

27. Given
$$f(x)=x\left(\frac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}\right),\ x\ne 0$$
$\quad =0,\quad x=0$
Show that $f(x)$ is not differentiable at $x = 0$.

28. Given
$$f(x)=xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)},\ x\ne 0$$
$\quad =0,\quad x=0$
Check continuity and differentiability at $x = 0$. {Ans. continuous but not differentiable at $x = 0$}

29. Discuss the limit, continuity and differentiability of the function

$$f(x) = x\left(\frac{3e^{\frac{1}{x}}+4}{2-e^{\frac{1}{x}}}\right),\ x \neq 0$$
$$= 0, \qquad x = 0$$

at $x = 0$. {Ans. limit exists, continuous but not differentiable at $x = 0$}

30. If $f(x) = \sqrt{|x-1|}$ and $g(x) = \sin x$ then calculate $(\text{fog})x$ and $(\text{gof})x$ and discuss differentiability of $(\text{gof})x$ at $x = 1$. {Ans. not differentiable at $x = 1$ }

31. If
$$f(x) = -1 + |x-1|, \quad -1 \le x \le 3$$
$$g(x) = 2 - |x+1|\ , \quad -2 \le x \le 2,$$
then calculate $(\text{fog})x$ and $(\text{gof})x$. Draw their graphs. Discuss the continuity of $(\text{fog})x$ at $x = 1$ and differentiability of $(\text{gof})x$ at $x = 1$. {Ans. continuous, not differentiable}

32. Given
$$f(x) = x^3, \quad x \ge 1$$
$$= ax + b, \quad x < 1.$$
Find the constants a & b such that $f'(1)$ exists. {Ans. $a = 3, b = -2$ }

33. If $f(x) = ax^2 + b, \quad x \le 1$
$$= bx^2 + ax + c, \quad x > 1,$$
where $b \neq 0$. Find a and c such that $f(x)$ is continuous and differentiable at $x = 1$. {Ans. $a = 2b$, $c = 0$ }

34. Given
$$f(x) = \ln x, \quad x \ge 1$$
$$= ax + b, \quad x < 1.$$
Find the constants a & b such that $f'(1)$ exists. {Ans. $a = 1, b = -1$ }

35. Given
$$f(x) = x^p \cos\frac{1}{x}, \quad x \neq 0$$
$$= 0, \quad x = 0$$
What conditions should be imposed on p so that
(i) $f(x)$ may be continuous at $x = 0$. {Ans. $p > 0$ }
(ii) $f(x)$ may be differentiable at $x = 0$. {Ans. $p > 1$ }

36. Given
$$f(x) = ax(x-1) + b, \quad x < 1$$
$$= x - 1, \quad 1 \le x \le 3$$
$$= px^2 + qx + 2, \quad x > 3$$
Find the constants a, b, p and q so that $f(x)$ is differentiable at $x = 1$ & $x = 3$. {Ans. $a = 1$, $b = 0$, $p = \frac{1}{3}$, $q = -1$}

37. Given

$$f(x) = b\sin^{-1}\left(\frac{x+c}{2}\right), \quad -\frac{1}{2} < x < 0$$
$$= \frac{1}{2}, \quad x = 0$$
$$= \frac{e^{\frac{ax}{2}} - 1}{x}, \quad 0 < x < \frac{1}{2}.$$

If $f(x)$ is differentiable at x = 0 and $|c| < \frac{1}{2}$, then find the value of 'a' and prove that $64b^2 = (4 - c^2)$.

{Ans. $a = 1$}

CATEGORY-4.2. FIRST DERIVATIVE FUNCTION OF A FUNCTION BY FIRST PRINCIPLES

38. Find $f'(x)$ by first principles:-

i. $f(x) = x^2$ {Ans. $2x$}

ii. $f(x) = \frac{1}{x}$ {Ans. $-\frac{1}{x^2}$}

iii. $f(x) = \sqrt{x}$ {Ans. $\frac{1}{2\sqrt{x}}$}

iv. $f(x) = x^a$ {Ans. ax^{a-1}}

v. $f(x) = e^x$ {Ans. e^x}

vi. $f(x) = a^x$ {Ans. $a^x \ln a$}

vii. $f(x) = \ln x$ {Ans. $\frac{1}{x}, x > 0$}

viii. $f(x) = \log_a x$ {Ans. $\frac{1}{x\ln a}, x > 0$}

ix. $f(x) = \sin x$ {Ans. $\cos x$}

x. $f(x) = \cos x$ {Ans. $-\sin x$}

xi. $f(x) = \tan x$ {Ans. $\sec^2 x$}

xii. $f(x) = cosecx$ {Ans. $-cosecx\cot x$}

xiii. $f(x) = \sec x$ {Ans. $\sec xtanx$}

xiv. $f(x) = \cot x$ {Ans. $-cosec^2 x$}

xv. $f(x) = xe^x$ {Ans. $(x+1)e^x$}

xvi. $f(x) = x\ln x$ {Ans. $1 + \ln x$}

xvii. $f(x) = x\sin x$ {Ans. $x\cos x + \sin x$}

xviii. $f(x) = \sin^2 x$ {Ans. $2\sin x\cos x$}

xix. $f(x) = \sqrt{\ln x}$ {Ans. $\frac{1}{2x\sqrt{\ln x}}$}

xx. $f(x) = \sin(\ln x)$ {Ans. $\frac{\cos(\ln x)}{x}$}

xxi. $f(x) = e^{\sin x}$ {Ans. $e^{Sinx}\cos x$}

xxii. $f(x) = x^x$ {Ans. $x^x(1 + \ln x)$}

xxiii. $f(x) = x^{\frac{1}{x}}$. {Ans. $\dfrac{x^{\frac{1}{x}}(1-\ln x)}{x^2}$}

39. If $f(x+y) = f(x) + f(y)\ \forall x, y$ & $f'(0) = 2$, then test the differentiability of $f(x)$. {Ans. Differentiable $\forall x$}
40. If $f(x+y) = f(x) \cdot f(y)\ \forall x, y$ & $f'(0) = 1$, then test the differentiability of $f(x)$. {Ans. Differentiable $\forall x$}
41. If $f(x+y) = f(x) + f(y)\ \forall x, y$ & $f'(1) = 3$, then test the differentiability of $f(x)$. {Ans. Differentiable $\forall x$}
42. If $f(xy) = f(x) + f(y)\ \forall x, y \neq 0$ & $f'(1) = 3$, then test the differentiability of $f(x)$. {Ans. Differentiable $\forall x \neq 0$}
43. If $f(xy) = f(x) \cdot f(y)\ \forall x, y$ & $f'(1) = 2$, then test the differentiability of $f(x)$. {Ans. Differentiable $\forall x \neq 0$}
44. If $f(x+y) = f(x)\, f(y)$ for all $x, y \in R,\ f(5) = 2,\ f'(0) = 3$, then find $f'(5)$. {Ans. 6}
45. Suppose the function *f* satisfies the conditions:
    (i) $f(x+y) = f(x) f(y)$ for all the *x* and *y*
    (ii) $f(x) = 1 + xg(x)$ where $\lim_{x \to 0} g(x) = 1$.
    Show that the derivative $f'(x)$ exists and $f'(x) = f(x)$ for all *x*.
46. Let $f(x+y) = f(x) + f(y)$ and $f(x) = x^2 g(x)$ for all $x, y \in R,$ where $g(x)$ is continuous function. Then find $f'(x)$. {Ans. 0}

CATEGORY-4.3. DIFFERENTIATING EXPLICIT FUNCTIONS

47. $y = (1+\sqrt[3]{x})^3$ {Ans. $\dfrac{(1+\sqrt[3]{x})^2}{\sqrt[3]{x^2}}$}
48. $y = a\tan\left(\dfrac{x}{k}+b\right)$ {Ans. $\dfrac{a}{k\cos^2\left(\dfrac{x}{k}+b\right)}$}
49. $y = \sqrt{1+\sqrt{2px}}$ {Ans. $\dfrac{p}{2\sqrt{1+\sqrt{2px}}\,\sqrt{2px}}$}
50. If $y = \tan^{-1}(x^2 - 3x + 2)$, find $\dfrac{dy}{dx}$, $\left(\dfrac{dy}{dx}\right)_{x=0}$, $\left(\dfrac{dy}{dx}\right)_{x=1}$. {Ans. $\dfrac{2x-3}{1+(x^2-3x+2)^2}$; $-\dfrac{3}{5}$; -1}
51. $y = \log_{10}(x - \cos x)$. {Ans. $\dfrac{1+\sin x}{(x-\cos x)\ln 10}$}
52. $y = 3\cos^2 x - \cos^3 x$ {Ans. $\dfrac{3}{2}\sin 2x(\cos x - 2)$}
53. $y = 5\tan\dfrac{x}{5} + \tan\dfrac{\pi}{8}$ {Ans. $\sec^2\dfrac{x}{5}$}

54. $y=\dfrac{1}{\sqrt[3]{x+\sqrt{x}}}$ {Ans. $-\dfrac{1+2\sqrt{x}}{6\sqrt{x}\sqrt[3]{\left(x+\sqrt{x}\right)^4}}$ }

55. $y=\sin\dfrac{x}{2}\sin 2x$ {Ans. $2\sin\dfrac{x}{2}\cos 2x+\dfrac{1}{2}\cos\dfrac{x}{2}\sin 2x$ }

56. $y=\sin x\cdot e^{\cos x}$ {Ans. $e^{\cos x}(\cos x-\sin^2 x)$ }

57. $y=x^5\sqrt[3]{x^6-8}$ {Ans. $\dfrac{x^4(7x^6-40)}{\sqrt[3]{(x^6-8)^2}}$ }

58. If $y=e^{-x^2}\ln x$, find $\dfrac{dy}{dx}$, $\left(\dfrac{dy}{dx}\right)_{x=1}$. {Ans. $e^{-x^2}\left(\dfrac{1}{x}-2x\ln x\right)$; $\dfrac{1}{e}$ }

59. $y=\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^{10}$ {Ans. $\dfrac{5(x-1)}{x\sqrt{x}}\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^9$ }

60. $y=\tan^{-1}\dfrac{x+1}{x-1}$ {Ans. $-\dfrac{1}{1+x^2}$ }

61. If $y=e^{2x+3}\left(x^2-x+\dfrac{1}{2}\right)$, find $\dfrac{dy}{dx}$, $\left(\dfrac{dy}{dx}\right)_{x=0}$. {Ans. $2x^2e^{2x+3}$; 0}

62. $y=\dfrac{2\sin^2 x}{\cos 2x}$ {Ans. $\dfrac{2\sin 2x}{\cos^2 2x}$ }

63. $y=\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{x\sqrt{3}}{1-x^2}$ {Ans. $\dfrac{1+x^2}{1+x^2+x^4}$ }

64. $y=\dfrac{\tan\frac{x}{2}+\cot\frac{x}{2}}{x}$ {Ans. $-\dfrac{2(x\cos x+\sin x)}{x^2\sin^2 x}$ }

65. $y=\sin^2\dfrac{x}{3}\cot\dfrac{x}{2}$ {Ans. $\dfrac{1}{3}\cot\dfrac{x}{2}\sin\dfrac{2x}{3}-\dfrac{1}{2}\sin^2\dfrac{x}{3}\operatorname{cosec}^2\dfrac{x}{2}$ }

66. $y=\dfrac{\sqrt[9]{4x^5+2}}{3x^4}$ {Ans. $-\dfrac{4(31x^5+18)}{27x^5\sqrt[9]{(4x^5+2)^8}}$ }

67. $y=\ln(x+\sqrt{a^2+x^2})$ {Ans. $\dfrac{1}{\sqrt{x^2+a^2}}$ }

68. $y=x\tan^{-1}\sqrt{x}$ {Ans. $\tan^{-1}\sqrt{x}+\dfrac{\sqrt{x}}{2(1+x)}$ }

69. $y=\sqrt{1+\tan^2 x+\tan^4 x}$ {Ans. $\dfrac{\tan x(1+2\tan^2 x)}{\cos^2 x\sqrt{1+\tan^2 x+\tan^4 x}}$ }

70. $y=\cos 2x\ln x$ {Ans. $\dfrac{\cos 2x}{x}-2\sin 2x\ln x$ }

71. $y=\dfrac{2}{3}\tan^{-1}x+\dfrac{1}{3}\tan^{-1}\dfrac{x}{1-x^2}$ {Ans. $\dfrac{1+x^4}{1+x^6}$ }

72. $y=\sin^{-1}(n\sin x)$ {Ans. $\dfrac{n\cos x}{\sqrt{1-n^2\sin^2 x}}$ }

73. $y=\sin^{-1}\sqrt{\sin x}$ {Ans. $\dfrac{\cos x}{2\sqrt{\sin x-\sin^2 x}}$ }

74. $y=\dfrac{1}{18}\sin^6 3x-\dfrac{1}{24}\sin^8 3x$ {Ans. $\sin^5 3x\cos^3 3x$ }

75. $y=x-\sqrt{1-x^2}\sin^{-1}x$ {Ans. $\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}$ }

76. $y=\cos\dfrac{\sin^{-1}x}{2}$ {Ans. $-\dfrac{1}{2}\sin\dfrac{\sin^{-1}x}{2}\dfrac{1}{\sqrt{1-x^2}}$ }

77. $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$ {Ans. $\dfrac{1+2\sqrt{x}+4\sqrt{x}\sqrt{x+\sqrt{x}}}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$ }

78. $y=\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$. {Ans. $-\dfrac{1}{2}$ }

79. $y=\ln\left\{e^x\left(\dfrac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$. {Ans. $\dfrac{x^2-1}{x^2-4}$ }

80. $y=\cos^{-1}\sqrt{1-3x}$ {Ans. $\dfrac{3}{2\sqrt{3x-9x^2}}$ }

81. $y=\sin^2\left(\dfrac{1-\ln x}{x}\right)$ {Ans. $\dfrac{\ln x-2}{x^2}\sin\left[2\left(\dfrac{1-\ln x}{x}\right)\right]$}

82. $y=\log_3(x^2-\sin x)$ {Ans. $\dfrac{2x-\cos x}{(x^2-\sin x)\ln 3}$ }

83. $y=\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$ {Ans. $-\dfrac{1}{2\sqrt{1-x^2}}$ }

84. $y=\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$. {Ans. $-\dfrac{2}{1+x^2}$ }

85. $y=\ln\dfrac{x+\sqrt{1-x^2}}{x}$ {Ans. $-\dfrac{1}{x\sqrt{1-x^2}\,(x+\sqrt{1-x^2})}$ }

86. $y=x\sin^{-1}(\ln x)$ {Ans. $\sin^{-1}(\ln x)+\dfrac{1}{\sqrt{1-\ln^2 x}}$ }

87. $y=\tan\dfrac{1-e^x}{1+e^x}$ {Ans. $-\dfrac{2e^x}{(1+e^x)^2}\sec^2\left(\dfrac{1-e^x}{1+e^x}\right)$}

88. $y=\cos x\sqrt{1+\sin^2 x}$ {Ans. $-\dfrac{2\sin^3 x}{\sqrt{1+\sin^2 x}}$ }

89. $y=0.4\left(\cos\frac{2x+1}{2}-\sin 0.8x\right)^2$ {Ans. $-0.8\left(\cos\frac{2x+1}{2}-\sin 0.8x\right)\left(\sin\frac{2x+1}{2}+0.8\cos 0.8x\right)$}

90. $y=x\cdot 10^{\sqrt{x}}$ {Ans. $10^{\sqrt{x}}\left(1+\frac{\sqrt{x}}{2}\ln 10\right)$}

91. $y=\frac{1}{\tan^2 2x}$ {Ans. $-\frac{4}{\tan 2x\sin^2 2x}$}

92. $y=\ln\tan^{-1}\frac{1}{1+x}$ {Ans. $-\frac{1}{(x^2+2x+2)\tan^{-1}\frac{1}{1+x}}$}

93. $y=\ln\frac{1}{x+\sqrt{x^2-1}}$ {Ans. $-\frac{1}{\sqrt{x^2-1}}$}

94. $y=\sqrt[3]{1+x\sqrt{x+3}}$ {Ans. $\frac{x+2}{2\sqrt{x+3}\sqrt[3]{(1+x\sqrt{x+3})^2}}$}

95. $y=x^2\sqrt{1+\sqrt{x}}$ {Ans. $\frac{x(8+9\sqrt{x})}{4\sqrt{1+\sqrt{x}}}$}

96. $y=\frac{1}{\sqrt{1+\sin^2 x}}$ {Ans. $-\frac{\sin 2x}{2\sqrt{(1+\sin^2 x)^3}}$}

97. $y=x^3\tan^{-1}x^3$ {Ans. $3x^2\tan^{-1}x^3+\frac{3x^5}{1+x^6}$}

98. $y=\log_{\cos x}\sin x$ {Ans. $\frac{\cot x\ln\cos x+\tan x\ln\sin x}{\ln^2\cos x}$}

99. If $y=\sin^{-1}x+\sqrt{1-x^2}$, find $\frac{dy}{dx}$, $\left(\frac{dy}{dx}\right)_{x=0}$. {Ans. $\sqrt{\frac{1-x}{1+x}}$; 1}

100. $y=\frac{\sin^{-1}4x}{1-4x}$ {Ans. $\frac{4}{(1-4x)^2}\left(\sqrt{\frac{1-4x}{1+4x}}+\sin^{-1}4x\right)$}

101. $y=e^{\frac{1}{\ln x}}$ {Ans. $-\frac{e^{\frac{1}{\ln x}}}{x\ln^2 x}$}

102. $y=\ln\left(\frac{e^x-1}{e^x}\right)$ {Ans. $\frac{1}{e^x-1}$}

103. $y=10^{x\tan x}$ {Ans. $10^{x\tan x}\ln 10\left(\tan x+\frac{x}{\cos^2 x}\right)$}

104. $y=a^{\log_{10}\operatorname{cosec}^{-1}x}$. {Ans. $-\frac{a^{\log_{10}(\operatorname{cosec}^{-1}x)}}{\operatorname{cosec}^{-1}x}\cdot\frac{1}{|x|\sqrt{x^2-1}}\log_{10}a$}

105. $y=\sin^2 x\cdot\sin x^2$ {Ans. $2\sin x(x\sin x\cos x^2+\cos x\sin x^2)$}

106. $y=\frac{2\cos x}{\sqrt{\cos 2x}}$ {Ans. $\frac{2\sin x}{\cos 2x\sqrt{\cos 2x}}$}

107. $y=x\sqrt{\dfrac{1-x}{1+x^2}}$ {Ans. $\dfrac{2-3x-x^3}{2(1-x)(1+x^2)}\sqrt{\dfrac{1-x}{1+x^2}}$ }

108. $y=\dfrac{1}{4}\ln\dfrac{1+x}{1-x}-\dfrac{1}{2}\tan^{-1}x$ {Ans. $\dfrac{x^2}{1-x^4}$ }

109. $y=2^{\frac{x}{\ln x}}$ {Ans. $2^{\frac{x}{\ln x}}\dfrac{\ln x-1}{\ln^2 x}\ln 2$ }

110. $y=\sqrt{(a-x)(x-b)}-(a-b)\tan^{-1}\sqrt{\dfrac{a-x}{x-b}}$ {Ans. $\sqrt{\dfrac{a-x}{x-b}}$ }

111. $y=\dfrac{\sin 3x}{2\sin^2 x\cos x}$ {Ans. $-\dfrac{2(2\cos^2 x+1)}{\sin^2 2x}$ }

112. $y=e^{\sqrt{\frac{1-x}{1+x}}}$ {Ans. $-\dfrac{1}{(1+x)\sqrt{1-x^2}}e^{\sqrt{\frac{1-x}{1+x}}}$ }

113. $y=\sqrt{a^2-x^2}-a\cos^{-1}\dfrac{x}{a}$ {Ans. $\sqrt{\dfrac{a-x}{a+x}}$ }

114. $y=\sqrt{x^2+1}-\ln\left(\dfrac{1}{x}+\sqrt{1+\dfrac{1}{x^2}}\right)$ {Ans. $\dfrac{\sqrt{x^2+1}}{x}$ }

115. $y=\dfrac{\sin^2 x}{1+\cot x}+\dfrac{\cos^2 x}{1+\tan x}$ {Ans. $-\cos 2x$ }

116. $y=\ln(x+\sqrt{x^2-1})-\dfrac{x}{\sqrt{x^2-1}}$ {Ans. $\dfrac{x^2}{\sqrt{(x^2-1)^3}}$ }

117. $y=e^{ax}(a\sin x-\cos x)$ {Ans. $(a^2+1)\sin xe^{ax}$ }

118. $y=xe^{1-\cos x}$ {Ans. $e^{1-\cos x}(1+x\sin x)$ }

119. $y=\dfrac{1}{\tan^{-1}e^{-2x}}$ {Ans. $\dfrac{2e^{-2x}}{(1+e^{-4x})(\tan^{-1}e^{-2x})^2}$ }

120. $y=e^x(\sin 3x-3\cos 3x)$ {Ans. $10e^x\sin 3x$ }

121. $y=3x^3\sin^{-1}x+(x^2+2)\sqrt{1-x^2}$ {Ans. $9x^2\sin^{-1}x$ }

122. $y=\dfrac{1}{\sqrt{1+e^{-\sqrt{x}}}}$ {Ans. $\dfrac{e^{-\sqrt{x}}}{4\sqrt{x}\sqrt{\left(1+e^{-\sqrt{x}}\right)^3}}$ }

123. $y=2\sin^{-1}\dfrac{x-2}{\sqrt{6}}-\sqrt{2+4x-x^2}$ {Ans. $\dfrac{x}{\sqrt{2+4x-x^2}}$ }

124. $y=\ln(e^x\cos x+e^{-x}\sin x)$ {Ans. $\dfrac{(\cos x-\sin x)(e^x+e^{-x})}{e^x\cos x+e^{-x}\sin x}$ }

125. $y=\dfrac{1+x\tan^{-1}x}{\sqrt{1+x^2}}$ {Ans. $\dfrac{\tan^{-1}x}{\sqrt{(1+x^2)^3}}$ }

126. $y=\dfrac{1}{\cos(x-\cos x)}$ {Ans. $\dfrac{\sin(x-\cos x)(1+\sin x)}{\cos^2(x-\cos x)}$ }

127. $y=e^x\sin x\cos^3 x$ {Ans. $e^x\sin x\cos^3 x(1+\cot x-3\tan x)$ }

128. $y=\sqrt[11]{9+6\sqrt[5]{x^9}}$ {Ans. $\dfrac{54\sqrt[5]{x^4}}{55\sqrt[11]{(9+6\sqrt[5]{x^9})^{10}}}$ }

129. $y=x-\ln(2e^x+1+\sqrt{e^{2x}+4e^x+1})$ {Ans. $\dfrac{1}{\sqrt{e^{2x}+4e^x+1}}$ }

130. $y=e^{\tan^{-1}\sqrt{1+\ln(2x+3)}}$ {Ans. $\dfrac{e^{\tan^{-1}\sqrt{1+\ln(2x+3)}}}{(2x+3)[2+\ln(2x+3)]\sqrt{1+\ln(2x+3)}}$ }

131. $y=\dfrac{e^{x^2}}{e^x+e^{-x}}$ {Ans. $\dfrac{e^{x^2}}{(e^x+e^{-x})^2}[2x(e^x+e^{-x})-(e^x-e^{-x})]$ }

132. $y=\ln\tan\dfrac{x}{2}-\cot x\ln(1+\sin x)-x$ {Ans. $\dfrac{\ln(1+\sin x)}{\sin^2 x}$ }

133. $y=2\ln(2x-3\sqrt{1-4x^2})-6\sin^{-1}2x$ {Ans. $\dfrac{40}{2x-3\sqrt{1-4x^2}}$ }

134. $y=\dfrac{3x^2-1}{3x^3}+\ln\sqrt{1+x^2}+\tan^{-1}x$ {Ans. $\dfrac{x^5+1}{x^4(x^2+1)}$ }

135. $y=\dfrac{1}{2}(3-x)\sqrt{1-2x-x^2}+2\sin^{-1}\dfrac{x+1}{\sqrt{2}}$ {Ans. $\dfrac{x^2}{\sqrt{1-2x-x^2}}$ }

136. $y=\ln(x\sin x\sqrt{1-x^2})$ {Ans. $\dfrac{1}{x}-\dfrac{x}{1-x^2}+\cot x$ }

137. $y=x\sqrt{1+x^2}\sin x$ {Ans. $\dfrac{(1+2x^2)\sin x+x(1+x^2)\cos x}{\sqrt{1+x^2}}$ }

138. $y=\dfrac{\sqrt{x+2}(3-x)^4}{(x+1)^5}$ {Ans. $\dfrac{(x^2-32x-73)(3-x)^3}{2(x+1)^6\sqrt{x+2}}$ }

139. $y=\sqrt[5]{(1+xe^{\sqrt{x}})^3}$ {Ans. $\dfrac{3e^{\sqrt{x}}(2+\sqrt{x})}{10\sqrt[5]{(1+xe^{\sqrt{x}})^2}}$ }

140. $y=\dfrac{1}{\sqrt{x}}e^{x^2-\tan^{-1}x+\frac{1}{2}\ln x+1}$ {Ans. $\left(2x-\dfrac{1}{1+x^2}\right)\dfrac{e^{x^2-\tan^{-1}x+\frac{1}{2}\ln x+1}}{\sqrt{x}}$ }

141. $y=\dfrac{\sin x}{4\cos^4 x}+\dfrac{3\sin x}{8\cos^2 x}+\dfrac{3}{8}\ln\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$ {Ans. $\dfrac{1}{\cos^5 x}$ }

142. $y=\dfrac{xe^x\tan^{-1}x}{\ln^5 x}$ {Ans. $\dfrac{e^x\tan^{-1}x}{\ln^5 x}\left[1+x+\dfrac{1}{(1+x^2)\tan^{-1}x}-\dfrac{5}{\ln x}\right]$}

143. $y=\dfrac{(1-x^2)e^{3x-1}\cos x}{(\cos^{-1}x)^3}$ {Ans. $\dfrac{(1-x^2)e^{3x-1}\cos x}{(\cos^{-1}x)^3}\left[\dfrac{3-2x-3x^2}{1-x^2}-\tan x+\dfrac{3}{\sqrt{1-x^2}\cos^{-1}x}\right]$}

144. $y = x\sqrt{(x^2+a^2)^3} + \frac{3a^2x}{2}\sqrt{x^2+a^2} + \frac{3a^4}{2}\ln(x+\sqrt{x^2+a^2})$ {Ans. $4\sqrt{(x^2+a^2)^3}$ }

145. $y = x(\sin^{-1}x)^2 - 2x + 2\sqrt{1-x^2}\sin^{-1}x$ {Ans. $(\sin^{-1}x)^2$ }

146. $y = \ln\cos\left(\tan^{-1}\frac{e^x-e^{-x}}{2}\right)$ {Ans. $\frac{e^{-x}-e^x}{e^{-x}+e^x}$ }

147. $y = \frac{1}{m\sqrt{ab}}\tan^{-1}\left(e^{mx}\sqrt{\frac{a}{b}}\right)$ {Ans. $\frac{1}{ae^{mx}+be^{-mx}}$ }

148. $y = \frac{1}{3}\ln\frac{x+1}{\sqrt{x^2-x+1}} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$ {Ans. $\frac{1}{x^3+1}$ }

149. $y = \ln\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} + 2\tan^{-1}\sqrt{\frac{1-x}{1+x}}$ {Ans. $\frac{1}{x}\sqrt{\frac{1-x}{1+x}}$ }

150. $y = \sqrt[3]{\frac{x-5}{\sqrt[5]{x^2+4}}}$ {Ans. $\frac{3x^2+10x+20}{15(x^2+4)\sqrt[3]{(x-5)^2\sqrt[5]{x^2+4}}}$ }

151. $y = \ln\sqrt[4]{\frac{x^2+x+1}{x^2-x+1}} + \frac{1}{2\sqrt{3}}\left(\tan^{-1}\frac{2x+1}{\sqrt{3}} + \tan^{-1}\frac{2x-1}{\sqrt{3}}\right)$ {Ans. $\frac{1}{x^4+x^2+1}$ }

152. $y = \cos^{-1}\frac{x^{2n}-1}{x^{2n}+1}$ {Ans. $-\frac{2nx^{n-1}}{x^{2n}+1}$ if n is even and $-\frac{2nx^n}{|x|(x^{2n}+1)}$, if n is odd.}

153. $y = -\frac{x}{1+8x^3} + \frac{1}{12}\ln\frac{(1+2x)^2}{1-2x+4x^2} + \frac{\sqrt{3}}{6}\tan^{-1}\frac{4x-1}{\sqrt{3}}$ {Ans. $\frac{24x^3}{(1+8x^3)^2}$ }

CATEGORY-4.4. LOGARITHMIC DIFFERENTIATION

154. $y = x^{x^2}$ {Ans. $x^{x^2+1}(2\ln x+1)$ }

155. $y = x^{x^x}$ {Ans. $x^{x^x}.x^x(\ln^2 x + \ln x + \frac{1}{x})$ }

156. $y = (\sin x)^{\cos x}$ {Ans. $(\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x\ln\sin x\right)$}

157. $y = (\tan 2x)^{\cot\frac{x}{2}}$ {Ans. $(\tan 2x)^{\cot\frac{x}{2}}\left(\frac{4\cot\frac{x}{2}}{\sin 4x} - \frac{\ln\tan 2x}{2\sin^2\frac{x}{2}}\right)$}

158. $y = (\ln x)^x$ {Ans. $(\ln x)^x\left(\frac{1}{\ln x} + \ln\ln x\right)$}

159. $y = (x+1)^{\frac{2}{x}}$ {Ans. $2\sqrt[x]{(x+1)^2}\left[\frac{1}{x(x+1)} - \frac{\ln(x+1)}{x^2}\right]$}

160. $y = x^3e^{x^2}\sin 2x$ {Ans. $x^2e^{x^2}\sin 2x(3+2x^2+2x\cot 2x)$}

161. $y = \frac{(x-2)^2\sqrt[3]{x+1}}{(x-5)^3}$ {Ans. $-\frac{2(x-2)(x^2+11x+1)}{3(x-5)^4\sqrt[3]{(x+1)^2}}$}

162. $y = x^{\ln x}$ {Ans. $2x^{\ln x - 1}\ln x$}

163. $y = \dfrac{(x+1)^3\sqrt[4]{x-2}}{\sqrt[5]{(x-3)^2}}$ {Ans. $\dfrac{57x^2-302x+361}{20(x-2)(x-3)}.\dfrac{(x+1)^2\sqrt[4]{x-2}}{\sqrt[5]{(x-3)^2}}$}

164. $y = \sqrt{x\sin x\sqrt{1-e^x}}$ {Ans. $\dfrac{1}{2}\sqrt{x\sin x\sqrt{1-e^x}}\left(\dfrac{1}{x}+\cot x-\dfrac{1}{2}.\dfrac{e^x}{1-e^x}\right)$}

165. $y = \sqrt{\dfrac{1-\sin^{-1}x}{1+\sin^{-1}x}}$ {Ans. $\dfrac{1}{\sqrt{1-x^2}[(\sin^{-1}x)^2-1]}\sqrt{\dfrac{1-\sin^{-1}x}{1+\sin^{-1}x}}$}

166. $y = x^{\frac{1}{x}}$ {Ans. $x^{\frac{1}{x}-2}(1-\ln x)$}

167. $y = x^{\sin x}$ {Ans. $x^{\sin x}\left(\cos x\ln x+\dfrac{\sin x}{x}\right)$}

168. $y = \left(\dfrac{x}{1+x}\right)^x$ {Ans. $\left(\dfrac{x}{x+1}\right)^x\left(\dfrac{1}{x+1}+\ln\dfrac{x}{x+1}\right)$

169. $y = 2x^{\sqrt{x}}$ {Ans. $x^{\sqrt{x}-\frac{1}{2}}(2+\ln x)$}

170. $y = (x^2+1)^{\sin x}$ {Ans. $(x^2+1)^{\sin x}\left[\dfrac{2x\sin x}{x^2+1}+\cos x\ln(x^2+1)\right]$}

171. $y = \sqrt[3]{\dfrac{x(x^2+1)}{(x^2-1)^2}}$ {Ans. $\dfrac{x^4+6x^2+1}{3x(1-x^4)}\sqrt[3]{\dfrac{x(x^2+1)}{(x^2-1)^2}}$}

CATEGORY-4.5. DIFFERENTIATING IMPLICIT FUNCTIONS

172. $x^3+x^2y+y^2=0$ {Ans. $-\dfrac{3x^2+2xy}{x^2+2y}$}

173. $\ln x+e^{-\frac{y}{x}}=c$ {Ans. $\dfrac{y}{x}+e^{\frac{y}{x}}$}

174. $x^2+y^2-4x-10y+4=0$ {Ans. $\dfrac{2-x}{y-5}$}

175. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ {Ans. $-\sqrt[3]{\dfrac{y}{x}}$}

176. $\tan^{-1}y-y+x=0$ {Ans. $1+\dfrac{1}{y^2}$}

177. $e^x-e^y=y-x$ {Ans. $\dfrac{1+e^x}{1+e^y}$}

178. $x+y=e^{x-y}$ {Ans. $\dfrac{e^{x-y}-1}{e^{x-y}+1}$}

179. $x^3-2x^2y^2+5x+y-5=0$ {Ans. $\dfrac{4xy^2-3x^2-5}{1-4x^2y}$}

180. $x+\sqrt{xy}+y=a$ {Ans. $\frac{2a-2x-y}{x+2y-2a}$}

181. $\tan^{-1}\left(\frac{y}{x}\right)=\ln\sqrt{x^2+y^2}$ {Ans. $\frac{x+y}{x-y}$}

182. $e^x\sin y-e^{-y}\cos x=0$ {Ans. $-\frac{e^x\sin y+e^{-y}\sin x}{e^x\cos y+e^{-y}\cos x}$}

183. $e^y+xy=e$ {Ans. $-\frac{y}{x+e^y}$}

184. $x^2+5xy+y^2-2x+y-6=0$ {Ans. $\frac{2-2x-5y}{5x+2y+1}$}

185. If $3\sin(xy)+4\cos(xy)=5$, then show that $\frac{dy}{dx}=-\frac{y}{x}$.

186. If $y=\sqrt{\sin x+y}$, then find $\left(\frac{dy}{dx}\right)_{\substack{x=0\\y=1}}$. {Ans. 1}

187. If $\sin y=x\sin(a+y)$, then find $\left(\frac{dy}{dx}\right)_{x=y=0}$. {Ans. $\sin a$}

188. If $2^x+2^y=2^{x+y}$, then find the value of $\left(\frac{dy}{dx}\right)_{x=y=1}$. {Ans. -1}

CATEGORY-4.6. DIFFERENTIATING PARAMETRIC FUNCTIONS

189. $x=a(t-\sin t),\ y=a(1-\cos t)$ {Ans. $\cot\frac{t}{2}$}

190. $x=k\sin t-\sin kt,\ y=k\cos t+\cos kt$ {Ans. $-\cot\left(\frac{k-1}{2}\right)t$}

191. $x=2\ln\cot t,\ y=\tan t+\cot t$ {Ans. $\cot 2t$}

192. $x=e^{ct},\ y=e^{-ct}$ {Ans. $-e^{-2ct}$}

193. $\begin{cases}x=a\cos^3 t\\ y=b\sin^3 t\end{cases}$ {Ans. $-\frac{b}{a}\tan t$}

194. $\begin{cases}x=t^3+3t+1\\ y=t^3-3t+1\end{cases}$ {Ans. $\frac{t^2-1}{t^2+1}$}

195. $\begin{cases}x=a(\cos t+t\sin t)\\ y=a(\sin t-t\cos t)\end{cases}$ {Ans. $\tan t$}

196. $\begin{cases}x=e^t\cos t\\ y=e^t\sin t\end{cases}$ {Ans. $\frac{\cos t+\sin t}{\cos t-\sin t}$}

197. $x=e^{-t},\quad y=t^3$ {Ans. $-3t^2e^t$}

198. $x=\sec t,\quad y=\tan t$ {Ans. $\operatorname{cosec} t$}

199. $x = \dfrac{a\sin t}{1+b\cos t}$, $y = \dfrac{c\cos t}{1+b\cos t}$ {Ans. $-\dfrac{c\sin t}{a(b+\cos t)}$ }

200. $x = \ln(1+t^2)$, $y = t - \tan^{-1} t$ {Ans. $\dfrac{t}{2}$ }

201. $x = t^2 + 2$, $y = \dfrac{t^3}{3} - t$ {Ans. $\dfrac{t^2-1}{2t}$ }

202. $x = e^{-t^2}$, $y = \tan^{-1}(2t+1)$ {Ans. $-\dfrac{e^{t^2}}{2t(2t^2+2t+1)}$ }

203. $x = 4\tan^2\dfrac{t}{2}$, $y = a\sin t + b\cos t$ {Ans. $\dfrac{(a\cos t - b\sin t)\cos^3\frac{t}{2}}{4\sin\frac{t}{2}}$ }

204. Given $x = \sin^{-1}(t^2 \text{ -}1)$, $y = \cos^{-1} 2t$, find $\left(\dfrac{dy}{dx}\right)_{t=\frac{1}{4}}$ {Ans. $-\dfrac{1}{2}\sqrt{\dfrac{3}{23}}$ }

205. Given $x = \sin^{-1} t$, $y = \sqrt{1-t^2}$, find $\left(\dfrac{dy}{dx}\right)_{t=\frac{1}{2}}$ {Ans. $-\dfrac{1}{2}$ }

CATEGORY-4.7. DIFFERENTIATING A FUNCTION W.R.T. ANOTHER FUNCTION

206. Differentiate $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$ w.r.t. $\tan^{-1} x$ {Ans. $\dfrac{1}{2}$ }

207. Differentiate $\tan^{-1}\dfrac{2x}{1-x^2}$ w.r.t. $\sin^{-1}\dfrac{2x}{1+x^2}$ {Ans. 1}

208. Differentiate $\cos^{1}\left(2x^2-1\right)$ with respect to $\cos^{-1} x$. {Ans. 2}

209. Differentiate $\sin^{-1}\dfrac{1-x}{1+x}$ w.r.t. $\sqrt{x}$ {Ans. $-\dfrac{2}{1+x}$ }

210. Differentiate $x^{\sin^{-1} x}$ w.r.t. $\sin^{-1} x$ {Ans. $x^{\sin^{-1}x}(\ln x + \sin^{-1} x \cdot \dfrac{\sqrt{1-x^2}}{x})$ }

211. Differentiate $\sec^{-1}\dfrac{1}{2x^2-1}$ w.r.t. $\sqrt{1-x^2}$ {Ans. $\dfrac{2}{|x|}$ }

212. Differentiate $\dfrac{\tan^{-1} x}{1+\tan^{-1} x}$ w.r.t. $\tan^{-1} x$. {Ans. $\dfrac{1}{(1+\tan^{-1} x)^2}$ }

213. Let $U = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ and $V = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, then find $\dfrac{dU}{dV}$. {Ans. $\dfrac{1-x^2}{\left|1-x^2\right|}$ }

CATEGORY-4.8. FIRST DERIVATIVE BY DIFFERENTIATION AND BY FIRST PRINCIPLES

214. Let $f(x) = \dfrac{x^2}{1-x^2}$, $x \neq 0, \pm 1$, then find $f'(2)$. {Ans. $\dfrac{4}{9}$ }

215. If $f(x)=(x+1)\tan^{-1}\left(e^{-2x}\right)$, then find $f'(0)$. {Ans. $\frac{\pi}{4}-1$}

216. If $f(x)=\log_{x^2}(\ln x)$, then find $f'(e)$. {Ans. $\frac{1}{2e}$}

217. Find the derivative of the function $\cot^{-1}\sqrt{\cos 2x}$ at $x=\frac{\pi}{6}$. {Ans. $\sqrt{\frac{2}{3}}$}

218. If $f(x)=\tan^{-1}\sqrt{\frac{1+\sin x}{1-\sin x}}, 0\le x\le\frac{\pi}{2}$, then find $f'\left(\frac{\pi}{6}\right)$. {Ans. $\frac{1}{2}$}

219. Find the derivative w.r.t. x of the function $(\log_{\cos x}\sin x)(\log_{\sin x}\cos x)^{-1}+\sin^{-1}\frac{2x}{1+x^2}$ at $x=\frac{\pi}{4}$. {Ans. $8\left(\frac{4}{\pi^2+16}-\frac{1}{\ln 2}\right)$}

220. Given $f(x)=|x-1|+|x-3|$, find $f'(2)$. {Ans. 0}

221. $f(x)=\ln x\cdot\cos^{-1}x$, find $f'(1)$. {Ans. 0}

222. $f(x)=(x^2-1)\cos^{-1}x$, find $f'(-1)$ & $f'(1)$. {Ans. -2π & 0}

223. Given $f(x)=|x|$, find $f'(x)$.
{Ans. $f'(x)=1, \quad x>0$
$=-1, \quad x<0$}

224. Given $f(x)=\operatorname{sgn} x$, find $f'(x)$.
{Ans. $f'(x)=0, \quad x\ne 0$}

225. Find the derivative of $\ln|x|$. {Ans. $\frac{1}{x}$}

226. Check the differentiability of the function & find $f'(x)$.
$f(x)=x^2, \qquad x\ge 0$
$=x, \qquad x<0$
{Ans. $f'(x)=2x, \quad x>0$
$=1, \quad x<0$}

227. Check the differentiability of the function & find $f'(x)$.
$f(x)=\sin x, \quad x\ge 0$
$=e^x-1, \quad x<0$
{Ans. $f'(x)=\cos x, \quad x\ge 0$
$=e^x, \quad x<0$}

228. Given
$f(x)=x^2\sin\frac{1}{x}, \qquad x\ne 0$
$=0, \qquad x=0$
find $f'(x)$.
{Ans. $f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}, \quad x\ne 0$

$= 0, \quad x = 0\}$

229. Given

$$f(x) = \frac{\sin x^2}{x}, \quad x \neq 0$$
$$= 0, \quad x = 0,$$

find $f'(x)$.

{Ans. $f'(x) = \dfrac{2x^2 \cos x^2 - \sin x^2}{x^2}, \quad x \neq 0$

$= 1, \quad x = 0\}$

230. Given $f(x) = \sqrt[3]{x - \sin x}$, find $f'(x)$.

{Ans.

$$f'(x) = \tfrac{1}{3}(x - \sin x)^{-\frac{2}{3}}(1 - \cos x), \; x \neq 0$$
$$= \tfrac{1}{\sqrt[3]{6}}, \; x = 0 \}$$

231. Given $f(x) = \sqrt{e^{x^4} - 1}$, find $f'(x)$.

{Ans.

$$f'(x) = \frac{2x^3 e^{x^4}}{\sqrt{e^{x^4} - 1}}, \; x \neq 0$$
$$= 0, \; x = 0 \}$$

232. Given $f(x) = (x^2 - 1)\sin^{-1} x$, find $f'(x)$.

{Ans.

$$f'(x) = 2x \sin^{-1} x - \sqrt{1 - x^2}, \; -1 < x < 1$$
$$= \pi, \; x = \pm 1 \}$$

233. Given $f(x) = \ln x \cdot \sin^{-1} x$, find $f'(x)$.

{Ans.

$$f'(x) = \frac{\sin^{-1} x}{x} + \frac{\ln x}{\sqrt{1 - x^2}}, \; 0 < x < 1$$
$$= \tfrac{\pi}{2}, \; x = 1 \}$$

234. Given $f(x) = \sin x \cdot \sqrt{x}$, find $f'(x)$.

{Ans.

$$f'(x) = \cos x \cdot \sqrt{x} + \frac{\sin x}{2\sqrt{x}}, \; x > 0$$
$$= 0, \; x = 0 \}$$

235. If $f(x) = \sqrt{x^2 - 2x + 1}$, then find $f'(x)$.

{Ans.

$$f'(x) = 1, \quad x > 1$$
$$= -1, \quad x < 1 \}$$

236. Given $f(x)=|x|+|x-1|$, find $f'(x)$.

{Ans.

$f'(x)=-2,\ x<0$

$=0,\ 0<x<1$

$=2,\ x>1$}

237. Given

$$f(x)=(x-1)^2\sin\left(\frac{1}{x-1}\right)-|x|,\ x\neq 1$$

$$=-1,\quad x=1$$

Find the set of points where $f(x)$ is not differentiable. {Ans. x = 0}

238. Discuss the continuity and differentiability of the function

$$f(x)=\frac{x}{1+|x|},\ |x|\geq 1$$

$$=\frac{x}{1-|x|},\ |x|<1$$

{Ans. Discontinuous at $x=\pm 1$, not differentiable at $x=\pm 1$}

239. Let

$$f(x)=\sqrt{x}\left(1+x\sin\frac{1}{x}\right)\quad,\quad x>0$$

$$=-\sqrt{-x}\left(1+x\sin\frac{1}{x}\right),\quad x<0$$

$$=0\quad,\quad x=0$$

Show that $f'(x)$ exists everywhere and is finite except at x = 0.

240. Draw the graph of the function $y=|x-1|+|x-2|$ in the interval [0, 3] and discuss the continuity and differentiability of the function in this interval. {Ans. continuous, not differentiable at x = 1 & 2}

241. Let $f(x)$ be defined in the interval [–2, 2] such that

$f(x)=-1,\quad -2\leq x\leq 0$

$=x-1,\quad 0<x\leq 2$

and $g(x)=f(|x|)+|f(x)|$. Test the differentiability of $g(x)$ in (–2, 2). {Ans. not differentiable at x = 0, 1}

242. Find a & b such that the function

$$f(x)=ax^2-b,\qquad |x|<1$$

$$=-\frac{1}{|x|},\qquad |x|\geq 1$$

is continuous and differentiable function. {Ans. $a=\frac{1}{2}, b=\frac{3}{2}$}

243. Test for continuity and differentiability of the function & find $f'(x)$

$f(x)=x,\quad x$ is rational

$=-x,\quad x$ is irrational.

{Ans. Continuous at $x=0$ only, not differentiable $\forall x$ }

244. Check continuity and differentiability of the function & find $f'(x)$

$f(x)=x^2$, x is rational

$= x^3$, x is irrational

{Ans. Continuous at $x = 0$ & 1 only, $f'(x)=0,\ x=0$ }

245. If $f(x)=e^x g(x)$, $g(0)=2$, $g'(0)=1$, then find $f'(0)$. {Ans. 3}

246. Discuss continuity & differentiability of $f(x)=a_0+a_1|x|+a_2|x|^2+\cdots\cdots+a_n|x|^n$, where a_0, a_1, ..., a_n are constants. {Ans. Continuous $\forall x$; differentiable $\forall x \neq 0$ and not differentiable at $x=0$ if $a_1 \neq 0$; differentiable $\forall x$ if $a_1=0$)

247. If the derivative of the function

$f(x)=ax^2+b,\quad x<-1$

$=bx^2+ax+4,\quad x\geq -1$

is everywhere continuous, then find a and b. {Ans. $a=2, b=3$}

248. Let R be the set of real numbers and $f: R \to R$ such that for all x and y in R, $|f(x)-f(y)| \leq |x-y|^3$. Prove that $f(x)$ is a constant.

249. Suppose $p(x)=a_0+a_1x+a_2x^2+.............+a_nx^n$. If $|p(x)| \leq |e^{x-1}-1|\ \forall x \geq 0$, then prove that $|a_1+2a_2+............+na_n| \leq 1$.

CATEGORY-4.9. HIGHER ORDER DERIVATIVES OF EXPLICIT FUNCTIONS BY DIFFERENTIATION

250. If $y=x^2-3x+2$, find $\dfrac{d^2y}{dx^2}$. {Ans. 2}

251. If $y=\left(x^2+1\right)^3$, find $\dfrac{d^2y}{dx^2}$. {Ans. $6\left(5x^4+6x^2+1\right)$}

252. If $y=\dfrac{a}{x^n}$, find $\dfrac{d^2y}{dx^2}$. {Ans. $\dfrac{an(n+1)}{x^{n+2}}$ }

253. If $y=xe^{x^2}$, find $\dfrac{d^2y}{dx^2}$. {Ans. $2e^{x^2}\left(3x+2x^3\right)$}

254. If $y=\dfrac{1}{1+x^3}$, find $\dfrac{d^2y}{dx^2}$. {Ans. $\dfrac{6x\left(2x^3-1\right)}{\left(x^3+1\right)^3}$ }

255. If $y=\left(1+x^2\right)\tan^{-1}x$, find $\dfrac{d^2y}{dx^2}$. {Ans. $\dfrac{2x}{1+x^2}+2\tan^{-1}x$ }

256. If $y=\sqrt{a^2-x^2}$, find $\dfrac{d^2y}{dx^2}$. {Ans. $-\dfrac{a^2}{\sqrt{\left(a^2-x^2\right)^3}}$ }

257. If $y=\ln\left(x+\sqrt{1+x^2}\right)$, find $\dfrac{d^2y}{dx^2}$. {Ans. $-\dfrac{x}{\sqrt{\left(1+x^2\right)^3}}$ }

258. If $y=\dfrac{1}{a+\sqrt{x}}$, find $\dfrac{d^2y}{dx^2}$. {Ans. $\dfrac{a+3\sqrt{x}}{4x\sqrt{x}\left(a+\sqrt{x}\right)^3}$ }

259. If $y=e^{\sqrt{x}}$, find $\dfrac{d^2y}{dx^2}$. {Ans. $\dfrac{e^{\sqrt{x}}\left(\sqrt{x}-1\right)}{4x\sqrt{x}}$}

260. If $y=\sqrt{1-x^2}\sin^{-1}x$, find $\dfrac{d^2y}{dx^2}$. {Ans. $-\dfrac{\sin^{-1}x+x\sqrt{1-x^2}}{\sqrt{\left(1-x^2\right)^3}}$ }

261. If $y=\sin^{-1}\left(a\sin x\right)$, find $\dfrac{d^2y}{dx^2}$. {Ans. $\dfrac{a\left(a^2-1\right)\sin x}{\sqrt{\left(1-a^2\sin^2 x\right)^3}}$ }

262. If $y=x^x$, find $\dfrac{d^2y}{dx^2}$. {Ans. $x^x\left[\left(\ln x+1\right)^2+\dfrac{1}{x}\right]$}

263. If $y=e^{2x-1}$, find $\left(\dfrac{d^2y}{dx^2}\right)_{x=0}$. {Ans. $\dfrac{4}{e}$ }

264. If $y=\tan^{-1}x$, find $\left(\dfrac{d^2y}{dx^2}\right)_{x=1}$. {Ans. $-\dfrac{1}{2}$}

265. If $y=1-x^2-x^4$, find $\dfrac{d^3y}{dx^3}$. {Ans. $-24x$ }

266. If $y=\cos^2 x$, find $\dfrac{d^3y}{dx^3}$. {Ans. $4\sin 2x$ }

267. If $y=\left(x+10\right)^6$, find $\left(\dfrac{d^3y}{dx^3}\right)_{x=2}$. {Ans. 207360}

268. If $f\left(x\right)=\ln\left(\ln x\right)$, find $\left(\dfrac{d^3y}{dx^3}\right)_{x=e}$. {Ans. $\dfrac{7}{e^3}$ }

269. If $y=x^3\ln x$, find $\dfrac{d^4y}{dx^4}$. {Ans. $\dfrac{6}{x}$ }

270. If $y=a\sin 2x$, find $\dfrac{d^4y}{dx^4}$. {Ans. $16a\sin 2x$ }

271. If $y=x^6-4x^3+4$, find $\left(\dfrac{d^4y}{dx^4}\right)_{x=1}$. {Ans. 360}

272. If $y=\dfrac{1}{1-x}$, find $\dfrac{d^5y}{dx^5}$. {Ans. $\dfrac{120}{(1-x)^6}$ }

273. If $y=\dfrac{1-x}{1+x}$, find $\left(\dfrac{d^5y}{dx^5}\right)_{x=1}$. {Ans. $-\dfrac{15}{4}$ }

CATEGORY-4.10. HIGHER ORDER DERIVATIVES OF IMPLICIT FUNCTIONS BY DIFFERENTIATION

274. If $ax^2+2hxy+by^2=1$, find $\frac{d^2y}{dx^2}$. {Ans. $\frac{h^2-ab}{(hx+by)^3}$ }

275. If $b^2x^2+a^2y^2=a^2b^2$, find $\frac{d^2y}{dx^2}$. {Ans. $-\frac{b^4}{a^2y^3}$ }

276. If $s=1+te^s$, find $\frac{d^2s}{dt^2}$. {Ans. $\frac{(3-s)e^{2s}}{(2-s)^3}$ }

277. If $y^3+x^3-3axy=0$, find $\frac{d^2y}{dx^2}$. {Ans. $-\frac{2a^3xy}{(y^2-ax)^3}$ }

278. If $y=\sin(x+y)$, find $\frac{d^2y}{dx^2}$. {Ans. $-\frac{y}{[1-\cos(x+y)]^3}$ }

279. If $e^{x+y}=xy$, find $\frac{d^2y}{dx^2}$. {Ans. $-\frac{y[(x-1)^2+(y-1)^2]}{x^2(y-1)^3}$}

280. If $e^y+xy=e$, find $\left(\frac{d^2y}{dx^2}\right)_{x=0}$. {Ans. $\frac{1}{e^2}$ }

281. If $y=\tan(x+y)$, find $\frac{d^3y}{dx^3}$. {Ans. $-\frac{2(3y^4+8y^2+5)}{y^8}$}

282. If $x^2+y^2=r^2$, find $\left(\frac{d^3y}{dx^3}\right)_{x=0}$. {Ans. 0}

CATEGORY-4.11. HIGHER ORDER DERIVATIVES OF PARAMETRIC FUNCTIONS BY DIFFERENTIATION

283. If $x=\phi(t), y=\psi(t)$, then find $\frac{d^2y}{dx^2}$. {Ans. $\frac{\phi'(t)\psi''(t)-\psi'(t)\phi''(t)}{(\phi'(t))^3}$}

284. If $x=at^2, y=2at$, find $\frac{d^2y}{dx^2}$. {Ans. $-\frac{1}{2at^3}$ }

285. If $x=at^2$, $y=bt^3$, find $\frac{d^2y}{dx^2}$. {Ans. $\frac{3b}{4a^2t}$ }

286. If $x=a\cos t$, $y=a\sin t$, find $\frac{d^2y}{dx^2}$. {Ans. $-\frac{1}{a\sin^3 t}$ }

287. If $x=a(t-\sin t)$, $y=a(1-\cos t)$, find $\frac{d^2y}{dx^2}$. {Ans. $-\frac{1}{a(1-\cos t)^2}$ }

288. If $x=a\cos^2 t$, $y=a\sin^2 t$, find $\frac{d^2y}{dx^2}$. {Ans. 0}

289. If $x=\ln t$, $y=t^2-1$, find $\frac{d^2y}{dx^2}$. {Ans. $4t^2$ }

290. If $x=\sin^{-1} t$, $y=\ln(1-t^2)$, find $\frac{d^2y}{dx^2}$. {Ans. $\frac{2}{t^2-1}$ }

291. If $x = at\cos t$, $y = at\sin t$, find $\left(\dfrac{d^2y}{dx^2}\right)_{t=0}$. {Ans. $\dfrac{2}{a}$}

292. If $x = a\cos^3 t$, $y = a\sin^3 t$, find $\dfrac{d^3y}{dx^3}$. {Ans. $\dfrac{\cos^2 t - 4\sin^2 t}{9a^2\cos^7 t\sin^3 t}$}

293. If $x = a\cos t$, $y = b\sin t$, find $\left(\dfrac{d^3y}{dx^3}\right)_{t=\frac{\pi}{2}}$. {Ans. 0}

CATEGORY-4.12. LEIBNIZ THEOREM

294. If $y = x^2\sin x$, find $\dfrac{d^{25}y}{dx^{25}}$. {Ans. $\left(x^2 - 600\right)\cos x + 50x\sin x$}

295. If $y = e^x\left(x^2 - 1\right)$, find $\dfrac{d^{24}y}{dx^{24}}$. {Ans. $e^x\left(x^2 + 48x + 551\right)$}

296. If $y = x^3\sin x$, find y_{20}. {Ans. $x^3\sin x - 60x^2\cos x - 1140x\sin x + 6840\cos x$}

297. If $y = e^x\left(3x^2 - 4\right)$, find y_n. {Ans. $e^x\left(3x^2 + 6nx + 3n^2 - 3n - 4\right)$}

298. If $y = \left(1 - x^2\right)\cos x$, find y_{2n}. {Ans. $(-1)^n\left[\left(4n^2 - 2n + 1 - x^2\right)\cos x - 4nx\sin x\right]$}

299. If $y = \dfrac{3x+2}{x^2 - 2x + 5}$, prove that $y_n(0) = \dfrac{2}{5}ny_{n-1}(0) - \dfrac{n(n-1)}{5}y_{n-2}(0)$ for $n \ge 2$.

CATEGORY-4.13. MISCELLANEOUS QUESTIONS ON DIFFERENTIATION

300. If $f(x) = 3e^{x^2}$, then find the value of $f'(x) - 2xf(x) + \dfrac{1}{3}f(0) - f'(0)$. {Ans. 1}

301. If $\sin^{-1}\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right) = \ln a$, then show that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

302. If $y = ce^{\frac{x}{(x-a)}}$, then show that $\dfrac{dy}{dx} = -\dfrac{ay}{(x-a)^2}$.

303. If $x^y = e^{x-y}$, then show that $\dfrac{dy}{dx} = \dfrac{\ln x}{(1 + \ln x)^2}$.

304. If $y = e^{x + e^{x + e^{x \ldots}}}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{1-y}$

305. If $y = x^{x^{x \ldots}}$, prove that $\dfrac{dy}{dx} = \dfrac{y^2}{(1 - y\ln x)x}$

306. If $y = a^{x^{a^{x \ldots \infty}}}$, show that $\dfrac{dy}{dx} = \dfrac{y^2\ln y}{x(1 - y\ln x\ln y)}$.

307. If $x = \sec\theta - \cos\theta$, $y = \sec^n\theta - \cos^n\theta$ then prove that $\left(x^2 + 4\right)(y')^2 = n^2\left(y^2 + 4\right)$.

308. If $y = a + bx^2$, where a and b are arbitrary constants, then prove that $x\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}$.

309. If $x^p y^q = (x+y)^{p+q}$, prove that $\frac{dy}{dx} = \frac{y}{x}$ & $\frac{d^2y}{dx^2} = 0$

310. If $x = a\cos nt - b\sin nt$, then show that $\frac{d^2x}{dt^2} + n^2 x = 0$.

311. If $y = ax^{n+1} + bx^{-n}$, then show that $x^2\frac{d^2y}{dx^2} = n(n+1)y$.

312. If $y = x\ln\left(\frac{x}{a+bx}\right)$, show that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$

313. If $y = e^x(a\cos x + b\sin x)$ prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

314. If $\cos^{-1}\left(\frac{y}{b}\right) = \ln\left(\frac{x}{n}\right)^n$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + n^2 y = 0$

315. If $y = x^n\left[a\cos(\ln x) + b\sin(\ln x)\right]$, then prove that $x^2\frac{d^2y}{dx^2} + (1-2n)x\frac{dy}{dx} + (1+n^2)y = 0$

316. Prove that $\frac{d^2x}{dy^2} = \frac{-\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$.

317. If $y = \sin x + e^x$, find $\frac{d^2x}{dy^2}$ in terms of x. {Ans. $\frac{\sin x - e^x}{(\cos x + e^x)^3}$ }

318. If $y = x + e^x$, find $\frac{d^2x}{dy^2}$ in terms of x. {Ans. $-\frac{e^x}{(1+e^x)^3}$ }

CATEGORY-4.14. HIGHER ORDER DERIVATIVES BY DIFFERENTIATION AND BY FIRST PRINCIPLES

319. Given $f(x) = \sin x, \quad x \ge 0$

$= x, \quad x < 0$,

find $f'(x), f''(x), f'''(x)$ & $f^{IV}(x)$.

{Ans.

$f'(x) = \cos x, \quad x \ge 0$

$= 1, \quad x < 0$;

$f''(x) = -\sin x, \quad x \ge 0$

$= 0, \quad x < 0$;

$f'''(x) = -\cos x, \quad x > 0$

$= 0, \quad x < 0$;

$f^{IV}(x) = \sin x, \quad x > 0$

$= 0, \quad x < 0$ }

320. Given $f(x) = x^4, \quad x \ge 0$

$= x^3, \quad x < 0$

find $f'(x), f''(x), f'''(x)$ & $f^{IV}(x)$.

{Ans.

$f'(x) = 4x^3, \quad x \geq 0$

$= 3x^2, \quad x < 0$;

$f''(x) = 12x^2, \quad x \geq 0$

$= 6x, \quad x < 0$;

$f'''(x) = 24x, \quad x > 0$

$= 6, \quad x < 0$;

$f^{IV}(x) = 24, \quad x > 0$

$= 0, \quad x < 0$ }

321. Given $f(x) = e^x, \quad x \geq 0$

$$= 1 + x + \frac{x^2}{2}, \quad x < 0,$$

find $f'(x), f''(x), f'''(x)$ & $f^{IV}(x)$.

{Ans.

$f'(x) = e^x, \quad x \geq 0$

$= 1 + x, \quad x < 0$;

$f''(x) = e^x, \quad x \geq 0$

$= 1, \quad x < 0$;

$f'''(x) = e^x, \quad x > 0$

$= 0, \quad x < 0$;

$f^{IV}(x) = e^x, \quad x > 0$

$= 0, \quad x < 0$ }

322. Given $f(x) = x|x|$, find $f'(x), f''(x), f'''(x)$ & $f^{IV}(x)$.

{Ans.

$f'(x) = 2x, \quad x \geq 0$

$= -2x, \quad x < 0$;

$f''(x) = 2, \quad x > 0$

$= -2, \quad x < 0$;

$f'''(x) = 0, \quad x \neq 0$;

$f^{IV}(x) = 0, \quad x \neq 0$ }

323. Given $f(x) = \ln x, \quad x \geq 1$

$= x - 1, \quad x < 1,$

find $f'(x), f''(x)$ & $f'''(x)$.

{Ans.

$f'(x) = \frac{1}{x}, \quad x \geq 1$

$= 1, \quad x < 1;$

$f''(x) = -\frac{1}{x^2}, \quad x > 1$

$= 0, \quad x < 1;$

$f'''(x) = \frac{2}{x^3}, \quad x > 1$

$= 0, \quad x < 1$ }

324. Let $f(x) = |x^3|$, then find $f'''(0)$. {Ans. does not exist}

325. Given $f(x) = \ln x, \quad x \ge 1$

$= ax^2 + bx + c, \quad x < 1,$

find the constants a, b, c such that $f''(1)$ exists. {Ans. $a = -\frac{1}{2}$, b = 2, $c = -\frac{3}{2}$ }

CATEGORY-4.15. ADDITIONAL QUESTIONS

326. Let $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = \ln x$. If $F(x) = (hogof)(x)$, then find $F''(x)$. {Ans. $-2\operatorname{cosec}^2 x$ }

327. If $f(x) = \sqrt{x^2 + 9}$, then find $\lim_{x \to 4} \frac{f(x) - f(4)}{x - 4}$. {Ans. $\frac{4}{5}$ }

328. Let $f(2) = 4$ and $f'(2) = 4$. Then find $\lim_{x \to 2} \frac{xf(2) - 2f(x)}{x - 2}$. {Ans. -4}

329. If $f(9) = 9$, $f'(9) = 4$, then find $\lim_{x \to 9} \frac{\sqrt{f(x)} - 3}{\sqrt{x} - 3}$. {Ans. 4}

330. If $f'(2) = 2$, $f''(2) = 1$, then find $\lim_{x \to 2} \frac{x^2 - 2f'(x)}{x - 2}$. {Ans. 2}

331. If $x = a\cos^3\theta$, $y = a\sin^3\theta$ then find $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$. {Ans. $|\sec\theta|$ }

332. If $y = e^{\sin^{-1} x}$ and $u = \ln x$, then find $\frac{d}{dx}\left(\frac{1}{y}\frac{dy}{du}\right)$. {Ans. $\frac{1}{(1 - x^2)^{\frac{3}{2}}}$ }

333. Find the derivative of $\sec^{-1}\left(\frac{1}{2x^2 + 1}\right)$ w.r.t. $\sqrt{1 + 3x}$ at $x = -\frac{1}{3}$. {Ans. does not exist}

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Mathematics for IIT-JEE

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PART-I
DIFFERENTIAL CALCULUS

CHAPTER-5
APPLICATIONS OF DERIVATIVES

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CHAPTER-5
APPLICATIONS OF DERIVATIVES

LIST OF THEORY SECTIONS

LIST OF QUESTION CATEGORIES

CHAPTER-5
APPLICATIONS OF DERIVATIVES

SECTION-5.1. RATE OF CHANGE OF A FUNCTION

1. Mean (Average) rate of change of a function in an interval

i. The mean (average) rate of change of a function $f(x)$ with respect to *x* in the interval $[a,b]$ is given by $\frac{f(b)-f(a)}{b-a}$, which is also equal to the slope of the chord joining the end points of the curve of the function in the interval $[a,b]$.

2. Instantaneous rate of change of a function at a point

i. The instantaneous rate of change of a function $f(x)$ with respect to *x* at a point $x=a$ is given by $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, which is $f'(a)$, which is also equal to the slope of tangent of the curve of the function at $x=a$.

3. Stationary point

i. *Stationary points* of a function are those points where it's first derivative is zero.

ii. The instantaneous rate of change of a function $f(x)$ with respect to *x* at the Stationary point is zero.

iii. Stationary points of graphical functions:- Points at which tangent to the curve of the function is parallel to *x*-axis are Stationary points.

iv. Stationary points of Basic functions.

4. Approximate value of a function

i. If $x\approx a$ then $f(x)\approx f(a)+(x-a)f'(a)$.

SECTION-5.2. MONOTONICITY OF A FUNCTION

1. Function increasing, decreasing, non-decreasing, non-increasing, constant in an open interval

i. A function $f(x)$ is said to be strictly increasing (or increasing) in an interval (a,b) if $x_1>x_2 \Rightarrow f(x_1)>f(x_2)\ \forall x_1, x_2\in(a,b)$.

ii. A function $f(x)$ is said to be strictly decreasing (or decreasing) in an interval (a,b) if $x_1>x_2 \Rightarrow f(x_1)<f(x_2)\ \forall x_1, x_2\in(a,b)$.

iii. A function $f(x)$ is said to be non-decreasing in an interval (a,b) if $x_1>x_2 \Rightarrow f(x_1)\geq f(x_2)\ \forall x_1, x_2\in(a,b)$.

iv. A function $f(x)$ is said to be non-increasing in an interval (a,b) if $x_1>x_2 \Rightarrow f(x_1)\leq f(x_2)\ \forall x_1, x_2\in(a,b)$.

v. A function $f(x)$ is said to be constant in an interval (a,b) if $x_1\neq x_2 \Rightarrow f(x_1)=f(x_2)\ \forall x_1, x_2\in(a,b)$.

vi. A function $f(x)$ may be neither increasing, decreasing or constant in an interval (a,b).

2. Function increasing and decreasing at a point

i. A function $f(x)$ is said to be increasing (decreasing) at a point *a* if there is a neighbourhood of point *a* in which $f(x)$ is increasing (decreasing), otherwise $f(x)$ is not increasing (decreasing) at the point *a*.

3. Function increasing and decreasing in a closed interval and semi-closed interval

i. A function $f(x)$ is said to be increasing (decreasing) in a closed interval $[a,b]$ if it is increasing (decreasing) (a,b) and also increasing (decreasing) at points *a* and *b*.

ii. A function $f(x)$ is said to be increasing (decreasing) in a semi-closed interval $(a,b]$ if it is increasing (decreasing) (a,b) and also increasing (decreasing) at point *b*.

iii. A function $f(x)$ is said to be increasing (decreasing) in a semi-closed interval $[a,b)$ if it is increasing (decreasing) (a,b) and also increasing (decreasing) at point *a*.

4. Intervals of monotonicity of a function

i. Intervals in which a function increases or decreases are called *intervals of monotonicity* of the function.

ii. A function increasing (decreasing) in its domain is called a monotonous function.

5. Intervals of monotonicity of a graphical function

i. A function is increasing in an interval if its curve moves upwards as *x* increases.

ii. A function is decreasing in an interval if its curve moves downwards as *x* increases.

6. Intervals of monotonicity of basic functions

7. Testing a defined analytical function for monotonicity

i. <u>Theorem:-</u> If $f(x)$ is continuous in an interval (a,b) and $f'(x)>0$ in the interval, may be zero or non-existent at finite number of isolated points, then $f(x)$ is increasing in the interval.

ii. <u>Theorem:-</u> If $f(x)$ is continuous in an interval (a,b) and $f'(x)<0$ in the interval, may be zero or non-existent at finite number of isolated points, then $f(x)$ is decreasing in the interval.

iii. <u>Theorem:-</u> If $f(x)$ is continuous in an interval (a,b) and $f'(x)=0$ in the interval, then $f(x)$ is constant in the interval.

SECTION-5.3. EXTREMUM (MAXIMA/ MINIMA) OF A FUNCTION

1. Definition of extremum (maxima/ minima or local maxima/ local minima) of a function

i. A point *a* belonging to the domain of the function $f(x)$ is called a point of *maxima* (or local maxima) if there is a deleted neighbourhood of point *a* in which $f(x)< f(a)$, otherwise the point *a* is not a point of maxima.

ii. A point *a* belonging to the domain of the function $f(x)$ is called a point of *minima* (or local minima) if there is a deleted neighbourhood of point *a* in which $f(x)> f(a)$, otherwise the point *a* is not a point of minima.

iii. A point may be neither maxima nor minima.

iv. The points of maxima and minima are called the extremum points of the function.

2. Extremum of a graphical function

3. Extremum of basic functions

4. Testing a point for extremum

i. By first principles

ii. First derivative sign change test

iii. Second derivative test

iv. Higher order derivative test

5. Testing a defined analytical function for extremum

i. <u>Theorem:-</u> If point *a* is an extremum of the function $f(x)$ then $f'(a)$ is either zero or does not exist.

ii. <u>Critical point:-</u> *Critical points* of a function are those points in its domain at which its first derivative is either zero or does not exist.

iii. Critical points of graphical function

iv. Critical points of Basic functions
v. To test a defined analytical function for extremum, find all critical points and test each critical point for extremum.

SECTION-5.4. GREATEST AND LEAST VALUE OF A FUNCTION IN AN INTERVAL

1. **Definition of Greatest (largest or maximum or global maximum) value & least (smallest or minimum or global minimum) value of a function in an interval**
   i. A function $f(x)$ has a *greatest value* G in an interval A if the function attains the value G within the interval A and $f(x) \le G\ \forall x \in A$.
   ii. A function $f(x)$ has a *least value* L in an interval A if the function attains the value L within the interval A and $f(x) \ge L\ \forall x \in A$.
   iii. A function may not have a greatest or least value in an interval.
   iv. A function may have same greatest and least value in an interval.
2. **Greatest & least value of a graphical function**
3. **Greatest & least value of basic functions**
4. **Greatest & least value of defined analytical functions**
   i. <u>Theorem:-</u> If a function $f(x)$ is continuous in a closed interval $[a,b]$ then $f(x)$ must have greatest value and least value in the interval $[a,b]$ and the greatest value and least value are attained either at the critical points within the interval or at the end-points of the interval.
5. **Intermediate value theorem**
   i. If $f(x)$ is continuous in a interval and has greatest value G and least value L in the interval then it assumes all intermediate values between L and G in the interval, i.e. if C is any value between L and G then there exists at least one point α in the interval such that $f(\alpha) = C$.

SECTION-5.5. CONCAVITY OF A FUNCTION

1. **Concavity of a function in an interval, concave upward, concave downward**
   i. The curve of the function $f(x)$ is said to be *concave upwards* in the interval (a,b) if the function $f(x)$ has non-vertical tangent at every point of the interval; and the curve of the function lies not below than any of its tangents within the interval.
   ii. The curve of the function $f(x)$ is said to be *concave downwards* in the interval (a,b) if the function $f(x)$ has non-vertical tangent at every point of the interval; and the curve of the function lies not above than any of its tangents within the interval.
   iii. A function may be neither concave upwards nor concave downwards in an interval.
2. **Concavity of a function at a point**
   i. A function $f(x)$ is said to be concave upwards (concave downwards) at a point a if there is a neighbourhood of point a in which $f(x)$ is concave upwards (concave downwards), otherwise $f(x)$ is not concave upwards (concave downwards) at the point a.
3. **Intervals of concavity of a function**
   i. Intervals in which a function is concave upwards or concave downwards are called *intervals of concavity* of the function.
4. **Concavity of a graphical function**
5. **Concavity of basic functions**
6. **Testing a defined analytical function for concavity**

i. Theorem:- If $f(x)$ is differentiable in an interval (a,b) and $f''(x)>0$ in the interval, may be zero or non-existent at finite number of isolated points, then $f(x)$ is concave upwards in the interval.

ii. Theorem:- If $f(x)$ is differentiable in an interval (a,b) and $f''(x)<0$ in the interval, may be zero or non-existent at finite number of isolated points, then $f(x)$ is concave downwards in the interval.

SECTION-5.6. POINT OF INFLECTION

1. Definition of point of inflection of a function

i. A point *a* is said to be a *point of inflection* of the function $f(x)$ if there is tangent at the point *a* and the function has different concavity on the left and on the right of the point a.

2. Point of infection of a graphical function

i. The point on the curve of the function at which tangent cuts through the curve, is a point of inflection.

3. Point of infection of basic functions

4. Testing a point for Point of inflection

i. If there is tangent at point *a* and $f''(x)$ has different signs on the left and on the right of the point *a*, then *a* is a point of inflection.

5. Testing a defined analytical function for Point of inflection

i. Theorem:- If point *a* is a point of inflection of the function $f(x)$ then $f''(a)$ is either zero or does not exist.

ii. To test a defined analytical function for Point of inflection, find all points where $f''(x)$ is either zero or does not exist and test each such point for Point of inflection.

SECTION-5.7. ROLLE'S THEOREM AND LAGRANGE'S MEAN VALUE THEOREM

1. Rolle's theorem

i. If $f(x)$ is continuous in $[a,b]$ and differential in (a,b) and if $f(a)=f(b)$, then there exists a point $c\in(a,b)$ such that $f'(c)=0$.

2. Lagrange's Mean Value theorem

i. If $f(x)$ is continuous in $[a,b]$ and differential in (a,b), then there exists a point $c\in(a,b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

EXERCISE-5

CATEGORY-5.1. RATE OF CHANGE OF A FUNCTION

1. Find the mean rate of change of the following functions in the interval $[1,2]$:-
   i. $f(x)=x^2$. {Ans. 3}
   ii. $f(x)=x^3$. {Ans. 7}
   iii. $f(x)=\sqrt{x}$. {Ans. 0.414}
   iv. $f(x)=\frac{1}{x}$. {Ans. –0.5}
   v. $f(x)=e^x$. {Ans. 4.67}
   vi. $f(x)=\ln x$. {Ans. 0.693}
   vii. $f(x)=\sin x$. {Ans. 0.068}
   viii. $f(x)=\cos x$. {Ans. –0.956}
2. For what value of *a*, the mean rate of change of the function $f(x)=x^2$ in the interval $[a,a+1]$ is 2? {Ans. $\frac{1}{2}$}
3. Find the instantaneous rate of change of the following functions at $x=1$ and also find Stationary points:-
   i. $f(x)=x^2$. {Ans. 2; 0}
   ii. $f(x)=x^3$. {Ans. 3; 0}
   iii. $f(x)=\sqrt{x}$. {Ans. $\frac{1}{2}$; no Stationary point}
   iv. $f(x)=\frac{1}{x}$. {Ans. -1; no Stationary point}
   v. $f(x)=e^x$. {Ans. 2.72; no Stationary point}
   vi. $f(x)=\ln x$. {Ans. 1; no Stationary point}
   vii. $f(x)=\sin x$. {Ans. 0.54; $(2n+1)\frac{\pi}{2}$}
   viii. $f(x)=\cos x$. {Ans. –0.84; $n\pi$}
4. For what value of *a*, the mean rate of change of the function $f(x)=x^3$ in the interval $[-1,a]$ is equal to the instantaneous rate of change at *a*? {Ans. $\frac{1}{2}$}
5. Find the approximate value of the following:-
   i. $\cos 31°$. {Ans. 0.851}
   ii. $\log 10.21$. {Ans. 1.009}
   iii. $\sqrt[5]{33}$. {Ans. 2.012}
   iv. $\cot 45°10'$. {Ans. 0.994}

CATEGORY-5.2. MONOTONICITY OF A FUNCTION

6. Test the following functions for monotonicity and find stationary points:-

i. $f(x)=2x^3-9x^2+12x+29$. {Ans. $(-\infty,1)$ increasing, $(1,2)$ decreasing, $(2,\infty)$ increasing; 1, 2}

ii. $y=x^3-3x^2+6x-17$. {Ans. $(-\infty,\infty)$ increasing}

iii. $f(x)=x^3-3x$. {Ans. $(-\infty,-1)$ increasing, $(-1,1)$ decreasing, $(1,\infty)$ increasing; -1, 1}

iv. $y=x^2(x-3)^2$. {Ans. $(-\infty,0)$ decreasing, $\left(0,\frac{3}{2}\right)$ increasing, $\left(\frac{3}{2},3\right)$ decreasing, $(3,\infty)$ increasing; 0, $\frac{3}{2}$, 3}

v. $f(x)=x^9+3x^7+64$. {Ans. $(-\infty,\infty)$ increasing; 0}

vi. $f(x)=x\ln x$. {Ans. $\left(0,\frac{1}{e}\right)$ decreasing, $\left(\frac{1}{e},\infty\right)$ increasing; $\frac{1}{e}$ }

vii. $f(x)=\frac{\ln x}{x}$. {Ans. $(0,e)$ increasing, (e,∞) decreasing; e}

viii. $f(x)=\frac{x}{\ln x}$. {Ans. $(0,1)$ decreasing, $(1,e)$ decreasing, (e,∞) increasing; e}

ix. $f(x)=x^x$. {Ans. $\left(0,\frac{1}{e}\right)$ decreasing, $\left(\frac{1}{e},\infty\right)$ increasing; $\frac{1}{e}$ }

x. $f(x)=x^{\frac{1}{x}}$. {Ans. $(0,e)$ increasing, (e,∞) decreasing; e}

xi. $f(x)=xe^x$. {Ans. $(-\infty,-1)$ decreasing, $(-1,\infty)$ increasing; -1}

xii. $f(x)=xe^{-x}$. {Ans. $(-\infty,1)$ increasing, $(1,\infty)$ decreasing; 1}

xiii. $f(x)=e^{\frac{1}{x}}$. {Ans. $(-\infty,0)$ decreasing, $(0,\infty)$ decreasing}

xiv. $f(x)=xe^{\frac{1}{x}}$. {Ans. $(-\infty,0)$ increasing, $(0,1)$ decreasing, $(1,\infty)$ increasing; 1}

xv. $f(x)=\log_x(\ln x)$. {Ans. $(1,e^e)$ increasing, (e^e,∞) decreasing; e^e }

xvi. $f(x)=\ln(1+x)-\frac{2x}{2+x}$. {Ans. $(-1,\infty)$ increasing; 0}

xvii. $f(x)=x+\sin x$. {Ans. increasing for all x; $(2n+1)\pi,\ n\in I$ }

xviii. $f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x},\quad 0<x<2\pi$. {Ans. $\left(0,\frac{\pi}{2}\right)$ increasing, $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ decreasing, $\left(\frac{3\pi}{2},2\pi\right)$ increasing; $\frac{\pi}{2}$, $\frac{3\pi}{2}$ }

xix. $f(x)=\tan^{-1}x-\frac{1}{2}\ln|x|$. {Ans. $(-\infty,0)$ increasing, $(0,\infty)$ decreasing, 1}

xx. $y=x-\cot^{-1}x-\ln\left(x+\sqrt{x^2+1}\right)$. {Ans. $(-\infty,\infty)$ increasing}

xxi. $f(x)=2x^2-\ln|x|$. {Ans. $\left(-\infty,-\frac{1}{2}\right)$ decreasing, $\left(-\frac{1}{2},0\right)$ increasing, $\left(0,\frac{1}{2}\right)$ decreasing, $\left(\frac{1}{2},\infty\right)$ increasing; $-\frac{1}{2}$, $\frac{1}{2}$ }

7. Show that the function
$f(x)=\sin x,\quad x$ is a rational no.

$= x, \quad x$ is an irrationalno.

has positive derivative at $x = 0$ but $f(x)$ is not increasing at $x = 0$.

8. Show that the function

$$f(x) = \frac{x}{2} + x^2 \sin\frac{1}{x}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

is continuous and differentiable in any neighbourhood of $x = 0$ and $f'(0)$ is positive but $f(x)$ is not increasing at $x = 0$.

9. For what values of b the function $f(x) = \sin x - bx + c$ is decreasing in the interval $(-\infty, \infty)$? {Ans. $b \geq 1$}

10. If $g(x) = f(x) + f(1-x)$ and $f''(x) < 0; 0 \leq x \leq 1$, show that $g(x)$ increases in $0 < x < \frac{1}{2}$ and decreases in $\frac{1}{2} < x < 1$.

11. Given $g(x) = f(x^2 - x - 10) + f(14 + x - x^2)$ and $f''(x) > 0$ for all real *x*, except at finite no. of real values of *x* for which $f''(x) = 0$. Discuss the monotonicity of the function $g(x)$. {Ans. $(-\infty, -3)$ decreasing, $\left(-3, \frac{1}{2}\right)$ increasing, $\left(\frac{1}{2}, 4\right)$ decreasing, $(4, \infty)$ increasing}

CATEGORY-5.3. EXTREMUM POINTS OF A FUNCTION

12. Investigate the following functions for extremum at $x = 0$:-

i. $f(x) = \sin x - x$. {Ans. no extremum}

ii. $f(x) = \sin x - x + \frac{x^3}{3!}$. {Ans. no extremum}

iii. $f(x) = \sin x - x + \frac{x^3}{3!} - \frac{x^4}{4!}$. {Ans. maxima}

iv. $f(x) = e^{\frac{1}{x}}, \quad x \neq 0$

$= 0, \quad x = 0$. {Ans. minima}

v. $f(x) = \cosh x + \cos x$. {Ans. minima}

vi. $f(x) = \cos x - 1 + \frac{x^2}{2!} - \frac{x^3}{3!}$. {Ans. no extremum}

vii. $f(x) = \cos x - 1 + \frac{x^2}{2}$. {Ans. minima}

viii. $f(x) = x + x^{\frac{2}{3}}$. {Ans. minima}

ix. $f(x) = x^2 + x^{\frac{1}{3}}$. {Ans. no extremum}

x. $f(x) = x^{\frac{4}{3}} + 2$. {Ans. minima}

xi. $f(x) = x^{\frac{5}{3}} - 3$. {Ans. no extremum}

xii. $f(x)=\sin\frac{1}{x}, \quad x\neq 0$

$=2, \quad x=0$. {Ans. maxima}

xiii. $f(x)=\left(2-\sin\frac{1}{x}\right)|x|, \quad x\neq 0$

$=0, \quad x=0$. {Ans. minima}

xiv. $f(x)=\left(\cos\frac{1}{x}-2\right)x^2, \quad x\neq 0$

$=0, \quad x=0$. {Ans. maxima}

xv. $f(x)=x^2, \quad x$ is a rational no.

$=x^4, \quad x$ is an irrational no. {Ans. minima}

13. Test the following functions for extremum:-

i. $f(x)=2x^3-15x^2-84x+8$. {Ans. maxima at -2, minima at 7}

ii. $f(x)=x^3-6x^2+9x-8$. {Ans. maxima at 1, minima at 3}

iii. $f(x)=-\frac{3}{4}x^4-8x^3-\frac{45}{2}x^2+105$. {Ans. maxima at 0, -5, minima at -3}

iv. $f(x)=\frac{3}{4}x^4-x^3-9x^2+7$. {Ans. maxima at 0, minima at -2, 3}

v. $f(x)=x^4-8x^3+22x^2-24x+12$. {Ans. maxima at 2, minima at 1, 3}

vi. $f(x)=x(x+1)^3(x-3)^2$. {Ans. maxima at $\frac{3+\sqrt{17}}{4}$, minima at $\frac{3-\sqrt{17}}{4}$,3}

vii. $f(x)=\frac{x^2-3x+2}{x^2+2x+1}$. {Ans. minima at $\frac{7}{5}$}

viii. $f(x)=3\sqrt[3]{x^2}-x^2$. {Ans. maxima at ± 1, minima at 0}

ix. $f(x)=\sqrt[3]{(x-1)^2}+\sqrt[3]{(x+1)^2}$. {Ans. maxima at 0, minima at ± 1}

x. $f(x)=-2x, \quad x<0$

$=3x+5, \quad x\geq 0$. {Ans. no extremum}

xi. $f(x)=2x^2+3, \quad x\neq 0$

$=4, \quad x=0$. {Ans. maxima at 0}

xii. $f(x)=\frac{50}{3x^4+8x^3-18x^2+60}$. {Ans. maxima at -3,1, minima at 0}

xiii. $f(x)=\sqrt{e^{x^2}-1}$. {Ans. minima at 0}

xiv. $f(x)=xe^x$. {Ans. minima at -1}

xv. $f(x)=x^4e^{-x^2}$. {Ans. maxima at $\pm\sqrt{2}$, minima at 0}

xvi. $f(x)=x^2e^{-x}$. {Ans. maxima at 2, minima at 0}

xvii. $f(x)=\frac{4x}{x^2+4}$. {Ans. maxima at 2, minima at -2}

xviii. $f(x)=-x^2\sqrt[5]{(x-2)^2}$. {Ans. maxima at 0, 2, minima at $\frac{5}{3}$}

xix. $f(x)=\dfrac{14}{x^4-8x^2+2}$. {Ans. maxima at ± 2, minima at 0}

xx. $f(x)=\sqrt[3]{2x^3+3x^2-36x}$. {Ans. maxima at -3, minima at 2}

xxi. $f(x)=x^2\ln x$. {Ans. minima at $e^{-\frac{1}{2}}$}

xxii. $f(x)=x\ln^2 x$. {Ans. maxima at e^{-2}, minima at 1}

xxiii. $f(x)=\dfrac{|x-1|}{x^2}$. {Ans. maxima at 2, minima at 1}

xxiv. $f(x)=|x|+|x-1|+|x-2|$. {Ans. minima at 1}

xxv. $f(x)=\sin^4 x+\cos^4 x,\ 0<x<\dfrac{\pi}{2}$. {Ans. minima at $\dfrac{\pi}{4}$}

xxvi. $f(x)=\sin 2x-x,\ -\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2}$. {Ans. maxima at $\dfrac{\pi}{6}$, minima at $-\dfrac{\pi}{6}$}

xxvii. $f(x)=\sin x+\dfrac{1}{2}\cos 2x,\ 0\le x\le\dfrac{\pi}{2}$. {Ans. maxima at $\dfrac{\pi}{6}$, minima at $\dfrac{\pi}{2}$}

14. The function $f(x)=a\sin x+\dfrac{1}{3}\sin 3x$ has maximum value at $x=\dfrac{\pi}{3}$. Find the value of a. {Ans. 2}

15. Given

$$f(x)=|x-2|+\ln(a^2-1),\quad x<2$$
$$=3x+5,\quad x\ge 2.$$

Find values of a for which $f(x)$ has local minima at $x=2$. {Ans. $a\in\left(-\infty,-\sqrt{e^{11}+1}\right]\cup\left[\sqrt{e^{11}+1},\infty\right)$}

16. Find the polynomial of degree 6 which satisfies $\lim\limits_{x\to 0}\left(1+\dfrac{f(x)}{x^3}\right)^{\frac{1}{x}}=e^2$ and has local maxima at $x=1$ and local minima at $x=0$ & $x=2$. {Ans. $\dfrac{2}{3}x^6-\dfrac{12}{5}x^5+2x^4$}

CATEGORY-5.4. GREATEST AND LEAST VALUE OF A FUNCTION

17. Find greatest & least value of the following functions in the indicated intervals:-

i. $f(x)=x^3-3x$ in $[0,2]$. {Ans. 2, –2}

ii. $f(x)=2x^3-3x^2-12x+1$ in $\left[-2,\dfrac{5}{2}\right]$. {Ans. $8,-19$}

iii. $f(x)=2x^3-24x+107$ in $[1,3]$. {Ans. 89, 75}

iv. $f(x)=x^2\ln x$ in $[1,e]$. {Ans. $e^2,0$}

v. $f(x)=\sqrt{(1-x^2)(1+2x^2)}$ in $[-1,1]$. {Ans. $\frac{3}{2\sqrt{2}},0$}

vi. $f(x)=\cos^{-1}(x^2)$ in $\left[-\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right]$. {Ans. $\frac{\pi}{2},\frac{\pi}{3}$}

vii. $f(x)=x+\sqrt{x}$ in $[0,4]$. {Ans. $6,0$}

viii. $f(x)=\frac{x^4}{4}-\frac{2x^3}{3}-\frac{3x^2}{2}+2$ in $[-2,4]$. {Ans. $\frac{16}{3},-\frac{37}{4}$ }

ix. $f(x)=\sqrt{4-x^2}$ in $[-2,2]$. {Ans. $2,0$ }

x. $f(x)=\tan^{-1}x-\frac{1}{2}\ln x$ in $\left[\frac{1}{\sqrt{3}},\sqrt{3}\right]$. {Ans. $\frac{\pi}{6}+\frac{\ln 3}{4},\frac{\pi}{3}-\frac{\ln 3}{4}$ }

xi. $f(x)=2\sin x+\sin 2x$ in $\left[0,\frac{3}{2}\pi\right]$. {Ans. $\frac{3\sqrt{3}}{2},-2$ }

xii. $f(x)=x-2\ln x$ in $[1,e]$. {Ans. $1,2-2\ln 2$ }

xiii. $f(x)=3x^2+6x+8$. {Ans. No greatest value, 5}

xiv. $f(x)=-|x-1|+5$. {Ans. 5, No least value}

xv. $f(x)=\sin 3x+4$. {Ans. 5, 3}

xvi. $f(x)=x^3+1$. {Ans. No greatest value, no least value}

xvii. $f(x)=\sin(\sin x)$. {Ans. $-\sin 1,\ \sin 1$}

xviii. $f(x)=|x+3|$. {Ans. No greatest value, 0}

xix. $$f(x)=\begin{cases}2x^2+\frac{2}{x^2}, & x\in[-2,0)\cup(0,2]\\ =1, & x=0\end{cases}$$

in $[-2,2]$. {Ans. no greatest value, 1}

18. Prove that:-

i. $e^x>1+x,\quad x\neq 0$.

ii. $x-\frac{x^3}{6}<\sin x<x,\quad x>0$.

iii. $\frac{x}{1+x}\le\ln(1+x)\le x,\quad x>-1$.

iv. $\frac{x}{1+x^2}<\tan^{-1}x<x,\quad x>0$.

v. $\ln x>\frac{2(x-1)}{x+1},\quad x>1$.

vi. $2x\tan^{-1}x\ge\ln(1+x^2)$.

vii. $\ln(1+x)>\frac{\tan^{-1}x}{1+x},\quad x>0$.

viii. $\sin x<x-\frac{x^3}{6}+\frac{x^5}{120},\quad x>0$.

ix. $\sin x+\tan x>2x,\quad 0<x<\frac{\pi}{2}$.

x. $\cosh x>1+\frac{x^2}{2},\quad x\neq 0$.

xi. $1+x\ln(x+\sqrt{x^2+1})\ge\sqrt{1+x^2}$.

xii. $2\sin x + \tan x \geq 3x$ for $0 \leq x < \dfrac{\pi}{2}$.

19. The function $f(x) = x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval $[0,2]$ at $x = 1$. Find the value of a. {Ans. 120}

20. Let

$$f(x) = -x^3 + \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}, \quad 0 \leq x < 1$$
$$= 2x - 3, \quad 1 \leq x \leq 3.$$

Find all possible real values of b such that $f(x)$ has the smallest value at x = 1. {Ans. $b \in (-2,-1) \cup [1,\infty)$ }

21. Use the function $f(x) = x^{\frac{1}{x}}, x > 0$ to determine the bigger of the two numbers e^{π} & π^{e}. {Ans. e^{π} }

CATEGORY-5.5. CONCAVITY OF A FUNCTION

22. Find the intervals of concavity of the following functions:-

i. $f(x) = x^4 + x^3 - 18x^2 + 24x - 12$. {Ans. concave upward in $(-\infty,-2)$, concave downward in $\left(-2, \dfrac{3}{2}\right)$, concave upward in $\left(\dfrac{3}{2}, \infty\right)$}

ii. $f(x) = 3x^5 - 5x^4 + 3x - 2$. {Ans. concave downward in $(-\infty,1)$, concave upward in $(1,\infty)$}

iii. $f(x) = x^6 - 10x^4$. {Ans. concave upward in $(-\infty,-2)$; concave downward in $(-2,2)$; concave upward in $(2,\infty)$}

iv. $f(x) = \ln(x^2 - 1)$. {Ans. concave downward in $(-\infty,-1)$ and $(1,\infty)$}

v. $f(x) = (x+1)^4 + e^x$. {Ans. concave upward in $(-\infty,\infty)$}

vi. $f(x) = x^2 \ln x$. {Ans. concave downward in $\left(0, e^{-\frac{3}{2}}\right)$; concave upward in $\left(e^{-\frac{3}{2}}, \infty\right)$}

vii. $f(x) = x + x^{\frac{4}{3}}$. {Ans. concave upward in $(-\infty,\infty)$}

viii. $f(x) = x + x^{\frac{5}{3}} + 1$. {Ans. concave downward in $(-\infty,0)$; concave upward in $(0,\infty)$}

ix. $f(x) = x + x^{\frac{2}{3}}$. {Ans. concave downward in $(-\infty,0)$; concave downward in $(0,\infty)$}

x. $f(x) = x^2, \quad x \leq 0$
$= x^3, \quad x > 0$. {Ans. concave upward in $(-\infty,\infty)$}

xi. $f(x) = x^3, \quad x \leq 0$
$= x^2, \quad x > 0$. {Ans. concave downward in $(-\infty,0)$; concave upward in $(0,\infty)$}

xii. $f(x) = x^3, \quad x \leq 1$
$= x^2, \quad x > 1$. {Ans. concave downward in $(-\infty,0)$; concave upward in $(0,1)$; concave upward in $(1,\infty)$}

CATEGORY-5.6. POINT OF INFLECTION

23. Test the indicated points for point of inflection:-

i. $f(x)=x^3-5x^2+3x-5$ at $x=1,\frac{5}{3},2$. {Ans. $\frac{5}{3}$ is point of inflection; 1, 2 are not points of inflection}

ii. $f(x)=x^4-12x^3+48x^2$ at $x=1,2,3,4$. {Ans. 2, 4 are points of inflection; 1, 3 are not points of inflection}

iii. $f(x)=x+x^{\frac{5}{3}}-2$ at $x=0$. {Ans. 0 is point of inflection}

iv. $f(x)=x^2+x^{\frac{4}{3}}+1$ at $x=0$. {Ans. 0 is not point of inflection}

v. $f(x)=x+x^{\frac{2}{3}}+4$ at $x=0$. {Ans. 0 is not point of inflection}

vi. $f(x)=x+x^{\frac{3}{5}}-3$ at $x=0$. {Ans. 0 is point of inflection}

vii. $f(x)=\sin x+\frac{x^3}{3!}-\frac{x^5}{5!}$ at $x=0$. {Ans. 0 is point of inflection}

viii. $f(x)=e^x-\frac{x^2}{2}-\frac{x^3}{6}$ at $x=0$. {Ans. 0 is not point of inflection}

ix. $f(x)=\sin x, \quad x\ge 0$

$= x-\frac{x^3}{6}, \quad x<0$

at $x=0$. {Ans. 0 is point of inflection}

24. Test the following functions for concavity & find points of inflection:-

i. $f(x)=x+36x^2-2x^3-x^4$. {Ans. concave downward in $(-\infty,-3)$, concave upward in $(-3,2)$, concave downward in $(2,\infty)$; –3, 2}

ii. $f(x)=3x^4-8x^3+6x^2+12$. {Ans. concave upward in $\left(-\infty,\frac{1}{3}\right)$, concave downward in $\left(\frac{1}{3},1\right)$, concave upward in $(1,\infty)$; $\frac{1}{3}$, 1}

iii. $f(x)=(x+2)^6+2x+2$. {Ans. concave upward in $(-\infty,\infty)$}

iv. $f(x)=\frac{x}{1+x^2}$. {Ans. concave downward in $(-\infty,-\sqrt{3})$, concave upward in $(-\sqrt{3},0)$, concave downward in $(0,\sqrt{3})$, concave upward in $(\sqrt{3},\infty)$; 0, $\pm\sqrt{3}$ }

v. $f(x)=\ln(1+x^2)$. {Ans. concave downward in $(-\infty,-1)$, concave upward in $(-1,1)$, concave downward in $(1,\infty)$; –1, 1}

vi. $f(x)=x^4(12\ln x-7)$. {Ans. concave downward in $(0,1)$, concave upward in $(1,\infty)$; 1}

vii. $f(x)=x\ln x$. {Ans. concave upward in $(0,\infty)$}

viii. $f(x)=\frac{\ln x}{x}$. {Ans. concave downward in $\left(0,e^{\frac{3}{2}}\right)$, concave upward in $\left(e^{\frac{3}{2}},\infty\right)$; $e^{\frac{3}{2}}$ }

ix. $f(x)=x^x$. {Ans. concave upward in $(0,\infty)$}

x. $f(x)=xe^x$. {Ans. concave downward in $(-\infty,-2)$, concave upward in $(-2,\infty)$; –2}

xi. $f(x)=xe^{-x}$. {Ans. concave downward in $(-\infty,2)$, concave upward in $(2,\infty)$; 2}

xii. $f(x)=e^{\frac{1}{x}}$. {Ans. concave downward in $\left(-\infty,-\frac{1}{2}\right)$, concave upward in $\left(-\frac{1}{2},0\right)$, concave upward in $(0,\infty)$; $-\frac{1}{2}$ }

xiii. $f(x)=xe^{\frac{1}{x}}$. {Ans. concave downward in $(-\infty,0)$, concave upward in $(0,\infty)$ }

xiv. $f(x)=x+x^{\frac{5}{3}}$. {Ans. concave downward in $(-\infty,0)$, concave upward in $(0,\infty)$; 0}

xv. $f(x)=\sqrt{(x-1)^5}+5\sqrt{(x-1)^3}$. {Ans. concave upward in $(1,\infty)$}

xvi. $f(x)=2-\left|x^5-1\right|$. {Ans. concave downward in $(-\infty,0)$, concave upward in $(0,1)$, concave downward in $(1,\infty)$; 0}

xvii. $f(x)=x-\sqrt[5]{(x-3)^2}$. {Ans. concave upward in $(-\infty,3)$ and $(3,\infty)$ }

xviii. $f(x)=2x+x^{\frac{4}{3}}+5$. {Ans. concave upward in $(-\infty,\infty)$ }

xix. $f(x)=3x+x^{\frac{3}{5}}-2$. {Ans. concave upward in $(-\infty,0)$, concave downward in $(0,\infty)$; 0}

xx. $f(x)=x^3+x^2+x+1, \quad x\le 0$

$=-x^2+x+1, \quad x>0$. {Ans. concave downward in $\left(-\infty,-\dfrac{1}{3}\right)$, concave upward in $\left(-\dfrac{1}{3},0\right)$, concave downward in $(0,\infty)$; $-\dfrac{1}{3}$, 0}

25. Given

$f(x)=e^x, \quad x\ge 0$

$=ax^3+bx^2+cx+d, \quad x<0$.

Find the constants *a*, *b*, *c*, *d* if $f''(0)$ exists and $f(x)$ has a point of inflection at $x=-1$.

{Ans. $a=\dfrac{1}{6}$, $b=\dfrac{1}{2}$, $c=d=1$ }

CATEGORY-5.7. ROLLE'S THEOREM

26. Verify Rolle's theorem for the following functions:-

i. $f(x)=2x^3+x^2-4x-2$ in $\left[-\sqrt{2},\sqrt{2}\right]$.

ii. $f(x)=\sin x$ in $[0,\pi]$.

iii. $f(x)=\tan x$ in $[0,\pi]$.

iv. $f(x)=\cos\dfrac{1}{x}$ in $[-1,1]$.

v. $f(x)=x(x+3)e^{-\frac{x}{2}}$ in $[-3,0]$.

vi. $f(x)=e^x(\sin x-\cos x)$ in $\left[\dfrac{\pi}{4},\dfrac{5\pi}{4}\right]$.

vii. $f(x)=|x|$ in $[-1,1]$.

viii. $f(x)=3+(x-2)^{\frac{2}{3}}$ in $[1,3]$.

ix. $f(x)=\ln\left(\dfrac{x^2+ab}{(a+b)x}\right)$ in $[a,b]$, $a>0$.

x. $f(x)=(x-a)^m(x-b)^n$ in $[a,b]$, where *m* and *n* are positive integers.

27. Show that the equation $x^3-3x+c=0$ cannot have two different roots in the interval $(0,1)$.
28. Prove that the equation $3x^5+15x-8=0$ has only one real solution.
29. Prove that the polynomial x^4-4x-1 has exactly two different real roots.
30. Show that the equation $xe^x=2$ has only one solution which lies in the interval $(0,1)$.
31. Show that the equation $x^4+2x-2=0$ has exactly one real solution in the interval $(0,1)$.
32. If the equation $a_nx^n+a_{n-1}x^{n-1}+......+a_1x=0$ has a positive solution *a*, then prove that the equation $na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+......+a_1=0$ also has a positive solution which is smaller than *a*.
33. If $2a+3b+6c=0$, then show that the equation $ax^2+bx+c=0$ has at least one real root between 0 and 1.
34. Let $\dfrac{a_0}{n+1}+\dfrac{a_1}{n}+\dfrac{a_2}{n-1}+..........+\dfrac{a_{n-1}}{2}+\dfrac{a_n}{1}=0$. Show that there exists at least one real *x* between 0 and 1 such that $a_0x^n+a_1x^{n-1}+......+a_{n-1}x+a_n=0$.

CATEGORY-5.8. LAGRANGE'S MEAN VALUE THEOREM

35. Show that $f(x)=x^2$ satisfies Lagrange's Mean value theorem in the interval $[0,1]$ and find the value of *c*.
36. Prove the validity of Lagrange's theorem for the function $y=\ln x$ in the interval $[1,e]$ and find the value of *c*.
37. With the aid of Lagrange's theorem prove the inequalities $\dfrac{a-b}{a}\le\ln\dfrac{a}{b}\le\dfrac{a-b}{b}$, for the condition $0<b\le a$.
38. With the aid of Lagrange's theorem prove the inequalities $\dfrac{a-b}{\cos^2 b}\le\tan a-\tan b\le\dfrac{a-b}{\cos^2 a}$, for the condition $0<b\le a<\dfrac{\pi}{2}$.
39. Using Mean value theorem, show that $|\cos a-\cos b|\le|a-b|$.
40. Use Lagrange's theorem to prove that $1+x<e^x<1+xe^x\ \forall x>0$.
41. If $f''(x)$ exists for all points in $[a,b]$ and $\dfrac{f(c)-f(a)}{c-a}=\dfrac{f(b)-f(c)}{b-c}$, where $a<c<b$, then there is a number α such that $a<\alpha<b$ and $f''(\alpha)=0$.
42. If $f(x)$ is differentiable and $\lim\limits_{x\to\infty}f(x)$ is finite and $\lim\limits_{x\to\infty}f'(x)$ is finite, then show that $\lim\limits_{x\to\infty}f'(x)=0$.
43. If $f''(x)\ge0\forall x\in[a,b]$, show that $f\left(\dfrac{x_1+x_2}{2}\right)\le\dfrac{f(x_1)+f(x_2)}{2}$ for $x_1, x_2\in[a,b]$.

CATEGORY-5.9. ADDITIONAL QUESTIONS

44. Suppose that $f(x)$ and $g(x)$ are non-constant differentiable real valued functions on R. If for every $x, y \in R$, $f(x+y) = f(x)f(y) - g(x)g(y)$ and $g(x+y) = g(x)f(y) + f(x)g(y)$ and $f'(0) = 0$, then prove that maximum and minimum values of the function $f^2(x) + g^2(x)$ are same for all real x.

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Mathematics for IIT-JEE

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PART-I
DIFFERENTIAL CALCULUS

CHAPTER-6
INVESTIGATION OF FUNCTIONS AND THEIR GRAPHS

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CHAPTER-6
INVESTIGATION OF FUNCTIONS AND THEIR GRAPHS

LIST OF THEORY SECTIONS

LIST OF QUESTION CATEGORIES

CHAPTER-6
INVESTIGATION OF FUNCTIONS AND THEIR GRAPHS

SECTION-6.1. CALCULATION OF DOMAIN OF ANALYTICAL FUNCTIONS

1. **Calculation of Domain of analytical functions**
   i. Write domain conditions and solve to get domain.

SECTION-6.2. PLOTTING THE GRAPHS OF ANALYTICAL FUNCTIONS BY PROPERTIES

1. **For investigating a function and sketching its graph, determine the following properties of the function:-**
   i. Find Domain of the function.
   ii. Check continuity and find discontinuities if any.
   iii. Check differentiability and find first derivative function. Find the points where first derivative function is positive, negative, zero and does not exist. Find the following properties:-
      a. Intervals of monotonicity of the function.
      b. Stationary points of the function.
      c. Critical points of the function.
      d. Maxima and minima of the function.
   iv. Find second derivative function. Find the points where second derivative function is positive, negative, zero and does not exist. Find the following properties:-
      a. Intervals of concavity of the function.
      b. Points of inflection of the function.
   v. Find value/limit of the function at important points by finding the following: properties:-
      a. Value/limit of the function at the end points of the domain.
      b. Right hand limit, left hand limit and value of the function at the points of discontinuity.
      c. Value of the function at critical points and points of inflection.
      d. Zeros of the function if any, i.e. points where the function cuts x-axis.
      e. Value of the function at $x = 0$ if any, i.e. point where the graph cuts y-axis.
   vi. Find slope of the tangent at important points by finding the following: properties:-
      a. Value/limit of the first derivative function at the end points of the domain.
      b. Right hand limit and left hand limit of the first derivative function at the points of discontinuity.
      c. Right hand limit and left hand limit of the first derivative function at the points where function is not differentiable.
      d. Value of the first derivative function at points of inflection.
      e. Value of the first derivative function at zeros of the function if any, i.e. slope of tangent at the points where the function cuts x-axis.
      f. Value of the first derivative function at $x = 0$ if any, i.e. slope of tangent at the point where the graph cuts y-axis.
   vii. Based on these properties, plot the graph of the function.
   viii. Write the Range of the function.
   ix. Write the greatest and least value of the function.
2. **Graph of the function** $f(x) = a^x$
3. **Graph of the function** $f(x) = \log_a x$

4. **Graph of the function** $f(x) = x^a$
5. **Graphs of trigonometric functions**
6. **Graphs of inverse trigonometric functions**
7. **Graphs of hyperbolic functions**

SECTION-6.3. SIMPLE TRANSFORMATIONS ON GRAPHS

1. **Plotting graphs by simple transformations**
   i. If graph of the function $f(x)$ is known then graphs of the functions $f(-x)$, $-f(x)$, $f(|x|)$, $|f(x)|$, $\alpha f(x)$, $f(\alpha x)$, $f(x)+\alpha$, $f(x+\alpha)$ can be plotted by simple transformations on the graph of the function $f(x)$ without finding its properties.
2. **Graphs of** $f(-x)$**,** $-f(x)$**,** $-f(-x)$
3. **Graphs of** $f(|x|)$**,** $|f(x)|$**,** $|f(|x|)|$
4. **Graphs of** $\alpha f(x)$**,** $f(\alpha x)$
5. **Graphs of** $f(x)+\alpha$**,** $f(x+\alpha)$
6. **Applications of transformations in graph plotting**

SECTION-6.4. EVEN AND ODD FUNCTIONS

1. **Definition of even and odd functions**
   i. A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x in the domain of $f(x)$; is said to be an odd function if $f(-x) = -f(x)$ for all x in the domain of $f(x)$; otherwise $f(x)$ is neither even nor odd function.
   ii. A function may be both even and odd.
   iii. Domain of an even/odd function must be symmetrical about origin.
2. **Checking a graphical function for being even or odd or neither even nor odd**
   i. If graph of the function is in itself symmetrical about y-axis then the function is even; if graph of the function is in itself symmetrical about origin then the function is odd; otherwise the function is neither even nor odd.
3. **Basic functions which are even or odd or neither even nor odd**
4. **Testing an analytical function for being even or odd or neither even nor odd**
   i. Check by definition.
   ii. If $f(x) + f(-x) = 2f(x)$ then the function is even; if $f(x) + f(-x) = 0$ then the function is odd; otherwise the function is neither even nor odd.
5. **Applications of even/odd property in graph plotting**
   i. If the graph of the function is to be plotted by properties and if the function is even or odd, then first plot the graph for positive values of x only and then plot the graph for negative values of x by symmetry to get the complete graph.

SECTION-6.5. PERIODIC FUNCTIONS

1. **Definition of periodic functions**
   i. A function $f(x)$ is called periodic if there exists a non-zero constant real number P such that $f(x+P) \equiv f(x)$ for all values of x within the domain of $f(x)$ and the number P is called a period of $f(x)$. All other functions are non-periodic.

ii. If P is a period then nP (n is any non-zero integer) is also a period. Hence, a periodic function has infinite periods.
iii. The least positive period is called the Fundamental period, denoted by T.
iv. If T is the Fundamental period then nT (n is any non-zero integer) is also a period of the function.
v. A periodic function may not have fundamental period.
vi. Domain of periodic functions must extend to $-\infty$ and to $+\infty$.

2. Checking a graphical function for being periodic

3. Basic functions which are periodic/non-periodic

4. Testing an analytical function for periodicity

i. By definition
ii. By method
iii. By Theorems

5. Applications of periodic/non-periodic property in graph plotting

SECTION-6.6. INVERSE FUNCTIONS

1. Definition of inverse functions

i. The function $f(x)$ has an inverse function, denoted by $f^{-1}(x)$, if $f(a) = b \Leftrightarrow f^{-1}(b) = a \ \forall a \in$ domain of $f(x)$ and $\forall b \in$ domain of $f^{-1}(x)$, i.e. $f^{-1}(f(x)) = x \ \forall x \in$ domain of $f(x)$ and $f(f^{-1}(x)) = x \ \forall x \in$ domain of $f^{-1}(x)$.

2. Properties of inverse functions

i. Domain of $f^{-1}(x)$ = Range of $f(x)$ and Range of $f^{-1}(x)$ = Domain of $f(x)$.
ii. A function has an inverse iff the function is one-one function (invertible function).
iii. Many-one functions may have different inverse functions for its different one-one parts.
iv. A function may be inverse to itself.

3. Finding inverse of graphical function

i. Graphs of inverse functions are symmetric about $y = x$ line.

4. Inverse functions of basic functions

5. Finding inverse of an explicit analytical function

i. Given an explicit analytical function $f(x)$, to find its inverse function y, solve the equation $f(y) = x$.

6. Inverse of implicit and parametric functions

7. Inverse of piecewise functions.

8. Derivative of inverse function

i. $\dfrac{d}{dx} f^{-1}(x) = \dfrac{1}{f'\left(f^{-1}(x)\right)}$.

9. If $f(x)$ is monotonously increasing (decreasing) then $f^{-1}(x)$ is also monotonously increasing (decreasing)

10. Applications of inverse function in graph plotting

SECTION-6.7. GREATEST INTEGER FUNCTION AND FRACTIONAL PART FUNCTION

1. Definition of Greatest Integer Function [*x*]

i. Greatest integer function, denoted by [x], is the greatest integer less than or equal to x for any real number x.

2. Graph and properties of Greatest Integer Function

3. Definition of Fractional Part Function {*x*}

i. Fractional part function, denoted by $\{x\}$, is the fractional part of *x* for any real number *x*, i.e. $\{x\} = x - [x]$.

4. Graph and properties of Fractional Part Function

5. Graphs of $[f(x)]$ and $f([x])$

6. Graphs of $\{f(x)\}$ and $f(\{x\})$

SECTION-6.8. FINDING RANGE OF ANALYTICAL FUNCTIONS

1. **By graph, if known or easily drawn by transformation**
2. **Range of Composite functions**
3. **If $f(x)$ is continuous & domain is closed interval, find least value (*L*) and greatest value (*G*), range is [*L*,*G*]**
4. **Find inverse function of $f(x)$ and find its domain, which is range of $f(x)$**
5. **Find values of *y* for which equation $f(x) = y$ has solution, then these values of *y* is the range of $f(x)$**
6. **If $f(x)$ is periodic with period *T* then find range in any interval of length *T***
7. **By properties (concavity, slope of tangent not required)**
8. **If $f(x)$ is piecewise function, find range of each part and take their union**
9. **Find an equivalent function of $f(x)$ and find its range**

SECTION-6.9. FUNCTION (MAPPING)

1. Definition of function (mapping)

i. If *A* and *B* are two non-empty sets, then a function *f* from set *A* to set *B* is a rule which associates each element of set *A* to a unique element of set *B*.

ii. *f* is a function from set *A* to set *B* is denoted by $f : A \to B$.

iii. If $A \subseteq R$ and $B \subseteq R$ then *f* is a real function.

2. Domain and Co-domain

i. Set *A* is known as the *domain* of *f* and set *B* is known as the *co-domain* of *f*.

3. Value (Image) and pre-image

i. If and element $a \in A$ is associated to an element $b \in B$ then it is written as $f(a) = b$ and *b* is called value or image of *a* and *a* is called the pre-image of *b*.

4. Range of a function

i. The set of all values (images) of elements of set *A* is called the range of *f* or image set of *A* and is denoted by $f(A)$, i.e. $f(A) = \{f(x) | x \in A\}$.

ii. $f(A) \subseteq B$.

5. Ways of representing functions

i. By diagram;

ii. By notation $f(x) = y$;

iii. By sets of ordered pairs;

iv. By formula.

6. One-one (injective) function and many-one function

i. A function $f : A \to B$ is said to be a one-one function or injective function if different elements of set *A* have different images in set *B*, i.e. $a \neq b \Rightarrow f(a) \neq f(b) \; \forall a, b \in A$.

ii. A function $f : A \to B$ is said to be a many-one function if two or more elements of set *A* have same image in set *B*, i.e. there exists $a, b \in A$ such that $a \neq b$ but $f(a) = f(b)$.

7. Into and Onto (Surjective) function

i. A function $f : A \to B$ is said to be an into function if there exists an element in set *B* having no pre-image in set *A*, i.e. $f(A) \subset B$.

ii. A function $f : A \to B$ is said to be an onto function or surjective function if every element of set *B* is the image of some element of set *A*, i.e. $f(A) = B$.

8. One-one and into functions; one-one and onto (Bijective) functions; many-one and into functions; many-one and onto functions

EXERCISE-6

CATEGORY-6.1. CALCULATION OF DOMAIN OF ANALYTICAL FUNCTIONS

1. $f(x)=\sqrt{x-x^2}+\sqrt{3x-x^2-2}$. {Ans. $[1]$}
2. $f(x)=\dfrac{1+2(x+4)^{-0\cdot 5}}{2-(x+4)^{0\cdot 5}}$. {Ans. $(-4,0)\cup(0,\infty)$}
3. $f(x)=\dfrac{1}{3-\log_3(x-3)}$. {Ans. $(3,30)\cup(30,\infty)$}
4. $f(x)=\dfrac{\sqrt{x+5}}{\log(9-x)}$. {Ans. $[-5,8)\cup(8,9)$}
5. $f(x)=\log_2\left(\dfrac{x-2}{x+2}\right)$. {Ans. $(-\infty,-2)\cup(2,\infty)$}
6. $f(x)=\sqrt{\log_{12}x^2}$. {Ans. $(-\infty,-1]\cup[1,\infty)$}
7. $f(x)=\sqrt{\log\dfrac{3-x}{x}}$. {Ans. $\left(0,\frac{3}{2}\right]$}
8. $f(x)=\dfrac{1}{\sqrt{|x|-x}}$. {Ans. $(-\infty,0)$}
9. $f(x)=\sqrt[4]{x-|x|}+\log(x+2)$. {Ans. $[0,\infty)$}
10. $f(x)=\sin^{-1}\left(\log_3\left(\dfrac{x}{3}\right)\right)$. {Ans. $[1,9]$}
11. $f(x)=\sin^{-1}\left(\dfrac{x-3}{2}\right)-\log(4-x)$. {Ans. $[1,4)$}
12. $f(x)=\sin^{-1}(|x-1|-2)$. {Ans. $[-2,0]\cup[2,4]$}
13. $f(x)=\sin^{-1}\log_2\left(\dfrac{1}{2}x^2\right)$. {Ans. $[-2,-1]\cup[1,2]$}
14. $f(x)=\sqrt{\log_{0.4}\left(\dfrac{x-1}{x+5}\right)}\times\dfrac{1}{x^2-36}$. {Ans. $(1,6)\cup(6,\infty)$}
15. $f(x)=\log_{\left(\frac{x-2}{x+3}\right)}\sqrt{16-x^2}$. {Ans. $(-4,-3)\cup(2,4)$}
16. $f(x)=\cos^{-1}\left(\dfrac{2-|x|}{4}\right)+(\log(3-x))^{-1}$. {Ans. $[-6,2)\cup(2,3)$}
17. $f(x)=\sqrt{\log_{10}\left(\dfrac{5x-x^2}{4}\right)}$. {Ans. $[1,4]$}
18. $f(x)=\sqrt{\dfrac{-\log_{0.3}(x-1)}{-x^2+3x+18}}$. {Ans. $[2,6)$}

19. $f(x)=\sqrt{\log(\log x)-\log(4-\log x)-\log 3}$. {Ans. $[10^3,10^4)$}
20. $f(x)=\log\left(\sqrt{x-4}+\sqrt{6-x}\right)$. {Ans. $[4,6]$}
21. $f(x)=\log\left(\log^2 x-5\log x+6\right)$. {Ans. $(0,100)\cup(1000,\infty)$}
22. $f(x)=\log\left(1-\log\left(x^2-5x+16\right)\right)$. {Ans. $(2,3)$}
23. $f(x)=\log_{0.5}\left\{-\log_2\left(\dfrac{3x-1}{3x+2}\right)\right\}$. {Ans. $\left(\dfrac{1}{3},\infty\right)$}
24. $f(x)=\log_{2x-5}\left(x^2-3x-10\right)$. {Ans. $(5,\infty)$}
25. Find the set of values of x for which the function $f(x)=\dfrac{1}{x}+2^{\sin^{-1}x}+\dfrac{1}{\sqrt{x-2}}$ is defined. {Ans. ϕ}
26. If a function $f(x)$ is defined for $x\in[0,1]$, then find the domain of the function $f(2x+3)$. {Ans. $\left[-\dfrac{3}{2},-1\right]$}
27. If $f(x)=x-1,\quad 0\le x\le 2$, find the domain of $\phi(x)=f(f(x))$. {Ans. $[1,2]$}

CATEGORY-6.2. PLOTTING GRAPH BY PROPERTIES

28. Plot graph of the following functions and write their range, greatest value and least value:-
    - i. $f(x)=x\ln x$.
    - ii. $f(x)=\dfrac{\ln x}{x}$.
    - iii. $f(x)=x^x$.
    - iv. $f(x)=xe^x$.
    - v. $f(x)=xe^{-x}$.
    - vi. $f(x)=e^{\frac{1}{x}}$.
    - vii. $f(x)=xe^{\frac{1}{x}}$.
    - viii. $f(x)=x-\ln(x+1)$.

CATEGORY-6.3. PLOTTING GRAPHS BY TRANSFORMATION

29. Plot graph:-
    - i. $f(x)=\left||x|-1\right|$.
    - ii. $f(x)=\left||x-2|-2\right|$.
    - iii. $f(x)=\left(|x|-1\right)^2$.
    - iv. $f(x)=\left|1-\dfrac{1}{|x|}\right|$.
    - v. $f(x)=\left|\sin\left|x-\dfrac{\pi}{2}\right|\right|$.

vi. $f(x) = \sin\left||x| - \frac{\pi}{4}\right|$.

vii. $f(x) = e^{|x+1|}$.

viii. $f(x) = e^{|x|+1}$.

ix. $f(x) = e^{|x+1|-1}$.

x. $f(x) = e^{||x|-1|}$.

xi. $f(x) = \left|e^{|x|} - 2\right|$.

xii. $f(x) = \left|\ln(|1-x|)\right|$.

xiii. $f(x) = \ln(1-|x|)$.

xiv. $f(x) = \left||\ln|x|| - 1\right|$.

xv. $f(x) = |\ln(1-x)|$.

xvi. $f(x) = |\ln|x+1||$.

xvii. $f(x) = \ln(|x|+1)$.

xviii. $f(x) = |\ln|x| - 1|$.

xix. $f(x) = \operatorname{sgn}(|x|+1)$.

xx. $f(x) = \operatorname{sgn}(|x+1|)$.

xxi. $f(x) = |\operatorname{sgn}(|x|-1)|$.

xxii. $f(x) = \operatorname{sgn}(|x-1|)$.

xxiii. $f(x) = \left||\tanh x| - \frac{1}{2}\right|$.

xxiv. $f(x) = \frac{x}{x+1}$.

xxv. $f(x) = \cos^{-1}(|x|-1)$.

xxvi. $f(x) = \left|\sqrt{|x|} - 1\right|$.

xxvii. $f(x) = \sqrt{||x|-1|}$.

xxviii. $f(x) = \frac{1}{2-|x|}$.

xxix. $f(x) = \frac{1}{|x-1|-1}$.

CATEGORY-6.4. EVEN AND ODD FUNCTIONS

30. Test the following functions for even, odd or neither:-

i. $f(x) = \log\left(x + \sqrt{1+x^2}\right)$. {Ans. odd}

ii. $f(x)=\log\frac{1-x}{1+x}$. {Ans. odd}

iii. $f(x)=2x^3-x+1$. {Ans. neither}

iv. $f(x)=x^4-2x^2$. {Ans. even}

v. $f(x)=x-x^2$. {Ans. neither}

vi. $f(x)=\sin x-\cos x$.{Ans. neither}

vii. $f(x)=2^{-x^2}$. {Ans. even}

viii. $f(x)=\frac{a^x+a^{-x}}{2}$. {Ans. even}

ix. $f(x)=\frac{a^x-a^{-x}}{2}$. {Ans. odd}

x. $f(x)=\frac{x}{a^x-1}$. {Ans. neither}

xi. $f(x)=2^{x-x^4}$. {Ans. neither}

xii. $f(x)=x\frac{a^x+1}{a^x-1}$. {Ans. even}

xiii. $f(x)=4-2x^4+\sin^2 x$. {Ans. even}

xiv. $f(x)=\sqrt{1+x+x^2}-\sqrt{1-x+x^2}$. {Ans. odd}

xv. $f(x)=\frac{1+a^{kx}}{1-a^{kx}}$. {Ans. odd}

xvi. $f(x)=\sin x+\cos x$. {Ans. neither}

xvii. $f(x)=\sqrt[3]{(1-x)^2}-\sqrt[3]{(1+x)^2}$. {Ans. odd}

xviii. $f(x)=x^2-|x|$. {Ans. even}

xix. $f(x)=x\sin^2 x-x^3$. {Ans. odd}

xx. $f(x)=\frac{(1+2^x)^2}{2^x}$. {Ans. even}

xxi. $f(x)=\frac{\cos x\sin x}{\tan x+\cot x}$. {Ans. even}

xxii. $f(x)=\sin^3 x+2\tan^5 x$. {Ans. odd}

xxiii. $f(x)=\frac{\sin^4 x+\cos^4 x}{x+x^2\tan x}$. {Ans. odd}

xxiv. $f(x)=\frac{\sec^4 x+\operatorname{cosec}^4 x}{x^3+x^4\cot x}$. {Ans. odd}

xxv. $f(x)=\frac{x}{e^x-1}+\frac{x}{2}+1$. {Ans. even}

31. For what values of a, the function $f(x)=(a^2+a-2)x+a^2+2a-3$ is (a) even, (b) odd? {Ans. Even for $a=1,-2$; odd for $a=1,-3$}

32. Prove that the product of two even or two odd functions is an even function, whereas the product of an even and an odd function is an odd function.

33. Prove that if the domain of the function $f(x)$ is symmetrical with respect to $x = 0$, then $f(x)+f(-x)$ is an even function and $f(x)-f(-x)$ is an odd function.

34. Prove that any function $f(x)$, whose domain is symmetrical about origin, can be presented as a sum of an even and an odd function. Rewrite the following functions in the form of a sum of an even and an odd function:-

i. $f(x)=\dfrac{x+2}{1+x^2}$. {Ans. $f(x)=\frac{2}{1+x^2}+\frac{x}{1+x^2}$ }

ii. $f(x)=a^x$. {Ans. $f(x)=\frac{a^x+a^{-x}}{2}+\frac{a^x-a^{-x}}{2}$ }

iii. $f(x)=x^2+3x+2$. {Ans. $f(x)=(x^2+2)+3x$ }

iv. $f(x)=1-x^3-x^4-2x^5$. {Ans. $f(x)=(1-x^4)+(-x^3-2x^5)$ }

v. $f(x)=\sin 2x+\cos\frac{x}{2}+\tan x$. {Ans. $f(x)=\cos\frac{x}{2}+(\sin 2x+\tan x)$ }

vi. $f(x)=(1+x)^{100}$. {Ans. $f(x)=\frac{(1+x)^{100}+(1-x)^{100}}{2}+\frac{(1+x)^{100}-(1-x)^{100}}{2}$ }

35. Extend the function $f(x)=x^2+x$ defined on the interval [0, 3] onto the interval [-3, 3] in an even and an odd way.

{Ans.

$$\begin{aligned} f(x)&=x^2+x, \quad 0\le x\le 3\\ &=x^2-x, \quad -3\le x<0; \end{aligned}$$

$$\begin{aligned} f(x)&=x^2+x, \quad 0\le x\le 3\\ &=-x^2+x, \quad -3\le x<0 \end{aligned}$$ }

36. Let the function $f(x)=x^2+x+\sin x-\cos x$ be defined on the interval $[0,1]$. Find the odd and even extensions of $f(x)$ in the interval $[-1,1]$.

{Ans.

$$\begin{aligned} f(x)&=x^2+x+\sin x-\cos x, \quad 0\le x\le 1\\ &=-x^2+x+\sin x+\cos x, \quad -1\le x<0; \end{aligned}$$

$$\begin{aligned} f(x)&=x^2+x+\sin x-\cos x, \quad 0\le x\le 1\\ &=x^2-x-\sin x-\cos x, \quad -1\le x<0 \end{aligned}$$ }

37. If $f(x)$ is an odd function and if $\lim\limits_{x\to 0} f(x)$ exists, prove that this limit must be zero.

38. Show that derivative of an even function is an odd function and derivative of an odd function is an even function.

39. Let $f(x)$ be an even function and if $f'(0)$ exists, find it's value. {Ans. 0}

40. If $f(x)$ is an even function defined on the interval $(-5,5)$ then find the real values of x satisfying the equation $f(x)=f\left(\dfrac{x+1}{x+2}\right)$. {Ans. $\dfrac{-1+\sqrt{5}}{2}$, $\dfrac{-1-\sqrt{5}}{2}$, $\dfrac{-3-\sqrt{5}}{2}$, $\dfrac{-3+\sqrt{5}}{2}$ }

41. If $f(x+y)+f(x-y)=2f(x)f(y)\ \forall\ x, y\in R$ and $f(0)\neq 0$, then determine that $f(x)$ is an even function or odd function or neither. {Ans. even}

CATEGORY-6.5. PERIODIC FUNCTIONS

42. Which of the following functions are periodic:-
   i. $f(x)=\cos x^2$. {Ans. non-periodic}
   ii. $f(x)=x+\sin x$. {Ans. non-periodic}
   iii. $f(x)=\cos\sqrt{x}$. {Ans. non-periodic}
43. Find the period of the following functions:-
   i. $f(x)=5\sin 4x$. {Ans. $\frac{\pi}{2}$ }
   ii. $f(x)=4\sin\left(3x+\frac{\pi}{4}\right)$. {Ans. $\frac{2\pi}{3}$ }
   iii. $f(x)=\tan 2x$. {Ans. $\frac{\pi}{2}$ }
   iv. $f(x)=\cot\frac{x}{2}$. {Ans. 2π }
   v. $f(x)=\sin 2\pi x$. {Ans. 1}
   vi. $f(x)=\sin^2 x$. {Ans. π }
   vii. $f(x)=\sin\left(\dfrac{2x+3}{6\pi}\right)$. {Ans. $6\pi^2$ }
   viii. $f(x)=\sin^4 x+\cos^4 x$. {Ans. $\dfrac{\pi}{2}$ }
   ix. $f(x)=|\cos x|$. {Ans. π }
   x. $f(x)=\sin 2x+\cos 3x$. {Ans. 2π }
   xi. $f(x)=3\sin\frac{x}{2}+4\cos\frac{x}{2}$. {Ans. 4π }
   xii. $f(x)=\tan^{-1}(\tan x)$. {Ans. π }
   xiii. $f(x)=2\cos\dfrac{x-\pi}{3}$. {Ans. 6π }
   xiv. $f(x)=\sin\left(\dfrac{\pi x}{2}\right)+\cos\left(\dfrac{\pi x}{2}\right)$. {Ans. 4}
   xv. $f(x)=\sin\dfrac{2\pi x}{3}+\cos\dfrac{\pi x}{2}$. {Ans. 12}
   xvi. $f(x)=\sin\dfrac{\pi x}{3}+\sin\dfrac{\pi x}{4}$. {Ans. 24}
   xvii. $f(x)=\sin\left(2\pi x+\dfrac{\pi}{3}\right)+2\sin\left(3\pi x+\dfrac{\pi}{4}\right)+3\sin 5\pi x$. {Ans. 2}
   xviii. $f(x)=\cos(\sin x)$. {Ans. π }
   xix. $f(x)=\cos(\sin x)+\cos(\cos x)$. {Ans. $\dfrac{\pi}{2}$ }
   xx. $f(x)=\dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)}$. {Ans. π }
   xxi. $f(x)=|\sin x|+|\cos x|$. {Ans. $\dfrac{\pi}{2}$ }
   xxii. $f(x)=\sin\dfrac{\pi x}{2}+2\cos\dfrac{\pi x}{3}-\tan\dfrac{\pi x}{4}$. {Ans. 12}

44. Plot the graph of a periodic function $f(x)$ with fundamental period $T = 1$ defined in the interval (0,1] if
   i. $f(x) = x$.
   ii. $f(x) = x^2$.
   iii. $f(x) = \ln x$.
45. Prove that the function
   $$f(x) = 1, \quad x \text{ is a rational no.}$$
   $$= 0, \quad x \text{ is a irrational no.}$$
   is periodic but has no fundamental period.
46. Prove that if $f(x)$ is a periodic function with period T, then the function $f(ax+b)$ is periodic with period $\frac{T}{|a|}$.
47. Prove that if the function $f(x) = \sin x + \cos ax$ is periodic, then a is a rational number.
48. Let $f(x)$ be a function and k be a positive real number such that $f(x+k) + f(x) = 0 \ \forall\ x \in R$. Prove that $f(x)$ is a periodic function with period $2k$.
49. Let f be a real valued function defined for all real numbers x such that for some fixed $a > 0$ $f(x+a) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2}\ \forall x$. Show that the function $f(x)$ is periodic with period $2a$.
50. Let $f(x)$ be a real valued function with domain R such that
   $f(x+p) = 1 + \left[2 - 3f(x) + 3(f(x))^2 - (f(x))^3\right]^{\frac{1}{3}}$ holds good for all $x \in R$ and some positive constant p, then prove that $f(x)$ is a periodic function.
51. For what integral value of n, the function $f(x) = \cos nx \sin \frac{5x}{n}$ is periodic with period 3π ? {Ans. $n = \pm 1, \pm 3, \pm 5, \pm 15$}

CATEGORY-6.6. MISCELLANEOUS QUESTIONS ON GRAPH PLOTTING

52. Plot graph of the following functions:-
   i. $f(x) = x + \frac{1}{x}$.
   ii. $f(x) = \ln(x^2 + 1)$.
   iii. $f(x) = \sin^{-1}(\sin x)$.
   iv. $f(x) = \cos^{-1}(\cos x)$.
   v. $f(x) = \tan^{-1}(\tan x)$.
53. Plot graph of the following functions:-
   i. $y = e^{\ln|x|}$.
   ii. $y = x^{\log_x 2}$.
   iii. $y = \operatorname{sgn}(\sin x)$.
   iv. $y = x^{\operatorname{sgn} x}$.
   v. $y = \operatorname{sgn}(\tan^{-1} x)$.
   vi. $y = (\operatorname{sgn} x)^{\operatorname{sgn} x}$.

vii. $y=\sin^{-1}(\operatorname{sgn} x)$.

viii. $y=\cos^{-1}(\operatorname{sgn} x)$.

ix. $y=|x-1|+|x-2|+x$.

x. $y=2^{\frac{|x|+x}{x}}$.

xi. $y=\log_x x^2$.

xii. $y=\ln\tan x+\ln\cot x$.

54. Plot the graphs of the functions $\dfrac{f(x)+|f(x)|}{2}, \dfrac{f(x)-|f(x)|}{2}$ & $\dfrac{|f(x)|}{f(x)}$ from the graph of $f(x)$ for the following functions:-

i. $f(x)=\ln x$.

ii. $f(x)=x^3$.

iii. $f(x)=\sin x$.

iv. $f(x)=\tan x$.

v. $f(x)=\sin^{-1} x$.

vi. $f(x)=\tan^{-1} x$.

55. Plot the graphs of the functions

i. $f(x)=\max\{x, x^2\}$.

ii. $f(x)=\min\{x, x^2\}$.

iii. $f(x)=\max\{\sin x, \cos x\}$.

iv. $f(x)=\min\{\sin x, \cos x\}$.

56. Determine the function $f(x)=\max\{(1-x), (1+x), 2\}$.

{Ans.

$$\begin{aligned} f(x) &= 1-x, \quad x\le -1 \\ &= 2, \quad -1<x<1 \\ &= 1+x, \quad x\ge 1\} \end{aligned}$$

57. Plot the graphs of the functions

i. $f(x)=\max\{\sin t : 0\le t\le x\}, \quad x\ge 0$
$\quad = \max\{\sin t : x\le t\le 0\}, \quad x<0$.

ii. $f(x)=\min\{\cos t : 0\le t\le x\}, \quad x\ge 0$
$\quad = \min\{\cos t : x\le t\le 0\}, \quad x<0$.

iii. $f(x)=\min\{\sin t : 0\le t\le x\}, \quad x\ge 0$
$\quad = \min\{\sin t : x\le t\le 0\}, \quad x<0$.

iv. $f(x)=\max\{\cos t : 0\le t\le x\}, \quad x\ge 0$
$\quad = \max\{\cos t : x\le t\le 0\}, \quad x<0$.

v. $f(x)=\max\{\tan t : 0\le t\le x\}$.

vi. $f(x)=\min\{\tan t : 0\le t\le x\}$.

58. If $f(x)=\min\{|x|, |x-2|, 2-|x-1|\}$, then draw the graph of $f(x)$ and also discuss its continuity and

differentiability. {Ans. Continuous $\forall x$; not differentiable at $x = -\frac{1}{2}, 0, 1, 2, \frac{5}{2}$ }

59. Let $f(x) = x^3 - x^2 + x + 1$ and

$$g(x) = \max\{f(t) : 0 \le t \le x\}, \quad 0 \le x \le 1$$
$$= 3 - x, \quad 1 < x \le 2.$$

Discuss the continuity and differentiability of the function $g(x)$ in the interval (0, 2). {Ans. not differentiable at x = 1}

60. Plot graph of the function

$$f(x) = (|x| - 1)^2 e^{\frac{1}{|x|-1}}, \quad x \ne \pm 1$$
$$= 0, \quad x = \pm 1.$$

Write

i. all the points of discontinuity of $f(x)$;
ii. all the points where $f(x)$ is not differentiable;
iii. all the stationary points of $f(x)$;
iv. intervals of monotonicity of $f(x)$;
v. all the critical points of $f(x)$;
vi. all the points of maxima of $f(x)$;
vii. all the points of minima of $f(x)$;
viii. intervals of concavity of $f(x)$;
ix. all the points of inflection of $f(x)$;
x. range of $f(x)$;
xi. greatest and least value of $f(x)$;

CATEGORY-6.7. INVERSE FUNCTIONS

61. Find the inverse of the following functions:-

i. $f(x) = x$. {Ans. $f^{-1}(x) = x$ }
ii. $f(x) = x^2$. {Ans. $f_1^{-1}(x) = \sqrt{x}$; $f_2^{-1}(x) = -\sqrt{x}$ }
iii. $f(x) = x^3$. {Ans. $f^{-1}(x) = \sqrt[3]{x}$ }
iv. $f(x) = \sqrt{x}$. {Ans. $f^{-1}(x) = x^2, \quad x \ge 0$ }
v. $f(x) = \frac{1}{x}$. {Ans. $f^{-1}(x) = \frac{1}{x}$ }
vi. $f(x) = \frac{1}{x^2}$. {Ans. $f_1^{-1}(x) = \frac{1}{\sqrt{x}}$; $f_2^{-1}(x) = -\frac{1}{\sqrt{x}}$ }
vii. $f(x) = e^x$. {Ans. $f^{-1}(x) = \ln x$}
viii. $f(x) = a^x$. {Ans. $f^{-1}(x) = \log_a x$ }
ix. $f(x) = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$. {Ans. $f^{-1}(x) = \sin^{-1} x$ }

x. $f(x)=\sin x,\quad \frac{\pi}{2}\le x\le\frac{3\pi}{2}$. {Ans. $f^{-1}(x)=\pi-\sin^{-1}x$ }

xi. $f(x)=\cos x,\quad 0\le x\le\pi$. {Ans. $f^{-1}(x)=\cos^{-1}x$ }

xii. $f(x)=\tan x,\quad -\frac{\pi}{2}<x<\frac{\pi}{2}$. {Ans. $f^{-1}(x)=\tan^{-1}x$ }

xiii. $f(x)=\sinh x$. {Ans. $f^{-1}(x)=\ln\left(x+\sqrt{x^2+1}\right)$}

xiv. $f(x)=\cosh x$. {Ans. $f_1^{-1}(x)=\ln\left(x+\sqrt{x^2-1}\right),\ f_2^{-1}(x)=\ln\left(x-\sqrt{x^2-1}\right)$}

xv. $f(x)=\tanh x$. {Ans. $f^{-1}(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$}

xvi. $f(x)=2x$. {Ans. $f^{-1}(x)=\frac{x}{2}$ }

xvii. $f(x)=1-3x$. {Ans. $f^{-1}(x)=\frac{1-x}{3}$ }

xviii. $f(x)=3x+5$. {Ans. $f^{-1}(x)=\frac{x-5}{3}$ }

xix. $f(x)=x^2+1$. {Ans. $f_1^{-1}(x)=\sqrt{x-1},\ f_2^{-1}(x)=-\sqrt{x-1}$ }

xx. $f(x)=x^2-x+1$. {Ans. $f_1^{-1}(x)=\frac{1+\sqrt{4x-3}}{2},\ f_2^{-1}(x)=\frac{1-\sqrt{4x-3}}{2}$ }

xxi. $f(x)=x^2-2x$. {Ans. $f_1^{-1}(x)=1+\sqrt{1+x},\ f_2^{-1}(x)=1-\sqrt{1+x}$ }

xxii. $f(x)=x^3+5$. {Ans. $f^{-1}(x)=(x-5)^{\frac{1}{3}}$ }

xxiii. $f(x)=\frac{1}{1-x}$. {Ans. $f^{-1}(x)=\frac{x-1}{x}$ }

xxiv. $f(x)=\sqrt[3]{x^2+1}$. {Ans. $f_1^{-1}(x)=\sqrt{x^3-1},\ f_2^{-1}(x)=-\sqrt{x^3-1}$ }

xxv. $f(x)=10^{x+1}$. {Ans. $f^{-1}(x)=\log x-1$}

xxvi. $f(x)=2^{x(x-1)}$. {Ans. $f_1^{-1}(x)=\frac{1+\sqrt{1+4\log_2 x}}{2},\ f_2^{-1}(x)=\frac{1-\sqrt{1+4\log_2 x}}{2}$ }

xxvii. $f(x)=5^{\log x}$. {Ans. $f^{-1}(x)=10^{\log_5 x}$ }

xxviii. $f(x)=1+\ln(x+2)$. {Ans. $f^{-1}(x)=e^{x-1}-2$ }

xxix. $f(x)=\log_x 2$. {Ans. $f^{-1}(x)=2^{\frac{1}{x}}$ }

xxx. $f(x)=\log_a\left(x+\sqrt{x^2+1}\right)$. {Ans. $f^{-1}(x)=\frac{a^x-a^{-x}}{2}$ }

xxxi. $f(x)=\frac{2^x}{2^x+1}$. {Ans. $f^{-1}(x)=\log_2\left(\frac{x}{1-x}\right)$}

xxxii. $f(x)=\frac{10^x-10^{-x}}{10^x+10^{-x}}$ {Ans. $f^{-1}(x)=\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$}

xxxiii. $f(x)=\left(1-(x-5)^3\right)^{\frac{1}{5}}$. {Ans. $f^{-1}(x)=5+\left(1-x^5\right)^{\frac{1}{3}}$ }

xxxiv. $f(x) = 2\sin 3x$. {Ans. $2\sin 3y = x$ }

xxxv. $f(x) = 1 + 2\sin\dfrac{x-1}{x+1}$. {Ans. $\sin\dfrac{y-1}{y+1} = \dfrac{x-1}{2}$ }

xxxvi. $f(x) = 4\sin^{-1}\sqrt{1-x^2}$. {Ans. $f_1^{-1}(x) = \cos\dfrac{x}{4},\quad 0 \le x \le 2\pi$; $f_2^{-1}(x) = -\cos\dfrac{x}{4},\quad 0 \le x \le 2\pi$ }

xxxvii. $y^2 - 1 + \log_2(x-1) = 0$. {Ans. $f_1^{-1}(x) = 1 + 2^{1-x^2},\quad x \ge 0$; $f_2^{-1}(x) = 1 + 2^{1-x^2},\quad x \le 0$ }

xxxviii. $f(x) = \sin(3x-1),\quad x \in \left[\dfrac{1}{3} - \dfrac{\pi}{6}, \dfrac{1}{3} + \dfrac{\pi}{6}\right]$. {Ans. $f^{-1}(x) = \dfrac{1+\sin^{-1}x}{3}$ }

xxxix. $f(x) = \sin^{-1}\dfrac{x}{3},\quad x \in [-3,3]$. {Ans. $f^{-1}(x) = 3\sin x,\quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ }

xl. $f(x) = \ln(x^2 + 3x + 1),\quad x \in [1,3]$. {Ans. $f^{-1}(x) = \dfrac{-3+\sqrt{5+4e^x}}{2},\quad \ln 5 \le x \le \ln 19$ }

xli.
$$\begin{aligned} f(x) &= x, \quad x < 1 \\ &= x^2, \quad 1 \le x \le 4 \\ &= 2^x, \quad x > 4. \end{aligned}$$

{Ans.
$$\begin{aligned} f^{-1}(x) &= x, \quad x < 1 \\ &= \sqrt{x}, \quad 1 \le x \le 16 \\ &= \log_2 x, \quad x > 16 \end{aligned}$$
}

xlii.
$$\begin{aligned} f(x) &= e^x, \quad x \le 0 \\ &= x+1, \quad 0 < x < 1 \\ &= 2\sqrt{x}, \quad x \ge 1. \end{aligned}$$

{Ans.
$$\begin{aligned} f^{-1}(x) &= \ln x, \quad 0 < x \le 1 \\ &= x-1, \quad 1 < x < 2 \\ &= \frac{x^2}{4}, \quad x \ge 2 \end{aligned}$$
}

xliii.
$$\begin{aligned} f(x) &= x^3 + 1, \quad x < 0 \\ &= x^2 + 1, \quad x \ge 0. \end{aligned}$$

{Ans.
$$\begin{aligned} f^{-1}(x) &= \sqrt[3]{x-1}, \quad x < 1 \\ &= \sqrt{x-1}, \quad x \ge 1 \end{aligned}$$
}

62. Prove that the function $y = \dfrac{k}{x}$ is inverse to itself.

63. Prove that the function $y = \dfrac{1-x}{1+x}$ is inverse to itself.

64. Prove that the function $y = \dfrac{x+2}{x-1}$ is inverse to itself.

65. Find the value of the parameter α, for which the function $f(x)=1+\alpha x, \quad \alpha \neq 0$ is the inverse of itself. {Ans. -1}

66. Let $f(x)=\dfrac{\alpha x}{x+1}, \quad x \neq -1$. For what value of α, $f(x)$ is the inverse of itself? {Ans. -1}

67. Prove that the inverse of the linear-fractional function $y=\dfrac{ax+b}{cx+d}$ $(ad-bc \neq 0)$ is also a linear-fractional function. Under what conditions does this function coincide with it's inverse? {Ans. $a=-d$ }

68. Show that the function $f(x)=\sqrt[n]{a-x^n}$, $x>0$ is inverse to itself.

69. If $f(x)=2x-3$ and $g(x)=x^3+5$, then find $(fog)^{-1}(x)$. {Ans. $(fog)^{-1}(x)=\left(\dfrac{x-7}{2}\right)^{\frac{1}{3}}$ }

70. Find the set of all values of *a* for which $f(x)=x^3+(a+2)x^2+3ax+5$ is invertible. {Ans. $a \in [1,4]$}

71. If $f(x)=x^3+x+1$, then find $\left(\dfrac{d}{dx} f^{-1}(x)\right)_{x=1}$. {Ans. 1}

72. If $f(x)=x+e^x$ and $g(x)$ be its inverse function, then find $g'(1)$. {Ans. $\dfrac{1}{2}$ }

73. If $f(x)=x+\cos x$ and $g(x)$ be its inverse function, then find $g'\left(\dfrac{3\pi}{2}\right)$ and $g'\left(\dfrac{\pi}{4}+\dfrac{1}{\sqrt{2}}\right)$. {Ans. $\dfrac{1}{2}$, $\dfrac{\sqrt{2}}{\sqrt{2}-1}$ }

74. Show that the function $f(x)=x^2-x+1, \quad x \geq \dfrac{1}{2}$ and $g(x)=\dfrac{1}{2}+\sqrt{x-\dfrac{3}{4}}$ are mutually inverse, and solve the equation $x^2-x+1=\dfrac{1}{2}+\sqrt{x-\dfrac{3}{4}}$. {Ans. $x=1$}

CATEGORY-6.8. GREATEST INTEGER FUNCTION

75. Find the domain of the function $f(x)=\dfrac{1}{[|x-1|]+[|7-x|]-6}$ ([ ] denotes greatest integer function). {Ans. $(-\infty,0]\cup(1,2)\cup(2,3)\cup(3,4)\cup(4,5)\cup(5,6)\cup(6,7)\cup[8,\infty)$}

76. $\lim\limits_{x\to\infty}\{x\}$. {Ans. no limit}

77. $\lim\limits_{x\to 2^-} x+(x-[x])^2$. {Ans. 3}

78. $\lim\limits_{x\to 1} 1-x+[x-1]+[1-x]$. {Ans. -1}

79. $\lim\limits_{x\to 4}\dfrac{x-\{x\}}{x-4}$. {Ans. ∞, $-\infty$}

80. $\lim\limits_{x\to 0}\dfrac{x-|x|}{x-[x]}$. {Ans. 0}

81. $\lim\limits_{x\to 0}\left[\dfrac{\sin(\operatorname{sgn} x)}{\operatorname{sgn} x}\right]$. {Ans. 0}

82. $\lim\limits_{x\to 0} x\left[\frac{1}{x}\right]$. {Ans. 1}

83. $\lim\limits_{x\to \infty} x\left[\frac{1}{x}\right]$. {Ans. 0}

84. $\lim\limits_{x\to n} (-1)^{[x]}$. {Ans. $(-1)^n$, $(-1)^{n-1}$ }

85. $\lim\limits_{x\to 0} \{x\}$. {Ans. 0, 1}

86. $\lim\limits_{n\to \infty} \frac{[nx]}{n}$. {Ans. x}

87. $\lim\limits_{n\to \infty} \frac{[x]+[2x]+.........+[nx]}{n^2}$. {Ans. $\frac{x}{2}$ }

88. $\lim\limits_{n\to \infty} \frac{[1^2x]+[2^2x]+[3^2x]+......+[n^2x]}{n^3}$. {Ans. $\frac{x}{3}$ }

89. $\lim\limits_{x\to 0} \frac{\tan\left(\left[-2\pi^2\right]x^2\right)-\left(\tan\left[-2\pi^2\right]\right)x^2}{\sin^2 x}$. {Ans. $\tan 20-20$ }

90. $\lim\limits_{x\to 0}\lim\limits_{n\to \infty} \frac{\left[1^2(\sin x)^x\right]+\left[2^2(\sin x)^x\right]+....+\left[n^2(\sin x)^x\right]}{n^3}$. {Ans. $\frac{1}{3}$ }

91. $\lim\limits_{x\to 0} \frac{\sin^{-1}(1-\{x\})\cdot\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}(1-\{x\})}$. {Ans. $\frac{\pi}{2}, \frac{\pi}{2\sqrt{2}}$ }

92. $\lim\limits_{x\to 0}\left(\frac{(1+\{x\})^{\frac{1}{\{x\}}}}{e}\right)^{\frac{1}{\{x\}}}$. {Ans. $e^{-\frac{1}{2}}, \frac{2}{e}$ }

93. Show that $[x]+\left[x+\frac{1}{2}\right]=[2x]\ \forall x$.

94. Show that $[x]+\left[x+\frac{1}{3}\right]+\left[x+\frac{2}{3}\right]=[3x]\ \forall x$.

95. Show that $\left[x+\frac{1}{2}\right]-\left[x-\frac{1}{2}\right]=1\ \forall x$.

96. Plot graphs:-
    i. $f(x)=[\sin x]$.
    ii. $f(x)=\sin^{-1}[x]$.
    iii. $y=\operatorname{sgn}[x]$.
    iv. $y=\ln[\sin x]$.
    v. $f(x)=\left|e^{\{x\}}-2\right|$.
    vi. $f(x)=[2x]-2[x]$.
    vii. $f(x)=|[x]|-[|x|]$.
    viii. $f(x)=[x]-\left[x-\frac{1}{2}\right]$.
    ix. $f(x)=(-1)^{[x]}$.
    x. $f(x)=\frac{1}{\{x\}}$.

xi. $f(x)=[x]+[-x]$.

xii. $f(x)=\left[x+\frac{1}{2}\right]-\left[x-\frac{1}{2}\right]$.

97. If $f(x)=1+x-[x]$, then find the function $g(x)=\operatorname{sgn}(f(x))$. {Ans. $g(x)=1$}

98. Show that the function $f(x)=[x]+[-x]$ has removable discontinuity for integral values of *x*.

99. Draw the graphs and discuss continuity and differentiability of the following functions:-

i. $f(x)=2x+3[x], \quad -2\le x\le 2$. {Ans. Discontinuous & not differentiable at $x=-1, 0, 1, 2$}

ii. $f(x)=x[x], \quad -2\le x\le 2$. {Ans. Discontinuous at $x=-1, 1, 2$ & not differentiable at $x=-1, 0, 1, 2$}

iii. $f(x)=[x^2], \quad -\frac{3}{2}\le x\le \frac{3}{2}$. {Ans. Discontinuous & not differentiable at $x=-\sqrt{2}, -1, 1, \sqrt{2}$}

iv. $f(x)=x[x^2], \quad -\frac{3}{2}\le x\le \frac{3}{2}$. {Ans. Discontinuous & not differentiable at $x=-\sqrt{2}, -1, 1, \sqrt{2}$}

v. $f(x)=\sin\pi[x], \quad -\frac{\pi}{2}<x<\frac{\pi}{2}$. {Ans. Differentiable everywhere}

vi. $f(x)=\sin\pi\{x\}, \quad -\frac{\pi}{2}<x<\frac{\pi}{2}$. {Ans. Continuous $\forall x$; not differentiable at $0, \pm 1$}

vii. $f(x)=[x]+|1-x|, \quad -1\le x\le 3$. {Ans. Discontinuous & not differentiable at 0, 1, 2, 3}

viii. $f(x)=x[x], \quad 0\le x<2$

$\quad = (x-1)[x], \quad 2\le x<3$. {Ans. Discontinuous at $x=1$ & not differentiable at $x=1, 2$}

ix. $f(x)=\left|\left(x+\frac{1}{2}\right)[x]\right|, \quad -2\le x\le 2$. {Ans. Discontinuous at $x=-1, 0, 1, 2$; not differentiable at $x=-1, -\frac{1}{2}, 0, 1, 2$}

x. $f(x)=\frac{\sin 4\pi[x]}{1+[x]^2}$. {Ans. Continuous and differentiable for all *x*)

100. If

$$f(x)=0, \quad x\in I$$
$$\quad = x^2, \quad x\notin I.$$

Discuss the continuity and differentiability of $[f(x)]$ and $f([x])$. {Ans. Discontinuous & not differentiable at $x=\pm\sqrt{n}, n\in N$. Continuous & differentiable $\forall x$.}

101. Prove that $f(x)=\frac{2x(\sin x+\tan x)}{2\left[\frac{x+2\pi}{\pi}\right]-3}$ is an odd function.

102. Check the function

$$f(x)=\frac{[x^2]-1}{x^2-1}, \quad x^2\ne 1$$
$$\quad =0, \quad x^2=1$$

for continuity at $x=1$. {Ans. Discontinuous}

103. For what values of *a* and *b*, the function

$$f(x)=\frac{a+3\cos x}{x^2},\quad x<0$$
$$=b\tan\frac{\pi}{[x+3]},\quad x\geq 0$$

is continuous at $x=0$. {Ans. $a=-3,\ b=-\frac{\sqrt{3}}{2}$}

104. Discuss the continuity of the function
$$f(x)=[x]+\sqrt{\{x\}},\quad x\geq 0$$
$$=\sin x,\quad x<0.$$ {Ans. Continuous $\forall x$}

105. If
$$f(x)=|x-1|\{x\},\quad x\neq 1$$
$$=0,\quad x=1.$$
Test the differentiability at $x=1$. {Ans. Not differentiable}

106. Draw the graph of the function
$$F(x)=x-[x],\quad 2n\leq x<2n+1$$
$$=\frac{1}{2}\quad,\quad 2n+1\leq x<2n+2$$
where n is an integer. Is the function periodic? If periodic, what is its period? What are the points of discontinuity of $F(x)$? {Ans. T = 2, discontinuous for all integer x}

107. Find the period of the function $f(x)=e^{\{x\}+|\cos\pi x|+|\cos 2\pi x|+|\cos 3\pi x|+......+|\cos n\pi x|}$. {Ans. 1}

108. For what values of a & b the function $f(x)=x+a[bx]$ is periodic? Find it's period. {Ans. $ab=-1, T=|a|$}

109. If $f(x)=x+[x]$, then find its inverse function. {Ans. $f^{-1}(x)=x-\frac{1}{2}[x],\quad 2I\leq x\leq 2I+1$}

110. For what values of the constant a, the function $f(x)=x+[ax]$ is inverse to itself and plot it's graph. {Ans. -2, 0}

111. For what values of a, $[x+a]-[x-a]=\text{constant}\,\forall x$. {Ans. $a=\frac{I}{2}=0,\pm\frac{1}{2},\pm 1,\pm\frac{3}{2},....$}

112. If f be function defined on set of non-negative integers and taking values in the same set. Given that:-

i. $x-f(x)=19\left[\frac{x}{19}\right]-90\left[\frac{f(x)}{90}\right]$ for all non-negative integers;

ii. $1900<f(1990)<2000$.

Find the possible values of $f(1990)$. ([ ] denotes greatest integer function). {Ans. 1904, 1994}

CATEGORY-6.9. FINDING RANGE OF ANALYTICAL FUNCTIONS

113. $f(x)=\frac{x}{|x|}$. {Ans. $[-1]\cup[1]$}

114. $f(x)=\sqrt{x-x^2}$. {Ans. $\left[0,\frac{1}{2}\right]$}

115. $f(x)=\sqrt{3x^2-4x+5}$. {Ans. $\left[\sqrt{\frac{11}{3}},\infty\right)$}

116. $f(x)=\log(3x^2-4x+5)$. {Ans. $\left[\log\frac{11}{3},\infty\right)$}

117. $f(x)=\log(5x^2-8x+4)$. {Ans. $\left[\log\frac{4}{5},\infty\right)$}

118. $f(x)=\sqrt{x-1}+2\sqrt{3-x}$. {Ans. $\left[\sqrt{2},\sqrt{10}\right]$}

119. $f(x)=\log_2\frac{\sin x-\cos x+3\sqrt{2}}{\sqrt{2}}$. {Ans. $[1,2]$}

120. $f(x)=\sqrt{2-x}+\sqrt{1+x}$. {Ans. $\left[\sqrt{3},\sqrt{6}\right]$}

121. $f(x)=\frac{x}{x+1}$. {Ans. $R-[1]$}

122. $f(x)=\frac{1}{2-\sin 3x}$. {Ans. $\left[\frac{1}{3},1\right]$}

123. $f(x)=\log\sqrt{x^2+6x+10}$. {Ans. $[0,\infty)$}

124. $f(x)=\sin^{-1}\left[\frac{1}{2}+x^2\right]$ ([ ] denotes Greatest integer function). {Ans. $[0]\cup\left[\frac{\pi}{2}\right]$}

125. $f(x)=\frac{e^x-e^{-|x|}}{e^x+e^{|x|}}$. {Ans. $\left[0,\frac{1}{2}\right)$}

126. $f(x)=\frac{1}{\sqrt{4+3\cos x}}$. {Ans. $\left[\frac{1}{\sqrt{7}},1\right]$}

127. $f(x)=\frac{1}{\sqrt{\{x\}}}$. {Ans. $(1,\infty)$}

128. $f(x)=3\sin\sqrt{\frac{\pi^2}{16}-x^2}$. {Ans. $\left[0,\frac{3}{\sqrt{2}}\right]$}

129. Find domain and range of $f(x)=\sin\log\left(\frac{\sqrt{4-x^2}}{1-x}\right)$. {Ans. Domain: $(-2,1)$; Range: $[-1,1]$}

CATEGORY-6.10. MISCELLANEOUS QUESTIONS ON FUNCTIONS

130. Let $A=\{1,2,3,\}$, $B=\{2,3,4\}$ then which of the following is function from A to B :-
   i. $f_1=\{(1,2),(1,3),(2,3),(3,3)\}$. {Ans. not a function}
   ii. $f_2=\{(1,3),(2,4)\}$. {Ans. not a function}
   iii. $f_3=\{(1,3),(2,2),(3,3)\}$. {Ans. function}
   iv. $f_4=\{(1,2),(2,3),(3,2),(3,4)\}$. {Ans. not a function}

131. Let $A=\{-2,-1,0,1,2\}$ and $B=\{0,1,2,3,4,5,6\}$ and a rule $f(x)=x^2$. Whether $f:A\to B$ is a function or not? If yes, find range of *f*. {Ans. Yes; $f(A)=\{0,1,4\}$}

132. Consider a rule $f(x)=2x-3$. Whether $f:N\to N$ is a function or not?. {Ans. No}

133. Let $A=\{-2,-1,0,1,2\}$ and $f:A\to I$ given by $f(x)=x^2-2x-3$. Find range of *f*. Also find pre-images of 6, -3 and 5. {Ans. $f(A)=\{-4,-3,0,5\}$; no pre-image of 6; 0 and 2 are pre-images of –3; -2 is the pre-image of 5}

134. Given $A=\{-1,0,2,5,6,11\}$ and $B=\{-2,-1,0,18,28,108\}$ and $f(x)=x^2-x-2$. Find $f(A)$. Whether $f(A)=B$ or not. {Ans. $f(A)=\{-2,0,18,28,108\}$; $f(A)\neq B$ }

135. Let $f:R\to R$ be given by $f(x)=x^2+3$. Find $\{x \mid f(x)=28\}$. Also find the pre-images of 39 and 2 under *f*. {Ans. $\{-5,5\}$; pre-images of 39 are –6 and 6; no pre-image of 2}

136. Express the following functions as sets of ordered pairs and determine their ranges:-
   - i. $f:A\to R$, $f(x)=x^2+1$, where $A=\{-1,0,2,4\}$. {Ans. $f=\{(-1,2),(0,1),(2,5),(4,17)\}$; $f(A)=\{1,2,5,17\}$}
   - ii. $g:A\to N$, $g(x)=2x$, where $A=\{x \mid x\in N, x\le 5\}$. {Ans. $g=\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$; $g(A)=\{2,4,6,8,10\}$}

137. Whether $g=\{(1,1),(2,3),(3,5),(4,7)\}$ a function or not? If *g* is defined by the rule $g(x)=ax+b$, then what values should be assigned to *a* and *b*? {Ans. $a=2, b=-1$}

138. If the function f and g are given by $f=\{(1,2),(3,5),(4,1)\}$ and $g=\{(2,3),(5,1),(1,3)\}$. Write *fog* and *gof* as set of ordered pairs. {Ans. $fog=\{(2,5),(5,2),(1,5)\}$; $gof=\{(1,3),(3,1),(4,3)\}$}

139. If $A=\{1,2,3,4\}$ and $B=\{2,4,6,8\}$ and $f:A\to B$ is given by $f(x)=2x$, then write f^{-1} as a set of ordered pairs. {Ans. $f^{-1}=\{(2,1),(4,2),(6,3),(8,4)\}$}

140. Let $f(x)=x^3$ be a function with domain $\{0,1,2,3\}$. Find the domain of f^{-1}. {Ans. $\{0,1,8,27\}$}

141. Which of the following functions are one-one, many-one, into, onto & bijective:-
   - i. $f:R\to R,\ f(x)=\sin x$. {Ans. many-one, into}
   - ii. $f:R\to[-1,1],\ f(x)=\sin x$. {Ans. many-one, onto}
   - iii. $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to R,\ f(x)=\sin x$. {Ans. one-one, into}
   - iv. $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to[-1,1],\ f(x)=\sin x$. {Ans. one-one, onto, bijective}
   - v. $f:[0,\pi]\to[-1,1],\ f(x)=\cos x$. {Ans. one-one, onto, bijective}
   - vi. $f:R^+\to R,\ f(x)=2\sqrt{x}+1$. {Ans. one-one, into}
   - vii. $f:R\to R,\ f(x)=x^3+3$. {Ans. one-one, onto, bijective}
   - viii. $f(x)=\ln x$. {Ans. one-one, onto, bijective}
   - ix. $f:(0,1)\to R,\ f(x)=\ln x$. {Ans. one-one, into}
   - x. $f:R\to I,\ f(x)=[x]$. {Ans. many-one, onto}
   - xi. $f:N\to N,\ f(x)=[x]$. {Ans. one-one, onto, bijective}
   - xii. $f:R\to I,\ f(x)=\operatorname{sgn} x$. {Ans. many-one, into}
   - xiii. $f:R\to[0,1),\ f(x)=\{x\}$. {Ans. many-one, onto}
   - xiv. $f:[0,1)\to[0,1),\ f(x)=\{x\}$. {Ans. one-one, onto, bijective}

xv. $f : N \to N$, $f(x) = 2x + 3$. {Ans. injective, into}

142. Which of the following are functions?
i. $\{(x, y) : y^2 = x,\ x, y \in R\}$. {Ans. not a function}
ii. $\{(x, y) : y = |x|,\ x, y \in R\}$. {Ans. function}
iii. $\{(x, y) : x^2 + y^2 = 1,\ x, y \in R\}$. {Ans. not a function}
iv. $\{(x, y) : x^2 - y^2 = 1,\ x, y \in R\}$. {Ans. not a function}
v. $\{(x, y) \mid x, y \in R,\ x^2 = y\}$. {Ans. function}
vi. $\{(x, y) \mid x, y \in R,\ y^2 = x\}$. {Ans. not a function}
vii. $\{(x, y) \mid x, y \in R,\ x = y^3\}$. {Ans. function}
viii. $\{(x, y) \mid x, y \in R,\ y = x^3\}$. {Ans. function}

143. If a function $f : [2, \infty) \to B$ defined by $f(x) = x^2 - 4x + 5$ is a bijection, then find *B*. {Ans. $[1, \infty)$}

CATEGORY-6.11. ADDITIONAL QUESTIONS

144. Find the minimum value of $\dfrac{\left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6}\right) - 2}{\left(x + \frac{1}{x}\right)^3 + x^3 + \frac{1}{x^3}}$ for $x > 0$. {Ans. 6}

145. If $f(x) = x$ and $g(x) = |x|$, then find the function $\phi(x)$ satisfying $[\phi(x) - f(x)]^2 + [\phi(x) - g(x)]^2 = 0$. {Ans. $\phi(x) = x,\ x \in [0, \infty)$}

146. If $f(x) = (ax^2 + b)^3$, then find the function g such that $f(g(x)) = g(f(x))$. {Ans. $g(x) = \left(\dfrac{x^{\frac{1}{3}} - b}{a}\right)^{\frac{1}{2}}$}

147. Draw a graph of the function $f(x) = x - |x - x^2|$, $-1 \le x \le 1$ and discuss its continuity or discontinuity in the interval $-1 \le x \le 1$. {Ans. continuous}

148. The function *f* is defined by $y = f(x)$ where $x = 2t - |t|,\ y = t^2 + t|t|,\ t \in R$. Draw the graph of *f* for the interval $-1 \le x \le 1$. Also discuss it's continuity and differentiability at $x = 0$. {Ans. continuous and differentiable}

149. Given $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, $x \ne 0$, $a \ne b$. Find $f(x)$. {Ans. $f(x) = \dfrac{\frac{a}{x} - bx}{a^2 - b^2} - \dfrac{5}{a + b}$}

150. If $b \ge 1$ and $f(x) = \dfrac{1}{|x|}$, then show that the conditions of L. M. V. theorem are not satisfied in the interval $[-1, b]$ but the conclusion of the theorem is true iff $b > 1 + \sqrt{2}$.

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Mathematics for IIT-JEE

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PART-II
ALGEBRA

CHAPTER-7
EQUATIONS AND INEQUALITIES

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CHAPTER-7
EQUATIONS AND INEQUALITIES

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CHAPTER-7
EQUATIONS AND INEQUALITIES

SECTION-7.1. EQUATION

1. Equation, Solution (root) of equation, Solution set of equation, Domain of equation

i. If $f(x)$ and $g(x)$ are functions, then $f(x)=g(x)$ is said to be an equation.

ii. A real number *a* is said to be a solution (root) of the equation $f(x)=g(x)$ iff

a. $f(a)$ is defined;

b. $g(a)$ is defined;

c. $f(a)$ and $g(a)$ are numerically equal, i.e. $f(a)=g(a)$;

otherwise *a* is not a solution of the equation $f(x)=g(x)$.

iii. The set of all the solutions of an equation is called its solution set, denoted by S_e. $S_e \subseteq R$.

iv. Domain of the equation $f(x)=g(x)$, denoted by D_e, is defined as $D_e = D_f \cap D_g$. $D_e \subseteq R$.

v. $S_e \subseteq D_e$.

2. Identity & contradiction

i. If $S_e = D_e$, then the equation is said to be an identity.

ii. If $S_e = \phi$, then the equation is said to be a contradiction.

iii. If $S_e \neq \phi$ and $S_e \subset D_e$, then the equation is neither an identity nor a contradiction.

3. Solving an equation

i. Solving an equation means finding its solution set.

4. Solving an equation graphically

i. Solutions of the equation $f(x)=g(x)$ are the *x*-coordinates of the points of intersection of the curves of $f(x)$ and $g(x)$ in their graphical representation. The number of solutions is the number of points of intersection of their curves. From the graphical sketch of the curves of $f(x)$ and $g(x)$, number of solutions of the equation $f(x)=g(x)$ and approximate values of solutions can be determined.

5. Simplest algebraic equations

i. If *a* is constant real number, then equations of type $x=a$ are said to be simplest algebraic equations having solution set $S_e = [a]$.

6. Equivalent equations

i. Two (or more) equations are said to be equivalent equations iff they have the same solution set, otherwise they are said to be non- equivalent equation.

ii. If the equations $f_1(x)=g_1(x)$ and $f_2(x)=g_2(x)$are equivalent then they are written as $f_1(x)=g_1(x) \equiv f_2(x)=g_2(x)$ and if they are non- equivalent then they are written as $f_1(x)=g_1(x) \not\equiv f_2(x)=g_2(x)$.

7. Equivalent transformations in equations

i. <u>Theorem:</u> Consider the equations $f(x)=g(x)$ and $f(x)+\phi(x)=g(x)+\phi(x)$. If *a* is a solution of the equation $f(x)=g(x)$ and *a* does not belong to the domain of $\phi(x)$, then *a* is not a solution of the equation $f(x)+\phi(x)=g(x)+\phi(x)$ and hence, both the equations are non-equivalent.

ii. Theorem: Consider the equations $f(x)=g(x)$ and $f(x)\times\phi(x)=g(x)\times\phi(x)$. If *a* is a solution of the equation $f(x)=g(x)$ and *a* does not belong to the domain of $\phi(x)$, then *a* is not a solution of the equation $f(x)\times\phi(x)=g(x)\times\phi(x)$ and hence, both the equations are non-equivalent.

iii. Theorem: Consider the equations $f(x)=g(x)$ and $f(x)\times\phi(x)=g(x)\times\phi(x)$. If *a* is not a solution of the equation $f(x)=g(x)$, *a* belongs to the domain of this equation and $\phi(a)=0$, then *a* is a solution of the equation $f(x)\times\phi(x)=g(x)\times\phi(x)$ and hence, both the equations are non-equivalent.

8. System & Collection of equations/ inequalities/ systems/ collections

i. A *system* of two or more equations/inequalities is written as

$$\left\{\begin{array}{c} f_1(x)=g_1(x) \\ f_2(x)=g_2(x) \\ \vdots \\ \vdots \end{array}\right\}$$ or $f_1(x)=g_1(x),\ f_2(x)=g_2(x),\ \cdots\cdots$ (separated by comma). The *solution set of a system* is defined as intersection of the solution sets of the equations/inequalities in the system, i.e every solution of the system satisfies each of the equations/inequalities in the system.

ii. A *collection* of two or more equations/inequalities is written as $f_1(x)=g_1(x);\ f_2(x)=g_2(x);\ \cdots\cdots$ (separated by semicolon). The *solution set of a collection* is defined as union of the solution sets of the equations/inequalities in the collection, i.e every solution of the collection satisfies at least one of the equations/inequalities in the collection.

iii. A system or collection may contain systems and collections.

9. Equivalent equations/ inequalities/ systems/ collections

i. Two or more equations/ inequalities/ systems/ collections are said to be equivalent if they have the same solution set.

10. Solving an equation analytically

i. The process of solving an equation analytically consists of certain analytical transformation leading to equivalent simplest algebraic equations.

11. Types of equations

i. Polynomial equations
ii. Rational equations
iii. Irrational equations
iv. Equations containing modulus function
v. Exponential equations
vi. Logarithmic equations
vii. Expo-logarithmic equations

SECTION-7.2. POLYNOMIAL FUNCTIONS

1. Polynomial function, degree, coefficients, real polynomial, complex polynomial

i. The function $P_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots\cdots+a_1x+a_0$ is called a polynomial function where $n\in N$; a_n, a_{n-1}, ..., a_1, a_0 are real/complex constants; $a_n\neq 0$.

ii. *n* is called the degree of the polynomial.

iii. a_n, a_{n-1}, ..., a_1, a_0 are called coefficients.

iv. If all the coefficients are real numbers then the polynomial is called a real polynomial. If at least one coefficient is non-real complex number then the polynomial is called a complex polynomial.

2. Theorem of roots

i. Any polynomial in the set of complex numbers can always be factored into a product of n linear factors, i.e.
$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_1 x + a_0$
$= a_n(x-\alpha_1)(x-\alpha_2)\cdots\cdots(x-\alpha_{n-1})(x-\alpha_n)$.

ii. α_1, α_2, ..., α_{n-1}, α_n are real/complex numbers and are called the n roots of the polynomial. A polynomial of degree n always has n roots and the set of roots are unique.

iii. Some of the roots may be equal to one another. If m roots are equal to α, then it is said that the number α is a root of order (multiplicity) m of the polynomial.

iv. If α is a root of $P(x)$, then $x-\alpha$ is a factor of $P(x)$.

v. If α is a root of order m of $P(x)$, then $(x-\alpha)^m$ is a factor of $P(x)$.

vi. If α is a root of $P(x)$, then $P(\alpha)=0$.

vii. Polynomials $P(x)$ and $kP(x)$ have same roots.

3. Theorem of equivalence

i. Two polynomials $P(x)$ and $Q(x)$ are equivalent functions if and only if their degrees are equal and the coefficients of same powers of x are also equal, otherwise they are non-equivalent. Thus, if
$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_1 x + a_0$
$Q_m(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots\cdots + b_1 x + b_0$
then $P_n(x) \equiv Q_m(x)$ iff $n = m$ and $a_i = b_i$ $\forall\ i = 0, 1, 2, \cdots\cdots, n$.

4. Theorem of conjugate complex roots

i. If a real polynomial has a complex root $a+ib$ then it is necessary that its conjugate $a-ib$ is also a root of the polynomial.

ii. A real polynomial function either has no non-real complex roots or has even number of non-real complex roots which are in conjugate pairs.

iii. If a real polynomial has a complex roots $a+ib$ and its conjugate $a-ib$, then the real quadratic $x^2-2ax+a^2+b^2$ is a factor of the polynomial.

iv. If a real polynomial has p real roots and q non-real complex roots then the polynomial has p real linear factors and $\frac{q}{2}$ real quadratic factors.

5. Theorem of roots of type $p \pm \sqrt{q}$ $(p \in I,\, q \in N)$

i. If a real polynomial has rational coefficients and if has a root $p+\sqrt{q}$ $(p \in I,\, q \in N)$ then it is necessary that its conjugate $p-\sqrt{q}$ $(p \in I,\, q \in N)$ is also a root of the polynomial.

6. Theorem of rational roots

i. If the real polynomial $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_1 x + a_0$ has integer coefficients and if a rational number $\frac{p}{q}$ is a root of the polynomial then p must be a divisor of a_0 and q must be a divisor of a_n.

7. Descartes' rule of signs

i. A real polynomial $P(x)$ cannot have more positive roots than there are changes in sign in $P(x)$ and cannot have more negative roots than there are changes of sign in $P(-x)$.

8. To find the roots of a real polynomial function

i. Polynomial of degree one (Linear function)
  a. The root of linear function $P_1(x) = ax + b$ is $\alpha_1 = -\frac{b}{a}$.

ii. Polynomial of degree two (Quadratic function)
  a. Given a quadratic function $P_2(x) = ax^2 + bx + c$, the number $\Delta = b^2 - 4ac$ is called the discriminant.
  b. The two roots of the quadratic function are $\alpha_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $\alpha_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$.
  c. If
     $\Delta > 0$ both the roots are real and different
     $\Delta = 0$ both the roots are real and equal
     $\Delta < 0$ both the roots are non-real complex and different.

iii. Higher degree polynomial function
  a. There is no formula for the roots of polynomials of degree greater than two.
  b. Find roots using Theorem of rational roots.
  c. Find the roots by substitution.
  d. Find the roots by factorization.

SECTION-7.3. POLYNOMIAL EQUATIONS

1. Linear equation

i. If $a \neq 0$ then $ax + b = 0$ is a linear equation.

ii. $ax + b = 0 \equiv x = -\frac{b}{a}$.

2. Quadratic equation

i. If $a \neq 0$ then $ax^2 + bx + c = 0$ is a quadratic equation.

ii. Type-I: If $\Delta = b^2 - 4ac > 0$

$$ax^2 + bx + c = 0$$
$$\equiv x = \frac{-b-\sqrt{b^2-4ac}}{2a};\ x = \frac{-b+\sqrt{b^2-4ac}}{2a}.$$

iii. Type-II: If $\Delta = b^2 - 4ac = 0$

$$ax^2 + bx + c = 0$$
$$\equiv x = \frac{-b}{2a}.$$

iv. Type-III: If $\Delta = b^2 - 4ac < 0$

$$ax^2 + bx + c = 0$$
$$\equiv x \in \phi.$$

3. Higher order polynomial equation

i. Type-I: Write the equation as $P(x) = 0$. Find real distinct roots of the polynomial $P(x)$ using Theorem of rational roots and these are the real solutions of the equation.

ii. Type-II: Substitution.

iii. Type-III: Factorization.

iv. Type-IV: Using Descartes' rule of signs and other theorems, if it can be shown that $P(x)$ has no real root, then the equation has no solution.

SECTION-7.4. RATIONAL EQUATIONS

1. Rational equations

i. An equation containing rational functions of *x* is called a rational equation.

2. Solving rational equations

i. Type-I:

$$\frac{P(x)}{Q(x)} = 0$$

$$\equiv \left\{\begin{matrix} Q(x) \neq 0 \\ P(x) = 0 \end{matrix}\right\}.$$

ii. Type-II: Substitution.

SECTION-7.5. IRRATIONAL EQUATIONS

1. Irrational equations

i. An equation containing irrational functions of *x* is called an irrational equation.

2. Solving irrational equations

i. Type-I: For equations containing square root, repeatedly square the equation to obtain a polynomial equation. Solve the polynomial equation. Back check each solution of the polynomial equation in the original equation to discard extraneous solutions.

ii. Type-II:

$$\sqrt{f(x)} \pm \sqrt{g(x)} = h(x) \cdots\cdots\cdots (1)$$

$$\Rightarrow \left(\sqrt{f(x)} \pm \sqrt{g(x)}\right)\left(\sqrt{f(x)} \mp \sqrt{g(x)}\right) = h(x)\left(\sqrt{f(x)} \mp \sqrt{g(x)}\right)$$

$$\Rightarrow \sqrt{f(x)} \mp \sqrt{g(x)} = \frac{f(x) - g(x)}{h(x)} \cdots\cdots\cdots (2).$$

Add equations (1) and (2) and solve.

iii. Type-III:

$$\sqrt[3]{f(x)} + \sqrt[3]{g(x)} = h(x)$$

$$\Rightarrow f(x) + g(x) + 3\sqrt[3]{f(x)}\sqrt[3]{g(x)}\left(\sqrt[3]{f(x)} + \sqrt[3]{g(x)}\right) = (h(x))^3$$

$$\Rightarrow f(x) + g(x) + 3\sqrt[3]{f(x)}\sqrt[3]{g(x)}h(x) = (h(x))^3$$

$$\Rightarrow 3\sqrt[3]{f(x)}\sqrt[3]{g(x)}h(x) = (h(x))^3 - f(x) - g(x).$$

Cube both sides to get polynomial equation and solve. Back check each solution of the polynomial equation in the original equation to discard extraneous solutions.

iv. Type-IV: Substitution.

SECTION-7.6. EQUATIONS CONTAINING MODULUS FUNCTION

1. Equations containing modulus function

i. An equation containing modulus of expressions of *x* is called an equation containing modulus function.

2. Solving equations containing modulus function

i. Type-I: Using definition of modulus.

ii. Type-II: Method of interval.

iii. Type-III: Substitution.

SECTION-7.7. EXPONENTIAL EQUATIONS

1. Exponential equations

i. An equation containing exponential functions of *x* is called an exponential equation.

2. Solving exponential equations

i. Type-I: Simplest exponential equations

a. $a^x = b \equiv x \in \phi, \quad b \le 0$.

b. $a^x = b \equiv x = \log_a b, \quad b > 0$.

ii. Type-II:

a. $a^{f(x)} = b \equiv x \in \phi, \quad b \le 0$.

b. $a^{f(x)} = b \equiv f(x) = \log_a b, \quad b > 0$.

iii. Type-III: $a^{f(x)} = a^{g(x)} \equiv f(x) = g(x)$.

iv. Type-IV: Substitution

$$\left\{\begin{array}{l} a^x / a^{f(x)} = y \\ \varphi(y) = 0 \end{array}\right\}.$$

v. Type-V: Homogeneous equation in two exponential functions *u* and *v*

$$a_n u^n + a_{n-1}u^{n-1}v + a_{n-2}u^{n-2}v^2 + \cdots\cdots + a_1 uv^{n-1} + a_0 v^n = 0$$

$$\equiv a_n\left(\frac{u}{v}\right)^n + a_{n-1}\left(\frac{u}{v}\right)^{n-1} + \cdots\cdots + a_1\left(\frac{u}{v}\right) + a_0 = 0$$

$$\equiv \left\{\begin{array}{l} \dfrac{u}{v} = y \\ a_n y^n + a_{n-1}y^{n-1} + \cdots\cdots + a_1 y + a_0 = 0 \end{array}\right\}.$$

SECTION-7.8. LOGARITHMIC EQUATIONS

1. Logarithmic equations

i. An equation containing logarithmic functions of *x* is called a logarithmic equation.

2. Solving logarithmic equations

i. Type-I: Simplest logarithmic equation

$\log_a x = b \equiv x = a^b$.

ii. Type-II:

$\log_a f(x) = b \equiv f(x) = a^b$.

iii. Type-III:

$$\log_a f(x) = \log_a g(x)$$

$$\equiv \left\{\begin{array}{l} f(x) > 0 \\ g(x) > 0 \\ f(x) = g(x) \end{array}\right\}.$$

iv. Type-IV:

$$\log_{a(x)} f(x) = \log_{a(x)} g(x)$$

$$\equiv \left\{ \begin{array}{l} f(x) > 0 \\ g(x) > 0 \\ a(x) > 0 \\ a(x) \neq 1 \\ f(x) = g(x) \end{array} \right\}.$$

v. Type-V: Substitution

$$\left\{ \begin{array}{l} \log_a x / \log_a f(x) / \log_{a(x)} f(x) = y \\ \varphi(y) = 0 \end{array} \right\}.$$

vi. Type-V: Homogeneous equation in two logarithmic functions u and v

$$a_n u^n + a_{n-1} u^{n-1} v + a_{n-2} u^{n-2} v^2 + \cdots\cdots + a_1 u v^{n-1} + a_0 v^n = 0$$

$$\equiv \left\{ \begin{array}{l} v^n = 0 \\ u^n = 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n \neq 0 \\ a_n \left(\frac{u}{v} \right)^n + a_{n-1} \left(\frac{u}{v} \right)^{n-1} + \cdots\cdots + a_1 \left(\frac{u}{v} \right) + a_0 = 0 \end{array} \right\}$$

$$\equiv \left\{ \begin{array}{l} v = 0 \\ u = 0 \end{array} \right\}; \left\{ \begin{array}{l} v \neq 0 \\ \frac{u}{v} = y \\ a_n y^n + a_{n-1} y^{n-1} + \cdots\cdots + a_1 y + a_0 = 0 \end{array} \right\}.$$

SECTION-7.9. EXPO-LOGARITHMIC EQUATIONS

1. Expo-logarithmic equations

i. An equation containing expo-logarithmic functions of x is called an expo-logarithmic equation.

2. Solving expo-logarithmic equations

i. Type-I:

$$f(x)^{g(x)} = \phi(x)^{\varphi(x)}$$

$$\equiv g(x) \log_a f(x) = \varphi(x) \log_a \phi(x).$$

Take log (of a suitable base) both sides and solve as logarithmic equation.

ii. Type-II: Write $f(x)^{g(x)} = a^{g(x) \log_a f(x)}$ for a suitable base a and solve as exponential equation.

iii. Type-III: Substitution

$$\left\{ \begin{array}{l} f(x)^{g(x)} = y \\ \varphi(y) = 0 \end{array} \right\}.$$

SECTION-7.10. INEQUALITY

1. Inequality, Solution of inequality, Solution set of inequality, Domain of inequality

i. If $f(x)$ and $g(x)$ are functions, then $f(x) > g(x)$ or $f(x) < g(x)$ is said to be an inequality.

ii. A real number a is said to be a solution of the inequality $f(x) > g(x)$ iff

a. $f(a)$ is defined;

b. $g(a)$ is defined;

c. $f(a)$ is numerically greater than $g(a)$, i.e. $f(a) > g(a)$;

otherwise a is not a solution of the inequality $f(x) > g(x)$.

iii. The set of all the solutions of the inequality is called its solution set, denoted by S_i. $S_i \subseteq R$.

iv. Domain of the inequality $f(x) > g(x)$, denoted by D_i, is defined as $D_i = D_f \cap D_g$. $D_i \subseteq R$.

v. $S_i \subseteq D_i$.

2. Strict & non-strict inequalities

i. $f(x) > g(x)$ or $f(x) < g(x)$ are called strict inequality and $f(x) \geq g(x)$ or $f(x) \leq g(x)$ are called non-strict inequality.

ii. $f(x) \geq g(x) \equiv f(x) > g(x); f(x) = g(x)$. $f(x) \leq g(x) \equiv f(x) < g(x); f(x) = g(x)$.

3. Identity & contradiction

i. If $S_i = D_i$, then the inequality is said to be an identity.

ii. If $S_i = \phi$, then the inequality is said to be a contradiction.

iii. If $S_i \neq \phi$ and $S_i \subset D_i$, then the inequality is neither an identity nor a contradiction.

4. Solving an inequality

i. Solving an inequality means finding its solution set.

5. Solving an inequality graphically

i. Inequalities can be solved from the graphical sketch of the curves of the functions contained in the inequality.

6. Simplest algebraic inequalities

i. If a is constant real number, then the following inequalities are said to be the simplest algebraic inequalities:-

a. $x > a$ having solution set $S_i = (a, \infty)$.

b. $x < a$ having solution set $S_i = (-\infty, a)$.

c. $x \geq a$ having solution set $S_i = [a, \infty)$.

d. $x \leq a$ having solution set $S_i = (-\infty, a]$.

7. Equivalent inequalities

i. Two (or more) inequalities are said to be equivalent inequalities iff they have the same solution set, otherwise they are said to be non- equivalent inequalities.

ii. If the inequalities $f_1(x) > g_1(x)$ and $f_2(x) > g_2(x)$ are equivalent then they are written as $f_1(x) > g_1(x) \equiv f_2(x) > g_2(x)$ and if they are non- equivalent then they are written as $f_1(x) > g_1(x) \not\equiv f_2(x) > g_2(x)$.

8. Equivalent transformations in inequalities

i. $\alpha > \beta \equiv \alpha + a > \beta + a$, where a is a constant.

ii. $\alpha > \beta \equiv a\alpha > a\beta$, if $a > 0$

$\alpha > \beta \equiv a\alpha < a\beta$, if $a < 0$.

iii. $\alpha > \beta \equiv \alpha^n > \beta^n$, if $\alpha > 0$, $\beta > 0$; n is even

$\alpha > \beta \equiv \alpha^n < \beta^n$, if $\alpha < 0$, $\beta < 0$; n is even.

iv. $\alpha > \beta \equiv \alpha^n > \beta^n$, if n is odd.

v. $\alpha > \beta \equiv \sqrt[n]{\alpha} > \sqrt[n]{\beta}$, if $\alpha > 0$, $\beta > 0$; n is even.

vi. $\alpha > \beta \equiv \sqrt[n]{\alpha} > \sqrt[n]{\beta}$, if n is odd.

vii. $\alpha > \beta \equiv \frac{1}{\alpha} < \frac{1}{\beta}$, if $\alpha > 0$, $\beta > 0$

$\alpha > \beta \equiv \frac{1}{\alpha} < \frac{1}{\beta}$, if $\alpha < 0$, $\beta < 0$

$\alpha > \beta \equiv \frac{1}{\alpha} > \frac{1}{\beta}$, if $\alpha > 0$, $\beta < 0$.

viii. $\alpha > \beta \equiv \alpha^a > \beta^a$, if $\alpha > 0, \beta > 0, a > 0$

$\alpha > \beta \equiv \alpha^a < \beta^a$, if $\alpha > 0, \beta > 0, a < 0$.

ix. $\alpha > \beta \equiv a^\alpha > a^\beta$, if $a > 1$

$\alpha > \beta \equiv a^\alpha < a^\beta$, if $0 < a < 1$.

x. $\alpha > \beta \equiv \log_a \alpha > \log_a \beta$, if $\alpha > 0$, $\beta > 0$, $a > 1$

$\alpha > \beta \equiv \log_a \alpha < \log_a \beta$, if $\alpha > 0$, $\beta > 0$, $0 < a < 1$.

9. System & Collection of equations/ inequalities/ systems/ collections

i. A *system* of two or more equations/inequalities is written as

$$\begin{cases} f_1(x) > g_1(x) \\ f_2(x) > g_2(x) \\ \vdots \\ \vdots \end{cases}$$ or $f_1(x) > g_1(x),\ f_2(x) > g_2(x), \cdots\cdots$ (separated by comma). The *solution set of a system* is defined as intersection of the solution sets of the equations/inequalities in the system, i.e every solution of the system satisfies each of the equations/inequalities in the system.

ii. A *collection* of two or more equations/inequalities is written as $f_1(x) > g_1(x);\ f_2(x) > g_2(x); \cdots\cdots$ (separated by semicolon). The *solution set of a collection* is defined as union of the solution sets of the equations/inequalities in the collection, i.e every solution of the collection satisfies at least one of the equations/inequalities in the collection.

iii. A system or collection may contain systems and collections.

10. Equivalent equations/ inequalities/ systems/ collections

i. Two or more equations/ inequalities/ systems/ collections are said to be equivalent if they have the same solution set.

11. Solving an inequality analytically

i. The process of solving an inequality analytically consists of certain analytical transformation leading to equivalent simplest algebraic inequalities.

12. Types of inequalities

i. Polynomial inequalities
ii. Rational inequalities
iii. Irrational inequalities
iv. Inequalities containing modulus function
v. Exponential inequalities
vi. Logarithmic inequalities
vii. Expo-logarithmic inequalities

SECTION-7.11. POLYNOMIAL INEQUALITIES

1. **Linear inequalities**
   i. Type-I: $ax+b>0$.
   ii. Type-II: $ax+b\geq 0$.
   iii. Type-III: $ax+b<0$.
   iv. Type-IV: $ax+b\leq 0$.
2. **Quadratic inequalities**
   i. Type-I: $ax^2+bx+c>0$.
   ii. Type-II: $ax^2+bx+c\geq 0$.
   iii. Type-III: $ax^2+bx+c<0$.
   iv. Type-IV: $ax^2+bx+c\leq 0$.
   v. Case-I: $a>0, \Delta>0$.
   vi. Case-II: $a>0, \Delta=0$.
   vii. Case-III: $a>0, \Delta<0$.
   viii. Case-IV: $a<0, \Delta>0$.
   ix. Case-V: $a<0, \Delta=0$.
   x. Case-VI: $a<0, \Delta<0$.
3. **Higher order polynomial inequalities**
   i. Type-I: Method of interval
      a. $P(x)>0$
      b. $P(x)\geq 0$
      c. $P(x)<0$
      d. $P(x)\leq 0$.
   ii. Type-II: Substitution.
   iii. Type-III: Factorization.

SECTION-7.12. RATIONAL INEQUALITIES

1. **Solving rational inequalities**
   i. Type-I: Method of interval
      a. $\frac{P(x)}{Q(x)}>0$
      b. $\frac{P(x)}{Q(x)}\geq 0$
      c. $\frac{P(x)}{Q(x)}<0$
      d. $\frac{P(x)}{Q(x)}\leq 0$.
   ii. Type-II: Substitution.
   iii. Type-III: Factorization.

SECTION-7.13. IRRATIONAL INEQUALITIES

1. **Solving irrational inequalities**
   i. Type-I:

a. $\sqrt{f(x)} > \sqrt{g(x)}$

$$\equiv \begin{Bmatrix} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) > g(x) \end{Bmatrix}.$$

b. $\sqrt{f(x)} \geq \sqrt{g(x)}$

$$\equiv \begin{Bmatrix} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) \geq g(x) \end{Bmatrix}.$$

c. $\sqrt{f(x)} < \sqrt{g(x)}$

$$\equiv \begin{Bmatrix} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) < g(x) \end{Bmatrix}.$$

d. $\sqrt{f(x)} \leq \sqrt{g(x)}$

$$\equiv \begin{Bmatrix} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) \leq g(x) \end{Bmatrix}.$$

ii. Type-II:

a. $\sqrt{f(x)} - \sqrt{g(x)} > \sqrt{h(x)}$

$\equiv \sqrt{f(x)} > \sqrt{h(x)} + \sqrt{g(x)}$

$$\equiv \begin{Bmatrix} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) > \left(\sqrt{g(x)} + \sqrt{h(x)}\right)^2 \end{Bmatrix}.$$

b. $\sqrt{f(x)} - \sqrt{g(x)} \geq \sqrt{h(x)}$

$\equiv \sqrt{f(x)} \geq \sqrt{h(x)} + \sqrt{g(x)}$

$$\equiv \begin{Bmatrix} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) \geq \left(\sqrt{g(x)} + \sqrt{h(x)}\right)^2 \end{Bmatrix}.$$

c. $\sqrt{f(x)} - \sqrt{g(x)} < \sqrt{h(x)}$

$\equiv \sqrt{f(x)} < \sqrt{h(x)} + \sqrt{g(x)}$

$$\equiv \begin{Bmatrix} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) < \left(\sqrt{g(x)} + \sqrt{h(x)}\right)^2 \end{Bmatrix}.$$

d. $\sqrt{f(x)}-\sqrt{g(x)}\le\sqrt{h(x)}$

$\equiv\sqrt{f(x)}\le\sqrt{h(x)}+\sqrt{g(x)}$

$$\equiv\left\{\begin{array}{l} f(x)\ge 0 \\ g(x)\ge 0 \\ f(x)\le\left(\sqrt{g(x)}+\sqrt{h(x)}\right)^2 \end{array}\right\}.$$

iii. Type-III:

a. $\sqrt{f(x)}>g(x)$

$$\equiv\left\{\begin{array}{l} f(x)\ge 0 \\ g(x)\ge 0 \\ f(x)>(g(x))^2 \end{array}\right\};\left\{\begin{array}{l} f(x)\ge 0 \\ g(x)<0 \end{array}\right\}.$$

b. $\sqrt{f(x)}\ge g(x)$

$$\equiv\left\{\begin{array}{l} f(x)\ge 0 \\ g(x)\ge 0 \\ f(x)\ge(g(x))^2 \end{array}\right\};\left\{\begin{array}{l} f(x)\ge 0 \\ g(x)<0 \end{array}\right\}.$$

iv. Type-IV:

a. $\sqrt{f(x)}<g(x)$

$$\equiv\left\{\begin{array}{l} f(x)\ge 0 \\ g(x)\ge 0 \\ f(x)<(g(x))^2 \end{array}\right\}$$

b. $\sqrt{f(x)}\le g(x)$

$$\equiv\left\{\begin{array}{l} f(x)\ge 0 \\ g(x)\ge 0 \\ f(x)\le(g(x))^2 \end{array}\right\}$$

v. Type-V:

a. $f(x)>g(x)$

$\equiv(f(x))^3>(g(x))^3$

b. $f(x)\ge g(x)$

$\equiv(f(x))^3\ge(g(x))^3$

c. $f(x)<g(x)$

$\equiv(f(x))^3<(g(x))^3$

d. $f(x)\le g(x)$

$\equiv(f(x))^3\le(g(x))^3$

vi. Type-VI: Substitution.

SECTION-7.14. INEQUALITIES CONTAINING MODULUS FUNCTION

1. Solving inequalities containing modulus function

i. Type-I: Using definition of modulus.
ii. Type-II: Method of interval.
iii. Type-III: Substitution.

SECTION-7.15. EXPONENTIAL INEQUALITIES

1. Solving exponential inequalities

i. Type-I: Simplest exponential inequalities

a. $a^x > b \equiv x \in R, \quad b \le 0$
$\equiv x > \log_a b, \quad b > 0, a > 1$
$\equiv x < \log_a b, \quad b > 0, 0 < a < 1.$

b. $a^x \ge b \equiv x \in R, \quad b \le 0$
$\equiv x \ge \log_a b, \quad b > 0, a > 1$
$\equiv x \le \log_a b, \quad b > 0, 0 < a < 1.$

c. $a^x < b \equiv x \in \phi, \quad b \le 0$
$\equiv x < \log_a b, \quad b > 0, a > 1$
$\equiv x > \log_a b, \quad b > 0, 0 < a < 1.$

d. $a^x \le b \equiv x \in \phi, \quad b \le 0$
$\equiv x \le \log_a b, \quad b > 0, a > 1$
$\equiv x \ge \log_a b, \quad b > 0, 0 < a < 1.$

ii. Type-II:

a. $a^{f(x)} > b \equiv x \in D_f, \quad b \le 0$
$\equiv f(x) > \log_a b, \quad b > 0, a > 1$
$\equiv f(x) < \log_a b, \quad b > 0, 0 < a < 1.$

b. $a^{f(x)} \ge b \equiv x \in D_f, \quad b \le 0$
$\equiv f(x) \ge \log_a b, \quad b > 0, a > 1$
$\equiv f(x) \le \log_a b, \quad b > 0, 0 < a < 1.$

c. $a^{f(x)} < b \equiv x \in \phi, \quad b \le 0$
$\equiv f(x) < \log_a b, \quad b > 0, a > 1$
$\equiv f(x) > \log_a b, \quad b > 0, 0 < a < 1.$

d. $a^{f(x)} \le b \equiv x \in \phi, \quad b \le 0$
$\equiv f(x) \le \log_a b, \quad b > 0, a > 1$
$\equiv f(x) \ge \log_a b, \quad b > 0, 0 < a < 1.$

iii. Type-III:

a. $a^{f(x)} > a^{g(x)} \equiv f(x) > g(x), \quad a > 1$
$\equiv f(x) < g(x), \quad 0 < a < 1.$

b. $a^{f(x)} \ge a^{g(x)} \equiv f(x) \ge g(x), \quad a > 1$

$\equiv f(x) \le g(x), \quad 0 < a < 1.$

c. $a^{f(x)} < a^{g(x)} \equiv f(x) < g(x), \quad a > 1$

$\equiv f(x) > g(x), \quad 0 < a < 1.$

d. $a^{f(x)} \le a^{g(x)} \equiv f(x) \le g(x), \quad a > 1$

$\equiv f(x) \ge g(x), \quad 0 < a < 1.$

iv. Type-IV: Substitution

v. Type-V: Homogeneous inequality in two exponential functions u and v

a. $a_n u^n + a_{n-1} u^{n-1} v + a_{n-2} u^{n-2} v^2 + \cdots\cdots + a_1 u v^{n-1} + a_0 v^n > 0$

$$\equiv a_n \left(\frac{u}{v}\right)^n + a_{n-1}\left(\frac{u}{v}\right)^{n-1} + \cdots\cdots + a_1\left(\frac{u}{v}\right) + a_0 > 0$$

$$\equiv \left\{ \begin{array}{l} \frac{u}{v} = y \\ a_n y^n + a_{n-1} y^{n-1} + \cdots\cdots + a_1 y + a_0 > 0 \end{array} \right\}.$$

b. $a_n u^n + a_{n-1} u^{n-1} v + a_{n-2} u^{n-2} v^2 + \cdots\cdots + a_1 u v^{n-1} + a_0 v^n \ge 0$

$$\equiv a_n \left(\frac{u}{v}\right)^n + a_{n-1}\left(\frac{u}{v}\right)^{n-1} + \cdots\cdots + a_1\left(\frac{u}{v}\right) + a_0 \ge 0$$

$$\equiv \left\{ \begin{array}{l} \frac{u}{v} = y \\ a_n y^n + a_{n-1} y^{n-1} + \cdots\cdots + a_1 y + a_0 \ge 0 \end{array} \right\}.$$

c. $a_n u^n + a_{n-1} u^{n-1} v + a_{n-2} u^{n-2} v^2 + \cdots\cdots + a_1 u v^{n-1} + a_0 v^n < 0$

$$\equiv a_n \left(\frac{u}{v}\right)^n + a_{n-1}\left(\frac{u}{v}\right)^{n-1} + \cdots\cdots + a_1\left(\frac{u}{v}\right) + a_0 < 0$$

$$\equiv \left\{ \begin{array}{l} \frac{u}{v} = y \\ a_n y^n + a_{n-1} y^{n-1} + \cdots\cdots + a_1 y + a_0 < 0 \end{array} \right\}.$$

d. $a_n u^n + a_{n-1} u^{n-1} v + a_{n-2} u^{n-2} v^2 + \cdots\cdots + a_1 u v^{n-1} + a_0 v^n \le 0$

$$\equiv a_n \left(\frac{u}{v}\right)^n + a_{n-1}\left(\frac{u}{v}\right)^{n-1} + \cdots\cdots + a_1\left(\frac{u}{v}\right) + a_0 \le 0$$

$$\equiv \left\{ \begin{array}{l} \frac{u}{v} = y \\ a_n y^n + a_{n-1} y^{n-1} + \cdots\cdots + a_1 y + a_0 \le 0 \end{array} \right\}.$$

SECTION-7.16. LOGARITHMIC INEQUALITIES

1. Solving logarithmic inequalities

i. Type-I: Simplest logarithmic inequalities

a. $\log_a x > b \equiv x > a^b, \quad a > 1$

$$\equiv \left\{\begin{matrix} x > 0 \\ x < a^b \end{matrix}\right\}, \quad 0 < a < 1.$$

b. $\log_a x \geq b \equiv x \geq a^b, \quad a > 1$

$$\equiv \left\{\begin{matrix} x > 0 \\ x \leq a^b \end{matrix}\right\}, \quad 0 < a < 1.$$

c. $\log_a x < b \equiv \left\{\begin{matrix} x > 0 \\ x < a^b \end{matrix}\right\}, \quad a > 1$

$$\equiv x > a^b, \quad 0 < a < 1.$$

d. $\log_a x \leq b \equiv \left\{\begin{matrix} x > 0 \\ x \leq a^b \end{matrix}\right\}, \quad a > 1$

$$\equiv x \geq a^b, \quad 0 < a < 1.$$

ii. Type-II:

a. $\log_a f(x) > b \equiv f(x) > a^b, \quad a > 1$

$$\equiv \left\{\begin{matrix} f(x) > 0 \\ f(x) < a^b \end{matrix}\right\}, \quad 0 < a < 1.$$

b. $\log_a f(x) \geq b \equiv f(x) \geq a^b, \quad a > 1$

$$\equiv \left\{\begin{matrix} f(x) > 0 \\ f(x) \leq a^b \end{matrix}\right\}, \quad 0 < a < 1.$$

c. $\log_a f(x) < b \equiv \left\{\begin{matrix} f(x) > 0 \\ f(x) < a^b \end{matrix}\right\}, \quad a > 1$

$$\equiv f(x) > a^b, \quad 0 < a < 1.$$

d. $\log_a f(x) \leq b \equiv \left\{\begin{matrix} f(x) > 0 \\ f(x) \leq a^b \end{matrix}\right\}, \quad a > 1$

$$\equiv f(x) \geq a^b, \quad 0 < a < 1.$$

iii. Type-III:

a. $\log_a f(x) > \log_a g(x) \equiv \left\{\begin{matrix} f(x) > 0 \\ g(x) > 0 \\ f(x) > g(x) \end{matrix}\right\}, \quad a > 1$

$$\equiv \left\{\begin{matrix} f(x) > 0 \\ g(x) > 0 \\ f(x) < g(x) \end{matrix}\right\}, \quad 0 < a < 1.$$

b. $\log_a f(x) \geq \log_a g(x) \equiv \left\{\begin{matrix} f(x) > 0 \\ g(x) > 0 \\ f(x) \geq g(x) \end{matrix}\right\}, \quad a > 1$

$$\equiv \begin{Bmatrix} f(x) > 0 \\ g(x) > 0 \\ f(x) \le g(x) \end{Bmatrix}, \quad 0 < a < 1.$$

iv. Type-IV:

a. $\log_{a(x)} f(x) > \log_{a(x)} g(x)$

$$\equiv \begin{Bmatrix} f(x) > 0 \\ g(x) > 0 \\ a(x) > 0 \\ a(x) < 1 \\ f(x) < g(x) \end{Bmatrix}; \begin{Bmatrix} f(x) > 0 \\ g(x) > 0 \\ a(x) > 1 \\ f(x) > g(x) \end{Bmatrix}$$

b. $\log_{a(x)} f(x) \ge \log_{a(x)} g(x)$

$$\equiv \begin{Bmatrix} f(x) > 0 \\ g(x) > 0 \\ a(x) > 0 \\ a(x) < 1 \\ f(x) \le g(x) \end{Bmatrix}; \begin{Bmatrix} f(x) > 0 \\ g(x) > 0 \\ a(x) > 1 \\ f(x) \ge g(x) \end{Bmatrix}$$

c. $\log_{a(x)} f(x) < \log_{a(x)} g(x)$

$$\equiv \begin{Bmatrix} f(x) > 0 \\ g(x) > 0 \\ a(x) > 0 \\ a(x) < 1 \\ f(x) > g(x) \end{Bmatrix}; \begin{Bmatrix} f(x) > 0 \\ g(x) > 0 \\ a(x) > 1 \\ f(x) < g(x) \end{Bmatrix}$$

d. $\log_{a(x)} f(x) \le \log_{a(x)} g(x)$

$$\equiv \begin{Bmatrix} f(x) > 0 \\ g(x) > 0 \\ a(x) > 0 \\ a(x) < 1 \\ f(x) \ge g(x) \end{Bmatrix}; \begin{Bmatrix} f(x) > 0 \\ g(x) > 0 \\ a(x) > 1 \\ f(x) \le g(x) \end{Bmatrix}$$

v. Type-V: Substitution

vi. Type-VI: Homogeneous inequality in two logarithmic functions u and v

a. $a_n u^n + a_{n-1}u^{n-1}v + a_{n-2}u^{n-2}v^2 + \cdots\cdots + a_1 uv^{n-1} + a_0 v^n > 0$

$$\equiv \begin{Bmatrix} v^n < 0 \\ a_n\left(\frac{u}{v}\right)^n + \cdots\cdots + a_1\left(\frac{u}{v}\right) + a_0 < 0 \end{Bmatrix}; \begin{Bmatrix} v^n = 0 \\ a_n u^n > 0 \end{Bmatrix}; \begin{Bmatrix} v^n > 0 \\ a_n\left(\frac{u}{v}\right)^n + \cdots\cdots + a_1\left(\frac{u}{v}\right) + a_0 > 0 \end{Bmatrix}$$

$$\equiv \left\{ \begin{array}{l} v^n < 0 \\ \dfrac{u}{v} = y \\ a_n y^n + \cdots\cdots + a_1 y + a_0 < 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n = 0 \\ a_n u^n > 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n > 0 \\ \dfrac{u}{v} = y \\ a_n y^n + \cdots\cdots + a_1 y + a_0 > 0 \end{array} \right\}.$$

b. $a_n u^n + a_{n-1} u^{n-1} v + a_{n-2} u^{n-2} v^2 + \cdots\cdots + a_1 u v^{n-1} + a_0 v^n \ge 0$

$$\equiv \left\{ \begin{array}{l} v^n < 0 \\ a_n\left(\dfrac{u}{v}\right)^n + \cdots\cdots + a_1\left(\dfrac{u}{v}\right) + a_0 \le 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n = 0 \\ a_n u^n \ge 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n > 0 \\ a_n\left(\dfrac{u}{v}\right)^n + \cdots\cdots + a_1\left(\dfrac{u}{v}\right) + a_0 \ge 0 \end{array} \right\}$$

$$\equiv \left\{ \begin{array}{l} v^n < 0 \\ \dfrac{u}{v} = y \\ a_n y^n + \cdots\cdots + a_1 y + a_0 \le 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n = 0 \\ a_n u^n \ge 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n > 0 \\ \dfrac{u}{v} = y \\ a_n y^n + \cdots\cdots + a_1 y + a_0 \ge 0 \end{array} \right\}.$$

c. $a_n u^n + a_{n-1} u^{n-1} v + a_{n-2} u^{n-2} v^2 + \cdots\cdots + a_1 u v^{n-1} + a_0 v^n < 0$

$$\equiv \left\{ \begin{array}{l} v^n < 0 \\ a_n\left(\dfrac{u}{v}\right)^n + \cdots\cdots + a_1\left(\dfrac{u}{v}\right) + a_0 > 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n = 0 \\ a_n u^n < 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n > 0 \\ a_n\left(\dfrac{u}{v}\right)^n + \cdots\cdots + a_1\left(\dfrac{u}{v}\right) + a_0 < 0 \end{array} \right\}$$

$$\equiv \left\{ \begin{array}{l} v^n < 0 \\ \dfrac{u}{v} = y \\ a_n y^n + \cdots\cdots + a_1 y + a_0 > 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n = 0 \\ a_n u^n < 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n > 0 \\ \dfrac{u}{v} = y \\ a_n y^n + \cdots\cdots + a_1 y + a_0 < 0 \end{array} \right\}.$$

d. $a_n u^n + a_{n-1} u^{n-1} v + a_{n-2} u^{n-2} v^2 + \cdots\cdots + a_1 u v^{n-1} + a_0 v^n \le 0$

$$\equiv \left\{ \begin{array}{l} v^n < 0 \\ a_n\left(\dfrac{u}{v}\right)^n + \cdots\cdots + a_1\left(\dfrac{u}{v}\right) + a_0 \ge 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n = 0 \\ a_n u^n \le 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n > 0 \\ a_n\left(\dfrac{u}{v}\right)^n + \cdots\cdots + a_1\left(\dfrac{u}{v}\right) + a_0 \le 0 \end{array} \right\}$$

$$\equiv \left\{ \begin{array}{l} v^n < 0 \\ \dfrac{u}{v} = y \\ a_n y^n + \cdots\cdots + a_1 y + a_0 \ge 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n = 0 \\ a_n u^n \le 0 \end{array} \right\}; \left\{ \begin{array}{l} v^n > 0 \\ \dfrac{u}{v} = y \\ a_n y^n + \cdots\cdots + a_1 y + a_0 \le 0 \end{array} \right\}.$$

SECTION-7.17. EXPO-LOGARITHMIC INEQUALITIES

1. Solving expo-logarithmic inequalities

i. Type-I:

a. $f(x)^{g(x)} > \phi(x)^{\varphi(x)}$.

b. $f(x)^{g(x)} \ge \phi(x)^{\varphi(x)}$.

c. $f(x)^{g(x)} < \phi(x)^{\varphi(x)}$.

d. $f(x)^{g(x)} \le \phi(x)^{\varphi(x)}$.

Take log (of a suitable base) both sides and solve as logarithmic inequality.

ii. Type-II: Write $f(x)^{g(x)} = a^{g(x)\log_a f(x)}$ for a suitable base *a* and solve as exponential inequality.

iii. Type-III: Substitution.

SECTION-7.18. PARAMETRIC EQUATIONS AND INEQUALITIES

1. Parametric equations and inequalities

i. An equation/inequality containing parameter is to be solved for each value of the parameter.

SECTION-7.19. PROVING INEQUALITIES

1. Property-1

If $a > b$ & $b > c$ then $a > c$.

2. Property-2

If $a_1 > b_1, a_2 > b_2, \ldots\ldots, a_n > b_n$ then $a_1 + a_2 + \ldots\ldots + a_n > b_1 + b_2 + \ldots\ldots + b_n$.

3. Property-3

For $a_i > 0$, $b_i > 0$, if $a_1 > b_1, a_2 > b_2, \ldots\ldots, a_n > b_n$ then $a_1 a_2 \ldots\ldots a_n > b_1 b_2 \ldots\ldots b_n$.

4. Theorem-1

For $a, b > 0$,

$$A \ge G \ge H \text{ or } \frac{a+b}{2} \ge \sqrt{ab} \ge \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

equality holds only when $a = b$.

5. Theorem-2

For $a_1, a_2, \ldots\ldots, a_n > 0$,

$$A \ge G \ge H \text{ or } \frac{a_1 + a_2 + \ldots\ldots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \ldots\ldots a_n} \ge \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots + \frac{1}{a_n}}$$

equality holds only when $a_1 = a_2 = \ldots\ldots = a_n$.

6. Theorem-3

If $a_i > 0$,

i. $$\frac{a_1^m + a_2^m + \ldots\ldots + a_n^m}{n} \ge \left(\frac{a_1 + a_2 + \ldots\ldots + a_n}{n}\right)^m \quad \text{if } m < 0 \text{ or } m \ge 1$$

ii. $$\frac{a_1^m + a_2^m + \ldots\ldots + a_n^m}{n} \le \left(\frac{a_1 + a_2 + \ldots\ldots + a_n}{n}\right)^m \quad \text{if } 0 < m < 1$$

equality holds only when $a_1 = a_2 = \ldots\ldots = a_n$.

SECTION-7.20. SYSTEM OF NON-LINEAR EQUATIONS IN MORE THAN ONE VARIABLES

1. Linear and non-linear equations

i. A linear equation in *n* variables $x_1, x_2, \ldots\ldots, x_n$ is

$a_1 x_1 + a_2 x_2 + \cdots\cdots + a_n x_n + a_0 = 0$, where $a_0, a_1, a_2, \ldots\ldots, a_n$ are constants.

All other equations in *n* variables $x_1, x_2, \ldots\ldots, x_n$ are non-linear equations.

2. System of non-linear equations in two variables

i. Equations in two variables *x* and *y* are of type $f(x, y) = g(x, y)$ or $f(x, y) = 0$.

ii. Solution and solution set

a. An ordered pair of numbers x_1, y_1 corresponding to the variables *x* and *y* respectively, which satisfies the equation is called a solution of the equation, written as $(x, y) = (x_1, y_1)$.

b. All the solutions of the equation is called solution set of the equation, written as $(x, y) = (x_1, y_1), (x_2, y_2), \cdots\cdots$.

iii. Solving a system of non-linear equations in two variables
A system containing at least one non-linear equation in two variables *x* and *y* is called a system of non-linear equation in two variables. Solving a system of non-linear equations in two variables means finding its solution set.

iv. Methods of solving a system of non-linear equations in two variables

a. Elimination
b. Substitution
c. Symmetric equations

3. System of non-linear equations in three or more variables

i. Equations in three variables *x*, *y* and *z* are of type $f(x, y, z) = g(x, y, z)$ or $f(x, y, z) = 0$.

ii. Solution and solution set

a. An ordered triplet of numbers x_1, y_1, z_1 corresponding to the variables *x*, *y* and *z* respectively, which satisfies the equation is called a solution of the equation, written as $(x, y, z) = (x_1, y_1, z_1)$.

b. All the solutions of the equation is called solution set of the equation, written as $(x, y, z) = (x_1, y_1, z_1), (x_2, y_2, z_2), \cdots\cdots$.

iii. Solving a system of non-linear equations in three variables
A system containing at least one non-linear equation in three variables *x*, *y* and *z* is called a system of non-linear equation in three variables. Solving a system of non-linear equations in three variables means finding its solution set.

iv. Methods of solving a system of non-linear equations in three variables

a. Elimination
b. Substitution

EXERCISE-7

CATEGORY-7.1. ROOTS OF POLYNOMIAL FUNCTIONS

1. x^3-5x^2+3x+9. {Ans. $-1,3,3$}
2. $2x^3-18x^2+108$. {Ans. $3-3\sqrt{3},3,3+3\sqrt{3}$}
3. $x^3+\frac{1}{2}x^2-\frac{1}{4}x-\frac{1}{8}$. {Ans. $-\frac{1}{2},-\frac{1}{2},\frac{1}{2}$}
4. x^3+x+1. {Ans. No rational root}
5. $x^4-12x^3+54x^2-108x+81$. {Ans. $3,3,3,3$}
6. $x^4-x^3-3x^2+5x-2$. {Ans. $-2,1,1,1$}
7. $2x^4-3x^3-x^2+3x-1$. {Ans. $-1,\frac{1}{2},1,1$}
8. $x^4+14x^3+71x^2+154x+120$. {Ans. $-5,-4,-3,-2$}
9. $x^4-\frac{7}{3}x^3-11x^2+\frac{49}{3}x+4$. {Ans. $-3,\frac{2-\sqrt{7}}{3},\frac{2+\sqrt{7}}{3},4$}
10. $x^4-4x^3+4x^2-1$. {Ans. $1,1,1-\sqrt{2},1+\sqrt{2}$}
11. $x^5-5x^4+10x^3-10x^2+5x-1$. {Ans. $1,1,1,1,1$}
12. $x^5-2x^4-2x^3+8x^2-7x+2$. {Ans. $-2,1,1,1,1$}
13. $4x^5-5x^4-11x^3+23x^2-13x+2$. {Ans. $-2,\frac{1}{4},1,1,1$}
14. $2x^5-9x^4+8x^3+15x^2-28x+12$. {Ans. $-\frac{3}{2},1,1,2,2$}
15. $3x^5-19x^4+9x^3+71x^2-84x+20$. {Ans. $-2,\frac{1}{3},1,2,5$}
16. $x^5-7x^4+16x^3-16x^2+7x-1$. {Ans. $1,1,1,2-\sqrt{3},2+\sqrt{3}$}

CATEGORY-7.2. DESCARTES' RULE OF SIGNS

17. Show that the polynomial $P(x)=x^5+x^3+2x+1$ cannot have a positive real root.
18. Show that the polynomial $P(x)=x^7+x^5-2x^4+x^3-3x^2+7x-5$ cannot have a negative real root.
19. Find the nature of roots of the polynomial $P(x)=x^3+x+1$. {Ans. One real root, two imaginary roots}
20. Find the nature of roots of the polynomial $P(x)=3x^4+12x^2+5x-4$. {Ans. One positive real root, one negative real root, two imaginary roots}
21. Find the nature of roots of the polynomial $P(x)=2x^4+5x^2+3$. {Ans. Four imaginary roots}
22. Find the nature of roots of the polynomial $P(x)=2x^8+3x^4+x^2+7$. {Ans. No real root, eight imaginary roots}
23. Find the nature of roots of the polynomial $P(x)=x^9+2x^5+3x^3+x$. {Ans. One root is 0, eight imaginary roots}
24. Show that the polynomial $P(x)=2x^7-x^4+4x^3-5$ has at least four imaginary roots.
25. Find the least possible number of imaginary roots of the polynomial $P(x)=x^9-x^5+x^4+x^2+1$. {Ans.

Six}

26. Show that the polynomial $P(x) = x^9 + 5x^8 - x^3 + 7x + 2$ has at least four imaginary roots.
27. Show that the polynomial $P(x) = x^{10} - 4x^6 + x^4 - 2x - 3$ has at least four imaginary roots.

CATEGORY-7.3. POLYNOMIAL EQUATIONS

28. $x^3 + 4x^2 + 6x + 3 = 0$ {Ans. $[-1]$}
29. $(x-1)^3 + (2x+3)^3 = 27x^3 + 8$ {Ans. $\left[-\frac{1}{2}\right] \cup \left[-\frac{2}{3}\right] \cup [3]$}
30. $2x^4 - x^3 - 9x^2 + 13x - 5 = 0$ {Ans. $[1] \cup \left[-\frac{5}{2}\right]$}
31. $8x^4 + 6x^3 - 13x^2 - x + 3 = 0$ {Ans. $\left[-\frac{1}{2}\right] \cup \left[\frac{3}{4}\right] \cup \left[\frac{-1+\sqrt{5}}{2}\right] \cup \left[\frac{-1-\sqrt{5}}{2}\right]$}
32. $x^4 - 4x^3 - 19x^2 + 106x - 120 = 0$ {Ans. $[2] \cup [3] \cup [4] \cup [-5]$}
33. $(x-4)(x-5)(x-6)(x-7) = 1680$ {Ans. $[-1] \cup [12]$}
34. $(6x+5)^2 (3x+2)(x+1) = 35$ {Ans. $\left[\frac{-5+\sqrt{21}}{6}\right] \cup \left[\frac{-5-\sqrt{21}}{6}\right]$}
35. $x^4 - 2x^3 + x - 132 = 0$ {Ans. $[-3] \cup [4]$}
36. $(x-1)(x+1)(x+2)\,x = 24$ {Ans. $[2] \cup [-3]$}
37. $(x-4)(x+2)(x+8)(x+14) = 304$ {Ans. $\left[-5-\sqrt{85}\right] \cup \left[-5+\sqrt{85}\right] \cup \left[-5-\sqrt{5}\right] \cup \left[-5+\sqrt{5}\right]$}
38. $\left(x^2 + x + 1\right)\left(2x^2 + 2x + 3\right) = 3\left(1 - x - x^2\right)$ {Ans. $[0] \cup [-1]$}

CATEGORY-7.4. RATIONAL EQUATIONS

39. $\dfrac{x+1}{x+3} + \dfrac{4}{x+7} = 1$ {Ans. $[1]$}
40. $\dfrac{6x-5}{4x-3} = \dfrac{3x+3}{2x+5}$ {Ans. $\left[\frac{16}{17}\right]$}
41. $\dfrac{1}{x-1} + \dfrac{4}{x+2} = \dfrac{3}{x}$ {Ans. $\varnothing$}
42. $\dfrac{x-5}{2} + \dfrac{2x-1}{2+3x} = \dfrac{5x-1}{10} - \dfrac{7}{5}$ {Ans. $[-3]$}
43. $\dfrac{3-5x}{x+2} = 2 + \dfrac{x-11}{x+4}$ {Ans. $\left[\frac{-5-\sqrt{61}}{4}\right] \cup \left[\frac{-5+\sqrt{61}}{4}\right]$}
44. $\dfrac{3}{x+1} + \dfrac{7}{x+2} = \dfrac{6}{x-1}$ {Ans. $[5] \cup \left[-\frac{5}{4}\right]$}
45. $\dfrac{7}{x^2 + x - 12} - \dfrac{6}{x^2 + 2x - 8} = 0$ {Ans. $\varnothing$}
46. $\dfrac{x^2 - 7x + 10}{x^2 - 7x + 12} = \dfrac{x^2 + 3x - 10}{x^2 + 3x - 8}$ {Ans. $[2]$}
47. $\dfrac{x-3}{x^2 - 3x - 4} = \dfrac{x-1}{x^2 - x - 2}$ {Ans. $\varnothing$}
48. $x^2 + 4x - \dfrac{7}{x^2 + 4x + 5} = 1$ {Ans. $\left[-2-\sqrt{6}\right] \cup \left[-2+\sqrt{6}\right]$}

49. $\dfrac{1}{x^2-3x+3}+\dfrac{2}{x^2-3x+4}=\dfrac{6}{x^2-3x+5}$. {Ans. $[1]\cup[2]$}

50. $\dfrac{1}{x-8}+\dfrac{1}{x-6}+\dfrac{1}{x+6}+\dfrac{1}{x+8}=0$ {Ans. $\left[-5\sqrt{2}\right]\cup[0]\cup\left[5\sqrt{2}\right]$}

51. $\dfrac{2}{x-14}-\dfrac{5}{x-13}=\dfrac{2}{x-9}-\dfrac{5}{x-11}$ {Ans. $[17]$}

52. $\dfrac{1}{x-1}-\dfrac{4}{x-2}+\dfrac{4}{x-3}-\dfrac{1}{x-4}=\dfrac{1}{30}$ {Ans. $[-2]\cup[-1]\cup[6]\cup[7]$}

CATEGORY-7.5. IRRATIONAL EQUATIONS

53. $\sqrt{x+1}=8-\sqrt{3x+1}$ {Ans. $[8]$}
54. $\sqrt{17+x}-\sqrt{17-x}=2$ {Ans. $[8]$}
55. $\sqrt{3x+7}-\sqrt{x+1}=2$ {Ans. $[-1]\cup[3]$}
56. $\sqrt{25-x}=2-\sqrt{9+x}$ {Ans. $\varnothing$}
57. $\left(x^2-4\right)\sqrt{x+1}=0$ {Ans. $[-1]\cup[2]$}
58. $\sqrt{4x-3}+\sqrt{5x+1}=\sqrt{15x+4}$ {Ans. $[3]$}
59. $\sqrt{x+5}+\sqrt{x+3}=\sqrt{2x+7}$ {Ans. $\varnothing$}
60. $\sqrt{4-x}+\sqrt{5+x}=3$ {Ans. $[-5]\cup[4]$}
61. $\sqrt{4x+2}+\sqrt{4x-2}=4$ {Ans. $\left[\frac{17}{16}\right]$}
62. $\sqrt{2x+5}-\sqrt{3x-5}=2$ {Ans. $[2]$}
63. $\sqrt{x^2+x-5}+\sqrt{x^2+8x-4}=5$ {Ans. $[2]$}
64. $\sqrt{x^2+x+1}=\sqrt{x^2-x+1}+1$ {Ans. $\varnothing$}
65. $\sqrt{x+\sqrt{x+11}}+\sqrt{x-\sqrt{x+11}}=4$ {Ans. $[5]$}
66. $\sqrt{x-\sqrt{x-2}}+\sqrt{x+\sqrt{x-2}}=2$ {Ans. $\varnothing$}
67. $\sqrt[3]{x+34}-\sqrt[3]{x-3}=1$ {Ans. $[-61]\cup[30]$}
68. $\sqrt[3]{x+5}+\sqrt[3]{x+6}=\sqrt[3]{2x+11}$ {Ans. $[-6]\cup[-5]\cup\left[-\frac{11}{2}\right]$}
69. $\sqrt[3]{x+1}+\sqrt[3]{3x+1}=\sqrt[3]{x-1}$ {Ans. $[-1]$}
70. $\sqrt[3]{x+1}+\sqrt[3]{x+2}+\sqrt[3]{x+3}=0$ {Ans. $[-2]$}
71. $\sqrt[3]{1+\sqrt{x}}+\sqrt[3]{1-\sqrt{x}}=2$ {Ans. $[0]$}

CATEGORY-7.6. EQUATIONS CONTAINING MODULUS FUNCTION

72. $|x+1|+x=3$ {Ans. $[1]$}
73. $|x|+|x+1|=1$ {Ans. $[-1,0]$}
74. $|x+1|+|x+2|=2$ {Ans. $\left[-\frac{1}{2}\right]\cup\left[-\frac{5}{2}\right]$}
75. $|x-1|-|x-2|=1$ {Ans. $[2,\infty)$}
76. $|x-2|+|4-x|=3$ {Ans. $\left[\frac{3}{2}\right]\cup\left[\frac{9}{2}\right]$}
77. $|x-1|+|x-2|=1$ {Ans. $[1,2]$}

78. $|x-2|+|x-3|+2|x-4|=9$ {Ans. $[1]\cup\left[\frac{11}{2}\right]$}
79. $|2x+1|-|3-x|=|x-4|$ {Ans. $\left[\frac{3}{2}\right]$}
80. $|x-1|+|1-2x|=2|x|$ {Ans. $[2]\cup\left[\frac{2}{5}\right]$}
81. $|x|-2|x+1|+3|x+2|=0$ {Ans. $[-2]$}
82. $|x+1|-|x|+3|x-1|-2|x-2|=|x+2|$ {Ans. $(-\infty,-2]\cup[2,\infty)$}
83. $|x|+2|x+1|-3|x-3|=0$ {Ans. $\left[\frac{7}{6}\right]$}
84. $|x|^3-3x^2+3|x|-2=0$. {Ans. $[-2]\cup[2]$}
85. $x^2+5|x|+4=0$. {Ans. ϕ}
86. $|x^2-9|+|x-2|=5$ {Ans. $[-3]\cup[2]\cup\left[\frac{-1+\sqrt{65}}{2}\right]$}
87. $|x^2-1|+x+1=0$ {Ans. $[-1]$}
88. $|x^2-4|-|9-x^2|=5$ {Ans. $(-\infty,-3]\cup[3,\infty)$}
89. $|x^2-9|+|x^2-4|=5$ {Ans. $[-3,-2]\cup[2,3]$}
90. $|x^2+2x|-|2-x|=|x^2-x|$ {Ans. $\left[\frac{-1+\sqrt{5}}{2}\right]$}
91. $|x^2+4x+3|+2x+5=0$. {Ans. $[-4]\cup\left[-1-\sqrt{3}\right]$}
92. $|2+|x-1||=1$ {Ans. $\varnothing$}
93. $|3-|x+2||=2$ {Ans. $[-7]\cup[-3]\cup[-1]\cup[3]$}
94. $|x+|x+1||+|x|=10$ {Ans. $[-9]\cup[3]$}
95. $||3-2x|-1|=2|x|$ {Ans. $\left[\frac{1}{2}\right]$}
96. $|x|+|x+|2x+4||=5$ {Ans. $\left[-\frac{9}{2}\right]\cup\left[\frac{1}{4}\right]$}
97. $\dfrac{|x^2-4x|+3}{x^2+|x-5|}=1$ {Ans. $\left[-\frac{2}{3}\right]\cup\left[\frac{1}{2}\right]\cup[2]$}
98. $|x|^2-3|x|+2=0$. {Ans. $[-2]\cup[-1]\cup[1]\cup[2]$}

CATEGORY-7.7. EXPONENTIAL EQUATIONS

99. $3^{2x^2-7x+7}=9$. {Ans. $[1]\cup\left[\frac{5}{2}\right]$}
100. $\sqrt{3^x}\cdot\sqrt{5^x}=225$ {Ans. [4]}
101. $2^{3x}\cdot 5^x=1600$ {Ans. [2]}
102. $\left(9^{3-5x}\right)\left(7^{5x-3}\right)=1$ {Ans. $\left[\frac{3}{5}\right]$}
103. $\left(3^{2x-1}\right)\left(5^{3x+2}\right)=\frac{9}{5}\left(5^{2x}\right)\left(3^{3x}\right)$ {Ans. [-3]}

104. $3\cdot 4^{x}+\frac{1}{3}\cdot 9^{x+2}=6\cdot 4^{x+1}-\frac{1}{2}\cdot 9^{x+1}$ {Ans. $\left[-\frac{1}{2}\right]$}

105. $7^{\frac{2x^2-5x-9}{2}}=\left(\sqrt{2}\right)^{3\log_2 7}$ {Ans. $\left[-\frac{3}{2}\right]\cup[4]$}

106. $4\cdot 3^{x+2}+5\cdot 3^{x}-7\cdot 3^{x+1}=40$ {Ans. $[\log_3 2]$}

107. $16^{\frac{x+5}{x-7}}=512\cdot 64^{\frac{x+17}{x-3}}$ {Ans. $[-5]\cup\left[\frac{93}{11}\right]$}

108. $5^{|4x-6|}=25^{3x-4}$ {Ans. $\left[\frac{7}{5}\right]$}

109. $\sqrt{3}\cdot\left(3^{\frac{x}{1+\sqrt{x}}}\right)\cdot\left(\frac{1}{3}\right)^{\frac{2+\sqrt{x}+x}{2(1+\sqrt{x})}}=81$ {Ans. [81]}

110. $\left(\frac{3}{5}\right)^{x}\cdot\left(\frac{25}{9}\right)^{x^2-12}=\left(\frac{27}{125}\right)^{3}$ {Ans. $\left[-\frac{5}{2}\right]\cup[3]$}

111. $2^{x+1}+3\cdot 2^{x-3}=76$ {Ans. [5]}

112. $3\cdot\sqrt[x]{81}-10\cdot\sqrt[x]{9}+3=0$ {Ans. $[-2]\cup[2]$}

113. $3^{4x+8}-4\cdot 3^{2x+5}+28=2\log_2\sqrt{2}$. {Ans. $[-1]\cup\left[-\frac{3}{2}\right]$}

114. $2^{3x-3}-5+6\cdot 2^{3-3x}=0$ {Ans. $\left[\frac{4}{3}\right]\cup\left[\frac{3+\log_2 3}{3}\right]$}

115. $3^{1-x}-3^{1+x}+9^{x}+9^{-x}=6$ {Ans. $\left[\log_3\frac{\sqrt{5}-1}{2}\right]\cup\left[\log_3\left(2+\sqrt{5}\right)\right]$}

116. $64^{\frac{1}{x}}-2^{3+\frac{3}{x}}+12=0$ {Ans. $[\log_6 8]\cup[3]$}

117. $4^{\log_9 x}-6\cdot 2^{\log_9 x}+2^{\log_3 27}=0$ {Ans. $[9]\cup[81]$}

118. $4^{\sqrt{3x^2-2x}+1}+2=9\cdot 2^{\sqrt{3x^2-2x}}$ {Ans. $\left[-\frac{1}{3}\right]\cup[1]$}

119. $7\cdot 4^{x^2}-9\cdot 14^{x^2}+2\cdot 49^{x^2}=0$ {Ans. $[-1]\cup[0]\cup[1]$}

120. $3\cdot 16^{x}+36^{x}=2\cdot 81^{x}$ {Ans. $\left[\frac{1}{2}\right]$}

121. $8^{x}+18^{x}=2\cdot 27^{x}$ {Ans. $[0]$}

122. $6\cdot\sqrt[x]{9}-13\cdot\sqrt[x]{6}+6\cdot\sqrt[x]{4}=0$ {Ans. $[-1]\cup[1]$}

123. $16^{x}-5\cdot 8^{x}+6\cdot 4^{x}=0$ {Ans. $[1]\cup[\log_2 3]$}

124. $27^{x}+12^{x}=2\cdot 8^{x}$ {Ans. $[0]$}

125. $\left(4+\sqrt{15}\right)^{x}+\left(4-\sqrt{15}\right)^{x}=62$ {Ans. $[-2]\cup[2]$}

126. $\left(\sqrt{5+2\sqrt{6}}\right)^{x}+\left(\sqrt{5-2\sqrt{6}}\right)^{x}=10$ {Ans. $[-2]\cup[2]$}

127. $\left(\sqrt{a+\sqrt{a^2-1}}\right)^x+\left(\sqrt{a-\sqrt{a^2-1}}\right)^x=2a$ {Ans. $[-2]\cup[2]$}

128. $5^{1+x^3}-5^{1-x^3}=24$ {Ans. $[1]$}

129. $5^{x-1}+5\cdot\left(\frac{1}{5}\right)^{x-2}=26$ {Ans. $[1]\cup[3]$}

130. $10^{\frac{2}{x}}+25^{\frac{1}{x}}=\frac{17}{4}\cdot 50^{\frac{1}{x}}$ {Ans. $\left[-\frac{1}{2}\right]\cup\left[\frac{1}{2}\right]$}

CATEGORY-7.8. LOGARITHMIC EQUATIONS

131. $7^{\log_7\left(x^2-4x+5\right)}=x-1$. {Ans. $[2]\cup[3]$}

132. $\log_3\left(x^2+4x+12\right)=2$. {Ans. $[-3]\cup[-1]$}

133. $\log_3 x+\log_9 x+\log_{27} x=\frac{11}{2}$ {Ans. $[27]$}

134. $\log_2(3-x)+\log_2(1-x)=3$ {Ans. $[-1]$}

135. $\log(x-3)+\log(x+6)=1$ {Ans. $[4]$}

136. $\log(x-4)+\log(x+3)=\log(5x+4)$ {Ans. $[8]$}

137. $\ln\left(x^3+1\right)-\frac{1}{2}\ln\left(x^2+2x+1\right)=\ln 3$ {Ans. $[2]$}

138. $\log_5(x-2)+2\log_5\left(x^3-2\right)+\log_5(x-2)^{-1}=4$ {Ans. $[3]$}

139. $2\log_3(x-2)+\log_3(x-4)^2=0$ {Ans. $[3]\cup\left[3+\sqrt{2}\right]$}

140. $\log_2(x+2)^2+\log_2(x+10)^2=4\log_2 3$ {Ans. $[-11]\cup\left[-6-\sqrt{7}\right]\cup[-1]\cup\left[-6+\sqrt{7}\right]$}

141. $\log_2\left(\frac{x-2}{x-1}\right)-1=\log_2\left(\frac{3x-7}{3x-1}\right)$ {Ans. $[3]$}

142. $2\log_2\left(\frac{x-7}{x-1}\right)+\log_2\left(\frac{x-1}{x+1}\right)=1$ {Ans. $[-17]$}

143. $\log_3(5x-2)-2\log_3\sqrt{3x+1}=1-\log_3 4$ {Ans. $[1]$}

144. $\log(3x-2)-2=\frac{1}{2}\log(x+2)-\log 50$ {Ans. $[2]$}

145. $\log^2\left(1+\frac{4}{x}\right)+\log^2\left(1-\frac{4}{x+4}\right)=2\log^2\left(\frac{2}{x-1}-1\right)$ {Ans. $\left[\sqrt{2}\right]\cup\left[\sqrt{6}\right]$}

146. $\log_2 x^4+\log_2 x^2=1$ {Ans. $\left[-2^{\frac{1}{6}}\right]\cup\left[2^{\frac{1}{6}}\right]$}

147. $\log\left(10x^2\right)\cdot\log x=1$ {Ans. $\left[\frac{1}{10}\right]\cup\left[\sqrt{10}\right]$}

148. $\dfrac{\log_2 x-1}{\log_2\frac{x}{2}}=2\log_2\sqrt{x}+3-\log_2^2 x$ {Ans. $\left[\frac{1}{2}\right]\cup[4]$}

149. $2\log_9 x+9\log_x 3=10$ {Ans. $[3]\cup\left[3^9\right]$}

150. $\log_x(125x)\cdot\log_{25}^2 x=1$ {Ans. $\left[\frac{1}{625}\right]\cup[5]$}

151. $\log_x \sqrt{5} + \log_x 5x = \frac{9}{4} + \log_x^2 \sqrt{5}$ {Ans. $\left[5^{\frac{1}{5}}\right] \cup [5]$ }

152. $\log(\log x) + \log(\log x^3 - 2) = 0$ {Ans. $[10]$ }

153. $\log_{3x+7}(9 + 12x + 4x^2) = 4 - \log_{2x+3}(6x^2 + 23x + 21)$ {Ans. $\left[-\frac{1}{4}\right]$ }

154. $\log^2(4-x) + \log(4-x) \cdot \log\left(x + \frac{1}{2}\right) - 2\log^2\left(x + \frac{1}{2}\right) = 0$ {Ans. $[0] \cup \left[\frac{7}{4}\right] \cup \left[\frac{3+2\sqrt{6}}{2}\right]$ }

155. $\log_x 2 - \log_4 x + \frac{7}{6} = 0$ {Ans. $\left[4^{-\frac{1}{3}}\right] \cup [8]$ }

156. $\log_{0\cdot 5x} x^2 - 14\log_{16x} x^3 + 40\log_{4x} \sqrt{x} = 0$ {Ans. $[1] \cup \left[\frac{1}{\sqrt{2}}\right] \cup [4]$ }

157. $4\log_{\frac{x}{2}} \sqrt{x} + 2\log_{4x} x^2 = 3\log_{2x} x^3$ {Ans. $\left[\frac{1}{8}\right] \cup [1] \cup [4]$ }

158. $\log_{3x}\left(\frac{3}{x}\right) + \log_3^2 x = 1$ {Ans. $\left[\frac{1}{9}\right] \cup [1] \cup [3]$ }

159. $\sqrt{1 + \log_{0.04} x} + \sqrt{3 + \log_{0.2} x} = 1$ {Ans. $[25]$ }

160. $\sqrt{2 - \log_x 9} = -\frac{\sqrt{12}}{\log_3 x}$ {Ans. $\left[\frac{1}{9}\right]$ }

161. $\log_x\left(x^2 + 1\right) = \sqrt{\log_{\sqrt{x}}\left(x^2\left(1 + x^2\right)\right) + 4}$ {Ans. $\left[\sqrt{\frac{1+\sqrt{5}}{2}}\right]$ }

162. $\sqrt{\log_2 x} - 0.5 = \log_2 \sqrt{x}$ {Ans. $[2]$ }

163. $\log 2 + \log(4^{x-2} + 9) = 1 + \log(2^{x-2} + 1)$ {Ans. $[2] \cup [4]$ }

164. $\log(6 \cdot 5^x + 25 \cdot 20^x) = x + \log 5$ {Ans. $\varnothing$ }

165. $|1 - \log_4 x| + 1 = |3 - \log_4 x|$ {Ans. $[8]$ }

CATEGORY-7.9. EXPO-LOGARITHMIC EQUATIONS

166. $(x+1)^{\log(x+1)} = 100(x+1)$ {Ans. $\left[-\frac{9}{10}\right] \cup [99]$ }

167. $x^{\log x} = 1000x^2$. {Ans. $\left[\frac{1}{10}\right] \cup [1000]$ }

168. $x^{\frac{\log x + 5}{3}} = 10^{5 + \log x}$ {Ans. $\left[10^3\right] \cup \left[10^{-5}\right]$ }

169. $|x-1|^{\log^2 x - \log x^2} = |x-1|^3$ {Ans. $\left[\frac{1}{10}\right] \cup [2] \cup [1000]$ }

170. $|x-3|^{\frac{x+1}{4}} = |x-3|^{\frac{x-2}{3}}$ {Ans. $[2] \cup [4] \cup [11]$ }

171. $|x-3|^{3x^2 - 10x + 3} = 1$ {Ans. $\left[\frac{1}{3}\right] \cup [2] \cup [4]$ }

172. $\left(\sqrt{x}\right)^{\log_5 x - 1} = 5$ {Ans. $\left[\frac{1}{5}\right] \cup [25]$ }

173. $x^{\log x + 7} = 10^{4(\log x + 1)}$ {Ans. $\left[10^{-4}\right] \cup [10]$ }

174. $x^{3\log x-\frac{1}{\log x}}=\sqrt[3]{10}$. {Ans. $\left[10^{-\frac{2}{3}}\right]\cup\left[10^{\frac{2}{3}}\right]$}

CATEGORY-7.10. QUADRATIC INEQUALITIES

175. $3x^2-7x+4\le 0$ {Ans. $\left[1,\frac{4}{3}\right]$}
176. $3x^2-7x+6<0$ {Ans. $\varnothing$}
177. $3x^2-7x-6<0$ {Ans. $\left(-\frac{2}{3},3\right)$}
178. $x^2-3x+5>0$ {Ans. R}
179. $x^2-14x-15>0$ {Ans. $(-\infty,-1)\cup(15,\infty)$}
180. $2-x-x^2\ge 0$ {Ans. $[-2,1]$}
181. $x^2-25<0$ {Ans. $(-5,5)$}
182. $x^2+10x\le 7x$ {Ans. $[-3,0]$}
183. $x^2-7x<3$ {Ans. $\left(\frac{7-\sqrt{61}}{2},\frac{7+\sqrt{61}}{2}\right)$}
184. $-x^2-16+8x\ge 0$ {Ans. $[4]$}
185. $x^2+5x+8>0$ {Ans. R}
186. $5x-1<(x+1)^2<7x-3$. {Ans. $(2,4)$}
187. $x^2+2\le 3x\le 2x^2-5$. {Ans. ϕ}

CATEGORY-7.11. POLYNOMIAL INEQUALITIES

188. $x(x-1)^2>0$ {Ans. $(0,1)\cup(1,\infty)$}
189. $x(x-1)^2\le 0$ {Ans. $(-\infty,0]\cup[1]$}
190. $(2-x)(3x+1)(2x-3)>0$ {Ans. $\left(-\infty,-\frac{1}{3}\right)\cup\left(\frac{3}{2},2\right)$}
191. $(3x-2)(x-3)^3(x+1)^3(x+2)^4<0$ {Ans. $(-\infty,-2)\cup(-2,-1)\cup\left(\frac{2}{3},3\right)$}
192. $x^3-64x>0$ {Ans. $(-8,0)\cup(8,\infty)$}
193. $x^4+8x^3+12x^2\ge 0$ {Ans. $(-\infty,-6]\cup[-2,\infty)$}
194. $(x-1)(x^2-3x+8)<0$ {Ans. $(-\infty,1)$}
195. $(x-1)(x^2-1)(x^3-1)(x^4-1)\le 0$ {Ans. $[-1]\cup[1]$}
196. $(16-x^2)(x^2+4)(x^2+x+1)(x^2-x-3)\le 0$ {Ans. $(-\infty,-4]\cup\left[\frac{1-\sqrt{13}}{2},\frac{1+\sqrt{13}}{2}\right]\cup[4,\infty)$}
197. $(x^2-4)(x^2-4x+4)(x^2-6x+8)(x^2+4x+4)<0$ {Ans. $(-2,2)\cup(2,4)$}
198. $(2x^2-x-5)(x^2-9)(x^2-3x)\le 0$ {Ans. $\left[-3,\frac{1-\sqrt{41}}{4}\right]\cup\left[0,\frac{1+\sqrt{41}}{4}\right]\cup[3]$}
199. $x^3-3x^2-10x+24>0$ {Ans. $(-3,2)\cup(4,\infty)$}
200. $x^3+4x^2+5x+2\le 0$ {Ans. $(-\infty,-2]\cup[-1]$}
201. $2x^3-3x^2+7x-3>0$ {Ans. $\left(\frac{1}{2},\infty\right)$}
202. $x^4-3x^3+x^2+3x-2\ge 0$ {Ans. $(-\infty,-1]\cup[1]\cup[2,\infty)$}
203. $x^4+6x^3+6x^2+6x+5<0$ {Ans. $(-5,-1)$}
204. $3x^4-10x^2+3>0$ {Ans. $\left(-\infty,-\sqrt{3}\right)\cup\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\cup\left(\sqrt{3},\infty\right)$}

CATEGORY-7.12. RATIONAL INEQUALITIES

205. $\dfrac{3x-2}{2x-3} < 3$ {Ans. $\left(-\infty, \frac{3}{2}\right) \cup \left(\frac{7}{3}, \infty\right)$}

206. $\dfrac{7x-4}{x+2} \geq 1$ {Ans. $(-\infty,-2) \cup [1, \infty)$}

207. $\dfrac{1}{x} < \dfrac{1}{3}$ {Ans. $(-\infty,0) \cup (3, \infty)$}

208. $\dfrac{(x-1)(3x-2)}{(5-2x)} > 0$ {Ans. $\left(-\infty, \frac{2}{3}\right) \cup \left(1, \frac{5}{2}\right)$}

209. $\dfrac{x^2-3x+4}{x+1} > 1$. {Ans. $(-1,1) \cup (3, \infty)$}

210. $\dfrac{x^2+2x+7}{2x+3} < 6$. {Ans. $\left(-\infty,-\dfrac{3}{2}\right) \cup (-1,11)$}

211. $\dfrac{x^2-5x+6}{x^2-12x+35} > 0$ {Ans. $(-\infty,2) \cup (3,5) \cup (7, \infty)$}

212. $\dfrac{x^2-5x+6}{x^2-x+1} > 0$. {Ans. $(-\infty,2) \cup (3, \infty)$}

213. $\dfrac{x^2-4x-2}{9-x^2} < 0$ {Ans. $(-\infty,-3) \cup \left(2-\sqrt{6},3\right) \cup \left(2+\sqrt{6}, \infty\right)$}

214. $\dfrac{2x^2+18x-4}{x^2+9x+8} > 2$ {Ans. $(-8,-1)$}

215. $\dfrac{10x^2+17x-34}{x^2+2x-3} < 8$. {Ans. $\left(-3,-\dfrac{5}{2}\right) \cup (1,2)$}

216. $\dfrac{8x^2+16x-51}{(2x-3)(x+4)} < 3$. {Ans. $(-4,-3) \cup \left(\dfrac{3}{2}, \dfrac{5}{2}\right)$}

217. $\dfrac{x+1}{x-2} > \dfrac{3}{x-2} - \dfrac{1}{2}$ {Ans. $(-\infty,2) \cup (2, \infty)$}

218. $\dfrac{2}{x-1} - \dfrac{1}{x+1} > 3$ {Ans. $\left(\frac{1-\sqrt{73}}{6},-1\right) \cup \left(1, \frac{1+\sqrt{73}}{6}\right)$}

219. $\dfrac{x-1}{x} - \dfrac{x+1}{x-1} < 2$ {Ans. $(-\infty,-1) \cup \left(0, \frac{1}{2}\right) \cup (1, \infty)$}

220. $\dfrac{x-2}{x+2} \geq \dfrac{2x-3}{4x-1}$ {Ans. $(-\infty,-2) \cup \left(\frac{1}{4},1\right] \cup [4, \infty)$}

221. $\dfrac{x}{x-1} - \dfrac{2}{x+1} > \dfrac{8}{x^2-1}$ {Ans. $(-\infty,-2) \cup (-1,1) \cup (3, \infty)$}

222. $\dfrac{(x+1)(x+2)(x+3)}{(2x-1)(x+4)(3-x)} > 0$ {Ans. $(-4,-3) \cup (-2,-1) \cup \left(\frac{1}{2},3\right)$}

223. $\dfrac{x^3+x^2+x}{9x^2-25} \geq 0$ {Ans. $\left(-\frac{5}{3},0\right] \cup \left(\frac{5}{3}, \infty\right)$}

224. $\frac{x^3-x^2+x-1}{x+8}\le 0$ {Ans. $(-8,1]$}

225. $\frac{(x+2)(x^2-2x+1)}{4+3x-x^2}\ge 0$. {Ans. $(-\infty,-2]\cup(-1,4)$}

226. $\frac{1}{x+1}+\frac{2}{x+3}>\frac{3}{x+2}$ {Ans. $(-3,-2)\cup(-1,1)$}

227. $\frac{1}{x-2}+\frac{1}{x-1}>\frac{1}{x}$ {Ans. $(-\sqrt{2},0)\cup(1,\sqrt{2})\cup(2,\infty)$}

228. $\frac{3}{x+1}+\frac{7}{x+2}\le\frac{6}{x-1}$ {Ans. $(-\infty,-2)\cup[-\frac{5}{4},-1)\cup(1,5]$}

229. $\frac{2x-1}{x+1}+\frac{3x-1}{x+2}<4+\frac{x-7}{x-1}$ {Ans. $(-2,-\frac{5}{4})\cup(-1,1)\cup(5,\infty)$}

230. $\frac{x^4+x^2+1}{x^2-4x-5}<0$ {Ans. $(-1,5)$}

231. $\frac{x^4-2x^2-8}{x^2+x-1}<0$ {Ans. $\left(-2,\frac{-1-\sqrt{5}}{2}\right)\cup\left(\frac{-1+\sqrt{5}}{2},2\right)$}

232. $\frac{16}{3x-2-x^2}>\frac{3}{7x-4-3x^2}$ {Ans. $(1,\frac{58}{45})\cup(\frac{4}{3},2)$}

233. $\frac{1}{x-8}+\frac{1}{x-6}+\frac{1}{x+8}+\frac{1}{x+6}\ge 0$ {Ans. $(-8,-5\sqrt{2}]\cup(-6,0]\cup(6,5\sqrt{2}]\cup(8,\infty)$}

234. $\frac{1}{x-1}-\frac{4}{x-2}+\frac{4}{x-3}-\frac{1}{x-4}\le\frac{1}{30}$ {Ans. $(-\infty,-2]\cup[-1,1)\cup(2,3)\cup(4,6]\cup[7,\infty)$}

CATEGORY-7.13. IRRATIONAL INEQUALITIES

235. $\sqrt{2x+1}<5$ {Ans. $[-\frac{1}{2},12)$}

236. $\sqrt{3x-2}>1$ {Ans. $(1,\infty)$}

237. $\sqrt{2x+10}<3x-5$ {Ans. $(3,\infty)$}

238. $\sqrt{(x-3)(x+1)}>3(x+1)$ {Ans. $(-\infty,-1)$}

239. $\sqrt{(x+4)(2x-1)}<2(x+4)$ {Ans. $[\frac{1}{2},\infty)$}

240. $\sqrt{(x+2)(x-5)}<8-x$ {Ans. $(-\infty,-2]\cup[5,\frac{74}{13})$}

241. $\sqrt{x^2-x-12}<x$ {Ans. $[4,\infty)$}

242. $\sqrt{9x-20}<x$ {Ans. $[\frac{20}{9},4)\cup(5,\infty)$}

243. $\sqrt{x^2-4x}>x-3$ {Ans. $(-\infty,0]\cup(\frac{9}{2},\infty)$}

244. $\sqrt{3x^2-22x}>2x-7$ {Ans. $(-\infty,0]$}

245. $\sqrt{x^2-5x+6}\le x+4$ {Ans. $[-\frac{10}{13},2]\cup[3,\infty)$}

246. $\sqrt{2x^2+7x+50}\ge x-3$ {Ans. $(-\infty,\infty)$}

247. $\sqrt{x+1}-\sqrt{x-2}\le 1$ {Ans. $[3,\infty)$}

248. $\sqrt{x+3}-\sqrt{x-4}\geq 2$ {Ans. $\left[4,\frac{73}{16}\right]$}

249. $\sqrt{x-1}+\sqrt{x+2}\leq 1$ {Ans. $\varnothing$}

250. $\sqrt{3x+1}+\sqrt{x-4}-\sqrt{4x+5}<0$ {Ans. $[4,5)$}

251. $2\sqrt{x+1}-\sqrt{x-1}\geq 2\sqrt{x-3}$ {Ans. $\left[3,\frac{15+16\sqrt{15}}{15}\right]$}

252. $\sqrt{x-3}+\sqrt{1-x}>\sqrt{8x-5}$ {Ans. $\varnothing$}

253. $\sqrt{17-4x}+\sqrt{x-5}\leq\sqrt{13x+1}$ {Ans. $\varnothing$}

254. $\sqrt{x+6}>\sqrt{x-1}+\sqrt{2x-5}$ {Ans. $\left[\frac{5}{2},\frac{-5+\sqrt{149}}{2}\right)$}

255. $\sqrt{x-2}-\sqrt{x+3}-2\sqrt{x}\geq 0$ {Ans. $\varnothing$}

256. $\sqrt[3]{x+5}+2>\sqrt[3]{x-3}$ {Ans. $(-\infty,\infty)$}

257. $\sqrt[3]{1+\sqrt{x}}<2-\sqrt[3]{1-\sqrt{x}}$ {Ans. $(0,\infty)$}

CATEGORY-7.14. INEQUALITIES CONTAINING MODULUS FUNCTION

258. $|2x-3|<1$. {Ans. $(1,2)$}

259. $|x|-1<1-x$. {Ans. $(-\infty,1)$}

260. $|x|+|x-1|<5$ {Ans. $(-2,3)$}

261. $|x-1|+|x+1|<4$. {Ans. $(-2,2)$}

262. $|x+1|+|x-2|>5$ {Ans. $(-\infty,-2)\cup(3,\infty)$}

263. $|2x+1|-|5x-2|\geq 1$ {Ans. $\left[\frac{2}{7},\frac{2}{3}\right]$}

264. $|3x-1|+|2x-3|-|x+5|<2$ {Ans. $\left(-\frac{1}{2},\frac{11}{4}\right)$}

265. $|x-1|+|2-x|>3+x$ {Ans. $(-\infty,0)\cup(6,\infty)$}

266. $|x-1|+|x-2|+|x-3|\geq 6$. {Ans. $(-\infty,0]\cup[4,\infty)$}

267. $x^2+2|x|-3\leq 0$ {Ans. $[-1,1]$}

268. $x^2+5|x|-24>0$ {Ans. $(-\infty,-3)\cup(3,\infty)$}

269. $|x^2-3x-15|<2x^2-x$ {Ans. $\left(-\infty,-\frac{5}{3}\right)\cup(3,\infty)$}

270. $|x^2+x+10|\leq 3x^2+7x+2$ {Ans. $(-\infty,-4]\cup[1,\infty)$}

271. $|2x^2+x+11|>x^2-5x+6$ {Ans. $(-\infty,-5)\cup(-1,\infty)$}

272. $|4x^2-9x+6|>-x^2+x-3$ {Ans. $(-\infty,\infty)$}

273. $|x-6|>|x^2-5x+9|$ {Ans. $(1,3)$}

274. $||2x+1|-5|>x+4$ {Ans. $\left(-\infty,-\dfrac{10}{3}\right)\cup(-2,0)\cup(8,\infty)$}

275. $||x-1|-1|\leq 1$. {Ans. $[-1,3]$}

276. $||x-3|+1|\geq 2$ {Ans. $(-\infty,2]\cup[4,\infty)$}

277. $\left||x-1|+x\right|<x^2$ {Ans. $(-\infty,-1)\cup(1,\infty)$}

278. $\left||x-2|-x+3\right|<|x|$ {Ans. $\left(\frac{5}{3},\infty\right)$}

279. $\left|2x-|3-x|-2\right|\le x^2$ {Ans. $\left(-\infty,\frac{-3-\sqrt{29}}{2}\right]\cup\left[\frac{-3+\sqrt{29}}{2},\infty\right)$}

CATEGORY-7.15. EXPONENTIAL INEQUALITIES

280. $2^{3-6x}>1$ {Ans. $\left(-\infty,\frac{1}{2}\right)$}

281. $6^{3-x}<216$ {Ans. $(0,\infty)$}

282. $2^{x^2-6x-\frac{5}{2}}>16\sqrt{2}$ {Ans. $(-\infty,-1)\cup(7,\infty)$}

283. $\left(\frac{1}{3}\right)^{-|x+2|}\ge 81$ {Ans. $(-\infty,-6]\cup[2,\infty)$}

284. $\left(\frac{1}{3}\right)^{\sqrt{x+2}}>3^{-x}$ {Ans. $(2,\infty)$}

285. $\left(\frac{2}{5}\right)^{\frac{6-5x}{2+5x}}<\frac{25}{4}$ {Ans. $(-\infty,-2)\cup\left(-\frac{2}{5},\infty\right)$}

286. $\sqrt{3^{x-54}}-7\sqrt{3^{x-58}}\le 162$ {Ans. $(-\infty,66]$}

287. $8^{x+1}-8^{2x-1}>30$ {Ans. $\left(\frac{2}{3},\log_8 60\right)$}

288. $4^x+2^{x+1}-6\le 0$ {Ans. $\left(-\infty,\log_2\left(\sqrt{7}-1\right)\right]$}

289. $25^{-x}+5^{-x+1}\ge 50$ {Ans. $(-\infty,-1]$}

290. $4^{x^2}-3\cdot 2^{x^2}+1\ge 0$ {Ans. $\left(-\infty,-\sqrt{\log_2\frac{3+\sqrt{5}}{2}}\right]\cup\left[\sqrt{\log_2\frac{3+\sqrt{5}}{2}},\infty\right)$}

291. $\frac{2^{x-1}-1}{2^{x+1}-1}<2$ {Ans. $\left(-1,\log_2\frac{2}{7}\right)$}

292. $\frac{1}{3^x+5}<\frac{1}{3^{x+1}-1}$ {Ans. $(-1,1)$}

293. $36^x-2\cdot 18^x-8\cdot 9^x>0$ {Ans. $(2,\infty)$}

294. $98-7^{x^2+5x-48}\ge 49^{x^2+5x-49}$ {Ans. $[-10,5]$}

295. $5\cdot 4^x+2\cdot 25^x\le 7\cdot 10^x$ {Ans. $[0,1]$}

296. $\sqrt{13^x-5}\le\sqrt{2\left(13^x+12\right)}-\sqrt{13^x+5}$ {Ans. $[\log_{13}5,1]$}

297. $\sqrt{9^x+3^x-2}\ge 9-3^x$ {Ans. $\left[\log_3\frac{83}{19},\infty\right)$}

298. $\sqrt{4^{x+1}+17}-5>2^x$ {Ans. $(2,\infty)$}

299. $\sqrt{2\left(5^x+24\right)}-\sqrt{5^x-7}\ge\sqrt{5^x+7}$ {Ans. $[\log_5 7,2]$}

CATEGORY-7.16. LOGARITHMIC INEQUALITIES

300. $\log_{\frac{1}{2}}(2x+3) > 0$ {Ans. $\left(-\frac{3}{2},-1\right)$}

301. $\log_2 \frac{x-3}{x+2} < 0$ {Ans. $(3,\infty)$}

302. $\log_{\frac{5}{8}}\left(2x^2 - x - \frac{3}{8}\right) \ge 1$ {Ans. $\left[-\frac{1}{2},-\frac{1}{4}\right) \cup \left(\frac{3}{4},1\right]$}

303. $\log_2^2 x + \log_2 x - 2 \le 0$ {Ans. $\left[\frac{1}{4},2\right]$}

304. $2\log_4(2x^2+3) < \log_2(x^2+6)$ {Ans. $\left(-\sqrt{3},\sqrt{3}\right)$}

305. $\log_2 \sqrt{x} - 2\log_{\frac{1}{4}}^2 x + 1 > 0$ {Ans. $\left(\frac{1}{2},4\right)$}

306. $\log(x-4) + \log x < \log 21$ {Ans. $(4,7)$}

307. $\log_7 x - \log_x \frac{1}{7} \ge 2$ {Ans. $(1,\infty)$}

308. $\log_{100} x^2 + \log_{10}^2 x < 2$ {Ans. $\left(\frac{1}{100},10\right)$}

309. $\log_3(7-x) \le \frac{9}{16}\log_{2\sqrt{2}}^2 \frac{1}{4} + \log_{7-x} 9$ {Ans. $[-2,6) \cup \left[\frac{20}{3},7\right)$}

310. $\log_3 x - \log_3^2 x \le \frac{3}{2}\log_{\frac{1}{2\sqrt{2}}} 4$ {Ans. $\left(0,\frac{1}{3}\right] \cup [9,\infty)$}

311. $(\log_5 x)^2 + (\log_5 x) < 2$. {Ans. $\left(\frac{1}{25},5\right)$}

312. $\frac{1-\log_4 x}{1+\log_2 x} \le \frac{1}{2}$ {Ans. $\left(0,\frac{1}{2}\right) \cup \left[\sqrt{2},\infty\right)$}

313. $\log_x(x-1) \ge 2$ {Ans. $\varnothing$}

314. $\log_x\left(x^2 - \frac{3}{16}\right) > 4$ {Ans. $\left(\frac{\sqrt{3}}{4},\frac{1}{2}\right) \cup \left(\frac{\sqrt{3}}{2},1\right)$}

315. $\log_x(16 - 6x - x^2) \le 1$ {Ans. $(0,1) \cup \left[\frac{-7+\sqrt{113}}{2},2\right)$}

316. $\log_x(x^3 - x^2 - 2x) < 3$ {Ans. $(2,\infty)$}

317. $\log_x \frac{(x+3)}{(x-1)} > 1$ {Ans. $(1,3)$}

318. $\log_x \sqrt{21-4x} > 1$ {Ans. $(1,3)$}

319. $\log_{2x}(x^2 - 5x + 6) < 1$ {Ans. $\left(0,\frac{1}{2}\right) \cup (1,2) \cup (3,6)$}

320. $\log_{x+4}(5x+20) \le \log_{x+4}(x+4)^2$ {Ans. $(-4,-3) \cup [1,\infty)$}

321. $\log_{2x+3} x^2 < 1$ {Ans. $\left(-\frac{3}{2},-1\right) \cup (-1,0) \cup (0,3)$}

322. $\log_{2x-x^2}\left(x - \frac{3}{2}\right)^4 > 0$ {Ans. $\left(\frac{1}{2},1\right) \cup \left(1,\frac{3}{2}\right) \cup \left(\frac{3}{2},2\right)$}

323. $\log_{\frac{x-1}{x+5}} \frac{3}{10} > 0$ {Ans. $(1,\infty)$}

324. $\log_{\frac{1}{x}} \frac{2(x-2)}{(x+1)(x-5)} \ge 1$ {Ans. $(1,2)$}

325. $\log_{x+\frac{1}{x}}\left(x^2+\frac{1}{x^2}-4\right)\geq 1$ {Ans. $\left(0,\frac{3-\sqrt{5}}{2}\right]\cup\left[\frac{3+\sqrt{5}}{2},\infty\right)$}

326. $\log_{\sqrt{x+1}-\sqrt{x-1}}\left(x^2-3x+1\right)\geq 0$ {Ans. $\left(\frac{3+\sqrt{5}}{2},3\right]$}

CATEGORY-7.17. EXPO-LOGARITHMIC INEQUALITIES

327. $x^{\log^2 x-3\log x+1}>1000$ {Ans. $(1000,\infty)$}

328. $\left(\frac{x}{10}\right)^{\log x-2}<100$ {Ans. $(1,1000)$}

329. $x^{\log x}>10\cdot x^{-\log x}+3$ {Ans. $\left(0,10^{-\sqrt{\log 5}}\right)\cup\left(10^{\sqrt{\log 5}},\infty\right)$}

330. $\left(x^2-x-1\right)^{x^2-1}<1$ {Ans. $\left(\frac{1+\sqrt{5}}{2},2\right)$}

CATEGORY-7.18. PARAMETRIC EQUATIONS AND INEQUALITIES

331. Solve the equation $2a(a-2)x=a-2$.

{Ans. $a=0,\quad x\in\phi$

$a=2,\quad x\in R$

$a\neq 0,2,\quad x\in\left[\frac{1}{2a}\right]$}

332. Solve the equation $(a-1)x^2+2(2a+1)x+(4a+3)=0$.

{Ans. $a<-\frac{4}{5},\quad x\in\phi$

$a=1,\quad x\in\left[-\frac{7}{6}\right]$

$a=-\frac{4}{5},\quad x\in\left[-\frac{1}{3}\right]$

$a>-\frac{4}{5}, a\neq 1,\quad x\in\left[\frac{-(2a+1)-\sqrt{5a+4}}{a-1}\right]\cup\left[\frac{-(2a+1)+\sqrt{5a+4}}{a-1}\right]$}

333. Solve the equation $\frac{x^2+1}{a^2x-2a}-\frac{1}{2-ax}=\frac{x}{a}$.

{Ans. $a=0,\quad x\in\phi$

$a=1,\quad x\in[-1]$

$a=-2,\quad x\in\left[\frac{1}{3}\right]$

$a\neq 0,1,-2,\quad x\in[-1]\cup\left[\frac{a+1}{a-1}\right]$}

334. Solve the inequality $\frac{7x-11}{a+3}>(1+3a)\frac{x}{4}$.

{Ans. $a=-3,\frac{5}{3},\quad x\in\phi$

$a<-5,-3<a<\frac{5}{3},\quad x\in\left(-\frac{44}{3a^2+10a-25},\infty\right)$

$-5<a<-3,a>\frac{5}{3},\quad x\in\left(-\infty,-\frac{44}{3a^2+10a-25}\right)$

$a=-5,\quad x\in R$}

335. Solve the inequality $x^2+ax+a>0$.

{Ans. $a\in(-\infty,0)\cup(4,\infty),\quad x\in\left(-\infty,\frac{-a-\sqrt{a^2-4a}}{2}\right)\cup\left(\frac{-a+\sqrt{a^2-4a}}{2},\infty\right)$

$a\in(0,4),\quad x\in R$

$a\in[0]\cup[4],\quad x\in\left(-\infty,-\frac{a}{2}\right)\cup\left(-\frac{a}{2},\infty\right)$}

336. Solve the inequality $ax^2-2x+4>0$.

{Ans. $a>\frac{1}{4},\quad x\in R$

$0<a\le\frac{1}{4},\quad x\in\left(-\infty,\frac{1-\sqrt{1-4a}}{a}\right)\cup\left(\frac{1+\sqrt{1-4a}}{a},\infty\right)$

$a<0,\quad x\in\left(\frac{1+\sqrt{1-4a}}{a},\frac{1-\sqrt{1-4a}}{a}\right)$

$a=0,\quad x\in(-\infty,2)$}

337. Find all values of a for which the inequality $|x+a^3|+|x+1|\le 1-a^3$ has no less than four different integral solutions. {Ans. $a\in\left(-\infty,-\sqrt[3]{2}\right]$}

CATEGORY-7.19. PROVING INEQUALITIES

338. $x+\frac{1}{x}\ge 2$ if $x>0$.

339. $x+\frac{1}{x}\le -2$ if $x<0$.

340. $\tan^2 x+\cot^2 x\ge 2$.

341. $10^x+10^{-x}\ge 2$.

342. $|\cos x+\sec x|\ge 2$.

343. $a^2+b^2+c^2\ge ab+bc+ca$.

344. $(ab+xy)(ax+by)>4abxy$.

345. $(b+c)(c+a)(a+b)>8abc$.

346. $(a+b+c)(bc+ca+ab)>9abc$.

347. $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)>9$.

348. $\left(\frac{a}{e}+\frac{b}{f}+\frac{c}{g}\right)\left(\frac{e}{a}+\frac{f}{b}+\frac{g}{c}\right)>9$.

349. $b^2c^2+c^2a^2+a^2b^2>abc(a+b+c)$.

350. $\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\geq a+b+c$.

351. $\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

352. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}$.

353. If $a^2+b^2+c^2=1$ then show that $-\frac{1}{2}\leq ab+bc+ca\leq 1$.

354. $2(a^3+b^3+c^3)\geq bc(b+c)+ca(c+a)+ab(a+b)$

355. $\frac{a^3+b^3+c^3}{3}>\left(\frac{a+b+c}{3}\right)\cdot\left(\frac{a^2+b^2+c^2}{3}\right)$.

356. If $x_i>0,\quad i=1,2,\dots n$, prove $(x_1+x_2+\dots x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\dots+\frac{1}{x_n}\right)\geq n^2$.

357. If $a+b+c=1$ then prove that $\frac{8}{27abc}>\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)>8$.

358. If $x+y+z=a$ then prove that $8xyz\leq(a-x)(a-y)(a-z)\leq\frac{8}{27}a^3$.

359. If a, b, c are positive real numbers such that $a+b+c=1$, prove that $\frac{b(1-b)}{ac}+\frac{c(1-c)}{ab}+\frac{a(1-a)}{bc}\geq 6$.

360. Prove that $abcd>81(s-a)(s-b)(s-c)(s-d)$ where $3s=a+b+c+d$.

361. Prove that $(S-a)(S-b)(S-c)(S-d)>81\,abcd$ where $S=a+b+c+d$.

362. If $a^2+b^2+c^2=1$ and $x^2+y^2+z^2=1$, show that $ax+by+cz\leq 1$.

363. If x and y are positive real numbers and m and n are any positive integers, then $\frac{x^ny^m}{(1+x^{2n})(1+y^{2m})}>\frac{1}{4}$.

(True/False) {Ans. False}

364. For $a\geq 0$, $b\geq 0$ and $x>y>0$, prove that $(a^x+b^x)^{\frac{1}{x}}\leq(a^y+b^y)^{\frac{1}{y}}$.

CATEGORY-7.20. SYSTEM OF NON-LINEAR EQUATIONS IN TWO VARIABLES

365. $5x-y=3$, $y^2-6x^2=25$. {Ans. $(2,7)$, $\left(-\frac{8}{19},-\frac{97}{19}\right)$}

366. $x^4+y^4=706$, $x+y=8$. {Ans. $(5,3)$, $(3,5)$}

367. $3x+4y=18$, $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}$. {Ans. $(2,3)$, $\left(\frac{18}{5},\frac{9}{5}\right)$}

368. $\frac{x^2}{y}+\frac{y^2}{x}=18$, $x+y=12$. {Ans. $(8,4)$, $(4,8)$}

369. $\left(3-\frac{6y}{x+y}\right)^2+\left(3+\frac{6y}{x-y}\right)^2=82$, $3x+7y=26$. {Ans. $(4,2)$, $\left(\frac{26}{17},\frac{52}{17}\right)$, $(-52,26)$, $\left(-\frac{26}{11},\frac{52}{11}\right)$}

370. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2}$, $x+y=10$. {Ans. $(8,2)$, $(2,8)$}

371. $(x+y)^{\frac{2}{3}}+2(x-y)^{\frac{2}{3}}=3\left(x^2-y^2\right)^{\frac{1}{3}}$, $3x-2y=13$. {Ans. $(9,7)$, $\left(\frac{13}{3},0\right)$}

372. $x^2+y(x+1)=17$, $y^2+x(y+1)=13$. {Ans. $(3,2)$, $\left(-\frac{23}{7},-\frac{19}{7}\right)$}

373. $xy+3y^2-x+4y-7=0$, $2xy+y^2-2x-2y+1=0$. {Ans. $x\in R$, $y=1$; $(2,-3)$}

374. $3x^2+xy+y^2=15$, $31xy-3x^2-5y^2=45$. {Ans. $(\pm1,\pm3)$, $(\pm2,\pm1)$}

375. $3x^2+2y^2=50$, $xy-3y^2=1$. {Ans. $(\pm4,\pm1)$, $\left(\pm\frac{38}{\sqrt{87}},\pm\frac{3}{\sqrt{87}}\right)$}

376. $x+y+xy=11$, $x^2y+xy^2=30$. {Ans. $(5,1)$, $(1,5)$, $(2,3)$, $(3,2)$}

377. $x^3-y^3=127$, $x^2y-xy^2=42$. {Ans. $(7,6)$, $(-6,-7)$}

378. $x^4+x^2y^2+y^4=2613$, $x^2+xy+y^2=67$. {Ans. $(\pm7,\pm2)$, $(\pm2,\pm7)$}

379. $\frac{1}{x^3}-\frac{1}{y^3}=91$, $\frac{1}{x}-\frac{1}{y}=1$. {Ans. $\left(-\frac{1}{5},-\frac{1}{6}\right)$, $\left(\frac{1}{6},\frac{1}{5}\right)$}

380. $(3x)^{\log 3}=(4y)^{\log 4}$, $4^{\log x}=3^{\log y}$. {Ans. $\left(\frac{1}{3},\frac{1}{4}\right)$}

381. $2^x+2^y=20$, $x+y=6$. {Ans. $(4,2)$, $(2,4)$}

382. $6^x\left(\frac{2}{3}\right)^y-3\cdot2^{x+y}-8\cdot3^{x-y}+24=0$, $xy=2$. {Ans. $(2,1)$, $(1,2)$, $(-1,-2)$}

383. Find the positive solutions of the system of equations $x^{x+y}=y^n$, $y^{x+y}=x^{2n}y^n$, where $n>0$. {Ans. $(1,1)$, $\left(\frac{\sqrt{1+8n}-1}{2},\frac{4n+1-\sqrt{1+8n}}{2}\right)$}

CATEGORY-7.21. SYSTEM OF NON-LINEAR EQUATIONS IN THREE OR MORE VARIABLES

384. $2x+y-2z=0$, $7x+6y-9z=0$, $x^3+y^3+z^3=1728$. {Ans. $(6,8,10)$}

385. $9x+y-8z=0$, $4x-8y+7z=0$, $yz+zx+xy=47$. {Ans. $(\pm3,\pm5,\pm4)$}

386. $x(x+y+z)=4$, $y(x+y+z)=9$, $z(x+y+z)=12$. {Ans. $\left(\pm\frac{4}{5},\pm\frac{9}{5},\pm\frac{12}{5}\right)$}

387. $x(y+z)=3$, $y(z+x)=4$, $z(x+y)=5$. {Ans. $\left(\pm\frac{\sqrt{6}}{3},\pm\frac{\sqrt{6}}{2},\pm\sqrt{6}\right)$}

388. $xy=x+y$, $yz=2(y+z)$, $zx=3(z+x)$. {Ans. $(0,0,0)$, $\left(\frac{12}{5},\frac{12}{7},-12\right)$}

389. $(x+y)^2-z^2=-9$, $(y+z)^2-x^2=15$, $(z+x)^2-y^2=3$. {Ans. $(-1,1,3)$, $(1,-1,-3)$}

390. $xy + x + y = 23$, $xz + z + x = 41$, $yz + y + z = 27$. {Ans. $(5,3,6)$, $(-7,-5,-8)$}
391. $x + y = 2$, $xy - z^2 = 1$. {Ans. $(1,1,0)$}
392. $x^2 + y^2 + z^2 = 84$, $x + y + z = 14$, $xz = y^2$. {Ans. $(8,4,2)$, $(2,4,8)$}
393. $x + y + z = 7$, $xy + xz = yz - 2$, $x^2 + y^2 + z^2 = 21$. {Ans. $(1,2,4)$, $(1,4,2)$}
394. $(x-2)^2 + (y-3)^2 + (z-1)^2 = 24$, $xy + yz + zx = 63$, $2x + 3y + z = 30$. {Ans. $(6,5,3)$, $\left(\frac{10}{3}, \frac{19}{3}, \frac{13}{3}\right)$}
395. $x + y - 4xy = 0$, $y + z - 6yz = 0$, $z + x - 8zx = 0$. {Ans. $(0,0,0)$, $\left(\frac{1}{3}, 1, \frac{1}{5}\right)$}
396. $xz + y = 7z$, $yz + x = 8z$, $x + y + z = 12$. {Ans. $\left(\frac{60}{7}, \frac{66}{7}, -6\right)$, $(4,6,2)$}

CATEGORY-7.22. SOLVING EQUATIONS AND INEQUALITIES GRAPHICALLY

397. Equation $\ln x = x$ has how many solutions? {Ans. No solution}
398. Equation $x \ln x = 1$ has how many solutions? {Ans. One}
399. $e^{|x|} = |x|$. {Ans. No solution}
400. Solve $3^x + 4^x = 5^x$. {Ans. $[2]$}
401. Solve $2^x < x + 1$. {Ans. $(0,1)$}
402. Solve $3^x + 4^x > 7$. {Ans. $(1, \infty)$}

CATEGORY-7.23. EQUATIONS AND INEQUALITIES CONTAINING GREATEST INTEGER FUNCTION

403. Solve $\text{Sgn}x = [x]$. {Ans. $[-1, 0] \cup [1, 2)$}
404. Solve $[x^2 + x + 1] = 2$ ([ ] denotes Greatest Integer Function). {Ans. $\left(-2, \frac{-1-\sqrt{5}}{2}\right] \cup \left[\frac{-1+\sqrt{5}}{2}, 1\right)$}
405. Let $\{x\}$ and $[x]$ denote the fractional and integral parts of a real number x respectively. Solve $4\{x\} = x + [x]$. {Ans. $[0] \cup \left[\frac{5}{3}\right]$}
406. Solve $y = 2[x] + 3$, $y = 3[x-2] + 5$ ([ ] denotes Greatest Integer Function). {Ans. $x \in [4,5)$, $y \in [11]$}
407. Solve $(x)^2 = [x]^2 + 2x$, where $[x]$ is the greatest integer $\le x$ and (x) is the least integer $\ge x$. {Ans. $[0] \cup \left[I + \frac{1}{2}\right]$}
408. Solve $[x^2] + 2[x] - 3x = 0, \quad 0 \le x \le 2$ ([ ] denotes Greatest Integer Function). {Ans. $[0] \cup [1]$}
409. Solve $y = 4 - [x]^2$, $[y] + y = 6$ ([ ] denotes Greatest Integer Function). {Ans. $x \in [-1,0) \cup [1,2)$, $y \in [3]$}
410. Solve the equation $\left[\frac{3x-1}{4}\right] + \left[\frac{3x+1}{4}\right] + \left[\frac{3x-1}{2}\right] = \frac{6x+3}{5}$ ([ ] denotes Greatest integer function). {Ans. $\frac{7}{6}$}
411. Solve the system of equations:-

$x+[y]+\{z\}=1.1$
$[x]+\{y\}+z=2.2$
$\{x\}+y+[z]=3.3$.
([ ] denotes Greatest integer function, { } denotes Fractional part function). {Ans. $x=0.1$, $y=1.2$, $z=2$ }

412. For every +ve integer n, prove that $\sqrt{4n+1}<\sqrt{n}+\sqrt{n+1}<\sqrt{4n+2}$. Hence, prove that $\left\lfloor\sqrt{n}+\sqrt{n+1}\right\rfloor=\left\lfloor\sqrt{4n+1}\right\rfloor$, where [ ] denotes greatest integer function.

CATEGORY-7.24. ADDITIONAL QUESTIONS

413. Find out whether the following numerical expressions are defined or not.
   i. $\sqrt{\log_2 1.4+\log_2 0.7}$. {Ans. Not defined}
   ii. $\sqrt{\log 15+\log 0.07}$. {Ans. Defined}
   iii. $\log\log\log 11$. {Ans. Defined}

414. Without using tables prove $\frac{1}{\log_3 \pi}+\frac{1}{\log_4 \pi}>2$.

415. Without using tables prove that $\log_2 17 \log_{\frac{1}{5}} 2 \log_3 \frac{1}{5}>2$.

416. Which is greater?
   i. $\log_2 3$ or $\log_{\frac{1}{2}} 5$. {Ans. $\log_2 3$ }
   ii. $\log_4 5$ or $\log_{\frac{1}{16}}\left(\frac{1}{25}\right)$. {Ans. equal}
   iii. $\log_7 11$ or $\log_8 5$. {Ans. $\log_7 11$}
   iv. $\log_2 3$ or $\log_3 11$. {Ans. $\log_3 11$}
   v. $\log_{\frac{1}{3}}\frac{1}{2}$ or $\log_{\frac{1}{2}}\frac{1}{3}$. {Ans. $\log_{\frac{1}{2}}\frac{1}{3}$ }

417. Find all pairs (x,y) of real numbers such that $16^{x^2+y}+16^{x+y^2}=1$. {Ans. $x=-\frac{1}{2}$; $y=-\frac{1}{2}$ }

418. Given $(2011a+2012b)(2012c-2011b)=(2011b+2012c)(2012b-2011a)$ and $0<a<b<c$ and $\sqrt{c}\cdot a^x+\sqrt{b}\cdot c^x<\left(\sqrt{c}+\sqrt{b}\right)\cdot b^x$, then find the interval of solution of x. {Ans. $x\in\left(0,\frac{1}{2}\right)$ }

419. For each integer $n\geq 1$, define $a_n=\left[\frac{n}{\left[\sqrt{n}\right]}\right]$, where [ ] denotes the Greatest Integer Function. Find the number of all n in the set $\{1,2,3,\cdots\cdots,2010\}$ for which $a_n>a_{n+1}$. {Ans. 43}

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Mathematics for IIT-JEE

By **Er. Sanjiva Dayal,** *B.Tech. (I.I.T. Kanpur)*

PART-II
ALGEBRA

CHAPTER-8
QUADRATIC EXPRESSIONS

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CHAPTER-8
QUADRATIC EXPRESSIONS

LIST OF THEORY SECTIONS

LIST OF QUESTION CATEGORIES

CHAPTER-8
QUADRATIC EXPRESSIONS

SECTION-8.1. RELATIONSHIP BETWEEN THE ROOTS AND COEFFICIENTS

1. Vieta's Theorem

i. If the polynomial $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_1 x + a_0$ has roots α_1, α_2, ..., α_{n-1}, α_n then the following equalities hold true:-

Sum of the roots = $\alpha_1 + \alpha_2 + \alpha_3 + \cdots\cdots + \alpha_n = -\dfrac{a_{n-1}}{a_n}$;

Sum of the product of the roots taken two at a time = $\alpha_1\alpha_2 + \cdots\cdots = \dfrac{a_{n-2}}{a_n}$;

Sum of the product of the roots taken three at a time = $\alpha_1\alpha_2\alpha_3 + \cdots\cdots = -\dfrac{a_{n-3}}{a_n}$;

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Product of the roots = $\alpha_1\alpha_2\alpha_3\cdots\cdots\alpha_n = (-1)^n \dfrac{a_0}{a_n}$.

2. Application of Vieta's Theorem in quadratic function

i. If the quadratic function $P_2(x) = ax^2 + bx + c$ has roots α and β then the following equalities hold true:-

$\alpha + \beta = -\dfrac{b}{a}$;

$\alpha\beta = \dfrac{c}{a}$.

3. Application of Vieta's Theorem in cubic function

i. If the cubic function $P_3(x) = ax^3 + bx^2 + cx + d$ has roots α, β and γ then the following equalities hold true:-

$\alpha + \beta + \gamma = -\dfrac{b}{a}$;

$\alpha\beta + \alpha\gamma + \beta\gamma = \dfrac{c}{a}$;

$\alpha\beta\gamma = -\dfrac{d}{a}$.

4. Application of Vieta's Theorem in bi-quadratic function

i. If the bi-quadratic function $P_4(x) = ax^4 + bx^3 + cx^2 + dx + e$ has roots α, β, γ and δ then the following equalities hold true:-

$\alpha + \beta + \gamma + \delta = -\dfrac{b}{a}$;

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \dfrac{c}{a}$;

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\dfrac{d}{a}$;

$\alpha\beta\gamma\delta = \dfrac{e}{a}$.

SECTION-8.2. FORMATION OF POLYNOMIAL EQUATION FROM GIVEN ROOTS

1. To find a polynomial from given roots

i. Given n numbers α_1, α_2, ..., α_{n-1}, α_n, then a polynomial of degree n having these numbers as roots is

$P_n(x) = x^n - S_1 x^{n-1} + S_2 x^{n-2} - S_3 x^{n-3} + \cdots\cdots + (-1)^n S_n$;

where

S_1 = Sum of the roots = $\alpha_1 + \alpha_2 + \alpha_3 + \cdots\cdots + \alpha_n$;

S_2 = Sum of the product of the roots taken two at a time = $\alpha_1\alpha_2 + \cdots\cdots$;

Sum of the product of the roots taken three at a time = $\alpha_1\alpha_2\alpha_3 + \cdots\cdots$;

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…………

Sum of the product of the roots taken k at a time = $\alpha_1\alpha_2\alpha_3 \cdots\cdots \alpha_k + \cdots\cdots$;

…………

…………

Product of the roots = $\alpha_1\alpha_2\alpha_3 \cdots\cdots \alpha_n$.

2. To find a quadratic function from given two roots

i. Given two numbers α and β, then a quadratic function having these numbers as roots is

$P_2(x) = x^2 - S_1 x + S_2$, i.e.

$P_2(x) = x^2 - (\alpha + \beta)x + \alpha\beta$.

3. To find a cubic function from given three roots

i. Given three numbers α, β and γ, then a cubic function having these numbers as roots is

$P_3(x) = x^3 - S_1 x^2 + S_2 x - S_3$, i.e.

$P_3(x) = x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma$.

4. To find a bi-quadratic function from given four roots

i. Given three numbers α, β, γ and δ, then a bi-quadratic function having these numbers as roots is

$P_4(x) = x^4 - S_1 x^3 + S_2 x^2 - S_3 x + S_4$, i.e.

$P_4(x) = x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + \alpha\beta\gamma\delta$.

SECTION-8.3. FORMATION OF POLYNOMIAL EQUATION FROM TRANSFORMATION

1. Formation of polynomial equation from transformation

i. Sometimes a given polynomial equation can be transformed into another polynomial equation whose roots bears a certain assigned relation with those of the given equation.

2. To transform a given polynomial equation into another polynomial equation whose roots are negative of the roots of the given polynomial equation

i. Substitute $y = -x$.

3. To transform a given polynomial equation into another polynomial equation whose roots are a given

multiple of the roots of the given polynomial equation

i. Substitute $y = kx$.

4. **To transform a given polynomial equation into another polynomial equation whose roots exceed the roots of the given polynomial equation by a given quantity**

   i. Substitute $y = x + k$.

5. **To transform a given polynomial equation into another polynomial equation whose roots are reciprocal of the roots of the given polynomial equation**

   i. Substitute $y = \frac{1}{x}$.

6. **To transform a given polynomial equation into another polynomial equation whose roots are square of the roots of the given polynomial equation**

   i. Substitute $y = x^2$.

7. **To transform a given polynomial equation into another polynomial equation whose roots are cube of the roots of the given polynomial equation**

   i. Substitute $y = x^3$.

SECTION-8.4. NATURE OF ROOTS AND SIGN OF QUADRATIC FUNCTION

1. **Nature of roots of quadratic function**

   i. To determine whether the roots are real, imaginary, rational, integer, even, odd etc.

2. **Sign of quadratic function**

   i. To determine the intervals where the quadratic function is positive or negative.

SECTION-8.5. COMMON ROOTS

1. **Common roots of quadratic functions**

   i. Two quadratic functions $a_1x^2 + b_1x + c_1$ and $a_2x^2 + b_2x + c_2$ have one root common if $(b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1) = (a_1c_2 - a_2c_1)^2$ and common root is $\frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$ or $\frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2}$.

   ii. Two quadratic functions $a_1x^2 + b_1x + c_1$ and $a_2x^2 + b_2x + c_2$ have both roots common if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

2. **Common roots of polynomial functions**

   i. If all roots of $P_m(x)$ are roots of $Q_n(x)$ $(n > m)$ then $P_m(x)$ is a factor of $Q_n(x)$, i.e. $Q_n(x) = \lambda_{n-m}(x) \cdot P_m(x)$.

SECTION-8.6. POSITION OF ROOTS

1. **Given a quadratic function $f(x) = ax^2 + bx + c$ and given a number *k*, to find the condition that one root is less than *k* and the other root is greater than *k*.**
2. **Given a quadratic function $f(x) = ax^2 + bx + c$ and given a number *k*, to find the condition that both the roots are less than *k*.**
3. **Given a quadratic function $f(x) = ax^2 + bx + c$ and given a number *k*, to find the condition that both the roots are greater than *k*.**
4. **Given a quadratic function $f(x) = ax^2 + bx + c$ and given a two numbers *k* and *l*, to find the condition**

that both the roots lie between *k* and *l*.

5. **Given a quadratic function $f(x) = ax^2 + bx + c$ and given a two numbers *k* and *l*, to find the condition that both the roots do not lie between *k* and *l*.**
6. **Given a quadratic function $f(x) = ax^2 + bx + c$ and given a two numbers *k* and *l*, to find the condition that exactly one of the roots lie between *k* and *l*.**

SECTION-8.7. RANGE OF RATIONAL FUNCTIONS

1. **To find the range of rational functions of type** $f(x) = \dfrac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}$

   i. Find the values of α for which the equation $\dfrac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2} = \alpha$ has a solution. These values of α are the range of the function.

EXERCISE-8

CATEGORY-8.1. RELATIONSHIP BETWEEN THE ROOTS AND COEFFICIENTS

1. If both the roots of ax^2+bx+c are zero, then find the value of *a*, *b*, *c*. {Ans. $a \neq 0, b=c=0$ }
2. If 8, 2 are the roots of $x^2+ax+\beta$ and 3, 3 are the roots of $x^2+\alpha x+b$, then find the roots of x^2+ax+b. {Ans. 1, 9}
3. If the product of the roots of the equation $mx^2+6x+(2m-1)=0$ is –1, then find *m*. {Ans. $\frac{1}{3}$ }
4. If the sum of the roots of the equation $(a+1)x^2+(2a+3)x+(3a+4)=0$ is -1, then find the product of the roots. {Ans. 2}
5. If *a*, *b* are the roots of the equation $x^2+x+1=0$, then find the value of a^2+b^2. {Ans. -1}
6. If α, β are the roots of the equation $4x^2+3x+7=0$, then find the value of $\frac{1}{\alpha}+\frac{1}{\beta}$. {Ans. $-\frac{3}{7}$ }
7. If α and β are the roots of $ax^2+bx+c=0$, find the values of the following:-
   i. $\frac{1}{a\alpha+b}+\frac{1}{a\beta+b}$. {Ans. $\frac{b}{ac}$ }
   ii. $\frac{\beta}{a\alpha+b}+\frac{\alpha}{a\beta+b}$. {Ans. $-\frac{2}{a}$ }
   iii. $(a\alpha+b)^{-3}+(a\beta+b)^{-3}$. {Ans. $\frac{b^3-3abc}{a^3c^3}$ }
   iv. $(a\alpha+b)^{-2}+(a\beta+b)^{-2}$. {Ans. $\frac{b^2-2ac}{a^2c^2}$ }
8. If α, β are roots of the equation $6x^2-6x+1=0$, then find the value of $\frac{1}{2}(a+b\alpha+c\alpha^2+d\alpha^3)+\frac{1}{2}(a+b\beta+c\beta^2+d\beta^3)$ in terms of *a*, *b*, *c*, *d*. {Ans. $\frac{1}{12}(12a+6b+4c+3d)$ }
9. If α and β are the roots of $x^2+ax+b=0$, then prove that $\frac{\alpha}{\beta}$ is a root of the equation $bx^2+(2b-a^2)x+b=0$.
10. If α and β are the roots of $x^2-p(x+1)-c=0$, show that $(\alpha+1)(\beta+1)=1-c$. Hence prove that $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}=1$.
11. If the roots of the equation $(x-a)(x-b)-k=0$ be *c* and *d*, then prove that the roots of the equation $(x-c)(x-d)+k=0$ are *a* and *b*.
12. If the roots of the equation $x^2+px+q=0$ are obtained –2 and –15 when the coefficient of x was misread 17 in place of 13, then find the correct roots of the equation. {Ans. –10, –3}
13. Two candidates attempt to solve a quadratic of the form $x^2+px+q=0$. One starts with a wrong value of *p* and finds the roots to be 2 and 6. The other starts with a wrong value of *q* and finds the roots to be 2 and –9. Find the correct roots. {Ans. –3, -4}
14. If α be a root of $4x^2+2x-1=0$, prove that $4\alpha^3-3\alpha$ is the other root.

15. Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equation $ax^2 + bx + c = 0$. Express the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α, β. {Ans. $\alpha^2\beta, \alpha\beta^2$}
16. If $\alpha + \beta = 3$ and $\alpha^3 + \beta^3 = 7$, then show that α and β are the roots of $9x^2 - 27x + 20 = 0$.
17. If α, β be the roots of $ax^2 + 2bx + c = 0$ and $\alpha + \delta$, $\beta + \delta$ be those of $Ax^2 + 2Bx + C = 0$, then prove that $\dfrac{b^2 - ac}{B^2 - AC} = \left(\dfrac{a}{A}\right)^2$.
18. The ratio of the roots of the equation $ax^2 + bx + c = 0$ is same as the ratio of the roots of the equation $Ax^2 + Bx + C = 0$. If D_1 and D_2 are the discriminants of $ax^2 + bx + c = 0$ and $Ax^2 + Bx + C = 0$ respectively, then show that $D_1 : D_2 = b^2 : B^2$.
19. If α and β be the roots of $x^2 + px - q = 0$ and γ, δ the roots of $x^2 + px + r = 0$, prove that $(\alpha - \gamma)(\alpha - \delta) = (\beta - \gamma)(\beta - \delta) = q + r$.
20. If α and β are the roots of $x^2 + px + 1 = 0$ and γ, δ the roots of $x^2 + qx + 1 = 0$, show that $q^2 - p^2 = (\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta)$.
21. If $\sin\alpha$ and $\cos\alpha$ are roots of the equation $px^2 + qx + r = 0$, then show that $p^2 - q^2 + 2pr = 0$.
22. If one root of the equation $5x^2 + 13x + k = 0$ is reciprocal of other, then find the value of k. {Ans. 5}
23. If the roots of $px^2 + qx + 2 = 0$ are reciprocal of each other, then find p. {Ans. 2}
24. Find the condition that the roots of the equation $ax^2 + bx + c = 0$ be such that
    i. One root is n times the other. {Ans. $nb^2 = ac(n+1)^2$}
    ii. One root is three times the other. {Ans. $3b^2 = 16ac$}
25. If the roots of the equation $ax^2 + bx + c = 0$ are of the form $\dfrac{k+1}{k}$ and $\dfrac{k+2}{k+1}$, prove that $(a + b + c)^2 = b^2 - 4ac$.
26. If one root of the equation $ax^2 + bx + c = 0$ be the square of the other, then prove that $b^3 + ac^2 + a^2c = 3abc$.
27. Find the relation between a, b, c such that one root of the equation $ax^2 + bx + c = 0$ may be double of the other. {Ans. $2b^2 = 9ac$}
28. If one root of the quadratic equation $ax^2 + bx + c = 0$ is equal to the nth power of the other root, then show that $(ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} + b = 0$.
29. If the equation $\dfrac{a}{x-a} + \dfrac{b}{x-b} = 1$ has roots equal in magnitude but opposite in sign, then find the value of $a + b$. {Ans. 0}
30. If the roots of the equation $\dfrac{1}{x+p} + \dfrac{1}{x+q} = \dfrac{1}{r}$ are equal in magnitude but opposite in sign, show that $p + q = 2r$ and the product of the roots is equal to $-\dfrac{1}{2}(p^2 + q^2)$.
31. If the equation $(k^2 - 5k + 6)x^2 + (k^2 - 3k + 2)x + (k^2 - 4) = 0$ is satisfied by more than two values of x, then determine the value of k. {Ans. 2}
32. If the sum of the roots of $ax^2 + bx + c = 0$ be equal to sum of their squares, prove that $2ac = ab + b^2$.

33. α, β are the roots of the equation $\lambda(x^2 - x) + x + 5 = 0$. If λ_1 and λ_2 be the two values of λ for which α and β are connected by the relation $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$, then find the value of $\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}$. {Ans. 254}
34. If the ratio of the roots of the equation $x^2 + px + q = 0$ be equal to the ratio of the roots of $x^2 + lx + m = 0$, then prove that $p^2 m = l^2 q$.
35. Find the value of p for which $x+1$ is a factor of $x^4 + (p-3)x^3 - (3p-5)x^2 + (2p-9)x + 6$. Find the remaining factors for this value of p. {Ans. p = 4, $x-1, x-2, x+3$}
36. If $x^2 - 3x + 2$ is a factor of $x^4 - px^2 + q$, prove p = 5, q = 4.
37. If the difference of the roots of the equation $x^2 + px + 12 = 0$ is 1, find the value of p. {Ans. ± 7}
38. Find the value of p for which the difference between the roots of the equation $x^2 + px + 8 = 0$ is 2. {Ans. ± 6}
39. If the roots of the equation $x^2 - px + q = 0$ differ by unity, then prove that $p^2 = 4q + 1$.
40. If a, b are roots of the equation $x^2 + ab = (a+1)x$, then find the value of b. {Ans. 1}
41. Knowing that 2 and 3 are the roots of the equation $2x^3 + mx^2 - 13x + n = 0$, determine m and n and find the third root of the equation. {Ans. $m = -5, n = 30, -\frac{5}{2}$}
42. Find the value of m for which the equation $\frac{a}{x+a+m} + \frac{b}{x+b+m} = 1$ has roots equal in magnitude but opposite in sign. {Ans. 0}
43. If p and q are roots of the quadratic equation $x^2 + mx + m^2 + a = 0$, then find the value of $p^2 + q^2 + pq$. {Ans. $-a$}
44. If α, β are the roots of the equation $x^2 + x + 1 = 0$ and $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are roots of the equation $x^2 + px + q = 0$, then find the value of p. {Ans. 1}
45. If α and β are the roots of $x^2 + px + q = 0$ and also of $x^{2n} + p^n x^n + q^n = 0$ and if $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are the roots of $x^n + 1 + (x+1)^n = 0$, then prove that n must be a even integer.

CATEGORY-8.2. FORMATION OF POLYNOMIAL EQUATION FROM GIVEN ROOTS

46. Find the polynomial whose roots are 1, 2 and 3. {Ans. $x^3 - 6x^2 + 11x - 6$}
47. Form a quadratic equation whose roots are $\frac{a}{\sqrt{a} \pm \sqrt{a-b}}$. {Ans. $bx^2 - 2a\sqrt{a}x + a^2 = 0$}
48. If α and β are the roots of $2x^2 - 3x - 6 = 0$, find the equation whose roots are $\alpha^2 + 2, \beta^2 + 2$. {Ans. $4x^2 - 49x + 118 = 0$}
49. Find the equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$, where α, β are the roots of $2x^2 + 2(m+n)x + (m^2 + n^2) = 0$. {Ans. $x^2 - 4mnx - (m^2 - n^2)^2 = 0$}
50. If α and β are the roots of $x^2 - 2x + 3 = 0$, find the equation whose roots are:-
    i. $\alpha + 2, \beta + 2$. {Ans. $x^2 - 6x + 11 = 0$}

ii. $\dfrac{\alpha-1}{\alpha+1}, \dfrac{\beta-1}{\beta+1}$. {Ans. $3x^2-2x+1=0$ }

iii. $\alpha^3-3\alpha^2+5\alpha-2, \beta^3-\beta^2+\beta+5$. {Ans. $x^2-3x+2=0$ }

51. If α and β are the roots of $ax^2+bx+c=0$, find the equation whose roots are given below:-

i. $\dfrac{1}{\alpha+\beta}, \dfrac{1}{\alpha}+\dfrac{1}{\beta}$. {Ans. $bcx^2+\left(b^2+ac\right)x+ab=0$ }

ii. $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$. {Ans. $acx^2-\left(b^2-2ac\right)x+ac=0$ }

iii. $\alpha+\dfrac{1}{\beta}, \beta+\dfrac{1}{\alpha}$. {Ans. $acx^2+b(a+c)x+(a+c)^2=0$ }

iv. $\alpha^2+\beta^2, \dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$. {Ans. $a^2c^2x^2-\left(b^2-2ac\right)\left(a^2+c^2\right)x+\left(b^2-2ac\right)^2=0$ }

v. $\dfrac{1}{a\alpha+b}, \dfrac{1}{a\beta+b}$. {Ans. $acx^2-bx+1=0$ }

52. If α, β are the roots of $ax^2+bx+c=0$, $\alpha_1, -\beta$ are the roots of $a_1x^2+b_1x+c_1=0$, show that α, α_1 are the roots of $\dfrac{x^2}{\frac{b}{a}+\frac{b_1}{a_1}}+x+\dfrac{1}{\frac{b}{c}+\frac{b_1}{c_1}}=0$.

CATEGORY-8.3. FORMATION OF POLYNOMIAL EQUATION FROM TRANSFORMATION

53. Form an equation whose roots are negative of the roots of the equation $x^3-5x^2-7x-3=0$. {Ans. $x^3+5x^2-7x+3=0$ }

54. Form an equation whose roots are double the roots of the equation $x^3+x+1=0$. {Ans. $x^3+4x+8=0$ }

55. Form an equation whose roots exceed the roots of the equation $x^3+x+1=0$ by 2. {Ans. $x^3-6x^2+13x-9=0$ }

56. If α, β are the roots of $ax^2+bx+c=0$ then find the equation whose roots are $2+\alpha, 2+\beta$. {Ans. $ax^2+x(b-4a)+4a-2b+c=0$ }

57. Form an equation whose roots are reciprocal of the roots of the equation $x^3+x+1=0$. {Ans. $x^3+x^2+1=0$ }

58. Find the quadratic equation whose roots are reciprocal of the roots of the equation $ax^2+bx+c=0$. {Ans. $cx^2+bx+a=0$ }

59. Form an equation whose roots are squares of the roots of the equation $x^3-6x^2+11x-6=0$. {Ans. $x^3-14x^2+49x-36=0$ }

60. Form an equation whose roots are cubes of the roots of the equation $ax^3+bx^2+cx+d=0$. {Ans. $a^3x^3+\left(3a^2d-3abc+b^3\right)x^2+\left(3ad^2-3bcd+c^3\right)x+d^3=0$ }

61. If α, β, γ are roots of $x^3+x+1=0$, then find the polynomial whose roots are $\dfrac{3}{\alpha^2+1}, \dfrac{3}{\beta^2+1}, \dfrac{3}{\gamma^2+1}$.

{Ans. $x^3+9x-27=0$ }

CATEGORY-8.4. NATURE OF ROOTS AND SIGN OF QUADRATIC FUNCTIONS

62. For what values of *a*, the roots of the equation $x^2 + a^2 = 8x + 6a$ are real. {Ans. $[-2,8]$}
63. For what values of *a* the function $x^2 - ax + 1$ has no real roots? {Ans. $a \in (-2,2)$}
64. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, then find the value of q. {Ans. $\frac{49}{4}$}
65. For what values of k, $x^2 - kx + k + 2$ has equal roots? {Ans. $2 \pm \sqrt{12}$}
66. Find the values of k for which the quadratic $(k+11)x^2 - (k+3)x + 1$ has real and equal roots. {Ans. $5,-7$}
67. If *a* and *b* are integers and $x^2 + ax + b = 0$ has discriminant as a perfect square then prove that its roots are integers.
68. Show that the expression $ax^2 + bx + c$ has always the same sign as c if $4ac > b^2$.
69. Find the values of *a* for which $(a^2 - 1)x^2 + 2(a-1)x + 2$ is positive for any x. {Ans. $a \in (-\infty,-3) \cup [1,\infty)$}
70. If the graph of the function $y = 16x^2 + 8(a+5)x - 7a - 5$ is strictly above the x-axis, then find *a*. {Ans. $-15 < a < -2$}
71. For what values of *m* the function $mx^2 - 9mx + 5m + 1$ is positive for all *x*? {Ans. $m \in [0, \frac{4}{61})$}
72. For what values of *m* the function $mx^2 - 9mx + 5m + 1$ is negative for all *x*? {Ans. $m \in \varnothing$}
73. Prove that the roots of $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$ are always real and they will be equal if and only if *a* = *b* = *c*.
74. Examine the nature of the roots of the quadratic $(b-x)^2 - 4(a-x)(c-x) = 0$ where *a*, *b*, *c* are real. {Ans. real}
75. Show that the equation $ax^2 + bx + c = 0$ where *a*, *b*, *c* are real numbers connected by the relation $4a + 2b + c = 0$ and $ab > 0$ has real roots.
76. If the roots of the equation $x^2 - ax + b = 0$ are real and differ by a quantity which is less than *c* (*c* > 0), then *b* lies between $\frac{a^2 - c^2}{4}$ and $\frac{a^2}{4}$.
77. If the two roots of the equation $(a-1)(x^2 + x + 1)^2 - (a+1)(x^4 + x^2 + 1) = 0$ are real and distinct then prove that *a* lies in the interval $(-\infty,-2) \cup (2,\infty)$.
78. Show that a polynomial of an odd degree has at least one real root.
79. Show that a polynomial of an even degree has at least two real roots if it attains at least one value opposite in sign to the coefficient of its highest degree term.

CATEGORY-8.5. COMMON ROOTS

80. For what value of a, $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$ have a common root? {Ans. 0, 24}
81. If $x^2 - ax - 21 = 0$ and $x^2 - 3ax + 35 = 0$ have a common root, then find the value of *a*. {Ans. ± 4}
82. Find *k* if the equations $4x^2 - 11x + 2k = 0$ and $x^2 - 3x - k = 0$ have a common root and obtain the common root for this value of *k*. {Ans. $k = 0 \text{ or } -\frac{17}{36}$, common root $0 \text{ or } \frac{17}{6}$}
83. If the equations $x^2 + 2x + 3\lambda = 0$ and $2x^2 + 3x + 5\lambda = 0$ have a non-zero common root, then find the value of λ. {Ans. -1}

84. If the equations $x^2+ax+b=0$ and $x^2+bx+a=0$ have a common root, then find the numerical value of $a+b$. {Ans. -1}

85. If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, then find the common root. {Ans. $-\left(\dfrac{q-q'}{p-p'}\right)$}

86. If α, β are the roots of $x^2+px+q=0$ and γ, δ are the roots of $x^2+rx+s=0$, evaluate $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$ in terms of p, q, r and s. Hence deduce the condition that the equations have a common root. {Ans. $q^2+s^2+p^2s+qr^2-prs-2qs-pqr$, $q^2+s^2+p^2s+qr^2-prs-2qs-pqr=0$}

87. If the equations $x^2+bx+ca=0$ and $x^2+cx+ab=0$ have a common root, then show that their other roots are the roots of the equation $x^2+ax+bc=0$.

88. Find the condition that a root of the equation $ax^2+bx+c=0$ be reciprocal of a root of the equation $a'x^2+b'x+c'=0$. {Ans. $(cc'-aa')^2=(ba'-cb')(ab'-bc')$}

89. If each pair of the three equations $x^2+p_1x+q_1=0$, $x^2+p_2x+q_2=0$ and $x^2+p_3x+q_3=0$ have a common root, then prove that $p_1^2+p_2^2+p_3^2+4(q_1+q_2+q_3)=2(p_1p_2+p_2p_3+p_3p_1)$

90. If every pair of equations $x^2+ax+bc=0$, $x^2+bx+ca=0$, $x^2+cx+ab=0$ have a common root, then find the sum and product of these common roots. {Ans. $-\dfrac{a+b+c}{2}, abc$}

91. If the three equations $x^2+ax+12=0$, $x^2+bx+15=0$ and $x^2+(a+b)x+36=0$ have a common positive root, then find a and b and the roots. {Ans. $a=-7, b=-8, 3,4; 3,5; 3,12$}

92. If the equations $ax^2+bx+c=0$ and $x^3+3x^2+3x+2=0$ have two common roots, then show that $a=b=c$.

CATEGORY-8.6. POSITION OF ROOTS

93. If the roots of the equation $(m-3)x^2-2mx+5m=0$ are real and positive then prove that $m\in\left[3,\dfrac{15}{4}\right]$.

94. For what values of a do both roots of the function $x^2-6ax+(2-2a+9a^2)$ exceed 3? {Ans. $a\in\left(\frac{11}{9},\infty\right)$}

95. For what values of a, the roots of the equation $x^2-2ax+a^2+a-3=0$ are real and less than 3? {Ans. $a<2$}

96. Find all the values of m for which both roots of the function $2x^2+mx+m^2-5$
   i. are less than 1. {Ans. $m\in\left[-\sqrt{\frac{40}{7}},\frac{-1-\sqrt{13}}{2}\right)\cup\left(\frac{-1+\sqrt{13}}{2},\sqrt{\frac{40}{7}}\right]$}
   ii. exceed –1. {Ans. $m\in\left[-\sqrt{\frac{40}{7}},\frac{1-\sqrt{13}}{2}\right)\cup\left(\frac{1+\sqrt{13}}{2},\sqrt{\frac{40}{7}}\right]$}

97. For what values of a does the function $x^2+2(a+1)x+9a-5$ has
   i. no real roots. {Ans. $a\in(1,6)$}
   ii. only negative roots. {Ans. $a\in\left(\frac{5}{9},1\right]\cup[6,\infty)$}
   iii. only positive roots. {Ans. $a\in\varnothing$}

98. If $b,c>0$, then show that roots of the equation $x^2+bx-c=0$ are of opposite sign.

99. Find the set of values of p for which the roots of the equation $3x^2+2x+p(p-1)=0$ are of opposite signs. {Ans. $(0,1)$}

100. If the roots of the equation $3x^2 + 2(k^2 + 1)x + (k^2 - 3k + 2) = 0$ be of opposite signs, then prove that $1 < k < 2$.
101. For what values of *a* does the function $2x^2 - (a^3 + 8a - 1)x + a^2 - 4a$ has the roots of opposite signs? {Ans. $a \in (0,4)$}
102. For what values of *a* does the function $(a^2 - a - 2)x^2 + 2ax + a^3 - 27$ has the roots of opposite signs? {Ans. $a \in (-\infty,-1) \cup (2,3)$}
103. For what values of *a* do both roots of the function $x^2 - ax + 2$ belong to the interval $[0,3]$? {Ans. $a \in \left[2\sqrt{2}, \frac{11}{3}\right]$}
104. For what values of *k* do both roots of the function $x^2 + 2(k-3)x + 9$ belong to the interval $(-6,1)$? {Ans. $k \in \left[6, \frac{27}{4}\right)$}
105. Find all the values of *k* for which one root of the function $x^2 - (k+1)x + k^2 + k - 8$ exceeds 2 and the other root is less than 2? {Ans. $k \in (-2,3)$}
106. For what values of *k*, one root of the function $(k-5)x^2 - 2kx + k - 4$ is smaller than 1 and the other root exceeds 2? {Ans. $k \in (5,24)$}
107. For what values of *a* does the function $x^2 + 2(a-1)x + a + 5$ has at least one positive root? {Ans. $a \in (-\infty,-1]$}
108. Let *a*, *b*, *c* be real. If $ax^2 + bx + c = 0$ has two real roots α and β, where $\alpha < -1$ and $\beta > 1$, then show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$.
109. Prove that the value of *a* for which $2x^2 - 2(2a+1)x + a(a+1) = 0$ may have one root less than *a* and the other root greater than *a*, are given by $a > 0$ or $a < -1$.
110. Find all values of the parameter *a* for which the roots of the function $x^2 + x + a$ are real and exceed *a*? {Ans. $a \in (-\infty,-2)$}
111. *a*, *b*, *c* are real numbers, $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then show that the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always lies between α and β.

CATEGORY-8.7. MISCELLANEOUS QUESTIONS ON QUADRATIC FUNCTIONS

112. For what values of *k*, the function $f(x) = kx^3 - 9x^2 + 9x + 3$ is monotonically increasing in every interval? {Ans. $k \geq 3$}
113. For what values of *k*, the function $f(x) = x^3 - 9kx^2 + 27x + 30$ is increasing on *R*? {Ans. $-1 \leq k \leq 1$}
114. Find the minimum value of $(x-a)(x-b)$. {Ans. $-\frac{(a-b)^2}{4}$}
115. If the function $f(x) = x^2 - kx + 5$ is increasing on $[2,4]$, then find the value of *k*. {Ans. $k \in (-\infty,4)$}
116. Find all numbers *a* for which the least value of the quadratic function $4x^2 - 4ax + a^2 - 2a + 2$ in the interval $[0,2]$ is equal to 3? {Ans. $a \in \left[1-\sqrt{2}\right] \cup \left[5+\sqrt{10}\right]$}
117. Find all real values of *m* for which the inequality $mx^2 + 4x + 3m + 1 > 0$ is satisfied for all positive *x*? {Ans. $m \in [0,\infty)$}
118. For what values of *a* does the inequality $4^x - a \cdot 2^x - a + 3 \leq 0$ has at least one solution? {Ans. $a \in [2,\infty)$}

119. For what values of *a* does the equation $2\log_3^2 x - |\log_3 x| + a = 0$ possess
   i. four solutions. {Ans. $a \in \left(0, \frac{1}{8}\right)$}
   ii. three solutions. {Ans. $a \in [0]$}
   iii. two solutions. {Ans. $a \in (-\infty, 0) \cup \left[\frac{1}{8}\right]$}
   iv. one solution. {Ans. $a \in \varnothing$}
   v. no solution. {Ans. $a \in \left(\frac{1}{8}, \infty\right)$}
120. Solve the equation $x^2 - |x| + a = 0$ for every real number *a*.
121. For what values of *a* does the equation $a \cdot 2^x + 2^{-x} = 5$ possess
   i. two solutions. {Ans. $a \in \left(0, \frac{25}{4}\right)$}
   ii. one solution. {Ans. $a \in (-\infty, 0] \cup \left[\frac{25}{4}\right]$}
   iii. no solution. {Ans. $a \in \left(\frac{25}{4}, \infty\right)$}

CATEGORY-8.8. RANGE OF RATIONAL FUNCTIONS

122. Find the range of the function $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1}$. {Ans. $\left[\frac{1}{3}, 3\right]$}
123. Find the range of the function $f(x) = \dfrac{x + 1}{x^2 + x + 1}$. {Ans. $\left[-\frac{1}{3}, 1\right]$}
124. Find the range of the function $\dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$. {Ans. $\left[\dfrac{1}{3}, 3\right]$}
125. For real values of *x*, prove that the value of the expression $\dfrac{11x^2 + 12x + 6}{x^2 + 4x + 2}$ cannot lie between -5 and 3.
126. If *x* be real, prove that the expression $\dfrac{x + 2}{2x^2 + 3x + 6}$ takes all values in the interval $\left[-\dfrac{1}{13}, \dfrac{1}{3}\right]$.
127. Prove that for real values of *x* the expression $\dfrac{(x-1)(x+3)}{(x-2)(x+4)}$ cannot lie between $\dfrac{4}{9}$ and 1.
128. If *x* is real, show that the expression $\dfrac{x^2 + 2x - 11}{x - 3}$ takes all values which do not lie between 4 and 12.
129. If *x* is real, find the maximum and minimum values of $\dfrac{x^2 + 14x + 9}{x^2 + 2x + 3}$. {Ans. 4, -5}
130. Show that $\dfrac{x^2 - 3x + 4}{x^2 + 3x + 4}$ can never be greater than 7 nor less than $\dfrac{1}{7}$ for real values of *x*.
131. Show that the expression $\dfrac{px^2 + 3x - 4}{p + 3x - 4x^2}$ will be capable of all real values when *x* is real, provided that *p* has any value between 1 and 7.
132. For what values of *a* is the inequality $\dfrac{x^2 + ax - 1}{2x^2 - 2x + 3} < 1$ fulfilled for all *x*? {Ans. $a \in (-6, 2)$}
133. For what values of *a* is the inequality $-6 < \dfrac{2x^2 + ax - 4}{x^2 - x + 1} < 4$ fulfilled for all *x*? {Ans. $a \in (-2, 4)$}

CATEGORY-8.9. ADDITIONAL QUESTIONS

134. Find the value of $\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}$. {Ans. 2}

135. If $f(x)$ is a polynomial function satisfying $f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$ and $f(3)=28$, then find $f(4)$. {Ans. 65}

136. If $l, n \in R$, $p, q > 0$ and the roots of the equation $lx^2+nx+n=0$ be real and the ratio of the roots be $p:q$, then prove that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}-\sqrt{\frac{n}{l}}=0$.

137. A function $f: R \to R$ is defined by $f(x)=\frac{\alpha x^2+6x-8}{\alpha+6x-8x^2}$, find the interval of values of α for which f is onto. Is the function one-to-one for $\alpha=3$? Justify your answer. {Ans. $\alpha \in [2,14]$, $f(x)$ is not one-to-one for $\alpha=3$}

138. Let c be a fixed real number. Show that a root of the polynomial $P(x)=x(x+1)(x+2)\cdot\cdots\cdot(x+2009)-c$ can have multiplicity at most 2.

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Mathematics for IIT-JEE

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PART-II
ALGEBRA

CHAPTER-9
PROGRESSIONS

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CHAPTER-9
PROGRESSIONS

LIST OF THEORY SECTIONS

9.1. Sequence And Series
9.2. Arithmetic Progressions
9.3. Geometric Progressions
9.4. Harmonic Progressions
9.5. Sum Of Standard Series

LIST OF QUESTION CATEGORIES

9.1. Arithmetic Progression
9.2. Arithmetic Series
9.3. Arithmetic Mean
9.4. Geometric Progression
9.5. Geometric Series
9.6. Geometric Mean
9.7. Harmonic Progression And Harmonic Series
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9.9. Polynomial Series
9.10. Difference Of Consecutive Terms In A.P.
9.11. Difference Of Consecutive Terms In G.P.
9.12. Arithmetic-Geometric Series
9.13. Method Of Difference
9.14. Exponential And Logarithmic Series
9.15. Additional Questions

CHAPTER-9
PROGRESSIONS

SECTION-9.1. SEQUENCE AND SERIES

1. Sequence

i. A sequence is an ordered arrangement of numbers (real/complex) according to a definite rule.

ii. Each number in the sequence is called a *term*.

2. Notation and representation of a sequence

i. A sequence is written as $a_1, a_2, a_3, \cdots\cdots\cdots, a_n, \cdots\cdots$, which is also denoted as $\{a_n\}$.

ii. The number a_n is called the *n*th term of the sequence.

3. Ways of defining sequences

i. By means of formula for its general term, i.e. $a_n = f(n)$.

ii. By a recursive relationship, i.e. a formula which expresses a_n in terms of some preceding terms of the sequence.

iii. By other ways.

4. Series

i. If $a_1, a_2, a_3, \cdots\cdots\cdots, a_n, \cdots\cdots$ is a sequence, then $a_1 + a_2 + a_3 + \cdots\cdots\cdots + a_n + \cdots\cdots$ is a *series*, generally denoted by *S*.

5. Partial sum of a series

i. If there is a series $S = a_1 + a_2 + a_3 + \cdots\cdots\cdots + a_n + \cdots\cdots$, then $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, ..., $S_n = a_1 + a_2 + a_3 + \cdots\cdots\cdots + a_n$, ... are called the partial sums of the series *S*.

6. Convergent and divergent series

i. Given a series *S* and its partial sum S_n, if $\lim_{n\to\infty} S_n = L$ exists, then the series is said to be *convergent* and *L* is its sum, otherwise the series is said to be *divergent*.

7. Sigma (Σ) notation, Pi (Π) notation and their properties

i. If *m* and *n* $(n > m)$ are given integers and $f(i)$ is an expression containing *i*, then the symbol $\sum_{i=m}^{n} f(i)$ means $f(m) + f(m+1) + f(m+2) + \cdots\cdots + f(n)$.

ii. $\sum_{i=m}^{n}\left(f(i) + g(i)\right) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$.

iii. $\sum_{i=m}^{n} kf(i) = k\sum_{i=m}^{n} f(i)$.

iv. $\sum_{i=m}^{n} k = (n - m + 1)k$.

v. If *m* and *n* $(n > m)$ are given integers and $f(i)$ is an expression containing *i*, then the symbol $\prod_{i=m}^{n} f(i)$ means $f(m) \cdot f(m+1) \cdot f(m+2) \cdots\cdots\cdots f(n)$.

vi. $\prod_{i=m}^{n}\left(f(i) \cdot g(i)\right) = \prod_{i=m}^{n} f(i) \cdot \prod_{i=m}^{n} g(i)$.

vii. $\prod_{i=m}^{n} kf(i) = k^{n-m+1} \prod_{i=m}^{n} f(i)$.

viii. $\prod_{i=m}^{n} k = k^{n-m+1}$.

SECTION-9.2. ARITHMETIC PROGRESSIONS

1. Definition of Arithmetic Progression (A.P.) and General Term of an A.P.

i. A sequence is called an Arithmetic Progression if the difference of its two consecutive terms is a constant.

ii. Terms of an A.P. are of the form $a, a+d, a+2d, a+3d, \cdots\cdots\cdots\cdots$.

iii. If $\{a_n\}$ is an A.P., then $a_n = a+(n-1)d$, where *a* and *d* are parameters and *a* is called the first term and *d* is called the common difference.

2. Properties of A.P.

i. If *a*, *b*, *c* are in A.P. then $2b = a+c$.

ii. If a constant *k* is added to all the terms of a given A.P. with first term *a* and common difference *d*, then the resulting sequence is also an A.P. with the first term $a+k$ and the same common difference *d*.

iii. If a constant *k* is multiplied to all the terms of a given A.P. with first term *a* and common difference *d*, then the resulting sequence is also an A.P. with the first term ak and the common difference dk .

3. Arithmetic series, partial sum of Arithmetic series and its convergence/ divergence

i. The sum of the terms of an A.P. is called an Arithmetic series.

ii. If $\{a_n\}$ is an A.P. with first term *a* and common difference *d* and the Arithmetic series

$S = a_1 + a_2 + a_3 + \cdots\cdots\cdots + a_n + \cdots\cdots$, then

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$= \frac{n}{2}\left[a_1 + a_n\right].$$

iii. All Arithmetic series diverge except when $a = d = 0$.

4. Arithmetic mean(s) between two numbers

i. When three numbers are in A.P., then the middle number is the arithmetic mean of the other two numbers.

ii. If *a* and *b* are two numbers and *A* is their arithmetic mean then *a*, *A*, *b* are in A.P., hence

$$b - A = A - a \Rightarrow A = \frac{a+b}{2}.$$

iii. A given number of arithmetic means can be inserted between two given numbers. Given two numbers *a* and *b* and *n* the number of means, let A_1, A_2, ..., A_n be the *n* arithmetic means such that

$a, A_1, A_2, \cdots\cdots, A_n, b$ are in A.P.. Then $A_i = a + i\left(\frac{b-a}{n+1}\right), i = 1, 2, \cdots\cdots, n$.

5. Arithmetic mean of two or more numbers

i. If $a_1, a_2, a_3, \cdots\cdots\cdots, a_n$ are *n* numbers then their arithmetic mean is $\frac{a_1 + a_2 + a_3 + \cdots\cdots\cdots + a_n}{n}$.

SECTION-9.3. GEOMETRIC PROGRESSIONS

1. Definition of Geometric Progression (G.P.) and General Term of a G.P.

i. A sequence of non-zero numbers is called a Geometric Progression if the ratio of its two consecutive terms is a constant.

ii. Terms of a G.P. are of the form $a, ar, ar^2, ar^3, \cdots\cdots\cdots\cdots$.

iii. If $\{a_n\}$ is a G.P., then $a_n = ar^{n-1}$, where *a* and *r* are non-zero parameters and *a* is called the first term and *r* is called the common ratio.

2. Properties of G.P.

i. If *a*, *b*, *c* are in G.P. then $b^2 = ac$.

ii. If a constant *k* is multiplied to all the terms of a given G.P. with first term *a* and common ratio *r*, then the resulting sequence is also a G.P. with the first term ak and the same common ratio *r*.

iii. If all the terms of a given G.P. with first term *a* and common ratio *r*, are raised to a constant power *k* then the resulting sequence is also a G.P. with the first term a^k and the common ratio r^k.

3. Geometric series, partial sum of Geometric series and its convergence/ divergence

i. The sum of the terms of an G.P. is called an Geometric series.

ii. If $\{a_n\}$ is an G.P. with first term *a* and common *r* and the Geometric series

$S = a_1 + a_2 + a_3 + \cdots\cdots\cdots + a_n + \cdots\cdots$, then

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right) = a\left(\frac{1 - r^n}{1 - r}\right), \quad r \neq 1$$

$$= na, \quad r = 1.$$

iii. When $|r| < 1$ then the Geometric series converges and $S = \frac{a}{1-r}$. When $|r| \geq 1$ then the Geometric series diverges.

4. Geometric mean(s) between two positive numbers

i. When three positive numbers are in G.P., then the middle number is the geometric mean of the other two numbers.

ii. If *a* and *b* are two numbers and *G* is their geometric mean then *a*, *G*, *b* are in G.P., hence

$$\frac{b}{G} = \frac{G}{a} \Rightarrow G = \sqrt{ab}.$$

iii. A given number of geometric means can be inserted between two given numbers. Given two numbers *a* and *b* and *n* the number of means, let G_1, G_2, ..., G_n be the *n* arithmetic means such that

$a, G_1, G_2, \cdots\cdots, G_n, b$ are in G.P.. Then $G_i = a\left(\frac{b}{a}\right)^{\frac{i}{n+1}}, i = 1, 2, \cdots\cdots, n$.

5. Geometric mean of two or more positive numbers

i. If $a_1, a_2, a_3, \cdots\cdots\cdots, a_n$ are *n* numbers then their geometric mean is $\sqrt[n]{a_1 a_2 a_3 \cdots\cdots\cdots a_n}$.

SECTION-9.4. HARMONIC PROGRESSIONS

1. Definition of Harmonic Progression (H.P.) and General Term of a H.P.

i. A sequence of is called a Harmonic Progression if the reciprocal of its terms are in A.P.

ii. Terms of a H.P. are of the form $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \cdots\cdots\cdots\cdots$.

iii. If $\{a_n\}$ is a H.P., then $a_n = \dfrac{1}{a+(n-1)d}$, where a and d are parameters such that $a+(n-1)d \neq 0 \ \forall n \in N$.

2. **Properties of H.P.**

   i. If a, b, c are in H.P. then $b = \dfrac{2ac}{a+c}$.

   ii. If a constant k is multiplied to all the terms of a given H.P. then the resulting sequence is also a H.P..

3. **Harmonic series, partial sum of Harmonic series and its convergence/ divergence**

   i. The sum of the terms of a H.P. is called a Harmonic series.

   ii. If $\{a_n\}$ is a H.P. and the Harmonic series $S = a_1 + a_2 + a_3 + \cdots\cdots\cdots + a_n + \cdots\cdots$, then there is no formula for S_n.

   iii. All Harmonic series diverge.

4. **Harmonic mean(s) between two numbers**

   i. When three numbers are in H.P., then the middle number is the harmonic mean of the other two numbers.

   ii. If a and b are two numbers and H is their harmonic mean then a, H, b are in H.P., hence $\dfrac{1}{a}, \dfrac{1}{H}, \dfrac{1}{b}$ are in A.P. therefore $H = \dfrac{2ab}{a+b}$.

   iii. A given number of harmonic means can be inserted between two given numbers. Given two numbers a and b and n the number of means, let H_1, H_2, ..., H_n be the n harmonic means such that $a, H_1, H_2, \cdots\cdots, H_n, b$ are in H.P.. Then $\dfrac{1}{H_i} = \dfrac{1}{a} + i\left(\dfrac{a-b}{(n+1)ab}\right), i = 1, 2, \cdots\cdots, n$.

5. **Harmonic mean of two or more numbers**

   i. If $a_1, a_2, a_3, \cdots\cdots\cdots, a_n$ are n numbers then their harmonic mean is $\dfrac{n}{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\cdots\cdots+\dfrac{1}{a_n}}$.

   ii. For two positive numbers, $AH = G^2$.

   iii. For two or more positive numbers, $A \geq G \geq H$.

SECTION-9.5. SUM OF STANDARD SERIES

1. **Polynomial series**

   i. Sum of first n natural numbers = $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$.

   ii. Sum of the squares of first n natural numbers = $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$.

   iii. Sum of the cubes of first n natural numbers = $\sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$.

2. **Difference of consecutive terms is in A.P.**
3. **Difference of consecutive terms is in G.P.**
4. **Arithmetic-Geometric series**
5. **Method of difference**

6. Exponential and logarithmic series

i. Standard exponential series:-

a. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\cdots + \frac{x^n}{n!} + \cdots\cdots\cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

b. $e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\cdots\cdots + \frac{x^{2n}}{(2n)!} + \cdots\cdots\cdots\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

c. $e^x - e^{-x} = 2\left(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots\cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots\cdots\cdots\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

d. $a^x = 1 + \frac{x \ln a}{1!} + \frac{x^2 (\ln a)^2}{2!} + \frac{x^3 (\ln a)^3}{3!} + \cdots\cdots\cdots + \frac{x^n (\ln a)^n}{n!} + \cdots\cdots\cdots = \sum_{n=0}^{\infty} \frac{x^n (\ln a)^n}{n!}$.

ii. Standard logarithmic series:-

a. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots\cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots\cdots\cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad -1 < x \le 1$.

b. $\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\cdots\cdots + \frac{x^{2n-1}}{2n-1} + \cdots\cdots\cdots\right) = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}, \quad -1 < x < 1$.

iii. Sum of certain infinite series can be obtained by expressing it in terms of standard exponential and logarithmic series.

EXERCISE-9

CATEGORY-9.1. ARITHMETIC PROGRESSION

1. If 7^{th} and 13^{th} terms of an AP be 34 and 64 respectively, then find its 18^{th} term. {Ans. 89}
2. Divide 28 into four parts in A.P. so that ratio of the product of first and third with the product of second and fourth is 8:15. {Ans. 4, 6, 8, 10}
3. If x, a, b, c are real and $(x-a+b)^2+(x-b+c)^2=0$, then show that a, b, c are in AP.
4. Prove that if *p, q, r* (*p*≠*q*) are terms (not necessarily consecutive) of an A.P., than there exists a rational number *k* such that $\frac{(r-q)}{(q-p)}=k$.
5. Prove that the numbers $\sqrt{2}, \sqrt{3}, \sqrt{5}$ cannot be the terms of a single A.P. with non-zero common difference.
6. Find the number of terms common to the two A.P.'s 3, 7, 11, ..., 407 and 2, 9, 16, ..., 709. {Ans. 14}
7. Prove that there are 17 identical terms in the two A.P.'s 2, 5, 8, 11, ...60 terms and 3, 5, 7, 9, ...50 terms.
8. If the roots of the equation $x^3-12x^2+39x-28=0$ are in A.P., then find their common difference. {Ans. ± 3}
9. The *m*th term of an A. P. is *n* and its *n*th term is *m*. Prove that its *p*th term is $m+n+p$. Also show that its $(m+n)$th term is zero.
10. If the *p*th , *q*th and *r*th terms of an A.P. be *a, b* and *c* respectively, then prove that $a(q-r)+b(r-p)+c(p-q)=0.$
11. If *a, b, c* are in A.P., then prove that $(a-c)^2=4(b^2-ac).$
12. If *a, b, c* are in A.P., prove that the following are also in A.P.:-
    i. $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}.$
    ii. $b+c, c+a, a+b.$
    iii. $a^2(b+c), b^2(c+a), c^2(a+b).$
    iv. $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right).$
    v. $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}.$
13. If a^2, b^2, c^2 are in A.P., then the following are also in A.P.:-
    i. $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}.$
    ii. $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}.$
14. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.
15. If $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P. then prove that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are also in A.P.
16. If $\log 2$, $\log(2^x-1)$ and $\log(2^x+3)$ be three consecutive terms of an A.P., then find *x*. {Ans. $\log_2 5$}
17. For what values of the parameter *a* are there values of *x* such that $5^{1+x}+5^{1-x}, \frac{a}{2}, 25^x+25^{-x}$ are three

consecutive terms of an A.P.? {Ans. $[12,\infty)$ }

18. If $x^{18}=y^{21}=z^{28}$, prove that 3, $3\log_y x$, $3\log_z y$, $7\log_x z$ form an A.P.

19. Each of the two triplets of numbers $\log a$, $\log b$, $\log c$ and $\log a-\log 2b$, $\log 2b-\log 3c$, $\log 3c-\log a$ are in A.P. Can the numbers *a, b, c* be the lengths of the sides of a triangle? {Ans. yes}

CATEGORY-9.2. ARITHMETIC SERIES

20. The fifth term of an A.P. is 1 whereas its 31st term is -77. Find its 20th term and sum of its first fifteen terms. Also find which term of the series will be -17 and sum of how many terms will be 20. {Ans. -44, -120, 11, 8}
21. The third term of an A.P. is 7 and its 7th term is 2 more than thrice of its 3rd term. Find the first term, common difference and the sum of its first 20 terms. {Ans. -1, 4, 740}
22. Find the number of terms in the series $20+19\frac{1}{3}+18\frac{2}{3}+\cdots\cdots$ of which the sum is 300. Explain the double answer. Also find the maximum sum of the series. {Ans. 25 or 36; 310}
23. How many terms of the sequence 54, 51, 48, ... be taken so that their sum is 513? Explain the double answer. {Ans. 18 or 19}
24. If the sum of three consecutive terms of an increasing AP is 51 and the product of the first and third of these terms is 273, then find the third term. {Ans. 21}
25. Find the sum of all 2 digit odd numbers. {Ans. 2475}
26. If the sum of the sequence 2, 5, 8, 11, ... is 60100, then find the number of terms. {Ans. 200}
27. The *n*th term of a series is given to be $\frac{3+n}{4}$, find the sum of 105 terms of this series. {Ans. 1470}
28. Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots\ldots$ if it is known that $a_1+a_5+a_{10}+a_{15}+a_{20}+a_{24}=225$. {Ans. 900}
29. Find $a_1+a_6+a_{11}+a_{16}$ if it is known that $a_1, a_2, \ldots$ is an A.P. and $a_1+a_4+a_7+\ldots+a_{16}=147$. {Ans. 98}
30. The first and last terms of an AP are 1 and 11. If the sum of its terms is 36, then find the number of terms. {Ans. 6}
31. If the sum of first 8 and 19 terms of an A.P. are 64 and 361 respectively, find the common difference and sum of its *n* terms. {Ans. 2, n^2}
32. A man arranges to pay off a debt of Rs. 3600 in 40 annual installments, which form an arithmetic series. When 30 of the installments are paid he dies leaving one-third of the debt unpaid. Find the value of the first installment. {Ans. Rs. 51}
33. A class consists of a number of boys whose ages are in A.P., the common difference being 4 months. If the youngest boy is just eight years old and if the sum of the ages is 168 years, find the number of boys. {Ans. 16}
34. The sum of *n* term of a series is $3n^2+4n$. Show that the series is an A.P. and find the first term and common difference. What will be its *n*th term? {Ans. 7, 6, 6*n*+1}
35. If the sum of n terms of an AP be $3n^2-n$, then find its first term and common difference. {Ans. 2, 6}
36. If the sum of n terms of an AP is $2n^2+5n$, then show that its nth term is $4n+3$.
37. Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3. {Ans. 156375}
38. If $\frac{3+5+7+\ldots\ldots+n\ terms}{5+8+11+\ldots\ldots+10\ terms}=7$, then find the value of n. {Ans. 35}

39. Show that the sum of all odd numbers between 1 and 1000, which are divisible by 3, is 83667.
40. Find the sum of first n odd natural numbers. {Ans. n^2}
41. Find the sum of all odd integers between 2 and 100 divisible by 3. {Ans. 867}
42. Find the sum of all two digit numbers which when divided by 4, yield unity as remainder. {Ans. 1210}
43. Find the sum of integers from 1 to 100 which are divisible by 2 or 5. {Ans. 3050}
44. The sum of n terms of the two series $3+10+17+\ldots\ldots$ and $63+65+67+\ldots\ldots$ are equal, then find the value of n. {Ans. 25}
45. The series of natural numbers is divided into groups (1), (2, 3, 4), (3, 4, 5, 6, 7), (4, 5, 6, 7, 8, 9, 10), ... Find the sum of the numbers in nth group. {Ans. $(2n-1)^2$ }
46. The series of natural numbers is dividend into groups (1); (2, 3, 4); (5, 6, 7, 8, 9); ...... and so on . Show that the sum of the numbers in the nth group is $(n-1)^3+n^3$.
47. N, the set of natural numbers, is partitioned into subsets $S_1 = \{1\}$, $S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}$, $S_4 = \{7, 8, 9, 10\}$. Find the sum of the elements in the subset S_{50}. {Ans. 62525}
48. Show that sum of the terms in the nth bracket (1); (3, 5); (7, 9, 11); .... is n^3.
49. The sum of three numbers in A.P. is 15 whereas sum of their squares is 83. Find the numbers. {Ans. 3, 5, 7 or 7, 5, 3}
50. The sum of three numbers in A.P. is 12 and the sum of their cubes is 288. Find the numbers. {Ans. 2, 4, 6 or 6, 4, 2}
51. Find four numbers in A.P. whose sum is 20 and sum of their squares is 120. {Ans. 2, 4, 6, 8 or 8, 6, 4, 2}
52. Find four numbers in A.P. whose sum is 32 and sum of squares is 276. {Ans. 5, 7, 9, 11 or 11, 9, 7, 5}
53. The number of terms of an A.P. is even; the sum of the odd terms is 24, of the even terms 30, and the last term exceeds the first by $10\frac{1}{2}$, find the number of terms and the series. {Ans. 8 terms, $\frac{3}{2}, 3, \frac{9}{2}, \ldots$ }
54. Find an A.P. in which sum of any number of terms is always three times the squared number of these terms. {Ans. 3, 9, 15, ...}
55. If $S_n = an^2 + bn$, prove that series is an A.P.
56. Sum of certain consecutive odd positive integers is $57^2 - 13^2$. Find them.
{Ans.
1539, 1541
767, 769, 771, 773
299, 301,......10 numbers
207, 209, ...14 numbers
135, 137, ...20 numbers
119, 121, ...22 numbers
83, 85, ...28 numbers
27, 29, ...44 numbers.}
57. 150 workers were engaged to finish a piece of work in a certain number of days. Four workers dropped the second day, four more workers dropped the third day and so on. It takes 8 more days to finish the work now. Find the number of days in which the work was completed. {Ans. 25}
58. Along a road lie an odd number of stones placed at intervals of 10 meters. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried the job with one of the end stones by carrying them in succession. In carrying all the stones he covered a distance of 3 km. Find the number of stones. {Ans. 25}
59. Balls are arranged in rows to from an equilateral triangle. The first row consists of one ball, the second row

of two balls and so on. If 669 more balls are added then all the ball can be arranged in the shape of a square and each of the sides then contains 8 balls less then each side of the triangle did. Determine the initial numbers of balls. {Ans. 1540}

60. Certain numbers appear in both arithmetic progressions 17, 21, 25, ... and 16, 21, 26, ... Find the sum of first hundred numbers appearing in both progressions. {Ans. 101100}

61. Let S_n denote the sum of first n terms of an A.P. If $S_{2n} = 3S_n$, then find the ratio $\frac{S_{3n}}{S_n}$. {Ans. 6}

62. If $S_n = Qn^2 + Pn$, where S_n denotes the sum of the first n terms of an A.P., then show that the third term is $P + 5Q$.

63. The first and last term of an A.P. are a and l respectively. If S be the sum of all the term of the A.P., show that the common difference is $\frac{l^2 - a^2}{2S - (l + a)}$.

64. Show that the sum of an A.P. whose first term is a, second term is b and the last term is c is equal to $\frac{(a + c)(b + c - 2a)}{2(b - a)}$.

65. If the ratio of the sum of m term and n terms of an A.P. be $m^2 : n^2$, prove that the ratio of its mth and nth terms will be $2m - 1 : 2n - 1$.

66. The ratio between the sum of n term of two A.P.'s is $3n + 8 : 7n + 15$. Find the ratio between their 12th terms. Also find the ratio of their common difference. {Ans. $\frac{7}{16}$, $\frac{3}{7}$}

67. The ratio between the sum of n term of two A.P.'s is $7n + 1 : 4n + 27$. Find the ratio between their nth terms. {Ans. $\frac{14n - 6}{8n + 23}$}

68. The sum of the first n term of two A.P.'s are as $3n + 5 : 5n - 9$. Prove that their 4th terms are equal.

69. The pth term of an A.P. is a and qth term is b. Prove that sum of its $(p + q)$ terms is $\frac{p + q}{2}\left[a + b + \frac{a - b}{p - q}\right]$.

70. If the sums of p, q and r terms of an A.P. be a, b and c respectively then prove that $\frac{a}{p}(q - r) + \frac{b}{q}(r - p) + \frac{c}{r}(p - q) = 0$.

71. If in an A.P the sum of p terms is equal to sum of q terms, then prove that the sum of $p + q$ terms is zero.

72. In an A.P., of which a is the first term, if the sum of the first p terms is zero, show that the sum of the next q terms is $-\frac{a(p + q)\, q}{p - 1}$.

73. The sum of first p terms of an A.P. is q and the sum of the first q terms is p. Find the sum of the first $p + q$ terms. {Ans. $-(p + q)$}

74. Prove that the sum of the latter half of $2n$ terms of any A.P. is one-third the sum of $3n$ terms of the same A.P.

75. The sums of n terms of three arithmetical progressions are S_1, S_2 and S_3. The first term of each is unity and the common differences are 1, 2 and 3 respectively. Prove that $S_1 + S_3 = 2S_2$.

76. The sums of n, $2n$, $3n$ terms of an A.P. are S_1, S_2, S_3 respectively. Prove that $S_3 = 3(S_2 - S_1)$.

77. If $S_n = n^2 p$ and $S_m = m^2 p$, $m \neq n$, in an A.P., prove that $S_p = p^3$.

78. There are n A.P.'s whose common difference are 1, 2, 3, ... n respectively, the first term of each being unity. Prove that sum of their nth terms is $\frac{1}{2}n\left(n^2+1\right)$.

79. If there be m A.P.'s beginning with unity whose common differences are 1, 2, 3, ... m respectively, show that the sum of their nth terms is $\frac{1}{2}m[mn-m+n+1]$.

80. The sum of n terms of m arithmetical progressions are S_1, S_2, S_3, ... S_m. The first term and common differences are 1, 2, 3, ..., m respectively. Prove that $S_1+S_2+S_3+\ldots+S_m=\frac{1}{4}mn\left(m+1\right)\left(n+1\right)$.

81. If S_1, S_2, S_3,...., S_m are the sums of n term of m A.P.'s whose first terms are 1, 2, 3, ..., m and common differences are 1, 3, 5, ..., $2m-1$ respectively. Show that $S_1+S_2+S_3+\ldots+S_m=\frac{1}{2}mn\left(mn+1\right)$.

82. If the sum of m terms of an A.P. is equal to sum of either the next n terms or the next p terms, prove that $\left(m+n\right)\left(\frac{1}{m}-\frac{1}{p}\right)=\left(m+p\right)\left(\frac{1}{m}-\frac{1}{n}\right)$.

83. Show that in arithmetical progression a_1, a_2, a_3, ..., $a_1^{\,2}-a_2^{\,2}+a_3^{\,2}-a_4^{\,2}+\ldots-a_{2k}^{\,2}=\frac{k}{2k-1}\left(a_1^{\,2}-a_{2k}^{\,2}\right)$.

84. If a_1, a_2, a_3, ..., a_n be an A.P. of non-zero terms prove that

i. $\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\frac{1}{a_3a_{n-2}}+\ldots+\frac{1}{a_na_1}=\frac{2}{a_1+a_n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$;

ii. $\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots+\frac{1}{a_{n-1}a_n}=\frac{n-1}{a_1a_n}$.

85. If a_1, a_2, a_3, ..., a_n are in A.P. where $a_i>0$ for all i, show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}.$$

86. Let the sequence a_1, a_2, a_3, ..., a_n from an A.P. and let $a_1=0$, prove that

$$\frac{a_3}{a_2}+\frac{a_4}{a_3}+\frac{a_5}{a_4}+\ldots+\frac{a_n}{a_{n-1}}-a_2\left(\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_{n-2}}\right)=\frac{a_{n-1}}{a_2}+\frac{a_2}{a_{n-1}}.$$

CATEGORY-9.3. ARITHMETIC MEAN

87. Find fifth of the ten arithmetic means inserted between 1 and 100. {Ans. 46}

88. If a, b, c, d, e, f are AM's between 2 and 12, then find the value of $a+b+c+d+e+f$. {Ans. 42}

89. The sum of two numbers is $2\frac{1}{6}$. An even number of arithmetic means are being inserted between them and their sum exceeds their number by 1. Find the number of means inserted. {Ans. 12}

90. Between 1 and 31 are inserted m arithmetic means so that the ratio of the 7th and $\left(m-1\right)$th means is 5:9. Find the value of m. {Ans. 14}

91. Prove that the sum of the n arithmetic means inserted between two quantities is n times the single arithmetic mean between them.

92. For what value of n, $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the arithmetic mean of a and b? {Ans. 0}

CATEGORY-9.4. GEOMETRIC PROGRESSION

93. If $x, 2x+2, 3x+3$ are in GP, then find the fourth term. {Ans. –13.5}
94. Find three numbers in G.P. whose sum is 65 and whose product is 3375. {Ans. 45, 15, 5 or 5, 15, 45}
95. The product of three numbers in G.P. is 125 and sum of their products taken in pairs is $87\frac{1}{2}$. Find them.
    {Ans. 10, 5, $\frac{5}{2}$ or $\frac{5}{2}$, 5, 10}
96. If the product of three numbers in G.P. is 216 and sum of the products taken in pairs is 156, find the numbers. {Ans. 18, 6, 2 or 2, 6, 18}
97. Three numbers are in G.P. whose sum is 70. If the extremes be each multiplied by 4 and the mean by 5, they will be in A.P. Find the numbers. {Ans. 10, 20, 40 or 40, 20, 10}
98. The sum of three numbers in G.P. is 14. If the first two terms are each increased by 1 and the third term decreased by 1, the resulting numbers are in A.P. Find the numbers. {Ans. 2, 4, 8 or 8, 4, 2}
99. Three numbers whose sum is 15 are in A.P. If 1, 4, 19 be added to them respectively, then they are in G.P. Find the numbers. {Ans. 26, 5, -16 or 2, 5, 8}
100. In a set of four numbers the first three are in G.P. and the last three in A.P. with common difference 6. If the first number is the same as the fourth, find the four numbers. {Ans. 8, -4, 2, 8}
101. Find four numbers in G.P. whose sum is 85 and product is 4096. {Ans. 1, 4, 16, 64 or 64, 16, 4, 1}
102. Does there exist a geometric progression containing 27, 8 and 12 as three of its terms? If it exists, how many such progressions are possible? {Ans. yes, infinite}
103. Show that the numbers 10, 11, 12 cannot be the terms of a single G.P. with common ratio not equal to 1.
104. The third term of a G.P. is 4. Find the product of first five terms. {Ans. 4^5 }
105. In a G.P. the first, third and fifth terms may be considered as the first, fourth and sixteenth terms of an A.P. Determine the fourth term of the A.P., knowing that its first term is 5 and determine T_1, T_3, T_5 of G.P. {Ans. 20, 5, 20, 80}
106. The sum of first ten terms of an A.P. is equal to 155, and sum of the first two terms of a G.P. is 9, find these progressions if the first term of A.P. is equal to common ratio of G.P. and the first term of G.P. is equal to common difference of A.P. {Ans. 2, 5, 8, 11, ... & 3, 6, 12, 24,..... or $\frac{25}{2},\frac{79}{6},\frac{83}{6},...$ &
    $\frac{2}{3},\frac{25}{3},\frac{625}{6},...$ }
107. Find the three numbers constituting a G.P. if it is known that the sum of the numbers is equal to 26 and that when 1, 6 and 3 are added to them respectively, the new numbers are obtained which from an A.P. {Ans. 2, 6, 18 or 18, 6, 2}
108. Three numbers from a G.P. If the 3rd term is decreased by 64, then the three numbers thus obtained will constitute an A.P. If the second term of this A.P. is decreased by 8, a G.P. will be formed again. Determine the numbers. {Ans. 4, 20, 100 or $\frac{4}{9},\frac{52}{9},\frac{676}{9}$ }
109. Find the numbers a, b, c between 2 and 18 such that (i) their sum is 25 (ii) the numbers 2, a, b are consecutive terms of an A.P and (iii) the numbers b, c, 18 are consecutive terms of a G.P. {Ans. 5, 8, 12}
110. In a G.P. if the $(m+n)$th term be p and $(m-n)$th term be q, then prove that its mth term is $\sqrt{pq}$.
111. The first and second terms of a GP are x^{-4} and x^n respectively. If x^{52} is the eight term of the same

progression, then find n. {Ans. 4}

112. If p, q, r are in AP, then show that pth, qth and rth terms of any GP are in GP.
113. If a, b, c are in AP, $b-a$, $c-b$ and a are in GP, then find $a:b:c$. {Ans. 1:2:3}
114. If the roots of $(a^2+b^2)x^2-2b(a+c)x+(b^2+c^2)$ are equal then show that a, b, c are in GP.
115. Let $\{a_n\}$ be a GP such that $\frac{a_4}{a_6}=\frac{1}{4}$ and $a_2+a_5=216$. Then find a_1. {Ans. 12 or $\frac{108}{7}$}
116. The rth, sth and tth terms of a certain G.P. are R, S and T respectively. Prove that $R^{s-t}\cdot S^{t-r}\cdot T^{r-s}=1$.
117. The fourth, seventh and tenth terms of a GP are p, q, r respectively, then show that $q^2=pr$.
118. If x, y, z be respectively the pth, qth and rth terms of a G.P., then prove that $(q-r)\log x+(r-p)\log y+(p-q)\log z=0$.
119. If the pth, qth, rth terms of an A.P. are in G.P. show that common ratio of the G.P. is $\frac{q-r}{p-q}$.
120. If a, b, c, d be in G.P., prove that
    i. $(a^2+ac+c^2)(b^2+bd+d^2)=(ab+bc+cd)^2$;
    ii. $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.
121. If a, b, c be in G.P., then prove that $\frac{a^2+ab+b^2}{bc+ca+ab}=\frac{b+a}{c+b}$.
122. If a, b, c are three distinct real numbers in G.P. and $a+b+c=xb$, then prove that either $x<-1$ or $x>3$.
123. Find all the numbers x and y and such that x, $x+2y$, $2x+y$ from an A.P. while the numbers $(y+1)^2$, $xy+5$, $(x+1)^2$ form a G.P. Write down the progressions. {Ans. x = 3, y = 1; 3, 5, 7, & 4, 8, 16}
124. If a, b, c, x are all real numbers, and $(a^2+b^2)x^2-2b(a+c)x+(b^2+c^2)=0$ then show that a, b, c are in G.P., and x is their common ratio.
125. If $a^{\frac{1}{x}}=b^{\frac{1}{y}}=c^{\frac{1}{z}}$ and a, b, c be in G.P. then prove that x, y, z are in A.P.
126. If a, b, c be in G.P. then prove that $\log a^n, \log b^n, \log c^n$ are in A.P.
127. If the mth, nth, and pth terms of an A.P. and G.P. be equal and be respectively x, y and z, then prove that $x^{y-z}\cdot y^{z-x}\cdot z^{x-y}=1$.
128. If a, b, c are in G.P. and x, y respectively be arithmetic means between a, b and b, c, then prove that $\frac{a}{x}+\frac{c}{y}=2$ and $\frac{1}{x}+\frac{1}{y}=\frac{2}{b}$.
129. If a, b, c be distinct positive and in G.P. and $\log_c a, \log_b c, \log_a b$ be in A.P. then show that the common difference of this A.P. is $\frac{3}{2}$.
130. Prove that the three successive terms of a G.P. will form the sides of a triangle if the common ratio r satisfies the inequality $\frac{1}{2}(\sqrt{5}-1)<r<\frac{1}{2}(\sqrt{5}+1)$.

CATEGORY-9.5. GEOMETRIC SERIES

131. The fifth term of a G.P. is 81 whereas its second term is 24. Find the series and sum of its first eight terms.

{Ans. 16, 24, 36, ...; $\frac{6305}{8}$}

132. The sum of first three of a G.P. is to the sum of the first six terms as 125:152. Find the common ratio of the G.P. {Ans. $\frac{3}{5}$}
133. For a sequence $\{a_n\}$, $a_1 = 2$ and $\frac{a_{n+1}}{a_n} = \frac{1}{3}$. Then find the value of $\sum_{r=1}^{20} a_r$. {Ans. $3\left(1-\frac{1}{3^{20}}\right)$}
134. Find the value of $9^{\frac{1}{3}} \cdot 9^{\frac{1}{9}} \cdot 9^{\frac{1}{27}} \cdots\cdots\cdots$ upto ∞. {Ans. 3}
135. Find the value of $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}} \cdot x^{\frac{1}{8}} \cdot x^{\frac{1}{16}} \cdots\cdots$ to ∞. {Ans. x}
136. If $S = 1+2+4+8+16+32+\ldots\ldots\infty$, then S is a positive number. Multiply both sides by 2, then it is found that $2S = S - 1$ which leads to conclusion $S = -1$ which is certainly negative. Do you agree with the conclusion? Your answer should be supported by explanation.
137. The first term of an infinite G.P. is 1 and any term is equal to the sum of all the succeeding terms. Find the series. {Ans. $1, \frac{1}{2}, \frac{1}{4}, \cdots\cdots$}
138. The sum of first two terms of an infinite G.P. is 5 and each term is three times the sum of succeeding terms. Find the series. {Ans. $4, 1, \frac{1}{4}, \cdots\cdots$}
139. Sum of a certain number of terms of the series $\frac{2}{9} - \frac{1}{3} + \frac{1}{2} - \cdots\cdots$ is $\frac{55}{72}$. Find the number. {Ans. 5}
140. How many terms of the series 1, 4, 16, ... must be taken to have their sum equal to 341 ? {Ans. 5}
141. In a G.P. sum of n terms is 255, the last term is 128 and common ratio is 2. Find n. {Ans. 8}
142. In an increasing G.P., the sum of the first and the last term is 66, the product of the second and the last but one term is 128, and the sum of all the terms is 126. How many terms are there in the progression? {Ans. 6}
143. In a G.P. sum of n terms is 364, first term is 1 and the common ratio is 3. Find n. {Ans. 6}
144. Express the recurring decimal $0.125\,125\,125\ldots\ldots$ as a rational number. {Ans. $\frac{125}{999}$}
145. Find the value of $0.1\overline{23}$ regarding it as a geometric series. {Ans. $\frac{61}{495}$}
146. Find the value of $0.4\overline{23}$. {Ans. $\frac{419}{990}$}
147. Find the value of $2.\overline{357}$. {Ans. $\frac{2355}{999}$}
148. After striking a floor a certain ball rebounds $\frac{4}{5}$ th of the height from which it has fallen. Find the total distance that it travels before coming to rest, if it is gently dropped from a height of 120 meters. {Ans. 1080m}
149. A ball is dropped from a height of 48 ft. and rebounds two-third of the distance it falls. If it continues to fall and rebound in this way, how far will it travel before coming to rest ? {Ans. 240 ft.}
150. A GP consists of an even number of terms. If the sum of all the terms is 5 times the sum of the terms occupying odd places, then find the common ratio. {Ans. 4}
151. If in an infinite GP, first term is equal to 10 times the sum of all successive terms, then find its common

ratio. {Ans. $\frac{1}{11}$}

152. If second term of a GP is 2 and the sum of its infinite terms is 8, then find its first term. {Ans. 4}
153. Find the sum of the term of an infinitely decreasing G.P. in which all the terms are positive, the first term is 4, and the difference between the third and fifth term is equal to $\frac{32}{81}$. {Ans. 6, $\frac{12}{3-2\sqrt{2}}$}
154. The sum of an infinite geometric progression is 2 and the sum of the geometric progression made from the cube of the terms of this infinite series is 24. Then find the series. {Ans. $3-\frac{3}{2}+\frac{3}{4}-\frac{3}{8}+\cdots\cdots$}
155. If the sum of an infinite GP be 3 and the sum of squares of its term is also 3, then find its first term and common ratio. {Ans. $\frac{3}{2},\frac{1}{2}$}
156. The length of a side of a square is a meters. A second square is formed by joining the middle points of this square. Then a third square is formed by joining the middle points of the second square and so on. The process is carried on ad-infinitum. Find the sum of the areas of the squares. {Ans. $2a^2$}
157. Show that the sum of n terms of a G.P. of common ratio r beginning with the p^{th} term is r^{p-q} times the sum of an equal number of terms of the same series beginning with q^{th} term.
158. Find the sum of $2n$ term of a series of which every even term is a times the term before it, and every odd term c times the term before it, the first term being unity. {Ans. $\left(\frac{1+a}{1-ac}\right)\left(1-a^nc^n\right)$}
159. If S is the sum to infinity of a GP, whose first term is a, then find the sum of the first n terms. {Ans. $S\left[1-\left(1-\frac{a}{S}\right)^n\right]$}
160. If S be the sum, P the product and R the sum of the reciprocals of n terms of a G.P., prove that $\left(\frac{S}{R}\right)^n = P^2$.
161. If $S = 1+a+a^2+\ldots\ldots$ to ∞ $(a<1)$, then show that $a = \frac{S-1}{S}$.
162. If the sum of the series $1+\frac{3}{x}+\frac{9}{x^2}+\frac{27}{x^3}+\ldots$ to ∞ is a finite number, then show that $x>3$.
163. Prove that $\sum_{r=1}^{n}\log\left(\frac{a^r}{b^{r-1}}\right)=\frac{n}{2}\log\left(\frac{a^{n+1}}{b^{n-1}}\right)$.
164. If $A = 1+r^a+r^{2a}+\ldots\ldots\ldots \text{to}\,\infty$ and $B = 1+r^b+r^{2b}+\ldots\ldots\ldots \text{to}\,\infty$, prove that $r=\left(\frac{A-1}{A}\right)^{1/a}=\left(\frac{B-1}{B}\right)^{1/b}$.
165. If S_1, S_2, S_3 be respectively the sums of n, $2n$, $3n$ terms of a G.P., then prove that
   i. $S_1(S_3-S_2)=(S_2-S_1)^2$
   ii. $S_1^2+S_2^2=S_1(S_2+S_3)$.
166. If S_1, S_2, ..., S_n are the sums of infinite geometric series whose first terms are 1, 2, 3, ...n and common ratios are $\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots,\frac{1}{n+1}$ respectively then prove that $S_1+S_2+S_3+\ldots+S_n=\frac{1}{2}n(n+3)$.

167. If S_p denotes the sum of series $1 + r^p + r^{2p} + \ldots\ldots$ to ∞ and s_p the sum of the series $1 - r^p + r^{2p} - \ldots\ldots\ldots$ to ∞, prove that $S_p + s_p = 2S_{2p}$.

168. If S_n represents the sum of n terms of a G.P. whose first term and common ratio are a and r respectively, then prove that

i. $S_1 + S_2 + S_3 + \ldots + S_n = \dfrac{na}{1-r} - \dfrac{ar(1-r^n)}{(1-r)^2}$;

ii. $S_1 + S_3 + S_5 + \ldots + S_{2n-1} = \dfrac{an}{1-r} - \dfrac{ar(1-r^{2n})}{(1-r)^2(1+r)}$.

169. The sum of the squares of three distinct real numbers, which are in G.P. is S^2. If their sum is αS, show that $\alpha^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$.

170. Prove the equality $\underbrace{(666\ldots\ldots\ldots 6)^2}_{n \text{ digits}} + \underbrace{888\ldots\ldots\ldots 8}_{n \text{ digits}} = \underbrace{444\ldots\ldots\ldots 4}_{2n \text{ digits}}$.

171. Find the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfies the relation $f(x+y) = f(x).f(y)$ for all natural numbers x, y and further $f(1) = 2$. {Ans. 3}

172. Sum the series $(a+b) + (a^2 + 2b) + (a^3 + 3b) + \ldots$ to n terms. {Ans. $a\left(\dfrac{a^n - 1}{a-1}\right) + b\dfrac{n(n+1)}{2}$ }

173. Sum the series $\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 \ldots \left(x^n + \frac{1}{x^n}\right)^2$. {Ans. $\left(\dfrac{x^{2n} - 1}{x^2 - 1}\right)\left(\dfrac{x^{2n+2} + 1}{x^{2n}}\right) + 2n$ }

174. Sum the series $1 + (1+x) + (1+x+x^2) + (1+x+x^2+x^3) + \cdots\cdots$ to n terms. {Ans. $\dfrac{1}{(1-x)^2}\left[n(1-x) - x(1-x^n)\right]$}

175. Sum the series $x(x+y) + x^2(x^2+y^2) + x^3(x^3+y^3) + \cdots\cdots$ to n terms. {Ans. $x^2\left(\dfrac{1-x^{2n}}{1-x^2}\right) + xy\left(\dfrac{1-x^n y^n}{1-xy}\right)$}

176. If $x = 1 + a + a^2 + a^3 + \ldots$ to ∞ $(|a| < 1)$ and $y = 1 + b + b^2 + b^3 + \ldots\ldots$ to ∞ $(|b| < 1)$. Prove that $1 + ab + a^2b^2 + a^3b^3 + \ldots$ to $\infty = \dfrac{xy}{x+y-1}$.

177. Sum the series $1 + \dfrac{\sqrt{2}-1}{2\sqrt{3}} + \dfrac{3-2\sqrt{2}}{12} + \dfrac{5\sqrt{2}-7}{24\sqrt{3}} + \dfrac{17-12\sqrt{2}}{144} + \ldots\infty$. {Ans. $\dfrac{2\sqrt{3}}{2\sqrt{3}-\sqrt{2}+1}$ }

178. Find the sum of the first n terms of the series $\dfrac{1}{2} + \dfrac{3}{4} + \dfrac{7}{8} + \dfrac{15}{16} + \cdots\cdots$. {Ans. $n + 2^{-n} - 1$}

179. Let r be the common ratio of the G.P. $a_1, a_2, \ldots,$ show that

$$\frac{1}{a_1^m + a_2^m} + \frac{1}{a_2^m + a_3^m} + \ldots + \frac{1}{a_{n-1}^m + a_n^m} = \frac{1 - r^{m(1-n)}}{a_1^m (r^m - r^{-m})}.$$

180. Find the sum of the infinite series $1+(1+a)r+\left(1+a+a^2\right)r^2+\left(1+a+a^2+a^3\right)r^3+.......,$ r and a being proper fractions. {Ans. $\frac{1}{(1-r)(1-ar)}$ }

181. Find $\lim_{x\to 1}\frac{x+x^2+........+x^n-n}{x-1}$. {Ans. $\frac{n(n+1)}{2}$ }

182. Find $\lim_{n\to\infty} n^{-n^2}\left((n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)\cdots\left(n+\frac{1}{2^{n-1}}\right)\right)^n$. {Ans. e^2 }

CATEGORY-9.6. GEOMETRIC MEAN

183. Insert five geometric means between 486 and $\frac{2}{3}$. {Ans. 162, 54, 18, 6, 2}

184. If A and G be the A.M. and G.M. between two numbers, prove that the numbers are $A\pm\sqrt{(A+G)(A-G)}$.

185. Construct a quadratic in x such that A.M. of its roots is A and G.M. is G. {Ans. $x^2-2Ax+G^2=0$ }

186. If one G.M. G and two arithmetic means p and q be inserted between any two given numbers then show that $G^2=(2p-q)(2q-p)$.

187. If one A.M. A and two geometric means p and q be inserted between any two given numbers then show that $p^3+q^3=2Apq$.

188. The A.M. between m and n and the G.M. between a and b are each equal to $(ma+nb)(m+n)$. Find m and n in terms of a and b. {Ans. $m=\frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}, n=\frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ }

189. If n geometric means be inserted between a and b, then prove that their product is $(ab)^{\frac{n}{2}}$.

190. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the geometric mean of a and b? {Ans. $-\frac{1}{2}$ }

191. If A is the AM of the roots of the equation $x^2-2ax+b=0$ and G is the GM of the roots of the equation $x^2-2bx+a^2=0$, then show that $A=G$.

192. If A_1, A_2 be two AM's and G_1, G_2 be two GM's between a and b, then prove that $\frac{A_1+A_2}{G_1G_2}=\frac{a+b}{ab}$.

193. If the A.M. between a and b is twice as great as their G.M. show that $a:b=\left(2+\sqrt{3}\right):\left(2-\sqrt{3}\right)$.

194. The A.M. of a and b is to their G.M as m to n, show $a:b=m+\sqrt{m^2-n^2}:m-\sqrt{m^2-n^2}$.

195. If A be the arithmetic mean of b and c and G_1, G_2, be the two geometric means between them, then prove that ${G_1}^3+{G_2}^3=2G_1G_2A$.

CATEGORY-9.7. HARMONIC PROGRESSION AND HARMONIC SERIES

196. The 7th term of a H.P. is $\frac{1}{10}$ and 12th term is $\frac{1}{25}$, find the 20th term. {Ans. $\frac{1}{49}$ }

197. Solve the equation $6x^3-11x^2+6x-1=0$ if its roots are in harmonic progression. {Ans. $1, \frac{1}{2}, \frac{1}{3}$ }

198. The value of $x+y+z$ is 15 if a, x, y, z, b are in A.P. while the value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is $\frac{5}{3}$ if a, x, y, z, b are

in H.P., find a and b. {Ans. 9 & 1}

199. If x, y, z are in H.P., prove that $\log(x+z)+\log(x+z-2y)=2\log(x-z)$.

200. Show that $\log_3 2, \log_6 2, \log_{12} 2$ are in HP.

201. Prove that a, b, c are in A.P., G.P or H.P. according as the value of $\dfrac{a-b}{b-c}$ is equal to $\dfrac{a}{a}, \dfrac{a}{b}$ or $\dfrac{a}{c}$ respectively.

202. If $a_1, a_2, a_3, ..., a_n$ are in harmonic progression, prove that $a_1a_2+a_2a_3+\ldots+a_{n-1}a_n=(n-1)\,a_1a_n$.

203. If a, b, c be respectively the pth, qth and rth terms of an H.P., then prove that $bc(q-r)+ca(r-p)+ab(p-q)=0$.

204. If pth term of an H.P. is qr and qth term is rp, prove that rth term is pq.

205. If the roots of the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ be equal, then prove that a, b, c are in H.P.

206. If the mth term of an H.P. is n and nth term be m, then prove that $(m+n)$th term is $\dfrac{mn}{m+n}$.

207. If a, b, c be in H.P., prove that
   i. $\dfrac{a-b}{b-c}=\dfrac{a}{c}$;
   ii. $\dfrac{1}{b-a}+\dfrac{1}{b-c}=\dfrac{2}{b}$;
   iii. $\dfrac{b+a}{b-a}+\dfrac{b+c}{b-c}=2.$;
   iv. $\left(\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}\right)\left(\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{a}\right)=\dfrac{4}{ac}-\dfrac{3}{b^2}$;
   v. $(bc+ca-ab)\,(ca+ab-bc)=ac\left(4b^2-3ac\right)$.

208. If a, b, c be in H.P. prove that
   i. $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in H.P.;
   ii. $\dfrac{a}{b+c-a}, \dfrac{b}{c+a-b}, \dfrac{c}{a+b-c}$ are in H.P.;
   iii. $\dfrac{1}{a}+\dfrac{1}{b+c}, \dfrac{1}{b}+\dfrac{1}{c+a}, \dfrac{1}{c}+\dfrac{1}{a+b}$ are in H.P.

209. If $\dfrac{1}{a(b+c)}, \dfrac{1}{b(c+a)}, \dfrac{1}{c(a+b)}$ be in H.P. then a, b, c are also in H.P.

210. If $b+c,\ c+a,\ a+b$ are in H.P. then prove that
   i. $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in A.P.;
   ii. a^2, b^2, c^2 are in A.P.

211. If a, b, c are in A.P., prove that $\dfrac{bc}{ca+ab}, \dfrac{ca}{bc+ab}, \dfrac{ab}{bc+ca}$ are in H.P.

212. First three of the four numbers are in A.P. & the last three in H.P. Prove that the four numbers are proportional.

213. If a, b, c be in A.P. b, c, a be in H.P. then prove that c, a, b are in G.P.

214. If x, y, z are in A.P., ax, by, cz in G.P. and a, b, c in H.P. prove that $\frac{x}{z}+\frac{z}{x}=\frac{a}{c}+\frac{c}{a}$.
215. If $a^x=b^y=c^z$ and a, b, c be in G.P. then prove that x, y, z are in H.P.
216. If a, b, c be in G.P. then prove that $\log_a n, \log_b n, \log_c n$ are in H.P.
217. Given $a^x=b^y=c^z=d^u$ and a, b, c, d are in G.P., show that x, y, z, u are in H.P.
218. If a, b, c, d, e be five numbers such that a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in H.P., prove that a, c, e are in G.P. and $e=\frac{(2b-a)^2}{a}$. If a = 2 and e = 18, find all possible values of b, c and d. {Ans. 4, 6, 9 or -2, -6, -18}
219. a, b, c are in H.P., b, c, d are in G.P. and c, d, e are in A.P. show that $e=\frac{ab^2}{(2a-b)^2}$.
220. If $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}$ and p, q, r be in A.P. then prove that x, y, z are in H.P.
221. If a, b, c are all positive and in H.P., then show that the roots of $ax^2+2bx+3c$ are imaginary.
222. If a, b, c be in A.P. and a^2, b^2, c^2 in H.P. then prove that either $-\frac{a}{2}, b, c$ are in G.P. or $a=b=c$.
223. p, q, r are three numbers in G.P. Prove that the first term of an A.P., whose pth, qth and rth terms are in H.P., is to the common difference as $(q+1):1$.
224. A G.P. and a H.P. have the same pth, qth and rth terms as a, b, c respectively. Show that $a(b-c)\log a+b(c-a)\log b+c(a-b)\log c=0.$
225. An A.P., a G.P and a H.P. have a and b for their first two terms. Show that their $(n+2)$th terms will be in G.P. if $\frac{b^{2n+2}-a^{2n+2}}{ab\left(b^{2n}-a^{2n}\right)}=\frac{n+1}{n}$.
226. An A.P. and a H.P., have the same first term, the same last term, and the same number of terms; prove that the product of the rth term from the beginning in one series and the rth term from the end in the other is independent of r.
227. α, β, γ are the geometric means between $ca, ab; ab, bc; bc, ca$ respectively. Prove that if a, b, c are in A.P., then $\alpha^2, \beta^2, \gamma^2$ are also in A.P., and $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$ are in H.P.
228. If the $(m+1)^{th}$, $(n+1)^{th}$ and $(r+1)^{th}$ terms of an A.P. are in G.P., m, n, r are in H.P. show that the ratio of the common difference to the first term in the A.P. is $-\frac{2}{n}$.
229. If S_1, S_2, S_3 denote the sums of n terms of three A.P.'s whose first terms are unity and common differences in H.P., prove that $n=\frac{2S_3S_1-S_1S_2-S_2S_3}{S_1-2S_2+S_3}$.
230. If a, b, c are in A.P., α, β, γ in H.P., $a\alpha, b\beta, c\gamma$ in G.P., (with common ratio not equal to 1.), then prove that $a:b:c=\frac{1}{\gamma}:\frac{1}{\beta}:\frac{1}{\alpha}$.

CATEGORY-9.8. HARMONIC MEAN

231. Insert six harmonic means between 3 and $\frac{6}{23}$. {Ans. $\frac{6}{5}, \frac{3}{4}, \frac{6}{11}, \frac{3}{7}, \frac{6}{17}, \frac{3}{10}$}
232. If the harmonic mean of two numbers is to their geometric mean as 12:13, prove that the numbers are the ratio of 4:9.
233. Let the harmonic mean and geometric mean of two positive numbers be in the ratio 4:5, then find the ratio of the two numbers. {Ans. 1:4}
234. If A.M between two numbers is 5 and their GM is 4, then find their HM. {Ans. $\frac{16}{5}$}
235. A.M and H.M. between two quantities are 27 and 12 respectively, find their G.M. {Ans. 18}
236. If $x, 1, z$ are in AP and $x, 2, z$ are in GP, then find the harmonic mean of x and z. {Ans. 4}
237. If H be the HM between a and b, then show that $\frac{H}{a}+\frac{H}{b}=2$.
238. The harmonic mean of two numbers is 4. Their A.M., A, and G.M., G, satisfy the relation $2A+G^2=27$. Find the two numbers. {Ans. 6 & 3}
239. The A.M. of two numbers exceeds their G.M. by 15 and H.M. by 27, find the numbers. {Ans. 120 & 30}
240. If the A.M. between two numbers exceeds their G.M. by 2 and the G.M. exceeds their H.M. by $\frac{8}{5}$; find the numbers. {Ans. 4 & 16}
241. If the A.M., the G.M and the H.M. of first and last terms of the sequence 25, 26, 27, ..., $N-1$, N are the term of this sequence, find the value of N. {Ans. 225 & 1225}
242. If 9 arithmetic and harmonic means be inserted between 2 and 3, prove that $A+\frac{6}{H}=5$ where A is any of the A.M.'s and H the corresponding H.M.
243. If H be the harmonic mean between a and b then prove that $\frac{1}{H-a}+\frac{1}{H-b}=\frac{1}{a}+\frac{1}{b}$.
244. If A, G, H be respectively the A.M., G.M. and H.M. between two given quantities a and b, then prove that A, G, H are in G.P.
245. If A be the A.M. and H the H.M. between two numbers a and b then $\frac{a-A}{a-H}\times\frac{b-A}{b-H}=\frac{A}{H}$.
246. If A_1, A_2; G_1, G_2; and H_1, H_2 be two A.M.'s and G.M.'s and H.M.'s between two quantities then prove that $\frac{G_1 G_2}{H_1 H_2}=\frac{A_1+A_2}{H_1+H_2}$.
247. If n harmonic means are inserted between 1 and r then show that $\frac{\text{1st mean}}{n\text{th mean}}=\frac{n+r}{nr+1}$.
248. If H_1, H_2, ... H_n be n harmonic means between a and b show that $\frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b}=2n$. If n be a root of the equation $x^2(1-ab)-x\left(a^2+b^2\right)-(1+ab)=0$, prove that $H_1-H_n=ab(a-b)$.
249. For what value of $n, \frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the harmonic mean of a and b? {Ans. -1}
250. If a be A.M. of b and c, b the G.M. of c and a, then prove that c is the H.M. of a and b.
251. If $2(y-a)$ is the H.M. between y-x and y-z, then show that $x-a$, $y-a$, $z-a$ are in G.P.
252. If p be the first of n arithmetic means between two numbers and q be the first of n harmonic means

between the same two numbers, prove that the value of q cannot be between p and $\left(\frac{n+1}{n-1}\right)^2 p$.

CATEGORY-9.9. POLYNOMIAL SERIES

253. Sum the series $1\cdot 2^2+2\cdot 3^2+3\cdot 4^2+\cdots\cdots$ to n terms. {Ans. $\frac{n(n+1)(n+2)(3n+5)}{12}$}
254. Sum the series $1\cdot 3^2+2\cdot 5^2+3\cdot 7^2+\cdots\cdots$ to 20 terms. {Ans. 188090}
255. Sum the series $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\cdots\cdots$ to n terms. {Ans. $\frac{n(n+1)(n+2)(n+3)}{4}$}
256. Sum the series $1\cdot 2\cdot 5+2\cdot 3\cdot 6+3\cdot 4\cdot 7+\cdots\cdots$ to n terms. {Ans. $\frac{n(n+1)(n+2)(3n+17)}{12}$}
257. Sum the series $1+(1+2)+(1+2+3)+(1+2+3+4)+\cdots\cdots$ to n terms. {Ans. $\frac{n(n+1)(n+2)}{6}$}
258. Sum the series $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\cdots\cdots$ to n terms. {Ans. $\frac{n(n+1)^2(n+2)}{12}$}
259. Sum the series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots\cdots$ to 16 terms. {Ans. 446}
260. Find the sum of the series $31^3+32^3+\cdots\cdots+50^3$. {Ans. 1409400}
261. Sum to n terms the series $1^2-2^2+3^2-4^2+5^2-6^2+\cdots\cdots$. {Ans. $-\frac{n(n+1)}{2}$, n even; $\frac{n(n+1)}{2}$, n odd }
262. Sum the series $n\cdot 1+(n-1)\cdot 2+(n-2)\cdot 3+\cdots\cdots+1\cdot n$. {Ans. $\frac{n(n+1)(n+2)}{6}$}
263. Find the sum of all possible products of the first n natural numbers taken two by two. {Ans. $\frac{n(n+1)(n-1)(3n+2)}{24}$}
264. Find the coefficient of x^{n-2} in the polynomial $(x-1)(x-2)(x-3)\cdots\cdots(x-n)$. {Ans. $\frac{n(n+1)(n-1)(3n+2)}{24}$}
265. Find the coefficient of x^{99} in the polynomial $(x-1)(x-2)(x-3)\cdots\cdots(x-100)$ {Ans. -5050}
266. If s and t are respectively the sum and the sum of the squares of n successive positive integers beginning with a then show that $nt-s^2$ is independent of a.
267. If S_1, S_2, S_3, ..., S_n are the sums of infinite geometric series whose first terms are 1, 2, 3, ….., n and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}$ respectively, then find the value of $S_1^2+S_2^2+S_3^2+\cdots\cdots+S_{2n-1}^2$. {Ans. $\frac{n(2n+1)(4n+1)}{3}-1$}
268. Consider n A.P.s whose first terms are 1, 2, 3, ………, n and their common differences are 1, 2, 3, ………, n respectively. If $S_{i,i}$ denotes sum of i terms of i^{th} A.P., then find $S_{1,1}+S_{2,2}+S_{3,3}+\cdots\cdots+S_{n,n}$. {Ans. $\frac{n(n+1)(n+2)(3n+1)}{24}$}
269. On the ground are placed n stones, the distance between the first and second is one yard, between the 2nd and 3rd is 3 yards, between the 3rd and 4th, 5 yards, and so on. How far will a person have to travel who

shall bring them one by one to a basket placed at the first stone? {Ans. $\frac{n(n-1)(2n-1)}{3}$ yards}

270. Find $\lim_{n\to\infty}\left\{\frac{1}{1-n^2}+\frac{2}{1-n^2}+\frac{3}{1-n^2}+\cdots\cdots+\frac{n}{1-n^2}\right\}$. {Ans. $-\frac{1}{2}$ }

271. Find $\lim_{n\to\infty}\frac{1^2+2^2+3^2+\cdots\cdots+n^2}{n^3}$. {Ans. $\frac{1}{3}$ }

CATEGORY-9.10. DIFFERENCE OF CONSECUTIVE TERMS IN A.P.

272. Sum the series $1+3+6+10+15+\ldots$ to n terms. {Ans. $\frac{n(n+1)(n+2)}{6}$ }

273. Sum the series $4+6+9+13+18+\ldots$ to n terms. {Ans. $\frac{n(n^2+3n+20)}{6}$ }

274. Sum the series $2+4+7+11+16+\ldots\ldots$ to n terms. {Ans. $\frac{n(n^2+3n+8)}{6}$ }

CATEGORY-9.11. DIFFERENCE OF CONSECUTIVE TERMS IN G.P.

275. Sum the series $1+3+7+15+31+\ldots$ to n terms. {Ans. $2^{n+1}-n-2$ }

276. Sum up to n terms the series $0.7+0.77+0.777+\ldots.$ {Ans. $\frac{7n}{9}-\frac{7}{81}\left(1-\frac{1}{10^n}\right)$}

277. Sum up to n terms the series $6+66+666+\ldots.$ {Ans. $\frac{6}{81}\left(10^{n+1}-9n-10\right)$}

278. Sum up to n terms the series $8+88+888+\ldots\ldots.$ {Ans. $\frac{8}{81}\left(10^{n+1}-9n-10\right)$}

CATEGORY-9.12. ARITHMETIC-GEOMETRIC SERIES

279. Sum the series $1+2.2+3.2^2+4.2^3+\ldots+100.2^{99}$. {Ans. $99\cdot 2^{100}+1$}

280. Find the sum of n terms of the series the rth term of which is $(2r + 1)\ 3^r$. {Ans. $n3^{n+1}$ }

281. Sum the series $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots$ n terms. {Ans. $6-\frac{2n+3}{2^{n-1}}$}

282. Sum the series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots$ to n terms and to ∞. {Ans. $\frac{35}{16}-\frac{12n+7}{16\cdot 5^{n-1}},\frac{35}{16}$ }

283. Sum the series $1+\frac{1}{3}+\frac{3}{3^2}+\frac{5}{3^3}+\ldots$ to ∞. {Ans. 2}

CATEGORY-9.13. METHOD OF DIFFERENCE

284. Sum to n terms the series and sum of infinite series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\ldots.$ {Ans. $\frac{n}{n+1}$, 1}

285. Sum to n terms the series and sum of infinite series $\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots\ldots.$ {Ans. $\frac{n}{3(2n+3)}$, $\frac{1}{6}$ }

286. Sum to n terms the series and sum of infinite series $1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots\ldots.$ {Ans. $\frac{2n}{n+1}$, 2}

287. Sum to n terms the series and sum of infinite series $\frac{1}{1\cdot3\cdot5}+\frac{1}{3\cdot5\cdot7}+\frac{1}{5\cdot7\cdot9}+.......$ {Ans. $\frac{n(n+2)}{3(2n+1)(2n+3)}$, $\frac{1}{12}$}

288. Sum to n terms the series and sum of infinite series $\frac{3}{1^2\cdot2^2}+\frac{5}{2^2\cdot3^2}+\frac{7}{3^2\cdot4^2}+.......$ {Ans. $\frac{n(n+2)}{(n+1)^2}$, 1}

289. Sum to n terms the series and sum of infinite series $\frac{1}{1\cdot3}+\frac{2}{1\cdot3\cdot5}+\frac{3}{1\cdot3\cdot5\cdot7}+.......$ {Ans. $\frac{1}{2}\left[1-\frac{1}{1\cdot3\cdot5\cdot............\cdot(2n+1)}\right]$, $\frac{1}{2}$}

290. Sum to n terms the series and sum of infinite series $\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+.......$ {Ans. $\sqrt{n+1}-1$, diverging}

291. Sum to n terms the series and sum of infinite series $\ln\left(1-\frac{1}{2^2}\right)+\ln\left(1-\frac{1}{3^2}\right)+\ln\left(1-\frac{1}{4^2}\right)+.......$ {Ans. $\ln\frac{n+2}{2(n+1)}$, $-\ln 2$}

292. Find $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{(r+2)\cdot r!}$. {Ans. $\frac{1}{2}$}

293. Prove that $1-\frac{1}{n+1}<\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots+\frac{1}{n^2}<2-\frac{1}{n}$.

CATEGORY-9.14. EXPONENTIAL AND LOGARITHMIC SERIES

294. Find $\sum_{n=1}^{\infty}\frac{1}{(n+1)!}$. {Ans. $e-2$}

295. Find $\sum_{n=1}^{\infty}\frac{1}{(n+2)!}$. {Ans. $e-\frac{5}{2}$}

296. Find $\sum_{n=1}^{\infty}\frac{1}{(2n-1)!}$. {Ans. $\frac{e-e^{-1}}{2}$}

297. Find $\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}$. {Ans. $\frac{e-e^{-1}-2}{2}$}

298. Find $\sum_{n=1}^{\infty}\frac{1}{(n-1)!}$. {Ans. e}

299. Find $\sum_{n=2}^{\infty}\frac{1}{(n-2)!}$. {Ans. e}

300. Sum the infinite series $1+\frac{3}{1!}+\frac{5}{2!}+\frac{7}{3!}+\cdots\cdots$. {Ans. $3e$}

301. Sum the infinite series $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\cdots\cdots$. {Ans. $2e$}

302. Sum the infinite series $1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\cdots\cdots$. {Ans. $5e$ }

303. Sum the infinite series $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\frac{8}{9!}+\cdots\cdots$. {Ans. $\frac{1}{e}$ }

304. Sum the infinite series $\frac{2}{1!}+\frac{4}{3!}+\frac{6}{5!}+\frac{8}{7!}+\cdots\cdots$. {Ans. e }

305. Sum the infinite series $1+\frac{1+a}{2!}+\frac{1+a+a^2}{3!}+\frac{1+a+a^2+a^3}{4!}+\cdots\cdots$. {Ans. $\frac{e^a-e}{a-1}$ }

306. Find $\sum_{n=1}^{\infty}\frac{n^2}{(n+1)!}$. {Ans. $e-1$}

307. Sum the infinite series $1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\cdots\cdots$. {Ans. $\frac{3e}{2}$ }

308. Sum the infinite series $\frac{1}{2!}+\frac{1+2}{3!}+\frac{1+2+3}{4!}+\cdots\cdots$. {Ans. $\frac{e}{2}$ }

309. Sum the infinite series $\frac{1^2\cdot 2}{1!}+\frac{2^2\cdot 3}{2!}+\frac{3^2\cdot 4}{3!}+\cdots\cdots$. {Ans. $7e$ }

310. Sum the infinite series $1+\frac{3}{2!}+\frac{6}{3!}+\frac{10}{4!}+\cdots\cdots$. {Ans. $\frac{3e}{2}$ }

311. Sum the infinite series $\frac{12}{2!}+\frac{28}{3!}+\frac{50}{4!}+\frac{78}{5!}+\cdots\cdots$. {Ans. $5e$ }

312. Sum the infinite series $\frac{1}{3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\frac{1}{7\cdot 3^7}+\cdots\cdots$. {Ans. $\frac{1}{2}\ln 2$ }

313. Sum the infinite series $1+\frac{1}{3\cdot 2^2}+\frac{1}{5\cdot 2^4}+\frac{1}{7\cdot 2^6}+\cdots\cdots$. {Ans. $\ln 3$ }

314. Sum the infinite series $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots\cdots$. {Ans. $\ln 2$ }

315. Sum the infinite series $\frac{1}{2\cdot 3}+\frac{1}{4\cdot 5}+\frac{1}{6\cdot 7}+\cdots\cdots$. {Ans. $1-\ln 2$ }

316. Sum the infinite series $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 7}+\frac{1}{4\cdot 9}+\cdots\cdots$. {Ans. $2-\ln 2$ }

317. Sum the infinite series $\frac{5}{1\cdot 2\cdot 3}+\frac{7}{3\cdot 4\cdot 5}+\frac{9}{5\cdot 6\cdot 7}+\cdots\cdots$. {Ans. $3\ln 2-1$ }

318. Sum the infinite series $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{3\cdot 4\cdot 5}+\frac{1}{5\cdot 6\cdot 7}+\cdots\cdots$. {Ans. $\ln 2-\frac{1}{2}$ }

CATEGORY-9.15. ADDITIONAL QUESTIONS

319. If the sum of first n natural numbers is $\frac{1}{5}$ times the sum of their squares, then find the value of n. {Ans. 7}

320. If $a_1=1$ and $a_{n+1}=\frac{4+3a_n}{3+2a_n}$, $n\geq 1$ and if $\lim_{n\to\infty}a_n=a$, then find the value of a. {Ans. $\sqrt{2}$ }

321. Prove that each number is the square of an odd integer in the sequence 49, 4489, 444889, ... in which every number is formed by inserting 48 in the middle of the previous number as indicated.

322. Find n for which the sum of the product of integers $0, \pm 1, \pm 2, \pm 3, \cdots\cdots\cdots\cdots, \pm n$ taken two at a time, lies between -50 to -100. {Ans. 5, 6}

323. Let $a_1, a_2, a_3, \cdots\cdots, a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, \cdots\cdots, 11$. If $\dfrac{a_1^{\ 2} + a_2^{\ 2} + \cdots\cdots + a_{11}^{\ 2}}{11} = 90$, then find the value of $\dfrac{a_1 + a_2 + \cdots\cdots + a_{11}}{11}$. {Ans. 0}

324. If $a > b > 0$ and $n \in N$, prove that $a^n - b^n \geq n(a-b)(ab)^{\frac{n-1}{2}}$.

325. Given $|x| < 1$, sum to infinite terms $\dfrac{1}{(1-x)(1-x^3)} + \dfrac{x^2}{(1-x^3)(1-x^5)} + \dfrac{x^4}{(1-x^5)(1-x^7)} + \cdots\cdots$. {Ans. $\dfrac{1}{(1+x)(1-x)^2}$ }

326. Let $f_1(x) = \dfrac{x}{3} + 10$ for all $x \in R$ and $f_n(x) = f_1(f_{n-1}(x))$ for $n \geq 2$. Then find $f_n(x)$. {Ans. $f_n(x) = \dfrac{x-15}{3^n} + 15$ }

Mathematics for IIT-JEE

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PART-II
ALGEBRA

CHAPTER-10
DETERMINANTS AND MATRICES

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CHAPTER-10
DETERMINANTS AND MATRICES

LIST OF THEORY SECTIONS

LIST OF QUESTION CATEGORIES

CHAPTER-10
DETERMINANTS AND MATRICES

SECTION-10.1. DETERMINANT AND ITS PROPERTIES

1. Definition of Determinant of order two

i. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ represents a determinant of order two.

ii. A determinant of order two has two horizontal rows denoted by R_1 & R_2 and has two vertical columns denoted by C_1 & C_2.

iii. a_{ij} $(i, j = 1, 2)$ are real/ complex expressions and they are called the elements of the determinant.

iv. a_{ij} denotes element of i^{th} row and j^{th} column.

v. Value of the determinant is $a_{11} \times a_{22} - a_{21} \times a_{12}$.

2. Higher order determinants

i. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ represents a determinant of order 3. It has 9 elements arranged in 3 rows and 3 columns.

ii. Likewise, a determinant of order n has n^2 elements arranged in n rows and n columns.

3. Minors

i. The determinant obtained by deleting the row and column of the element a_{ij} is called the minor of a_{ij} and is denoted by M_{ij}.

ii. Order of the determinant M_{ij} is one less than the order of the original determinant.

4. Cofactor

i. $(-1)^{i+j} M_{ij}$ is called the cofactor of a_{ij} and is denoted by C_{ij}.

5. Value of the determinant of order 3 or more

The value of a determinant of order 3 or more is the sum of the products of elements of any row (column) with their corresponding cofactors.

For example, let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$$\Delta = \sum_{j=1}^{3} a_{1j} C_{1j} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= \sum_{j=1}^{3} a_{2j} C_{2j}$$

$$= \sum_{j=1}^{3} a_{3j} C_{3j}$$

$$= \sum_{i=1}^{3} a_{i1} C_{i1} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= \sum_{i=1}^{3} a_{i2} C_{i2}$$
$$= \sum_{i=1}^{3} a_{i3} C_{i3}$$

6. Properties of Determinants

i. If rows and columns are interchanged, then the value of the determinant remains unchanged
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

ii. If two rows (columns) are identical then the value of the determinant is zero
$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

iii. If two rows (columns) are interchanged then the sign of the determinant changes
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

iv. If all the elements of a row (column) are multiplied by a factor then the value of the determinant gets multiplied by that factor
$$\begin{vmatrix} \alpha a & \alpha b \\ c & d \end{vmatrix} = \alpha \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

v. If all the elements of a row (column) are written as the sum of two factors, then the determinant can be written as the sum of two determinants
$$\begin{vmatrix} a+\alpha & b \\ c+\beta & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix}$$

vi. If to the elements of a row (column), any multiple of the elements of any other row (column) is added, then the value of the determinant remains unchanged
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a+\alpha c & b+\alpha d \\ c & d \end{vmatrix}$$

vii. Multiplication of two determinants of the same order
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \begin{vmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{vmatrix}$$

viii. Derivative of a function expressed in determinant form
$$\frac{d}{dx}\begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix} = \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}$$

ix. Sum of the products of the elements of any row (column) with the cofactor of the corresponding elements of any other row (column) is zero.
$$\sum_{j=1}^{n} a_{pj} C_{qj} = \sum_{i=1}^{n} a_{pj} C_{qj} = 0 \quad (p \neq q)$$

x. If Δ is a determinant of order n and Δ^c denotes the determinant of cofactors then

$$\Delta \cdot \Delta^c = \begin{vmatrix} \Delta & 0 & \cdots & 0 \\ 0 & \Delta & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \Delta \end{vmatrix} = \Delta^n \Rightarrow \Delta^c = \Delta^{n-1}.$$

For example, if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then $\Delta^c = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2$.

SECTION-10.2. SYSTEM OF LINEAR EQUATIONS IN MORE THAN ONE VARIABLES

1. **A linear equation in more than one variable**
   i. Solution
   ii. Solution set
2. **System of linear equations in more than one variable**
   i. Solution
   ii. Solution set
   iii. Consistent and determinate
   iv. Consistent and indeterminate
   v. Inconsistent
3. **No. of equations = no. of variables**
   i. Unique solution set: Cramer's rule
   ii. No solution set
   iii. Infinite solution set: parametric representation
   iv. Consider the system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

We define $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Case-1: If $\Delta \neq 0 \Rightarrow$ The system has unique solution given by

$$\left\{ \begin{aligned} x &= \frac{\Delta_x}{\Delta} \\ y &= \frac{\Delta_y}{\Delta} \\ z &= \frac{\Delta_z}{\Delta} \end{aligned} \right\} \text{ (Cramer's rule)}$$

The system is said to be consistent and determinate.

Case-2: If $\Delta = 0$ but at least one $\Delta_i \neq 0 \Rightarrow$ The system has no solution.

The system is said to be inconsistent

Case-3: If $\Delta = 0 = \Delta_x = \Delta_y = \Delta_z \Rightarrow$ The system has infinite solutions; represented parametrically.

The system is said to be consistent and indeterminate.

v. When $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$, there may be no solution in some cases.

4. Homogeneous equations, trivial and non trivial solutions

i. A linear equation whose free term is 0, is called a homogeneous linear equation.

ii. All variables simultaneously equal to 0 is always a solution of any homogeneous linear equation and this solution is called trivial solution.

iii. Consider the system of homogeneous linear equations

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

For the system of homogeneous equations $\Delta_x = \Delta_y = \Delta_z = 0$.

Case-1: If $\Delta \neq 0 \Rightarrow$ The system has unique solution given by $(0,0,0)$, i.e. trivial solution is the only solution.

Case-2: If $\Delta = 0 \Rightarrow$ The system has infinite solutions, trivial solution $(0,0,0)$ is a solution and other solutions are called non-trivial solutions.

5. No. of equations = no. of variables + 1

i. Consider the system of 3 linear equations in 2 unknowns

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$
$$a_3x + b_3y = c_3$$

This system is consistent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ and inconsistent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ (Consistency condition).

6. No. of equations < no. of variables

i. Unique solution not possible.

SECTION-10.3. MATRIX AND ITS PROPERTIES

1. Definition of Matrix

i. Matrix is a rectangular arrangement of real/ complex expressions called elements.

ii. A matrix is written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

iii. Matrix having *m* rows and *n* columns is said to be of the order $m \times n$ $(m, n \geq 1)$.

iv. A matrix is represented by $A_{m\times n} = \lfloor a_{ij} \rfloor_{m\times n}$, where a_{ij} denotes element of i^{th} row & j^{th} column, *m* denotes no. of rows and n denotes no. of columns.

2. **Row matrix**
   A matrix having only one row is called a row matrix.
3. **Column matrix**
   A matrix having only one column is called a column matrix.
4. **Square matrix**
   A matrix in which number of rows and columns are equal, is called a square matrix.
5. **Principal diagonal of a square matrix**
   The elements a_{ii} lying on the diagonal constitute the principal diagonal.
6. **Trace of a square matrix**
   Sum of the diagonal elements of a square matrix A is called trace of A, i.e.
   Trace of $A = a_{11} + a_{22} + \cdots\cdots + a_{nn}$.
7. **Diagonal matrix**
   A square matrix in which all the elements, except those in the principal diagonal, are zero, is called a diagonal matrix, i.e. in a diagonal matrix $a_{ij} = 0 \ \forall \ i \neq j$. A diagonal matrix of order $n \times n$ having d_1, d_2, …….., d_n as diagonal elements is also denoted by $diag[d_1, d_2, \ldots\ldots.., d_n]$.
8. **Scalar matrix**
   A diagonal matrix in which all the elements in the principal diagonal are equal, is called a scalar matrix.
9. **Unit (Identity) matrix**
   A diagonal matrix in which all the elements in the principal diagonal are 1, is called a unit or identity matrix. A unit matrix of order n is denoted by I_n. Thus for an identity matrix $a_{ii} = 1$ and $a_{ij} = 0 \ \forall \ i \neq j$.
10. **Null (Zero) matrix**
    A matrix in which all the elements are zero, is called a null or zero matrix and is denoted by O or $O_{m\times n}$.
11. **Upper triangular matrix**
    A square matrix $A = \lfloor a_{ij} \rfloor_{m\times n}$ is called an upper triangular matrix if $a_{ij} = 0 \ \forall \ i > j$.
12. **Lower triangular matrix**
    A square matrix $A = \lfloor a_{ij} \rfloor_{m\times n}$ is called a lower triangular matrix if $a_{ij} = 0 \ \forall \ i < j$.

SECTION-10.4. MATHEMATICAL OPERATIONS ON MATRICES

1. **Equality of matrices**
   Two matrices $A = \lfloor a_{ij} \rfloor_{m\times n}$ and $B = \lfloor b_{ij} \rfloor_{r\times s}$ are said to be equal, denoted by $A = B$, iff $m=r$, $n=s$ and $a_{ij} = b_{ij} \ \forall \ i, j$, otherwise they are not equal, denoted by $A \neq B$.
2. **Multiplication of a matrix by a scalar**
   The product of a scalar k and a matrix A, denoted by kA, is a matrix defined as $kA = \lfloor ka_{ij} \rfloor$.

   <u>Properties:-</u>
   i. $1A = A$
   ii. $0A_{m\times n} = O_{m\times n}$
   iii. $(kl)A = k(lA) = l(kA)$
3. **Negative of a matrix**
   $-1A$, denoted by $-A$, is said to be the negative of matrix A, i.e. $-A = \lfloor -a_{ij} \rfloor$.
4. **Addition of two matrices**

Two matrices *A* and *B* are conformable for addition if they are of the same order. Sum of two matrices *A* and *B*, denoted by $A+B$, is defined as a matrix obtained by adding the corresponding elements of *A* and *B* and their sum matrix is of the same order as *A* and *B*, i.e. $A_{m\times n}+B_{m\times n}=\left[a_{ij}+b_{ij}\right]_{m\times n}$.

Properties:-

i. $A+B=B+A$
ii. $A+(B+C)=(A+B)+C=A+B+C$
iii. $A+O=A=O+A$
iv. $A+C=B+C\Rightarrow A=B$
v. $k(A+B)=kA+kB$
vi. $(k+l)A=kA+lA$

5. Subtraction of two matrices

Two matrices *A* and *B* are conformable for subtraction if they are of the same order. Subtraction of two matrices *A* and *B*, denoted by $A-B$, is a matrix defined as $A-B=A+(-B)$. Therefore, the difference of two matrices *A* and *B* is a matrix obtained by subtracting the corresponding elements of *A* and *B* and their difference matrix is of the same order as *A* and *B*, i.e. $A_{m\times n}-B_{m\times n}=\left[a_{ij}-b_{ij}\right]_{m\times n}$.

Properties:-

i. $A-A=O$
ii. $A-O=A$
iii. $O-A=-A$

6. Multiplication of two matrices

Two matrices *A* and *B* are conformable for multiplication if *A* is of the order $m\times n$ and *B* is of the order $n\times p$. Their product matrix, denoted by *AB*, is of the order $m\times p$ and is defined as $AB=\left[C_{ij}\right]$ where $C_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}$. The matrix *A* is called pre-multiplier and matrix *B* is called post-multiplier.

Properties:-

i. In general, $AB\neq BA$.
ii. *AB* and *BA* are both defined iff *A* is of the order $m\times n$ and *B* is of the order $n\times m$.
iii. *AB* and *BA* are both defined and are of the same order iff *A* and *B* are square matrices.
iv. If *AB* and *BA* are both defined and are of the same order, even then *AB* and *BA* may not be equal.
v. If $AB=BA$, then *A* and *B* must be square matrices and *A* and *B* are said to be commuting matrices.
vi. *AA* is defined iff *A* is a square matrix.
vii. $k(AB)=(kA)B=A(kB)$
viii. $(AB)C=A(BC)$
ix. $A(B+C)=AB+AC$
x. $I_mA_{m\times n}=A_{m\times n}=A_{m\times n}I_n$
xi. $I_mI_m=I_m$
xii. $A_{m\times n}O_{n\times p}=O_{m\times p}$
xiii. $O_{p\times m}A_{m\times n}=O_{p\times n}$
xiv. If $AB = O$, then it does not imply that either $A = O$ or $B = O$.
xv. If $AB = O$, then it does not imply that $BA = O$.
xvi. If $AB = AC$, then it does not imply that $B = C$.

7. Natural powers of a square matrix

If A is a square matrix, then its natural power is defined as

i. $A^1 = A$

ii. $A^{n+1} = A^n A$

iii. $A^n = AAA\cdots\cdots(n \text{ times})$

Properties:-

i. $A^m A^n = A^{m+n}$

ii. $(A^m)^n = A^{mn}$

iii. ${I_m}^n = I_m$

8. Transpose of a matrix

A matrix obtained by interchanging rows and columns of a matrix A is called transpose of A and is denoted by A^T or A'. If A is a $m \times n$ matrix then A^T is a $n \times m$ matrix. $A^T = [b_{ij}]_{n\times m}$, $b_{ij} = a_{ji}$.

Properties:-

i. $(A^T)^T = A$

ii. $(kA)^T = kA^T$

iii. $(A+B)^T = A^T + B^T$

iv. $(AB)^T = B^T A^T$

v. $(ABC)^T = C^T B^T A^T$

vi. ${I_n}^T = I_n$

9. Symmetric matrix and Skew symmetric matrix

i. A square matrix is said to be symmetric if $a_{ij} = a_{ji}$ or $A^T = A$.

ii. I_n is a symmetric matrix.

iii. A square matrix is said to be skew symmetric if $a_{ij} = -a_{ji}$ or $A^T = -A$.

iv. The diagonal elements of a skew symmetric matrix are zero.

10. Determinant of a square matrix

Determinant of a square matrix A, denoted by $\det A$ or $|A|$, is the determinant formed by the elements of matrix A. If $A_{1\times 1}$ then $|A|$ is defined as $|A| = a_{11}$.

Properties:-

i. $|kA_{n\times n}| = k^n |A|$

ii. If A & B are square matrices then $|AB| = |A||B|$

iii. If A is a square matrix then $|A^n| = |A|^n$

iv. If A is a square matrix then $|A^T| = |A|$

v. $|O_{n\times n}| = 0$

vi. $|I_n| = 1$

11. Singular matrix and Non-singular matrix

i. A square matrix A is said to be singular if $|A| = 0$.

ii. A square matrix A is said to be non-singular if $|A| \neq 0$.

12. Adjoint of a square matrix

Adjoint of a square matrix $A = \lfloor a_{ij} \rfloor_{n \times n}$ is the matrix $\left(\left[C_{ij}\right]_{n\times n}\right)^T$, where C_{ij} denotes cofactor of a_{ij}. Adjoint of A is denoted by *adj A*.

Properties:-

i. $A(adj\ A) = |A|I = (adj\ A)A$

ii. If A is non-singular square matrix of order n then $|adj\ A| = |A|^{n-1}$

iii. If A & B are non-singular square matrices of same order then $adj\ AB = (adj\ B)(adj\ A)$

iv. If A is non-singular square matrix then $adj\ A^T = (adj\ A)^T$

v. If A is non-singular square matrix of order n then $adj\ (adj\ A) = |A|^{n-2} A$

vi. $adj\ I_n = I_n$

13. Inverse of a square matrix

A square matrix A is invertible if there exists a square matrix B of the same order such that $AB = I = BA$, then B is said to be the inverse matrix of matrix A, denoted by A^{-1}, otherwise matrix A is not invertible.

Properties:-

i. A square matrix is invertible iff it is non-singular. Singular matrices are non-invertible.

ii. Every non-singular matrix is invertible and possesses a unique inverse.

iii. $A^{-1} = \dfrac{1}{|A|}(adj\ A)$.

iv. $A^{-1}A = AA^{-1} = I$.

v. $\left(A^{-1}\right)^{-1} = A$.

vi. $\left|A^{-1}\right| = \dfrac{1}{|A|}$.

vii. If A, B, C are square matrices of same order and if A is non-singular then $AB = AC \Rightarrow B = C$ and $BA = CA \Rightarrow B = C$.

viii. $(AB)^{-1} = B^{-1}A^{-1}$.

ix. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

x. If A is invertible then A^T is also invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

xi. $I^{-1} = I$.

14. Orthogonal matrix

A square matrix is called an orthogonal matrix if $AA^T = A^T A = I$ or $A^{-1} = A^T$.

15. Indempotent matrix

i. A square matrix is called an indempotent matrix if $A^2 = A$.

ii. If A is an indempotent matrix then $A^n = A\ \forall\ n \geq 2$.

16. Involutary matrix

i. A square matrix is called an involutary matrix if $A^2 = I$.

ii. If A is an involutary matrix then $A^{-1} = A$.

iii. If A is an involutary matrix then $A^n = A$ if n is odd and $A^n = I$ if n is even.

17. Periodic matrix

i. A square matrix is called a periodic matrix if $A^{k+1} = A$ for some natural number *k*. The least value of *k* is said to be its period.

ii. An indempotent matrix is a periodic matrix of period 1.

18. Nilpotent matrix

i. A square matrix is called a nilpotent matrix if $A^k = O$ for some natural number *k*. The least value of *k* is said to be its index.

ii. If *A* is a nilpotent matrix of index *k* then $A^n = O \ \forall \ n \geq k$.

iii. A null square matrix is a nilpotent matrix of index 1.

19. Sub-matrix

A matrix obtained by leaving some rows or columns or both of matrix *A* is called a submatrix of *A*. The matrix *A* is itself a sub-matrix of *A*.

20. Rank of a matrix

Rank of a matrix, denoted by *r*(*A*), is the order of the highest order non-singular square submatrix. Rank of a null matrix is not defined.

Properties:-

i. $r(I_n) = n$.

ii. $r(A^T) = r(A)$.

iii. $r(A_{m\times n}) \leq \min\{m, n\}$.

iv. $r(A+B) \leq r(A) + r(B)$.

v. $r(AB) \leq \min\{r(A), r(B)\}$.

SECTION-10.5. SOLVING SYSTEM OF LINEAR EQUATIONS BY MATRIX

1. Consider the following system of *n* non-homogeneous linear equations in *n* unknowns x_1, x_2, ……., x_n

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
…………………………………
…………………………………
$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

This system of equation can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

or *AX* = *B*,

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$

The matrix *A* is called the coefficient matrix, matrix *X* is called the variable matrix and matrix *B* is called the

constant matrix.
Solution of system of non-homogeneous linear equations $AX = B$

Case-1: If $|A| \neq 0 \Rightarrow$ The system has unique solution, $X = A^{-1}B$

Case-2: If $|A| = 0$ & $(adj\ A)B \neq O \Rightarrow$ The system has no solution

Case-3: If $|A| = 0$ & $(adj\ A)B = O \Rightarrow$ The system has infinite solutions

2. Solution of system of homogeneous linear equations $AX = O$

Case-1: If $|A| \neq 0 \Rightarrow$ The system has unique solution, $X = O$ (Trivial solution)

Case-2: If $|A| = 0 \Rightarrow$ The system has infinite solutions (Trivial solution & non-trivial solutions)

EXERCISE-10

CATEGORY-10.1. EVALUATING DETERMINANTS

1. $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 8 & 14 & 20 \end{vmatrix}$ {Ans. 0}

2. $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$ {Ans. 0}

3. $\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$ {Ans. 4}

4. $\begin{vmatrix} 11 & 12 & 13 \\ 12 & 13 & 14 \\ 13 & 14 & 15 \end{vmatrix}$ {Ans. 0}

5. $\begin{vmatrix} 38 & 7 & 63 \\ 16 & 3 & 29 \\ 27 & 5 & 46 \end{vmatrix}$ {Ans. 0}

6. $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$ {Ans. 0}

7. $\begin{vmatrix} 18 & 40 & 89 \\ 40 & 89 & 198 \\ 89 & 198 & 440 \end{vmatrix}$ {Ans. -1}

8. $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$ {Ans. $5\sqrt{3}\left(\sqrt{6}-5\right)$ }

9. $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$ {Ans. 0}

10. $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$ {Ans. 0}

11. $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 69 \end{vmatrix}$ {Ans. 0}

12. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ {Ans. 0}

13. $\begin{vmatrix} a-b & b-c & c-a \\ x-y & y-z & z-x \\ p-q & q-r & r-p \end{vmatrix}$ {Ans. 0}

14. $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ {Ans. 0}

15. $\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$ {Ans. 0}

16. $\begin{vmatrix} \sin^2 x & \cos^2 x & 1 \\ \cos^2 x & \sin^2 x & 1 \\ -10 & 12 & 2 \end{vmatrix}$. {Ans. 0}

17. $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ {Ans. 0}

18. $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$ {Ans. 0}

19. $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$ {Ans. 0}

20. $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix}$ {Ans. 0}

21. $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$ {Ans. 0}

22. $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$ {Ans. 0}

23. $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+abd \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$ {Ans. 0}

24. If $D_r = \begin{vmatrix} r & x & \frac{n(n+1)}{2} \\ 2r-1 & y & n^2 \\ 3r-2 & z & \frac{n(3n-1)}{2} \end{vmatrix}$ then prove that $\sum_{r=1}^{n} D_r = 0$.

25. If $D_r = \begin{vmatrix} 2^{r-1} & 2(3^{r-1}) & 4(5^{r-1}) \\ x & y & z \\ 2^n-1 & 3^n-1 & 5^n-1 \end{vmatrix}$ then prove that $\sum_{r=1}^{n} D_r = 0$.

26. Let $\Delta_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^2 & 3n^2-3n \end{vmatrix}$ show that $\sum_{a=1}^{n} \Delta_a = c$, a constant.

27. Evaluate $\sum_{n=1}^{N} U_n$ if $U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N \end{vmatrix}$. {Ans. 0}

28. Express $\begin{vmatrix} 1 & 2 & -3 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix}^2$ in determinant form and find its value also. {Ans. $\begin{vmatrix} -1 & -5 & -4 \\ 6 & 8 & -4 \\ 10 & 10 & -2 \end{vmatrix}$, 196}

CATEGORY-10.2. PROVING IDENTITIES BY DETERMINANTS

29. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$.

30. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

31. $\begin{vmatrix} ka & k^2+a^2 & 1 \\ kb & k^2+b^2 & 1 \\ kc & k^2+c^2 & 1 \end{vmatrix} = k(a-b)(b-c)(c-a)$.

32. $\begin{vmatrix} 1 & (m+n-l-p)^2 & (m+n-l-p)^4 \\ 1 & (n+l-m-p)^2 & (n+l-m-p)^4 \\ 1 & (l+m-n-p)^2 & (l+m-n-p)^4 \end{vmatrix} = 64(l-m)(l-n)(l-p)(m-n)(m-p)(n-p)$.

33. $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix} = -(a+b+c)(a-b)(b-c)(c-a)$.

34. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix}$

35. $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \end{vmatrix} = (b-c)(c-a)(a-b)$.

36. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$.

37. $\begin{vmatrix} -2a & a+b & c+a \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+c)(c+a)(a+b)$.

38. $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$.

39. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

40. $\begin{vmatrix} x & y & z \\ -x & y & z \\ -x & -y & z \end{vmatrix} = 4xyz$

41. $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

42. $\begin{vmatrix} 2ab & a^2 & b^2 \\ a^2 & b^2 & 2ab \\ b^2 & 2ab & a^2 \end{vmatrix} = -\left(a^3+b^3\right)^2$

43. $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$

44. $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 4xyz$

45. $\begin{vmatrix} \dfrac{a^2+b^2}{c} & c & c \\ a & \dfrac{b^2+c^2}{a} & a \\ b & b & \dfrac{c^2+a^2}{b} \end{vmatrix} = 4abc$

46. $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = \left(b^2-ac\right)\left(ax^2+2bxy+cy^2\right).$

47. $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$

48. $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = \begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$

49. $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc.$

50. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$

51. $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$

52. $\begin{vmatrix} 1+a_1 & a_2 & a_3 & a_4 \\ a_1 & 1+a_2 & a_3 & a_4 \\ a_1 & a_2 & 1+a_3 & a_4 \\ a_1 & a_2 & a_3 & 1+a_4 \end{vmatrix} = 1+a_1+a_2+a_3+a_4.$

53. $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$

54. $\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix} = x^3(x+10).$

55. $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x+3a)(x-a)^3.$

56. $\begin{vmatrix} a^2+1 & ab & ac & ad \\ ab & b^2+1 & bc & bd \\ ac & bc & c^2+1 & cd \\ ad & bd & cd & d^2+1 \end{vmatrix} = \begin{vmatrix} a^2+1 & b^2 & c^2 & d^2 \\ a^2 & b^2+1 & c^2 & d^2 \\ a^2 & b^2 & c^2+1 & d^2 \\ a^2 & b^2 & c^2 & d^2+1 \end{vmatrix} = 1+a^2+b^2+c^2+d^2.$

57. $\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix}.$

58. $\begin{vmatrix} x^3 & 3x^2 & 3x & 1 \\ x^2 & x^2+2x & 2x+1 & 1 \\ x & 2x+1 & x+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (x-1)^6$

59. $\begin{vmatrix} x^2+2x & 2x+1 & 1 \\ 2x+1 & x+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (x-1)^3$

60. $\begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix} = -8.$

61. $\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & b \\ -y & -c & 0 & a \\ -z & -b & -a & 0 \end{vmatrix} = (ax - by + cz)^2.$

62. $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)\left(a^2+b^2+c^2\right).$

63. $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = \begin{vmatrix} (a+b)^2 & ca & cb \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3$

64. $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3.$

65. $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$

66. $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (bc+ca+ab)^3.$

67. $\begin{vmatrix} ax-by-cz & ay+bx & cx+az \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix}. = \left(x^2+y^2+z^2\right)\left(a^2+b^2+c^2\right)(ax+by+cz).$

68. $\begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & ca+bd & c^2a^2+b^2d^2 \\ 1 & ab+cd & a^2b^2+c^2d^2 \end{vmatrix} = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$

69. $\begin{vmatrix} x^2 & x^2-(y-z)^2 & yz \\ y^2 & y^2-(z-x)^2 & zx \\ z^2 & z^2-(x-y)^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)\left(x^2+y^2+z^2\right).$

70. $\begin{vmatrix} a_1\alpha_1+b_1\beta_1 & a_1\alpha_2+b_1\beta_2 & a_1\alpha_3+b_1\beta_3 \\ a_2\alpha_1+b_2\beta_1 & a_2\alpha_2+b_2\beta_2 & a_2\alpha_3+b_2\beta_3 \\ a_3\alpha_1+b_3\beta_1 & a_3\alpha_2+b_3\beta_2 & a_3\alpha_3+b_3\beta_3 \end{vmatrix} = 0.$

71. $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$

72. $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$, where $r^2 = x^2 + y^2 + z^2$ and $u^2 = yz + zx + xy$.

73. $\begin{vmatrix} \cos^2 x & \cos x \sin x & -\sin x \\ \cos x \sin x & \sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix} = 1$

74. If $a^{-1} + b^{-1} + c^{-1} = 0$, prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc$.

75. If $p + q + r = 0$, prove that $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

76. If $2s = a + b + c$, prove that $\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3(s-a)(s-b)(s-c)$.

77. Without expanding at any stage show that $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = xA + B$, where A and B are determinants of order 3 not involving x.

78. If $y = \sin px$ and y_r means rth derivative of y then prove that $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix} = 0$.

79. If a, b, c (all +ive) are the pth, qth, rth, terms respectively of a geometric progression, then prove that $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$.

80. If a, b, c are given to be in A.P., prove that $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$.

81. Given that $A + B + C = \pi$, prove that $\begin{vmatrix} \sin^2 A & \sin A \cos A & \cos^2 A \\ \sin^2 B & \sin B \cos B & \cos^2 B \\ \sin^2 C & \sin C \cos C & \cos^2 C \end{vmatrix} = -\sin(A-B)\sin(B-C)\sin(C-A)$.

CATEGORY-10.3. EQUATIONS CONTAINING DETERMINANTS

82. Find the roots of the equation $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$. {Ans. –1. 2}

83. $\begin{vmatrix} a & a & x \\ x & x & x \\ b & x & b \end{vmatrix} = 0$. {Ans. x = 0, a, b}

84. $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0$. {Ans. $0, -(a+b+c)$}

85. $\begin{vmatrix} x+a & a^2 & a^3 \\ x+b & b^2 & b^3 \\ x+c & c^2 & c^3 \end{vmatrix} = 0$. {Ans. $x = -\dfrac{abc}{ab+bc+ac}$}

86. $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$. {Ans. x = 4}

87. $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$ {Ans. x = 4}

88. $\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0$ {Ans. $x = -\frac{11}{97}$}

89. $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$ {Ans. $x = \frac{2}{3}, \frac{11}{3}$}

90. $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$ {Ans. x = -2, -1}

91. $\begin{vmatrix} x+2 & x+6 & x-1 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$ {Ans. $x = -\dfrac{7}{3}$}

92. Show that x = 2 is a root of $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$ and solve it completely. {Ans. x = -3, 1, 2}

93. Solve $\begin{vmatrix} 1 & 3 & 9 \\ 1 & x & x^2 \\ 4 & 6 & 9 \end{vmatrix} = 0$. {Ans. $x = 3, \frac{3}{2}$ }

94. Show that x= -9 is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ and find the other two roots. {Ans. x = 2, 7}

95. Given $a + b + c = 0$, solve $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ {Ans. $x = 0, \pm\sqrt{\frac{3}{2}\left(a^2+b^2+c^2\right)}$ }

96. If $(b-c)^2 \neq (a-b)(c-a)$, solve for x $\begin{vmatrix} a+x & b+x & c+x \\ b+x & c+x & a+x \\ c+x & a+x & b+x \end{vmatrix} = 0$ {Ans. $x = -\frac{a+b+c}{3}$ }

97. If $a \neq b \neq c$, solve for x $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$. {Ans. x = 0}

CATEGORY-10.4. MISCELLANEOUS QUESTIONS ON DETERMINANTS

98. If $a \neq p, b \neq q, c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, then find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$. {Ans. 2}

99. Suppose three digit numbers A28, 3B9 and 62C where A, B and C are integers between 0 and 9, are divisible by a fixed integer k. Prove that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is also divisible by k.

100. For a fixed positive integer n, if $D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$ then show that $\left[\frac{D}{(n!)^3} - 4\right]$ is divisible by n.

101. If x,y and z are all different and given that $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, prove that 1+$xyz$ = 0

102. If x, y and z are all different and given that $\begin{vmatrix} x & x^3 & x^4-1 \\ y & y^3 & y^4-1 \\ z & z^3 & z^4-1 \end{vmatrix} = 0$, prove that $xyz(xy+yz+zx) = x+y+z$.

103. Let a, b, c be positive and not all equal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.

104. Let α be a repeated root of quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$, $C(x)$ be polynomials of degree 3, 4 and 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where $'$ denotes the derivative.

CATEGORY-10.5. SOLVING SYSTEM OF LINEAR EQUATIONS BY DETERMINANTS

105. Solve

$x + 2y + 3z = 6$

$2x + 4y + z = 7$

$3x + 2y + 9z = 14.$

{Ans. (1, 1, 1)}

106. Solve

$x + 2y + 3z - 6 = 0$

$2x + 4y + z = 17$

$3x + 9z + 2y = 2.$

{Ans. (1, 4, -1)}

107. Solve

$x + y + z = 11$

$2x - 6y - z = 0$

$3x + 4y + 2z = 0.$

{Ans. (-8, -7, 26)}

108. Solve

$x + y + z = 6$

$x - y + z = 2$

$2x + y - z = 1.$

{Ans. (1, 2, 3)}

109. Solve

$5x - 6y + 4z = 15$

$7x + 4y - 3z = 19$

$2x + y + 6z = 46.$

{Ans. (3, 4, 6)}

110. Solve

$x + y + z = 1$

$3x + 5y + 6z = 4$

$9x + 2y - 36z = 17.$

{Ans. $\left(\frac{1}{3}, 1, -\frac{1}{3}\right)$}

111. Find a, b, c when $f(x) = ax^2 + bx + c$ and $f(0) = 6$, $f(2) = 11$, $f(-3) = 6$. Determine $f(x)$ and find the value of $f(1)$. {Ans. $a = \frac{1}{2}, b = \frac{3}{2}, c = 6$, $f(x) = \frac{1}{2}x^2 + \frac{3}{2}x + 6$, $f(1) = 8$}

112. Solve

$(b+c)(y+z) - ax = b - c$

$(c+a)(z+x) - by = c - a$

$(a+b)(x+y) - cz = a - b$

where $a+b+c \neq 0$. {Ans. $\left(\frac{c-b}{a+b+c}, \frac{a-c}{a+b+c}, \frac{b-a}{a+b+c}\right)$}

113. For what value of k, the system of linear equations $x + y + z = 2,\ 2x + y - z = 3,\ 3x + 2y + kz = 4$ has a unique solution? {Ans. $k \neq 0$}

114. For what value of k, the system of simultaneous equations $kx + 2y - z = 1,\ (k-1)y - 2z = 2$ and $(k+2)z = 3$ have a unique solution? {Ans. $k \neq -2, 0, 1$}

115. Solve

$x + 4y - 2z = 3$

$3x + y + 5z = 7$

$2x + 3y + z = 5.$

{Ans. no solution}

116. Solve

$2x + 3y - 2z = 3$

$x + 2y + z = 4$

$5x + 9y + z = 15$

{Ans. $\left(t, \frac{11-4t}{7}, \frac{6+t}{7}\right)$}

117. Find k for which the set of equations

$x + y - 2z = 0$

$2x - 3y + z = 0$

$x - 5y + 4z = k$

are consistent and find the solution for all such values of k. {Ans. $k = 0$, (t, t, t)}

118. Show that the system of equations

$3x - y + 4z = 3$

$x + 2y - 3z = -2$

$6x + 5y + \lambda z = -3$

has at least one solution for any real number λ. Find the set of solutions if $\lambda = -5$. {Ans. $\left(\frac{4-5t}{7}, \frac{13t-9}{7}, t\right)$}

119. Solve

$3x - y + z = 0$

$-15x + 6y - 5z = 0$

$5x - 2y + 2z = 0$

{Ans. (0, 0, 0)}

120. Solve

$x + 3y - 2z = 0$

$2x - y + 4z = 0$

$x - 11y + 14z = 0$

{Ans. $\left(-\frac{10k}{7}, \frac{8k}{7}, k\right)$}

121. Find all values of k for which the following system possesses a non-trivial solution

$x + ky + 3z = 0$

$kx + 2y + 2z = 0$

$2x + 3y + 4z = 0$

{Ans. 2, $\frac{5}{4}$ }

122. For what value of k do the following system of equations possess a non-trivial solution over the set of rationals

$x + ky + 3z = 0$

$3x + ky - 2z = 0$

$2x + 3y - 4z = 0.$

For that value of k, find all the solutions of the system. {Ans. $k = \frac{33}{2}$, $\left(-\frac{15t}{2}, t, -3t\right)$}

123. Given

$x = cy + bz$

$y = az + cx$

$z = bx + ay$

where x, y, z are not all zero prove that $a^2 + b^2 + c^2 + 2abc = 1.$

124. If $a = \frac{x}{y - z}$, $b = \frac{y}{z - x}$ and $c = \frac{z}{x - y}$, where x, y, z are not all zero, prove that $1 + ab + bc + ca = 0.$

125. Let α_1, α_2, and β_1, β_2 be the roots of $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$ respectively. If the system of equations

$\alpha_1 y + \alpha_2 z = 0$

$\beta_1 y + \beta_2 z = 0$

has non-trivial solution, then prove that $\frac{b^2}{q^2} = \frac{ac}{pr}$.

126. If a, b, c are in G.P. with common ratio r_1 and α, β, γ also form in G.P.with common ratio r_2, then find the conditions that r_1 and r_2 must satisfy so that the equations

$ax + \alpha y + z = 0$

$bx + \beta y + z = 0$

$cx + \gamma y + z = 0$
have only zero solution. {Ans. $r_1 \neq r_2, r_1 \neq 1, r_2 \neq 1$}

127. Solve
$2x + 3y = 2$
$x - y = 1$
$x + 2y = 2.$
{Ans. no solution}

128. Solve
$x + 2y = 3$
$2x - y = 1$
$x - 2y = -1.$
{Ans. (1, 1)}

129. Find the value of λ if the following equations are consistent
$x + y - 3 = 0$
$(1+\lambda)x + (2+\lambda)y - 8 = 0$
$x - (1+\lambda)y + (2+\lambda) = 0$
{Ans. 1, $-\frac{5}{3}$}

130. If the equations
$ax + hy + g = 0$
$hx + by + f = 0$
$gx + fy + c = \lambda$
are consistent, show that $\lambda = \dfrac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2}$.

131. If the equations
$(b+c)x + (c+a)y + (a+b) = 0$
$cx + ay + b = 0$
$ax + by + c = 0$
are consistent then show that either $a + b + c = 0$ or $a = b = c$.

132. If *a*, *b*, *c* are all different and the equations
$ax + a^2y + (a^3 + 1) = 0$
$bx + b^2y + (b^3 + 1) = 0$
$cx + c^2y + (c^3 + 1) = 0$
are consistent then prove that $abc + 1 = 0$.

133. Find the value of *a* if the three equations are consistent
$(a+1)^3x + (a+2)^3y = (a+3)^3$
$(a+1)x + (a+2)y = (a+3)$
$x + y = 1.$
{Ans. $a = -2$}

134. Are the equations

$x + ay = b + c$

$x + by = c + a$

$x + cy = a + b$

where a, b and c are real numbers such that $a^2 + b^2 + c^2 = 1$, consistent? {Ans. consistent}

CATEGORY-10.6. SYSTEM OF LINEAR EQUATIONS WITH PARAMETERS

135. $\begin{Bmatrix} ax + y = 2 \\ x + ay = 2a \end{Bmatrix}$

$$\begin{Bmatrix} Ans. & \\ a \neq \pm 1, & (0,2) \\ a = 1, & (t, 2-t) \\ a = -1, & (t, 2+t) \end{Bmatrix}$$

136. $\begin{Bmatrix} x + ay - 1 = 0 \\ ax - 3ay - (2a+3) = 0 \end{Bmatrix}$

$$\begin{Bmatrix} Ans. & \\ a \neq 0, -3, & \left(2, -\frac{1}{a}\right) \\ a = -3, & \left(t, \frac{t-1}{3}\right) \\ a = 0 & \text{no solution} \end{Bmatrix}$$

137. $\begin{Bmatrix} 3x + ay = 5a^2 \\ 3x - ay = a^2 \end{Bmatrix}$

$$\begin{Bmatrix} Ans. & \\ a \neq 0, & (a^2, 2a) \\ a = 0, & (0, t) \end{Bmatrix}$$

138. $\begin{Bmatrix} (a+5)x + (2a+3)y - (3a+2) = 0 \\ (3a+10)x + (5a+6)y - (2a+4) = 0 \end{Bmatrix}$

$$\begin{Bmatrix} Ans. & \\ a \neq 0, 2, & \left(\frac{11a+14}{2-a}, \frac{7a+22}{a-2}\right) \\ a = 0, & \left(t, \frac{2-5t}{3}\right) \\ a = 2, & \text{no solution} \end{Bmatrix}$$

139. $\left\{\begin{array}{l} a(a-1)x + a(a+1)y = a^3+2 \\ (a^2-1)x + (a^3+1)y = a^4-1 \end{array}\right\}$

$\left\{\begin{array}{l} Ans. \\ a \neq 0, \pm 1, 2, \quad \left(\dfrac{a^3+a^2+1}{a^2(1-a)}, \dfrac{a^3+a+1}{a^2}\right) \\ a=-1, \quad \left(\dfrac{1}{2}, t\right) \\ a=2, \quad \left(t, \dfrac{5-t}{3}\right) \\ a=0,1, \quad \text{no solution} \end{array}\right\}$

140. $\left\{\begin{array}{l} ax - y = b \\ bx + y = a \end{array}\right\}$

$\left\{\begin{array}{l} Ans. \\ a+b \neq 0, \quad (1, a-b) \\ a+b=0, \quad (t, a-bt) \end{array}\right\}$

141. $\left\{\begin{array}{l} (a^2+b^2)x + (a^2-b^2)y = a^2 \\ (a+b)x + (a-b)y = a \end{array}\right\}$

$\left\{\begin{array}{l} Ans. \\ a \neq b \neq 0, \quad \left(\dfrac{1}{2}, \dfrac{1}{2}\right) \\ a=0, b \neq 0, \quad (t,t) \\ a \neq 0, b=0, \quad (t, 1-t) \\ a=b \neq 0, \quad \left(\dfrac{1}{2}, t\right) \\ a=b=0, \quad (t_1, t_2) \end{array}\right\}$

142. For what values of m does the system of equations

$3x + my = m$

$2x - 5y = 20$

has a solution satisfying the conditions $x > 0,\ y > 0$. {Ans. $m \in \left(-\infty, -\frac{15}{2}\right) \cup (30, \infty)$ }

143. For what values of a does the system

$a^2x + (2-a)y = 4 + a^2$

$ax + (2a-1)y = a^5 - 2$

possess no solution? {Ans. a = -1, 1}

144. For what values of the parameter a does the system of equations

$ax - 4y = a + 1$

$2x + (a+6)y = a + 3$

possess no solutions? {Ans. a = -4}

145. For what values of the parameter a does the system

$2x + ay = a + 2$

$(a+1)x + 2ay = 2a + 4$

possess infinitely many solutions? {Ans. a = 3}

146. Find the values of the parameters m and p such that the system

$(3m - 5p + 1)x + (8m - 3p - 1)y = 1$

$(2m - 3p + 1)x + (4m - p)y = 2$

possesses infinite solutions. {Ans. $p = \frac{5}{16}, m = \frac{19}{64}$ }

147. Find the parameters a and b such that the system

$a^2x - ay = 1 - a$

$bx + (3 - 2b)y = 3 + a$

possesses a unique solution x = 1, y = 1. {Ans. a = 1, b = -1}

148. For what values of a and b does the system

$a^2x - by = a^2 - b$

$bx - b^2y = 2 + 4b$

possesses an infinite number of solutions? {Ans. (1, -1), (1, -2), (-1, -1), (-1, -2)}

149. Find the values of λ and μ so that the system of equation

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

has

i. Unique solution {Ans. $\lambda \neq 3, \mu \in R$ }

ii. Infinite solutions {Ans. $\lambda = 3, \mu = 10$ }

iii. No solution {Ans. $\lambda = 3, \mu \neq 10$ }

150. For what values of a, b and c, the system

$ax - by = 2a - b$

$(c+1)x + cy = 10 - a + 3b$

has infinitely many solutions and x = 1, y = 3 is one of the solutions? {Ans. $\left(0, 0, \frac{9}{4}\right), (2, -1, 1)$ }

CATEGORY-10.7. MATRIX AND ITS PROPERTIES

151. Find a 3×4 matrix A given that $a_{ij} = \frac{(i+j)^2}{2}$. {Ans. $A = \begin{bmatrix} 2 & \frac{9}{2} & 8 & \frac{25}{2} \\ \frac{9}{2} & 8 & \frac{25}{2} & 18 \\ 8 & \frac{25}{2} & 18 & \frac{49}{2} \end{bmatrix}$ }

152. Find a matrix $A_{2\times3}$ given that $a_{ij}=\left[\frac{i}{j}\right]$, where [ ] denotes greatest integer function. {Ans. $A=\begin{bmatrix}1&0&0\\2&1&0\end{bmatrix}$}

153. If $A=\begin{bmatrix}1&-5&7\\0&7&9\\11&8&9\end{bmatrix}$, then find the trace of matrix A. {Ans. 17}

CATEGORY-10.8. EQUALITY OF MATRICES

154. Find the value of x, y, z and a which satisfy the matrix equation $\begin{bmatrix}x+3&2y+x\\z-1&4a-6\end{bmatrix}=\begin{bmatrix}0&-7\\3&2a\end{bmatrix}$ {Ans. x = -3, y = -2, z = 4, a = 3}

155. Find x and y if $\begin{bmatrix}x+y&2\\1&x-y\end{bmatrix}=\begin{bmatrix}3&2\\1&7\end{bmatrix}$ {Ans. x = 5, y = -2}

156. Find x, y, z, w if $\begin{bmatrix}x-y&2x+z\\2x-y&3z+w\end{bmatrix}=\begin{bmatrix}-1&5\\0&13\end{bmatrix}$ {Ans. x = 1, y = 2, z = 3, w = 4}

157. Find x, y, z if $\begin{bmatrix}x+y&y-z\\z-2x&y-x\end{bmatrix}=\begin{bmatrix}3&-1\\1&1\end{bmatrix}$ {Ans. x = 1, y = 2, z = 3}

158. Find a, b, c, d if $\begin{bmatrix}a+3&2b-8\\c+1&4d-6\end{bmatrix}=\begin{bmatrix}0&-6\\-3&2d\end{bmatrix}$ {Ans. a = -3, b = 1, c = -4, d = 3}

CATEGORY-10.9. MULTIPLICATION BY SCALAR, ADDITION, SUBTRACTION OF MATRICES

159. If $A=\begin{bmatrix}2&3&1\\0&-1&5\end{bmatrix}$ and $B=\begin{bmatrix}1&2&-1\\0&-1&3\end{bmatrix}$ evaluate $3A-4B$. {Ans. $\begin{bmatrix}2&1&7\\0&1&3\end{bmatrix}$}

160. If $P=\begin{bmatrix}1&2&3\\0&5&7\\6&8&9\end{bmatrix}$, $Q=\begin{bmatrix}2&0&3\\3&0&5\\5&7&0\end{bmatrix}$, evaluate $2P-3Q$. {Ans. $\begin{bmatrix}-4&4&-3\\-9&10&-1\\-3&-5&18\end{bmatrix}$}

161. If $A=\begin{bmatrix}1&-2\\3&0\end{bmatrix}$, $B=\begin{bmatrix}-1&4\\2&3\end{bmatrix}$, $C=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$, then find $5A-3B+2C$. {Ans. $\begin{bmatrix}8&-20\\7&-9\end{bmatrix}$}

162. If $A=\begin{bmatrix}1&2\\-1&0\\1&-3\end{bmatrix}$, $B=\begin{bmatrix}4&5\\-1&0\\2&1\end{bmatrix}$, $C=\begin{bmatrix}-1&-2\\-1&2\\-1&-2\end{bmatrix}$, find $A-2B+3C$. {Ans. $\begin{bmatrix}-10&-14\\-2&6\\-6&-11\end{bmatrix}$}

163. If $\begin{bmatrix}x&0\\1&y\end{bmatrix}+\begin{bmatrix}-2&1\\3&4\end{bmatrix}=\begin{bmatrix}3&5\\6&3\end{bmatrix}-\begin{bmatrix}2&4\\2&1\end{bmatrix}$, then find x, y. {Ans. $x=3,\ y=-2$}

164. Given $A=\begin{bmatrix}1&2&-3\\5&0&2\\1&-1&1\end{bmatrix}$ and $B=\begin{bmatrix}3&-1&2\\4&2&5\\2&0&3\end{bmatrix}$. Find the matrix C such that $A+2C=B$. {Ans.

$\begin{bmatrix} 1 & -\frac{3}{2} & \frac{5}{2} \\ -\frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$}

165. Find A and B if $A+B=\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ and $A-B=\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$. {Ans. $A=\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$, $B=\begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$}

166. If $A+B=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $A-2B=\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$, then find A. {Ans. $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$}

167. Find A and B if $2A-B=\begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$ and $2B+A=\begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$. {Ans. $A=\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$,

$B=\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$}

168. If $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, J=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then show that $B=I\cos\theta+J\sin\theta$.

CATEGORY-10.10. MULTIPLICATION OF MATRICES, POWER OF A MATRIX

169. What is the order of $\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. {Ans. 1×1}

170. Given $A=\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, find AB and BA. {Ans. $AB=\begin{bmatrix} 0 & 3 & 1 \\ 4 & -1 & 4 \end{bmatrix}$, BA is not defined}

171. Given $A=\begin{bmatrix} 1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, find AB and BA. {Ans. $AB=\begin{bmatrix} 5 \end{bmatrix}$, $BA=\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$}

172. If $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then find AB and BA and show that $AB\neq BA$. {Ans. $AB=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$,

$BA=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$}

173. If $A=\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, obtain the product AB and BA and show that $AB\neq BA$. {Ans.

$AB = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$, $BA = \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$}

174. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then find A^2 and A^3. {Ans. $A^2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$, $A^3 = \begin{bmatrix} 19 & 18 \\ -18 & 1 \end{bmatrix}$}

175. If $A_{1\times 2} = [a \quad b]$, $B_{1\times 2} = [-b \quad -a]$ and $C_{2\times 1} = \begin{bmatrix} a \\ -a \end{bmatrix}$, then show that $AC = BC$.

176. Given $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, find *AB*. {Ans. $AB = O_{2\times 2}$ }

177. Given $A = [1 \quad 0]$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find *AB* and *BA*. {Ans. $AB = O_{1\times 1}$, $BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$}

178. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$, evaluate $A^2 - B^2$. {Ans. not defined}

179. If $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$, show that $(A+B)^2 = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$

180. If $A = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$. Verify that $(A+B)^2 = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$.

181. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, show that $(A+B)(A-B) \neq A^2 - B^2$.

182. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$. Show that $A(B+C) = AB + AC$.

183. The matrix ***R*(*t*)** is defined by $R(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. Show that ***R*(s) *R*(*t*) = *R*(s + *t*)**.

184. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, $B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$. Show that *AB* = *BA*.

185. If *a*, *b*, *c*, *d* are real numbers and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, prove that $A^2 - (a+d)A + (ad-bc)I = O$.

186. Show that $E^2F + F^2E = E$, where $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

187. Find *x* so that $[1 \quad x \quad 1]\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = O$. {Ans. $\frac{1}{2}\left(-9 \pm \sqrt{53}\right)$}

188. If $A = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$, find A^3. {Ans. $\begin{bmatrix} -0.352 & 0.936 \\ -0.936 & -0.352 \end{bmatrix}$}

189. If $X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then find X^3.

190. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then find A^{11}. {Ans. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$}

191. If $J_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $J_2 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$, then find $J_1^2 + J_2^2$.

192. If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 2 & 1 \\ 1 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & -1 \end{bmatrix}$. Show that $(AB)C = A(BC)$.

193. If $f(x) = x^2 - 5x + 6I$, find $f(A)$ if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$. {Ans. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$}

194. If $A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$, $X = \begin{bmatrix} n \\ 1 \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$ and $AX = B$, then find n. {Ans. 2}

195. If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, then show that $\alpha = a^2 + b^2, \beta = 2ab$.

196. If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $A^2 - 4A - nI_2 = O$, then find the value of n. {Ans. -3}

197. If $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ and $A^2 - kA - 5I_2 = O$, then find the value of k. {Ans. 5}

198. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, I is the unit matrix of order 2 and a, b are arbitrary constants, then show that $(aI + bA)^2 = a^2 I + 2abA$.

CATEGORY-10.11. TRANSPOSE OF A MATRIX

199. If $A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$, then find $A + A^T$. {Ans. $\begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$}

200. Verify that $(AB)^T = B^T A^T \neq A^T B^T$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 4 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & -1 \\ -3 & 2 & 4 \\ 1 & 1 & 0 \end{bmatrix}$.

201. If $A = \begin{bmatrix} 4 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 4 \end{bmatrix}$, show that $(AB)^T = B^T A^T$.

202. If $A=\begin{bmatrix}1 & -1 & 0\\ 2 & 1 & 3\\ 4 & 1 & 8\end{bmatrix}$ and $B=\begin{bmatrix}4 & 1 & 0\\ 2 & -3 & 1\\ 1 & 1 & -1\end{bmatrix}$, then verify that $(AB)^T = B^T A^T$.

203. If matrix $A=\begin{bmatrix}a & b & c\\ b & c & a\\ c & a & b\end{bmatrix}$, where *a*, *b*, *c* are real positive numbers, *abc* = 1 and $A^T A = I$, then find the value of $a^3+b^3+c^3$. {Ans. 4}

CATEGORY-10.12. SYMMETRIC AND SKEW-SYMMETRIC MATRIX

204. If $A=\begin{bmatrix}5 & x\\ y & 0\end{bmatrix}$ is a symmetric matrix, then show that $x=y$.
205. If $A=\begin{bmatrix}4 & x+2\\ 2x-3 & x+1\end{bmatrix}$ is symmetric, then find *x*. {Ans. 5}
206. If the matrix $\begin{bmatrix}0 & a & 5\\ 3 & 0 & b\\ c & 2 & 0\end{bmatrix}$ is skew-symmetric, then find *a*, *b*, *c*. {Ans. $a=-3, b=-2, c=-5$}
207. If A is a skew-symmetric matrix, then find trace of A. {Ans. 0}
208. If for a square matrix $A=[a_{ij}], a_{ij}=i^2-j^2$, then show that A is a skew-symmetric matrix.
209. Let *A* be a square matrix, then prove that
    i. $A+A^T$ is a symmetric matrix
    ii. $A-A^T$ is a skew-symmetric matrix
    iii. AA^T and $A^T A$ are symmetric matrices.
210. Prove that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.
211. If *A* and *B* are symmetric matrices, then show that *AB* is symmetric iff *AB* = *BA*.
212. Show that the matrix $B^T AB$ is symmetric or skew-symmetric according as *A* is symmetric or skew-symmetric.
213. Show that all natural powers of a symmetric matrix are symmetric.
214. Show that odd natural powers of a skew-symmetric matrix are skew-symmetric and even natural powers of a skew-symmetric matrix are symmetric.
215. If *A* and *B* be symmetric matrices of the same order, then show that
    i. $A+B$ is a symmetric matrix
    ii. $AB-BA$ is a skew-symmetric matrix
    iii. $AB+BA$ is a symmetric matrix

CATEGORY-10.13. DETERMINANT OF A SQUARE MARIX, SINGULAR AND NON-SINGULAR MATRIX

216. If $A=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}$, find det A. {Ans. 42}

217. For what value of k, $\begin{bmatrix}1 & 2 & -3\\ 5 & 0 & k\\ 1 & -1 & 1\end{bmatrix}$ is a singular matrix. {Ans. $-\frac{5}{3}$ }

218. If A and B are two square matrices of order 3 such that $|A|=-1$, $|B|=3$, then find $|3AB|$. {Ans. -81}

219. Prove that a skew-symmetric matrix of odd order must be a singular matrix.

CATEGORY-10.14. ADJOINT OF A SQUARE MARIX

220. If $A=\begin{bmatrix}-5 & 2\\ 1 & -3\end{bmatrix}$, then find $adjA$. {Ans. $\begin{bmatrix}-3 & -2\\ -1 & -5\end{bmatrix}$}

221. Find the adjoint of $\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$. {Ans. $\begin{bmatrix}3 & -4 & -5\\ -9 & 1 & 4\\ -5 & 3 & 1\end{bmatrix}$}

222. Find the adjoint of the matrix $A=\begin{bmatrix}1 & 2\\ 3 & -5\end{bmatrix}$ and verify that $A(adjA)=|A|I_2=(adjA)A$. {Ans. $\begin{bmatrix}-5 & -2\\ -3 & 1\end{bmatrix}$}

223. Find the adjoint of the matrix $A=\begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\\ 0 & -6 & -7\end{bmatrix}$ and verify that

i. $A(adjA)=|A|I_3=(adjA)A$

ii. $|adjA|=|A|^2$

iii. $adjA^T=(adjA)^T$

iv. $adj(adjA)=|A|A$.

{Ans. $\begin{bmatrix}2 & 6 & 4\\ 21 & -7 & -8\\ -18 & 6 & 4\end{bmatrix}$}

224. For the matrix A, prove that $A(adjA)=O$, where $A=\begin{bmatrix}1 & -1 & 1\\ 2 & 3 & 0\\ 18 & 2 & 10\end{bmatrix}$.

225. If $A=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 1 & 2 & 0\end{bmatrix}$, then verify that $adjAB=(adjB)(adjA)$.

226. If $A=\begin{bmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{bmatrix}$ and $A(adjA)=k\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, then find the value of k. {Ans. 1}

CATEGORY-10.15. INVERSE OF A SQUARE MATRIX

227. Find $\begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}^{-1}$. {Ans. $\begin{bmatrix} 10 & -3 \\ -3 & 1 \end{bmatrix}$}

228. If $A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$, then find A^{-1}. {Ans. $\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$}

229. Find A^{-1}, if the matrix A is given by $A = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$. {Ans. $\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$}

230. For what value of k the matrix $A = \begin{bmatrix} 2 & k \\ 3 & 5 \end{bmatrix}$ has no inverse. {Ans. $\frac{10}{3}$}

231. If $A = \begin{bmatrix} 2x & 0 \\ x & x \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, then find the value of x. {Ans. $\frac{1}{2}$}

232. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that $A^{-1} = kA$, then find the value of k. {Ans. $\frac{1}{19}$}

233. X is an unknown square matrix satisfying the equation $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Determine the matrix X.

{Ans. $\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$}

234. If $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then find the matrix A. {Ans. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$}

235. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. Verify that $|A^{-1}| = \frac{1}{|A|}$. {Ans. $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$}

236. If $A = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}$, find A^{-1} and verify that $A^{-1} = -\frac{1}{7}A + \frac{8}{7}I$. {Ans. $\frac{1}{7}\begin{bmatrix} 6 & -5 \\ -1 & 2 \end{bmatrix}$}

237. If $A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

238. Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$ and verify that $A^{-1}A = I$. {Ans. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{9}{2} & -\frac{5}{2} & -\frac{1}{2} \end{bmatrix}$}

239. Find the inverse of the matrix $A = \begin{bmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ and verify that $A^{-1}A = I$. {Ans. $-\frac{1}{19}\begin{bmatrix} -1 & 6 & -1 \\ 5 & 8 & -14 \\ -3 & -1 & -3 \end{bmatrix}$}

240. Find the inverse of the matrix $A=\begin{bmatrix}2 & -2 & 4\\ 2 & 3 & 2\\ -1 & 1 & -1\end{bmatrix}$ and verify that $A^{-1}A=I$. {Ans. $\frac{1}{10}\begin{bmatrix}-5 & 2 & -16\\ 0 & 2 & 4\\ 5 & 0 & 10\end{bmatrix}$}

241. If $F(\alpha)=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$ then show that $F(x)F(y)=F(x+y)$. Hence prove that $[F(x)]^{-1}=F(-x)$.

242. If $A=\begin{bmatrix}2 & 0 & -1\\ 5 & 1 & 0\\ 0 & 1 & 3\end{bmatrix}$, prove that $A^{-1}=A^2-6A+11I$.

243. If $A=\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}$ and $B=\begin{bmatrix}4 & 5\\ 3 & 4\end{bmatrix}$, verify that $(AB)^{-1}=B^{-1}A^{-1}$.

244. If $A=\begin{bmatrix}4 & 5\\ 2 & 1\end{bmatrix}$, then show that $A-3I=2I+6A^{-1}$.

245. Let $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$, prove that $A^2-4A-5I=O$. Hence obtain A^{-1}. {Ans. $\frac{1}{5}\begin{bmatrix}-3 & 2 & 2\\ 2 & -3 & 2\\ 2 & 2 & -3\end{bmatrix}$}

246. If $\begin{bmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{bmatrix}\begin{bmatrix}1 & \tan\theta\\ -\tan\theta & 1\end{bmatrix}^{-1}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$, then show that $a=\cos 2\theta, b=\sin 2\theta$.

247. Show that the inverse of a diagonal matrix is a diagonal matrix.

CATEGORY-10.16. ORTHOGONAL MATRIX

248. If $A=\frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$, prove that A is an orthogonal matrix.

249. Show that if A is an orthogonal matrix, then A^T is also orthogonal.

250. If A, B be orthogonal matrices, then prove that AB and BA are also orthogonal matrices.

CATEGORY-10.17. INDEMPOTENT, INVOLUTARY, PERIODIC, NILPOTENT MATRIX

251. Prove that the matrix $A=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}$ is idempotent.

252. Find all idempotent diagonal matrices of order 3. {Ans. $\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$}

253. If B is idempotent, show that $A=I-B$ is also idempotent and that $AB=BA=O$.

254. If A and B are idempotent and A and B commute, then show that AB is also idempotent.

255. If A and B are idempotent and $AB=BA=O$, then show that $A+B$ is also idempotent.

256. Show that $\begin{bmatrix} -6 & 5 \\ -7 & 6 \end{bmatrix}$ is an involutary matrix.

257. Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$ is an involutary matrix.

258. Prove that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutary.

259. Determine the condition that the matrix $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ is involutary. {Ans. $a^2 + bc = 1$}

260. Prove that the matrix $A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 1 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ is periodic whose period is 2.

261. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, find A^2 and show that $A^2 = A^{-1}$. Is A a periodic matrix? If yes, find its period. {Ans. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, Period is 3}

262. Show that the matrix $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is nilpotent of index 2.

263. Show that the matrix $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is nilpotent of index 2.

264. Show that the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent of index 3.

CATEGORY-10.18. RANK OF A MATRIX

265. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. {Ans. 3}

266. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 4 & 5 \end{bmatrix}$. {Ans. 2}

267. Find the rank of the matrix $A=\begin{bmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{bmatrix}$. {Ans. 2}

268. Find the rank of the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 3 & 6 & 9\\ 1 & 2 & 3\end{bmatrix}$. {Ans. 1}

CATEGORY-10.19. SOLVING SYSTEM OF LINEAR EQUATIONS BY MATRIX

269. Solve
$x+y+z=3$
$2x-y+z=2$
$x-2y+3z=2$
{Ans. (1, 1, 1)}

270. Solve
$x-y+z=4$
$2x+y-3z=0$
$x+y+z=2$
{Ans. (2, -1, 1)}

271. Solve
$5x+3y+z=16$
$2x+y+3z=19$
$x+2y+4z=25$
{Ans. (1, 2, 5)}

272. Solve
$2x+8y+5z=5$
$x+y+z=-2$
$x+2y-z=2$
{Ans. (-3, 2, -1)}

273. Solve
$x+2y-z=6$
$3x-y-2z=3$
$4x+3y+z=9$
{Ans. (1, 2, -1)}

274. Solve
$x+y+z=6$
$x+2y+3z=10$
$x+2y+4z=1$
{Ans. (-7, 22, -9)}

275. Solve

$x+y+z=4$

$2x+5y-2z=3$

$x+7y-7z=5$

{Ans. No solution}

276. Solve

$x+y+z=6$

$x+2y+3z=14$

$x+4y+7z=30$

{Ans. $(k-2,8-2k,k)$}

277. Solve

$5x+3y+7z=4$

$3x+26y+2z=9$

$7x+2y+10z=5$

{Ans. $\left(\frac{7-16k}{11},\frac{k+3}{11},k\right)$}

278. Solve

$x-3y-8z=-10$

$3x+y-4z=0$

$2x+5y+6z=13$

{Ans. $(-1+2k,3-2k,k)$}

279. Solve

$3x-y+z=0$

$-15x+6y-5z=0$

$5x-2y+2z=0$

{Ans. (0, 0, 0)}

280. Solve

$x+3y-2z=0$

$2x-y+4z=0$

$x-11y+14z=0$

{Ans. $\left(-\frac{10k}{7},\frac{8k}{7},k\right)$}

281. Solve

$3x+2y+7z=0$

$4x-3y-2z=0$

$5x+9y+23z=0$

{Ans. $(-k,-2k,k)$}

282. Using matrix method, find the values of λ and μ so that the system of equation

$2x-3y+5z=12$

$3x+y+\lambda z=\mu$

$x-7y+8z=17$

has

i. Unique solution {Ans. $\lambda \neq 2$}

ii. Infinite solutions {Ans. $\lambda = 2,\ \mu = 7$}

iii. No solution {Ans. $\lambda = 2,\ \mu \neq 7$}

CATEGORY-10.20. ADDITIONAL QUESTIONS

283. If every element of a third order determinant of value Δ is multiplied by 5, then find the value of new determinant. {Ans. 125Δ}
284. If *M* is a 3×3 matrix, where $M^T M = I$ and $\det(M)=1$, then prove that $\det(M-I)=0$.
285. If *A* and *B* are two non-zero square matrices of the same order and $AB=O$, then show that both *A* and *B* must be singular.

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Mathematics for IIT-JEE

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PART-III
TRIGONOMETRY

CHAPTER-11
TRIGONOMETRIC IDENTITIES AND EXPRESSIONS

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CHAPTER-11
TRIGONOMETRIC IDENTITIES AND EXPRESSIONS

LIST OF THEORY SECTIONS

LIST OF QUESTION CATEGORIES

CHAPTER-11
TRIGONOMETRIC IDENTITIES AND EXPRESSIONS

SECTION-11.1. ANGLES AND THEIR MEASUREMENTS

1. Definition of angle

i. An angle is the amount of rotation of a revolving line with respect to a fixed line.

2. Measurement of angles

i. 1 full rotation = 360°; 1 degree (1°) = 60 minutes (60'); 1 minute (1') = 60 seconds (60").

ii. Angle (in radians) = $\frac{arc\ length}{radius}$; 1 full rotation = $2\pi^{c}$.

iii. 1 radian (1^{c}) = $\frac{180}{\pi}$ degrees = 57°17'45" (approx.).

3. Positive and negative angles

i. If the revolving line makes anti-clockwise rotation, then the angle is defined as positive and if the revolving line makes clockwise rotation, then the angle is defined as negative.

ii. When rotated by angle θ or $\theta+(2\pi)n$ or $\theta^{\circ}+360^{\circ}\times n\ \ (n\in I)$, the revolving line is in the same position.

SECTION-11.2. STANDARD TRIGONOMETRIC IDENTITIES

1. Sign of trigonometric ratios in quadrants

i. $\sin\theta$, $\operatorname{cosec}\theta$, $\cos\theta$, $\sec\theta$, $\tan\theta$, $\cot\theta$ are positive when θ is in Ist quadrant.

ii. $\sin\theta$, $\operatorname{cosec}\theta$ are positive and $\cos\theta$, $\sec\theta$, $\tan\theta$, $\cot\theta$ are negative when θ is in IInd quadrant.

iii. $\tan\theta$, $\cot\theta$ are positive and $\sin\theta$, $\operatorname{cosec}\theta$, $\cos\theta$, $\sec\theta$ are negative when θ is in IIIrd quadrant.

iv. $\cos\theta$, $\sec\theta$ are positive and $\sin\theta$, $\operatorname{cosec}\theta$, $\tan\theta$, $\cot\theta$ are negative when θ is in IVth quadrant.

2. Relationship between trigonometric ratios

i. $\operatorname{cosec}\theta=\frac{1}{\sin\theta}$; $\sec\theta=\frac{1}{\cos\theta}$; $\tan\theta=\frac{\sin\theta}{\cos\theta}$; $\cot\theta=\frac{\cos\theta}{\sin\theta}$.

ii. $\sin^2\theta+\cos^2\theta=1$; $1+\tan^2\theta=\sec^2\theta$; $1+\cot^2\theta=\operatorname{cosec}^2\theta$.

3. Trigonometric ratios of some important angles

i. $\sin 0^{\circ}=0;\quad \cos 0^{\circ}=1$.

ii. $\sin 30^{\circ}=\frac{1}{2};\quad \cos 30^{\circ}=\frac{\sqrt{3}}{2}$.

iii. $\sin 45^{\circ}=\cos 45^{\circ}=\frac{1}{\sqrt{2}}$.

iv. $\sin 60^{\circ}=\frac{\sqrt{3}}{2};\quad \cos 60^{\circ}=\frac{1}{2}$.

v. $\sin 90^{\circ}=1;\quad \cos 90^{\circ}=0$.

vi. $\sin 15^{\circ}=\frac{\sqrt{3}-1}{2\sqrt{2}};\quad \cos 15^{\circ}=\frac{\sqrt{3}+1}{2\sqrt{2}}$.

vii. $\sin 18^{\circ}=\frac{\sqrt{5}-1}{4};\quad \cos 18^{\circ}=\frac{\sqrt{10+2\sqrt{5}}}{4}$.

viii. $\sin 22\frac{1}{2}^\circ = \frac{\sqrt{2-\sqrt{2}}}{2}$; $\cos 22\frac{1}{2}^\circ = \frac{\sqrt{2+\sqrt{2}}}{2}$.

ix. $\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$; $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$.

4. Trigonometric ratios of related angles

i. $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$; $\tan(-\theta) = -\tan\theta$.

ii. $\sin(90^\circ-\theta) = \cos\theta$; $\cos(90^\circ-\theta) = \sin\theta$; $\tan(90^\circ-\theta) = \cot\theta$.

iii. $\sin(90^\circ+\theta) = \cos\theta$; $\cos(90^\circ+\theta) = -\sin\theta$; $\tan(90^\circ+\theta) = -\cot\theta$.

iv. $\sin(180^\circ-\theta) = \sin\theta$; $\cos(180^\circ-\theta) = -\cos\theta$; $\tan(180^\circ-\theta) = -\tan\theta$.

v. $\sin(180^\circ+\theta) = -\sin\theta$; $\cos(180^\circ+\theta) = -\cos\theta$; $\tan(180^\circ+\theta) = \tan\theta$.

vi. $\sin(270^\circ-\theta) = -\cos\theta$; $\cos(270^\circ-\theta) = -\sin\theta$; $\tan(270^\circ-\theta) = \cot\theta$.

vii. $\sin(270^\circ+\theta) = -\cos\theta$; $\cos(270^\circ+\theta) = \sin\theta$; $\tan(270^\circ+\theta) = -\cot\theta$.

viii. $\sin(360^\circ-\theta) = -\sin\theta$; $\cos(360^\circ-\theta) = \cos\theta$; $\tan(360^\circ-\theta) = -\tan\theta$.

ix. $\sin(360^\circ+\theta) = \sin\theta$; $\cos(360^\circ+\theta) = \cos\theta$; $\tan(360^\circ+\theta) = \tan\theta$.

5. Sum and difference of angles

i. $\sin(A+B) = \sin A\cos B + \cos A\sin B$.

ii. $\sin(A-B) = \sin A\cos B - \cos A\sin B$.

iii. $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

iv. $\cos(A-B) = \cos A\cos B + \sin A\sin B$.

v. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$.

vi. $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$.

vii. $\cot(A+B) = \frac{\cot A\cot B - 1}{\cot A + \cot B}$.

viii. $\cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A}$.

ix. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$.

x. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$.

6. Sum and difference into product

i. $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$.

ii. $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$.

iii. $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$.

iv. $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$.

v. $\tan A + \tan B = \frac{\sin(A+B)}{\cos A\cos B}$.

vi. $\tan A-\tan B=\dfrac{\sin(A-B)}{\cos A\cos B}$.

vii. $\cot A+\cot B=\dfrac{\sin(A+B)}{\sin A\sin B}$.

viii. $\cot A-\cot B=\dfrac{\sin(B-A)}{\sin A\sin B}$.

7. Product into sum and difference

i. $2\sin A\cos B=\sin(A+B)+\sin(A-B)$.

ii. $2\cos A\sin B=\sin(A+B)-\sin(A-B)$.

iii. $2\cos A\cos B=\cos(A+B)+\cos(A-B)$.

iv. $2\sin A\sin B=\cos(A-B)-\cos(A+B)$.

8. Double angle formulae

i. $\sin 2A=2\sin A\cos A=\dfrac{2\tan A}{1+\tan^2 A}$.

ii. $\cos 2A=\cos^2 A-\sin^2 A=1-2\sin^2 A=2\cos^2 A-1=\dfrac{1-\tan^2 A}{1+\tan^2 A}$.

iii. $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$.

9. Triple angle formulae

i. $\sin 3A=3\sin A-4\sin^3 A$.

ii. $\cos 3A=4\cos^3 A-3\cos A$.

iii. $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$.

10. Half angle formulae

i. $\sin A=2\sin\dfrac{A}{2}\cos\dfrac{A}{2}=\dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$.

ii. $\cos A=\cos^2\dfrac{A}{2}-\sin^2\dfrac{A}{2}=2\cos^2\dfrac{A}{2}-1=1-2\sin^2\dfrac{A}{2}=\dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$.

iii. $\tan A=\dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$.

11. Trigonometric ratios of the sum of three or more angles

i. $\sin(A+B+C)=\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C$.

ii. $\cos(A+B+C)=\cos A\cos B\cos C-\cos A\sin B\sin C-\sin A\cos B\sin C-\sin A\sin B\cos C$.

iii. $\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$.

iv. $\tan(A_1 + A_2 + A_3 + ... + A_n) = \dfrac{s_1 - s_3 + s_5 - s_7 + ...}{1 - s_2 + s_4 - s_6 + ...}$

where $s_1 = \sum \tan A_i$, $s_2 = \sum \tan A_i \tan A_j$, $s_3 = \sum \tan A_i \tan A_j \tan A_k$, ……..and so on.

v. $\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \cdots\cdots\cdots$

vi. $\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + \cdots\cdots\cdots$

vii. $\tan nA = \dfrac{{}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - ....}{1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - ....}$; where ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$; ${}^nC_0 = {}^nC_n = 1$.

12. Conditional identities

i. Identities valid between two or more angles, which are related by a mathematical relationship, are called conditional identities.

ii. If $A + B + C = \pi$, then

a. $\sin(A+B) = \sin C$; $\sin(B+C) = \sin A$; $\sin(A+C) = \sin B$,

b. $\cos(A+B) = -\cos C$; $\cos(B+C) = -\cos A$; $\cos(A+C) = -\cos B$,

c. $\sin\left(\dfrac{A+B}{2}\right) = \cos\dfrac{C}{2}$; $\sin\left(\dfrac{B+C}{2}\right) = \cos\dfrac{A}{2}$; $\sin\left(\dfrac{A+C}{2}\right) = \cos\dfrac{B}{2}$,

d. $\cos\left(\dfrac{A+B}{2}\right) = \sin\dfrac{C}{2}$; $\cos\left(\dfrac{B+C}{2}\right) = \sin\dfrac{A}{2}$; $\cos\left(\dfrac{A+C}{2}\right) = \sin\dfrac{B}{2}$,

e. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$,

f. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$,

g. $\tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} + \tan\dfrac{A}{2}\tan\dfrac{B}{2} = 1$,

h. $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$,

i. $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$,

j. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$,

k. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$,

l. $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$,

m. $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

13. Standard Trigonometric series

i. $$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ... + \sin(\alpha+(n-1)\beta) = \frac{\sin\left(\alpha + (n-1)\dfrac{\beta}{2}\right)}{\sin\dfrac{\beta}{2}}\sin\frac{n\beta}{2}, \quad \beta \neq 2m\pi$$
$$= n\sin\alpha, \quad \beta = 2m\pi.$$

ii. $$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ... + \cos(\alpha+(n-1)\beta) = \frac{\cos\left(\alpha + (n-1)\dfrac{\beta}{2}\right)}{\sin\dfrac{\beta}{2}}\sin\frac{n\beta}{2}, \quad \beta \neq 2m\pi$$

$$= n\cos\alpha, \quad \beta = 2m\pi .$$

14. Product of cosines of double angles

i. $$\cos\alpha\cos 2\alpha\cos 2^2\alpha\cdots\cdots\cdots\cos 2^{n-1}\alpha = \frac{\sin 2^n\alpha}{2^n\sin\alpha}, \quad \alpha \neq m\pi$$
$$= 1, \quad \alpha = 2m\pi$$
$$= -1, \quad \alpha = (2m+1)\pi .$$

SECTION-11.3. TRIGONOMETRIC VALUES AS ROOTS OF POLYNOMIALS

1. Certain trigonometric values are roots of polynomial with integer coefficients.

SECTION-11.4. ELIMINATION

1. If (n + 1) equations containing n unknowns are given, then these n unknowns can be eliminated, i.e. an equation, called eliminant, can be obtained which does not contain these n unknowns.

EXERCISE-11

CATEGORY-11.1. RELATIONSHIP BETWEEN TRIGONOMETRIC RATIOS

1. $\cos^4 A - \sin^4 A + 1 = 2\cos^2 A$
2. $(\sin A + \cos A)(1 - \sin A \cos A) = \sin^3 A + \cos^3 A$
3. $\dfrac{\sin A}{1+\cos A} + \dfrac{1+\cos A}{\sin A} = 2\operatorname{cosec} A$
4. $\cos^6 A + \sin^6 A = 1 - 3\sin^2 A \cos^2 A$
5. $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = \sec A - \tan A$
6. $\dfrac{\operatorname{cosec} A}{\operatorname{cosec} A - 1} + \dfrac{\operatorname{cosec} A}{\operatorname{cosec} A + 1} = 2\sec^2 A$
7. $\dfrac{\operatorname{cosec} A}{\cot A + \tan A} = \cos A$
8. $(\sec A + \cos A)(\sec A - \cos A) = \tan^2 A + \sin^2 A$
9. $\dfrac{1}{\cot A + \tan A} = \sin A \cos A$
10. $\dfrac{1}{\sec A - \tan A} = \sec A + \tan A$
11. $\dfrac{1-\tan A}{1+\tan A} = \dfrac{\cot A - 1}{\cot A + 1}$
12. $\dfrac{1+\tan^2 A}{1+\cot^2 A} = \dfrac{\sin^2 A}{\cos^2 A}$
13. $\dfrac{\sec A - \tan A}{\sec A + \tan A} = 1 - 2\sec A \tan A + 2\tan^2 A$
14. $\dfrac{\tan A}{1-\cot A} + \dfrac{\cot A}{1-\tan A} = \sec A \operatorname{cosec} A + 1$
15. $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \sin A + \cos A$
16. $(\sin A + \cos A)(\cot A + \tan A) = \sec A + \operatorname{cosec} A$
17. $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A$
18. $\cot^4 A + \cot^2 A = \operatorname{cosec}^4 A - \operatorname{cosec}^2 A$
19. $\sqrt{\operatorname{cosec}^2 A - 1} = \cos A \operatorname{cosec} A$
20. $\sec^2 A \operatorname{cosec}^2 A = \tan^2 A + \cot^2 A + 2$
21. $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A$
22. $(1 + \cot A - \operatorname{cosec} A)(1 + \tan A + \sec A) = 2$
23. $\dfrac{1}{\operatorname{cosec} A - \cot A} - \dfrac{1}{\sin A} = \dfrac{1}{\sin A} - \dfrac{1}{\operatorname{cosec} A + \cot A}$
24. $\dfrac{\cot A \cos A}{\cot A + \cos A} = \dfrac{\cot A - \cos A}{\cot A \cos A}$

25. $\dfrac{\cot A+\tan B}{\cot B+\tan A}=\cot A\tan B$

26. $\left(\dfrac{1}{\sec^2 A-\cos^2 A}+\dfrac{1}{\operatorname{cosec}^2 A-\sin^2 A}\right)\cos^2 A\sin^2 A=\dfrac{1-\cos^2 A\sin^2 A}{2+\cos^2 A\sin^2 A}$

27. $\sin^8 A-\cos^8 A=\left(\sin^2 A-\cos^2 A\right)\left(1-2\sin^2 A\cos^2 A\right)$

28. $\dfrac{\cos A\operatorname{cosec} A-\sin A\sec A}{\cos A+\sin A}=\operatorname{cosec} A-\sec A$

29. $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1}=\dfrac{1+\sin A}{\cos A}$

30. $(\tan A+\operatorname{cosec} B)^2-(\cot B-\sec A)^2=2\tan A\cot B(\operatorname{cosec} A+\sec B)$

31. $2\sec^2 A-\sec^4 A-2\operatorname{cosec}^2 A+\operatorname{cosec}^4 A=\cot^4 A-\tan^4 A$

32. $(\sin A+\operatorname{cosec} A)^2+(\cos A+\sec A)^2=\tan^2 A+\cot^2 A+7$

33. $(1+\cot A+\tan A)(\sin A-\cos A)=\dfrac{\sec A}{\operatorname{cosec}^2 A}-\dfrac{\operatorname{cosec} A}{\sec^2 A}$

34. If the angle α is in the third quadrant and $\tan\alpha=2$, then find the value of $\sin\alpha$. {Ans. $-\dfrac{2}{\sqrt{5}}$}

35. If θ is an acute angle and $\tan\theta=\dfrac{1}{\sqrt{7}}$, then find the value of $\dfrac{\operatorname{cosec}^2\theta-\sec^2\theta}{\operatorname{cosec}^2\theta+\sec^2\theta}$. {Ans. $\dfrac{3}{4}$}

36. If $\tan\theta=\dfrac{p}{q}$, show that $\dfrac{p\sin\theta-q\cos\theta}{p\sin\theta+q\cos\theta}=\dfrac{p^2-q^2}{p^2+q^2}$.

37. If $\tan^2\theta=1-a^2$, prove that $\sec\theta+\tan^3\theta\operatorname{cosec}\theta=\left(2-a^2\right)^{\frac{3}{2}}$.

38. If $\sec x=p+\dfrac{1}{4p}$, show that $\sec x+\tan x=2p$ *or* $\dfrac{1}{2p}$.

CATEGORY-11.2. TRIGONOMETRIC RATIOS OF SOME IMPORTANT ANGLES AND RELATED ANGLES

39. $\sin(\pi+\theta)\sin(\pi-\theta)\operatorname{cosec}^2\theta=-1$

40. $\tan\theta\sin\left(\dfrac{\pi}{2}+\theta\right)\cos\left(\dfrac{\pi}{2}-\theta\right)=\sin^2\theta$

41. $\sin 75^\circ+\cos 75^\circ=\sqrt{\dfrac{3}{2}}$

42. $\sin 105^\circ+\cos 105^\circ=\cos 45^\circ$

43. $\cos 255^\circ+\sin 165^\circ=0$

44. $\sin 75^\circ-\sin 15^\circ=\cos 105^\circ+\cos 15^\circ$

45. $\sin^2 72^\circ-\sin^2 60^\circ=\frac{\sqrt{5}-1}{8}$

46. $\sin 36^\circ\sin 72^\circ\sin 108^\circ\sin 144^\circ=\dfrac{5}{16}$.

47. $\sin\dfrac{\pi}{10}+\sin\dfrac{13\pi}{10}=-\dfrac{1}{2}$

48. $\sin\frac{\pi}{10}\sin\frac{13\pi}{10}=-\frac{1}{4}$

49. $\sin^2 5°+\sin^2 10°+\sin^2 15°+.........+\sin^2 90°=9\frac{1}{2}$.

50. $\cos^2 5°+\cos^2 10°+\cos^2 15°+.........+\cos^2 90°=8\frac{1}{2}$.

51. Find the value of $\sin 10°+\sin 20°+\sin 30°+\cdots\cdots+\sin 360°$. {Ans. 0}

52. If $x=y\cos\frac{2\pi}{3}=z\cos\frac{4\pi}{3}$, then show that $xy+yz+zx=0$.

CATEGORY-11.3. SUM AND DIFFERENCE OF ANGLES

53. $\cos(45°-A)\cos(45°-B)-\sin(45°-A)\sin(45°-B)=\sin(A+B)$

54. $\sin(45°+A)\cos(45°-B)+\cos(45°+A)\sin(45°-B)=\cos(A-B)$

55. $\frac{\sin(A-B)}{\cos A\cos B}+\frac{\sin(B-C)}{\cos B\cos C}+\frac{\sin(C-A)}{\cos C\cos A}=0$

56. $\cos A\cos(B-A)-\sin A\sin(B-A)=\cos B$

57. $\cos(A+B)\cos C-\cos(B+C)\cos A=\sin B\sin(C-A)$

58. $\sin(n+1)A\sin(n-1)A+\cos(n+1)A\cos(n-1)A=\cos 2A$

59. $\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A=\cos A$

60. $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)=\frac{1}{\sqrt{2}}\sin A$

61. $\cos 2\theta\cos 2\phi+\sin^2(\theta-\phi)-\sin^2(\theta+\phi)=\cos(2\theta+2\phi)$

62. $\cos^2 48^\circ-\sin^2 12^\circ=\frac{\sqrt{5}+1}{8}$

63. $\sin^2 24°-\sin^2 6°=\frac{\sqrt{5}-1}{8}$.

64. $\cot 16°\cot 44°+\cot 44°\cot 76°-\cot 76°\cot 16°=3$.

65. If $\frac{\sin(x+y)}{\sin(x-y)}=\frac{a+b}{a-b}$, then find the value of $\frac{\tan x}{\tan y}$ in terms of a and b. {Ans. $\frac{a}{b}$}

66. If $\tan A=\frac{1}{2}$, $\tan B=\frac{1}{3}$, show that $A+B=45°$.

67. If $\tan A=\frac{m}{m+1}$, $\tan B=\frac{1}{2m+1}$, show that $A+B=45°$.

CATEGORY-11.4. SUM AND DIFFERENCE INTO PRODUCT

68. $\frac{\sin 7\theta-\sin 5\theta}{\cos 7\theta+\cos 5\theta}=\tan\theta$

69. $\frac{\cos 6\theta-\cos 4\theta}{\sin 6\theta+\sin 4\theta}=-\tan\theta$.

70. $\frac{\sin A+\sin 3A}{\cos A+\cos 3A}=\tan 2A$.

71. $\dfrac{\sin 7A-\sin A}{\sin 8A-\sin 2A}=\cos 4A\sec 5A.$

72. $\dfrac{\cos 2B+\cos 2A}{\cos 2B-\cos 2A}=\cot(A+B)\cot(A-B).$

73. $\dfrac{\sin 2A+\sin 2B}{\sin 2A-\sin 2B}=\dfrac{\tan(A+B)}{\tan(A-B)}.$

74. $\dfrac{\sin A+\sin 2A}{\cos A-\cos 2A}=\cot\dfrac{A}{2}.$

75. $\dfrac{\sin 5A-\sin 3A}{\cos 3A+\cos 5A}=\tan A.$

76. $\dfrac{\cos 2B-\cos 2A}{\sin 2B+\sin 2A}=\tan(A-B).$

77. $\cos(A+B)+\sin(A-B)=2\sin(45^\circ+A)\cos(45^\circ+B).$

78. $\dfrac{\cos 3A-\cos A}{\sin 3A-\sin A}+\dfrac{\cos 2A-\cos 4A}{\sin 4A-\sin 2A}=\dfrac{\sin A}{\cos 2A\cos 3A}.$

79. $\dfrac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)}=\tan(A+B).$

80. $\dfrac{\cos 3\theta+2\cos 5\theta+\cos 7\theta}{\cos\theta+2\cos 3\theta+\cos 5\theta}=\cos 2\theta-\sin 2\theta\tan 3\theta.$

81. $\dfrac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}=\tan 4A.$

82. $\dfrac{\sin(\theta+\phi)-2\sin\theta+\sin(\theta-\phi)}{\cos(\theta+\phi)-2\cos\theta+\cos(\theta-\phi)}=\tan\theta.$

83. $\dfrac{\sin A+2\sin 3A+\sin 5A}{\sin 3A+2\sin 5A+\sin 7A}=\dfrac{\sin 3A}{\sin 5A}.$

84. $\dfrac{\sin(A-C)+2\sin A+\sin(A+C)}{\sin(B-C)+2\sin B+\sin(B+C)}=\dfrac{\sin A}{\sin B}$

85. $\dfrac{\sin A-\sin 5A+\sin 9A-\sin 13A}{\cos A-\cos 5A-\cos 9A+\cos 13A}=\cot 4A$

86. $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\tan\dfrac{A+B}{2}\cot\dfrac{A-B}{2}.$

87. $\dfrac{\cos A+\cos B}{\cos B-\cos A}=\cot\dfrac{A+B}{2}\cot\dfrac{A-B}{2}.$

88. $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\dfrac{A+B}{2}.$

89. $\dfrac{\sin A-\sin B}{\cos B-\cos A}=\cot\dfrac{A+B}{2}.$

90. $\dfrac{\cos(A+B+C)+\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)}{\sin(A+B+C)+\sin(-A+B+C)-\sin(A-B+C)+\sin(A+B-C)}=\cot B$

91. $\cos 3A+\cos 5A+\cos 7A+\cos 15A=4\cos 4A\cos 5A\cos 6A.$

92. $\sin a+\sin 2a+\sin 4a+\sin 5a=4\cos\frac{a}{2}\cos\frac{3a}{2}\sin 3a.$

93. $\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)+\cos(A+B+C)=4\cos A\cos B\cos C.$

94. $\sin(A+B+C+D)+\sin(A+B-C-D)+\sin(A+B-C+D)+\sin(A+B+C-D)=4\sin(A+B)\cos C\cos D$

95. $\cos 70°-\cos 10°+\sin 40°=0$.

96. $\sin 50^\circ-\sin 70^\circ+\sin 10^\circ=0.$

97. $\sin 10^\circ+\sin 20^\circ+\sin 40^\circ+\sin 50^\circ=\sin 70^\circ+\sin 80^\circ.$

98. $\cos 20°+\cos 100°+\cos 140°=0$.

99. $\cos 12^\circ+\cos 60^\circ+\cos 84^\circ=\cos 24^\circ+\cos 48^\circ$

100. $[\sin 55°-\sin 19°]+[\sin 53°-\sin 17°]=\cos 1°$.

CATEGORY-11.5. PRODUCT INTO SUM AND DIFFERENCE

101. $\sin\frac{\theta}{2}\sin\frac{7\theta}{2}+\sin\frac{3\theta}{2}\sin\frac{11\theta}{2}=\sin 2\theta\sin 5\theta.$

102. $\cos 2\theta\cos\frac{\theta}{2}-\cos 3\theta\cos\frac{9\theta}{2}=\sin 5\theta\sin\frac{5\theta}{2}.$

103. $\sin A\sin(A+2B)-\sin B\sin(B+2A)=\sin(A-B)\sin(A+B).$

104. $(\sin 3A+\sin A)\sin A+(\cos 3A-\cos A)\cos A=0.$

105. $\frac{2\sin(A-C)\cos C-\sin(A-2C)}{2\sin(B-C)\cos C-\sin(B-2C)}=\frac{\sin A}{\sin B}.$

106. $\frac{\sin A\sin 2A+\sin 3A\sin 6A+\sin 4A\sin 13A}{\sin A\cos 2A+\sin 3A\cos 6A+\sin 4A\cos 13A}=\tan 9A.$

107. $\frac{\cos 2A\cos 3A-\cos 2A\cos 7A+\cos A\cos 10A}{\sin 4A\sin 3A-\sin 2A\sin 5A+\sin 4A\sin 7A}=\cot 6A\cot 5A.$

108. $\cos(36^\circ-A)\cos(36^\circ+A)+\cos(54^\circ+A)\cos(54^\circ-A)=\cos 2A.$

109. $\cos A\sin(B-C)+\cos B\sin(C-A)+\cos C\sin(A-B)=0.$

110. $\sin(45^\circ+A)\sin(45^\circ-A)=\frac{1}{2}\cos 2A.$

111. $\sin A\sin(A+120°)+\sin(A+120°)\sin(A+240°)+\sin(A+240°)\sin A=-\frac{3}{4}$.

112. $\sin(A-B)\cos(A+B)+\sin(B-C)\cos(B+C)+\sin(C-D)\cos(C+D)+\sin(D-A)\cos(D+A)=0.$

113. $\sin(\beta-\gamma)\cos(a-\delta)+\sin(\gamma-a)\cos(\beta-\delta)+\sin(a-\beta)\cos(\gamma-\delta)=0.$

114. $\sin(A+B-2C)\cos B-\sin(A+C-2B)\cos C=\sin(B-C)\{\cos(B+C-A)+\cos(C+A-B)+\cos(A+B-C)\}$

115. $\sin A\sin B\sin(A-B)+\sin B\sin C\sin(B-C)+\sin C\sin A\sin(C-A)+\sin(A-B)\sin(B-C)\sin(C-A)=0.$

116. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0.$

117. $\tan 70°-\tan 20°=2\tan 50°$.

118. $\cos 20°\cos 100°+\cos 100°\cos 140°-\cos 140°\cos 200°=-\frac{3}{4}$.

119. $\sin 12^\circ \sin 48^\circ \sin 54^\circ = \frac{1}{8}$.

120. $\cos 36^\circ \cos 42^\circ \cos 78^\circ = \frac{1}{8}$.

121. $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$.

122. $\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ = \frac{1}{16}$.

123. $\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ = \frac{1}{16}$.

124. $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1$.

CATEGORY-11.6. DOUBLE ANGLE FORMULAE

125. $\frac{\sin 2A}{1+\cos 2A} = \tan A$.

126. $\frac{\sin 2A}{1-\cos 2A} = \cot A$.

127. $\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$.

128. $\tan A + \cot A = 2\operatorname{cosec} 2A$.

129. $\tan A - \cot A = -2\cot 2A$.

130. $\operatorname{cosec} 2A + \cot 2A = \cot A$.

131. $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4\cos 2\theta \cos 4\theta$.

132. $\frac{\sec 8A - 1}{\sec 4A - 1} = \frac{\tan 8A}{\tan 2A}$.

133. $\frac{1+\tan^2(45^\circ - A)}{1-\tan^2(45^\circ - A)} = \operatorname{cosec} 2A$.

134. $\frac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} = \tan(A+B)$.

135. $\tan\left(\frac{\pi}{4}+\theta\right) - \tan\left(\frac{\pi}{4}-\theta\right) = 2\tan 2\theta$.

136. $\frac{\cos A + \sin A}{\cos A - \sin A} - \frac{\cos A - \sin A}{\cos A + \sin A} = 2\tan 2A$.

137. $\cot(A+15^\circ) - \tan(A-15^\circ) = \frac{4\cos 2A}{1+2\sin 2A}$.

138. $\frac{\sin\theta + \sin 2\theta}{1+\cos\theta + \cos 2\theta} = \tan\theta$.

139. $\sin(2n+1)A \sin A = \sin^2(n+1)A - \sin^2 nA$.

140. $\frac{\sin(A+3B)+\sin(3A+B)}{\sin 2A + \sin 2B} = 2\cos(A+B)$.

141. $\sin 3A + \sin 2A - \sin A = 4\sin A \cos\frac{A}{2}\cos\frac{3A}{2}$.

142. $\tan 2A = (\sec 2A + 1)\sqrt{\sec^2 A - 1}$.

143. $\cos^3 2\theta + 3\cos 2\theta = 4(\cos^6\theta - \sin^6\theta)$.

144. $1 + \cos^2 2\theta = 2(\cos^4\theta + \sin^4\theta)$.

145. $\sec^2 A(1 + \sec 2A) = 2\sec 2A$.

146. $\operatorname{cosec} A - 2\cot 2A \cos A = 2\sin A$.

147. $\cos 4a = 1 - 8\cos^2 a + 8\cos^4 a$

148. $\sin 4A = 4\sin A\cos^3 A - 4\cos A \sin^3 A$

149. $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$

150. $\sqrt{2 + \sqrt{2 + 2\cos 4\theta}} = 2\cos\theta$.

151. $\sec\left(\frac{\pi}{4} + \theta\right)\sec\left(\frac{\pi}{4} - \theta\right) = 2\sec 2\theta$

152. $\cos^2 a + \cos^2(a + 120^\circ) + \cos^2(a - 120^\circ) = \frac{3}{2}$

153. $(\tan 4A + \tan 2A)(1 - \tan^2 3A \tan^2 A) = 2\tan 3A \sec^2 A$

154. $\frac{2\cos 2^n\theta + 1}{2\cos\theta + 1} = (2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\cdots\cdots\cdots\cdots\cdot(2\cos 2^{n-1}\theta - 1)$

155. $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}$

156. $\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8} = \frac{3}{2}$

157. $\left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right) = \frac{1}{8}$.

158. $4\sin 27^\circ = \sqrt{5 + \sqrt{5}} - \sqrt{3 - \sqrt{5}}$.

159. $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4$.

160. $\sqrt{3}\operatorname{cosec} 20^\circ - \sec 20^\circ = 4$.

161. $\tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ = 4$.

162. $\tan 10^\circ + \tan 70^\circ - \tan 50^\circ = \sqrt{3}$.

163. $\tan^2\frac{\pi}{16} + \tan^2\frac{2\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{4\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{6\pi}{16} + \tan^2\frac{7\pi}{16} = 35$.

164. $\cot 7\frac{1}{2}^\circ = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$ or $\tan 82\frac{1}{2}^\circ = (\sqrt{2} + \sqrt{3})(1 + \sqrt{2})$.

165. If $\cos 2\alpha = \frac{3\cos 2\beta - 1}{3 - \cos 2\beta}$, then show that $\tan\alpha = \sqrt{2}\tan\beta$.

166. If $\tan\theta = \frac{b}{a}$, prove that $a\cos 2\theta + b\sin 2\theta = a$.

167. If $\tan\theta\tan\phi=\sqrt{\dfrac{a-b}{a+b}}$, prove that $(a-b\cos 2\theta)(a-b\cos 2\phi)$ is independent of θ and ϕ.

168. If $\tan^2\theta=2\tan^2\phi+1$, then prove that $\cos 2\theta+\sin^2\phi=0$.

169. If $\tan\beta=\dfrac{\tan\alpha+\tan\gamma}{1+\tan\alpha\tan\gamma}$, prove that $\sin 2\beta=\dfrac{\sin 2\alpha+\sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$.

CATEGORY-11.7. TRIPLE ANGLE FORMULAE

170. $\sin a\sin\left(60^\circ-a\right)\sin\left(60^\circ+a\right)=\dfrac{1}{4}\sin 3a$

171. $\cos a\cos\left(60^\circ-a\right)\cos\left(60^\circ+a\right)=\dfrac{1}{4}\cos 3a$

172. $\cot a+\cot\left(60^\circ+a\right)-\cot\left(60^\circ-a\right)=3\cot 3a$

173. $\cos 6a=32\cos^6 a-48\cos^4 a+18\cos^2 a-1$

174. $\cos^3\theta+\cos^3(120^\circ+\theta)+\cos^3(240^\circ+\theta)=\dfrac{3}{4}\cos 3\theta$.

175. If $\cos\theta=\dfrac{1}{2}\left(a+\dfrac{1}{a}\right)$, then prove that $\cos 3\theta=\dfrac{1}{2}\left(a^3+\dfrac{1}{a^3}\right)$.

CATEGORY-11.8. HALF ANGLE FORMULAE

176. $(\cos\alpha+\cos\beta)^2+(\sin\alpha-\sin\beta)^2=4\cos^2\dfrac{\alpha+\beta}{2}$

177. $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2=4\cos^2\dfrac{\alpha-\beta}{2}$

178. $(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=4\sin^2\dfrac{\alpha-\beta}{2}$

179. $\tan\left(45^\circ+\dfrac{A}{2}\right)=\sqrt{\dfrac{1+\sin A}{1-\sin A}}=\sec A+\tan A$

180. $\left(1+\tan\dfrac{\alpha}{2}-\sec\dfrac{\alpha}{2}\right)\left(1+\tan\dfrac{\alpha}{2}+\sec\dfrac{\alpha}{2}\right)=\sin\alpha\sec^2\dfrac{\alpha}{2}$

181. $\dfrac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)}=\tan\dfrac{A}{2}\cot\dfrac{B}{2}$.

182. $\dfrac{\cos A}{1\mp\sin A}=\tan\left(45^\circ\pm\dfrac{A}{2}\right)$.

183. $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\tan\dfrac{\theta}{2}$.

184. $\cot A=\dfrac{1}{2}\left(\cot\dfrac{A}{2}-\tan\dfrac{A}{2}\right)$.

185. $\dfrac{\sin(n+1)A-\sin(n-1)A}{\cos(n+1)A+2\cos nA+\cos(n-1)A}=\tan\dfrac{A}{2}$.

186. $\dfrac{\sin(n+1)A+2\sin nA+\sin(n-1)A}{\cos(n-1)A-\cos(n+1)A}=\cot\dfrac{A}{2}$.

187. $\dfrac{1-\tan^2 15°}{1+\tan^2 15°}=\dfrac{\sqrt{3}}{2}$.

188. $1+\cos 56°+\cos 58°-\cos 66°=4\cos 28°\cos 29°\sin 33°$.

189. If $\cos\theta=\dfrac{a\cos\phi+b}{a+b\cos\phi}$, prove that $\tan\dfrac{\theta}{2}=\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{\phi}{2}$.

190. If $\tan\theta=\dfrac{\sin\alpha\sin\beta}{\cos\alpha+\cos\beta}$, prove that one of the values of $\tan\dfrac{\theta}{2}$ is $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$.

191. If $\cos A=\dfrac{3}{5}$, $\cos B=\dfrac{5}{13}$, prove that $\sin^2\dfrac{A-B}{2}=\dfrac{1}{65}$, $\cos^2\dfrac{A-B}{2}=\dfrac{64}{65}$.

192. If $\cos\theta=\dfrac{2\cos\phi-1}{2-\cos\phi}$, prove that $\tan\dfrac{\theta}{2}=\sqrt{3}\tan\dfrac{\phi}{2}$, and hence show that $\sin\phi=\dfrac{\sqrt{3}\sin\theta}{2+\cos\theta}$.

193. If $\sin A+\sin B=a$ and $\cos A+\cos B=b$, then find the value of $\cos(A+B)$ in terms of a and b. {Ans. $\dfrac{b^2-a^2}{a^2+b^2}$}

CATEGORY-11.9. TRIGONOMETRIC RATIOS OF THE SUM OF THREE OR MORE ANGLES

194. Prove that $\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma=\dfrac{\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}$.

CATEGORY-11.10. CONDITIONAL IDENTITIES

195. If $A+B=45°$, prove that $(\cot A-1)(\cot B-1)=2$.

196. If $A+B=225°$, prove that $\left(\dfrac{\cot A}{1+\cot A}\right)\left(\dfrac{\cot B}{1+\cot B}\right)=\dfrac{1}{2}$.

197. If $A+B+C=180^\circ$, prove that

i. $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C$.

ii. $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$.

iii. $\sin A+\sin B-\sin C=4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$.

iv. $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C$.

v. $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C$.

vi. $\sin^2\dfrac{A}{2}+\sin^2\dfrac{B}{2}+\sin^2\dfrac{C}{2}=1-2\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

vii. $\sin^2\dfrac{A}{2}+\sin^2\dfrac{B}{2}-\sin^2\dfrac{C}{2}=1-2\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$.

viii. $\sin(B+2C)+\sin(C+2A)+\sin(A+2B)=4\sin\dfrac{B-C}{2}\sin\dfrac{C-A}{2}\sin\dfrac{A-B}{2}$.

ix. $\sin\dfrac{A}{2}+\sin\dfrac{B}{2}+\sin\dfrac{C}{2}-1=4\sin\dfrac{\pi-A}{4}\sin\dfrac{\pi-B}{4}\sin\dfrac{\pi-C}{4}$.

x. $\cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}=4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi-C}{4}$.

xi. $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

xii. $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)=4\sin A\sin B\sin C$.

198. If $A+B+C=\frac{\pi}{2}$, prove that

i. $\sin^2 A+\sin^2 B+\sin^2 C=1-2\sin A\sin B\sin C$

ii. $\cos^2 A+\cos^2 B+\cos^2 C=2+2\sin A\sin B\sin C$

iii. $\cot A+\cot B+\cot C=\cot A\cot B\cot C$

iv. $\tan A\tan B+\tan B\tan C+\tan A\tan C=1$

199. If $A+B+C=2\pi$, prove that

i. $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$

ii. $1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C=0$

iii. $\sin^3 A+\sin^3 B+\sin^3 C=3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}-\sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2}$

200. If $A+B+C=2S$, prove that

i. $\sin(S-A)\sin(S-B)+\sin S\sin(S-C)=\sin A\sin B$.

ii. $4\sin S\sin(S-A)\sin(S-B)\sin(S-C)=1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C$.

iii. $\sin(S-A)+\sin(S-B)+\sin(S-C)-\sin S=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

iv. $\cos^2 S+\cos^2(S-A)+\cos^2(S-B)+\cos^2(S-C)=2+2\cos A\cos B\cos C$.

v. $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1+4\cos S\cos(S-A)\cos(S-B)\cos(S-C)$.

201. If $\alpha+\beta+\gamma=0$, prove that $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=2(\sin\alpha+\sin\beta+\sin\gamma)(1+\cos\alpha+\cos\beta+\cos\gamma)$

202. If $A+B=C$, prove that

i. $\cos^2 A+\cos^2 B+\cos^2 C=1+2\cos A\cos B\cos C$.

ii. $\tan A\tan B\tan C=\tan C-\tan B-\tan A$.

203. If $A+C=2B$, prove that $\cot B=\frac{\sin A-\sin C}{\cos C-\cos A}$.

204. If $\alpha+\beta+\gamma+\delta=2\pi$, prove that

i. $\cos\alpha+\cos\beta+\cos\gamma+\cos\delta+4\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2}=0$

ii. $\sin\alpha-\sin\beta+\sin\gamma-\sin\delta+4\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2}=0$

iii. $\tan\alpha+\tan\beta+\tan\gamma+\tan\delta=\tan\alpha\tan\beta\tan\gamma\tan\delta(\cot\alpha+\cot\beta+\cot\gamma+\cot\delta)$.

205. If the sum of four angles be 180^0, prove that the sum of the products of their cosines taken two and two together is equal to the sum of the products of their sines taken similarly.

CATEGORY-11.11. STANDARD TRIGONOMETRIC SERIES

206. $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$.

207. $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}=\frac{1}{2}$.

208. $\cos 0^\circ+\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{6\pi}{7}=1$.

209. $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{6\pi}{7}+\sin\frac{8\pi}{7}+\sin\frac{10\pi}{7}+\sin\frac{12\pi}{7}=0$.

210. If n is an integer greater than 2, prove that $\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\cos\frac{6\pi}{n}+.........to\ n\ terms=0$.

211. Prove that $\sin^2\alpha+\sin^2 2\alpha+\sin^2 3\alpha+.......\ to\ n\ terms=\frac{1}{4}\left[(2n+1)\sin\alpha-\sin(2n+1)\alpha\right]\operatorname{cosec}\alpha$

212. Prove that

$\sin^3\alpha+\sin^3 2\alpha+\sin^3 3\alpha+.......\ to\ n\ terms$

$=\frac{3}{4}\sin\frac{(n+1)\alpha}{2}\sin\frac{n\alpha}{2}\operatorname{cosec}\frac{\alpha}{2}-\frac{1}{4}\sin\frac{3(n+1)\alpha}{2}\sin\frac{3n\alpha}{2}\operatorname{cosec}\frac{3\alpha}{2}$.

213. Prove that the sum of the product of sines of the angles $\alpha, 2\alpha, 3\alpha,....., n\alpha$ taking two at a time is

$$\frac{1}{2}\left[\frac{\sin^2\frac{(n+1)\alpha}{2}\sin^2\frac{n\alpha}{2}}{\sin^2\frac{\alpha}{2}}+\frac{\cos(n+1)\alpha\sin n\alpha}{2\sin\alpha}-\frac{n}{2}\right].$$

CATEGORY-11.12. PRODUCT OF COSINES OF DOUBLE ANGLES

214. $\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ=\frac{1}{16}$

215. $16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15}=1$

216. $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}=\frac{1}{64}$.

217. If $\theta=\frac{\pi}{2^n-1}$, prove that $\cos\theta\cos 2\theta\cos 2^2\theta............\cos 2^{n-1}\theta=-\frac{1}{2^n}$.

218. If $\theta=\frac{\pi}{2^n+1}$, prove that $\cos\theta\cos 2\theta\cos 2^2\theta............\cos 2^{n-1}\theta=\frac{1}{2^n}$.

219. $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{2^7}$

220. $\cos\frac{\pi}{20}\cos\frac{3\pi}{20}\cos\frac{7\pi}{20}\cos\frac{9\pi}{20}+\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}=0$.

221. $(1+\sec 2\theta)(1+\sec 4\theta)(1+\sec 8\theta).............(1+\sec 2^n\theta)=\tan 2^n\theta\cot\theta$.

222. $\tan 20^\circ\tan 40^\circ\tan 80^\circ=\sqrt{3}$.

223. $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}=\frac{1}{8}$.

224. $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}=\frac{1}{64}$.

225. If $A=\frac{\pi}{2^{n+1}}$, $B=\frac{\pi}{2^{n+2}}$, then prove that

$$(\cos A+\cos B)(\cos 2A+\cos 2B)(\cos 2^2 A+\cos 2^2 B).......(\cos 2^n A+\cos 2^n B)=\frac{1}{2^{n+1}}\left(\cos\frac{\pi}{2^{n+2}}-\cos\frac{\pi}{2^{n+1}}\right)^{-1}.$$

CATEGORY-11.13. TRIGONOMETRIC VALUES AS ROOTS OF POLYNOMIALS

226. Prove that the roots of the equation $8x^3-4x^2-4x+1=0$ are $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$ and $\cos\frac{5\pi}{7}$ and hence prove that

i. $\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}$

ii. $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}\cos\frac{5\pi}{7}+\cos\frac{3\pi}{7}\cos\frac{5\pi}{7}=-\frac{1}{2}$

iii. $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7}=-\frac{1}{8}$

iv. $\left(1-\cos\frac{\pi}{7}\right)\left(1-\cos\frac{3\pi}{7}\right)\left(1-\cos\frac{5\pi}{7}\right)=\frac{1}{8}$

v. the equation whose roots are $\cos^2\frac{\pi}{7}$, $\cos^2\frac{3\pi}{7}$ and $\cos^2\frac{5\pi}{7}$ is $64x^3-80x^2+24x-1=0$

vi. $\cos^2\frac{\pi}{7}+\cos^2\frac{3\pi}{7}+\cos^2\frac{5\pi}{7}=\frac{5}{4}$

vii. the equation whose roots are $\sec\frac{\pi}{7}$, $\sec\frac{3\pi}{7}$ and $\sec\frac{5\pi}{7}$ is $x^3-4x^2-4x+8=0$

viii. $\sec\frac{\pi}{7}+\sec\frac{3\pi}{7}+\sec\frac{5\pi}{7}=4$

ix. the equation whose roots are $\sec^2\frac{\pi}{7}$, $\sec^2\frac{3\pi}{7}$ and $\sec^2\frac{5\pi}{7}$ is $x^3-24x^2+80x-64=0$

x. $\sec^2\frac{\pi}{7}+\sec^2\frac{3\pi}{7}+\sec^2\frac{5\pi}{7}=24$

xi. the equation whose roots are $\tan^2\frac{\pi}{7}$, $\tan^2\frac{3\pi}{7}$ and $\tan^2\frac{5\pi}{7}$ is $x^3-21x^2+35x-7=0$

xii. $\tan^2\frac{\pi}{7}+\tan^2\frac{3\pi}{7}+\tan^2\frac{5\pi}{7}=21$

xiii. $\tan\frac{\pi}{7}\tan\frac{3\pi}{7}\tan\frac{5\pi}{7}=-\sqrt{7}$

xiv. the equation whose roots are $\cot^2\frac{\pi}{7}$, $\cot^2\frac{3\pi}{7}$ and $\cot^2\frac{5\pi}{7}$ is $7x^3-35x^2+21x-1=0$

xv. $\cot^2\frac{\pi}{7}+\cot^2\frac{3\pi}{7}+\cot^2\frac{5\pi}{7}=5$

227. Find the equation whose roots are $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$ and $\cos\frac{6\pi}{7}$. {Ans. $8x^3+4x^2-4x-1=0$ }

228. $\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7}=\sqrt{7}$.

229. $\left(\tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7}\right)\left(\cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}\right)=105$.

230. Find the value of $\sin^2\frac{\pi}{7}+\sin^2\frac{2\pi}{7}+\sin^2\frac{3\pi}{7}$. {Ans. $\frac{7}{4}$ }

231. $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}=\frac{\sqrt{7}}{8}$.

232. Find the value of $\operatorname{cosec}^2\frac{\pi}{7}+\operatorname{cosec}^2\frac{2\pi}{7}+\operatorname{cosec}^2\frac{3\pi}{7}$. {Ans. 8}

233. $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\frac{\sqrt{7}}{2}$.

234. Show that $\sin\frac{\pi}{14}$ is a root of the equation $8x^3-4x^2-4x+1=0$ and find the other roots. {Ans. $-\sin\frac{3\pi}{14}$, $\sin\frac{5\pi}{14}$ }

CATEGORY-11.14. ELIMINATION

235. If $\frac{\cos\alpha}{\cos\beta}=n$, $\frac{\sin\alpha}{\sin\beta}=m$, show that $\left(m^2-n^2\right)\sin^2\beta=1-n^2$.

236. If $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2-b^2$ and $\frac{ax\sin\theta}{\cos^2\theta}-\frac{by\cos\theta}{\sin^2\theta}=0$, show that $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=\left(a^2-b^2\right)^{\frac{2}{3}}$.

237. If $\sec\theta-\cos\theta=a$ and $\operatorname{cosec}\theta-\sin\theta=b$, prove that $a^2b^2\left(a^2+b^2+3\right)=1$.

238. If $\operatorname{cosec}\theta-\sin\theta=a^3$, $\sec\theta-\cos\theta=b^3$, prove that $a^2b^2\left(a^2+b^2\right)=1$.

239. If $\tan\theta+\sin\theta=m$ and $\tan\theta-\sin\theta=n$, prove that $m^2-n^2=\pm4\sqrt{mn}$.

240. If $a\cos\theta+b\sin\theta=p$ and $a\sin\theta-b\cos\theta=q$, prove that $a^2+b^2=p^2+q^2$.

241. If $\cot\theta(1+\sin\theta)=4m$ and $\cot\theta(1-\sin\theta)=4n$, prove that $\left(m^2-n^2\right)^2=mn$.

242. If $c\cos^3\theta+3c\cos\theta\sin^2\theta=m$ and $c\sin^3\theta+3c\cos^2\theta\sin\theta=n$, prove that $(m+n)^{\frac{2}{3}}+(m-n)^{\frac{2}{3}}=2c^{\frac{2}{3}}$.

243. If $\sin\theta+\sin2\theta=a$ and $\cos\theta+\cos2\theta=b$, prove that $\left(a^2+b^2\right)\left(a^2+b^2-3\right)=2b$.

244. If $\cos^2\theta=\frac{1}{3}\left(a^2-1\right)$ and $\tan^2\frac{\theta}{2}=\tan^{\frac{2}{3}}\alpha$, prove that $\cos^{\frac{2}{3}}\alpha+\sin^{\frac{2}{3}}\alpha=\left(\frac{2}{a}\right)^{\frac{2}{3}}$.

245. If $\frac{\tan(\theta+x)}{a}=\frac{\tan(\theta+y)}{b}=\frac{\tan(\theta+z)}{c}$ prove that
$\frac{a+b}{a-b}\sin^2(x-y)+\frac{b+c}{b-c}\sin^2(y-z)+\frac{c+a}{c-a}\sin^2(z-x)=0$.

246. If $a\sin x^2 = b\cos x^2 = \dfrac{2c\tan x^2}{1-\tan^2 x^2}$, prove that $\left(a^2-b^2\right)^2 = 4c^2\left(a^2+b^2\right)$.

247. If $\cot\alpha = \left(x^3+x^2+x\right)^{\frac{1}{2}}$, $\cot\beta = \left(x+x^{-1}+1\right)^{\frac{1}{2}}$ and $\cot\gamma = \left(x^{-3}+x^{-2}+x^{-1}\right)^{-\frac{1}{2}}$, prove that $\alpha+\beta=\gamma$.

248. If $\dfrac{\cos^3\theta}{\cos(\alpha-3\theta)} = \dfrac{\sin^3\theta}{\sin(\alpha-3\theta)} = p$, show that $\cos\alpha = \dfrac{2p^2-1}{p}$.

249. If $\cot\theta+\tan\theta = x$, $\sec\theta-\cos\theta = y$, eliminate θ. {Ans. $(xy)^{\frac{2}{3}}\left(x^{\frac{2}{3}}-y^{\frac{2}{3}}\right)=1$}

250. If $a\sec\theta = 1-b\tan\theta$, $a^2\sec^2\theta = 5+b^2\tan^2\theta$, eliminate θ. {Ans. $a^2b^2+4a^2-9b^2=0$}

251. If $a\sec\theta+b\tan\theta+c=0$, $p\sec\theta+q\tan\theta+r=0$, eliminate θ. {Ans. $(br-cq)^2-(cp-ar)^2=(aq-bp)^2$}

252. If $a\tan\alpha+b\cot 2\alpha = c$, $a\cot\alpha-b\tan 2\alpha = c$, eliminate α. {Ans. $(b-a)\sqrt{b} = c\sqrt{2a-b}-a\sqrt{2a}$}

253. If $\dfrac{\cos x}{a} = \dfrac{\cos(x+y)}{b} = \dfrac{\cos(x+2y)}{c} = \dfrac{\cos(x+3y)}{d}$, prove that $b(b+d)=c(c+a)$.

254. If $a\cot^2\alpha+b\cot^2\beta=1$, $a\cos^2\alpha+b\cos^2\beta=1$ and $a\sin\alpha=b\sin\beta$, then prove that $\left(a^2-b^2\right)^2+ab=0$.

255. Let $\cos\alpha=\cos\beta\cos\phi=\cos\gamma\cos\theta$ and $\sin\alpha = 2\sin\dfrac{\phi}{2}\sin\dfrac{\theta}{2}$, prove that $\tan^2\dfrac{\alpha}{2}=\tan^2\dfrac{\beta}{2}\tan^2\dfrac{\gamma}{2}$.

256. If $\cos\theta=\cos\alpha\cos\beta$ and $\cos\phi=\cos\gamma\cos\beta$ where $\cos\beta\neq 0$ and $\tan\dfrac{\beta}{2}=\tan\dfrac{\theta}{2}\tan\dfrac{\phi}{2}$, prove that $\sin^2\beta = (\sec\alpha-1)(\sec\gamma-1)$.

CATEGORY-11.15. APPLICATIONS OF TRIGONOMETRIC IDENTITIES

257. Find the value of $\tan\dfrac{2\pi}{5}-\tan\dfrac{\pi}{15}-\sqrt{3}\tan\dfrac{2\pi}{5}\tan\dfrac{\pi}{15}$. {Ans. $\sqrt{3}$}

258. If $f(x)=\cos(\log x)$, then evaluate $f(x)f(y)-\dfrac{1}{2}\left[f\left(\dfrac{x}{y}\right)+f(xy)\right]$. {Ans. 0}

259. If $\cos\alpha+\cos\beta=0=\sin\alpha+\sin\beta$, then show that $\cos 2\alpha+\cos 2\beta = -2\cos(\alpha+\beta)$.

260. If $\sec\theta+\tan\theta=p$, obtain the values of $\sec\theta$, $\tan\theta$ and $\sin\theta$ in terms of p. {Ans. $\dfrac{p^2+1}{2p}, \dfrac{p^2-1}{2p}, \dfrac{p^2-1}{p^2+1}$}

261. If $\sin x+\sin^2 x=1$, show that $\cos^2 x+\cos^4 x=1$.

262. If $\sin x+\sin^2 x=1$, then find the value of $\cos^8 x+2\cos^6 x+\cos^4 x$. {Ans. 1}

263. If $\cos\theta+\sin\theta=\sqrt{2}\cos\theta$, show that $\cos\theta-\sin\theta=\sqrt{2}\sin\theta$.

264. If $\cos 3x = -\dfrac{3\sqrt{6}}{8}$, show that the three values of $\cos x$ are $\dfrac{\sqrt{6}}{2}\sin\dfrac{\pi}{6}$, $\dfrac{\sqrt{6}}{2}\sin\dfrac{\pi}{10}$, $-\dfrac{\sqrt{6}}{2}\sin\dfrac{3\pi}{10}$.

265. If $0<\alpha, \beta<\pi$ and $\cos\alpha+\cos\beta-\cos(\alpha+\beta)=\dfrac{3}{2}$, prove that $\alpha=\beta=\dfrac{\pi}{3}$.

266. If $a\cos\theta-b\sin\theta=c$, show that $a\sin\theta+b\cos\theta=\pm\sqrt{a^2+b^2-c^2}$.

267. If $3\sin\theta+5\cos\theta=5$, show that $5\sin\theta-3\cos\theta=3$ *or* -3.

268. If $(1+\sin A)(1+\sin B)(1+\sin C)=(1-\sin A)(1-\sin B)(1-\sin C)$ prove that each side is equal to $\pm\cos A\cos B\cos C$.

269. Prove that $\left(\dfrac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\dfrac{\sin A+\sin B}{\cos A-\cos B}\right)^n=2\cot^n\left(\dfrac{A-B}{2}\right)$ or 0 according as n is even or odd positive integer.

270. If $\cos(A+B)\sin(C+D)=\cos(A-B)\sin(C-D)$, prove that $\cot A\cot B\cot C=\cot D$.

271. If $m\tan(\theta-30°)=n\tan(\theta+120°)$, show that $\cos 2\theta=\dfrac{m+n}{2(m-n)}$.

272. If $m\cos(\theta+\alpha)=n\cos(\theta-\alpha)$, show that $(m-n)\cot\theta=(m+n)\tan\alpha$.

273. If $\cot^2\theta=\cot(\theta-\alpha)\cot(\theta-\beta)$, show that $\cot 2\theta=\dfrac{1}{2}(\cot\alpha+\cot\beta)$.

274. If $\tan\left(\dfrac{\pi}{4}+\dfrac{y}{2}\right)=\tan^3\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$, prove that $\sin y=\dfrac{\sin x(3+\sin^2 x)}{1+3\sin^2 x}$.

275. If $\dfrac{\sin(\theta+A)}{\sin(\theta+B)}=\sqrt{\dfrac{\sin 2A}{\sin 2B}}$, prove that $\tan^2\theta=\tan A\tan B$.

276. If $0<\alpha,\beta,\gamma<\pi$, prove that
   i. $\sin\alpha+\sin\beta+\sin\gamma>\sin(\alpha+\beta+\gamma)$
   ii. $\sin\alpha+\sin\beta+\sin\gamma>3\sin\alpha\sin\beta\sin\gamma$

277. If $\cos x=\tan y$, $\cos y=\tan z$, $\cos z=\tan x$, prove that $\sin x=\sin y=\sin z=2\sin 18°$.

278. If $\sqrt{2}\cos A=\cos B+\cos^3 B$ and $\sqrt{2}\sin A=\sin B-\sin^3 B$, show that $\sin(A-B)=\pm\dfrac{1}{3}$.

279. If $\dfrac{\sin^4 A}{a}+\dfrac{\cos^4 A}{b}=\dfrac{1}{a+b}$, prove that $\dfrac{\sin^8 A}{a^3}+\dfrac{\cos^8 A}{b^3}=\dfrac{1}{(a+b)^3}$.

280. If $\sin(y+z-x)$, $\sin(z+x-y)$, $\sin(x+y-z)$ be in A.P., prove that $\tan x$, $\tan y$, $\tan z$ are also in A.P.

281. If $\sec(\phi-\alpha)$, $\sec\phi$, $\sec(\phi+\alpha)$ be in A.P. prove that $\cos\phi=\sqrt{2}\cos\dfrac{\alpha}{2}$.

282. If $\sin x+\sin y=a$, $\cos x+\cos y=b$, show that
   i. $\cos(x-y)=\dfrac{a^2+b^2-2}{2}$
   ii. $\tan\dfrac{x-y}{2}=\pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$.

283. If $\cos(\alpha+\beta)=\dfrac{4}{5}$ and $\sin(\alpha-\beta)=\dfrac{5}{13}$ and α, β lie between 0 and $\dfrac{\pi}{4}$, find $\tan 2\alpha$. {Ans. $\dfrac{56}{33}$}

284. If $\tan A-\tan B=x$ and $\cot B-\cot A=y$, prove that $\cot(A-B)=\dfrac{1}{x}+\dfrac{1}{y}$.

285. Prove that $\sin^2(\theta+\alpha)+\sin^2(\theta+\beta)-2\cos(\alpha-\beta)\sin(\theta+\alpha)\sin(\theta+\beta)$ is independent of θ.

286. If $\frac{\cos^4 x}{\cos^2 y}+\frac{\sin^4 x}{\sin^2 y}=1$, prove that $\frac{\cos^4 y}{\cos^2 x}+\frac{\sin^4 y}{\sin^2 x}=1$.

287. If $\sin x+\sin y=3(\cos y-\cos x)$, prove that $\sin 3x+\sin 3y=0$.

288. If $x=\sin\theta(1+\sin\theta)+\cos\theta(1+\cos\theta)$ and $y=\sin\theta(1-\sin\theta)+\cos\theta(1-\cos\theta)$, prove that $x^2-2x-\sin 2\theta=y^2+2y-\sin 2\theta=0$.

289. If $\cos(A+B+C)=\cos A\cos B\cos C$, show that $8\sin(B+C)\sin(C+A)\sin(A+B)=-\sin 2A\sin 2B\sin 2C$.

290. If $\cos A+\cos B+\cos C=0$, prove that $\cos 3A+\cos 3B+\cos 3C=12\cos A\cos B\cos C$.

291. If $\tan\beta=\cos\theta\tan\alpha$, then prove that $\sin(\alpha-\beta)=\tan^2\frac{\theta}{2}\sin(\alpha+\beta)$.

292. If $2\tan\beta+\cot\beta=\tan\alpha$, then $\cot\beta=2\tan(\alpha-\beta)$.

293. If $\sin\theta=n\sin(\theta+2\alpha)$, show that $\tan(\theta+\alpha)=\frac{1+n}{1-n}\tan\alpha$.

294. If an angle θ be divided into two parts such that the tangent of one part is m times the tangent of the other, then prove that their difference ϕ is obtained by the equation $\sin\phi=\frac{m-1}{m+1}\sin\theta$.

295. If $x^2\sin^2(\alpha+\beta)=\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\cos(\alpha-\beta)$, show that $\tan\alpha=\frac{1\pm x}{1\mp x}\tan\beta$.

296. If $\cos A=m\cos B$, then prove that $\cot\left(\frac{A+B}{2}\right)=\left(\frac{m+1}{m-1}\right)\tan\left(\frac{B-A}{2}\right)$.

297. Given that the angles α,β,γ are connected by the relation $2\tan^2\alpha\tan^2\beta\tan^2\gamma+\tan^2\alpha\tan^2\beta+\tan^2\beta\tan^2\gamma+\tan^2\gamma\tan^2\alpha=1$, find the value of $\sin^2\alpha+\sin^2\beta+\sin^2\gamma$. {Ans. 1}

298. If $\sin x+\cos x=a$, evaluate $\sin^6 x+\cos^6 x$. {Ans. $1-\frac{3}{4}\left(a^2-1\right)^2$ }

299. If $\sec(x+y)+\sec(x-y)=2\sec x$, then prove that $\cos x=\pm\sqrt{2}\cos\frac{y}{2}$.

300. If $\tan y=\frac{n\sin x\cos x}{1-n\sin^2 x}$, prove that $\tan(x-y)=(1-n)\tan x$.

301. If $\sec\alpha\sec\beta+\tan\alpha\tan\beta=\tan\theta$, prove that $\cos 2\theta\le 0$.

302. Suppose $\sin^3 x\sin 3x=\sum_{m=0}^{n}C_m\cos^m x$, is an identity in x, where $C_0, C_1, C_2, \ldots\ldots\ldots.C_n$ are real constants and $C_n\neq 0$. Find the value of n. Also find $C_0, C_1, C_2, \ldots\ldots\ldots.C_n$. {Ans. n = 6}

303. If $\alpha=\frac{2\pi}{7}$, prove that $\tan\alpha\tan 2\alpha+\tan 2\alpha\tan 4\alpha+\tan 4\alpha\tan\alpha=-7$.

304. If $4n\alpha=\pi$, prove that $\tan\alpha\tan 2\alpha\tan 3\alpha\ldots\ldots\ldots.\tan(2n-2)\alpha\tan(2n-1)\alpha=1$.

305. Prove that $x^2-x\cos(A+B)+1$ is a factor of $2x^4+4x^3\sin A\sin B-x^2(\cos 2A+\cos 2B)+4x\cos A\cos B-2$. Also find the other factor. {Ans. $2x^2+2x\cos(A-B)-2$ }

306. Show that $3\left(\sin^4\left(\frac{3\pi}{2}-x\right)+\sin^4(3\pi+x)\right)-2\left(\sin^6\left(\frac{\pi}{2}+x\right)+\sin^6(5\pi-x)\right)$ is independent of *x*.

307. If $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$. Prove that

i. $\cot\frac{\alpha+\beta}{2}=\cot\gamma$

ii. $\cos\frac{\alpha-\beta}{2}=\pm\frac{1}{2}$.

iii. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\frac{3}{2}$.

308. If $a_{r+1}=\sqrt{\frac{1+a_r}{2}}$, prove that $\cos\left(\frac{\sqrt{1-a_0^{\ 2}}}{a_1\cdot a_2\cdots\cdots\cdots to\ \infty}\right)=a_0$.

CATEGORY-11.16. ADDITIONAL QUESTIONS

Mathematics for IIT-JEE

By **Er. Sanjiva Dayal,** *B.Tech. (I.I.T. Kanpur)*

PART-III
TRIGONOMETRY

CHAPTER-12
TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS, EQUATIONS AND INEQUALITIES

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CHAPTER-12
TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS, EQUATIONS AND INEQUALITIES

LIST OF THEORY SECTIONS

LIST OF QUESTION CATEGORIES

CHAPTER-12
TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS, EQUATIONS AND INEQUALITIES

SECTION-12.1. MAXIMUM AND MINIMUM VALUES OF TRIGONOMETRIC FUNCTIONS

1. Standard results

i. $|\sin x| \le 1$; $|\cos x| \le 1$; $|\operatorname{cosec} x| \ge 1$; $|\sec x| \ge 1$.

ii. $\sin x < x < \tan x, \quad 0 < x < \frac{\pi}{2}$.

iii. $-\sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}$.

SECTION-12.2. TRIGONOMETRIC EQUATIONS

1. Simplest Trigonometric equations

i. $\sin x = a \equiv x = n\pi + (-1)^n \sin^{-1} a, \quad |a| \le 1$

$\equiv x \in \phi, \quad |a| > 1$

$\sin x = 1 \equiv x = 2n\pi + \frac{\pi}{2}$

$\sin x = -1 \equiv x = 2n\pi + \frac{3\pi}{2}$

ii. $\cos x = a \equiv x = 2n\pi \pm \cos^{-1} a, \quad |a| \le 1$

$\equiv x \in \phi, \quad |a| > 1$

iii. $\tan x = a \equiv x = n\pi + \tan^{-1} a$

iv. $\cot x = a \equiv x = n\pi + \cot^{-1} a$

2. Equations of type $\sin f(x) = a$**, etc.**

3. Substitution

4. Homogeneous equation in $\sin x$ **and** $\cos x$

5. Equation of type $a\sin^2 x + b\sin x\cos x + c\cos^2 x = d$

6. Equation of type $a\sin x + b\cos x = c$

7. Equation containing Rational functions of $\sin x$ **and** $\cos x$

8. Lowering high even powers of $\sin x$ **and** $\cos x$

9. Change/break into simpler equations using trigonometric identities

SECTION-12.3. INVERSE TRIGONOMETRIC IDENTITIES

1.

i. $\sin(\sin^{-1} x) = x, \qquad -1 \le x \le 1$

ii. $\cos(\sin^{-1} x) = \sqrt{1-x^2}, \quad -1 \le x \le 1$

iii. $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}, \quad -1 < x < 1$

iv. $\cos(\cos^{-1} x) = x, \quad -1 \le x \le 1$

v. $\sin(\cos^{-1} x) = \sqrt{1-x^2}, \quad -1 \le x \le 1$

vi. $\tan(\cos^{-1} x) = \dfrac{\sqrt{1-x^2}}{x}, \quad x \in [-1,0) \cup (0,1]$

vii. $\tan(\tan^{-1} x) = x$

viii. $\sin(\tan^{-1} x) = \dfrac{x}{\sqrt{1+x^2}}$

ix. $\cos(\tan^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$

2.

i. $\sin^{-1}(-x) = -\sin^{-1} x, \ -1 \le x \le 1$

ii. $\tan^{-1}(-x) = -\tan^{-1} x$

iii. $\cos^{-1}(-x) = \pi - \cos^{-1} x, \ -1 \le x \le 1$

iv. $\cot^{-1}(-x) = \pi - \cot^{-1} x$

3.

i. $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}, \ -1 \le x \le 1$

ii. $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$

iii. $\sec^{-1} x + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2}, \ x \in (-\infty,-1] \cup [1,\infty)$

4.

i. $\sin^{-1}(\sin x) = x, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

ii. $\cos^{-1}(\cos x) = x, \quad 0 \le x \le \pi$

iii. $\tan^{-1}(\tan x) = x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

5.

i. $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}, \quad 0 \le x \le 1$

$= -\cos^{-1}\sqrt{1-x^2}, \quad -1 \le x < 0$

ii. $\sin^{-1} x = \tan^{-1} \dfrac{x}{\sqrt{1-x^2}}, \quad -1 < x < 1$

iii. $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}, \quad 0 \le x \le 1$

$= \pi - \sin^{-1}\sqrt{1-x^2}, \quad -1 \le x < 0$

iv. $\cos^{-1} x = \tan^{-1} \dfrac{\sqrt{1-x^2}}{x}, \quad 0 < x \le 1$

$= \pi + \tan^{-1} \dfrac{\sqrt{1-x^2}}{x}, \quad -1 \le x < 0$

v. $\tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{1+x^2}}$

vi. $\tan^{-1} x = \cos^{-1} \frac{1}{\sqrt{1+x^2}}, \quad x \ge 0$

$= -\cos^{-1} \frac{1}{\sqrt{1+x^2}}, \quad x < 0$

6.

i. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}, \quad xy < 1$

$= \pi + \tan^{-1} \frac{x+y}{1-xy}, \quad xy > 1, x > 0, y > 0$

$= -\pi + \tan^{-1} \frac{x+y}{1-xy}, \quad xy > 1, x < 0, y < 0$

$= \frac{\pi}{2}, \quad xy = 1, x > 0, y > 0$

$= -\frac{\pi}{2}, \quad xy = 1, x < 0, y < 0$

ii. $2\tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}, \quad -1 < x < 1$

$= \pi + \tan^{-1} \frac{2x}{1-x^2}, \quad x > 1$

$= -\pi + \tan^{-1} \frac{2x}{1-x^2}, \quad x < -1$

$= \frac{\pi}{2}, \quad x = 1$

$= -\frac{\pi}{2}, \quad x = -1$

SECTION-12.4. INVERSE TRIGONOMETRIC EQUATIONS

1. Simplest inverse trigonometric equations

i. $\sin^{-1} x = a \equiv x = \sin a, \quad -\frac{\pi}{2} \le a \le \frac{\pi}{2}$

$\equiv x \in \phi, \quad a < -\frac{\pi}{2}; a > \frac{\pi}{2}$

ii. $\cos^{-1} x = a \equiv x = \cos a, \quad 0 \le a \le \pi$

$\equiv x \in \phi, \quad a < 0; a > \pi$

iii. $\tan^{-1} x = a \equiv x = \tan a, \quad -\frac{\pi}{2} < a < \frac{\pi}{2}$

$\equiv x \in \phi, \quad a \le -\frac{\pi}{2}; a \ge \frac{\pi}{2}$

2. **Equations of type** $\sin^{-1} f(x) = a$**, etc.**
3. **Equations of type** $\sin^{-1} f(x) = \sin^{-1} g(x)$**, etc.**
   i. $\sin^{-1} f(x) = \sin^{-1} g(x) \equiv \begin{Bmatrix} -1 \le f(x) \le 1 \\ -1 \le g(x) \le 1 \\ f(x) = g(x) \end{Bmatrix}$
4. **Substitution.**
5. **Take sin, cos or tan both the sides converting to non-equivalent algebraic equation. Solve the algebraic equation, back check its each solution in the original equation and discard extraneous solutions, if any, to get the solutions of the original equation.**
6. **Change/break into simpler equations using inverse trigonometric identities**

SECTION-12.5. TRIGONOMETRIC INEQUALITIES

1. **Simplest trigonometric inequalities**
   i. $\sin x > a, \sin x < a, \sin x \ge a, \sin x \le a$
   ii. $\cos x > a, \cos x < a, \cos x \ge a, \cos x \le a$
   iii. $\tan x > a, \tan x < a, \tan x \ge a, \tan x \le a$
   iv. $\cot x > a, \cot x < a, \cot x \ge a, \cot x \le a$
2. **Inequalities of type** $\sin f(x) > a$**, etc.**
3. **Substitution**
4. **Homogeneous inequality in** $\sin x$ **and** $\cos x$
5. **Inequality of type** $a\sin^2 x + b\sin x\cos x + c\cos^2 x < d$
6. **Inequality of type** $a\sin x + b\cos x < c$
7. **Inequalities containing Rational functions of** $\sin x$ **and** $\cos x$
8. **Lowering high even powers of** $\sin x$ **and** $\cos x$
9. **Change/break into simpler inequalities using trigonometric identities**

SECTION-12.6. INVERSE TRIGONOMETRIC INEQUALITIES

1. **Simplest inverse trigonometric inequalities**
   i. $\sin^{-1} x > a, \sin^{-1} x < a, \sin^{-1} x \ge a, \sin^{-1} x \le a$
   ii. $\cos^{-1} x > a, \cos^{-1} x < a, \cos^{-1} x \ge a, \cos^{-1} x \le a$
   iii. $\tan^{-1} x > a, \tan^{-1} x < a, \tan^{-1} x \ge a, \tan^{-1} x \le a$
2. **Inequalities of type** $\sin^{-1} f(x) > a$**, etc.**
3. **Inequalities of type** $\sin^{-1} f(x) > \sin^{-1} g(x)$**, etc.**
4. **Substitution.**
5. **Take sin, cos or tan both sides converting to equivalent algebraic inequalities.**
6. **Change/break into simpler inequalities using inverse trigonometric identities.**

EXERCISE-12

CATEGORY-12.1. MAXIMUM AND MINIMUM VALUES OF TRIGONOMETRIC FUNCTIONS

1. Which of the following statements are correct/incorrect?
   i. $\sin\theta = -\frac{1}{5}$. {Ans. correct}
   ii. $\cos\theta = 1$. {Ans. correct}
   iii. $\sec\theta = \frac{1}{2}$. {Ans. incorrect}
   iv. $\tan\theta = 20$. {Ans. correct}
2. Find the maximum and minimum values of $7\cos\theta + 24\sin\theta$. {Ans. –25, 25}
3. Show that the maximum and minimum values of $8\cos\theta - 15\sin\theta$ are 17 and –17 respectively.
4. Find the maximum and minimum values of $3\cos x + 4\sin x + 5$. {Ans. 0, 10}
5. For what value of *x* in the interval $\left(0, \frac{\pi}{2}\right)$, the maximum value of $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$ is attained? {Ans. $\frac{\pi}{12}$}
6. Show that $\left|\sin x + \cos x\right| \le \sqrt{2}$.
7. Prove that $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ lies between –4 and 10.
8. Find *a* and *b* such that the inequality $a \le 3\cos x + 5\sin\left(x - \frac{\pi}{6}\right) \le b$ holds good for all *x*. {Ans. $a \le -\sqrt{19}, b \ge \sqrt{19}$}
9. Find the maximum and minimum values of $6\sin x\cos x + 4\cos 2x$. {Ans. $5, -5$}
10. Show that for all values of θ, the expression $a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$ lies between $\frac{a+c}{2} - \frac{\sqrt{b^2 + (a-c)^2}}{2}$ and $\frac{a+c}{2} + \frac{\sqrt{b^2 + (a-c)^2}}{2}$.
11. Express $6\cos^2\alpha + 8\sin\alpha\cos\alpha$ as $A + B\cos(2\alpha - \beta)$ and hence show that the greatest and the least values of the expression are 8 and –2 respectively.
12. Find the maximum and minimum values of $\cos 2x + 9\sin x$. {Ans. 8, -10}
13. Prove that $-4 \le \cos 2x + 3\sin x \le \frac{17}{8}$.
14. If $a \le \cos 2x + 5\sin x + 6 \le b$, find *a* and *b*. {Ans. *a* = 0, *b* = 10}
15. Show that the value of $\sec^2\theta + \cos^2\theta$ is never less than 2.
16. Prove that $\frac{\sin\alpha}{\sin\beta} + \frac{\sin\beta}{\sin\alpha} = \frac{(\sin\alpha - \sin\beta)^2}{\sin\alpha\sin\beta} + 2$. Hence deduce that if $0 < \alpha, \beta < \pi$, $\frac{\sin\alpha}{\sin\beta} + \frac{\sin\beta}{\sin\alpha} \ge 2$.
17. Find the greatest and least values of $\cos A\cos B$ when $A + B = 90°$. {Ans. $-\frac{1}{2}, \frac{1}{2}$}
18. If *A* > 0, *B* > 0 and $A + B = \frac{\pi}{3}$, find the maximum value of $\tan A\tan B$. {Ans. $\frac{1}{3}$}

19. Prove that $\dfrac{\cot 3x}{\cot x}$ never lies between $\dfrac{1}{3}$ and 3.

20. Prove that $\tan\left(x+\dfrac{\pi}{6}\right)\cot x$ cannot lie between $\dfrac{1}{3}$ and 3.

21. Show that $\dfrac{3+\cos\theta}{\sin\theta}$ cannot have any value between $-2\sqrt{2}$ and $2\sqrt{2}$. What are the limits of $\dfrac{\sin\theta}{3+\cos\theta}$.
{Ans. $\left[-\dfrac{1}{2\sqrt{2}},\dfrac{1}{2\sqrt{2}}\right]$}

22. Prove that the expression $\cos\theta\left(\sin\theta+\sqrt{\sin^2\alpha+\sin^2\theta}\right)$ lies between $-\sqrt{1+\sin^2\alpha}$ and $\sqrt{1+\sin^2\alpha}$, $0<\theta<\dfrac{\pi}{2}$.

23. Prove that $\cos(\cos\theta)>0$ and $\cos(\sin\theta)>0$ for all θ.

24. For all θ in $\left[0,\dfrac{\pi}{2}\right]$ show that $\cos(\sin\theta)\geq\sin(\cos\theta)$.

25. If $0<\theta<\pi$, prove that $\cot\dfrac{\theta}{4}-\cot\theta>2$ and $\cot\dfrac{\theta}{2}-\cot\theta\geq 1$.

CATEGORY-12.2. TRIGONOMETRIC EQUATIONS

26. $\left\{\begin{array}{l}\cos\theta=-\dfrac{1}{\sqrt{2}}\\ \tan\theta=1\end{array}\right\}$. {Ans. $\left[(2n+1)\pi+\dfrac{\pi}{4}\right]$}

27. $\tan 3x=1$. {Ans. $\left[\dfrac{n\pi}{3}+\dfrac{\pi}{12}\right]$}

28. $\dfrac{\cos x}{1+\cos 2x}=0$ {Ans. ϕ}

29. $\dfrac{\sin x+\cos x}{\cos 2x}=0$ {Ans. ϕ}

30. $\cos x\tan 3x=0$ {Ans. $\left[\dfrac{n\pi}{3}\right]$}

31. $\sin 4x\cos x\tan 2x=0$ {Ans. $\left[\dfrac{n\pi}{2}\right]$}

32. $(1+\cos x)\left(\dfrac{1}{\sin x}-1\right)=0$ {Ans. $\left[2n\pi+\dfrac{\pi}{2}\right]$}

33. $(1+\cos x)\tan\dfrac{x}{2}=0$ {Ans. $[2n\pi]$}

34. $2\sin^2\theta-3\sin\theta-2=0$. {Ans. $n\pi+(-1)^n\dfrac{7\pi}{6}$}

35. $\sin^2 3x-5\sin 3x+4=0$ {Ans. $\left[\dfrac{2n\pi}{3}+\dfrac{\pi}{6}\right]$}

36. $8\sec^2\theta - 6\sec\theta + 1 = 0$. {Ans. ϕ}

37. $\tan^3 x + \tan^2 x - 3\tan x = 3$ {Ans. $\left[n\pi \pm \frac{\pi}{3}\right] \cup \left[k\pi - \frac{\pi}{4}\right]$}

38. $8\cos^4 x - 8\cos^2 x - \cos x + 1 = 0$ {Ans. $\left[\frac{2n\pi}{3}\right] \cup \left[2k\pi \pm \cos^{-1}\left(\frac{-1 \pm \sqrt{5}}{4}\right)\right]$}

39. $2\sin^3 x - \cos 2x - \sin x = 0$ {Ans. $\left[\frac{n\pi}{2} + \frac{\pi}{4}\right] \cup \left[2k\pi - \frac{\pi}{2}\right]$}

40. $2\cos^2 x + 5\sin x - 4 = 0$ {Ans. $\left[n\pi + (-1)^n \frac{\pi}{6}\right]$}

41. $3\sin^2 2x + 7\cos 2x = 3$ {Ans. $\left[\frac{n\pi}{2} + \frac{\pi}{4}\right]$}

42. $2\cos^2 x + \sin x = 2$ {Ans. $[n\pi] \cup \left[k\pi + (-1)^k \frac{\pi}{6}\right]$}

43. $\sqrt{2}\sin^2 x + \cos x = 0$ {Ans. $\left[2n\pi \pm \frac{3\pi}{4}\right]$}

44. $\sin 2x + \cos 2x = \sin x + \cos x$ {Ans. $[2n\pi] \cup \left[\frac{2k\pi}{3} + \frac{\pi}{6}\right]$}

45. $\sqrt{2}\cos 2x = \cos x + \sin x$ {Ans. $\left[\frac{2n\pi}{3} + \frac{\pi}{12}\right] \cup \left[2k\pi - \frac{\pi}{4}\right]$}

46. $\sin 3x = \cos 2x$ {Ans. $\left[\frac{2n\pi}{5} + \frac{\pi}{10}\right]$}

47. $\cos 5x = \sin 15x$ {Ans. $\left[\frac{n\pi}{10} + \frac{\pi}{40}\right] \cup \left[\frac{k\pi}{5} + \frac{\pi}{20}\right]$}

48. $\sin(5\pi - x) = \cos(2x + 7\pi)$ {Ans. $\left[2n\pi + \frac{\pi}{2}\right] \cup \left[2k\pi + (-1)^{k+1} \frac{\pi}{6}\right]$}

49. $4\sin^2 x + \sin^2 2x = 3$ {Ans. $\left[\frac{n\pi}{2} + \frac{\pi}{4}\right]$}

50. $4\cos^2 2x + 8\cos^2 x = 7$ {Ans. $\left[n\pi \pm \frac{\pi}{6}\right]$}

51. $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{3}\right) = 1 + \cos 2x$ {Ans. $\left[n\pi + \frac{\pi}{2}\right] \cup \left[2k\pi \pm \frac{\pi}{3}\right]$}

52. $8\sin^6 x + 3\cos 2x + 2\cos 4x + 1 = 0$ {Ans. $\left[\frac{n\pi}{2} + \frac{\pi}{4}\right]$}

53. $3(1 - \sin x) = 1 + \cos 2x$ {Ans. $\left[2n\pi + \frac{\pi}{2}\right] \cup \left[k\pi + (-1)^k \frac{\pi}{6}\right]$}

54. $\sin x = \frac{3}{4}\cos x$ {Ans. $\left[n\pi + \tan^{-1}\frac{3}{4}\right]$}

55. $3\sin x = 2\cos x$ {Ans. $\left[n\pi + \tan^{-1}\dfrac{2}{3}\right]$}

56. $3\sin^2 x + 3\sin x\cos x - 6\cos^2 x = 0$ {Ans. $\left[n\pi + \dfrac{\pi}{4}\right] \cup \left[k\pi - \tan^{-1} 2\right]$}

57. $\sin^2 x + 3\cos^2 x - 2\sin 2x = 0$ {Ans. $\left[n\pi + \dfrac{\pi}{4}\right] \cup \left[k\pi + \tan^{-1} 3\right]$}

58. $3\sin^2 x + 2\sin x\cos x = 2$ {Ans. $\left[n\pi + \tan^{-1}\left(-1 \pm \sqrt{3}\right)\right]$}

59. $2\cos^2 x - 3\sin x\cos x + 5\sin^2 x = 3$ {Ans. $\left[n\pi + \tan^{-1}\dfrac{3 \pm \sqrt{17}}{4}\right]$}

60. $\cos^2 x + \sin x + 1 = 0$.

61. $\tan 2x \tan x = 1$. {Ans. $\left[n\pi \pm \dfrac{\pi}{6}\right]$}

62. $\left(\sqrt{3} - 1\right)\sin x + \left(\sqrt{3} + 1\right)\cos x = 2$. {Ans. $\left[2n\pi \pm \dfrac{\pi}{4} + \dfrac{\pi}{12}\right]$}

63. $\sin 7x = \sin 4x - \sin x$. {Ans. $\left[\dfrac{n\pi}{4}\right] \cup \left[\dfrac{2n\pi}{3} \pm \dfrac{\pi}{9}\right]$}

64. $\sin 2x = \cos 3x$.

{Ans. $\left[2n\pi + \theta\right]$, $\theta = \sin^{-1}\dfrac{\sqrt{5}-1}{4}, \dfrac{\pi}{2}, \pi - \sin^{-1}\dfrac{\sqrt{5}-1}{4}, \pi + \sin^{-1}\dfrac{\sqrt{5}+1}{4}, \dfrac{3\pi}{2}, 2\pi - \sin^{-1}\dfrac{\sqrt{5}+1}{4}$ }

65. $4\sin x\cos x - 2\cos x - 2\sqrt{3}\sin x + \sqrt{3} = 0$. {Ans. $\left[2n\pi + \dfrac{\pi}{6}\right] \cup \left[2n\pi + \dfrac{5\pi}{6}\right] \cup \left[2n\pi + \dfrac{11\pi}{6}\right]$}

66. $\tan 3x = \tan 5x$. {Ans. $\left[n\pi\right]$}

67. $\sin 5x\cos 3x = \sin 9x\cos 7x$ {Ans. $\left[\dfrac{n\pi}{4}\right] \cup \left[\dfrac{k\pi}{12} + \dfrac{\pi}{24}\right]$}

68. $\sin 6x\cos 2x = \sin 5x\cos 3x - \sin 2x$ {Ans. $\left[\dfrac{n\pi}{3}\right] \cup \left[k\pi + \dfrac{\pi}{2}\right]$}

69. $\sin^6 x + \cos^6 x = \dfrac{7}{16}$ {Ans. $\left[\dfrac{n\pi}{2} \pm \dfrac{\pi}{6}\right]$}

70. $2\cos^2 x + \cos 5x = 1$ {Ans. $\left[\dfrac{2n\pi}{7} + \dfrac{\pi}{7}\right] \cup \left[\dfrac{2k\pi}{3} + \dfrac{\pi}{3}\right]$}

71. $\sin x + \sin 2x + \sin 3x = 0$ {Ans. $\left[\dfrac{n\pi}{2}\right] \cup \left[2k\pi \pm \dfrac{2\pi}{3}\right]$}

72. $\sin x + \sin 3x + \cos x + \cos 3x = 0$ {Ans. $\left[n\pi + \dfrac{\pi}{2}\right] \cup \left[\dfrac{k\pi}{2} - \dfrac{\pi}{8}\right]$}

73. $\sin x + \cos x = 1$. {Ans. $\left[n\pi + (-1)^n\dfrac{\pi}{4} - \dfrac{\pi}{4}\right]$}

74. $\cos x+\sqrt{3}\sin x=2$. {Ans. $\left[2n\pi+\frac{\pi}{3}\right]$}

75. $\sqrt{3}\sin 2x+\cos 2x=\sqrt{2}$ {Ans. $\left[n\pi+\frac{\pi}{6}\pm\frac{\pi}{8}\right]$}

76. $\frac{1}{2}\sin 3x+\frac{\sqrt{3}}{2}\cos 3x=\sin 5x$ {Ans. $\left[n\pi+\frac{\pi}{6}\right]\cup\left[\frac{k\pi}{4}+\frac{\pi}{12}\right]$}

77. $2\cos 3x+\sqrt{3}\sin x+\cos x=0$ {Ans. $\left[\frac{n\pi}{2}+\frac{\pi}{3}\right]$}

78. $\sin 5x+\cos 5x=\sqrt{2}\cos 13x$ {Ans. $\left[\frac{n\pi}{4}-\frac{\pi}{32}\right]\cup\left[\frac{k\pi}{9}+\frac{\pi}{72}\right]$}

79. $\sin^2 x-\cos 2x=2-\sin 2x$ {Ans. $\left[n\pi+\frac{\pi}{2}\right]\cup\left[k\pi+\tan^{-1}\frac{3}{2}\right]$}

80. $\sin^2 x\cos^2 x-10\sin x\cos^3 x+21\cos^4 x=0$ {Ans. $\left[n\pi+\frac{\pi}{2}\right]\cup\left[k\pi+\tan^{-1}7\right]\cup\left[m\pi+\tan^{-1}3\right]$}

81. $8\sin^2\frac{x}{2}-3\sin x-4=0$ {Ans. $\left[n\pi-\tan^{-1}\frac{4}{3}\right]$}

82. $\sin^4 x+\cos^4 x=\cos 4x$ {Ans. $\left[\frac{n\pi}{2}\right]$}

83. $\cos^4 x+\sin^4 x-\sin 2x+\frac{3}{4}\sin^2 2x=0$ {Ans. ϕ}

84. $3\tan\frac{x}{2}+\cot x=\frac{5}{\sin x}$ {Ans. ϕ}

85. $\cos 2x-3\cos x+1=\frac{1}{(\cot 2x-\cot x)\sin(x-\pi)}$ {Ans. ϕ}

86. $\cos x=\frac{\tan x}{1+\tan^2 x}$ {Ans. ϕ}

87. $\cot x+\frac{\sin x}{1+\cos x}=2$ {Ans. $\left[n\pi+(-1)^n\frac{\pi}{6}\right]$}

88. $2\sin x-3\cos x=3$ {Ans. $[2n\pi+\pi]\cup\left[2k\pi+2\tan^{-1}\frac{3}{2}\right]$}

89. $3\sin 2x+\cos 2x=2$ {Ans. $\left[n\pi+\tan^{-1}\frac{3\pm\sqrt{6}}{3}\right]$}

90. $\cos 4x+2\sin 4x=1$ {Ans. $\left[\frac{n\pi}{2}\right]\cup\left[k\pi+\frac{1}{2}\tan^{-1}2\right]$}

91. $\sin 2x+\tan x=2$ {Ans. $\left[n\pi+\frac{\pi}{4}\right]$}

92. $\frac{1+\sin x}{1+\cos x}=\frac{1}{2}$ {Ans. $[2n\pi]\cup\left[2k\pi-2\tan^{-1}2\right]$}

93. $\sin^3 x + \cos^3 x = 1$ {Ans. $\left[2n\pi + \frac{\pi}{2}\right] \cup [2k\pi]$}

94. $4\sin^4 3x - 3\cos x + 5 = 0$ {Ans. ϕ}

95. $\sin x \cos x - 6\sin x + 6\cos x + 6 = 0$ {Ans. $\left[2n\pi + \frac{\pi}{2}\right] \cup [2k\pi + \pi]$}

96. $4 - 4(\cos x - \sin x) - \sin 2x = 0$ {Ans. $[2n\pi] \cup \left[2k\pi - \frac{\pi}{2}\right]$}

97. $5\sin 2x - 11(\sin x + \cos x) + 7 = 0$ {Ans. $\left[2n\pi + \frac{\pi}{4} \pm \cos^{-1}\frac{\sqrt{2}}{10}\right]$}

98. $\left(2\sin^4 \frac{x}{2} - 1\right)\frac{1}{\cos^4 \frac{x}{2}} = 2$ {Ans. $\left[2n\pi \pm \frac{2\pi}{3}\right]$}

99. $\cos x \cos 2x \cos 4x \cos 8x = \frac{1}{16}$ {Ans. $\left[\frac{2n\pi}{15}\right] \cup \left[\frac{2k\pi}{17} + \frac{\pi}{17}\right]; \left(k \neq \frac{15l}{2}, n \neq 17m + 8\right)$}

100. $2\sin 17x + \sqrt{3}\cos 5x + \sin 5x = 0$ {Ans. $\left[\frac{n\pi}{11} - \frac{\pi}{66}\right] \cup \left[\frac{k\pi}{6} + \frac{\pi}{9}\right]$}

101. $4\cos^3 \frac{x}{2} + 3\sqrt{2}\sin x = 8\cos\frac{x}{2}$ {Ans. $[2n\pi + \pi] \cup \left[2k\pi + (-1)^k \frac{\pi}{2}\right]$}

102. $\frac{7}{4}\cos\frac{x}{4} = \cos^3 \frac{x}{4} + \sin\frac{x}{2}$ {Ans. $[4n\pi + 2\pi] \cup \left[4k\pi + (-1)^k \frac{2\pi}{3}\right]$}

103. $4\sin 2x - \tan^2\left(x - \frac{\pi}{4}\right) = 4$ {Ans. $\left[n\pi + \frac{\pi}{4}\right]$}

104. $\left(\sin 2x + \sqrt{3}\cos 2x\right)^2 - 5 = \cos\left(\frac{\pi}{6} - 2x\right)$ {Ans. $\left[n\pi + \frac{7\pi}{12}\right]$}

105. $\cos\frac{4x}{3} = \cos^2 x$ {Ans. $[3n\pi] \cup \left[\frac{3k\pi}{2} \pm \frac{\pi}{4}\right]$}

106. $\sin x + 2\cos x = \cos 2x - \sin 2x$ {Ans. $\left[n\pi - \frac{\pi}{2}\right] \cup \left[2k\pi + \frac{\pi}{4} \pm \cos^{-1}\frac{1}{2\sqrt{2}}\right]$}

107. $32\cos^6 x - \cos 6x = 1$ {Ans. $\left[n\pi + \frac{\pi}{2}\right] \cup \left[k\pi \pm \frac{1}{2}\cos^{-1}\left(-\frac{1}{4}\right)\right]$}

108. $\tan x + \cot x - \cos 4x = 3$ {Ans. $\left[n\pi + \frac{\pi}{4}\right] \cup \left[\frac{k\pi}{2} + (-1)^k \frac{1}{2}\sin^{-1}\left(\frac{\sqrt{5} - 1}{2}\right)\right]$}

109. $2(1 - \sin x - \cos x) + \tan x + \cot x = 0$ {Ans. $\left[n\pi - \frac{\pi}{4}\right] \cup \left[2k\pi + \frac{\pi}{4} \pm \cos^{-1}\left(\frac{\sqrt{2} - \sqrt{10}}{4}\right)\right]$}

110. $\sin^5 x - \cos^5 x = \frac{1}{\cos x} - \frac{1}{\sin x}$ {Ans. $\left[n\pi + \frac{\pi}{4}\right]$}

111. $\sin^8 2x+\cos^8 2x=\dfrac{41}{128}$ {Ans. $\left[\dfrac{n\pi}{4}\pm\dfrac{\pi}{12}\right]$}

112. $\sin^{10} x+\cos^{10} x=\dfrac{29}{64}$ {Ans. $\left[\dfrac{n\pi}{4}+\dfrac{\pi}{8}\right]$}

113. $\sin^{10} x+\cos^{10} x=\dfrac{29}{16}\cos^4 2x$ {Ans. $\left[\dfrac{n\pi}{4}+\dfrac{\pi}{8}\right]$}

114. $|\cos x|=\cos x-2\sin x$ {Ans. $[2n\pi]\cup\left[2k\pi+\dfrac{5\pi}{4}\right]$}

115. $|\cot x|=\cot x+\dfrac{1}{\sin x}$ {Ans. $\left[2n\pi+\dfrac{2\pi}{3}\right]$}

116. $\sqrt{5-2\sin x}=6\sin x-1$ {Ans. $\left[n\pi+(-1)^n\dfrac{\pi}{6}\right]$}

117. $\sqrt{2+4\cos x}=\dfrac{1}{2}+3\cos x$ {Ans. $\left[2n\pi\pm\dfrac{\pi}{3}\right]$}

118. $\sqrt{3+2\tan x-\tan^2 x}=\dfrac{1+3\tan x}{2}$ {Ans. $\left[n\pi+\dfrac{\pi}{4}\right]$}

119. $\sqrt{-3\sin 5x-\cos^2 x-3}+\sin x=1$ {Ans. ϕ}

120. $\tan x+\dfrac{1}{9}\cot x=\sqrt{\dfrac{1}{\cos^2 x}-1}-1$ {Ans. $\left[n\pi-\tan^{-1}\dfrac{1}{6}\right]\cup\left[k\pi-\tan^{-1}\dfrac{1}{3}\right]$}

121. $(1+\cos x)\sqrt{\tan\dfrac{x}{2}-2+\sin x}=2\cos x$ {Ans. $\left[2n\pi+\dfrac{\pi}{2}\right]$}

122. $\sqrt{\cos^2 x+\dfrac{1}{2}}+\sqrt{\sin^2 x+\dfrac{1}{2}}=2$ {Ans. $\left[2n\pi\pm\dfrac{\pi}{4}\right]$}

123. $\sqrt{1-2\tan x}-\sqrt{1+2\cot x}=2$ {Ans. $\left[n\pi-\dfrac{3\pi}{8}\right]$}

124. $\sqrt{3}\sin x-\sqrt{2\sin^2 x-\sin 2x+3\cos^2 x}=0$ {Ans. $\left[2n\pi+\dfrac{\pi}{4}\right]\cup\left[(2k+1)\pi-\tan^{-1}3\right]$}

125. $\cos x+\sqrt{\sin^2 x-2\sin 2x+4\cos^2 x}=0$ {Ans. $\left[2n\pi+\dfrac{5\pi}{4}\right]\cup\left[(2k+1)\pi+\tan^{-1}3\right]$}

126. $\sqrt{\cos 2x}+\sqrt{1+\sin 2x}=2\sqrt{\sin x+\cos x}$ {Ans. $\left[n\pi-\dfrac{\pi}{4}\right]\cup[2k\pi]$}

127. $2\cot 2x-3\cot 3x=\tan 2x$ {Ans. ϕ}

128. $6\tan x+5\cot 3x=\tan 2x$ {Ans. $\left[n\pi\pm\dfrac{1}{2}\cos^{-1}\dfrac{1}{3}\right]\cup\left[k\pi\pm\dfrac{1}{2}\cos^{-1}\left(-\dfrac{1}{4}\right)\right]$}

129. $\tan\left(x-\dfrac{\pi}{4}\right)\tan x\tan\left(x+\dfrac{\pi}{4}\right)=\dfrac{4\cos^2 x}{\tan\dfrac{x}{2}-\cot\dfrac{x}{2}}$ {Ans. ϕ}

130. $\sin^2 5x\left(\sin 7x\cos x-\sin\frac{x}{2}\cos\frac{3x}{2}\right)=\frac{\sin\frac{3x}{2}\cos\frac{x}{2}+\sin x\cos 7x}{1+\cot^2 5x}$ {Ans. $\left[n\pi+\frac{\pi}{2}\right]\cup\left[\frac{k\pi}{4}+\frac{\pi}{8}\right]$}

131. $\sin^6 x+\sin^4 x+\cos^6 x+\cos^4 x+\sin\frac{x}{2}=3$ {Ans. $[4n\pi+\pi]$}

132. $1+\cos 2x\cos 3x=\frac{1}{2}\sin^2 3x$ {Ans. $[2n\pi+\pi]$}

133. $\sin 5x+\sin x=2+\cos^2 x$ {Ans. $\left[2n\pi+\frac{\pi}{2}\right]$}

134. $3\sin^2\frac{x}{3}+5\sin^2 x=8$ {Ans. $\left[3n\pi+\frac{3\pi}{2}\right]$}

135. $\left(\sin x+\sqrt{3}\cos x\right)\sin 3x=2$ {Ans. $\left[n\pi+\frac{\pi}{6}\right]$}

136. $2\sin\left(\frac{2}{3}x-\frac{\pi}{6}\right)-3\cos\left(2x+\frac{\pi}{3}\right)=5$ {Ans. ϕ }

137. $\sin\frac{x}{4}+2\cos\frac{x-2\pi}{3}=3$ {Ans. $[24n\pi+2\pi]$}

138. $\sin 18x+\sin 10x+\sin 2x=3+\cos^2 2x$ {Ans. $\left[n\pi+\frac{\pi}{4}\right]$}

139. $\cos 2x\left(1-\frac{3}{4}\sin^2 2x\right)=1$ {Ans. $[n\pi]$}

140. $\sin x+\cos x=\sqrt{2}+\sin^4 4x$ {Ans. $\left[2n\pi+\frac{\pi}{4}\right]$}

141. $\cos^6 2x=1+\sin^4 x$ {Ans. $[n\pi]$}

142. $\cot\left(\frac{\pi}{3}\cos 2\pi x\right)=\sqrt{3}$ {Ans. $\left[n\pm\frac{1}{6}\right]$}

143. $2\sin^2\left(\frac{\pi}{2}\cos^2 x\right)=1-\cos(\pi\sin 2x)$ {Ans. $\left[n\pi+\frac{\pi}{2}\right]\cup\left[k\pi\pm\tan^{-1}\frac{1}{2}\right]$}

144. If $\sin(\pi\cos x)=\cos(\pi\sin x)$, prove that $\cos\left(x\pm\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$.

145. If $\sin(\pi\cot x)=\cos(\pi\tan x)$, prove that either $\operatorname{cosec} 2x$ or $\cot 2x$ is equal to $n+\frac{1}{4}$, where n is a positive or negative integer.

146. If $\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$, prove that $\cos\left(\theta-\frac{\pi}{4}\right)=\pm\frac{1}{2\sqrt{2}}$.

147. Find the values of $\theta\ (0<\theta<360°)$ satisfying $\operatorname{cosec}\theta+2=0$. {Ans. $210°, 330°$ }

148. Find the solution set of $(2\cos x-1)(3+2\cos x)=0$ in the interval $0\le x\le 2\pi$. {Ans. $\left[\frac{\pi}{3}\right]\cup\left[\frac{5\pi}{3}\right]$}

149. Find the smallest value of θ satisfying the equation $\sqrt{3}(\cot\theta+\tan\theta)=4$. {Ans. $\frac{\pi}{6}$}

150. If $\cos 20°=k$ and $\cos x=2k^2-1$, then find the possible values of x between $0°$ and $360°$. {Ans. 40° and $320°$}

151. If α and β are the solutions to the equation $a\tan\theta+b\sec\theta=c$, then show that $\tan(\alpha+\beta)=\frac{2ac}{a^2-c^2}$.

152. If α, β be unequal values of θ satisfying the equation $a\tan\theta+b\sec\theta=1$, find a and b in terms of α and β and prove that $\sin\alpha+\cos\alpha+\sin\beta+\cos\beta=\frac{2b(1-a)}{(1+a^2)}$. {Ans. $a=\frac{\cos\alpha-\cos\beta}{\sin\alpha-\sin\beta}$, $b=\frac{\sin(\alpha-\beta)}{\sin\alpha-\sin\beta}$}

153. If $a\cos 2\theta+b\sin 2\theta=c$ has α and β as its solutions, then prove that $\tan\alpha+\tan\beta=\frac{2b}{c+a}$ and $\tan\alpha\tan\beta=\frac{c-a}{c+a}$.

154. If α and β are the solutions of $a\cos\theta+b\sin\theta=c$, then show that
   i. $\cos\alpha+\cos\beta=\frac{2ac}{a^2+b^2}$
   ii. $\cos\alpha\cos\beta=\frac{c^2-b^2}{a^2+b^2}$
   iii. $\sin\alpha+\sin\beta=\frac{2bc}{a^2+b^2}$
   iv. $\sin\alpha\sin\beta=\frac{c^2-a^2}{a^2+b^2}$.
   v. $\cos(\alpha+\beta)=\frac{a^2-b^2}{a^2+b^2}$.

155. If α and β are the roots of the equation $a\sin^2\theta+b\sin\theta+c=0$, show that $\cos(\alpha+\beta)\cos(\alpha-\beta)=\frac{a^2-b^2+2ac}{a^2}$.

156. If α and β are distinct roots of the equation $a\cos\theta+b\sin\theta=c$, between 0 and 2π, and if $\alpha+\beta$ also satisfies the equation, show that $a = c$.

157. If $\theta_1, \theta_2, \theta_3$ are the values of θ which satisfy the equation $\tan 2\theta=\lambda\tan(\theta+\alpha)$, and if no two of these values differ by a multiple of π, then show that $\theta_1+\theta_2+\theta_3+\alpha$ is a multiple of π.

158. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $\tan\left(\frac{\pi}{4}+\theta\right)=3\tan 3\theta$, no two which have equal tangents, show that $\tan\alpha+\tan\beta+\tan\gamma+\tan\delta=0$.

159. If θ_1 and θ_2 are two distinct values of θ, $0\le\theta_1, \theta_2\le 2\pi$, satisfying the equation $\sin(\theta+\alpha)=\frac{1}{2}\sin 2\alpha$

prove that $\dfrac{\sin\theta_1 + \sin\theta_2}{\cos\theta_1 + \cos\theta_2} = \cot\alpha$.

160. Prove that the equation $x + \dfrac{1}{x} = \sin\theta$ is not possible for any real value of x.

161. Find the values of $\cos\theta$ for which the equation $2\cos\theta = x + \dfrac{1}{x}$ is possible, x being real. {Ans. ± 1}

162. Equation $2\sin e^{x} = 5^{x} + 5^{-x}$ has how many solutions? {Ans. No solution}

163. Prove that the equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ is possible for real values of x and y only if $x = y$.

164. For what values of c, the equation $\sec\theta + \operatorname{cosec}\theta = c$ has two real roots between 0 and 2π? {Ans. $c^2 > 8$}

165. If $|k| < 5$ and $0° \le \theta \le 360°$, then what is the number of different solutions of $3\cos\theta + 4\sin\theta = k$? {Ans. Two}

166. If $\sin x + \operatorname{cosec} x = 2$, then find the value of $\sin^n x + \operatorname{cosec}^n x$. {Ans. 2}

167. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then find $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$. {Ans. 3}

168. Equation $\sin x = \ln x$ has how many solutions? {Ans. One}

169. Equation $\sin x = \log x$ has how many solutions? {Ans. Three}

170. Equation $x\sin x = 1$ has how many solutions? {Ans. Infinite solutions}

171. Equation $\sec e^{x} = \tanh x^3$ has how many solutions? {Ans. No solution}

172. Equation $|\ln|x|| = \cot^{-1} x$ has how many solutions? {Ans. Four}

CATEGORY-12.3. VALUES OF INVERSE TRIGONOMETRIC FUNCTIONS

173. Evaluate:-

i. $\cos^{-1}\dfrac{1}{2} + 2\sin^{-1}\dfrac{1}{2}$. {Ans. $\dfrac{2\pi}{3}$}

ii. $2\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right) + \cot^{-1}(-1) + \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) + \dfrac{1}{2}\cos^{-1}(-1)$ {Ans. $\dfrac{5\pi}{6}$}

iii. $\tan\left(5\tan^{-1}\dfrac{1}{\sqrt{3}} - \dfrac{1}{4}\sin^{-1}\dfrac{\sqrt{3}}{2}\right)$ {Ans. -1}

iv. $\sin\left(3\tan^{-1}\sqrt{3} + 2\cos^{-1}\dfrac{1}{2}\right)$ {Ans. $-\dfrac{\sqrt{3}}{2}$}

v. $\cos\left(3\sin^{-1}\dfrac{\sqrt{3}}{2} + \cos^{-1}\left(-\dfrac{1}{2}\right)\right)$ {Ans. $\dfrac{1}{2}$}

vi. $\cos^{-1}\left(\cos\dfrac{\pi}{4}\right)$ {Ans. $\dfrac{\pi}{4}$}

vii. $\cos^{-1}\left(\cos\left(\dfrac{5\pi}{4}\right)\right)$. {Ans. $\dfrac{3\pi}{4}$}

viii. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$. {Ans. $\frac{5\pi}{6}$ }

ix. $\cos^{-1}\left(-\cos\frac{3\pi}{4}\right)$ {Ans. $\frac{\pi}{4}$ }

x. $\cos^{-1}(\cos 6)$. {Ans. $2\pi - 6$ }

xi. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$. {Ans. $\frac{\pi}{3}$ }

xii. $\sin^{-1}\left(-\sin\frac{7\pi}{3}\right)$ {Ans. $-\frac{\pi}{3}$ }

xiii. $\sin^{-1}(\sin 2)$. {Ans. $\pi - 2$ }

xiv. $\sin^{-1}(\sin 10)$. {Ans. $3\pi - 10$ }

xv. $\tan^{-1}\left(\tan\frac{3\pi}{10}\right)$ {Ans. $\frac{3\pi}{10}$ }

xvi. $\tan^{-1}\left(-\tan\frac{2\pi}{3}\right)$ {Ans. $\frac{\pi}{3}$ }

xvii. $\tan^{-1}(\tan 5)$ {Ans. $5 - 2\pi$ }

xviii. $\sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\left(\cos\frac{46\pi}{7}\right)$ {Ans. $\frac{6\pi}{7}$ }

xix. $\tan^{-1}\left(-\tan\frac{13\pi}{8}\right) + \cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$ {Ans. π }

xx. $\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right)$. {Ans. $\frac{\pi}{4}$ }

xxi. $\sin^{-1}\left(\cos\left(\frac{33\pi}{5}\right)\right)$. {Ans. $-\frac{\pi}{10}$ }

xxii. $\sin\left(\frac{1}{2}\sin^{-1}\left(-\frac{2\sqrt{2}}{3}\right)\right)$ {Ans. $-\frac{1}{\sqrt{3}}$ }

xxiii. $\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{5}{13}\right)\right)$ {Ans. $\frac{1}{5}$ }

xxiv. $\tan\left(\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)\right)$. {Ans. $\frac{3-\sqrt{5}}{2}$ }

xxv. $\cot\left(\frac{1}{2}\cos^{-1}\left(-\frac{4}{7}\right)\right)$ {Ans. $\sqrt{\frac{3}{11}}$ }

xxvi. $\tan\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right)$. {Ans. $\frac{17}{6}$ }

xxvii. $\sin\left(\tan^{-1}\frac{8}{15} - \sin^{-1}\frac{8}{17}\right)$ {Ans. 0} xxviii. $\sin\left(2\sin^{-1}(0.8)\right)$. {Ans. 0.96}

xxix. $\cos\left(2\tan^{-1}\frac{1}{4}+\cos^{-1}\frac{3}{5}\right)$ {Ans. $\frac{13}{85}$}

xxx. $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$. {Ans. $-\frac{7}{17}$}

xxxi. $\sin\left(2\left(\sin^{-1}\frac{\sqrt{5}}{3}-\cos^{-1}\frac{\sqrt{5}}{3}\right)\right)$ {Ans. $\frac{8\sqrt{5}}{81}$}

174. If $\sin^{-1}x=\frac{\pi}{5}$, find the value of $\cos^{-1}x$. {Ans. $\frac{3\pi}{10}$}

175. If $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$, then find the value of $\cos^{-1}x+\cos^{-1}y$. {Ans. $\frac{\pi}{3}$}

176. If $\frac{\pi}{2}\le x\le\frac{3\pi}{2}$, then find the value of $\sin^{-1}(\sin x)$. {Ans. $\pi-x$}

177. If $\pi\le x\le 2\pi$, then find the value of $\cos^{-1}(\cos x)$. {Ans. $2\pi-x$}

178. Prove that:-

i. $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$.

ii. $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{15}{17}=\pi-\sin^{-1}\frac{77}{85}$.

iii. $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}=\cos^{-1}\frac{253}{325}$.

iv. $\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{3}{5}=\tan^{-1}\frac{27}{11}$.

v. $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$.

vi. $2\cos^{-1}\frac{3}{\sqrt{13}}+\cot^{-1}\frac{16}{63}+\frac{1}{2}\cos^{-1}\frac{7}{25}=\pi$.

vii. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=45^\circ$.

viii. $\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=45^\circ$.

ix. $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\frac{\pi}{4}$.

x. $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$.

xi. $\tan^{-1}\frac{2}{3}=\frac{1}{2}\tan^{-1}\frac{12}{5}$.

xii. $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}$.

xiii. $\cos^{-1}\left(\frac{15}{17}\right)+2\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\frac{171}{140}$.

xiv. $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

xv. $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}$.

xvi. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

xvii. $3\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{20}=\frac{\pi}{4}-\tan^{-1}\frac{1}{1985}$.

xviii. $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}$.

xix. $\sin^{-1}\frac{3}{5}+\mathrm{cosec}^{-1}\frac{5}{4}=\frac{\pi}{2}$.

xx. $\tan^{-1}\frac{120}{119}=2\sin^{-1}\frac{5}{13}$.

xxi. $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$.

xxii. $\tan^{-1}t+\tan^{-1}\frac{2t}{1-t^2}=\tan^{-1}\frac{3t-t^3}{1-3t^2},\ t\in(-\infty,-1)\cup\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\cup(1,\infty)$

$$=\pi+\tan^{-1}\frac{3t-t^3}{1-3t^2},\ t\in\left(\frac{1}{\sqrt{3}},1\right)$$

$$=-\pi+\tan^{-1}\frac{3t-t^3}{1-3t^2},\ t\in\left(-1,-\frac{1}{\sqrt{3}}\right)$$

179. Prove that $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=2\tan^{-1}x,\quad x\ge 0$

$$=-2\tan^{-1}x,\quad x<0.$$

CATEGORY-12.4. INVERSE TRIGONOMETRIC EQUATIONS

180. $\tan^{-1}(x^2-3x+3)=\frac{\pi}{4}$. {Ans. $[1]\cup[2]$}

181. $\tan^{-1}3x-\cot^{-1}3x=\frac{\pi}{4}$. {Ans. $\left[\frac{1+\sqrt{2}}{3}\right]$}

182. $2(\sin^{-1}x)^2-5\sin^{-1}x+2=0$. {Ans. $\sin\frac{1}{2}$}

183. $4\tan^{-1}x-6\cot^{-1}x=\pi$. {Ans. $\tan\frac{2\pi}{5}$}

184. $2\sin^{-1}x+\cos^{-1}(1-x)=0$ {Ans. 0}

185. $\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x$. {Ans. $[0] \cup \left[\frac{1}{2}\right]$}

186. $2\sin^{-1} x = \cos^{-1} 2x$ {Ans. $\frac{-1+\sqrt{3}}{2}$}

187. $\sin^{-1}\frac{x}{2} + \cos^{-1}\left(x + \frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$. {Ans. 0}

188. $\cos^{-1} x = \tan^{-1} x$. {Ans. $\sqrt{\frac{\sqrt{5}-1}{2}}$}

189. $\sin^{-1} x - \cos^{-1} x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. {Ans. $\left[\frac{\sqrt{3}}{2}\right]$}

190. $\sin^{-1}\frac{2}{3\sqrt{x}} - \sin^{-1}\sqrt{1-x} = \sin^{-1}\frac{1}{3}$. {Ans. $\frac{2}{3}$}

191. $\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \frac{\pi}{4}$. {Ans. ± 1}

192. $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$. {Ans. $\frac{1}{6}$}

193. $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$. {Ans. $\pm\frac{1}{\sqrt{2}}$}

194. $\tan^{-1}(x+1) + \cot^{-1}(x-1) = \sin^{-1}\frac{4}{5} + \cos^{-1}\frac{3}{5}$. {Ans. $\pm 4\sqrt{\frac{3}{7}}$}

195. $\tan^{-1}(1+x) + \tan^{-1}(1-x) = \frac{\pi}{2}$. {Ans. 0}

196. $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$. {Ans. $\frac{1}{4}$}

197. $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$. {Ans. $\left[n\pi + \frac{\pi}{4}\right]$}

198. $\tan^{-1} x + 2\cot^{-1} x = \frac{2}{3}\pi$. {Ans. $\sqrt{3}$}

199. $\tan\cos^{-1} x = \sin\cot^{-1}\frac{1}{2}$. {Ans. $\frac{\sqrt{5}}{3}$}

200. $\cot^{-1} x - \cot^{-1}(x+2) = 15^\circ$. {Ans. $\left[\sqrt{3}\right] \cup \left[-2-\sqrt{3}\right]$}

201. $\cos^{-1}\frac{x^2-1}{x^2+1} + \tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3}$. {Ans. $\left[\sqrt{3}\right] \cup \left[2-\sqrt{3}\right]$}

202. $\cot^{-1} x + \cot^{-1}(10-x) = \cot^{-1} 2$. {Ans. $[3] \cup [7]$}

203. $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$. {Ans. $\frac{1}{2}\sqrt{\frac{3}{7}}$}

204. $\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$. {Ans. 13}

205. $\sec^{-1}\frac{x}{4}-\sec^{-1}\frac{x}{3}=\sec^{-1}3-\sec^{-1}4$. {Ans. 12}

206. $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$. {Ans. $\frac{1}{5}$}

207. If $4\sin^{-1}x+\cos^{-1}x=\pi$, then find x. {Ans. $\frac{1}{2}$}

208. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z+\cos^{-1}u=4\pi$, then find the value of $x^{1999}+y^{2000}+z^{2001}+u^{2002}$. {Ans. 0}

209. If $\sin^{-1}x_1+\sin^{-1}x_2+\cdots\cdots+\sin^{-1}x_{2n}=n\pi$, then find the value of $x_1+x_2+\cdots\cdots+x_{2n}$. {Ans. $2n$}

210. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, find the value of $x^{100}+y^{100}+z^{100}-\frac{9}{x^{101}+y^{101}+z^{101}}$. {Ans. 0}

CATEGORY-12.5. TRIGONOMETRIC INEQUALITIES

211. $\sin x>-\frac{1}{2}$ {Ans. $\left(2n\pi-\frac{\pi}{6},2n\pi+\frac{7\pi}{6}\right)$}

212. $\cos x<\frac{\sqrt{3}}{2}$ {Ans. $\left(2n\pi+\frac{\pi}{6},2n\pi+\frac{11\pi}{6}\right)$}

213. $\tan x\geq-\frac{1}{\sqrt{3}}$ {Ans. $\left[n\pi-\frac{\pi}{6},n\pi+\frac{\pi}{2}\right)$}

214. $\cot x\leq-1$ {Ans. $\left[n\pi+\frac{3\pi}{4},n\pi+\pi\right)$}

215. $\sin x<\frac{1}{5}$ {Ans. $\left(2n\pi+\pi-\sin^{-1}\frac{1}{5},2n\pi+2\pi+\sin^{-1}\frac{1}{5}\right)$}

216. $\cos x\geq-0.7$ {Ans. $\left[2n\pi-\cos^{-1}(-0.7),2n\pi+\cos^{-1}(-0.7)\right]$}

217. $\tan x\leq 5$ {Ans. $\left(n\pi-\frac{\pi}{2},n\pi+\tan^{-1}5\right]$}

218. $\cot x>-\frac{\sqrt{3}}{4}$ {Ans. $\left(n\pi,n\pi+\cot^{-1}\left(-\frac{\sqrt{3}}{4}\right)\right)$}

219. $\begin{Bmatrix}\sin x<\frac{1}{2}\\ \cos x<\frac{1}{2}\end{Bmatrix}$ {Ans. $\left(2n\pi+\frac{5\pi}{6},2n\pi+\frac{5\pi}{3}\right)$}

220. $\begin{Bmatrix}\sin x>-\frac{\sqrt{3}}{2}\\ \tan x\leq 0\end{Bmatrix}$ {Ans. $\left(2n\pi-\frac{\pi}{3},2n\pi\right]\cup\left(2k\pi+\frac{\pi}{2},2k\pi+\pi\right]$}

221. $\begin{Bmatrix}\cos x\leq\frac{1}{\sqrt{2}}\\ \cot x>-\sqrt{3}\end{Bmatrix}$ {Ans. $\left[2n\pi+\frac{\pi}{4},2n\pi+\frac{5\pi}{6}\right)\cup\left(2k\pi+\pi,2k\pi+\frac{7\pi}{4}\right]$}

222. $\begin{Bmatrix} \tan x < 1 \\ \cot x \geq -\dfrac{1}{\sqrt{3}} \end{Bmatrix}$ {Ans. $\left(n\pi, n\pi + \dfrac{\pi}{4}\right) \cup \left(k\pi + \dfrac{\pi}{2}, k\pi + \dfrac{2\pi}{3}\right]$}

223. $\begin{Bmatrix} \sin x > \dfrac{1}{5} \\ \cos x < \dfrac{1}{5} \end{Bmatrix}$ {Ans. $\left(2n\pi + \cos^{-1}\dfrac{1}{5}, 2n\pi + \pi - \sin^{-1}\dfrac{1}{5}\right)$}

224. $\begin{Bmatrix} \cos x \geq -\dfrac{3}{5} \\ \tan x < 3 \end{Bmatrix}$

{Ans. $\left(2n\pi - \dfrac{\pi}{2}, 2n\pi + \tan^{-1} 3\right) \cup \left(2k\pi + \dfrac{\pi}{2}, 2k\pi + \cos^{-1}\left(-\dfrac{3}{5}\right)\right] \cup \left[2m\pi + \pi + \cos^{-1}\dfrac{3}{5}, 2m\pi + \pi + \tan^{-1} 3\right)$}

225. $\begin{Bmatrix} \sin x < \dfrac{4}{7} \\ \cot x < 2 \end{Bmatrix}$

{Ans. $\left(2n\pi + \cot^{-1} 2, 2n\pi + \sin^{-1}\dfrac{4}{7}\right) \cup \left(2k\pi + \pi - \sin^{-1}\dfrac{4}{7}, 2k\pi + \pi\right) \cup \left(2m\pi + \pi + \cot^{-1} 2, 2m\pi + 2\pi\right)$}

226. $\begin{Bmatrix} \tan x > 0.23 \\ \cot x \leq 0.3 \end{Bmatrix}$ {Ans. $\left[n\pi + \cot^{-1} 0.3, n\pi + \dfrac{\pi}{2}\right)$}

227. $\begin{Bmatrix} \cos x < 0 \\ \sin \dfrac{3}{5} x > 0 \end{Bmatrix}$

{Ans.

$$\left(10n\pi + \frac{\pi}{2}, 10n\pi + \frac{3\pi}{2}\right) \cup \left(10k\pi + \frac{10\pi}{3}, 10k\pi + \frac{7\pi}{2}\right)$$
$$\cup \left(10m\pi + \frac{9\pi}{2}, 10m\pi + 5\pi\right) \cup \left(10l\pi + \frac{20\pi}{3}, 10l\pi + \frac{25\pi}{3}\right)$$
}

228. $\begin{Bmatrix} \sin \dfrac{x}{2} < \dfrac{1}{2} \\ \cos 2x > -\dfrac{1}{2} \end{Bmatrix}$ {Ans. $\left[4n\pi, 4n\pi + \dfrac{5\pi}{12}\right) \cup \left(4n\pi + \dfrac{5\pi}{3}, 4n\pi + 2\pi\right)$}

229. $\sin x > \dfrac{2}{3}$; $\cos x < 0$ {Ans. $\left(2n\pi + \sin^{-1}\dfrac{2}{3}, 2n\pi + \dfrac{3\pi}{2}\right)$}

230. $\cos x < \dfrac{1}{2}$; $\tan x > -3.5$ {Ans. $(-\infty, +\infty)$}

231. $\sin x < -\dfrac{\sqrt{3}}{2}$; $\cot x \leq 7$ {Ans. $\left[n\pi + \cot^{-1} 7, n\pi + \pi\right)$}

232. $\tan x < \frac{1}{\sqrt{3}}; \cot x < \sqrt{2}$ {Ans. $\left[n\pi, n\pi + \frac{\pi}{6}\right) \cup \left(n\pi + \cot^{-1}\sqrt{2}, n\pi + \pi\right)$}

233. $\cos x^2 \geq \frac{1}{2}$ {Ans. $\left[-\sqrt{\frac{\pi}{3}}, \sqrt{\frac{\pi}{3}}\right] \cup \left[\sqrt{2n\pi - \frac{\pi}{3}}, \sqrt{2n\pi + \frac{\pi}{3}}\right] \cup \left[-\sqrt{2k\pi + \frac{\pi}{3}}, -\sqrt{2k\pi - \frac{\pi}{3}}\right], n \geq 1$}

234. $\sin\theta - \cos\theta < 0$. {Ans. $\left(2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right)$}

235. $\sqrt{3}\sin 2x + \cos 2x < 1$ {Ans. $\left(n\pi + \frac{\pi}{3}, n\pi + \pi\right)$}

236. $\cos 3x + \sqrt{3}\sin 3x < -\sqrt{2}$ {Ans. $\left(\frac{2n\pi}{3} + \frac{13\pi}{36}, \frac{2n\pi}{3} + \frac{19\pi}{36}\right)$}

237. $\cos 2x + \cos x > 0$ {Ans. $\left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right)$}

238. $\frac{\cos x}{1 + \cos 2x} < 0$ {Ans. $\left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right)$}

239. $\sin 3x > \cos 3x$ {Ans. $\left(\frac{2n\pi}{3} + \frac{\pi}{12}, \frac{2n\pi}{3} + \frac{5\pi}{12}\right)$}

240. $\tan x + 3\cot x - 4 > 0$ {Ans. $\left(n\pi, n\pi + \frac{\pi}{4}\right) \cup \left(k\pi + \tan^{-1} 3, k\pi + \frac{\pi}{2}\right)$}

241. $\sin^2 x - \cos^2 x - 3\sin x + 2 < 0$ {Ans. $\left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{\pi}{2}\right) \cup \left(2k\pi + \frac{\pi}{2}, 2k\pi + \frac{5\pi}{6}\right)$}

242. $2\sin^2 \frac{x}{2} + \cos 2x < 0$ {Ans. $\left(2n\pi - \frac{\pi}{2}, 2n\pi - \frac{\pi}{3}\right) \cup \left(2k\pi + \frac{\pi}{3}, 2k\pi + \frac{\pi}{2}\right)$}

243. $\tan^3 x + 3 > 3\tan x + \tan^2 x$} {Ans. $\left(n\pi - \frac{\pi}{3}, n\pi + \frac{\pi}{4}\right) \cup \left(k\pi + \frac{\pi}{3}, k\pi + \frac{\pi}{2}\right)$}

244. $\frac{\sin 3x - \cos 3x}{\sin 3x + \cos 3x} < 0$ {Ans. $\left(\frac{n\pi}{3} - \frac{\pi}{12}, \frac{n\pi}{3} + \frac{\pi}{12}\right)$}

245. $5\sin^2 x - 3\sin x\cos x - 36\cos^2 x > 0$ {Ans. $\left(n\pi + \cot^{-1}\frac{1}{3}, n\pi + \cot^{-1}\left(-\frac{5}{12}\right)\right)$}

246. $2\sin^2 x - 4\sin x\cos x + 9\cos^2 x > 0$ {Ans. $(-\infty, +\infty)$}

247. $\cos^2 x + 3\sin^2 x + 2\sqrt{3}\sin x\cos x < 1$ {Ans. $\left(n\pi - \frac{\pi}{3}, n\pi\right)$}

248. $3\sin^2 x + \sin 2x - \cos^2 x \geq 2$ {Ans. $\left(n\pi + \frac{\pi}{4}, n\pi + \cot^{-1}\left(-\frac{1}{3}\right)\right)$}

249. $\sqrt{3}\cos^{-2} x < 4\tan x$ {Ans. $\left(n\pi + \frac{\pi}{6}, n\pi + \frac{\pi}{3}\right)$}

250. $\sin 4x + \cos 4x \cot 2x > 1$ {Ans. $\left(\frac{n\pi}{2}, \frac{n\pi}{2} + \frac{\pi}{8}\right)$}

251. $2+\tan 2x+\cot 2x<0$ {Ans. $\left(\frac{n\pi}{2}-\frac{\pi}{4},\frac{n\pi}{2}-\frac{\pi}{8}\right)\cup\left(\frac{n\pi}{2}-\frac{\pi}{8},\frac{n\pi}{2}\right)$}

252. $2\cos x\left(\cos x-\sqrt{8}\tan x\right)<5$ {Ans. $\left(2n\pi-\frac{\pi}{4},2n\pi+\frac{\pi}{2}\right)\cup\left(2k\pi+\frac{\pi}{2},2k\pi+\frac{5\pi}{4}\right)$}

253. $\sin x+\cos x<\frac{1}{\cos x}$ {Ans. $\left(2k\pi+\frac{\pi}{4},2k\pi+\frac{\pi}{2}\right)\cup\left(2m\pi+\pi,2m\pi+\frac{5\pi}{4}\right)\cup\left(2n\pi+\frac{3\pi}{2},2n\pi+2\pi\right)$}

254. $\sin^6 x+\cos^6 x<\frac{7}{16}$ {Ans. $\left(\frac{n\pi}{2}+\frac{\pi}{6},\frac{n\pi}{2}+\frac{\pi}{3}\right)$}

255. $\cot x+\frac{\sin x}{\cos x-2}\geq 0$ {Ans. $\left[2n\pi-\frac{\pi}{3},2n\pi\right)\cup\left[2k\pi+\frac{\pi}{3},2k\pi+\pi\right)$}

256. $\cos^2 2x+\cos^2 x\leq 1$ {Ans. $\left[n\pi+\frac{\pi}{6},n\pi+\frac{5\pi}{6}\right]$}

257. $8\sin^2\frac{x}{2}+3\sin x-4>0$ {Ans. $\left(2n\pi+2\cot^{-1}2,2n\pi+2\cot^{-1}\left(-\frac{1}{2}\right)\right)$}

258. $\sin x+\cos x>\sqrt{2}\cos 2x$ {Ans. $\left(2n\pi+\frac{\pi}{12},2n\pi+\frac{3\pi}{4}\right)\cup\left(2k\pi+\frac{17\pi}{12},2k\pi+\frac{7\pi}{4}\right)$}

259. $\tan x+\tan 2x+\tan 3x>0$

{Ans.

$$\left(n\pi-\frac{1}{2}\cos^{-1}\frac{1}{3},n\pi-\frac{\pi}{6}\right)\cup\left(k\pi,k\pi+\frac{\pi}{6}\right)\cup\left(m\pi+\frac{1}{2}\cos^{-1}\frac{1}{3},m\pi+\frac{\pi}{4}\right)$$
$$\cup\left(p\pi+\frac{2\pi}{3},p\pi+\frac{3\pi}{4}\right)\cup\left(l\pi+\frac{\pi}{3},l\pi+\frac{\pi}{2}\right)$$
}

260. $\cos 2x\cos 5x<\cos 3x$

{Ans.

$$\left(2n\pi-\frac{\pi}{5},2n\pi\right)\cup\left(2k\pi,2k\pi+\frac{\pi}{5}\right)\cup\left(2m\pi+\frac{2\pi}{5},2m\pi+\frac{\pi}{2}\right)\cup\left(2p\pi+\frac{3\pi}{5},2p\pi+\frac{4\pi}{5}\right)$$
$$\cup\left(2l\pi+\frac{6\pi}{5},2l\pi+\frac{7\pi}{5}\right)\cup\left(2r\pi+\frac{3\pi}{2},2r\pi+\frac{8\pi}{5}\right)$$
}

261. $\sin 2x\sin 3x-\cos 2x\cos 3x>\sin 10x$ {Ans. $\left(\frac{2n\pi}{5}-\frac{\pi}{10},\frac{2n\pi}{5}-\frac{\pi}{30}\right)\cup\left(\frac{2k\pi}{5}+\frac{\pi}{10},\frac{2k\pi}{5}+\frac{7\pi}{30}\right)$}

262. $\cot x+\cot\left(x+\frac{\pi}{2}\right)+2\cot\left(x+\frac{\pi}{3}\right)>0$ {Ans. $\left(n\pi-\frac{\pi}{3},n\pi-\frac{\pi}{9}\right)\cup\left(k\pi,k\pi+\frac{2\pi}{9}\right)\cup\left(m\pi+\frac{\pi}{2},m\pi+\frac{5\pi}{9}\right)$}

263. $2\sin^2 x-\sin x+\sin 3x<1$ {Ans. $\left(2n\pi-\frac{\pi}{4},2n\pi+\frac{\pi}{6}\right)\cup\left(2k\pi+\frac{\pi}{4},2k\pi+\frac{3\pi}{4}\right)\cup\left(2m\pi+\frac{5\pi}{6},2m\pi+\frac{5\pi}{4}\right)$}

264. $4\sin x\sin 2x\sin 3x>\sin 4x$ {Ans. $\left(n\pi-\frac{\pi}{8},n\pi\right)\cup\left(k\pi+\frac{\pi}{2},k\pi+\frac{5\pi}{8}\right)\cup\left(m\pi+\frac{\pi}{8},m\pi+\frac{3\pi}{8}\right)$}

265. $\frac{\cos^2 2x}{\cos^2 x}\geq 3\tan x$ {Ans. $\left[n\pi-\frac{7\pi}{12},n\pi-\frac{\pi}{2}\right)\cup\left(k\pi-\frac{\pi}{2},k\pi+\frac{\pi}{12}\right]$}

266. $\dfrac{\cos x+2\cos^2 x+\cos 3x}{\cos x+2\cos^2 x-1}>1$ {Ans. $\left(2n\pi-\dfrac{\pi}{3},2n\pi+\dfrac{\pi}{3}\right)$}

CATEGORY-12.6. INVERSE TRIGONOMETRIC INEQUALITIES

267. $\sin^{-1} x\le 5$ {Ans. $[-1,1]$}

268. $\sin^{-1} x\ge -2$ {Ans. $[-1,1]$}

269. $\cos^{-1} x\le \cos^{-1}\dfrac{1}{4}$ {Ans. $\left[\dfrac{1}{4},1\right]$}

270. $\cos^{-1} x>\dfrac{\pi}{6}$. {Ans. $\left[-1,\dfrac{\sqrt{3}}{2}\right)$}

271. $\tan^{-1} x>-\dfrac{\pi}{3}$. {Ans. $\left(-\sqrt{3},\infty\right)$}

272. $\cot^{-1} x>2$ {Ans. $(-\infty,\cot 2)$}

273. $\left(\cot^{-1} x\right)^2-5\cot^{-1} x+6>0$ {Ans. $(-\infty,\cot 3)\cup(\cot 2,\infty)$}

274. $\left(\tan^{-1} x\right)^2-4\tan^{-1} x+3>0$ {Ans. $(-\infty,\tan 1)$}

275. $\log_2\left(\tan^{-1} x\right)>1$ {Ans. ϕ}

276. $2^{\tan^{-1} x}+2^{-\tan^{-1} x}\ge 2$ {Ans. $(-\infty,+\infty)$}

277. $4\left(\cos^{-1} x\right)^2-1\ge 0$ {Ans. $\left[-1,\cos\dfrac{1}{2}\right]$}

278. $\sin^{-1} x>\cos^{-1} x$ {Ans. $\left(\dfrac{1}{\sqrt{2}},1\right]$}

279. $\sin^{-1} x<\cos^{-1} x$ {Ans. $\left[-1,\dfrac{1}{\sqrt{2}}\right)$}

280. $\cos^{-1} x>\cos^{-1} x^2$ {Ans. $[-1,0)$}

281. $\tan^{-1} x>\cot^{-1} x$ {Ans. $(1,\infty)$}

282. $\sin^{-1} x<\sin^{-1}(1-x)$ {Ans. $\left[0,\dfrac{1}{2}\right)$}

283. $\tan^2\left(\sin^{-1} x\right)>1$ {Ans. $\left(\dfrac{1}{\sqrt{2}},1\right)\cup\left(-1,-\dfrac{1}{\sqrt{2}}\right)$}

CATEGORY-12.7. LIMIT OF INVERSE TRIGONOMETRIC FUNCTIONS

284. $\lim\limits_{x\to\infty} x\left(\tan^{-1}\dfrac{x+1}{x+2}-\dfrac{\pi}{4}\right)$ {Ans. $-\frac{1}{2}$}

285. $\lim\limits_{x\to\infty} x\left(\tan^{-1}\dfrac{x+1}{x+2}-\tan^{-1}\dfrac{x}{x+2}\right)$ {Ans. $\frac{1}{2}$}

286. $\lim\limits_{x\to 0}\dfrac{\sin^{-1} x-\tan^{-1} x}{x^3}$ {Ans. $\frac{1}{2}$}

287. $\lim\limits_{x\to\infty}(x+2)\tan^{-1}(x+2)-x\tan^{-1}x$ {Ans. π }

CATEGORY-12.8. SERIES CONTAINING TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

288. Sum to n terms the series and sum of infinite series

$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+......+\tan^{-1}\frac{1}{n^2+n+1}+.......$ {Ans. $\tan^{-1}\left(\frac{n}{n+2}\right)$, $\frac{\pi}{4}$ }

289. Prove that $\sin\theta\sec 3\theta=\frac{1}{2}(\tan 3\theta-\tan\theta)$ and hence find the sum to n terms of the series

$\sin\theta\sec 3\theta+\sin 3\theta\sec 3^2\theta+\sin 3^2\theta\sec 3^3\theta+...............$ {Ans. $\frac{1}{2}\left(\tan 3^n\theta-\tan\theta\right)$}

290. Prove that $\tan\alpha=\cot\alpha-2\cot 2\alpha$. Hence show that the sum to n terms of the series $\tan\alpha+2\tan 2\alpha+2^2\tan 2^2\alpha+..........$ is $\cot\alpha-2^n\cot 2^n\alpha$.

291. If $a_1,a_2,a_3,\ldots\ldots a_n$ are in AP with common difference d, then prove that the sum of the series $\sin d[\sec a_1\sec a_2+\sec a_2\sec a_3+\ldots+\sec a_{n-1}\sec a_n]$, is $\tan a_n-\tan a_1$.

292. If $a_1,a_2,a_3,\ldots\ldots a_n$ are in AP with common difference d, then prove that the sum of the series $\sin d[\operatorname{cosec} a_1\operatorname{cosec} a_2+\operatorname{cosec} a_2\operatorname{cosec} a_3+\ldots+\operatorname{cosec} a_{n-1}\operatorname{cosec} a_n]=\cot a_1-\cot a_n$.

293. If $U_n=\sin n\theta\sec^n\theta$, $V_n=\cos n\theta\sec^n\theta$, n = 0, 1, 2, ..., prove that $V_n-V_{n-1}=-U_{n-1}\tan\theta$. Hence deduce that $U_1+U_2+.......+U_n=\cot\theta\sec^{n+1}\theta\left(\cos^{n+1}\theta-\cos(n+1)\theta\right)$.

CATEGORY-12.9. ADDITIONAL QUESTIONS

294. If $A+B+C=\pi\ (A, B, C>0)$ and the angle C is obtuse, then show that $\tan A\tan B<1$.

295. If $y=\sec^{-1}\left(\frac{x+1}{x-1}\right)+\sin^{-1}\left(\frac{x-1}{x+1}\right)$ then find $\frac{dy}{dx}$. {Ans. 0}

296. For what value of a the function $f(x)=\sin x-\cos x-ax+b$ decreases for all real values of x? {Ans. $a\ge\sqrt{2}$ }

297. For what value of K the function $f(x)=\frac{K\sin x+2\cos x}{\sin x+\cos x}$ is increasing for all values of x? {Ans. $K>2$}

298. Show that $\sin^p\theta\cos^q\theta$ attains a maxima when $\theta=\tan^{-1}\sqrt{\frac{p}{q}}$.

299. Find the values of x for which the function $f(x)=1+2\sin x+3\cos^2 x,\quad 0\le x\le\frac{2\pi}{3}$ has maxima or minima. Also find the values of the function at these extremum. {Ans. minima at $x=\frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right)=3$, maxima at $x=\sin^{-1}\frac{1}{3}$, $f\left(\sin^{-1}\frac{1}{3}\right)=\frac{13}{3}$ }

300. If $\frac{1}{6}\sin x,\cos x,\tan x$ are in GP, then find the value of x. {Ans. $2n\pi\pm\frac{\pi}{3}$ }

301. If $\tan p\theta-\tan q\theta=0$, then show that the values of θ form a series in A.P.

302. For what value of b, will the roots of the equation $\cos x = b, -1 \le b \le 1$ when arranged in ascending order of their magnitudes, form an AP. {Ans. -1}

303. If $xy + yz + zx = 1$, then find the value of $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$. {Ans. $\frac{\pi}{2}$}

304. Find the greatest and least values of $(\sin^{-1} x)^3 + (\cos^{-1} x)^3$. {Ans. $\frac{\pi^3}{32}, \frac{7\pi^3}{8}$}

305. What is the number of all possible triplets (a_1, a_2, a_3) such that $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$ for all x? {Ans. infinite}

306. If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4} - \tan^{-1} z$ and $x + y + z = 1$ then arithmetic mean of odd powers of x, y, z is $\frac{1}{3}$. (True/False) {Ans. True}

307. Find the intervals of monotonicity of the function $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3, 0 \le x \le \pi$. {Ans. $\left(0, \frac{\pi}{2}\right)$ decreasing, $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ increasing, $\left(\frac{2\pi}{3}, \pi\right)$ decreasing}

308. Consider the system of linear equations in x, y, z

$$(\sin 3\theta)x - y + z = 0$$

$$(\cos 2\theta)x + 4y + 3z = 0$$

$$2x + 7y + 7z = 0.$$

Find the value of θ for which this system has non-trivial solutions. {Ans. $[n\pi] \cup \left[m\pi + (-1)^m \frac{\pi}{6}\right]$}

309. Let λ and α be real. Find the set of all values of λ for which the system of linear equations

$$\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0$$

$$x + (\cos\alpha)y + (\sin\alpha)z = 0$$

$$-x + (\sin\alpha)y - (\cos\alpha)z = 0.$$

has a non-trivial solution. For $\lambda = 1$, find all values of α. {Ans. $\lambda = \sin 2\alpha + \cos 2\alpha$, $\alpha \in [n\pi] \cup \left[m\pi + \frac{\pi}{4}\right]$}

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Mathematics for IIT-JEE

By **Er. Sanjiva Dayal,** *B.Tech. (I.I.T. Kanpur)*

PART-III
TRIGONOMETRY

CHAPTER-13
PROPERTIES OF TRIANGLES

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CHAPTER-13
PROPERTIES OF TRIANGLES

LIST OF THEORY SECTIONS

LIST OF QUESTION CATEGORIES

CHAPTER-13
PROPERTIES OF TRIANGLES

SECTION-13.1. IDENTITIES RELATED TO SIDES AND ANGLES OF A TRIANGLE

1. Convention

i. Vertices of a triangle are denoted by *A*, *B*, *C*; internal angles of vertices *A*, *B*, *C* are denoted by angles *A*, *B*, *C* respectively; sides opposite to angles *A*, *B*, *C* are denoted by *a*, *b*, *c* respectively.

2. Neccessary conditions

i. $a < b + c$, $b < a + c$, $c < a + b$.

ii. $A + B + C = \pi$.

3. Semi-perimeter (*s*)

i. $s = \dfrac{a+b+c}{2}$.

4. Area (Δ)

i. $\Delta = \dfrac{1}{2}ap_1 = \dfrac{1}{2}bp_2 = \dfrac{1}{2}cp_3$.

ii. $\Delta = \dfrac{1}{2}ab\sin C = \dfrac{1}{2}bc\sin A = \dfrac{1}{2}ac\sin B$.

iii. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

5. Sine rule

i. $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{2\Delta}{abc}$.

ii. $a : b : c = \sin A : \sin B : \sin C$.

6. Cosine rule

i. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

7. Projection rule

i. $a = b\cos C + c\cos B, b = a\cos C + c\cos A, c = a\cos B + b\cos A$.

8. Half angle formulae

i. $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-a)(s-c)}{ac}}$, $\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$.

ii. $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$, $\cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ac}}$, $\cos\dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$.

iii. $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, $\tan\dfrac{B}{2} = \sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}$, $\tan\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$.

9. Napier's analogy

i. $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$, $\tan\dfrac{C-A}{2} = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$, $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$.

SECTION-13.2. SOLVING A TRIANGLE

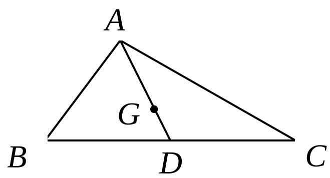
1. Three sides given: Angles obtained by cosine rule.

2. Two sides, a & b, and included angle, C, given: c obtained by cosine rule, $A-B$ obtained by Napier's analogy.
3. Two sides, a & b, and opposite angle, A, given
   i. A is acute angle
      a. $a < b\sin A \Rightarrow$ No triangle, $\sin B = \dfrac{b\sin A}{a}$ has no solution
      b. $a = b\sin A \Rightarrow$ Unique Right-angled triangle, $B = 90°$, $\sin B = 1$
      c. $b\sin A < a < b \Rightarrow$ Two triangles (Ambiguous case)
         I. There are two set of values of c, B, C; c_1, B_1, C_1 and c_2, B_2, C_2.
         II. B_1 and B_2 are solutions of $\sin B = \dfrac{b\sin A}{a}$ and $B_1 + B_2 = 180°$.
         III. Also, c_1 and c_2 are roots of quadratic $c^2 - (2b\cos A)c + (b^2 - a^2) = 0$, and so $c_1 + c_2 = 2b\cos A$ and $c_1 c_2 = b^2 - a^2$.
      d. $a \ge b \Rightarrow$ Unique triangle, B is acute, $\sin B = \dfrac{b\sin A}{a}$
   ii. A is right angle or obtuse angle
      a. $a \le b \Rightarrow$ No triangle
      b. $a > b \Rightarrow$ Unique triangle, B is acute, $\sin B = \dfrac{b\sin A}{a}$
4. One side and two angles given: obtain third angle, obtain other two sides by sine rule.
5. Three angles given: Infinite similar triangles, Ratios of sides is fixed, i.e. $a : b : c = \sin A : \sin B : \sin C$.

SECTION-13.3. PROPERTIES OF MEDIAN, ALTITUDE AND INTERNAL BISECTOR

1. Property of median and centroid (*G*)

i. Centroid (G) divides median AD in the ratio 2:1, i.e. AG:DG = 2:1.



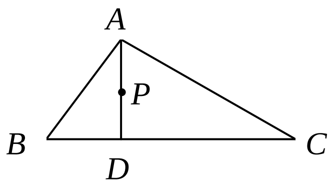
2. Property of altitude and orthocenter (*P*)

i. $p_1 = b\sin C = c\sin B = \dfrac{2\Delta}{a}$, $p_2 = a\sin C = c\sin A = \dfrac{2\Delta}{b}$,

$p_3 = a\sin B = b\sin A = \dfrac{2\Delta}{c}$.

ii. $AP = a\cot A$.

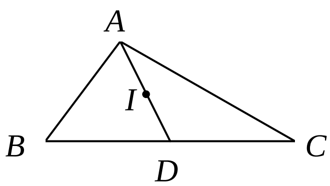
iii. $DP = \dfrac{a\cos B\cos C}{\sin A}$.



3. Property of internal bisector and incenter (*I*)

i. Internal bisector AD divides side BC in the ratio c:b, i.e. BD:CD = c:b.

ii. $BD = \dfrac{ac}{b+c}$ and $DC = \dfrac{ab}{b+c}$.

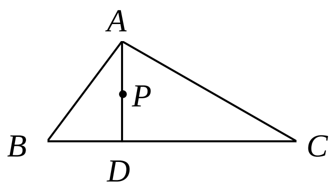


SECTION-13.4. IDENTITIES RELATED TO CIRCUMRADIUS

1. Circum-radius (*R*)

i. $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C} = \dfrac{abc}{4\Delta}$.

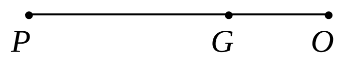
2. Distance of orthocenter (*P*) from vertex and side



i. $AP = 2R\cos A$.

ii. $DP = 2R\cos B\cos C$.

3. Position of circumcenter (*O*), centroid (*G*) and orthocenter (*P*)



i. Circumcenter (*O*), centroid (*G*), and orthocenter (*P*) are collinear & *G* divides *OP* in the ratio 1:2, i.e. *OG*:*GP* = 1:2.

ii. In an isosceles triangle, *O*, *G*, *P*, *I* are collinear.

iii. In an equilateral triangle, *O*, *G*, *P*, *I* are coincident.

SECTION-13.5. IDENTITIES RELATED TO IN-RADIUS

1. **In-radius (*r*)**

i. $r = \dfrac{\Delta}{s}$.

ii. $r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}$.

iii. $r = \dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \dfrac{b\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = \dfrac{c\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}} = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

SECTION-13.6. IDENTITIES RELATED TO EX-RADII

1. **Ex-radii (r_1, r_2, r_3)**

i. $r_1 = \dfrac{\Delta}{s-a} = s\tan\dfrac{A}{2} = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$.

ii. $r_2 = \dfrac{\Delta}{s-b} = s\tan\dfrac{B}{2} = \dfrac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}} = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$.

iii. $r_3 = \dfrac{\Delta}{s-c} = s\tan\dfrac{C}{2} = \dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}} = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$.

SECTION-13.7. PROPERTIES OF A QUADRILATERAL

1. Convention

i. Vertices of a quadrilateral are denoted by *A*, *B*, *C* *D* in anti-clockwise manner; internal angles of vertices *A*, *B*, *C*, *D* are denoted by angles *A*, *B*, *C*, *D* respectively; sides adjacent to vertices *A*, *B*, *C*, *D* in anti-clockwise manner are denoted by *a*, *b*, *c*, *d* respectively.

2. Properties of a general quadrilateral

i. $A + B + C + D = 2\pi$.

ii. $s = \dfrac{a+b+c+d}{2}$.

iii. $\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{B+D}{2}\right)}$

$= \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{A+C}{2}\right)}$.

3. Properties of a cyclic quadrilateral

i. $B+D=\pi; \quad A+C=\pi$.

ii. $\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

iii. $\cos B = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$.

iv. **Circumradius:** $R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$.

v. **Ptolemy's theorem:** $AC \cdot BD = AB \cdot CD + BC \cdot AD = ac+bd$.

4. Properties of a quadrilateral with an inscribed circle

i. $a+c=b+d=s$.

ii. $\Delta = \sqrt{abcd}\sin\left(\frac{B+D}{2}\right) = \sqrt{abcd}\sin\left(\frac{A+C}{2}\right)$.

iii. $r = \frac{\Delta}{s}$.

SECTION-13.8. PROPERTIES OF A POLYGON

1. Properties of a general polygon of *n* sides

i. Sum of external angle = 2π.

ii. Sum of internal angles = $(n-2)\pi$.

2. Properties of a regular polygon of *n* sides and side length *a*

i. Internal angle = $\left(\frac{n-2}{n}\right)\pi$.

ii. $s = \frac{na}{2}$.

iii. $R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$.

iv. $r = \frac{a}{2}\cot\frac{\pi}{n}$.

v. $\Delta = n\frac{a^2}{4}\cot\frac{\pi}{n} = nr^2\tan\frac{\pi}{n} = n\frac{R^2}{2}\sin\frac{2\pi}{n}$.

vi. $r = \frac{\Delta}{s}$.

EXERCISE-13

CATEGORY-13.1. PROVING IDENTITIES RELATED TO SIDES AND ANGLES OF A TRIANGLE

1. $\sin\frac{B-C}{2}=\frac{b-c}{a}\cos\frac{A}{2}$.
2. $b^2\sin 2C+c^2\sin 2B=2bc\sin A$.
3. $a(b\cos C-c\cos B)=b^2-c^2$.
4. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$.
5. $a(\cos B+\cos C)=2(b+c)\sin^2\frac{A}{2}$.
6. $a(\cos C-\cos B)=2(b-c)\cos^2\frac{A}{2}$.
7. $\frac{\sin(B-C)}{\sin(B+C)}=\frac{b^2-c^2}{a^2}$.
8. $\frac{a+b}{a-b}=\tan\frac{A+B}{2}\cot\frac{A-B}{2}$,
9. $a\sin\left(\frac{A}{2}+B\right)=(b+c)\sin\frac{A}{2}=a\cos\left(\frac{B-C}{2}\right)$.
10. $\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}=0$.
11. $(b+c-a)\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=2a\cot\frac{A}{2}$.
12. $a^2+b^2+c^2=2(bc\cos A+ca\cos B+ab\cos C)$.
13. $(-a^2+b^2+c^2)\tan A=(a^2-b^2+c^2)\tan B=(a^2+b^2-c^2)\tan C$.
14. $c^2=(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2}$.
15. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$.
16. $\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{c^2-a^2}=\frac{c\sin(A-B)}{a^2-b^2}$.
17. $a\sin\frac{A}{2}\sin\frac{B-C}{2}+b\sin\frac{B}{2}\sin\frac{C-A}{2}+c\sin\frac{C}{2}\sin\frac{A-B}{2}=0$.
18. $a^2(\cos^2 B-\cos^2 C)+b^2(\cos^2 C-\cos^2 A)+c^2(\cos^2 A-\cos^2 B)=0$.
19. $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$.
20. $\frac{(a+b+c)^2}{a^2+b^2+c^2}=\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}$.
21. $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc$.
22. $a^2\cos 2B+b^2\cos 2A+2ab\cos(A-B)=c^2$.

23. $(a+b+c)(\cos A+\cos B+\cos C)=2\left(a\cos^2\frac{A}{2}+b\cos^2\frac{B}{2}+c\cos^2\frac{C}{2}\right).$

24. $\left(b^2-c^2\right)\cot A+\left(c^2-a^2\right)\cot B+\left(a^2-b^2\right)\cot C=0.$

25. $a^2=(b-c)^2+4bc\sin^2\frac{A}{2}.$

26. $\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\dfrac{a^2+b^2}{a^2+c^2}.$

27. $a(\cos B\cos C+\cos A)=b(\cos C\cos A+\cos B)=c(\cos A\cos B+\cos C).$

28. $\dfrac{c}{a-b}=\dfrac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}-\tan\frac{B}{2}}.$

29. $\dfrac{c}{a+b}=\dfrac{1-\tan\frac{A}{2}\tan\frac{B}{2}}{1+\tan\frac{A}{2}\tan\frac{B}{2}}.$

30. $\dfrac{a^2\sin B\sin C}{2\sin A}=\Delta.$

31. $\dfrac{s}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=2\sqrt[3]{\dfrac{abc}{\sin A\sin B\sin C}}$

32. $\dfrac{1}{(b+c)^2}\cos^2\left(\dfrac{B-C}{2}\right)+\dfrac{1}{(b-c)^2}\sin^2\left(\dfrac{B-C}{2}\right)=\dfrac{1}{a^2}.$

33. $(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}=0.$

34. $a^2-2ab\cos(60^\circ+C)=c^2-2bc\cos(60^\circ+A).$

35. $a^3\sin(B-C)+b^3\sin(C-A)+c^3\sin(A-B)=0.$

36. $\dfrac{b^2-c^2}{\cos B+\cos C}+\dfrac{c^2-a^2}{\cos C+\cos A}+\dfrac{a^2-b^2}{\cos A+\cos B}=0.$

37. $\dfrac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}=\sin^2 A.$

38. $(a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=2c\cot\frac{C}{2}.$

39. $\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\left(a\sin^2\frac{B}{2}+b\sin^2\frac{A}{2}\right)=c\cot\frac{C}{2}.$

40. $1-\tan\frac{A}{2}\tan\frac{B}{2}=\dfrac{2c}{a+b+c}.$

41. $2abc\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=2s\Delta.$

42. $bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}=s^2.$

43. $\frac{b-c}{a}\cos^2\frac{A}{2}+\frac{c-a}{b}\cos^2\frac{B}{2}+\frac{a-b}{c}\cos^2\frac{C}{2}=0.$

44. $\frac{\tan\frac{A}{2}}{(a-b)(a-c)}+\frac{\tan\frac{B}{2}}{(b-c)(b-a)}+\frac{\tan\frac{C}{2}}{(c-a)(c-b)}=\frac{1}{\Delta}.$

45. $\sin^3 A\cos(B-C)+\sin^3 B\cos(C-A)+\sin^3 C\cos(A-B)=3\sin A\sin B\sin C.$

46. $a^3\cos B\cos C+b^3\cos C\cos A+c^3\cos A\cos B=abc(1-2\cos A\cos B\cos C).$

CATEGORY-13.2. SOLVING A TRIANGLE

47. Solve the triangle, given

i. $a=\sqrt{3}$, $b=\sqrt{2}$ and $c=\frac{\sqrt{6}+\sqrt{2}}{2}$. {Ans. A = 60°, B = 45°, C = 75°}

ii. $b=\sqrt{3}$, $c=1$ and A = 30°. {Ans. a = 1, B = 120°, C = 30°}

iii. a = 5, b = 7 and A = 60°. {Ans. no triangle}

iv. a = 1, c = 2 and A = 30°. {Ans. $b=\sqrt{3}$, B = 60°, C = 90°}

v. $a=2$, $c=\sqrt{3}+1$ and A = 45°. {Ans. $b_1=\sqrt{2}$, $B_1=30°$, $C_1=105°$ & $b_2=\sqrt{6}$, $B_2=60°$, $C_2=75°$}

vi. $a=\sqrt{3}$, $b=\sqrt{2}$ and A = 60°. {Ans. B = 45°, C = 75°, $c=\frac{\sqrt{3}+1}{\sqrt{2}}$}

vii. a = 4, b = 5 and A = 120°. {Ans. no triangle}

viii. $a=\sqrt{3}$, $b=\sqrt{2}$ and A = 120°. {Ans. B = 45°, C = 15°, $c=\frac{\sqrt{3}-1}{\sqrt{2}}$}

ix. a = 2, B = 60° and C = 45°. {Ans. A = 75°, $b=\frac{2\sqrt{6}}{\sqrt{3}+1}$, $c=\frac{4}{\sqrt{3}+1}$}

x. A = 45°, B = 60° and C = 75°. {Ans. $a:b:c=2:\sqrt{6}:\sqrt{3}+1$}

48. In a ΔABC, if $A=45°$, $b=\sqrt{6}$, $a=2$, then find B. {Ans. $60°$ or $120°$}

49. In triangle ABC, $A=30°$, $b=8$, $a=6$, then find B. {Ans. $\sin^{-1}\frac{2}{3}$}

50. If $A=30°$, $a=7$, $b=8$ in ΔABC, then how many values of B are possible? {Ans. Two}

51. If the data given to construct a triangle ABC are $a=5$, $b=7$, $\sin A=\frac{3}{4}$, then how many triangles can be constructed? {Ans. none}

CATEGORY-13.3. RELATIONSHIP BETWEEN SIDES AND ANGLES OF A TRIANGLE

52. In a triangle whose sides are 3, 4 and $\sqrt{38}$ meters respectively, prove that the largest angle is greater then 120°.

53. If in any triangle the angle be to one another as 1:2:3, prove that the corresponding sides are as 1:$\sqrt{3}$:2.

54. In any triangle, if $\tan\frac{A}{2}=\frac{5}{6}$ and $\tan\frac{B}{2}=\frac{20}{37}$, find $\tan\frac{C}{2}$ and prove that in this triangle $a+c = 2b$. {Ans. $\frac{2}{5}$ }

55. In a ΔABC, $a=13$, $b=14$, $c=15$, then find $\sin\frac{A}{2}$. {Ans. $\frac{1}{\sqrt{5}}$}

56. If $a = 4$, $b = 5$ and $\cos(A-B)=\frac{31}{32}$, prove that $c = 6$.

57. If a, b and c be in A.P. prove that

i. $\cot\frac{A}{2}$, $\cot\frac{B}{2}$ and $\cot\frac{C}{2}$ are in A.P.

ii. $\cos A\cot\frac{A}{2}$, $\cos B\cot\frac{B}{2}$ and $\cos C\cot\frac{C}{2}$ are in A.P.

iii. $a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{3b}{2}$.

iv. $\tan\frac{A}{2}+\tan\frac{C}{2}=\frac{2}{3}\cot\frac{B}{2}$.

v. $\cot\frac{A}{2}\cot\frac{C}{2}=3$.

58. If a^2, b^2 and c^2 be in A.P., prove that

i. $\cot A$, $\cot B$ and $\cot C$ are in A.P.

ii. $\frac{\sin 3B}{\sin B}=\left[\frac{a^2-c^2}{2ac}\right]^2$.

59. If a, b and c are in H.P., prove that $\sin^2\frac{A}{2}, \sin^2\frac{B}{2}$ and $\sin^2\frac{C}{2}$ are also in H.P.

60. The sides of a triangle are in A.P. and the greatest and least angles are θ and ϕ; prove that $4(1-\cos\theta)(1-\cos\phi)=\cos\theta+\cos\phi$.

61. The sides of a triangle are in A.P. and the greatest angle exceeds the least by 90°; prove that the sides are proportional to $\sqrt{7}+1$, $\sqrt{7}$ and $\sqrt{7}-1$.

62. If $C = 60^0$, then prove that $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$.

63. In any triangle prove that, if θ be any angle, then $b\cos\theta=c\cos(A-\theta)+a\cos(C+\theta)$.

64. If $A = 45°$, $B = 75°$, prove that $a+c\sqrt{2}=2b$.

65. If $(a+b+c)(b+c-a)=kbc$, then prove that $k\in(0,4)$.

66. The sides of a triangle are a, b, $\sqrt{a^2+ab+b^2}$, prove that the greatest angle is 120°.

67. In any triangle ABC, if $\sin^2 A+\sin^2 B=\sin^2 C$, then show that the triangle is right angled.

68. In any ΔABC if $2\cos B=\frac{a}{c}$, then show that the triangle is isosceles.

69. If in a ΔABC, $a\sin A=b\sin B$, then show that the triangle is isosceles.

70. If $\cos A=\frac{\sin B}{2\sin C}$, prove that the triangle is isosceles.

71. If $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$, prove that the triangle is either isosceles or right angled.

72. If $\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$, find the value of A. {Ans. 90°}

73. If A, B, C are in A.P., show that $2\cos\frac{A-C}{2} = \frac{a+c}{\sqrt{a^2 - ac + c^2}}$.

74. If the angles of a triangle are in the ratio 1:2:4, then prove that $a^2b^2c^2 = (b^2 - a^2)(c^2 - b^2)(c^2 - a^2)$.

75. If $a\cos A = b\cos B$, prove that the triangle is either isosceles or right angled.

76. If $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A-B)}{\sin(A+B)}$, prove that the triangle is either isosceles or right angled.

77. If $\cos^2 A + \cos^2 B + \cos^2 C = 1$, prove that the triangle is right angled.

78. If $\cot A + \cot B + \cot C = \sqrt{3}$, prove that the triangle is equilateral.

79. If $\cos A\cos B + \sin A\sin B\sin C = 1$, show that $a : b : c = 1 : 1 : \sqrt{2}$.

80. If $\cos A + 2\cos B + \cos C = 2$, prove that the sides of the triangle are in A.P.

81. If $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, prove that a^2, b^2, c^2 are in A.P.

82. If $b + c = 3a$, prove that $\cot\frac{B}{2}\cot\frac{C}{2} = 2$.

83. In a triangle ABC, $\angle B = \frac{\pi}{3}$ and $\angle C = \frac{\pi}{4}$. Let D divide BC internally in the ratio 1:3, then find the value of $\frac{\sin\angle BAD}{\sin\angle CAD}$. {Ans. $\frac{1}{\sqrt{6}}$ }

84. The sides of a triangle are three consecutive natural numbers and it's largest angle is twice the smallest one. Determine the sides of the triangle. {Ans. 4, 5, 6}

85. Find the greatest angle of the triangle whose sides are $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. {Ans. 120°}

86. If $a\tan A + b\tan B = (a+b)\tan\left(\frac{A+B}{2}\right)$, prove that $A = B$.

87. If $c(a+b)\cos\frac{B}{2} = b(a+c)\cos\frac{C}{2}$, prove that the triangle is isosceles.

88. If $B = 3C$, prove that $\cos C = \sqrt{\frac{b+c}{4c}}$, $\sin C = \sqrt{\frac{3c-b}{4c}}$ and $\sin\frac{A}{2} = \frac{b-c}{2c}$.

89. If $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.

90. If a, b, c be 5, 4, 3 respectively and D and E are the points of trisection of side BC, then prove that $\tan\angle CAE = \frac{3}{8}$.

91. $ABCD$ is a trapezium such that AB is parallel to CD and CB is perpendicular to them. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, show that $AB = \frac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta}$.

92. If the tangents of the angles of a triangle are in A.P., prove that the squares of the sides are in the ratio

$x^2(x^2+9):(3+x^2)^2:9(1+x^2)$, where x is the tangent of the least or greatest angle.

93. If p and q be perpendiculars from the angular points A and B on any line passing through the vertex C of the triangle ABC, then prove that $a^2p^2+b^2q^2-2abpq\cos C=a^2b^2\sin^2 C$.

94. In the triangle ABC, lines OA, OB and OC are drawn so that the angles OAB, OBC and OCA are each equal to ω; prove that cot ω = cot A + cot B + cot C and cosec^2 ω = cosec^2 A + cosec^2 B + cosec^2 C.

95. In any triangle ABC if D be any point of the base BC, such that $BD:DC = m:n$, and if ∠ BAD = α, ∠ DAC = β, and ∠ CDA = θ and $AD = x$, prove that $(m+n)\cot\theta = m\cot\alpha - n\cot\beta = n\cot B - m\cot C$ and $(m+n)^2\cdot x^2 = (m+n)(mb^2+nc^2)-mna^2$.

96. Two straight roads intersect at an angle of $60°$. A bus on one road is 2 km. away from the intersection and a car on the other is 3 km. away from the intersection. Find the direct distance between the two vehicles. {Ans. $\sqrt{7}$ km}

97. A ring, 10 cm. in diameter, is suspended from a point 12 cm. above its center by 6 equal strings attached to its circumference at equal intervals. Find the cosine of the angle between consecutive strings. {Ans. $\frac{313}{338}$}

98. The side of a base of a square pyramid is a meters and it's vertex is at a height of h meters above the center of the base. If θ & ϕ be respectively the inclinations of any face to the base and of any two faces to one another, prove that $\tan\theta = \dfrac{2h}{a}$ and $\cot\dfrac{\phi}{2} = \sqrt{1+\dfrac{a^2}{2h^2}}$.

CATEGORY-13.4. AREA OF A TRIANGLE

99. Find the area of the triangle having sides 13, 14 and 15 cm. {Ans. 84 sq. cm.}

100. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)$ cm., then prove that the area of the triangle is $\dfrac{1}{2}(\sqrt{3}+1)$ sq. cm.

101. If B = 45°, $a = 2(\sqrt{3}+1)$ units and $\Delta = 6+2\sqrt{3}$ sq. units. Determine the side b. {Ans. 4}

102. If one angle of a triangle be 60°, the area $10\sqrt{3}$ sq. cm. and the perimeter 20 cm., find the length of the sides. {Ans. 5 cm., 7 cm., 8 cm.}

103. If a = 6, b = 3 and $\cos(A-B)=\dfrac{4}{5}$, then find it's area. {Ans. 9 sq. units}

104. If $\Delta = a^2-(b-c)^2$, find $\tan\dfrac{A}{2}$. {Ans. $\dfrac{1}{4}$}

105. If C = 60°, A = 75° and D is a point on AC such that the area of the ΔBAD is $\sqrt{3}$ times the area of the ΔBCD, find the $\angle ABD$. {Ans. 30°}

106. If the sides of a triangle are in A.P. and it's area is $\dfrac{3}{5}$ th of an equilateral triangle of same perimeter. Prove that it's sides are in the ratio 3:5:7.

107. If p_1, p_2, p_3 are altitudes of a triangle ABC from the vertices A, B, C and Δ the area of the triangle, then prove that $p_1^{-2}+p_2^{-2}+p_3^{-2} = \dfrac{a^2+b^2+c^2}{4\Delta^2}$.

108. If $\dfrac{a}{1+m^2n^2} = \dfrac{b}{m^2+n^2} = \dfrac{c}{(1-m^2)(1+n^2)}$, then prove that $\tan\dfrac{A}{2}=\dfrac{m}{n}$, $\tan\dfrac{B}{2}=mn$ and $\Delta = \dfrac{mnbc}{m^2+n^2}$.

109. The sides *a*, *b*, *c* of a triangle are the roots of the equation $x^3 - px^2 + qx - r = 0$. Prove that $\Delta = \frac{1}{4}\sqrt{p(4pq - p^3 - 8r)}$.

110. Through the angular points of a triangle are drawn straight lines which make the same angle α with the opposite sides of the triangle. Prove that the area of the triangle formed by them is to the area of the original triangle as $4\cos^2\alpha : 1$.

111. In the sides *BC*, *CA* and *AB* are taken three points *A*', *B*', *C*' such that *BA*':*A*'*C* = *CB*':*B*'*A* = *AC*':*C*'*B* = *m*:*n*. Prove that if *AA*', *BB*' and *CC*' be joined, they will form by their intersections a triangle whose area is to that of the triangle *ABC* as $(m-n)^2 : m^2 + mn + n^2$.

CATEGORY-13.5. AMBIGUOUS CASE

112. In a triangle ABC, $a = 4, b = 3, \angle A = 60°$. Then show that c is the root of the equation $c^2 - 3c - 7 = 0$.

113. In the ambiguous case, given *a*, *b*, *A* and c_1, c_2 are the two value of side *c*, then show that
    i. $c_1 + c_2 = 2b\cos A$.
    ii. $c_1c_2 = b^2 - a^2$.
    iii. $c_1 \sim c_2 = 2\sqrt{a^2 - b^2\sin^2 A}$.
    iv. $c_1^2 - 2c_1c_2\cos 2A + c_2^2 = 4a^2\cos^2 A$.
    v. $(c_1 - c_2)^2 + (c_1 + c_2)^2\tan^2 A = 4a^2$.
    vi. $\frac{(a+b)^2}{1+\cos C_1} + \frac{(b-a)^2}{1-\cos C_1} = \frac{(a+b)^2}{1+\cos C_2} + \frac{(b-a)^2}{1-\cos C_2}$.
    vii. $\cos B_1CB_2 = \frac{2c_1c_2}{c_1^2 + c_2^2}$ if *A* = 45°.

114. In the ambiguous case, given *a*, *c*, *A* and $b_2 = 2b_1$, where b_1, b_2 are the two value of side *b*, then prove that $3a = c\sqrt{1 + 8\sin^2 A}$.

115. In the ambiguous case, given *a*, *b*, *A*, if the remaining angles of the triangles formed be B_1, C_1 and B_2, C_2, then prove that $\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = 2\cos A$.

116. In the ambiguous case, given *a*, *b*, *A*, prove that the sum of the areas of the two triangles formed is $\frac{1}{2}b^2\sin 2A$.

117. If $2b = 3a$ and $\tan^2 A = \frac{3}{5}$, prove that there are two values of *c*, one of which is double the other.

118. If $2b = (m+1)a$ and $\cos A = \frac{1}{2}\sqrt{\frac{(m-1)(m+3)}{m}}$, where $1 < m < 3$, then prove that there are two values of the third side, one of which is *m* times the other.

CATEGORY-13.6. MEDIAN OF A TRIANGLE

119. Determine the lengths of medians in terms of the sides. {Ans. $AD = \frac{\sqrt{2b^2 + 2c^2 - a^2}}{2}$}

120. If D is the mid-point of BC, then prove that $\sin\angle CAD = \dfrac{a\sin C}{\sqrt{2b^2+2c^2-a^2}}$,

$\sin\angle BAD = \dfrac{a\sin B}{\sqrt{2b^2+2c^2-a^2}}$, $\sin\angle ADB = \dfrac{2b\sin C}{\sqrt{2b^2+2c^2-a^2}}$ and $\cot\angle ADB = \dfrac{\left(b^2-c^2\right)}{4\Delta}$.

121. In a triangle ABC, the median to the side BC is of length $\dfrac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides angle A into angles of $30°$ and $45°$. Prove that side BC is of length 2 units.

122. In an isosceles right-angled triangle a straight line is drawn from the middle point of one of the equal sides to the opposite angle. Show that it divides the angle into parts whose cotangents are 2 and 3.

123. If in a triangle the median through A is perpendicular to the side AB, prove that $\tan A + 2\tan B = 0$.

124. If D is the mid-point of BC and AD is perpendicular to AC, then prove that $\cos A\cos C = \dfrac{2\left(c^2-a^2\right)}{3ac}$.

125. Prove that the median through A divides it into angles whose cotangents are $2\cot A + \cot C$ and $2\cot A + \cot B$ and makes with the base an angle whose cotangent is $\dfrac{1}{2}\left(\cot C \sim \cot B\right)$.

CATEGORY-13.7. ALTITUDE OF A TRIANGLE

126. The sides of a right angled triangle are 21 and 28 cm.; find the length of the perpendicular drawn to the hypotenuse from the right angle. {Ans. 16.8 cm.}

127. The perpendicular AD to the base of a triangle ABC divides it into segments such that BD, CD and AD are in the ratio of 2, 3 and 6; prove that the vertical angle of the triangle is $45°$.

128. If $\sin A$, $\sin B$, $\sin C$ are in A.P., then prove that the altitudes are in H.P.

129. If AD is the altitude from A, $b>c$, $C = 23°$ and $AD = \dfrac{abc}{b^2-c^2}$, find B. {Ans. 113°}

130. The base angles of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$. Show that the base is equal to twice the height.

131. Prove that the perpendicular from A divides BC into portions which are proportional to the cotangents of the adjacent angles and that it divides the angle A into portions whose cosines are inversely proportional to the adjacent sides.

132. Prove that the distance between the middle point of BC and the foot of the perpendicular from A is $\dfrac{b^2 \sim c^2}{2a}$.

133. If p, q, r are the altitudes of a triangle ABC, prove that $\dfrac{1}{p^2}+\dfrac{1}{q^2}+\dfrac{1}{r^2} = \dfrac{\cot A+\cot B+\cot C}{\Delta}$.

134. If p_1, p_2, p_3 are altitudes of a triangle ABC from the vertices A, B, C and Δ the area of the triangle, then prove that $p_1^{-1}+p_2^{-1}-p_3^{-1} = \dfrac{s-c}{\Delta}$.

135. If p, q, r are the altitudes of a triangle from the vertices A, B, C respectively, prove that $\dfrac{1}{p}+\dfrac{1}{q}-\dfrac{1}{r} = \dfrac{ab}{s\Delta}\cos^2\dfrac{C}{2}$.

136. If x, y, z are respectively the distance of the vertices from orthocentre, prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$.

137. In a triangle of base a, the ratio of the other sides is r ($r < 1$). Show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$.

CATEGORY-13.8. INTERNAL BISECTOR OF A TRIANGLE

138. Prove that the length of internal bisector AD is $\frac{2bc}{b+c}\cos\frac{A}{2}$.

139. If p, q, r are the lengths of internal bisectors of the angles A, B, C respectively, then prove that $\frac{1}{p}\cos\frac{A}{2}+\frac{1}{q}\cos\frac{B}{2}+\frac{1}{r}\cos\frac{C}{2}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

140. In a right angled triangle ABC, the bisector of the right angle C divides AB into segments p and q and if $\tan\frac{A-B}{2}=t$, then show that $p:q=(1-t):(1+t)$.

141. If the bisectors of the angles of a triangle ABC meet the opposite sides in A', B' and C', prove that the ratio of the areas of the triangles $A'B'C'$ and ABC is $2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}:\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}$.

CATEGORY-13.9. PROVING IDENTITIES RELATED TO CIRCUMRADIUS

142. $\Delta=2R^2\sin A\sin B\sin C$.

143. $4R\sin A\sin B\sin C=a\cos A+b\cos B+c\cos C$.

144. $\sin A+\sin B+\sin C=\frac{s}{R}$

CATEGORY-13.10. CIRCUMCIRCLE, CIRCUMCENTER, CIRCUMRADIUS

145. The sides of a triangle are 18, 24 and 30 cm. Find R. {Ans. 15 cm.}

146. Find the circumradius of the equilateral triangle of side $2\sqrt{3}$ cm. {Ans. 2 cm}

147. In the ambiguous case of the triangle, prove that the circumradius of the two triangles are equal.

148. Prove that
   i. $OP^2=9R^2-(a^2+b^2+c^2)$.
   ii. $OG^2=R^2-\frac{1}{9}(a^2+b^2+c^2)$.

149. If x, y, z are respectively the perpendiculars from the vertices A, B, C to the opposite sides, prove that
   i. $xyz=\frac{a^2b^2c^2}{8R^3}$
   ii. $\frac{\cos A}{x}+\frac{\cos B}{y}+\frac{\cos C}{z}=\frac{1}{R}$
   iii. $\frac{bx}{c}+\frac{cy}{a}+\frac{az}{b}=\frac{a^2+b^2+c^2}{2R}$

150. If x, y, z are respectively the perpendiculars from the circumcentre to the sides, prove that

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}.$$

151. If $8R^2=a^2+b^2+c^2$, prove that the triangle is right angled.
152. Prove that the radii of the circles circumscribing the triangles *BPC*, *CPA*, *APB* and *ABC* are all equal.
153. *D*, *E* and *F* are the middle points of the sides of the triangle *ABC*; prove that the centroid of the triangle *DEF* is the same as that of *ABC* and that it's orthocentre is the circumcentre of *ABC*.
154. If R_1, R_2 and R_3 are respectively the radii of the circumcircles of the triangle *OBC*, *OCA* and *OAB*, prove that $\frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3}=\frac{abc}{R^3}$.
155. At the points *A*, *B*, *C*, tangents are drawn to the circumcircle. These tangents enclose a triangle *PQR*. Prove that its angles and sides are respectively $180°-2A$, $180°-2B$, $180°-2C$ and $\frac{a}{2\cos B\cos C}$, $\frac{b}{2\cos C\cos A}$, $\frac{c}{2\cos A\cos B}$.
156. The legs of a tripod are each 10 cm. in length and their points of contact with a horizontal table on which the tripod stands form a triangle whose sides are 7, 8 and 9 cm. in length. Find the inclination of the legs to the horizontal and the height of the apex. {Ans. 62°, 8.83 cm.}

CATEGORY-13.11. PROVING IDENTITIES RELATED TO IN-RADIUS

157. $\Delta=4Rr\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.
158. $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{1}{2Rr}$.
159. $\cos A+\cos B+\cos C=1+\frac{r}{R}$
160. $a\cot A+b\cot B+c\cot C=2(R+r)$
161. $(b+c)\tan\frac{A}{2}+(c+a)\tan\frac{B}{2}+(a+b)\tan\frac{C}{2}=4(R+r)$
162. $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+\frac{r}{2R}$

CATEGORY-13.12. INCIRCLE, INCENTER, IN-RADIUS

163. Find the in-radius of the triangle having sides 13, 14, 15. {Ans. 4}
164. The sides of a triangle are 18, 24 and 30 cm. Find *r*. {Ans. 6 cm.}
165. Given $a:b:c=4:5:6$. Find the ratio of the radius of the circumcircle to that of the incircle. {Ans. $\frac{16}{7}$}
166. If *C* = 90°, prove that $R+r=\frac{1}{2}(a+b)$ and $\frac{c}{r}=\frac{c+a}{b}+\frac{c+b}{a}$.
167. Prove that
    i. $AI=r\,\mathrm{cosec}\frac{A}{2}$.

ii. $IA \cdot IB \cdot IC = abc \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = 4Rr^2$.

iii. $IO^2 = R^2(3 - 2\cos A - 2\cos B - 2\cos C)$.

iv. $IP^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$.

v. Area of $\Delta IOP = 2R^2 \sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$.

vi. Area of $\Delta IPG = \frac{4}{3}R^2 \sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$.

168. DEF is the triangle formed by joining the points of contact of the incircle with the sides of the triangle ABC; prove that:-

i. it's sides are $2r\cos\frac{A}{2}, 2r\cos\frac{B}{2}, 2r\cos\frac{C}{2}$

ii. it's angles are $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}, \frac{\pi}{2}-\frac{C}{2}$

iii. it's area is $\frac{2\Delta^3}{abcs}$, i.e. $\frac{r\Delta}{2R}$.

169. *AD*, *BE* and *CF* are the perpendiculars from the angular points of a triangle *ABC* upon the opposite sides. Prove that the diameters of the circumcircles of the triangles *AEF*, *BDF* and *CDE* are respectively $a\cot A$, $b\cot B$ and $c\cot C$, and that the perimeters of the triangles *DEF* and *ABC* are in the ratio $r : R$.

170. If *x*, *y*, *z* are respectively the perpendiculars from the vertices *A*, *B*, *C* to the opposite sides, prove that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{r}$.

171. If *x*, *y*, *z* are respectively the distance of the vertices from orthocentre, prove that $x+y+z=2(R+r)$.

172. If *x*, *y*, *z* are the distances of the vertices of a triangle from the corresponding points of contact with the incircle, prove that $\frac{xyz}{(x+y+z)} = r^2$.

173. In a ΔABC, the line joining the circumcentre to the incentre is parallel to BC, then find the value of $\cos B + \cos C$. {Ans. 1}

174. Let *ABC* be a triangle and let BB_1, CC_1 be respectively the bisectors of $\angle B$, $\angle C$ with B_1 on *AC* and C_1 on *AB*. Let *E*, *F* be the feet of perpendiculars drawn from *A* on BB_1, CC_1 respectively. Suppose *D* is the point at which the incircle of *ABC* touches *AB*. Prove that *AD* = *EF*.

175. Let *ABC* be a triangle having *O* and *I* as its circumcentre and incentre respectively. If *R* and *r* are the circumradius and the inradius respectively, then prove that $(IO)^2 = R^2 - 2Rr$. Further show that the triangle *BIO* is right-angled triangle if and only if *b* is the arithmetic mean of *a* and *c*.

176. If R_1, R_2 and R_3 are respectively the radii of the circumcircles of the triangle *IBC*, *ICA* and *IAB*, prove that $R_1R_2R_3 = 2R^2r$.

177. If *x*, *y*, *z* be the lengths of the tangents drawn to the incircle parallel to the sides *BC*, *AC*, *AB* respectively and intercepted between the other two sides, then prove that $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

178. Prove that the area of the incircle is to the area of the triangle itself is $\pi : \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

179. C is right-angled and a perpendicular CD is drawn to AB. The radii of the circles inscribed into the triangles ACD and BCD are equal to x and y respectively. Find the radius of the circle inscribed into the triangle ABC. {Ans. $\sqrt{x^2+y^2}$ }

180. If a circle be drawn touching the incircle and circumcircle of a triangle and the side BC externally, prove that it's radius is $\frac{\Delta}{a}\tan^2\frac{A}{2}$.

CATEGORY-13.13. PROVING IDENTITIES RELATED TO EX-RADII

181. $\frac{rr_1}{r_2r_3}=\tan^2\frac{A}{2}$.

182. $rr_1r_2r_3=\Delta^2$.

183. $r_1r_2r_3=r^3\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}$.

184. $rr_1\cot\frac{A}{2}=\Delta$.

185. $r_2r_3+r_3r_1+r_1r_2=s^2$.

186. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$.

187. $a(rr_1+r_2r_3)=b(rr_2+r_3r_1)=c(rr_3+r_1r_2)$.

188. $(r_1+r_2)\tan\frac{C}{2}=(r_3-r)\cot\frac{C}{2}=c$.

189. $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$.

190. $r_1+r_2+r_3-r=4R$.

191. $(r_1-r)(r_2-r)(r_3-r)=4Rr^2$.

192. $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}-\frac{1}{2R}$.

193. $r^2+r_1^2+r_2^2+r_3^2=16R^2-a^2-b^2-c^2$.

194. $\frac{(r_1-r)}{a}+\frac{(r_2-r)}{b}=\frac{c}{r_3}$

195. $\frac{r_1(r_2+r_3)}{a}=\frac{r_2(r_3+r_1)}{b}=\frac{r_3(r_1+r_2)}{c}$

196. $\frac{(ab-r_1r_2)}{r_3}=\frac{(bc-r_2r_3)}{r_1}=\frac{(ca-r_3r_1)}{r_2}=r$

197. $r_1+r_2-r_3+r=4R\cos C$

198. $(r_1-r)(r_2+r_3)=a^2$

199. $\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)=\frac{16R}{r^2(a+b+c)^2}$

200. $\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right)=\frac{64R^3}{a^2b^2c^2}$

201. $(r_1+r_2)(r_2+r_3)(r_3+r_1)=4Rs^2$

202. $R=\frac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{4(r_1r_2+r_2r_3+r_3r_1)}$

203. $\frac{r_1}{(s-b)(s-c)}+\frac{r_2}{(s-c)(s-a)}+\frac{r_3}{(s-a)(s-b)}=\frac{3}{r}$

204. $(r+r_1)\tan\frac{B-C}{2}+(r+r_2)\tan\frac{C-A}{2}+(r+r_3)\tan\frac{A-B}{2}=0$

205. $\frac{(b-c)}{r_1}+\frac{(c-a)}{r_2}+\frac{(a-b)}{r_3}=0$

206. $\frac{(r_2+r_3)}{(1+\cos A)}=\frac{(r_3+r_1)}{1+\cos B}=\frac{(r_1+r_2)}{1+\cos C}$

207. $\frac{bc}{r_1}+\frac{ca}{r_2}+\frac{ab}{r_3}=2R\left[\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)-3\right]$

CATEGORY-13.14. EX-CIRCLES, EX-CENTERS, EX-RADII

208. The sides of a triangle are 18, 24 and 30 cm. Find r_1, r_2 and r_3. {Ans. 12, 18 and 36 cm.}

209. If a = 13, b = 4 and $\cos C=-\frac{5}{13}$, find R, r, r_1, r_2 and r_3. {Ans. $8\frac{1}{8}$, $1\frac{1}{2}$, 8, 2 and 24}

210. If $r_1=2r_2=3r_3$ then prove that $a:b=5:4$.

211. If r_1, r_2, r_3 are in H.P., area is 24 sq. cm. and perimeter is 24 cm., find a, b, c. {Ans. 6 cm., 8 cm., 10 cm.}

212. Prove that

i. $AI_1=r_1\operatorname{cosec}\frac{A}{2}$

ii. $II_1=a\sec\frac{A}{2}$

iii. $I_2I_3=a\operatorname{cosec}\frac{A}{2}=4R\cos\frac{A}{2}$

iv. $II_1\cdot II_2\cdot II_3=16R^2r$

v. $I_2{I_3}^2=4R(r_2+r_3)$

vi. $\angle I_3I_1I_2=\frac{B+C}{2}=\frac{\pi}{2}-\frac{A}{2}$

vii. $II_1{}^2+I_2I_3{}^2=II_2{}^2+I_3I_1{}^2=II_3{}^2+I_1I_2{}^2$

viii. Area of $\Delta I_1I_2I_3=8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{abc}{2r}=2Rs$

ix. $\frac{II_1\cdot I_2I_3}{\sin A}=\frac{II_2\cdot I_3I_1}{\sin B}=\frac{II_3\cdot I_1I_2}{\sin C}$

213. Prove that the cubic equation whose roots are r_1, r_2 and r_3 is $x^3-(r+4R)x^2+s^2x-s^2r=0$.

214. If *u*, *x*, *y*, *z* are respectively the areas of the incircle and ex-circles, prove that $\frac{1}{\sqrt{u}}=\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}$.

215. If $r_1=r_2+r_3+r$, prove that the triangle is right angled.

216. If $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)=2$, prove that the triangle is right angled.

217. If $(a-b)(s-c)=(b-c)(s-a)$, prove that r_1, r_2, r_3 are in A.P.

218. If r_1, r_2, r_3 are in H.P., show that *a*, *b*, *c* are in A.P.

219. If *x*, *y*, *z* be the lengths of the tangents from the centers of ex-circles to the circumcircle, prove that $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{2s}{abc}$.

220. If Δ_0 be the area of the triangle formed by joining the points of contact of the incircle with the sides of the given triangle and Δ_1, Δ_2 and Δ_3 the corresponding areas for the ex-circles, prove that $\Delta_1+\Delta_2+\Delta_3-\Delta_0=2\Delta$.

221. The triangle *DEF* circumscribes the three ex-circles of the triangle *ABC*, prove that $\frac{EF}{a\cos A}=\frac{FD}{b\cos B}=\frac{DE}{c\cos C}$.

CATEGORY-13.15. SYSTEM OF CIRCLES

222. Two circles, of radii *a* and *b*, cut each other at an angle *θ*. Prove that the length of the common chord is $\frac{2ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}$.

223. Three equal circles touch one another. Find the radius of the circle which touches all three. {Ans. $\left(\frac{2}{\sqrt{3}}\pm 1\right)r$ }

224. Three circles, whose radii are *a*, *b* and *c*, touch one another externally and the tangents at their points of contact meet in a point. Prove that the distance of this point from either of their points of contact is $\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}$.

225. Three circles touch one another externally. The tangents at their points of contact meet at a point whose distance from any point of contact is 4. Find the ratio of the product of the radii to the sum of the radii of circles. {Ans. 16:1}

CATEGORY-13.16. MAXIMUM AND MINIMUM VALUES IN A TRIANGLE

226. In an acute angled triangle, prove that $\tan A+\tan B+\tan C\ge 3\sqrt{3}$. If $\tan A+\tan B+\tan C=3\sqrt{3}$, prove that the triangle is equilateral.

227. Prove that $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\ge 3\sqrt{3}$.

228. Prove that $1 < \cos A + \cos B + \cos C \le \frac{3}{2}$. In case of equality, triangle will be equilateral.

229. Prove that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$.

230. Prove that $\Delta < \frac{s^2}{4}$.

CATEGORY-13.17. GENERAL QUADRILATERAL

231. The sides of a quadrilateral are respectively 3, 4, 5 and 6 cm., and the sum of a pair of opposite angles is 120°. Find the area. {Ans. $3\sqrt{30}$ sq. cm.}

232. Prove that the area of any quadrilateral is one-half the product of the two diagonals and the sine of the angle between them.

233. *a*, *b*, *c* and *d* are the sides of a quadrilateral taken in order and α is the angle between the diagonals opposite to *b* or *d*, prove that the area of the quadrilateral is $\frac{1}{4}\left(a^2 - b^2 + c^2 - d^2\right)\tan\alpha$.

234. If *a*, *b*, *c* and *d* be the sides and *x* and *y* the diagonals of a quadrilateral, prove that it's area is $\frac{1}{4}\sqrt{4x^2y^2 - \left(b^2 + d^2 - a^2 - c^2\right)^2}$.

235. The sides of a quadrilateral are divided in order in the ratio *m*:*n* and a new quadrilateral is formed by joining the points of division. Prove that it's area is to the area of the original figure as $m^2 + n^2$ to $(m+n)^2$.

236. If *a*, *b*, *c* and *d* be the sides of a quadrilateral, taken in order, prove that $d^2 = a^2 + b^2 + c^2 - 2ab\cos B - 2bc\cos C - 2ca\cos\gamma$, where γ denotes the angle between the sides *a* and *c*.

CATEGORY-13.18. CYCLIC QUADRILATERAL

237. Sides of a cyclic quadrilateral *ABCD* are *a* = 3 cm., *b* = 5 cm., *c* = 7 cm. and *d* = 9 cm. Find Δ, *R*, *A* and *D*. {Ans. $3\sqrt{105}$ sq. cm., $\sqrt{\frac{4433}{210}}$ cm., $\cos^{-1}\frac{4}{31}$, $\cos^{-1}\frac{8}{13}$ }

238. In a cyclic quadrilateral, prove that the angle between its diagonals is $\sin^{-1}\left[\frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ac+bd)}\right]$.

239. If *ABCD* be a cyclic quadrilateral, prove that $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-b)}{(s-c)(s-d)}}$.

240. If *ABCD* be a cyclic quadrilateral, prove that the product of the segments into which one diagonal is divided by the other diagonal is $\frac{abcd(ac+bd)}{(ab+cd)(ad+bc)}$.

CATEGORY-13.19. QUADRILATERAL WITH AN INSCRIBED CIRCLE

241. The sides of a quadrilateral with an inscribed circle are 7, 10, 5 and 2 cm. and the sum of a pair of opposite angles is 120°. Find area and radius of inscribed circle. {Ans. $5\sqrt{21}$ sq. cm., $\frac{5\sqrt{21}}{12}$ cm.}

242. A quadrilateral *ABCD* is circumscribed about a circle, prove that $a\sin\frac{A}{2}\sin\frac{B}{2}=c\sin\frac{C}{2}\sin\frac{D}{2}$.
243. The sides of a cyclic quadrilateral are 3, 3, 4 and 4 cm.. Find the radii of the incircle and circumcircle and the area and the angles. {Ans. $\frac{12}{7}$ cm., 2.5 cm., 12 sq. cm., 90°, 90°, $\cos^{-1}\frac{7}{25}$, $180°-\cos^{-1}\frac{7}{25}$ }
244. If a cyclic quadrilateral can be circumscribed about another circle, prove that it's area is $\sqrt{abcd}$ and that the radius of the inscribed circle is $\frac{2\sqrt{abcd}}{a+b+c+d}$.
245. If a cyclic quadrilateral can be circumscribed about a circle, prove that the angle between its diagonals is $\cos^{-1}\left(\frac{ac-bd}{ac+bd}\right)$.
246. Let *ABC* be a triangle with incentre *I* and inradius *r*. Let *D*, *E*, *F* be the feet of the perpendiculars from *I* to the sides *BC*, *CA* and *AB* respectively. If r_1, r_2 and r_3 are the radii of circles inscribed in the quadrilateral *AFIE*, *BDIF* and *CEID* respectively, prove that $\frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}$.

CATEGORY-13.20. REGULAR POLYGON

247. Find the length of the perimeter of a regular decagon which surrounds a circle of radius 12 cm. {Ans. 77.98 cm.}
248. Find the length of the side of a regular polygon of 12 sides which is circumscribed to a circle of unit radius. {Ans. 0.536 units}
249. Find the area of a pentagon, a hexagon, an octagon and a decagon, each being a regular figure of side 1 meter. {Ans. 1.72 sq. m., 2.6 sq. m., 4.8 sq. m., 7.7 sq. m.}
250. Find the difference between the areas of a regular octagon and a regular hexagon if the perimeter of each is 24 cm. {Ans. 1.887 sq. cm.}
251. A square, whose side is 2 cm., has it's corners cut away so as to form a regular octagon, find it's area. {Ans. 3.3 sq. cm.}
252. Compare the areas and perimeters of octagons which are respectively inscribed in and circumscribed to a given circle. {Ans. $2+\sqrt{2}:4$, $\sqrt{2+\sqrt{2}}:2$ }
253. Show that the areas of the hexagon and octagon inscribed to a given circle are as $\sqrt{27}:\sqrt{32}$.
254. If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as 2:3.
255. If a regular pentagon and a regular decagon have the same perimeter, prove that their areas are as $2:\sqrt{5}$.
256. Given that the area of a polygon of *n* sides circumscribed about a circle is to the area of the circumscribed polygon of 2*n* sides as 3:2, find *n*. {Ans. 3}
257. The area of a regular polygon of *n* sides inscribed in a circle is to that of the same number of sides circumscribing the same circle as 3:4. Find the value of *n*. {Ans. 6}
258. The interior angles of a polygon are in A.P., the least angle is 120° and the common difference is 5°. Find the number of sides. {Ans. 9}
259. There are two regular polygons, the number of sides in one being double the number in the other, and an angle of one polygon is to an angle of the other as 9:8, find the number of sides of each polygon. {Ans. 20 & 10}
260. Show that there are eleven pairs of regular polygons such that the number of degrees in the angle of one is

to the number in the angle of the other as 10:9. Find the number of sides in each. {Ans. 6 & 5, 12 & 8, 18 & 10, 22 & 11, 27 & 12, 42 & 14, 54 & 15, 72 & 16, 102 & 17, 162 & 18, 342 & 19}

261. Prove that the area of the circle and the area of a regular polygon of n sides and of perimeter equal to that of the circle are in the ratio of $\tan\frac{\pi}{n} : \frac{\pi}{n}$.

262. Prove that the sum of the radii of the circles, which are respectively in and circumscribed about a regular polygon of n sides, is $\frac{a}{2}\cot\frac{\pi}{2n}$, where a is a side of the polygon.

263. Of two regular polygons of n sides, one circumscribed and the other is inscribed in a given circle, prove that the perimeters of the circumscribing polygon, the circle and the inscribed polygon are in the ratio $\sec\frac{\pi}{n} : \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n} : 1$ and that the areas of the polygons are in the ratio $\cos^2\frac{\pi}{n} : 1$.

264. Prove that the area of a regular polygon of $2n$ sides inscribed in a circle is a mean proportional between the areas of the regular inscribed and circumscribed polygons of n sides.

265. A pyramid stands on a regular hexagon as base. The perpendicular from the vertex of the pyramid on the base passes through the center of the hexagon and it's length is equal to that of a side of the base. Find the tangent of the angle between the base and any face of the pyramid and also of half the angle between any two side faces. {Ans. $\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{6}}$ }

266. A regular pyramid has for it's base a polygon of n sides and each slant face consists of an isosceles triangle of vertical angle 2α. If the slant faces are each inclined at an angle β to the base and at an angle 2γ to one another, show that $\cos\beta = \tan\alpha\cot\frac{\pi}{n}$ and $\cos\gamma = \sec\alpha\cos\frac{\pi}{n}$.

CATEGORY-13.21. ADDITIONAL QUESTIONS

267. If A, B, C are the angles of a triangle, then show that the system of equations

$$-x + y\cos C + z\cos B = 0$$
$$x\cos C - y + z\cos A = 0$$
$$x\cos B + y\cos A - z = 0.$$

has non-zero solution.

268. If A, B, C are the angles of a triangle show that system of equations

$$x\sin 2A + y\sin C + z\sin B = 0$$
$$x\sin C + y\sin 2B + z\sin A = 0$$
$$x\sin B + y\sin A + z\sin 2C = 0.$$

possesses non-trivial solution.

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