IAS/PARK CITY MATHEMATICS SERIES

Volume 2

Nonlinear Partial Differential Equations in Differential Geometry

Robert Hardt
Michael Wolf
Editors

American Mathematical Society
Institute for Advanced Study

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IAS/Park City Mathematics Institute runs mathematics education programs that bring together high school mathematics teachers, researchers, graduate students, and undergraduates to participate in distinct but overlapping programs of research and education. This volume contains the lecture notes from the Graduate Summer School program on Nonlinear Partial Differential Equations in Differential Geometry, held June 20–July 12, 1992, in Park City, Utah.

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Preface

The IAS/Park City Mathematics Institute (PCMI) was founded in 1991 as part of the "Regional Geometry Institute" initiative of the National Science Foundation. In mid 1993 it found an institutional home at the Institute for Advanced Study (IAS) in Princeton. The PCMI will continue to hold summer programs in both Park City and in Princeton.

The IAS/Park City Mathematics Institute encourages research and education in mathematics and fosters interaction between the two. The month-long summer institute offers programs for researchers and postdoctoral scholars, graduate students, undergraduates, and high school teachers. One of our main goals is to make all of the participants aware of the total spectrum of activities that occur in mathematics education and research: we wish to involve professional mathematicians in education and to bring modern concepts in mathematics to the attention of educators. To that end the summer institute features general sessions designed to encourage interaction among the various groups. In-year activities at sites around the country form an integral part of the Program for High School Teachers.

Each summer a different topic is chosen as the focus of the Research Program and Graduate Summer School. (Activities in the Undergraduate Program deal with this topic as well.) Lecture notes from the Graduate Summer School are being published each year in this series. The first volume contained notes from the 1991 Summer School on the *Geometry and Topology of Manifolds and Quantum Field Theory*. This second volume is from the 1992 Summer School *Nonlinear Partial Differential Equations in Differential Geometry*. The 1993 Summer School *Higher Dimensional Algebraic Geometry* and the 1994 Summer School *Gauge Theory and the Topology of Four-Manifolds* are in preparation. The 1995 Research Program and Graduate Summer School topic is *Nonlinear Wave Phenomena*.

We plan to publish material from other parts of the IAS/Park City Mathematics Institute in the future, including the interactive activities which are a primary focus of the PCMI. At the summer institute late afternoons are devoted to seminars of common interest to all participants. Many deal with current issues in education; others treat mathematical topics at a level which encourages broad participation. Several popular evening programs are also well-attended. These include lectures, panel discussions, computer demonstrations, and videos. The PCMI has also spawned interactions between universities and high schools at a local level. We hope to share these activities with a wider audience in future volumes.

Dan Freed, Series Editor
June, 1995

Introduction

That differential geometry involves differential equations is virtually a tautology; yet, what distinguishes differential geometry in the last half of the twentieth century from its earlier history is the use of non-linear partial differential equations in the study of curved manifolds, submanifolds, mapping problems and function theory on manifolds, among other topics. Here the differential equations appear both as a tool and as an object of study, with analytic and geometric advances fueling each other in the current explosion of progress in this area of geometry in the last twenty years.

Against this background, the second Summer Geometry Institute in 1992 at Park City, Utah was devoted to exploring some topics at the frontier of this area of non-linear PDE and geometry. Of course, given the current breadth of this topic (see the three volume summary of the 1989 UCLA Summer Institute, [AMS Proc. Symp. Pure Math. vol. 54, no. 1, ed. R. Greene and S.-T. Yau] or even the decade earlier influential Yau articles in the Annals of Mathematics Study 102), these lectures could not begin to comprehensively cover the field: many very important topics and researchers were ignored. Still, the five lecturers whose notes are contained in this volume did an outstanding job of presenting breaking research in a number of representative and rapidly developing areas in geometric analysis. The lecturers were Luis Caffarelli, Sun-Yang Alice Chang, Richard Schoen, Leon Simon, and Michael Struwe.

At Park City, each lecturer delivered 8-10 lectures, and these formed the bases for the present articles. As indicated in the Table of Contents, some of the lecture series have been organized approximately by lecture while others have been organized into larger groupings as Part 1, Part 2, etc.

Luis Caffarelli lectured on regularity theory of fully nonlinear equations that are uniformly elliptic or of Monge-Ampère type. These include, for example, the equation for a graph of a convex function of prescribed scalar curvature. Caffarelli has, over the last few years, established beautiful very general facts about the regularity of bounded weak or viscosity solutions of Monge-Ampère type equations. These include many strong extensions and simplifications of previous regularity work, and this lecture series is a good source for many key ideas. His results include strong, often optimal, interior estimates in $W^{2,p}$ (for $p > n$) and in $C^{1,\alpha}$ and $C^{2,\alpha}$. The proofs involve several geometric constructions along with some ideas from the Alexandrov-Bakelman-Pucci estimate, Krylov-Safonov Harnack's inequality, and Calderon-Zygmund decomposition.

The lectures of Sun-Yang Alice Chang treat the Moser-Trudinger type inequality and their uses in many conformally invariant problems. After some nice background on linear P.D.E. on manifolds including Weyl's asymptotic formula, the Ray-Singer/Polykov log determinal formula for surfaces is presented. The Trudinger-Moser-Onofri inequality is proven and leads to the compactness of isospectral families of metrics on a surface. Extremal functions for the Moser inequality exist by a symmetrization argument and an interesting one dimensional integral inequality. Finding the formulations and proofs of suitable higher dimensional generalizations of these inequalities is a very important problem. The recent work presented here of Branson-Chang-Yang involves Beckner-Adams inequalities and log-determinants. Finally, many of these inequalities are related to works on the solvability of various prescribing curvature equations in conformal geometry, such as the "Yamabe problem" or the zeta-function determinant extremal metrics of Chang-Yang.

Richard Schoen's lectures dealt with the effect of curvature of the domain on harmonic functions or of curvature of the range for harmonic mappings. The first two lectures deal with comparison theorems, whose hypotheses involve curvature conditions and whose conclusion involves derivatives of the distance function or of a positive harmonic function. A corollary is the Cheng-Yau Harnack inequality for a positive harmonic function on a ball in a manifold with Ricci curvature bounded below. The third lecture contains some nice discussion and questions concerning harmonic functions on negatively curved spaces. The remaining six lectures treat the more recent beautiful work of Gromov-Schoen on harmonic maps into non-positively curved metric spaces. Strong motivation comes from rigidity questions for lattices in semi-simple Lie groups. Some of these applications are described in the last lecture. The intermediate lectures give an excellent introduction to the analytic machinery, the Bochner identity, monotonicity of normalized energy and order, homogeneous minimizers, and intrinsic differentiability.

At Park City Leon Simon gave essentially two parallel lecture series. One series covered the basic partial regularity theorem (of Schoen-Uhlenbeck and Giaquinta-Giusti) for energy minimizing harmonic maps. This work from the early 80's has proven to be fundamental to developments in many variational problems. Simon's proof here is complete, clear, and direct. The now basic notions of small energy regularity, blowing-up, monotonicity, tangent maps, and Federer dimension reduction are all well exposed in Simon's lectures. The partial regularity theorem provides the optimal estimate on the size of the singular set of a minimizer. The second lecture series discusses the structure of and behavior near the singular set. It includes an exposition of Simon's very important recent work on rectifiability of singularities and uniqueness of tangent maps with line singularities.

Michael Struwe discusses evolution equations associated with geometric variational problems. His lectures at Park City have here been greatly expanded and organized into three parts which now give a very comprehensive clear introduction to these equations. The first part deals with the evolution of harmonic maps including discussions and proofs of the Bochner identity, Eells-Sampson existence results, finite-time blow-up examples of Coron-Ghidaglia and Chang-Ding-Ye, Struwe's $2d$ existence and uniqueness theory, applications to the existence of harmonic 2-spheres, the higher dimensional existence theory of Chen-Struwe, and the parabolic monotonicity inequality. The second part concerns the mean curvature flow of hypersurfaces including discussions of flows of convex hypersurfaces (Huisken), Lipschitz graphs (Ecker-Huisken), and level surfaces (Evans-Spruck, Chen-Giga-Goto) as well as the relevant area monotonicity formula. The third part treats recent work of Struwe and Shatah-Struwe on harmonic maps of Minkowsky space. The equations here are hyperbolic and give rise to many interesting new phenomena.

In editting this record of the lectures at Park City, we are struck not only by how well these fine speakers have written clear and comprehensive descriptions of broad ranges of mathmatices, but also by how well the written accounts convey the excitement of these frontiers of nonlinear partial differential equations and geometry.

Robert Hardt and Michael Wolf, Volume Editors
June, 1995

A Priori Estimates and the Geometry of the Monge Ampère Equation

Luis A. Caffarelli

IAS/Park City Mathematics Series
Volume 2, 1996

A Priori Estimates and the Geometry of the Monge Ampère Equation

Luis A. Caffarelli

Introduction

In this series of lectures, we will develop interior a priori estimates and regularity theory for solutions of fully non-linear elliptic equations, first uniformly elliptic and then, for the Monge Ampère equations.

There are two important aspects to the approach followed here, that we would like to stress; and that apply to many questions on elliptic equations (minimal surface theory, free boundary problems, the behavior of solutions to elliptic equations near a boundary, etc.)

The first one is to study regularity properties in the correct class of invariances. For instance, when studying an equation

$$a_{ij}D_{ij}u = 0$$

or

$$D_i a_{ij} D_j u = 0$$

classes of coefficients invariant under dialations are constant coefficients, BMO coefficients, bounded measurable coefficients, etc.

Classes of right hand sides invariant under affine transformations for $\log\det a_{ij}$, are constant bounded, BMO, etc., and each of these "linear" problems must be understood to develop a general theory for non-linear ones. The second is that once the invariant families are understood (be it of the linear or non-linear equations) one can try to treat dependence in X as a perturbation, just with very weak tools.

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PART 1
Interior A Priori Estimates for Solutions of Fully Non-linear Equations

The classic interior *a priori* estimates for solutions of linear second order elliptic equations establish that, since for harmonic functions one may estimate the function's derivative in the interior of a domain by the oscillation of the function itself, the same property remains true for small perturbations in appropriate functional spaces.

More precisely, we have

(a) **(Cordes-Nirenberg type estimate).** Let $0 < \alpha < 1$. If in the unit ball B_1 of R^n u is a bounded solution of

$$Lu = a_{ij}D_{ij}u = f,$$

with

$$|a_{ij} - \delta_{ij}| \leq \delta_0$$

for small $\delta_0 = \delta_0(\alpha)$, and if f is (say) bounded, then $u|_{B_{1/2}}$ is of class $C^{1,\alpha}$ and

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty}).$$

(b) **(Calderon-Zygmund).** Assume above that $f \in L^p$ for some $1 < p < \infty$ and $\delta_0 = \delta_0(p)$ is small. Then $u|_{B_{1/2}}$ belongs to $W^{2,p}$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p}).$$

(c) **(Schauder).** If the coefficients a_{ij} are of class C^α and f is of class C^α, then $u|_{B_{1/2}}$ is of class $C^{2,\alpha}$ and

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha}).$$

In this work we present a different approach to the above theory that allows us to extend the perturbation theory above to solutions of second order, uniformly elliptic, fully non-linear equations

$$F(D^2u, x) = f(x).$$

In fact, even in the linear case, these techniques provide new results in that the closeness of a_{ij} to δ_{ij} is measured in L^n instead of L^∞.

1. Statement of main theorems

All equations under consideration in this chapter will be uniformly elliptic in D^2; that is, for any matrix N and for any positive definite symmetric matrix M, there exist λ, Λ such that

$$\begin{aligned} 0 < \lambda\|M\| &< F(N+M, x) - F(N, x) \\ &\leq \Lambda\|M\|. \end{aligned}$$

In particular, F is uniformly Lipschitz continuous in D^2.

In order to measure the oscillation of F in the variable x we will consider the function

$$\beta(x) = \sup_{M\in S} \frac{|F(M, x) - F(M, 0)|}{\|M\|}.$$

Here S is the subspace of symmetric matrices, S^+ denoting the non-negative ones.

The theory can be written in terms of "viscosity solutions", a very weak notion of solution, introduced by Crandall and Lions, and that is, to non-divergence equations, akin to the "energy"-definition of weak solutions with divergence structure.

Definition. The continuous function u is a (C^2)-viscosity solution of

$$F(D^2u, x) = f(x)$$

if for any classical C^2 subsolution (resp. supersolution) φ, $u - \varphi$ cannot have an interior minimum (resp. maximum).

That is, if $F(D^2\varphi, x) \geq f(x)$ (resp. $\leq$), $u - \varphi$ cannot have corresponding interior extremum.

For instance, in one dimension we may say:

Definition. The continuous function u is a viscosity solution of

$$u_{xx} = 0$$

if whenever the C^2 function φ satisfies $\varphi_{xx} > 0$ in a subinterval (α, β) (resp. $\varphi_{xx} < 0$), $\varphi - u$ cannot have an interior minimum (respectively maximum) in the subinterval (α, β).

Exercise. u is linear.

Or in R^n:

Definition. The continuous function u is subharmonic in Ω, if for any small ball $B_r \subset \Omega$, for any C^2 function φ satisfying $\Delta\varphi < 0$ in B_r, $\varphi - u$ cannot have an interior minimum.

Exercise. a) u satisfies the mean value theorem.
b) $\Delta u \geq 0$ in the sense of distributions, i.e.

$$\int u\Delta\varphi \geq 0$$

for any non-negative C_0^∞ function φ.

If we insist on considering C^2-viscosity solutions, we have to ask continuous dependence in x, but will obtain *a priori* estimates independently of the modulus of continuity.

If we are willing to look at the more restricted class of $W^{2,p}(p > n)$ solutions, that is, if one requires the maximum principle to hold when u is tested against $W^{2,p}$ sub- and super-solutions φ, one may work directly with discontinuous dependence in x.

Nevertheless, the reader, at least in a first approach, should always think of u as a C^2 solution of the equation under consideration.

The results we want to discuss are mainly perturbation, in x, theorems:

That is, assume that solutions of $F(D^2w, 0) = 0$ are in some functional space, then if $\beta(x)$ is small in the appropriate class, also solutions of

$$F(D^2u, x) = f(x)$$

are in the same functional space.

More precisely:

Theorem 1. ($W^{2,p}$ estimates):

Let u be a bounded viscosity solution of

$$F(D^2u, x) = f(x)$$

in B_1.

Assume further that solutions w of the Dirichlet problem

$$\begin{cases} F(D^2w, 0) = 0 & \text{in } B_r \\ w = w_0 & \text{on } \partial B_r \end{cases}$$

satisfy the interior a priori estimate

$$\|w\|_{C^{1,1}(B_{r/2})} \leq Cr^{-2}\|w\|_{L^\infty(\partial B_r)}.$$

Let $n < p < \infty$, assume that $f \in L^p$, and for some $\theta = \theta(p)$ sufficiently small

$$\sup_{B_1} \beta(x) \leq \theta(p).$$

Then $u|_{B_{1/2}}$ is in $W^{2,p}$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C\left(\sup_{\partial B_1}|u| + \|f\|_{L^p}\right).$$

Remark. For $p < n$ consider the family of functions

$$u_{\varepsilon,\alpha}\begin{cases} 1 - r^\alpha & \text{for } r > \varepsilon \\ 1 - \alpha\varepsilon^{\alpha-2}r^2 - (1-\alpha)\varepsilon^\alpha & \text{for } r \leq \varepsilon. \end{cases}$$

Then

$$u_{rr} = \begin{cases} \alpha(1-\alpha)r^{\alpha-2} & \text{for } r > \varepsilon \\ -2\alpha\varepsilon^{\alpha-2} & \text{for } r < \varepsilon, \end{cases}$$

$$\frac{1}{r}u_r = \begin{cases} -\alpha r^{\alpha-2} & \text{for } r > \varepsilon \\ -2\alpha\varepsilon^{\alpha-2} & \text{for } r \leq \varepsilon. \end{cases}$$

Consider for $\alpha < 1$ the non-linear operator

$$F(D^2u) = \frac{1}{1-\alpha}\left(\sum_{\lambda_j \geq 0} \lambda_j\right) + \frac{1}{n-1}\sum_{\lambda_j \leq 0} \lambda_j.$$

This operator, being a convex function of λ_j (the eigenvalues of D^2u), has interior $C^{2,\alpha}$ estimates (see Evans [**E**]), and

$$F(D^2u) = -C(\alpha)\varepsilon^{\alpha-2}\chi_{B_\varepsilon}.$$

Hence ($\alpha < 1$, fixed)

$$\|F(D^2n)\|_{L^p}^p = \varepsilon^{p(\alpha-2)}\varepsilon^n$$

goes to zero (with ε) for any

$$p < \frac{n}{2-\alpha}.$$

This shows that $W^{2,p}$ *a priori* estimates for fully non-linear equations cannot hold for $p < n$.

Theorem 2. ($C^{1,\alpha}$ results)*: Suppose* $0 < \overline{\alpha} < 1$.
Assume now that solutions w to the equation

$$F(D^2w, 0) = 0$$

in B_r satisfy the a priori estimate

$$\|w\|_{C^{1,\overline{\alpha}}(B_{r/2})} \leq Cr^{-(1+\overline{\alpha})}\|w\|_{L^\infty(B_r)}.$$

Then, for any $0 < \alpha < \overline{\alpha}$ there exists $\theta = \theta(\alpha)$ so that if

$$\fint_{B_r} \beta^n(x)dx \leq \theta$$

and

$$\fint_{B_r} |f(x)|^n dx \leq C_1 r^{(\alpha-1)n},$$

then any bounded solution u of

$$F(D^2u, x) = f(x)$$

in B_{r_0}, is $C^{1,\alpha}$ at the origin. That is, there exist a linear function ℓ such that for $r < r_0$

$$|u - \ell| \leq C_2 r^{1+\alpha}$$

and

$$\|\ell\|_{C^1} \leq C_3,$$

with

$$C_2, C_3 \leq C(\alpha)r_0^{-(1+\alpha)}\sup_{B_{r_0}}|u| + C_1^{1/n}.$$

Theorem 3. ($C^{2,\alpha}$ estimates):
Assume now the existence of $C^{2,\overline{\alpha}}$ interior a priori estimates for solutions of

$$F(D^2w + M, 0) = 0,$$

for any M satisfying

$$F(M, 0) = F(0, 0) = 0.$$

Then, if $0 < \alpha < \overline{\alpha}$,

$$\fint_{B_r} \beta^n dx \le Cr^{\alpha n},$$

$$\fint_{B_r} |f(x)|^n dx \le Cr^{\alpha n},$$

and u is a solution u of

$$F(D^2u, x) = f(x),$$

u is $C^{2,\alpha}$ at the origin (in the same sense as above).

Remark. If $f(0) \neq 0$, we must translate F to obtain the appropriate statement.

Remark. F convex provides a family of examples for Theorem 3 (see [E]). Any F gives an example for Theorem 2 with some $\overline{\alpha} > 0$.

Remark. In Theorems 2 and 3 we may keep the dependence in x continuous, and obtain *a priori* estimates, or require solutions to be $W^{2,p}$-viscosity.

2. Preliminary tools

In this section we establish the main three techniques used in the rest of the work.

These are some facts hidden in Pucci's proof [**Pu**] of Alexandrov-Bakelman-Pucci's $L^n - L^\infty$ *a priori* estimate, Krylov-Safonov [**K-S**] Harnack's inequality, and an elementary consequence of Calderon-Zygmund decomposition lemma.

In order to avoid talking of a specific operator, given $0 < \lambda \le \Lambda < \infty$ we introduce Pucci's extremal operators

$$\mathcal{M}^-(D^2u) = \lambda \left(\sum_{\lambda_j > 0} \lambda_j \right) + \Lambda \left(\sum_{\lambda_j < 0} \lambda_j \right)$$

$$\mathcal{M}^+(D^2u) = \Lambda \left(\sum_{\lambda_j > 0} \lambda_j \right) + \lambda \left(\sum_{\lambda_j < 0} \lambda_j \right)$$

(λ_j the eigenvalues of D^2u).

These are called "extremal" operators because their solutions (to a given Dirichlet problem) are supersolutions to *any* elliptic operator with λ and Λ ellipticity bounds:

Consider for instance a solution u of an equation

$$a_{ij}D_{ij}u = f$$

with $A = a_{ij}(x)$ satisfying

$$\lambda|\xi|^2 \leq \xi^T A\xi \leq \Lambda\xi^2$$

and some Dirichlet data, and w a solution of

$$\mathcal{M}^+(D^2 w) = f$$

with the same data.

Assuming all the necessary smoothness, at every point we may rewrite

$$a_{ij} D_{ij} w$$

as

$$A_{ii}\lambda_i(D^2 w)$$

(λ_i the eigenvalues of $D^2 w$). From the ellipticity hypothesis

$$\lambda \leq A_{ii} \leq \Lambda.$$

Therefore,

$$a_{ij} D_{ij} w \leq \mathcal{M}^+ w$$

and hence w is a supersolution of

$$a_{ij} D_{ij} w \leq f.$$

It thus majorizes u.

On the other hand, w is itself (always heuristically) a solution of

$$a^*_{ij} D_{ij} w = f$$

with a^*_{ij} defined at every point as the transformed matrix

$$A^* = \mathcal{O}\begin{bmatrix} \Lambda & & & & & \\ & \Lambda & & & & \\ & & \ddots & & & \\ & & & \lambda & & \\ & & & & \ddots & \\ & & & & & \lambda \end{bmatrix}\mathcal{O}^T.$$

with O the orthonomial change of coordinates between the basis of eigenvalues of $D^2 w$ and the standard basis.

With this in mind we now define the class of "all viscosity solutions of all elliptic equations with ellipticity between λ and Λ":

We will say that u belongs to the class $S(f) = S(\lambda, \Lambda, f)$ if for any C^2 supersolution φ of $\mathcal{M}^+(D^2\varphi) \leq f$, $u - \varphi$ cannot have a local maximum, and for any C^2 subsolution φ of $\mathcal{M}^-(D^2\varphi) \geq f$, $u - \varphi$ cannot have a local minimum.

The usefulness of the class $S(\lambda, \Lambda, f)$ is in avoiding linearization.

Note. If u is of class C^2 and belongs to $S(\lambda, \Lambda, f)$ one may find pointwise an a_{ij}, with eigenvalues between λ and Λ, such that

$$a_{ij}D_{ij}u = f.$$

We will say that u is a subsolution $\big(u \in \underline{S}(\lambda,\ \Lambda,\ f) = \underline{S}(f)\big)$ or a supersolution $\big(u \in \overline{S}(\lambda,\ \Lambda,\ f) = \overline{S}(f)\big)$ if only the first or second condition is satisfied.

Finally we denote by $S^*(f) = \overline{S}(|f|) \cap \underline{S}(-|f|) \supset S(f)$.

Lemma 1 (A-B-P). *Let v belong to $\overline{S}(\lambda,\ \Lambda,\ f)$ in B_1, for f bounded continuous. Assume that $v|_{\partial B_1} \geq 0$. Then*

$$|\inf(v,\ 0)| = \sup v^- \leq C(\lambda,\ \Lambda) \left[\int_{v=\Gamma(v)} f^n dx\right]^{1/n},$$

where $\Gamma(v)$ denotes the convex envelope of $\min(v,\ 0) = -v^-$, *in the ball B_2.*

The proof will follow the usual way, provided we have shown sufficient regularity for the convex envelope of $-v^-$ in B_1.

Lemma 2. *Let u belong to $\overline{S}(\lambda,\ \Lambda,\ f)$ in B_1, with $f \leq$ a constant μ. Assume that there exists a convex function w satisfying*

(i) $w < u$ *in* B_1,
(ii) $w = u$ *at* 0.

Then

(a) $\mu \geq 0$ *and*
(b) *there exists a constant* $C = C(\lambda,\ \Lambda)$ *and a linear function* $\ell(x) \leq w \leq \ell(x) + C\mu|x|^2$.

Proof. By subtracting from w the supporting plane for w at the origin, we may prove the theorem with $\ell \equiv 0$. First, we prove (a). Indeed, if $\mu < 0$, then $h = -\varepsilon|x|^2$ satisfies $\mathcal{M}^- h > \mu \leq f$ for ε small enough and $u - h$ has a minimum at zero; a contradiction in virtue of the definition of viscosity solution.

To prove (b), let $\alpha(\rho) = \sup_{B_\rho} w(\rho < 1)$. Assume that such a sup is attained at ρe_n.

Then, at ρe_n, any supporting plane for w is of the form $A + Bx_n$ and, w being convex, we get that, for any $(x_1, \dots, x_{n-1})$

$$w(x_1, \dots, x_{n-1},\ \rho) \geq \alpha(\rho).$$

Consider now a large box

$$R = \{|(x_1, \dots, x_{n-1})| \leq M\rho,\ |x_n| < \rho\}$$

with

$$M = (4n(1 + \Lambda)/\lambda)^{1/2} \qquad \text{and the function}$$

$$P = (1 + \Lambda)(x_n + \rho)^2 - \frac{\lambda}{n}\left(\sum_1^{n-1} x_j^2\right).$$

The following properties of P are readily verifiable:

(1) $\mathcal{M}^-(D^2P) > \lambda$,

(2) $P \leq 0$ on $\partial R \backslash \{x_n = \rho\}$,
(3) $P \leq 4(1+\Lambda)\rho^2$ on the side $\{x_n = \rho\}$.

Therefore we have, on ∂R,

$$\frac{\mu}{\lambda} P \leq w \leq u$$

provided that $\alpha(\rho) \geq \frac{4(1+\Lambda)}{\lambda}\mu\rho^2$. This is a contradiction to the definition, since

$$\mathcal{M}^-\left(\frac{\mu D^2 P}{\lambda}\right) > \mu \text{ and } P(0) > u(0).$$

Thus we must have

$$\sup_{B_\rho} w < \frac{4(1+\Lambda)}{\lambda}\mu\rho^2$$

and (b) follows. □

Corollary-exercise. *The convex envelope $\Gamma(v)$ is a $C^{1,1}$ function.*

Idea of the proof. The set $\{v = \Gamma(v)\} \subset B_1$, contains the extremal points of $\Gamma(v)$. Thus, for any other point X, Γ is linear in some simplex through X generated by points in $\{v = \Gamma(v)\}$ and ∂B_2.

See what happens when you "wiggle" the convex combinations around these points.

Sketch of the proof of Lemma 1. We are now going to apply Lemma 2, taking as w the convex envelope of $\min(v, 0)$ in B_1, $\Gamma(v)$. Lemma 2 shows that $\Gamma(v)$ is locally a $C^{1,1}$ function, so the usual proof (see [G-T], §9.1) applies.

We study the properties of the map $X \to \nabla\Gamma(v)$, more precisely the volume of the image under this transformation.

On one hand, we have that

$$B_{\frac{-\min(v,0)}{4}} \subset \nabla\Gamma(v)(B_2).$$

Indeed consider any plane

$$P(x) = \langle y, x\rangle + \lambda$$

with slope $|y| \leq \frac{-\min(v,0)}{4}$. For λ very negative, P stays below $\Gamma(v)$ in B_2.

Raise it (increase λ) until P crosses the graph of $\Gamma(v)$.

Since $|y| \leq \frac{-\min(v,0)}{4}$, P will touch the graph of $\Gamma(v)$ for the first time in the interior of B_2 (before becoming zero at ∂B_2).

(Convince yourself, i.e. prove it!!)

On the other hand

$$\operatorname{vol}\left(\nabla(\Gamma(v)(B_2))\right) = \int_{B_2} \det D(\nabla(\Gamma(v)) = \int_{v=\Gamma(v)} \det D(\nabla(\Gamma(v))$$

(because everywhere else det $\equiv 0$ by the corollary-exercise) and

$$\int_{v=\Gamma(v)} \det D(\nabla(\Gamma(v)) \leq \int_{v=\Gamma(v)} \det D^2 v$$

(because at the points where $v = \Gamma(v)$

$$0 \leq D^2\Gamma(v) \leq D^2 v$$

and

$$\det D^2 v \leq C(\lambda, \Lambda)(Lv)^n$$

because

$$0 \leq D^2 v.$$

Putting all together

$$(\sup v^-)^n \leq C(\lambda, \Lambda) \int_{v=\Gamma(v)} (f^+)^n.$$

□

Lemma 3. *(an auxiliary barrier):*

Let Q^1, Q^2, Q^3 be three cubes in R^n of sides $\ell^1 = 1$, $\ell^2 < 2$, and ℓ^3 respectively. Assume that

(i) $Q^1 \subset Q^2 \subset Q^3$

and

(ii) $d(Q^1, \partial Q^3) \geq 2\sqrt{n}$.

Then, given λ, Λ, there exists a function p defined in Q^3 such that

(a) $p|_{\partial Q^3} \geq 0$.

(b) $\mathcal{M}^+(D^2 p) \leq 0$ *outside* Q^1.

(c) $p \leq -1$ *on* Q^2.

(d) $\|p\|_{C^2} \leq C(\lambda, \Lambda)$.

Proof. Let S be a convex subset of Q^1 with smooth boundary and let $d(x)$ be the distance function from x to S. A candidate for p will be a function defined outside Q_1 as a multiple of $(2\sqrt{n})^{-M} - d^{-M}$ and then smoothly continued inside Q_1 (M is a positive number to be chosen conveniently). Let us show that such a function has the required properties; (a), (b), (c) have a local character, then let us verify them locally. Fix x_0 sufficiently close to the boundary ∂S and $y_0 \in \partial S$ such that $|x_0 - y_0| = d(x_0)$. Then, in terms of a principal coordinate system at y_0 (i.e., a system of coordinates where e_n is parallel to $\nabla d(x_0)$) we have (see [G-T], §14.6 for details)

$$[D^2 d(x_0)] = \operatorname{diag}\left[\frac{-k_1}{1-k_1 d}, \dots, \frac{-k_{n-1}}{1-k_{n-1}d}, 0\right]$$

where k_i are the principal curvatures of ∂S at y_0. Then we get, at x_0,

$$D_{nn} d^{-M} = M(M+1)d^{-M-2}$$

and, for i, $j \neq n$

$$|D_{ij}d^{-M}| \leq MC(k)d^{-M-2}$$

where $C(k)$ is some constant depending on the curvatures of ∂S. Then if we choose ∂S close to ∂Q^1 first, then M sufficiently large, we see that properties (a), (b), (c) hold at x_0. This M will depend on λ, Λ; then it is clear that also (d) is satisfied.□

We now describe Calderon-Zygmund decomposition of a set A contained in a cube Q of R^n.

Let Q be a cube in $\mathbb{R}^n$; by bisection of the edges of Q we subdivide Q into 2^n congruent sub-cubes with disjoint interiors. We can repeat the process indefinitely, obtaining a family of so-called dyadic cubes. Given two dyadic cubes Q^i, $\tilde{Q}^i$, we will say that $\tilde{Q}^i$ is the predecessor of Q^i if Q^i is one of the 2^n cubes obtained from $\tilde{Q}^i$. Let $A \subset Q$ be a measurable set and δ a positive number satisfying

$$|A| \leq \delta |Q|.$$

If we split Q again and again, and keep the largest cubes for which $|A \cap Q^i| > \delta Q^i$, we obtain an $a \cdot e$ covering of A by disjoint dyadic cubes Q^i, such that (see [G-T] §9.2 for details):

$$|A \cap Q^i| > \delta |Q^i|$$

and for $\tilde{Q}^i$ (the predecessor of Q^i)

$$|A \cap \tilde{Q}^i| \leq \delta |\tilde{Q}^i|.$$

Lemma 4. *Let $A \subset B \subset Q^1$ be two measurable sets contained in the unit cube. Assume that there is a δ such that*

(a) $|A| \leq \delta$,

(b) *For any dyadic cube Q^i satisfying*

$$|A \cap Q^i| > \delta |Q^i|$$

then $\tilde{Q}^i \subset B$.

We conclude that

$$|A| \leq \delta |B|.$$

Proof. By Calderon-Zygmund decomposition, there is a disjoint family of dyadic cubes Q^i such that

(1) $A \subset \cup Q^i$ a.e.,

(2) $|A \cap Q^i| > \delta |Q^i|$,

(3) for any predecessor $\tilde{Q}^i$, $|A \cap \tilde{Q}^i| \leq \delta |\tilde{Q}^i|$.

Then, selecting a disjoint family of predecessors $\{\tilde{Q}^i\}$, we have

$$|A = \sum |A \cap \tilde{Q}_i| \leq \delta \sum |\tilde{Q}^i| \leq \delta |B|. \qquad \square$$

We are now in the position to prove the (Krilov-Safanov) Harnack inequality for viscosity solutions.

3. Harnack inequality for functions in $S^*(f)$

Let Q^k denote the cube of side k and center origin.

Theorem 4. *Let u be a nonnegative function in $S^*(f)$ in Q^2, then*

$$\sup_{Q^1} u \leq c \cdot \left(\inf_{Q^1} u + \|f\|_{L^n(Q^2)} \right)$$

with $c = c(\lambda, \Lambda)$.

The Hölder continuity of the solutions at interior points is a well-known consequence of Theorem 1.

Corollary 1. *Let u be a bounded function in $S^*(f)$ in Q^2. Then for some $\alpha = \alpha(\lambda, \Lambda)$, $u \in C^\alpha\left(\overline{Q}^1\right)$ and*

$$\|u\|_{C^\alpha(\overline{Q}^1)} \leq c \cdot (\|u\|_{L^\infty(Q^2)} + \|f\|_{L^n(Q^2)})$$

with $c = c(\lambda, \Lambda)$.

In the proof of Theorem 1 we may assume $\inf_{Q_1} u = 1$ and $\|f\| \leq \varepsilon_0$ (otherwise we change u by $\frac{u + \frac{\|f\|}{\varepsilon_0}}{(\inf u) + \frac{\|f\|}{\varepsilon_0}}$ and show that $\sup_{Q_1} u \leq c(\lambda, \Lambda)$. Moreover, we can assume that the solution is defined in a large cube Q^ℓ such that $d(Q_1, \partial Q^\ell) \geq 2\sqrt{n}$. In this way the geometrical situation is the one described in Lemma 3.

The proof is carried out in four steps:

(I) To prove that if in some point in Q_2, u is less than one, then u is bounded by a universal constant in a large portion of Q_1, (Lemma 5).

(II) Iteration of the result in (I) gives a polynomial decay for the distribution function of u in Q_2, (Lemma 6). The corollary of the Calderon-Zygmund decomposition lemma is the key tool to connect each step in the iteration with the next one.

(III) To prove that if u is large at a point well inside Q_2, then u is larger at some point nearby (Lemma 7).

(IV) The final step can be done in two equivalent ways: by constructing a negative power of the distance function $d(x, \partial Q_2)$, $x \in Q_2$, which controls u from above or by showing that if u were larger than a certain constant, it would be possible to construct a sequence of points $\{x_k\} \subset Q_2$ such that $u(x_k) \to \infty$ (Lemma 6, (a), (b)). (II to IV follow closely [K-S]).

Lemma 5. *Let u in $S^*(f)$ satisfy*

(a) $u \geq 0$ *in* Q^ℓ,

(b) $\inf_{Q_2} u \leq 1$.

(c) $\|f\|_{L^n} \leq \varepsilon_0(\lambda, \Lambda)$ *(f continuous).*

Then there exist two positive constants M, μ, depending only on λ, Λ such that

$$|\{u < M\} \cap Q_1| > \mu.$$

Proof. We may assume $\ell \geq 4\sqrt{n}$. Let $v = \min(u + 2p, 0)$ and apply Lemma 1 of Alexandrov-Bakelman-Pucci to v in a ball B of radius $\geq 4\sqrt{n}$ containing Q^ℓ. Since

$$\min_{Q_2} v \leq \min_{Q_2} u + 2 \sup_{Q_2} p \leq 1 - 2 = -1, \quad \text{we get}$$

$$1 \leq \left(\max_B v^-\right)^n \leq C(\lambda, \Lambda) \int_{[v=\Gamma(v)]} (g^+)^n dx$$

for any continuous majorant g of Lv.

On the set $\{v = \Gamma(v)\}$, $\Gamma(v)$ is locally $C^{1,1}$ and $\mathcal{M}^-(\Gamma(v)) \leq \mathcal{M}^-(2p) + C|f| \leq c(\lambda, \Lambda)\chi_{\{v=\Gamma(v)\}\cap Q_1} + C|f|$ (outside Q_1, $\mathcal{M}^-(2p) < 0$).
Hence: if a is small enough,

$$1 \leq C(\lambda, \Lambda) \cdot |\{v = \Gamma(v)\} \cap Q_1|.$$

Finally, we observe that, when $v = \Gamma(v)$, v is negative and therefore

$$u \leq 2p \leq M,$$

with some M universal constant. □

Remark. We may take any $Q_1(x) \subset Q_2$.

Lemma 6. *Let u be as in Lemma 5. Then*

$$(*) \qquad |\{u > M^k\} \cap Q_1| \leq (1 - \mu)^k \quad (k = 1, 2, \dots).$$

Proof. By iteration. The case $k = 1$ is just Lemma 5. Suppose now that $(*)$ is true for $k - 1$. Let $A = \{u > M^k\}$ and $B = \{u > M^{k-1}\}$. Clearly $A \subset B$ and $|A| \leq 1 - \mu$. We want to show that $|A| \leq (1 - \mu)|B|$, using Lemma 4. Let $\{Q^j\}$ denote a Calderon-Zygmund decomposition of A, so that, for each cube Q^j we have

$$(**) \qquad |A \cap Q^j| > (1 - \mu)|Q^j|.$$

We have to show that if $\tilde{Q}^j$ is a predecessor of some cube Q^j then $\tilde{Q}^j \subset B$.

Suppose there exists a point $\bar{x} \in \tilde{Q}^j$ such that $u(\bar{x}) < M^{k-1}$. Normalizing u to $\bar{u} = u/M^{k-1}$ and scaling suitably, we are under the hypotheses of Lemma 5, which gives

$$|\{\bar{u} < M\} \cap Q^j| > \mu|Q^j|;$$

that is,

$$|\{u > M^k\} \cap Q^j| \leq (1 - \mu)|Q^j|,$$

in contradiction with $(**)$. Note that under the scaling $\tilde{u} = au(bX)$, $\|\mathcal{M}^\pm(D^2\tilde{u})\|_{L^n(Q)} = ab\|\mathcal{M}^\pm u\|_{L^n(bQ)}$. □

Lemma 7. *Let u be as in Lemma 5. There exists two constants k_0 and c, depending only on λ, Λ, such that if $k > k_0$,*

(a) $u(x_0) \geq M^k$,

(b) $d(x_0, CQ_1) \geq c(1 - \mu)^{k/n}$

then (defining $2\rho = c(1 - \mu)^{k/n}$)

$$(***) \qquad \sup_{B_\rho(x_0)} u \geq u(x_0)\left(1 + \frac{1}{M}\right).$$

Proof. By contradiction; suppose $(***)$ is not true. Consider

$$w = \frac{u(x_0)(1+\frac{1}{M}) - u}{\frac{u(x_0)}{M}},$$

and let Q^* be a cube with center x_0 and side $\frac{\rho}{8\sqrt{n}}$. From Lemma 6 we have

$$|A_1| = |\{u \geq M^{k-1}\} \cap Q_1| \leq (1-\mu)^{k-1}.$$

On the other hand, $w(x_0) = 1$, and if $(***)$ is not true, then $w > 0$ on B_ρ; so in Q^* we are under the hypotheses of Lemma 5 (verifying the scaling for $\mathcal{M}^\pm(D^2 w)$) which gives

$$|A_2| = |\{w \geq M\} \cap Q^*| \leq (1-\mu)|Q^*|.$$

Now observe that $A_1 \cup A_2 \supset Q^*$; in fact if $x \in Q^*$ and $u(x) < u(x_0)/M$ then $w(x) \geq M$, so $x \in A_2$.
Therefore

$$|Q^*| \leq (1-\mu)|Q^*| + (1-\mu)^{k-1}$$

or

$$|Q^*| \leq \frac{(1-\mu)^{k-1}}{\mu},$$

which implies

$$c \leq \frac{16}{[(1-\mu)\mu]^{1/n}}.$$

To get a contradiction, chose first $c > \frac{16}{\left((1-\mu)\mu\right)^{1/n}}$ and then k_0 to keep 2ρ small enough. □

Proof of Theorem 4. We will show that there exist $D_0 = D_0(\lambda, \Lambda)$ and $\delta = \delta(\lambda, \Lambda)$ such that, if u is as in Lemma 5, then

$$u(x) \leq D_0 d^{-\delta}(x, \partial Q_1)$$

for $x \in Q_1$.

A covering argument gives the theorem in the form stated originally. Define δ by $M^{-1} = (1-\mu)^{\delta/n}$. Let D be the smallest constant for which the inequality is valid (which at this point depends on u) and let x_0 be a point where

$$u(x_0) = D d^{-\delta}(x_0, \partial Q_1).$$

Define k by

$$M^k \leq u(x_0) \leq M^{k+1}.$$

Then

$$d(x_0, \partial Q_1) \geq \left(\frac{M^{k+1}}{D}\right)^{-1/\delta} = \left(\frac{D}{M}\right)^{1/\delta} (M^{-k/\delta}) = \left(\frac{D}{M}\right)^{1/\delta} (1-\mu)^{k/n}.$$

If D is large enough, then $k > k_0$, where k_0 is defined in Lemma 7, and so we are under the hypotheses of this lemma, which gives

$$\sup_{B_\rho} u \geq u(x_0)\left(1 + \frac{1}{M}\right) = D d^{-\delta}(x_0)\left(1 + \frac{1}{M}\right).$$

On the other hand, by definition of D,

$$\sup_{B_\rho} u \le Dd^{-\delta}(B_\rho, CQ_1) \le D\big(d(x_0)-\rho\big)^{-\delta}$$

$$\le D\left[d(x_0)\left(1-\frac{c}{2}\left(\frac{M}{D}\right)^{1/\delta}\right)\right]^{-\delta}.$$

This is a contradiction if

$$\left[1-\frac{c}{2}\left(\frac{M}{D}\right)^{1/\delta}\right]^{-\delta} \le \left(1+\frac{1}{M}\right)$$

that is, if $D = D(\lambda, \Lambda)$ is too large. $\square$

4. $W^{2,p}$ estimates

In this section we prove Theorem 1.

For that purpose we will use Harnack inequality in its scaled version

$$\sup_{B_R} u \le c\left\{\inf_{B_R} u + R^2\left(\fint_{B_{2R}} |f|^n\right)^{1/n}\right\}.$$

We will also drop the λ, Λ from $S(\lambda, \Lambda, f)$. Let now $Q_{r_1}(x_1)$, $Q_{r_2}(x_2) \subset B_{1-\eta}(0) = B_{1-\eta}$ and denote by p the function defined in Lemma 3 of Section 2. We start with an easy consequence of Lemma 5:

Lemma 8. *Assume that u is a nonnegative function on $S^*(f)$ in B_1 such that $\inf_{Q_{r_1}(x_1)} u \le 1$, $\|f\|_{L^n} \le \varepsilon_0 = \varepsilon_0(\lambda\Lambda)$. Then, the set $S = \{x : (u+2p) = \Gamma(u+2p)\}$ satisfies*

$$|S \cap Q_{r_2}(x_2)| \ge \mu = \mu(\Lambda, \lambda, r_1, r_2, \eta) > 0.$$

Let us point out three consequences of the above result.

(a) For any point $y_1 \in S$, $u + 2p$ stays above its tangent plane at y_1 in all of B_1; that is

$$[u+2p](x) \ge \bar{\ell}(x) \text{ in } B_1 \qquad (\bar{\ell} \text{ linear})$$
$$[u+2p](y_1) = \bar{\ell}(y_1).$$

Since p is C^2

(i) $u(x) \ge \ell(x) - M(|x-y_1|^2)$ in B_1.

(ii) $u(y_1) = \ell(y_1)$

for some M controlled by $C^{1,1}$ norm of p and therefore depending only on $\Lambda, \lambda, r_1, r_2, \eta$. In case (i) and (ii) hold we will say that u *has a tangent paraboloid of aperture M by below at y_1*.

(b) Suppose $\Omega \supset B_1$, $u \ge 0$ in a solution of $Lu = f$ in Ω with $\|f\|_{L^n} \le \delta$.

Moreover, suppose u has a tangent paraboloid by below of aperture 1, at a point x_0 with respect to all of Ω.

We may assume that this paraboloid is $1 - |x|^2$. Let now $y_1 \in B_1$ as in Remark (a) above.

Then

$$\ell(x) - M|x - y_1|^2 \le \bar{\ell}(x) - 2p(x),\ x \in B_1.$$

Consequently $\ell(x) - M|x - y_1|^2 \le 0$ on ∂B_1, since on $\partial B_1 p \ge 0$ and $\bar{\ell} \le 0$. On the other hand

$$\ell(y_1) \ge 1 - |y_1|^2$$

and therefore

$$\ell(x) - M|x - y_1|^2 \le 1 - |x|^2 \le u(x)$$

in all of Ω.

(c) If u has a tangent paraboloid of aperture M by below at y_1, then any second differential quotient of u centered at y_1 is controlled from below by $-M$ and from above by $C[M + \sup_r(⨍_{B_r(y_1)} f^n)^{1/n}]$ by Harnack inequality $\big(c = c(\lambda, \Lambda)\big)$.

Define

$$D_\lambda = \{x \in B_1; u \text{ has a tangent paraboloid of aperture } \lambda \text{ by below at } x\},$$

and

$$A_\lambda = \complement D_\lambda.$$

From consequences (a) and (b), the function

$$g(x) = \sup_{x \in D_\lambda} \lambda$$

is a pointwise majorant for the supermum of all the second differential quotients of u centered at x and its distribution function is exactly $|A_\lambda|$; that is

$$|\{x \in B_1,\ g(x) < \lambda\}| \equiv |A_\lambda|.$$

Therefore if $g \in L^p$, all the $D_{ij}u$ are in L^p.

The consequences (a), (b), (c) set the strategy to prove the $W^{2,p}$ estimates: to study the distribution function of g, $|A_\lambda|$.

This will be accomplished in several steps:

(i) To show that if $x_0 \in D_1 \cap Q_{r_1}(x_1)$ then a fixed portion of $Q_{r_1}(x_1)$ is contained in D_M for a suitable M (Lemma 9).

(ii) Iteration of the result in (i) gives a polynomial decay for $|A_\lambda| : |A_{M^k}| < \varepsilon^k +$ term with f for a suitable $\varepsilon > 0$, (Corollary 2).

A simple consequence is that the second derivatives of u belong to L^q, with q small.

(iii) To force ε to be as small as we want: in this step we use the hypotheses on the coefficients of the operator and show that in $Q_{1/2}$ u has a tangent paraboloid of aperture $M/2$ (say) in as large a set as we wish (Lemma 12).

This step is carried over by means of approximation by solutions of the homogeneous problem (Lemma 11).

A variation of Lemma 2 is the following:

Lemma 9. *Let u be an element of $S(f)$ in $\Omega \supset B_1$. Assume that $D_1 \cap Q_{r_1}(x_1) \neq \phi$ and that $\|f\|_{L^n(B_1)} \leq \delta$ (small enough), then there exist M and μ, depending on $\Lambda, \lambda, r_1, r_2, \eta$, such that*

$$|D_M \cap Q_{r_2}(x_1)| \geq \mu |Q_{r_2}(x_1)| > 0.$$

Proof. Let $x_0 \in D_1 \cap Q_{r_1}(x_1)$. Subtracting from u a linear function, we may suppose that the paraboloid at x_0 is $1-|x|^2$. So $u \geq 0$ in B_1 and $u(x_0) = 1-|x_0|^2 \leq 1$ and we are under the hypotheses of Lemma 8. The conclusion follows easily. □

The next lemma shows that the sets $A_{M\lambda}$ and A_λ satisfy, for a proper choice of M, the hypotheses on A and B in Lemma 4.
Note that, if $\bar{u}(x) = \frac{u(x)}{\lambda}$, $A_M(\bar{u}) = A_{\lambda M}(u) = A_{\lambda M}$, so that, by Lemma 8,

$$|A_{\lambda M} \cap Q_{r_2}(x_1)| < (1-\mu)|Q_{r_2}(x)|.$$

Denote by Q, $\tilde{Q}$ a pair cube-predecessor in the Calderon-Zygmund decomposition of $A_{M\lambda}; Q, \tilde{Q} \subset B_{1./2}$.

Lemma 10. *Let u be an element of $S(f)$ in $\Omega \supset B_1$. Let δ be the small constant in Lemma 9. Assume that, for some $\lambda > 0$*

$$\fint_{\tilde{Q}} |f|^n \leq (\delta\lambda)^n.$$

Then there exists M, μ such that, whenever

$$(1) \qquad |Q \cap A_{\lambda M}| > (1-\mu)|Q|$$

then $\tilde{Q} \subset A_\lambda$.

Proof. Let Q satisfy (1) and μ the constant in Lemma 9. If h is the side length of $\tilde{Q}$, define

$$w(x) = \frac{u(hx)}{h^2 \cdot \lambda}.$$

Then w is defined in a large ball and $w \in S(\frac{f}{\lambda})$. Furthermore,

$$\begin{aligned} A_\lambda &= A_1(w), \\ A_{\lambda M} &= A_M(w), \\ |Q_{1/2} \cap A_M(w)| &> (1-\mu)|Q_{1/2}| \qquad \text{(by (1))}. \end{aligned}$$

Let $x_0 \in \tilde{Q}$, so that $y_0 = hx_0 \in Q_1$. If $y_0 \notin A_1(w)$, by Lemma 9 we should have

$$|D_M(w) \cap Q_{1/2}| \geq \mu |Q_{1/2}|,$$

or

$$|A_M(w) \cap Q_{1/2}| \leq (1-\mu)|Q_{1/2}|$$

which is a contradiction.
Therefore $Q_1 \subset A_1(w)$ or $\tilde{Q} \subset A_\lambda$. Notice that we have used implicitly Remark (b) after Lemma 8. □

Corollary 2. *Let $u \in S(f)$ in $\Omega \supset B_2\sqrt{n}$ with $f \in L^n$. Assume that*

$$Q_1 \cap D_{\lambda_0} \cap \{m(f^n) \le (\delta\lambda_0)^n\} \ne \phi$$

where $m(f^n)$ is the Hardy-Littlewood maximal function of f^n. (That is $\sup_{\chi\in Q} \fint_Q |f|^n$). Then, if $\varepsilon = 1-\mu$, denoting for simplicity $\overline{m}(f^n) = m(f^n \cdot \chi_{Q_1})$,

(a) $|A_{\lambda_0 M^k} \cap Q_1| =\le \varepsilon(|A_{\lambda_0 M^{k-1}} \cap Q_1| + |\{\overline{m}(f^n) \ge (\delta\lambda_0 M^{k-1})^n\}|)$

(b) $A|_{A_0 M^k} \cap Q_1| \le \varepsilon^k + \sum_{j=1}^k \varepsilon^{k-j+1}|\{\overline{m}(f^n) \ge (\delta\lambda_0 M^{j-1})^n\}|$.

Proof. Set

$$A = A_{\lambda_0 M^k} \cap Q_1 \;\; (= A_{\lambda_0 M^k} \text{ for simplicity})$$
$$B = A_{\lambda_0 M^{k-1}} \cup \{\overline{m}(f^n) > (\delta\lambda_0 M^{k-1})^n\}.$$

Apply Lemma 10 with $\lambda = \lambda_0 M^{k-1}$ to any subcube Q and its predecessor $\tilde{Q}$:

If $x_0 \in \tilde{Q}$ and $x_0 \notin B$, then $x_0 \in D_\lambda \cap \{\overline{m}(f^n) < (\delta\lambda)^n\}$. After a scaling we are under the hypotheses of Lemma 10 and conclude that

$$|A_{\lambda M} \cap Q| < (1-\mu)|Q|,$$

which is a contradiction. That $|A| < 1-\mu$ follows from Lemma 3 for $k = 1$.

So, A and B satisfy the hypotheses of Lemma 4, and therefore (a) is proven.

To prove (b) by induction we get

$$|A_{\lambda_0 M^k}| \le \varepsilon\Bigg[\varepsilon^{k-1} + \sum_{j=1}^{k-1} \sigma^{k-j}|\{\overline{m}(f^n) \ge (\delta\lambda_0 M^{j-1})^n\}| \\ + |\{\overline{m}f^n \ge (\delta\lambda_0 M^{k-1})^n\}|\Bigg],$$

which is (b). □

Corollary 3. *There exists γ such that for $t > 1$*

$$|A_{\lambda_0 t} \cap Q_1| \le c \cdot t^{-\gamma}.$$

Proof. Since $Q_1 \cap \{m(f^n) \le (\delta\lambda_0)^n\} \ne \phi$,

$$\int_{Q_1} f^n \le (\delta\lambda_0)^n.$$

Hence

$$|\{\overline{m}(f^n) \ge (\delta\lambda_0 t)^n\} \le \frac{c}{t^n}.$$

We then have in the formula above

$$\sum_{j=1}^k \varepsilon^{k-j+1}|\{\overline{m}(f^n)\cdots\}| \le c \cdot \sum_{j=1}^k \varepsilon^{k-j+1}(M^{j-1})^{-n} \\ \le c \cdot k \max(\varepsilon, M^{-n})^k \le c \cdot M^{-\gamma k}$$

with $\gamma = \gamma(\varepsilon, M)$ small.

Remark. Recall that $|A_\lambda|$ is the distribution function of $g(x) = \sup_{x\in D_\lambda} \lambda$. Hence

$$\|g\|^p_{L^p(Q_1)} \leq \sum_k |A_{\lambda_0 M^k} \cap Q_1| \cdot M^{kp}.$$

If $\|f\|_{L^p(Q_1)} < \delta$, $p > n$, then $m(f^n) \in L^{p/n}$ and $\|m(f^n)\|_{L^{p/n}} < c\cdot\delta$.

Therefore if we define

$$a_j = (\delta\lambda_0 M^{j-1})^p \cdot |\{\overline{m}(f^n) \geq (\delta\lambda_0 M^{j-1})^n\}|$$

we have

$$\sum_j a_j < c\cdot\delta^p.$$

We now substitute in the estimate (b) of Corollary 2:

$$|Q_1 \cap A_{\lambda_0 M^k}| \leq \varepsilon^k + \sum_{j=1}^{k} \varepsilon^{k-j+1}\cdot(\delta\lambda_0 M^{j-1})^{-1}\cdot a_j.$$

Therefore

$$\begin{aligned}\sum_k |A_{\lambda_0 M^k} \cap Q_1| \cdot M^{kp} \leq& \sum_k (\varepsilon M^p)^k \\ &+ \sum_k \sum_{j=1}^{k} (\varepsilon M^p)^{k-j+1}\cdot(\delta\lambda_0)^{-p}\cdot a_j\end{aligned} \tag{2}$$

and, if $\varepsilon M^p < 1$,

$$\|g\|^p_{L^p(Q_1)} \leq c\cdot\left(1 + (\delta\lambda_0)^{-p}\cdot\sum_j a_j\right) \leq c.$$

Conclusion. In order to show that $g \in L^p$ with a p large, according to how small $\beta(x)$ is, it suffices to show that we may keep M fixed and let ε go to zero. □

Here we need an approximation lemma. First an observation.

Remark. Let u be a viscosity supersolution of $F(D^2u, x) \leq f(x)$ and $\tilde\varphi$ in the class of test functions (C^2) satisfying

$$F(D^2\tilde\varphi, x) \geq g(x).$$

Then $u - \tilde\varphi \in \overline{S}\left(\frac{\lambda}{C}, C\lambda, |f-g|\right)$ with $C = C(n)$.

Proof. Let φ be in the class of test functions and satisfy

$$\mathcal{M}^-(\varphi) > |f-g|.$$

It is enough to show that

$$F\big(D^2(\tilde\varphi + \varphi), x\big) \geq f.$$

We decompose $D^2\varphi$ as $(D^2\varphi)^+ + (D^2\varphi)^-$.

Then

$$\begin{aligned}F\big(D^2(\tilde\varphi+\varphi),x\big) &= F(D^2\tilde\varphi,x)+F\big(D^2\tilde\varphi+(D^2\varphi)^+,x\big)-F(D^2\tilde\varphi,x)\\&\quad+F(D^2\tilde\varphi+D^2\varphi,x)-F\big(D^2\tilde\varphi+(D^2\varphi)^+,x\big)\\&\ge g+\lambda\|(D^2\varphi)^+\|-\Lambda\|(D^2\varphi)^-\|\\&\ge g+C|f-g|\\&\ge f.\end{aligned}$$

We now prove

Lemma 11. *Let u be a bounded solution of*

$$F(D^2u,x)=f(x)$$

in the $C^{1,1}$-viscosity sense, with $\|u\|_{L^\infty}\le 1$ and

$$\int_{B_1}\beta^n(x)dx\le\varepsilon^n.$$

Assume that $F(D^2w,0)=0$ has $C^{1,1}$ interior estimates (as in Theorem 1). Then there exists a $\gamma>0$ and a $C^{1,1}$ function h in $B_{1/2}$ with

$$\|h\|_{C^{1,1}(B_{1/2})}\le C(\lambda,\Lambda)$$

and

$$\|u-h\|_{L^\infty(B_{1/2})}\le C(\varepsilon^\gamma+\|f\|_{L^n}).$$

Proof. We construct h by solving the problem (see Ishii [I])

$$\begin{cases}F(D^2h,0)=0 \text{ in } B_{3/4}\\ h|_{\partial B_{3/4}}=u\end{cases}$$

From Theorem 4 in Section 2,

$$\|u\|_{B^\alpha(C_{3/4})}\le C(\|u\|_{L^\infty(B_1)}+\|f\|_{L^n(B_1)}).$$

Therefore h is of class $C^\alpha(\overline{B}_{3/4})$ and for any $\delta>0$

$$\|u-h\|_{L^\infty(\partial B_{3/4-\delta})}\le C\delta^\alpha.$$

From the hypothesis on interior estimates for $F(D^2w,0)=0$ properly scaled, we obtain

$$\|D_{ij}h\|_{L^\infty(B_{3/4-\delta})}\le C\delta^{\alpha-2}.$$

In particular

$$\|F(D^2h,x)\|_{L^\infty(B_{3/4-\delta})}\le\beta(x)C\delta^{\alpha-2}.$$

Hence $u-h$ belongs to $S^*(|f|+\beta(x)C\delta^{\alpha-2})$ and

$$\|u-h\|_{L^\infty(B_{3/4-\delta})}\le C(\delta^\alpha+\|f\|_{L^n}+C\delta^{\alpha-2}\|\beta(x)\|_{L^n}).$$

We now choose $\delta=\varepsilon^{1/2}$. $\square$

Remark. For h above, there exists an M_0, depending only in the interior $C^{1,1}$ estimates, for which

$$A_{M_0}(h) \cap B_{1/2} = \phi.$$

The following final lemma gives the proof of Theorem 1 (see remark after Corollary 3).

Lemma 12. *Assume $F(D^2u, x) = f$ in $B_1, |u| < 1$. Given $\varepsilon > 0$, there exist M independent of ε, and $\theta = \theta(\varepsilon)$ such that, if $\sup \beta(x) < \theta$, then*

$$|Q_{1/2} \cap A_M| < \varepsilon.$$

Proof. Let $\sup \beta(x) < \delta\theta_0, \delta$ small as in Lemma 9; we may suppose $\|f\|_{L^n} \leq \delta\theta_0^\gamma$.

Decompose u as $u - h + h = w + h$ where h is as in Lemma 7. From the previous remark

$$A_{2M_0}(u) \cap Q_{1/2} \subset (A_{M_0}(w) \cap Q_{1/2}) \cup (A_{M_0}(h) \cap Q_{1/2}) = A_{M_0}(w) \cap Q_{1/2}.$$

From Lemma 11:

$$|w| < c\theta_0^\gamma$$

and

$$\|F(D^2w, x)\|_{L^n(Q_{1/2})} \leq \|F(D^2u, x)\|_{L^n(Q_{1/2})} + |F(D^2h, x)|_{L^n(Q_{1/2})} \leq c \cdot \delta\theta_0^\gamma.$$

It follows that

$$\overline{w} = \frac{w}{c\theta_0^\gamma}$$

satisfies the hypotheses of Corollary 3.

Therefore

$$|A_t(\overline{w}) \cap Q_{1/2}| \leq c \cdot t^{-\gamma}$$

that is

$$|A_{c\theta_0^\gamma t}(w) \cap Q_{1/2}| \leq c \cdot t^{-\gamma}.$$

Choosing $t = \frac{M_0}{c\theta_0^\gamma}$ the lemma follows. □

5. Hölder estimates

In order to prove Hölder estimates we have to substitute the approximation lemma (Lemma 7 of Section 3) by a compactness argument.

We use uniqueness theorems for problem (*) to prove the following lemma. Uniqueness results for viscosity solutions are in Jensen [J]:

Lemma 13. *Let us fix the modulus of continuity γ of boundary data u_0 on ∂B_1 and the ellipticity constant Λ/λ of the operators $F(D^2, x)$.*

Then, given ε, there exist a $\delta = \delta(\varepsilon, \gamma, \Lambda/\lambda)$ such that if

$$\int_{B_1} \beta^n(x)dx \leq \delta^n$$

and

$$\int_{B_1} |f(x)|^n dx \leq \delta^n,$$

then two solutions u and w of respectively

$$F(D^2u, x) = f(x) \qquad \text{in } B_1$$
$$u = u_0 \qquad \text{on } \partial B_1$$

and

$$(*) \qquad F(D^2w, 0) = 0 \qquad \text{in } B_1$$
$$w = u_0 \qquad \text{on } \partial B_1$$

satisfy

$$\|u - w\|_{L^\infty(B_1)} \le \varepsilon.$$

Proof. If not we consider a uniformly convergent subsequence of u_k's, solutions of equations $F_k(D^2_{u_k}, x) = f_k(x)$, such that $f_k \to 0$, $\beta_k \to 0$ in $L^n(B_1)$. $F_k(\cdot, 0)$ locally uniformly convergent to $F(\cdot, 0)$ and show that the limit u_∞ is the solution of the problem

$$F(D^2w, 0) = 0 \text{ in } B_1, w = u_0 \text{ on } \partial B_1.$$

For this purpose given a C^2 supersolution φ in an open set $A \subset \Omega$ of

$$F(D^2\varphi, 0) \le 0.$$

We must show that $u_\infty - \varphi$ cannot have a local maximum. Suppose that in $A, |F_k(D^2\varphi, 0) - F(D^2\varphi, 0) < \varepsilon_k$, with $\varepsilon_k \downarrow 0$. We perturb φ to a supersolution of $F_k(D^2\tilde{\varphi}, x) \le -f_k^-(x)$, by adding to φ a solution, ψ_k, of the extremal equation

$$\sum \lambda_j^+(D^2\psi_k) + \varepsilon \sum \lambda_j^-(D^2\psi_k) = -|f_k(x)| - |\beta_k(x)| - \varepsilon_k \text{ in } B_1$$
$$\psi_k = 0 \qquad \text{on } \partial B_1.$$

Here $\varepsilon = \varepsilon(\max \|D^2\varphi\|, \Lambda/\lambda)$, where the λ_j^+ (resp. λ_j^-) are the positive (resp. negative) eigenvalues of $D^2\psi_k$.

Then if we separate the matrix in its positive and negative parts

$$M_k = D^2\psi_k = M_k^+ - M_k^-$$

we have

$$\|M_k^-\| \ge c \cdot \varepsilon^{-1}(|f_k| + |g_k| + \|M_k^+\| + \varepsilon_k).$$

Then, using the ellipticity condition and the above inequalities

$$\begin{aligned}
F_k(D^2[\varphi + \psi_k], x) &\le F_k(D^2\varphi, x) + \Lambda\|M_k^+\| - \lambda\|M^-\| \\
&\le F_k(D^2\varphi, 0) + c \cdot \beta_k(x) \cdot \|D^2\varphi\| + \Lambda\|M_k^+\| - \lambda\|M_k^-\| \\
&\le F(D^2\varphi, 0) + \varepsilon_k + c \cdot \beta_k(x) \cdot \|D^2\varphi\| + \Lambda\|M_k^+\| - \lambda\|M_k^-\| \\
&\le F(D^2\varphi, 0) + \varepsilon_k + c \cdot \beta_k(x)\|D^2\varphi\| + \Lambda\|M_k^+\| \\
&\qquad - \frac{c}{\varepsilon}(|f_k| + |\beta_k| + \|M_k^+\| + \varepsilon_k) \\
&\le -|f_k|
\end{aligned}$$

if ε is properly chosen.

Since $\varepsilon_k \downarrow 0$ and f_k and β_k go to zero in L^n, ψ_k goes to zero uniformly and we recover the maximum principle for u_∞. □

Proof of Theorem 2. We prove it by induction. By expanding eventually the variables $\tilde{u} = \lambda^{-2}u$, we may assume that $r_0 = 1$,

$$\sup_r \left(r^{-\alpha n} \int_{B_r} |f|^n \right)^{1/n} = \|f\|_{\alpha-1,n} \leq \delta(\alpha),$$

and that $\|u\|_{L^\infty} \leq 1$ (Note that F also changes).

Claim. There exists $\lambda = \lambda(\alpha)$ and a sequence of linear functions

$$\ell^k(x) = a^k + \langle b^k, x \rangle$$

such that

(3.5) $$\sup_{B_{\lambda^k}} |u - \ell^k| \leq (\lambda^k)^{1+\alpha}$$

and

(3.6) $$|a^k - a^{k-1}|, \lambda^k |b^k - b^{k-1}| \leq c\lambda^{k(1+\alpha)}.$$

Once the claim is proved we define

$$\ell = \lim_{n\to\infty} \ell^n \qquad \text{(uniform limit)},$$

which is well defined by (3.6).
Then if $r = \lambda^k$, we have, on B_r

$$\begin{aligned} |u - \ell| &\leq |u - \ell^k| + \sum_{k+1}^{\infty} |\ell^{s+1} - \ell^s| \\ &\leq c \cdot (\lambda^k)^{1+\alpha} = cr^{1+\alpha}. \end{aligned}$$

We have to prove the claim.

Choose $\ell^0 \equiv 0$ and assume the k^{th} step to be correct. Consider

$$w(x) = \frac{[u - \ell^k](\lambda^k x)}{\lambda^{k(1+\alpha)}}, \text{ for } x \in B_1.$$

We have:

$$F^k(D^2w, x) = \frac{\lambda^{2k} F(D^2u(\lambda^k x), \lambda^k x)}{\lambda^{k(1+\alpha)}} = \lambda^{k(1-\alpha)} f(\lambda^k x)$$

and

$$\|\lambda^{k(1-\alpha)} f(\lambda^k x)\|_{L^n(B_1)} \leq \|f\|_{\alpha-1,n}.$$

Applying Lemma 13 to w we obtain (with $F(D^2h) = 0$)

$$\|w - h\|_{L^\infty(B_{1/2})} \leq c \cdot \varepsilon(\delta_1 \theta).$$

If $\bar{\ell}(x)$ is the linear part of h at the origin, we get

$$\|w - \bar{\ell}\|_{L^\infty(B_\lambda)} \leq \varepsilon(\delta, \theta) + \lambda^{1+\bar{\alpha}} \qquad \left(\lambda \leq \frac{1}{2}\right)$$

since $|w| \le 1$.

We now choose first $\lambda(\alpha)$ and then δ, θ and hence $\varepsilon(\alpha)$ so that

$$\|w - \bar{\ell}\|_{L^\infty(B_\lambda)} \le \lambda^{1+\alpha}.$$

Rescaling, we get

$$\|u(x) - \ell^k(x) - \lambda^{k(1+\alpha)}\bar{\ell}\left(\frac{x}{\lambda^k}\right)\|_{L^\infty(B_{\lambda^{k+1}})} \le \lambda^{(k+1)(1+\alpha)}.$$

So, we put

$$\ell^{k+1}(x) = \ell^k(x) + \lambda^{k(1+\alpha)}\bar{\ell}\left(\frac{x}{\lambda^k}\right).$$

Therefore

$$|a_k - a_{k+1}| \le c \cdot \lambda^{k(1+\lambda)}(1 + \|f\|_{n-1,\alpha})$$

while

$$\lambda^k|b_k - b_{k+1}| \le c \cdot \lambda^{k(1+\alpha)}(1 + \|f\|_{n-1,\alpha})$$

since $\bar{\ell}$ is the linear part of h at the origin. □

Proof of Theorem 3. The hypotheses are dialation invariant, therefore we may assume, by expanding the variable X, that $\|u\|_{L^\infty} \le 1$ and that the constants controlling β and $f(x)$ are as small as we want, say δ.

Claim. There exists a$\lambda(\alpha)$, a $\delta(\alpha)$ and a family of second order polynomials $P_k = a^k + (b^k, x) + \frac{1}{2}x^tC^kx$ with

$$F(D^2P_k, 0) = 0$$

so that

(i) $\|u - P^k\|_{L^\infty(B_{\lambda^k})} \le \lambda^{k(2+\alpha)}$.
(ii) $|a^k - a^{k+1}|, \lambda^k|b^k - b^{k+1}|, \lambda^{2k}|C^k - C^{k+1}| < C \cdot \lambda^{k(2+\alpha)}$.

We start with $P^0 \equiv 0$ and prove the claim by induction. Set

$$w = \frac{[u - P^k](\lambda^k x)}{\lambda^{k(2+\alpha)}} \quad \text{for } x \in B_1.$$

We have, for $F^k(M, x) = \frac{1}{\lambda^{k\alpha}}F(\lambda^{k\alpha}M + C^k, \lambda^k x)$

$$\begin{aligned} F^k(D^2w, x) &= \frac{1}{\lambda^{\alpha k}}F(D^2u, \lambda^k x) \\ &= \frac{1}{\lambda^{\alpha k}}f(\lambda^k x) \end{aligned}$$

and

$$\begin{aligned} \beta_k(x) &= \sup \frac{|F^k(M, x) - F^k(M, 0)|}{|M|} \\ &= \frac{1}{\lambda^{\alpha k}}\beta(\lambda^k x). \end{aligned}$$

Therefore, if h is the solution of

$$\begin{cases} F(D^2h, 0) = 0 & \text{in } B_{3/4} \\ h = w & \text{in } \partial B_{3/4} \end{cases}$$

we get

$$\|w - h\|_{L^\infty(B_{1/2})} \leq C(\delta + \varepsilon^\gamma).$$

If $\overline{P}$ is the quadratic part of h at the origin,

$$\|w - \overline{P}\|_{L^\infty(B_\lambda)} \leq c \cdot [\delta + \varepsilon^\gamma + \lambda^{2+\overline{\alpha}}].$$

We first choose λ, then δ and ε so that

$$\|w - \overline{P}\|_{L^\infty(B_\lambda)} \leq \lambda^{2+\alpha}.$$

Rescaling back, we get

$$\|u - P^k - \lambda^{k(2+\alpha)}\overline{P}(\lambda^{-k}x)\|_{L^\infty(B_{\lambda^{k+1}})} < \lambda^{(k+1)(2+\alpha)}.$$

So we put

$$P^{k+1} = P^k + \lambda^{k(2+\alpha)}\overline{P}(\lambda^{-k}x).$$

Since $P^{k+1} - P^k = \lambda^{k(2+\alpha)}\overline{P}(\lambda^{-k}x)$ it is easy to prove (i) and (ii). Therefore, the sequence P^k converges uniformly to a polynomial P with the desired properties.□

PART 2
Geometric Properties of the Monge Ampère Equation

In the remain of these lectures we will study issues concerning solutions to the Monge Ampère equation, $\det D_{ij}u = f$.

This equation appears naturally in may problems of geometry, and applied mathematics as a "Lagrangian" version of Laplaces equation.

Laplaces equation, $\Delta u = 0$ represents for instance, a potential of an "irrational, incompressible velocity field $\vec{v}$" or a "conservative force field", while the Monge Ampère equation

$$\det D^2 u = 1$$

represents the potential of an "incompressible" map $Y(x)$, irrational in the Cauchy-Novozhilov sense.

While Laplaces equation is linear and invariant under dialations and rigid motions, Monge Ampère equation is invariant under dialations (with the appropriate renormalization) $\frac{1}{\lambda^2}u(\lambda x)$, rigid motions, and (with the proper renormalization *any affine transformation*: i.e., if u is a solution,

$$\frac{1}{(\det T)^{2/n}}u(TX)$$

is also a solution.

For instance, if u is a solution

$$u(\varepsilon x_1, \frac{1}{\varepsilon}x_2)$$

is also a solution, i.e. one may "stretch" the graph of u in one direction and "squeeze" it in the orthogonal one and again we get a solution, of the same type of equation.

Thus, it becomes almost unavoidable for this equation to have singular solutions.

In fact in three or more dimensions one can construct, following Pogorelov, examples of solutions which are only $C^{2-2/n}$ or even just Lipschitz.

The purpose of the first part of these lectures will be to develop an understanding of how these singular solutions form, and to give general geometric conditions under which solutions are expected to be regular.

As in all of elliptic P.D.E. the understanding of such a phenomena comes from studying the different invariant classes.

As mentioned in the introduction, in second order uniformly elliptic equations, the first class of invariances under rigid motions and dialations are equations with constant coefficients of which equations with regular coefficients are a perturbation, next invariant class is equations with bounded measurable coefficients and finally equations with BMO coefficients.

In minimal surface theory, we have hyperplanes, and next global minimal cones, of which any minimal surface is a perturbation.

Finally, in boundary value problems, the first class is half spaces, of which smooth domains are a perturbation and next is Lipschitz domains (and associated to it harmonic measures with doubling properties, a sort of "exponential" of BMO).

For the Monge Ampère equation, invariant under any affine transformation, the natural classes of right hand sides are constant, next measurable coefficients, bounded away from zero and infinity, and third measures doubling in any direction.

Our geometric discussion will be thus centered on convex viscosity solutions of

$$0 < \lambda_1 \leq Mu = \det D^2 u \leq \lambda_2$$

(we denote from now on $Mv = \det D^2 v$).

We recall that the inequality above is satisfied in the viscosity sense "u is convex (hence Lipschitz continuous in the interior) and for any C^2 convex function v with $u(X_0) = v(X_0)$, $u \leq v$ (resp. $u \geq v$) near X_0, we must have $Mv(X_0) \geq \lambda_1$ (resp. $M_v(X_0) \leq \lambda_2$)".

We also point out that a weak solution of (1) (in the sense that, for any open set $\mathcal{O}$

$$\lambda_1|\mathcal{O}| \leq |\{\nabla u(\mathcal{O})\}| \leq \lambda_2|\mathcal{O}|)$$

is also a viscosity solution of (1). (Here we remark that for any point $x \in \mathcal{O}$, $\nabla u(x)$ denotes the set of gradients of all supporting plains of u at x, namely,

$$\nabla u(x) = \{p \in R^n | u(y) \geq u(x) + p(y-x) \forall y\}.$$

Furthermore $\nabla u(\mathcal{O}) = \cup_{x \in \mathcal{O}} \nabla u(x)$.)

Indeed, we may arrange so that $u - v = 0$ at X_0 and $u - v < 0$ nearby, and apply the definition to $\mathcal{O} = \{u - v = -\varepsilon\}$ since $\{\nabla u(\mathcal{O})\} \subset \{\nabla v(\mathcal{O})\}$ and hence $|\{\nabla u(\mathcal{O})\}| \leq |\{\nabla v(\mathcal{O})\}| = \int_{\mathcal{O}} Mv < \lambda_1|\mathcal{O}|$ if $\varepsilon > 0$ is small enough, a contradiction.

Related results can be found in Urbas [U], where the history of the topic is also discussed. Further regularity theory can be found in [**G-T**].

A localization property

The purpose of this section is to show that given a solution u of the above inequality, the convex set where u is tangent to a linear function, if not a point, does not have interior extremal points in the domain of definition of u.

Lemma 1. *If u is convex in the domain $\Omega \subset B_1$, and*

a) $\det(D^2 u) \leq 1$

and

b) $u|_{\partial\Omega} \geq 0$.

then

$$u(X) \geq -Cd^{2/n}(X, \partial\Omega) \qquad (n \geq 3\ X \in \Omega).$$

Where $C = C(n)$ is some positive constant depending only on the dimension n.

Proof. We construct a lower barrier for u.

Let

$$h(X) = [|x'|^2 - C]y^{2/n}(X') = (x_1, \dots, x_{n-1}, y = x_n).$$

Then $D_{ij}h = 2\delta_{ij}y^{2/n}$ for $i < n$, $j < n$,

$$D_{in} = \frac{4}{n}x_i y^{\frac{2-n}{n}},$$

and

$$D_{nn}h = \frac{4(2-n)}{n^2}(|x'|^2 - C)y^{2\frac{(1-n)}{n}}.$$

Since the first $n-1$ minors are positive, it is enough to show that

$$\det D^2 h > 0$$

to show that h is convex.

In fact

$$\det D^2 h = 2^{n+1}\frac{(2-n)}{n^2}(|x'|^2 - C)$$
$$- |x'|^2 \frac{2^{n+2}}{n^2} > 1 \text{ on } \{|x'|^2 < 2\}$$

if $C(n)$ is chosen large enough.

We now use h as a lower barrier for u, choosing coordinates so that a given point X_0 on $\partial\Omega$ becomes the origin and $\{y = 0\}$ a supporting plane for $\partial\Omega$ at X_0. Notice that we have used the following fact which follows immediately from the definition of the viscosity solutions given above: Assume that u is convex in Ω, $\det(D^2u) \leq 1$ in the viscosity sense, $u|_{\partial\Omega} = 0$, h is C^2 inside $\Omega, \det(D^2h) > 1$, $h|_{\partial\Omega} \leq 0$. Then $u \leq h$ in Ω.

Lemma 2. *Let $u \leq 0$ on ∂B_1, $\det D^2 u \geq 1$. Then $\inf_{B_1} u \leq -1/2$.*

Proof. $u \leq \frac{1}{2}(|x|^2 - 1)$. Here it is easy to see that the right hand side is a barrier and the inequality follows as in the proof of Lemma 1.

Corollary 1. *If a) $0 < \lambda_1 \leq Mu \leq \lambda_2$ in Ω, with $B_1 \subset \Omega \subset B_K$.*
b) $u = 0$ on $\partial\Omega$.
c) $\frac{u(X_0)}{\inf u} \geq \delta > 0$.
Then distance $(X_0, \partial\Omega) \geq \mu(\delta, K, \lambda_1, \lambda_2) > 0$.

Theorem 1. *Assume $u \geq 0$ satisfies (1) and that the convex set $\{u = 0\}$ is not a point. Then $\{u = 0\}$ cannot have extremal points in the interior of the domain of definition of u.*

Remark. A well known example by Pogorelov shows that $\{u = 0\}$ may be a line.

Proof. Assume not.

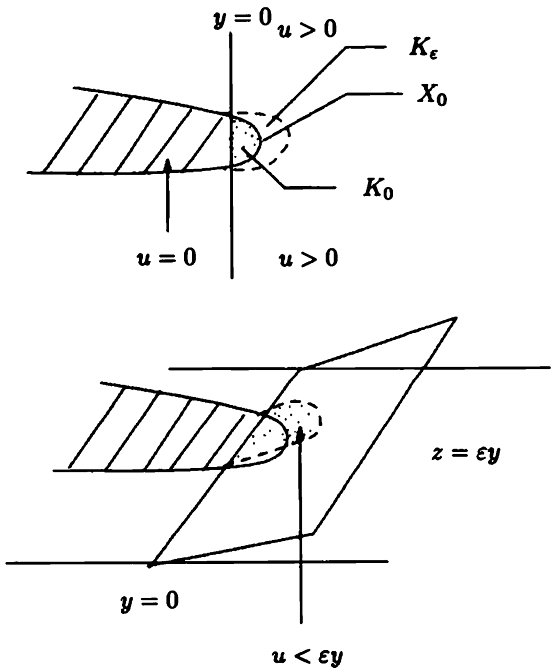
Then there is a half space, say $\{y \geq 0\}$, such that $\{y \geq 0\} \cap \{u = 0\} = K_0$ is compactly contained in the domain of definition of u, and $\theta \neq \{y = 0\} \cap \{u = 0\} \neq K_0$.

In particular the family

$$K\varepsilon = \{u \leq \varepsilon y\} \cap \{y \geq 0\}, \varepsilon > 0,$$

converges for ε going to zero to K_0 (see fig. 1).

Figure 1



Finally, let X_0 be the point in K_0 where y attains its maximum.

The set K_0 may have empty interior, but the (convex) set K_ε has non empty interior since on $K_0 \backslash \{y = 0\}$, $u < \varepsilon y$. We recall a lemma of F. John that says that any convex set, K, with non empty interior can be mapped by an affine transformation T into a new set $T(K)$ "equivalent" to the unit ball, i.e.

$$B_j \subset T(K) \subset B_n \qquad (n\text{-dimension}).$$

Thus, given a solution u of $\lambda_1 \leq Mu \leq \lambda_2$, and a convex "section",

$$K = \{u < \ell(x)\} \subset\subset \Omega, \quad (\ell \text{ a linear function}).$$

We can renormalize them into

$$K^* = T(K),$$

with

$$B_1 \subset K^* \subset B_n$$

and

$$(\tilde{T} = T^{-1}u^* = \frac{1}{(\det \tilde{T})^{2/n}}(u - \ell)(\tilde{T}(x))$$

solution in K^* of $\lambda_1 \le Mu^* \le \lambda_2$, with $u^* = 0$ on ∂K^*.

(In particular, Lemmas 1, 2 and Corollary 1 apply to u^*).

We now will center our attention on K_ε, $u_\varepsilon = u - \varepsilon y$, and their renormalizations

$$K_\varepsilon^*, u_\varepsilon^*.$$

and two quantities invariant under renormalization:

Ratios of u_ε, i.e.

$$\frac{u_\varepsilon(X)}{u_\varepsilon(Y)} = \frac{u_\varepsilon^*(T_\varepsilon X)}{u_\varepsilon^*(T_\varepsilon Y)}$$

and ratios between distances of parallel planes and their images

$$\frac{d(\Pi_1, \Pi_2)}{d(\Pi_1, \Pi_3)} = \frac{d(\Pi_1^*, \Pi_2^*)}{d(\Pi_1^*, \Pi_3^*)}.$$

The first observation is that

$$\frac{u_\varepsilon^*(X_0^*)}{\inf_{K_\varepsilon^*} u_\varepsilon^*} = \frac{u_\varepsilon(X_0)}{\inf_{K_\varepsilon u_\varepsilon}} \ge \frac{\varepsilon y_0}{\varepsilon \sup_{K_\varepsilon} y} \ge \frac{1}{2}$$

for ε small, since

a) K_ε converges to K_0,

and

b) $y_0 = \sup_{K_0} y$.

Next, we consider the three parallel planes

$$\begin{aligned} \Pi_1 &= \{y = 0\} \\ \Pi_2 &= \{y = y_0\} \\ \Pi_3^\varepsilon &= \{y = \sup_{K_\varepsilon} y = y_0^\varepsilon\}. \end{aligned}$$

The ratio

$$\frac{d(\Pi_1, \Pi_3^\varepsilon)}{d(\Pi_1, \Pi_3^\varepsilon)} = \frac{y_0^\varepsilon - y_0}{y_0^\varepsilon}$$

is preserved under the transformation T_ε.

But Π_1^* and $\Pi_3^{\varepsilon *}$ are parallel, opposite supporting planes to K_ε^* and $B_1 \subset K_\varepsilon^* \subset B_n$.

Hence $2 < d(\Pi_1^*, \Pi_3^{\varepsilon *}) < 2n$. It follows that

$$d(\Pi_2^*, \Pi_3^{\varepsilon *}) \le 2n \left(\frac{y_0^\varepsilon - y_0}{y_0^\varepsilon} \right)$$

becomes as small as we please.

In particular

$$d(X_0^*, \partial K_\varepsilon^*) \le d(\Pi_2^*, \Pi_3^*)$$

also becomes as small as we please. This contradicts Corollary 1, since $\frac{u^*(X_0)}{\inf U^*} \geq 1/2$.

Corollary 2. *Assume that u is strictly convex, then u has a unique supporting plane at each point.*

Proof. If not we may assume that

a) $u \geq \alpha x_n^+$ $(\alpha > 0)$,
b) $u(0) = 0$,
c) $u(-te_n) = 0(t)$ $(t > 0)$.

We now consider the auxiliary function

$$u_{\sigma,\tau} = u - \tau(X_n + \sigma).$$

From the strict convexity of u, the set $u_{\sigma,\tau} \leq 0$ becomes compactly contained in the domain of definition of u for σ, τ positive and small.

Also for $\tau < \alpha, u_{\sigma,\tau}$ attains its minimum at $x = 0$.

Finally, we estimate the location of the two supporting planes to the set

$$u_{\sigma,\tau} \leq 0$$

of the form $\Pi_1 = \{x_n = C^-\}, \Pi_3 = \{x_n = C^+\}$.

On the one hand, for $x_n > 0$

$$u_{\sigma,\tau} \geq (\alpha - \tau)x_n - \tau\sigma \geq 10$$

$$\text{for } x_n > \frac{\tau\sigma}{\alpha - \tau}. \text{ Hence}$$

$$C^+ < \frac{\tau\sigma}{\alpha - \tau}.$$

On the other hand, for $x = -te_n$, we choose σ small and set

$$\tau = \frac{2}{\sigma} u\left(-\frac{1}{2}\sigma e_n\right)$$

(note that $\tau = 0(1)$ for σ going to zero).

For $t < \frac{\sigma}{2}$, we get

$$\begin{aligned} u_{\sigma,\tau}(-te_n) &= u(-te_n) \\ &\quad - \frac{2u(\frac{1}{2}\sigma e_n)}{\sigma}(-t + \sigma) \\ &\leq u(-te_n) - u\left(-\frac{\sigma}{2}e_n\right) \\ &\leq 0. \end{aligned}$$

Hence $C^- \geq -\frac{\sigma}{2}$.

As in the proof of Theorem 1, we get a contradiction from the fact that $\frac{C^+}{C^-}$ goes to zero, by considering the three planes Π_1, Π_3 and $\Pi_2 = \{x_n = 0\}$. □

Corollary 3. *Let u be a (convex) solution of*

$$0 < \lambda_1 \leq \det D^2 u \leq \lambda_2$$

in Ω bounded,

$$u|_{\partial\Omega} = f \text{ be continuous}$$

and $\Gamma(f)$ be the convex envelope of f in Ω, that is

$$\Gamma(f) = \sup_{\substack{L \leq f \text{ on } \partial\Omega \\ L \text{ affine}}} L.$$

Finally, assume that

$$u(X_0) < \Gamma(f)(X_0).$$

Then u is strictly convex at X_0 with a strict "modulus of convexity"; that is, if L_{X_0} is the plane of support of u at X_0,

$$\operatorname{diam}\{u \leq L_{X_0} + \rho\} \leq \sigma(\rho)$$

where $\sigma(o^+) = 0$ and σ depends only on the modulus of continuity of f, $\Gamma(f)(X_0) - u(X_0)$, $\operatorname{dist}(X_0, \partial\Omega)$, and on λ_1, λ_2.

Proof. For $\varepsilon_0 > 0$ fixed, we define for $\rho > 0$

$$\sigma(\rho) = \sup \operatorname{diam}\{u \leq L_{X_0} + \rho\}$$

where sup is taken over all $X_0 \in \Omega$, $\operatorname{dist}(X_0, \partial\Omega) \geq \varepsilon_0$, u belongs to the "admissable class" as in the statement of Corollary 2 (remember that f is controlled by a fixed modulo of continuity). We claim that the σ defined above satisfies $\sigma(0^+) = 0$ and therefore we have established Corollary 2. By contradiction, if not, then for some $\delta_0 > 0, \sigma(0^+) \geq \delta_0 > 0$, therefore there is a sequence u_k covering to a limiting function u_∞ for which

$$\{u_\infty - L_0 = 0\} \text{ is not a point.}$$

Since this set cannot have extremal points, this implies

$$u_\infty(X_0) = \Gamma(f_\infty)(X_0)$$

a contradiction. (Note that X_0 cannot be on $\partial\Omega_\infty$, since, f being uniformly continuous,

$$u - \Gamma(f)$$

goes uniformly to zero when X converges to $\partial\Omega$). □

As an application to the Minkowski problem, we state

Corollary 4. *Let Γ be a convex set in R^n, (normalized so that $B_1 \subset \Gamma \subset B_n$). Assume that for $\mathcal{G}$ the Gauss curvature in the viscosity sense*

$$0 < \lambda_1 \leq \mathcal{G} \leq \lambda^2.$$

Then Γ is a strictly convex, C^1 surface, the strict convexity and the C^1 modulus of continuity depending only on λ_1, λ_2.

Proof. We recall that being of bounded Gauss curvature translates locally, in Euclidean coordinates, in the surface, as a graph $x_n = u(x_1, \dots, x_{n-1})$, satisfying a Monge Ampère equation with bounded right hand side.

Suppose, by contradiction that we can find a limiting surface Γ_∞, that satisfies the hypothesis of the Corollary and still has a nontrivial piece of hyperplane contained on its graph.

Locally, such a portion cannot have extremal points, thus it is unbounded.

Corollary 5. *If $\partial\Omega$ is $C^{1,\alpha}$ and f is $C^{1,\alpha}$ for $\alpha > 1 - \frac{2}{n}$, then u is strictly convex on Ω.*

Proof. If u is not strictly convex it must coincide with a supporting plane along a segment $[X_1, X_2]$ with $X_i \in \partial\Omega$.

By substracting the hyperplane we may assume that $u \geq 0$ in Ω and $u \equiv 0$ on $[X_1, X_2]$.

By expanding and rotating, assume $X_1 = -e_n$, $X_2 = e_n$, and write $X = (x^1, x_n) = (x_1, \dots, x_{n-1}, x_n)$.

Then, near X_i, $u \leq C|x^1|^{1+\alpha}$ and thus, by convexity, the same is true in the cylinder

$$C = \{|x^1| < \varepsilon\} \times \{|x_n| < 1\}.$$

We construct an upper barrier:

$$h = C[\varepsilon^{1+\alpha}|x_n|^2 + \varepsilon^{\alpha-1}|x^1|^2].$$

Indeed, $Mh = C\varepsilon^{n\alpha-n+2} = C\varepsilon^\theta$ for $\theta > 0 \quad (\theta = n(\alpha - (2 - \frac{2}{n}))$, so

a) If C is large $h > u$ on $\partial\xi$

b) If ε is small h is a strict supersolution

and

c) $h(0) = 0 = u(0)$.

A contradiction.

PART 3
A Priori Estimates of solutions to Monge Ampére Equations

The main purpose of this chapter is to show that solutions of the Monge Ampère inequality

$$0 < \lambda_1 \le \operatorname{Det} D_{ij}u \le \lambda_2 < \infty$$

when strictly convex are $C^{1,\alpha}$.

In fact, the result applies to solutions of

$$\operatorname{Det} D_{ij}u = d\mu$$

with μ an appropriately invariant measure.

Although elementary, this result allows us to avoid the $W^{2,p}$ theory developed in Part 4 (see also [C-2]), when studying problems of Minkoski type or those recently announced by David Jerison [Je], on gauss curvature and harmonic measure. It also implies partial regularity results for the rearrangement of C^1 vector fields as treated by Brenier in [B]. This subject will be treated later.

The correct invariant measure

In [Je], Jerison points out that in order to carry on the methods of the previous section for an equation of the type

$$\det D_{ij}u = d\mu,$$

one needs the following "doubling like" geometric property of μ:

Property $P1$: Let Ω be a section of the graph of u, i.e. a set of the form $\Omega = \{x/(u-\ell)(x) < 0 \text{ for some linear function } \ell\}$.

Then

$$\mu(\Omega) \le C\mu\left(\frac{1}{2}\Omega\right)$$

(taking as origin the center of mass of Ω).

In fact Jerison proves that Harmonic measure on graph u has this property.

This is exactly what is needed to prove the following variation of the strict convexity theorems in the previous section.

Theorem 1. *Let u be a (locally Lipschitz, convex) solution of*

$$\det D_{ij}u = d\mu$$

in the Alexandrov weak sense. Assume that μ has property P1. Then if ℓ_{X_0} is a supporting linear function to u at X_0 we have

(i) $$\{u = \ell_{X_0}\} = \{X_0\} \text{ or}$$

(ii) $$\{u = \ell_{X_0}\}$$

has no extremal points in the domain of definition of u.

Proof. The proof follows exactly that of the previous section. Since condition P_1 is invariant under affine transformations of the variable X and homogeneous in u, it is enough to prove the following normalized statement.

Lemma 1. *("u^* goes to zero uniformally near $\partial\Omega$"). Let u^* be a (weak, convex) solution of* $\det D_{ij}u^* = d\mu^*$ *on Ω^* $u \equiv 0$ on $\partial\Omega^*$.*

$B_1 \subset \Omega^* \subset B_R$ with center of mass at zero.

Suppose that for some $0 < \delta$, $\lambda < 1$, $\mu^*(\lambda\Omega^*) \geq \delta\mu^*(\Omega^*) > 0$.

Then

$$|u^*(X)| \leq Cd^\alpha(X, \partial\Omega^*)|\inf_{\Omega^*} u^*|$$
$$\sim Cd^\alpha(X, \partial\Omega^*)\mu^*(\Omega^*),$$

with C, α depending only on δ, λ, R.

Proof of Lemma 1. On one hand, by repeating carefully classical Alexandrov estimates ([**A**] Theorem B) as in Lemma 1 of "preliminary tools",

$$|u(X_0)|^n \leq Cd(X_0, \partial\Omega)\mu^*(\Omega^*)$$

(with $C = C(R)$).

Indeed note (from fig. 2), that any vector v, with components (in the appropriate system of coordinates)

$$|v'| \leq Cu(X_0),$$
$$-C\frac{u(X_0)}{d(X_0, \partial\Omega)} < v_n < 0$$

belongs to image $\nabla u(\Omega)$, since the plane

$$\langle u, x\rangle + \lambda$$

will hit the graph of u in the interior of Ω when λ grows from $-\infty$. (see fig. 2)

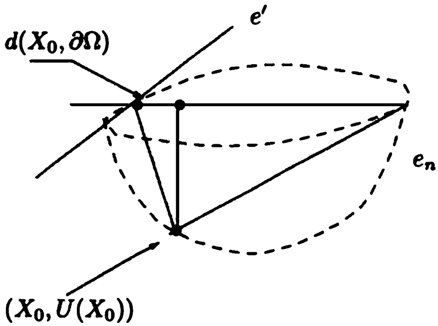
On the other hand, since on $\lambda\Omega^*$, $(1-\lambda)|\nabla u(X)| \leq |u(X)| \leq |\inf_{\Omega^*} u|$, we have $\mu^*(\lambda\Omega^*) = \mathrm{Vol}\left(\mathrm{Image}\nabla u(\lambda\Omega^*)\right) \leq (1-\lambda)^{-n}|\inf_{\Omega^*}(u)|^n$.

This completes the proof of Lemma 1, and the proof of Theorem 1 follows.

In fact, one can prove

Theorem 1$'$. *Let u be a uniform limit of solutions u_k as in Theorem 1, Then, same conclusion holds for u.*

Figure 2



"any vector v with $|v'| \le Cu(X_0)$ and

$$-\frac{u(X_0)}{2d(X_0,\partial\Omega)} \le v_n \le 0$$

$$|v'| \le \frac{1}{2}u(X_0)$$

will belong to the image of $\nabla u(\Omega)$".

Proof of Theorem 1′. The proof of Theorem 1 consists of renormalizing appropriate sections Ω_ε of the graph of u, so contradicting Lemma 1 for ε small. If u is now the limit of a sequence u_k, the section Ω_ε is the uniform limit of the corresponding sections $\Omega_\varepsilon(u_k)$. (Indeed, the section Ω_ε is open, i.e. $\Omega_\varepsilon = \{u < \ell_\varepsilon\}$, and by convexity u is transversal, i.e. $|\nabla(u - \ell_\varepsilon)| > 0$, along $\partial\Omega_\varepsilon$). We then get the same contradiction to Lemma 1 from the renormalization of $\Omega_\varepsilon(u_k)$ for k large enough.

We next show how a careful normalization of the argument in proving strict convexity implies $C^{1,\alpha}$ regularity.

The main lemma is the following.

Lemma 2. *Let u be a solution of*

$$\det D_{ij}u = d\mu \text{ on } \Omega$$

normalized as follows

a) $u = 1$ *on* $\partial\Omega, B_1 \subset \Omega \subset B_n$

b) $\inf_\Omega u = u(X_0) = 0$

c) μ *satisfies property* $P1$ *and hence* $\mu(\Omega) \sim 1$ *(from Lemma 1).*

Let h_α be the cone generated by X_0 and the level surface $u = \alpha$, i.e.

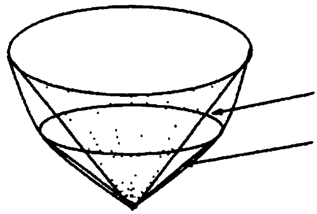
a) $h_\alpha(X - X_0)$ is homogeneous of degree one and

b) $h_\alpha(X - X_0) = \alpha$ for $X \in \{u = \alpha\}$.

Then, there exists a $\delta < 1$, such that

$$h_{1/2}(X - X_0) < \delta h_1(X - X_0).$$

Figure 3



"$h_{1/2}(X) \leq \delta h_1(X)$ for some universal $\delta < 1$."

(see fig. 3).

Proof. The lemma follows by compactness, i.e. Theorem 1′.

Indeed, consider a sequence u_k for which

$$\sup_{X \neq X_0} \frac{h_{1/2}(X - X_0)}{h_1(X - X_0)} \geq 1 - 1/k.$$

From Lemma 1, $X_0, \{u_k = 1/2\}$ and $\{u_k = 1\}$ stay uniformly away from each other. In particular $0 < C_1 < C_2$.

$$C_1|X - X_0| \leq h_{1/2}(X - X_0) \leq h_1(X - X_0) \leq C_2|X - X_0|.$$

uniformly in k.

We may hence choose a subsequence for which $\{X_0\} = \{u_k = C\}$ remains fixed, $\Omega_k = \{u_k = 0\}$ converge to $\Omega = \{u = 1\}$ and u_k converges uniformly to u in any compact subset of Ω.

The graph of u has, then, a segment starting at X_0, that contradicts Theorem 1′.

This completes the proof.

It is now easy to prove the $C^{1,\alpha}$ regularity of a strictly convex solution u.

Theorem 2. *A strictly convex solution u of*

$$\det D_{ij}u = d\mu$$

($d\mu$ satisfying P_1) is $C^{1,\alpha}$, with the $C^{1,\alpha}$ norm depending only on the Lipschitz norm of u, and on its strict convexity.

Proof. Subtract from u is supporting plane at $X_0 = 0$. The conclusion of Lemma 2 is invariant under affine transformations, and therefore, it follows after renormalization of the level surface $u = 2^{-k}$ and iteration of the lemma that

$$h_{2^{-k}} \leq \gamma^k h_1.$$

Since u is Lipschitz,

$$h_1(X) \leq C(X).$$

Define α, by $2^{-\alpha} = \gamma$.

Then

$$h_{2^{-k}}(X) \le (2^{-k})^{\alpha}|X| \le C2^{-k}.$$

provided that

$$|X| \le (2^{-k})^{(1-\alpha)}.$$

Since $u(X) \le h_{2^{-k}}(X)$ as long as $h_{2^{-k}}(X) \le 2^{-k}$, it follows that $u(X) \le 2^{-k}$ provided that

$$|X| \le C2^{-k(1-\alpha)}.$$

This allows to estimate the slope, s, of a supporting plane at say

$$|Y| \le \frac{1}{2}C2^{-k(1-\alpha)}$$

by

$$|s| \le C2^{-k\alpha} \le |X|^{1-\alpha}.$$

This is because the vertical component of such a plane must remain smaller than 2^{-k} on the ball of radius $|Y|$ around Y.

PART 4

Interior $W^{2,p}$ Estimates for Solutions of the Monge Ampère Equation

In this chapter we adapt the techniques developed in Part 1 to prove interior estimates for solutions of perturbations of Monge Ampère equation.

Our two main results are as follows:

Theorem 1. *Let u be a (convex) viscosity solution of*

$$Mu = \det D_{ij}u = f$$

on the convex set $\Omega \subset R^n$.

Normalize Ω so that

$$B_1 \subset \Omega \subset B_n.$$

(see Lemma 1 below) and assume further $u|_{\partial\Omega} \equiv 0$.

Then:

a) Given $p < \infty$, there exists an $\varepsilon = \varepsilon(p)$ such that if

$$|f - 1| \leq \varepsilon$$

then

$$u \in W^{2,p}(B_{1/2})$$

and

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C(\varepsilon).$$

b) If f is continuous and strictly positive, $u \in W^{2,p}(B_{1/2})$ for any $p < \infty$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C(p, \sigma)$$

with σ the modulus of continuity of f.

The previous result in this direction is due to Pogorelov **[P1]** but valid only for $n = 2$.

A simple consequence of our techniques is

Theorem 2. *If $f \in C^\alpha$, then $u \in C^{2,\alpha}$.*

In this direction, Urbas has similar results provided that f is Lipschitz and $D_{ij}u$ satisfies some integrability conditions (see **[U]**).

1. Preliminary Results

We first recall a result on solvability and a few *a priori* estimates.

Let Ω be a convex domain in R^n. We state a normalization theorem of John Cordoba and Gallegos (see [dG]).

Lemma 1. *Let Ω be a convex bounded set with non-empty interior. Let E be the ellipsoid of smallest volume containing Ω.*

Then $\frac{1}{n}E \subset \Omega$. In particular after an affine transformation $T, B_1 \subset T(\Omega) \subset B_n$.

We also recall some interior estimates and solvability of Monge Ampère equation (see [G-T]).

Theorem 3. *Let Ω be a strictly convex smooth (normalized) domain in R^n (that is, $B_1 \subset \Omega \subset B_n$).*

Let $f > 0, g$ be smooth (C^4) functions on $\Omega, \partial\Omega$, then the following problem has a unique smooth (C^2) solution w:

$$\begin{cases} w & \text{convex in } \Omega \\ w|_{\partial\Omega} = g & \\ \det D_{ij}(w) = f & \text{in } \Gamma. \end{cases}$$

Furthermore, if $f \equiv C, g \equiv 0$,

$$\|w\|_{C^{2,1}}(\Omega_{-\delta}) \leq C(\delta) \quad (\Omega_{-\delta} = \{x \in \Omega / d(X, \partial\Omega) > \delta\})$$

with $C(\delta)$ **independent** *of the smoothness of Ω.*

Proof of the last estimate (Pogorelov's).

By taking a slightly smaller section ($\Omega' = \{X | w < -C\delta^n\}$) we may assume that ∇w is bounded in Ω', and by taking a multiple, we assume

$$|w| < 1.$$

Thus,

$$|\nabla w| \leq C\frac{\inf w}{\delta} \leq \frac{C}{\delta}.$$

We bound the maximum of

$$\begin{aligned} &(C\delta^n - w)D_{11}we^{1/2}(D_w)^2 \\ &= |u|D_{11}ue^{1/2(D_1u)^2} = e^h \end{aligned}$$

by a constant M, depending only on $\sup|u|$ and $\sup|D_1u|$.

Let M be such a maximum, taken at an interior point, say zero.

We repeat Pogorelov's computation:

By an affine transformation

$$\bar{x}_1 = x_1 - \frac{D_{1i}u(0)}{D_{11}u(0)}x_i$$

which leaves h invariant, we may assume that $D_{ij}u$ is diagonal at zero.

Since $h = \log u_{11} + \frac{1}{2}u_1^2 + \log(-u)$ has a maximum at zero, the linearized operator $L = \sum \frac{1}{u_{ii}(0)} D_{ii}h$ cannot be positive, i.e.

$$Lh \leq 0.$$

That is

$$\begin{aligned}&\frac{1}{u_{11}}L(u_{11}) - \frac{1}{(u_{11})^2}\left(\frac{1}{u_{ii}}(u_{112})^2\right)\\ &+ u_1 L u_1 + \frac{(u_{1i})^2}{u_{ii}} + \frac{1}{u}Lu - \frac{(u_i)^2}{u_{ii}} \leq 0.\end{aligned}$$

Now

$$Lu = n, Lu_1 = 0, Lu_{11} = \frac{(u_{1k\ell})^2}{u_{kk}u_{\ell\ell}}.$$

Thus, we get

$$\frac{1}{u_{11}}\left[\frac{(u_{1k\ell})^2}{u_{kk}u_{\ell\ell}}\right] - \frac{1}{(u_{11})^2}\left(\frac{1}{u_{ii}}(u_{11i})^2\right) + u_{11} + \frac{n}{u} - \frac{(u_i)^2}{u_{ii}} \leq 0.$$

We now use the factors $k = 1$ to cancel the i terms, i.e.

$$\frac{1}{u_{11}}\sum_{k\neq 1}\frac{(u_{1k\ell})^2}{u_{kk}u_{\ell\ell}} + u_{11} + \frac{n}{u}\frac{(u_i)^2}{u_{ii}} \leq 0$$

and, to control $\frac{(u_i)^2}{u_{ii}}$, we use that, at zero

$$\frac{u_{11i}}{u_{11}} + u_1 u_{1i} + \frac{u_i}{u} = 0.$$

For $i \neq 1$, this means

$$\frac{u_i^2}{u_{ii}} = \frac{u^2(u_{11i})^2}{(u_{11})^2 u_{ii}}.$$

Since $|u| < 1$, this leaves us with

$$u_{11} \leq \frac{u_1^2}{u_{11}} + \frac{n}{u},$$

or multiplying by $u^2 u_{11} e^{(u_1)^2}$ □

$$M^2 \leq \frac{(u_1)^2}{(u_{11})^2}M^2 + e^{1/2(u_1)^2}nCM.$$

Finally, an approximation lemma on the image of the gradient mapping:

Lemma 2. *Let $\delta > \varepsilon > 0$, and u, v two convex functions in B_1 with $\|u - v\|_{L^\infty} \leq \varepsilon$. Then (with $N_{2\delta}$ the 2δ-neighborhood)*

$$\{\nabla u|_{B_{1-\varepsilon/\delta}}\} \subset N_{2\delta}\{\nabla v|_{B_1}\}.$$

Proof. Consider $v^* = v + \varepsilon - \delta(1 - r^2)$.
Then
$$v^*|_{\partial B_1} \geq u$$
and
$$v^*|_{B_{1-\varepsilon/\delta}} \leq u.$$
Therefore any plane of support of u in $B_{1-\varepsilon/\delta}$ is a plane of support of v^* in B_1.
That is
$$\left\{\nabla u|_{B_{1-\varepsilon/\delta}}\right\} \subset N_{2\delta}\left\{\nabla v|_{B_1}\right\}.$$

Remark. (Pogorelov, see [P2]. If $u_m \to u$ uniformly on compact sets of B_1,
$$\text{meas}\left(\nabla u(B_1)\right) \leq \underline{\lim}\,\text{meas}\left(\nabla u_n(B_1)\right).$$

Proof.
$$\nabla u(B_1) \subset \left[\underline{\lim}(\nabla u_n(B_1))\right] \cup \mathcal{D}$$
where $\underline{\lim}(\nabla u_n(B_1)) = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} (\nabla u_k(B_1))$ and $\mathcal{D}$ is the set of
$$\{\vec{v} = \nabla L\}$$
with L a supporting function that coincides with u in a segment $[X_1, X_2]$ with $|X_1| < 1.$, $|X_2| = 1$.
We show that
$$\mathcal{D}_\varepsilon = \mathcal{D} \cap \nabla u(B_{1-\varepsilon})$$
has no Lebesgue density point.
We may assume $L = 0$, and $X_2 = -e_1$.
In particular $u \geq 0$ on B_1 and $u = 0$ on $-e_1$.
In particular, for any X in $B_{1-\varepsilon}, u(y) \geq u(X) + \langle\nabla u(X), y - X\rangle$ for any $y \in B_1$. Taking $y = -e_1$ and noticing that $u(-e_1) = 0 = 0, u(X) \geq 0$, we know that $(\nabla u(X), X + e_1)$ must be nonnegative.
Since $|X| < 1$ and $\langle X + e_1, e_1\rangle > \varepsilon$, if u_1 is negative, it must satisfy $\varepsilon|u_1| < 2|\nabla u|$.
That excludes a cone (of aperture $\sim \varepsilon$) around $-e_n$, and 0 is not a density point for $\mathcal{D}_\varepsilon$. Since we have shown that $\mathcal{D}_\varepsilon$ has no Lebesgue density point, so $\mathcal{D}_\varepsilon$ has zero Lebesgue measure and therefore $\mathcal{D} = \cup_{k=2}^{\infty}\mathcal{D}_{1/k}$ also has zero Lebesgue measure. It follows immediately that
$$\text{meas}\left(\nabla u(B_1)\right) \leq \text{meas}\left(\underline{\lim}(\nabla u_n(B_1))\right) \leq \underline{\lim}\,\text{meas}\left(\nabla u_n(B_1)\right).$$

2. Approximation lemmas

We first recall the definitions of weak and viscous solutions, and compare them.

Definitions. Let $f \in L^\infty, u$ be a convex function in Ω. Then (see Pogorelov [**P2**])

a) u is a weak solution of $Mu = f$ if the Jacobian of the transformation

$$T : X \to \nabla u,$$

coincides, as a measure, with f, that is

$$\text{meas}\, \nabla u(\mathcal{O}) = \int_{\mathcal{O}} f dx.$$

b) u is a viscosity solution of $Mu = f$, as before, if it is a sub and super solution, that is: u is convex and for any $C^{2,\alpha}$ convex function v, with $u(X_0) = v(X_0), u \le v$ (resp. $u \ge v$) near X_0, we have $Mv(X_0) \ge f(X_0)$ (resp. $Mv(X_0) \le f$).

We should remark that the above definition is valid if f is continuous. If f is not continuous, we only care about the case when $|f - 1| \le \varepsilon$ and $Mu = f$ should be understood as $1 - \varepsilon \le Mu \le 1 + \varepsilon$ and this is what we really need.

Lemma 3.

a) *A weak solution is a viscosity solution*

b) *if $f < f_0$ (a constant) in a ball B_ε and u is a viscous solution*

$$\text{meas}(\nabla u|_{B_\varepsilon}) \le f_0 |B_\varepsilon.$$

(In particular, both definitions are equivalent for continuous f).

Proof. a) Suppose that $u - v$ has a strict local maximum at X_0. We may assume $u > v$ in a neighborhood $\mathcal{O}$ and $u = v$ on $\partial\Omega$. Then

$$\nabla u(\mathcal{O}) \subset \nabla v(\mathcal{O}) \text{ and}$$
$$\text{meas}\left(\nabla u(\mathcal{O})\right) \le \text{meas}\left(\nabla v(\mathcal{O})\right).$$

On the other hand

$$\begin{aligned}\text{meas}\left(\nabla u(\mathcal{O})\right) &= \int_{\mathcal{O}} Mu \int_{\mathcal{O}} Mv \\ &= \text{meas}\left(\nabla v(\mathcal{O})\right),\end{aligned}$$

a contradiction.

b) Heuristically, we compare in B_ε, u with w with

$$w\Big|_{\partial B_\varepsilon} = u\Big|_{\partial B_\varepsilon}$$

and $Mw = f_0$. Then $w \le u$ and

$$\text{meas}\left(\nabla u(B_\varepsilon)\right) \le \text{meas}\left(\nabla w(B_\varepsilon)\right) = f_0|B_\varepsilon|.$$

Since w is not necessarily smooth, we smooth the boundary values:

$$\begin{cases} w_\sigma\Big|_{\partial B_\varepsilon} = u_\sigma\Big|_{\partial B_\varepsilon} & \text{a smoothing of } u \\ Mw_\sigma = f_0 \end{cases}$$

where u_σ is chosen so that $u_\sigma|_{\partial B_\varepsilon} \le u|_{\partial B_\varepsilon}$.

We compare u with w_σ in B_ε where

$$w_\sigma|_{\partial B_\varepsilon} = u_\sigma|_{\partial B_\varepsilon} \le u|_{\partial B_\varepsilon}$$

and $Mw_\sigma = f_0$. Then $w_\sigma \le u$ and

$$\text{meas}\,(\nabla u(B_\varepsilon)) \le \text{meas}(\nabla w_\sigma(B_\varepsilon)) = f_0|B_\varepsilon|.$$

First an obvious lemma.

Lemma 4. *("Close data imply close solutions"). Let f_1 continuous, f_2 smooth, and u a viscosity solution of*

$$\det D_{ij}u = Mu = f_1$$

with

$$f_2 < f_1 < f_2(1+\varepsilon)$$

and

$$u = 0$$

on $\partial\Omega$.

Let w be a $C^{2,\alpha}$ solution of

$$\begin{cases} Mw = f_2 \\ w = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Then

$$(1+\varepsilon)^{1/n} w \le u \le w.$$

Proof. By definition of viscosity solution.

Next a remark about the convex envelope of $u-v$ with v a $C^{2,\alpha}$ convex function.

Lemma 5. *("An equation for the difference of solutions") u as before ($Mu \le 1+\varepsilon$ in the viscosity sense).*

Let $\Gamma(u-v)$ denote the convex envelope of $(u-v)$ where v is a convex $C^{2,\alpha}$ function.

Then (χ the indicator function)

$$M^{1/n}(\Gamma) \le [(1+\varepsilon)^{1/n} - M^{1/n}(v)] \cdot \chi_{\Gamma(u-v)=u-v}.$$

Proof. Heuristically this follows from the fact that a) $M(\Gamma)$ is always supported at the contact set $\Gamma(u-v) = (u-v)$, and b) that at such points both $D_{ij}(u-v)$ and $D_{ij}(u-v)$ and $D_{ij}v$ are non-negative matrices and that $\det^{1/n}$ is a concave function of non-negative matrices.

Hence

$$M^{1/n}(v) + M^{1/n}(u-v) \le M^{1/n}(u).$$

For a more formal argument, we first point out that is is well known that

$$\det D_{ij}\Gamma(u-v)$$

is (as defined by the gradient map) a measure supported on the set

$$\{\Gamma(u-v) = u-v\}.$$

(See the proof of the remark after Lemma 2.)

Therefore, it is enough to prove that for any ball $B_\varepsilon(X_0)$,

$$\mathrm{Vol}\left(\mathrm{Im}\,\nabla\Gamma(u-v)\right)(B_\varepsilon) \leq \{1+0(1)\}|B_\varepsilon|[(1+\varepsilon)^{1/n} - M^{1/n}(v)].$$

If we substitute u by a minorizing sequence $u^{(k)}$ of solutions with smooth right hand side $1+\varepsilon$ and smooth boundary date $g_k|_{\partial B_\varepsilon} \to u|_{\partial B_\varepsilon}$ the result is true for any X_0 from the heuristic discussion above.

Since u^k converges uniformly to u^∞, with $u^\infty \leq u, u^\infty|_{\partial B_\varepsilon} = u, \Gamma(u^k - v)$ does so to $\Gamma(u^\infty - v)$ and therefore, for any ball B_ε

$$\mathrm{Im}\,\left(\nabla\Gamma(u-v)\right)(B_\varepsilon) \subset \mathrm{Im}\left(\nabla\Gamma(u^\infty - v)\right) \subset \overline{\lim}\,\mathrm{Im}\left(\nabla\Gamma(u_k - v)\right)(\overline{B}_\varepsilon).$$

3. Tangent paraboloids in measure and level surface estimates

The next lemma and its corollary are crucial for our estimate. They give us the tool to control second derivatives.

Lemma 6. *Let u be as above, namely, u convex, $Mu = f$, $u|_{\partial\Omega} = 0$, $B_1 \subset \Omega \subset B_n$. Assume that $1 < f < 1+\varepsilon$. Let w be the solution of*

$$\begin{cases} Mw = 1 & \text{on } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Then,

$$\frac{|\{\Gamma(u - \frac{1}{2}w) = u - \frac{1}{2}w\} \cap B_{1/2}|}{|B_{1/2}|} \geq 1 - \delta(\varepsilon)$$

with $\delta(\varepsilon) = C\varepsilon^{1/2}$. Where $C = C(n) > 0$ depends only on the dimension n.

Proof. We will estimate

$$\left|\left\{\Gamma\left(u - \frac{1}{2}w\right) = u - \frac{1}{2}w\right\}\right|,$$

through the image of the gradient mapping.

Since

$$\left[(1+\varepsilon)^{1/n} - 1/2\right] w \leq u - 1/2w \leq \frac{1}{2}w$$

(from Lemma 4), and w is convex, it follows that

$$\left[(1+\varepsilon)^{1/n} - 1/2\right] w \leq \Gamma(u - 1/2w) \leq \frac{1}{2}w.$$

From Lemma 2

$$N_{-\delta}\left\{\nabla\frac{1}{2}w\Big|_{B_{1/2(1-\varepsilon/\delta)}}\right\} \subset N_\delta\left\{\nabla\Gamma\left(u - \frac{1}{2}w\right)\Big|_{B_{1/2}}\right\}.$$

But

$$\nabla\frac{1}{2}w\Big|_{B_{1/2(1-\varepsilon/\delta)}}$$

is an *a priori* smooth set of volume exactly

$$\frac{1}{2^n}|B_{1/2(1-\varepsilon/\delta)}|.$$

Therefore

$$\mathrm{Vol}\{\nabla\Gamma(u-1/2w)|_{B_{1/2}}\} \geq \frac{1}{2^n}|B_{1/2(1-\varepsilon/\delta)}| - C\delta.$$

Here we have used the fact that

$$|N_{-\delta}\left\{\nabla\frac{1}{2}w\Big|_{B_{1/2(1-\varepsilon/\delta)}}\right\}| \geq |\left\{\nabla\frac{1}{2}w\Big|_{B_{1/2(1-\varepsilon/\delta)}}\right\}| - C\delta$$

with $C = C(n)$ depending only on dimension n. The above inequality relies on the fact that $\{\nabla\frac{1}{2}w|_{B_{1/2}}(1-\varepsilon/\delta)\}$ is a smooth deformation of B_1 since we have all the derivative estimates of w.

We now use Lemma 5 to conclude that

$$\begin{aligned}\frac{1}{2^n}|B_{1/2(1-\varepsilon/\delta)}| - C\delta &\leq \mathrm{Vol}\left(\mathrm{Im}\,\nabla\Gamma\left(u-\frac{1}{2}w\right)|_{B_{1/2}}\right)\\ &\leq \int_{B_{1/2}} M\left(\Gamma\left(u-\frac{1}{2}w\right)dx\right)\end{aligned}$$

from Lemma 5,

$$\begin{aligned}&\leq \left\{(1+\varepsilon)^{1/n} - \frac{1}{2}\right\}^n \left|\left\{\Gamma\left(u-\frac{1}{2}w\right) = u-\frac{1}{2}w\right\}\cap B_{1/2}\right|\\ &\leq \left(\frac{1}{2^n}+C\varepsilon\right)\left|\left\{\Gamma\left(u-\frac{1}{2}w\right) = u-\frac{1}{2}w\right\}\cap B_{1/2}\right|.\end{aligned}$$

We now choose $\delta = \varepsilon^{1/2}$ and the lemma is complete.

Remark. At any point X_0 in $\{\Gamma(u-1/2w) = u-1/2w\}$, there exists by definition a linear function L_{X_0} such that in all of Ω

$$\begin{cases} L_{X_0}(X) & \leq \left(u-\frac{1}{2}w\right)(X)\\ L_{X_0}(X_0) & = \left(u-\frac{1}{2}w\right)(X_0).\end{cases}$$

Therefore

$$\begin{cases} L_{X_0}(X)+\frac{1}{2}w(X) & \leq u(X)\\ L_{X_0}(X_0)+\frac{1}{2}w(X_0) & = u(X_0).\end{cases}$$

Corollary 1. *For any point $X_0 \in B_{1/2}$ in $\{\Gamma(u-\frac{1}{2}w) = u-\frac{1}{2}w\}, u$ has a global tangent paraboloid by below of the form*

$$L_{X_0} + \frac{1}{N}(X-X_0)^2$$

where N is a universal constant.

Our next objective is to study how the level surfaces of u deteriorate around a point.

The idea is the following:

Suppose that u is a non negative solution, with $u(0) = 0$, and let us look at the level sets say $\{u < 2^{-k}\} = S_k$.

From the general renormalization theory S_k looks like an ellipsoid of volume approximately equal to $(2^{-k})^{n/2}$, since when we renormalize S_k to S_k^*, $\inf u^*$ becomes of order one (thus, from the renormalization formula, $T = T_k$).

$$(\det T)^{2/n} \sim 2^k.$$

Since we do not expect u to be $C^{1,1}$, that means that the eccentricity of T_k will deteriorate.

By approximating u, and iterating, we will prove that the eccentricity of T_k deteriorates as slowly as we want if f is closer and closer to a constant.

Lemma 7. *Let u be a non negative solution of*

$$Mu = f, \quad |f - 1| \leq \varepsilon$$

with

$$\alpha)\ u(0) = 0.$$
$$\beta)\ u|_{\partial\Omega} = 1.$$
$$\gamma)\ \Omega \sim B_1.$$

Let

$$S_\mu = \{X : u < \mu\}, \varepsilon < \mu_0 \textit{small.}$$

Then, there exists a quadratic polynomial

$$P = \sum \frac{(x_i)^2}{\alpha_i}$$

with

a) $\Pi \frac{\alpha_i}{2} = 1$

b) $0 < C_1 < \alpha_i < C_2$

such that $u - P|_{S_\mu} \leq C(\mu^{3/2} + (\mu\varepsilon)^{1/2} + \varepsilon)$.

If furthermore we assume the more precise bound

$$B_{\sqrt{2}-\theta}(X_0) \subset \Omega \subset B_{\sqrt{2}+\theta}(X_0).$$

Then, we get the more precise bounds

b′)

$$|\alpha_i - 2| \leq C\theta$$

and

$$|u - P| \leq C[\theta\mu^{3/2} + (\mu\varepsilon)^{1/2} + \varepsilon].$$

Proof. We use the approximation of u by w, solution of

$$w|_{\partial\Omega} = 1, \ Mw = 1.$$

Then $|w - u| < \varepsilon$, and w is $C^{1,1}$ and thus $C^{2,\alpha}$ (since M becomes uniformly elliptic) and thus C^3 by Shauder estimates, in say the section $\{X : w \leq 7/8\}$.

Thus, near the origin.

$$w = w(0) + \langle \nabla w(0), X\rangle + \sum \frac{1}{\alpha_i} x_i^2 + 0(|X|^3)$$

P will be precisely $\sum \frac{1}{\alpha_i} x_i^2$. We estimate

$$|w(0)| = |w(0) - u(0) \leq \varepsilon$$

and

$$|\nabla w(0)| \leq \varepsilon^{1/2}$$

since $w \geq -\varepsilon$ in all of Ω.

In particular $S_\mu \subset \{X : w < \mu + \varepsilon\}$ that for μ small is contained in $B_{C(\mu+\varepsilon)}$.

Substituting in X, we get the desired estimate.

To improve the estimate in case γ', we notice that, with that hypothesis

$$\frac{1}{2}|X - X_0|^2 C\theta \leq w \leq \frac{1}{2}|X - X_0|^2 + C\theta.$$

Therefore

$$v = w - \frac{1}{2}|X - X_0|^2$$

satisfies a concave, fully non linear uniformly elliptic equation

$$\det(D^2 v + 1) = 1$$

that carries $C^{2,\alpha}$, and thus C^3, estimates:

$$\|w - \frac{1}{2}|X - X_0|^2\|_{C^3(\{w<7/8\})} \leq C\theta.$$

In particular, we can improve the estimate of the term $0(|X|^3)$ by $\theta(\mu+\varepsilon)^{3/2}$.

We now use the previous estimates to produce $C^{1,\alpha}$ and $C^{1,1}$ (thus $C^{2,\alpha}$, thus C^3) estimates.

Lemma 8. *Assume that* $|f - 1||_{S_\mu} \leq \varepsilon(\mu)$ *where we are interested in the two cases.*

a) $\varepsilon(\mu) \leq \varepsilon_0$*, and b)* $\varepsilon(\mu) \leq \mu^s$ *for some* s.

Then, ($\mu < \mu_0$*, small but independent of* ε) *there exists polynomials*

$$P^k = \sum \frac{1}{\alpha_i(k)} x_i^2,$$

such that for $\mu = (\mu_0)^k$*, we have*

$$|u - P^k|_{S_\mu} \leq C\mu(\varepsilon(\mu)^s)$$

(for some σ*) and*

$$\frac{\sup \alpha_i(k)}{\sup \alpha_i(k-1)} \leq 1 + C\varepsilon^\sigma.$$

Proof. There is a distinct first step, before starting induction to carry the shape of Ω to be almost a ball.

Step 1. We are in hypothesis γ, and thus on $S\mu$,

$$|u-P| \leq C(\mu^{3/2} + (\mu\varepsilon_0)^{1/2} + \varepsilon_0)$$

with

$$0 \leq C_1 \leq \alpha_i \leq C_2.$$

We normalize $S\mu$ to a unit section by first making $P = \frac{1}{2}x_i^2$, i.e. by changing $x_i \to \sqrt{\frac{2}{\alpha_i}}x_i$ and then we expand the new $\overline{u}$, $\overline{P}$ by $u^* = \frac{1}{\mu}\overline{u}^*(\mu X)$. $P^* = \overline{P} = \frac{1}{2}|X|^2$.

Then, on $\partial(S\mu)^*$ we now have

$$u^* = 1, \text{ and } |(u^* - \frac{1}{2}|X|^2) =$$

$$|1 - \frac{1}{2}|X|^2| \leq C[\mu^{1/2} + \frac{\varepsilon_0^{1/2}}{\mu^{1/2}} + \frac{\varepsilon_0}{\mu}].$$

We chose $\mu = \varepsilon_0^{1/4}$, and we get now our starting configuration: (we can forget about $*$'s) $u =$ (old u^*) $= 1$ on ∂S_1 (old ∂S_μ^*).

$$B_{\sqrt{2}-C\varepsilon_0^{1/4}} \subset S_1 \subset B_{\sqrt{2}+C\varepsilon_0^{1/4}}.$$

Now we do the inductive step. Assume that we have proven step k. As before, we renormalize in two steps, first by making $P = \frac{1}{2}|X|^2$ i.e. $x_i \to \sqrt{\frac{2}{\alpha_i(k)}}x_i$ and then expanding by $\mu^{1/2}$, i.e. $u^* = \frac{1}{\mu}\overline{u}(\mu^{1/2}X)$, (this does not change $\overline{P} = P^* = \frac{1}{2}|X|^2$).

From the inductive hypothesis

$$B_{\sqrt{2}-C\varepsilon^\sigma(\mu)} \subset S^* \subset B_{\sqrt{2}+C\varepsilon^\sigma(\mu)},$$

and

$$|(f-1)| \leq \varepsilon(\mu).$$

Thus, we are under hypothesis γ', and on $S_{\mu_0}^* = \{u^* < \mu_0\}$ we have

$$|u^* - \hat{P}| \leq C[\varepsilon^\sigma(\mu)\mu_0^{3/2} + \mu_0^{1/2}\varepsilon_{(\mu)}^{1/2} + \varepsilon(\mu)]$$

and

$$|\hat{\alpha}_i - 2| \leq C[\varepsilon^\sigma(\mu)].$$

We renormalize and we get for $\overline{\mu} = \mu_0\mu$

$$|u - P_{k+1}| \leq \mu\overline{C}[\varepsilon^\sigma(\mu)\mu_0^{3/2} + \mu_0^{1/2}\varepsilon^{1/2}(\mu) + \varepsilon(\mu)].$$

If ε_0 is much smaller than μ_0, we can say

$$|u - P_{k+1}| \leq C\mu\mu_0\varepsilon_0^\sigma$$

in case 1, and

$$|u - P_{k+1}| \leq C\mu\mu_0(\mu\mu_0)^\sigma$$

in case 2.

About the coefficients of P_{k+1}, they are the renormalization of $\hat{P}$ by the transformation that carries $\frac{1}{2}|x|^2$ into P_k and thus the new largest $\alpha_i(k+1)$ is smaller or equal than

$$\sup_i \alpha_i(k+1) \leq \sup_i(\hat{\alpha}_i - 2)\sup_j \alpha_j(k).$$

Corollary. *a) The affine transformation of determinant one that makes $S\mu$ equivalent to a ball of radius $\mu^{1/2}$ has norm (for $\mu_0^k \le \mu \le \mu_0^{k-1}$)*

$$\|T\mu\| \le \Pi_{\ell=1}^{k}(1 + C\varepsilon^{\sigma}(\mu_0^{\varepsilon})).$$

b) If the infinite product above is convergent, for instance if $\varepsilon(\mu) \le \mu^{s}$, u is $C^{1,1}$ at the origin.

c) Given $\alpha < 1$ there exists $\varepsilon_0(\alpha)$, so that if $\varepsilon(\mu) \le \varepsilon_0$, u at the origin is $C^{1,\alpha}$.

Finally, before entering into the $W^{2,P}$ estimates; a consequence about the geometry of sections.

We recall that if L_{X_0} is the supporting plane to u at X_0, we define the section

$$S_{t,X_0} = \{u \le L_{x_0} + t\}$$

so S_μ before was really $S_{\mu,0}$.

The previous approximation theorem, says that as ε becomes small, sections look like ellipsoids whose eccentricity may deteriorate, but at a slower and slower geometric rate as we let ε become smaller.

We are now ready to prove the $W^{2,p}$ estimates.

At this point it is convenient to review the geometric properties of the graph of u, solution of

a) $Mu = f$

b) $|f - 1| < \varepsilon(p)$

c) $u|_{\partial\Omega} = 0$, $B_1 \subset \Omega \subset B_n$.

We recall that we denote by S_{μ,X_0} the section

$$\{X : u - L_{X_0} < \mu\}.$$

S^*, u^* denote the corresponding renormalizations

$$S^* = T(S).$$
$$u^* = \frac{1}{\mu}u(T^{-1}(X)).$$

where

$$S^* \sim B_1$$

and

$$Mu^* = f(T^{-1}(X)) \sim 1.$$

We also can decompose

$$T = \mu^{1/2}Q$$

where

$$\det Q = 1.$$

By now, we have proven the following properties (for ε small, some power σ, small, independent of ε).

Property 1.

$$B_{\sqrt{2}-C\varepsilon^{\sigma}} \subset T(S) = S^* \subset B_{\sqrt{2}-C\varepsilon^{\sigma}}.$$

Property 2.

$$||Q||, ||Q^{-1}|| \le \mu^{-\tau(\varepsilon)}$$

with $\tau(\varepsilon)$ as small as we want for ε small. (lemma 8).

Property 3. (Rescaling of property 2.)
If $\eta < \mu$,

$$||Q_\mu Q_\eta^{-1}||, ||Q_n^{-1} Q_\mu|| \le (\frac{n}{\mu})^{-\tau(\varepsilon)}.$$

Property 4. (Adjacent sections.)
If $y \in S_\mu(X)$,

$$||Q_\mu(y) Q_\mu^{-1}(x)||, ||Q_\mu(X) Q_\mu^{-1}(y)|| \le (1 + C\varepsilon^\sigma).$$

(This follows from the fact that the quadratic approximation to u is $S_{K\mu}(X_0)$ determines, for K a large, constant, the shape of $S_\mu(Y_0)$ as being almost a circle.)

Property 5. (Corollary of Property 3.)
If we normalize S_{μ,X_0} to be a unit ball and $\eta < \mu$,

$$B_{(\frac{\eta}{\mu})^{1/2+\tau(\varepsilon)}} \subset T_\mu(S_\eta) \subset B_{(\frac{\eta}{\mu})^{1/2-\tau(\varepsilon)}}$$

Property 6. If u has a tangent paraboloid by below, (λ large)

$$u \ge \frac{1}{\lambda}|X|^2$$

then u has a tangent paraboloid

$$u \le \lambda^{n-1}|X|^2$$

by above.

Proof. The paraboloid by below puts a uniform bound

$$||Q_\mu|| \le \lambda.$$

Since $\det Q = 1$, this gives a bound by below.

We finally recall lemma 6 and Corollary 1 combined.

Property 7. If we renormalize S_μ to $S^* \sim B_1$ and u to u^*, then u^* has a tangent paraboloid by below

$$u^* \ge L_{X_0} + \frac{1}{\lambda_0}|X - X_0|^2$$

in a set $E \subset S_\mu$ of measure

$$|S_\mu \backslash E| \le C\varepsilon^{1/2}$$

for some universal λ_0, and the last tool is a finite overlapping theorem:

Property 8. Let A be a measurable subset of Ω, so that for every X in A we chose a section $S_{\mu,X}$. For any δ, there is an ε such that, we can extract a countable covering of A, S_{μ_k,X_k}, so that $(|f-1|<\varepsilon)$

$$S_{\mu_k(1-\delta),X_k}$$

only overlaps a finite number of times depending only on dimension.

Proof. We extract the standard subcovering by sections S_{μ_k,X_k} of decreasing parameter μ_k, such that X_k does not belong to the previous sections in the covering.

Let now X_0 be a point that belongs to $S_{\mu(1-\delta),X}$ for a sequence of sets which we relabel

$$S_{\mu_1,X_1},\dots,S_{\mu_k,X_k}.$$

We renormalize the first one to make it a ball, by the affine transformation T_{μ_1,X_1}, and transform all the other by the same T_{μ_1,X_1}, let us call these new sections $S^*_{\mu,X}$, where now

$$S^*_{\mu_1,X_1}\sim B_1$$

and all the $\mu_j<1$ (the new μ_j are the old μ_j/μ_1.)

If all the sections $S^*_{\mu_j X_j}$ would satisfy, say

$$B_{r_j/4}\subset S^*_{\mu_j,X_j}\subset B_{4r_j},$$

for some r_j, they would overlap a finite number of times

$$M\sim 4^n$$

according to standard differentiation theory.

From property 4 and property 5, this will happen for those μ, such that

$$\mu^{-\tau(\varepsilon)}\le 4.$$

Now, since

$$X_0\in S^*_{\mu_1(1-\delta),X_1}$$

$$S^*_{\mu_1,X_1}\sim B_1$$

and

$$X_j\notin S^*_{\mu_1,X_1}$$

any other section that contains X_0 must have $\operatorname{diam} S^*\sim\delta$.

From properties 4 and 5, this means

$$\mu^{1/2-\tau(\varepsilon)}\ge\delta.$$

So, given δ, if we chose $\tau(\varepsilon)$ small, the inequality

$$\mu^{1/2-\tau(\varepsilon)}\ge\delta$$

guarantees that

$$\mu^{-\tau(\varepsilon)}\le 4$$

and the property is proven.

We are now ready to prove theorem 1.

Proof. Our starting configuration is a renormalized section $S_{1,0}$, such that for any point Y such that $S_{1,Y}$ touches $S_{1,0}$, all properties P_1 to P_7 are satisfied.

(This is achieved by renormalizing to size one a small section around any given point X well inside Ω).

At this point, we retrieve the N from Corollary 1, and define $\lambda = N^n$.

Our good sets B_{λ^k}, will be those X_0 for which the transformations needed to renormalize S_{μ,X_0} are all of norm less than λ^k, i.e.

$$B_{\lambda^k} = \{X_0 : ||Q_\mu|| \leq \lambda^k, \text{ for every } \mu < \mu_0\}.$$

In particular for such an X_0, u has tangent paraboloids by above and below of the form

$$L_{X_0} + \frac{1}{\lambda^k}|X - X_0|^2 \leq u \leq L_{X_0} + \lambda^{(n-1)^k}|X - X_0|^2.$$

The bad set A_{λ^k} will be the complement of B_{λ^k}, and we will show fast polynomial decay of the measure of A_{λ^k}.

First we point out that if a section $S_{\mu,y}$ has the property that $||Q_\mu|| \geq t$ then $\mu^{-\tau(\varepsilon)} \geq t$ (property 2), i.e. $\mu \leq t^{-\frac{1}{\tau(\varepsilon)}} = t^{-M(\varepsilon)}$ with M large.

In particular

$$\operatorname{diam}(S\mu) \leq \mu^{1/2}\mu^{-\tau(\varepsilon)} \leq t^{-\frac{M(\varepsilon)}{4}}.$$

We now find an inductive relation between

$$A_{\lambda^{k+1}} \cap B_{k+1} \text{ and } A_{\lambda^k} \cap B_k.$$

where B_k is the ball with radius

$$\sqrt{1} = 1/2, r_{k+1} = r_k - C\lambda^{-\frac{kM(\varepsilon)}{4}}.$$

For $X \in A_{\lambda^{k+1}} \cap B_{k+1}$, let $S_{\mu,X}$ be the first (largest) section for which

$$||Q_\mu|| \geq \lambda^{k-1}.$$

We note the following properties:

If we renormalize $S_{4\mu}$, after renormalization we have that every point Y^* in S^*_μ, except for a set of measure $\varepsilon^{1/2}$, u has a tangent paraboloid by below of the form

$$L_{Y^*} + \frac{1}{N}|Y - Y^*|^2 = L_{Y^*} + \frac{1}{\lambda^{1/n}}|Y - Y^*|^2$$

and by above

$$L_{Y^*} + \lambda|Y - Y^*|^2.$$

This means after renormalizing back that, except for a set of measure $\varepsilon^{1/2}|S\mu|$ every point Y, in $S\mu$ belongs to

$$A_{k-2} \backslash A_{k+1}.$$

Indeed, the renormalization $Q_{\mu,X}$ transforms already $S_{\eta,Y}$ into a set trapped between two circles

$$\lambda^{-1} \leq |Y - Y^*|^2 \leq \lambda^{1/n}.$$

Therefore the renormalization $Q_{\eta,Y}$ differs from $Q_{\mu,X}$ by at most a factor of λ that is

$$\lambda^{k-2} \leq ||Q_{n,Y}|| \leq \lambda^{k+1}.$$

As for those sections $S_{\eta,Y}$ with η not in 2μ, that implies that $\eta \geq 2\mu$ and then, property 4 says that

$$\|Q_\eta(Y)Q_\eta^{-1}(X)\|, \|Q_\eta(X)Q_\eta^{-1}(Y)\| \leq (1 + C\varepsilon^\sigma),$$

thus $\|Q_\eta(Y)\| \leq \|Q_\eta(X)\|(1 + C\varepsilon^\sigma) \leq \lambda^{k-1}(1 + C\varepsilon^\sigma)$.

Let us now extract a covering of $B_{r_{k+1}\cap A_{k+1}}$ by these sections $S_{\mu,X}$, so that $S_{\mu(1-\delta)}$ has finite overlapping.

Notice that $S_{\mu,X}$ is contained in B_{r_k} since

$$\operatorname{diam} S_{\mu,X} \leq \lambda^{-\frac{kM\varepsilon}{4}} \leq \sqrt{k} - \sqrt{k+1}$$

and that $r_k \searrow \Gamma_\infty > \frac{1}{4}$ if we chose ε small (remember that λ is universal).

Then, the standard computation yields

$$\begin{aligned}
|A_{k+1} \cap B_{r_{k+1}}| &\leq \sum_j |A_{k+1} \cap B_{r_{k+1}} \cap S_{\mu_j,X_j}| \\
&\leq \sum \varepsilon^{1/2} |S_{\mu_j,X_j}| \leq \sum \varepsilon^{1/2}(1+\delta)^n
\end{aligned}$$

$$\begin{aligned}
&|S_{\mu_j(1-\delta),X_j}| \leq \sum \varepsilon^{1/2}(1+\delta)^n(1+\varepsilon^{1/2}) \\
&|S_{\mu_j(1-\delta),X_j} \cap (A_{k-2} \cap B_{r_{k-2}})| \\
&\leq C\varepsilon^{1/2}(1+\delta)^n(1+\varepsilon^{1/2})4^n |A_{k-2} \cap B_{r_{k-2}}
\end{aligned}$$

(from finite overlapping, property 7).

This completes the proof of the theorem.

Theorem 4. *Let Γ be a convex set in R^n*

$$B_1 \subset \Gamma \subset B_n.$$

Assume that, in the viscosity sense, $\partial\Gamma$ satisfies

$$\mathcal{G}(X) = f(X, \nu)$$

for $\mathcal{G}$ the Gauss curvature of $\partial\Gamma$ at X, where

a) $0 < \lambda_1 < f < \lambda_2$

and

b) *f is continuous in X and ν.*

Then, $\partial\Gamma$ is locally a $W^{2,p}$ graph for every p.

If

b') *f is C^α in X and ν, then $\partial\Gamma$ is locally a $C^{2,\alpha}$ graph.*

Proof. $\partial\Gamma$ is strictly convex and C^1 from **[C2]**.

Theorem 5. *In Theorems 1 and 2, replace*

$$u|_{\partial\Omega} = 0$$

by

$$u|_{\partial\Omega} = g$$

with $\partial\Omega \in C^{1+\alpha}$ and $g \in C^{1+\alpha}$, with $\alpha > 1 - \frac{2}{n}$; then, the same conclusion holds.

Proof. We may localize, by Corollary 4 of Part 2.

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The Moser-Trudinger Inequality and Applications to Some Problems in Conformal Geometry

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Introduction

This is the set of lecture notes for a graduate course which the author gave at the Summer Geometry Institute at Park City, Utah in the summer of 1992. The main theme of the notes is the study of extremals of Sobolev inequalities with applications to some non-linear partial differential equations arising from problems in conformal geometry. Some effort has been made to make the notes self-contained. However, the intention is to keep the material at basic and introductory level, thus many recent research articles in the subject are not covered or mentioned.

The notes are organized as follows: In Lecture 1, we cover some background material concerning the notion of eigenvalue, the heat kernel and Weyl's asymptotic formula. In Lecture 2, we discuss the trace of the heat kernel, the concept of log-determinant of the Laplacian operator and the Ray-Singer/Polyakov formula for the zeta-function determinant on compact surfaces. In Lecture 3, we discuss a limiting form of the Sobolev inequality called the Trudinger-Moser-Onofri inequality, we also have as an application of this inequality the isospectral problem on compact surfaces. In Lecture 4, we establish the existence of extremals for Moser's inequality. In Lecture 5, we discuss the higher dimensional analogue of Moser's inequality and application to the extremal and compactness problem for zeta-function determinant of 4-manifolds. In Lecture 6, we return to the "isospectral" problem on three dimensional manifolds, we discuss the relation of the problem to the "Yamabe" problem, and the role played by the first eigenvalue in the spectral compactness problem. Finally in Lecture 7 we discuss the role of Moser-Trudinger type inequalities to the problem of prescribing curvature equations in conformal geometry.

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LECTURE 1
Some Background Material

1. Preliminaries

Let M be a compact Riemannian manifold with boundary ∂M (∂M may be empty). In terms of local coordinates $(x^1, \ldots, x^n)$ the metric can be expressed as $ds^2 = \sum g_{ij} dx^i dx^j$ and the Laplace operator is defined by

$$\Delta = \frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial x^i} (\sqrt{g} g^{ij} \frac{\partial}{\partial x^j})$$

where $(g^{ij}) = (g_{ij})^{-1}$, $g = \det(g_{ij})$.

Given $f \in C^\infty(M)$, define a Hilbert space norm by

$$\{\int_M |f|^2 dx + \int_M |\nabla f|^2 dx\}^{\frac{1}{2}},$$

and denote by $W^{1,2}(M)$ the Sobolev space obtained by taking the closure of $C^\infty(M)$ in the above norm, and by $W_0^{1,2}(M)$ the closure of $C_0^\infty(M)$ (C^∞ functions with compact support in M) in the above norm.

(a). If ∂M is empty, then Δ is a self-adjoint operator on $W^{1,2}(M)$. According to the spectral theory of self-adjoint operators, Δ has discrete eigenvalues,

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \to \infty$$

with corresponding eigenfunctions ϕ_i satisfying

$$-\Delta \phi_i = \lambda_i \phi_i$$

where $\phi_i \in C^\infty(M) \bigcap W^{1,2}(M)$ can be chosen to form an orthonormal basis of $W^{1,2}(M)$ (Parseval Theorem).

(b). When ∂M is not empty, we should specify some boundary conditions so that Δ is self-adjoint. There are two standard conditions:

(1) Dirichlet boundary conditions, in which case $\mathrm{Dom}(\Delta) = W_0^{1,2}(M)$, and the eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

and the corresponding eigenfunctions satisfy

$$-\Delta\phi_i = \lambda_i\phi_i$$
$$\phi_i|_{\partial M} = 0 \quad \phi_i \in C^\infty(M).$$

(2) Neumann boundary conditions, in which case $\mathrm{Dom}(\Delta) = W^{1,2}(M)$, and the eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

and the corresponding eigenfunctions satisfy

$$-\Delta\phi_i = \lambda_i\phi_i$$
$$\frac{\partial\phi_i}{\partial\nu}|_{\partial M} = 0, \quad \phi_i \in C^\infty(M).$$

Some Basic Examples.

1. Let $M = [0, a]$. Then the eigenvalues are $\lambda_i = \frac{i^2\pi^2}{a^2}$, $i = 1, 2, \ldots$, and the corresponding eigenfunctions are $\sin\frac{i\pi x}{a}$ for the Dirichlet boundary conditions; and the eigenvalues are $\lambda_i = \frac{i^2\pi^2}{a^2}$, $i = 0, 1, \ldots$, and the corresponding eigenfunctions are $\cos\frac{i\pi x}{a}$ for the Neumann boundary conditions.
2. If M is a domain in R^n, then in Euclidean and polar coordinates,

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} = r^{1-n}\frac{\partial}{\partial r}(r^{n-1}\frac{\partial}{\partial r}) + \frac{1}{r^2}\Delta_{S^{n-1}}.$$

In the special case when $M = D$ the unit disk in R^2, using separation of variables, we consider $u(r,\theta) = f(r)h(\theta)$. Then $\Delta u + \lambda u = 0$ is equivalent to the following equation:

$$\frac{r^2(f''(r) + \frac{1}{r}f'(r) + \lambda f(r))}{f(r)} = -\frac{h''(\theta)}{h(\theta)} = c = constant.$$

Since $h(\theta)$ is periodic in θ with period 2π, $c = n^2$ and $h = a\cos n\theta + b\sin n\theta$. Set $f(r) = y$; we have

$$(*) \qquad r^2y'' + ry' + (r^2\lambda - n^2)y = 0$$

To find λ we solve $(*)$ with y continuous at the origin and $y(1) = 0$. Let $r\sqrt{\lambda} = \rho$ $(\lambda \neq 0)$. We set $\lambda = k^2$, then $rk = \rho$ and

$$(**) \qquad \frac{d^2y}{d\rho^2} + \frac{1}{\rho}\frac{dy}{d\rho} + (1 - \frac{n^2}{\rho^2})y = 0$$

Solutions of (**) are called Bessel functions $J(\rho)$. Actually all eigenfunctions are of the form $J(\sqrt{\lambda}r)(a\cos n\theta + b\sin n\theta)$.

Remark. It is a classical result of Faber-Krahn that the ball realizes the smallest λ_1 among domains (in R^n) of the same volume. A recent result of Ashbaugh-Berguria [A-B], using special properties of the Bessel functions, proved that $\frac{\lambda_2}{\lambda_1}$ (for the Dirichlet problem) is maximal if and only if the domain Ω is a ball.

3. Let $S^n(r) = \{x \in R^{n+1}, |x| = r\}$, and $S^n = S^n(1)$. Given $x \in R^{n+1}$ write $x = r\xi$, for $r \in [0,\infty)$ and $\xi \in S^n$. It is simple to show that

$$\Delta_{R^{n+1}}F = r^{-n}\partial_r(r^n\partial_r F) + \Delta_{S^n(r)}(F|_{S^n(r)}).$$

See [C, p 34], or [S-W, p 137].

If $F(x) = r^k g(\xi)$ (F is homogeneous of degree k), then

$$\Delta_{R^{n+1}}F = r^{k-2}(\Delta_{S^n}g + k(k+n-1)g)$$

Thus F is harmonic on R^{n+1} if and only if g is an eigenfunction on S^n with eigenvalues $k(k+n-1)$. It turns out that the space of homogeneous harmonic polynomials of degree k when restricted to S^n is the eigenspace corresponding to the eigenvalue $k(k+n-1)$ of S^n with dimension $C_k^{n+k} - C_{k-2}^{n+k-2}$ [S-W].

Corollary. $\lambda_1(S^n) = n$ *with eigenfunctions*

$$\{x^1|_{S^n}, x^2|_{S^n}, \ldots, x^{n+1}|_{S^n}\}.$$

2. Rayleigh Quotients

We define the space H in the following way:

(1) If $\partial M = \emptyset$, then $H = \{f \in W^{1,2}(M) |\ \int_M f dx = 0\}$.

(2) If $\partial M \neq \emptyset$, then for the Dirichlet problem take $H = W_0^{1,2}(M)$, and for the Neumann problem take H as in (1).

For

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \to \infty$$

we have

$$\lambda_1 = \inf_{f\in H} \frac{\int_M |\nabla f|^2 dx}{\int_M f^2 dx}$$

$$\lambda_i = \inf\{\frac{\int_M |\nabla f|^2 dx}{\int_M f^2 dx} | f \in H, \int_M f\phi_j dx = 0, \text{for } j = 1,2,\ldots,i-1\}.$$

In particular, λ_1 is the smallest value c such that ;

$$c\int_M f^2 dx \le \int_M |\nabla f|^2 dx, \text{ for any } f \in H,$$

which is called the Poincare inequality [C, p.16].

In general, one likes to estimate λ_1 from below by simple geometric quantities of the manifold, such as the isoperimetric constant, diameter of the manifold, volume of the manifold, and so on. The reader is referred to [C, Lecture 3] for some examples in this direction.

Examples.

1. For the dumb-bell region, λ_1 is very small when the neck is either thin or long.
2. (Hersch Theorem) For (S^2, g) with any metric g with Area$(M) = 4\pi$ we have $\lambda_1(g) \leq 2$ and $\lambda_1(g) = 2$ if and only if g is isometric to the standard metric on S^2.

Exercise. If ϕ is a conformal map between compact surfaces (M, g_1) and (M, g_2), then

$$\int_M < \nabla_1 f, \nabla_1 g > dv_1 = \int_M < \nabla_2(f \circ \phi^{-1}), \nabla_2(g \circ \phi^{-1}) > dv_2.$$

Exercise. Verify the following:

$$(*) \qquad \sum_{k=1}^{N} \frac{1}{\lambda_k} = \sup \sum_{k=1}^{N} \frac{\int_M u_k^2 dv}{\int_M |\nabla u_k|^2 dv}$$

where the sup is taken over functions $u_1, u_2, \cdots, u_k$ satisfying the conditions $\int_M u_k dv = 0$ and $\int_M < \nabla u_k, \nabla u_l > dv = 0$ if $l \neq k$. Equality holds if and only if u_k is the $k-$th eigenfunction.

<u>**Proof of the Hersch Theorem**</u>. We take the standard coordinate functions x_1, x_2, x_3 as the test functions in $(*)$. However, to meet the restrictions it will be necessary to find a conformal transformation ϕ of S^2 so that $\int_{S^2}(x_k \circ \phi)dv = 0$ for $k = 1, 2, 3$. This can be done by a topological argument. The remaining orthogonality condition follows from conformal invariance of the Dirichlet product. Thus we find, using $\int_{S^2} |\nabla x_k|^2 dv = \frac{8\pi}{3}$ for each k,

$$\frac{3}{\lambda_1} \geq \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \geq \frac{3}{8\pi} \int_{S^2} (\sum_{k=1}^{3} x_k^2) dv = \frac{3}{2}.$$

Hence $\lambda_1 \leq 2$. On the other hand, if $\lambda_1 = 2$, equality must hold everywhere in the formula above. Hence x_1, x_2, x_3 must be the eigenfunctions of the metric. This means that the metric is the standard sphere.

Remark. N. Korevaar [Ko] has recently proved $\lambda_k \leq Ck$ on compact surfaces, and also an extension of this result to conformal metrics on higher dimensional manifolds.

3. Weyl's Asymptotic Formula

Let M be a compact Riemannian manifold (or bounded domain in R^n). Let's denote $N(\lambda) = \#$ of eigenvalues $\leq \lambda$, where we count multiplicity. Then

$$N(\lambda) \sim \omega_n \mathrm{Vol}(M) \frac{\lambda^{\frac{n}{2}}}{(2\pi)^n}$$

as $\lambda \to \infty$, where ω_n is the volume of the unit ball in R^n. If $\lambda = \lambda_k$, then we get

$$(\lambda_k)^{\frac{n}{2}} \sim \frac{(2\pi)^n k}{\omega_n \mathrm{Vol}(M)}$$

Remarks.

(1) In the case M is a bounded smooth domain, Weyl's formula can be derived by using the max-min principle (Rayleigh quotient). [C, p 32–33].

(2) Polya conjecture: $(\lambda_k)^{\frac{n}{2}} \geq \frac{(2\pi)^n k}{\omega_n \mathrm{Vol}(M)}$ and $(\mu_k)^{\frac{n}{2}} \leq \frac{(2\pi)^n k}{\omega_n \mathrm{Vol}(M)}$ for plane domains, where μ_k denotes the Neumann eigenvalues. Polya proved his conjecture for "plane-tiling" domains in 1961 [P].

(3) The formula means that

$$\lim_{n \to \infty} \frac{N(\lambda)}{\lambda^{\frac{n}{2}}} = \frac{\omega_n}{(2\pi)^n} \mathrm{Vol}(M)$$

i.e., $\mathrm{Vol}(M)$ is determined by the spectral counting function $N(\lambda)$. A sharper study of the asymptotic distribution of the eigenvalues is through the study of the trace of the heat kernel.

Let's write

$$Z(t) = \sum_i e^{-\lambda_i t} = \mathrm{Trace}\, e^{-t\Delta}.$$

Theorem ([MP, MS]). *When $t \to 0^+$, $Z(t) \to (4\pi t)^{-\frac{n}{2}}$ Vol(M). Actually,*

$$Z(t) \sim (4\pi t)^{-\frac{n}{2}} (a_0 + a_{\frac{1}{2}} t^{\frac{1}{2}} + a_1 t + \cdots)$$

as $t \to 0^+$, where $a_0 =$ Vol(M), and a_i are given by the integrals of the local invariants of g.

We will return to this theorem in Lecture 2.

If we write

$$Z(t) = \int_0^\infty e^{-t\lambda} dN(\lambda)$$

then we see that Weyl's formula implies $a_0 = \mathrm{Vol}(M)$. It turns out that $a_0 = \mathrm{Vol}(M)$ also implies Weyl's asymptotic formula through an application of Karamata's Tauberian Theorem.

Remark. One cannot mention this subject without mentioning the problem of M. Kac, "Can one hear the shape of a drum?" [K]. Let Ω be a domain in R^2 with smooth boundary. Is Ω determined by its spectrum $\lambda_1, \lambda_2, \cdots,$? i.e., can we hear Ω (a membrane of a drum) if we had perfect pitch? Weyl's formula shows that the area of Ω can be heard. Kac [K] proved that if Ω is convex, then $L =$ the length of boundary of Ω can be heard. In fact, L appears in the coefficient $a_{\frac{1}{2}}$ in the asymptotic formula of $Z(t)$:

$$Z(t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4}\frac{1}{(2\pi t)^{\frac{1}{2}}} + \cdots$$

for $t \to 0$.

Since a disk is the only region with $L^2 = 4\pi|\Omega|$, we have

Corollary. *Circular drums can be heard.*

Remark. Kac's originally problem was answered last year in the negative by C. Gordon, D. Webb and S. Wolpert [G-W-W] based on a method of Sunada [S]. The main statement is that there is a pair of non-congruent simply connected domains in the Euclidean space which have the same spectrum for both the Dirichlet and Neumann problems.

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LECTURE 2
Ray-Singer-Polyakov Formula on Compact Surfaces

1. Heat kernel

In chapter 1 we have mentioned Weyl's formula:

$$\lambda_k \sim C(n)\left(\frac{k}{\mathrm{Vol}(M)}\right)^{\frac{2}{n}}$$

as $k \longrightarrow \infty$. If we let $Z(t) = \sum_{k=0}^{\infty} e^{-\lambda_k t}$, $Z(t)$ tends to $(4\pi t)^{-\frac{n}{2}}(a_0)$ as $t \longrightarrow 0^+$, where $a_0 = \mathrm{Vol}(M)$. $Z(t)$ is called "the trace of the heat kernel". Assume $\{\phi_i\}$ is a sequence of eigenfunctions which forms an orthonormal basis for $L^2(M)$, then

$$\begin{aligned} Z(t) &= \int_M \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(x)dV(x) \\ &= \int_M K(x,x,t)dV(x) \end{aligned}$$

where $K(x,y,t) = \sum e^{-\lambda_i t}\phi_i(x)\phi_i(y)$ is called the heat kernel.

Proposition. *$K(x,y,t)$ is the unique fundamental solution of the heat equation on M. It is continuous in $M \times M \times (0,\infty)$, C^2 in x, y, and C^1 in t. That is, for any given bounded continuous function f on M, the function $u(x) = \int K(x,y,t)f(y)$ $dV(y)$ satisfies the following equation:*

$$\begin{aligned} &\frac{\partial}{\partial t}u(x) = \Delta u(x) \\ &\lim_{t\longrightarrow 0^+} u(x,t) = f(x) \ \textit{on}\ \ M. \end{aligned}$$

Examples.

1. On $S^1 = \{z \in C : |z| = 1\}$, the Laplacian has spectrum $\{n^2, e^{in\theta}\}_{n=-\infty}^{\infty}$ and $dV = \frac{d\theta}{2\pi}$. So

$$K(x,y,t) = \sum_{n=-\infty}^{\infty} e^{-n^2 t} e^{inx} e^{-iny}.$$

   In particular, $K(x,x,t) = Z(t) = \sum_{-\infty}^{\infty} e^{-n^2 t}$ which tends to $\frac{1}{\sqrt{4\pi t}}$ as $t \longrightarrow 0^+$.

2. On R^n, $K(x,y,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$.

   Estimates of the heat kernel for general manifolds can be found in Davies [D], and in Li-Yau's paper [L-Y] for the case when Ric ≥ 0.

2. Asymptotic behavior of the trace of the heat kernel

The main tool for studying the asymptotic behavior of the trace of the heat kernel is the Pseudo-differential calculus (see, e.g., P. Gilkey [G, Lecture 3]). In the following we will briefly mention this approach.

Recall the Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} d\xi$$

where f is analytic in a neighborhood of the closed curve C and z is inside C. We apply this formula to the elliptic operator $D : L^2(M) \longrightarrow L^2(M)$. Define

$$f(D) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi I - D} d\xi$$

where C is a curve avoiding the spectrum of D. Then f is analytic outside a neighborhood of the spectrum of D.

Example. $f_s(z) = z^s$, $s \in C$ and $D = -\frac{d^2}{d\theta^2}$ on S^1. Let $\phi = \sum a_n e^{inx}$ be a function in $L^2(S^1)$. We have

$$\begin{aligned}(D^s\phi)(x) &= \sum a_n n^{2s} e^{inx} \\ &= \sum \Big(\int \phi(y) e^{-iny} dy\Big) n^{2s} e^{inx} \\ &= \int \Big(\sum n^{2s} e^{in(x-y)}\Big) \phi(y) dy.\end{aligned}$$

So kernel of D^s is $K_s(x,y) = \sum n^{2s} e^{in(x-y)}$ and in particular, $K_s(x,x) = \sum n^{2s}$ which converges when $\mathrm{Re}(s) < -\frac{1}{2}$. K_s as a function of s, which is closely related to the Riemann zeta function, is meromorphic in all C with only a simple pole at $s = -\frac{1}{2}$.

Take $f(\xi) = e^{-t\xi}$. It turns out that e^{-tD} is a infinitely "smoothing" operator. Taking a "good" approximation D_ξ of $(\xi I - D)^{-1}$, and using the pseudo differential calculus on manifolds, we can get

Theorem (Minakshisundaram-Pleijel [M-P], McKean-Singer [M-S]). *For any positive self-adjoint elliptic operator D of order 2, we have*

$$K(x,x,t) \sim \sum_0^\infty B_k(x) t^{\frac{(k-\dim M)}{2}} \quad as \quad t \to 0^+$$

where B_k is computable in terms of the symbol σ_D of D and derivatives of σ_D. If $\partial M = \emptyset$, then $B_k \equiv 0$ when k is odd.

Let us denote $a_k = \int B_{2k}(x) dV(x)$. Applying the above theorem to $D = \Delta$ on a manifold, we get

Theorem [M-K]. *Suppose (M, g) is a compact Riemannian manifold without boundary. Then*

$$\begin{aligned} Z(t) &\sim (4\pi t)^{-\frac{n}{2}}(a_0 + a_1 t + a_2 t^2 + \cdots) \quad as \quad t \to 0^+, \quad where \\ a_0 &= Vol(M) \\ a_1 &= \frac{1}{3}\int_M K dV \\ a_2 &= \frac{1}{180}\int_M (10A - B + 2C) dV \end{aligned}$$

where K is the Gaussian curvature of the metric (M, g), and where A, B, C are polynomials of degree 2 in the curvature tensor R_{ijkl}.

The formula which we will use in this lecture is the following:

Corollary. *When M is a compact closed surface ($\dim(M) = 2$), then*

$$\int_M K = 2\pi\chi(M) = 2\pi(2 - 2g)$$

hence

$$K(x,x,t) \sim \frac{1}{4\pi t} + \frac{K(x)}{12\pi} + O(t) \ as \ t \to 0^+$$

and

$$Z(t) \sim \frac{area(M)}{4\pi t} + \frac{\chi(M)}{6} + \frac{\pi t}{60}\int_M K^2 + O(t^2) \ as \ t \to 0^+.$$

Corollary. *$\chi(M)$ is audible.*

Remark. A computation of a_1 can also be found in [G, Ch 6].

3. Ray-Singer-Polyakov log determinant formula on a compact surface without boundary

Recall Weyl's formula $\lambda_k^{\frac{n}{2}} \sim C_n(\frac{k}{\mathrm{Vol}(M)})$. Then $\lambda_k^{-s} \sim C_n k^{-\frac{2}{n}s}$. In particular $\sum_0^\infty \lambda_k^{-s}$ converges if $\mathrm{Re}(s) > \frac{\dim M}{2} = \frac{n}{2}$.

Now let $n = 2$ and define the zeta function with $\mathrm{Re}(s) > 1$ as $\zeta(s) = \sum_{k>0} \lambda_k^{-s}$. By using $\frac{d}{ds}\lambda^{-s} = -\lambda^{-s}\log\lambda$, we can formally write

$$\zeta'(s) = -\sum \lambda_k^{-s}\log\lambda_k$$

and

$$-\zeta'(0) \text{``}=\text{''} \sum \log\lambda_k$$

where we must notice that the last sum diverges in general. Now we would like to formally define $\det\Delta = e^{-\zeta'(0)} \text{``}=\text{''} \Pi_{k=1}^\infty \lambda_k$ if $\zeta'(0)$ exists. To see $\zeta'(0)$ is well-defined, with $\mathrm{Re}(s) > 1, x > 0$ we argue as follows:

$$x^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty e^{-xt}t^{s-1}dt$$

where $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$ is the gamma function. So $\lambda_k^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty e^{-\lambda_k t}t^{s-1}dt$ and hence

$$\begin{aligned}\zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty (\sum_{k=1}^\infty e^{-\lambda_k t})t^{s-1}dt \\ &= \frac{1}{\Gamma(s)}\int_0^\infty (\mathrm{Tr}\, e^{-t\Delta} - 1)t^{s-1}dt\end{aligned}$$

To see the behavior as $t \longrightarrow 0^+$ of the above integral, we apply Theorem 2 and get

$$\mathrm{Tr}\, e^{-t\Delta} \sim \frac{A}{4\pi t} + \frac{\chi(M)}{6} + O(t) \text{ as } t \longrightarrow 0^+$$

Thus for $\mathrm{Re}(s) > -1$ we have

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^1 (\frac{A}{4\pi t} + \frac{\chi(M)}{6} + O(t) - 1)t^{s-1}dt + \text{ analytic function in } s$$

For s close to zero, we may write

$$\zeta(s) = \frac{1}{\Gamma(s)}(\frac{A}{4\pi(s-1)} + (\frac{\chi(M)}{6} - 1)\frac{1}{s} + \frac{1}{s+1}) + \text{ analytic function in } s$$

Here we have used the property that $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt = \frac{1}{s} - \frac{1}{s+1} + \cdots = \frac{1}{s} +$ analytic function in s around $s = 0$. Thus $\zeta(s)$ is well-defined for $\mathrm{Re} s > -1$ and meromorphic with pole at $s = +1$ (it is analytic at $s = 0$). We know $\zeta(0) = \frac{\chi(M)}{6} - 1$, so we define

$$\zeta'(s)|_{s=0} = \lim_{s \longrightarrow 0} \frac{\zeta(s) - \zeta(0)}{s}$$

which is our definition of $-\log \det \Delta$.

Ray-Singer-Polyakov Formula [P1] [P2] [R-S]. *Suppose (M, g_0) is a compact closed surface, and $g_1 = e^{2u} g_0$ is a metric conformal to g_0 with $vol(M, g_1) = vol(M, g_0)$. Then we have*

$$\log \frac{\det \Delta_1}{\det \Delta_0} = -\frac{1}{12\pi} \int_M (2K_0 u + |\nabla_0 u|^2) dV_0$$

where Δ_i, $(i = 1, 2)$ denotes the Laplacian operators with respect to g_i respectively.

Proof. Since $g_1 = e^{2u} g_0$, we have $\Delta_1 = e^{-2u} \Delta_0$ with volume $A = \int_M dV_1 = \int_M e^{2u} dV_0$ and

$$\zeta_0(s) - \zeta_1(s) = \frac{1}{\Gamma(s)} \int_0^\infty (Tr(e^{-t\Delta_0}) - Tr(e^{-t\Delta_1})) t^{s-1} dt.$$

Thus

$$\begin{aligned} I = \log \frac{\det \Delta_1}{\det \Delta_0} &= \lim_{s \to 0} \frac{\zeta_0(s) - \zeta_1(s)}{s} \\ &= \lim_{s \to 0} \frac{1}{s\Gamma(s)} \int_0^\infty (Tr(e^{-t\Delta_0}) - Tr(e^{-t\Delta_1})) t^{s-1} dt \end{aligned}$$

Consider the one parameter family of metrics $g_\alpha = e^{2u_\alpha} g_0$, where $\rho_\alpha = e^{2u_\alpha} = (1 - \alpha) + \alpha e^{2u}$, $0 \leq \alpha \leq 1$. Denote $\Delta_\alpha = e^{-2u_\alpha} \Delta_0$, then the curvature K_α of the metric $g_\alpha = \rho_\alpha g_0$ satisfies

$$\Delta_0 u_\alpha + K_\alpha e^{2u_\alpha} = K_0.$$

Observe that Δ_α and $\tilde{\Delta_\alpha} = \rho_\alpha^{-\frac{1}{2}} \Delta_0 \rho_\alpha^{-\frac{1}{2}}$ have the same set of eigenvalues (hence the same trace of the heat kernel) with the advantage that the domain of $\tilde{\Delta_\alpha}$ is

$L^2(dV_0)$. Hence

$$
\begin{aligned}
I &= \int_0^\infty (Tr(e^{-t\Delta_0}) - Tr(e^{-t\Delta_1}))\frac{dt}{t} \\
&= -\int_0^\infty \frac{1}{t}\int_0^1 \frac{d}{d\alpha} Tr(e^{-t\tilde{\Delta}_\alpha}) d\alpha dt \\
&= \int_0^\infty \int_0^1 Tr(e^{-t\tilde{\Delta}_\alpha}\frac{d\tilde{\Delta}_\alpha}{d\alpha}) d\alpha dt \\
&= -\int_0^\infty \int_0^1 Tr(\frac{\rho'_\alpha}{\rho_\alpha} e^{-t\tilde{\Delta}_\alpha}\tilde{\Delta}_\alpha) d\alpha dt \quad (\text{ where } \rho'_\alpha = \frac{d}{d\alpha}\rho_\alpha \text{ }) \\
&= \int_0^1 d\alpha \int_0^\infty Tr(\frac{d}{dt}\frac{\rho'_\alpha}{\rho_\alpha} e^{-t\tilde{\Delta}_\alpha}) dt \\
&= \int_0^1 d\alpha \int_0^\infty \frac{d}{dt} Tr(\frac{\rho'_\alpha}{\rho_\alpha} e^{-t\tilde{\Delta}_\alpha}) dt \\
&= \int_0^1 d\alpha \, Tr(\frac{\rho'_\alpha}{\rho_\alpha} e^{-t\tilde{\Delta}_\alpha})|_0^\infty \\
&= -\int_0^1 d\alpha (\lim_{\epsilon\to 0} Tr(\frac{\rho'_\alpha}{\rho_\alpha} e^{-\epsilon\tilde{\Delta}_\alpha})).
\end{aligned}
$$

To compute $\lim_{\epsilon\to 0} Tr(\frac{\rho'_\alpha}{\rho_\alpha} e^{-\epsilon\tilde{\Delta}_\alpha})$, we apply the asymptotic formula for the trace of heat-kernel for $\tilde{\Delta}_\alpha$, and get

$$
\begin{aligned}
I_\epsilon(\alpha) &= \lim_{\epsilon\to 0} Tr(\frac{\rho'_\alpha}{\rho_\alpha} e^{-\epsilon\tilde{\Delta}_\alpha}) \\
&= \lim_{\epsilon\to 0}\sum_k \int_M \frac{\rho'_\alpha}{\rho_\alpha} e^{-\epsilon\tilde{\lambda}_k^{(\alpha)}} (\tilde{\phi}_k^{(\alpha)}(x))^2 d\tilde{V}_\alpha
\end{aligned}
$$

where $\tilde{\lambda}_k^{(\alpha)}$, and $\tilde{\phi}_k^{(\alpha)}(x)$ are eigenvalues and eigenfunctions of $\tilde{\Delta}_\alpha$. Notice that $\tilde{\lambda}_k^{(\alpha)} = \lambda_k^{(\alpha)}$, and $\phi_k^{(\alpha)}(x) = \rho_\alpha^{-1/2}\tilde{\phi}_k^{(\alpha)}(x)$ are eigenvalues and eigenfunctions for Δ_α. We get

$$
I_\epsilon(\alpha) = \lim_{\epsilon\longrightarrow 0}\int_M \rho'_\alpha(x)\frac{1}{4\pi\epsilon}(1 + \frac{\epsilon}{3}K_\alpha(x) + O(\epsilon^2)) dV_0(x).
$$

Notice that by our assumption $\int \rho_\alpha(x)dV_0(x) = A$ is fixed, thus $\int_M \rho'_\alpha(x)dV_0(x) = 0$. Hence

$$\begin{aligned}
I &= -\int_0^1 d\alpha I_\epsilon(\alpha) = -\int_0^1 \lim_{\epsilon\longrightarrow 0}\int_M \rho'_\alpha(x)\cdot\frac{1}{12\pi}K_\alpha(x)dV_0(x)\\
&= -\frac{1}{12\pi}\int_0^1 d\alpha\int_M \frac{\rho'_\alpha(x)}{\rho_\alpha(x)}(K_0-\Delta_0 u_\alpha)(x)dV_0(x)\\
&= -\frac{1}{12\pi}\int_0^1 d\alpha\int_M 2u'_\alpha(K_0-\Delta_0 u_\alpha)dV_0(x)\\
&= -\frac{1}{12\pi}\int_0^1 d\alpha\int_M \frac{d}{d\alpha}(2K_0u_\alpha+|\nabla u_\alpha|^2)dV_0(x)\\
&= -\frac{1}{12\pi}\int_M (2K_0u+|\nabla u|^2)dV_0.
\end{aligned}$$

Which finishes the proof of the formula.

We will now state a easy corollary of the formula

Corollary. *On (M, g_0), with $K_0 \equiv$ non-positive constant, log det Δ_0 is extremal among all metrics g_0 on M with $Vol(g_1) = Vol(g_0)$.*

Proof. Denote $g_1 = e^{2u}g_0$, then $Vol(g_1) = Vol(g_0)$ implies that

$$\exp\left(\int 2udV_0/\int dV_0\right) \le \int e^{2u}dV_0/\int dV_0 = 1,$$

hence $\int 2udV_0 \le 0$. Thus if $K_0 \equiv$ non-positive constant, we have $\int 2K_0udV_0 \ge 0$. Apply the Ray-Singer-Polyakov formula above, we have

$$\log\frac{det\Delta_1}{det\Delta_0} \le 0 \quad \text{for} \quad \Delta_1 = \Delta_{g_1};$$

i.e. log det Δ_0 is maximum among all metrics g_1 with $Vol(g_1) = Vol(g_0)$.

In the next chapter, we will discuss the case when $K_0 \equiv$ positive constant, i.e. when $M =$ the unit sphere S^2.

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LECTURE 3
Moser-Onofri Inequality and Applications

Let's recall the Ray-Singer-Polyakov's formula on a compact surface. Suppose that $g_1 = e^{2u} g_0$ and g_0 are two conformal metrics on a compact surface M without boundary with $\Delta_1 = e^{-2u}\Delta_0$. Then we have

$$\log \frac{\det \Delta_1}{\det \Delta_0} = -\frac{1}{12\pi} \int_M (2K_0 u + |\nabla_0 u|^2) dV_0$$

In the special case with $M = S^2$ and g_0 the standard metric of constant Gaussian curvature $K_0 = 1$, for any u with $\int_{S^2} e^{2u} dV_0 = 4\pi$ and $\Delta_u = e^{-2u}\Delta_0$,

$$\log \frac{\det \Delta_u}{\det \Delta_0} = -\frac{1}{12\pi} \int_{S^2} (2u + |\nabla u|^2) dV_0$$

Onofri's inequality.

(1) $$\frac{1}{4\pi} \int_{S^2} e^{2u} \leq \exp(\frac{1}{4\pi} \int_{S^2} (2u + |\nabla u|^2))$$

with equality iff $e^{2u} g_0$ is isometric to g_0.

Corollary. *Among all metrics on S^2, $\log \det \Delta_0$ is the maximum.*

Remark. This should be compared to result of Hersch that $\lambda_1 \leq 2$.

There are now many different proofs of Onofri's inequality. The author knows at least four: Onofri's original proof [O] and [O-V], an independent proof by Hong [H], subsequent proof by O-P-S [O-P-S-1] and a recent proof by Beckner. Here we will present the original proof by Onofri as it connects the inequality to conformal geometry on S^2. All these proofs depend on the following inequality of Moser:

Moser ([M]). *If u is smooth on S^2 then there is some constant C_1 such that*

(2) $$\int_{S^2} exp\left(\frac{\alpha(u - \bar{u})^2}{\int_{S^2} |\nabla u|^2 dV_0}\right) \leq C_1$$

for any $\alpha \le 4\pi$, *with* 4π *being the best constant, i.e., if* $\alpha > 4\pi$, *the integral can be made arbitrarily large by appropriate choice of* u. *In the above inequality* $\bar{u} = \frac{1}{4\pi}\int_{S^2} u dV_0$.

Remark 1. Trudinger first proved that there exists some constant α so that the above inequality (2) holds. Inequality (2) should be considered as the limiting case of Sobolev imbedding Theorem: $W_0^{1,q} \hookrightarrow L^p$ with $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ for $1 < q < n$. If $q > n$, then $W_0^{1,q}$ can be identified with Hölder continuous function with Hölder exponent $1 - \frac{n}{q}$. When $q = n$, $W_0^{1,n} \not\subset L^\infty$, but $W_0^{1,n}$ is contained in functions of the exponential class: $W_0^{1,n}(D) \hookrightarrow \exp \frac{n}{n-1}$ class, i.e., for any bounded domain D in R^n, there is a constant $\alpha = \alpha(D)$ and constant $C = C(D)$ so that if u is in $W_0^{1,n}(D)$ with $\int_D |\nabla u|^n \le 1$ then

$$\int_D \exp(\alpha |u|^{\frac{n}{n-1}}) \le C\text{meas}(D).$$

Later Moser improved Trudinger's inequality and obtained the value of the best constant α, in the case of plane domains $\alpha = 4\pi$. .

Remark 2. Moser's proof is quite sharp and delicate. We will not include his original proof here. (An easier proof is included in D. Adams ([A]) given by A. Garsia). But the key elements in the proof are:

Step 1. Apply "symmetrization" based on "isoperimetric inequality" and reduce the function to one variable, i.e.,

Exercise. Let u^* be symmetric decreasing function with same distribution function as u. Then $\int |\nabla u^*|^2 \le \int |\nabla u|^2$.

Step 2. The one-variable inequality after the change of variable $t = logr$ becomes the following: Given w smooth on $(-\infty, \infty)$ with $\dot{w} \ge 0$, $\int_{-\infty}^{\infty} \dot{w}^2 dt \le 1$, $\int_{-\infty}^{\infty} w\rho dt = 0$ for some positive function ρ with $\int_{-\infty}^{\infty} \rho = 1$, $\rho(t) \le C_0 e^{-|t|}$; then $\int_{-\infty}^{\infty} e^{w^2(t)}\rho(t)dt$ is bounded independent of w.

We will present in Lecture 4 a proof of Moser's inequality based on step 2 above given by [C-C], the proof also indicates that an extremal function for the inequality exists.

Remark 3. We call u defined on S^2 an even function if $u(\xi) = u(-\xi)$ for all $\xi \in S^2$. The best constant α in Moser's inequality for functions in the even class is $\alpha = 8\pi$.

To see this we have the following lemma.

Lemma [C-Y-5]. *Assume that* D *is a piecewise* C^2 *domain in* R^2 *with* $\int_D |\nabla u|^2 = 1$ *and* $\int_D u = 0$. *Let* L_M = *length of* $\{x \in D | u(x) = M\}$ *and* A_M = *area of* $\{x \in D | u(x) \ge M\}$. *If* $\alpha_u = \sup_{M \to \infty, |A_M| \to 0} \frac{L_M^2}{A_M}$, *then there exists a constant* C_2 *(independent of* u*) so that*

$$\int_D e^{\alpha_u u^2} \le C_2 |D|.$$

In particular,

(a) when u is smooth function with compact support in D, then $\alpha_u \geq 4\pi$.

(b) When $D = S^2$, we have the isoperimetric inequality $L^2 \geq A(4\pi - A)$, hence $\alpha_u \geq 4\pi$.

(c) If u is even, for two small regions on S^2 symmetric under antipodal map, $L = L_1 + L_2$ with $L_1 = L_2$, $A = A_1 + A_2$ with $A_1 = A_2$, and $L_i^2 \geq A_i(4\pi - A_i)$, thus $L^2 = 4L_i^2 \geq 4A_i(4\pi - A_i) = A(8\pi - A)$. So $\alpha_u \geq 8\pi$.

Corollary.

$$\frac{1}{4\pi}\int_{S^2} e^{2u} \leq C_2 \exp\left(\frac{1}{4\pi}\int_{S^2} |\nabla u|^2 + 2\frac{1}{4\pi}\int_{S^2} u\right) \tag{4}$$

with $C_2 \leq C_1$ in (2).

Proof.

$$2(u - \bar{u}) \leq \frac{4\pi(u-\bar{u})^2}{\int_{S^2} |\nabla u|^2 dV_0} + \frac{1}{4\pi}\int_{S^2} |\nabla u|^2 dV_0$$

by using $2ab \leq a^2 + b^2$.

Notice the best constants C_1 in (2) is greater than 1, while the statement in Onofri's inequality indicates the best constant C_2 in (4) is 1.

Proof of Onofri's inequality. We will use the following fact:

Lemma 1. *Let $S[u] = \frac{1}{4\pi}\int_{S^2} |\nabla u|^2 + 2\frac{1}{4\pi}\int_{S^2} u$. Then $S[u]$ is conformally invariant in the following sense: Given ϕ a conformal transformation of S^2, let $T_\phi u = v$ denote the induced conformal factor given by $\phi^*(e^{2u}g_0) = e^{2v}g_0$, of more explicitly, $T_\phi u = (u \circ \phi) + \frac{1}{2}\log|J_\phi|$. Then $S[T_\phi u] = S[u]$.*

Proof. Since log determinant is an intrinsic quantity of the metric it is invariant under conformal transformation, $S[T_\phi u] = S[u]$ as a consequence of Polyakov's formula.

Remark. This lemma also can be proved directly without applying Polykov's formula, but based on the following corollary.

Corollary. *If $u = \frac{1}{2}\log|J_\phi|$ for some conformal transformation ϕ, then $S[u] = 0$.*

Definition. We say $u \in S$ if $\int_{S^2} e^{2u}x_j = 0$ for $j = 1, 2, 3$.

Lemma 2. *Given a C^1 function u defined on S^2, there exists a conformal transformation ϕ such that $T_\phi u \in S$.*

Proof. Consider the following family of conformal transformations of S^2. Given $P \in S^2$, $t \in [1, \infty)$, rotate the coordinates so that P corresponds to the north pole $(0, 0, 1)$. Using the stereographic projection from P mapping the sphere to the equatorial plane on which we have the complex coordinate z, we denote the transformation $\phi_{P,t}(z) = tz$. Observe that $\phi_{P,1} = id, \phi_{P,t} = \phi_{-P,t^{-1}}$, hence this set of conformal transformations can be parametrized by the unit ball $B^3 = S^2 \times [1,\infty)/S^2 \times \{1\}$. Notice that for $\phi = \phi_{P,t}$ we may write with a change of variable

$$\int e^{2T_\phi u} x_j = \int e^{2u}(x_j \circ \phi_{P,t}^{-1}).$$

Define the center of mass of the mass distribution e^{2u} by $C.M.(\phi_{P,t}) = \frac{\int e^{2u} x_j \circ \phi_{P,t}^{-1}}{\int e^{2u}}$; $j = 1, 2, 3$. The center of mass map may be considered as a map from B^3 to B^3, it has a continuous extension to the boundary mapping P to $-P$. It follows from the Brouwer's fixed point theorem that there exists $P \in S^2$, $t \in [1,\infty)$ for which $C.M.(\phi_{P,t}) = 0$.

Lemma 3 (Aubin [Au]). *If $u \in S$, for all $\epsilon > 0$ there is a constant C_ϵ such that*

$$\frac{1}{4\pi}\int_{S^2} e^{2u} \le C_\epsilon \exp((\frac{1}{2}+\epsilon)\frac{1}{4\pi}\int_{S^2}|\nabla u|^2 + 2\frac{1}{4\pi}\int_{S^2} u)$$

We will now finish the proof of Onofri's inequality based on Lemmas 1, 2 and 3.

Proof. Consider the functional

$$F[u] = \log\frac{1}{4\pi}\int_{S^2} e^{2u} - S[u]$$

defined for all u on $W^{1,2}(S^2)$. Apply Moser's inequality, we get

$$\log\frac{1}{4\pi}\int_{S^2} e^{2u} \le S[u] + C_1$$

so $F[u] \le C_1$. Choose $u_k \in W^{1,2}(S^2)$ so that $F[u_k] \longrightarrow M = \max F[u]$.

Apply Lemma 1, we know $F[u] = F[T_\phi u]$ for any conformal transformation. Apply Lemma 2, we may assume w.l.o.g. that $u_k \in S$ and $\int u_k = 0$. Apply Lemma 3, then

$$\log\frac{1}{4\pi}\int_{S^2} e^{2u_k} \le C_\epsilon + (\frac{1}{2}+\epsilon)\frac{1}{4\pi}\int_{S^2}|\nabla u_k|^2 + 2\frac{1}{4\pi}\int_{S^2} u_k$$

Hence

$$M - \epsilon \le F[u_k] \le C_\epsilon - (\frac{1}{2}-\epsilon)\frac{1}{4\pi}\int_{S^2}|\nabla u_k|^2$$

for all $\epsilon < \frac{1}{2}$. i.e.,

$$\frac{1}{4\pi}\int_{S^2}|\nabla u_k|^2 \le C_\epsilon + \epsilon - M$$

is bounded. Choose $\int u_k = 0$. Thus $u_k \longrightarrow u$ weakly in $W^{1,2}$ and $\int |\nabla u|^2 \leq \int |\nabla u_k|^2$, $F[u] \geq F[u_k]$ and $\int e^{2u_k} \longrightarrow \int e^{2u}$. So $F[u]$ attains the maximum value and u satisfies

$$\Delta u + e^{2u} = 1$$

with $u \in S$. This implies that $e^{2u} g_0$ has Gauss curvature 1, which is equivalent to $u = \frac{1}{2} \log |J_{\phi_{p,t}}|$ for some conformal transformation ϕ, but among this class of functions only the constant function $u = 0$ belongs to the symmetric class S.

Exercise. Prove that if $\psi_{p,t} = \frac{1}{2} \log |J_{\phi_{p,t}}|$, then $\int |\nabla \psi_{p,t}|^2 \longrightarrow +\infty$ as $t \longrightarrow \infty$ and $\int \psi_{p,t} \longrightarrow -\infty$ as $t \longrightarrow \infty$. While $\int |\nabla \psi_{p,t}|^2 + 2\int \psi_{p,t} = 0$.

As an application, we have the following result due to Osgood-Phillips-Sarnak:

Theorem [O-P-S-2]. *An isospectral family of metrics on a compact surface without boundary form a compact set in the C^∞- topology.*

Sketch of Proof. We will only prove here the genus zero case, i.e., $M = S^2$. In this case the theorem says given $\{e^{2u_k}\}$ with $\Delta_k = e^{-2u_k}\Delta_0$ having the same spectral data, then the conformal class $[u_k]$ of u_k forms a C^∞- compact family.

Recall the heat kernel asymptotics:

$$\operatorname{Tr} e^{-\Delta t} \longrightarrow (4\pi t)^{-1}(a_0 + a_1 t + a_2 t^2 + \cdots) \text{ as } t \to 0^+$$

with

$a_0 = \int dV = \int e^{2u} dV_0$,

$a_1 = \frac{1}{3}\int_M K dV = \frac{4\pi}{6}\chi(M) = \frac{4\pi}{3}$,

$a_2 = \frac{\pi}{60}\int K^2 dV = \frac{\pi}{60}\int (1 - \Delta_0 u)^2 e^{-2u} dV_0$.

For $k \geq 3$, $a_k = \int_M B_{2k} dV$ where B_{2k} are universal polynomials of weight $2k$ in K and Δ with K, Δ each counting as degree 2. a_0 is fixed while a_2 gives some information of $\Delta_0 u$. Thus we need some information about the 1st derivative of u (which is missing — which a_1 does not provide); while leading term a_k provides some informations about $W^{k,2}$ norm of u.

The key idea is to use the log determinant formula to bound $\int |\nabla u|^2$.

$$\log \frac{\det \Delta_u}{\det \Delta_0} = -\frac{1}{3} S[u]$$

here $\int e^{2u} = 4\pi$ say. W.l.o.g., assume that $u \in S$, then Aubin's inequality implies that $\int |\nabla u|^2 \leq$ bdd.

Remark. In the papers of [O-P-S-1], they have also treated the higher genus case, and also the class of simply connected plane domains. In the case of plane domain (via conformal map, the unit disc), the role played by Moser's inequality is replaced by Beurling's inequality, the role of Onofri's inequality is replaced by the Lebedev-Milin inequality and the analogue of the Ray-Singer-Polyakov formula is proved by Alvarez [Al].

Beurling's inequality. *Let D be the unit disc and f be an analytic function on D. If $\int\int_D |f'(z)|^2 dxdy \leq \pi$, $f(0) = 0$, then*

$$|\{\theta \in \partial D : |f(e^{i\theta})| > t\}| \leq 2\pi e^{1-t^2}.$$

Lebedev-Milin's inequality.

$$\log \int_{\partial D} e^{2u} \frac{d\theta}{2\pi} \leq \int_D |\nabla u|^2 \frac{dxdy}{\pi} + 2 \int_{\partial D} u \frac{d\theta}{2\pi}$$

where u is harmonic in D.

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LECTURE 4
Existence of Extremal Functions for Moser Inequality

In this chapter, we will describe a proof to establish the existence of extremal functions for Moser-Trudinger inequality as stated in the previous lecture. The proof has appeared in [C-C], see also [St].

Theorem 1. *Let B_n denote the unit ball in R^n, $n \geqq 2$, then*

$$\sup \int_{B_n} \exp(\alpha_n |u|^{\frac{n}{n-1}})\, dx$$

is attained for functions $u \in W_0^{1,n}(B_n)$ with $\int_{B_n} |\nabla u|^n dx \leqq 1$, where $\alpha_n = n(\omega_{n-1})^{\frac{1}{n-1}}$, ω_{n-1} is the area of $(n-1)$ spheres.

We first remark that the result is surprising in the following sense: Moser's inequality is the limiting case of the Sobolev Embedding Theorem $W_0^{1,q}(D) \subset L^p(D)$ with $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ for $1 \leq q < n$ and domain D in R^n. In the limiting case for $q = n$, we have $W_0^{1,n}(D) \subset \exp \frac{n}{n-1}$ class, while the embedding with exponent α_n is not compact. In another well-known case of Sobolev Embedding Theorem $W_0^{1,2}(D) \subset L^{\frac{2n}{n-2}}$, while the embedding is not compact, the extremal function for the embedding does not exist unless $D = R^n$. In the latter case the study of the noncompactness phenomenon leads to the solution of "Yamabe" problem ([A] [S-1]), and many other interesting problems in non-linear PDE (cf [B-N]). In contrast, in the case of the Moser functional, the extremal function is attained. We also want to point out that the proof we shall present below is an existence proof, and the explicit expression of the extremal function is unknown and remains an open question.

Our second remark is that we have obtained the existence result only for $D = B_n$: the unit ball in R^n. The reason for this constraint is that only in this case can we apply the symmetrization technique mentioned in chapter 3 to reduce Theorem 1 to the following equivalent form by a change of variables as in [M].

Theorem 2. *Let K denote the class of C^1 functions $\omega(t)$ defined on $0 \leqq t \leq \infty$ satisfying*

$$\omega(0) = 0, \ \dot{\omega}(t) \geqq 0 \ \textit{and} \int_0^\infty \dot{\omega}^n(t)\, dt \leqq 1,$$

then

$$\sup_{\omega \in K} \int_0^\infty e^{\omega^{\frac{n}{n-1}}(t)-t}\, dt \tag{1}$$

is attained by some function $\omega^* \in K$.

In a recent paper by M. Flucher [F], using an ingenious method in complex analysis, he was able to generalize the existence part of Theorem 1 to general domains in R^2. In another related development, T.L. Soong [So] has proved the analogous results of Theorem 1 for functions u defined on S^2 with $\int_{S^2} u = 0$ and $\int_{S^2} |\nabla u|^2\, dx \leqq 1$, and some partial results for functions v defined on S^4, with $\int_{S^4} v = 0$ and $< Pv, v > \leqq 1$ where $P = (-\Delta)^2 - 2\Delta$ is the Panitz operator on S^4.

In the remaining part of this chapter, we will prove Theorem 2 for the special case $n = 2$. The proof for general n is only more complicated in the technical sense.

Proof of Theorem 2, for n=2.

The existence Theorem was established by contradiction. This is accomplished through the following two steps:

Step 1 Assume there is no extremal function attaining the supremum in (1), then

$$\sup_{\omega \in K} \int_0^\infty e^{\omega^2(t)-t}\, dt \leqq 1 + e.$$

Step 2 There exists some $\omega \in K$ with

$$\sup_{\omega \in K} \int_0^\infty e^{\omega^2(t)-t}\, dt > 1 + e.$$

The proofs of both step 1 and step 2 are motivated by the existence of a sequence of "broken line" functions $\{\omega_a\}_{a>0}$ defined as follows: $\omega_1(t) = t$ when $0 \leqq t \leqq 1$, $\omega_1(t) = 1$ when $t \geqq 1$, and $\omega_a(t) = \sqrt{a}\,\omega_1(\frac{t}{a})$. This is the sequence of functions which has been used by Moser [M] in establishing that $\alpha_2 = 4\pi$ is the best exponent in the embedding $W_0^{1,2} \subset L^2$. Notice that for this sequence of functions $\{\omega_a\}$, $\omega_a \in K$, $\omega_a(x) \to 0$ as $a \to \infty$ uniformly in compact subsets, and $I(\omega_a) = \int_0^\infty e^{\omega_a^2(t)-t}\, dt \nrightarrow I(0) = 1$. The intuitive idea in the proof of the theorem is to prove that if the extremal of $I(\omega)$ for $\omega \in K$ does not exist, the maximal sequence would behave quite like the sequence $\{\omega_a\}$.

Proof of step 2

Define

$$\omega(t) = \begin{cases} \frac{t}{2} & \text{when } 0 \leqq t \leqq 2, \\ (t-1)^{\frac{1}{2}} & \text{when } 2 \leqq t \leqq e^2+1, \\ e & \text{when } t > e^2+1 \end{cases}$$

then

$$I(\omega) = 2\int_0^1 e^{t^2-2t}\,dt + e$$

$$= \frac{2}{e}\int_0^1 e^{s^2}\,ds + e > \frac{2.723}{e} + e > 1+e,$$

based on lower Riemann sum estimate of the integral $\int_0^1 e^{s^2}\,ds$.

Proof of step 1

Choose a sequence $\omega_m \in K$ with $I(\omega_m) \to M = \sup_{\omega\in K} I(\omega)$. Assume that there does not exist any $\omega \in K$ attaining the supremum M, we claim that the sequence ω_m has the following properties:

(a) For each $A > 0$, $\int_0^A (\dot{\omega}_m)^2\,dt \to 0$ as $m \to \infty$,

(b) Let a_m be the first in $[1,\infty)$ with $\omega_m^2(a_m) = a_m - 2\log a_m$ (if such a_m exists). Then $a_m \to \infty$ as $m \to \infty$.

(c) $\lim_{m\to\infty} \int_0^{a_m} e^{\omega_m^2(t)-t}\,dt = 1$,

(d) $\limsup_{m\to\infty} \int_{a_m}^{\infty} e^{\omega_m^2(t)-t}\,dt \leqq e$.

(a) (b) can be established via argument by contradiction and elementary calculus.

To prove (c), we notice that it follows from (a) that $\omega_m \to 0$ uniformly on any compact subset of $[0,\infty)$. Thus for each $\epsilon > 0$, A and m large with $\omega_m^2(t) \leqq \epsilon$ for $t \leqq A$, using the property that a_m is the first point with $\omega_m^2(t) \geqq t - 2\log^+ t$ we have

$$\int_0^{a_m} e^{\omega^2(t)-t}\,dt = \int_0^A + \int_A^{a_m}$$

$$\leqq e^{\epsilon}\int_0^A e^{-t}\,dt + \int_A^{a_m} e^{-2\log^+ t}\,dt$$

$$= e^{\epsilon}(1-e^{-A}) + \left(\frac{1}{A} - \frac{1}{a_m}\right) \leqq 1,$$

as $\epsilon \to 0$, $A \to \infty$. On the other hand

$$\int_0^{a_m} e^{\omega_m^2(t)-t}\,dt \geqq \int_0^{a_m} e^{-t}\,dt = 1 - e^{-a_m},$$

which tends to one as $a_m \to \infty$, which finishes the proof of (c).

To prove (d), we first establish a lemma.

Lemma 1. *Let $K_\delta = \{\phi : C^1$ functions defined on $0 \leqq t < \infty$, $\phi(0) = 0$, $\int_0^\infty \dot{\phi}^2\, dt \leqq \delta\}$, then for each $c > 0$ we have*

$$\sup_{\phi \in K_\delta} \int_0^\infty e^{c\phi(t)-t}\, dt < e \cdot e^{\frac{c^2}{4}\delta}. \tag{2}$$

Also when $c^2\delta \to \infty$, the inequality in (2) tends asymptotically to an equality.

Proof of Lemma 1. We first remark that since $\int_N^\infty e^{c\phi(t)-t}\, dt$ is uniformly small for all $\phi \in K_\delta$ as $N \to \infty$, it is easy to verify that the extremal function for $\sup_{\phi\in K_\delta} \int_0^\infty e^{c\phi(t)-t}\, dt$ exists.

Suppose $\phi \in K_\delta$ is such an extremal function, then via variational method, ϕ satisfies the following differential equation:

$$e^{c\phi(t)-t} = A\,\ddot{\phi}, \text{ for some constant } A. \tag{3}$$

Let $k(t) = c\phi(t) - t$, we may rewrite (3) into

$$e^{k(t)} = A\,\ddot{k}(t), \tag{4}$$

with $k(t)$ satisfying

$$\int_0^\infty (\dot{k} + 1)^2\, dt = c^2\delta, k(0) = 0; \tag{5}$$

hence $\dot{k}(\infty) = -1$, $k(\infty) = -\infty$. Multiply (4) by $\dot{k}$ and integrate, we get

$$e^{k(t)} = A(\frac{1}{2}(\dot{k}+1)^2(t) - (\dot{k}+1)(t) + C). \tag{6}$$

Letting $t \to \infty$ and using (5), we find $C = 0$. Compare (4) and (6) we get

$$\ddot{k}(t) = \frac{1}{2}(\dot{k}+1)^2(t) - (\dot{k}+1)(t). \tag{7}$$

After integrating (7), we have

$$1 + \dot{k}(t) = \frac{2}{1 + Be^t},$$

where the constant B is determined by the equality

$$c^2\delta = \int_0^\infty (\dot{k}+1)^2\, dt = 4(\log\frac{1+B}{B} - \frac{1}{1+B}). \tag{8}$$

Evaluating (6) at $t = 0$ we get

$$1 = A\left(\frac{2}{(1+B)^2} - \frac{2}{1+B}\right). \tag{9}$$

On the other hand, integrating (4) directly we get from (9)

$$J = \int_0^\infty e^{k(t)}\,dt = A\dot{k}(t)|_0^\infty = -\frac{2A}{1+B} = \frac{1+B}{B}.$$

Since $1 + \dot{k}(t) = c\phi(t) \geqq 0$ we have $1 + B \geqq 0$. Thus $B \geqq 0$. It follows from (9) that $B \to 0$ when $c^2\delta \to \infty$ and $J \leqq \exp(\frac{1}{4}c^2\delta + 1)$ with the inequality approaching an equality when $c^2\delta \to \infty$ as claimed in the statement of this lemma.

We may now apply the above lemma to finish the proof of (d) as follows: Let $x = t - a_m$, $\phi_m = \omega_m(x + a_m) - \omega_m(a_m)$, then $\omega_m(t) = \phi_m(x) + \omega_m(a_m)$ for all $x > 0$,

$$\omega_m^2(t) = \omega_m^2(a_m) + 2\omega_m(a_m)\phi_m(x) + \phi_m^2(x).$$

Since $\phi_m(0) = 0$, $\int_0^\infty \dot{\phi}_m^2(x)\,dx = \int_{a_m}^\infty \dot{\omega}_m^2(t)\,dt = \delta_m$. Thus if we set $y = (1-\delta)x$, $c = 2\omega_m(a_m)$ and $\phi(y) = \phi_m(x)$ we get

$$\int_{a_m}^\infty e^{\omega_m^2(t)-t}\,dt \leqq e^{\omega_m^2(a_m)-a_m}\frac{1}{1-\delta_m}\int_0^\infty e^{c\phi(y)-y}\,dy. \tag{10}$$

Applying the above lemma, we get

$$\int_{a_m}^\infty e^{\omega_m^2(t)-t}\,dt \leqq e^{k_m}\frac{1}{1-\delta_m}e,$$

where $k_m = \omega_m^2(a_m) - a_m + \frac{\delta_m}{1-\delta_m}\omega_m^2(a_m)$. Since $\omega_m \in K$, $\omega_m^2(a_m) \leqq (1-\delta_m)a_m$, from this using the definition of a_m we can check $\delta_m \leqq \frac{2\log^+ a_m}{a_m} \to 0$ as $m \to \infty$ and $k_m \to 0$ as $m \to \infty$. Thus it follows from (10) we have

$$\limsup_{m\to\infty}\int_{a_m}^\infty e^{\omega_m^2(t)-t}\,dt \leqq e,$$

which establishes (d). We have finished the proof of Theorem 2 for $n = 2$ case.

Remark. The author has learned from B. Beckner that Lemma 3 is actually equivalent to the Onofri's inequality on S^2 as stated in Lecture 3. This fact is not an obvious one.

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LECTURE 5
Beckner-Adams Inequalities and Extremal Log-Determinants in 4-D

In this chapter, we will discuss some attempts to generalize the results of Onofri and [O-P-S] to higher-dimensional manifolds.

Recall that Polyakov's formula depends highly on the invariant property of the Laplacian operator, i.e. if $g = e^{2w} g_0$, then

$$\int_M |\nabla u|^2 dV_g = \int_M |\nabla_0 u|^2 dV_0$$

for every u defined on a compact closed surface M. Rewrite

$$\int_M |\nabla u|^2 dV_g = -\int_M (\Delta u) u dV_g.$$

We see that this invariant property follows from the relation that $\Delta_g = e^{-2w} \Delta_{g_0}$.

Definition. We call an operator A a conformally covariant operator if $g = e^{2w} g_0$ implies that $A = e^{-bw} A_0 e^{aw}$ with $b - a = 2$.

Example 1. $A = \Delta$ then $b = 2$, $a = 0$ in the case dim $M = 2$.

Example 2. M is a closed manifold of dimension n and let

$$A = L = -c_n \Delta + R, \quad \text{where } c_n = \frac{4(n-1)}{n-2},$$

R is the scalar curvature of M. L is called the conformal Laplacian. Then L has the following conformal covariance property

$$L_\omega(\phi) = e^{-\frac{n+2}{2}\omega} L_0 (e^{\frac{n-2}{2}\omega} \phi)$$

for every $\phi \in C^\infty(M)$.

Exercise. Prove the conformally covariant property of the conformal Laplacian.

Hint: Let u denote $e^{\frac{n-2}{2}\omega}$, then u satisfies the equation $Lu = Ru^{\frac{n+2}{n-2}}$, where R is the scalar curvature of g.

In [B-O-1] (see also [P-R]), it was proved that for the conformal Laplacian operator L, the $\frac{n}{2}$-coefficient $a_{\frac{n}{2}}$ of the heat kernel for L is invariant under conformal change of metrics. And when n is odd, the log determinant of L is an invariant quantity.

Restricting our attention to $n = 4$, for a compact 4-manifold, the a_2 coefficient of the trace of the kernel for the conformal Laplacian operator is conformally invariant. The work of [O-P-S 1,2] thus suggests that the role of a_2 should be replaced by $\log\det L$ in the study of isospectral compactness problem. In this chapter, we shall discuss some aspects in this direction. First we recall the work of Branson-Orsted [B-O-2], where they obtained some generalized form of the Ray-Singer-Polyakov formula for log-determinants of conformally covariant operators in four dimension on compact locally symmetric Einstein manifolds. A special case of their formula is when the covariant operator is the conformal Laplacian L.

Suppose $V(M,g) = \int_M e^{4\omega} dV_0 = v_0$, then

$$F(\omega) = \log\frac{\det L_\omega}{\det L_0} = -2\{\frac{1}{4}k(M,g_0,L)\log\frac{1}{v_0}\int_M e^{4(\omega-\bar{\omega})}$$

$$-\int_M (\Delta\omega)^2 - R_0\int|\nabla\omega|^2\} - 4\{R_0\int_M|\nabla\omega|^2 - \frac{1}{2}\int_M\left(\frac{\Delta e^\omega}{e^\omega}\right)^2\}$$

where $k(M,g_0,L) = -\frac{3}{2}v_0 c^2 + 16\pi^2\chi$, and c^2 denotes the square-norm of the Weyl conformal curvature of g_0. Indeed, compactness information similar to those in two dimension can be obtained for metric $e^{2\omega}g_0$ with $F(\omega)$ bounded. This is formulated as the following result.

Theorem 1 [B-C-Y]. *If $k(M,g_0,L) < 32\pi^2$, and with normalized volume $\int_M e^{4\omega}dV_0 = v_0$, then $\|\omega\|_{2,2}$ is bounded by a constant depending only on $F(\omega)$.*

Examples of manifolds with $k(M,g_0,L) < 32\pi^2$: any manifold M which is compact and locally symmetric, Einstein but not (S^4,g_0), or a hyperbolic space form. $k(S^4,g_0,L) = 32\pi^2$. But RP^4, CP^2, $S^2\times S^2$, T^4 and compact quotients of a polydisc with standard metric all have $k(M) < 32\pi^2$.

Theorem 1'. *The conclusion of Theorem 1 holds on (S^4,g_0) with ω replaced by $T_\phi\omega$ for some ϕ a conformal transformation of S^4. ($T_\phi(\omega) = \omega_0\phi + \frac{1}{4}\ln|J_\phi|$, $e^{2T_\phi\omega}g_0 = \phi^*(e^{2\omega}g_0)$).*

The role $32\pi^2$ plays here is the same as that of 4π in Moser's inequality. Actually there is a higher dimensional analogue of Moser's inequality when $n \geq 2$ due to Adams, which we will now describe below.

Recall by the classical Sobolev embedding theorem, we have $W^{\alpha,q} \subset L^p$, when $\frac{1}{p} = \frac{1}{q} - \frac{\alpha}{n}$ if $q > 1$ and $\alpha q < n$. The following result deals with the limiting case $\alpha q = n$.

Theorem 2 (Adams). *Suppose $m < n$ are positive integers, Ω is a bounded domain in R^n. Then there are constants $C_0 = C(m,n)$, $\beta_0 = \beta(m,n)$ such that if $u \in W^{m,q}$ of compact support in Ω with $\|\nabla^m u\|_q \leq 1$, where $qm = n$, then for $\beta \leq \beta_0$,*

$$\int_\Omega \exp(\beta|u(x)|^{q'}) \leq C_0|\Omega|,$$

where $q' = \frac{q}{q-1}$ and β_0 is the best constant for the inequality to hold.

When $n = 4$, $q = m = q' = 2$ we have $\beta_0(2,4) = 32\pi^2$. L. Fontana [F] has generalized the above inequality to compact n-manifolds without boundary.

In particular we have

Theorem 2'. *Suppose that (M,g) is a compact 4-manifold without boundary, then there exists a constant C_0 such that for every $u \in C^2(M)$*

$$\int_M \exp\left(\frac{32\pi^2(u-\bar{u})^2}{\int_M(\Delta u)^2}\right)dV_g \leq C_0 Vol(M).$$

Corollary. *If M is a compact 4-manifold without boundary, then for every $\omega \in C^2(M)$,*

$$\log\frac{1}{Vol(M)}\int \exp(4(\omega-\bar{\omega})) \leq \log C_0 + \frac{1}{8\pi^2}\int_M(\Delta\omega)^2, \quad \textit{where } \bar{\omega} = \frac{1}{Vol(M)}\int\omega.$$

Suppose $k(M, g_0, L) < 32\pi^2$, then when $F_A(\omega)$ is bounded, we may apply the corollary above to show

$$\int_M(\Delta\omega)^2 + \int_M|\nabla\omega|^4$$

is bounded. The proof is in the same spirit as the proof of Onofri's inequality that if $u \in S$ then $S[u]$ is bounded implies that $\int_M|\nabla u|^2$ is bounded. Details of the argument are contained in [B-C-Y].

Restricting our attention to (S^4, g_0), where g_0 is the standard metric and $v_0 = Vol(S^4)$, we have

$$\begin{aligned} F(\omega) &= \text{constant}\cdot\log\frac{\det L_0}{\det L_\omega} \\ &= \left[\log\frac{1}{v_0}\int_{S^4}e^{4(\omega-\bar{\omega})} - \frac{1}{3}\frac{1}{v_0}\int_{S^4}(\Delta\omega)^2 - \frac{2}{3}\frac{1}{v_0}\int_{S^4}|\nabla\omega|^2\right] \\ &\quad + \frac{2}{3}\{4\frac{1}{v_0}\int_{S^4}|\nabla\omega|^2 - \frac{1}{v_0}\int_{S^4}\left(\frac{\Delta e^\omega}{e^\omega}\right)^2\} = I + \frac{2}{3}II. \end{aligned}$$

Theorem 3 [B-C-Y]. *On (S^4, g_0), $F(\omega) \leq 0$, and $F(\omega) = 0$ if and only if $e^{2\omega} g_0 = \phi^*(g_0)$ for some conformal transformation on S^4. That is, on S^4 the log determinant L is extremal if and only if g is isometric to the standard metric.*

It is interesting to see that the proof of the Theorem 3 can be formulated into two extremal inequalities.

Lemma 1. *$I \leq 0$ on S^4 and $I = 0$ iff $e^{2\omega} g_0 = \phi^*(g_0)$.*

Lemma 2. *$II \leq 0$ on S^4 and $II = 0$ iff $e^{2\omega} g_0 = \phi^*(g_0)$.*

Lemma 1 is a special case of an inequality of Beckner which holds for general n. It is a linearized version of the Adams inequality just as Onofri's inequality is a linearized version of Moser's inequality.

Theorem 4 (Beckner [B]). *If $f \in C^\infty(S^n)$ has an expansion $\sum_{k=0}^{\infty} Y_k$ in spherical harmonic functions Y_k, then*

$$\log \frac{1}{V(S^n)} \int_{S^n} e^{(f-\bar{f})} \leq \frac{1}{2n} \sum_{k=1}^{\infty} B(n,k) \frac{1}{V(S^n)} \int_{S^n} |Y_k|^2$$

where $B(n,k) = \Gamma(n+k)/\Gamma(n)\Gamma(k)$ and equality holds iff $e^{2f/n} g_0 = \phi^*(g_0)$ for some conformal transformation ϕ of S^n.

The proof of the Beckner inequality is quite delicate and depends on the Fourier analysis method (Lieb-Young inequality).

Proof of Lemma 1. Take $n = 4$ in Beckner's inequality, in this case,

$$B(4,k) = \frac{\Gamma(3+k)}{\Gamma(3)\Gamma(k)} = \frac{k(k+1)(k+2)(k+3)}{6}$$

Recall that the k-th eigenvalue of Δ on S^4 is $\lambda_k = k(k+3)$. Thus if Y_k is a harmonic polynomial of degree k, then $\Delta Y_k + k(k+3) Y_k = 0$.

Letting $f = e^{4\omega}$ we get :

$$\begin{aligned}
\log \frac{1}{V(S^4)} \int_{S^4} e^{4(\omega-\bar{\omega})} &\leq \frac{16}{8 \cdot 6} \sum_{k=0}^{\infty} \int_{S^4} k(k+1)(k+2)(k+3)|Y_k|^2 \\
&= \frac{1}{3} \sum_{0}^{\infty} \langle -\Delta(-\Delta+2) Y_k, Y_k \rangle \\
&= \frac{1}{3} \int_{S^4} \omega(-\Delta(-\Delta+2)\omega), \\
&= \frac{1}{3} \left(\int_{S^4} (\Delta\omega)^2 + 2 \int_{S^4} |\nabla\omega|^2 \right).
\end{aligned}$$

Thus Lemma 1 is equivalent to Beckner's inequality on S^4.

Remark. Beckner's proof also indicates that

$$S_4(\omega) = \int (\Delta\omega)^2 + 2|\nabla\omega|^2 + 4\bar{\omega}$$

is a conformal invariant and $S_4(\omega) \geq 0$ if $\int e^{4\omega} = Vol(S^4)$. In particular we have

$$P_4\omega + 4!e^{4\omega} = 4! \text{ on } S^4 \iff \omega = \frac{1}{4}\log|J_\phi|$$

w here the operator $P_4(\omega) = -\Delta(-\Delta + 2)\omega$, P_4 has appeared in the literature previously and is called Paneitz's operator.

Proof of the Lemma 2. We will present a geometric proof here. Recall that Lemma 2 is equivalent to the inequality:

$$4\int_{S^4} |\nabla\omega|^2 \leq \int_{S^4} \left(\frac{\Delta e^\omega}{e^\omega}\right)^2$$

with "=" holding iff $e^{2\omega}g_0 = \phi^*(g_0)$.

Let $u = e^\omega$. Then u satisfies the Yamabe equation

$$6\Delta u + Ru^3 = R_0 u = 12u$$

where R is the scalar curvature of the metric $e^{2\omega}g$.

Thus

$$\frac{\Delta e^\omega}{e^\omega} = \frac{\Delta u}{u} = 2 - \frac{R}{6}u^2.$$

We also observe that

$$\frac{\Delta e^\omega}{e^\omega} = \Delta\omega + |\nabla\omega|^2, \text{and hence} \int \left(\frac{\Delta e^\omega}{e^\omega}\right) = \int |\nabla\omega|^2$$

Denote $V = Vol(S^4)$, we have

$$\begin{aligned}
-II &= \frac{1}{V}\int \left(\frac{\Delta e^\omega}{e^\omega}\right)^2 - \frac{4}{V}\int |\nabla\omega|^2 \\
&= \frac{1}{V}\int (2 - \frac{R}{6}u^2)^2 - 4\frac{1}{V}\int (2 - \frac{R}{6}u^2) = -4 + \frac{1}{V}\int_{S^4} \frac{R^2}{36}u^4.
\end{aligned}$$

Recall the Yamabe functional defined as:

$$\begin{aligned}
Q(u) &= Q(S^4, g_0, u) \\
&= V^{1/2}\frac{\int |\nabla u|^2 + 2u^2}{(\int u^4)^{1/2}},
\end{aligned}$$

The infimum of $Q(u)$ is achieved only by conformal factors u of the form $u^2 g_0 = \phi^* g_0$ where ϕ is a conformal transformation of S^4. We observe that inequality II is scale invariant, thus we may assume w.l.o.g that $\int u^4 = V$. Hence applying Schwartz inequality and the Sobolev inequality in the Yamabe functional above, we get

$$-II = -4 + \frac{1}{V} \int \frac{R^2}{36} u^4 \geq -4 + \frac{1}{V^2} \left[\int \frac{R}{6} u^4 \right]^2 \geq 0,$$

with equality holding if and only if $u^2 g_0 = \phi^* g_0$ for some conformal transformation ϕ. This finishes the proof of lemma 2 and hence the proof of Theorem 3.

Beckner's inequality highly suggests that the results above for 4-manifolds holds for all even dimensional manifolds. Yet so far there is difficulty to compute the precise logdeterminant formula for higher-dimensional manifold.

Remark added after the lecture: More recently, results in this section have been extended substantially in many directions. In [B-G], Polyakov-Ray-Singer type formula for functional determinant has been worked out for 4-manifolds with boundary. In [Br], Theorem 3 in this chapter has been extended to (S^6, g_0). And in [C-Y], the zeta functional determinant F[w] as defined in this chapter has been studied for general compact 4-manifolds. Under conditions similar to that of Theorem 1 (i.e. $k < 32\pi^2$), existence for the extremal metric for F[w] has been established and proved to satisfy some sharp Moser-Trudinger type inequality. Also in [C-Y], a different proof of Beckner's inequality (Theorem 4 in this chapter) was given,the proof relies on the conformal invariant property of the Paneitz operator and Adam's inequality instead of the sharp Lieb-Young inequality.

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LECTURE 6
Isospectral Compactness on 3-manifolds and Relation to the Yamabe Problem

In Lecture 3 and 5, we have discussed the compactness results for isospectral families of conformal metrics for compact, closed manifolds of dimension 2 and 4 ([O-P-S-1] and [B-C-Y]) and the extremal metric of the log-determinant of the (conformal) Laplacian operator on S^2 and S^4 ([O] [B-C-Y]). In this chapter, we will briefly discuss some progress which has been made for these problems on 3-manifolds.

Recall that we have mentioned in the previous chapter that [P-R] when n is odd, L_g = conformal Laplacian operator w.r.t. the metric g, then $\log\det L_g$ is a conformal invariant quantity when one changes the metric g in the same conformal class. Thus in particular, $F[\omega] = \log\frac{\det L_{g_\omega}}{\det L_g}$ for $g_\omega = e^{2\omega}g$ does not carry any information on ω.

Recall also that in the case of dimension $n = 2$, to achieve the C^∞ compactness for isospectral family of conformal metrics $\{g_\omega = e^{2\omega}g\}$ on compact surfaces, one of the chief strategies in [O-P-S-1] is to use the spectral information in $F[\omega] = \log\frac{\det\Delta_{g_\omega}}{\det\Delta_g}$ to control the $W^{1,2}$ norm of ω, then use the information in the k-th coeffient a_k of the heat kernel expansion $e^{-\Delta t}\sim\sum_{k=0}^{\infty}a_k t^{k-\frac{n}{2}}$ to control the $W^{k,2}$ norm of ω for $k \geqq 2$.

It turns in the case of dimension 2 when restricting metrics to a fixed conformal class, one can also give an alternative argument (using λ_1 to replace the log-determinant of the Laplacian) to show that isospectral conformal metrics on compact surfaces form a compact set in the C^∞-topology

Theorem 1 *Suppose (M, g_0) is a compact surface, $\{e^{2u_j}\}$ is a sequence of conformal factors on M with*

(1) $\int e^{2u_j}\, dV_0 = a_0$

(2) $\int K_j^2 e^{2u_j} dV_0 = a_2 < \infty$ *where K_j = Gaussian curvature of the metric $e^{2u_j}g_0$.*

And assume in addition that, the first eigenvalue λ_1 of the Laplacian w.r.t. to the metrics $e^{2u_j}g_0$ are bounded from below by $\Lambda > 0$, i.e.

(3) *For each function ϕ defined on M*

$$\int_M \phi^2 e^{2u_j}\, dV_0 \le \left(\int_M \phi e^{2u_j}\, dV_0\right)^2 \Big/ \left(\int_M e^{2u_j}\, dV_0\right) + \frac{1}{\Lambda}\int_M |\nabla_0\, phi|^2\, dV_0.$$

Then either (a) and (b): in case K_0 (Gaussian curvature of the metric g_0) is < 0, or $= 0$ respectively; $\{u_j\}$ forms a bounded family in W_2^1 (i.e. $\sup_n \int_M |\nabla u_j|^2 dV_0$ is finite) or (c): in case $K_0 = 1$ and $(M, g_0) = (S^2, g_0)$ with g_0 = surface measure on S^2, then the isometry class of u_j forms a bounded family in $W^{1,2}$. In the setting of Theorem 1 one can generalize the compactness result to compact manifolds of dimension 3 ([C-Y-2]).

Theorem 2 *Let $g_j = u_j^4 g_0$ be a sequence of conformal metrics satisfying the following conditions:*

(i) $Vol(M, g_j) = \alpha_0$ *for some positive constant* α_0.

(ii) $\int R^2(g_j) + |\rho(g_j)|^2\, dV_j \le \alpha_2$, *for some positive constant α_2 where $R(g_j)$ is the scalar curvature of g_j and ρ is the Ricci tensor of g_j, and $dV_j = u_j^6\, dV_0$,*

(iii) $\lambda_1(g_j)$, *the lowest eigenvalue of the Laplacian of the metric g_j, has a positive lower bound: $\lambda_1(g_j) \ge \Lambda > 0$; i.e. For each ϕ defined on M, we have*

$$\left(\int_M \phi^2 dV_j\right) \le \left(\int_M \phi\, dV_j\right)^2 \Big/ \left(\int_M dV_j\right) + \frac{1}{r}\int_M |\nabla_j \phi|^2\, dV_0.$$

Then there exists constants c_1, c_2 so that

(a) $c_1 \le u(x) \le \dfrac{1}{c_1}$,

(b) $||u_j||_{2,2} \le c_2$;

except in the case where (M, g_0) is the standard 3-sphere. In the latter case, the isometry class of u_j satisfy (a) and (b).

Theorem 2 was proved in the special case when R_0 is negative by [B-C-Y] and when $(M, g_0) = (S^3, g_0)$ in [C-Y-1] [C-Y-3], see also Theorem 3 below. Notice that in dimension $n = 3$ (actually for $3 \leqq n \leqq 6$, cf [K], page 3), one can make a explicit computation in the formula of McKean and Singer [M-S] in Lecture 2, and obtain on (M, g_0)

$$\begin{aligned} a_0 &= \text{volume of } g_0 \\ a_1 &= \text{constant} \int R dV_0 \\ a_2 &= A_2 \int R^2 dV_0 + B_2 \int |\rho|^2 dV_0 \;\; A_2, B_2 > 0 \end{aligned}$$

Furthermore [8] for $k \geqq 3$

$$a_k = A_k \int |\nabla^{k-2} R|^2 dV_0 + B_k \int |\nabla^{k-2}\rho|^2 dV_0 + \text{lower order terms},$$

with $A_k, B_k > 0$. Thus the condition (i) (ii) (iii) in Theorem 2 all are spectral in formation, and once one controls $W^{2,2}$-norm of the conformal factor $\{u_j\}$ for an isospectral sequence of metrics $g_j = u_j^4 g_0$, one can gain the control of $W^{k,2}$-norm of $\{u_j\}$ through the information in a_k, and we get as a corollary of Theorem 2.

Corollary 3 *An isospectral set of conformal metrics on a compact 3-manifolds is compact in the C^∞-topology.*

Since conformal transformations preserv the spectral information another immediate corollary of Theorem 2 is the following result of Obata for the special case $n = 3$.

Corollary 4 *The conformal group of a compact 3-manifold is non-compact if and only if it is comformally equivalent to the standard 3-sphere.*

Before we discuss the ideas in the proof of Theorem 2, we would like to mention a result of Gursky [G] which generalize Theorem 2 to general dimensions. First we remark that, in the statement of Theorem 1 (for compact surface), it is well-known that the "local information" (i.e. those computed using local coordinates in the metric g) like that in conditions (1) and (2) in Theorem 1 are not enough to conclude the $W^{1,2}$ compactness of $\{\omega_j\}$ for the metrics $\{e^{2\omega_j} g_0\}$. This can be seen through the famous "Dumbell" surface with the neck getting thinner and thinner. Notice that for such surfaces, using Rayleigh quotients, it is easy to see that $\lambda_1 \to 0$. Thus it comes as a surprise that when $n \geqq 3$, some "local information" (e.g. size of curvature tensor of the metric) suffice to provide the compactness of the metrics. This is the content of the following theorem of Gursky [G].

Theorem 3 *Let (M, g_0) be a compact manifold without boundary. Suppose that $g_j = u_j^{\frac{4}{n-2}} g_0$ is a sequence of metrics satisfying*
(1) $\int u_j^{\frac{2n}{n-2}} dV_0 = Vol(g_j) \leqq \alpha_0$
(2) $\int |Rm(g_j)|^p dV_j \leqq \beta$ for some $p > \frac{n}{2}$,
where $Rm(g_j)$ is the full curvature tensor of g_j. Then there exist $c_1, c_2 > 0$ so that $\frac{1}{c_1} \leqq u_j \leqq c_1$ and $\|u_j\|_{W^{2,p}} \leqq c_2$, unless $(M, g_0) = (S^n, g_0)$, in that case the conclusion holds in the isometry class of u_j.

One would like to point out that the exponent $p > \frac{n}{2}$ is necessary in Theorem 3. In a recent article [C-G-W], two examples have been constructed to indicate that Theorem 3 fails when $p = \frac{n}{2}$.

Gursky's proof of Theorem 3 is quite ingenious. One of the key idea is to apply condition (2) to Bochner's formula to start Nash-Moser iteration process and to establish "Harnack type" inequality for sequence $\{u_j\}$. In this lecture, instead of proving the general result [8], we give a complete proof some special cases of Theorem 2 to convey the idea how to use the condition $\lambda_1 \geqq \Lambda > 0$ in the compactness result. We start with a definition.

Definition *We say a sequence of positive function $\{u_j\}$ satisfies condition (*), if there exist $l_0, r_0 > 0$ so that for all j*

$$(*) \qquad \int_{\{u_j(x) \geqq r_0\}} dV_0 \geqq l_0 \int_M dV_0$$

Theorem 2' *Suppose $\{u_j\}$ is a sequence on (M^3, g_0) satisfying*
(1) $vol(g_j) = \int u_j^6 dV_0 = \alpha_0$
(2)' $\int_M R_j^2 u_j^6 \leqq \alpha_2$ R_j = scalar curvature w.r.t. g_j
(3) $\lambda_1(g_j) \geqq \Lambda > 0$ where λ_1 is the first eigenvalue of the Laplacian operator.
And if in addition, $\{u_j\}$ satisfies condition (*) then there exist some $\epsilon_0 > 0$ and a constant C_0 depending only on the data $\alpha_0, \alpha_2, \Lambda, l_0, r_0$ so that

$$(4) \qquad \int_M u_j^{6+\epsilon_0} \leqq C_0$$

Remark
1. The underlying analysis of the compactness results in Theorem 2 and Theorem 2' is the optimal Sobolev inequality:

$$Q(M)\left(\int_M u^6 \, dV_0\right)^{1/3} \leq 8 \int_M |\nabla u|^2 \, dV_0 + \int_M R_0 u^2 \, dV_0.$$

The optimal constant $Q(M)$ is an invariant of the conformal class of M. For a conformal metric $g = u^4 g_0$, its scalar curvature R is given by the equation

$$(5) \qquad 8\Delta u + Ru^5 = R_0 u \quad \text{on} \quad M.$$

Thus the Sobolev x_0 quotient

$$Q[u] = \frac{\int (8|\nabla u|^2 + R_0 u^2) \, dV_0}{(\int u^6 dV_0)^{1/3}}$$

is exactly given by $\int Ru^6 dV_0$ if the volume is held to be 1, (i.e. $\int u^6 dV_0 = 1$). The celebrated recent solution of Yamabe's problem ([A], [S]) asserts that (a) $Q(M) < Q(S^3)$ unless M is conformally S^3 and (b) a minimizing sequence for $Q[u]$ is compact if $Q(M) < Q(S^3)$. Thus in our compactness assertion, we have substituted an L^2 bound for the curvature in place of the condition $Q[u_j] < Q(S^3)$, and substituted the condition $\lambda_1(g_j) \geq \Lambda > 0$ in place of the minimizing property for $Q[u]$.
2. As we have mentioned in Lecture 4, one fact which has a key role in Yamabe problem is that the embedding $W^{1,2} \subset L^{\frac{2n}{n-2}}$ is not compact. On (S^3, g_0), if we adopt coordinates on S^3 through its stereographic projection mapping with pole of

S^3 to $O = (0,0,0)$ in R^3, then in this coordinates system, volume form dV_0 on S^3 is defined by

$$dV_0 = (\frac{2}{1+|x|^2})^3\, dx$$

For each $a > 0$, consider the function

$$u_a(x) = (\frac{a(1+|x|^2}{|x|^2+|a|^2})^{\frac{1}{2}}$$

then $\{u_a\}$ is a family of function with

$$6(2\pi^2) = Q(S^3)(\int_{S^3} u_a^6 dV_0)^{\frac{1}{3}} = 8\int_{S^3} |\nabla u_a|^2 dV_0 + 6\int_{S^3} u_a^2 dV_0,$$

where $Q(S^3) = 6(2\pi^2)^{\frac{2}{3}}$. Since $u_a(x) \to 0$ uniformly on compact subset off $x = 0$, $\{u_a\}$ is a typical example of family of functions which indicate the failure of compactness of the inclusion $W^{1,2} \subset L^6$ on S^3. Notice that $\{u_a\}$ does not satisfy comdition $(*)$ in our Theorem 2', and also for all $\epsilon > 0$, $\int_{S^3} u_a^{6+\epsilon} dV_0 \to \infty$ as $a \to 0$. Thus it is convincing that a sequence $\{u_j\}$ satisfies condition (1), (2), (3), (4) in Theorem 2' is a sequence which is bounded in the sense of the conclusions (a) and (b) in Theorem 2; i.e. there exist constants c_1, c_2 so that $c_1 \leqq u_j \leqq \frac{1}{c_1}$ and $\|u_j\|_{2,2} \leqq c_2$. Indeed, using an iteration argument (Lemma 3.4 in [C-Y-1]) one can verify this latter fact. Thus Theorem 2' is a special case of Theorem 2.

3. Examples of (M, g_0) for which conditiopn $(*)$ automaticallyfollows from assumptions (1), (2) and (3).

(a) When $R_0 < 0$ (we may assume w.r.l.g. R_0 is a constant), then

$$\int_M R_j^2 u_j^6 dV_0 = \int_M (\frac{8\,\Delta\, u_j - R_0 u_j}{u_j^5})^2 u_j^6 dV_0$$

$$= \int_M (\frac{\Delta u_j}{u_j})^2 dV_0 - 16 R_0 \int_M \frac{\Delta u_j}{u_j^3} dV_0 + R_0^2 \int_M u_j^{-2} dV_0$$

Since $\int_M \frac{\Delta u_j}{u_j^3} dV_0 = 3\int_M |\nabla u_j|^2 u_j^{-4} dV_0 \leqq 0$, we have

$$\int_M R_j^2 u_j^6 dV_0 \leqq \alpha_2, \text{so } R_0^2 \int_M u_j^{-2} dV_0 \leqq \alpha_2.$$

And it is easy to check that for $\{u_j\}$ satisfies $\int_M u_j^{-2} dV_0 \leqq \beta$, $\{u_j\}$ satisfies condition $(*)$.

(b) On (S^3, g_0), given a sequence $\{u_j\}$ satisfying (1), (2), (3) we can find v_j in the same isometry class as u_j with v_j satisfying (1), (2) ,(3) and condition $(*)$. To see this, given u_j, we apply Lemma 2 in Lecture 3 to obtain $v_j = T_{\phi_j} u_j$ for some conformal transformation ϕ_j of S^3 and with $v_j \in S$, then $\lambda_1(u_j^4 g_0) = \lambda_1(v_j g_0)$, and $\{v_j\}$ satisfies (1), (2), (3) with the same constants α_0, α_2. To see $\{v_j\}$ satisfies condition $(*)$ we apply $\phi = x_k, k = 1,2,3,4$ in the Rayleigh quotient of $\lambda_1(v_j g_0)$ and get $dV_j = v_j dV_0$

$$\int_{S^3} x_k^2 dV_j \leqq \frac{1}{\lambda_1}\int_{S^3} |\nabla_j x_k|^2 dV_j = \frac{1}{\lambda_1}\int_{S^3} |\nabla_0 x_k v_j^2 dV_0 \tag{6}$$
$$\leqq dV_j = \frac{1}{\Lambda}\int_{S^3} |\nabla_0 x_k v_j^2 dV_0$$

Sum (6) over $k = 1,2,3,4$ we get

$$\alpha_0 \leqq \frac{3}{\Lambda}\int_{S^3} v_j^2 dV_0$$

Thus $\{v_j\}$ satisfies condition $(*)$.

Proof of Theorem 2'

Denote $u = u_j$, we will show that u satisfies (4) with constant depending only on $\alpha_0\alpha_2, \Lambda, l_0, r_0$.

Multiplying the equation (5) by u^β ($\beta > 1$ to be chosen later) we have (denote $\int = \int_M dV_0$) for $w = u^{\frac{1+\beta}{2}}$

$$8\frac{4\beta}{(1+\beta)^2}\int |\nabla w|^2 + R_0 \int w^2 = \int Ru^4 w^2 \tag{7}$$

We will now apply our assumptions (1), (2), (3) to estimate the term $I = \int Ru^4 w^2$. Taking a suitably large number b (again to be chosen later) on the region $|R| \geqq b$ we have

$$b^2 \int_{|R|\geqq b} u^2 dV_0 \leqq \int_{|R|\geqq b} R^2 u^6 dV_0 \leqq \alpha_2$$

Thus

$$\int_{|R|\geqq b} Ru^4 w^2 \leqq \left(\int R^2 u^6\right)^{1/2}\left(\int_{|R|\geqq b} u^6\right)^{1/6}\left(\int w^6\right)^{1/3} \tag{8}$$
$$\leqq \alpha_2^{1/2}\left(\frac{\alpha_2}{b^2}\right)^{1/6}\left(\int w^6\right)^{1/3}.$$

For the remaining part of the proof, we will apply condition $(*)$ in the statement of Theorem 2'.

For $dV = v^6 dV_0$, we have from the Rayleigh-Ritz characterization for λ_1,

$$\int_M \Phi^2 dV \leqq \left(\int_M \Phi dV\right)^2 \Big/ \left(\int dV\right) + \frac{1}{\lambda_1}\int_M |\nabla_u \Phi|^2 dV \tag{9}$$

where $|\nabla_u \Phi|^2 dV = |\nabla\Phi|^2 u^2 dV_0$. We will denote $E_\gamma = \{x \in M, u(x) \geqq \gamma\}$ and $|E_\gamma| = \int_{E_\gamma} dV_0$. By assumption $(*)$, there exist some $\gamma_0, l_o > 0$ so that $|E_{\gamma_0}| \geqq l_0 |\int_M dV_0|$. Applying (9) and (3) to $\Phi = u^\epsilon$ with $\beta = 1 + 2\epsilon$ and ϵ small, we have

$$\text{(10)} \qquad \int u^{6+2\epsilon} dV_0 \leqq \left(\int u^{6+\epsilon} dV_0\right)^2 \Big/ \left(\int u^6 dV_0\right) + \frac{1}{\Lambda}\int |\nabla u^\epsilon|^2 u^2 dV_0.$$

For simplicity, we will now normalize u and assume that $\alpha_0 = \int u^6 dV_0 = 1$. We may then estimate the term $\int^{6+\epsilon} dV_0$ as

$$\begin{aligned}
\int u^{6+\epsilon} dV_0 &= \int_{E_{\gamma_0}} u^{6+\epsilon} dV_0 + \int_{E^c_{\gamma_0}} u^{6+\epsilon} dV_0 \\
&= \int_{E_{\gamma_0}} (u^6 - \gamma_0^6) u^\epsilon dV_0 + \int_{E_{\gamma_0}} \gamma_0^6 u^\epsilon dV_0 + \int_{E^c_{\gamma_0}} u^{6+\epsilon} dV_0 \\
&\leqq \left(\int_{E_{\gamma_0}} (u^6 - \gamma_0^6) u^{2\epsilon} dV_0\right)^{1/2} \left(\int_{E_{\gamma_0}} (u^6 - \gamma_0^6) dV_0\right)^{1/2} + C(\gamma_0),
\end{aligned}$$

where $C(\gamma_0)$ is a constant depending only on γ_0 and $\int dV_0$. Thus, for each $\eta > 0$ we have

$$\text{(11)} \quad \begin{aligned}
\left(\int u^{6+\epsilon} dV_0\right)^2 &\leqq (1+\eta)\left(\int_{E_{\gamma_0}} (u^6 - \gamma_0^6) u^{2\epsilon} dV_0\right)\left(\int_{E_{\gamma_0}} (u^6 - \gamma_0^6) dV_0\right) \\
&\quad + \left(1 + \frac{1}{\eta}\right) C^2(\gamma_0) \\
&\leqq (1+\eta)(1 - \gamma_0^6 |E_{\gamma_0}|)\left(\int u^{6+2\epsilon} dV_0\right) \\
&\quad + \left(1 + \frac{1}{\eta} C^2(\gamma_0)\right)
\end{aligned}$$

(we may assume w.l.o.g. that γ_0 is small and $\gamma_0 |E_{\gamma_0}| \ll 1$).

Since by our assumption on $|E_{\gamma_0}|$ we have $\gamma_0^6 |E_{\gamma_0}| \geqq \gamma_0^6 l_0 > 0$, we may choose η so that $(1+\eta)(1 - \gamma_0^6 |E_{\gamma_0}|) \leqq 1 - \delta$ for some positive δ, $\delta = \delta(\gamma_0, l_0)$ and obtain from (10), (11)

$$\text{(12)} \qquad \delta \int u^{6+2\epsilon} dV_0 \leqq C(\gamma_0, l_0) + \frac{1}{\Lambda}\int |\nabla u^\epsilon|^2 u^2 dV_0,$$

where again $C(\gamma_0, l_0)$ is a constant depending only on γ_0, l_0.

From this point on, we may estimate the term $\int |\nabla u^\epsilon|^2 dV_0$ as follows:

$$\int |\nabla u^\epsilon|^2 u^2 dV_0 = \frac{\epsilon^2}{(1+\epsilon)^2}\int |\nabla u^{1+\epsilon}|^2 dV_0$$

and notice that for $\beta = 1+2\epsilon$, $w = u^{\frac{1+\beta}{2}} = u^{1+\epsilon}$. Thus, combining (7) and (12) we have

$$\int u^{6+2\epsilon} dV_0 \leqq \frac{\epsilon^2}{\delta\Lambda(1+\epsilon)^2} \frac{(1+\epsilon)^2}{8(1+2\epsilon)} I + L, \tag{13}$$

where

$$I = \int R u^4 w^2 \text{ and } L = 0\Big(\frac{R_0}{\delta}\int u^{2+2\epsilon} dV_0\Big) + \frac{1}{\delta} C(\gamma_0, l_0).$$

Combining (11), (13) with (8), we find

$$\begin{aligned} I = \int R u^4 w^2 &\leqq \Big(\frac{\alpha_2^2}{b}\Big)^{1/3} \Big(\int w^6 dV_0\Big)^{1/3} + b\int u^4 w^2 dV_0 \\ &\leqq \Big(\frac{\alpha_2^2}{b}\Big)^{1/3} \Big(\int w^6 dV_0\Big)^{1/3} + \frac{b\epsilon^2}{8\Lambda} I + bL \end{aligned} \tag{14}$$

so that

$$\Big(1 - \frac{b\epsilon^2}{8\Lambda}\Big) I \leqq \Big(\frac{\alpha_2^2}{b}\Big)^{1/3} \Big(\int w^6 dV_0\Big)^{1/3} + bL. \tag{15}$$

Now choosing b sufficiently large so that $(\frac{\alpha_2^2}{b}) < \frac{1}{2}Q$, and then choosing ϵ sufficient small, we get

$$\begin{aligned} \frac{3}{4} Q\Big(\int w^6 dV_0\Big)^{1/3} &\leq I + |R_0| \int w^2 dV_0 \\ &\leq \frac{2}{3} Q\Big(\int w^6 dV_0\Big)^{1/3} + \frac{4}{3}\, bL + |R_0| \int w^2 \, dV_0. \end{aligned}$$

Recall $w = u^{1+\epsilon}$, hence

$$\begin{aligned} \Big(\int u^{6+6\epsilon} dV_0\Big)^{1/3} &= \Big(\int w^6 dV_0\Big)^{1/3} < 16\, bL + 12\, |R_0| \int u^{2+2\epsilon}\, dV_0 \\ &\leq C(b, |R_0|) \Big(\int u^6 dV_0\Big)^{(2+2\epsilon)/6} \Big(\int dV_0\Big)^{(4-2\epsilon)/6} \\ &= C_0 < \infty. \end{aligned}$$

This proves Theorem 2' with $\varepsilon_0 = 6\varepsilon$.

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LECTURE 7
Prescribing Curvature Function on S^n

In this chapter we will describe an application of the Sobolev-Trudinger-Moser type inequalities to a problem in prescribing curvature functions on the n-sphere.

We will start with a question originally proposed by L. Nirenberg: what function is allowed to be the Gaussian curvature function on the 2-sphere S^2?

Applying the uniformization theorem, the question can be described in analytic terms in the following way. Denote the metric with Gaussian curvature K as $g = e^{2w} g_0$, then K and w satisfy the following equation

$$(1) \qquad \Delta w + K e^{2w} = 1 \quad \text{on } S^2$$

where Δ denote the Laplacian operator related to the standard metric. The question is: For which functions K does there exist a solution to (1) ?

In this chapter we will discuss some basic facts, techniques and progress made in answering the above question on the 2-sphere. Before we do that, we will mention some other related questions of prescribing curvature functions. It turns out many of the techniques for the S^2 problem, which we will mention below, can be applied to these problems. We do not intend to give a survey of the progress made in this fast developing field, and will mention only a few relevant results related to the techniques of the 2-sphere problem discussed here. The reader is referred to [S-2] and [C-Y-G] for more complete and up to date references on the subject.

On general compact manifolds M^n ($n \geqq 3$) without boundary, an analogous question to (1) is that of prescribing scalar curvature problem. In this case, when restricting metrics conformal to a fixed one g_0, denote the metric with scalar curvature R as $g = u^{\frac{4}{n-2}} g_0$, then R and u satisfy the following equation

$$(2) \qquad c_n \Delta u + R u^{\frac{n+2}{n-2}} = R_0 u$$

where $c_n = \frac{4(n-1)}{n-2}$, Δ denote the Laplacian operator w.r.t. the metric g_0 and R_0 is the scalar curvature of g_0. The question is: Which function R can be prescribed so that equation (2) has a solution? When R = constant, this is the Yamabe problem mentioned earlier, and the question has been solved in the affirmative way by [A],[S-1].

Another analogous question, whose precise geometric meaning is well-fomulated only on 4-manifolds at this moment, is the problem of prescribing "curvature" for the Paneitz operator. The Paneitz operator was introduced by Paneitz ([P], see also [Br]) on 4-manifolds, it is a fourth order differential operator P_g defined as

$$P_g = (-\Delta)^2 + \delta(\frac{2}{3}Rg_{ij} - 2R_{ij})d$$

where R_{ij} is the Ricci tensor, δ is the divergence, d is the differential w.r.t. the metric $g = (g_{ij})$. This operator enjoys the following good properties:

(a) It is "conformally invariant", in the sense that if $g_w = e^{2w}g$ is a metric conformal to g, then $P_{g_w} = e^{-4w}P_g$.

(b) Let Q denote the "curvature" function

$$Q = \frac{1}{6}(-3|\rho|^2 + R^2 - \Delta R)$$

where $|\rho|^2$ = square norm of Ricci tensor, the P_g and Q_g are related as

$$(3) \qquad -P_g w + Q_{g_w} e^{4w} = Q_g \quad \text{on } M.$$

It is in the context of equation (3) when comparing equation (1) and (2) that a natural question to propose is that of prescribing "curvature" Q as defined by (3).

When the manifold M is of dimension not equal to four, in general the existence and precise form of the Paneitz operator with properties (a) and (b) is unknown. But in a recent article [G-J-M-S], it is proved that when n is even, an operator of order n with conformal invariant property as described in (a) exists; also when $M = R^n$ with the Euclidean metric, the operator $P_n = (-\Delta)^{\frac{n}{2}}$. Using stereographical projection pulling S^n to R^n, one can then compute the form of P_n on S^n. It turns out that this is precisely the operator which has appeared in Branson [Br, p 231] and Beckner [B] (which we have discussed in Lecture 5). And the precise form of P_n on S^n is

$$\begin{aligned} P_n &= \prod_{k=0}^{\frac{n}{2}} (-\Delta + k(n-k-1)) \quad \text{when } n \text{ is even} \\ &= (-\Delta + (\frac{n-1}{2})^2)^{\frac{1}{2}} \prod_{k=0}^{\frac{n-3}{2}} (-\Delta + k(n-k-1)) \quad \text{when } n \text{ is odd.} \end{aligned}$$

In view of (1), (2) and (3), an analogous question to ask is that of prescribing Q in the following equation

$$(4) \qquad -P_n w + Qe^{nw} = (n-1)! \quad \text{on } S^n.$$

It turns out that equation (4) is a more natural extension of equation (1) than that of equation (3) of the problem of prescribing scalar curvature. This point of view is explained in a recent article [C-Y].

For the rest of this chapter, we will return to equation (1) -the problem of prescribing Gaussian curvature on S^2. We first remark that integrating equation (1) over S^2, we get (we denote $⨍ = \frac{1}{4\pi}\int_{S^2} dV_0$)

$$⨍ Ke^{2w} = 1$$

Thus $K > 0$ somewhere is a necessary condition for (1) to be solvable. To explore other properties of K, we will use a variational approach. Consider the variation functional F_K associated with (1):

$$F_K[w] = \log ⨍ Ke^{2w} - S[w]$$

where $S[w] = ⨍|\nabla w|^2 + 2⨍ w$ as defined in Lecture 2.

Basic facts about $F_K[w]$

(i)
$$\begin{aligned} F_K[w] &\leqq \log \max K + F_1[w] \\ &\leqq \log \max K, \end{aligned}$$

via Onofri's inequality in Lecture 3.

(ii) When $K > 0$, $\sup\limits_{w \in W^{1,2}} F_K[w]$ is attained iff $K \equiv$ constant.

Proof. When $u = \frac{1}{2}\log|J_\phi|$ for some conformal transformation ϕ of S^2, then u satisfies $\Delta u + e^{2u} = 1$ and $K \equiv 1$, thus, when $K = c > 0$, $u = \frac{1}{2}\log|J_\phi| + \log c$ is a nontrivial solution of (1) for which $F_c[u] = \log c$.

In general, suppose u is a local maximum of the functional $F_K[u]$, a direct computation yields:

$$0 \geqq \frac{d^2}{dt^2}F_K[u+tv]\Big|_{t=0} = 2\left[⨍ Ke^{2u}v^2 - (⨍ Ke^{2u}v)^2\right] - ⨍ |\nabla v|^2$$

for all $v \in W^{1,2}$. This implies that the first non-zero eigenvalue λ_1 of the Laplacian of the metric $Ke^{2u}g_0$ satisfies $\lambda_1 \geqq 2$. While the result of Hersch in Lecture 1 says $\lambda_1 \leqq 2$ with equality if and only if the metric $Ke^{2u}g_0$ has constant curvature 1. This coupled with the assumption that K, u satisfy (1) is equivalent to $K \equiv$ constant.

Thus the only way to solve equation (1) using the variational approach is to locate saddle point of the functional $F_K[w]$.

(iii) Kazdan-Warner condition ([K-W]): if K, u satisfy (1) then

$$\fint < \nabla K, \nabla x_j > e^{2u} = 0 \quad \text{for} \quad j = 1, 2, 3. \tag{5}$$

Proof. ([C-Y-4]) Recall Lemma 1 in Lecture 3, $S[w]$ is a conformally invariant quantity in the sense that $S[T_\phi(u)] = S[u]$, where $v = T_\phi(u)$ denote the induced conformal factor given by $\phi^*(e^{2u}g_0) = e^{2v}g_0$, and ϕ is a conformal transformation of S^2.

Suppose u is a critical point of $F_K[u]$, hence

$$\frac{d}{dt}F_K[T_{\phi_{P,t}}]\Big|_{t=0} = 0$$

where $P \in S^2, t \geqq 1$, $\phi_{P,t}$ is the conformal transformation on S^2 as defined in Lemma 3 of Lecture 3. But

$$\begin{aligned} F_K[T_{\phi_{P,t}}(u)] &= \log \fint K e^{2T_{\phi_{P,t}}(u)} - S[T_{\phi_{P,t}}(u)] \\ &= \log \fint K e^{2T_{\phi_{P,t}}(u)} - S[u] \\ &= \log \fint K e^{2u \circ \phi_{P,t}} \det |J_{\phi_{P,t}}| - S[u] \\ &= \log \fint (K \circ \phi_{P,t}^{-1}) e^{2u} - S[u]. \end{aligned}$$

Thus

$$\frac{d}{dt}F[T_{\phi_{P,t}}(u)]\Big|_{t=1} = \fint \frac{d}{dt}(K \circ \phi_{P,t}^{-1})\Big|_{t=1} e^{2u}.$$

Now a simple calculation shows that

$$\frac{d}{dt}K \circ \phi_{P,t}^{-1} = < \nabla K, \nabla x_3 >, \quad \text{if} \quad x_3 = \vec{x} \cdot P.$$

This gives the desired condition.

(iv) K, u satisfy equation (1) if and only if $K \circ \phi, T_\phi(u)$ satisfy equation (1) for any conformal transformation ϕ of S^2.

From the geometric interpretation of K, this is an obvious fact. It can also be checked directly.

(v) (Moser [M-2]) When K is an even function (i.e. $K(\xi) = K(-\xi)$ for all $\xi \in S^2$), (1) has an even solution.

Proof. Recall that for even functions u defined on S^2, the best constant in Moser's inequality is $\alpha = 8\pi$ (Remark 3 in Lecture 3). Thus following the argument as in Lecture 3 after Remark 3, we have

$$\text{(6)} \qquad \text{for } u \text{ even,} \quad \log \fint e^{2u} \leqq \frac{1}{2} \fint |\nabla u|^2 + 2 \fint u + C_2,$$

for some constant C_2.

From (6), the same argument as in the derivation of Onofri's inequality as in Lecture 3 gives an even solution for equation (1) in case K is an even function.

We now describe an a priori estimate for solutions of (1), which we think is a key step to the prescribing curvature problem.

Theorem 1. *Suppose K, w satisfy equation* (1).

(a) *Assume that $0 < m \leqq K \leqq M$, then*

$$S[w] \leqq C(m, M).$$

(b) *Assume further that K is a nondegenerate function in the following sense*

$$\text{(nd)} \qquad \Delta K(Q) \neq 0 \quad \textit{whenever} \quad \nabla K(Q) = 0$$

then there exists a constant $C = C(m, M, (nd)) > 0$ so that

$$\frac{1}{C} \leqq w \leqq C, \quad \textit{and} \quad \|w\|_{W^{4,2}} \leqq C.$$

Remarks.

1. Considering the fact (iv), bounds for $S[w]$ is the best which one can hope for (instead of the energy bound $\int |\nabla w|^2$) for a given K. For the sequence $w_t = \frac{1}{2} \log |J_{\phi_{P,t}}|$ for some fixed $P \in S^2$, $K \equiv 1, S[w_t] \equiv 0$ while $\int |\nabla w_t|^2 \to \infty$ as $t \to 0$.
2. Part (a) of Theorem 1 has appeared in [C-Y-6], [Ch] part (b) is a special case of a result in [C-Y-G], where a similar a priori estimate for the prescribing scalar curvature equation on S^3 was also obtained. One should also mention, using a different approach (i.e. local estimate), the result analogous to Theorem 1 on S^3 has been proved in [Z].
3. Result with a similar statement as in part (a) also holds for the Paneitz equation (4) on S^n, with $S[w]$ replaced by a suitably defined conformal invariant term $S_n[w]$ ([C-Y]).

Proof of part (a). What we actually will establish is:

(a)' If $w \in S$ and satisfies the equation (1), then there exists a constant $C = C(m, M)$ with

$$\frac{1}{C} \leqq w \leqq C, \quad \text{and} \quad \fint |\nabla w|^2 \leqq C.$$

In view of Lemma 2 in Lecture 3 and (iv) above, statements (a) and (a)' are equivalent. To begin the proof of (a)' we will first state the following result, which is a modified form of Aubin's result (Lemma 3 in Lecture 3) (cf also [Ch]).

Lemma 2. *Given $w \in S$, there exists a constant $C(K)$ such that*

$$\fint e^{cw} \leqq C(K) \exp\left(\frac{c^2}{8} \fint |\nabla w|^2 + c \fint w\right) \tag{7}$$

where c *is a real number,* (w, K) *satisfies equation* (1).

Proof of (a)'. Aubin's Lemma shows that for all $\varepsilon > 0$

$$\fint e^{cw} \leq C(\varepsilon) \exp\left(\left(\frac{c^2}{8} + \varepsilon\right) \fint |\nabla w|^2 + c \fint w\right) \tag{8}$$

for some constant $C(\varepsilon)$. But when one examines the proof of Lemma 3 in Lecture 3 ([A]), one notices that the reason the ε-term $\varepsilon \fint |\nabla w|^2$ appeared in (8) is to absorb the term $\fint |\nabla w|$. Thus (7) is a consequence of (8) along with the observation that when w, K satisfies (1) one can easily verify $\fint |\nabla w| \leq C(K)$.

We multiply equation (1) by $2w$ and integrate and apply Jensen's inequality (recall $\fint K e^{2w} = 1$)

$$\begin{aligned} 2\fint |\nabla w|^2 + 2\fint w &= 2\fint K e^{2w} w \\ &\leq \log \fint K e^{2w}\, e^{2w} \\ \text{(by (7) with } C = 4)) &\leq \log \max K + \log C(K) + 2\fint |\nabla w|^2 + 4\fint w \end{aligned} \tag{9}$$

hence $-\fint w \leq C(K)$. From this we conclude w is bounded below by using the Green's identity:

$$\begin{aligned} -w(y) + \fint w &= \fint_{S^2} (\Delta w)(x) G(y,x) d\mu(x) \\ &= \fint (1 - K e^{2w})(x)(G(y,x) + \text{ constant}) d\mu(x) \\ \text{(since } K > 0) &\leq \fint G(y,x) d\mu(x) + \text{ constant.} \end{aligned}$$

But once w is bounded from below, we may modify (9) as

$$\begin{aligned} 2 ⨍ |\nabla w|^2 &= 2 ⨍ K e^{2w}(w - \overline{w}) \\ &= 2 ⨍ (K e^{2w} - \delta)(w - \overline{w}) \\ &= (1-\delta) \log ⨍ \frac{(K e^{2w} - \delta)}{1-\delta} e^{2(w-\overline{w})} \\ &\le (1-\delta)\left[\log \max K + 2 ⨍ |\nabla w|^2 + 2 ⨍ w - \log(1-\delta) + \log C(K)\right] \end{aligned}$$

where $\delta = \min K e^{2w}$. From this, we conclude that $⨍ |\nabla w|^2 \le C(K)$.

To obtain a pointwise bound of w, we notice that

$$\exp\left(⨍ w\right) \leqq ⨍ e^{2w} \leqq \frac{1}{m} ⨍ K e^{2w} = \frac{1}{m}$$

hence

$$\left| ⨍ w \right| \leqq C(m, M) \tag{10}$$

Notice that for any $p > 1$, we may then apply Onofri's inequality to conclude

$$⨍ e^{pw} \le \exp\left(\frac{p^2}{4} ⨍ |\nabla w|^2 + p ⨍ w\right) \le c(m, M, p),$$

hence

$$\begin{aligned} |-w(Q) + ⨍ w| &= |\int_{S^2} G(Q,P) \Delta w(P) dV_0(P)| \\ &\le \left(\int |G(Q,P)|^2 dV_0(P)\right)^{\frac{1}{2}} \left(\int (K e^{2w} - 1)^2\right)^{\frac{1}{2}} \\ &\le c(m, M), \end{aligned} \tag{11}$$

where $G(\xi, \cdot)$ is the Green's function on S^2 with pole at ξ. Combining the estimates in (10), (11), we get the estimates in (a)'.

Proof of (b). We will prove the result by contradiction. Given $K > 0$ satisfying the non-degeneracy condition (nd), suppose the statement of part (b) does not hold. Then there exists a sequence w_k satisfying

$$\Delta w_k + K e^{2w_k} = 1 \text{ on } S^2$$

with $\max_{S^2} w_k \to +\infty$. Applying Lemma 2 in Lecture 3, we get a sequence of conformal transformations $\phi_k = \phi_{P_k,t_k}$, with $e^{2v_k} g_0 = \phi_k^*(e^{2w_k} g_0)$, $v_k \in \mathcal{S}$ satisfying

$$\Delta v_k + K \circ \phi_{P_k,t_k} \, e^{2v_k} = 1 \text{ on } S^2. \tag{12}$$

Applying Lemma 2, we have $\|v_k\|_\infty \leq c(m, M)$ and $⨍|\nabla v_k|^2 \leq c(m, M)$. We may then conclude that some subsequence of $t_k \to +\infty$. For if not, i.e., $t_k \leq t_0$ for all k, for some t_0, then $w_k \circ \phi_k = v_k - \frac{1}{2} \log \, det \, d\phi_k$ is uniformly bounded, which contradicts our assumption that $\max_{S^2} w_k \to +\infty$. Thus, after passing to a subsequence we may assume that $t_k \to +\infty, P_k \to P \in S^2$, and $v_k \to v_\infty$ in $C^{1,\alpha}$ for some $\alpha \in (0, 1)$; the last fact following from the pointwise estimates on v_k and (12) along with the Sobolev imbedding. Notice that $K \circ \phi_K \to K(P)$ uniformly on compact subsets of $S^2 - \{-P\}$, hence v_∞ satisfies

$$\Delta v_\infty + K(P) e^{2v_\infty} = 1, \tag{13}$$

at least weakly on $S^2 - \{-P\}$. But after applying standard arguments from elliptic theory one sees that in fact v_∞ satisfies (13) on all of S^2. By the uniqueness of solutions of (13) belonging to $\mathcal{S}$ we conclude that $v_0 \equiv -\frac{1}{2} \log K(P)$. Normalizing v_k (by rotating P_k to P and adding a suitable constant), we may assume that $K(P) = 1$ and that v_k satisfies

$$\Delta v_k + (K \circ \phi_k) \, e^{2v_k} = 1 \tag{14}$$

with $\phi_k = \phi_{P,t_k}$. Also, by our work above we have

$$\|v_k\|_\infty = o(1) \quad \text{as } k \to \infty \tag{15}$$

$$\|\nabla v_k\|_\infty = o(1) \quad \text{as } k \to \infty. \tag{16}$$

Applying the Kazdan-Warner condition to (14) we have

$$\int (\nabla K \circ \phi_k, \nabla x_j) e^{2v_k} = 0, \ \ j = 1, 2, 3 \tag{17}$$

where $\phi_k = \phi_{P,t_k}$ and $t_k \to \infty$. We now claim that (17) implies

$$\nabla K(P) = 0 \text{ and } \Delta K(P) = 0, \tag{18}$$

thus contradicting our assumption that K is a non-degenerate function and finishing the proof of part (b).

To verify (18) we will begin to compare the expression in (17) to $\int\langle\nabla K\circ\phi_k,\nabla x_j\rangle$, and then compute the asymptotic behavior of the latter term as $t_k\to\infty$. To this end we define

$$A_k=\int<\nabla(K\circ\phi_k),\nabla\vec{x}>e^{2v_k},$$
$$B_k=\int<\nabla K\circ\phi_k,\nabla\vec{x}>=2\int(K\circ\phi_k-1)\vec{x}.$$

Lemma 3. *Under the assumptions (15), (16) with $\phi_k=\phi_{P,t_k},t_k\to\infty,K(P)=1$ we have*

$$A_k=B_k+C_k\quad\textit{with } |C_k|=\begin{cases}o(\frac{1}{t_k})\ \textit{if}\ \nabla K(P)\neq 0,\\ o(\frac{1}{t_k^2}\log\frac{1}{t_k})\ \textit{if}\ \nabla K(P)=0.\end{cases}\tag{19}$$

Let $B_k^{(i)}$ denote the i-th component of B_k, $1\le i\le 3$, then

$$\begin{cases}B_k^{(i)}=c_1a_i\frac{1}{t_k}+O(\frac{1}{t_k^2}),\ \ i=1,2;\\ \\ B_k^{(3)}=c_2(b_{11}+b_{22})\frac{1}{t_k^2}\log t_k+O(\frac{1}{t_k^2})\end{cases}\tag{20}$$

where c_1,c_2 are dimensional constants.

Clearly (18) is a consequence of (17), (19) and (20).

Proof of Lemma 3. To prove (2.12) we first write

$$\begin{aligned}A_k&=\int<\nabla K\circ\phi_k,\nabla\vec{x}>e^{2v_k}\\&=\int<\nabla(K\circ\phi_k-1),\nabla\vec{x}>e^{2v_k}\\&=-\int(K\circ\phi_k-1)\Delta\vec{x}e^{2v_k}-\int(K\circ\phi_k-1)<\nabla\vec{x},\nabla e^{2v_k}>\\&=2\int(K\circ\phi_k-1)\vec{x}+2\int(K\circ\phi_k-1)\vec{x}(e^{2v_k}-1)\\&\quad-2\int(K\circ\phi_k-1)<\nabla\vec{x},\nabla v_k>e^{2v_k}.\end{aligned}$$

Thus $A_k=B_k+C_k$ where

$$C_k=2\int(K\circ\phi_k-1)\vec{x}(e^{2v_k}-1)-2\int(K\circ\phi_k-1)<\nabla\vec{x},\nabla v_k>e^{2v_k}.$$

We will see that the desired estimate of C_k follows from the same asymptotic computation of B_k as in (20) below together with the assumption that $\|v_k\|_\infty = o(1), \|\nabla v_k\|_\infty = o(1)$.

To verify (20), we will use the stereographic projection coordinates of S^2 to compute B_k in terms of the Taylor series expansion of K. To do this, denote $Q = (x_1, x_2, x_3) \in S^2$ and let $y = (y_1, y_2)$ be the stereographic projection from S^2 to the equatorial plane R^2 sending the north pole $N_0 = (0,0,1)$ to ∞. We can also w.l.o.g. identify the point P as the north pole N. Thus $x_i = \frac{2y_i}{1+|y|^2}$ for $1 \le i \le 2$, and $x_3 = \frac{|y|^2-1}{|y|^2+1}$. We assume the Taylor series expansion of K around N is given by

$$K(x_1,x_2,x_3) = K(x_1,x_2) = K(N) + \sum_{i=1}^{2} a_i x_i + \sum_{i,j=1}^{2} b_{ij} x_i x_j + o(\sum_{i=1}^{2} x_i^2) \tag{21}$$

and (2.14) holds in a neighborhood $\tilde{D} = \{y \in R^2, |y| \ge M\}$ of N, for some $M > 0$ large. Notice in this notation $\phi_k(y) = t_k y$. Denote $D_k = \{y \in R^2, |y| \ge \frac{M}{t_k}\}$, then $\phi_k(D_k) = \tilde{D}$. To estimate B_k, let $dA(y) = \frac{1}{\pi}\frac{d|y|^2}{(1+|y|^2)^2} d\theta$ denote the area form, then

$$\int_{D_k^c} dA(y) = 2\int_0^{\frac{M}{t_k}} \frac{d|y|^2}{(1+|y|^2)^2} = \frac{M^2}{t_k^2 + M^2} = O(\frac{1}{t_k^2}) \text{ as } t_k \to \infty. \tag{22}$$

Thus

$$B_k = 2\int (K \circ \phi_k - 1)\vec{x}$$

$$= 2\int_{D_k} (K \circ \phi_k - 1)\vec{x} + O(\frac{1}{t_k^2}).$$

Next we notice that by circular symmetry,

$$\int_{D_k} x_i(t_k y) x_j(y) dA(y) = 0 \quad \text{if } i \ne j;\ 1 \le i,j \le 3.$$

Hence,

$$B_k^{(i)} = \int_{D_k} a_i x_i(t_k y) x_i(y) dA(y) + 2\int_{D_k} \left(\sum_{j,\ell=1}^{2} b_{j\ell} x_j(t_k y) x_\ell(t_k y) \right) x_i(y) dA(y)$$
$$+ E_k^i + O(\frac{1}{t_k^2}) \quad \text{for } i = 1,2,3,$$

where

$$E_k^{(i)} = O\left(\int_{D_k} \left(\frac{|t_k y|}{1+|t_k y|^2} \right)^3 |x_i(y)| dA(y) \right), \ i = 1,2,3,$$

and

$$B_k^{(3)} = 2\int\left(\sum_{j,\ell=1}^{2} b_{j\ell}x_j(t_ky)x_\ell(t_ky)\right)x_3(y)dA(y) + E_k^{(3)} + O(\frac{1}{t_k^2}).$$

Similarly we define $C_k^{(i)}$ for $i = 1, 2, 3$ to be the components of C_k, and $B_k^{(i)}$ for $i = 1, 2, 3$ to be the components of B_k. (19) and (20) will be consequences of the following lemma which can be verified by direct computation.

Lemma 4.

(23)
$$\int_{D_k} x_i(t_ky)x_i(y)dA(y) \sim \frac{1}{t_k} \text{ as } k \to \infty$$

(24)
$$\int_{D_k} x_j(t_ky)x_\ell(t_ky)x_i(y)dA(y) = \begin{cases} 0, & 1 \le i,j,\ell \le 2; \\ 0, & i = 3,\ j \ne \ell; \\ \frac{1}{t_k^2}\log t_k, & i = 3,\ 1 \le j = \ell \le 2. \end{cases}$$

$$|E_k^i| = O(\frac{1}{t_k^2}),\ i = 1,2,3.$$

We conclude from (23), (24) that

(25)
$$\begin{cases} B_k^{(i)} = c_1 a_i \frac{1}{t_k} + O(\frac{1}{t_k^2}),\ i = 1,2, \\ B_k^{(3)} = c_2(b_{11} + b_{22})\frac{1}{t_k^2}\log t_k + O(\frac{1}{t_k^2}), \end{cases}$$

with $c_1 > 0$, $c_2 > 0$ dimensional constants.

By a similar argument we can prove

(26)
$$C_k^{(i)} = o(|a|)\frac{1}{t_k} + o(|b|)\frac{\log t_k}{t_k^2} + O(\frac{1}{t_k^2})$$

where $|a| = \sum |a_i|$, $|b| = \sum_{i,j} |b_{ij}|$. Then (19), (20) are direct consequences of (25), (26). We have thus finished the proof of part (b) of Theorem 1.

In the rest of this chapter, we will describe some sufficient conditions on K ($n = 2$) or R ($n \geqq 3$) for equations (1) and (2) to have solutions.

Recall that in the proof of Kazdan-Warner condition (iii) above, we have, for a solution w of (1), $\frac{d}{dt}\Big|_{t=1} F_K[T_{\phi_{P,t}}(w)] = 0$, for any $P \in S_x^2$, and

$$\frac{d}{dt}\Big|_{t=1} F_K[T_{\phi_{P,t}}(w)] = \fint < \nabla K, \nabla(\overrightarrow{x} \cdot P) > e^{2w} = 0$$

A relevant observation here is that if e^{2w} is close to the constant 1, then the integral is comparable to

$$2 \fint K \overrightarrow{x} \cdot \overrightarrow{P} dA.$$

A parallel argument works also for the higher dimensional setting. Thus given a potential curvature function R (or K) we form the following map $G : B \to R^{n+1}$ given by

$$G\left(\frac{t-1}{t}P\right) = \int_{S^n} (R \circ \phi_{P,t}) \cdot \overrightarrow{x}. \tag{27}$$

It is natural to investigate the asymptotic behavior of the map $G : (P,t) \mapsto \fint (R \circ \phi_{P,t})\overrightarrow{x}$ as the parameter $t \to \infty$. We choose coordinates $x_1, \ldots, x_{n+1}$ so that P is the south pole $P = (0, \ldots 0, -1)$ and let $y_1 \ldots y_n$ be the stereographic coordinates

$$\begin{cases} x_i = \frac{2y_i}{1+|y|^2} & 1 \le i \le n\,, \\ x_n = \frac{|y|^2-1}{|y|^2+1}\,. \end{cases}$$

(In the coordinate system y, $\phi_t(y) = \frac{1}{t}y$.)

Expand R in Taylor series in y coordinates around P:

$$R(y) = R(P) + \sum_{k=1}^{\infty} R_k(y)$$

where R_n is a homogenous polynomial in $y_1, \ldots, y_n$ of degree k, so that

$$(R \circ \phi_{P,t})(y) = R(P) + \sum_{k=1}^{\infty} R_k(y)t^{-k}.$$

We denote by $R^{(\alpha)}(y) = R(P) + \sum_{k=1}^{\alpha} R_k(y)$, the truncated Taylor polynomial of order α.

Definition. We say R is non-degenerate at P of order α if

(i) the truncated Taylor polynomial of order α has the form

$$R^{(\alpha)}(y) = R(P) + R_\alpha(y),$$

(ii) for t sufficiently large, the map

$$G(P,t) = \fint (R^{(\alpha)} \circ \phi_{P,t}) \vec{x}$$

satisfies a lower bound:

$$|G(P,t)| \geq \begin{cases} c\frac{1}{t^\alpha} & \alpha < n \\ c\frac{1}{t^n} \log t & \alpha = n\,. \end{cases}$$

At any point $P \in S^n$, one can examine this notion of non-degeneracy with some simple computations (as in introduction of [C-Y-6]), we can see for example:

(1) At a point P where $\nabla R(P) \neq 0$, it is non-degenerate of order 1.

(2) At a point P where $\nabla R(P) = 0$ but $\Delta R(P) \neq 0$, it is non-degenerate of order 2.

We observe that for a function R uniformly non-degenerate of order $\alpha \leqq n$, the map $G(P,t)$ is non-zero for $t \geqq t_0$, hence the restriction of G to the sphere $\{(P,t), P \in S^n\}$ are mutually homotopic for $t \geqq t_0$, and the degree $\deg(G,B,0)$ is well defined.

Theorem 2. *There exists constants $\epsilon(n)$ such that if K or R is a smooth function satisfying:*

(i) $||K-1||_\infty \leq \epsilon(2)$ or $||R - R_0||_\infty \leq \epsilon(n)$;

(ii) K or R is a uniformly nondegenerate function of order α, where $\alpha \leq n$ when n is even and $\alpha \leq n-1$ when n is odd, i.e.

$$(*) \qquad |G(P,t)| \geq \begin{cases} \frac{C}{t^\alpha}, & \text{when } \alpha < n, \\ \frac{C}{t^n \log t}, & \text{if } \alpha = n \end{cases}$$

for $t \geq t_0$, uniformly in P; for all P in S^n.

(iii) $deg(G,B,0) \neq 0$;

then the equation (1) has a solution.

Remarks.

1. In the case $\alpha = 2$, uniformity in $(*)$ is a consequence of the nondegenerate assumption on the critical points of the function K or R. In the general case, the uniformity requirement does not follow from the nondegenerate requirement on the critical points of K or R alone. In principle, it is possible to reduce this requirement to algebraic criteria on the Taylor coefficients of the function R at its critical points. For example we determine necessary and sufficient conditions on the Taylor coefficients of R at its critical points when $\alpha = 3$ in [C-Y-6a].
2. In dimension 2 (and 3), result in Theorem 2 is optimal in the sense that if the differential equation (1) admits a solution then it is necessarily captured

by the variational scheme used in the proof. This amounts to an a priori estimate for solutions of (1) when $\|K-1\|_\infty$ is sufficiently small

It turns out in the case of $n=2$ and 3, one can drop the smallness assumptionin the statement of Theorem 2.

Theorem 3 [C-Y-G]. *On S^2 (S^3), suppose $K>0$ ($R>0$) is a smooth function satisying the non-degeneracy condition that $\nabla K(Q)=0$ implies $\Delta K(Q)\neq 0$ (respectively for R) and* $\deg(G|_{t=t_0}0)\neq 0$, *then the equation (1) (respectively (2)) has a solution.*

Theorem 2 generalizes previous existence results of Chang-Yang ([C-Y-4] [C-Y-5] and [H]) on S^2 and Bahri-Coron ([B-C]) and Schoen-Zhang ([S-Z]) on S^3 where K (respectively R) is assumed positive, having only isolated non-degenerate critical points and in addition satisfying $\Delta K(Q)\neq 0$ at critical points, and the index count condition:

$$\sum_{Q\ \mathrm{critical},\Delta K(Q)<0} (-1)^{ind(Q)} \neq (-1)^n. \tag{28}$$

In the appendix of [C-Y-G], it was shown that the index count condition (28) implies $deg(G,0)\neq 0$. On the other hand, the nondegeneracy condition (nd) allows K (or R) to have non-isolated critical point sets. In dimension 4, there is an interesting example of Bianchi-Egnell ([B-E]) which indicates that further complications arise and it will be necessary to understand the interactions of more than one concentrated masses. The validity of Theorem 3 remains open for $n\geqq 5$.

Theorem 3 is a consequence of a topological degree argument. The earlier perturbation result (Theorem 2) provides the initial step of a continuity argument. One then verifies that $deg(G,0)$ gives in fact the Leray Schauder degree of a nonlinear map whose zeroes correspond to solutions of the differential equations when R satisfies the close to constant condition. Then the a priori estimates as in Theorem 1 provide the continuity argument needed to verify the invariance of the Leray Schauder degree as one moves along the parameter in the continuity scheme. We refer interested reader to [C-Y-G] for details.

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The Effect of Curvature on the Behavior of Harmonic Functions and Mappings

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Introduction

The purpose of this lecture series is to explain the role that curvature plays in the analysis of solutions of geometric partial differential equations, especially those which arise from variational principles. We do this by considering two concrete situations. The first of these is the study of harmonic functions on Riemannian manifolds. This subject occupies the first three lectures. The second problem we consider is the harmonic mapping system which arises from the variational problem of extremizing the L^2 energy for maps between Riemannian manifolds. Although we consider two specific problems here, much of the theory we describe has been generalized to other problems, and the general ideas involved have broad application to the study of geometric partial differential equations of elliptic type. In this introduction we describe in general terms the content of the various lectures, and give references for various extensions of the theory which have been obtained.

The first two lectures give the proof of the gradient estimate and Harnack inequality for harmonic functions on Riemannian manifolds with constants depending only on a lower bound of the Ricci curvature of the manifold, the dimension, and the radius of a ball on which the function is defined. This result in global form; that is, for positive harmonic functions defined on complete manifolds, is due to S. T. Yau[Y]. The local version is due to Cheng and Yau[CY]. An important point of this estimate, is that it does not require a global coordinate system, nor a bound on the injectivity radius of the manifold. Standard Harnack inequalities in PDE require bounds on the coefficients of the operator in some fixed coordinate chart, so these are not suitable for our purpose. There are two basic ingredients in the proof of the gradient estimate. The first is the use of comparison theorems for the Riemannian distance function. It is important in many PDE estimates to have an appropriate exhaustion function for your manifold or domain. The basic properties of the Riemannian distance function which are needed for the Harnack inequality

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are that it is uniformly Lipschitz (true on any metric space), and that its Laplacian has an upper bound depending on the lower bound of the Ricci curvature. This lower bound of the Laplacian holds in the distributional sense beyond the injectivity radius, and so does not require the distance function to be smooth. Geometrically, the Laplacian of the distance function equals the mean curvature of the geodesic sphere, so this comparison result says that the mean curvature of the geodesic sphere may be bounded from above in terms of a lower bound of the Ricci curvature of the manifold. The second main ingredient is the Bochner formula which yields a lower bound of the Laplacian of $|\nabla u|^2$ for a harmonic function u in terms of a lower bound of the Ricci curvature. The proof then relies on a clever application of the maximum principle. The outline of this proof may be used to obtain important results for other problems as well. For eigenfunctions, a modification of this proof was used to obtain sharp lower bounds for the first eigenvalue of a manifold by P. Li[Li], and more generally by Li and Yau[LY1]. Li and Yau[LY2] have obtained an analogous estimate for the heat equation. S. Y. Cheng[C1] used a similar argument to obtain a gradient estimate for harmonic mappings into simply connected manifolds of nonpositive curvature in terms of a bound on the oscillation of the map. This result was improved by H. I. Choi[Cho] to handle mappings whose image falls in a convex coordinate chart. The reader may also see the recent book [SY] for more details of these results.

In the third lecture we prove an existence theorem for harmonic functions with prescribed boundary data on complete, simply connected, manifolds with bounded negative curvature; that is, all sectional curvatures bounded between two negative constants. Such manifolds have a geometrically natural compactification as a closed ball. This is given by adding a sphere at infinity, which may be thought of as the sphere of directions of geodesics from a point. This boundary is referred to as the geometric boundary. We then give an elementary proof of the unique solvability of the Dirichlet problem for continuous boundary data on such manifolds. This theorem is due to Anderson[A] and Sullivan[Su], and the proof given here is due to the author, and appears in [AS]. The basic idea is that the theory of harmonic functions on manifolds of this type is similar to that on bounded domains in euclidean space. The condition of bounded negative curvature corresponds to a very weak smoothness assumption on the boundary, so one should imagine doing analysis with a very singular boundary. The result of Anderson and the author[AS] is much more general than the solvability of the Dirichlet problem because it constructs with estimates the Poisson kernel functions of such a manifold, showing that there is a unique kernel function for each point of the geometric boundary, and that these kernel functions are Hölder continuous on the boundary. The main ingredient which is established in [AS] is a Harnack inequality at infinity. These results were generalized and improved by A. Ancona[An]. Results for the special case of coverings of compact manifolds which weaken the curvature assumption were proven by Lyons and Sullivan[LS], and by W. Ballman[Ba]. A result of Cheng[C2] establishes solvability of the Dirichlet problem under the assumption of *pointwise* bounded negative curvature; that is, the curvature is less than a negative constant, and the ratio of sectional curvatures at any given point is bounded. This asumption removes the lower bound in two dimensions, and allows the curvature to be unbounded in higher dimensions. It is not known whether the Dirichlet problem is solvable without the assumption of a lower curvature bound. The reader may see [SY] for more discussion, and questions in this direction.

The last six lectures deal with questions involving harmonic mappings. Here the key geometric ingredient is the curvature of the target space. There have been recent developments which extend much of the existing harmonic mapping theory to singular target spaces for which the curvature is bounded from above in the sense of triangle comparison hypotheses. We discuss here only the first part of this theory which was developed by Gromov and the author[GS]. This theory has been extended substantially by N. Korevaar and the author[KS], and many existence results were also treated from a different point of view by J. Jost[J]. These new results handle arbitrary metric space targets, and in case the target space is a length space of nonpositive curvature, the results of [KS] prove that the harmonic mappings are Lipschitz continuous in the interior. T. Serbinowski[Se] has shown that harmonic maps are C^α for any $\alpha < 1$ provided the boundary map is C^α. In a recent joint work with Korevaar and Serbinowski we have also extended the regularity theory of Uhlenbeck and the author[SU] to work under the hypothesis that the target be a length space with curvature bounded from above by a constant. This result implies that minimizing mappings are Lipschitz continuous on an open set whose complement has Hausdorff codimension at least three. In these lectures, we deal only with the case considered in [GS] where the target is a Riemannian simplicial complex of nonpositive curvature. The results of Lectures 4,5, and 6 extend to harmonic mappings to arbitrary length spaces of nonpositive curvature. Much of this extension appears in [KS]. The major new ingredient which is needed is a detailed construction of the nonlinear Sobolev space of maps with finite energy given in [KS].

Lecture 4 gives the definition of metric spaces of nonpositive curvature, together with examples and motivation. One of the basic motivational issues is that the classical theory developed by Eells and Sampson[ES] makes use, in a crucial way, of a Bochner formula (analagous to the one used above for harmonic functions). It is necessary to find a replacement for this formula in the case of singular targets in order to prove the Lipschitz bound which is critical for the theory. Lecture 6 derives a monotonicity inequality which is crucial for the theory of [GS] concerning the structure of harmonic maps to euclidean buildings. This inequality extends to harmonic maps to arbitrary length spaces of nonpositive curvature; however this extension is not carried out in [KS], as the Lipschitz result is derived in [KS] by a different method, closer to the Eells-Sampson method. In [GS], the Lipschitz bound was derived from the monotonicity inequality. The extension of the monotonicity inequality for maps to general targets, and its consequences will be discussed in a future paper of Korevaar and the author. Lectures 7 and 8 describe the detailed results of [GS] for harmonic mappings to euclidean buildings. This theory uses special properties of the target space to derive very special structure of harmonic mappings to such spaces. Finally Lecture 9 describes the application [GS] to proving the arithmeticity of lattices in rank 1 Lie groups. This involves the use of the structure theorem to prove a vanishing theorem for harmonic mappings to euclidean buildings.

Finally we mention that in light of the direction which is pursued here of studying harmonic mappings to singular spaces, it seems to be natural to propose a similar direction for the material in the first three lectures. Since the gradient estimate of Lecture 1 depends only on the dimension and a lower bound on the curvature, one might guess that it is true for harmonic functions on length spaces with curvature bounded from below. This is not known in full generality, but some interesting partial results have been obtained by J. Chen[Ch], and recently by J. Chen and H. Pei. The main issue is similar to that for harmonic mappings, namely making sense of a Bochner formula (and proving it).

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LECTURE 1
Gradient Estimate and Comparison Theorems

Let (M, g) be a n-dimensional complete Riemannian manifold. We will be interested in several higher order quantities. Consider first the Riemannian curvature tensor $\text{Riem}(g)$. In terms of a local orthonormal basis $\{e_i\}_{i=1}^n$ for the tangent bundle of M, the coefficients of $\text{Riem}(g)$ are written R_{ijkl}. Using these, we may describe the following important geometric components:

1. Sectional Curvature, $K(\pi)$. Given the 2-dimensional subspace $\pi \subseteq T_xM$ spanned by e_1, e_2, we get the sectional curvature $K(\pi) = R_{1212}$. This is equal to the Gauss curvature at x of the two-dimensional image of π under the exponential map $\exp_x$.
2. Ricci Curvature, $\text{Ric}(g)$. A diagonal component $R_{ii} = \sum_{k=1}^n R_{ikik}$ = sum of the sectional curvatures of the coordinate 2-planes containing e_i. The off-diagonal components R_{ij} may be obtained by polarization.
3. Scalar Curvature, $R(g)$. $R = \sum_{i,j=1}^n R_{ijij}$ = sum of the sectional curvatures of all the coordinate planes.

Among the interesting geometric operators on (M, g), we have the most fundamental operator, the Laplace-Bertrami operator Δ on (M, ds^2), which is defined in local coordinates by

$$\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_j} \right)$$

where $ds^2 = g_{ij} dx^i dx^j$, $(g^{ij}) = (g_{ij})^{-1}$, $g = \det(g_{ij})$. This operator is associated to the Dirichlet energy $E(f) = \int_M |\nabla f|^2 d\mu$ where $f \in C_c^\infty(M)$ and $d\mu$ the volume form of M; in local coordinates, $d\mu = \sqrt{\det(g_{ij})} dx$.

The relationship between harmonic functions, curvature, and spectral theory is quite interesting. Basic intuition for this comes from 2-dimensional case. Here we have that a Riemannian surface is a *Riemann surface* and we can invoke uniformization. Suppose (M^2, g) is simply connected so that the possible conformal types are $\mathbb{S}^2$, $\mathbb{C}$, $\mathbb{D}$. Moreover, from the point of view of harmonic functions, a surface of nonnegative curvature behaves like $\mathbb{C}$, and a surface of nonpositive curvature behaves like $\mathbb{D}$. Therefore, one might expect that, in higher dimensions, manifolds of sectional curvature ≥ 0 behave like $\mathbf{R}^n$ and manifolds of sectional curvature ≤ 0 behave like $\mathbf{H}^n$. Unfortunately, there are few theorems like this. Here is one:

Theorem 1.1 (Gradient Estimate). *If u is a positive harmonic function on a ball $B_a(x) \subseteq M$, $k \geq 0$, and* $\mathrm{Ric} \geq -(n-1)k^2$, *then*

$$\sup_{B_{a/2}(x)} \frac{|\nabla u|}{u} \leq C_n \left(\frac{1+ka}{a}\right) .$$

Corollary 1.2. *Suppose* $\mathrm{Ric}_M \geq 0$. *Then any positive harmonic function u on M is constant.*

In order to prove the gradient estimate theorem, we first need to review several important comparison theorems.

Recall first, for $x \in M$, the exponential map $\exp_x : T_xM \to M$. For $X \in T_xM$ with $|X| = 1$, $\gamma(t) = \exp_x(tX)$ is the unique unit speed geodesic that starts from x and goes in the direction X. When t is small, γ is the unique minimal unit speed geodesic joining x and $\exp_x(tX)$. Note that the differential $d\exp_x|_t X : T_{tX}(T_xM) \to T_{\gamma(t)}M$ is a linear isomorphism. Let $t_0 = \sup\{t > 0 : \gamma$ be the unique minimal geodesic joining p and $\gamma(t)\}$. If $t_0 < +\infty$, then $\gamma(t_0)$ is called a *cut point* of x. Let $\mathrm{Cut}(x)$ = set of all cut points of x. Observe that the distance function $d(\cdot, x)$, while only Lipschitz on M, is actually smooth on $M \setminus C(x)$. In particular, we may take derivatives, which will involve various curvatures of M.

Theorem 1.3 (Hessian Comparison). *Let M_1 and M_2 be two n-dimensional complete Riemannian manifolds. Assume that $\gamma_i : [0,a] \to M_i$ ($i = 1, 2$) are two geodesics parametrized by arc length and that γ_i doesn't intersect the cut locus of $\gamma_i(0)$. Let ρ_i be the distance function from $\gamma_i(0)$ on M_i, and let $K_i(\cdot,\cdot)$ denote a sectional curvature of M_i. Assume that $0 \leq t \leq a$ and that at $\gamma_1(t)$ and $\gamma_2(t)$, we have*

$$K_1\left(X_1, \frac{\partial}{\partial\gamma_1}\right) \geq K_2\left(X_2, \frac{\partial}{\partial\gamma_2}\right)$$

where X_i is any unit vector in $T_{X_i(t)}M_i$ perpendicular to $\frac{\partial}{\partial\gamma_i}$. Then

$$H(\rho_1)(X_1, X_1) \leq H(\rho_2)(X_2, X_2)$$

where $X_i \in T_{\gamma_i(a)}M_i$ with $\left\langle X_i \frac{\partial}{\partial\gamma_i}\right\rangle(\gamma_i(a)) = 0$, and $H(\rho_i)$ denotes the Riemannian Hessian of the distance function ρ_i.

Proof. We start with the definition $H(f)(X,Y) = \tilde{X}(\tilde{Y}(f)) - (\nabla_{\tilde{X}}\tilde{Y})f$ where $f \in C^2(M)$, $X, Y \in T_xM$ are two vectors, and $\tilde{X}$ (or $\tilde{Y}$) are two extended vector fields of X (or Y) near x, and ∇ denotes the Riemannian connection of M. Since $\gamma_i \cap \mathrm{Cut}(\gamma_i(0)) = \emptyset$, we may extend $X_i \in T_{\gamma_i(a)}M_i$) with $\left\langle \frac{\partial}{\partial\gamma_i}, X_i\right\rangle = 0$ to a Jacobi field $\tilde{X}_i$ along γ_i satisfying $\tilde{X}_i(\gamma_i(0)) = 0$, $\tilde{X}_i(\gamma_i(a)) = X_i$, and $\left[\tilde{X}_i, \frac{\partial}{\partial\gamma_i}\right] = 0$. Since

$\operatorname{grad} \rho_i = \frac{\partial}{\partial \gamma_i}$, we have that

$$\begin{aligned} H(\rho_i)(X_i, X_i) &= \widetilde{X}_i \widetilde{X}_i \rho_i - \left(\nabla_{\widetilde{X}_i} \widetilde{X}_i\right) \rho_i \\ &= X_i \left\langle \widetilde{X}_i, \frac{\partial}{\partial \gamma_i} \right\rangle - \left\langle \nabla_{\widetilde{X}_i} \widetilde{X}_i, \frac{\partial}{\partial \gamma_i} \right\rangle \\ &= \left\langle \widetilde{X}_i, \nabla_{\widetilde{X}_i} \frac{\partial}{\partial \gamma_i} \right\rangle \\ &= \left\langle \widetilde{X}_i, \nabla_{\frac{\partial}{\partial \gamma_i}} \widetilde{X}_i \right\rangle \left(\text{because} \left[\widetilde{X}_i, \frac{\partial}{\partial \gamma_i}\right] = 0\right) \\ &= \int_0^a \frac{d}{dt} \left\langle \widetilde{X}_i, \nabla_{\frac{\partial}{\partial \gamma_i}} \widetilde{X}_i \right\rangle \\ &= \int_0^a \left| \nabla_{\frac{\partial}{\partial \gamma_i}} \widetilde{X}_i \right| + \left\langle \widetilde{X}_i, \nabla_{\frac{\partial}{\partial \gamma_i}} \nabla_{\frac{\partial}{\partial \gamma_i}} \widetilde{X}_i \right\rangle . \end{aligned}$$

Since $\widetilde{X}_i$ is a Jacobi field, it satisfies

$$\nabla_{\frac{\partial}{\partial \gamma_i}} \nabla_{\frac{\partial}{\partial \gamma_i}} \widetilde{X}_i + R_i \left\langle \widetilde{X}_i, \frac{\partial}{\partial \gamma_i} \right\rangle \frac{\partial}{\partial \gamma_i} = 0 ,$$

and we get

$$H(\rho_i)(X_i, X_i) = \int_0^a \left| \nabla_{\frac{\partial}{\partial \gamma_i}} \widetilde{X}_i \right|^2 - \left\langle R_i \left(\widetilde{X}_i, \frac{\partial}{\partial \gamma_i}\right) \frac{\partial}{\partial \gamma_i}, \widetilde{X}_i \right\rangle .$$

We may now compare these two Hessians as follows: Let $E_1^i, \ldots, E_n^i$ be orthonormal parallel vector fields along γ_i such that $E_n^i = \frac{\partial}{\partial \gamma_i}$. Since $\left\langle X_i, \frac{\partial}{\partial \gamma_i} \right\rangle (\gamma_i(0)) = 0$, we have $\left\langle \widetilde{X}_i, \frac{\partial}{\partial \gamma_i} \right\rangle = 0$ at each point of γ_i. Set $\widetilde{X}_2 = \sum_{j=1}^{n-1} \lambda_j(t) E_j^2$. We may assume that $\widetilde{X}_1(a) = \sum_{j=1}^{n-1} \lambda_j(a) E_j^1(\gamma_1(a))$. Define now a vector field Z along γ_i by

$$Z = \sum_{j=1}^{n-1} \lambda_j(t) E_j^1 .$$

Then $Z(0) = \widetilde{X}_1(0)$, $Z(a) = \widetilde{X}_1(a)$. Moreover $|Z| = |\widetilde{X}_2|$ and

$$\left| \nabla_{\frac{\partial}{\partial \gamma_1}} Z \right| = \left| \sum_{j=1}^{n-1} \lambda_j'(t) E_j^1 \right| = \left| \sum_{j=1}^{n-1} \lambda_j'(t) E_j^2 \right| = \left| \nabla_{\frac{\partial}{\partial \gamma_2}} \widetilde{X}_2 \right| .$$

Since a Jacobi field minimizes the index form among all vector fields along the same geodesic with the same boundary data, we have

$$\begin{aligned}
H(\rho_1)(X_1,X_1) &= \int_0^a \left|\nabla_{\frac{\partial}{\partial\gamma_1}}\tilde{X}_1\right|^2 - \left\langle R_1\left(\tilde{X}_1\frac{\partial}{\partial\gamma_1}\right)\frac{\partial}{\partial\gamma_1},\tilde{X}_1\right\rangle \\
&\le \int_0^a \left|\nabla_{\frac{\partial}{\partial\gamma_1}}Z\right|^2 - \left\langle R_1\left(Z,\frac{\partial}{\partial\gamma_1}\right)\frac{\partial}{\partial\gamma_1},Z\right\rangle \\
&= \int_0^a \left|\nabla_{\frac{\partial}{\partial\gamma_2}}\tilde{X}_2\right| - \left\langle R_1\left(Z,\frac{\partial}{\partial\gamma_1}\right)\frac{\partial}{\partial\gamma_1},Z\right\rangle \\
&\le \int_0^a \left|\nabla_{\frac{\partial}{\partial\gamma_2}}\tilde{X}_2\right| - \left\langle R_2\left(\tilde{X}_2,\frac{\partial}{\partial\gamma_2}\right)\frac{\partial}{\partial\gamma_2},\tilde{X}_2\right\rangle \\
&= H(\rho_2)(X_2,X_2). \qquad \square
\end{aligned}$$

Theorem 1.4 (Laplacian Comparison). *Let M be an n-dimensional complete Riemannian manifold with* $\mathrm{Ric}(M) \ge -(n-1)k^2$. *Let $\rho_M(\cdot,\tilde{0})$ be the distance on M from a fixed point $\tilde{0}$. If $x \in M$ and $\rho_M(\cdot,0)$ is smooth at x, then*

$$\Delta\rho_M(\cdot,\tilde{0})\Big|_x \le \frac{n-1}{\rho_M(x,\tilde{0})}(1+k\rho_M(x,\tilde{0})).$$

Proof. Let N be a space form of constant curvature $-k^2$, and 0 be a fixed point of N. We start to compute the Hessian of distance function $\rho_N(\cdot,0)$ on N. Let γ be a geodesic in N starting at 0, $\rho > 0$, and $X \in T_{\gamma(\rho)}N$ with $X \perp \gamma(\rho)$. For $0 \le t \le \rho$, let $X(t)$ be the parallel displacement of X along γ. Then the Jacobi field $Y(t)$ along γ with $Y(0)=0$, $Y(\rho)=X$ has the form $Y(t)=f(t)X(t)$ where $f(t)$ satisfies

$$\begin{cases} \frac{d^2}{dt^2}f(t) - k^2 f(t) = 0\,, \\ f(0)=0,\ f(\rho)=1. \end{cases}$$

It is easy to see that $f(t) = \frac{\sinh\, kt}{\sin k\rho}$. Now, if $\left\{\frac{\partial}{\partial\gamma}, X_1,\ldots,X_{n-1}\right\}$ is an orthonormal basis of $T_{\gamma(\rho)}M$ and $\tilde{X}_i(t)=f(t)X_i(t)$ is the Jacobi field with $\tilde{X}_i(\rho)=X_i$, then we have

$$\begin{aligned}
\Delta\rho_N(\cdot,0) &= \sum_{i=1}^{n-1} H(\rho)(X_i,X_i) \\
&= (n-1)\int_0^\rho \left[\left|\frac{d}{dt}f(t)\right|^2 + k^2 f^2(t)\right]dt \\
&= (n-1)k\coth k\rho.
\end{aligned}$$

Applying the Hessian Comparison Theorem, we have that

$$\Delta\rho_M(\cdot,\tilde{0})\Big|_x \le \Delta\rho_N(\cdot,0)\Big|_y$$

where y is point on N with $\rho_N(y,0) = \rho_M(x,\tilde{0})$. Therefore

$$\Delta\rho_M(\cdot,\tilde{0})\Big|_x \leq (n-1)k\coth(k\rho_M(x,\tilde{0})).$$

It is easy to check that

$$k\rho\coth k\rho \leq 1 + k\rho\ ,$$

so

$$\Delta\rho_M(\cdot,\tilde{0})\Big|_x \leq \frac{n-1}{\rho_M(x,\tilde{0})}(1 + k\rho_M(x,\tilde{0})).$$

Remark 1.5. From the Laplacian Comparison Theorem, we see that if $\mathrm{Ric}(M) \geq -(n-1)k^2$ and $y \in M \setminus \mathrm{Cut}(x)$, then

$$\Delta\rho_M(\cdot,x)\Big|_y \leq \frac{n-1}{\rho_M(y,x)}(1 + k\rho_M(y,x)).$$

Moreover, we can show that a corresponding global inequality holds in a distribution sense, i.e.,

$$(*)\ \int_M \rho_M(\cdot,\tilde{0})\Delta\varphi \leq \int_M \frac{n-1}{\rho_M(\cdot,\tilde{0})}(1+k\rho_M(\cdot,\tilde{0}))\varphi \quad \forall\, \varphi \in C_0^\infty(M,R) \text{ and } \varphi \geq 0.$$

1.6 Proof of (*). Let $\Omega = M \setminus \mathrm{Cut}(0)$, and $\varphi \in C_0^\infty(M)$ with $\varphi \geq 0$. Since $\mathrm{Cut}(0)$ has Hausdorff H^n measure zero, we have

$$\int_M \rho_M(\cdot,\tilde{0})\Delta\varphi = \int_\Omega \rho_M(\cdot,\tilde{0})\Delta\varphi.$$

Defining $\Omega_\epsilon \subseteq \Omega$ as $\Omega_\epsilon = \{x \in \Omega\ :\ d(x,\mathrm{Cut}(\tilde{0})) \geq \epsilon\}$, we know Ω_ϵ converges to Ω as $\epsilon \downarrow 0$, and hence, applying Stokes formula and the Green formula gives

$$\begin{aligned}
&\int_M \rho_M(\cdot,\tilde{0})\Delta\varphi \\
&\qquad = -\int_M \nabla\rho_M(\cdot,\tilde{0})\nabla\varphi \\
&\qquad = \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} \nabla\rho_M(\cdot,\tilde{0})\nabla\varphi \\
&\qquad = +\lim_{\epsilon\to 0}\int_{\Omega_\epsilon} \Delta\rho_M(\cdot,\tilde{0})\varphi - \lim_{\epsilon\to 0}\int_{\partial\Omega_\epsilon} \varphi\frac{\partial\rho_M}{\partial\nu} \\
&\qquad \leq \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} \frac{n-1}{\rho_M(\cdot,\tilde{0})}(1 + k\rho_M(\cdot,\tilde{0}))\varphi - \lim_{\epsilon\to 0}\int_{\partial\Omega_\epsilon} \varphi\frac{\partial\rho_M}{\partial\nu} \\
&\qquad = \int_M \frac{n-1}{\rho_M(\cdot,\tilde{0})}(1 + k\rho_M(\cdot,\tilde{0}))\varphi - \lim_{\epsilon\to 0}\int_{\partial\Omega_\epsilon} \varphi\frac{\partial\rho_M}{\partial\nu}.
\end{aligned}$$

We also note that Ω is star-shaped and hence so is Ω_ϵ. In particular, $\frac{\partial \rho_M}{\partial \nu} \geq 0$ and $\int_{\partial\Omega_\epsilon} \varphi \frac{\partial \rho_M}{\partial \nu} \geq 0$. Therefore,

$$\int_M \rho_M(\cdot, 0)\Delta\varphi \leq \int_M \frac{n-1}{\rho_M(\cdot, \bar{0})}(1 + k\rho_M(\cdot, \bar{0}))\varphi.$$

Now, we return to the proof of the Gradient Estimate Theorem. First, we calculate the Laplacian of the energy density of u, namely

$$\begin{aligned}\Delta\left(\frac{1}{2}|\nabla u|^2\right) &= |\nabla\nabla u|^2 + \nabla u \nabla(\Delta u) + \mathrm{Ric}(\nabla u, \nabla u) \\ &\geq |\nabla\nabla u|^2 - (n-1)k^2|\nabla u|^2.\end{aligned}$$

Since, by hypothesis, $\mathrm{Ric}(M) \geq -(n-1)k^2$ and $\Delta u = 0$, we see that

$$\Delta\left(\frac{1}{2}|\nabla u|^2\right) = |\nabla u|\Delta|\nabla u| + |\nabla|\nabla u||^2.$$

In particular, we have

$$|\nabla u|\Delta|\nabla u| + (n-1)k^2|\nabla u|^2 \geq |\nabla\nabla u|^2 - |\nabla|\nabla u||^2. \tag{1.1}$$

Now, we want to calculate the right hand side in some detail. Since the calculation can be taken locally, we fixed a point $p \in M$ and chose a suitable orthonormal frame near p such that $\frac{\partial u}{\partial x_i}(p) = 0$ for $i \geq 2$ and $\frac{\partial u}{\partial x_1} = |\nabla u|(p)$ then at p

$$\nabla_j(|\nabla u|) = \nabla_j(\sqrt{\langle u_i, u_i\rangle}) = \sum_i \frac{u_i \cdot u_{i,j}}{|\nabla u|} = u_{1,j}$$

and

$$\begin{aligned}\nabla|\nabla u| \cdot \nabla|\nabla u| &= \sum_j u_{1,j}^2 |\nabla\nabla u|^2 - |\nabla|\nabla u||^2 \\ &= \sum_{i,j} u_{i,j}^2 - \sum_j u_{1,j}^2 \\ &= \sum_{i\geq 2,\, j} u_{i,j}^2 \\ &\geq \sum_{i\geq 2} u_{i,1}^2 + \sum_{i\geq 2,\, j\geq 2} u_{i,j}^2.\end{aligned}$$

Since $\Delta u = 0$, we have $u_{1,1}^2 = \left(\sum_{i\geq 2} u_{i,i}\right)^2$. So by the Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
|\nabla\nabla u|^2 - |\nabla|\nabla u||^2 &\geq \sum_{i\geq 2} u_{i,1}^2 + \frac{1}{n-1}\left(\sum_{i\geq 2} u_{i,i}\right)^2 \\
&\geq \sum_{i\geq 2} u_{i,1}^2 + \frac{1}{n-1} u_{1,1}^2 \\
&\geq \frac{1}{n-1}\sum_{i\geq 1} u_{i,1}^2 \\
&= \frac{1}{n-1}|\nabla|\nabla u||^2 .
\end{aligned}
\tag{1.2}
$$

Thus, combining (1.1) and (1.2), we get

$$
|\nabla u|\Delta|\nabla u| \geq \frac{1}{n-1}|\nabla|\nabla u||^2 - (n-1)k^2|\nabla u|^2. \tag{1.3}
$$

LECTURE 2
Gradient Estimate Proof and Corollaries

We continue our proof of the gradient estimate theorem. Using inequality (1.3) from last lecture, we introduce the function $\varphi = \frac{|\nabla u|}{u}$. To use the maximum principle, we compute

$$
\begin{aligned}
\nabla\varphi &= \frac{\nabla|\nabla u|}{u} - \frac{|\nabla u|\nabla u}{u^2}\,, \\
\Delta\varphi &= \frac{\Delta|\nabla u|}{u} - \frac{2\nabla|\nabla u|\cdot\nabla u}{u^2} - \frac{|\nabla u|\Delta u}{u^2} + \frac{2|\nabla u|^3}{u^3} \\
&= \frac{\Delta|\nabla u|}{u} - \frac{2\nabla|\nabla u|\cdot\nabla u}{u^2} + \frac{2|\nabla u|^3}{u^3} \\
&= \frac{\Delta|\nabla u|}{u} - \frac{2\nabla\varphi\cdot\nabla u}{u} \\
&\geq \frac{1}{|\nabla u|u}\left(\frac{1}{n-1}|\nabla|\nabla u||^2 - (n-1)k^2|\nabla u|^2\right) - \frac{2\nabla\varphi\cdot\nabla u}{u} \\
&= \frac{|\nabla|\nabla u||^2}{(n-1)|\nabla u|u} - (n-1)k^2\frac{|\nabla u|}{u} - \frac{2\nabla\varphi\cdot\nabla u}{u}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\nabla|\nabla u| &= u\nabla\varphi + \frac{|\nabla u|\nabla u}{u}\,, \\
|\nabla|\nabla u||^2 &= u^2|\nabla\varphi|^2 + \frac{|\nabla u|^4}{u^2} + 2|\nabla u|\nabla u\cdot\nabla\varphi\,,
\end{aligned}
$$

so that

$$
\frac{|\nabla|\nabla u||^2}{|\nabla u|u} \geq \frac{|\nabla u|^3}{u^3} + 2\nabla\varphi\cdot\frac{\nabla u}{u}.
$$

Therefore,

$$
\Delta\varphi \geq \frac{1}{(n-1)}\varphi^3 - (n-1)k^2\varphi - \left(2 - \frac{2}{n-1}\right)\nabla\varphi\cdot\frac{\nabla u}{u}. \tag{2.1}
$$

Note that if φ attains its maximum inside $B_a(x)$, say at x_0, then we know that $\Delta\varphi(x_0) \le 0$, $(\nabla\varphi)(x_0) = 0$, and hence,

$$\frac{1}{(n-1)}\varphi^3(x_0) - (n-1)k^2\varphi(x_0) \le 0 \quad \text{or} \quad \varphi(x_0) \le (n-1)k^2.$$

Unfortunately, φ can also attain its maximum on $\partial B_a(x)$. We introduce a new function for $B_a(x)$ as $F(y) = \varphi(a^2 - \rho^2(y))$, where $\rho(y) = d(y,x)$, and note that $F(\partial B_a(x)) = 0$ and $F \ge 0$ on $B_a(x)$. We may assume that there exists $x_0 \in B_a(x)$ such that $|\nabla u|(x_0) \ne 0$ (otherwise, the conclusion holds automatically). Suppose that $x_1 \in B_a(x)$ is a maximum point of F on $B_a(x)$.

Case I: $x_1 \notin \mathrm{Cut}(x)$. In this case, $F \in C^2$ on $B_a(x)$. Therefore, $\nabla F(x_1) = 0$, $\Delta F(x_1) \le 0$. At x_1, this implies

$$\begin{aligned} &\frac{\nabla\phi}{\phi} = \frac{\nabla\rho^2}{a^2-\rho^2} \\ &\frac{\Delta\varphi}{\varphi} - \frac{\Delta\rho^2}{a^2-\rho^2} - \frac{2\nabla\rho^2\nabla\varphi}{(a^2-\rho^2)\varphi} \le 0\,, \end{aligned} \tag{2.2}$$

hence,

$$\frac{\Delta\varphi}{\varphi} - \frac{\Delta\rho^2}{a^2-\rho^2} - \frac{2|\nabla\rho^2|^2}{(a^2-\rho^2)^2} \le 0\,.$$

Since $\mathrm{Ric}(M) \ge -(n-1)k^2$, we have that

$$\begin{aligned} \Delta\rho^2 &= 2\rho\Delta\rho + 2|\nabla\rho|^2 = 2 + 2\rho\Delta\rho \\ &\le 2 + 2\rho\frac{n-1}{\rho}(1+k\rho) \\ &= 2 + 2(n-1)(1+k\rho). \end{aligned}$$

Substituting this into the previous inequality, we get

$$\frac{\Delta\varphi}{\varphi} - \frac{2+2(n-1)(1+k\rho)}{a^2-\rho^2} - \frac{8\rho^2}{(a^2-\rho^2)^2} \le 0.$$

We can also plug the inequality (2.1) for $\Delta\varphi$ into it

$$0 \ge \frac{\phi^2}{(n-1)} - (n-1)k^2 - \left(2 - \frac{2}{n-1}\right)\frac{\nabla\phi\cdot\nabla u}{\phi u}\frac{(1+k\rho)}{a^2-\rho^2} - \frac{8\rho^2}{(a^2-\rho^2)^2}.$$

At x_1, we have, by (2.2),

$$-\frac{\nabla\phi}{\phi}\cdot\frac{\nabla u}{u} = -\frac{2\rho\nabla\rho\cdot\nabla u}{(a^2-\rho^2)u} \ge -\frac{2\rho\varphi}{a^2-\rho^2}$$

so

$$0 \geq \frac{\phi^2}{n-1} - (n-1)k^2 - \frac{4(n-2)}{n-1}\frac{\rho}{a^2-\rho^2}\phi - \frac{c(1+k\rho)}{a^2-\rho^2} - \frac{8\rho^2}{(a^2-\rho^2)^2}$$

or

$$\begin{aligned} 0 &\geq \frac{F^2}{n-1} - (n-1)k^2(a^2-\rho^2) - c(1+k\rho)(a^2-\rho^2) - 8\rho^2 - 4\frac{n-2}{n-1}\rho F \\ &\geq \frac{F^2}{n-1} - (n-1)k^2a^4 - 8a^2 - 2c_1aF - c(1+k\rho)a^2 \end{aligned}$$

(since $a^2 - \rho^2 \leq a^2$)

$$\geq \frac{F^2}{n-1} - 2c_1aF - c_2(1+ka)^2a^2.$$

Therefore $F(x_1) = \sup_{B_a(x)} F \leq c_n a(1+ka)$ restricted on $B_{\frac{a}{2}}(x)$, we will have

$$\frac{3}{4}a^2 \sup_{B_{\frac{a}{2}}(x)} \frac{|\nabla u|}{u} \leq C_n a(1+ka)$$

i.e.,

$$\sup_{B_{\frac{a}{2}}(x)} \frac{|\nabla u|}{u} \leq C_n\left(\frac{1+ka}{a}\right).$$

Case II: $x_1 \in \mathrm{Cut}(x)$. In this case, even though the inequality of $\Delta\rho$ holds distributionally, we don't have a pointwise estimate of $\Delta\rho$ at x_1. In order to overcome this difficulty, we employ a "support function". Let γ denote a minimal geodesic joining x and x_1, and choose $\bar{x}$ on γ which is very close to x. Then $x_1 \notin \mathrm{Cut}(\bar{x})$. If we denote $\epsilon = d(\bar{x}, x)$, then by triangular inequality, we have

$$\begin{aligned} &\epsilon + d(y, \bar{x}) \geq d(y, x) \quad \forall\, y \in N_{\bar{x}}, \quad \text{a small neighborhood of } \bar{x}, \\ &\epsilon + d(x_1, \bar{x}) = d(x_1, x). \end{aligned}$$

We consider $\overline{F} = (a^2 - (\epsilon + \rho_{\bar{x}})^2)\phi$. Then $\overline{F} \leq F$ near x_1 and $\overline{F}(x_1) = F(x_0)$ so that x_1 is maximum point of $\overline{F}$. We can apply the above argument to $\overline{F}$, and then let $\epsilon \to 0$. We finally get the same conclusion.

This method, or modification of it, have been used to prove many results in geometry. For example:

1. (Li-Yau [LY1],[LY2]) Obtain lower bounds on eigenvalues of compact manifolds. Gradient estimate for a positive solution of the heat equation on a manifold with a lower Ricci curvature bound. The heat equation then gives information on the eigenvalues.

3. (S.Y. Cheng [C1] and H.I. Choi [Cho]) Used a similar method to obtain gradient bounds for harmonic mappings to special target manifolds in terms of a bound on the oscillation of the mapping.

Now, we draw some direct corollaries of the Gradient Estimate Theorem.

Corollary 2.1. *Suppose u is harmonic on $B_a(x) \subseteq M$, where $\mathrm{Ric}(M) \geq -(n-1)k^2$. Then*

$$\sup_{B_{a/2}} |\nabla u| \leq c\left(\frac{1+ka}{a}\right) \sup |u|.$$

Proof. Let $A = \sup_{B_a} |u|$. Then $v = u + A + \epsilon > 0$ on B_a. Applying the theorem to v, we get

$$\begin{aligned}\sup_{B_{a/2}} |\nabla u| = \sup_{B_{a/2}} |\nabla v| &\leq c\frac{1+ka}{a} \sup_{B\frac{a}{2}}(u + A + \epsilon) \\ &\leq c\frac{1+ka}{a}\left(2\sup_{B_a}|u| + \epsilon\right).\end{aligned}$$

Letting $\epsilon \downarrow 0$, we get

$$\sup_{B_{a/2}} |\nabla u| \leq c\frac{1+ka}{a} \sup |u|.$$

Corollary 2.2 (Harnack inequality). *Let M be a complete Riemannian manifold with $\mathrm{Ric}(M) \geq -(n-1)k^2$. Suppose that u is a positive harmonic function on the geodesic ball $B_a \subseteq M$. Then*

$$\sup_{B_{a/2}} u \leq c(n,a,k) \inf_{B_{a/2}} u.$$

Proof. By the gradient estimate theorem, we have $\sup_{B_{a/2}} \frac{|\nabla u|}{u} \leq c(n,a,k)$. Let $x_1, x_2 \in B_a$ be such that $\sup_{B_{a/2}} u(x) = u(x_1)$, $\inf_{B_{a/2}} u(x) = u(x_2)$. Let γ be the minimal geodesic joining x_1 and x_2. Then

$$\int_\gamma \frac{|\nabla u|}{u} ds \leq c(n,a,k) \int_\gamma ds \leq c(n,a,k)2a.$$

On the other hand

$$\begin{aligned}\log\frac{u(x_1)}{u(x_2)} &= \left| \int_\gamma \frac{d\log u(\gamma(s))}{ds} ds \right| \\ &\leq \int_\gamma \frac{|\nabla u|}{u} ds \leq 2ac(n,a,k).\end{aligned}$$

Therefore, $u(x_1) \leq e^{2ac(n,a,k)} u(x_2)$.

The following Liouville theorem now follows readily.

Corollary 2.3. *Suppose M is a complete Riemannian manifold with* $\mathrm{Ric}(M) \geq 0$, *then any positive harmonic function is constant.*

Proof. Applying the theorem with $k = 0$, we have

$$\sup_{B_a} \frac{|\nabla u|}{u} \leq \frac{c}{a}.$$

Letting a go to infinity, we have that $|\nabla u| \equiv 0$, hence, $u \equiv \text{const}$.

Remark 2.4. Here is an example which indicates when the Harnack inequality goes bad. Consider the catenoids in $\mathbf{R}^3$,

$$M_\epsilon = \{(x, y, z) \in \mathbf{R}^3 : x = \epsilon \cosh(\frac{z}{\epsilon}) \cos\theta,\ y = \epsilon \cos(\frac{z}{\epsilon}) \sin\theta,\ z \in \mathbf{R},\ \theta \in (0, 2\pi]\}.$$

Truncate these catenoids by the planes $z = \pm 1$ to obtain the catenoidal regions $\widetilde{M}_\epsilon$. Then, as ϵ becomes smaller, $\widetilde{M}_\epsilon$ looks more like a pair of discs. The harmonic function solving the Dirichlet problem with $u \mid \partial_0 = 1$ and $u \mid \partial_1 = 0$ will have wide oscillation on the neck. The reason is that the curvature K_ϵ of $\widetilde{M}_\epsilon$ approaches ∞ as $\epsilon \to 0$.

Now, we have seen that M with $\mathrm{Ric} \geq -(n-1)k^2$ looks like $\mathbf{R}^n$ from the point of view of harmonic function. Finally we mention the

Conjecture 2.5 (Yau, 1974). *Let (M, g) be complete with* $\mathrm{Ric}(M) \geq 0$, *then, for each $\ell > 0$*

$$\dim\{u : \Delta u = 0, \sup_{B_R} |u| \leq cR^\ell\} < +\infty.$$

LECTURE 3
Harmonic Functions on Negatively Curved Manifolds

Let (M, g) be Cartan-Hadamand manifold, i.e., M is simply connected, complete, and all sectional curvatures are nonpositive. By the Cartan-Hadamand Theorem the exponential map $\exp_p : T_pM \to M$ is, for every $p \in M$, a diffeomorphism.

We will now further assume that every sectional curvature $K_x(M)$ satisfies the strict inequalities

$$-b^2 \leq K_x(\pi) \leq -a^2 < 0$$

for some constants $0 < a \leq b$.

Question: Do such manifolds behave like $\mathbf{H}^n$?

The answer is roughly yes. In order to see this, we wish to compactify M by adding a "visibility boundary" ∂M to M. In other words, we will add an end point to each ray so that we may identify asymptotic directions. Here a *ray* from p is a geodesic

$$\gamma : [0, \infty) \to M, \quad \gamma(t) = \exp_p tv$$

corresponding to some $v \in T_pM$. We say that two rays γ_1, γ_2 are equivalent provided that $\sup_t \operatorname{dist}_M(\gamma_1(t), \gamma_2(t)) < \infty$, and we define $\partial M = \{$equivalence classes of rays$\}$.

How should we think of this? Consider the spheres $S_p(M) = \{v \in T_p(M) : |v| = 1\}$.

For each $p \in M$, a vector $v \in S_p(M)$ gives a unique representative in ∂M, and we have an identification $\partial M \approx S_p(M)$. Using a different point $q \in M$ determines similarly an identification $\partial M \approx S_q(M)$, and hence a homeomorphism $\psi_{p,q} : S_p(M) \to S_q(M)$. Some distance estimates using the Rauch and Toponogov Comparison Theorems shows that $\psi_{p,q}$ is actually Hölder continuous of order a/b [SY, Prop. II.1.3]. Thus ∂M has a natural Hölder structure. For real hyperbolic space $\mathbb{H}^n$, $\psi_{p,q}$ is a conformal transformation. For $\mathbb{R}^n$ (where one would have $a = b = 0$) $\psi_{p,q}$ is induced by a translation of $\mathbf{R}^n$. In general, the transformation $\psi_{p,q}$ is badly behaved. For example, for complex hyperbolic space $\mathbb{CH}^n$, it is not even quasi-conformal.

The existence of many non-trivial bounded harmonic functions on M is guaranteed by the following solution of the Dirichlet problem at ∞.

Theorem 3.1 [A], [Su]. *For any $\varphi \in C^0(\overline{M})$, there exists a unique $u \in C^\infty(M) \cup C^0(\overline{M})$ satisfying*

$$\begin{aligned} \Delta u &= 0 \quad \text{on } M \\ u &= \varphi \quad \text{on } \partial M. \end{aligned}$$

Proof. We give a simplified argument which appears in [AS].

Fix $p \in M$ and write $\varphi = \varphi(\xi)$ for $\xi \in S_p(M)$. We first assume φ is Lipschitz in ξ. We may use polar coordinates (ρ, ξ) on T_pM to describe the corresponding points on M, via the diffeomorphism $\exp_p$. Thus

$$\rho(x) = \rho_p(x) = \operatorname{dist}(x,p) \qquad \text{for } x \in M.$$

Consider now the unit ball $B_1(x)$, centered at x. From the comparison theorem we find that $B_1(x)$, when viewed from p, subtends an angle no more than $ce^{-a\rho(x)}$. Thus the oscillation of $\varphi(\xi)$ as $y = \exp_p(\rho\xi)$ varies in $B_1(x)$ satisfies

$$\operatorname{osc}_{B_1(x)}(\varphi) = \varphi_{\max} - \varphi_{\min} \le ce^{-a\rho(x)}$$

Here $\varphi(\xi)$ may be viewed as a function on $M \setminus \{p\}$ extending φ. To obtain suitable decay also on the derivatives we need to cut-off and mollify this function. This will use the lower bound on curvature, which implies that unit balls are basically the same everywhere on M. Consider the expression $\chi(\operatorname{dist}^2(x,y))$ where

$$\chi \in C_0^\infty(\mathbb{R}, [0,1])$$

is a fixed function that is identically one near 0. We define

$$\tilde{\varphi}(x) = \frac{\int_M \chi(\operatorname{dist}^2(x,y))\varphi(y)d\mu(y)}{\int_M \chi(\operatorname{dist}^2(x,y))d\mu(y)}.$$

Then, $\tilde{\varphi}$ also extends φ and, for x near ∞, the denominator is bounded away from 0. Moreover, since

$$|\nabla_x \operatorname{dist}^2(x,y)| \le C, \quad |\Delta_x \rho^2(x,y)| \le C,$$

we have the desired decay

$$\begin{aligned} \operatorname{osc}_{B_1(x)} \tilde{\varphi} &\le Ce^{-a\rho(x)} \\ |\nabla\tilde{\varphi}(x)| &\le Ce^{-a\rho(x)} \\ |\Delta\tilde{\varphi}(x)| &\le Ce^{-a\rho(x)}. \end{aligned}$$

Recall that in $\mathbb{H}^n$, the distance function $\rho = \rho_p$ is strictly convex, the Hessian of ρ is strictly positive definite, and the Laplacian of ρ is bounded away from 0. Similarly here in M we find that

$$\Delta\rho \geq \frac{n-1}{\rho}(1 + a\rho) \geq c_0 > 0,$$

hence,

$$\begin{aligned}\Delta e^{-\delta\rho} &= -\delta e^{-\delta\rho}(\Delta\rho - \delta|\nabla\rho|^2) \\ &\leq -|C_1|\delta e^{-\delta\rho}\end{aligned}$$

because $|\nabla\rho| = 1$. Thus

$$\begin{aligned}\Delta(\tilde{\varphi} + ce^{-\delta\rho}) &\leq 0, \\ \Delta(\tilde{\varphi} - c'e^{-\delta\rho}) &\geq 0,\end{aligned}$$

and we have super- and sub-harmonic functions which agree with φ on ∂M. We may now use either the standard Perron process or solve a suitable Dirichlet problem on the bounded domain $B_R(p)$ and let $R \uparrow +\infty$. We find that there exists a $u : M \to \mathbb{R}$ satisfying

$$\Delta u = 0 \quad \text{and} \quad \tilde{\varphi} - c'e^{-\delta\rho} \leq u \leq \tilde{\varphi} + ce^{-\delta\rho},$$

hence $u \mid \partial M = \tilde{\varphi} \mid \partial M = \varphi$.

Finally, dropping the Lipschitz assumption on φ, we choose $\varphi_j \in \mathrm{Lip}(\partial M)$ to approximate φ uniformly as $j \to \infty$. The solutions u_j for φ_j satisfy

$$\sup_M |u_j - u_k| = \max_{\partial M} |\varphi_j - \varphi_k|.$$

Thus the $\{u_j\}$ are uniformly Cauchy in $C^0(\overline{M})$ and converge uniformly to $u \in C^0(\overline{M})$ with $u \mid \partial M = \varphi$. Standard local representation formulas show that u is harmonic and smooth.

To conclude this chapter we will mention a few related results and questions. First a more refined result on the Dirichlet problem involving Poisson kernel functions is given in the paper [AS]. W. Ballman [B] discusses the Dirichlet problem for the universal covering of a compact manifold M with nonpositive sectional curvature. He also assumes that M is rank one, where here the rank of M is the largest dimension of a flat totally geodesic submanifold of $\widetilde{M}$, the universal cover of M. His proof involves a certain amount of dynamical systems theory. Some interesting problems are to extend results to finite volume manifolds or higher rank manifolds. S.Y. Cheng [C2] showed that the Dirichlet problem is solvable if every sectional curvature $K_x(\pi) \leq -a^2$ and, for each $x \in M$,

$$\max_\pi(-K_x(\pi)) \leq C \min_\pi(-K_x(\pi)).$$

This allows cases where $\inf_{x\in M} \min_\pi K_x(\pi) = -\infty$.

The DeGiorgi-Nash-Moser machinery for regularity of nonlinear elliptic equations may be treated in a Riemannian manifold (M, g) in a coordinate free way. Think of M possibly having a boundary. There are three crucial hypotheses involving positive constants c_1, c_2, c_3 depending only on M.

H1. *Isoperimetric inequality.*

$$\mathrm{vol}(\Omega)^{n-1/n} \leq c_1 \, \mathrm{vol}(\partial\Omega)$$

for any $\Omega \Subset M$.

H2. *Volume bound.*

$$\mathrm{vol}(B_\sigma(x)) \leq c_2\sigma^n$$

for any ball $B_\sigma(x) \Subset M$.

H3. *Poincaré inequality.*

$$\inf_{\alpha\in\mathbb{R}} \int_{B_{\sigma/2}(x)} |f - \alpha|^2 d\mu \leq c_3\sigma^2 \int_{B_\sigma(x)} |\nabla f|^2 d\mu.$$

Note that these are all lower order quantities. They involve no curvature hypotheses and may make sense if the metric is not smooth or even if M is not a manifold.

These hypotheses have some useful consequences. The *Harnack inequality* states that a positive harmonic function on a ball $B_\sigma \subset M$ satisfies

$$\sup_{B_{\sigma/2}} u \leq c(n, c_1, c_2, c_3) \inf_{B_{\sigma/2}} u.$$

The *mean-value inequality*, which is implied by H1, says that a nonnegative subsolution (i.e., $\Delta u \geq 0$) on B_σ satisfies

$$\sup_{B_{\sigma/2}} u^p \leq c(n, p, c_1)\sigma^{-n} \int_{B_\sigma} u^p d\mu \quad \text{for any} \quad p > 0.$$

More generally, a solution of $\Delta u + fu = 0$ with $\int_{B_\sigma} |f|^{\epsilon+\frac{n}{2}} d\mu \leq \tau$ satisfies

$$\sup_{B_{\sigma/2}} u^p \leq c(n, \tau, p, c_1)\sigma^{-n} \int_{B_\sigma} u^p d\mu \quad \text{for any} \quad p > 0.$$

To understand hypothesis H3, consider a family of catenoids with shrinking necks. Here H1 and H2 continue to hold with fixed c_1, c_2, the Poincaré inequality H3 eventually fails for any fixed c_3. That is, the first eigenvalue of a small ball centered on the neck approaches 0.

LECTURE 4
Harmonic Mapping into Singular Spaces

4.1 Motivation

The motivation for us to establish a theory of harmonic mappings into singular spaces is some new application of harmonic mapping in proving "Rigidity Theorems". These applications, which will be described in the next chapter, rely heavily on both existence and vanishing results, and the so-called Bochner Method. This has two points.

(i) *Analysis.* We seek to represent a topological object by a geometric one. By analogy, in Hodge theory, we represent a cohomology class by a differential form that is harmonic. The general issue is that we need to establish the existence and regularity for a "canonical representative" of a topological object.

(ii) *Vanishing Theorems.* Under certain conditions, we try to prove that some quantities vanish by using a Bochner formula for these quantities. This has a more algebraic flavor.

Roughly speaking, the combination of an analysis and an algebraic result will imply a geometric result. To illustrate this idea, we'll examine a theorem due to Preissmann [P].

Theorem. *If X is a compact manifold with negative sectional curvature, then any abelian subgroup of the fundamental group $\pi_1(X)$ is cyclic.*

Proof. Assume that a and b are two commuting elements in $\pi_1(X)$, i.e., $aba^{-1}b^{-1} \sim 1$. Then the homotopy mapping between $aba^{-1}b^{-1}$ and 1 will give a smooth mapping ϕ of the torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ to X. Note that X is compact with negative sectional curvature. We claim that ϕ is homotopic to a mapping ψ whose image lies in a closed geodesic of X.

In order to prove the claim, we need the following analytic contributions.

(i) The Eells-Sampson theorem: (see [ES]). *Any $\varphi : M \to X$ is homotopic to a harmonic mapping, provided that X is compact with sectional curvature $K_X < 0$.*

(ii) The vanishing result: Suppose $u : T^2 \to X$ is the harmonic map homotopic to ϕ. Then we have the Bochner formula (see [ES])

$$\begin{aligned}\frac{1}{2}\Delta(|\nabla u|^2) = |\nabla^2 u|^2 + \sum_{\alpha=1}^{2} \mathrm{Ric}_{T^2}(\nabla_\alpha u, \nabla_\alpha u) \\ - \sum_{\alpha,\beta=1}^{2} \langle R_X(\nabla_\alpha u, \nabla_\beta u)\nabla_\alpha u, \nabla_\beta u\rangle\end{aligned}$$

where Δ denotes the laplace operator on T^2, $\nabla^2 u$ the Hessian of u, Ric_{T^2} the Ricci operator on T^2, and R_X the curvature tensor of X. Since T^2 is endowed with flat metric, $\mathrm{Ric}_{T^2} = 0$; $K_X < 0$, and the third term has a positive sign. Therefore

$$\frac{1}{2}\Delta(|\nabla u|^2) \geq |\nabla^2 u|^2 \geq 0$$

i.e., $|\nabla u|^2$ is subharmonic and hence constant by the maximum principle. Moreover, $|\nabla^2 u| = 0$, that is, u is a totally geodesic mapping. By using the strict negativity of K_X, we know that the rank

$$\dim(u_*(T_x T^2)) \leq 1 \qquad \forall\, x \in T^2.$$

This implies $\mathrm{Im}(u)$ is contained in some closed geodesic. Therefore a and b are homotopic to multiples of some closed geodesic.

4.2 Nonpositively curved metric spaces

In order to study more complicated group actions by the theory of harmonic maps, we want to allow the image space X to be singular. In fact, we will consider a singular metric space X with nonpositive curvature. Suppose $X \subseteq \mathbb{R}^k$ is a closed set, and Ω is a bounded, smooth domain in M. We want to find energy minimizing maps from Ω to X with specified boundary data $\varphi : \partial\Omega \to X$.

Let $H^1(\Omega, X) = \{v \in H^1(\Omega, \mathbb{R}^k) \;:\; v(x) \in X \text{ a.e. } x \in \Omega\}$ where $H^1(\Omega, \mathbb{R}^k)$ is the Hilbert space of $\mathbb{R}^k$-vector-valued L^2 functions on Ω with first derivatives in L^2. Recall that the energy of $u \in H^1(\Omega, X)$ is given by

$$E(u) = \int_\Omega |\nabla u|^2 \, d\mu$$

where $d\mu$ is the volume element of M. The relation $u = \varphi$ on $\partial\Omega$ is to be understood in the $H^1(\Omega, X)$ trace sense.

Lemma. *Suppose $\varphi : \partial\Omega \to X$ occurs as the trace of some map in $H^1(\Omega, X)$. Then there exists $u \in H^1(\Omega, X)$ such that $u = \varphi$ on $\partial\Omega$, and $E(u) \leq E(v)$ for all $v \in H^1(\Omega, X)$ with $v = \varphi$ on $\partial\Omega$.*

Proof. (direct method) Let $\{u_i\}$ be a minimizing sequence of maps in $H^1(\Omega, X)$ with $u_i = \varphi$ on $\partial\Omega$. Since a bounded subset of $H^1(\Omega, X)$ is weakly compact, there is a subsequence again denoted $\{u_i\}$ which converges weakly to a map $u \in H^1(\Omega, X)$. On the other hand, E is sequentially weakly lower-semicontinuous. Therefore

$$E(u) = \inf\{E(v) \ : \ v \in H^1(\Omega, X), \quad v = \varphi \ \text{ on } \ \partial\Omega\}.$$

Note that if every pair of points in X can be joined by at least one Lipschitz curve, then Lemma 4.3 implies that any pair of points x, y in X can be joined by an energy minimizing curve γ parametrized by arclength. Using the infemum of the length of such curves γ one obtains an *intrinsic* metric on the space X. This should not be confused with the topologically equivalent,but smaller extrinsic distance $|x - y|$.

Next, we explain the generalized notion of *nonpositive curvature* which we use. We assume the X is metrized as above so that any two points x, y in X may be joined by a unique curve in X whose length in $\mathbb{R}^k$ is $\operatorname{dist}(x, y)$. Consider a geodesic triangle in X, wxy and a corresponding triangle in $\mathbb{R}^2$, $0\bar{x}\bar{y}$, with the same side lengths. Let $\gamma(s)$ denote the minimizing curve parametrized by arclength between x and y. Then the squared distance $D(s) = \operatorname{dist}^2(\gamma(s), w)$ corresponds to $\overline{D}(s) = |s\bar{y} + (1-s)\bar{x}|^2 = s^2 + \alpha s + \beta$ which is the unique quadratic polynomial determined by boundary conditions $\overline{D}(0) = |\bar{x}|^2 = |d(x, w)|^2$ and $\overline{D}(\ell) = |\bar{y}|^2 = d(y, w)|^2$. Note that $\overline{D}''(s) = 2$. The nonpositive curvature condition of X is that the corresponding inequality $D''(s) \geq 2$ holds, in a weak sense. Since $D(s)$ is only Lipschitz, $x = \gamma(0)$, and $y = \gamma(\ell)$, the condition $D''(s) \geq 2$ means that

$$\int_0^\ell \zeta''(s) D(s) ds \geq 2 \int_0^\ell \zeta(s) ds \quad \forall\, \zeta \in C_c^\infty(0, \ell), \ \zeta \geq 0.$$

Definition. X has *nonpositive curvature* if, for any three points w, x, $y \in X$, the inequality $D''(s) \geq 2$ holds.

There are two elementary examples of non-positively curved singular spaces:

(1) Trees. A tree is a connected and simply connected graph. One determines an intrinsic metric on a tree by assigning a length to each edge.

(2) Surfaces with cone metric: On the disk, the metric is smooth and flat away from the origin. The sign of the curvature at 0 is determined by the cone angle α at 0; namely,

1. $\alpha < 2\pi \iff$ positive curvature at 0.
2. $\alpha = 2\pi \iff$ zero curvature at 0.
3. $\alpha > 2\pi \iff$ negative curvature at 0.

One can often understand these two examples by approximating them by a sequence of smooth surfaces $\{X_i\}$, and hope that some properties of harmonic mappings into X_i can be inherited. In fact, here is a question due to M. Wolf [W].

Degeneration question: Consider a sequence of smooth manifolds $X_j \to X$ where convergence is in the sense of the distance functions. If $u_j : \Omega \to X_j$ are harmonic maps with bounded energy, $E(u_j) \leq c$, what do the limits of the u_j's look like?

Suppose each X_j has nonpositive curvature. As before, by virtue of the Bochner formula,

$$\frac{1}{2}\Delta(|\nabla u_j|^2) \geq \mathrm{Ric}_\Omega(\nabla u_j, \nabla u_j) \geq -c|\nabla u_j|^2.$$

$|\nabla u_j|^2$ is thus a subsolution for the operator $\Delta + 2C$. We can apply the DeGiorgi-Nash-Moser mean-value inequality ([GT]) to conclude that

$$\sup_{\Omega_1} |\nabla u_j|^2 \leq C(M, \Omega_1) \int_\Omega |\nabla u_j|^2 d\mu \leq C(M, \Omega_1)$$

where $\Omega_1 \Subset \Omega \subset M$, which gives some information about the limit.

4.3 A two-dimensional result

Consider a map u from disk D^2 into X. If u is smooth, we may define its Hopf differential $\Phi = \phi dz^2$ by letting

$$\phi = |u_x|^2 - |u_v|^2 - 2i(u_x, u_y).$$

If u is also harmonic, then ϕ is holomorphic. If u is simply energy minimizing, not necessarily smooth (e.g. X isn't smooth), then ϕ is nevertheless still holomorphic and hence smooth.

Here, we sketch a proof of the latter (see [S]).

Proof. Consider a variation of domain $F_t = D \rightarrow D$ for $|t|$ small such that $F_0 =$ identity, $F_t =$ identity near ∂D. Then, by virtue of minimality of u,

$$E(u \circ F_t) \geq E(u) \qquad \forall\, t.$$

On the other hand, the map $t \rightarrow E(u \circ F_t)$ is C^1 in t. To see this, let $y = F_t(x)$ and perform a change of variables in the integral to get

$$E(u \circ F_t) = \int_D \sum_{i,j,k} \frac{\partial y_j}{\partial x_i} \frac{\partial y_k}{\partial x_i} \left(\frac{\partial u}{\partial y_i}, \frac{\partial u}{\partial y_k} \right) \det \left(\frac{\partial x_p}{\partial y_q} \right) dy.$$

Therefore, $\frac{d}{dt}\big|_{t=0} E(u \circ F_t) = 0$.

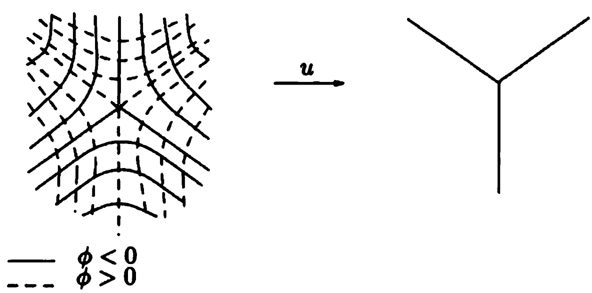
Now, for $\eta(x, y)$ and $\zeta(x, y)$ any smooth functions with compact support in D^2, consider the variation given by

$$F_t(x, y) = (x + t\zeta(x, y), y)$$
$$G_t(x, y) = (x, y + t\eta(x, y)).$$

The first variation of these shows that ϕ satisfies the Cauchy-Riemannian equations weakly. Linear regularity theory (e.g., Weyl's lemma) shows that ϕ is thus holomorphic and smooth.

Harmonic maps from surfaces to an $\mathbf{R}$-tree X were recently studied by M. Wolf [W]. By involving the appropriate definition of harmonic mapping, M. Wolf used holomorphic quadratic forms Φ on $\mathcal{R}$, whose profile can be viewed as the Hopf differential of some harmonic map. In particular, such Φ are used to construct harmonic maps into trees. To indicate his idea, we suppose $u : D^2 \to \mathbf{R}$ is a harmonic *function*. Then its Hopf differential $\Phi = \left(\frac{\partial u}{\partial z}\right)^2 dz^2$. If $\Phi(0) \neq 0$, we can do a change of coordinates so that u is a projection map, i.e., $u(x,y) = x$ in a suitable coordinate. Changing coordinates so that $\phi \sim 1$ near 0, the map u contracts (or expands) maximally on regions where $\Phi < 0$ (or $\Phi > 0$). e.g., If $\Phi = zdz^2$, then the map which projects, in a suitable way, the regions between the maximal negative curves onto the maximal positive curves defines a harmonic map from the disk to the tree. See Figure 1.

Figure 1



Suppose X is a surface with a negatively curved cone metric. e.g., We may obtain such an X in $\mathbf{R}^3$ by choosing any curve (embedded) in S^2 of length $\theta > 2\pi$ and taking a cone over it with vertex at the origin. This gives a cone angle θ. Let g be a cone metric on X, u^*g is a real symmetric $(2,0)$ tensor on D^2 and $u^*g = \Phi + \overline{\Phi} + Fdzd\bar{z}$. Here $\Phi = \varphi(z)dz^2$ is the Hopf differential of u, and F is the energy density of u. We can view $Fdzd\bar{z}$ as a conformal metric; its sketch factor is constant in all directions. On the other hand, $\Phi(\nu \cdot \nu) < 0$ along the direction ν of minimal sketch, and $\Phi(\omega \cdot \omega) > 0$ along the direction ω of maximal sketch. Moreover, $\nu \perp \omega$. The direction fields ν, ω, define the positive and negative foliations of Φ away from the discrete set $|\phi|^{-1}\{0\}$.

Recent work of E. Kuwert[K] analyzes the possible collapse of u along the negative foliation of Φ.

Theorem ([K]). *Assume that the boundary map φ is a diffeomorphism of S^1 and u is $1-1$. Then $u^{-1}(0)$ is a curve along a segment of the negative foliation of Φ. Moreover, if $u^{-1}(0)$ is a point, then u is quasiconformal and $\Phi = 0$ at $u^{-1}(0)$.*

Thus, in general, $u^{-1}(0)$ is 1-dimensional, e.g., if we place a large number of cone points on a Riemannian surface of genus g, Σ^g and consider a harmonic map u homotopic to the identity, where $u : \Sigma^g \to \Sigma^q$ cone. Then the number of zeros of Φ is controlled by the genus of Σ. Most preimages of cone points are arcs on the negative foliation of Φ. In particular, u isn't in general quasiconformal.

LECTURE 5
Energy Convexity of Maps to an NPC Metric Space

We want to deduce the convexity of energy for a map into an NPC metric space X. First, we assume that X is smooth NPC. Let $u_t : \Omega \to X$ and $F : \Omega \times [0,1] \to X$ be a smooth map with $F\big|_{\Omega\times\{t\}} = u_t$. Then we have

$$\frac{d^2}{dt^2}E(u_t)$$
$$= 2\int_\Omega \left\{ \sum_\alpha \|\nabla e'_\alpha v\|^2 - \langle R_X(v, F^* e_\alpha)v, F^* e_\alpha\rangle + \left\langle \nabla e'_\alpha \nabla \frac{\partial'}{\partial t} v, F^* e_\alpha \right\rangle \right\} d\mu$$

where $v = F * \left(\frac{\partial}{\partial t}\right)$ is the velocity of the variation, $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $T \times \Omega$, $\nabla \frac{\partial'}{\partial t}\sigma$ is the acceleration vector, and ∇' is the connection on u^*TX gotten by pulling back the Levi-Civita connection on Ω. For details on this formula, one can refer to [EL].

Since we are assuming nonpositive curvature, we may disregard the second term. We also want to disregard the third term. In order to do that, let u_0, u_1 be two given maps, and u_t be a homotopy from u_0 to u_1. This is a homotopy u_t such that for each x $u_t(x)$ is a constant speed geodesic in X parametrized on $[0,1]$. This is done by choosing $\{u_t(x) \ : \ t \in [0,1]\}$ to be the unique geodesic between $u_0(x)$ and $u_1(x)$.

For such a *geodesic homotopy*, the acceleration term vanishes; thus

$$\frac{d^2}{dt^2}E(u_t) \geq 2\int_\Omega |\nabla' v|^2 d\mu.$$

Now, we need to modify this argument so that it works when X is NPC Riemannian simplicial complex. This is a simplicial complex whose faces are endowed with Riemannian metrics which extend smoothly to the closure in such a way that the lower-dimensional faces have the metric induced from the higher dimensional faces. We may also assume $X \subset \mathbb{R}^N$ in such a way that each face has the metric induced from $\mathbb{R}^N$. The following properties can be derived for a NPC metric space X. First, any two points in X may be joined by a unique length-minimizing path. Secondly, if p_0, p_1 and θ_0, θ_1 are two pairs of points in X, and if we parametrize the geodesic paths from p_0 to p_1 and from θ_0 to θ_1 by $p(t)$, $\theta(t)$ for $t \in [0,1]$, then the function $d(p(t),\theta(t))$ is a convex function of t. The second property implies that geodesics from a point spread apart more quickly than in Euclidean space, since we may take $p_0 = \theta_0$ and conclude that

$$d(p(t),\theta(t)) \geq \frac{t}{s} d(p(s),\theta(s)) \quad \text{for} \quad s < t.$$

By virtue of the convexity of distance function, we have the following proposition concerning energy convexity.

Proposition 5.1. *Suppose X is a Riemannian simplicial complex of nonpositive curvature, and u_0 and u_1 are Lipschitz maps from Ω to X. Then if u_t is the geodesic homotopy from u_0 to u_1 we have*

$$\frac{d^2}{dt^2} E(u_t) \geq 2 \int_\Omega |\nabla d(u_0,u_1)|^2 d\mu$$

weakly on $[0,1]$, i.e.,

$$\int_0^1 E(u_t)\zeta''(t)dt \geq 2 \int_\Omega |\nabla d(u_0,u_1)|^2 \int_0^1 \zeta(t)dt \quad \forall\, \zeta \in C_0^\infty([0,1],R) \quad \zeta \geq 0.$$

Proof. (See [GS], also). First consider the one-dimensional case in which we have Lipschitz curves γ_0, $\gamma_1 : (-\delta,\delta) \to X$ and a geodesic homotopy γ_t for $0 \leq t \leq 1$. Assume that $\delta = 0$ is a point of differentiability for γ_0, γ_1 and γ_t for a.e. $t \in (0,1)$. We fix $s = 0$ and calculate $\frac{d\gamma_t}{ds}\big|_{s=0}$. We can replace γ_0, γ_1 by the corresponding constant speed geodesics. Let $\ell(s)$ be the length of the curve $t \mapsto \gamma_t(s)$ and observe that $\ell(s) = d(\gamma_0(s),\gamma_1(s))$ is a Lipschitz function of s, and reparametrize the homotopy by setting $\bar\gamma_\tau(s) = \gamma(\tau s/\ell(s)$ for $\tau \in [0,\ell(s)]$. Thus $\tau \mapsto \bar\gamma_\tau(s)$ is now a unit speed geodesic. For any h the function $\tau \mapsto d^2(\bar\gamma_\tau(h),\bar\gamma_\tau(0))$ is convex because X has nonpositive curvature. At any τ for which $\frac{d}{ds}\gamma_\tau(0)$ exists we have

$$\lim_{h\to 0} h^{-2} d^2\left(\bar\gamma_\tau(h),\bar\gamma_\tau(0)\right) = \left| \frac{d\bar\gamma_\tau}{ds}(0) \right|^2.$$

Since $\frac{d\bar\gamma_\tau}{ds}(0)$ exists at $\tau = 0$, and $\tau = \ell(0)$, it follows that there is a sequence h_i tending to zero such that $\tau \mapsto h_i^{-2} d^2(\bar\gamma_\tau(h_i),\bar\gamma_\tau(0))$ converges uniformly on $[0,\ell(0)]$ to a convex function which agrees a.e. with the function $\tau \mapsto \left|\frac{d\bar\gamma_\tau}{ds}\right|^2(0)$. In particular, we can assume that $\left|\frac{d\bar\gamma_\tau}{ds}\right|^2(0)$ is convex in τ. Now by the chain rule we have

$$\frac{d\bar\gamma_\tau}{ds}(0) = -\tau\ell(0)^{-2}\frac{d\ell}{ds}(0)\frac{\partial\gamma_\tau/\ell(0)}{\partial\tau}(0) + \frac{d}{ds}\gamma_\tau/\ell(0)(0)$$

or in terms of t

$$\frac{d}{ds}\gamma_t(0) = \frac{d\bar\gamma_t}{ds}(0) + t\ell(0)^{-1}\frac{d\ell(0)}{ds}\frac{d\gamma_t}{dt}(0).$$

For any τ_1, $\tau_2 \in (0, \ell(0))$ with $\tau_1 < \tau_2$ we have $d(\bar{\gamma}_{\tau_1}(s), \bar{\gamma}_{\tau_2}(s)) = \tau_2 - \tau_1$. Differentiating with respect to s, we then conclude that

$$\frac{d\bar{\gamma}_\tau}{ds}\frac{\partial\bar{\gamma}_\tau}{\partial\tau}\bigg|_{\tau=\tau_1} = \frac{d\bar{\gamma}_\tau}{ds}\frac{\partial\bar{\gamma}_\tau}{\partial\tau}\bigg|_{\tau=\tau_2}.$$

Therefore, for a.e. $t \in [0,1]$,

$$\frac{d\gamma_t}{ds}(0) = V(t) + \left(a + t\ell(0)^{-1}\frac{d\ell}{ds}(0)\right)\frac{\partial\gamma_t}{\partial t}(0).$$

For a constant a where $V(t) \cdot \frac{\partial\gamma_t}{\partial t}(0) = 0$ for a.e. t and

$$\begin{aligned}
\left|\frac{d\gamma_t}{ds}(0)\right|^2 &= |V(t)|^2 + \left| a + t\ell^{-1}(0)\frac{d\ell}{ds}(0)\right|^2 |\ell(0)|^2 \\
|V(t)|^2 &= \left|\frac{d\bar{\gamma}_\tau}{ds}(0)\right|^2 + a^2\ell^2(0) \\
\frac{d^2}{dt^2}\left|\frac{d\gamma_t}{ds}\right|^2 &= \frac{d^2}{dt^2}|V(t)|^2 + 2\left|\frac{d\ell}{ds}(0)\right|^2 \\
&\geq 2\left|\frac{d\ell}{ds}(0)\right|^2 = 2\left(\frac{d}{ds}d(\gamma_0(s), \gamma_1(s))\right)^2\bigg|_{s=0}
\end{aligned}$$

since $|V(t)|^2$ is convex.

Now to prove the result in higher dimensions, observe that the map $(x,t) \mapsto u_t(x)$ is Lipschitz. Thus, for almost every line parallel to the t-axis, it is differentiable at a.e. point of the line. At such points of differentiability the previous results tell us

$$\frac{d^2}{dt^2}|\nabla u_t|^2 \geq 2|\nabla d(u_0, u_1)|^2$$

in the weak sense. Thus, if $\zeta \in C_0^\infty(0,1)$ and $\zeta \geq 0$ we have, for a.e. $x \in M$

$$\int_0^1 |\nabla u_t|^2(x)\zeta''(t)dt \geq 2\int_0^1 |\nabla d(u_0(x), u_1(x))|^2\zeta(t)dt.$$

Integrating w, t, and x we get

$$\int_0^1\int_\Omega |\nabla u_t|^2\zeta''(t)\,d\mu\,dt \geq 2\int_\Omega |\nabla d(u_\sigma, u_1)|^2 d\mu \int_0^1 \zeta(t)dt.$$

Corollary 5.2. *Assume $X \subseteq \mathbb{R}^n$ is the same as the above proposition. Suppose u_0 and u_1 are Lipschitz and $\epsilon > 0$ is small enough so that $E(u_i) \leq E_0 + \epsilon$ for $i = 0$, 1 where $E_0 = \inf\{E(u) \ : \ u \in H^1(\Omega, X), u = \varphi \text{ on } \partial\Omega\}$. Then $\int_\Omega d^2(u_0, u_1) \leq c\epsilon$ with a constant c depending only on Poincaré constant Ω.*

Proof. Apply the proposition and the Poincaré inequality (since $d(u_0, u_1) = 0$ on $\partial\Omega$) to get

$$\frac{d^2}{dt^2}E(u_t) \geq c \int_\Omega d^2(u_0, u_1)d\mu$$

weakly on $[0,1]$. By the convexity of the energy, $\forall \alpha \in [0,1]$

$$E_0 \leq E(u_\alpha) \leq \alpha E(u_0) + (1-\alpha)E(u_1) \leq E_0 + \epsilon.$$

Now, use the fundamental theorem of calculus and an appropriate test function to get the required result.

Corollary 5.3. *Suppose X is simply connected NPC. There is a unique energy minimizing map $u : \Omega \to X$ with given Lipschitz boundary data.*

Proof. Suppose u_0 and u_1 are two minimizers with the same boundary data φ. Applying Corollary 5.2 with $\epsilon = 0$, we can conclude that $\int_M |d(u_0, u_1)|^2 d\mu = 0$. It follows that $u_0 \equiv u_1$. This completes the proof.

5.4 Monotonicity

We may choose a smooth variation of the domain of the form $u_\tau = u \circ F_\tau$, where $F_\tau(x) = (1 + \tau\xi(x))x$, where τ is small and ξ is a smooth, compactly supported approximation to the characteristic function of a ball $B_\sigma(0)$. Since u is a minimizer, we can deduce from a first variation argument, as in L. Simon [Si], that

$$(5.1) \qquad (2-n)\int_{B_\sigma(0)} |\nabla u|^2 d\mu + \sigma \int_{\partial B_\sigma(0)} |\nabla u|^2 d\Sigma = 2\sigma \int_{\partial B_\sigma} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma$$

for a.e. σ.

One can integrate the identity w.r.t σ to get the usual monotonicity formula for the normalized energy.

To derive the monotonicity of another useful quantity, we will combine (5.1) with an inequality having to do with the convexity of the distance function in the target.

For example, if X is a smooth NPC space and u is a smooth harmonic map, then we have, by the chain rule, for $Q \in X$, fixed

$$\begin{aligned} \Delta d^2(u(x), Q) &= \operatorname{tr} \nabla d d^2(u(x), Q)(\nabla u, \nabla u) \\ &= \sum_{\alpha=1}^{n} \operatorname{Hess} d^2(u, Q)(\nabla_{e_\alpha} u, \nabla_{e_\alpha} u) \geq 2|\nabla u|^2. \end{aligned}$$

Since $\operatorname{Hess}_p d^2(p.Q)(v,v) \geq 2|v|^2$ for all $v \in T_pN$ i.e., $d^2(u(x), \theta)$ is a strongly subharmonic function.

For NPC metric space X, we need to make this proof variational since this is the only tool available.

Fix $Q \in X$. Given $p \in X$. There exist a unique geodesic $\gamma(t)$ parametrized with constant speed on $[0,1]$ such that $\gamma(0) = Q$, and $\gamma(1) = p$.

Define $R_{\lambda,Q} : X \to X$ by $R_{\lambda,Q}(p) = \gamma(\lambda)$. Then $R_{\lambda,Q}$ is a Lipschitz and contracting map. In fact

$$\begin{aligned} &d(R_{\lambda,Q}(p_1), R_{\lambda,Q}(p_2)) \leq \lambda d(p_1,p_2) \\ &R_{1,Q} = \text{ Identity} \\ &R_{0,Q}(p) = \theta \quad \forall\, p \in X. \end{aligned}$$

So $R_{\lambda,Q}$ is a retraction map of X onto Q with finite Lipschitz constant.

Proposition 5.5. *Suppose X is a NPC metric space. If $u \in H^1(\bar{\Omega}, X)$ is minimizing and $u(\bar{\Omega})$ is a compact subset of X, then*

$$\Delta d^2(u(x), Q) \geq 2|\nabla u|^2 \quad \textit{weakly.}$$

Proof. For $\xi \in C_0^\infty(\Omega)$ with $\xi \geq 0$, consider the deformation $u_\tau(x) = R_{1-\tau\xi(x),Q}(u(x))$ where $\tau \geq 0$. Then $u_\tau = u$ near $\partial\Omega$, and $E(u_\tau)$ has a minimum at $\tau = 0$. Therefore

$$\frac{d}{dt}\Big|_{\tau=0} E(u_\tau) \geq 0\ .$$

On the other hand,

$$\begin{aligned} \frac{\partial u_\tau}{\partial x_i} &= D\frac{\partial u}{\partial x_i} R_{1-\tau\xi(x),Q}(u(x)) - \tau\frac{\partial \xi}{\partial x_i}\frac{\partial}{\partial \lambda} R_{1-\tau\xi(x),Q}(u(x)) \\ \left|\frac{\partial u_\tau}{\partial x_i}\right|^2 &= \left| D\frac{\partial u}{\partial x_i}(x) R_{1-\tau\xi,Q}(u)\right|^2 - 2\tau\frac{\partial \xi}{\partial x_i} D\frac{\partial u}{\partial x_i}(x) R_{1-\tau\xi,Q}(u(x)) \\ &\quad \frac{\partial}{\partial x} R_{1-\tau\xi,Q}(u(x)) + \tau^2\left(\frac{\partial \xi}{\partial x_i}\right)^2 \left|\frac{\partial}{\partial \lambda} R_{1-\tau\xi(x),Q}(u(x))\right|^2 . \end{aligned}$$

Note that $\frac{\partial}{\partial\lambda} R_{\lambda,Q}(p) = d(R_{\lambda,Q}(p), Q)\gamma^1(R_{\lambda,Q}(p))$ where γ is the unit speed geodesic from Q to p. We have that

$$D_v R_{\lambda,Q}(p) \cdot \frac{\partial}{\partial \lambda} R_{\lambda,Q}(p) = \frac{1}{2} D_v d^2(g_{\lambda,Q}(p), Q).$$

Using the contracting property of $R_{\lambda,Q}$, we also have

$$\left| D\frac{\partial u}{\partial x_i}(x) R_{1-\tau\xi,Q}(u)\right| \leq (1-\tau\xi)^2 \left|\frac{\partial u}{\partial x_i}\right|^2 .$$

Therefore

$$E(u_\tau) \leq \int_\Omega \left[(1-\tau\xi)^2|\nabla u|^2 - \sum_i \tau\frac{\partial \xi}{\partial x_i}\frac{\partial}{\partial x_i}\left(d^2(R_{1-\tau\xi,Q}(u), Q)\right)\right] d\mu + O(\tau^2).$$

Letting $\tau \downarrow 0$ and using the minimality of u, we have

$$0 \leq -2 \int_\Omega \xi |\nabla u|^2 d\mu + \int_\Omega \Delta \xi d^2(u(x), Q)\, d\mu$$

i.e.,

$$\int_\Omega \left[(\Delta \xi) d^2(u(x), Q) - 2\xi |\nabla u|^2 \right] d\mu \geq 0.$$

Remark. Since the function involved is only Lipschitz, our derivative is taken almost everywhere.

LECTURE 6
The Order Function

We define the *order* function for $u \in H^1(\Omega, X)$ as follows: Let $x \in \Omega$ and $0 < \sigma < \operatorname{dist}(x, \partial\Omega)$. If $u \not\equiv Q$ on $\partial B_\sigma(x)$, then

$$\operatorname{ord}(x, \sigma, Q) = \frac{\sigma \int_{B_\sigma(x)} |\nabla u|^2 d\mu}{\int_{\partial B_\sigma}(x) d^2(u, Q) d\Sigma}.$$

Proposition 6.1. *Suppose X is a NPC space and $u \in H^1(\Omega, X)$ is a locally minimizing map. Then, for any $x \in \Omega$, either $u \equiv Q$ near x, or* $\operatorname{ord}(x, \sigma, Q)$ *is monotonically increasing.*

Proof. For $x \in \Omega$ given, we assume that $\operatorname{ord}(x, \sigma, Q)$ is defined for σ small; otherwise, from Proposition 5.4 it follows that $d^2(u(y), Q)$ is subharmonic, and $u \equiv Q$ near x.

Define $E(\sigma) = \int_{B_\sigma(x)} |\nabla u|^2 d\mu$, $I(\sigma) = \int_{\partial B_\sigma(x)} d^2(u, \theta) d\Sigma$. From the monotonicity formula (5.1), we compute the logarithmic derivative of E_σ

$$\frac{E'(s)}{E(s)} = \frac{n-2}{\sigma} + \frac{2}{E(\sigma)} \int_{\partial B_\sigma(x)} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma. \tag{6.1}$$

Recall from Proposition 5.4 we have that $\Delta d^2(u, Q) \geq 2|\nabla u|^2$. Integrating this inequality over $B_\sigma(x)$ using a smooth approximation of the characteristic function of $B_\sigma(x)$, we get that

$$2 \int_{B_\sigma(x)} |\nabla u|^2 d\mu \leq \int_{\partial B_\sigma(x)} \frac{\partial}{\partial r} d^2(u(\cdot), Q) d\Sigma$$

i.e.,

$$2E(\sigma) \leq \int_{\partial B_\sigma(x)} \frac{\partial}{\partial r} d^2(u(\cdot), Q) d\Sigma.$$

The logarithmic derivative of $I(\sigma)$ is

$$\frac{I'(\sigma)}{I(\sigma)} = \frac{n-1}{\sigma} + \frac{1}{I(\sigma)} \int_{\partial B_\sigma(x)} \frac{\partial}{\partial r} d^2(u(\cdot), Q) d\Sigma. \tag{6.2}$$

Combining (6.1) with (6.2), we compute

$$\begin{aligned} \frac{d}{d\sigma} \log\left(\frac{\sigma E(\sigma)}{I(\sigma)}\right) = & \frac{2}{E(\sigma)} \int_{\partial B_\sigma(x)} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma \\ & - \frac{1}{I(\sigma)} \int_{\partial B_\sigma(x)} \frac{\partial}{\partial r} d^2(u(\cdot), Q) d\Sigma. \end{aligned}$$

Noting that $\left| \frac{d}{dr} d(u, Q) \right| \leq \left| \frac{\partial u}{\partial r} \right|$, we use the Schwartz inequality and (5.1) to verify that the lefthand side is nonnegative. Thus

$$\operatorname{ord}(x, \sigma, Q) = \frac{\sigma E(\sigma)}{I(\sigma)} \quad \text{is monotonically increasing.}$$

Remark. The monotonicity of $\operatorname{ord}(x, \sigma, Q)$ was also studied by F. H. Lin [L] and R. Hardt [HL] for energy-minimizing harmonic maps into round cones.

Recall that $d^2(u, Q)$ is a convex function of Q. Consider $Q \mapsto \int_{\partial B_\sigma(x)} d^2(u, Q)\, d\Sigma$. Then there is a unique maximum point $Q_{x,\sigma}$. Let $\operatorname{ord}(x, \sigma) = \operatorname{ord}(x, \sigma, Q_{x,\sigma})$. Then $\operatorname{ord}(x, \sigma)$ is still monotonically increasing in σ.

Definition. $\operatorname{ord}(x) = \lim_{\sigma \downarrow 0} \operatorname{ord}(x, \sigma)$. Note that the function $x \mapsto \operatorname{ord}(x)$ is upper semicontinuous because it is a decreasing limit of continuous functions.

Proposition 6.2. *If X is a Riemannian simplical complex of nonpositive curvature, $u \in H^1(\bar{\Omega}, X)$ is a locally minimizing map (with compact image), then u is locally Lipschitz and moreover*

$$\sup_{\Omega_1} |\nabla u|^2 \leq c(\Omega, \Omega_1) \int_\Omega |\nabla u|^2 d\mu \tag{6.3}$$

for any $\Omega_1 \subset\subset \Omega$.

Remark. For harmonic maps into a smooth NPC, the conclusion of Proposition 6.2 is a well-known result of Eells and Sampson [ES]. The usual proof of this is based on the Bochner formula for $|\nabla u|^2$ which relies heavily on the smoothness of X.

Proof. First, let's recall a measure theoretic property of $H^1(\Omega, \mathbb{R}^k)$ function u (see [Z]); namely that u is approximately differentiable a.e. in Ω. This means that at a.e. $x_0 \subset \Omega$ there exists a linear map $L(x) = A(x - x_0) + B$ which approximates u

in the sense that

$$\lim_{\sigma\downarrow 0}\left\{\sigma^{-2-n}\int_{B_\sigma(x_0)}|u-L|^2d\mu+\sigma^{-n}\int_{B_\sigma(x_0)}|\nabla u-\nabla L|^2d\mu\right\}=0.$$

An asymptotic calculation (see Lemma 1.3 in [GS]) then show that if $A\neq 0$, then $\mathrm{ord}(x)=1$ and thus by upper-semicontinuity of $\mathrm{ord}(x)$, $\mathrm{ord}(x)\geq 1$ whenever x_0 is in the closure of the set of points at which u has a nonzero approximate derivative. Let $\alpha=\mathrm{ord}(x_0)\geq 1$ and fix $\sigma_0>0$ so that $B_{\sigma_0}(x_0)\subset\Omega$. Let $\sigma_1\in(0,\sigma_0)$ and note that the monotonicity of the order implies

$$\sigma\int_{B_\sigma(x_0)}|\nabla u|^2d\mu\geq\alpha\int_{\partial B_\sigma(x_0)}d^2(u,Q_1)d\Sigma$$

for all $\sigma\in[\sigma_1,\sigma_0)$ where $Q_1=Q_{x_0,\sigma_1}$. Combining this with (6.2) yields

$$\begin{aligned}\alpha I(\sigma)&\leq\frac{1}{2}\sigma\int_{\partial B_\sigma(x_0)}\frac{d}{dr}d^2(u(x),Q_1)d\Sigma\\&\leq\frac{1}{2}(\sigma I'(\sigma)-(n-2)I(\sigma)).\end{aligned}$$

This implies

$$\frac{I'(\sigma)}{I(\sigma)}\geq\frac{n-1+2\alpha}{\sigma}\qquad\text{for }\ \sigma\in[\sigma_1,\sigma_0).$$

Integrating from σ_1 and σ_0 and fixing σ_0, we obtain

$$\sigma_1^{1-n}I(\sigma_1)\leq c\sigma_1^{2\alpha}I(\sigma_0)\leq c(\sigma_0)\sigma_1^{2\alpha}.$$

The subharmonic property of $d^2(u,Q_1)$ on Ω implies

$$\sup_{B_{\frac{\sigma_1}{2}}(x_0)}d^2(u(x),Q_1)\leq c\sigma_1^{2\alpha}.$$

Then, for all $x\in B_{\frac{\sigma_0}{2}}(x_0)$, choose $\sigma_1=2|x_0-x_0|$ and by triangle inequality

$$d(u(x),u(x_0))\leq d(u(x),Q_1)+d(u(x_0),Q_1)\leq \mathrm{c}|x-x_0|^\alpha$$

since $\alpha\geq 1$, this implies that for any x_0 at which approximate derivative exists and is non-zero we have $\mid\frac{\partial u}{\partial x_i}\mid\leq c$. Thus $\mid\frac{\partial u}{\partial x_i}\mid\leq$ c a.e., and u is locally Lipschitz.

We can also apply the above argument to show the interior estimate (6.3). See Theorem 2.4 in [GS] for details.

Estimate (4) can actually lead to the following Eells-Sampson type of Existence theorem.

Theorem 6.3. *Let M be a compact Riemannian manifold without boundary, and let X be a Riemannian simplicial complex and its universal covering $\tilde{X}$ has non-positive curvature. Then any Lipschitz map $\varphi : M \to X$ is freely homotopic to a Lipschitz map $u : M \to X$ which minimizes energy in the sense that*

$$E(u) = \inf\{E(v) \ : \ v : M \longrightarrow X \text{ Lipschitz, } v \text{ homotopic to } \varphi\}$$

Proof. We use a local replacement idea. Let $E_0 = \inf\{E(\sigma) : v : M \to X$ Lipschitz, v homotopic to $\varphi\}$. Let $\{u_i\}$ be a sequence of Lipschitz maps homotopic to φ with $E(u_i) \to E_0$. Then we can extract a subsequence, again denoted $\{u_i\}$, which converges weakly to a map $u \in H^1(\Omega, X)$ such that $E(u) \leq E$. Now we claim that u is Lipschitz and homotopic to φ. Let $x_0 \in M$ and consider a small ball B. We lift a map $u_i : B \to X$ to $\tilde{X}$. Denote this lift by $\tilde{u}_i : B \to \tilde{X}$. Let v_i be a minimizing map from $B \to \tilde{X}$ which is equal to $\tilde{u}_i$ on ∂B. We then defined a replaced map $\hat{u}_i$ by

$$\hat{u}_i(x) = \begin{cases} \pi(v_i(x)) & x \in B \\ u_i(x) & x \in M \setminus B \end{cases}$$

where π is the projection of $\tilde{X}$ onto X. Since $E(\hat{u}_i) \leq E(u_i)$, $\{\hat{u}_i\}$ is again a minimizing sequence. Applying Corollary 5.3 in B we have, for all $\epsilon > 0$ and i sufficiently large,

$$\int_B d^2(\hat{u}_i, u_i)d\mu \leq c\epsilon.$$

On the other hand, $\hat{u}_i$ is uniformly Lipschitz on compact subsets interior to B. Thus a subsequence of $\{\hat{u}_i\}$ converges uniformly near x_0 to a Lipschitz map which must be u by uniqueness of limit. In order to prove u is homotopic to φ, we note that geodesic homotopy $v_{i,t}$ with $v_{i,0} = \varphi$, $v_{i,1} = \hat{u}_i$ converges to a geodesic homotopy v_t with $v_0 = \varphi$, $v_1 = u$ near x_0. Then, by a covering argument, we have a global geodesic homotopy from φ to u.

LECTURE 7
Approximation and Smoothness Results for Harmonic Maps

To apply harmonic map theory to singular spaces, we need to improve its regularity in certain cases. First, we consider approximation of harmonic map by homogeneous minimizing (tangent) maps and note that the linearity of the approximation will indicate differentiability in a strong sense. Let X be a geometric cone in $\mathbb{R}^n$ (i.e., $\lambda X \subseteq X$, $\forall \lambda \in R_+$), and suppose X has nonpositive curvature. We say a minimizing map $u : \mathbf{B}_1(0) \to X$ is intrinsically homogeneous of degree α ($\alpha \geq 1$) if u maps each ray through origin onto a geodesic in X parametrized in such a way that $d(u(t\xi), u(0)) = t^\alpha d(u(\xi), u(0))$. Then the following lemma gives us a criterion of intrinsically homogeneous map.

Lemma 7.1. *If $u : \mathbb{B}_1(0) \to X$ is a minimizing map such that* $\operatorname{ord}(0, \sigma, Q_0\sigma) = \alpha$ *for all $\sigma \in (0,1)$ with a fixed $\alpha \geq 1$, then u is intrinsically homogeneous.*

Proof. First, we note that $\operatorname{ord}(0, \sigma, Q_0\sigma) \equiv \alpha$ implies that $\operatorname{ord}(0, \sigma, u(0)) \equiv \alpha$. Then all inequalities in the proof of monotonicity of order must be equalities. It follows that

$$\frac{d}{dr} d(u, u(0)) = \left| \frac{du}{dr} \right| \quad \text{a.e.,}$$

and there is a function $h : (0,1) \to R$ such that

$$\frac{d}{dr} d(u, u(0)) = h(\sigma) d(u, u(0)) \text{ on } \partial\mathbb{B}_\sigma(0).$$

Integrating the first equality along ray $\gamma : (\sigma, 1) \to \mathbf{B}_1^n$ given by $\gamma(r) = r\xi$ for some $\xi \in \partial\mathbb{B}_1(0)$, we have

$$\int_\sigma^1 \left| \frac{du}{dr} \right| ds \;=\; d(u(\xi), u(0)) - d(u(\sigma\xi), u(0)) \leq d(u(\xi), u(\sigma\xi)).$$

In particular it follows that $u(\gamma)$ is a geodesic path in X. We also have

$$E(\sigma) = \int_{\partial B_\sigma(0)} d(u, u(0)) \frac{d}{dr} d(u, u(0)) d\Sigma = h(\sigma) I(\sigma)$$

so we conclude from $E(\sigma) = \alpha\sigma^{-1} I(\sigma)$ that $h(\sigma) = \alpha\sigma^{-1}$. Then integrate along the ray from $\sigma\xi$ to ξ to obtain

$$d(u(\sigma\xi), u(0)) = \sigma^\alpha d(u(\xi), u(0)) \qquad \forall\, \xi \in \partial\mathbf{B}_1(0).$$

Remark. (1) If $0 \in \Omega$ and u is a harmonic *function*, then $u(x) - u(0) = p_\alpha(x) + 0(|x|^{\alpha+1})$ where $p_\alpha(x)$ is a spherical harmonic of degree α with

$$\alpha = \lim_{\sigma \downarrow 0} \frac{\sigma E(\sigma)}{I(\sigma)} \quad \text{and} \quad Q = u(0).$$

(2) The converse of the above lemma is also true. In fact, if $u : B_1(0) \to X$ is intrinsically homogeneous of degree α, then it follows immediately that $\operatorname{ord}(0, \sigma, u(0))$ is constant. Moreover, the constant is α and $\operatorname{ord}(0) = \alpha$. On the other hand, homogeneity of u implies that $\operatorname{ord}(0, \sigma, Q_0\sigma)$ is independent of σ, i.e., $\operatorname{ord}(0, \sigma, Q_0\sigma) = \alpha$.

An intrinsically homogeneous map of degree 1 is very special. Roughly speaking, it looks like a map of affine type. More precisely, let $\zeta : \mathbf{R}^\ell \to X$ be an isometric and totally geodesic embedding. This means that $d(\zeta(x), \zeta(y)) = |x - y|$, and the straight line between x and y is mapped to a unique geodesic between $\zeta(x)$ and $\zeta(y)$. Take $\nu : \mathbf{R}^n \to \mathbf{R}^\ell$ given by a harmonic function of degree α. We consider $u = \zeta \circ \nu : \mathbf{R}^n \to X$ and note that u is a minimizing homogeneous map of degree α. We call this u a *regular homogeneous map*.

Proposition 7.2. *Any homogeneous minimizing map $u : \mathbf{R}^n \to X$ is regular if it is of degree 1.*

Proof. (Sketch). We divide the proof into several steps.

Step 1. $\operatorname{ord}(0) = 1$ and $\operatorname{ord}(0, \sigma, Q_0\sigma) = 1$. This follows from (6.1) of the above Remark.

Step 2. $\operatorname{ord}(x) = 1$ for all $x \in \mathbf{R}^n$. In fact, $x \mapsto \operatorname{ord}(x)$ is upper semicontinuous and the degree 1 homogeneity of u implies that $\operatorname{ord}(x)$ is homogeneous of degree 0. Therefore $\operatorname{ord}(x) = 1\ \forall x \in \mathbf{R}^n$.

Step 3. $\operatorname{ord}(x, \sigma, u(x)) = 1$ for all $x \in \mathbf{R}^n$ and all $\sigma > 0$. To verify this, note that

$$\operatorname{ord}(x, \sigma, u(x)) = \operatorname{ord}(\lambda x, \lambda\sigma, \lambda u(x)) \quad \forall \lambda > 0.$$

So we take $\lambda = \sigma^{-1}$ and conclude that

$$\lim_{\sigma \to \infty} \operatorname{ord}(x, \sigma, u(x)) = 1.$$

On the other hand, $\lim_{\sigma \to 0} \operatorname{ord}(x, \sigma, u(x)) = \operatorname{ord}(x) = 1$. It follows that for all $x \in \mathbf{R}^n$ and all $\sigma > 0$ we have $\operatorname{ord}(x, \sigma, u(x)) = 1$.

Step 4. From Lemma 7.1, we conclude that u is an intrinsically homogeneous of degree one about every point. It follows that the restriction of u to any line parametrizes a geodesic in X with constant speed and from the equality $\Delta_x d^2(u(x), u(x_0)) = 2|\nabla u|^2$ and the degree 2 homogeneity about x_0 of the function $x \mapsto d^2(u(x), u(x_0))$ we also conclude that $|\nabla u|^2$ is homogeneous of degree 0. In particular $|\nabla u|^2$ is constant and $d^2(u(x), u(x_0))$ is a quadratic polynomial with positive value. Thus a linear algebraic computation implies that u can factor through an isometric, totally geodesic embedding of $\mathbb{R}^\ell$, for some ℓ, into X.

Now we return to the general situation of a minimizing map $u : \Omega \to X$. Given $x_0 \in \Omega$ we approximate u near x_0 by homogeneous map. We choose coordinates so that $x_0 = 0 = u(x_0)$. We define a rescaled map of u, $u_{\lambda,\kappa} : B_\lambda^{-1}\sigma_0 \to \kappa^{-1}X$ by $u_{\lambda,u}(x) = \kappa^{-1}u(\lambda, x)$. Note that $\kappa^{-1}X$ has nonpositive curvature since distances are multiplied by a constant factor. We have, by a change of variables,

$$\int_{B_\sigma} |\nabla u_{\lambda,\kappa}|^2 d\mu = \mu^{-2}\lambda^{2-n} \int_{B_{\lambda\sigma}(0)} |\nabla u|^2 d\mu$$

$$\int_{\partial B_\sigma} d^2_{\mu^{-1}X}(u_{\lambda,u}, 0) d\Sigma = \mu^{-2}\lambda^{1-n} \int_{\partial B_{\lambda\sigma}} d^2_X(u, 0) d\Sigma.$$

In particular, $\operatorname{ord} u_{\lambda,\kappa}(0, \sigma, 0) = \operatorname{ord} u(0, \lambda\sigma, 0)$ for $\sigma \in (0, \lambda^{-1}\sigma_0)$. For small $\lambda > 0$, let $\kappa = (\lambda^{1-n} I(\lambda))^{1/2}$, so that we have

$$\int_{\partial B_1} d^2_{\kappa^{-1}X}(u_{\lambda,\kappa}, 0) d\Sigma = 1$$

Since $\operatorname{ord} u_{\lambda,\kappa}(0) \to \operatorname{ord} u(0) = \alpha$ as $\lambda \to 0$. We thus have, for λ small

$$\int_{B_1(0)} |\nabla u_{\lambda,\kappa}|^2 d\mu \leq 2\alpha.$$

Thus $u_{\lambda,\kappa}$ has uniformly bounded energy and then by Proposition 6.2. has uniformly bounded Lipschitz constant on compact subsets of $B_1(0)$. We can extract a subsequence $\{\lambda_i\}$ tending to zero, so that the corresponding u_i converges uniformly to a Lipschitz limit $u_* : \mathbb{B}_1(0) \to T_0X$ where T_0X denotes the tangent cone of X at 0.

Proposition 7.3. *The map u_* is a nonconstant energy minimizer that is homogeneous of degree α.*

Proof. See [GS], Proposition 3.3.

For simplicity, we will focus on the smoothness of harmonic map into an F-connected space. This is a higher dimensional generalization of a tree space.

Definition. We say that a nonpositively curved Riemannian simplical complex X is *F-connected* if any two adjacent simplices are contained in the image of a totally geodesic embedding of a Euclidean space into X. We will assume that X is finite dimensional and locally compact.

Notice that (1-dimensional) trees are F-connected. The most important higher dimensional F-connected complexes are the Euclidean buildings of Bruhat and Tits (see [Br]). The importance of F-connected complexes is that there are lots of interesting groups which act isometrically on these spaces.

Definition. A *k-flat* F of X is the image of a isometric totally geodesic embedding of R^k into X. A point $x_0 \in \Omega$ is called a *regular point of* u if there is an $\epsilon > 0$ such that $u(B_\epsilon(x_0))$ lies in a k-flat. Let $\mathcal{R}(u) = \{x \in \Omega : x \text{ is a regular point of } u\}$. The singular set is the complement

$$\mathcal{S}(u) = \Omega \setminus R(u).$$

Notice that $\mathcal{R}(u)$ is an open subset of Ω by definition, and that $\mathcal{S}(u)$ is a relatively closed set.

The main smoothness theorem concerning minimizing map into F-connected complexes is the following.

Theorem 7.4. *Suppose X is an F-connected complex. If $u : \Omega \to X$ a locally minimizing map, then:*

1. $\dim_{\mathcal{H}} \mathcal{S}(u) \leq n-2$ *where* $\dim_{\mathcal{H}}$ *denotes the Hausdorff dimension.*
2. *For any $\Omega_1 \subset\subset \Omega$, there exists a sequence of Lipschitz cut-off function $\{\psi_i\}$ such that $\psi_i \equiv 0$ in a neighborhood of $\mathcal{S}(u) \cap \overline{\Omega}_1$, $0 \leq \psi_i \leq 1$, $\psi_i(x) \to 1$ for all $x \in \Omega_1 \setminus \mathcal{S}(u)$, and*

$$\lim_{i\to\infty} \int_\Omega |\nabla u||\nabla \psi_i| d\mu = 0.$$

The idea is that we want to show that almost all points x in Ω has $\operatorname{ord}(x) = 1$ and that, at such an x, the degree 1 homogeneous map u_* has rank equal to $\dim X$; moreover, such an x is a regular point of u. In order to carry out the details we need several preliminary results.

First, we generalize the notion of homogeneous map without the cone structure of the target. Let $x^1, \ldots, x^n$ be a normal coordinate system centered at x_0, and let $r = |x|$, $\xi = \frac{x}{|x|}$ denote polar coordinates in $B_{r_0}(x_0)$. We will say that a Lipschitz map $\ell : B_{r_0}(x_0) \to X$ is *essentially homogeneous of degree 1* if there is a nonnegative function $\lambda : S^{n-1} \to R$ and an assignment γ_ξ to each $\xi \in S^{n-1}$ of unit speed geodesic in X with $\gamma_\xi(0) = \ell(0)$, $\gamma_\xi(\lambda(\xi)\gamma) = \ell(\gamma\xi)$ for $x = \gamma\xi \in B_{\gamma_0}(x_0)$. In other words, a map is essentially homogeneous of degree 1 if the restriction of u to each ray is a constant speed geodesic in X. For $x_0 \in \Omega$ and $\sigma > 0$ such that $B_\sigma(x_0) \subset\subset \Omega$, we consider the error with which u can be approximated by degree 1 essentially homogeneous maps. Define

$$R(x_0, \sigma) = \inf\left\{ \sup_{x \in B_\sigma(x_0)} d(u(x), \ell(x)) : \ell \text{ is essentially homogeneous of degree 1} \right\}.$$

Note that $R(x_0, \sigma) \leq \sup_{x \in B_\sigma(x_0)} d(u(x), u(x_0)) \leq \operatorname{Lip}(u)\sigma$.

Definition. A minimizing map $u : \Omega \to X$ is *intrinsically differentiable* on a compact subset $K \subset \Omega$ provided there exist $r_0 > 0$, $c > 0$ and $\beta \in (0,1]$ such that $R(x,\sigma) \subseteq c\sigma^{1+\beta} R(x,r_0)$ for all $x \in K$ and $\sigma \in (0,r_0]$. The constants c, β, r_0 depend only on K, Ω, X and the total energy of u.

Definition. A subset $S \subseteq X$ is *essentially regular* if for any minimizing map $u : \Omega \to X$ with $u(\Omega) \subseteq S$, the restriction of u to any compact subset of Ω is intrinsically differentiable.

Suppose that X_0 is a totally geodesic subcomplex of X and $\ell : \mathbb{R}^n \to X_0$ is an essentially homogeneous degree 1 map, we have the following concept.

Definition. ℓ is said to be *effectively contained in* X_0 if $\ell^{-1}(X_1)$ is of codimension at least one in $\mathbf{R}^n$ where X_1 is the subcomplex of X_0 consisting of simplices which are faces of a simplex in X but not in X_0. Notice that $X_1 \subseteq X_0$ is of at least codimension one.

We are ready to state the following theorem:

Theorem 7.5. *Let $u : \Omega \to X$ be a minimizing map. Let $x_0 \in \Omega$ and $r_0 > 0$ be such that $B_{r_0}(x_0) \subset\subset \Omega$. Let $X_0 \subseteq X$ be a totally geodesic subcomplex, and let $\ell : B_{r_0}(x_0) \to X_0$ be an essentially homogeneous degree 1 map. Assume that X_0 is essentially regular near $p = \ell(x_0)$. There exists $\delta_0 > 0$ depending only on ℓ, Ω, X, X_0 such that if ℓ is effectively contained in X_0 and $\sup_{B_{\sigma_0}(x_0)} d(u(x),\ell(x)) \leq \delta_0$, then u is intrinsically differentiable near x_0, and there exist $\sigma < \delta_0 < r_0$ such that $u(B_{\delta_0}(x_0)) \subseteq X_0$.*

The proof is rather technical. We omit it, and refer to [GS].

LECTURE 8
Order 1 Points and Partial Regularity

The next result convinces us that a point $x \in \Omega$ where the order of u at x, $\mathrm{ord}^u(x) = 1$ and where its approximating map u_* has rank $k = \dim X$ is a regular point, provided that a little regularity condition holds on X. Roughly, this regularity condition means that there exist an isometric totally geodesic embedding $i : B_r(x) \subseteq \Omega \to X$ corresponding to that of u_*. Notice that if X is an F-connected complex, then this condition holds.

Theorem 8.1. *If X is an F-connected complex, and $u : \Omega \to X$ is a minimizing map, then any point $x \in \Omega$ where $\mathrm{ord}^u = 1$ and where u_* has rank $k = \dim X$ is a regular point.*

We also omit the proof and refer to [GS].

Suppose that X is F-connected, denote X_{p_0} as the tangent cone of X at p_0 which is also F-connected. Let $J : \mathbb{R}^m \to X_{p_0}$ be an isometric totally geodesic embedding for $1 \le m \le k$. It can be easily seen that $J(\mathbb{R}^m)$ is contained in at least one k-flat. We need the following geometric result.

Theorem 8.2. *Let X^k be F-connected, and let X_0 be the union of all k-dimensional flats in X_{p_0} which contains $J(\mathbb{R}^m)$. Then the subcomplex X_0 is totally geodesic and is isometric to $\mathbb{R}^m \times X_1$ where X_1 is F-connected of dimension $k-n$. Moreover J is effectively contained in X_0.*

Combining these theorems we can then prove the simplification result.

Theorem 8.3. *Let X be F-connected. Then the following properties hold:*

1. *For any positive integer u and a compact set $K_0 \subseteq X$ there exist $\epsilon_0 = \epsilon_0(K_0, u) > 0$ such that for any minimizing map $u : \Omega^n \to X$ with $u(\Omega) \subseteq K_0$, we have either $\mathrm{ord}(x) = 1$ or $\mathrm{ord}(x) \ge 1 + \epsilon_0$ for all $x \in \Omega$.*
2. *Let $u : \Omega \to X$ be a minimizing map, and let $x_0 \in$ with $\mathrm{ord}(x_0) = 1$. There exists a totally geodesic subcomplex X_0 of $X_u(x_0)$ which is isometric to $\mathbb{R}^m \times X_1^{k-m}$ for some $1 \le m \le \min\{n, k\}$ and some F-connected X_1 of dimension $k - m$ such that $u(B_\sigma(x_0)) \subseteq X_0$ for some $\sigma > 0$. Moreover, if we write $u = (u_1, u_2) = B_{\sigma_0}(x_0) \to \mathbb{R}^m \times X_1$, then u_1 is harmonic with rank m at x_0, and $\mathrm{ord}^{u_2}(x_0) > 1$*
3. *X is essentially regular.*

Proof. (Sketch) (i) Denote $k = \dim X$. If $k = 1$, i.e., X is a tree, we will show explicitly that ϵ_0 only depends on n. In fact, if $\mathrm{ord}^u(x_0) > 1$ and $u(x_0)$ is not a vertex of X, then u is smooth near x_0 and thus $\mathrm{ord}^u(x_0) \geq 2$ since $\mathrm{ord}^u(x_0)$ is integer. If $u(x_0)$ is a vertex with p edges emanating from $p_0 = u(x_0)$, then $p \geq 2$ by the maximum principle. If $p = 2$, then, again, $\mathrm{ord}^u(x_0) \geq 2$. We assume that $p \geq 3$, and consider every homogeneous approximating map $u_* : \mathbf{R}^n \to X_p$. If we choose an edge e emanating from p_0 and introduce an arc length parameter s along e, then on $O_e = u_*^{-1}\{e\}$ the function $h_e = s(u_*)$ is a homogeneous harmonic function of degree $\alpha = \mathrm{ord}^u(x_0)$. Of course, O_e is the cone over a region $D_e \subseteq S^{n-1}$. It follows that $h_e|D_e$ is a first eigenfunction of De and the corresponding first eigenvalue $\lambda_1(D_e) = \alpha(\alpha + n - 2)$. We thus decompose S^{n-1} into p disjoint regions all with the same eigenvalues. We can choose one such region D_e that $\mathrm{vol}(D_e) \leq \frac{\mathrm{vol}(S^{n-1})}{p}$. Standard results about eigenvalues then imply that there exist $\delta_n > 0$ such that $\lambda_1(D_e) = \alpha(\alpha + n - 2) \geq n - 1 + \delta_n$. In particular, $\alpha \geq 1 + \epsilon_0$ with $\epsilon_0 = \epsilon_0(n) > 0$.

Property 2 is an easy consequence of Theorem 8.1 and Theorem 8.2. To establish 3 we work by induction on k. For $k = 1$ the result follows easily. Assume that $k \geq 2$ and all F-connected complexes of dimension less than k are essentially regular. By a compactness argument, it suffices to prove the following result for any $x_0 \in \Omega$. There exists $r_0 > 0$ such that for $\sigma \in (0, r_0]$

$$R(x_0, \sigma) \leq c\sigma^{1+\beta} R(x_0, r_0)$$

for constants c, r_0, β depending on x_0, $E(u)$, Ω, X_0. There are two cases to consider. First suppose $\mathrm{ord}^u(x_0) > 1$. Then from 1 we know that $\mathrm{ord}^u(x_0) \geq 1 + \epsilon_0$. This implies, by lecture 6, that

$$\sup_{x \in B_\sigma(x_0)} d(u(x), u(x_0)) \leq c\sigma^{1+\epsilon_0} \sup_{B_{r_0}(x_0)} d(u(x), u(x_0))$$

for some constant c and $r_0 > 0$. Therefore, the desired decay on $R(x_0, \sigma)$ follows. In the remaining case $\mathrm{ord}(x_0) = 1$, the result follows immediately from 2, and the inductive assumption.

8.4 Proof of Singular Set Estimate

Now, we start to prove the main Theorem 7.4. The estimate on the Hausdorff dimension of $\mathcal{S}(u)$ is an application of the basic argument of Federer dimensional reduction [F2]. For any subset $E \subseteq \Omega$ and any real number $s \in (0, n]$, we recall the definition of Hausdorff (outer) measure $\mathcal{H}^s(\cdot)$

$$\mathcal{H}^s(E) = \inf\left\{\sum_{i=1}^{s} r_i^s \ :\ \bigcup_{i=1}^{\infty} B_{r_i}(x_0) \supseteq E\right\}$$

and of the Hausdorff dimension

$$\dim_{\mathcal{H}} E = \inf\{s : \mathcal{H}^s(E) = 0\}.$$

We observe that $\mathcal{S}(u) = \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_k$ where $k_0 = \min\{n, k-1\}$ and $\mathcal{S}_j$ consists of those singular points having rank j, where the rank at x_0 is the rank of the approximating map u_* if $\operatorname{ord}(x_0) = 1$, and the rank is zero if $\operatorname{ord}(x_0) > 1$. We will first show $\dim_{\mathcal{H}} \mathcal{S}_0 \leq n-2$. In fact, if we define $\tilde{\mathcal{S}}_0 = \{x \in \Omega, \operatorname{ord}^u(x) > 1\}$. Then we actually show that $\dim_{\mathcal{H}} \tilde{\mathcal{S}}_0 \leq n-2$. Notice that $\mathcal{S}_0 \subseteq \tilde{\mathcal{S}}_0$. In order to do this, we need the following lemma whose proof is very standard.

Lemma 8.5. *If $\{u_i\}$ is a sequence of minimizing maps from B_1 to X with $E(u_i)$ and Image (u_i) uniformly bounded, then a subsequence of $\{u_i\}$ converges uniformly on compact subsets of $B_1(0)$ to a minimizing map $u : B_1 \to X$, and*

$$\mathcal{H}^s(\tilde{\mathcal{S}}_0(u) \cap \overline{B}_r(0)) \geq \overline{\lim_{i\to\infty}} \mathcal{H}^s(\tilde{\mathcal{S}}_0(u_i) \cap B_r(0))$$

for all $r \in (0,1)$. In particular, $\dim_{\mathcal{H}}(\tilde{\mathcal{S}}_0(u)) \geq \overline{\lim_{i\to\infty}} \dim_{\mathcal{H}}(\tilde{\mathcal{S}}_0(u))$.

We now show that $\dim \tilde{\mathcal{S}}_0(u) \leq n-2$. Suppose $s \in [0,n]$ with $\mathcal{H}^s(\tilde{\mathcal{S}}_0(u)) > 0$. Then by [F1] we may find $x_0 \in \Omega$ such that

$$\varliminf_{\sigma\to 0} \sigma^{-s}\mathcal{H}^s(\tilde{\mathcal{S}}_0(u) \cap B_\sigma(x)) \geq 2^{-s}.$$

Let $u_* : \mathbb{R}^n \to X_{u(x_0)}$ be a homogeneous approximating map for u at x_0. Let $\alpha = \operatorname{ord}^u(x_0)$ so that u_* is of degree α. Since $x_0 \in \hat{\mathcal{S}}_0(u)$, we have $\alpha \geq 1 + \epsilon_0$. We may apply Lemma 8.5 to suitable rescalings $\{u_i\}$ of u near x_0 to conclude that $\mathcal{H}^s(\tilde{\mathcal{S}}_0(u_*)) > 0$. Since $\tilde{\mathcal{S}}_0(u_*)$ is a cone, it follows that there is $x \in S^{n-1} \cap \tilde{\mathcal{S}}_0(u_*)$ such that

$$\varliminf_{\sigma\to 0} \sigma^{-s}\mathcal{H}^s(\tilde{\mathcal{S}}_0(u_*) \cap B_\sigma(x_1)) \geq 2^{-s}.$$

Let u_1 be a homogeneous approximating map for u_* at x_1. Then $\operatorname{ord}^{u_1}(x_1) \geq 1+\epsilon_0$. It is easy to see that derivative of u_1 is zero along ray $t \mapsto tx_1$. If we choose coordinates in which $x_1 = (0,\ldots,0,1)$, then $\frac{\partial u_1}{\partial x^n} = 0$. Therefore the restriction of u_1, denoted $\tilde{u}_1$, to $\mathbb{R}^{n-1}$ is a homogeneous map of degree $\alpha \geq 1+\epsilon_0$. We then have

$$\tilde{\mathcal{S}}_0(u_1) = \tilde{\mathcal{S}}_0(\tilde{u}_1) \times \mathbb{R}$$

and thus $\mathcal{H}^{s-1}(\tilde{\mathcal{S}}_0(\tilde{u}_1)) > 0$. If $s > n-2$, we may repeat this argument inductively and produce finally an $\bar{\epsilon}_0 > 0$ and a minimizing map $\nu : \mathbb{R}^2 \to X_{u(x_0)}$ homogeneous of degree $\bar{\alpha} \geq 1 + \bar{\epsilon}_0$ such that

$$\mathcal{H}^{s-(n-2)}(\tilde{\mathcal{S}}_0(\nu)) > 0.$$

Thus repeating the argument again will produce a geodesic ω with degree > 1 and contradict with $\operatorname{ord}^\omega(x_0) = 1$ for all x_0.

We now show by induction on $k = \dim X$ that $\dim \mathcal{S}(u) \leq n-2$. For $k=1$ we have $\mathcal{S} = \mathcal{S}_0$, and we have established this case. Assume that $k \geq 2$ that the conclusion is true for F-connected complexes of dimension less than k. Let

$x_0 \in \mathcal{S}_m - \mathcal{S}_0$ for a minimizing map $u : \Omega \to X^k$. We then have $\mathrm{ord}^u(x_0) = 1$, and by 2 in Theorem 8.3 there is a $\delta_0 > 0$ such that $u(B_{\delta_0}(x_0)) \subseteq X_0 = \mathbb{R}^m \times X_1$ with X_1 an F-connected of dimension $k - m$. Thus we have $u = (u_1, u_2)$ where $u_1 : B_{\sigma_0}(x_0) \to \mathbb{R}^m$, $u_2 : B_{\sigma_0}(x_0) \to X_1$ are both minimizing. $\mathcal{S}_m(u) \cap B_0(x_0) \subseteq \mathcal{S}(u_2) \cap B_\sigma(x_0)$. By inductive assumption we then have $\dim(\mathcal{S}_m(u) \cap B_{\sigma_0}(x_0)) \leq n - 2$ and thus $\dim_{\mathcal{H}} \mathcal{S}(u) \leq n - 2$.

To prove 2 we use induction on $k = \dim X$ again. For $k = 1$, we have $\mathcal{S} = \mathcal{S}_0$ and $\dim_{\mathcal{H}} \mathcal{S} \leq n - 2$. Let $\epsilon > 0$ and $d > n - 2$. Let Ω_2 be a fixed domain of Ω with $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$, and choose a finite covering $\{B_{r_j}(x_j)\}_{j=1}$ of $\mathcal{S}_0 \cap \overline{\Omega}_1$ satisfying $x_j \in \mathcal{S}_0$ and $\Sigma r_j^d \leq \epsilon$, and $B_{4r_j}(x_j) \subset \Omega_2$. Let

$$\varphi_j \equiv \begin{cases} 1 & \text{on } \Omega \setminus B_{2r_j}(x_j) \\ \sigma \leq S_j \leq 1 & \text{on } B_{2r_j}(x_j) \setminus B_{r_j}(x_j) \\ 0 & B_{r_j}(x_j) \end{cases}.$$

Then $|\nabla S_j| \leq 2r_j^{-1}$. Define $\varphi = \min\{\varphi_j \; : \; j = 1 \ldots \ell\}$ and observe that $\varphi \equiv 0$ near $\mathcal{S}_0 \cap \overline{\Omega}_1$ and $\varphi_1 \equiv 1$ on $\Omega \setminus \bigcup_{j=1}^{\ell} B_{2r_j}(x_j)$. Now let $\varphi_0 = \varphi^2$ and observe

$$\begin{aligned} &\int_\Omega |\nabla\nabla u||\nabla\psi_0| d\mu = 2 \int_{\bigcup_{j=1}^{\ell} B_{2r_j}(x_j)} \varphi |\nabla\nabla u||\nabla\varphi| d\mu \\ &\leq 2 \left(\int_{\bigcup_{j=1}^{\ell} B_{2r_j}(x_j)} \varphi |\nabla\nabla u||\nabla u|^{-1} \varphi^2 d\mu \right)^{1/2} \left(\int_{\bigcup_{j=1}^{\ell} B_{2r_j}(x_j)} |\nabla u||\nabla\varphi|^2 d\mu \right)^{1/2} \end{aligned}$$

by Schwartz inequality. On the other hand, a result for harmonic maps (see [ES]) implies that on regular set we have

$$\frac{1}{2}\Delta|\nabla u|^2 \geq |\nabla\nabla u|^2 - c|\nabla u|^2.$$

For $j = 1, \ldots, \ell$ let $\rho_j \equiv \begin{cases} 1 & \text{on } B_{2r_j}(x_j) \\ \sigma \leq \rho_j \leq 1 & \text{on } B_{3r_j}(x_j) \setminus B_{2r_j}(x_j) \\ 0 & \Omega \setminus B_{4r_j}(x_j) \end{cases}$ with $|\nabla\rho_j| \leq cr_j^{-1}$. Define

$$\rho = \max\{\rho_j, j = 1, \ldots, \ell\}$$

and observe that $\rho \equiv 1$ on $\bigcup_{j=1}^{\ell} B_{2r_j}(x_j)$, and $\rho \equiv 0$ outside $\bigcup_{j=1}^{\ell} B_{3r_j}(x_j)$. We therefore have

$$\int_{\bigcup_{j=1}^{\ell} B_{2r_j}(x_j)} |\nabla\nabla u|^2 |\nabla u|^{-1} \varphi^2 d\mu \leq \int_\Omega |\nabla\nabla u|^2 |\nabla u|^{-1} \varphi^2 \rho^2 d\mu.$$

An equality in [SY] implies that on regular set we have $(1-\epsilon_n)|\nabla\nabla u|^2 \geq |\nabla||\nabla u|^2$ for some $\epsilon_n > 0$. Therefore we have

$$\Delta|\nabla u| \geq \epsilon_n |\nabla\nabla u|^2|\nabla u|^{-1} - c|\nabla u|$$

on $\Omega \setminus \mathcal{S}_0$. Using integration by parts, we have

$$\begin{aligned}&\epsilon_n \int_\Omega |\nabla\nabla u|^2|\nabla u|^{-1}\rho^2\varphi^2 d\mu \\ &\leq -2\int_\Omega \rho\varphi\,\langle\nabla|\nabla u|\nabla(\rho\varphi)\rangle\, d\mu + c\int_\Omega |\nabla u|^2\rho^2\varphi^2 d\mu.\end{aligned}$$

This implies

$$\int_\Omega |\nabla\nabla u|^2|\nabla u|^{-1}\rho^2\varphi^2 d\mu \leq c\int_\Omega |\nabla u|(\varphi^2|\nabla\rho|^2 + |\nabla\varphi|^2\rho^2 + \rho^2\varphi^2)d\mu.$$

Combining this with the above estimates, we have

$$\int_\Omega |\nabla\nabla u||\nabla\psi_0|d\mu \leq c\sum_{j=1}^{\ell} r_j^{-2}\int_{B_{3r_j}(x_j)} |\nabla u|d\mu \leq \sum_{j=1}^{\ell} r_j^{n-2} \sup_{B_{3r_j}(x_j)} |\nabla u|.$$

On the other hand since $x_j \in \mathcal{S}_0$ we have $\mathrm{ord}^u(x_j) \geq 1+\epsilon_0$ and therefore $\sup_{B_{3r_j}(x_j)}$ $|\nabla u| \leq cr_j\epsilon_0$. Thus we have

$$\int_\Omega |\nabla\nabla u||\nabla\psi_0|d\mu \leq c\sum_{j=1}^{\ell} r_j^{n-2+\epsilon_0} \leq \epsilon$$

provided $n-2 < d < n-2+\epsilon_0$. Now we can assume that 2 holds for maps into F-connected complexes of dimension less than k. We cover $\left(\mathcal{S} - \bigcup_{i=1}^{\ell} B_{r_i}(x_i)\right) \cap \overline{\Omega}_1$ with balls $\{B_{r_p}(y_p) : p = 1, \dots\}$ such that in $B_{r_p}(y_p)$ the map can be written $u = (u_1, u_2)$ as in (ii) of Theorem 8.3. By the inductive assumption, there exists a function ψ_p vanishing near $\varphi \cap \overline{B_{r_p}(y_p)}$ and identically one outside a slightly larger neighborhood with

$$\int_\Omega |\nabla\nabla u||\nabla\psi_p|d\mu \leq 2^{-p}\epsilon.$$

We finally set $\psi = \min\{\psi_0, \psi_1, \dots, \psi_\nu\}$ and conclude

$$\int_\Omega |\nabla\nabla u||\nabla\psi|d\mu \leq \sum_{p=0}^{\infty}\int_\Omega |\nabla\nabla u||\nabla\psi_p|d\mu \leq 2\epsilon.$$

LECTURE 9
Rigidity Results via Harmonic Maps

In this chapter we prove some rigidity theorems for discrete groups with the help of the theory developed in the previous lectures. In particular, we prove our p-adic superrigidity results for lattices in some groups of rank one.

First, let's review briefly the history of rigidity. We assume that (M, g) is complete (compact or non-compact with finite volume) locally symmetric space of non-compact type, i.e., in a neighborhood of any point, M is isometric to $\tilde{M}$, a simply-connected globally symmetric space with nonpositive sectional curvature. Note that the isometry group of $\tilde{M}$ is a semi-simple Lie group of non-compact type. A typical example of non-compact type locally symmetric space is hyperbolic space X (i.e., $K_X \equiv -1$, and consequently X is locally isometric to the hyperbolic disc $(\mathbb{H}^n, g_{\mathbb{H}})$). For M a given smooth manifold, we consider $\mathcal{M}_0 = \{g : (M, g)$ is locally symmetric$\}$ and define an equivalence relation in $\mathcal{M}_0$ by saying that, for g_1, $g_2 \in \mathcal{M}_0$, $g_1 \sim g_2$ if and only if there exist a diffeomorphism $F : M \to M$ such that $g_1 = F^* g_2$. We thus consider the moduli space $\mathcal{M}_0 / \operatorname{Diff}(M)$. For example, if $\dim M = 2$, then it is well-known that $\mathcal{M}_0 / \operatorname{Diff}(M)$ is a $(6g - 6)$ dimensional space where $g =$ genus of M.

The situation is different for locally symmetric manifolds of dimension larger than two whose universal cover is irreducible. In 1960 Calabi-Vesentini ([CV]) proved the local rigidity result (in the Kähler case) which says that there is no nontrivial curve in $\mathcal{M}_0$. In 1970, Mostow ([M]) proved the rigidity theorem (for the compact case) which says any two locally symmetric structures in $\mathcal{M}_0$ are equivalent. In particular, the moduli space $\mathcal{M}_0 / \operatorname{Diff}(M)$ is a point. In the 1970's, G. Marqulis ([Ma]) proved his celebrated "superrigidity" for lattices in groups of real rank at least two.

One analytic approach to prove Mostow rigidity theorem is to use harmonic map theory. In fact, in 1979, Y.T. Siu ([Siu]) used harmonic maps to prove rigidity in the Hermitian locally symmetric case; in the 1980's, Sampson ([Sa]) proved a vanishing theorem of harmonic maps from Kähler manifold into an arbitrary manifold with a certain negative curvature assummption.

Now, we formulate both Archimedian superrigidity and p-adic superrigidity in our setting. One can refer to [Ma] or [Zim] for details on superrigidity.

(i) Archimedian Case: Let M be locally symmetric, N be symmetric space of non-compact type. Denote the universal cover of M as $\tilde{M}$ so that $M = \tilde{M}/\Gamma$ with $\Gamma = \pi_1(M)$. Let $\rho : \Gamma \to \text{isom}(N)$ be a homomorphism. Consider a ρ-equivariant map $\varphi : \tilde{M} \to N$ i.e., $\varphi \circ \gamma = \rho(\gamma) \circ \varphi \ \forall \gamma \in \Gamma$. Then Archimedian superrigidity concerns whether there exists a totally geodesic ρ-equivariant map $u : \tilde{M} \to N$.

(ii) p-adic Case. Let M be as above. We replace N by an F-connected complex X such that $\text{isom}(X) = p$-adic Lie group. (Notice that there exists a procedure due to Bruhat-Tits (see [Br]) for starting with a p-adic Lie group G and constructing a building X such that $G \subseteq \text{isom}(X)$). Then p-adic superrigidity concerns whether there exists a constant ρ-equivariant map, i.e., whether $\rho(\Gamma)$ lies in the isotropy subgroup of a point.

Theorem 9.1 (Margulis). *If rank $(\tilde{M}) \geq 2$, then both superrigidity properties stated in (i) and (ii) are true.*

Note that rank $(\tilde{M})$ is the dimension of a maximal flat in $\tilde{M}$, i.e., a totally geodesic submanifold isometric to $\mathbf{R}^k$. Therefore, $\mathbf{H}^2$ has rank 1; $\mathbf{H}^2 \times \mathbf{H}^2$ has rank 2, $SL(n,R)/SO(n,R)$ has rank $n-1$. Margulis also showed that (i) and (ii) imply the arithmeticity of Γ (see [Ma] or [Zim]).

Margulis's result left questions unanswered for rank $(\widetilde{M}) = 1$. In fact, it is known that superrigidity fails for lattices in the isometry groups of real and complex hyperbolic space(see Introduction in [GS]). A result of K. Corlette ([C]) showed that Archimedian superrigidity (i) holds for quaternionic hyperbolic $\mathbf{H}^n_Q$ and Cayley hyperbolic spaces $\mathbf{H}^n_{\mathrm{Ca}}$. Here, we show that p-adic superrigidity (ii) holds for $\mathbf{H}^n_Q$ and $\mathbf{H}^n_{\mathrm{Ca}}$.

Theorem 9.2 (Gromov-Schoen). *In the case of $\widetilde{M} = \mathbf{H}^n_Q$ or $\mathbf{H}^n_{\mathrm{Ca}}$, Corlett's vanishing theorem can be derived to prove p-adic superrigidity (ii). Consequently the corresponding Γ are arithmetic.*

We will sketch the proof of this theorem in the remainder of this lecture. First, we need to prove the existence of a finite energy equivariant harmonic map. Let X be a Euclidean building associated to a p-adic Lie group $H(\equiv \text{isom}(X))$. Then X has a compactification $\overline{X} = X \cup \partial X$ such that any $h \in H$ acting isometrically on X extends as a homeomorphism to $\overline{X}$. Moreover, if $\{P_i\}$ is a sequence from X with $\lim_{i\to\infty} P_i = P \in \partial X$, and if $\{Q_i\}$ is another sequence from X with $d(P_i, Q_i) \leq c$ independent of i, then $\lim_i Q_i = P$. Finally, the isotropy group of $P \in \partial X$ is a proper algebraic subgroup of H (see [Br]).

Let M be a complete Riemannian manifold, $\widetilde{M}$ be its universal covering manifold such that $M = \widetilde{M}/\Gamma$. Suppose we have a homomorphism $\rho : \Gamma \to H$. Then we have the following existence result.

Theorem 9.3. *Suppose $\rho(\Gamma)$ is Zariski dense in H (i.e., $\rho(\Gamma)$ is not contained in a proper algebraic subgroup of H), and suppose there exists a Lipschitz ρ-equivariant map $\nu : \widetilde{M} \to X$ with finite energy. Then there is a Lipschitz equivariant map u of least energy and the restriction of u to a small ball about any point is minimizing.*

Proof. See Theorem 7.1 in [GS].

Using the result of our main theorem, we now prove the following extension of Corlette's vanishing theorem [C].

Theorem 9.4. *Let ω be a parallel p-form on $\widetilde{M}$, and assume that u is a finite energy equivariant harmonic map into an F-connected complex X. In a neighborhood of any regular point of u, the form $\omega \wedge du$ satisfies $d^*(\omega \wedge du) \equiv 0$.*

Proof. Suppose $x_0 \in \widetilde{M}$ is regular point for u. Then there exists $\delta_0 > 0$ such that $u(B_{\delta_0}(x_0)) \subseteq k$-flat F. The calculation of [C] then implies $dd^*(\omega \wedge du) \equiv 0$ in $B_{\delta_0}(x_0)$. Note that sets $\mathcal{R}(u)$ and $\mathcal{S}(u)$ are Γ-invariant, and we define $\mathcal{R}_0 = \mathcal{R}(u)/\Gamma$, $\mathcal{S}_0 = \mathcal{S}(u)/\Gamma$. We then have from Theorem 7.4 that $\dim \mathcal{S}_0 \leq n-2$, and for any compact subdomain $\Omega_1 \subseteq M$ there exists a sequence of nonnegative Lipschitz functions $\{\psi_i\}$ which vanish in a neighborhood of $\mathcal{S}_0 \cap \overline{\Omega}_1$ and tend to 1 on $M \setminus (\mathcal{S}_0 \cap \overline{\Omega}_1)$ such that

$$\lim_{i\to\infty} \int_M |\nabla\nabla u||\nabla\psi_i| d\mu = 0.$$

Let ρ be a nonnegative Lipschitz function which is one on $B_R(x_0)$ and zero outside $B_{2R}(x_0)$ with $|\nabla\rho| \leq 2R^{-1}$. We then apply Stoke's theorem on M using the identity $\psi_i\rho^2 \langle \omega \wedge du, dd^*(\omega \wedge du)\rangle = 0$. Thus we obtain

$$\int_M \psi_i\rho^2 \|d^*(\omega \wedge du)\|^2 d\mu = \pm \int_M \big\langle *d(\psi_i\rho^2) \wedge *(\omega \wedge du), d^*(\omega \wedge du)\big\rangle\, d\mu.$$

This implies

$$\int_M \psi_i\rho^2 \|d^*(\omega \wedge du)\|^2 du \leq c \int_M \left(\psi_i\rho(\nabla\rho| + |\nabla\psi_i|\rho^2\right) |\nabla u| \|d^*(\omega \wedge du)\|.$$

Using Young's inequality, we then have

$$\int_M \psi_i\rho^2 \|d^*(\omega \wedge du)\|^2 du \leq c \int_M \psi_i |\nabla\rho|^2 |\nabla u|^2 d\mu + c \int_M \rho^2 |\nabla\psi_i||\nabla u| d\mu |\nabla\nabla u|.$$

Therefore,

$$\int_{B_R(x_0)} \psi_i \|d^*(\omega \wedge du)\|^2 d\mu \leq cR^{-2}E(u) + e(R) \int_M |\nabla\psi_i||\nabla\nabla u| d\mu.$$

By first choosing R large and then taking i to infinity, we finally have that $d^*(\omega \wedge du) \equiv 0$ on $\mathcal{R}(u)$.

We now derive a consequence of this result.

Theorem 9.5. *A finite energy equivariant harmonic map from either $\mathbb{H}^n_Q$ or $\mathbb{H}^n_{Ca}$ into an F-connected complex is constant.*

See Theorem 7.4 in [GS] for the proofs.

Let $\widetilde{M}$ be $\mathbb{H}^n_Q$ (or $\mathbb{H}^n_{Ca}$), so that isom($\widetilde{M}$) is $(n, 1)$ (or F_4^{-20}). We then have

Lemma 9.6. *There exists a finite energy Lipschitz equivariant map.*

See Lemma 8.1 in [GS] for the proofs. Combining all these results will give Theorem 9.2.

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Singularities of Geometric Variational Problems

Leon Simon

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LECTURE 1
Basic Introductory Material

Introductory Remarks

These lectures are intended as a brief introduction, at graduate level, to the techniques (principally analytic and measure-theoretic) needed in the study of regularity and singularity of minimal surfaces and energy minimizing maps (sometimes loosely referred to as harmonic maps—see the discussion of terminology in 1.1 and 1.3 below).

Since it is technically simpler, we concentrate almost exclusively on energy minimizing maps, but the reader should keep in mind that essentially all the results discussed in these lectures have very close analogues for minimal surfaces.

The first 3 lectures are meant to be essentially self-contained, assuming no prior knowledge about harmonic maps; the main analytic tool used in the first 3 lectures is the Schoen-Uhlenbeck regularity theorem. We defer the proof of this until the second series of lectures (given in week 3 of the RGI). The last lecture in this present series touches on more recent work. This is too lengthy to be covered in any detail in the available time, but we do state the main results in a self-contained way, and prove a few things which give at least some hint of the kinds of techniques which are involved.

1.1 Definition of Energy Minimizing Map

Assume that Ω is an open subset of $\mathbf{R}^n$, $n \geq 2$, and that N is a smooth compact Riemannian manifold of dimension $p \geq 2$ which is isometrically embedded in some Euclidean space $\mathbf{R}^p$. We look at maps u of Ω into N; such a map will always be thought of as a map $u = (u^1, \ldots, u^p) : \Omega \to \mathbf{R}^p$ with the additional property that $u(\Omega) \subset N$. Consider such a map $u = (u^1, \ldots, u^p)$. We do not assume that u is

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smooth—in fact we make only the minimal assumption necessary to ensure that the energy of u is well-defined. Thus we assume only that $Du \in L^2_{\text{loc}}(\Omega)$, and then the energy $\mathcal{E}_{B_\rho(Y)}(u)$ of u in a ball $B_\rho(Y) \equiv \{X : |X - Y| < \rho\}$ with $\overline{B}_\rho(Y) \subset \Omega$ is defined by

$$\mathcal{E}_{B_\rho(Y)}(u) = \int_{B_\rho(Y)} |Du|^2.$$

Notice that here Du means the $n \times p$ matrix with entries $D_i u^j (\equiv \partial u^j / \partial x^i)$, and $|Du|^2 = \sum_{i=1}^n \sum_{j=1}^p (D_i u^j)^2$. We study maps which minimize energy in Ω in the sense that, for each ball $B_\rho(Y) \subset \Omega$,

$$\mathcal{E}_{B_\rho(Y)}(u) \leq \mathcal{E}_{B_\rho(Y)}(w),$$

for every $w : B_\rho(Y) \to \mathbf{R}^p$ with $Dw \in L^2(B_\rho(Y))$, with $w(B_\rho(Y)) \subset N$, and with $w \equiv u$ in a neighbourhood of $\partial B_\rho(Y)$. Such u will be called an energy minimizing map into N.

1.2 Definition of Regular and Singular Set

Given an energy minimizing map u in the sense of 1.1 above, the regular set $\operatorname{reg} u$ of u is defined simply as the set of points $Y \in \Omega$ such that u is smooth in some neighbouhood of Y; thus $\operatorname{reg} u$ is an open subset of Ω by definition.

The singular set $\operatorname{sing} u$ of u is then defined to be the complement of $\operatorname{reg} u$ in Ω. Thus

$$\operatorname{sing} u = \Omega \setminus \operatorname{reg} u,$$

and $\operatorname{sing} u$ is a closed subset of Ω.

1.3 The Variational Equations

Suppose u is energy minimizing as in 1.1, suppose $\overline{B}_\rho(Y) \subset \Omega$, and suppose that for some $\delta > 0$ we have a 1-parameter family $\{u_s\}_{s\in(-\delta,\delta)}$ of maps of $B_\rho(Y)$ into N such that $Du_s \in L^2(\Omega)$ and $u_s \equiv u$ in a neighbourhood of $\partial B_\rho(Y)$ for each $s \in (-\delta, \delta)$, and $u_0 = u$. Then by definition of minimizing we have $\mathcal{E}_{B_\rho(Y)}(u_s)$ takes a minimum at $s = 0$, and hence

$$* \qquad \frac{d\mathcal{E}(u_s)}{ds}\Big|_{s=0} = 0$$

whenever the derivative on the left exists. The derivative on the left is called the first variation of $\mathcal{E}_{B_\rho(Y)}$ relative to the given family; the family $\{u_s\}$ itself is called an (admissible) variation of u. There are two important kinds of variations of u:

Class 1: Variations of the form

$$\text{(i)} \qquad u_s = \Pi \circ (u + s\zeta),$$

where $\zeta = (\zeta^1, \ldots, \zeta^p)$ with each $\zeta^j \in C_c^\infty(B_\rho(Y))$ where Π is the nearest point projection onto N. (Here and subsequently $C_c^\infty(B_\rho(Y))$ denotes the C^∞ functions with compact support in $B_\rho(Y)$.) Notice that this nearest point projection onto N is well-defined and smooth in some tubular neighbourhood $\{x \in \mathbf{R}^p : \operatorname{dist}(X, N) < \sigma_0\}$ for some $\sigma_0 > 0$, and hence u_s defined in (i) is an admissible variation for $|s| < \sigma_0$. We recall the general facts that the induced linear map $d\Pi_Y$ gives orthogonal projection of $\mathbf{R}^p$ onto the tangent space of N at $Y \in N$, and the Hessian $\operatorname{Hess}\Pi_Y$ has the properties that $v_1 \cdot \operatorname{Hess}\Pi_Y(v_2, v_3)$ is a symmetric function of $v_1, v_2, v_3 \in \mathbf{R}^p$ and is related to the second fundamental form of N via the identity $v_1 \cdot \operatorname{Hess}\Pi_Y(v_2, v_3) = -\frac{1}{2}\sum v_{\sigma_1} \cdot A_Y(v_{\sigma_2}^T, v_{\sigma_3}^T)$, where the sum is over all permutations $\sigma_1, \sigma_2, \sigma_3$ of the integers 1,2,3 and where v^T means orthogonal projection onto the tangent space of N at Y. On the other hand by using a Taylor series expansion for Π it is straightforward to check that $D_i u_s = D_i u + s((D_i\zeta)^T + \zeta \cdot \operatorname{Hess}\Pi_u(D_i u, \cdot)) + O(s^2)$, where $(\ )^T$ means orthogonal projection into the tangent space at the image point $u(X)$, and hence for such a variation $*$ implies the integral identity

1.3(i)
$$\int_\Omega \sum_{i=1}^n (D_i u \cdot D_i\zeta - \zeta \cdot A_u(D_i u, D_i u) = 0$$

for any ζ as above. Notice that if u is C^2 we can integrate by parts here and use the fact that ζ is an arbitrary C^∞ function in order to deduce the equation

1.3(i)′
$$\Delta u + \sum_{i=1}^n A_u(D_i u, D_i u) = 0,$$

where Δu means simply $(\Delta u^1, \ldots, \Delta u^p)$. The identity 1.3(i) is called the weak form of the equation 1.3(i)′; of course if u is not C^2 the equation 1.3(i)′ makes no sense classically, and must be interpreted in the weak sense 1.3(i). It is worth noting (although we make no specific use of it here), that, in case $u \in C^2$, 1.3(i) says simply

$$(\Delta u)^T = 0$$

at a given point $X \in B_\rho(Y)$, where $(\Delta u)^T$ means orthogonal projection of $\Delta u(X)$ onto the tangent space $T_{u(X)}N$ of N at the image point $u(X)$.

Class 2: Variations of the form

$$u_s(X) = u(X + s\zeta(X)),$$

where $\zeta = (\zeta^1, \ldots, \zeta^n)$ with each $\zeta^j \in C_c^\infty(B_\rho(Y))$.

Then $D_i u_s(X) = \sum_{j=1}^n D_i u(X + s\zeta) + sD_i\zeta^j D_j u(X + s\zeta)$, and hence after making the change of variable $\xi = X + s\zeta$ (which gives a C^∞ diffeomorphism of

$B_\rho(Y)$ onto itself in case $|s|$ is small enough) in this case $*$ implies

$$\text{1.3(ii)} \qquad \int_{B_\rho(Y)} \sum_{i,j=1}^{n} (|Du|^2\delta_{ij} - 2D_iu\cdot D_ju)D_i\zeta^j = 0.$$

The identities 1.3(i), (ii) are of great importance in the study of energy minimizing maps. Notice that if $u \in C^2$ we can integrate by parts in 1.3(ii) in order to deduce that 1.3(i) implies 1.3(ii); it is however false that 1.3(i) implies 1.3(ii) in case Du is merely in L^2 (and there are simple examples to illustrate this). One calls a map u into N which satisfies 1.3(i) a "weakly harmonic map", while a map which satisfies both 1.3(i) and 1.3(ii) is usually referred to as a "stationary harmonic map". Thus the above discussion thus proves that energy minimizing implies stationary harmonic. We shall not here discuss weakly harmonic maps, but we do mention that such maps admit far worse singularities (see e.g. [RT1, 2]) than the energy minimizing maps . (Except in the case $n = 2$ when there are no singularities at all—we show this below in the case of minimizing maps, and refer to recent work of F. Hélein [HF] for the general case of weakly harmonic maps.)

1.4 The Monotonicity Formula

An important consequence of the variational identity 1.3(ii) is the "monotonicity identity"

$$\text{1.4(i)} \qquad \rho^{2-n}\int_{B_\rho(Y)} |Du|^2 - \sigma^{2-n}\int_{B_\sigma(Y)} |Du|^2 = 2\int_{B_\rho(Y)\setminus B_\sigma(Y)} R^{2-n}\left|\frac{\partial u}{\partial R}\right|^2,$$

valid for any $0 < \sigma < \rho < \rho_0$, provided $\overline{B}_{\rho_0}(Y) \subset \Omega$, where $R = |X - Y|$ and $\partial/\partial R$ means directional derivative in the radial direction $|X-Y|^{-1}(X-Y)$. Since it is a key tool in the study of energy minimizing maps, we give the proof of this identity.

Proof. First recall a general fact from analysis—Viz. if a_j are L^1 functions on $B_{\rho_0}(Y)$ and if $\int_{B_{\rho_0}(Y)}\sum_{j=1}^n a^jD_j\zeta = 0$ for each ζ which is C^∞ with compact support in $B_{\rho_0}(Y)$, then, for almost all $\rho \in (0,\rho_0)$, $\int_{B_\rho(Y)}\sum_{j=1}^n a_jD_j\zeta = \int_{\partial B_\rho(Y)}\eta\cdot a\zeta$ for any $\zeta \in C^\infty(\overline{B}_\rho(Y))$, where $a = (a^1,\ldots,a^n)$ and $\eta(\equiv \rho^{-1}(X-Y))$ is the outward pointing unit normal of $\partial B_\rho(Y)$. (This fact is easily checked by approximating the characteristic function of the ball $B_\rho(Y)$ by C^∞ functions with compact support.) Using this in the identity 1.3(ii), we obtain (for almost all $\rho \in (0,\rho_0)$) that

$$\int_{B_\rho(Y)} \sum_{i,j=1}^{n} (|Du|^2\delta_{ij} - 2D_iu\cdot D_ju)D_i\zeta^j =$$
$$\int_{\partial B_\rho(Y)} \sum_{i,j=1}^{n} (|Du|^2\delta_{ij} - 2D_iu\cdot D_ju)\rho^{-1}(X^i - Y^i)\zeta^j.$$

In this identity we choose $\zeta^j(X) \equiv X^j - Y^j$, so $D_i\zeta^j = \delta_{ij}$ and we obtain

$$(n-2)\int_{B_\rho(Y)} |Du|^2 = \rho^{-1}\int_{\partial B_\rho(Y)} (|Du|^2 - 2|\partial u/\partial R|^2).$$

Now by multiplying through by the factor ρ^{1-n} and noting that $\int_{\partial B_\rho} f = \frac{d}{d\rho}\int_{B_\rho} f$ for almost all ρ, we obtain the differential identity

$$\frac{d}{d\rho}\left(\rho^{2-n}\int_{B_\rho(Y)} |Du|^2\right) = 2\frac{d}{d\rho}\left(\int_{B_\rho(Y)} R^{2-n}\left|\frac{\partial u}{\partial R}\right|^2\right)$$

for almost all $\rho \in (0, \rho_0)$. Since $\int_{B_\rho} f$ is an absolutely continuous function of ρ (for any L^1-function f), we can now integrate to give the required monotonicity identity.

Notice that since the right side of 1.4(i) is non-negative, we have in particular that

1.4(ii) $$\rho^{2-n}\int_{B_\rho(Y)} |Du|^2 \text{ is an increasing function of } \rho \text{ for } \rho \in (0, \rho_0),$$

and hence that the limit as $\rho \to 0$ of $\rho^{2-n}\int_{B_\rho(Y)} |Du|^2$ exists.

1.5 The Density Function

We define the density function Θ_u of u on Ω by

1.5(i) $$\Theta_u(Y) = \lim_{\rho\downarrow 0} \rho^{2-n}\int_{B_\rho(Y)} |Du|^2.$$

(As we mentioned above, this limit always exists at each point of Ω for a minimizing map u.) We shall give a geometric interpretation of this below.

For the moment, notice that the density Θ_u is upper semi-continous on Ω; that is

1.5(ii) $$Y_j \to Y \in \Omega \Rightarrow \Theta_u(Y) \ge \limsup_{j\to\infty} \Theta_u(Y_j).$$

Proof. Let $\epsilon > 0, \rho > 0$ with $\rho + \epsilon < \operatorname{dist}(Y, \partial\Omega)$. By the monotonicity 1.4(ii) we have $\Theta_u(Y_j) \le \rho^{2-n}\int_{B_\rho(Y_j)} |Du|^2$ for j sufficiently large to ensure $\rho < \operatorname{dist}(Y_j, \partial\Omega)$. Since $B_\rho(Y_j) \subset B_{\rho+\epsilon}(Y)$ for all sufficiently large j, we then have the inequality $\Theta_u(Y_j) \le \rho^{2-n}\int_{B_{\rho+\epsilon}(Y)} |Du|^2$ for all sufficiently large j, so $\limsup_{j\to\infty} \Theta_u(Y_j) \le \rho^{2-n}\int_{B_{\rho+\epsilon}(Y)} |Du|^2$. Letting $\epsilon \downarrow 0$, we then conclude that $\limsup_{j\to\infty} \Theta_u(Y_j) \le \rho^{2-n}\int_{B_\rho(Y)} |Du|^2$, and the required inequality follows by taking the limit as $\rho \downarrow 0$.

1.6 The Regularity Theorem

Here we want to state the regularity theorem (due to Schoen-Uhlenbeck [SU]). This result is fundamental to the subsequent discussion in these lectures, but its proof is rather lengthy, so we defer discussion of that until our our later lectures (see [SL1]). The theorem is quite easy to state: it says (for any dimension $n \geq 2$) that if Λ is any given fixed positive constant, then there is $\epsilon > 0$, depending only on n, N, Λ, such that if u is an energy minimizing map into N as in 1.1, if $\overline{B}_\rho(Y) \subset \Omega$, if $\rho^{2-n} \int_{B_\rho(Y)} |Du|^2 \leq \Lambda$, and if

$$\min_{\lambda \in \mathbf{R}^p} \rho^{-n} \int_{B_\rho(Y)} |u - \lambda|^2 < \epsilon^2,$$

then $Y \in \operatorname{reg} u$, and furthermore for each $j \geq 0$

$$\sup_{B_{\rho/2}(Y)} \rho^j |D^j u| \leq C\epsilon,$$

where C depends only on n, N, j.

Thus, roughly speaking, and subject to a fixed bound on the energy of u on the ball $B_\rho(Y)$, the theorem says that if the mean square deviation of u away from some constant vector is sufficiently small then Y is a regular point of u and all the derivatives of u are controlled in the ball of radius $\rho/2$ and center Y.

We mention here that there are analogous theorems for various classes of minimal surfaces, due to De Giorgi, Reifenberg, Allard, Almgren, and Schoen-Simon. Discussion of these, and abundant references to related work, can be found for example in the references [DeG], [R], [AW], [A], [G], [SS].

1.7 Corollaries of the Regularity Theorem

Corollary 1. *There exists $\epsilon > 0$, depending only on n, N such that if $B_\rho(Y) \subset \Omega$ and if $\rho^{2-n} \int_{B_\rho(Y)} |Du|^2 < \epsilon$, then $Y \in \operatorname{reg} u$ and $\sup_{B_{\rho/2}(Y)} \rho^j |D^j u| \leq C$ for each $j \geq 1$, where C depends only on j, n, N.*

Proof. The Poincaré inequality for arbitrary functions f on $B_\rho(Y)$ with $Df \in L^2$ asserts that $\inf_\lambda \int_{B_\rho(Y)} |f - \lambda|^2 \leq C\rho^2 \int_{B_\rho(Y)} |Du|^2$, where C depends only on the dimension n. (See e.g. [GT] for a proof.) Since this is trivially also valid for $\mathbf{R}^p$-valued functions $f = (f^1, \ldots, f^p)$ we can apply it to u, thus giving $\inf_{\lambda \in \mathbf{R}^p} \rho^{-n} \int_{B_\rho(Y)} |u - \lambda|^2 \leq C\rho^{2-n} \int_{B_\rho(Y)} |Du|^2 \leq C\epsilon$, and hence Corollary 1 is a direct consequence of the regularity theorem 1.7.
A second corollary gives a nice characterization of the regular set:

Corollary 2. $\Theta_u(Y) = 0 \iff Y \in \operatorname{reg} u$.

Proof. "$\Leftarrow$" follows trivially from the fact that u smooth near Y implies $|Du|$ is bounded near Y, while "$\Rightarrow$" follows directly from Corollary 1.

The third corollary shows that the singular set of the energy-minimizing map u is actually quite small:

Corollary 3. $\mathcal{H}^{n-2}(\operatorname{sing} u) = 0$. *(In particular* $\operatorname{sing} u = \emptyset$ *in case* $n = 2$.*)*

Remark. Here $\mathcal{H}^j$ is the j-dimensional Hausdorff measure; recall that $\mathcal{H}^j(A) = 0$, for a given subset $A \subset \mathbf{R}^n$, means that for each $\eta > 0$ there is a covering of A by a countable collection $B_{\rho_i}(Y_i)$ of balls such that $\sum_i \rho_i^j < \eta$.

Proof of Corollary 3. Let K be a compact subset of Ω, $\delta_0 \in (0, \operatorname{dist}(K, \partial\Omega))$ and $\delta \in (0, \delta_0)$. For $Y \in \operatorname{sing} u \cap K$ we know by Corollary 2 that

$$\int_{B_\rho(Y)} |Du|^2 \geq \epsilon \rho^{n-2} \tag{1}$$

for all $\rho < \delta$. Choose a maximal pairwise disjoint collection $\{B_{\delta/2}(Y_j)\}_{j=1\ldots Q}$ of balls with $Y_j \in K \cap \operatorname{sing} u$. (Here maximal means that we take such a cover with maximum Q.) Then the balls $B_\delta(Y_j)$ cover all of $K \cap \operatorname{sing} u$ and by summing over j in (1) with Y_j in place of Y and $\delta/2$ in place of ρ we get

$$Q(\delta/2)^{n-2} \leq \epsilon^{-1} \int_{\cup B_{\delta/2}(Y_j)} |Du|^2 \leq \epsilon^{-1} \int_{(K \cap \operatorname{sing} u)_\delta} |Du|^2, \tag{2}$$

where $(K \cap \operatorname{sing} u)_\delta = \{X : \operatorname{dist}(X, K \cap \operatorname{sing} u) < \delta\}$. This evidently implies that

$$Q\omega_n \delta^n \leq 2^n \omega_n \delta^2 \epsilon^{-1} \int_{(K \cap \operatorname{sing} u)_{\delta_0}} |Du|^2,$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. Since $B_\delta(Y_j)$ cover all of $\operatorname{sing} u \cap K$, and since we can let $\delta \downarrow 0$, this last inequality shows that $\operatorname{sing} u \cap K$ has Lebesgue measure zero. But then $\int_{(\operatorname{sing} u \cap K)_\delta} |Du|^2 \to 0$ as $\delta \downarrow 0$ by the dominated convergence theorem, and hence (2) implies that $\mathcal{H}^{n-2}(\operatorname{sing} u \cap K) = 0$. Since K was an arbitrary compact subset of Ω, this completes the proof of Corollary 3.

There is also a nice compactness theorem for energy minimizing maps, as follows:

Lemma 1. *If* $\{u_j\}$ *is a sequence of energy minimizing maps from* Ω *into* N *with* $\sup_j \int_{B_\rho(Y)} |Du_j|^2 < \infty$ *for each ball* $B_\rho(Y)$ *with* $\overline{B}_\rho(Y) \subset \Omega$, *then there is a subsequence* $\{u_{j'}\}$ *and a minimizing harmonic map* u *of* Ω *into* N *such that* $u_{j'}$, $Du_{j'}$ *converge in* L^2 *locally on* Ω *to* u, Du *respectively.*

Remarks. **(1)** In particular the energy $\int_{B_\rho(Y)} |Du_{j'}|^2$ converges to $\int_{B_\rho(Y)} |Du|^2$ for each ball $\overline{B}_\rho(Y) \subset \Omega$.

(2) Notice that, by the Rellich compactness theorem for sequences of functions with gradient bounded in the L^2 norm (see e.g. [GT]), there is an $L^2_{\text{loc}}(\Omega)$ function u with $Du \in L^2_{\text{loc}}(\Omega)$ such that $u_{j'}$ converges in L^2 to u on compact subsets of Ω and

$Du_{j'}$ converges locally weakly in L^2 to Du in Ω. Of course then u maps into N (in the sense that $u(X) \in N$ a.e. $X \in \Omega$) because a subsequence of the subsequence $u_{j'}$ converges pointwise a.e. to u. Thus the content of Lemma 1 is that $Du_{j'}$ converges in L^2 and that u is minimizing.

To give the proof of these facts we shall need the following lemma, which is a consequence of the main result of Luckhaus [Luck1]. (The fact that $Du_{j'}$ converges in L^2 locally on Ω is originally due to Schoen-Uhlenbeck, who used the regularity theorem to establish it. This approach however does not establish the fact that u is energy minimizing; this was not proved in full generality until the paper [Luck1].)

Lemma 2. *Let $\Lambda > 0$ be given. There are constants $\epsilon_0 = \epsilon_0(n, N, \Lambda) > 0$ and $C = C(n, N, \Lambda)$ such that the following holds for any $\rho > 0$, $\epsilon \in (0, 1]$:*
If $\overline{B}_{(1+\epsilon)\rho}(Y) \subset \Omega$, $u, v : B_{(1+\epsilon)\rho}(Y) \backslash B_\rho(Y) \to \mathbf{R}^p$, if u, v have L^2 gradients Du, Dv with $\rho^{2-n} \int_{B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)}(|Du|^2 + |Dv|^2) \le \Lambda$, $u(X), v(X) \in N$ for $X \in B_{(1+\epsilon)\rho}(Y))\backslash B_\rho(Y)$, and if $\epsilon^{-2n}\rho^{-n} \int_{B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)} |u - v|^2 < \epsilon_0^2$, then there is w on $B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)$ such that $w = u$ in a neighbourhood of $\partial B_\rho(Y)$, $w = v$ in a neighbourhood of $\partial B_{(1+\epsilon)\rho}(Y)$, $w(B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)) \subset N$, and

$$\int_{B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)} |Dw|^2 \le$$
$$C \int_{B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)} (|Du|^2 + |Dv|^2) + C\epsilon^{-2} \int_{B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)} |u - v|^2.$$

We shall give the proof of Lemma 2 in the second series of lectures—see Lecture 3 of [SL1].

Proof of Lemma 1. Suppose $\overline{B}_{\rho_0}(Y) \subset \Omega$ and $u = \lim_j u_{j'}$ in the L^2 norm on $B_{\rho_0}(Y)$ as in the Remark (2) above. Also let $\delta \in (0, \frac{1}{4})$ and $\theta \in (\frac{1}{2}, 1)$ be given. (We shall eventually choose δ small and θ close to 1.) The main step in the proof (and the step which uses Lemma 2) is to prove that we can find $\rho \in (\theta\rho_0, \rho_0)$, $\epsilon \in (0, \rho_0/\rho - 1)$, a subsequence $\tilde{j} \subset \{j'\}$ and a sequence $w_j : B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y) \to N$ with L^2 gradients such that $w_j = u$ on $\partial B_\rho(Y)$ $w_j = u_{\tilde{j}}$ on $\partial B_{(1+\epsilon)\rho}(Y)$ and

$$\limsup_{j\to\infty} \int_{B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)} |Dw_j|^2 \le C\delta, \quad C = C(n, N, \Lambda). \tag{1}$$

We proceed to prove this. (Once we have (1) the remainder of the proof is quite straightforward.)

First choose Λ such that $\sup_j \rho_0^{2-n} \int_{B_{\rho_0}(Y)} |Du_j|^2 \le \Lambda$ and note that if $\epsilon \in (0, \delta(1-\theta)/2(1+\Lambda))$ then there must be an integer $\ell \in (2, 2(1+\Lambda)/\delta)$ such that

$$\int_{B_{(\theta+\ell\epsilon)\rho_0}(Y)\backslash B_{(\theta+(\ell-1)\epsilon)\rho_0}(Y)} |Du_j|^2 \le \delta \tag{2}$$

for infinitely many j, since otherwise by summation of the reverse inequality over integers $\ell \in (2, 2(1+\Lambda)/\delta)$ we would get $\int_{B_{\rho_0}(Y)} |Du_j|^2 > \Lambda$ for sufficiently large j, thus contradicting the definition of Λ. Thus select such an ℓ and let $\rho = (\theta + (\ell - 1)\epsilon)\rho_0$. Since $(\theta + \ell\epsilon)\rho_0 = \rho + \epsilon\rho_0 \geq (1+\epsilon)\rho$, we then have from (2)

$$\rho^{2-n} \int_{B_{(1+\epsilon)\rho}(Y)\setminus B_\rho(Y)} |Du_{\tilde{j}}|^2 \leq \delta \tag{3}$$

for some subsequence $\{\tilde{j}\}$ of $\{j'\}$, and of course then also (by lower semicontinuity of $\int_{B_{(1+\epsilon)\rho}(Y)\setminus B_\rho(Y)} |Du_{\tilde{j}}|^2$ with respect to weak convergence)

$$\int_{B_{(1+\epsilon)\rho}(Y)\setminus B_\rho(Y)} |Du|^2 \leq \delta. \tag{4}$$

Since $\int_{B_{(1+\epsilon)\rho}(Y)\setminus B_\rho(Y)} |u - u_{\tilde{j}}|^2 \to 0$, we can then apply Lemma 2 with $\widetilde{u}_{\tilde{j}}$ in place of v to deduce that there are $w_j : B_{(1+\epsilon)\rho}(Y)\setminus B_\rho(Y) \to \mathbf{R}^p$ such that $w_j(B_{(1+\epsilon)\rho}(Y)\setminus B_\rho(Y)) \subset N$, $w_j|\partial B_{(1+\epsilon)\rho}(Y) = u_{\tilde{j}}$, $w_j|\partial B_\rho(Y) = u$, and

$$\begin{aligned} \rho^{2-n} \int_{B_{(1+\epsilon)\rho}\setminus B_\rho(Y)} |Dw_j|^2 &\leq C\rho^{2-n} \int_{B_{(1+\epsilon)\rho}(Y)\setminus B_\rho(Y)} (|Du|^2 + |Du_{\tilde{j}}|^2 \\ &\qquad + \epsilon^{-2}\rho^{-2} \int_{B_{(1+\epsilon)\rho}(Y)\setminus B_\rho(Y)} |u - u_{\tilde{j}}|^2) \\ &\leq C\delta \end{aligned}$$

for sufficiently large j by (3) and (4), thus establishing (1) as claimed.

Now let $v : B_{\rho_0}(Y) \to \mathbf{R}^p$ with $v(B_{\rho_0}(Y)) \subset N$, $Dv \in L^2$, and $v = u$ in a neighbourhood of $\partial B_{\rho_0}(Y)$. Assuming θ is close enough to 1 to ensure that $v = u$ in $B_{\rho_0}(Y)\setminus B_{\theta\rho_0}(Y)$ we then let $\widetilde{u}_j$ be defined by

$$\widetilde{u}_j = \begin{cases} u_{\tilde{j}} \text{ on } B_{\rho_0}(Y)\setminus B_{(1+\epsilon)\rho}(Y) \\ w_j \text{ on } B_{(1+\epsilon)\rho}(Y)\setminus B_\rho(Y) \\ v \text{ on } B_\rho(Y). \end{cases} \tag{4}$$

Then by (1) and the minimizing property of u_j we have

$$\begin{aligned} &\int_{B_{\rho_0}(Y)} |Du_{\tilde{j}}|^2 \leq \int_{B_{\rho_0}(Y)} |D\tilde{u}_j|^2 \equiv \\ &\int_{B_{\rho_0}(Y)\setminus B_{(1+\epsilon)\rho}(Y)} |Du_{\tilde{j}}|^2 + \int_{B_{(1+\epsilon)\rho}(Y)\setminus B_\rho(Y)} |Dw_j|^2 + \int_{B_\rho(Y)} |Dv|^2, \end{aligned} \tag{5}$$

and hence

$$\liminf_{j\to\infty}\int_{B_{(1+\epsilon)\rho}(Y)}|Du_{\tilde{j}}|^2 \le \int_{B_\rho(Y)}|Dv|^2 + C\delta, \quad C = C(n,N,\Lambda). \tag{7}$$

Since $\delta > 0$ was arbitrary and $\int_{B_{(1+\epsilon)\rho}(Y)}|Du|^2 \le \liminf_{j\to\infty}\int_{B_{(1+\epsilon)\rho}(Y)}|Du_{\tilde{j}}|^2$, this shows that u

$$\int_{B_{\theta\rho_0}(Y)}|Du|^2 \le \int_{B_{\rho_0}(Y)}|Dv|^2,$$

so that (letting $\theta \uparrow 1$) we conclude that u is energy minimizing on $B_{\rho_0}(Y)$.

Finally to prove that the convergence is strong we note that if we use (7) with $v = u$ and (1) then we have

$$\liminf_{j\to\infty}\int_{B_\rho(Y)}|Du_{\tilde{j}}|^2 \le \int_{B_\rho(Y)}|Du|^2 + C\delta.$$

Thus using the weak convergence of $Du_{\tilde{j}}$ to Du we have

$$\liminf\int_{B_\rho(Y)}|Du_{\tilde{j}} - Du|^2 \equiv \liminf\int_{B_\rho(Y)}(|Du_{\tilde{j}}|^2 - |Du|^2) \le C\delta.$$

The subsequence $\{\tilde{j}\}$ depends on δ, but in any case (by a diagonal process) we can prove that this last inequality implies that there is a new subsequence $\tilde{j}$ such that $Du_{\tilde{j}}$ converges in L^2 to Du on $B_{\theta\rho_0}(Y)$. In view of the arbitrariness of θ and ρ_0 this gives the desired conclusion.

1.8 A Further Remark on Upper Semicontinuity of the Density

Notice that if we use the result of Lemma 1 above, a very minor modification of the argument used in 1.5 to prove the upper-semicontinuity of $\Theta_u(Y)$ as a function of Y shows that $\Theta_u(Y)$ is actually upper-semicontinuous with respect to the joint variables u, Y in the sense that if $Y_j \to Y \in \Omega$ and if u_j are a sequence of energy minimizing maps from Ω into N with locally bounded energy in the sense of Lemma 1 converging in L^2 locally to u, then $\Theta_u(Y) \ge \limsup_{j\to\infty}\Theta_{u_j}(Y_j)$.

LECTURE 2
Tangent Maps

Here u continues to denote an energy minimizing map of Ω into N, with Ω an open subset of $\mathbf{R}^n$.

2.1 Definition of Tangent Map

Take any ball $B_{\rho_0}(Y)$ with $\overline{B}_{\rho_0}(Y) \subset \Omega$ and for any $\rho > 0$ consider the scaled function $u_{Y,\rho}$ defined by

$$u_{Y,\rho}(X) = u(Y + \rho X).$$

Notice that $u_{Y,\rho}$ is well-defined on the ball $B_{\rho_0/\rho}(0)$; furthermore, if $\sigma > 0$ is arbitrary and $\rho < \rho_0/\sigma$, we have (using $Du_{Y,\rho}(X) = \rho(Du)(Y + \rho X)$, and making a change of variable $\widetilde{X} = Y + \rho X$ in the energy integral for $u_{Y,\rho}$)

$$\text{2.1(i)} \qquad \sigma^{2-n} \int_{B_\sigma(0)} |Du_{Y,\rho}|^2 = (\sigma\rho)^{2-n} \int_{B_{\sigma\rho}(Y)} |Du|^2 \le \rho_0^{2-n} \int_{B_{\rho_0}(Y)} |Du|^2,$$

where in the last inequality we used the monotonicity 1.4(i) from Lecture 1. Thus if $\rho_j \downarrow 0$ then $\limsup_{j\to\infty} \int_{B_\sigma(0)} |Du_{Y,\rho_j}|^2 < \infty$ for each $\sigma > 0$, and hence by the compactness theorem (Lemma 1 of 1.7) there is a subsequence $\rho_{j'}$ such that $u_{Y,\rho_{j'}} \to \varphi$ locally in $\mathbf{R}^n$ both with respect to the L^2-norm and in energy, where $\varphi : \mathbf{R}^n \to N$ is a minimizing harmonic map (in the sense of 1.1 with $\Omega = \mathbf{R}^n$). Any φ which is obtained in this way is called a <u>tangent</u> <u>map</u> <u>of</u> u <u>at</u> Y. In general it is <u>not</u> true that such tangent maps need be unique (see [WB])—that is, if we choose different sequences ρ_j (or different subsequences $\rho_{j'}$) then we may get a different limit map. We discuss this further below.

2.2 Key Properties of Tangent Maps

First note that by the equation 2.1(i) above, with $\rho_{j'}$ in place of ρ (keeping in mind that $u_{Y,\rho_{j'}}$ converges in energy to φ) we have, after taking limits on each side of 2.1(i) as $j \to \infty$,

$$\sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2 = \Theta_u(Y),$$

where we used the definition $\Theta_u(Y) = \lim_{\rho\downarrow 0}\rho^{2-n}\int_{B_\rho(Y)}|Du|^2$. Thus in particular $\sigma^{2-n}\int_{B_\sigma(0)}|D\varphi|^2$ is a constant function of σ, and since by definition $\Theta_\varphi(0) = \lim_{\sigma\downarrow 0}\sigma^{2-n}\int_{B_\sigma(0)}|D\varphi|^2$, we thus have

2.2(i) $$\Theta_u(Y) = \Theta_\varphi(0) \equiv \sigma^{2-n}\int_{B_\sigma(0)}|D\varphi|^2 \quad \forall\sigma > 0.$$

Thus any tangent map of u at Y has scaled energy constant and equal to the density of u at Y; this is also a nice interpretation of the density of u at Y.

Furthermore if we apply the monotonicity formula 1.4(i) to φ then we get the identity

$$0 = \sigma^{2-n}\int_{B_\sigma(0)}|D\varphi|^2 - \tau^{2-n}\int_{B_\tau(0)}|D\varphi|^2 = \int_{B_\sigma(0)\setminus B_\tau(0)} R^{2-n}\left|\frac{\partial\varphi}{\partial R}\right|^2,$$

so that $\partial\varphi/\partial R = 0$ a.e., and since φ has L^2 gradient it is correct to conclude from this, by integration along rays, that

2.2(ii) $$\varphi(\lambda X) \equiv \varphi(X) \quad \forall\lambda > 0,\ X \in \mathbf{R}^n.$$

This is a key property of tangent maps, and enables us to use the further properties of homogeneous degree zero minimizers. (See 2.3 below.)

We conclude this section with another nice characterization of the regular set of u:

2.2(iii) $Y \in \operatorname{reg} u \iff \exists$ a <u>constant</u> tangent map φ of u at Y

(and in this case of course the tangent map is unique). To prove 2.2(iii), note that by Corollary 2 of 1.7 we have $Y \in \operatorname{reg} u \iff \Theta_u(Y) = 0$, but $\Theta_u(Y) = 0 \iff \varphi \equiv$ const. by 2.2(i).

2.3 Properties of Homogeneous Degree Zero Minimizers

Suppose $\varphi : \mathbf{R}^n \to N$ is a homogeneous degree zero minimizer (e.g. a tangent map of u at some point Y); thus $\varphi(\lambda X) \equiv \varphi(X)$ for all $\lambda > 0$, $X \in \mathbf{R}^n$.

We first observe that the density $\Theta_\varphi(Y)$ is maximum at $Y = 0$; in fact by the monotonicity formula 1.4(i), for each $\rho > 0$ and each $Y \in \mathbf{R}^n$,

$$2\int_{B_\rho(Y)} R_Y^{2-n}\left|\frac{\partial\varphi}{\partial R_Y}\right|^2 + \Theta_\varphi(Y) = \rho^{2-n}\int_{B_\rho(Y)}|D\varphi|^2,$$

where $R_Y(X) \equiv |X - Y|$ and $\partial/\partial R_Y = |X - Y|^{-1}(X - Y) \cdot D$. Now $B_\rho(Y) \subset B_{\rho+|Y|}(0)$, so that

$$\begin{aligned}\rho^{2-n} \int_{B_\rho(Y)} |D\varphi|^2 &\le \rho^{2-n} \int_{B_{\rho+|Y|}(0)} |D\varphi|^2 \\ &= (1 + \frac{|Y|}{\rho})^{n-2}((\rho + |Y|)^{2-n} \int_{B_{\rho+|Y|}(0)} |D\varphi|^2) \equiv (1 + \frac{|Y|}{\rho})^{n-2}\Theta_\varphi(0),\end{aligned}$$

because φ is homogeneous of degree zero (which guarantees that $\tau^{2-n} \int_{B_\tau(0)} |D\varphi|^2 \equiv \Theta_\varphi(0)$). Thus letting $\rho \uparrow \infty$, we get

$$2 \int_{\mathbf{R}^n} R_Y^{2-n} \left|\frac{\partial \varphi}{\partial R_Y}\right|^2 + \Theta_\varphi(Y) \le \Theta_\varphi(0),$$

which establishes the required inequality

2.3(i) $$\Theta_\varphi(Y) \le \Theta_\varphi(0).$$

Notice also that this shows that <u>equality</u> can hold in 2.3(i) only if $\partial\varphi/\partial R_Y = 0$ a.e., that is only if $\varphi(Y + \lambda X) \equiv \varphi(Y + X)$ for each $\lambda > 0$. Since we also have (by assumption) $\varphi(X) \equiv \varphi(\lambda X)$ we can then compute for any $\lambda > 0$ and $X \in \mathbf{R}^n$ that

$$\begin{aligned}\varphi(X) = \varphi(\lambda X) &= \varphi(Y + (\lambda X - Y)) = \varphi(Y + \lambda^{-2}(\lambda X - Y)) \\ &= \varphi(\lambda(Y + \lambda^{-2}(\lambda X - Y))) = \varphi(X + tY),\end{aligned}$$

where $t = \lambda - \lambda^{-1}$ is an arbitrary real number. So let $S(\varphi)$ be defined by

2.3(ii) $$S(\varphi) = \{Y \in \mathbf{R}^n : \Theta_\varphi(Y) = \Theta_\varphi(0)\}.$$

Then we have shown that $\varphi(X) \equiv \varphi(X + tY)$ for all $X \in \mathbf{R}^n$, $t \in \mathbf{R}$, and $Y \in S(\varphi)$. Combining this with the fact that if $Z \in \mathbf{R}^n$ and $\varphi(X + Z) \equiv \varphi(X)$ for all $X \in \mathbf{R}^n$, then $\Theta_\varphi(Z) = \Theta_\varphi(0)$ (and hence then $Z \in S(\varphi)$ by definition of $S(\varphi)$), we conclude

$$S(\varphi) \text{ is a linear subspace of } \mathbf{R}^n \text{ and } \varphi(X + Y) \equiv \varphi(X),\ X \in \mathbf{R}^n,\ Y \in S(\varphi).$$

(Thus φ is invariant under composition with translation by elements of $S(\varphi)$.) Notice of course that

2.3(iii) $$\dim S(\varphi) = n \iff \varphi = \text{const}.$$

Also a homogeneous degree zero map which is not constant clearly cannot be continuous at 0, so we always have $0 \in \operatorname{sing}\varphi$ if φ is non-constant, and hence, since $\varphi(X + Z) \equiv \varphi(X)$ for any $Z \in S(\varphi)$, we have

2.3(iv) $$S(\varphi) \subset \operatorname{sing}\varphi$$

for any non-constant homogeneous degree zero minimizer φ.

2.4 Further Properties of sing u

For any $Y \in \Omega$ and any tangent map φ of u at Y we shall let $S(\varphi)$ be the linear subspace of points Y such that $\Theta_\varphi(Y) = \Theta_\varphi(0)$, as discussed in the previous section. Notice that then by 2.2(iii) we have

2.4(i) $\quad Y \in \operatorname{sing} u \iff \dim S(\varphi) \le n-1$ for every tangent map φ of u at Y.

Now for each $j = 0, 1, \ldots, n-1$ we define

$$\mathcal{S}_j = \{Y \in \operatorname{sing} u : \dim S(\varphi) \le j \text{ for } \underline{\text{all}} \text{ tangent maps } \varphi \text{ of } u \text{ at } Y\}.$$

Then we have

2.4(ii) $$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{n-3} = \mathcal{S}_{n-2} = \mathcal{S}_{n-1} = \operatorname{sing} u.$$

To see this first note that $\mathcal{S}_{n-1} = \operatorname{sing} u$ is just 2.4(i), and the inclusion $\mathcal{S}_{j-1} \subset \mathcal{S}_j$ is true by definition. Also, if $\mathcal{S}_{n-3}$ is not equal to both $\mathcal{S}_{n-2}$ and $\mathcal{S}_{n-1}$, then we can find $Y \in \operatorname{sing} u$ at which there is a tangent map φ with $n-2 \le \dim S(\varphi) \le n-1$; but then $\mathcal{H}^{n-2}(S(\varphi)) = \infty$ and hence (since $S(\varphi) \subset \operatorname{sing}\varphi$ by 2.3(iv)) we have $\mathcal{H}^{n-2}(\operatorname{sing}\varphi) = \infty$, contradicting the fact that $\mathcal{H}^{n-2}(\operatorname{sing}\varphi) = 0$ by Corollary 3 of 1.7.

The subsets $\mathcal{S}_j$ are mainly important because of the following lemma, due to F. Almgren [A]; the lemma can be thought of as a refinement of the "dimension reducing" argument of Federer (for this see the discussion in the appendix of [SL3]):

Lemma 1. *For each $j = 0, \ldots, n-3$, $\dim \mathcal{S}_j \le j$, and, for each $\alpha > 0$, $\mathcal{S}_0 \cap \{X : \Theta_u(X) = \alpha\}$ is a discrete set.*

Remark. Here "dim" means Hausdorff dimension; thus $\dim \mathcal{S}_j \le j$ means simply that $\mathcal{H}^{j+\epsilon}(\mathcal{S}_j) = 0$ for each $\epsilon > 0$.

Before we give the proof of this lemma, we note the following corollary.

Corollary. $\dim \operatorname{sing} u \le n-3$.

Proof. By 2.4(ii), $\operatorname{sing} u = \mathcal{S}_{n-3}$, hence the lemma with $j = n-3$ gives precisely $\dim \operatorname{sing} u \le n-3$ as claimed.

Proof of Lemma 1. We first prove that $\mathcal{S}_0 \cap \{X : \Theta_u(X) = \alpha\}$ is a discrete set for each $\alpha > 0$. Suppose this fails for some $\alpha > 0$. Then there are Y, $Y_j \in \mathcal{S}_0 \cap \{X : \Theta_u(X) = \alpha\}$ such that $Y_j \ne Y$ for each j, and $Y_j \to Y$. Let $\rho_j = |Y_j - Y|$ and consider the scaled maps u_{Y,ρ_j}. By the discussion of 2.1 there is a subsequence $\rho_{j'}$ such that $u_{Y,\rho_{j'}} \to \varphi$, where φ is (by definition) a tangent map of u at Y; also, by 2.2 we have $\Theta_\varphi(0) = \Theta_u(Y) = \alpha$.

Let $\xi_j = |Y_j - Y|^{-1}(Y_j - Y)(\in S^{n-1})$. We can suppose that the subsequence j' is such that $\xi_{j'}$ converges to some $\xi \in S^{n-1}$. Also (since the transformation

$X \mapsto Y + \rho_j X$ takes Y_j to ξ_j) $\Theta_u(Y_j) = \Theta_{u_{Y,\rho_j}}(\xi_j) = \alpha$ for each j, hence by the upper semi-continuity of the density (as in 1.8) we have $\Theta_\varphi(\xi) \geq \alpha$. Thus since $\Theta_\varphi(X)$ has maximum value at 0 (by 2.3(i)), we have $\Theta_\varphi(\xi) = \Theta_\varphi(0) = \alpha$, and hence $\xi \in S(\varphi)$, contradicting the fact that $S(\varphi) = \{0\}$ by virtue of the assumption that $Y \in \mathcal{S}_0$.

Before we give the proof of the the fact that $\dim \mathcal{S}_j \leq j$, we need a preliminary lemma, which is of some independent interest. In this lemma and subsequently we use $\eta_{Y,\rho}$ to be the map of $\mathbf{R}^n$ which translates Y to the origin and homotheties by the factor ρ^{-1}; thus

$$\eta_{Y,\rho}(X) = \rho^{-1}(X - Y).$$

Lemma 2. *For each $Y \in \mathcal{S}_j$, and each $\delta > 0$ there is an $\epsilon > 0$ (depending on u, Y, δ) such that for each $\rho \in (0, \epsilon]$*

$$\eta_{Y,\rho}\{X \in B_\rho(Y) : \Theta_u(X) \geq \Theta_u(Y) - \epsilon\} \subset \textit{ the } \delta\textit{-neighbourhood of } L_{Y,\rho}$$

for some j-dimensional subspace $L_{Y,\rho}$ of $\mathbf{R}^n$.

Proof. If this is false, then there exists $\delta > 0$ and $Y \in \mathcal{S}_j$ and sequences $\rho_k \downarrow 0$, $\epsilon_k \downarrow 0$ such that

$$\text{(1)} \qquad \{X \in B_1(0) : \Theta_{u_{Y,\rho_k}}(X) \geq \Theta_u(Y) - \epsilon_k\} \not\subset \text{ the } \delta\text{-neighbourhood of } L$$

for every j-dimensional subspace L of $\mathbf{R}^n$. But $u_{Y,\rho_{k'}} \to \varphi$, a tangent map of u at Y, and $\Theta_u(Y) = \Theta_\varphi(0)$. Since $Y \in \mathcal{S}_j$, we have $\dim S(\varphi) \leq j$, so (since $S(\varphi)$ is the set of points where Θ_φ takes its maximum value $\Theta_\varphi(0)$), there is a j-dimensional subspace $L_0 \supset S(\varphi)$ ($L_0 = S(\varphi)$ in case $\dim S(\varphi) = j$) and an $\alpha > 0$ such that

$$\text{(2)} \qquad \Theta_\varphi(X) < \Theta_\varphi(0) - \alpha \quad \text{for all } X \in \overline{B}_1(0) \text{ with } \operatorname{dist}\{X, L_0\} \geq \delta.$$

Then we must have, for all sufficiently large k', that

$$\text{(3)} \qquad \Theta_{u_{Y,\rho_{k'}}}(X) < \Theta_\varphi(0) - \alpha \quad \forall X \in B_1(0) \text{ with } \operatorname{dist}\{X, L_0\} \geq \delta.$$

Because otherwise we would have a subsequence $\{\tilde{k}\} \subset \{k'\}$ with $\Theta_{u_{Y,\rho_{\tilde{k}}}}(X_{\tilde{k}}) \geq \Theta_\varphi(0) - \alpha$ for some sequence $X_{\tilde{k}} \in B_1(0)$ with $\operatorname{dist}\{X_{\tilde{k}}, L_0\} \geq \delta$. Taking another subsequence if necessary and using the upper semi-continuity result of 1.8, we get $X_{\tilde{k}} \to X$ with $\Theta_\varphi(X) \geq \Theta_\varphi(0) - \alpha$, contradicting (2).

Thus (3) is established. But (3) says precisely that, for all sufficiently large k',

$$\{X \in B_1(0) : \Theta_{u_{Y,\rho_{k'}}}(X) \geq \Theta_\varphi(0) - \alpha\} \subset \text{ the } \delta\text{-neighbourhood of } L_0,$$

thus contradicting (1).

Completion of the proof of Lemma 1: We decompose $\mathcal{S}_j$ into subsets $\mathcal{S}_{j,i}$, $i \in \{1, 2, \dots\}$, defined to be the set of points Y in $\mathcal{S}_j$ such that the conclusion of

Lemma 2 above holds with $\epsilon = i^{-1}$. Then, by Lemma 2, $\mathcal{S}_j = \cup_{i\geq 1}\mathcal{S}_{j,i}$. Next, for each integer $q \geq 1$ we let

$$\mathcal{S}_{j,i,q} = \{X \in \mathcal{S}_{j,i} \,:\, \Theta_u(X) \in (\tfrac{q-1}{i}, \tfrac{q}{i}]\},$$

and note that $\mathcal{S}_j = \cup_{i,q}\mathcal{S}_{j,i,q}$. For any $Y \in \mathcal{S}_{j,i,q}$ we have trivially that

$$\mathcal{S}_{j,i,q} \subset \{X \,:\, \Theta_u(X) > \Theta_u(Y) - 1/i\},$$

and hence, by Lemma 2 (with $\epsilon = i^{-1}$), for each $\rho \leq i^{-1}$

$$\eta_{Y,\rho}(\mathcal{S}_{j,i,q} \cap B_\rho(Y)) \subset \text{ the } \delta\text{-neighbourhood of } L_{Y,\rho}$$

for some j-dimensional subspace $L_{Y,\rho}$ of $\mathbf{R}^n$.

Thus each of the sets $A = \mathcal{S}_{j,i,q}$ has the "δ-approximation property" that there is ρ_0 ($= i^{-1}$ in the present case) such that, for each $Y \in A$ and for each $\rho \in (0, \rho_0]$,

$$* \qquad \eta_{Y,\rho}(A \cap B_\rho(Y)) \subset \text{ the } \delta\text{-neighbourhood of } L_{Y,\rho}$$

for some j-dimensional subspace $L_{Y,\rho}$ of $\mathbf{R}^n$.

In view of the arbitrariness of δ the proof is now completed by virtue of the following lemma:

Lemma 3. *There is a function $\beta : (0,\infty) \to (0,\infty)$ with $\lim_{t\downarrow 0}\beta(t) = 0$ such that if $\delta > 0$ and if A is an arbitrary subset of $\mathbf{R}^n$ having the property $*$ above, then $H^{j+\beta(\delta)}(A) = 0$.*

This lemma is quite easy to prove, using the fact that there is a fixed constant C_n such that for each $\sigma \in (0,1)$ we can cover the closed unit ball $\overline{B}_1(0)$ of $\mathbf{R}^j$ with a finite collection of balls $\{B_\sigma(Y_k)\}_{k=1,\ldots,Q}$ in $\mathbf{R}^j$ with radii σ and centers $Y_k \in \overline{B}_1(0)$ such that $Q\sigma^j < C_n$. In view of the arbitrariness of σ, it then follows that for each $\beta > 0$ we can find $\sigma = \sigma(\beta) \in (0,1)$ such that there is a cover of $\overline{B}_1(0)$ by balls $\{B_\sigma(Y_k)\}_{k=1,\ldots,\tilde{Q}}$ such that $\tilde{Q}\sigma^{j+\beta} < \frac{1}{2}$. More generally, if L is any j-dimensional subspace of $\mathbf{R}^n$ and $\delta \in (0, 1/8)$ there is $\beta(\delta)$ (depending only on n, δ) with $\beta(\delta) \downarrow 0$ as $\delta \downarrow 0$ such that the 2δ-neighbourhood of $L \cap \overline{B}_1(0)$ can be covered by balls $B_\sigma(Y_k)$, $k = 1, \ldots, \tilde{Q}$ with centers in $L \cap B_1(0)$ and with $\tilde{Q}\sigma^{j+\beta(\delta)} < \frac{1}{2}$. By scaling this means that for each $R > 0$ a $2\delta R$-neighbourhood of $L \cap B_R(0)$ can be covered by balls $B_{\sigma R}(Y_k)$ with centers $Y_k \in L \cap \overline{B}_R(0)$, $k = 1, \ldots, \tilde{Q}$ such that $\tilde{Q}(\sigma R)^{j+\beta(\delta)} < \frac{1}{2}R^{j+\beta(\delta)}$. The above lemma follows easily from this general fact by using successively finer covers of A by balls. The details are as follows: Supposing without loss of generality that A is bounded, we first take an intial cover of A by balls $B_{\rho_0/2}(Y_k)$ with $A \cap B_{\rho_0/2}(Y_k) \neq \emptyset$, $k = 1, \ldots, Q$, and let $T_0 = Q(\rho_0/2)^{j+\beta(\delta)}$. For each k pick $Z_k \in A \cap B_{\rho_0/2}(Y_k)$. Then by $*$ with $\rho = \rho_0$ there is a j-dimensional affine space L_k such that $A \cap B_{\rho_0}(Z_k)$ is contained in the δ-neighbourhood of L_k. Notice that $L_k \cap B_{\rho_0/2}(Y_k)$ is a j-disk of radius $< \rho_0/2$, and so by the above discussion we can cover its $\delta\rho_0$-neighbourhood by balls $B_{\sigma\rho_0/2}(Z_{j,\ell})$, $\ell = 1, \ldots, P$, such that $P(\sigma\rho_0/2)^{j+\beta(\delta)} \leq \frac{1}{2}(\rho_0/2)^{j+\beta(\delta)}$. Thus A can be covered by balls $B_{\sigma\rho_0/2}(W_\ell)$, $k = 1, \dots, M$, such that $M(\sigma\rho_0/2)^{j+\beta(\delta)} \le \frac{1}{2}T_0$. Proceeding iteratively we can thus for each q find a cover by balls $B_{\sigma^q\rho_0/2}(W_k)$, $k = 1, \dots, R_q$, such that $R_q(\sigma^q\rho_0/2)^{j+\beta(\delta)} \le 2^{-q}T_0$.

LECTURE 3
The Top-Dimensional Part of Sing u

Here u continues to denote an energy minimizing map from $\Omega \subset \mathbf{R}^n$ into $N \subset \mathbf{R}^p$; the discussion is mainly only relevant when there are actually genuine "$(n-3)$-dimensional parts" of singular set in the sense that there are points $Y \in \operatorname{sing} u$ at which there are tangent maps φ with $\dim S(\varphi) = n-3$. But the reader should keep in mind that all the discussion here carries over with an integer $m \le n-4$ in place of $n-3$ if the target manifold N happens to be such that all tangent maps φ of energy minimizing maps into N have $\dim S(\varphi) = m \le n-4$ (one such case is in fact mentioned later in this lecture, when $\dim N = 2$ and N has genus ≥ 1). Since the discussion is essentially identical in this case, there is no conceptual loss of generality in adopting the definition of top dimensional part in the following section.

3.1 Definition of Top-dimensional Part of the Singular Set

We define the top dimensional part $\operatorname{sing}_* u$ to be the set of points $Y \in \operatorname{sing} u$ such that <u>some</u> tangent map φ of u at Y has $\dim S(\varphi) = n-3$.

Notice that then by definition we have $\operatorname{sing} u \backslash \operatorname{sing}_* u \subset \mathcal{S}_{n-4}$, and hence by Lemma 1 of the last lecture we have

3.1(i) $$\dim(\operatorname{sing} u \backslash \operatorname{sing}_* u) \le n-4.$$

To study $\operatorname{sing}_* u$ further, we first examine the properties of homogeneous degree zero minimizers $\varphi : \mathbf{R}^n \to N$ with $\dim S(\varphi) = n-3$.

3.2 Homogeneous Degree Zero φ with $\dim S(\varphi) = n-3$

Let $\varphi : \mathbf{R}^n \to N$ be any homogeneous degree zero minimizer with $\dim S(\varphi) = n-3$. Then, modulo a rotation of the X-variables which takes $S(\varphi)$ to $\{0\} \times \mathbf{R}^{n-3}$, we have

3.2(i) $$\varphi(x,y) \equiv \varphi_0(x),$$

where we use the notation $X = (x, y)$, $x \in \mathbf{R}^3$, $y \in \mathbf{R}^{n-3}$, and where φ_0 is a homogeneous degree zero map from $\mathbf{R}^3$ into N. We in fact claim that

3.2(ii) $$\operatorname{sing}\varphi_0 = \{0\} \quad \text{and hence } \varphi_0|S^2 \in C^\infty,$$

so that $\varphi_0|S^2$ is a smooth harmonic map of S^2 into N. To see this, first note that $\operatorname{sing}\varphi_0 \supset \{0\}$, otherwise φ_0, and hence φ, would be constant, thus contradicting the hypothesis $\dim S(\varphi) = n-3$. On the other hand if $\xi \neq 0$ with $\xi \in \operatorname{sing}\varphi_0$, then by homogeneity of φ_0 we would have $\{\lambda\xi : \lambda > 0\} \subset \operatorname{sing}\varphi_0$, and hence

$$\{(\lambda\xi, y) : \lambda > 0,\ y \in \mathbf{R}^{n-3}\} \subset \operatorname{sing}\varphi.$$

But the left side here is a half-space of dimension $(n-2)$, and hence this would give $\mathcal{H}^{n-2}(\operatorname{sing}\varphi) = \infty$, thus contradicting the fact that $\mathcal{H}^{n-2}(\operatorname{sing}\varphi) = 0$ by Corollary 3 of Lecture 1. Thus 3.2(ii) is established.

We also note that if $\varphi^{(j)}$ is any sequence of homogeneous degree zero minimizers with $\varphi^{(j)}(x, y) \equiv \varphi_0^{(j)}(x)$ for each j, and if $\limsup_{j\to\infty} \int_{B_1(0)} |D\varphi^{(j)}|^2 < \infty$, then

$$\limsup_{j\to\infty} \sup_{S^2} |D^\ell \varphi_0^{(j)}| < \infty$$

for each $\ell \geq 0$. Indeed this follows easily from the compactness theorem (Lemma 1 of 1.7) and from the fact that all singular sets of minimizers have dimension $\leq n-3$; the details are left as an exercise. It then follows that for a particular φ with $\int_{S^2} |D\varphi_0|^2 \leq \Lambda$, we have

3.2(iii) $$\sup_{S^2} |D^\ell \varphi_0| \leq C,$$

where C depends only on ℓ, N, Λ.

Finally we note the important fact that the set of densities of such homogeneous degree zero minimizers φ with $\dim S(\varphi) = n-3$ is a discrete subset of $(0, \infty)$ if either the target manifold N is real-analytic or if $\dim N = 2$ or if a certain integrability condition (see 3.2(vi) below) holds. We first check the case $\dim N = 2$. In this case we use the fact that all non-constant smooth harmonic maps ψ from S^2 (into any target) are conformal (except at finitely many points where $|D\psi| = 0$). (See e.g. [JJ].) Then for such a ψ, we have $\frac{1}{2}|D\psi|^2 \equiv +J$ or $\frac{1}{2}|D\psi|^2 \equiv -J$, where J is the (signed) Jacobian. Therefore by the area formula (see e.g. [SL3, §8]) the energy of ψ is exactly $2d(\psi)\operatorname{area} N$, where $d(\psi)$ is the topological degree of $\psi : S^2 \to N$. Since this all applies to $\varphi_0(x)$, where $\varphi(x, y) \equiv \varphi_0(x)$ is any tangent map of u with $\dim S(\varphi) = n-3$, this implies that the set of densities $\Theta_\varphi(0)$ of such φ forms a discrete set; but this set of densities is exactly the set of densities $\{\Theta_u(Y) : Y \in \operatorname{sing}_* u\}$ by definition of $\operatorname{sing}_* u$. This completes the discussion for the case $\dim N = 2$.

For $\dim N \geq 3$ and N real analytic we can also establish the discreteness as follows. Let $\mathcal{M}(\psi)$ by the harmonic map operator on S^2. Thus $\mathcal{M}(\psi) = \Delta\psi + \sum_{i=1}^3 A_\psi(\nabla_i\psi, \nabla_i\psi)$, where $\Delta\psi = (\Delta\psi^1, \ldots, \Delta\psi^p)$ and $\nabla_i\psi = (\nabla_i\psi^1, \ldots, \nabla_i\psi^p)$, with Δ denoting the Laplacian on S^2, and with $\nabla_i f$ such that $(\nabla_1 f, \nabla_2 f, \nabla_3 f) =$ the S^2-gradient of f for any C^1 scalar function f on S^2; thus ψ is a smooth solution of $\mathcal{M}(\psi) = 0$ on S^2 if and only if the homogeneous degree zero extension of ψ to $\mathbf{R}^3\setminus\{0\}$ is a smooth solution of the variational equation 1.3(i) of Lecture 1, and in fact $\mathcal{M}(\psi) = 0$ is exactly the variational equation (analogous to 1.3(i)) corresponding to the the energy functional $\mathcal{E}_{S^2}(\psi) = \int_{S^2} |\nabla\psi|^2$, where $|\nabla\psi|^2 = \sum_{i=1}^3 |\nabla_i\psi|^2$.

Now let ψ_0 be any smooth solution of $\mathcal{M}(\psi) = 0$ on S^2 and $v \in \mathcal{T}$, where $\mathcal{T}$ denotes set of smooth $\mathbf{R}^p$-valued function on S^2 with the property $v(x) \in T_{\psi_0(x)}N$ for each $x \in S^2$. (Thus $\mathcal{T}$ is a the set of smooth sections of the pull-back by ψ_0 of the tangent bundle of N.)

The linearized operator $\mathcal{L}_{\psi_0}v$ of $\mathcal{M}(\psi)$ at $\psi = \psi_0$ is defined by

$$\mathcal{L}_{\psi_0}(v) = \frac{d}{ds}\mathcal{M}(\psi_s))|_{s=0},$$

where ψ_s is any 1-parameter family of smooth maps of S^2 into N with $\psi_s(x)$ varying smoothly in the joint variables $(x, s) \in S^2 \times (-\epsilon, \epsilon)$ for some $\epsilon > 0$, and with $v = \frac{d}{ds}\psi_s|_{s=0}$. (Of course such a family always exists for any given v and the derivative on the right is independent of which particular family is used, so long as $\frac{d}{ds}\psi_s|_{s=0} = v$.) The linearized operator $\mathcal{L}_{\psi_0}$ always has non-trivial kernel; in fact if ψ_s is any 1-parameter family of smooth harmonic maps of S^2 into N (that is, $\mathcal{M}(\psi_s) = 0 \ \forall s$) with $\psi_s(x)$ varying smoothly in the joint variables $(x, s) \in S^2 \times (-\epsilon, \epsilon)$ for some $\epsilon > 0$, and if $v = \frac{d}{ds}\psi_s|_{s=0}$, then v is automatically such a solution, by definition of $\mathcal{L}_{\psi_0}$. In particular by using $\psi_s = e^{sA}\psi_0$, where A is any skew-symmetric transformation of $\mathbf{R}^3$, we get a linear space of solutions spanned by the special solutions

3.2(iv) $$v(x) \equiv x^i D_j\psi_0(x) - x^j D_j\psi_0(x), \quad i, j = 1, 2, 3$$

(In computing $D_i\psi_0$ we assume ψ_0 is extended as homogeneous degree zero to $\mathbf{R}^3\setminus\{0\}$.) Similarly by considering the homotheties of S^2 (which are conformal and hence preserve harmonicity), we get a family generated by the special solutions

3.2(v) $$D_i\psi_0(x), \quad i = 1, 2, 3.$$

Now if K denotes the L^2 projection of $\mathcal{T}$ onto the kernel of $\mathcal{L}_{\psi_0}$, then the operator $\mathcal{N}(v) = \mathcal{M}(\Pi(\psi_0 + v)) + K(v)$, for $v \in \mathcal{T}$ with $|v| < \delta$ where $\delta > 0$ is small enough to ensure that the nearest point projection Π onto N is smooth in the δ-neighbourhood of N, then the linearization of $\mathcal{N}$ at 0 is just $\mathcal{L}_{\psi_0} + K$, which has trivial kernel. Then using the implicit function theorem (applied on the appropriate Hölder spaces—we refer to the the discussion of [SL2, pp.537–540] for the details) together with the fact that, by smoothness of N, there is $\delta_1 \in (0, \delta)$ such that any smooth map $\psi : S^2 \to N$ with $|\psi - \psi_0| < \delta_1$ can be uniquely represented in the form $\psi = \Pi(\psi_0 + v)$ for $v \in \mathcal{T}$, we can find a real analytic embedding Ψ of a ball $B_\sigma(0) \subset \ker \mathcal{L}_{\psi_0}$ into $\mathcal{T}$ such that all solutions of $\mathcal{M}(\psi) = 0$ are contained in

$\Psi(B_\sigma(0))$, and the set of such solutions is precisely Ψ applied to the set of points $\xi \in B_\sigma(0)$ such that $\nabla f(\xi) = 0$, where f is the real analytic function on $B_\sigma(0)$ defined by $f(\psi) \equiv \mathcal{E}_{S^2}(\Psi(\psi))$ for $\psi \in B_\sigma(0)$. (In fact Ψ is just the restriction to $B_\sigma(0)$ of the local inverse $\mathcal{N}^{-1}$, which exists by the inverse function theorem.) Then since real analytic functions f have the property that f is constant on the connected components of the set where $\nabla f = 0$, and since there is $\sigma' \leq \sigma$ such that at most one connected component of the set of points where $\nabla f = 0$ intersects the ball $B_{\sigma'}(0)$, we deduce that the set of energies of smooth harmonic maps $\psi : S^2 \to N$ is discrete.

We actually note here that the above construction of the embedding Ψ and the map f makes sense in the smooth case, and shows that the solutions near ψ_0 are contained in the manifold $\Psi(B_\sigma(0))$ which has dimension equal to the dimension of the kernel $\ker \mathcal{L}_{\psi_0}$. We shall want to refer to the following integrability condition:

3.2(vi) $$f \equiv \text{const.}$$

Notice that by the above discussion this is equivalent to saying that <u>all</u> of the manifold $\Psi(B_\sigma(0))$ corresponds to solutions of $\mathcal{M}(\Pi(\psi_0 + \psi)) = 0$ and the energy $\mathcal{E}$ is constant on $\Psi(B_\sigma(0))$. That is, it is equivalent to the requirement that there is $\sigma > 0$ such that the set of solutions of $\mathcal{M}(v) = 0$ with $|v - \psi_0|_{C^3} < \sigma$ forms a manifold of dimension equal to the dimension of $\ker \mathcal{L}_{\psi_0}$. (This explains why 3.2(vi) is called an integrability condition.) Notice that by the Schauder theory for elliptic equations the condition $|v - \psi_0|_{C^3} < \sigma$ is equivalent to the condition $|v - \psi_0|_{C^2} < \sigma$ modulo a fixed multiplicative constant.

Using the definition of $\operatorname{sing}_* u$, the above discussion in particular implies

3.2(vii) $$\{\Theta_u(Y) : Y \in \operatorname{sing}_* u\} \text{ is discrete}$$

whenever either $\dim N = 2$ or N is real-analytic or when the integrability condition 3.2(vi) holds.

3.3 The Geometric Picture Near Points of $\operatorname{sing}_* u$

Let K be a compact subset of Ω and $Y \in \operatorname{sing}_* u \cap K$ and let φ be a tangent map of u at Y with $\dim S(\varphi) = n - 3$. As in 3.2, we can assume without loss of generality (after making an orthogonal transformation of the X variables which takes $S(\varphi)$ to $\{0\} \times \mathbf{R}^{n-3}$), that

3.3(i) $$\varphi(x, y) \equiv \varphi_0(x), \qquad x \in \mathbf{R}^3,\ y \in \mathbf{R}^{n-3}.$$

By definition of $\operatorname{sing}_* u$, there is a sequence $\rho_j \downarrow 0$ such that

3.3(ii) $$\lim_{j\to\infty} \rho_j^{-n} \int_{B_{\rho_j}(Y)} |u - \varphi|^2 = 0,$$

so for $\rho = \rho_j$ with j sufficiently large we can make the scaled L^2-norm $\rho^{-n}\int_{B_\rho(Y)}|u-\varphi|^2$ as small as we wish. On the other hand we claim that for any homogeneous degree zero minimizing maps $\varphi : \mathbf{R}^n \to N$ as in 3.3(i) and any ball $B_{\rho_0}(Y)$ with $\overline{B}_{\rho_0}(Y) \subset \Omega$ we have the estimate

3.3(iii) $\operatorname{sing} u \cap B_{\rho/2}(Y) \subset \{X : \operatorname{dist}(X, (Y + \{0\} \times \mathbf{R}^{n-3})) < \delta(\rho)\rho\} \quad \forall \rho \le \rho_0,$

$$\delta(\rho) = C\Big(\rho^{-n}\int_{B_\rho}|u-\varphi|^2\Big)^{1/n},$$

where C depends only on n, N, Λ with Λ any upper bound for $\rho_0^{2-n}\int_{B_{\rho_0}(Y)}|Du|^2$. In view of 3.3(ii), this perhaps suggests that the possibility that the top dimensional part of the singular set is contained in a C^1 manifold (or at least a Lipschitz manifold) of dimension $n-3$. But there is a problem in that 3.3(ii) only guarantees that $\delta(\rho)$ is small when ρ is proportionally close to one of the ρ_j, and, without further input, we cannot conclude very much about the structure of $\operatorname{sing}_* u$ from this—see the discussion in 3.4 below.

We conclude this section with the simple proof of 3.3(iii). We assume $Y = 0$.

Proof of 3.3(iii). Let $\rho < \rho_0$ and $Z = (\xi, \eta) \in \operatorname{sing}_* u \cap B_{\rho/2}(0)$. Take $\sigma = \beta_0|\xi|$, with $\beta_0 \le \frac{1}{2}$ to be chosen. By the Schoen-Uhlenbeck regularity theorem there is $\epsilon_0 = \epsilon_0(n, N, \Lambda) > 0$ such that

$$\begin{aligned} \epsilon_0 &\le \sigma^{-n}\int_{B_\sigma(Z)}|u-\varphi(Z)|^2 \\ &\le 2\sigma^{-n}\int_{B_\sigma(Z)}|u-\varphi|^2 + 2\sigma^{-n}\int_{B_\sigma(Z)}|\varphi-\varphi(Z)|^2. \end{aligned} \tag{1}$$

By virtue of 3.2(iii) we know that $|D\varphi_0(x)| \le C|x|^{-1}$, where C depends only on N, Λ, and hence $|\varphi(X) - \varphi(Z)| \le C|\xi|^{-1}\sigma \le C\beta_0$ for $X \in B_\sigma(Z)$, where C depends only on N, Λ. Then (1) gives

$$\epsilon_0 \le 2\beta_0^{-n}|\xi|^{-n}\int_{B_\rho(0)}|u-\varphi|^2 + C\beta_0^2.$$

Then selecting $C\beta_0^2 \le \frac{1}{2}\epsilon_0$ and multiplying through by $|\xi|^n$, we have

$$|\xi|^n \le C\Big(\rho^{-n}\int_{B_\rho(0)}|u-\varphi|^2\Big)\rho^n,$$

with C depending only on n, N, Λ. Taking n^{th} roots of each side, we then get the required inequality.

3.4 Consequences of Uniqueness of Tangent Maps

We want to show here that the geometric picture established in the previous section does give good information about the structure of the top dimensional part $\text{sing}_* u$ if the tangent map at each point $Y \in \text{sing}_* u$ is unique; because then for each $\delta > 0$ and each $Y \in \text{sing}_* u$ there is φ as above, an orthogonal transformation Q of $\mathbf{R}^n$ and a $\rho_{Y,\delta} > 0$ such that 3.3(iii) holds for all $\rho \le \rho_{Y,\delta}$, with Q independent of ρ. We claim that such a property implies that $\text{sing}_* u$ is contained in a countable union of $(n-3)$-dimensional Lipschitz graphs: To be precise, we could apply the case $j = n-3$ of the following lemma:

Lemma. *Let $j \in \{1, \dots, n-1\}$. Suppose $\delta \in (0, \frac{1}{2}]$ and A is a subset of $\mathbf{R}^n$ such that at each point $Y \in A$ there is a j-dimensional subspace L_Y of $\mathbf{R}^n$ and $\rho_Y > 0$ such that*

$$* \qquad A \cap B_\rho(Y) \subset \{X : \text{dist}(A \cap B_\rho(Y), Y + L_Y)) \le \delta\rho\} \qquad \forall \rho \le \rho_Y,$$

then $A \subset \cup_{i=1}^\infty \Sigma_i$, where each Σ_i is the finite union of graphs of Lipschitz functions, each the graph over some j-dimensional subspace (in the sense that there is an open subset U_i of some j-dimensional subspace $L_i \subset \mathbf{R}^n$ and an $L_i^\perp$-valued Lipschitz function f_i on U_i such that $\Sigma_i = \{X + f_i(X) : X \in U_i\}$).

Remark. In standard terminology, this says that A is countably j-rectifiable.

Proof. We decompose $A = \cup_{i=1}^\infty A_i$, where $A_{i+1} \subset A_i$ is the set of points $Y \in A$ such that $*$ holds with $\rho_Y = i^{-1}$. Notice that then $*$ holds for all $Y \in A_i$ with $\rho_Y \equiv i^{-1}$, and A_i satisfies a uniform cone condition—in the sense that

$$A_i \cap B_{i^{-1}}(Y) \subset K_Y, \qquad \forall Y \in A_i,$$

where K_Y is the cone given by $K_Y = \{X : \text{dist}(X, Y+L_Y) < \delta|X-Y|\}$. Now select j-dimensional subspaces $L_1, \dots, L_Q$ of $\mathbf{R}^n$ such that for each j-dimensional subspace $L \subset \mathbf{R}^n$ there is one of the L_j such that $\|L_j - L\| \equiv \sup_{x \in L \cap S^{n-1}} \text{dist}(x, L_j \cap S^{n-1}) < (1-\delta)/2$. Then we can decompose $A_i = \cup_{j=1}^Q A_{i,j}$, where

$$A_{i,j} = \{Y \in A_i : \|L_Y - L_j\| < (1-\delta)/2\}.$$

Then each $A_{i,j}$ has the uniform cone property that

$$A_{i,j} \cap B_{i^{-1}}(Y) \subset Y + K_j, \qquad \forall\, Y \in A_{i,j},$$

where $K_j = \{X : \text{dist}(X, L_j) < (1+\delta)|X|/2\}$. It is standard that such a uniform cone condition implies that, for each given $Y \in A_{i,j}$, $A_{i,j} \cap B_{i^{-1}}(Y)$ is contained the graph of a Lipschitz function with domain $B_{i^{-1}}(Y') \cap L_j$, where Y' is the orthogonal projection of Y on L_j. (See e.g. [SL3,§5].) The lemma is thus proved.

Notice also that the argument above actually shows that the following is true:

Corollary. *If A satisfies the same hypotheses as in the lemma, except that we can choose $\rho_Y \equiv \rho_0$, with $\rho_0 > 0$ independent of Y, then $A \cap B_{\rho_0}(Y)$ is contained in the finite union of j-dimensional Lipschitz graphs for each $Y \in A$.*

Notice that, by 3.3(iii), such a uniform choice of ρ_Y can be made for $\operatorname{sing}_* u \cap K$, K any compact subset of Ω, provided that for each $\delta > 0$ and each compact $K \subset \Omega$, there is $\rho(\delta, K) \in (0, \operatorname{dist}(K, \partial\Omega))$ and a homogeneous degree zero $\varphi_Y : \mathbf{R}^n \to N$ with $\operatorname{sing} \varphi_Y = L_Y$ (= some $(n-3)$-dimensional linear subspace) such that

$$** \qquad \rho^{-n} \int_{B_\rho(Y)} |u - \varphi_Y|^2 < \delta, \quad \rho < \rho(\delta, K), \; Y \in \operatorname{sing}_* u \cap K.$$

In fact by 3.3(iii) if ** holds with suitably small $\delta = \delta(n, N, K, \Lambda) > 0$ (Λ is any upper bound for $\sup d^{2-n} \int_{B_d(Y)} |Du|^2$ over all $Y \in K$ with $d \in (0, \operatorname{dist}(K, \partial\Omega))$), then we have the hypotheses of the above corollary with $\delta =$ (for example) $\frac{1}{2}$. Notice also that in this case it follows automatically from **, 3.3(iii), and the triangle inequality that the hypotheses of the corollary hold not only with ρ_Y independent of Y by also with L_Y independent of Y, and hence we deduce the following:

Corollary. *There is $\delta = \delta(n, N, K, \Lambda)$ such that if for each $Y \in \operatorname{sing}_* u \cap K$ there is φ_Y such that ** holds, then there is $\rho > 0$ such that $\operatorname{sing}_* u \cap B_\rho(Y)$ is contained in an $(n-3)$-dimensional Lipschitz graph for each $Y \in \operatorname{sing}_* u \cap K$.*

We see in the next lecture that there are stronger conditions on the L^2-norm which guarantee much stronger results in certain cases.

LECTURE 4
Recent Results Concerning sing u

Recall that in the previous lecture we identified a "top dimensional part" $\operatorname{sing}_* u$ of the singular set $\operatorname{sing} u$. Here we further refine this set by defining

$$\operatorname{sing}_\alpha u = \{Y \in \operatorname{sing}_* u : \Theta_u(Y) = \alpha\}.$$

Notice that by 3.2(vii) of the previous lecture we know that if either $\dim N = 2$ or if N is real-analytic then for each compact $K \subset \Omega$ there is is a finite set $\mathcal{F} = \{\alpha_1, \dots, \alpha_Q\}$ such that

$$* \qquad K \cap \operatorname{sing}_\alpha u = \emptyset, \quad \alpha \notin \mathcal{F}.$$

Thus

$$\operatorname{sing} u \cap K = (\cup_{j=1}^{Q} \operatorname{sing}_{\alpha_j} u) \cup \Gamma_0,$$

with $\dim \Gamma_0 \le n - 4$; our aim is to give conditions which ensure that each $\operatorname{sing}_\alpha u$ has closure which is $(n-3)$-rectifiable in a neighbourhood of each of its points.

4.1 Statement of Main Known Results

If $\dim N = 2$, we can show that each of the sets closure $\operatorname{sing}_\alpha u$ is locally $(n-3)$-rectifiable (i.e. countably $(n-3)$-rectifiable with locally finite $(n-3)$-dimensional Hausdorff measure (so that for each $Y \in$ closure $\operatorname{sing}_\alpha u$ there is $\sigma > 0$ such that $\mathcal{H}^{n-3}(B_\sigma(Y) \cap \text{closure} \operatorname{sing}_\alpha u) < \infty$). As a matter of fact, a bit more can be proved:

Theorem 1. *If* $\dim N = 2$, *then* $\operatorname{sing} u$ *is countably* $(n-3)$*-rectifiable, and, for each* $\alpha > 0$, S_α *is* $(n-3)$*-rectifiable in a neighbourhood of each of its points, where* $S_\alpha = \{X \in \operatorname{sing} u : \Theta_u(X) \ge \alpha\} (\supset \text{closure } \operatorname{sing}_\alpha u)$. *($S_\alpha$ is closed by upper semicontinuity 1.5(ii).)*
Furthermore, if $N = S^2$ *with its standard metric, or if* N *is* S^2 *with a metric which is sufficiently close to the standard metric of* S^2 *in the* C^3 *sense, then* $\operatorname{sing} u$ *can be written as the disjoint union of a properly embedded* $(n-3)$*-dimensional* $C^{1,\mu}$*-manifold and a closed set* S *with* $\dim S \le n - 4$. *If* $n = 4$ *then* S *is discrete and*

the $C^{1,\mu}$ curves making up the rest of the singular set have locally finite length in compact subsets of Ω.

Remark. In case $n = 4$ and $N = S^2$, Hardt & Lin [HL] have proved, by different methods than those to be described in these lectures, that the singular set is a union of $C^{0,\alpha}$ arcs with endpoints forming a discrete set.

In case $\dim N \geq 3$ we unfortunately can only get information about the part of the set $\text{sing}_* u$ consisting of points $Y \in \text{sing}_* u$ such that all tangent maps φ of u at Y with $\dim S(\varphi) = n - 3$ have the following "integrability property", in which we assume that we have made an orthogonal transformation of the X variables to ensure, as in 3.2(i), that $\varphi(x, y) = \varphi_0(x)$. Then we require:

‡ the condition 3.2(vi) holds with φ_0 in place of ψ_0.

Unfortunately this integrability condition is not always satisfied in case $\dim N \geq 3$, so the following theorem in general fails to establish rectifiability of the entire singular set, even if N is real analytic.[2]

Theorem 2. *Suppose* $\dim N \geq 3$ *and suppose that the integrability property ‡ holds for all tangent maps φ of u with* $\dim S(\varphi) = n - 3$. *Then* $\text{sing}\, u$ *is countably $(n-3)$-rectifiable, and the set $S_\alpha = \{X : \Theta_u(X) \geq \alpha\}$ has finite measure in a neighbourhood of each of its points for each $\alpha > 0$.*

We want to give some brief indications of the kinds of techniques which are needed to prove such results. Without explaining the terminology (for which we refer to [SL4]), we want to mention here again that there are analogous results (with analogous techniques of proof) for minimal surfaces. For example, the following theorem about mod 2 minimizing surfaces is proved in [SL4]:

Theorem. *If M is an n-dimensional mod 2 minimizing current in an open subset Ω of some $(n+k)$-dimensional smooth Riemannian manifold, and if Ω has zero mod 2 boundary in Ω, then the singular set* $\text{sing}\, M$ *of M is $(n-2)$-rectifiable, and can be decomposed* $\text{sing}\, M = \cup_{j=0}^{J} S_j$ *where $\mathcal{H}^{n-2}(S_0) = 0$ and where each S_j, $j \geq 1$ has locally finite $\mathcal{H}^{n-2}$-measure.*

4.2 Preliminary Remarks on the Method of Proof: "Blowing Up"

We initially suppose $0 \in \text{sing} u$ and work in balls $B_\rho(0)$. To begin, we note that the argument used in 3.3 to prove that $\text{sing}\, u \cap B_{\rho/2}(0))$ is contained in a $(\delta\rho)$-neighbourhood of $\{0\} \times \mathbf{R}^{n-3}$, where $\delta = C(\rho^{-n} \int_{B_\rho(0)} |u - \varphi|^2)^{1/n}$, assuming that $S(\varphi) = \{0\} \times \mathbf{R}^{n-3}$, actually gives more information. Namely, we can use that argument together with the regularity theorem 1.6 and the inequality 3.2(iii) in order to deduce that for any ϵ_0 small enough (depending only on n, N, Λ) we have for each $\ell \geq 0$

$$\text{4.2(i)} \qquad \rho^\ell |D^\ell(u - \varphi)| \leq C_\ell \sqrt{\epsilon_0} \text{ in } B_{\rho/2}(0) \backslash \{X : \text{dist}(X, \{0\} \times \mathbf{R}^{n-3}) < \delta\rho\}$$

[2]In the meantime, this integrability condition ‡ has been shown to be unnecessary in case the target N is real analytic; see [SL6]

$\forall \ell \geq 0$, where $\delta = C(\epsilon_0^{-1}\rho^{-n}\int_{B_\rho(0)}|u-\varphi|^2)^{1/n}$, with C depending only on n, N, Λ, provided δ is small enough depending on n, N, Λ. Thus φ smoothly approximates u away (at least distance $\delta\rho$) from the singular axis $\{0\} \times \mathbf{R}^{n-3}$.

Now analogous to the discussion of the linearized operator $\mathcal{L}_{\varphi_0}$ on S^2 (in the previous lecture), we can also discuss linearizing the harmonic map operator $\mathcal{M}(u) \equiv \Delta u + \sum_{i=1}^n A_u(D_i u, D_i u)$ at φ:

4.2(ii) $$\mathcal{L}_\varphi(v) = \frac{d}{ds}\mathcal{M}(\psi_s)|_{s=0},$$

whenever ψ_s is a family of smooth maps of $B_{R_s}(0)\backslash\{X : \operatorname{dist}(X, \{0\}\times\mathbf{R}^{n-3}) < \delta_s\}$, varying smoothly in both X, s, where $R_s \uparrow \infty$ and $\delta_s \downarrow 0$ as $|s| \downarrow 0$, with $\psi_0 = \varphi$, and where $v = \frac{d}{ds}\psi_s|_{s=0}$. Notice that then $v(X) \in T_{\varphi(X)}N$ for each $X \in \mathbf{R}^n\backslash\{0\} \times \mathbf{R}^n$. Then in view of the definition 4.2(ii) and the inequalities 4.2(i) (with $\ell = 0, 1, 2$) it is clear that the difference $u - \varphi$ satisfies, in the "good" region $B_{\rho/2}(0)\backslash\{X : \operatorname{dist}(X, \{0\} \times \mathbf{R}^{n-3}) < \delta\rho\}$, an equation of the form

4.2(iii) $$\mathcal{L}_\varphi(u-\varphi)^T = E,$$

where $v^T(X)$ means orthogonal projection of v onto $T_{\varphi(X)}N$, where $|E| \leq C(|D(u-\varphi)|^2 + \rho^{-2}|u-\varphi|^2) \leq C\epsilon_0$ on $\{X \in B_{\rho/2}(0) : \operatorname{dist}(X, \{0\}\times\mathbf{R}^{n-3}) \geq \delta\rho\}$, and where C depends only on n, N, Λ. In view of the fact that $|E|$ is small relative to $u - \varphi$, this suggests that we should try to approximate $u - \varphi$ by solutions ψ of the linear equation $\mathcal{L}_\varphi = 0$; this is essentially the idea of "blowing up", going back to De Giorgi (in his work on oriented boundaries of least area).

In fact our initial aim is to show that such an approximation can be made provided we have the following "no δ-gaps at radius ρ" hypothesis for the part $S_+ = \{X \in \operatorname{sing} u : \Theta_u(X) \geq \Theta_\varphi(0)\}$ of the singular set:

4.2(iv) $$(\{0\} \times \mathbf{R}^{n-3}) \cap B_\rho(0) \subset \cup_{Z\in S_+} B_{\delta\rho}(Z)$$

notice in particular that if this hypothesis <u>fails</u> (for a given $\delta > 0$), then there is a point $Y_0 = (0, y_0) \in (\{0\} \times \mathbf{R}^{n-3}) \cap B_\rho(0)$ such that $S_+ \cap B_{\delta\rho}(Y_0) = \emptyset$. Thus, if $|Y_0| < \rho/2$ and if $\rho^{-n}\int_{B_\rho(0)}|u-\varphi|^2 < \epsilon^2$ with ϵ small enough (depending only on δ, n, N, Λ), then by 3.3(iii) we have

4.2(v) $$S_+ \cap p^{-1}(B_{\delta\rho}(Y_0)) \cap B_{\rho/2}(0) = \emptyset,$$

where p is the orthogonal projection of $\mathbf{R}^n$ onto $\{0\} \times \mathbf{R}^{n-3}$. This explains the terminology "no δ-gaps at radius ρ". (We explain how to handle the alternative when there <u>are</u> δ-gaps at radius ρ as in 4.2(v) later.)

Subject to the above no δ-gaps hypothesis 4.2(iv), it is in fact possible to prove the approximation

4.2(vi) $$(\theta\rho)^{-n}\int_{B_{\theta\rho}(0)}|u-\varphi-\psi|^2 \leq \tfrac{1}{4}\rho^{-n}\int_{B_\rho(0)}|u-\varphi|^2,$$

where θ is fixed $\in (0, \frac{1}{2})$, and where ψ is a solution of $\mathcal{L}\psi = 0$ of very explicit (and controlled) type—we show that such an inequality holds with ψ of the special form

$$\psi(x, y) = \sum_{j=1}^{n-3} \sum_{i=1}^{2} a_{ij} y^j D_i \varphi_0(x) + \psi_0(\omega),$$

where a_{ij} are constants, $\omega = |x|^{-1}x \in S^2$, where ψ_0 is a smooth solution on S^2 of the equation $\mathcal{L}_{\varphi_0}\psi = 0$ as in 3.2, and where $|a_{ij}| + \sup_{S^2} |\psi_0| \le C\rho^{-n} \int_{B_\rho(0)} |u - \varphi|^2$. An outline of the methods needed to establish such an approximation are described below.

In any case, notice first that the information in 4.2(i), (ii), (iii) could not possibly be sufficient to establish such an approximation, because 4.2(i), (ii), (iii) say nothing about the behaviour of $u - \varphi$ in the "bad" region $\{X \in B_{\rho/2}(0) : \operatorname{dist}(X, \{0\} \times \mathbf{R}^{n-3}) < \delta\rho\}$. So we need information about the behaviour of $u - \varphi$ in this region. For example, we at least need to know that the contribution to the quantity $\rho^{-n} \int_{B_\rho(0)} |u - \varphi|^2$ from the bad region is relatively small. This motivates the discussion of L^2-estimates in the next section.

4.3 L^2 Estimates

We continue to assume that φ is a homogeneous degree zero energy minimizing map from $\mathbf{R}^n$ to N with $S(\varphi) = \{0\} \times \mathbf{R}^{n-3}$, and $u : \Omega \to N$ is energy minimizing. Later we assume that in fact φ (modulo composition with a rotation of the X-variables) is one of the tangent maps of u at a point $Y \in \operatorname{sing}_* u$, but for the moment φ, u have no particular connection, and Y is an arbitrary point of Ω. We define Ω_0 to be a bounded subset of $\mathbf{R}^n$ such that $\overline{\Omega}_0 \subset \Omega$, and Λ, d are positive constants such that $d < \operatorname{dist}(\Omega_0, \partial\Omega)$ with $d^{2-n} \int_{\Omega_0} |Du|^2 \le \Lambda$. (Notice that then by the monotonicity formula we have $\rho^{2-n} \int_{B_\rho(Y)} |Du|^2 \le \Lambda$ for any $Y \in \Omega_0$ and any $\rho \in (0, d)$.)

The main L^2-estimate is the following:

Lemma 1. *If $\overline{B}_\rho(Z) \subset \Omega_0$, if $\theta \in (0,1)$, and if $\Theta_u(Z) \ge \Theta_\varphi(0)$, then*

$$\rho^{2-n} \int_{B_{\theta\rho}(Z)} |D_y u|^2 + \int_{B_{\theta\rho}(Z)} R_Z^{2-n} \left|\frac{\partial u}{\partial R_Z}\right|^2 \le C\rho^{-n} \int_{B_\rho(Z)} |u - \varphi_Z|^2,$$

where $\varphi_Z(X) \equiv \varphi(X+Z)$, $X \in \mathbf{R}^n$, $R_Z = |X - Z|$, $\partial/\partial R_Z = |X-Z|^{-1}(X-Z)\cdot D$, and C depends only on n, N, Λ.

We give the proof of this lemma at the end of this section, but for the moment we record some corollaries.

Corollary 1. *Under the same hypotheses, for any $\alpha \in (0,1)$,*

$$\int_{B_{\theta\rho}(Z)} \frac{|u - \varphi_Z|^2}{R^{n-\alpha}} \le C\rho^{-n+\alpha} \int_{B_\rho(Z)} |u - \varphi_Z|^2,$$

where C depends only on n, N, Λ, α.

This corollary is a direct consequence Lemma 1 together with the calculus inequality

$$\int_0^\rho R^{\alpha-1} f^2(R)\, dR \le C_\alpha \int_{\rho/2}^\rho R^{\alpha-1} f^2(R)\, dR + C_\alpha \int_0^\rho R^{1+\alpha}(f'(R))^2\, dR,$$

valid for any bounded C^1 function on $(0,\rho)$. We apply this to $f(R) = u(Z+R\omega) - \varphi_Z(\omega)$, where $\omega = |X-Z|^{-1}(X-Z) \in S^{n-1}$.

Corollary 2. *Under the same hypotheses, if $Z = (\xi,\eta) \in B_{\theta\rho}$, and if $\overline{B}_\rho(0) \subset \Omega_0$ we have*

$$\rho^{-2}|\xi|^2 + \rho^{-n} \int_{B_{\theta\rho}(Z)} |u-\varphi_Z|^2 \le C\rho^{-n} \int_{B_\rho(0)} |u-\varphi|^2,$$

where C depends only on n, N, θ, Λ.

This corollary is is easy to prove once we note that, by 3.2(iii) of Lecture 3 and a compactness argument, we have a fixed constant C, depending only on n, N, Λ, such that $\int_{x\in\mathbf{R}^3 \cdot |x|\le 1} |D_a\varphi_0|^2 \ge C|a|^2$, for all $a \in \mathbf{R}^3$. For the details we refer to [SL5].

Corollary 3. *There is $\delta_0 \in (0,1)$, depending only on n, N, such that if $\delta \in (0,\delta_0]$, if $B_\rho(0) \subset \Omega_0$, and if $\rho^{-n}\int_{B_\rho(0)} |u-\varphi|^2 \le \delta^4$, and if there are no δ-gaps at radius ρ (i.e., if 4.2(iv) above holds), then for each $\theta, \alpha \in (0,1)$*

$$\rho^{-n+1-\alpha} \int_{B_{\rho/2}(0)} \frac{|u-\varphi|^2}{r_{\delta\rho}^{1-\alpha}} \le C\rho^{-n} \int_{B_\rho(0)} |u-\varphi|^2,$$

where C depends only on n, N, Λ.

Remark. Notice that Corollary 3 implies that

$$\rho^{-n} \int_{\{(x,y)\in B_{\rho/2}(0)\,:\,|x|<\sigma\rho\}} |u-\varphi|^2 \le C\sigma^{1-\alpha}\rho^{-n} \int_{B_\rho(0)} |u-\varphi|^2,$$

for each $\sigma \ge \delta$, and hence (if we choose δ small) we see that very little is contributed to the quantity $\rho^{-n}\int_{B_\rho(0)} |u-\varphi|^2$ from the region $B_{\rho/2}(0) \cap \{(x,y) : |x| \le \delta\rho\}$.

Proof of Corollary 3: For $\sigma \in (\delta\rho, \rho/4)$, we choose points $Z_1, \ldots, Z_Q$ in $B_{\rho/2}(0)$ with $\Theta_u(Z_j) \ge \Theta_\varphi(0)$ and such that any given point X lies in no more than C_n of the balls $B_\sigma(Z_j)$, and such that

$$\{(x,y) \in B_{\rho/2}(0) : |x| < \sigma/2\} \subset \cup_{j=1}^Q B_\sigma(Z_j). \tag{1}$$

By virtue of the no δ-gaps hypothesis at radius ρ, this can evidently be done with

$$Q \le C(\rho/\sigma)^{n-3}, \tag{2}$$

where C depends only on n. Now by virtue of Corollary 1 we have for each $Z = Z_j$

$$\int_{B_{\theta\rho}(Z)} \frac{|u-\varphi_Z|^2}{R^{n-\alpha}} \le C\rho^{-n}\int_{B_{\rho/2}(Z)} |u-\varphi_Z|^2,$$

and hence by Corollary 2

$$(3)\quad \sigma^{-n+\alpha}\int_{B_\sigma(Z)} |u-\varphi_Z|^2 \le C\rho^{-n+\alpha}\int_{B_{\rho/2}(Z)} |u-\varphi_Z|^2 \le C\rho^{-n+\alpha}\int_{B_\rho(0)} |u-\varphi|^2.$$

Notice also that

$$(4)\qquad |u-\varphi_Z|^2 \le 2|u-\varphi|^2 + 2|\varphi-\varphi_Z|^2,$$

and, by the same computation (based on the inequality 3.2(iii)) that we used in the proof of 3.3(iii), we know that

$$|\varphi(X)-\varphi_Z(X)| \le C|x|^{-1}\sigma, \qquad X\in B_\sigma(Z),$$

where C depends only on N, Λ, and hence (4) together with another application of Corollary 2 gives

$$|u-\varphi_Z|^2 \le 2|u-\varphi|^2 + C|x|^{-2}\rho^{-n}\int_{B_\rho(0)} |u-\varphi|^2.$$

Therefore, by integrating this over $B_\sigma(Z)$, we see that (3) yields

$$\sigma^{2-n+\alpha}\int_{B_\sigma(Z)} |u-\varphi|^2 \le C\rho^{2-n+\alpha}\int_{B_\rho(0)} |u-\varphi|^2,$$

where C depends only on N, Λ. Now summing over $Z\in\{Z_1,\dots,Z_Q\}$ and using (1) and (2), we obtain

$$\rho^{-n+1-\alpha}\sigma^{-1+\alpha}\int_{\{(x,y)\in B_{\rho/2}(0)\,:\,|x|\le\sigma\}} |u-\varphi|^2 \le C\rho^{-n}\int_{B_\rho(0)} |u-\varphi|^2.$$

This evidently implies the required inequality.

Notice that at one point in the above proof we "gave up" two powers of σ; by being a little more careful, and noting that if $Z = (\xi,\eta)$ then (by 3.2(iii))

$$\varphi(x,y)-\varphi_Z(x,y) \equiv \varphi_0(x)-\varphi_0(x+\xi) = \sum_{j=1}^{3}\xi^j D_j\varphi_0(x) + E,$$

with $|E| \le C|x|^{-2}|\xi|^2$, we could have proved a more precise inequality. In fact (see again [SL5] for the detailed discussion):

Corollary 4. *Under the same hypotheses as Corollary 3,*

$$\int_{B_{\rho/2}(0)} \frac{|u-\varphi-\sum_{j=1}^{3} a^j(r,y)D_j\varphi_0(x)|^2}{r_{\delta\rho}^{3-\alpha}} \le C\rho^{-3+\alpha}\int_{B_\rho(0)} |u-\varphi|^2,$$

where $|a_j| \le C\rho^{-n}\int_{B_\rho(0)}|u-\varphi|^2$.

There is one final L^2 estimate, which follows from the variational equation 1.3(ii) and from the part of Lemma 1 which bounds the L^2-norm of $|D_y u|^2$, as follows:

Corollary 5. *For each $i = 1, 2, 3$ let $v_i(r,y) = r\int_{S^2} D_i\varphi(\omega)\cdot(u-\varphi)(r\omega,y)$. Then for each smooth function $\zeta \in C^\infty(\{(r,y) \in \mathbf{R}\times\mathbf{R}^{n-3} : r^2+|y|^2 < \rho^2\})$ with support in $\{(r,y) : r^2+|y|^2 < \theta^2\rho^2\}$ and with $\partial\zeta/\partial r \equiv 0$ in a neighbourhood of the $(n-3)$-dimensional space $r=0$, there is a constant $C = C_{\rho,\zeta,\theta,n,N,\Lambda}$ such that*

$$\Big|\int_{(0,\rho)\times\mathbf{R}^{n-3}} \Big(D_r v_i D_r D_{y^\ell}\zeta + \sum_{j=1}^{n-3} D_{y^j} v_i D_{y^j} D_{y^\ell}\zeta\Big)\Big|$$
$$\le C\Big(\int_{B_\rho(0)} |u-\varphi|^2\Big)^{1/2}\Big(\int_{B_\rho(0)} |D(u-\varphi)|^2\Big)^{1/2},$$

for each $\ell = 1,\dots,n-3$.

Remark. In view of the fact that it is not necessary here that ζ vanish in a neighbourhood of the axis $r=0$, this says that the even extension of $(\rho^{-n}\int_{B_\rho(0)}|u-\varphi|^2)^{-1/2}D_{y^\ell}v_i$ is close (in a certain sense—see below) to being harmonic with respect to the variables $r, y^1,\dots,y^{n-3}$ in the region $\{(r,y)\in\mathbf{R}\times\mathbf{R}^{n-3} : r^2+|y|^2 < \rho^2\}$.

4.4 Special Solutions of the Linearized Equation

Now recall that $\mathcal{L}_\varphi$ is the linearized harmonic map operator at φ, where $\varphi(x,y) \equiv \varphi_0(x)$ is as in the previous section. Notice that then, in the coordinates $(x,y) \equiv (r\omega,y)$ with $\omega = |x|^{-1}x \in S^2$ and $r = |x|$, $\mathcal{L}_\varphi v$ has the form

4.4(i)
$$r^{-2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}v\right) + \sum_{j=1}^{n-3} D^2_{y^j}v + r^{-2}\mathcal{L}_{\varphi_0}v,$$

where $\mathcal{L}_{\varphi_0}$ is the linearized harmonic map operator on S^2 as discussed in 3.2 of Lecture 3.

As we discussed briefly in 4.2, we want to show that $u-\varphi$ is well-approximated by special solutions of this, provided $\rho^{-n}\int_{B_\rho(0)}|u-\varphi|^2$ is small enough. To see this the crucial ingredient is the following lemma concerning solutions of the linear equation $\mathcal{L}_\varphi v = 0$ which satisfy L^2 bounds analogous to those satisfied by $u-\varphi$ in 4.3 above:

Lemma. *If $v \in L^2(B_1(0)) \cap C^2(B_1(0)\backslash\{0\} \times R^{n-3})$ satisfies the equation $\mathcal{L}_\varphi v = 0$ on $B_1(0)\backslash\{0\} \times \mathbf{R}^{n-3}$, if, for some $\alpha \in (0,1)$, $\beta > 0$, v also satisfies the L^2 inequalities*

$$\int_{B_{1/2}(0)} r^{-1+\alpha}|v|^2, \quad \int_{B_{1/2}(0)} r^{-3+\alpha}|v - \sum_{j=1}^{3} a^j(r,y)D_j\varphi_0(x)|^2 < \beta(\int_{B_1(0)} |v|^2)^{1/2},$$

with $\sup|a^j| \le \beta(\int_{B_1(0)} |v|^2)^{1/2}$, and if each $v_i(r,y) \equiv \int_{S^2} rD_i\varphi_0(\omega) \cdot v(r\omega, y)\, d\omega$ satisfies

$$* \qquad \int_{(0,\infty)\times\mathbf{R}^{n-3}} (D_r v_i D_r D_{y^\ell}\zeta + \sum_{j=1}^{n-3} D_{y^j} v_i D_{y^j} D_{y^\ell}\zeta) = 0$$

for any C^∞-function ζ on $\{(t,y) \in \mathbf{R} \times \mathbf{R}^{n-3}\}$ with ζ having compact support in $t^2 + |y|^2 < 1$, then

$$v(r\omega, y) = \sum_{i=1}^{3}\sum_{j=1}^{n-3} a_{ij} y^j D_{x^i}\varphi_0(x) + \psi_0(\omega) + E,$$

where $(\rho^{-n}\int_{B_\rho(0)} |E|^2)^{1/2} \le C(\int_{B_1(0)} |v|^2)^{1/2}\rho^\mu$ for $\rho \in (0, \frac{1}{2})$, where $\mu \in (0,1)$ depends on n, N, Λ, and where a_{ij} are constants and ψ_0 is a solution of $\mathcal{L}_{\varphi_0}\psi_0 = 0$ (as in 3.2) on S^2 with $|a_{ij}| + \sup_{S^2}|\psi_0| \le C(\int_{B_1(0)} |v|^2)^{1/2}$. Here $C = C(\alpha, \beta, n, N, \Lambda)$.

Remark. Notice that v_i automatically satisfies the strong form of $*$ on $\{(r,y) : r^2 + |y|^2 < 1,\ r > 0\}$, as one easily checks by direct computation based on 4.4(i) and the facts that $D_i\varphi_0(r\omega) = r^{-1}D_i\varphi_0(\omega)$ and that $D_i\varphi_0(\omega)$ is a solution of $\mathcal{L}_{\varphi_0}(v) = 0$ on S^2 (see 3.2(v)). The point about $*$ is that ζ need not vanish along $r = 0$, so that the solution v_i satisfies $\partial^2 v_i/\partial r\partial y^\ell = 0$ along $r = 0$ in the weak sense, hence in the classical sense.

4.5 Brief Sketch of the Proof of the Results of 4.1

We here only discuss the case when the integrability condition ‡ holds at each point of $\operatorname{sing}_* u$ (as for example in the case $\dim N = 2$). Only minor modifications are needed to prove the more general case stated in Theorem 2. By virtue of the L^2-estimates of 4.3 and the above lemma, it is now quite easy to check the approximation result 4.2(vi), provided δ is chosen sufficiently small (depending only on n, N, Λ) and provided $\rho^{-n}\int_{B_\rho(Y)} |u - \varphi|^2 \le \epsilon^2$ with ϵ sufficiently small compared with δ (e.g. $\epsilon \le \delta^3$ with δ sufficiently small); the details of this can be found in [SL5].

On the other hand the solutions $y^j D_{x^i}\varphi_0(x)$ are generated (analogous to the solutions in 3.2(iv), (v)) by the 1-parameter family $\varphi(e^{sA})$, where A is the skew-symmetric matrix such that e^{sA} is a rotation by angle s in the y^j–x^i-plane, while the integrablilty hypothesis ‡ guarantees that there is a 1-parameter family φ_s of smooth harmonic maps of S^2 into N with initial velocity ψ_0. Then we deduce that corresponding to the linear solution ψ in 4.2(vi) there is such a $\widetilde{\varphi}$ with $\psi = \widetilde{\varphi} - \varphi + E$, where $\widetilde{\varphi}$ is a homogeneous degree zero map which modulo a rotation satisfies $\widetilde{\varphi}(x,y) \equiv \widetilde{\varphi}_0(x)$, with $\widetilde{\varphi}_0|S^2$ a smooth harmonic map into N, and where $(\theta\rho)^{-n}\|E\|^2_{L^2(B_{\theta\rho}(Y))} \le C\theta^\alpha\rho^{-n}\int_{B_\rho(0)}|u-\varphi|^2$, where $C > 0$, $\alpha \in (0,1)$ depend only on n, N, Λ, provided ϵ is small enough, depending only on $n, N, \Lambda, \theta, \delta$. Then it follows from 4.2(vi) (assuming the integrability condition and assuming ϵ, δ, θ small enough) that

4.5(i) $$(\theta\rho)^{-n}\int_{B_{\theta\rho}(Y)}|u-\widetilde{\varphi}|^2 \le \tfrac{1}{2}\rho^{-n}\int_{B_\rho(Y)}|u-\varphi|^2.$$

Notice that by the triangle inequality in 4.5(i) we get

4.5(ii) $$\|\widetilde{\varphi}-\varphi\|^2_{L^2(B_1(0))} \le C\rho^{-n}\int_{B_\rho(Y)}|u-\varphi|^2.$$

Provided there are no δ-gaps at the new radius $\theta\rho$, the inequality 4.5(i) can now be iterated.

Remark. Actually a technical point here is that the new $\widetilde{\varphi}$ need not be minimizing, although it is harmonic and has distance (in the L^2-sense) less that $C\epsilon$ from the original φ, and hence it can be shown that all the discussion of 3.2 and 4.3 applies with such a $\widetilde{\varphi}$ in place of the minimizer φ; we should therefore duplicate the entire discussion above starting with a homogeneous harmonic φ which lies in a suitably small neighbourhood of a minimizer and which (modulo a rotation) has $\varphi(x,y) \equiv \varphi_0(x)$, with φ homogeneous degree zero harmonic and $\operatorname{sing}\varphi = \{0\}\times\mathbf{R}^{n-3}$. Since the inequalities 4.5(i), (ii) ensure that we always remain close to the same minimizer, this point turns out to cause no serious difficulty. Notice in particular that the above argument only used the equation 1.3(i)$'$ for φ_j and not 1.3(ii).

Implementing such an iterative procedure, based on 4.5(i), 4.5(ii), then gives a homogeneous φ^* such that

4.5(iii) $$\sigma^{-n}\int_{B_\sigma(Y)}|u-\varphi^*|^2 \le C(\sigma/\rho)^\alpha\rho^{-n}\int_{B_\rho(Y)}|u-\varphi|^2, \quad 0<\sigma\le\rho$$

provided there are no δ-gaps at any radius $\sigma = \theta^j\rho$, where $\|\varphi^*-\varphi\|^2_{L^2(B_1(0))} \le C\rho^{-n}\int_{B_\rho(Y)}|u-\varphi|^2$. Notice that we continue to assume in this discussion that Y is an arbitrary point of Ω with $\overline{B}_\rho(Y) \subset \Omega$; 4.5(iii) shows that we must encounter δ-gaps at some radius if either $Y \notin \operatorname{sing}_* u$ or if $\Theta_u(Y) \neq \Theta_\varphi(0)$, because 4.5(iii) guarantees in particular that φ^* is the unique tangent map of u at Y.

Now at the points Y where 4.5(iii) fails, we must have a δ-gap in S_+ at either the initial radius ρ, or else at some $\sigma = \theta^j\rho$; notice that in the latter case we can select the smallest such j, so that 4.5(iii) still holds at least for $\theta^j\rho \le \sigma \le \rho$.

Keeping in mind that $\rho^{-n}\int_{B_\rho(Y)}|u-\varphi|^2\le\epsilon^2$ implies that $\rho^{-n}\int_{B_{\rho/2}(Z)}|u-\varphi|^2\le C\epsilon^2$ for any $Z\in B_{\rho/2}(Y)$, the rest of the proof involves using the above alternatives to show that (assuming the ball $B_\rho(Y)$ is a sufficiently small neighbourhood of some initial point Z_0 at which u has a tangent map $\varphi^{(0)}$ with $\dim S(\varphi^{(0)})=n-3$ and $\Theta_{\varphi^{(0)}}(0)=\Theta_\varphi(0)$) we can decompose $S_+\equiv B_\rho(Y)\cap\{Z:\Theta_u(Z)\ge\Theta_\varphi(0)\}$ into a part which is contained in an embedded $C^{1,\mu}$ manifold and a part which can be covered by a countable collection of balls $B_{\sigma_k}(Y_k)$ with $\sum_k\sigma_k^{n-3}\le(1-\beta)\rho^{n-3}$, and such that a similar decomposition can be made starting with any of the balls $B_{\sigma_k}(Y_k)$ in place of $B_\rho(Y)$. Here β is a fixed constant $\in(0,1)$, and we need to take ϵ small enough depending only on n,N,Λ. Thus after j iterations we get we get that S_+ is contained in the union of j embedded $C^{1,\mu}$ manifolds together with a set which is covered by a family of balls $B_{\sigma_k}(Y_k)$ with $\sum_k\sigma_k^{n-3}\le(1-\beta)^j\rho^{n-3}$. Thus S is contained in the countable union of $C^{1,\mu}$ manifolds together with a set of $(n-3)$-measure zero.

This iterative procedure has the additional property that it controls the sum of the measures of the embedded manifolds, thus giving the local finiteness result stated in the theorems of 4.1.

Finally to prove the additional conclusions of Theorem 1 in case N is S^2 or metrically sufficiently close to S^2, we need to use a result of Brezis, Coron, & Lieb [BCL] which asserts that a homogeneous minimizer ψ from $\mathbf{R}^3$ into the standard S^2 is such that $\psi|S^2$ is an orthogonal transformation. This result is easily seen (for topological reasons) to imply that there we can encounter no δ-gaps (at any radius) in the iterative argument described above, and hence we obtain an inequality like 4.5(iii) uniformly for non-isolated points Y of S_+, assuming that $Y\in B_{\rho_0/2}(Y_0)$, where $\rho_0^{-n}\int_{B_{\rho_0}(Y_0)}|u-\varphi^{(0)}|^2$ is sufficiently small, where $\varphi^{(0)}$ with $\dim S(\varphi^{(0)})=n-3$ as above. This evidently implies the stated $C^{1,\mu}$ property of S_+.

The fact that the exceptional set $\operatorname{sing} u\setminus\operatorname{sing}_* u$ is discrete in the case $n=4$ involves a simple scaling and compactness argument together with several applications of the result in the previous paragraph. The details are given in [SL5] (see also [HL]).

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LECTURE 1
Analytic Preliminaries

This second series of lectures is meant as an elementary introduction to the basic regularity theorem for harmonic maps, which we used frequently in the first lecture series [SL1]. We deal here with energy-minimizing maps rather than minimal surfaces, because the basic regularity theory for these (while being very analogous to the theory for minimal surfaces) is technically simpler than the corresponding theorems for minimal surfaces. Interestingly enough though, the regularity theory for harmonic maps into compact Riemannian manifolds was not established until much later than the basic regularity theory for minimal surfaces. (The work of Giaquinta-Giusti and Schoen-Uhlenbeck was done in the early 1980's, whereas the regularity theorem for area minimizing hypersurfaces was proved by De Giorgi [DG] in the early 1960's.) The explanation for this (apart from the difficulty in proving the energy inequality discussed in Lecture 3 below) is perhaps that the close parallel between regularity theory for area-minimizing surfaces and regularity theory for energy minimizing maps (e.g. monotonicity and harmonic approximation as a key ingredients in the proof of the main regularity lemma) was not widely appreciated until the work of Schoen-Uhlenbeck [SU] and Giaquinta-Giusti [GG].

The plan of this series of 4 lectures is as follows:

Lecture 1: Analytic Preliminaries.

Lecture 2: Proof of the Schoen-Uhlenbeck theorem modulo a "reverse Poincaré" inequality.

Lecture 3: Proof of the reverse Poincaré inequality used in Lecture 2.

Lecture 4: Higher regularity and other mopping-up operations.

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1.1 Hölder Continuous Functions

Recall that if $\Omega \subset \mathbf{R}^n$ is open and if $\alpha \in (0,1]$, we say that $u : \Omega \to \mathbf{R}$ is uniformly Hölder continuous with exponent α on $\overline{\Omega}$ (written $u \in C^{0,\alpha}(\overline{\Omega})$), if there is a constant C such that $|u(X) - u(Y)| \le C|X - Y|^\alpha$ for every $X, Y \in \Omega$.

There are various reasons why Hölder continuity turns out to be so important in geometric analysis and PDE. We mention two reasons here:

(i) (Scaling.) Notice that if $|u(X)-u(Y)| \le \beta|X-Y|^\alpha$ for every $X, Y \in \Omega$ and if for given $R > 0$ we define the scaled function $\tilde{u}(X) = R^{-\alpha}u(RX)$ for $X \in \tilde{\Omega} \equiv \{R^{-1}Y : Y \in \Omega\}$, then $|\tilde{u}(X) - \tilde{u}(Y)| \le \beta|X - Y|^\alpha$ for every $X, Y \in \tilde{\Omega}$.

(ii) (Dyadic decay implies Hölder continuity.) If $u : B_{2R}(X_0) \to \mathbf{R}$ and if there is a fixed $\theta \in (\frac{1}{2}, 1)$ such that $\operatorname{osc}_{B_{\rho/2}(Y)} u \le \theta \operatorname{osc}_{B_\rho(Y)} u$ ($\operatorname{osc}_A u = \sup_{x,y\in A} |u(x) - u(y)|$) for every $Y \in B_R(X_0)$ and every $\rho < R$, then $u \in C^{0,\alpha}(\overline{B}_R(X_0))$ with $\alpha = \log_2(1/\theta)$, where $\log_2$ means the base 2 logarithm. Indeed by iteration the given inequality implies $\operatorname{osc}_{B_{\rho/2^j}(Y)} u \le \theta^j \operatorname{osc}_{B_\rho(Y)} u \equiv (2^{-j})^{\log_2(1/\theta)} \operatorname{osc}_{B_\rho(Y)} u$

The following result of Campanato will be needed in the sequel:

Lemma (Campanato). *Suppose $u \in L^2(B_{2R}(X_0))$, $\alpha \in (0,1)$, $\beta > 0$ and*

$$* \qquad \inf_{\lambda\in\mathbf{R}} \rho^{-n} \int_{B_\rho(Y)} |u - \lambda|^2 \le \beta^2(\rho/R)^{2\alpha}$$

for every ball $B_\rho(Y)$ such that $Y \in B_R(X_0)$ and $\rho \le R$. Then there is a Hölder continuous representative $\overline{u}$ for the L^2-class of u with

$$|\overline{u}(X) - \overline{u}(Y)| \le C\beta(|X - Y|/R)^\alpha, \quad \forall X, Y \in B_R(X_0),$$

where C depends only on n, α.

Remarks: Notice that, since $\int_{B_\rho(Y)} |u-\lambda|^2 = \int_{B_\rho(Y)}(u^2 - 2\lambda u + \lambda^2)$, it is easy to check that the infimum on the left of $*$ is attained when λ has the average value $\lambda_{Y,\rho} \equiv (\omega_n\rho^n)^{-1} \int_{B_\rho(Y)} u$, where ω_n is the volume of the unit ball in $\mathbf{R}^n$. It will be convenient to use this fact from time to time in what follows.

Proof of the lemma. First note that

$$(1) \qquad (\rho/2)^{-n} \int_{B_{\rho/2}(Y)} |u - \lambda_{Y,\rho}|^2 \le 2^n\rho^{-n} \int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2 \le 2^n\beta^2(\rho/R)^{2\alpha},$$

where $\lambda_{Y,\rho}$ is the average of u over $B_\rho(Y)$ as in the above remark. Using the given inequality with $\rho/2$ in place of ρ, we also have

$$(2) \qquad (\rho/2)^{-n} \int_{B_{\rho/2}(Y)} |u - \lambda_{Y,\rho/2}|^2 \le \beta^2(\rho/R)^{2\alpha}.$$

Adding (1) and (2) and using the squared triangle inequality $|a-b|^2 \le 2|a-c|^2 + 2|b-c|^2$, we conclude that

$$|\lambda_{Y,\rho} - \lambda_{Y,\rho/2}| \le 2^{n+2}\omega_n^{-1/2}\beta(\rho/R)^\alpha \tag{3}$$

provided that $\rho \le R$ and $Y \in B_R(X_0)$.

Now for any integer $\nu \in \{0,1,2\dots\}$ we can choose $\rho = 2^{-\nu}R$, whereupon (3) gives

$$|\lambda_{Y,R/2^{\nu+1}} - \lambda_{Y,R/2^\nu}| \le 2^{n+2}\beta 2^{-\nu\alpha}. \tag{4}$$

Since $2^{-\nu\alpha}$ is the ν^{th} term of a geometric series, we see that the series s defined by $s = \sum_{\nu=0}^\infty(\lambda_{Y,R/2^{\nu+1}} - \lambda_{Y,R/2^\nu})$ is absolutely convergent. But the j^{th} partial sum s_j is just $\lambda_{Y,2^{-j}R} - \lambda_{Y,R}$, so we have $\lim_{\nu\to\infty}\lambda_{Y,2^{-\nu}R}$ exists, and we denote this limit by λ_Y. Since $\lambda_{Y,2^{-\nu}R} - \lambda_Y = -\sum_{j=\nu}^\infty(\lambda_{Y,R/2^{j+1}} - \lambda_{Y,R/2^j})$ we see

$$|\lambda_{Y,2^{-\nu}R} - \lambda_Y| \le C\beta 2^{-\nu\alpha}, \tag{5}$$

where C depends only on n, α. Then combining (1) (with $\rho = 2^{-\nu}R$) with (5) and using the squared triangle inequality again, we conclude

$$\rho^{-n}\int_{B_\rho(Y)} |u - \lambda_Y|^2 \le C\beta(\rho/R)^{2\alpha}, \tag{6}$$

for $\rho = 2^{-\nu}R$, $\nu = 0,1,\dots.$ On the other hand for <u>any</u> $\rho \in (0,R]$ there is an integer $\nu \ge 0$ such that $2^{-\nu-1}R < \rho \le 2^{-\nu}R$, and it evidently follows (since $B_{2^{-\nu}R}(Y) \supset B_\rho(Y)$ for such ν) that (6) holds (with 2^nC in place of C) for <u>every</u> $\rho \le R/2$.

Now take any pair of points $Y, Z \in B_R(X_0)$ with $|Y-Z| \le R/4$, and apply (6) with $\rho = 2|Y-Z|$ on balls with centers Y, Z and add the resultant inequalities. Since $B_{\rho/2}((Y+Z)/2) \subset B_\rho(Y) \cap B_\rho(Z)$ this gives

$$\rho^{-n}\int_{B_{\rho/2}((Y+Z)/2)} (|u-\lambda_Y|^2 + |u-\lambda_Z|^2) \le 2C\beta(\rho/R)^{2\alpha}.$$

Since $|\lambda_Y - \lambda_Z|^2 \le 2|u-\lambda_Y|^2 + 2|u-\lambda_Z|^2$, this gives

$$|\lambda_Y - \lambda_Z| \le 2C\beta(\rho/R)^\alpha \equiv 2C\beta 2^{2+\alpha}(|Y-Z|/R)^\alpha. \tag{7}$$

Now for any pair of points $Y, Z \in B_R(X_0)$ we can pick points $Y = Z_1, \dots, Z_8 = Z$ on the line segment joining Y, Z such that $|Z_i - Z_{i+1}| \le R/4$, and applying (7) to each of the pairs Z_i, Z_{i+1} and adding, we deduce finally that

$$|\lambda_Y - \lambda_Z| \le C\beta(|Y-Z|/R)^\alpha, \qquad \forall Y, Z \in B_R(X_0). \tag{8}$$

On the other hand by letting $\rho \downarrow 0$ in (6) and using the Lebesgue lemma, we have $\lambda_Y = u(Y)$ for almost all $Y \in B_R(X_0)$, so the proof is complete (because then $\overline{u}(Y) \equiv \lambda_Y$ is a representative for the L^2 class of u which by (8) satisfies the required Hölder estimate.)

1.2 Functions with L^2 Gradient

Recall that if $u \in L^2(\Omega)$, then we say that $Du = (D_1u, \ldots, D_nu)$ is an L^2 gradient for u in Ω if D_iu are $L^2(\Omega)$ and if

1.2(i) $$\int_\Omega D_iu\varphi = -\int_\Omega uD_i\varphi, \quad \forall \varphi \in C_c^\infty(\Omega),$$

where, here and subsequently, C_c^∞ denotes the set of C^∞ functions with compact support in Ω. Of course such functions D_iu as in 1.2(i) are uniquely determined by u if they exist at all; further if $u \in C^1(\Omega)$ then such an identity holds by integration by parts, so this notion of L^2 gradient really does generalize the classical notion of partial derivatives.

So from now on, when we say $Du \in L^2$ we mean that $u \in L^2$ and that there are $D_1u, \ldots, D_nu \in L^2$ such that 1.2(i) holds. In this section we present some of the important facts about such functions.

Lemma 1 (Rellich Compactness Lemma). *Suppose u_k is a sequence of L^2 functions on a ball $B_R(X_0)$ with $Du_k \in L^2$ for each k and*

$$\limsup_{k\to\infty} \int_{B_R(X_0)} (|u_k|^2 + |Du_k|^2) < \infty.$$

Then there is a subsequence $u_{k'}$ and $u \in L^2(B_R(X_0))$ with $Du \in L^2(B_R(X_0))$ such that $\lim \int_{B_R(X_0)} |u_{k'} - u|^2 = 0$, $Du_{k'} \rightharpoonup Du$ (weak convergence in $L^2(B_R(X_0))$), and $\int_{B_R(X_0)} |Du|^2 \leq \liminf_{k'\to\infty} \int_{B_R(X_0)} |Du_{k'}|^2$.

For a proof see e.g. [GT Chapter 7].

We shall also need the following Poincaré Inequality:

Lemma 2. *If $Du \in L^2(B_R(X_0))$, then*

$$R^{-n} \int_{B_R(X_0)} |u - \lambda|^2 \leq CR^{2-n} \int_{B_R(X_0)} |Du|^2,$$

where $\lambda = (\omega_n R^n)^{-1} \int_{B_R(X_0)} u$ and C is a constant depending only on n; here and subsequently ω_n denotes the measure of the unit ball in $\mathbf{R}^n$.

There are various proofs of the Poincaré inequality—one nice way to prove it is to use the above Rellich compactness theorem. (By a translation and change of scale, it suffices to prove the lemma with $R = 1$ and $X_0 = 0$, subject to the assumption $\int_{B_1(0)} u = 0$. Then if the theorem fails with $C = k$, $k = 1, 2, \ldots$, there is a sequence u_k with $\int_{B_1(0)} |Du_k|^2 \to 0$, $\int_{B_1(0)} u_k = 0$, and $\int_{B_1(0)} u_k^2 = 1$ for each k; now apply Rellich to get a contradiction.)

Finally we want to state and prove Morrey's lemma.

Lemma 3 (Morrey's Lemma). *Suppose $Du \in L^2(B_R(X_0))$, $\beta > 0$ is a constant, and*

$$\rho^{2-n} \int_{B_\rho(Y)} |Du|^2 \leq \beta^2 (\rho/R)^{2\alpha}, \qquad \forall Y \in B_{R/2}(X_0),\ \rho \in (0, R/2).$$

Then $u \in C^{0,\alpha}(\overline{B}_{R/2}(X_0))$, and in fact

$$|u(X) - u(Y)| \leq C\beta(|X - Y|/R)^\alpha, \qquad \forall X, Y \in B_{R/2}(X_0)$$

Proof. Let $\lambda_{Y,\rho} = (\omega_n \rho^n)^{-1} \int_{B_\rho(Y)} u$. The Poincaré inequality gives

$$\rho^{-n} \int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2 \leq C\rho^2 (\rho/R)^{2\alpha}$$

for each $Y \in B_{R/2}(X_0)$ and each $\rho \in (0, R/2)$. Using the Campanato lemma of §1.1, we then have the required result.

1.3 Harmonic Functions

Recall that a C^∞ function u on a domain $\Omega \subset \mathbf{R}^n$ is said to be harmonic if $\Delta u = 0$, where $\Delta u \equiv \sum_{j=1}^n D_j D_j u$, with $D_j = \frac{\partial}{\partial x^j}$ in classical notation. Thus we can write $\Delta u = \operatorname{div}(Du)$, where $Du = (D_1 u, \ldots, D_n u)$ as usual denotes the gradient of u, and $\operatorname{div} \Phi$ means $\sum_j D_j \Phi^j$ for any smooth vector function $\Phi = (\Phi^1, \ldots, \Phi^n)$.

If we choose a ball $B = B_R(X_0)$ with closure contained in Ω, and if we integrate the identity $\operatorname{div} Du = 0$ over B and apply the divergence theorem, then we can check rather easily (see e.g. [GT] for the details) that harmonic functions have the mean-value property

$$u(X_0) = (\omega_n R^n)^{-1} \int_{B_R(X_0)} u \tag{1.3(i)}$$

for any such ball $B_R(X_0)$. This property is quite fundamental; for example using it together with the fact that harmonicity of u trivially implies harmonicity of each partial derivative (of any order), one can easily check (see e.g. [GT] for the details) that

$$\sup_{B_{R/2}(X_0)} R^j |D^j u| \leq C^{j+1} j! \left(R^{-n} \int_{B_R(X_0)} u^2 \right)^{1/2}, \quad \forall j = 0, 1, 2, \ldots, \tag{1.3(ii)}$$

where C is a constant depending only on n.

1.4 Weakly Harmonic Functions

If $Du \in L^2(B)$, where $B = B_R(X_0)$ is an arbitrary ball in $\mathbf{R}^n$, we say that u is weakly harmonic on the ball if

$$\int_B Du \cdot D\varphi = 0 \quad \forall \varphi \in C_c^\infty(B). \tag{1.4(i)}$$

Notice this formally generalizes the notion of smooth harmonic, because if u is smooth harmonic then we could integrate by parts in the identity $\int \Delta u \, \varphi = 0$, thus establishing 1.4(i), and, conversely, if $u \in C^2(B)$, 1.4(i) evidently implies $\Delta u = 0$ by virtue of the arbitrariness of φ.

In fact H. Weyl proved that the two notions (weakly harmonic and classical harmonic) are the same:

Lemma (H. Weyl). *Suppose $Du \in L^2(B)$ and that 1.4(i) holds. Then the L^2 class of u has a C^∞ representative which is harmonic.*

We shall not give the proof, but we do want to point out that the proof is elementary, based on smoothing and the mean-value property 1.3(i) of the previous section. (See e.g. [GT].)

1.5 Harmonic Approximation Lemma

The following harmonic approximation (or "blow up") lemma will be of fundamental importance:

Lemma (Harmonic Approximation Lemma). *Let $B = B_1(0)$, the open ball of radius 1 and center 0 in $\mathbf{R}^n$. For each $\epsilon > 0$ there is $\delta = \delta(n, \epsilon) > 0$ such that if $Df \in L^2(B)$, $\int_B |Df|^2 \le 1$, and*

$$\left|\int_B Df \cdot D\varphi\right| \le \delta \sup_B |D\varphi|, \qquad \forall \varphi \in C_c^\infty(B),$$

then there is a harmonic u on B such that $\int_B |Du|^2 \le 1$ and

$$\int_B |f - u|^2 \le \epsilon^2.$$

Proof. If this fails for some $\epsilon > 0$, then there is a sequence f_k such that $Df_k \in L^2(B)$, $\int_B |Df_k|^2 \le 1$,

$$\left|\int_B Df_k \cdot D\varphi\right| \le k^{-1} \sup_B |D\varphi| \tag{1}$$

for each $\varphi \in C_c^\infty(B)$, and such that

$$\int_B |f_k - u|^2 \ge \epsilon^2 \tag{2}$$

for every harmonic u on B with $\int_B |Du|^2 \le 1$.

Notice that since the same holds with $\widetilde{f}_k = f_k - \lambda_k$ for any choice of constants λ_k, we can assume without loss of generality that $\int_B f_k = 0$ for each k. But then by the Poincaré inequality of 1.2 we conclude

$$\limsup_{k\to\infty} \int_B (|f_k|^2 + |Df_k|^2) < \infty,$$

and hence by the Rellich compactness theorem of 1.2 we have a subsequence $f_{k'}$ and an f with $Df \in L^2$ on B such that

$$\lim \int_B |f - f_{k'}|^2 = 0 \tag{3}$$

and $Df_{k'} \rightharpoonup Df$ (weak convergence in $L^2(B)$). But using this weak convergence in (1) we deduce that $\int_B Df \cdot D\varphi = 0$ for each $\varphi \in C_c^\infty(B)$, so that f is weakly harmonic on B, and Weyl's lemma (in 1.4 above) guarantees that f is smooth harmonic on B, and hence (since $\int_B |Df|^2 \le \liminf \int_B |Df_{k'}|^2 \le 1$) we see that (3) contradicts (2).

LECTURE 2
A General Regularity Lemma

Here we shall prove a general regularity lemma which establishes the Schoen-Uhlenbeck regularity theorem modulo the proof of a "reverse Poincaré" inequality, the proof of which we shall give in Lecture 3.

2.1. Statement of Main Regularity Lemma and Remarks

Lemma 1 (Regularity lemma). *Suppose $u = (u^1, \dots, u^p) : B_R(X_0) \to \mathbf{R}^p$, and $\alpha \in (0,1)$, $\beta \geq 1$ are given, with $Du \in L^2(B_R(X_0))$. Suppose further that u satisfies the equation*

$$* \qquad \Delta u = F \quad \text{weakly in } B_R(X_0),$$

where F is a measurable function satisfying

$$\text{(a)} \qquad |F| \leq \beta |Du|^2 \text{ on } B_R(X_0),$$

and that

$$\text{(b)} \qquad (\rho/2)^{2-n} \int_{B_{\rho/2}(Y)} |Du|^2 \leq \beta \rho^{-n} \int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2$$

whenever $Y \in B_{R/2}(X_0)$ and $\rho \leq R/2$.
Then there is $\delta_0 = \delta_0(n, \alpha, \beta) \in (0,1)$ such that the inequality $R^{-n} \int_{B_R(X_0)} |u - \lambda_{X_0,R}|^2 \leq \delta_0^2$ implies that u is Hölder continuous with exponent α on $B_{R/2}(X_0)$ and

$$|u(X) - u(Y)| \leq C(|X - Y|/R)^\alpha \left(R^{-n} \int_{B_R(X_0)} |u - \lambda_{X_0,R}|^2\right)^{1/2}, \quad X, Y \in B_{R/2}(X_0),$$

where C is a constant depending only on n, α, β.

Remarks: **(1)** Notice that $\Delta u = F$ weakly means that

$$\int_{B_R(X_0)} Du \cdot D\varphi = -\int_{B_R(X_0)} \varphi \cdot F, \quad \varphi \in C_c^\infty(B_R(X_0)).$$

(Here $Du \cdot D\varphi \equiv \sum_{i=1}^n \sum_{j=1}^p D_i u^j D_i \varphi^j$.)

(2) If N is a smooth compact manifold embedded in $\mathbf{R}^p$, and if $u : B_R(X_0) \to \mathbf{R}^p$ with $u(B_R(X_0)) \subset N$ is energy minimizing in the sense that $\int_{B_R(X_0)} |Du|^2 \leq \int_{B_R(X_0)} |Dv|^2$ whenever $v : B_R(X_0) \to \mathbf{R}^p$ with $v(B_R)(X_0) \subset N$ and $v = u$ in a neighbourhood of $\partial B_R(X_0)$, then (see the discussion in Lecture 1 of [SL1]) u satisfies an equation of the form $*$, (a). Thus to apply the above lemma to energy minimizing maps it is enough to check that the "reverse Poincaré inequality" (b) automatically holds for such maps. This we shall do in the next lecture.

(3) Actually, as we show in Lecture 4 below, the hypotheses of the above lemma ensure that $u \in C^{1,\alpha}$ rather than in $C^{0,\alpha}$.

Before we begin the proof of this lemma, we need to mention the appropriately scaled version of the harmonic approximation lemma 1.5.

2.2 Rescaled Version of the Harmonic Approximation Lemma

Lemma. *For any given $\epsilon > 0$ there is $\delta = \delta(n, \epsilon) > 0$ such that if $Df \in L^2(B_\rho(Y))$, if $\rho^{2-n}\int_{B_\rho(Y)} |Df|^2 \leq 1$, and if $|\rho^{2-n}\int_{B_\rho(Y)} Df \cdot D\varphi| \leq \delta\rho \sup_{B_\rho(Y)} |D\varphi|$ for every $\varphi \in C_c^\infty(B_\rho(Y))$, then there is a harmonic function u on $B_\rho(Y)$ satisfying the inequalities $\rho^{2-n}\int_{B_\rho(Y)} |Du|^2 \leq 1$ and $\rho^{-n}\int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2 \leq \epsilon^2$.*

Notice that this easily follows from the unscaled version of 1.5; in fact one just checks that 1.5 applies to the rescaled function $f_\rho(X) \equiv f(Y + \rho(X - Y))$.

2.3 Proof of the Regularity Lemma

As in Remark (1) of 2.2 we have

$$(1) \qquad \int_{B_R(X_0)} Du \cdot D\varphi = -\int_{B_R(X_0)} \varphi \cdot F, \quad \varphi \in C_c^\infty(B_R(X_0)).$$

By using this identity with $\varphi \in C_c^\infty(B_{\rho/2}(Y))$, where $B_\rho(Y) \subset B_R(X_0)$ is an arbitrary ball, and using also the hypotheses (a), (b), we have

$$|(\rho/2)^{2-n}\int_{B_{\rho/2}(Y)} Du \cdot D\varphi| \leq \beta\rho^{-n}\int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2 \sup_{B_{\rho/2}(Y)} |\varphi|,$$

and since $\sup_{B_{\rho/2}(Y)} |\varphi| \leq \rho \sup_{B_{\rho/2}(Y)} |D\varphi|$ (by 1-dimensional calculus along line segments in $B_{\rho/2}(Y)$), we thus have

$$(2) \qquad |(\rho/2)^{2-n}\int_{B_{\rho/2}(Y)} Dv \cdot D\varphi| \leq \left(\rho^{-n}\int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2\right)^{1/2} \rho \sup_{B_{\rho/2}(Y)} |D\varphi|,$$

where $v = \ell^{-1}u$, with $\ell = \beta(\rho^{-n}\int_{B_\rho(Y)}|u - \lambda_{Y,\rho}|^2)^{1/2}$.

Also, we have trivially by (b) and the definition of v that

$$(3) \qquad (\rho/2)^{2-n}\int_{B_{\rho/2}(Y)}|Dv|^2 \le 1.$$

Let $\epsilon > 0$ be arbitrary. By (2), (3) we can apply the harmonic approximation lemma in 2.2 above in order to conclude that there is a harmonic function w on $B_{\rho/2}(Y)$ such that

$$(4) \qquad (\rho/2)^{2-n}\int_{B_{\rho/2}(Y)}|Dw|^2 \le 1 \text{ and } (\rho/2)^{-n}\int_{B_{\rho/2}(Y)}|v - w|^2 \le \epsilon^2,$$

<u>assuming</u> that $\rho^{-n}\int_{B_\rho(Y)}|u - \lambda_{Y,\rho}|^2 \le \delta^2$, where $\delta(n,\epsilon)$ is as in the harmonic approximation lemma of 2.2 above.

Now take $\theta \in (0, \frac{1}{4}]$ and note that by the squared triangle inequality

$$(5) \qquad (\theta\rho)^{-n}\int_{B_{\theta\rho}(Y)}|v - w(Y)|^2 \le 2(\theta\rho)^{-n}\Big(\int_{B_{\theta\rho}(Y)}(|v - w|^2 + |w - w(Y)|^2\Big).$$

Now using 1-dimensional calculus along line segments with end-point at Y together with the estimate 1.3(ii) with $j = 0$ (applied to $D_i w$), we have

$$\sup_{B_{\theta\rho}(Y)}|w - w(Y)|^2 \le (\theta\rho \sup_{B_{\theta\rho}(Y)}|Dw|)^2 \le C\theta^2\rho^{2-n}\int_{B_{\rho/2}(Y)}|Dw|^2.$$

Using this together with (4) in (5), we conclude that

$$(\theta\rho)^{-n}\int_{B_{\theta\rho}(Y)}|v - w(Y)|^2 \le \theta^{-n}\epsilon^2 + C\theta^2,$$

where C depends only on n. Recalling that $v = \ell^{-1}u$, we have

$$(\theta\rho)^{-n}\int_{B_{\theta\rho}(Y)}|u - \lambda|^2 \le \beta^2(\theta^{-n}\epsilon^2 + C\theta^2)\rho^{-n}\int_{B_\rho(Y)}|u - \lambda_{Y,\rho}|^2,$$

where $\lambda = \ell w(Y)$ is a fixed vector in $\mathbf{R}^p$. Now we choose θ and ϵ: first select $\theta \in (0, \frac{1}{4}]$ so that $C\beta^2\theta^2 \le \frac{1}{2}\theta^{2\alpha}$; notice that such θ can be chosen to depend only on n, α, β. Having so chosen θ, now choose $\epsilon > 0$ such that $\beta^2\theta^{-n}\epsilon^2 < \frac{1}{2}\theta^{2\alpha}$. Therefore

$$(\theta\rho)^{-n}\int_{B_{\theta\rho}(Y)}|u - \lambda_{Y,\theta\rho}|^2 \le \theta^{2\alpha}\rho^{-n}\int_{B_\rho(Y)}|u - \lambda_{Y,\rho}|^2.$$

Thus, to summarize, we have shown that if $\alpha \in (0,1)$ if $B_\rho(Y) \subset B_R(X_0)$, and if $\rho^{-n}\int_{B_\rho(Y)} |u-u(Y)|^2 \le \delta_0^2$, with $\delta_0 = \delta_0(n,\beta,\alpha) \in (0,1)$ sufficiently small, then

$$(6) \qquad (\theta\rho)^{-n}\int_{B_{\theta\rho}(Y)} |u-\lambda_{Y,\theta\rho}|^2 \le \theta^{2\alpha}\rho^{-n}\int_{B_\rho(Y)} |u-\lambda_{Y,\rho}|^2,$$

with $\theta = \theta(n,\alpha,\beta) \in (0,\frac{1}{4}]$.

Next, let $I_0 = R^{-n}\int_{B_R(X_0)} |u-\lambda_{X_0,R}|^2$ and note that if $I_0 \le 2^{-n}\delta_0^2$, then, with $\rho = R/2$, we have $B_\rho(Y) \subset B_R(X_0)$ and

$$(7) \qquad \rho^{-n}\int_{B_\rho(Y)} |u-\lambda_{X_0,R}|^2 \le 2^n I_0 < \delta_0^2 \quad \forall\, Y \in B_{R/2}(X_0).$$

But notice that (6), (7) in particular imply that we have the "starting hypothesis" $\rho^{-n}\int_{B_\rho(Y)} |u-\lambda_{Y,\rho}|^2 < \delta_0^2$ is satisfied with $\theta\rho$ in place of ρ, and hence (6) holds with $\theta\rho$ in place of ρ ($\rho = R/2$ still). Continuing inductively, we deduce that

(8)

$$(\theta^j R/2)^{-n}\int_{B_{\theta^j R/2}(Y)} |u-\lambda_{Y,\theta^j R/2}|^2 \le \theta^{2j\alpha}(R/2)^{-n}\int_{B_{R/2}(Y)} |u-\lambda_{Y,R/2}|^2 \le 2^n I_0$$

for each $j = 0, 1, 2\ldots$, provided only that the inequality $2^n I_0 < \delta_0^2$ does hold. Now on the other hand if $\sigma \in (0, R/2]$, there is a unique $j \in \{0,1,\ldots\}$ such that $\theta^{j+1}R/2 < \sigma \le \theta^j R/2$, and it is then easy to check that (8) actually implies

$$\sigma^{-n}\int_{B_\sigma(Y)} |u-\lambda_{Y,\sigma}|^2 \le 2^{n+2\alpha}\theta^{-n-2\alpha}(\sigma/R)^{2\alpha} I_0 \qquad \forall Y \in B_{R/2}(X_0),\ \sigma \in (0,R/2],$$

provided $2^n I_0 < \delta_0^2$. Then the required Hölder continuity is established by virtue of the Campanato Lemma of 1.1.

LECTURE 3
The Reverse Poincaré Inequality

3.1 Statement of Main Inequality

Let N be a smooth compact Riemannian manifold (without boundary) isometrically embedded in $\mathbf{R}^p$, let $\Omega \subset \mathbf{R}^n$ be open, and let $u : \Omega \to \mathbf{R}^p$ with $Du \in L^2$ locally in Ω, be energy minimizing amongst maps into N in the sense that $u(\Omega) \subset N$ and $\int_B |Du|^2 \le \int_B |Dv|^2$ whenever B is a ball with $\overline{B} \subset \Omega$, $v : B \to \mathbf{R}^p$, $v(B) \subset N$ and $v = u$ in a neighbourhood of ∂B.

Then we have the following:

Lemma 1. *If u is energy minimizing as described above, if Λ is a given constant, and if $R^{2-n} \int_{B_R(X_0)} |Du|^2 \le \Lambda$ for some ball $B_R(X_0)$ with closure contained in Ω, then*

$$\rho^{2-n} \int_{B_{\rho/2}(Y)} |Du|^2 \le C\rho^{-n} \int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2$$

for each $Y \in B_{R/2}(X_0)$, $\rho \le R/2$. Here $C = C(n, N, \Lambda) > 0$.

Remark: Notice that, once we have established this, we will have completed the proof that $u \in C^{0,\alpha}(B_{R/4}(X_0))$—subject to the same hypotheses as in the above lemma—by virtue of the Remark (2) following the regularity lemma of 2.1.

3.2 A Lemma of Luckhaus, and Some Corollaries

The following lemma is due to Luckhaus [Luck1] (see also [Luck2]), and extends the Lemma 4.3 of [SU].

Lemma 2. *Suppose N is an arbitrary compact subset of $\mathbf{R}^p$, $n \ge 2$ and $u, v : S^{n-1} \to \mathbf{R}^p$ with ∇u, $\nabla v \in L^2$ and $u(S^{n-1})$, $v(S^{n-1}) \subset N$. Then for each $\epsilon \in (0,1)$ there is a $w : S^{n-1} \times [0,\epsilon] \to \mathbf{R}^p$ such that $\overline{\nabla} w \in L^2$, $w|S^{n-1} \times \{0\} = u$, $w|S^{n-1} \times \{\epsilon\} = v$,*

$$\int_{S^{n-1}\times[0,\epsilon]} |\overline{\nabla} w|^2 \le C\epsilon \int_{S^{n-1}} (|\nabla u|^2 + |\nabla v|^2) + C\epsilon^{-1} \int_{S^{n-1}} |u - v|^2$$

and

$$\operatorname{dist}^2(w(x,s),N) \le C\epsilon^{1-n}\Big(\int_{S^{n-1}} |\nabla u|^2 + |\nabla v|^2\Big)^{1/2}\Big(\int_{S^{n-1}} |u-v|^2\Big)^{1/2} + C\epsilon^{-n}\int_{S^{n-1}} |u-v|^2$$

for a.e. $(x,s) \in S^{n-1}\times[0,\epsilon]$. *Here* ∇ *is the gradient on* S^{n-1} *and* $\overline{\nabla}$ *is the gradient on the product space* $S^{n-1}\times[0,\epsilon]$.

We give the proof of this in the next lecture, but for the moment we want to establish two useful corollaries:

Corollary 1. *Suppose* N *is a smooth compact manifold embedded in* $\mathbf{R}^p$ *and* $\Lambda > 0$. *There is* $\epsilon_0 = \epsilon_0(n,N,\Lambda) > 0$ *such that the following holds:*
If $\epsilon \in (0,1]$, $u : S^{n-1} \to \mathbf{R}^p$ *with* $\nabla u \in L^2$, $\int_{S^{n-1}} |\nabla u|^2 \le \Lambda$, $u(S^{n-1}) \subset N$, *and if there is* $\lambda \in \mathbf{R}^p$ *such that* $\epsilon^{-2n}\int_{S^{n-1}} |u-\lambda|^2 \le \epsilon_0^2$, *then there is a* $w : S^{n-1}\times[0,\epsilon] \to \mathbf{R}^p$ *such that* $\overline{\nabla} w \in L^2$, $w(S^{n-1}\times[0,\epsilon]) \subset N$,

$$w|S^{n-1}\times\{0\} = u, \quad w|S^{n-1}\times\{\epsilon\} \equiv const.,$$

$$\int_{S^{n-1}\times[0,\epsilon]} |\overline{\nabla} w|^2 \le C\epsilon\int_{S^{n-1}} |\nabla u|^2 + C\epsilon^{-1}\int_{S^{n-1}} |u-\lambda|^2, \quad C = C(n,N)$$

Proof. First note that, since $u(S^{n-1}) \subset N$, $\operatorname{dist}^2(\lambda,N) \le |u(x)-\lambda|^2$ for each $x \in S^{n-1}$. Thus, assuming for the moment that $\epsilon_0 \in (0,1]$ is arbitrary, by integrating over S^{n-1} we obtain

$$|S^{n-1}|\operatorname{dist}^2(\lambda,N) \le \int_{S^{n-1}} |u-\lambda|^2 \le \epsilon_0^2\epsilon^{2n},$$

so that $\operatorname{dist}(\lambda,N) \le C\epsilon_0\epsilon^{2n} \le C\epsilon_0$, where C depends only on n. Thus assuming that $C\epsilon_0 < \alpha$, where $\alpha > 0$ is such that the nearest point projection Π onto N is well-defined and smooth in $N_\alpha \equiv \{X \in R^p : \operatorname{dist}(X,N) \le \alpha\}$, we can apply Lemma 2 with $N_{C\epsilon_0}$ in place of N and with $v \equiv \lambda$ in order to deduce that there is a $w_0 : S^{n-1} \to \mathbf{R}^p$ such that $w_0|S^{n-1}\times\{0\} = u$, $w_0|S^{n-1}\times\{\epsilon\} \equiv \lambda$,

$$\int_{S^{n-1}\times[0,\epsilon]} |\overline{\nabla} w_0|^2 \le C\epsilon\int_{S^{n-1}} |\nabla u|^2 + C\epsilon^{-1}\int_{S^{n-1}} |u-\lambda|^2$$

and $\operatorname{dist}(w_0(x,s), N_{C\epsilon_0}) \le C(1+\Lambda^{1/4})\epsilon_0^{1/2}$. Thus if $C(1+\Lambda^{1/4})\epsilon_0^{1/2} \le \alpha$, then we can define $w = \Pi\circ w_0$. Since $dw = d\Pi_{w_0(x,s)}\circ dw_0$ at each point $(x,s) \in S^{n-1}\times[0,\epsilon]$, w is then a suitable function.

Corollary 2. *With the same hypotheses as in Corollary 1, there is* $w : B_1(0) \to \mathbf{R}^p$ *with* $w|S^{n-1} = u$, $w(B_1) \subset N$, *and*

$$\int_{B_1(0)} |Dw|^2 \le C\epsilon \int_{S^{n-1}} |\nabla u|^2 + C\epsilon^{-1} \int_{S^{n-1}} |u-\lambda|^2.$$

Proof. Let w_0 be the w from Corollary 1, let $\lambda_0 \in N$ be the constant such that $w_0|S^{n-1} \times \{\epsilon\} \equiv \lambda_0$, and define

$$w(r\omega) = \begin{cases} w_0(\omega, 1-r) \text{ if } r \in (1-\epsilon, 1) \\ \lambda_0 \text{ if } r \in (0, 1-\epsilon), \end{cases}$$

where $\omega = |X|^{-1}X$, $r = |X|$ for $X \in B_1(0)\backslash\{0\}$. Then w has the required properties.

There is a scaled version of Corollary 2, proved by applying Corollary 2 to the scaled function $u_\rho(X) = u(Y + \rho X)$ as follows:

Corollary 2 (Scaled version). *If* $\epsilon \in (0,1]$, *if* $u : \partial B_\rho(Y) \to \mathbf{R}^p$ *with* $\nabla^T u \in L^2$ *and with* $\rho^{3-n} \int_{\partial B_\rho(Y)} |\nabla^T u|^2 \le \Lambda$ *(*∇^T *the gradient on* $\partial B_\rho(Y)$*), and if also* $\epsilon^{-2n}\rho^{1-n} \int_{\partial B_\rho(Y)} |u-\lambda|^2 \le \epsilon_0^2$, *then there is* $w : B_\rho(Y) \to \mathbf{R}^p$ *with* $Dw \in L^2$, $w(B_\rho(Y)) \subset N$, $w|\partial B_\rho(Y) = u$, *and*

$$\rho^{2-n} \int_{B_\rho(Y)} |Dw|^2 \le C\epsilon\rho^{3-n} \int_{\partial B_\rho(Y)} |\nabla^T u|^2 + C\epsilon^{-1}\rho^{1-n} \int_{\partial B_\rho(Y)} |u-\lambda|^2.$$

Finally we want to prove the lemma used in Lecture 1 of the first lecture series. For this we need a further corollary of Lemma 2 above. In the proof we shall need the following important general fact about slicing by the radial distance function. Suppose $g \ge 0$ is integrable on $B_\rho(Y)$. Since $\int_{B_\rho(Y)} g = \int_0^\rho (\int_{\partial B_\sigma(Y)} g)\, d\sigma$, for each $\theta \in (0,1)$ we have

3.2(i)
$$\int_{\partial B_\sigma(Y)} g \le 2\theta^{-1}\rho^{-1} \int_{B_\rho(Y)} g$$

for all $\sigma \in (\rho/2, \rho)$ with the exception of a set of measure $\theta\rho/2$. (Indeed otherwise the reverse inequality would hold on a set of measure $> \theta\rho/2$ and by integration this would give $\int_{B_\rho(Y)} g < \int_0^\rho (\int_{\partial B_\sigma(Y)} g)\, d\sigma$.)

Furthermore, if w a function with L^2 gradient in Ω and if $\widetilde{w}$ is any representative for the L^2 class of w, then for each ball $\overline{B}_\rho(Y) \subset \Omega$

3.2(ii) $\widetilde{w}(\sigma)$ has gradient in $L^2_{\mathrm{loc}}(\mathbf{R}^n)$ and $\sigma^{n-1} \int_{S^{n-1}} |D\widetilde{w}(\sigma)|^2 = \int_{\partial B_\sigma(Y)} |D\widetilde{w}|^2$

for a.e. $\sigma \in (\rho/2, \rho)$. where $\widetilde{w}(\sigma)$ is the homogeneous degree zero function (relative to origin at Y) on $\mathbf{R}^n$ defined by $\widetilde{w}(\sigma)(X) \equiv \widetilde{w}(Y + \sigma\omega)$, $\omega = |X-Y|^{-1}(X-Y)$.

We can now state the third corollary. (This is Lemma 2 of Lecture 1 of the lecture series [SL1]).

Corollary 3. *Let $\Lambda > 0$ be given. There are constants $\epsilon_0 = \epsilon_0(n, N, \Lambda) > 0$ and $C = C(n, N, \Lambda)$ such that the following holds for any $\rho > 0$, $\epsilon \in (0, \epsilon_0)$:*
If $\overline{B}_{(1+\epsilon)\rho}(Y) \subset \Omega$, $u, v : B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y) \to \mathbf{R}^p$, if u, v have L^2 gradients Du, Dv with $\rho^{2-n}\int_{B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)}(|Du|^2 + |Dv|^2) \le \Lambda$, $u(X), v(X) \in N$ for $X \in B_{(1+\epsilon)\rho}(Y))\backslash B_\rho(Y)$, and if $\epsilon^{-2n}\rho^{-n}\int_{B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)} |u - v|^2 < \epsilon_0^2$, then there is w on $B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)$ such that $w = u$ in a neighbourhood of $\partial B_\rho(Y)$, $w = v$ in a neighbourhood of $\partial B_{(1+\epsilon)\rho}(Y)$, $w(B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)) \subset N$, and

$$\int_{B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)} |Dw|^2 \le C\int_{B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)} (|Du|^2 + |Dv|^2) + C\epsilon^{-2}\int_{B_{(1+\epsilon)\rho}(Y)\backslash B_\rho(Y)} |u - v|^2$$

Proof. By scaling and translation, we may evidently assume that $\rho = 1$ and $Y = 0$. We abbreviate $B_\sigma(0) = B_\sigma$. First note that by 3.2(i) with $\theta = \epsilon/4$ we have a set of $\sigma \in (1, 1 + \epsilon/4)$ of positive measure such that

(1)
$$\int_{\partial B_\sigma} (|Du|^2 + |Dv|^2), \int_{\partial B_{(1+\epsilon/2)\sigma}} (|Du|^2 + |Dv|^2) \le C\epsilon^{-1}\int_{B_{1+\epsilon}\backslash B_1} (|Du|^2 + |Dv|^2)$$

and

$$\int_{\partial B_\sigma} |u - v|^2 \le C\epsilon^{-1}\int_{B_{1+\epsilon}\backslash B_1} |u - v|^2 \le C\epsilon_0^2. \tag{2}$$

Also, by 3.2(ii) we know that a.e. these σ can be selected such that $u|\partial B_\sigma$, $v|\partial B_\sigma$ have L^2 gradients $\nabla^T u$, $\nabla^T v$ on ∂B_σ.

Now we can apply the Luckhaus lemma (with $\epsilon/2$ in place of ϵ) to the functions $\tilde{u}(\omega) \equiv u(\sigma\omega)$ and $\tilde{v}(\omega) = v(\sigma\omega)$, thus giving $\tilde{w}$ on $S^{n-1} \times [0, \epsilon/2]$ with $\tilde{w}|S^{n-1} \times \{0\} = \tilde{u}$, $\tilde{w}|S^{n-1} \times \{\epsilon/2\} = \tilde{v}$ and

$$\int_{S^{n-1}\times[0,\epsilon/2]} |\overline{\nabla}\tilde{w}|^2 \le C\epsilon\int_{S^{n-1}} (|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2) + C\epsilon^{-1}\int_{S^{n-1}} |\tilde{u} - \tilde{v}|^2$$

$$\operatorname{dist}^2(\tilde{w}(\omega, s), N) \le C\Big(\int_{S^{n-1}} (|\nabla\tilde{u}|^2 + |\nabla\tilde{v}|^2)\Big)^{1/2}\Big(\epsilon^{2-2n}\int_{S^{n-1}} |\tilde{u} - \tilde{v}|^2\Big)^{1/2} + C\epsilon^{-n}\int_{S^{n-1}} |\tilde{u} - \tilde{v}|^2 \le C\epsilon_0,$$

and in view of (1) and (2) we get

$$\int_{S^{n-1}\times[0,\epsilon/2]} |\nabla\tilde{w}|^2 \le C\int_{B_{1+\epsilon}\setminus B_1} (|Du|^2+|Dv|^2) + C\epsilon^{-2}\int_{B_{1+\epsilon}\setminus B_1} |u-v|^2$$

$$\operatorname{dist}^2(\tilde{w}(\omega,s),N) \le C\Big(\int_{B_{1+\epsilon}\setminus B_1} (|Du|^2+|Dv|^2)\Big)^{1/2}\Big(\epsilon^{-2n}\int_{B_{1+\epsilon}\setminus B_1} |u-v|^2\Big)^{1/2}$$
$$+ C\epsilon^{-n-1}\int_{B_{1+\epsilon}\setminus B_1} |u-v|^2 \le C\epsilon_0$$

with $C = C(n,N,\Lambda)$. Taking ϵ_0 (depending on n,N,Λ) small enough so that $C\epsilon_0 < \sigma_0^2$, where σ_0 is the width of a tubular neighbourhood of N in which the nearest point projection Π is smooth, we see that now the lemma is established if we define w as follows:

$$w(X) = \begin{cases} v(\varphi(|X|)X), & (1+\epsilon/2)\sigma \le |X| \le (1+\epsilon) \\ \Pi\circ\tilde{w}(|X|^{-1}X, |X|/\sigma - 1), & \sigma \le |X| \le (1+\epsilon/2)\sigma \\ u(X), & 1 \le |X| \le \sigma, \end{cases}$$

where $\varphi : [(1+\epsilon/2)\sigma, 1+\epsilon] \to [1+\epsilon/2, 1+\epsilon]$ is a C^1 function such that $\varphi(t) \equiv \sigma$ for t close to $(1+\epsilon/2)\sigma$, $\varphi(t) \equiv 1+\epsilon$ for t close to $1+\epsilon$, and $0 \le \varphi'(t) < 9$ for all t. (Notice that $(1+\epsilon/2)\sigma \le 1+7\epsilon/8$ because $\sigma \le 1+\epsilon/4$, and hence there exists such a function φ.)

3.3 Proof of the Reverse Poincaré Inequality

For any $\rho \le R/2$ and $Y \in B_{R/2}(X_0)$, we then have by the monotonicity (see 1.4 of the lectures [SL1]) that

(1)
$$\rho^{2-n}\int_{B_\rho(Y)} |Du|^2 \le (R/2)^{2-n}\int_{B_{R/2}(Y)} |Du|^2 \le 2^{n-2}R^{-n}\int_{B_R(X_0)} |Du|^2 \le 2^n\Lambda.$$

Take a fixed $\rho \in (0, R/2]$. We want to prove $\rho^{2-n}\int_{B_\rho(Y)} |Du|^2 \le C\rho^{-n}\int_{B_\rho(Y)} |u-\lambda_{Y,\rho}|^2$. By (1), there is no loss of generality in assuming that

(2)
$$\rho^{-n}\int_{B_\rho(Y)} |u-\lambda_{Y,\rho}|^2 \le \delta_0^2,$$

where δ_0 is to be chosen depending only on n, N, Λ. (Because if the reverse inequality to (2) holds, then by (1) we would trivially have the reverse Poincaré inequality with constant $C = 2^n\Lambda/\delta_0^2$.)

Applying the general fact 3.2(i) above with $g = |u - \lambda_{Y,\rho}|^2$ and $g = |Du|^2$, in view of (2) we then have

$$\tag{3} \begin{aligned} \sigma^{1-n} \int_{\partial B_\sigma(Y)} |u - \lambda_{Y,\rho}|^2 \le C\rho^{-n} \int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2 \le C\delta_0^2 \\ \sigma^{3-n} \int_{\partial B_\sigma(Y)} |Du|^2 \le C\rho^{2-n} \int_{B_\rho(Y)} |Du|^2 \le C\Lambda \end{aligned}$$

for all $\sigma \in (\rho/2, \rho)$ except for a set of measure $\rho/4$, where C depends only on n. Now, assuming $\sigma \in (\rho/2, \rho)$ is as in (3) and that $\delta_0/C\epsilon_0 \le 1$, where ϵ_0 is as in Corollary 2, we can apply the scaled version of Corollary 2 with $\lambda = \lambda_{Y,\rho}$ and $\epsilon = (\delta_0/C\epsilon_0)^{1/2n}$ to give $w : B_\sigma(Y) \to \mathbf{R}^p$ with $w|\partial B_\sigma(Y) = u|\partial B_\sigma(Y)$, $w(B_\sigma(Y)) \subset N$, and

$$\tag{4} \sigma^{2-n} \int_{B_\sigma(Y)} |Dw|^2 \le C\epsilon\rho^{2-n} \int_{B_\rho(Y)} |Du|^2 + C\epsilon^{-1}\rho^{-n} \int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2.$$

But since u is energy minimizing we have $\int_{B_\sigma(Y)} |Du|^2 \le \int_{B_\sigma(Y)} |Dw|^2$, and hence, since $\sigma > \rho/2$, (4) gives

$$\tag{5} \rho^{2-n} \int_{B_{\rho/2}(Y)} |Du|^2 \le C\epsilon\rho^{2-n} \int_{B_\rho(Y)} |Du|^2 + C\epsilon^{-1}\rho^{-n} \int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2.$$

Thus in particular

$$\tag{6} \rho^{2-n} \int_{B_{\rho/2}(Y)} |Du|^2 \le C\epsilon\Lambda + C\epsilon^{-1}\delta_0^2 \le C\delta_0^{1/2n}, \quad C = C(n, N, \Lambda).$$

In view of monotonicity (Cf. (1) above) we see that then also $\sigma^{2-n} \int_{B_\sigma(Z)} |Du|^2 \le C\delta_0^{1/2n}$ provided $B_\sigma(Z) \subset B_{\rho/2}(Y)$. Then the ordinary Poincaré inequality implies then that for any such ball $B_\sigma(Z)$ we have

$$\sigma^{-n} \int_{B_\sigma(Z)} |u - u_{Z,\sigma}|^2 \le C\delta_0^{1/2n}$$

provided only that (2) holds with suitably small δ_0, depending only on n, N, Λ. Hence if $\epsilon \in (0,1)$ is given and if we now choose $\delta_0 = \delta_0(n, N, \Lambda, \epsilon)$ small enough in (2) then we also have (5) (with this ϵ) uniformly for such balls $B_\sigma(Z)$. That is,

$$\tag{7} \begin{aligned} \sigma^2 \int_{B_{\sigma/2}(Z)} |Du|^2 &\le C\epsilon\sigma^2 \int_{B_\sigma(Z)} |Du|^2 + C\epsilon^{-1} \int_{B_\sigma(Z)} |u - \lambda_{Z,\sigma}|^2 \\ &\le C\epsilon\sigma^2 \int_{B_\sigma(Z)} |Du|^2 + C\epsilon^{-1} I_1 \end{aligned}$$

whenever $B_\sigma(Z) \subset B_{\rho/2}(Y)$, where $I_1 = \int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2$. (Here we used $\int_{B_\sigma(Z)} |u - \lambda_{Z,\sigma}|^2 \le \int_{B_\sigma(Z)} |u - \lambda_{Y,\rho}|^2 \le \int_{B_\rho(Y)} |u - \lambda_{y,\rho}|^2$.) Notice that ϵ is still to be chosen to depend only on n, N, Λ; for the moment we have (7) for arbitrary $\epsilon \in (0,1)$ provided only that (2) holds for suitable $\delta_0 = \delta_0(n, N, \Lambda, \epsilon)$.

Now in view of the arbitrariness of the balls $B_\sigma(Z)$ we claim that (7) implies the required reverse Poincaré inequality. To see this, first define

$$Q = \sup_{\{B_\sigma(Z) : B_{2\sigma}(Z) \subset B_{\rho/2}(Y)\}} \sigma^2 \int_{B_\sigma(Z)} |Du|^2,$$

and then take an arbitrary ball $B_\sigma(Z)$ with $B_{2\sigma}(Z) \subset B_{\rho/2}(Y)$. Notice that such a ball can be covered by balls $B_{\sigma/2}(Z_i)$, $i = 1, \dots, S$, with $Z_i \in B_\sigma(Z)$ and with $B_\sigma(Z_i) \subset B_{\rho/2}(Y)$; further we can evidently bound the number S by a fixed constant depending only on n. Now using (7) with Z_i in place of Z and summing over i we have

$$\sigma^2 \int_{B_\sigma(Z)} |Du|^2 \le C\epsilon S Q + C S \epsilon^{-1} I_1.$$

Taking sup on the left we thus have

$$Q \le C\epsilon S Q + C S \epsilon^{-1} I_1,$$

whereupon choosing ϵ (depending on n, N, Λ) such that $C\epsilon S \le \frac{1}{2}$, we have

$$\sigma^2 \int_{B_\sigma(Z)} |Du|^2 \le C I_1$$

for each ball $B_\sigma(Z)$ with $B_{2\sigma}(Z) \subset B_{\rho/2}(Y)$, where C depends only on n, N, Λ. Taking $Z = Y$ and $\sigma = \rho/4$ we thus have the reverse Poincaré $\rho^{2-n} \int_{B_{\rho/4}(Y)} |Du|^2 \le C\rho^{-n} \int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2$ for any ball $B_\rho(Y)$ with $\rho \le R/2$ and $Y \in B_{R/2}(X_0)$. On the other hand for any $Y \in B_{R/2}(X_0)$ and $\rho \le R/2$ we can cover $B_{\rho/2}(Y)$ by a collection of $Q = Q(n)$ balls $B_{\rho/8}(Y_j)$ with $Y_j \in B_{\rho/2}(Y)$. Since the above shows that $\rho^{2-n} \int_{B_{\rho/8}(Y_j)} |Du|^2 \le C\rho^{-n} \int_{B_{\rho/2}(Y_j)} |u - \lambda_{Y_j,\rho/2}|^2 \le C\rho^{-n} \int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2$ for each $j = 1 \dots Q$, the required reverse Poincaré $\rho^{2-n} \int_{B_{\rho/2}(Y)} |Du|^2 \le C\rho^{-n} \int_{B_\rho(Y)} |u - \lambda_{Y,\rho}|^2$ follows by summing over j.

LECTURE 4
Completion of the Regularity Proof

Here we want to tidy up the loose ends. Specifically, we want first to prove the Luckhaus lemma stated in §3.2 of the previous lecture (this will then complete the proof that an energy minimizer u is $C^{0,\alpha}$ in $B_{\rho/2}(Y)$ assuming that it satisfies $\rho^{-n}\int_{B_\rho(Y)}|u-\lambda_{Y,\rho}|^2 \le \delta_0^2$ with δ_0 sufficiently small), then we want to show that $u \in C^{1,\alpha}$ under the same hypotheses; standard linear PDE then shows that $u \in C^\infty$. We also want to point out that the modifications which are needed to establish the regularity theorem in case the domain Ω is equipped with an arbitrary smooth Riemannian metric are <u>very</u> minor.

4.1 A Further Property of Functions with L^2 Gradient

We note an important property for L^2 functions ψ on a cube $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ with $D\psi \in L^2$: there is a representative $\overline{\psi}$ for the L^2 class of ψ such that, for each $j = 1, \ldots, n$, $\overline{\psi}(x^1, \ldots, x^{j-1}, x^j, x^{j+1}, \ldots, x^n)$ is an absolutely continuous function of x^j for almost all fixed values of $(x^1, \ldots, x^{j-1}, x^{j+1}, \ldots, x^n)$. Here of course "almost all" is with respect to $(n-1)$-dimensional Lebesgue measure on the $(n-1)$-dimensional cube $[a_1, b_1] \times \ldots \times [a_{j-1}, b_{j-1}] \times [a_{j+1}, b_{j+1}] \times \ldots \times [a_n, b_n]$. Furthermore, the <u>classical</u> partial derivatives $D_j\overline{\psi}$ (defined in the usual way by $D_j\overline{\psi}(X) = \lim t^{-1}(\overline{\psi}(X+te_j) - \overline{\psi}(X))$ whenever this exists) agree a.e. with the L^2 derivatives $D_j\psi$. A discussion of these properties can be found in e.g. [GT] or [MCB]. We here make one further point: one procedure for constructing such a representative (see e.g. [MCB]) $\overline{\psi}$ is to define $\overline{\psi}(X) = \lambda_X$ at all points where there exists a λ_X such that $\lim_{\rho\downarrow 0}\rho^{-n}\int_{B_\rho(Y)}|\lambda_X - u(Y)|\,dY = 0$, and to define $\overline{\psi}$ arbitrarily (e.g. $\overline{\psi}(X) = 0$) at points where the limit does not exist. Thus in particular if $\psi = (\psi^1, \ldots, \psi^p) : Q \to \mathbf{R}^p$, and if N is any closed subset of $\mathbf{R}^p$ with the property that $\psi(X) \in N$ for almost all $X \in \Omega$, then we can select the representative $\overline{\psi}$ to have the property $\overline{\psi}(X) \in N$ for <u>every</u> $X \in \Omega$, in addition to the absolute continuity properties mentioned above.

4.2 Proof of Luckhaus' Lemma

In the case $n = 2$ the functions u, v have an L^2 gradient on S^1 and hence have absolutely continuous representatives $\overline{u}$, $\overline{v}$ such that $\nabla\overline{u} = \nabla u$, $\nabla\overline{v} = \nabla v$ a.e., where ∇ denotes the gradient on S^1. Furthermore, by 1-dimensional calculus on S^1 and by the Cauchy-Schwarz inequality we have

$$\begin{aligned}\sup_{S^1}|\overline{u}-\overline{v}|^2 &\le \int_{S^1}|\nabla|\overline{u}-\overline{v}|^2| + (2\pi)^{-1}\int_{S^1}|\overline{u}-\overline{v}|^2\\ &\le C\Big(\int_{S^1}|\nabla(\overline{u}-\overline{v})|^2\Big)^{1/2}\Big(\int_{S^1}|\overline{u}-\overline{v}|^2\Big)^{1/2} + C\int_{S^1}|\overline{u}-\overline{v}|^2,\end{aligned}$$

and hence

$$\sup_{S^1}|\overline{u}-\overline{v}| \le C\Big(\int_{S^1}|\nabla(\overline{u}-\overline{v})|^2\Big)^{1/4}\Big(\int_{S^1}|\overline{u}-\overline{v}|^2\Big)^{1/4} + C\Big(\int_{S^1}|\overline{u}-\overline{v}|^2\Big)^{1/2}, \tag{1}$$

If we now define

$$w(\omega, s) = \overline{u}(\omega) + \frac{s}{\epsilon}(\overline{v}(\omega) - \overline{u}(\omega)),$$

then, letting $\overline{\nabla}w$ denote the gradient of w on $S^1 \times [0,\epsilon]$, we have

$$|\overline{\nabla}w| \le |\nabla\overline{u}| + |\nabla(\overline{v}-\overline{u})| + \frac{1}{\epsilon}|\overline{v}-\overline{u}|,$$

and hence

$$|\overline{\nabla}w|^2 \le 8(|\nabla\overline{u}|^2 + |\nabla\overline{v}|^2) + 2\epsilon^{-2}|\overline{v}-\overline{u}|^2. \tag{2}$$

Notice that, since $\overline{u}(S^1) \subset N$, (1) implies that for each $\omega \in S^1$, $s \in [0,\epsilon]$,

$$\operatorname{dist}(w(\omega,s),N) \le C\Big(\int_{S^1}|\nabla(\overline{u}-\overline{v})|^2\Big)^{1/4}\Big(\int_{S^1}|\overline{u}-\overline{v}|^2\Big)^{1/4} + C\Big(\int_{S^1}|\overline{u}-\overline{v}|^2\Big)^{1/2},$$

and by integrating (2) over $S^1 \times [0,\epsilon]$ the remaining inequality also follows.

This completes the proof in the case $n = 2$, so from now on assume $n \ge 3$, and again choose absolutely continuous representatives $\overline{u}$, $\overline{v}$ such that $\nabla\overline{u} = \nabla u$, $\nabla\overline{v} = \nabla v$ a.e. on S^{n-1}.

Without changing notation, we extend $\overline{u}$, $\overline{v}$ to give homogeneous degree zero functions on $\mathbf{R}^n$; thus $\overline{u}(r\omega) \equiv \overline{u}(\omega)$ for $r > 0$ and $\omega \in S^{n-1}$, and $D\overline{u} = \nabla u$ on S^{n-1}, $D\overline{v} = \nabla v$ on S^{n-1}. Notice also that $D\overline{u}(X) = |X|^{-1}D\overline{u}(\omega)$, with $\omega = |X|^{-1}X$, and hence (since $n \ge 3$) we have in particular that

$$\int_{[-1,1]^n}(|D\overline{u}|^2 + |D\overline{v}|^2) \le C\int_{S^{n-1}}(|\nabla u|^2 + |\nabla v|^2), \tag{3}$$

and also of course

(4)
$$\int_{[-1,1]^n} |\overline{u} - \overline{v}|^2 \le C \int_{S^{n-1}} |u - v|^2.$$

Now for $\epsilon \in (0, \frac{1}{8})$ and $i = (i_1, \dots, i_n) \in Z^n$, we let $Q_{i,\epsilon}$ denote the cube $[i_1\epsilon, (i_1 + 1)\epsilon] \times \cdots \times [i_n\epsilon, (i_n + 1)\epsilon]$, and for a given non-negative measurable function $f : [-1, 1]^n \to \mathbf{R}$ we let $\widetilde{f} : Q_{i,\epsilon} \to \mathbf{R}$ be defined by

$$\widetilde{f}(X) = \sum_{\{i : Q_{i,\epsilon} \subset [-\frac{1}{2}, \frac{1}{2}]^n\}} f(X + \epsilon i), \quad X \in Q_{i,\epsilon}.$$

Then

$$\int_{Q_{0,\epsilon}} \widetilde{f}(X)\, dX = \int_{\cup\{Q_{i,\epsilon} : Q_{i,\epsilon} \subset [-\frac{1}{2}, \frac{1}{2}]^n\}} f(X)\, dX \le \int_{[-1,1]^n} f(X)\, dX,$$

and hence for any $K \ge 1$ we have

$$\epsilon^n \sum_{\{i : Q_{i,\epsilon} \subset [-\frac{1}{2}, \frac{1}{2}]^n\}} f(X + \epsilon i) \le K \int_{[-1,1]^n} f(X)\, dX$$

for all $X \in Q_{0,\epsilon}$ with the exception of a set of measure $\le C\epsilon^n/K$, where $C = C(n)$.

Similarly, since by Fubini's theorem

$$\int_{Q_{0,\epsilon}} \int_0^\epsilon \widetilde{f}(X + te_n)\, dt dX \le \epsilon \int_{Q_{0,\epsilon}} \widetilde{f}(X)\, dX \le \epsilon \int_{[-1,1]^n} f(X)\, dX,$$

we have

$$\epsilon^{n-1} \sum_{\{i : Q_{i,\epsilon} \subset [-\frac{1}{2}, \frac{1}{2}]^n\}} \int_0^\epsilon f(X + te_n + \epsilon i)\, dt \le K \int_{[-1,1]^n} f(X)\, dX$$

for all $X \in Q_{0,\epsilon}$ with the exception of a set of measure $\le C\epsilon^n/K$, and generally, for any $\ell \in \{0, \dots, n\}$,

$$\epsilon^{n-\ell} \sum_{\{i : Q_{i,\epsilon} \subset [-\frac{1}{2}, \frac{1}{2}]^n\}} \int_{F^{(\ell)}} f(X + Y + \epsilon i)\, d\mathcal{H}^\ell(Y) \le K \int_{[-1,1]^n} f(X)\, dX$$

for all ℓ-faces $F^{(\ell)}$ of $Q_{0,\epsilon}$ and all $X \in Q_{0,\epsilon}$ with the exception of a set of measure $\le C\epsilon^n/K$. Notice that this last inequality implies

(5)
$$\epsilon^{n-\ell} \sum_{\{i : Q_{i,\epsilon} \subset [-\frac{1}{2}, \frac{1}{2}]^n\}} \sum_{\ell\text{-faces } F^{(\ell)} \text{ of } X + Q_{i,\epsilon}} \int_{F^{(\ell)}} f(Y)\, d\mathcal{H}^\ell(Y) \le K \int_{[-1,1]^n} f(X)\, dX$$

for all $\ell \in \{0,\dots,n\}$ and all $X \in Q_{0,\epsilon}$ with the exception of a set of measure $\le C\epsilon^n/K$.

Now by the absolute continuity properties of 4.1, we can select representatives $\overline{u}$, $\overline{v}$, $D\overline{u}$, $D\overline{v}$, for the L^2 classes of u, v, Du, Dv such that for almost all $X \in Q_{0,\epsilon}$ all of the functions $\overline{u}$, $\overline{v}$, $D\overline{u}$, $D\overline{u}$ are defined $\mathcal{H}^\ell$-a.e. on each of the ℓ-dimensional faces F^ℓ of each of the cubes $X + Q_{i,\epsilon}$ with $Q_{i,\epsilon} \subset [-\frac{1}{2}, \frac{1}{2}]^n$ for each $\ell = 0,\dots,n$, and such that furthermore, on each such ℓ-dimensional face, $\overline{u}$, $\overline{v}$ have L^2-gradients which coincide $\mathcal{H}^\ell$-a.e. with the tangential parts of $D\overline{u}$, $D\overline{v}$. Now by applying (5) with $f = |\overline{u} - \overline{v}|^2$ and $f = |D\overline{u}|^2 + |D\overline{v}|^2$ we see that we can select $X = a \in Q_{0,\epsilon}$ such that the above properties hold and also such that

(6)
$$\epsilon^{n-\ell} \sum_{\{i : Q_{i,\epsilon} \subset [-\frac{1}{2},\frac{1}{2}]^n\}} \sum_{\ell\text{-faces } F^{(\ell)} \text{ of } a+Q_{i,\epsilon}} \int_{F^{(\ell)}} f(Y)\, d\mathcal{H}^\ell(Y)$$
$$\le C \int_{[-1,1]^n} f(X)\, dX \text{ with } f = |\overline{u} - \overline{v}|^2 \text{ or } f = |D\overline{u}|^2 + |D\overline{v}|^2,$$

for each $\ell \in \{0,\dots,n\}$, where $C = C(n)$.

Next, let Q be any one of the cubes $a + Q_{i,\epsilon}$ with $Q_{i,\epsilon} \subset [-\frac{1}{2}, \frac{1}{2}]^n$, and we proceed to define a $W^{1,2}$ function $w = w^{(i,\epsilon)}$ on $Q \times [0,\epsilon]$ which agrees with $\overline{u}$ on $Q \times \{0\}$ at all points of Q where $\overline{u}$ exists, agrees with $\overline{v}$ on $Q \times \{\epsilon\}$ at all points of Q where $\overline{v}$ exists, and which is such that

$$\text{(7)} \quad \int_{Q\times[0,\epsilon]} |Dw|^2 \le C \sum_{j=1}^{n-1} \epsilon^{n-j+1} \sum_{\text{all } j\text{-faces } F^{(j)} \text{ of } Q} \int_{F^{(j)}} (|D\overline{u}|^2 + |D\overline{v}|^2)$$
$$+ C\epsilon^{n-2} \sum_{\text{all 1-faces } F^{(1)}} \int_{F^{(1)}} |u - v|^2$$

(8) $\operatorname{dist}^2(w(X), N) \le$
$$C \max_{\text{1-faces } F^{(1)} \text{ of } Q} \Big(\Big(\int_{F^{(1)}} |D(\overline{u} - \overline{v})|^2\Big)^{1/2} \Big(\int_{F^{(1)}} |\overline{u} - \overline{v}|^2\Big)^{1/2} + C\epsilon^{-1} \int_{F^{(1)}} |\overline{u} - \overline{v}|^2\Big).$$

Let E be any one of the edges (i.e. 1-dimensional faces) of Q. By 1-dimensional calculus along the line segment E, we have (since the length of E is ϵ)

$$\text{(9)} \qquad \sup_E |\overline{u} - \overline{v}|^2 \le \int_E |D|\overline{u} - \overline{v}|^2| + \epsilon^{-1} \int_E |\overline{u} - \overline{v}|^2.$$

Hence by using the Cauchy-Schwarz inequality we obtain

$$\text{(10)} \quad \sup_E |\overline{u} - \overline{v}|^2 \le C\Big(\int_E |D(\overline{u} - \overline{v})|^2\Big)^{1/2} \Big(\int_E |\overline{u} - \overline{v}|^2\Big)^{1/2} + C\epsilon^{-1} \int_E |\overline{u} - \overline{v}|^2,$$

where C is a constant depending only on n.

Then we can define an $\mathbf{R}^p$-valued function w on $Q \times [0,\epsilon]$ by the following inductive procedure. We first define w on $Q \times \{0\}$ and $Q \times \{\epsilon\}$ by

$$w(X,0) = \overline{u}(X), \qquad w(X,\epsilon) = \overline{v}(X), \qquad X \in Q. \tag{11}$$

Next we extend w to each $F^{(1)} \times [0,\epsilon]$, where $F^{(1)}$ is any 1-dimensional face (i.e. edge) of Q, by defining

$$w(X,s) = (1 - \frac{s}{\epsilon})\overline{u}(X) + \frac{s}{\epsilon}\overline{v}(X), \qquad X \in F^{(1)},\ s \in [0,\epsilon].$$

Notice that then by (10) and the fact that $\overline{u}(\mathbf{R}^n) \subset N$ we have
(12)

$$\operatorname{dist}^2(w(X,s),N) \le \max_{\text{1-faces } F^{(1)} \text{ of } Q} \sup_{F^{(1)}} |\overline{v} - \overline{u}|^2 \le$$

$$C \max_{\text{1-faces } F^{(1)} \text{ of } Q} ((\int_{F^{(1)}} |D(\overline{u} - \overline{v})|^2)^{1/2} (\int_{F^{(1)}} |\overline{u} - \overline{v}|^2)^{1/2} + C\epsilon^{-1} \int_{F^{(1)}} |\overline{u} - \overline{v}|^2).$$

Also notice that by direct computation

$$\sup_{s \in [0,\epsilon]} |\overline{D}w(X,s)|^2 \le 8(|D\overline{u}(X)|^2 + |D\overline{v}(X)|^2) + \frac{2}{\epsilon^2}|\overline{u}(X) - \overline{v}(X)|^2 \tag{13}$$

at any X in any edge $F^{(1)}$ of Q (where $\overline{D} = (\frac{\partial}{\partial X}, \frac{\partial}{\partial s})$, $\frac{\partial}{\partial X}$ = gradient on $F^{(1)}$), hence

$$\int_{F^{(1)} \times [0,\epsilon]} |\overline{D}w|^2 \le C\epsilon \int_{F^{(1)}} (|D\overline{u}|^2 + |D\overline{v}|^2) + C\epsilon^{-1} \int_{F^{(1)}} |\overline{u} - \overline{v}|^2. \tag{14}$$

For $\ell \ge 2$ we now proceed inductively by homogeneous extension into faces of larger and larger dimension. More precisely, assume $\ell \ge 2$, and that w is already defined (with L^2 gradient) on all $F^{(\ell-1)} \times [0,\epsilon]$ and $w(X,0) \equiv \overline{u}(X)$, $w(X,\epsilon) \equiv \overline{v}(X)$ on $F^{(\ell)}$. Since $\partial(F^{(\ell)} \times [0,\epsilon])$ is the union of $F^{(\ell-1)} \times [0,\epsilon]$ (over the $\ell - 1$ faces $F^{(\ell-1)}$ of $F^{(\ell)}$) together with $F^{(\ell)} \times \{0\}$ and $F^{(\ell)} \times \{\epsilon\}$, we then have that w is already well defined $\mathcal{H}^\ell$-a.e. on $\partial(F^{(\ell)} \times [0,\epsilon])$. We can thus use homogeneous degree zero extension of $w|\partial(F^{(\ell)} \times [0,\epsilon])$ into $F^{(\ell)} \times [0,\epsilon]$ with origin at the point $(q,\epsilon/2)$, where q is the center point of $F^{(\ell)}$. Then by direct computation we have

$$\int_{F^{(\ell)} \times [0,\epsilon]} |\overline{D}w|^2 \le C\epsilon \int_{F^{(\ell)}} (|D\overline{u}|^2 + |D\overline{v}|^2) + C\epsilon \sum_{\text{all } F^{(\ell-1)}} \int_{F^{(\ell-1)} \times [0,\epsilon]} |\overline{D}w|^2, \tag{15}$$

where $\overline{D} = (\frac{\partial}{\partial X}, \frac{\partial}{\partial s})$, $\frac{\partial}{\partial X}$ = gradient on $F^{(\ell)}$ on the left and on $F^{(\ell-1)}$ on the right. So by mathematical induction based on (15) we conclude that, for all $\ell \in \{2, \dots, n\}$,

w can be extended to all of $F^{(\ell)} \times [0,\epsilon]$ ($F^{(\ell)}$ = any ℓ-face of Q) such that w has L^2 gradient $\overline{D}w$ on all $F^{(\ell)} \times [0,\epsilon]$ with

(16)

$$\sum_{\text{all } \ell\text{-faces } F^{(\ell)} \text{ of } Q} \int_{F^{(\ell)}\times[0,\epsilon]} |\overline{D}w|^2 \le C\epsilon^{\ell-1} \sum_{\text{all 1-faces } F^{(1)} \text{ of } Q} \int_{F^{(1)}\times[0,\epsilon]} |\overline{D}w|^2$$
$$+ C\sum_{j=1}^{\ell} \epsilon^{\ell-j+1} \sum_{\text{all } j\text{-faces } F^{(j)} \text{ of } Q} \int_{F^{(j)}} (|D\overline{u}|^2 + |D\overline{v}|^2)$$

Furthermore notice that the homogeneous degree zero extension preserves the bound (12). Thus by (12), (14), (16) (with $\ell = n$) we conclude the existence of $w : Q \to \mathbf{R}^p$ as in (5) and (6). Since $Q = a + Q_{i,\epsilon}$, we should write $w = w^{(i,\epsilon)}$. Since the construction of $w^{(i,\epsilon)}$ is such as to ensure that $w^{(i,\epsilon)} = w^{(j,\epsilon)}$ $\mathcal{H}^{\ell+1}$-a.e. on $F^{(\ell)} \times [0,\epsilon]$ for any common ℓ-face $F^{(\ell)}$ of two different cubes $a + Q_{i,\epsilon}$ and $a + Q_{j,\epsilon}$, we can then define a $W^{1,2}$ function w on $[-\frac{1}{4},\frac{1}{4}]^n$ by setting $w(X,s) = w^{(i,\epsilon)}(X,s)$ for $(X,s) \in (a + Q_{i,\epsilon}) \times [0,\epsilon]$. (We are assuming $\epsilon \in (0,\frac{1}{8})$ and $a \in Q_{0,\epsilon}$, hence we automatically have that $[-\frac{1}{4},\frac{1}{4}]^n \subset \cup\{Q_{i,\epsilon} : a + Q_{i,\epsilon} \subset [-\frac{1}{2},\frac{1}{2}]^n.\}$) Notice that then by summing for i in (7) and also using (6) and (8) we get

$$\text{(17)} \quad \int_{[-\frac{1}{4},\frac{1}{4}]^n\times[0,\epsilon]} |\overline{D}w|^2 \le C\epsilon \int_{[-1,1]^n} (|Du|^2 + |Dv|^2) + C\epsilon^{-1} \int_{[-1,1]^n} |u-v|^2$$

and

(18)

$$\operatorname{dist}^2(w(X),N) \le$$
$$C \max_{\text{1-faces } F^{(1)}\in\mathcal{F}} \Big(\int_{F^{(1)}} |D(\overline{u}-\overline{v})|^2\Big)^{1/2}\Big(\int_{F^{(1)}} |\overline{u}-\overline{v}|^2\Big)^{1/2} + C\epsilon^{-1}\Big(\int_{F^{(1)}} |\overline{u}-\overline{v}|^2\Big)$$
$$\le C\epsilon^{1-n}\Big(\int_{[-1,1]^n} (|Du|^2 + |Dv|^2)\Big)^{1/2}\Big(\int_{[-1,1]^n} |u-v|^2\Big)^{1/2} + C\epsilon^{-n}\int_{[-1,1]^n} |u-v|^2,$$

where $\mathcal{F}$ denotes the collection of all 1-faces of cubes in the collection $\{a + Q_{i,\epsilon} : Q_{i,\epsilon} \subset [-\frac{1}{2},\frac{1}{2}]^n\}$.

Define $\widetilde{w} = w|([-\frac{1}{4},\frac{1}{4}]^n\backslash[-\frac{1}{8},\frac{1}{8}]^n) \times [0,\epsilon]$, and using 4.1 choose $\rho \in [\frac{1}{8},\frac{1}{4}]$ such that w has L^2-gradient on $\partial([-\rho,\rho]^n) \times [0,\epsilon]$ and such that

$$\int_{\partial([-\rho,\rho]^n)\times[0,\epsilon]} |\overline{D}w|^2 \le C\int_{[-\frac{1}{4},\frac{1}{4}]^n\times[0,\epsilon]} |\overline{D}w|^2.$$

Then finally let Ψ be the radial map from 0 taking S^{n-1} to $\partial([-\rho,\rho]^n)$ (notice that this is a Lipschitz piecewise C^1 map with a Lipschitz piecewise C^1 inverse). Thus we can define $\hat{w}$ on $S^{n-1} \times [0,\epsilon]$ by $\hat{w}(\omega,s) = \widetilde{w}(\Psi(\omega),s)$, and one then readily checks that this map $\hat{w}$ has the properties claimed for w in the statement of the Luckhaus lemma. (In particular, since $w(X,0) \equiv \overline{u}(X)$ for $X \in \partial([-\rho,\rho]^n)$ and since u is homogeneous of degree zero in $\mathbf{R}^n$, we then have by definition that $\hat{w}(\omega,0) \equiv u(\omega)$ a.e. on S^{n-1}. Similarly $\hat{w}(\omega,\epsilon) \equiv v(\omega)$ a.e. on S^{n-1}.)

This completes the proof of the Luckhaus lemma, and hence the proof that u is $C^{0,\alpha}$.

4.3 Proof of $C^{1,\alpha}$ and Higher Regularity

Here we assume the hypotheses are as in the abstract regularity lemma 2.1. First let $B_\rho(Y) \subset B_{R/2}(X_0)$ be arbitrary. We assume without loss of generality that $R = 1$, so that the Hölder continuity for u proved above gives

$$|u(X_1) - u(X_2)| \le C\rho^\alpha, \qquad X_1, X_2 \in \overline{B}_\rho(Y),\ 0 < \rho < 1/2. \tag{1}$$

Since $u \in W^{1,2}(B_\rho(Y)) \cap C^0(\overline{B}_\rho(Y))$, we know (see e.g. [GT]) that there is a $v \in C^2(B_\rho(Y); \mathbf{R}^p) \cap C^0(\overline{B}_\rho(Y); \mathbf{R}^p) \cap W^{1,2}(B_\rho(Y))$ which is harmonic on $B_\rho(X_0)$ and which agrees with u on $\partial B_\rho(Y)$. Of course u satisfies the weak form of Laplace's equation on $B_\rho(Y)$; that is,

$$\int_{B_\rho(Y)} \sum_{j=1}^n D_j v D_j \varphi = 0, \quad \varphi \in C_c^\infty(B_\rho(Y); \mathbf{R}^p).$$

Taking the difference of this equation and the weak form of the equation for u, we thus get that

$$\int_{B_\rho(Y)} \sum_{j=1}^n D_j(u - v) D_j \varphi = \int_{B_\rho(Y)} \varphi \cdot F, \quad \varphi \in C_c^\infty(B_\rho(Y); \mathbf{R}^p).$$

Now since u, v agree on the boundary it is easy to check that this is also valid with the choice $\varphi = v - u$; here we use the general fact that if a $C^0(\overline{B}_\rho(Y)) \cap W^{1,2}(B_\rho(Y))$ function is zero on $\partial B_\rho(Y)$, then it is the limit in the $W^{1,2}(B_\rho(Y))$ norm of a sequence of $C_c^\infty(B_\rho(Y))$ functions. Thus we obtain

$$\begin{aligned}\int_{B_\rho(Y)} |D(u-v)|^2 &= \int_{B_\rho(Y)} (u - v) \cdot F \\ &\le \beta \sup|u - v| \int_{B_\rho(Y)} |Du|^2 \le C\rho^\alpha \int_{B_\rho(Y)} |Du|^2,\end{aligned} \tag{2}$$

where we used the fact that $\sup_{B_\rho(Y)} |u - u(X_0)| \le C\rho^\alpha$ (by (1)) for any $Y_0 \in \overline{B}_\rho(Y)$ and that, for $Y_0 \in \partial B_\rho(Y)$, $\sup_{B_\rho(Y)} |v - u(Y_0)| \equiv \sup_{B_\rho(Y)} |v - v(Y_0)| \le p \sup_{\partial B_\rho(Y)} |u - u(X_0)| \le C\rho^\alpha$ by applying the maximum principle to each component v^j of $v = (v^1, \ldots, v^p)$. Now on the other hand we have, by using the reverse Poincaré inequality on the right of (2) and again using (1),

$$\rho^{-n} \int_{B_\rho(Y)} |D(u - v)|^2 = C\rho^{3\alpha - 2}, \qquad \rho \in (0, 1/4). \tag{3}$$

Now let us agree that α was chosen in the first place so that $3\alpha > 2$, and that hence $3\alpha = 2 + 2\mu$ for some $\mu > 0$. Thus we have

$$\rho^{-n} \int_{B_\rho(Y)} |D(u-v)|^2 = C\rho^{2\mu}, \quad \rho \in (0, 1/4)$$

with $2\mu = 3\alpha - 2$ depending only on n, β, ϵ_0, and this is all valid uniformly for $Y \in B_{1/4}(X_0)$. Thus we have

$$\begin{aligned}\sigma^{-n} \int_{B_\sigma(Y)} &|Du - Dv(Y)|^2 \\ &\leq 2\sigma^{-n} \int_{B_\sigma(Y)} |D(u-v)|^2 + 2\sigma^{-n} \int_{B_\sigma(Y)} |Dv - Dv(Y)|^2 \\ &\leq C(\rho/\sigma)^n \rho^{2\mu} + 2\sigma^2 \sup_{B_\sigma(Y)} |D^2 v|^2.\end{aligned}$$

Now by the estimates 1.3(ii) for harmonic functions we have

$$\begin{aligned}2\sigma^2 \sup_{B_\sigma(Y)} |D^2 v|^2 \leq 2\sigma^2 \sup_{B_{\rho/2}(Y)} |D^2 v|^2 &\leq C\sigma^2 \rho^{-2} \sup_{B_{3\rho/4}(Y)} |Dv - \Lambda_{Y,\rho}|^2 \\ &\leq C\sigma^2 \rho^{-n-2} \int_{B_\rho(Y)} |Dv - \Lambda_{Y,\rho}|^2,\end{aligned}$$

with $\Lambda_{Y,\rho} = |B_\rho(Y)|^{-1} \int_{B_\rho(Y)} Du$. (Note that the last estimates would be valid with any constant Λ in place of $\Lambda_{Y,\rho}$.) Now on the other hand

$$\begin{aligned}\rho^{-n} \int_{B_\rho(Y)} |Dv - \Lambda_{Y,\rho}|^2 &\leq 2\rho^{-n} \int_{B_\rho(Y)} |D(u-v)|^2 + 2\rho^{-n} \int_{B_\rho(Y)} |Du - \Lambda_{Y,\rho}| \\ &\leq C\rho^{2\mu} + C\rho^{-n} \int_{B_\rho(Y)} |Du|^2,\end{aligned}$$

and by the reverse Poincaré and the Hölder estimate for u we have

$$\rho^{2-n} \int_{B_\rho(Y)} |Du|^2 \leq C\rho^{-n} \int_{B_{2\rho}(Y)} |u - \lambda_{Y,2\rho}|^2 \leq C\rho^{2\alpha}.$$

Thus the above inequalities combine to give

$$I_\sigma \leq C(\rho/\sigma)^n \rho^{3\alpha-2} + C(\sigma/\rho)^2 \rho^{-2+2\alpha},$$

where $I_\sigma = \sigma^{-n} \int_{B_\sigma(Y)} |Du - \Lambda_{Y,\sigma}|^2$, and this is valid for all $\sigma < \rho < 1/4$. Choosing $\sigma = \rho^\kappa$ with $\kappa = 1 + \alpha/(n+2)$, we then have

$$I_\sigma \leq C\sigma^{2\gamma}$$

for any $\gamma < 2/(\kappa(n+2))$, provided α is chosen sufficiently close to 1, and this holds for any $\sigma \le R/4$ and any $Y \in B_{1/4}(X_0)$, with C depending on α, n and the constant β. Therefore the Campanato lemma of 1.1 can be applied to give $Du \in C^{0,\gamma}$ on $B_{1/4}(X_0)$. In particular this gives the boundedness of Du on $B_{1/4}(X_0)$, and hence the original equation has the form $\Delta u = f$, with f bounded on $B_{1/4}(X_0)$. But then the standard $C^{1,\alpha}$-Schauder estimates (see e.g. [GT]) give that u is $C^{1,\alpha}$ for each $\alpha < 1$, as required, on $B_{R/4}(X_0)$.

But now we can "bootstrap"; we have shown that u is a $C^{1,\alpha}$ weak solution of the equation

$$* \qquad \Delta u = -\sum_{j=1}^{n} A_u(D_j u, D_j u)$$

in $B_{R/4}(X_0)$. But since $u \in C^{1,\alpha}$, it is clear that the right side here is in $C^{0,\alpha}$ on $B_{R/4}(X_0)$ and hence the equation has the form $\Delta u = f$ with $\in C^{0,\alpha}$. But the standard Schauder theory (see [GT]) for equations $\Delta u = f$ now implies that $u \in C^{2,\alpha}$ on $B_{R/4}(X_0)$, and then the right side in $*$ is of class $C^{1,\alpha}$, so now u satisfies an equation of the form $\Delta u = f$ with $f \in C^{1,\alpha}$, and the standard Schauder theory this time gives $u \in C^{3,\alpha}$. Continuing in this way we deduce inductively that $u \in C^{k,\alpha}$ on $B_{R/4}(X_0)$ for each integer $k \ge 1$. We here used the qualitative part of the Schauder theory (that if $\Delta u = f$ in a domain Ω with $f \in C^{k,\alpha}(\Omega)$, then $u \in C^{2,\alpha}(\Omega)$); of course we can also use the quantitative part of the Schauder theory (i.e. the relevant Schauder estimates) at each stage in order to deduce the bounds

$$(4) \qquad R^j \sup_{B_{R/8}(X_0)} |D^j u| \le C\epsilon$$

(assuming $R^{2-n}\int_{B_R(X_0)} |Du|^2 \le \Lambda$ and $R^{-n}\int_{B_R(X_0)} |u - \lambda_{X_0,R}|^2 < \epsilon(n, N, \Lambda)^2$). This completes the proof of the regularity theory claimed in 1.6 of the first lecture series, except that we have the required estimates only over $B_{R/8}(X_0)$ rather that $B_{R/2}(X_0)$ as originally claimed. But this evidently follows, because the original hypotheses $R^{2-n}\int_{B_R(X_0)} |Du|^2 \le \Lambda$ and $R^{-n}\int_{B_R(X_0)} |u - \lambda_{X_0,R}|^2 < \epsilon(n, N, \Lambda)^2$ imply that $(R/2)^{2-n}\int_{B_{R/2}(Z)} |Du|^2 \le 2^{n-2}\Lambda$ and $(R/2)^{-n}\int_{B_{R/2}(Z)} |u - \lambda_{Z,R/2}|^2 < 2^n\epsilon^2$ uniformly for $Z \in B_{R/2}(X_0)$, and so the original hypotheses on $B_R(X_0)$ with new $\epsilon(= 2^{-n}\epsilon(n, N, 2^{n-2}\Lambda))$ gives us the required hypotheses on each of the balls $B_{R/2}(Z)$, and hence (4) holds with Z, $R/4$ in place of X_0, R:

$$R^j \sup_{B_{R/16}(Z)} |D^j u| \le C\epsilon.$$

Since $B_{R/2}(X_0)$ can be covered by a finite collection (with cardinality depending only on n) this gives inequalities like (4) on $B_{R/2}(X_0)$ as required.

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Geometric Evolution Problems

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Introduction

Variational principles play a fundamental role in geometry. Geodesics, minimal surfaces, or Einstein metrics can be characterized as stationary points of suitable variational integrals; moreover, extremals of variational problems for mappings between manifolds are tools for their differential-topological classification. A particular such class of maps is the family of harmonic maps.

Due to their geometric nature, these variational problems in general give rise to *non-linear* partial differential equations which rarely can be attacked by standard p.d.e. methods. Instead, it is more natural to use a variational approach. However, in many instances also the well established direct methods fail because the functional concerned is not (weakly) lower semi-continuous or because the space of admissible functions is not (weakly) closed. In addition, more sophisticated techniques derived from Morse theory or Ljusternik-Schnirelman theory cannot be applied because of lack of differentiability or failure of the Palais-Smale condition.

The latter is a compactness condition, and its failure in the context of geometric variational problems quite often *naturally* arises from invariance of the problem with respect to a non-compact group of manifest or hidden symmetries like gauge-invariance, conformal invariance, etc. Hence, lack of compactness reflects an important geometric aspect of the problem and is not simply a technical shortcoming. Moreover, special solutions which exhibit that symmetry often reveal highly interesting features of the problem.

Thus, while it is not possible to apply Morse or Ljusternik-Schnirelman theory in a standard way, it seems desirable to follow the ideas underlying these theories as closely as possible. Outstanding among these ideas is the concept of a gradient-like flow, deforming the sub-level sets of a given functional to its set of critical points and connecting orbits. In the sequel, we will investigate various geometric variational problems and the corresponding 'gradient flows'. In contrast to the setting of a C^2-functional on a Hilbert space V, where the gradient flow is defined by an ordinary differential equation in V satisfying a Lipschitz condition, most

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of the time we will find ourselves in a non-smooth environment and our 'gradient flows' will involve unbounded operators and possibly singular behavior. However, as in case of the heat equation, the unbounded operators that we encounter will often have a smoothing effect. Most remarkably, moreover, the singularities that we observe often can be related to the special 'symmetric' solutions mentioned earlier by means of certain 'monotonicity estimates'. Thus, the flows we consider not only will provide us with existence results but will also provide some geometric understanding when and why existence results will fail to hold.

We will pursue these questions in some detail in two model situations: the evolution of harmonic maps and the mean curvature flow. The latter is related to the Ricci-Hamilton flow, but can be stated in a simpler context better suited for the purpose of exposition as the analysis is less technical and more transparent. The evolution of harmonic maps in turn seems related to the evolution of Yang-Mills connections, and many results valid for the first have parallels in the second case.

Finally, Ye's Yamabe flow [**153**] is a beautiful example why an evolution approach might be preferred to a direct one.

A different aspect of geometric evolution problems will be the subject of the third part of these notes. This is the vast field of hyperbolic equations and systems arising from variational problems on a Minkowski space-time. Here we can only briefly discuss recent progress and various open questions concerning harmonic maps of Minkowski space. Similar results should be expected to hold true for Yang-Mills fields on principal bundles over Minkowski space. Important progress towards an understanding of the 'hyperbolic' Einstein equations has recently been achieved by Christodoulou-Klainerman[**26**].

PART 1
The Evolution of Harmonic Maps

1.1. Harmonic maps

Let M^m be an m-dimensional Riemannian manifold with metric γ, N^l a compact l-dimensional manifold with a metric g, respectively. By Nash's embedding theorem we may assume that $N \subset \mathbb{R}^n$ isometrically for some n. For a C^1-map $u = (u^1, \dots, u^n)\colon M \to N \subset \mathbb{R}^n$ let

$$e(u) = \frac{1}{2}\gamma^{\alpha\beta}(x)u^i_{x^\alpha}u^i_{x^\beta} =: \frac{1}{2}|\nabla u|^2_M$$

be the energy density, written in local coordinates $x = (x^\alpha)_{1\le\alpha\le m}$ on M, with $\gamma = (\gamma_{\alpha\beta}), (\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$. Repeated Greek indices tacitly will be summed from 1 to m, repeated Latin indices from 1 to n. Moreover, $u^i_{x^\alpha} = \frac{\partial}{\partial x^\alpha}u^i$, etc. A C^1-variation of u is a family (u_ϵ) of C^1-maps $u_\epsilon\colon M \to N \subset \mathbb{R}^n$ smoothly depending on a parameter $|\epsilon| < \epsilon_0$, and such that $u_0 = u$. A variation (u_ϵ) of u is compactly supported if there exists a compact set $\Omega \subset\subset M$ such that $u_\epsilon = u$ on $M \setminus \Omega$ for all $|\epsilon| < \epsilon_0$.

Definition 1.1. A C^1-map $u\colon M \to N \subset \mathbb{R}^n$ is harmonic if it is stationary for Dirichlet's energy

$$E(u) = \int_M e(u)\, dvol_M$$

with respect to compactly supported variations.

Note that in local coordinates $dvol_M = \sqrt{|\gamma|}dx$, where $|\gamma| = |\det(\gamma_{\alpha\beta})|$.

In order to derive the Euler-Lagrange equation satisfied by a harmonic map u, let $U \subset \mathbb{R}^n$ be a tubular neighborhood of N and $\pi_N\colon U \to N$ the (smooth) nearest-neighbor projection. Denote $T_pN \subset T_p\mathbb{R}^n$ the tangent space to N at a point $p \in N$. Let $\phi \in C^1_0(M, \mathbb{R}^n)$ satisfy

$$\phi(x) \in T_{u(x)}N$$

for all $x \in M$. ϕ induces a C^1-variation

$$u_\epsilon = \pi_N \circ (u + \epsilon\phi).$$

Since $d\pi_N(p)\big|_{T_pN} = \mathrm{id}$ for $p \in N$, clearly we have

$$\frac{d}{d\epsilon}u_\epsilon\big|_{\epsilon=0} = (d\pi_N \circ u)\phi = \phi.$$

Suppose ϕ has support in a single coordinate chart. Then

$$\begin{aligned}\frac{d}{d\epsilon}E(u_\epsilon)\Big|_{\epsilon=0} &= \int_M \gamma^{\alpha\beta}\sqrt{|\gamma|}u^i_{x_\beta}\phi^i_{x_\alpha}\,dx \\ &= -\int_M \frac{1}{\sqrt{|\gamma|}}\frac{\partial}{\partial x^\alpha}(\gamma^{\alpha\beta}\sqrt{|\gamma|}\frac{\partial}{\partial x^\beta}u^i)\phi^i\sqrt{|\gamma|}\,dx \\ &= -\int_M \Delta_M u^i\phi^i\,dvol_M,\end{aligned}$$

where $\Delta_M = \frac{1}{\sqrt{|\gamma|}}\frac{\partial}{\partial x^\alpha}(\gamma^{\alpha\beta}\sqrt{|\gamma|}\frac{\partial}{\partial x^\beta}\cdot)$ denotes the Laplace-Beltrami operator on M.

Thus, if $u \in C^2$ is harmonic, u satisfies

$$\Delta_M u \perp T_uN \tag{1.1}$$

and conversely. To obtain a more explicit form of (1.1), let $\nu_{l+1},\dots,\nu_n$ denote a local orthonormal frame for $(T_pN)^\perp$, the orthogonal complement of T_pN in $\mathbb{R}^n$, near $p = u(x) \in N$. Then, by (1.1) there exist scalar functions $\lambda^{l+1},\dots,\lambda^n$ such that

$$-\Delta_M u = \sum_{k=l+1}^{n}\lambda^k(\nu_k \circ u).$$

Multiplying by ν_k (k fixed), since $u_{x^\alpha}\cdot\nu_k(u) = 0$ for all α, we obtain

$$\begin{aligned}\lambda^k &= -\Delta_M u\cdot(\nu_k \circ u) = -\operatorname{div}(\nabla u\cdot\nu_k(u)) + \gamma^{\alpha\beta}u_{x^\alpha}\cdot\frac{\partial}{\partial x^\beta}(\nu_k \circ u) \\ &= \gamma^{\alpha\beta}A^k(u)(u_{x^\alpha},u_{x^\beta}),\end{aligned}$$

where $A^k = d\nu_k$ denotes the second fundamental form with respect to ν_k. Hence (1.1) is equivalent to

$$-\Delta_M u = A(u)(\nabla u,\nabla u)_M, \tag{1.2}$$

where in local coordinates

$$A(u)(\nabla u,\nabla u)_M = \sum_{k=l+1}^{n}\gamma^{\alpha\beta}A^k(u)(u_{x^\alpha},u_{x^\beta})(\nu_k \circ u).$$

Example 1.1. If $M = T^m = \mathbb{R}^m/\mathbb{Z}^m$, $N = S^n \subset \mathbb{R}^{n+1}$, equation (1.2) simply becomes

$$\Delta u = |\nabla u|^2 u.$$

Harmonic maps generalize the concept of harmonic functions. Harmonic maps $S^1 \to N$ uniquely correspond to closed geodesics on N. Important applications of harmonic maps are in Teichmüller theory in understanding the Weil-Peterson metric on Teichmüller space, see Earle-Eells [**32**], [**33**], Eells [**35**], Fisher-Tromba [**47**], Tromba [**148**], or in proving rigidity theorems for Kähler manifolds (Mostow [**113**], Mostow-Siu [**114**], Siu [**133**]). A comprehensive survey of harmonic maps and their applications is given in Eells-Lemaire [**36**], [**37**]; see also Hildebrandt [**73**], [**74**], Jost [**87**], [**88**].

In these notes for simplicity we shall mostly adopt the above parametric point of view. Intrinsically, the concept of harmonic map can also be defined as follows. A map $u \colon M \to N$ is harmonic if and only if its tension field

$$\tau(u) := \operatorname{trace}(\nabla du) = 0, \tag{1.3}$$

where ∇ denotes the pull-back covariant derivative in the bundle $T^*M \otimes u^{-1}TN$. In local coordinates on M and N, the analogue of (1.2) then is

$$-\Delta_M u^k = \gamma^{\alpha\beta}(x)\tilde{\Gamma}^k_{ij}(u(x))u^i_{x^\alpha}u^j_{x^\beta}, \tag{1.4}$$

where $(\tilde{\Gamma}^k_{ij})$ are the Christoffel symbols on N. However, we prefer (1.2) to this intrinsic equation for its simple geometric interpretation (1.1).

Bochner identity

A very useful identity is the differential equation satisfied by the energy density $e(u)$ of a harmonic map $u \colon M \to N$. Let $R^{(M)}$, $Ric^{(M)}$, $R^{(N)}$ denote the Riemann curvature tensor and Ricci curvature on M, respectively the Riemann curvature tensor on N. Given any point $x_0 \in M$ let $R^{(M)} = (R_{\alpha\mu\beta\nu})$, $Ric^{(M)} = (R_{\alpha\beta})$, $R^{(N)} = (\tilde{R}_{ikjl})$ be the coordinate representations of $R^{(M)}$, $Ric^{(M)}$, $R^{(N)}$ in normal coordinates $x = (x^\alpha)$ about $x_0 \in M$, respectively in normal coordinates $u = (u^i)$ around $u_0 = u(x_0) \in N$. We use the notation of Jost [89]; (1.2.6), (1.2.7), (1.2.13).

Proposition 1.1. *If $u \in C^3(M;N)$ is harmonic, then in local coordinates as above there holds*

$$-\Delta_M e(u) + |\nabla du|^2 + R_{\alpha\beta}u^i_{x^\alpha}u^i_{x^\beta} = \tilde{R}_{ikjl}u^i_{x^\alpha}u^j_{x^\alpha}u^k_{x^\beta}u^l_{x^\beta}, \tag{1.5}$$

where ∇ denotes the covariant derivative on $T^*M \otimes u^{-1}TN$. (See Jost [89]; p. 96f. for an equivalent expression in general coordinates and an invariant form of (1.5).)

Proof. The proof relies on the following identities valid at $x_0 \in M$ in normal coordinates around x_0 on M, respectively around $u(x_0)$ on N. Let $\gamma^{\mu\nu}_{,\alpha} = (\gamma^{\mu\nu})_{x^\alpha}$, etc. Then we have

$$\gamma^{\mu\nu}_{,\alpha\beta} = -\gamma_{\mu\nu,\alpha\beta};$$

$$\begin{aligned}(\Delta_M u)_{x^\mu} - \Delta_M(u_{x^\mu}) &= (\gamma^{\alpha\beta}\sqrt{|\gamma|})_{x^\alpha x^\mu}u_{x^\beta} = \gamma^{\alpha\beta}_{,\alpha\mu}u_{x^\beta} + \frac{1}{2}\gamma_{\rho\rho,\alpha\mu}u_{x^\alpha} \\ &= -\gamma_{\alpha\beta,\alpha\mu}u_{x^\beta} + \frac{1}{2}\gamma_{\rho\rho,\alpha\mu}u_{x^\alpha};\end{aligned}$$

$$R_{\alpha\beta} = R^\mu_{\alpha\mu\beta} = \Gamma^\mu_{\alpha\beta,\mu} - \Gamma^\mu_{\alpha\mu,\beta} = \frac{1}{2}(\gamma_{\alpha\mu,\beta\mu} + \gamma_{\beta\mu,\alpha\mu}) - \frac{1}{2}(\gamma_{\alpha\beta,\mu\mu} + \gamma_{\mu\mu,\alpha\beta}).$$

Hence, at x_0,

$$\begin{aligned}
\Delta_M e(u) &= \Delta_M\big(\frac{1}{2}\gamma^{\mu\nu} g_{ij}(u) u^i_{x^\mu} u^j_{x^\nu}\big) \\
&= (\Delta_M u^i_{x^\mu}) u^i_{x^\mu} + \frac{1}{2}\gamma^{\mu\nu}_{,\alpha\alpha} u^i_{x^\mu} u^i_{x^\nu} + u^i_{x^\alpha x^\mu} u^i_{x^\alpha x^\mu} + \frac{1}{2} g_{ij,kl}(u) u^i_{x^\mu} u^j_{x^\mu} u^k_{x^\alpha} u^l_{x^\alpha} \\
&= (\Delta_M u^i)_{x^\mu} u^i_{x^\mu} + u^i_{x^\alpha x^\mu} u^i_{x^\alpha x^\mu} + \gamma_{\alpha\beta,\alpha\mu} u^i_{x^\beta} u^i_{x^\mu} \\
&\quad - \frac{1}{2}(\gamma_{\mu\nu,\alpha\alpha} + \gamma_{\rho\rho,\mu\nu}) u^i_{x^\mu} u^i_{x^\nu} + \frac{1}{2} g_{ij,kl}(u) u^i_{x^\mu} u^j_{x^\mu} u^k_{x^\alpha} u^l_{x^\alpha} \\
&= (\Delta_M u^i)_{x^\mu} u^i_{x^\mu} + u^i_{x^\alpha x^\mu} u^i_{x^\alpha x^\mu} + R_{\alpha\beta} u^i_{x^\alpha} u^i_{x^\beta} \\
&\quad + \frac{1}{2}(g_{ij,kl} + g_{il,jk} - g_{jl,ik}) u^i_{x^\mu} u^j_{x^\mu} u^k_{x^\alpha} u^l_{x^\alpha}.
\end{aligned}$$

Finally, note that by (1.4) we have

$$(\Delta_M u^i)_{x^\mu} u^i_{x^\mu} = -\tilde{\Gamma}^i_{kl,j}(u) u^k_{x^\alpha} u^l_{x^\alpha} u^i_{x^\mu} u^j_{x^\mu}$$

at x_0, since $\tilde{\Gamma}(u(x_0)) = 0$ by our choice of coordinates. Moreover,

$$\begin{aligned}
\tilde{R}_{iljk} &= \tilde{\Gamma}^i_{kl,j} - \tilde{\Gamma}^i_{jl,k}, \\
\tilde{\Gamma}^i_{jl,k} &= \frac{1}{2}(g_{ij,kl} + g_{il,jk} - g_{jl,ik}),
\end{aligned}$$

whence

$$\Delta_M e(u) = u^i_{x^\alpha x^\mu} u^i_{x^\alpha x^\mu} + R_{\alpha\beta} u^i_{x^\alpha} u^i_{x^\beta} - \tilde{R}_{iljk} u^i_{x^\mu} u^j_{x^\mu} u^k_{x^\alpha} u^l_{x^\alpha}.$$

Since $\tilde{R}_{iljk} = \tilde{R}_{ikjl}$ the claim follows. □

Note the following corollary as a consequence of (1.5); see e.g. [89].

Proposition 1.2. *If M is compact, $Ric^M \geq 0$, and if the sectional curvature of N is non-positive, then any harmonic map $u \in C^\infty(M;N)$ is totally geodesic in the sense that $\nabla du \equiv 0$; that is, du is parallel with respect to the pull-back covariant derivative on $T^*M \otimes u^{-1}TN$. Moreover, if $Ric^M > 0$ at a point of M, then $u \equiv$ const. If the sectional curvature K^N of N is negative, then $u \equiv$ const or $u(M)$ is covered by a closed geodesic.*

Proof. Integrate (1.5) over M to obtain $|\nabla du|^2 \equiv 0$, $\mathrm{Ric}^{(M)}(du, du) \equiv 0$, $\langle \tilde{R}^{(N)}(du, du)du, du\rangle \equiv 0$ on M under the above assumptions. □

For most of our purposes it suffices to note a weaker Bochner-type estimate. To state this we return to the setting of maps $u\colon M \to N \subset \mathbb{R}^N$. Differentiating (1.2) in direction x^μ and taking the scalar product with u_{x^ν} with respect to γ, on account of (1.1) we obtain

$$-\Delta_M e(u) + |\nabla^2 u|^2 \leq |Ric^M| e(u) + C\big(e(u)\big)^2, \tag{1.6}$$

where the second term on the right arises from estimating

$$\begin{aligned}
\gamma^{\mu\nu} &\big(A^k(u)(\nabla u, \nabla u)_M (\nu_k \circ u)\big)_{x^\mu} \cdot u_{x^\nu} \\
&= A^k(u)(\nabla u, \nabla u)_M \gamma^{\mu\nu} \big(d\nu_k(u) u_{x^\nu} \cdot u_{x^\nu}\big) \\
&= \big(A(u)(\nabla u, \nabla u)_M\big)^2 \leq C\big(e(u)\big)^2.
\end{aligned}$$

Weakly harmonic maps

Let

$$H^{1,2}(M;N) = \{u \in H^{1,2}(M;\mathbf{R}^n); u(x) \in N \quad \text{for almost every } x \in M.$$

where $H^{1,2}(M;\mathbb{R}^n)$ is the standard Sobolev space of L^2-mappings $u\colon M \to \mathbb{R}^n$ with distributional derivative $\nabla u \in L^2$. That is, $H^{1,2}(M;N)$ is the space of maps $u\colon M \to N$ with finite energy $E(u)$. It was observed by Schoen-Uhlenbeck [**124**] that in general $H^{1,2}(M;N)$ as defined above is larger than the weak closure of $C^\infty(M;N)$ in the $H^{1,2}$-norm

$$\|u\|^2_{H^{1,2}} = \int_M (|u|^2 + |\nabla u|^2)\, dvol_M,$$

which in turn is larger than the strong closure of $C^\infty(M;N)$ in $H^{1,2}(M;N)$. However, if $m = \dim M = 2$ these spaces all coincide. By a result of Bethuel [7] the same is true if $\pi_2(N) = 0$. The respective relations between these spaces for general M and N were analyzed by Bethuel-Zheng [9] and Bethuel [7].

Definition 1.2. A map $u \in H^{1,2}(M;N)$ is weakly harmonic if u satisfies (1.2) in the distribution sense.

Example 1.2. The map $u\colon B_1(0) \subset \mathbf{R}^m \to S^{m-1}$, given by

$$u(x) = \frac{x}{|x|},$$

belongs to $H^{1,2}(B_1(0);S^{m-1})$ for $m \geq 3$ and weakly solves (1.2), that is

$$-\Delta u = |\nabla u|^2 u.$$

Existence of harmonic maps

As in Hodge theory, where one seeks to realize a de Rham cohomology class by a harmonic differential form, a basic existence problem for harmonic maps is the following:

Homotopy problem: *Given a map $u_0\colon M \to N$ is there a harmonic map u homotopic to u_0?*

This question, as we shall see below, has an affirmative answer if the sectional curvature K^N of N is non-positive (Eells-Sampson [38]), or if $m = 2$ and $\pi_2(N) = 0$ (Lemaire [**102**], Sacks-Uhlenbeck [**121**]). However, for $N = S^2$ and $m = 2$, we have the following counterexamples:

Example 1.3. (Lemaire [**102**], Wente [**151**]): If $u\colon B_1(0) \subset \mathbf{R}^2 \to S^2$ is harmonic and $u_{|\partial B_1(0)} \equiv \text{const.}$, then $u \equiv \text{const.}$

Example 1.4. (Eells-Wood [39]): If $u\colon T^2 \to S^2$ is harmonic, then $\deg u \neq \pm 1$.

In higher dimensions ($m \geq 3$), hardly any result is known for the homotopy problem unless $K^N \leq 0$. However, there are various existence results for the Dirichlet problem.

Dirichlet problem: *Suppose $\partial M \neq 0$ and let $u_0\colon M \to N$ be given. Is there a harmonic map $u\colon M \to N$ such that $u = u_0$ on ∂M?*

The Dirichlet problem can be attacked using variational methods. By minimizing E among the class

$$H^{1,2}_{u_0}(M;N) = \{u \in H^{1,2}(M;N); u = u_0 \text{ on } \partial M\}$$

one obtains a weakly harmonic map u satisfying the desired boundary condition.

Similarly, one could attempt to solve the homotopy problem by minimizing E in a given homotopy class. However, Lemaire's example shows that in general homotopy classes are not weakly closed in $H^{1,2}(M;N)$.

This is made explicit by the following example, whose construction relies on the fact that the conformal group on S^2 acts non-compactly on $H^{1,2}(S^2,S^2)$.

Example 1.5. The mappings

$$u_\lambda = \pi_p^{-1} \circ D_\lambda \circ \pi_p \colon S^2 \to S^2,$$

where $\pi_p\colon S^2\setminus\{p\} \to \mathbb{R}^2$ denotes stereographic projection and $D_\lambda x = \lambda x$ is dilation by a factor λ, are homotopic to the identity $id = u_1\colon S^2 \to S^2$. But

$$u_\lambda \rightharpoonup u_\infty(x) \equiv p \quad (\lambda \to \infty),$$

weakly in $H^{1,2}(S^2;S^2)$. (Incidentally, by conformal invariance of E in dimension $m = 2$, all u_λ are harmonic!)

Regularity

The direct methods in general only yield weakly harmonic maps. In fact, the (singular) map $x \mapsto \frac{x}{|x|}$ of Example 1.2 is minimizing from $B_1(0) \subset \mathbb{R}^m$ into S^{m-1} for its boundary values, if $m \geq 3$ (Brezis-Coron-Lieb [**13**], Lin [**105**]). A partial regularity theory for energy minimizing maps was developed by Schoen-Uhlenbeck [**123**], [**124**] showing that in general an energy minimizing map u is smooth on an open set whose complement, the singular set Sing(u), has Hausdorff-dimension $\leq m-3$ and is discrete if $m = 3$, which is best possible. Even if we regard the map $u(x) = \frac{x}{|x|}$ as a map $u\colon B_1(0) \subset \mathbb{R}^n \to S^{m-1} \subset N = S^m \subset \mathbb{R}^{m+1}$, for example, whence there is no topological reason for a singularity, u is still minimizing if $m \geq 7$ (Jäger-Kaul [**82**], Baldes [**6**]). Moreover, Hardt-Lin-Poon [**71**] have constructed examples of minimizing harmonic maps $u\colon B_1(0) \subset \mathbb{R}^3 \to S^2$ with cylindrical symmetry whose boundary data $u_{|\partial B_1(0)}\colon \partial B_1(0) \cong S^2 \to S^2$ have degree 0 and such that u possesses an arbitrarily large number of singular points on the axis of symmetry, and Rivière [**119**] has exhibited weakly harmonic maps $\mathbb{R}^3 \to S^2$ with line singularities.

By contrast, if $m = 2$, energy-minimizing harmonic maps are smooth (Morrey [**110**], [**111**], Giaquinta-Giusti [**51**]). Moreover, Grüter [**66**] proved smoothness of conformal weakly harmonic maps. This result was extended by Schoen [**122**] to harmonic maps which are stationary with respect to variations of parameters in the domain, hence possessing a holomorphic Hopf differential

$$(\partial u)^2 dz^2 = \left(|u_x|^2 - |u_y|^2 - 2iu_x \cdot u_y\right) dz^2$$

in suitable conformal parameters $z = x + iy$ on M. Finally Hélein [**72**] recently has shown regularity of weakly harmonic maps in general.

Theorem 1.1 (Hélein [**72**]). *Let $m = 2$ and let $u \in H^{1,2}(M,N)$ be weakly harmonic. Then $u \in C^\infty(M,N)$.*

Proof. For $N = S^{n-1}$, $M = B_1(0) \subset \mathbb{R}^2$ his proof exploits the equivalence of (1.2) and

$$-\Delta u^i = \sum_{j=1}^n \nabla u^j (u^i \nabla u^j - u^j \nabla u^i),\ i = 1,\dots,n,$$

because $0 = \nabla|u|^2 = 2\sum_j u^j \nabla u^j$. He then observes that for any $1 \le i, j \le n$ there holds

$$\operatorname{div}(u^i \nabla u^j - u^j \nabla u^i) = u^i \Delta u^j - u^j \Delta u^i = 0. \tag{1.7}$$

Hence there is a potential $a^{ij} \in H^{1,2}$ such that

$$\operatorname{rot} a^{ij} = u^i \nabla u^j - u^j \nabla u^i$$

and (1.2) takes the form

$$-\Delta u^i = \sum_{j=1}^{n} (u_x^j a_y^{ij} - u_y^j a_x^{ij}),$$

where the right hand side is the sum of Jacobians of $H^{1,2}$-mappings. Continuity of u (and hence smoothness) then follows from results of Wente [**150**] and Brezis-Coron [**12**].

Realizing that (1.7) is a consequence of Noether's theorem and the symmetries of S^{n-1}, Hélein then generalized this simple and beautiful idea to arbitrary target manifolds by an ingenious choice of rotated frame fields on $u^{-1}TN$. □

Inspired by Hélein's result, Evans [**41**] for $m \ge 3$ has obtained partial regularity results for "stationary", weakly harmonic maps into spheres.

1.2. The Eells-Sampson result

By Examples 1.3, 1.4 and 1.5 above we know that it may be difficult (if not altogether impossible) to solve the homotopy problem for harmonic maps by direct variational methods. To overcome these difficulties, Eells - Sampson [**38**] proposed to study the evolution problem

$$u_t - \Delta_M u = A(u)(\nabla u, \nabla u)_M \quad \text{on } M \times [0, \infty[\tag{1.8}$$

with initial and boundary data

$$u = u_0 \quad \text{at } t = 0 \text{ and on } \partial M \times [0, \infty[\tag{1.9}$$

for maps $u\colon M \times [0, \infty[\to N \subset \mathbb{R}^n$, the idea behind this strategy of course being that a continuous deformation $u(\cdot, t)$ of u_0 will remain in the given homotopy class. Moreover, the "energy inequality" (see Lemma 1.1 below) shows that (1.8) is the (L^2)-gradient flow for E, whence one may hope that the solution $u(\cdot, t)$ for $t \to \infty$ will come to rest at a critical point of E; that is, a harmonic map. For suitable targets, this program is successful.

Theorem 1.2 (Eells-Sampson [**38**]). *Suppose M is compact, $\partial M = \emptyset$ and that the sectional curvature K^N of N is non-positive. Then for any $u_0 \in C^\infty(M; N)$ the Cauchy problem* (1.8), (1.9) *admits a unique, global, smooth solution $u\colon M \times [0, \infty[\to N$ which, as $t \to \infty$ suitably, converges smoothly to a harmonic map $u_\infty \in C^\infty(M; N)$ homotopic to u_0.*

The proof uses three ingredients.

Lemma 1.1 (Energy inequality). *For a smooth solution u of* (1.8), (1.9) *and any $T \ge 0$ there holds*

$$E(u(T)) + \int_0^T |u_t|^2 \, dvol_M \, dt \le E(u_0).$$

Proof. Recall that $A(u)(\nabla u, \nabla u) \perp T_u N$. Hence, upon multiplying (1.8) by u_t and integrating by parts, we obtain

$$\int_M |u_t|^2 \, dvol_M + \frac{d}{dt} E(u(t)) = 0$$

for any $t \geq 0$, and the desired estimate (in fact, with equality) follows upon integrating in t. $\square$

Lemma 1.2 (Bochner inequality). *If $K^N \leq 0$, then for any smooth solution u of (1.8) with energy density $e(u)$ there holds*

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) e(u) \leq C e(u) \tag{1.10}$$

with a constant C depending only on the Ricci curvature of M.

Proof. To derive this estimate we use the equivalent intrinsic form

$$u_t - \operatorname{trace}(\nabla du) = 0 \tag{1.11}$$

of (1.8), where now $u = (u^1, \dots, u^l)$ denotes the representation of u in suitable local coordinates on N. As in deriving (1.5) from (1.4) in the stationary case, we then conclude that (in normal coordinates around x_0 on M)

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) e(u) + |\nabla du|^2 + R_{\alpha\beta} u^i_{x^\alpha} u^i_{x^\beta} = \tilde{R}_{ikjl} u^i_{x^\alpha} u^j_{x^\alpha} u^k_{x^\beta} u^l_{x^\beta}$$

at (x_0, t), where ∇, $R_{\alpha\beta}$, $\tilde{R}_{ikjl}$, respectively, denote the pull-back covariant derivative on $T^*M \otimes u^{-1}TN$, the Ricci curvature on M, and the Riemann curvature tensor on N. From this identity, the claim follows. $\square$

Note that in the parametric setting of (1.8) and for general targets, upon differentiating (1.8) in direction x^α, multiplying by u_{x^α} and summing over $1 \leq \alpha \leq m$, by orthogonality $A(u)(\nabla u, \nabla u) \perp T_u N$ we obtain

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) e(u) + |\nabla^2 u|^2 \leq C_M e(u) + C_N \big(e(u)\big)^2, \tag{1.12}$$

where $|\nabla^2 u|^2 = \gamma^{\alpha\beta}\gamma^{\mu\nu} u^i_{x^\alpha x^\mu} u^i_{x^\beta x^\nu}$ and C_M, C_N, respectively, denote constants depending only on the Ricci curvature of M and the second fundamental form of N.

The final ingredient is Moser's [112] sup-estimate for sub-solutions of parabolic equations. Let

$$\mathcal{L} = \frac{\partial}{\partial t} - \Delta_M.$$

Denote $P_R(z_0)$ the cylinder

$$P_R(z_0) = \{z = (x,t); |x - x_0| < R, t_0 - R^2 < t < t_0\},$$

where $z_0 = (x_0, t_0)$, $R > 0$ in local coordinates on M. Note that $\gamma^{\alpha\beta} \in C^\infty(P_R(z_0))$ satisfies the uniform ellipticity condition

$$\lambda |\xi|^2 \leq \sqrt{|\gamma|}\gamma^{\alpha\beta}(z)\xi_\alpha \xi_\beta \leq \Lambda |\xi|^2$$

for any $\xi \in \mathbb{R}^m$ with constants $0 < \lambda \leq \Lambda$, uniformly in $z \in P_R(t_0)$.

Lemma 1.3. *Suppose* $0 \le v \in C^\infty(P_R(z_0))$ *satisfies* $\mathcal{L}v \le C_1 v$ *in* $P_R(z_0)$, *where* $C_1 \in \mathbf{R}$, $R \le 1$. *Then for some constant* $C_2 = C_2(\lambda, \Lambda, C_1)$ *there holds*

$$v(z_0) \le C_2 R^{-(m+2)} \int_{P_R(z_0)} v\,dz.$$

Proof of Theorem 1.2. Local existence can be inferred from the a-priori estimates for uniformly parabolic equations (see Ladyženskaya-Solonnikov-Uralceva [**101**]) and the (standard) implicit function theorem; see Hamilton [**69**].

Let $u \in C^\infty(M \times [0,T[; N)$ solve (1.8). By Lemma 1.10 the energy density is a subsolution to the equation

$$(\partial_t - \Delta_M)e(u) \le Ce(u).$$

Let ι_M be the convexity radius on M. Choose $R < \min\{1, \sqrt{T}, \iota_M\}$ and apply Lemmas 1.1 and 1.3 to conclude that

$$\begin{aligned}(e(u))(z_0) &\le CR^{-(m+2)} \int_{P_R(z_0)} e(u)\,dz \\ &\le CR^{-(m+2)} \int_{t_0-R^2}^{t_0} E(u(t))\,dt \\ &\le CR^{-m}E(u_0)\end{aligned}$$

for any $z_0 = (x_0, t_0)$, $t_0 \ge R^2$, where $C = C(M)$. Hence $|\nabla u|$ is uniformly bounded on $M \times [0,T]$. By a boot-strap regularity argument we obtain uniform bounds for all derivatives of u up to time T in terms of the initial data u_0, T, and the manifolds M and N. The solution thus can be continued as a smooth solution to (1.8)) on $M \times [0, \infty[$. The preceding argument, applied with $t_0 \ge 1$, $R = \min\{1, \iota_M\}$, then gives uniform a-priori bounds for u and its derivatives on $M \times [1, \infty[$ depending only on M, N and $E(u_0)$. Hence, by Arzéla-Ascoli's theorem, the flow $(u(\cdot,t))_{t\ge 1}$ is relatively compact in any C^k-Topology. Finally, by Lemma 1.1, for any sequence $t_k \to \infty$ we have $u_t(\cdot, t_k) \to 0$ in $L^2(M)$, whence a sub-sequence $(u(\cdot, t_k))$ converges smoothly to a harmonic map u_∞. As $u(\cdot, t_k)$ is homotopic to u_0 through the flow $(u(\cdot,t))_{0\le t\le t_k}$ for any k, so is u_∞. If $K^N < 0$, the uniqueness of u_∞ and hence convergence of the flow $u(\cdot,t) \to u_\infty$ follows from a maximum principle due to Jäger-Kaul [**81**]. □

The Eells-Sampson result was extended to harmonic maps with boundary by Hamilton [**69**]. The condition $K^N \le 0$ can be replaced by the condition that the image of u_0 (and hence of u) should support a uniformly strictly convex function; see Jost [**86**], von Wahl [**149**]. However, none of these results can be applied in case $u_0 \colon T^2 \to S^2$ has degree 1 and, indeed, Example 1.4 above shows that the flow (1.8), (1.9) in this case cannot exist for all time *and* converge smoothly as $t \to \infty$.

1.3. Finite time blow-up

The question remains whether in this case the heat flow (1.8) will develop singularities in finite or infinite time. Reversing the order of the historical developments, we first address this question and later present the existence results for weak solutions that prompted these investigations.

The heat flow (1.8) is a quasi-linear parabolic system and therefore hard to deal with explicitly. With enough symmetry, however, (1.8) can be reduced to a scalar equation in only two variables. Consider equivariant maps

$$u_0(x) = \left(\frac{x}{|x|}\sin h_0(r), \cos h_0(r)\right)$$

of $B_1(0) \subset \mathbb{R}^m$ into $S^m \subset \mathbb{R}^{m+1}$, where $r = |x|$, $h_0(0) = 0$, and let $u\colon B_1(0) \times [0,T[\to S^m \subset \mathbb{R}^{m+1}$ be the corresponding smooth solution of (1.8), (1.9), defined on a maximal time interval $[0,T[$. By uniqueness, also u is equivariant and can be written

$$u(x,t) = \left(\frac{x}{|x|}\sin h(r,t), \cos h(r,t)\right) \tag{1.13}$$

in terms of a smooth map $h\colon [0,1] \times [0,T[\to \mathbb{R}$ satisfying the initial and boundary conditions

$$\begin{aligned} h(r,0) &= h_0(r) && \text{for} \quad 0 \le r \le 1, \\ h(0,t) &= h_0(0) = 0 && \text{for} \quad 0 \le t \le T, \\ h(1,t) &= h_0(1) =: b && \text{for} \quad 0 \le t \le T. \end{aligned} \tag{1.14}$$

In terms of h, the equation (1.8), that is

$$u_t - \Delta u = |\nabla u|^2 u,$$

becomes

$$h_t - h_{rr} - \frac{m-1}{r} h_r + \frac{\sin 2h}{2r^2} = 0. \tag{1.15}$$

If $m \ge 3$, it was shown by Coron-Ghidaglia [28] that for suitable h_0 the solution h of (1.14), (1.15)—and hence the solution u of (1.8), (1.9)—cannot be smoothly continued beyond some finite time. We will later see a deeper reason for this; see Theorem 1.11.

The case $m = 2$ is more interesting. In dimension $m = 2$ a family of stationary solutions h of (1.15) with $h(0) = 0$ is obtained by stereographic projection from the south pole; see Example 1.5. In terms of $r = |x|$ and the polar angle $\theta = \phi(r)$ the stereographic projection is given by

$$\frac{\cos\theta}{r - \sin\theta} = -\frac{1}{r};$$

that is,

$$\phi(r) = \arccos\left(\frac{1-r^2}{1+r^2}\right).$$

Composing with a dilation $r \to r/\lambda$, we obtain the family

$$\phi_\lambda(r) = \phi\left(\frac{r}{\lambda}\right) = \arccos\left(\frac{\lambda^2 - r^2}{\lambda^2 + r^2}\right), \qquad \lambda > 0,$$

of stationary solutions of (1.15), with $\phi_\lambda(0) = 0$.

Theorem 1.3 (Grayson-Hamilton [61], Chang-Ding [15]). *Let $m = 2$. Suppose $|h_0| \le \pi$. Then the solution h of* (1.14), (1.15) *exists for all time.*

Proof. The idea is to construct a barrier, preventing h from becoming discontinuous in finite time. Smoothness of h then follows from general results on quasilinear parabolic systems; see [101].

First note that Lemma 1.1 translates into the uniform energy bound

$$\pi \int_0^1 |h_r|^2 r\, dr \le E(u(t)) \le E(u_0).$$

Hence, by Sobolev's embedding theorem, $h(\cdot, t)$ is locally Hölder continuous on $]0,1]$ uniformly in t, and a singularity can only develop at the origin.

We assume $0 \le h_0 \le \pi$ for simplicity. Let $h_1 \ge h_0$ satisfy $\frac{\pi}{2} < h_1 \le \pi$, $h_1(0) = \pi$, $h_1(a) < \pi$ for some $a \in]0,1]$ That is, h_1 maps into the (convex) lower hemi-sphere. Therefore, equation (1.8) and hence (1.15) with initial data $h(\cdot, 0) = h_1$ possesses a global, smooth solution $\tilde{h}$. Moreover by the maximum principle, $\tilde{h}(r,t) < \pi$ for $0 < r < 1$, $t > 0$. Choose a strictly increasing function $\lambda(t)$ such that $\phi_{\lambda(0)} > h_0$ on $]0, a]$ and $\phi_{\lambda(t)} > \tilde{h}(a,t)$ for all $t \ge 0$. Let

$$\bar{h}(r,t) = \begin{cases} \inf\{\phi_{\lambda(t)}(r), \tilde{h}(r,t)\}, & 0 \le r \le a \\ \tilde{h}(r,t), & a \le r \le 1 \end{cases}$$

be our barrier. Note that $\bar{h}$ is a supersolution to (1.15). Similarly, $\underline{h} \equiv 0$ is a subsolution. Hence $0 \le h(r,t) \le \bar{h}(r,t)$ on $[0,1] \times [0,T]$, by the maximum principle, and $h(\cdot, t)$ is continuous, hence smooth, at $r = 0$ for any $t \ge 0$. □

Quite surprisingly, Theorem 1.3 is sharp.

Theorem 1.4 (Chang-Ding-Ye[16]). *Suppose $|b| > \pi$. Then the solution h to (1.14), (1.15) blows up in finite time.*

Proof. Let $b > \pi$. We show the existence of a sub-solution f to (1.15) with $f(0,t) = 0 \le f \le f(1,t) = b$ such that $f_r(0,t) \to \infty$ as $t \to T$ for some $T < \infty$. Letting $h_0 = f(\cdot, 0)$ and h the corresponding solution of (1.14), (1.15), the maximum principle then implies that $h \ge f$ on $[0,1] \times [0,T[$. Consequently, also h must blow up before time T. (For general initial data the proof is somewhat more complicated.)

We make the following ansatz for f:

$$f(r,t) = \phi_{\lambda(t)}(r) + \phi_\mu(r^{1+\epsilon}),$$

where $\epsilon > 0, \mu > 0$ and $\lambda = \lambda(t)$ will be suitably determined. Note that for any $\epsilon > 0$ we have

$$\phi_\mu(r^{1+\epsilon}) = \arccos\left(\frac{\mu^2 - r^{2+2\epsilon}}{\mu^2 + r^{2+2\epsilon}}\right) \to 0 \quad (\mu \to \infty)$$

uniformly in $r \in [0,1]$. Hence, given $\epsilon > 0$, we can choose $\mu > 0$ such that

$$\cos \phi_\mu(r^{1+\epsilon}) \ge \frac{1}{1+\epsilon} \quad \text{for } r \in [0,1].$$

Next observe that for any μ, $\epsilon > 0$ the function $\theta(r) = \phi_\mu(r^{1+\epsilon})$ satisfies

$$-\theta_{rr} - \frac{1}{r}\theta_r + \frac{(1+\epsilon)^2 \sin 2\theta}{2r^2} = 0.$$

Hence

$$\begin{aligned}
\tau(f) &:= f_{rr} + \frac{1}{r} f_r - \frac{\sin 2f}{2r^2} \\
&= \big(-\sin 2(\phi_\lambda + \theta) + \sin 2\phi_\lambda + (1+\epsilon)^2 \sin 2\theta\big)/(2r^2) \\
&= \Big(\sin\big((2\phi_\lambda + \theta) - \theta\big) - \sin\big((2\phi_\lambda + \theta) + \theta\big) + (1+\epsilon)^2 \sin 2\theta\Big) / \left(2r^2\right) \\
&= \left(-2\cos(2\phi_\lambda + \theta)\sin\theta + 2(1+\epsilon)^2 \cos\theta \sin\theta\right) / \left(2r^2\right) \\
&\geq \frac{(1+\epsilon) - \cos(2\phi_\lambda + \theta)}{r^2} \sin\theta \\
&\geq \frac{\epsilon}{r^2} \sin\theta = \frac{\epsilon}{r^2} \frac{2\mu r^{1+\epsilon}}{\mu^2 + r^{2+2\epsilon}} \geq \epsilon_1 r^{\epsilon - 1} \quad \text{for } \epsilon_1 = \frac{2\mu\epsilon}{\mu^2 + 1} > 0.
\end{aligned}$$

On the other hand, we have

$$\frac{\partial}{\partial t} f(r,t) = \frac{\partial}{\partial t} \phi_{\lambda(t)}(r) = -\frac{2r}{\lambda^2 + r^2} \frac{d}{dt} \lambda(t).$$

Let $\lambda(t)$ solve $\frac{d}{dt}\lambda = -\delta\lambda^\epsilon$, $\delta > 0$ to be determined; that is

$$\lambda(t) = \left[\lambda_0^{1-\epsilon} - (1-\epsilon)\delta t\right]^{1/(1-\epsilon)}.$$

Then $f_r(0,t) \to \infty$ as $t \to T = \frac{\lambda_0^{1-\epsilon}}{(1-\epsilon)\delta}$. Finally

$$\frac{\partial}{\partial t} f - \tau(f) \leq -\epsilon_1 r^{\epsilon - 1} + \frac{2\delta\lambda^\epsilon r}{\lambda^2 + r^2} = \left[\frac{2\delta\lambda^\epsilon r^{2-\epsilon}}{\lambda^2 + r^2} - \epsilon_1\right] r^{\epsilon - 1}.$$

But by Young's inequality

$$\lambda^\epsilon r^{2-\epsilon} \leq C(\epsilon)(\lambda^2 + r^2),$$

and hence f is a sub-solution of (1.15) for sufficiently small $\delta = \delta(\epsilon) > 0$, as desired. Since $\epsilon > 0$ is arbitrary, $\sup|f(\cdot, 0) - \pi|$ can be made as small as we please. □

Similarly, Chang-Ding-Ye [16] construct solutions of (1.8) for $M = S^2 = N$ that blow up in finite time.

1.4. Global existence and uniqueness of partially regular weak solutions for $m = 2$

Theorem 1.4 shows that in general smooth, global solutions to (1.8), (1.9) do not exist. However, is there maybe a "weak" analogue of Theorem 1.2 that will still provide a satisfactory means towards deciding the homotopy problem? The following result is due to Struwe [138]; the result was extended to the case $\partial M \neq \emptyset$ by Chang [14].

Theorem 1.5. *Suppose M is a compact Riemann surface, possibly with boundary, $N \subset \mathbb{R}^n$ is compact. Then for any $u_0 \in H^{1,2}(M;N)$ with smooth "trace" $u_0|_{\partial M} = \hat{u}_0|_{\partial M}$ for some smooth function $\hat{u}_0 \in C^\infty(M;N)$ there exists a global weak solution $u\colon M \times [0,\infty[\to N$ of (1.8), (1.9) satisfying the energy inequality and of class C^∞ on $\bar{M} \times]0,\infty[$ away from finitely many points $(\bar{x}_k, \bar{t}_k)$, $1 \leq k \leq K$. The solution u is unique in this class.*

At a singularity $(\bar{x}, \bar{t})$, a (non-constant) harmonic sphere $\bar{u}\colon S^2 \to N$ separates in the sense that for suitable sequences $x_k \to \bar{x}$, $t_k \nearrow \bar{t}$, $R_k \searrow 0$ the rescaled maps

$$u_k(x) = u(x_k + R_k x, t_k)\colon D_k \subset \mathbb{R}^2 \to N$$

(in a local conformal chart around $\bar{x}$) converge in $H^{2,2}_{loc}(\mathbb{R}^2; N)$ to a non-constant harmonic limit $\tilde{u}\colon \mathbb{R}^2 \to N$. $\tilde{u}$ has finite energy and extends to a smooth harmonic map $\bar{u}\colon S^2 \cong \overline{\mathbb{R}^2} \to N$.

Finally, for a suitable sequence $t_k \to \infty$, the sequence $(u(\cdot, t_k))$ converges weakly in $H^{1,2}(M; N)$ to a smooth harmonic map $u_\infty\colon M \to N$. The convergence is strong away from finitely many points x_l^∞, $1 \le l \le L$, where again harmonic spheres separate in the above sense, and there holds $K + L \le \epsilon_0^{-1} E(u_0)$, where

$$\epsilon_0 = \inf\{E(\bar{u}); \bar{u}\colon S^2 \to N \text{ is a non-constant smooth harmonic map}\} > 0$$

is a constant depending only on N.

The proof of this result is based on the following ingredients.

Lemma 1.4 (Ladyženskaya [100] Lemma 1, p.8). *For any $v \in H_0^{1,2}(\mathbb{R}^2)$ there holds $v \in L^4(\mathbb{R}^2)$ and*

$$\|v\|^4_{L^4} \le 4\|v\|^2_{L^2}\|\nabla v\|^2_{L^2}.$$

The same estimate holds for L^2-functions v on a half-space with $\nabla v \in L^2$. (See also Friedman [48] or Struwe [144]; Lemma 3.5.7].)

We now proceed with the proof of Theorem 1.5.

Positivity of ϵ_0: Applied to $v = |\nabla u|\phi$, where u solves (1.1) and $\phi \in C^\infty(S^2)$ is a smooth cut-off function $0 \le \phi \le 1$ with support in a coordinate neighborhood on S^2, Lemma 1.4 gives

(1.16)
$$\int_{S^2} |\nabla u|^4 \phi^2\, dx \le C \int_{\operatorname{supp}\phi} |\nabla u|^2\, dx \cdot \left(\int_{S^2} |\nabla^2 u|^2 \phi^2\, dx + \int_{S^2} |\nabla u|^2 |\nabla \phi|^2\, dx \right).$$

In particular, if (ϕ_j^2) is a smooth partition of unity subordinate to a finite cover (U_j) of S^2 by local coordinate charts $U_j, 1 \le j \le \mathcal{J}$, we obtain from (1.2), (1.16) and the Calderón-Zygmund inequality that

$$\begin{aligned}\int_{S^2} |\nabla^2 u|^2\, dvol_{S^2} &\le C \int_{S^2} |\Delta u|^2\, dvol_{S^2} + C \int_{S^2} |u|^2\, dvol_{S^2} \\ &\le C \int_{S^2} |\nabla u|^4\, dvol_{S^2} + C \\ &\le C\, E(u)\Big(\int_{S^2} |\nabla^2 u|^2\, dvol_{S^2} + E(u)\Big) + C\end{aligned}$$

with constants $C = C(N)$. Hence the number ϵ_0, defined in Theorem 1.5, is strictly positive. Indeed, if $E(u_k) \to 0$ for a sequence of harmonic maps $u_k\colon S^2 \to N$ we obtain that $u_k \to$ *const* in $H^{2,2}(S^2; N) \hookrightarrow C^0(S^2; N)$. In particular, $u_k(S^2)$ lies in a convex geodesic ball on N for large k. Contradiction.

L^2-estimates for $\nabla^2 u$: By (1.16) it is important to control the energy locally. For $\Omega \subset M$ denote

$$E(u; \Omega) = \int_\Omega e(u)\, dvol_M.$$

Lemma 1.5. *Let $u\colon M \times [0, T] \to N$ be a smooth solution of (1.8) (1.9). Let $x_0 \in M$ and let $B_R(x_0)$ be balls in a local conformal chart around x_0, $0 < R \le 2R_0$. Then for $R \le R_0$ there holds*

$$E\big(u(T); B_R(x_0)\big) \le E\big(u_0; B_{2R}(x_0)\big) + C\frac{T}{R^2} E(u_0),$$

with $C = C(M,N)$.

Proof. Let $\phi \in C_0^\infty(B_{2R}(x_0))$ satisfy $0 \le \phi \le 1$, $\phi \equiv 1$ on $B_R(x_0)$, $|\nabla\phi| \le \frac{2}{R}$, and test (1.8) with $u_t\phi^2$ to obtain

$$\int_M |u_t|^2\phi^2\,dx + \frac{d}{dt}\left(\int_M e(u)\phi^2\,dx\right) \le C\int_M |\nabla u||u_t||\nabla\phi|\phi\,dx$$
$$\le \int_M |u_t|^2\phi^2\,dx + C\int_M |\nabla u|^2|\nabla\phi|^2\,dx.$$

Hence

$$\frac{d}{dt}\left(\int_M e(u)\phi^2\,dx\right) \le CR^{-2}E(u(t)) \le CR^{-2}E(u_0),$$

and the lemma follows by integration. □

In particular, given $\epsilon_1 > 0$, $u_0 \in H^{1,2}(M;N)$ there exists a number $T_1 > 0$, depending only on a maximal number $R_1 > 0$ such that

$$\sup_{x_0\in M} E(u_0; B_{2R_1}(x_0)) < \epsilon_1$$

and the geometry of M and N, with the property that any (smooth) solution u of (1.8), (1.9) satisfies

$$\sup_{x_0\in M, 0\le t\le T_1} E(u(t); B_{R_1}(x_0)) < 2\epsilon_1.$$

Indeed, we may let $T_1 = \frac{\epsilon_1 R_1^2}{CE(u_0)}$, where C is the constant in Lemma 1.5.

Similarly, given $\epsilon_1 > 0$ let $R_1 > 0$ be determined as above and let ϕ_i be smooth cut-off functions subordinate to a cover of M by balls $B_{2R_1}(x_i)$ with finite overlap and such that $0 \le \phi_i \le 1$, $|\nabla\phi_i| \le \frac{2}{R_1}$, $\sum_i \phi_i^2 = 1$. Then by (1.16) we have

$$\begin{aligned}\int_M |\nabla u|^4\,dvol_M &= \sum_i \int_M |\nabla u|^4\phi_i^2\,dvol_M\\ &\le C\sup_i E\Big(u(t); B_{2R_1}(x_i)\Big)\cdot\left(\int_M |\nabla^2 u(t)|^2\,dvol_M + R_1^{-2}E(u_0)\right)\\ &\le C\epsilon_1\left(\int_M |\nabla^2 u(t)|^2\,dvol_M + R_1^{-2}E(u_0)\right)\end{aligned}$$

for any t. Moreover, similar to our Bochner-type estimate (1.12), upon multiplying (1.8) by $\Delta_M u$ and integrating by parts we obtain

$$\frac{d}{dt}(E(u(t))) + \int_M |\Delta_M u|^2\,dvol_M \le C\int_M |\nabla u|^4\,dvol_M.$$

Upon integrating over $[0,T_1]$ and combining the above estimates and the Calderón-Zygmund inequality, we obtain that

$$\int_0^{T_1}\int_M |\nabla^2 u|^2\,dvol_M \le C\int_0^{T_1}\int_M |\Delta_M u|^2\,dvol_M dt + CT_1\left(1 + \|\hat{u}_0\|^2_{H^{2,2}(M)}\right)$$
$$\le C\epsilon_1\int_0^{T_1}\int_M |\nabla^2 u(t)|^2\,dvol_M + C(1 + T_1R_1^{-2})E(u_0) + CT_1\|\hat{u}_0\|^2_{H^{2,2}(M)}$$

with constants $C = C(M, N)$. Thus, for sufficiently small $\epsilon_1 > 0$ we obtain an a-priori bound for u in the norm

$$\|u\|^2_{V^T(M)} = \sup_{0\leq t\leq T} E(u(t)) + \int_0^T \int_M \left(|\nabla^2 u|^2 + |u_t|^2\right) dvol_M\, dt$$

with $T = T_1$, of the form

$$\|u\|^2_{V^{T_1}(M)} \leq C\left(1 + \frac{T_1}{R_1^2}\right) E(u_0) + CT_1 \|\hat{u}_0\|^2_{H^{2,2}}.$$

Without proof we remark that $V^T(M)$ is a regularity class for (1.8) in the sense that, if $u \in V^T(M)$ solves (1.8) with finite energy initial data $u_0 \in H^{1,2}(M; N)$ and coinciding with a smooth function $\hat{u}_0$ on ∂M, then $u \in C^\infty(\bar{M}\times]0, T]; N)$.

Local existence: $R_1 > 0$ can be chosen uniformly for a set of initial data which is compact in $\{\hat{u}_0\} + H_0^{1,2}(M; N)$. In particular, if $u_{0m} \in C^\infty(M; N)$ converges to u_0 in $H^{1,2}(M; N)$, and if (u_m) is the corresponding sequence of local solutions (1.8) for initial and boundary data (u_{0m}), by the above a-priori estimate we have $\|u_m\|^2_{V^{T^1}(M)} \leq C\left(E(u_0) + \|\hat{u}_0\|^2_{H^{2,2}}\right)$ and the sequence (u_m) weakly accumulates at a function $u \in V^{T_1}(M)$. By Rellich's theorem, moreover, $\nabla u_m \to \nabla u$ in $L^2(M)$ for almost every t, and it is easy to pass to the limit in equation (1.8); by the preceding remarks about regularity, in fact, u solves (1.8) classically in $\bar{M}\times]0, T_1]$. Finally, by Lemma 1.1, u achieves its initial data continuously in $H^{1,2}(M; N)$.

Uniqueness: The space of functions with bounded $V^T(M)$-norm is a uniqueness class. Indeed, if $u, v \in V^T(M)$ weakly solve (1.8) with $u(0) = u_0 = v(0)$, their difference $w = u - v$ satisfies

$$|w_t - \Delta_M w| \leq C|w|(|\nabla u|^2 + |\nabla v|^2) + C|\nabla w|(|\nabla u| + |\nabla v|).$$

Testing with w and integrating by parts, we obtain for almost every $t \geq 0$

$$\begin{aligned}
&\frac{1}{2}\int_M |w(t)|^2\, dvol_M + \int_0^t \int_M |\nabla w|^2 dvol_M\, ds \\
&\quad\leq C\int_0^t \int_M |w|^2(|\nabla u|^2 + |\nabla v|^2)\, dvol_M\, ds \\
&\qquad + C\int_0^t \int_M |w||\nabla w|(|\nabla u| + |\nabla v|)\, dvol_M\, ds \\
&\quad\leq C\left(\int_0^t \int_M |w|^4\, dvol_M\, ds\right)^{1/2} \left(\int_0^t \int_M (|\nabla u|^4 + |\nabla v|^4)\, dvol_M\, ds\right)^{1/2} \\
&\qquad + C\left(\int_0^t \int_M |w|^4\, dvol_M\, ds\right)^{1/4} \left(\int_0^t \int_M |\nabla w|^2\, dvol_M\, ds\right)^{1/2} \\
&\qquad\quad \cdot \left(\int_0^t \int_M (|\nabla u|^4 + |\nabla v|^4)\, dvol_M\, ds\right)^{1/4} \\
&\quad\leq C\epsilon(t)\left[\left(\int_0^t \int_M |w|^4\, dvol_M\, ds\right)^{1/2} + \int_0^t \int_M |\nabla w|^2 dvol_M\, ds\right] \\
&\quad\leq C\epsilon(t)\left[\sup_{0\leq s\leq t} \int_M |w(s)|^2\, dvol_M + \int_0^t \int_M |\nabla w|^2\, dvol_M\, ds\right].
\end{aligned}$$

on account of Lemma 1.4, where $\epsilon(t) \to 0$ as $t \to 0$. Choosing $t > 0$ small enough and such that $\|w(t)\|_{L^2(M)} = \sup_{0\le s\le t}\|w(s)\|_{L^2(M)}$, uniqueness follows.

Global continuation: The local solution u constructed above can be extended until the first time $T = \bar{t}_1$ such that

$$\limsup_{t\nearrow T}\left(\sup_{x_0} E\big(u(t); B_R(x_0)\big)\right) \geq \epsilon_1$$

for all $R > 0$. By Lemma 1.1, $u_t \in L^2(M \times [0,T])$. Hence, the L^2-limit $u_1 = \lim_{t\nearrow T} u(t)$ exists. Let v be the local solution of (1.8) with initial data $v = u_1$ at time T and boundary data $\hat{u}_0$. The composed function

$$w(t) = \begin{cases} u(t), & 0 \le t \le T \\ v(t), & T \le t \end{cases}$$

then is a weak solution of (1.8). By iteration we obtain a weak solution u on a maximal time interval $[0,\bar{T}[$. If $\bar{T} < \infty$, the above arguments again permit us to extend u beyond $\bar{T}$, contradicting our assumption that $\bar{T}$ was maximal. Hence $\bar{T} = \infty$.

Finiteness of the singular set: Let $\bar{t} = \bar{t}_1 > 0$ be the first singular time, and let

$$\mathrm{Sing}(\bar{t}) = \{x_0 \in M; \forall R > 0\colon \limsup_{t\nearrow\bar{t}} E(u(t); B_R(x_0)) > \epsilon_1\}.$$

$\mathrm{Sing}(\bar{t})$ is finite. Indeed, let $x_1, \dots, x_K \in \mathrm{Sing}(\bar{t})$. Choose $R > 0$ such that $B_{2R}(x_i) \cap B_{2R}(x_j) = \emptyset$ $(i \neq j)$, and fix $\tau \in \left[\bar{t} - \frac{\epsilon_1 R^2}{2CE(u_0)}, \bar{t}\right[$, where C is the constant in Lemma 1.5. Then by Lemma 1.5

$$\begin{aligned} K\epsilon_1 &\le \sum_{i=1}^{K} \limsup_{t\nearrow\bar{t}} E(u(t); B_R(x_i)) \\ &\le \sum_{i=1}^{K}\left(E(u(\tau); B_{2R}(x_i)) + \frac{\epsilon_1}{2}\right) \le E(u_0) + \frac{K\epsilon_1}{2}, \end{aligned}$$

and $K = K_1 = \#\,\mathrm{Sing}(\bar{t}_1) \le 2E(u_0)\epsilon_1^{-1}$. Moreover, for $u_1 = \lim_{t\nearrow\bar{t}_1} u(t)$ we have

$$\begin{aligned} E(u_1) &= \lim_{R\to 0} E\left(u_1; M \setminus \bigcup_{i=1}^{K_1} B_{2R}(x_i)\right) \\ &\le \lim_{R\to 0}\limsup_{t\nearrow\bar{t}_1} E\left(u(t); M \setminus \bigcup_{i=1}^{K_1} B_{2R}(x_i)\right) \\ &\le \lim_{R\to 0}\limsup_{t\nearrow\bar{t}_1}\left(E(u(t)) - \sum_{i=1}^{K_1} E\big(u(t); B_{2R}(x_i)\big)\right) \\ &\le \lim_{R\to 0}\left(E(u_0) - \liminf_{t\nearrow\bar{t}_1}\sum_{i=1}^{K_1} E\big(u(t); B_{2R}(x_i)\big)\right) \\ &\le E(u_0) - \sum_{i=1}^{K_1}\lim_{R\to 0}\limsup_{t\nearrow\bar{t}_1} E\big(u(t); B_R(x_i)\big) \\ &\le E(u_0) - K_1\epsilon_1. \end{aligned}$$

Similarly, let K_2, K_3, ... be the number of concentration points at consecutive times $\bar{t_2} < \bar{t_3} < \dots$, and let $u_j = \lim_{t \nearrow \bar{t}_j} u(t)$ for $j = 2, 3, \dots$ Then by induction we obtain

$$E(u_j) \le E(u_{j-1}) - K_j\epsilon_1 \le \cdots \le E(u_0) - (K_1 + \cdots + K_j)\epsilon_1,$$

and it follows that the total number K of concentration points, hence also the number of concentration times t_j, is finite; in fact, $K \le E(u_0)\epsilon_1^{-1}$.

Smoothness: Let $\bar{t} = \bar{t}_j$ for some j. To see that u is smooth up to time $\bar{t}$ away from $\operatorname{Sing}(\bar{t})$ we present an argument based on scaling, as proposed by Schoen [**122**] in the stationary case. By working in a local conformal chart we may assume that M is the unit disc B or half disc

$$B^+ = \{(x^1, x^2) \in B; x^2 > 0\};$$

moreover, by scaling we may assume $\bar{t} \ge 1$. Finally, we shift time so that $\bar{t} = 0$. The solution u then is defined on a domain containing $M \times [-1, 0]$. For $R > 0$, $z_0 = (x_0, t_0)$ denote

$$P_R^{(+)}(z_0) = \{z = (x, t); x \in B^{(+)}, |x - x_0| < R, t_0 - R^2 < t < t_0\},$$

and denote $P_R^{(+)}(0) = P_R^{(+)}$, $P_1^{(+)} = P^{(+)}$. For ease of notation, for a function $u \colon \Omega \subset \mathbb{R}^2 \to N$ and $r > 0$, $x \in \mathbb{R}^2$ we define $E(u; B_r(x)) := E(u; B_r(x) \cap \Omega)$.

Proposition 1.3. *Suppose $u \in C^\infty(P^{(+)}; N)$ solves (1.8) with smooth boundary data $\hat{u}_0$. There exist constants C, $\epsilon_2 > 0$ depending only on N and $\hat{u}_0$ such that, if*

$$\sup_{-1 \le t \le 0} E(u(t); B) < \epsilon_2,$$

then there holds

$$\sup_{P_{\frac{1}{2}}^{(+)}} |\nabla u(x, t)| \le C$$

(and corresponding bounds for higher derivatives).

Proof. Choose $0 \le \rho < 1$ such that

$$(1 - \rho)^2 \sup_{P_\rho^{(+)}} e(u) = \max_{0 \le \sigma \le 1} \left\{ (1 - \sigma)^2 \sup_{P_\sigma^{(+)}} e(u) \right\}$$

and let $z_0 \in P_\rho^{(+)}$ satisfy

$$(e(u))(z_0) = \sup_{P_\rho^{(+)}} e(u) =: e_0.$$

Then either $e_0(1 - \rho)^2 \le 4$, in which case

$$\sup_{P_{\frac{1}{2}}^{(+)}} e(u) \le 4(1 - \rho)^2 e_0 \le 16,$$

or $e_0^{-\frac{1}{2}} \le \frac{1-\rho}{2}$. In the latter case, the scaled function

$$v(x, t) = u(x_0 + e_0^{-\frac{1}{2}} x, t_0 + e_0^{-1} t)$$

is well defined on $D = P$, respectively on

$$D^+ = \left\{ z = (x, t) \in P; x_0^2 + e_0^{-\frac{1}{2}} x^2 > 0 \right\}$$

with boundary data

$$\hat{v}_0(x) = \hat{u}_0(x_0 + e_0^{-\frac{1}{2}} x).$$

Moreover, $(e(v))(0) = 1$ and

$$\sup_{D^{(+)}} e(v) \leq e_0^{-1} \sup_{P^{(+)}_{\frac{1+\rho}{2}}} e(u) \leq e_0^{-1} \frac{(1-\rho)^2 \sup_{P_\rho^{(+)}} e(u)}{(\frac{1-\rho}{2})^2} = 4.$$

Thus, v satisfies a linear differential inequality

$$|v_t - \Delta v| \leq C|\nabla v|,$$

and from the L^p-theory for the heat equation, with $p > 4$ we get

$$\begin{aligned} 1 \leq \sup_{D^{(+)} \cap P_{\frac{1}{2}}} |\nabla v| &\leq C \left(\int_{D^{(+)}} |\nabla v|^p \, dz \right)^{1/p} + C \left(\|\nabla \hat{v}_0\|_{L^\infty} + \|\nabla^2 \hat{v}_0\|_{L^\infty} \right) \\ &\leq C_2 \sup_t \left(E\big(u(t); B^{(+)}\big) \right)^{1/p} + C \left(e_0^{-\frac{1}{2}} + e_0^{-1} \right); \end{aligned}$$

see [**101**]. That is, $e_0 \leq C$ if $C_2 \epsilon_2^{1/p} \leq \frac{1}{2}$. (If $D = P$, we can use Bochner's inequality (1.12) and the sup-estimate Lemma 1.3 to achieve a slightly simpler proof.) □

Blow-up of singularities: As in Struwe [**138**] we now use the scaling technique to analyze the singularities in more detail. Let $(\bar{x}, \bar{t})$ be a singularity of the solution u constructed above. Introduce a local conformal chart around $\bar{x}$ and shift time so that $(\bar{x}, \bar{t}) = (0, 0)$. Moreover, after scaling we may assume that $u \in C^\infty(P^{(+)} \setminus \{0\}; N)$ where $P_R^{(+)}(z_0)$ is defined as above. Let (R_k) be a sequence of numbers $R_k \in]0, 1[, R_k \searrow 0$ $(K \to \infty)$. Let $\epsilon_1 > 0$ be as above and define sequences (x_k), (t_k) such that $x_k \to \bar{x}$ while

$$E\big(u(t_k); B_{R_k}(x_k)\big) = \sup_{\substack{(x,t) \in P^{(+)}; \\ -1 \leq t \leq t_k}} E\left(u(t); B_{R_k}(x)\right) = \frac{\epsilon_1}{L},$$

where L is the number of unit discs needed to cover $B_2(0)$. We may assume $t_k - 4R_k^2 \geq -1$. Scale

$$v_k(x,t) = u\left(x_k + R_k x, t_k + R_k^2 t\right),$$

$v_k \colon D_k^{(+)} = \left\{(x,t) \in \mathbb{R}^2 \times [-4, 0]; (x_k + R_k x, t_k + R_k^2 t) \in P^{(+)}\right\} \to N$. Note that by Lemma 1.1 we have

$$\int_{D_k^{(+)}} |(v_k)_t|^2 \, dx \, dt \leq \int_{t_k - R_k^2}^{t_k} \int_M |u_t|^2 \, dvol_M \, dt \to 0 \qquad (k \to \infty),$$

$$E\big(v_k(t)\big) \leq E(u_0), \qquad -4 \leq t \leq 0, \qquad k \in N.$$

Moreover, we have

$$\begin{aligned} \sup_{\substack{(x,t) \in D_k^{(+)}; \\ -4 \leq t \leq 0}} E\big(v_k(t); B_2(x)\big) &\leq L \sup_{\substack{(x,t) \in D_k^{(+)}; \\ -4 \leq t \leq 0}} E\big(v_k(t); B_1(x)\big) \\ &\leq L \sup_{\substack{(x,t) \in P^{(+)}; \\ -1 \leq t \leq t_k}} E\big(u(t); B_{R_k}(x)\big) \leq L E\left(u(t_k); B_{R_k}(x_k)\right) = \epsilon_1. \end{aligned}$$

Hence, by Proposition 1.3, the sequence (v_k) is locally a priori bounded in C^l for any l, and a sub-sequence converges strongly in $H^{1,2}_{\mathrm{loc}}\big(\mathbb{R}^2 \times [-1, 0]; N\big)$ or in

$H^{1,2}_{\rm loc}(\mathbb{R}^2_+ \times [-1,0]; N)$ to a smooth solution v of (1.8) on $\mathbb{R}^2 \times [-1,0]$ or $\mathbb{R}^2_+ \times [-1,0]$, respectively. Moreover, $v_t = 0$, whence $v(\cdot, t) =: \tilde{u}$ is, in fact, harmonic. Finally

$$E(\tilde{u}; B) = \lim_{k\to\infty} E(v_k(0); B) = \lim_{k\to\infty} E\big(u(t_k); B_{R_k}(x)\big) \geq \frac{\epsilon_1}{L},$$

and $\tilde{u}$ is non-constant. If $\tilde{u}\colon \mathbb{R}^2 \to N$, by conformal equivalence $\mathbb{R}^2 \cong S^2 \setminus \{p\}$, $\tilde{u}$ induces a weakly harmonic map $\bar{u} : S^2 \to N$. By Hélein's result $\bar{u}$ is smooth. If $\tilde{u}\colon \mathbb{R}^2_+ \to N$, we also have $\tilde{u}|_{\partial\mathbb{R}^2_+} \equiv const. = u_0(\bar{x})$. Since $\mathbb{R}^2_+$ is conformal to the unit disc B, $\tilde{u}$ induces a non-constant harmonic map $\bar{u}\colon B \to N$ with $\bar{u}|_{\partial B} \equiv const.$ But this is impossible by Lemaire's result, Example 1.3. Thus, singularities at small scales look like harmonic spheres. In particular, we obtain the estimate $\epsilon_1 \geq \epsilon_0$.

A similar analysis is possible at concentration points at "$\bar{t} = \infty$"; see Struwe [**144**], Lecture III.5, for details. Moreover, it seems possible to iterate the above procedure and decompose u in the limit $t \nearrow \bar{t}$ into its weak limit $u(\bar{t})$ and a finite sum of harmonic spheres $\bar{u}_1, \ldots, \bar{u}_L$, similar to Struwe [**137**] for a related problem, and in such a way that energy is conserved; that is,

$$E\big(u(\bar{t})\big) + \sum_{l=1}^{L} E(\bar{u}_l) = \lim_{t\nearrow\bar{t}} E\big(u(t)\big).$$

Applications

As a first application we present a proof of the following theorem by Lemaire [**102**], and Sacks-Uhlenbeck [**121**].

Theorem 1.6. *If* $\dim M = m = 2$ *and if* $\pi_2(N) = 0$, *then any map* $u_0\colon M \to N$ *is homotopic to a smooth harmonic map.*

Proof. Let

$$\epsilon_0 = \inf\{E(\bar{u});\ \bar{u}\colon S^2 \to N \quad \text{is a non-constant, smooth harmonic map}\} > 0$$

as above. We may assume that $u_0 \in C^\infty(M; N)$ and

$$E(u_0) < \inf\big\{E(u); u \in C^\infty(M, N),\ u \text{ is homotopic to } u_0\big\} + \frac{\epsilon_0}{4}.$$

Let $u\colon M \times [0, \infty] \to N$ be the solution of (1.8), (1.9) constructed in Theorem 1.5 above. We claim that u is globally smooth and converges smoothly as $t \to \infty$ to a harmonic limit u_∞, solving the homotopy problem. Thus, assume by contradiction that u develops a first singularity at a point $(\bar{x}, \bar{t})$, $\bar{t} \leq \infty$, and let

$$u_k(x) = u(\bar{x} + R_k x, t_k)$$

be a suitably scaled sequence for $t_k \nearrow \bar{t}$, $R_k \searrow 0$, defined on larger and larger balls in $\mathbb{R}^2$ and converging in $H^{2,2}_{\rm loc}(\mathbb{R}^2; N)$ as $k \to \infty$ to a non-constant, smooth harmonic map $\tilde{u}\colon \mathbb{R}^2 \cong S^2 \setminus \{p\} \to N$. Replace the "large" part of $u_k \sim \tilde{u}$ by the "small" part of $\tilde{u}$ as follows. Fix $\rho > 0$ such that

$$E\big(\tilde{u}; \mathbb{R}^2 \setminus B_\rho(0)\big) < \frac{\epsilon_0}{4} < \frac{\epsilon_0}{2} < E\big(\tilde{u}; B_\rho(0)\big) = \limsup_{k\to\infty} E\big(u_k; B_\rho(0)\big)$$

and choose $\phi \in C_0^\infty(\mathbb{R}^2)$ such that $\phi \equiv 1$ on $B_\rho(0)$. Then, as $k \to \infty$ we have

$$v_k := \phi u_k + (1-\phi)\tilde{u} \to \tilde{u} \quad \text{in } H^{2,2}(\mathbb{R}^2; N)$$

and uniformly. Let $\pi\colon U_\delta \to N$ be the smooth nearest-neighbor projection of a tubular neighborhood U_δ of N in $\mathbf{R}^n$ and let $w_k = \pi \circ v_k$. Then $w_k \in C^\infty(\mathbf{R}^2; N)$ and $w_k \to \tilde{u}$ in $H^{2,2}(\mathbf{R}^2; N)$.

Invert w_k along the circle of radius ρ to obtain

$$\tilde{w}_k(x) = w_k\left(x\frac{\rho^2}{|x|^2}\right).$$

Then by conformal invariance of Dirichlet's integral

$$\limsup_{k\to\infty} E(\tilde{w}_k; B_\rho(0)) = \limsup_{k\to\infty} E\big(w_k; \mathbf{R}^2 \setminus B_\rho(0)\big) = E\big(\tilde{u}; \mathbf{R}^2 \setminus B_\rho(0)\big) < \frac{\epsilon_0}{4}.$$

Moreover

$$\tilde{w}_k = w_k = u_k \qquad \text{on } \partial B_\rho,$$

and upon replacing u_k by $\tilde{w}_k$ on $B_\rho(0)$ for some sufficiently large k we obtain a new map $v : M \to N$ such that

$$E\big(u(t_k)\big) - E(v) = E\big(u_k; B_\rho(0)\big) - E\big(\tilde{w}_k; B_\rho(0)\big) \geq \frac{\epsilon_0}{4}.$$

That is,

$$E(v) \leq E\big(u(t_k)\big) - \frac{\epsilon_0}{4} \leq E(u_0) - \frac{\epsilon_0}{4} < \inf\{E(u); u \sim u_0\}.$$

On the other hand, since $\pi_2(N) = 0$ by assumption, $u(t_k)$ and v are homotopic. The contradiction shows that the flow cannot develop singularities in finite or infinite time. Hence the proof is complete. $\square$

As a second application we present a key step in the proof of a result of Micallef-Moore on a generalization of the Berger-Klingenberg-Rauch-Topogonov sphere theorem.

Theorem 1.7. *Let N be a compact, simply-connected, n-dimensional Riemannian manifold and let $k > 0$ be the first integer such that $\pi_k(N) \neq 0$. Then there exists a non-constant harmonic sphere $\bar{u}\colon S^2 \to N$ having Morse index $MI(\bar{u}) \leq k - 2$.*

Recall that the Morse index of a non-degenerate critical point x of a C^2-functional f on a Hilbert space X is the maximal dimension of a linear sub-space $V \subset X$ such that $d^2 f(x)\big|_{V\times V} < 0$.

Proof. Represent

$$\begin{aligned} S^k &= \{(\xi,\eta) \in \mathbf{R}^3 \times \mathbf{R}^{k-2}; |\xi|^2 + |\eta|^2 = 1\} \\ &\cong S^2 \times B^{k-2}, \end{aligned}$$

where $B^{k-2} = B_1(0; \mathbf{R}^{k-2})$ and $S^2 \times \{\eta\}$ is collapsed for $\eta \in \partial B^{k-2}$. Let $h_0\colon S^k \to N$ represent a non-trivial homotopy class $[h_0] \in \pi_k(N)$. With respect to the above decomposition of S^k, for every $\eta \in B^{k-2}$ this induces a map

$$u_0(\eta) := h_0(\cdot, \eta)\colon S^2 \to N.$$

Note that $u_o(\eta) \equiv \text{const.}$ for $\eta \in \partial B^{k-2}$. Let $u(\cdot;\eta)$ be the corresponding solutions to the Cauchy problem (1.8) (1.9). First suppose

$$\sup_\eta E\big(u_0(\eta)\big) < \epsilon_0.$$

Then by Theorem 1.5 the flows $u(\cdot;\eta)$ are globally smooth and converge smoothly to constant maps as $t \to \infty$. The convergence is uniform in η. Indeed, for any $t \geq 0$ let

$$\mu(t) = \sup_{\eta} E\big(u(t;\eta)\big) \geq 0.$$

Note that the map $t \mapsto \mu(t)$ is non-increasing by Lemma 1.1. For a sequence $t_l \to \infty$ select $\eta_l \in B^{k-2}$ such that

$$E\big(u(t_l;\eta_l)\big) = \sup_{\eta} E\big(u(t_l;\eta)\big).$$

Then a sub-sequence $\eta_l \to \bar{\eta}$ and, by locally smooth dependence of $u(\cdot;\eta)$ on η, for any $t < \infty$ we have

$$E\big(u(t;\bar{\eta})\big) = \lim_{l\to\infty} E\big(u(t;\eta_l)\big).$$

For large l there holds $t \leq t_l$, and Lemma 1.1 gives

$$E\big(u(t;\eta_l)\big) \geq E\big(u(t_l;\eta_l)\big) = \mu(t_l).$$

Hence

$$\lim_{t\to\infty} \mu(t) \leq \lim_{t\to\infty} E\big(u(t;\bar{\eta})\big) = 0.$$

Thus $E(u(t;\eta)) \to 0$ as $t \to \infty$, uniformly in $\eta \in B^{k-2}$. By Proposition 1.3 this implies smooth convergence. But then for large t the map $h_t\colon S^k \simeq S^2 \times B^{k-2} \ni (\xi,\eta) \mapsto u(\xi,t;\eta) \in N$ is homotopic to a map $\bar{h}\colon (\xi,\eta) \mapsto \bar{u}(\eta) \in N$ where $\bar{u} \in C^\infty(B^{k-2},N)$. Since B^{k-2} is contractible, $\bar{u}$ and therefore also h_0 is homotopic to a constant map, contradicting our assumption about h_0. Hence

$$\epsilon_k = \inf_{h\sim h_0} \sup_{\eta} E\big(h(\cdot,\eta)\big) \geq \epsilon_0 > 0,$$

where we take the infimum over all h homotopic to h_0. Consider the set

$$\mathcal{H} = \{\bar{u} : S^2 \to N; \quad \bar{u} \text{ is non-constant, smooth and harmonic}, \quad E(\bar{u}) \leq 2\epsilon_k\}.$$

$\mathcal{H}$ is compact modulo separation of harmonic spheres $\bar{u} \in \mathcal{H}$. Suppose any $\bar{u} \in \mathcal{H}$ has Morse index $> k-2$. Then for each such $\bar{u}$ there exists a maximal sub-space $V \in H^{2,2}(S^2;\bar{u}^{-1}TN)$ of dimension $> k-2$ such that

$$d^2E(\bar{u})\big|_{V\times V} < 0.$$

Moreover, there exist numbers $\rho_0 > 0$, $\epsilon > 0$ with the following property: If $\pi\colon U \to N$ denotes nearest-neighbor projection in a tubular neighborhood U of N onto N, and if we denote

$$\bar{V}_\rho = \big\{\pi(\bar{u}+v); v \in B_\rho(0;V)\big\} \subset H^{2,2}\left(S^2;N\right)$$

then

$$\sup_{u\in\partial\bar{V}_\rho} E(u) =: \mu_\rho$$

is strictly decreasing for $0 < \rho < \rho_0$ and

$$\mu_{\rho_0} \leq E(\bar{u}) - \epsilon.$$

By the above compactness property of $\mathcal{H}$, the numbers ϵ and ρ_0 can be chosen independent of $\bar{u}$. Hence any path in $H^{2,2}(S^2;N)$ of dimension $\leq k-2$ that approaches an element $\bar{u} \in \mathcal{H}$ can be deformed within the corresponding set $\bar{V}_{\rho_0}$ to a path of strictly lower energy.

Now suppose h_0 is chosen such that

$$\sup_{\eta} E\big(h_0(\cdot,\eta)\big) < \epsilon_k + \frac{\epsilon}{2}$$

and let $u(\cdot;\eta)$ be the corresponding family of solutions to (1.8) with $u(0;\eta) = h_0(\cdot,\eta)$. If for some $\bar{\eta} \in B^{k-2}$ the flow $u(\cdot;\bar{\eta})$ converges to a nontrivial limit, or if $u(\cdot;\bar{\eta})$ develops a singularity at $(\bar{x},\bar{t})$ or some $\bar{t} \leq \infty$, a harmonic sphere$\bar{u}$ forms. By a deformation of the path $\eta \mapsto u(t;\eta)$ for $t < \bar{t}$ close to $\bar{t}$ we achieve a path $\eta \mapsto \tilde{u}(t;\eta)$ homotopic to $u(t;\eta)$, hence to h_0, with

$$E\big(\tilde{u}(t;\bar{\eta})\big) < \epsilon_k$$

for η near $\bar{\eta}$. By a covering argument we thus find a comparison path $h_1 \sim h_0$ such that

$$\sup_{\eta\in B} E\big(h_1(\cdot,\eta)\big) < \epsilon_k,$$

contradicting the definition of ϵ_k. □

Next Micallef-Moore establish that a strict point-wise $\frac{1}{4}$-pinching condition

$$\frac{1}{4}\kappa(p) < K(\sigma) < \kappa(p)$$

for all two-plane sections $\sigma \subset T_pN$, $p \in N$, where K denotes the sectional curvature, implies the lower bound

$$MI(\bar{u}) \geq \frac{n}{2} - \frac{3}{2}$$

for the index of any non-constant harmonic 2-sphere in N. In particular, if $\bar{u}$ is a harmonic sphere as constructed in Theorem 1.7, we obtain

$$k \geq \frac{n}{2} + \frac{1}{2};$$

that is, $H^k(N) = \pi_k(N) = 0$ for $0 < k \leq \frac{n}{2}$. By Poincaré duality

$$H_{n-k}(N) \cong H^k(N) = 0, \qquad 0 < k \leq \frac{n}{2}.$$

Moreover, if $k > 0$ is the first integer such that $\pi_k(N) = 0$, by the Hurewicz isomorphism theorem we also have

$$\pi_k(N) \cong H_k(N).$$

Hence $\pi_k(N) = 0$ for all $0 < k < n$, and N is a homotopy sphere. If $n \geq 4$, by the resolution of the generalized Poincaré conjecture therefore N is homeomorphic to a sphere.

Extensions and generalizations

Theorem 1.5 has been extended to target manifolds N with boundary by Chen-Musina [**20**]. The same technique can be used to study evolution problems related to other two-dimensional variational problems. For instance, in Struwe [**139**], Rey [**118**] the evolution problem for surfaces of prescribed mean curvature is investigated; Ma Li [**103**] has studied the evolution of harmonic maps with free boundaries.

1.5. Existence of global, partially regular weak solutions for $m \geq 3$.

Earlier we observed that singularities must be expected even for energy-minimizing weakly harmonic maps, if $m \geq 3$, and hence for the evolution problem (1.8). The following result was obtained by Chen-Struwe [**21**].

Theorem 1.8. *Suppose M is a compact m-manifold, $\partial M = \emptyset$. For any $u_0 \in H^{1,2}(M;N)$ there exists a distribution solution $u\colon M \times [0,\infty[\to N$ of (1.8),(1.9), satisfying the energy inequality and smooth away from a closed set Σ such that for each t the slice $\Sigma(t) = \Sigma \cap (M \times \{t\})$ is of co-dimension ≥ 2. As $t \to \infty$ suitably, $u(t)$ converges weakly to a weakly harmonic limit u_∞ which is smooth away from a closed set $\Sigma(\infty)$ of co-dimension ≥ 2.*

Originally, the estimate on the co-dimension of Σ was obtained in space-time, the above improvement is due to X. Cheng [**23**]. For manifolds M with boundary $\partial M \neq \emptyset$ a similar existence and interior partial regularity result holds; see Chen [**18**]. Boundary regularity is open. The proof of Theorem 1.8 rests on two pillars: A penalty approximation scheme for (1.8), developed independently by Chen [**17**], Keller-Rubinstein-Sternberg [**93**] and Shatah [**127**], and a monotonicity estimate for (1.8), due to Struwe [**140**].

Penalty approximation

Consider the case $N = S^{n-1} \subset \mathbb{R}^n$. Given $u_0 \in H^{1,2}(M; S^{n-1})$, $K \in \mathbb{N}$ consider the Cauchy problem

$$u_t - \Delta_M u + Ku(\|u\|^2 - 1) = 0, \tag{1.17}$$

$$u_{|t=0} = u_0 \tag{1.18}$$

for maps $u\colon M \times [0,\infty[\to \mathbb{R}^n$. That is, we "forget" the target constraint and regard all maps $u\colon M \to \mathbb{R}^n$ as admissible; however, we "penalize" violation of the constraint $|u|^2 = 1$ more and more severely, as $K \to \infty$. (1.17) is the L^2-gradient flow for the functional

$$E_K(u) = E(u) + K \int_M \frac{(|u|^2 - 1)^2}{4}\, dvol_M.$$

Indeed, we have

Lemma 1.6. *If $u \in C^\infty(M\times]0,T];\mathbb{R}^n)$ solves (1.17), (1.18), then there holds*

$$E_K(u(T)) + \int_0^T \int_M |\partial_t u|^2\, dvol_M dt = E_K(u_0) = E(u_0);$$

in particular, u attains its initial data continuously in $H^{1,2}(M;N)$

Proof. Multiply (1.17) by u_t and integrate to obtain the energy estimate. Since $\partial_t u \in L^2(M \times [0,T])$, clearly $u(t) \to u_0$ in $L^2(M)$ and weakly in $H^{1,2}(M,N)$ as $t \to 0$. Since also

$$\limsup_{t\to 0} E(u(t)) \leq \limsup_{t\to 0} E_K(u(t)) \leq E(u_0),$$

we, in fact, also have strong $H^{1,2}$-convergence. □

Moreover, we have an L^∞ a-priori bound.

Lemma 1.7. *If $u \in C^\infty(M\times]0,T];\mathbf{R}^n)$ solves* (1.17),(1.18), *then*

$$\|u\|_{L^\infty} \le 1.$$

Proof. Multiply (1.17) by u to obtain

$$\left(\frac{d}{dt}-\Delta\right)\frac{|u|^2}{2} + |\nabla u|^2 + K(|u|^2-1)|u|^2 = 0;$$

in particular,

$$\left(\frac{d}{dt}-\Delta+2K|u|^2\right)(|u|^2-1) \le 0.$$

The claim now follows from the parabolic maximum principle, since $|u(0)| = |u_0| \le 1$. □

Thus for any $K \in \mathbf{N}$ we have a unique, global, smooth solution u_K of (1.17), (1.18). Moreover,

$$\|\partial_t u_K\|^2_{L^2(M\times[0,\infty[)} \le E(u_0), \quad \sup_t E(u_K(t)) \le E(u_0),$$

$$|u_K| \le 1, \quad \sup_t \||u_K(t)|^2 - 1\|^2_{L^2(M)} \le \frac{4E(u_0)}{K} \to 0.$$

Thus, passing to a sub-sequence, if necessary, we may assume that

$$\begin{aligned} u_K &\to u \quad \text{in } L^2_{\mathrm{loc}}(M\times[0,\infty[;\mathbf{R}^n),\\ \nabla u_K &\to \nabla u \quad \text{weakly-* in } L^\infty([0,\infty[;L^2(M)), \\ \partial_t u_K &\to \partial_\tau \quad \text{weakly in } L^2(M\times[0,\infty[), \end{aligned} \tag{1.19}$$

and $|u| = 1$. Relations (1.19) are not sufficient to pass to the limit $K \to \infty$ in (1.17), directly. In case $N = S^{n-1}$, however, a clever manipulation of (1.17) will do the trick.

Lemma 1.8. *Suppose $N = S^{n-1} \subset \mathbf{R}^n$. Then* (1.8) *is equivalent to the relations*

$$|u| = 1 \tag{1.20}$$

and

$$u_t \wedge u - \operatorname{div}(\nabla u \wedge u) = 0, \tag{1.21}$$

where we identify the vector fields u, u_t, etc. with one-forms in $\mathbf{R}^n$ and where "$\wedge$" denotes the exterior product.

Proof. For smooth u satisfying (1.20) equation (1.21) is equivalent to asserting

$$u_t - \Delta_M u \perp T_u S^{n-1},$$

which in turn is equivalent to (1.8); that is, $u_t - \Delta u = |\nabla u|^2 u$. If u is a weak solution of (1.8), moreover, an approximation argument justifies testing (1.8) with $u \wedge \psi$, where $\psi \in C_0^\infty(M\times[0,\infty[;\Lambda^2(\mathbf{R}^n))$, whence we obtain (1.21). Conversely, suppose u weakly solves (1.20), (1.21). Note that by (1.20) for any $\tau \in C_0^\infty(M\times[0,\infty[;\mathbf{R})$ the relation

$$(u_t - \Delta u - |\nabla u|^2 u)\cdot u\tau = \left\{\left(\frac{d}{dt}-\Delta\right)\frac{|u|^2}{2} + |\nabla u|^2 - |\nabla u|^2|u|^2\right\}\tau = 0 \tag{1.22}$$

is automatically satisfied. Moreover, by (1.20) again, for any pair of vector fields $v, \phi \in C^\infty(M\times[0,\infty[;\Lambda^1(\mathbf{R}^n))$ there holds

$$v\cdot\phi = (v\cdot u)(u\cdot\phi) - *[(v\wedge u) * (u\wedge\phi)]$$

almost everywhere on $M \times [0, \infty[$, where "$*$" denotes the Hodge star-operator and where again we identify vector fields and 1-forms. By an approximation argument, $\psi = *(u \wedge \phi)$ and $\tau = u \cdot \phi$ are admissible as testing functions in (1.21), (1.22), respectively. From the resulting equations, we obtain the weak form of (1.8). □

Now, taking the exterior product of (1.17) with u_K, the nonlinear term from (1.17) drops out and we obtain (1.21). Because of the divergence structure of this equation and since by (1.19) we have

$$\begin{aligned} \partial_t u_K \wedge u_K &\to \partial_t u \wedge u \\ \nabla u_K \wedge u_K &\to \nabla u \wedge u \end{aligned}$$

weakly in $L^2_{\mathrm{loc}}(M \times [0, \infty[)$ we may pass to the limit $K \to \infty$ and find that also u satisfies (1.21). That is, by Lemma 1.8, u is a weak solution of (1.8), (1.9).

Note that the geometry of the sphere was used in an essential way. Moreover, the regularity of the approximating maps u_K may be lost in the limit.

The monotonicity formula

Monotonicity estimates first were introduced as a tool in regularity theory for minimal hypersurfaces. Schoen-Uhlenbeck [**123**] and Giaquinta-Giusti [**52**] observed that similar estimates hold for energy-minimizing harmonic maps and can be used to obtain partial regularity. Consider for instance a harmonic map $u\colon B = B_1(0; \mathbb{R}^m) \to N$.

Theorem 1.9. *If u is energy-minimizing (for its boundary values), then for any $0 < \rho < r < 1$ there holds*

$$\rho^{2-m} \int_{B_\rho} |\nabla u|^2 dx \le r^{2-m} \int_{B_r} |\nabla u|^2 dx.$$

Proof. Note that the quantity

$$\Phi(\rho) = \frac{1}{2}\rho^{2-m} \int_{B_\rho} |\nabla u|^2 dx$$

is invariant under scaling $u \to u_R(x) = u(Rx)$, which (in case $M = B$) also leaves (1.1) invariant. This observation allows us to give a simple proof of Theorem 1.9 for *smooth* harmonic maps, as follows. Note that $\Phi(\rho) = \Phi(\rho; u) = \Phi(1; u_\rho)$. Hence, for instance at $\rho = 1$, we have

$$\begin{aligned} \frac{d}{d\rho}\Phi(\rho) = \frac{d}{d\rho}\Phi(1; u_\rho) &= \int_B \nabla u \cdot \nabla \left(\frac{d}{d\rho} u_\rho\right) dx \\ &= \int_{\partial B} (x \cdot \nabla u) \cdot \frac{d}{d\rho} u_\rho \, do - \int_B \Delta u \cdot \frac{d}{d\rho} u_\rho \, dx. \end{aligned}$$

Since $\frac{d}{d\rho} u_\rho = x \cdot \nabla u \in T_u N$, by (1.1) the last term vanishes and the boundary integral simply becomes

$$\frac{d}{d\rho} \Phi(\rho)\Big|_{\rho=1} = \int_{\partial B} |x \cdot \nabla u|^2 dw \ge 0,$$

proving the theorem for smooth u. □

Moreover, since $\Phi(\rho)$ scales as a dimension-less quantity, smallness of $\Phi(\rho)$ for some $\rho > 0$ yields a-priori bounds for u near the origin.

A similar result holds in the time-dependent setting. For simplicity we consider smooth solutions $u \in C^\infty(\mathbb{R}^m \times [-1,0[;N)$ of (1.8) with $E(u(t)) \leq E_0 < \infty$ uniformly in $-1 \leq t < 0$. Denote

$$G(x,t) = \frac{1}{\left(\sqrt{4\pi|t|}\right)^m} \exp\left(-\frac{|x|^2}{4|t|}\right), \quad t < 0,$$

the fundamental solution to the backward heat equation on $\mathbb{R}^m \times \mathbb{R}$. Using G as a weight function, we define

$$\Phi(\rho) = \frac{1}{2}\rho^2 \int_{\mathbb{R}^m \times \{-\rho^2\}} |\nabla u|^2 G\, dx,$$

Then we obtain:

Theorem 1.10 ([140]). *For u as above and any $0 < \rho < r \leq 1$ there holds*

$$\Phi(\rho) \leq \Phi(r).$$

Proof. As in the stationary case, we use invariance of (1.8) under scaling $u \to u_R(x,t) = u(Rx, R^2t)$. Note that $\Phi(\rho) = \Phi(\rho; u) = \Phi(1; u_\rho)$. Hence, at $\rho = 1$ we compute

$$\frac{d}{d\rho}\Phi(\rho) = \frac{d}{d\rho}\Phi(1; u_\rho) = \int_{\mathbb{R}^m \times \{-1\}} \nabla u \cdot \nabla\left(\frac{d}{d\rho}u_\rho\right) G\, dx.$$

Integrate by parts and use (1.8) and the relation $\nabla G(x,t) = \frac{x}{2t}G$ to obtain

$$\begin{aligned} \frac{d}{d\rho}\Phi(\rho) &= \int_{\mathbb{R}^m \times \{-1\}} \left(-\Delta u - \frac{x \cdot \nabla u}{2t}\right) \cdot \frac{d}{d\rho}u_\rho G dx \\ &= -\int_{\mathbb{R}^m \times \{-1\}} \frac{2tu_t + x \cdot \nabla u}{2t} \cdot \frac{d}{d\rho}u_\rho G dx \qquad (1.23) \\ &= \frac{1}{2}\int_{\mathbb{R}^m \times \{-1\}} |2tu_t + x \cdot \nabla u|^2 G dx \geq 0. \end{aligned}$$

Since $E(u(t)) \leq E_0 < \infty$ no boundary terms appear. □

Remark 1.1. It was pointed out by R.Kohn that (1.23) is the energy inequality for (1.8) in similarity coordinates $s = -\log|t|, y = \frac{x}{\sqrt{|t|}}$, as introduced by Giga-Kohn [53] for a different problem.

By a scaling argument as in the proof of Proposition 1.3, smallness of $\Phi(\rho)$ can be turned into an a-priori gradient bound for u.

Proposition 1.4 ([140]). *There exists $\epsilon_0 = \epsilon_0(m, N) > 0$ such that for any solution $u \in C^\infty(\mathbb{R}^m \times [-1,0[; N)$ of (1.8) above, if $\Phi(R) < \epsilon_0$ for some $R > 0$, then*

$$\sup_{P_{\delta R}} |\nabla u| \leq \frac{C}{R}$$

with constants $\delta = \delta(m, N, E_0) > 0$ and $C = C(m, N, E_0)$.

Proof. Scaling with R, we may assume $R = 1$. For $\delta > 0$ fix $\rho \in]0, \delta[$, $z_0 = (x_0, t_0) \in P_\rho$ such that

$$(\delta - \rho)^2 \sup_{P_\rho} e(u) = \max_{0 \leq \sigma \leq \delta}\{(\delta - \sigma)^2 \sup_{P_\sigma} e(u)\},$$

$$(e(u))(z_0) = \sup_{P_\rho} e(u) = e_0.$$

First assume $e_0^{-1} \leq (\frac{\delta-\rho}{2})^2$ and scale $v(x,t) = u(x_0 + e_0^{-1/2}x, t_0 + e_0^{-1}t)$. Note that $v \in C^\infty(P; N)$ and

$$\begin{aligned}\sup_P e(v) &= e_0^{-1} \sup_{P_{e_0^{-1/2}}(z_0)} e(u) \leq e_0^{-1} \sup_{P_{\frac{\delta+\rho}{2}}} e(u) \\ &\leq 4e_0^{-1} \sup_{P_\rho} e(u) = 4,\end{aligned}$$

while

$$(e(v))(0) = 1.$$

By the Bochner inequality (1.12) therefore we have

$$\left(\frac{d}{dt} - \Delta\right) e(v) \leq Ce(v) \quad \text{in} \quad P$$

and Lemma 1.3 gives

$$1 = (e(v))(0) \leq C \int_P e(v)dz = Ce_0^{m/2} \int_{P_{e_0^{-1/2}}(z_0)} e(u)\, dx\, dt.$$

Denote $G_{\bar{z}}(z) = G(z - \bar{z})$ the fundamental solution with singularity at $\bar{z} = (\bar{x}, \bar{t})$ and choose $\bar{z} = z_0 + (0, e_0^{-1})$. Then

$$1 \leq C \int_{P_{e_0^{-\frac{1}{2}}}(z_0)} e(u) G_{\bar{z}}\, dx\, dt,$$

and by applying Theorem 1.10 for each $t \in [t_0 - e_0^{-1}, t_0]$, the latter is

$$\leq C \int_{\mathbb{R}^m \times \{-1\}} e(u) G_{\bar{z}}\, dx.$$

Now $|\bar{x}| \leq \delta, |\bar{t}| \leq \delta^2$. Thus at $t = -1$ we may estimate

$$\begin{aligned}|G_{\bar{z}} - G| &\leq \frac{1}{\sqrt{4\pi}^m} \left(\left| 1 - \frac{1}{\sqrt{|1+\bar{t}|}^m} \right| + \left| \exp\left(-\frac{|x|^2}{4}\right) - \exp\left(-\frac{|x-\bar{x}|^2}{4|1+\bar{t}|}\right) \right| \right) \\ &\leq C\delta,\end{aligned}$$

and we obtain that

$$1 \leq C\delta \int_{\mathbb{R}^m \times \{-1\}} e(u)\, dx + C \int_{\mathbb{R}^m \times \{-1\}} e(u) G\, dx \leq C_1 \delta E_0 + C_1 \Phi(1)$$

with a uniform constant $C_1 = C_1(m, N)$. Choosing $\delta = \frac{1}{2C_1 E_0}, \epsilon_0 = \frac{1}{2C_1}$, the above inequality will lead to a contradiction. Thus $(\delta - \rho)^2 e_0 \leq 4$, and we have

$$\sup_{P_{\frac{\delta}{2}}} e(u) \leq 16\delta^{-2}.$$

The proof is complete. □

As an application we establish the following result from [140].

Proposition 1.5. *Suppose $u_k \in C^\infty(\mathbb{R}^m \times [-1,0[;N)$ is a sequence of solutions to (1.8) with $E(u_k(t)) \le E_0 < \infty$ uniformly in t for $k \in \mathbb{N}$. Moreover, suppose $u_k(-1) \to u_0$ in $H^{1,2}_{loc}(\mathbb{R}^m;N)$ $(k \to \infty)$ and*

$$\begin{aligned} u_k &\longrightarrow u \quad \text{in } L^2_{loc}(\mathbb{R}^m \times [-1,0]),\\ \partial_t u_k &\longrightarrow \partial_t u \quad \text{in } L^2_{loc}(\mathbb{R}^m \times [-1,0]),\\ \nabla u_k &\longrightarrow \nabla u \quad \text{in } L^2_{loc}(\mathbb{R}^m \times [-1,0]).\end{aligned}$$

Then u weakly solves (1.8),(1.9) and u is smooth away from a closed set Σ of co-dimension ≥ 2; moreover, for any $R > 0$ we have

$$\Phi(R) + \int_{-1}^{-R^2} \int_{\mathbb{R}^m} \frac{|2tu_t + x \cdot \nabla u|^2}{2|t|} G\, dx\, dt \le \Phi(-1). \tag{1.24}$$

Proof. Let

$$\Sigma = \bigcap_{R>0} \{z; \liminf_{k\to\infty} \Phi_z^k(R) \ge \epsilon_0\},$$

where

$$\Phi_{z_0}^k(r) = \frac{1}{2} r^2 \int_{\mathbb{R}^m \times \{t_0 - r^2\}} |\nabla u_k|^2 G_{z_0}\, dx, \qquad \text{if} \qquad t_0 - r^2 \ge -1,$$

and $\Phi_{z_0}^k = \Phi_{z_0}^k(\sqrt{1-|t_0|})$, else. Σ is relatively closed. Indeed, if $\bar z \in \bar\Sigma$, let $z_l \in \Sigma, z_l \to \bar z$. By definition of Σ we have

$$\liminf_{l\to\infty} \liminf_{k\to\infty} \left(\frac{1}{2} R^2 \int_{\mathbb{R}^m \times \{\bar t - R^2\}} |\nabla u_k|^2 G_{z_l}\, dx \right) \ge \epsilon_0$$

for any $R > 0$. Since $G_{z_l} \to G_{\bar z}$ uniformly away from $\bar z$ and since $E(u_k(t)) \le E_0 < \infty$, the limits $l \to \infty, k \to \infty$ may be interchanged for fixed $R > 0$, whence

$$\liminf_{k\to\infty} \Phi_{\bar z}^k(R) \ge \epsilon_0,$$

for all $R > 0$; that is, $\bar z \in \Sigma$.

Next observe that for $z_0 \notin \Sigma$ there is a sequence (u_k) and some $R > 0$ such that

$$\Phi_{z_0}^k(R) < \epsilon_0.$$

Proposition 1.4 implies that

$$\sup_{P_{\delta R}(z_0)} |\nabla u_k| \le \frac{C}{R}$$

for $\delta = \delta(m, N, E_0) > 0$, uniformly in k, and similar bounds for higher derivatives. Thus we may pass to the limit $k \to \infty$ in (1.8) and find that u is a smooth solution of (1.8) away from Σ.

In order to be able to assert that u extends to a weak solution across Σ we need to estimate the "capacity" of Σ, respectively its m-dimensional Hausdorff measure with respect to the parabolic metric

$$\delta((x,t),(y,s)) = |x-y| + \sqrt{|t-s|}.$$

For a set $S \subset \mathbb{R}^m \times \mathbb{R}$ the latter is defined as

$$\mathcal{H}^m(S;\delta) = c(m) \sup_{R>0} \left\{ \inf \left\{ \sum_i r_i^m ; S \subset \bigcup_i Q_{r_i}(z_i), z_i \in S, r_i < R \right\} \right\},$$

where $c(m)$ is a normalizing constant and

$$Q_r(z_0) = \{z; |x - x_0| < r, |t - t_0| < r^2\}.$$

Fix a compact set $Q \subset \mathbb{R}^m \times]-1, 0[$ and let $S = Q \cap \Sigma$. Fix $R > 0$ and let $Q_{r_i}(z_i), r_i < R$, be a cover of S. Since S is compact, we may assume that the cover is finite. Moreover, a simple variant of Vitali's covering lemma shows that there is a disjoint sub-family $Q_{r_i}(z_i)$, $i \in \mathcal{J}$, such that $S \subset \cup_{i\in\mathcal{J}} Q_{5r_i}(z_i)$. Let $\bar{z}_i = y_i + (0, r_i^2)$, $i \in J$. Since $\mathcal{J}$ is finite, there exists $k \in \mathbb{N}$ such that

$$\begin{aligned}
\epsilon_0 \le \Phi^k_{z_i}(\delta r_i) &\le C \int_{t_i - 4\delta^2 r_i^2}^{t_i - \delta^2 r_i^2} \int_{\mathbb{R}^m} |\nabla u_k|^2 G_{z_i}\, dx\, dt \\
&\le C(\delta) r_i^{-m} \int_{Q_{r_i}(z_i)} |\nabla u_k|^2\, dx\, dt + C\delta^{-m} \exp\left(-\frac{1}{32\delta^2}\right) \\
&\quad \cdot \int_{t_i - 4\delta^2 r_i^2}^{t_i - \delta^2 r_i^2} \int_{\mathbb{R}^m} |\nabla u_k|^2 G_{\bar{z}_i}\, dx\, dt \\
&\le C(\delta) r_i^{-m} \int_{Q_{r_i}(z_i)} |\nabla u_k|^2\, dx\, dt + C\delta^{2-m} \exp\left(-\frac{1}{32\delta^2}\right) E_0,
\end{aligned}$$

for all $i \in \mathcal{J}$, where we used the fact that

$$G_{z_i} \le \delta^{-m} \exp\left(-\frac{1}{32\delta^2}\right) G_{\bar{z}_i} \qquad \text{on } \mathbb{R}^m \times [t_i - 4\delta^2 r_i^2, t_i - \delta^2 r_i^2] \setminus Q_{r_i}(z_i)$$

and Theorem 1.10 to derive the last inequality. If $\delta > 0$ is sufficiently small, we have

$$C\delta^{2-m} \exp\left(-\frac{1}{32\delta^2}\right) E_0 < \frac{\epsilon_0}{2}$$

and hence

$$r_i^m \le C \int_{Q_{r_i}(z_i)} |\nabla u_k|^2\, dx\, dt.$$

Summing over $i \in \mathcal{J}$, we obtain

$$\begin{aligned}
\sum_{i\in\mathcal{J}} r_i^m &\le C \sum_{i\in\mathcal{J}} \int_{Q_{r_i}(z_i)} |\nabla u_k|^2\, dx\, dt \\
&= C \int_{\cup_{i\in\mathcal{J}} Q_{r_i}(z_i)} |\nabla u_k|^2\, dx\, dt \le C(Q) E_0,
\end{aligned}$$

with constants independent of $R > 0$. That is, the m-dimensional Hausdorff measure of Σ is locally finite.

In particular, for a suitable cover $(Q_{r_i}(z_i))_{i\in\mathcal{J}}$ of $S, r_i < R$, we can achieve that

$$\mathcal{L}^{m+1}\left(\bigcup_{i\in\mathcal{J}} Q_{r_i}(z_i)\right) \to 0 \qquad \text{as } R \to 0,$$

where $\mathcal{L}^{m+1}$ denotes Lebesgue measure on $\mathbb{R}^m \times \mathbb{R}$. Now let $\phi \in C_0^\infty(Q_2(0))$ satisfy $0 \le \phi \le 1, \phi \equiv 1$ on $Q_1(0)$ and scale $\phi_i(z) = \phi(\frac{x - x_i}{r_i}, \frac{t - t_i}{r_i^2}) \in C_0^\infty(Q_{2r_i}(z_i))$. Given $\psi \in C_0^\infty(Q; \mathbb{R}^n)$, then $\tau = \psi \inf_i(1 - \phi_i)$ is a Lipschitz function and $\tau(z) \to \psi(z)$

a.e. as $R \to 0$. Testing (1.8) with τ, we obtain

$$\begin{aligned}
&\int_{-1}^{0}\int_{\mathbb{R}^m}(u_t - \Delta u - A(u)(\nabla u, \nabla u))\psi\,dx\,dt \\
&\leq C\int_{-1}^{0}\int_{\mathbb{R}^m}|\nabla u||\nabla \inf_i(1-\phi_i)|\,dx\,dt + o(1) \\
&\leq C\|\nabla u\|_{L^2(\bigcup_{i\in\mathcal{J}} Q_{r_i}(z_i))}\left(\int_{-1}^{0}\int_{\mathbb{R}^m}|\nabla \inf_i(1-\phi_i)|^2\,dx\,dt\right)^{1/2} + o(1) \\
&\leq o(1)\left(\sum_i\int_{Q_{r_i}(z_i)} r_i^{-2}\,dx\,dt\right)^{1/2} + o(1) \\
&= o(1)\left(\sum_i r_i^m\right)^{1/2} + o(1) \to 0 \qquad (R \to 0),
\end{aligned}$$

where $o(1) \to 0$ $(R \to 0)$, and u weakly solves (1.8). Finally, (1.24) follows from (1.23) and the fact that $\Phi(R;u) \leq \liminf_{k\to\infty}\Phi(R;u_k)$, $\Phi(1;u) = \Phi(1;u_0) = \lim_{k\to\infty}\Phi(1;u_k)$. □

The proof of Theorem 1.8 uses the fact that results similar to Theorem 1.10 and Proposition 1.4 hold for solutions u to (1.8) on a compact manifold M, where Φ is defined with reference to a local coordinate chart V and where we truncate the integrand with a smooth cut-off function $\tau \in C_0^\infty(V)$. Moreover, if U_δ is a 3δ-tubular neighborhood on N in $\mathbb{R}^n$ and if we define

$$E_K(u) = E(u) + K\int_M \chi(\mathrm{dist}^2(u,N))\,dx,$$

where $\chi(s) = s$ for $s \leq \delta$, $\chi'(s) \geq 0$, $\chi(s) \equiv 2\delta$ for $s \geq 3\delta$, for maps $u : M \to \mathbb{R}^n$, then the sequence of approximate solutions (u_K) to (1.8) defined by the gradient flow of E_k again satisfies an analogue of Theorem 1.10 and Proposition 1.4. Similar to Proposition 1.5 we then establish that a sub-sequence (u_K) converges weakly to a partially regular weak solution u of (1.8),(1.9). Moreover, inequality (1.24) holds. See Chen-Struwe [**21**] for details.

Let us now turn to some further consequences of the monotonicity formula.

Nonuniqueness

Coron [**27**] observed that for certain weakly harmonic maps $u_0 : B^3 \to S^2$ the stationary weak solution $u(x,t) = u_0(x)$ of (1.8) does not satisfy (1.24), hence must be different from the solution constructed in Theorem 1.8.

Slightly modified, we repeat his construction. Suppose $u_0 \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^3;S^2)$ is weakly harmonic, $u_0(x) = u_0\left(\frac{x}{|x|}\right)$, and consider $u(x,t) = u_0(x)$. Then u weakly solves (1.8) and

$$\Phi_{\bar z}(\rho) = \frac{1}{2\sqrt{4\pi}^3\rho}\int_{\mathbb{R}^3}|\nabla u_0|^2\exp\left(-\frac{|x-\bar x|^2}{4\rho^2}\right)dx < \infty$$

for any $\bar z \in \mathbb{R}^3 \times \mathbb{R}$, any $\rho > 0$. Suppose that u satisfies (1.24). This implies

$$\frac{1}{\rho}\int_{\mathbb{R}^3}|\nabla u_0|^2\exp\left(-\frac{|x-\bar x|^2}{4\rho^2}\right)dx \leq \frac{1}{r}\int_{\mathbb{R}^3}|\nabla u_0|^2\exp\left(-\frac{|x-\bar x|^2}{4r^2}\right)dx \tag{1.25}$$

for any $0 < \rho < r < \infty$. We show that (1.25) is violated for a suitable map u_0. The map u_0 is obtained as follows. Let $\pi: S^2 \setminus \{p\} \to \mathbb{R}^2 \cong \mathbb{C}$ be stereographic projection from the north pole $(0,0,1)$ of S^2 and let $g : \mathbb{C} \to \mathbb{C}$ be a rational map. Composing the weakly harmonic map $u : x \to \frac{x}{|x|}$ from Example 1.2 with π and g we obtain a map

$$u_0(x) = \pi^{-1}\left(g\left(\pi\left(\frac{x}{|x|}\right)\right)\right).$$

Regarding $u_0(x) = u_0\left(\frac{x}{|x|}\right)$ as a map $u_0 : S^2 \to S^2$, by conformal invariance u_0 is harmonic; hence $u_0 : \mathbb{R}^3 \to S^2$ is weakly harmonic. By suitable choice of g (for instance, $g(z) = \lambda z$ with $\lambda \in \mathbb{R}, \lambda > 1$), moreover, we can achieve that the center of mass

$$q = \int_{S^2} |\nabla u_0(x)|^2 x \, dvol_{S^2} \neq 0.$$

(Hence the map u_0 is not minimizing for its boundary values on $B_1(0) \subset \mathbb{R}^3$; see Brezis-Coron-Lieb [13], Remark 7.6.)

Denote

$$\phi(\rho, \bar{x}) = \frac{1}{\rho} \int_{\mathbb{R}^3} |\nabla u_0|^2 \exp\left(-\frac{|x - \bar{x}|^2}{4\rho^2}\right) dx$$

for brevity. Note that

$$\phi(\rho, 0) = \int_0^\infty \left(\int_{S^2} |\nabla u_0(\xi)|^2 \, dvol_{S^2}\right) \exp\left(-\frac{s^2}{4\rho^2}\right) \frac{ds}{\rho} = a_0$$

is independent of $\rho > 0$. Moreover, compute

$$\begin{aligned} \nabla_{\bar{x}} \phi(\rho, 0) &= \int_{\mathbb{R}^3} |\nabla u_0|^2 \frac{x}{2\rho^3} \exp\left(-\frac{|x|^2}{4\rho^2}\right) dx \\ &= \int_0^\infty \left(\int_{S^2} |\nabla u_0(\xi)|^2 \xi \, dvol_{S^2}\right) \cdot \frac{\exp\left(-\frac{s^2}{4\rho^2}\right)}{2\rho^3} s \, ds \\ &= q \int_0^\infty \frac{\exp(-\sigma) \, d\sigma}{\rho} = \frac{q}{\rho}. \end{aligned}$$

Hence for $\bar{x} = tq$, $0 < \rho < r$, if $t > 0$ is sufficiently small we obtain

$$\phi(\rho, \bar{x}) = a_0 + t\frac{|q|^2}{\rho} + O(t^2) > \phi(r, \bar{x}) = a_0 + t\frac{|q|^2}{r} + O(t^2),$$

contradicting (1.25). On the other hand, as in Theorem 1.8 we can construct weak solutions $\tilde{u}$ to (1.8) for initial data u_0 satisfying (1.24), showing that $u \neq \tilde{u}$ and hence showing nonuniqueness in the energy class of weak solutions to (1.8), (1.9).

Note that we have spontaneous symmetry breaking, since u cannot be of the form $u(x,t) = v\left(\frac{x}{|x|}, t\right)$. The latter map v would solve (1.8), (1.9) on $S^2 \times [0, \infty[$. Since $u_0 : S^2 \to S^2$ is smooth and harmonic, by local unique solvability of (1.8),(1.9) on $S^2 \times [0, \infty[$ for smooth data this would imply $v(t) \equiv u_0$.

It remains an open problem to exhibit a class of functions within which (1.8), (1.9) possesses a unique solution. Certainly, the class of functions satisfying the strong monotonicity formula

$$\Phi_{\bar{z}}(\rho) \leq \Phi_{\bar{z}}(r)$$

for all $\bar{z}$ and all $0 < \rho < r \leq \sqrt{\bar{t}}$ is a likely candidate.

Development of singularities

The most surprising aspect of the monotonicity formula is that it may be used to prove that (1.8), (1.9) in general will develop singularities in arbitrarily short time. The existence of singularities was first established by Coron-Ghidaglia [**28**]; see also Grayson-Hamilton [**61**]. These results were based on comparison principles for the reduced harmonic map evolution problem (1.15) in the equivariant setting. A deeper reason for the formation of singularities was worked out by Chen-Ding [**19**]. This is related to a result by White [**152**].

Theorem 1.11. *Let M, N be compact Riemannian manifolds and consider a smooth map $u_0 \colon M \to N$. Then*

$$\inf\{E(u); u \in C^\infty(M,N), \quad u \text{ is homotopic to } u_0\} > 0$$

if and only if the restriction of u_0 to a 2-skeleton of M is not homotopic to a constant.

Remark 1.2. In particular, there are examples of non-trivial homotopy classes of maps $u_0 \colon M \to N$ such that

$$\inf\{E(u); u \text{ is homotopic to } u_0\} = 0.$$

Example 1.6. Let $u_1 = \mathrm{id} \colon S^3 \to S^3$. Let $\pi \colon S^3 \setminus \{p\} \to \mathbb{R}^3$ be stereographic projection, and let $D_\lambda \colon \mathbb{R}^3 \to \mathbb{R}^3$, $D_\lambda(x) = \lambda x$ be dilation with $\lambda > 0$. Then define

$$u_\lambda = \pi^{-1} \circ D_\lambda \circ \pi \colon S^3 \to S^3.$$

Clearly, $u_\lambda \sim u_1 = id$ for all $\lambda > 0$ and $E(u_\lambda) \to 0 \quad (\lambda \to \infty)$.

The construction of Chen-Ding can be vastly simplified by combining Remark 1.10 with Proposition 1.4, as in Struwe [**143**] or [**145**]. Let M, N be compact manifolds, $\dim M = m \geq 3$.

Theorem 1.12. *For any $T > 0$ there exists a constant $\epsilon = \epsilon(M,N,T) > 0$ such that for any map $u_0 \colon M \to N$ which is not homotopic to a constant and satisfies $E(u_0) < \epsilon$ the solution u to* (1.8), (1.9) *must blow up before time $2T$.*

Proof. Suppose $u \in C^\infty(M \times [0,2T]; N)$ solves (1.8), (1.9). For $\bar{z} = (\bar{x}, \bar{t})$, $T \leq \bar{t} < 2T$, $R^2 = T$ estimate

$$\Phi_{\bar{z}}(R) \leq CR^{2-m} E\big(u(\bar{t} - R^2)\big) \leq CR^{2-m} E(u_0) < \epsilon_0$$

if $\epsilon < \frac{\epsilon_0 T^{(m-2)/2}}{C}$, where $C = C(M,N)$ and $\epsilon_0 = \epsilon_0(M,N) > 0$ is the constant in Proposition 1.4. Hence by Proposition 1.4 for $\bar{z} = (\bar{x}, \bar{t}), T \leq \bar{t} \leq 2T$, we have the uniform a-priori bound

$$|\nabla u(\bar{z})| \leq \frac{C}{R} = C(M,N,T).$$

By the Bochner-type inequality (1.12) therefore we have

$$\left(\frac{d}{dt} - \Delta_M\right) e(u) \leq Ce(u) \quad \text{on} \quad M \times [T, 2T]$$

and Lemma 1.10 implies that

$$\sup_x \big(e(u)\big)(x, 2T) \leq CE(u_0) \leq C\epsilon,$$

where $C = C(M,N,T)$. Hence, if $\epsilon = \epsilon(M,N,T) > 0$ is sufficiently small, the image of $u(2T)$ is contained in a convex, hence contractible, coordinate neighborhood on

N and $u(2T)$ is homotopic to a constant. But then also u_0 is homotopic to a constant map, a contradiction. Therefore u must blow up before time $2T$. □

Singularities of first and second kind

Let $u \in C^\infty(\mathbb{R}^m \times [-1,0[; N)$ be a solution to (1.8) with an isolated singularity at the origin, and satisfying (1.24). If

$$|\nabla u(x,t)|^2 \leq \frac{C}{|t|}, \tag{1.26}$$

the rescaled sequence

$$u_R(x,t) = u(Rx, R^2 t), \quad R > 0$$

satisfies the same estimate and hence a sub-sequence converges smoothly locally on $\mathbb{R}^m \times]-\infty, 0[$ to a smooth limit $\bar{u}$ as $R \to 0$. $\bar{u}$ satisfies (1.8). Moreover, $\bar{u} \not\equiv$ const; otherwise $\Phi(R;u) = \Phi(1,u_R) < \epsilon_0$ for some $R > 0$ and u is regular at 0. Since by (1.24) there holds

$$\begin{aligned}\int_{-T}^{-\tau}\int_{\mathbb{R}^m} \frac{|2t\bar{u}_t + x\cdot\nabla\bar{u}|^2}{2|t|} G\,dx\,dt &\leq \lim_{R\to 0}\int_{-T}^{-\tau}\int_{\mathbb{R}^m} \frac{|2tu_{Rt} + x\cdot\nabla u_R|^2}{2|t|} G\,dx\,dt \\ &= \lim_{R\to 0}\int_{-TR^2}^{-\tau R^2}\int_{\mathbb{R}^m} \frac{|2tu_t + x\cdot\nabla u|^2}{2|t|} G\,dx\,dt = 0,\end{aligned}$$

for any $0 < \tau < T < \infty$, it follows that $\bar{u}$ satisfies

$$2t\bar{u}_t + x\nabla\bar{u} \equiv 0;$$

that is

$$\bar{u}(x,t) = v\left(\frac{x}{\sqrt{|t|}}\right)$$

or

$$\bar{u}(x,t) = w\left(\frac{x}{|x|}\right).$$

We call singularities satisfying (1.26) of first kind. It is not known whether self-similar solutions $\bar{u}(x,t) = v\left(\frac{x}{\sqrt{|t|}}\right)$ actually may exist. All other singularities are said to be of second kind. Since singularities in case $m = 2$ by Theorem 1.5 are related to time-independent harmonic maps $\bar{u}\colon S^2 \to N$ of finite energy and since a non-constant, radially homogeneous map $\bar{u}(x) = w\left(\frac{x}{|x|}\right)$ in $m = 2$ dimensions has infinite energy, Theorem 1.4 of Chang-Ding-Ye above shows that singularities of second kind, in fact, exist for the evolution problem (1.8).

Extensions and generalizations

Further current developments include the evolution problem for harmonic maps on general complete, non-compact manifolds (Li-Tam [**104**]) and for harmonic maps with symmetry (Grotowsky [**65**]).

PART 2
The Evolution of Hypersurfaces by Mean Curvature

2.1. Mean curvature flow

Let M be a complete oriented m-dimensional manifold without boundary, $F_0: M \to \mathbb{R}^n$ a smooth embedding into $\mathbb{R}^n, n = m+1$. Denote $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$. Let ν be a smooth unit normal vector field on $M_0 = F_0(M) \subset \mathbb{R}^n$ and denote

$$H = \sum_i \langle e_i, \nabla_{e_i} \nu \rangle$$

(m times) the mean curvature of M_0 with respect to ν, evaluated in a local orthonormal frame $e_1, \dots, e_m$ on M_0. Also denote

$$\underline{H} = -H\nu$$

the mean curvature vector. We say $M_0 = F_0(M)$ is moved by mean curvature if there is a family $F(\cdot, t)$ of smooth embeddings $F(\cdot, t): M \to \mathbb{R}^n$ with corresponding hypersurfaces $M_t = F(M, t)$ such that

$$\frac{\partial}{\partial t} F(p,t) = \underline{H}(p,t) \quad \text{for all } p \in M,\ t \geq 0, \tag{2.1}$$

and $F(\cdot, 0) = F_0$. Here we denote by $\underline{H}(p,t)$ the mean curvature vector of M_t at a point $x = F(p,t) \in \mathbb{R}^n$.

Equation (2.1) corresponds to the negative gradient flow for the volume of M_t. Indeed, if $\mu_t = F(\cdot, t)^* \mu$ denotes the pull-back of the first fundamental form on M_t induced by the Euclidean metric μ , we have

$$\frac{d}{dt} \mu_t = -H^2 \mu_t.$$

Recall that the first variation of volume

$$\operatorname{Vol} M_t = \int_{M_t} d\mathcal{H}^m = \int_M d\mu_t$$

of the hypersurface $M_t \subset \mathbb{R}^n$, when deformed in direction of a vector field ψ, is given by

$$\int_{M_t} \langle \psi, \nu \rangle H \, d\mathcal{H}^m = -\int_{M_t} \langle \psi, \underline{H} \rangle \, d\mathcal{H}^m,$$

where ν denotes a unit normal vector field and H the corresponding mean curvature on M_t.

Moreover, if Δ denotes the Laplacian in the pull-back metric μ_t, we have

$$\underline{H} = \Delta F$$

and (2.1) takes the form of a heat equation on M.

The mean curvature flow was first investigated by Brakke [11], motivated by a study of grain boundaries in annealing metal, and later by Huisken [76]. While Brakke considered the problem in the general context of varifolds, Huisken approached the problem from the classical, parametric point of view (2.1). A weak form of (2.1) in terms of motion of level sets was later proposed by Osher-Sethian [116] and investigated in detail by Evans-Spruck [43], [44], [45], [46] and independently by Chen-Giga-Goto [22].

Finally, Ilmanen [79] has been able to relate the level-set flow and the Brakke motion of varifolds in the frame-work of geometric measure theory.

We will trace a part of these developments. First we consider the parametric point of view. We will consider two model cases: the case when M_0 is the boundary of a bounded region $U_0 \subset \mathbb{R}^n$ and the case when M_0 is represented as an entire graph

$$M_0 = \{(x', u_0(x')); \quad x' \in \mathbb{R}^{n-1}\}.$$

Another special case is the case of plane curves or, more generally, curves on Riemannian surfaces. In the 80's Gage [49] and Gage-Hamilton [50] established that convex curves in the plane evolve smoothly to a nearly circular shape before they shrink to a 'round' point. Grayson [58], [59] extended these results to general closed embedded curves in the plane and on surfaces. In the latter case, another possible long-time behavior is the convergence towards a closed geodesic. Abresch-Langer [1], Angenent [4] and Grayson [60] also studied immersed curves in the plane and obtained highly interesting examples of singular, self-similar behavior.

However, in these notes we will study only the higher-dimensional case $m \geq 2$, where we will be able to observe singularities even if the original hypersurface is smoothly embedded, in contrast to the one-dimensional case.

2.2. Compact surfaces

Let U_0 be a compact set in $\mathbb{R}^n$ with smooth boundary M_0 and, say, outer unit normal ν. If U_0 is convex, then with our sign convention the mean curvature vector $\underline{H}$ is pointing inside U_0, and M_0, evolving by (2.1), is contracting.

Example 2.1. If $U_0 = B_R(0)$, then M_t is a sphere of radius $R(t)$, satisfying

$$\frac{d}{dt}R = -\frac{n-1}{R}, \quad R(0) = R_0;$$

that is,

$$R(t) = \sqrt{R_0^2 - 2(n-1)t},$$

and M_0 shrinks to a point in finite time $T = \frac{R_0^2}{2(n-1)}$.

Example 2.2 (Angenent [5]). There exists a torus-like surface in $\mathbb{R}^3$ shrinking to a point by self-similar motion.

The evolution of spheres is in certain ways characteristic; moreover, their evolution gives nice "barriers" to control the evolution of more general hypersurfaces, due to the following

Theorem 2.1 (Comparison principle). *If M_0 and $\tilde{M}_0$ are boundaries of smooth, relatively compact regions $U_0 \subset \tilde{U}_0 \subset\subset \mathbb{R}^n$, respectively, and if M_0 and $\tilde{M}_0$ evolve by mean curvature through families $M_t = \partial U_t$, $\tilde{M}_t = \partial \tilde{U}_t$, respectively, then for $t > 0$ there holds $U_t \subset \tilde{U}_t$. If $U_0 \neq \tilde{U}_0$, we even have $U_t \subset\subset \tilde{U}_t$.*

Heuristically, the comparison principle follows from the observation that if M_t were to "touch" $\tilde{M}_t$ from "inside" at some point x, then the mean curvature of M_t at x would exceed the mean curvature of $\tilde{M}_t$ at x, drawing M_t and $\tilde{M}_t$ apart. A formal derivation, based on the maximum principle, will be given shortly.

As a consequence of the comparison principle any compact hypersurface $M_0 = \partial U_0$ evolving by mean curvature will become extinct in finite time T. Indeed, if $U_0 \subset B_{R_0}(x_0)$ for some $x_0, R_0 > 0$, we have $T \leq T_0 = \frac{R_0^2}{2(n-1)}$.

On the other hand, later we shall see that spheres may also be used as barriers to ensure long-time existence of entire graphs by mean curvature flow.

Local existence

Observe that (2.1) is only weakly parabolic, as the equation is invariant under diffeomorphisms of M. Local existence for (2.1) thus cannot be derived by the standard theory for parabolic systems; instead, more sophisticated methods, like the Nash-Moser scheme, have been used.

However, as observed by Bollow [10], a simple alternative proof for local existence can be given based on a representation of the evolving surfaces M_t as graphs over M_0.

Since the general case is rather technical, let us derive this result only in the case that M_0 is strictly star-shaped with respect to the origin, whence M_0 may be represented as a graph

$$M_0 = \{u_0(\xi)\xi;\ \xi \in S^{n-1}\}$$

over the standard sphere.

Lemma 2.1. *Let M_0 be given as above with a smooth, strictly positive function $u_0 : S^{n-1} \to \mathbb{R}$. Then (2.1) is equivalent to the evolution problem*

$$\frac{\partial}{\partial t}u - \frac{\sqrt{u^2 + |\nabla u|^2}}{u^2} \operatorname{div}\left(\frac{\nabla u}{\sqrt{u^2 + |\nabla u|^2}}\right) = -\frac{n-1}{u} \tag{2.2}$$

on $S^{n-1} \times [0, \infty[$ with initial condition $u|_{t=0} = u_0$, in the sense that for the solution of (2.1) there holds $M_t = \{u(\xi, t)\xi;\ \xi \in S^{n-1}\}$, where u solves (2.2). Conversely, if u solves (2.2) then M_t, given as above, may be reparametrized by tangential diffeomorphisms to achieve a solution $F(\cdot, t)$ of (2.1).

Proof. Let $\xi \in S^{n-1}$, and let e_i, $1 \leq i \leq n-1$, be an orthonormal basis for $T_\xi S^{n-1}$, and let $x = u(\xi, t)\xi$ be a parametrization of M_t. Denote $u_{,i} = e_i(u) = \nabla_{e_i} u$, etc., the derivative in direction e_i. Then at the point ξ we have the following expressions for ν, H and the first and second fundamental forms $(g_{ij}), (h_{ij})$ on M_t, respectively:

$$\begin{aligned} x_{,i} &= u_{,i}\xi + ue_i, \\ g_{ij} &= \langle x_{,i}, x_{,j}\rangle = u_{,i}u_{,j} + u^2\delta_{ij}, \\ \nu &= \frac{u\xi - u_{,i}e_i}{\sqrt{u^2+|\nabla u|^2}}, \\ h_{ij} &= \langle x_{,i}, \nu_{,j}\rangle = \frac{2u_{,i}u_{,j} + u^2\delta_{ij} - u_{,ij}}{\sqrt{u^2+|\nabla u|^2}}, \\ H &= \operatorname{trace}(g^{ij}h_{jk}) = g^{ij}h_{ji} \\ &= \frac{n-1}{\sqrt{u^2+|\nabla u|^2}} - \frac{1}{u}\operatorname{div}\left(\frac{\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right). \end{aligned}$$

(To see the latter we may orient e_i such that $u_{,i} = 0$ for $i \neq 1$.) Hence if M_t evolves according to (2.1), the radial component $(\frac{\partial}{\partial t}F)^{\perp}$ of its velocity satisfies

$$\begin{aligned} \frac{u^2\frac{\partial}{\partial t}u}{u^2+|\nabla u|^2} &= \left\langle \frac{\partial}{\partial t}F, \xi\right\rangle = \langle \underline{H}, \xi\rangle = -\frac{Hu}{\sqrt{u^2+|\nabla u|^2}} \\ &= -(n-1)\frac{u}{u^2+|\nabla u|^2} + \frac{1}{\sqrt{u^2+|\nabla u|^2}}\operatorname{div}\left(\frac{\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right). \end{aligned}$$

That is, u is a solution of (2.2). Conversely, if u solves (2.2) and M_t is the graph of u above S^{n-1}, then M_t evolves by mean curvature. □

Now if $u_0 \geq c > 0$, then for u near u_0 the linearized operator (2.2) is uniformly parabolic and we obtain local existence and uniqueness of solutions of (2.2) and hence (2.1). Moreover, the comparison principle stated above for strictly star-shaped hypersurfaces can be deduced from the strong maximum principle for uniformly parabolic equations.

Finally, if M_0 is convex, then M_t remains convex until M_0 becomes extinct at a 'round' point, as was proved by Huisken [76]. Although one might expect a similar result to hold for strictly star-shaped initial surfaces, the corresponding result is false in this case. To see this, consider the boundary of a fattened double-cone bounding two large spheres at the ends and circled by a self-similarly shrinking torus at the waist.

2.3. Entire graphs

Suppose M_0 can be represented as a graph

$$M_0 = \{(x', u_0(x')); x' \in \mathbb{R}^{n-1}\},$$

with a smooth function $u_0 : \mathbb{R}^{n-1} \to \mathbb{R}$. By a result of Ecker and Huisken [34] then M_0 generates a global smooth evolution by mean curvature. To see this it is useful to derive a non-parametric form of the evolution problem (2.1).

Lemma 2.2. *Suppose M_0 is given by the graph of a smooth function $u_0 : \mathbb{R}^{n-1} \to \mathbb{R}$. Then* (2.1) *is equivalent to the evolution problem*

$$\frac{\partial}{\partial t}u - \sqrt{1+|\nabla u|^2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{in } \mathbb{R}^{n-1} \times [0,\infty[, \tag{2.3}$$

with initial condition $u_{|t=0} = u_0$, *in the following sense. For any* $t \geq 0$ *the set* M_t *evolving by* (2.1), *starting from* M_0 *at* $t = 0$, *is an entire graph*

$$M_t = \left\{(x', u(x', t)); x' \in \mathbf{R}^{n-1}\right\},$$

where u *solves* (2.3). *Conversely, if* u *satisfies* (2.3), *then* M_t, *given as above, up to tangential diffeomorphism evolves by* (2.1).

Proof. Let $x = (x', u(x', t))$, $e_1, \ldots, e_n$ the standard orthonormal basis for $\mathbf{R}^n$. Then we have

$$\begin{aligned}
x_{,i} &= e_i + u_{,i}e_n, \\
g_{ij} &= u_{,i}u_{,j} + \delta_{ij}, \\
\nu &= \frac{e_n - u_{,i}e_i}{\sqrt{1 + |\nabla u|^2}}, \\
h_{ij} &= \langle x_{,i}, \nu_{,j}\rangle = \frac{\langle e_i + u_{,i}e_n, -u_{,ij}e_i\rangle}{\sqrt{1 + |\nabla u|^2}} \\
&= \frac{-u_{,ij}}{\sqrt{1 + |\nabla u|^2}}
\end{aligned}$$

for $1 \leq i, j \leq n - 1$, and thus

$$H = g^{ij}h_{ij} = -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right).$$

Hence if $M_t = F(M_0, t)$ evolves according to (2.1), its vertical velocity satisfies

$$\frac{\frac{\partial}{\partial t}u}{1 + |\nabla u|^2} = \left\langle \frac{\partial}{\partial t}F, e_n \right\rangle = -\langle H\nu, e_n\rangle = \frac{1}{\sqrt{1 + |\nabla u|^2}} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla|^2}}\right),$$

as claimed, and conversely. □

Again, (2.3) is uniformly parabolic at smooth functions u satisfying a global Lipschitz condition. Hence, for smooth, globally Lipschitz initial data it is not hard to show that (2.3) admits a smooth solution, which, in fact, is defined for all time. That is, an entire Lipschitz graph M_0 evolves through smooth Lipschitz graphs M_t for all time. Surprisingly, Ecker and Huisken were able to show that a solution to (2.3) and hence to (2.1) exists globally even without any growth condition on u_0 near infinity. This result—somewhat reminiscent of results on removable singularities for the minimal surface equation—is obtained by first using spheres "above" and "below" M_0 as barriers to control the "height" of M_t locally and then combining the local L^∞-estimate thus obtained with the following interior gradient estimate for solutions of (2.3).

Theorem 2.2 ([34], Theorem 2.3 and Theorem 5.2). *Suppose* u *is a smooth solution of* (2.3) *on* $B_R(x_0') \times [0, T]$ *with initial data* u_0. *Then for all* $t \in [0, T]$ *there holds*

$$\sqrt{1 + |\nabla u(x_0', t)|^2} \leq C \sup_{B_R(x_0')} \sqrt{1 + |\nabla u_0|^2} \cdot \exp\left\{CR^{-2} \sup_{x' \in B_R(x_0'), 0 \leq s, t \leq T} |u(x', s) - u(x_0', t)|^2\right\}$$

with constants $C = C(n)$.

The proof of this result uses techniques of [98], adapted to a parabolic setting.

Theorem 2.3 ([34], Theorem 5.1). *Let $M_0 = \{(x', u_0(x')); x' \in \mathbb{R}^{n-1}\}$ be a locally Lipschitz entire graph over $\mathbb{R}^{n-1}$. Then the mean curvature flow (2.1) has a unique smooth solution $(M_t)_{t>0}$; moreover, each M_t is an entire graph $M_t = \{(x', u(x', t)); x' \in \mathbb{R}^{n-1}\}$ given by a smooth global solution u of (2.3).*

Proof. Given $x'_0 \in \mathbb{R}^{n-1}$, $R > 0$, $T > 0$ let $R_0^2 > R^2 + 2(n-1)T$,

$$a = \sup_{B_{R_0}(x'_0)} |u_0| + R_0$$

and choose balls $B_0^{\pm} = B_{R_0}((x'_0, \pm a))$ "above" and "below" M_0 as "barriers" to control u. Note that by time t the mean curvature flow will shrink the spheres $S_0^{\pm} = \partial B_0^{\pm}$ to the concentric spheres $S_t^{\pm} = \partial B_t^{\pm}$, where $B_t^{\pm} = B_{R(t)}((x'_0, \pm a))$, with $R(t) = \sqrt{R_0^2 - 2(n-1)t} > R$ for $0 \le t \le T$.

By the comparison principle, therefore, M_t is confined "between" S_t^- and S_t^+ for $t \le T$, whence we obtain the uniform a-priori bound $|u(x', t)| \le a$ on $B_R(x'_0) \times [0, T]$.

By Theorem 2.2 above, this L^∞-bound translates into a uniform gradient bound

$$|\nabla u(x'_0, t)| \le C \left(\sup_{B_R(x'_0)} \left(|\nabla u_0| + \frac{|u_0|}{R} \right) \right)$$

for $0 \le t \le T$. Varying x'_0, we obtain a-priori gradient bounds on $\mathbb{R}^{n-1} \times [0, T]$. By a boot-strap argument similar a-priori bounds for higher derivatives of u may be obtained. Approximating u_0 by smooth functions u_{0k} linear at infinity and such that $u_{0k} \to u_0$ uniformly on any compact subset $\Omega \subset \mathbb{R}^{n-1}$ with $|\nabla u_{0k}|$ bounded uniformly in Ω, uniformly in k, the corresponding solutions u_k to (2.3) for initial data u_{0k} converge smoothly to a solution u of (2.3), locally on $\mathbb{R}^{n-1} \times]0, \infty[$. $\square$

Another use of the non-parametric form (2.3) of (2.1) is to give rigorous proof of the comparison principle Theorem 2.1: Suppose M_{t_0} and $\tilde{M}_{t_0}$ touch at a point $x_0 \in \mathbb{R}^n$. We may suppose $0 \le t_0$ is minimal with that property and that x_0 lies on the boundary of $M_t \cap \tilde{M}_t$, relative to M_t. Represent M_t and $\tilde{M}_t$ as graphs above their common tangent plane at (x_0, t_0) and apply the maximum principle to equation (2.3) in a backward neighborhood of (x_0, t_0).

2.4. Generalized motion by mean curvature

Above we have seen that an initial surface near a double cone or, simpler, a long dumbbell-shaped surface develops a singularity before it becomes extinct under mean curvature flow.

In order to be able to follow the evolution past the singularity, we now define generalized motion by mean curvature as developed by Evans-Spruck [43], [44], [45], [46], and, independently, by Chen-Giga-Goto [22]. The idea is to represent a hypersurface M_0 in $\mathbb{R}^n$ as the level-set $\{u_0(x) = 0\}$ for some function u_0 and such that for a suitable family $u_t = u(\cdot, t)$, all level surfaces $\{u(x, t) = \text{const}\}$, wherever sufficiently regular, evolve by mean curvature. To derive the differential equation that the family u_t will have to satisfy in order for its level surfaces to evolve according to (2.1) we fix a point $x \in \Gamma_t^\gamma = \{u(x, t) = \gamma\}$ and follow its path as Γ_t^γ is evolving by (2.1).

Note that we have $\nu = \frac{\nabla u}{|\nabla u|}$, $H = \operatorname{div} \nu$; hence if we parametrize the particle path of x by $x(s), s \geq t$, we have

$$\dot{x} = \frac{d}{ds}x(s) = -H\nu = -\frac{\nabla u}{|\nabla u|} \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

at the initial time $s = t$. On the other hand, the requirement $u(x(s), s) = \gamma$ for all s leads to

$$0 = \frac{d}{ds}u(x(s), s) = \langle \dot{x}, \nabla u \rangle + \frac{\partial}{\partial t}u = -|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \frac{\partial}{\partial t}u.$$

Thus we say that the motion of level sets of u defines a generalized motion by mean curvature if $u : \mathbb{R}^n \times [0, \infty[\to \mathbb{R}$ satisfies

$$\frac{\partial}{\partial t}u - \left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\right) u_{x_i x_j} = 0, \tag{2.4}$$

where i and j are now summed from 1 to n. Note that in the language of fluid mechanics, equation (2.4) corresponds to the Eulerian viewpoint while the parametric problem (2.1) corresponds to the Lagrangian one. In the present context, equation (2.4) seems to have first been proposed by Osher-Sethian [**116**].

For smooth functions u with $\nabla u \neq 0$, equations (2.4) and (2.1) are in fact equivalent. However, if we want to model the motion of a closed hypersurface M_0 bounding a compact region in $\mathbb{R}^n$ by the evolving level sets $\Gamma_t = \{u(x,t) = 0\}$ of a solution of (2.4), it is clear that ∇u must vanish at some point in the interior of Γ_t for each t and hence (2.4) cannot be interpreted classically on all of $\mathbb{R}^n \times [0, \infty[$.

Moreover, by the geometric interpretation of (2.4), if u is a solution of (2.4), and if Ψ is any continuous, one-to-one function, then $\Psi(u)$ should be a solution of (2.4), as u and $\Psi(u)$ have the same level sets. To be able to include these cases a notion of weak solution is required.

The appropriate notion of generalized solution is that of viscosity solution as in [**29**], [**30**], [**40**], [**80**], [**83**], and [**106**].

Definition 2.1. A function $u \in C^0 \cap L^\infty(\mathbb{R}^n \times [0, \infty[)$ is a weak sub- (super-) solution of (2.4) if the following holds: For each $\phi \in C^\infty(\mathbb{R}^{n+1})$ and each $(x_0, t_0) \in \mathbb{R}^n \times]0, \infty[$ such that $u - \phi$ has a maximum (minimum) at (x_0, t_0) we have

$$\phi_t - (\delta_{ij} - \eta_i\eta_j)\phi_{x_i x_j} \leq 0 \qquad (\geq 0)$$

for some $\eta \in \mathbb{R}^n$, $|\eta| \leq 1$, and

$$\eta = \frac{\nabla\phi(x_0, t_0)}{|\nabla\phi(x_0, t_0)|}, \quad \text{if } \nabla\phi(x_0, t_0) \neq 0.$$

In particular, a weak sub-solution forces any smooth function whose graph "touches" the graph of u from "above" at (x_0, t_0) to be a sub-solution at (x_0, t_0), and similarly for super-solutions. In fact, for more general classes of equations one usually requires that $\phi(x_0, t_0) = u(x_0, t_0)$ for comparison functions ϕ. However, since translations $u \to u + c$ leaves (2.4) invariant, here this condition can be waived.

Often it may be convenient to represent admissible comparison functions as

$$\phi(x,t) = u(x_0, t_0) + p(x - x_0) + q(t - t_0) + \frac{1}{2}(z - z_0)^T R(z - z_0) + o(|z - z_0|^2)$$

as $z \to z_0$, for $p \in \mathbb{R}^n, q \in \mathbb{R}$, $R = (r_{ij})$ a symmetric $(n+1) \times (n+1)$-matrix, and $z = (x, t)$.

Definition 2.2. A function $u \in C^0 \cap L^\infty(\mathbb{R}^n \times [0,\infty[)$ is a weak solution of (2.4) if u is simultaneously a weak sub- and super-solution.

Let us now turn to some simple special cases.

Example 2.3. (The motion of spheres) Let $u_0(x) = R_0^2 - |x|^2$. Then

$$u(x,t) = u_0(x) - 2(n-1)t$$

satisfies (2.4) classically at all points where $\nabla u \neq 0$, and the level surface

$$\Gamma_t = \{x \in \mathbb{R}^n; u(x,t) = 0\} \quad \text{is given by } \Gamma_t = \partial B_{R(t)}(0)$$

with $R(t) = \sqrt{R_0^2 - 2(n-1)t}$ as in the classical, parametric setting. To verify that u solves (2.4) also at $x_0 = 0$ it suffices to consider as comparison functions

$$\phi^\pm(x,t) = u(x,t) \pm \frac{1}{2}(z - z_0)^T R(z - z_0),$$

where $z = (x,t), z_0 = (0,t_0), R = (r_{ij}) \geq 0$. Then $\phi^+ \geq u \geq \phi^-$ near $z_0, \nabla\phi^\pm(z_0) = 0$, and, letting η denote an arbitrary unit vector in $\mathbb{R}^n$,

$$\begin{aligned}\phi_t^\pm - (\delta_{ij} - \eta_i\eta_j)\phi^\pm_{x_ix_j} &= u_t - (\delta_{ij} - \eta_i\eta_j)u_{x_ix_j} \mp (\delta_{ij} - \eta_i\eta_j)r_{ij} \\ &= \mp(\delta_{ij} - \eta_i\eta_j)r_{ij}\end{aligned}$$

has the desired sign.

2.5. Uniqueness, comparison principles, global existence

Thus, in a simple case we reobtain the classical motion by mean curvature. Note that in the example above we simultaneously recover the evolution of *all* spheres centered at the origin; moreover, the solution u remains smooth and is in fact globally defined even past the extinction time $T_0 = R_0^2/2(n-1)$ of any individual sphere.

However, it may not be obvious that this solution is unique; conceivably, after passing through a "geometric" singularity a level surface might evolve in any one of several different ways, as is illustrated by examples of Brakke **[11]** for the mean curvature flow of varifolds. In fact, it is quite surprising that solutions to (2.4) *are* unique. Moreover, we shall see that the motion of a level surface Γ_0 depends only on the surface, not on the function u_0 defining it. Finally, solutions persist for all time. Thus, (2.4) defines a unique global extension of the mean curvature flow of hypersurfaces to arbitrary level surfaces of continuous functions in $\mathbb{R}^n$.

We begin by the following observation.

Theorem 2.4 (Naturality; **[43]**, Theorem 2.8). *If $u \in C^0 \cap L^\infty(\mathbb{R}^n \times [0,\infty[)$ is a weak solution of* (2.4) *and if $\Psi\colon \mathbb{R} \to \mathbb{R}$ is continuous, then*

$$v \equiv \Psi(u)$$

is a weak solution of (2.4).

Proof. First consider the case where $\Psi \in C^\infty$ with $\Psi' > 0$ on $\mathbb{R}$. Let $\psi \in C^\infty(\mathbb{R}^{n+1})$ and suppose

$$0 = \psi(x_0,t_0) - v(x_0,t_0) \leq \psi(x,t) - v(x,t) \quad \text{near } (x_0,t_0).$$

Then $\phi = \Psi^{-1}(\psi) \in C^\infty(\mathbb{R}^{n+1})$ and

$$0 = \phi(x_0,t_0) - u(x_0,t_0) \leq \phi(x,t) - u(x,t) \quad \text{near } (x_0,t_0),$$

whence for some $\eta \in \mathbb{R}^n, |\eta| \leq 1$, we have

$$\phi_t - (\delta_{ij} - \eta_i\eta_j)\phi_{x_ix_j} \leq 0 \quad \text{at } (x_0, t_0).$$

Moreover, $\eta = \nabla\phi/|\nabla\phi| = \nabla\psi/|\nabla\psi|$, evaluated at (x_0, t_0), if $\nabla\phi(x_0, t_0) \neq 0$. But then

$$\psi_t - (\delta_{ij} - \eta_i\eta_j)\psi_{x_ix_j} = \Psi'(\phi_t - (\delta_{ij} - \eta_i\eta_j)\phi_{x_ix_j}) + \Psi''(|\nabla\phi|^2 - |\eta \cdot \nabla\phi|^2) \leq 0,$$

and v is a weak sub-solution of (2.4). Similarly, v is a weak super-solution, which proves the theorem in case $\Psi \in C^\infty$ satisfies $\Psi' > 0$. The case $\Psi' < 0$ is similar.

More generally, in order for $\Psi(u)$ to be a weak solution of (2.4) near (x_0, t_0) it suffices that Ψ is monotone in a neighborhood of $u(x_0, t_0)$. Since an arbitrary continuous function can be uniformly approximated by functions which are locally monotone – that is, regions where Ψ is increasing or decreasing, respectively, are separated by intervals where Ψ is constant – the assertion of the theorem for general Ψ is a consequence of the next result. □

Theorem 2.5 (Closedness of solution set; [43], Theorem 2.7). *Let $u_k \in C^0 \cap L^\infty(\mathbb{R}^n \times [0, \infty[)$ weakly solve (2.4) and suppose $u_k \to u$ boundedly and locally uniformly on $\mathbb{R}^n \times [0, \infty[$. Then u is a weak solution of (2.4).*

Proof. Let $\phi \in C^\infty(\mathbb{R}^{n+1})$ and suppose that $u - \phi$ has a local maximum at $z_0 = (x_0, t_0)$. Replacing ϕ by the function $\psi(z) = \phi(z) + |z - z_0|^4$, if necessary, we may assume that $u - \phi$ has a *strict* local maximum at z_0.

By locally uniform convergence $u_k \to u$, for large k also $u_k - \phi$ has a local maximum at a point z_k, $z_k \to z_0 \quad (k \to \infty)$. If $\nabla\phi(z_0) \neq 0$ we also have $\nabla\phi(z_k) \neq 0$ for large k and thus

$$\phi_t - \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\phi_{x_ix_j} \leq 0 \text{ at } z_k.$$

Passing to the limit $k \to \infty$, we obtain the same relation also at z_0.

If $\nabla\phi(z_0) = 0$ we have

$$\phi_t - \left(\delta_{ij} - \eta_i^{(k)}\eta_j^{(k)}\right)\phi_{x_ix_j} \leq 0 \quad \text{at } z_k$$

for a sequence of vectors $\eta^{(k)} \in \mathbb{R}^n$, $|\eta^{(k)}| \leq 1$. (Of course, if $\nabla\phi(z_k) \neq 0$ we let $\eta^{(k)} = \frac{\nabla\phi(z_k)}{|\nabla\phi(z_k)|}$). By passing to a sub-sequence, if necessary, we assume that $\eta^{(k)} \to \eta \quad (k \to \infty)$. It follows that

$$\phi_t - (\delta_{ij} - \eta_i\eta_j)\,\phi_{x_ix_j} \leq 0 \quad \text{at } z_0,$$

and u is a weak sub-solution, as claimed. The proof that u is a weak super-solution is similar. □

Theorem 2.6 (Global existence; [43], Theorem 4.2). *Assume $u_0 \colon \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies $u_0(x) \equiv \gamma_0 \in \mathbb{R}$ for $|x| \geq R_0$ for some constant $R_0 > 0$. Then there exists a weak solution $u \in C^0 \cap L^\infty(\mathbb{R}^n \times [0, \infty[)$ of (2.4) with initial data $u(\cdot, 0) = u_0$ and*

$$u(x, t) = \gamma_0 \qquad \text{for } |x|^2 + 2(n-1)t \geq R_0^2.$$

Moreover, if u_0 is Lipschitz, so is $u(\cdot, t)$ and

$$\|\nabla u\|_{L^\infty(\mathbb{R}^n \times [0,\infty[)} \leq \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}.$$

Proof. Assume for simplicity that u_0 is smooth. Extend u_0 to $\mathbb{R}^{n+1}$ by letting

$$\tilde{u}_0^\epsilon(x, x_{n+1}) = u_0(x) - \epsilon x_{n+1}.$$

Denote points in $\mathbb{R}^{n+1}$ by $y = (x, x_{n+1})$. The level surface

$$\tilde{\Gamma}_0 = \{y \in \mathbb{R}^{n+1}; \tilde{u}_0^\epsilon(y) = 0\}$$

is a smooth graph

$$\tilde{\Gamma}_0 = \left\{\left(x, \frac{u_0(x)}{\epsilon}\right); \ x \in \mathbb{R}^n\right\}.$$

Hence, by Theorem 2.3 above, $\tilde{\Gamma}_0$ generates a smooth family of graphs

$$\tilde{\Gamma}_t = \left\{\left(x, \frac{u^\epsilon(x,t)}{\epsilon}\right); \ x \in \mathbb{R}^n\right\},$$

evolving by mean curvature. Any other level surface

$$\tilde{\Gamma}_0^\gamma = \{y; \tilde{u}_0^\epsilon(y) = \gamma\}$$

is a parallel translate of $\tilde{\Gamma}_0$ and evolves through parallel translates of $\tilde{\Gamma}_t$. Hence

$$\tilde{u}^\epsilon(x, x_{n+1}, t) = u^\epsilon(x,t) - \epsilon x_{n+1}$$

is a (classical) solution to (2.4) on $\mathbb{R}^{n+1} \times [0, \infty[$. Computing explicitly, thus u^ϵ satisfies

$$u_t^\epsilon - \left(\delta_{ij} - \frac{u_{x_i}^\epsilon u_{x_j}^\epsilon}{|\nabla u^\epsilon|^2 + \epsilon^2}\right) u_{x_i x_j}^\epsilon = 0 \tag{2.5}$$

in $\mathbb{R}^n \times [0, \infty[$ with initial condition $u^\epsilon(\cdot, 0) = u_0$.

Moreover, for fixed $\epsilon > 0$ by Theorem 2.3 we have local bounds for $|\nabla u^\epsilon|$ on any slice $\mathbb{R}^n \times [0, T]$. Thus, by the maximum principle for uniformly parabolic equations, we have uniform a-priori bounds

$$\|u^\epsilon\|_{L^\infty(\mathbb{R}^n \times [0,\infty[)} = \|u_0\|_{L^\infty(\mathbb{R}^n)}.$$

Finally, differentiating (2.5) with respect to x_l, we have

$$\left| u_{x_l t}^\epsilon - \left(\delta_{ij} - \frac{u_{x_i}^\epsilon u_{x_j}^\epsilon}{|\nabla u^\epsilon|^2 + \epsilon^2}\right) u_{x_l x_i x_j} \right| \le C(\epsilon) \left|u_{x_i x_j}\right| |\nabla u_{x_l}|$$

and another application of the maximum principle gives the Lipschitz bound

$$\|\nabla u^\epsilon\|_{L^\infty(\mathbb{R}^n \times [0,\infty[)} \le \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)},$$

uniformly in $\epsilon > 0$. Thus, passing to a suitable sequence $\epsilon \to 0$ if necessary, $u^\epsilon \to u$ boundedly and locally uniformly on $\mathbb{R}^n \times [0, \infty[$. A simple variation in the proof of the preceding compactness result shows that u weakly solves (2.4). The characterization of the support of $(u - \gamma_0)_+$ results from comparing the solutions $\tilde{u}^\epsilon$ above to the solutions $\tilde{v}^\epsilon(x,t)$ corresponding to the shrinking of spheres. □

Theorem 2.7 (Comparison principle and uniqueness; [43], Theorem 3.2). *If* u, $v \in C^0 \cap L^\infty(\mathbb{R}^n \times [0, \infty[)$ *are weak solutions of* (2.4) *with initial data* $u(\cdot, 0) = u_0 \le v_0 = v(\cdot, 0)$ *and if for some* $R > 0$ *both* u *and* v *are constant on* $\mathbb{R}^n \times [0, \infty[\cap \{|x| + t \ge R\}$, *then* $u \le v$. *In fact,*

$$\|u - v\|_{L^\infty(\mathbb{R}^n \times [0,\infty[)} \le \|u_0 - v_0\|_{L^\infty(\mathbb{R}^n)}.$$

Remark 2.3. The construction used above to prove global existence yields weak solutions $u \le v$ if the initial data are ordered correspondingly. This follows by applying the maximum principle to the approximate equations (2.5). However, the proof of the corresponding ordering relation among *all* weak solutions of (2.4) with initial data $u_0 \le v_0$ is much more involved. An important concept in the proof is that of sup- and inf-convolution, defined as follows. For $w \in C^0 \cap L^\infty(\mathbf{R}^n \times [0,\infty[)$, $\epsilon > 0$ let

$$w^\epsilon(z) = \sup_{\xi \in \mathbf{R}^n \times [0,\infty[} \left(w(\xi) - \frac{1}{\epsilon}|z-\xi|^2 \right)$$
$$w_\epsilon(z) = \inf_{\xi \in \mathbf{R}^n \times [0,\infty[} \left(w(\xi) + \frac{1}{\epsilon}|z-\xi|^2 \right).$$

Then $\inf w \le w_\epsilon \le w \le w^\epsilon \le \sup w$, and $w_\epsilon, w^\epsilon \to w$ locally uniformly on $\mathbf{R}^n \times [0,\infty[$.

Moreover, sup- and inf-convolutions are regularizing in the sense that w_ϵ, w^ϵ are Lipschitz and almost everywhere twice differentiable. The latter assertion follows from the fact that, for instance, the function

$$z \to w^\epsilon(z) + \frac{1}{\epsilon}|z|^2 = \sup_\xi \left\{ w^\epsilon(\xi) + \frac{1}{\epsilon}\left(|z|^2 - |z-\xi|^2\right) \right\},$$

being the supremum of a family of affine functions, is convex, hence almost everywhere twice differentiable by a theorem of Alexandroff.

Finally, if w is a super-solution of (2.4), w^ϵ again is a super-solution. Similarly, if w is a sub-solution of (2.4), so is w_ϵ. Now the idea in the proof of the comparison principle is as follows: Suppose by contradiction that $u(z) > v(z)$ for some $z \in \mathbf{R}^n \times]0,\infty[$. By hypothesis, u and v are constants for large $|x| + t$. Thus

$$\max_{\mathbf{R}^n \times [0,\infty[} (u - v) = a > 0$$

is attained in $\mathbf{R}^n \times]0,\infty[$. By uniform local convergence $u_\epsilon \to u$, $v^\epsilon \to v$ then also

$$\max_{\mathbf{R}^n \times [0,\infty[} (u_\epsilon - v^\epsilon) \ge \frac{a}{2} > 0$$

is attained in $\mathbf{R}^n \times]0,\infty[$ for sufficiently small $\epsilon > 0$. Finally, if we choose $\alpha > 0$ small enough, also

$$\max_{\mathbf{R}^n \times [0,\infty[} (u_\epsilon - v^\epsilon - \alpha t) \ge \frac{a}{4} > 0$$

will be attained at some interior point $z_0 \in \mathbf{R}^n \times]0,\infty[$. Suppose for simplicity that v^ϵ is twice differentiable near z_0 with $\nabla v^\epsilon(z_0) \ne 0$. Then a suitable extension of $(v^\epsilon + \alpha t)$ is admissible as a comparison function ϕ for u_ϵ. Since $u_\epsilon - \phi$ by choice of z_0 has a local maximum at z_0 and u_ϵ is a weak sub-solution of (2.4), we find

$$0 \ge \phi_t - \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2} \right) \phi_{x_i x_j} = (v_t^\epsilon + \alpha) - \left(\delta_{ij} - \frac{v_{x_i}^\epsilon v_{x_j}^\epsilon}{|\nabla v^\epsilon|^2} \right) v_{x_i x_j}^\epsilon,$$

contradicting the fact that v^ϵ is a super-solution.

2.6. Monotonicity formula

In order to obtain partial regularity results for (2.4) we may try to carry over the estimates obtained for harmonic maps by means of the monotonicity formula. In the following, we derive an analogous monotonicity estimate for the generalized mean curvature flow. In fact, in the parametric setting of problem (2.1) a monotonicity formula analogous to Theorem 1.10 was independently obtained by Huisken [77] in 1990.

For a given point $z_0 = (x_0, t_0) \in \mathbb{R}^n \times [0, \infty[$ let

$$G_{z_0}(x,t) = \frac{1}{\sqrt{4\pi(t_0-t)}^{\,n-1}} \exp\left(-\frac{|x-x_0|^2}{4(t_0-t)}\right), \qquad \text{if } t < t_0,$$

and $G_{z_0}(x,t) = 0$ else. Observe that G_{z_0} is the fundamental solution to the backward heat equation in any $n-1$- dimensional hyperplane through x_0. (Any such hypersurface is a stationary solution of (2.1).)

Consider a smooth function $u_0 \colon \mathbb{R}^n \to \mathbb{R}$ which is constant for large $|x|$ and let u^ϵ be the unique global smooth solutions to the approximating problem (2.5) with initial data $u^\epsilon = u_0$ constructed in the proof of Theorem 2.6 above.

Proposition 2.1. *For any $z_0 \in \mathbb{R}^n \times [0, \infty[$ the function*

$$\Phi^\epsilon(t) = \int_{\mathbb{R}^n \times \{t\}} \sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}\, G_{z_0}\, dx$$

is non-increasing and

$$\frac{d}{dt}\Phi^\epsilon(t) \le -\int_{\mathbb{R}^n \times \{t\}} \frac{|2\tau u_t^\epsilon + \xi \cdot \nabla u^\epsilon|^2}{4|\tau|^2(|\nabla u^\epsilon|^2 + \epsilon^2)} \sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}\, G_{z_0}\, dx,$$

where $\tau = t - t_0$, $\xi = x - x_0$.

Proof. Translate z_0 to $(0,0)$ and for $\lambda > 0$ scale

$$u \to u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$$
$$\epsilon \to \epsilon_\lambda = \lambda\epsilon.$$

Then u_λ^ϵ solves (2.5) for ϵ_λ with $u_\lambda^\epsilon(\cdot, 0) = u_{0\lambda}$ and $\Phi^\epsilon(-\lambda^2; u^\epsilon) = \Phi^{\lambda\epsilon}(-1; u_\lambda^\epsilon)$. Hence

$$\frac{d}{dt}\Phi^\epsilon(t) = \frac{d}{dt}\Phi^\epsilon(t; u^\epsilon) = \frac{-1}{2\sqrt{|t|}} \frac{d}{d\lambda}\left(\Phi^{\lambda\epsilon}(-1; u_\lambda^\epsilon)\right)\Big|_{\lambda=\sqrt{|t|}}. \tag{2.6}$$

Since u^ϵ is smooth, moreover

$$\begin{aligned}
\frac{d}{d\lambda}\Phi^{\lambda\epsilon}(-1; u_\lambda^\epsilon) &= \int_{\mathbb{R}^n \times \{-1\}} \frac{\nabla u_\lambda^\epsilon \nabla(\frac{d}{d\lambda} u_\lambda^\epsilon) + \lambda\epsilon^2}{\sqrt{|\nabla u_\lambda^\epsilon|^2 + \lambda^2\epsilon^2}}\, G\, dx \\
&\ge \int_{\mathbb{R}^n \times \{-1\}} \frac{\nabla u_\lambda^\epsilon\, \nabla(\frac{d}{d\lambda} u_\lambda^\epsilon)}{\sqrt{|\nabla u_\lambda^\epsilon|^2 + \lambda^2\epsilon^2}}\, G\, dx,
\end{aligned}$$

where $G = G_{(0,0)}$ and we used the fact that $\lambda \ge 0$. Integrating by parts and using the scaled equation (2.5), that is,

$$\operatorname{div}\left(\frac{\nabla u_\lambda^\epsilon}{\sqrt{|\nabla u_\lambda^\epsilon|^2 + \lambda^2\epsilon^2}}\right) = \frac{u_{\lambda t}^\epsilon}{\sqrt{|\nabla u_\lambda^\epsilon|^2 + \lambda^2\epsilon^2}},$$

and the relation $\nabla G(x,t) = \frac{x}{2t} G(x,t)$, we thus obtain

$$\begin{aligned}\frac{d}{d\lambda}\Phi^{\lambda\epsilon}(-1;u_\lambda^\epsilon) &\geq \int_{\mathbf{R}^n\times\{-1\}} -\frac{2tu^\epsilon_{\lambda t} + x\cdot\nabla u^\epsilon_\lambda}{2t\sqrt{|\nabla u^\epsilon_\lambda|^2+\lambda^2\epsilon^2}}\Big(\frac{d}{d\lambda}u^\epsilon_\lambda\Big) G\,dx \\ &= \int_{\mathbf{R}^n\times\{-1\}} \frac{|2tu^\epsilon_{\lambda t} + x\cdot\nabla u^\epsilon_\lambda|^2}{2\lambda|t|\sqrt{|\nabla u^\epsilon_\lambda|^2+\lambda^2\epsilon^2}} G\,dx \\ &= \int_{\mathbf{R}^n\times\{-\lambda^2\}} \frac{|2tu^\epsilon_t + x\nabla u^\epsilon|^2}{2|t|^{3/2}\sqrt{|\nabla u^\epsilon|^2+\epsilon^2}} G\,dx.\end{aligned}$$

Thus, finally, we deduce from (2.6) that

$$\frac{d}{dt}\Phi^\epsilon(t) \leq -\int_{\mathbf{R}^n\times\{t\}} \frac{|2tu^\epsilon_t + x\cdot\nabla u^\epsilon|^2}{4|t|^2(|\nabla u^\epsilon|^2+\epsilon^2)}\sqrt{|\nabla u^\epsilon|^2+\epsilon}\, G\,dx, \tag{2.7}$$

as claimed. □

Passing to the limit $\epsilon \to 0$ in Proposition 2.1, from Fatou's lemma we thus obtain:

Theorem 2.8. *Let* $u \in C^0 \cap L^\infty(\mathbf{R}^n \times [0,\infty[)$, $u \equiv$ const *for large* $|x| + t$, *weakly solve* (2.4), *and let* $z_0 \in \mathbf{R}^n \times [0,\infty[$. *Then for any* $T \geq 0$ *we have*

$$\int_{\mathbf{R}^n\times\{T\}} |\nabla u| G_{z_0}\,dx \leq \int_{\mathbf{R}^n} |\nabla u_0| G_{z_0}\,dx.$$

A similar result was obtained by Ilmanen [**78**], approximating the level-set motion u by solutions to the Allen-Cahn equation.

2.7. Consequences of the monotonicity estimate

As a first consequence we obtain bounds on the volume of hypersurfaces evolving under the mean curvature flow.

Note that by the co-area formula ([**57**]; Theorem 1.32) for any $\Omega \subset \mathbf{R}^n$, any function $f \in BV(\Omega)$, and any $s \in \mathbf{R}$, letting $\Omega_s = \{x \in \Omega; f(x) < s\}$ and denoting χ_{Ω_s} the characteristic function, we have

$$\int_\Omega |Df| = \int_{-\infty}^{\infty} ds \int_\Omega |D\chi_{\Omega_s}|.$$

The monotonicity estimate thus gives a bound on the average of the volume of the level sets of u, weighted with G_{z_0}. To narrow in on a particular level surface, say $\Gamma_t = \{u(x,t) = 0\}$, choose smooth functions Ψ_k converging locally uniformly away from $s = 0$ to the signum function

$$\Psi(s) = \begin{cases} +1\,, & s > 0 \\ 0\,, & s = 0 \\ -1\,, & s < 0 \end{cases}.$$

Since the monotonicity estimate holds for $\Psi_k(u)$ as well, from lower semi-continuity of the BV-norm ([**57**]; Theorem 1.9) we deduce the estimate

$$\int_{\mathbf{R}^n\times\{T\}} |Dv| G_{z_0} \leq \int_{\mathbf{R}^n} |Dv_0| G_{z_0}$$

for $v = \Psi(u)$, $v_0 = \Psi(u_0)$, provided $\nabla u_0 \neq 0$ on $\Gamma_0 = \{u_0(x) = 0\}$. Multiply by $t_0^{(n-1)/2}$ and let $t_0 \to \infty$ to conclude

$$\int_{\mathbf{R}^n \times \{T\}} |Dv| \leq \int_{\mathbf{R}^n} |Dv_0| = 2\mathcal{H}^{n-1}(\Gamma_0),$$

the $(n-1)$-dimensional Hausdorff-measure of the initial surface. In particular, $\Gamma_T = \{x; u(x,T) = 0\}$ has finite perimeter; that is, Γ_t is a Caccioppoli set (see [**57**]; p.6). By De Giorgi's structure theorem for Caccioppoli sets (see [**57**]; Theorem 4.4), therefore $\partial\Gamma_T$ possesses a unit normal

$$\nu(x,T) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} Dv(\cdot,T)}{\int_{B_\rho(x)} |Dv(\cdot,T)|},$$

and hence a tangent hyperplane, $|Dv(\cdot,T)|$-almost everywhere. Thus we can also define a generalized mean curvature $H = \operatorname{div} \nu$ and generalized mean curvature vector $|Dv(\cdot,T)|$-almost everywhere on $\partial\Gamma_T$, for any $T \geq 0$.

Of course, ν and H coincide with the classical notions as long as Γ_T is a smooth hypersurface. In the latter case, from the monotonicity estimate (2.7) for $v_k = \Psi_k(u)$, upon passing to the limit $k \to \infty$, we recover the sharper estimate

$$\int_{\mathbf{R}^n \times \{T\}} |Dv| G_{z_0} + \int_0^T \int_{\mathbf{R}^n} \left| H + \frac{\xi}{2\tau}\nu \right|^2 |Dv| G_{z_0}\, dt \leq \int_{\mathbf{R}^n} |Dv_0| G_{z_0}, \tag{2.8}$$

corresponding to Huisken's formula [**77**], Theorem 3.1.

On the other hand, we can recover some of the estimates of [**45**]; see for instance Theorem 4.2. If Γ_T is the closure of an open set with smooth boundary $\partial\Gamma_T$ then

$$\int_{\mathbf{R}^n \times \{T\}} |Dv| = \mathcal{H}^{n-1}(\partial\Gamma_T) \leq 2\mathcal{H}^{n-1}(\Gamma_0).$$

Examples show that an initial (non-smooth) hypersurface may indeed "double up", as suggested by this formula. (Take a pair of crossed diagonals in the plane, for example.)

As a further consequence we can obtain improved estimates on extinction times for compact hypersurfaces of finite $(n-1)$-dimensional measure, analogous to [**45**], Theorem 4.1.

Theorem 2.9. *Suppose* $u \in C^0 \cap L^\infty(\mathbf{R}^n \times [0,\infty[)$, $u \equiv$ const *for large* $|x| + t$, *solves* (2.4) *with initial data* u_0. *Let* $\Gamma_t = \{x; u(x,t) = 0\}$ *for* $t \geq 0$ *and suppose* $\Gamma_T \neq \emptyset$. *Then we have* $T \leq C(\mathcal{H}^{n-1}(\Gamma_0))^{2/(n-1)}$ *with a constant* $C = C(n)$.

Proof. Let x_0 be a point of density for the $(n-1)$-dimensional Hausdorff measure on $\partial\Gamma_T$. Then with constants $C_i = C_i(n) > 0$ we have

$$\lim_{t_0 \searrow T} \int_{\mathbf{R}^n \times \{T\}} |Dv| G_{z_0} \geq C_1 \lim_{\rho \to 0} \rho^{1-n} \int_{B_\rho(x_0)} |Dv(\cdot,T)| \geq C_2 > 0;$$

see [**57**], Lemma 3.5. On the other hand, the left hand side of this inequality is bounded by

$$\lim_{t_0 \searrow T} \int_{\mathbf{R}^n} |Dv_0| G_{z_0} \leq \frac{\mathcal{H}^{n-1}(\Gamma_0)}{\sqrt{4\pi T}^{\,n-1}}.$$

$\square$

Similarly, a result analogous to Brakke's "clearing out" lemma can be deduced.

Proposition 2.2 ([11], §6.3; [45],Theorem 3.1). *Suppose $C_0 = \mathcal{H}^{n-1}(\Gamma_0) < \infty$. Given $\rho > 0$ there exist constants $\alpha, \eta > 0$, possibly depending on C_0, such that, if $\mathcal{H}^{n-1}(\Gamma_0 \cap B_{2\rho}(\xi)) \leq \eta\rho^{n-1}$, then we have*

$$\partial\Gamma_t \cap B_\rho(\xi) = \emptyset \quad \text{for } t \in [\alpha\rho^2, 2\alpha\rho^2].$$

Proof. After scaling we may assume that $\rho = 1$. Let $t \in [\alpha\rho^2, 2\alpha\rho^2], \alpha > 0$ to be determined, and let $x_0 \in \partial\Gamma_t \cap B_\rho(\xi)$ be a point of density for the $(n-1)$-dimensional Hausdorff measure. Then

$$\begin{aligned}
0 < C_2 &\leq \lim_{t_0 \searrow t} \int_{\mathbb{R}^n \times \{t\}} |Dv| G_{z_0} \\
&\leq \lim_{t_0 \searrow t} \int_{\mathbb{R}^n} |Dv_0| G_{z_0} \leq 2 \int_{\Gamma_0} \frac{\exp\left(-\frac{|x-x_0|^2}{4t}\right)}{\sqrt{4\pi|t|}^{n-1}} \, d\mathcal{H}^{n-1} \\
&\leq \frac{2}{\sqrt{4\pi\alpha}^{n-1}} \left\{ \int_{\Gamma_0 \cap B_2(\xi)} d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}(\Gamma_0) \exp\left(-\frac{1}{8\alpha}\right) \right\} \\
&\leq \frac{2}{\sqrt{4\pi\alpha}^{n-1}} \left(\eta + C_0 \exp\left(-\frac{1}{8\alpha}\right)\right).
\end{aligned}$$

Now choose $\alpha = \alpha(C_0) > 0$ and then $\eta = \eta(\alpha) > 0$ to achieve that the right of the above inequality is smaller than C_2. □

In view of the above results, it seems reasonable to extend the classsical motion of surfaces M_t by mean curvature beyond the first singular time T by letting $M_t = \partial\Gamma_t$ for $t > T$.

2.8. Singularities

A point $z_0 = (x_0, t_0)$ with $x_0 \in \Gamma_{t_0}$ is a singular point of a C^2-solution u of (2.4) if $\nabla u(z_0) = 0$. (Conversely, if z_0 is regular, that is, $\nabla u(z_0) \neq 0$, then Γ_{t_0} is a C^2-surface near x_0.)

Suppose, for instance, that $A = \nabla^2 u(z_0)$ is positive definite. Then by (2.4) we have $\alpha := u_t = (\delta_{ij} - \eta_i\eta_j)u_{x_ix_j} > 0$ at z_0 and hence for $t < t_0$ close to t_0 there holds

$$\Gamma_t \approx \left\{ x; \frac{1}{2}(x - x_0)^T A(x - x_0) = \alpha(t_0 - t) \right\}.$$

Hence we expect the 0-level surfaces of the rescaled functions

$$u_R(x,t) = u\left(x_0 + \frac{x}{R}, t_0 + \frac{t}{R^2}\right)$$

at $t = -1$ to approach a smooth limit.

More generally, consider the signum function Ψ above and for a weak solution u of (2.4) let $v = \Psi(u)$. Moreover, for $z_0 = (x_0, t_0) \in \mathbb{R}^n \times [0, \infty[$, $R > 0$, let

$$v_R(x,t) = v\left(x_0 + \frac{x}{R}, t_0 + \frac{t}{R^2}\right).$$

Suppose the level surfaces $\Gamma_t^R = \{v_R(\cdot,t) = 0\}$ are smooth. Then by the monotonicity estimate (2.8) for any $T > 0$ we have

$$\int_{-T}^{0}\int_{\mathbb{R}^n}\left|H_R + \frac{x}{2t}\nu_R\right|^2 |Dv_R| G\,dt$$
$$= \int_{t_0-TR^{-2}}^{t_0}\int_{\mathbb{R}^n}\left|H + \frac{\xi}{2\tau}\nu\right|^2 |Dv| G_{z_0}\,dt \quad \to \quad 0 \quad (R \to \infty),$$

where ν denotes the normal $\nu = \frac{Dv}{|Dv|}$ and H the mean curvature. Generalizing [77], we call a singularity of *Type 1* if the surfaces Γ_t^R smoothly converge to a smooth surface Γ_t^∞ for every $t < 0$, as $R \to \infty$. Let H_∞, ν_∞ denote the mean curvature and normal on Γ_t^∞. Then by the above we have

$$2|t|H_\infty = \langle x, \nu_\infty\rangle.$$

Represent $\Gamma_t^\infty = F_t(\Gamma_{-1}^\infty)$. Then by (2.1) it follows that

$$\frac{d}{dt}F_t = -H_\infty \nu_\infty = -\frac{\langle F, \nu_\infty\rangle}{2|t|}\nu_\infty,$$

whence up to tangential diffeomorphism Γ_t^∞ is contracting by self-similar motion. All smooth star-shaped hypersurfaces in $\mathbb{R}^n$ satisfying

$$H = \langle x, \nu\rangle \geq 0,$$

and hence giving rise to self-similar solutions of (2.1) or (2.4), have been classified by Huisken [77], Theorem 5.1. In particular, we have:

Theorem 2.10 ([77], Theorem 4.1). *If $M^{n-1} \subset \mathbb{R}^n$ is compact with non-negative mean curvature $H = \langle x, \nu\rangle$, then M is a sphere of radius $\sqrt{n-1}$.*

There seems to be a large variety of self-similar solutions of (2.1) that do not satisfy $H \geq 0$; in particular, the torus-shaped hypersurface of Example 2.2. Moreover, there are singularities that evolve at a different (faster) rate, which we call of Type 2. The study of singularities for the mean curvature flow is a very active field of current research.

PART 3
Harmonic maps of Minkowsky space

3.1. The Cauchy problem for harmonic maps

Consider $M^{m+1} = \mathbb{R} \times \mathbb{R}^m$, equipped with the Minkowsky metric

$$g = (g_{\alpha\beta}) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix},$$

and let N be a compact Riemannian manifold, isometrically embedded in $\mathbb{R}^n$. By analogy with harmonic maps of Riemannian manifolds, we define a map $u\colon M^{m+1} \to N \subset \mathbb{R}^n$ to be harmonic if it is stationary for the Lagrangian

$$L(u; R) = \int_R l(u)\, dz,$$

where

$$l(u) = \frac{1}{2} g^{\alpha\beta} \partial_\alpha u^i \partial_\beta u^i = \frac{1}{2}\left(|\nabla u|^2 - |u_t|^2\right) ,$$

and where $z = (t, x) = (x^0, x^1, \ldots, x^m)$. Here, $\nabla = \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\right)$ denotes the spatial derivatives and, for brevity, $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$, $\alpha = 0, \ldots, m$. Moreover, it will be convenient to denote $D = (\partial_0, \ldots, \partial_m) = \left(\frac{\partial}{\partial t}, \nabla\right)$ and $\partial^\alpha = g^{\alpha\beta}\partial_\beta$. For $\phi \in C_0^\infty(M^{m+1}; u^{-1}TN)$ supported on a space-time domain $R \subset M^{m+1}$ we have

$$\frac{d}{d\epsilon} L(\pi_N(u + \epsilon\phi); R)\Big|_{\epsilon=0} = \int_R \Box u \cdot \phi\, dz = 0,$$

where

$$\Box = -\partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial t^2} - \Delta$$

is the wave operator, acting component-wise on u; that is, u is harmonic iff

$$\Box u \perp T_u N. \tag{3.1}$$

Suppose $\nu_{l+1},\dots,\nu_n$ is a smooth (local) orthonormal frame field for the normal bundle $T^\perp N \subset T\mathbf{R}^n$. Then in the spirit of 1.1 we may write

$$\Box u = \sum_k \lambda^k \nu_k \circ u,$$

where $\lambda^k\colon M^{m+1} \to \mathbf{R}$ is given by

$$\begin{aligned}\lambda^k &= \langle \Box u, \nu_k \circ u\rangle = -\partial_\alpha \langle \partial^\alpha u, \nu_k \circ u\rangle + \langle \partial^\alpha u, \partial_\alpha(\nu_k \circ u)\rangle \\ &= A^k(u)(\partial_\alpha u, \partial^\alpha u) = A^k(u)(\nabla u, \nabla u) - A^k(u)(u_t, u_t),\end{aligned}$$

with $\langle\cdot,\cdot\rangle$ denoting the scalar product in $\mathbf{R}^n$. Hence

$$\Box u = \sum_k (\nu_k \circ u) A^k(u)\,(\partial_\alpha u, \partial^\alpha u) = A(u)(Du, Du). \tag{3.2}$$

Introducing the "null form"

$$Q(\phi,\psi) = \nabla\phi\nabla\psi - \phi_t\psi_t,$$

note that (3.2) also may be written in the form

$$\Box u^k = a^k_{ij}(u) Q(u^i, u^j), \quad k = 1,\dots,n. \tag{3.3}$$

Finally, in local coordinates $\left(u^1,\dots,u^l\right)$ on N, (3.2) may be written

$$\Box u^k = \tilde\Gamma^k_{ij}(u) Q(u^i, u^j), \tag{3.4}$$

where $\tilde\Gamma^k_{ij}$ denote the Christoffel symbols on N. Note that

$$Q(u,u) = 2l(u),$$

and Q is associated with the wave operator in the same way as the Dirichlet energy density is associated with the Laplacian.

For hyperbolic problems it is natural to consider the Cauchy problem: Given initial data u_0, u_1 at $t=0$ we seek $u\colon M^{m+1} \to N \subset \mathbf{R}^n$ satisfying (3.2) and the initial condition

$$u|_{t=0} = u_0, \qquad u_t|_{t=0} = u_1 \qquad \text{on } \mathbf{R}^m. \tag{3.5}$$

The questions we consider are local and global existence of classical or weak solutions, and development of singularities.

We will not consider the problem of determining the asymptotic behavior of solutions or scattering theory.

Equation (3.2) shares the property of the homogeneous wave equation $\Box\phi = 0$ that initial disturbances propagate with speed ≤ 1. Thus, as far as the above topics are concerned, we may restrict our attention to initial data with "compact support" in the sense that

$$u_0 \equiv \text{const}, \qquad u_1 \equiv 0 \qquad \text{on } \mathbf{R}^m \setminus \Omega_0 \tag{3.6}$$

for some compact set $\Omega_0 \subset\subset \mathbf{R}^m$. In what follows we discuss various mechanisms for proving existence for (3.2), (3.5).

3.2. Local existence

Existence via the fundamental solution

Consider the Cauchy problem for $u\colon M^{m+1} \to \mathbb{R}$ satisfying

$$\Box u = f \quad \text{in } M^{m+1}$$

with initial data u_0 and u_1. If $m = 1$, the solution is given by

$$u(t,x) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,dy + \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(t-s,y)\,dy\,ds.$$

If $m = 3$, we have

$$u(t,x) = \frac{d}{dt}\left(\frac{t}{4\pi}\int_{S^2} u_0(x+t\xi)\,do\right) + \frac{t}{4\pi}\int_{S^2} u_1(x+t\xi)\,do + \int_0^t \frac{s}{4\pi}\int_{S^2} f(t-s,x+s\xi)\,do\,ds,$$

whereas, if $m > 3$, the representation formula involves also derivatives of f transverse to the backward light cone $M(z)$ from $z = (t,x)$, given by

$$M(z) = \{(s,y)\,;\, t-s = |x-y|\}.$$

Note that, even in dimension $m = 1$, there is no "gain" in derivatives. Thus, already in dimension $m = 1$, for nonlinearities $f = f(u, Du)$ as in (3.2) a simple-minded iteration procedure will fail to yield a local existence result due to loss of derivatives.

Energy method

First consider a smooth solution u of the homogeneous wave equation

$$\Box u = 0 \qquad \text{in } M^{m+1}$$

having compact support on any slice $\{t = const.\}$ Multiplying by u_t we obtain the conservation law

$$0 = u_{tt}u_t - \Delta u u_t = \frac{d}{dt}\left(\frac{|u_t|^2 + |\nabla u|^2}{2}\right) - \operatorname{div}(\nabla u u_t). \tag{3.7}$$

Denote

$$e(u) = \frac{|u_t|^2 + |\nabla u|^2}{2} = \frac{1}{2}|Du|^2$$

the energy density of u and let

$$E(u(t)) = \int_{\{t\}\times\mathbb{R}^m} e(u)\,dx.$$

Then, since $u(t)$ has compact support, upon integrating the above conservation identity we obtain $\frac{d}{dt}E(u(t)) = 0$; in particular

$$E\big(u(t)\big) \le E\big(u(0)\big).$$

Similarly, since derivatives $v = D^\alpha u = \partial_0^{\alpha_0}\dots\partial_m^{\alpha_m} u$ for any multi-index $\alpha = (\alpha_0,\dots,\alpha_m)$ again solve $\Box v = 0$ with $\operatorname{supp}(v(t)) \subset\subset \mathbb{R}^m$, we have

$$E\big(D^\alpha u(t)\big) \le E\big(D^\alpha u(0)\big).$$

Note that $D^\alpha u(0)$ can be computed entirely in terms of u_0 and u_1, in view of the equation $u_{tt} = \Delta u$.

Observe that (3.7) remains true if instead of $\Box u = 0$ we only require $\Box u \cdot u_t = 0$. On account of (3.1), the latter condition is satisfied for harmonic maps, and we obtain

Lemma 3.1 (Energy inequality). *Let u be a smooth solution of the Cauchy problem* (3.2), (3.5) *having compact support in the sense of* (3.6). *Then for all t we have*

$$E(u(t)) \le E(u(0)).$$

(In fact, for smooth u equality holds.)

In order to obtain corresponding bounds for higher derivatives, however, we have to consider also inhomogeneous equations. It is useful to introduce the space-time norms

$$\|f\|_{L^{q,p}} = \left(\int_{-\infty}^{\infty} \|f(t)\|^q_{L^p(\mathbb{R}^m)}\, dt\right)^{1/q},$$

$1 \le p, q \le \infty$, and the corresponding spaces $L^{q,p}(M^{m+1}) = L^q(\mathbb{R}; L^p(\mathbb{R}^m))$. On a finite space-time cylinder $\Omega^T = [0,T] \times \Omega$ corresponding norms may be defined. Consider now a smooth solution $u\colon M^{m+1} \to \mathbb{R}$ of

$$\Box u = f$$

with initial data u_0, u_1, where $u(t)$ has compact support for any t. Multiplying by u_t we obtain

$$\frac{d}{dt} e(u) - \operatorname{div}(\nabla u u_t) = f u_t \le |f||u_t|.$$

Integrating over $\{t\} \times \mathbb{R}^m$, by the Cauchy-Schwartz inequality this gives

$$\frac{d}{dt} E(u(t)) \le \int_{\{t\}\times\mathbb{R}^m} |f||u_t|\, dx \le \|f(t)\|_{L^2} \|u_t(t)\|_{L^2} \le \|f(t)\|_{L^2} \big(2E(u(t))\big)^{1/2}.$$

That is,

$$\frac{d}{dt} \sqrt{E(u(t))} \le \|f(t)\|_{L^2},$$

neglecting a factor $\sqrt{2}$. Hence we obtain

$$E(u(t)) \le 2E(u(0)) + 2\|f\|^2_{L^{1,2}}$$

for all t, and similarly

$$E(D^\alpha u(t)) \le 2E(D^\alpha u(0)) + 2\|D^\alpha f\|^2_{L^{1,2}}, \tag{3.8}$$

for all $\alpha = (\alpha_0, \dots, \alpha_m)$.

Local existence for the Cauchy problem for harmonic maps

We now apply (3.8) to solve (3.2) and (3.5) near $t = 0$. This result is quite standard. A survey of local existence results is given in Kato [**91**].

Denote

$$H_c^{s+1,2}(\mathbb{R}^m; TN) = \Big\{(u_0, u_1); u_0 \in H^{s+1,2}_{\text{loc}}(\mathbb{R}^m; N),$$
$$u_1 \in H^{s,2}_{\text{loc}}(\mathbb{R}^m; u_0^{-1}TN), (u_0, u_1) \text{satisfy } (3.6)\Big\}.$$

where $H^{m,2}(\mathbb{R}^m; N)$, etc., is defined as in Part 1, sec 1, and let $H_c^{s+1} = H_c^{s+1}(\mathbb{R}^m; T\mathbb{R}^n)$.

Then we have:

Theorem 3.1. *Suppose $(u_0, u_1) \in H_c^{s+1}(\mathbb{R}^m; TN)$, where $s > \frac{m}{2}$. Then for some $T > 0$ problem (3.2), (3.5) admits a unique solution $u: [-T,T] \times \mathbb{R}^m \to N$ such that*

$$\sup_{|t| \leq T} \|Du(t)\|_{H^s} < \infty.$$

Proof. Let $u^{(0)} \in L^\infty(\mathbb{R}; H_c^{s+1})$ solve $\Box u^{(0)} = 0$ with Cauchy date (u_0, u_1). We claim there exists $T > 0$ and a convex neighborhood $\mathcal{U}$ of $u^{(0)}$ in $L^\infty\left([-T,T]; H_c^{s+1}\right)$ such that the map $S: v \to u = S(v)$ given by solving

$$\Box u = A(v)(Dv, Dv)$$

with initial data u_0, u_1 is a strictly contracting map of $\mathcal{U}$ to itself. Indeed, by (3.8) we have the estimate

$$\begin{aligned}\sup_t \|(Du - D\tilde{u})(t)\|_{H^s}^2 &\leq C \sup_t \sum_{|\alpha| \leq s} E(D^\alpha(u - \tilde{u})(t)) \\ &\leq C \sum_{|\alpha| \leq s} \|D^\alpha \left(A(v)(Dv, Dv) - A(\tilde{v})(D\tilde{v}, D\tilde{v})\right)\|_{L^{1,2}}^2\end{aligned} \tag{3.9}$$

for the difference of any two solutions $u = S(v)$, $\tilde{u} = S(\tilde{v})$.

Moreover, for any $|\alpha| \leq s$, any $|t| \leq T$, by Leibniz' rule we can estimate

$$\begin{aligned}F(t) &:= |D^\alpha \left(A(v)(Dv, Dv) - A(\tilde{v})(D\tilde{v}, D\tilde{v})\right)| \\ &\leq C \sum_{|\beta| \leq |\alpha|+1} \left\{ |D^\beta(v - \tilde{v})| \left(\prod_i \left(|D^{\gamma^{(i)}} v| + |D^{\gamma^{(i)}} \tilde{v}| \right) \right) \right\},\end{aligned}$$

where in the product $\gamma = \sum_i \gamma^{(i)}$ ranges over all multi-indices such that $|\gamma| \leq 2 + |\alpha| - |\beta|$ and $|\gamma^{(i)}| \leq |\alpha| + 1$. We use Sobolev's embedding theorem

$$H^{k,2}(\mathbb{R}^m) \hookrightarrow H^{l,q}(\mathbb{R}^m), \quad \text{if } 2(k - l) < m,$$

where

$$\frac{1}{q} = \frac{1}{2} - \frac{k - l}{m},$$

and

$$H^{k,2}(\mathbb{R}^m) \hookrightarrow H^{l,\infty}(\mathbb{R}^m), \quad \text{if } 2(k - l) > m.$$

Choosing $2s \notin \mathbb{N}$ we can achieve that $2((s + 1) - l) \neq m$ for all l; otherwise we may use the extended Hölder-Sobolev inequalities (see for example Friedman [48] or Mazja [107]) to justify the following estimates. Note that by Hölder's inequality

$$\|F(t)\|_{L^2} \leq C \sum_{|\beta| \leq |\alpha|+1} \|D^\beta(v - \tilde{v})\|_{L^{r^{(0)}}} \prod_i \left(\|D^{\gamma^{(i)}} v\|_{L^{r^{(i)}}} + \|D^{\gamma^{(i)}} \tilde{v}\|_{L^{r^{(i)}}} \right),$$

where $\gamma = \sum_i \gamma^{(i)}$, $|\gamma| \leq 2 + |\alpha| - |\beta|$, $|\gamma^{(i)}| \leq |\alpha| + 1$, and where for each $|\beta| \leq |\alpha| + 1$ and each γ we assume

$$\frac{1}{2} \geq \frac{1}{r^{(0)}} + \sum_i \frac{1}{r^{(i)}}.$$

Let $\beta = \gamma^{(0)}$ and for $i = 0, 1, \ldots$ set

$$\frac{1}{r^{(i)}} = \frac{1}{2} - \frac{(s+1) - |\gamma^{(i)}|}{m} \quad \text{if } 2((s+1) - |\gamma^{(i)}|) < m$$

and set $r^{(i)} = \infty$ in the remaining cases. Note that $r^{(i)} \geq 2$ for all i. Hence we may assume that at least two distinct numbers $r^{(i)}$ are finite. Since

$$\gamma^{(i)} \leq 1 + |\alpha| \leq 1 + s, \qquad \sum_{i \geq 0} |\gamma^{(i)}| \leq 2 + |\alpha| \leq 2 + s$$

then, if $s > m/2$, indeed we have

$$\sum_{i \geq 0} \frac{1}{r^{(i)}} \leq \sum_{\substack{i \geq 0 \\ r^{(i)} < \infty}} \left(\frac{1}{2} - \frac{\gamma^{(i)} - (s+1)}{m} \right) \leq 1 + \frac{\sum_{i \geq 0} \gamma^{(i)} - 2(s+1)}{m} \leq 1 - \frac{s}{m} \leq \frac{1}{2}.$$

Hence we obtain

$$\|F\|_{L^2} \leq C \|(Dv - D\tilde{v})\|_{H^s} \left(1 + \|Dv\|_{H^s}^{2+s} + \|D\tilde{v}\|_{H^s}^{2+s}\right).$$

Define

$$\mathcal{U} = \left\{ v;\ \sup_t \|(Dv - Du^{(0)})(t)\|_{H^s} \leq 1 \right\}.$$

Integrating the above inequality and combining with (3.9) we obtain

$$\begin{aligned}
\sup_t \|(Du - D\tilde{u})(t)\|_{H^s}^2 &\leq C\|F\|_{L^{1,2}}^2 \leq CT^2 \sup_t \|F(t)\|_{L^2}^2 \\
&\leq CT^2 \sup_t \|(Dv - D\tilde{v})(t)\|_{H^s}^2 \left(1 + \sup_t \left(\|Dv(t)\|_{H^s}^{2+s} + \|D\tilde{v}(t)\|_{H^s}^{2+s}\right)\right)^2 \\
&\leq \delta \sup_t \|(Dv - D\tilde{v})(t)\|_{H^s}^2
\end{aligned}$$

with $\delta < 1$ if $v, \tilde{v} \in \mathcal{U}$, and if $T > 0$ is sufficiently small. Similarly, we have

$$\begin{aligned}
\sup_t \|(Du - Du^{(0)})(t)\|_{H^s}^2 &\leq C \sup_t \sum_{|\alpha| \leq s} \|D^\alpha (A(v)(Dv, Dv))\|_{L^{1,2}}^2 \\
&\leq CT^2 \left(1 + \sup_t \|Dv(t)\|_{H^s}^{2+s}\right)^2,
\end{aligned}$$

and this is < 1, if $v \in \mathcal{U}$ and $T > 0$ is sufficiently small.

It follows that S has a unique fixed point $u = S(u) \in \mathcal{U}$ which is the local solution we seek. □

In the derivation of Theorem 3.1 neither the geometric interpretation (3.1) of the harmonic map equation nor its special form (3.3) or (3.4) were used. An example of how this structure may be exploited is given by the following result of Klainerman-Machedon [97].

Theorem 3.2. *Let $u, v \colon M^{3+1} \to \mathbf{R}$ be solutions to*

$$\Box u = F, \qquad \Box v = G \qquad \text{in } M^{3+1}$$

with initial conditions

$$u|_{t=0} = u_0, \qquad u_t|_{t=0} = u_1,$$
$$v|_{t=0} = v_0, \qquad v_t|_{t=0} = v_1,$$

respectively, and let $Q(u,v) = \nabla u \cdot \nabla v - u_t v_t$ be a null-form as above. Then for any $T > 0$ we have

$$\|DQ(u,v)\|_{L^2} \le C\left(\|Du(0)\|_{H^1} + \int_0^T \|F(t)\|_{H^1}\,dt\right)\cdot\left(\|Dv(0)\|_{H^1} + \int_0^T \|G(t)\|_{H^1}\,dt\right),$$

with an absolute constant C.

Q may be replaced by other "null-forms"; however, the result is false for general bilinear forms in Du and Dv. As a consequence they derive the following slight improvement of Theorem 3.1.

Theorem 3.3. *Let $m \le 3$ and suppose $(u_0, u_1) \in H_c^2$. Then there exists $T > 0$ and a unique local solution $u\colon [-T,T] \times \mathbf{R}^3 \to N$ of (3.2), (3.5), satisfying*

$$\sup_t \|Du(t)\|_{H^1} \le C\|Du(0)\|_{H^1}.$$

A simple proof of this fact, based, however, on the *geometric* condition (3.1) and energy estimates, can be given als follows.

Proof of Theorem 3.3. For any $\alpha \in 0,\ldots,m$ we have

$$\begin{aligned}\frac{d}{dt}E(\partial_\alpha u) &\le \int_{\mathbf{R}^m} \partial_\alpha(A(u)(Du,Du))\cdot \partial_\alpha u_t\,dx \\ &\le C\int_{\mathbf{R}^m} |Du|^3|D^2u|\,dx + 2\int_{\mathbf{R}^m} A(u)(D\partial_\alpha u, Du)\cdot\partial_\alpha u_t\,dx.\end{aligned}$$

But $A(u)(\cdot,\cdot) \perp T_uN$; thus the integrand in the second term on the right may be rewritten as

$$-(\partial_\alpha A(u))(D\partial_\alpha u, Du)\cdot u_t \le C|Du|^3|D^2u|$$

and we obtain

$$\frac{d}{dt}E(Du) \le C\int_{\mathbf{R}^m} |Du|^3|D^2u|dx \le C\|Du\|_{L^6}^3\|D^2u\|_{L^2}.$$

Finally, since $Du(t)$ has compact support for any t, by Sobolev's embedding theorem and since $m \le 3$ this implies

$$\frac{d}{dt}E(Du) \le C\|D^2u\|_{L^2}^4 \le C(E(Du))^2$$

for $0 \le t \le T$, where C may depend on T and the size of the support of the initial data. Local existence thus follows from Gronwall's inequality.

To obtain uniqueness, note that by (3.1) the difference of equations (3.2) for two H^2-solutions u and v after multiplication by $u_t - v_t$ gives

$$\begin{aligned}\Box(u-v)\cdot(u-v)_t &= \big(A(u)(Du,Du) - A(v)(Dv,Dv)\big)(u_t - v_t) \\ &= -A(u)(Du,Du)v_t - A(v)(Dv,Dv)u_t \\ &= \big(A(u)-A(v)\big)(Dv,Dv)u_t - \big(A(u)-A(v)\big)(Du,Du)v_t \\ &\le C|u-v||D(u-v)|\big(|Du|^2 + |Dv|^2\big).\end{aligned}$$

Integrating over $\mathbb{R}^m$, for $m \le 3$ by Sobolev's embedding theorem therefore we obtain

$$\begin{aligned}\frac{d}{dt}E(u-v) &\le C\|u-v\|_{L^6}\|D(u-v)\|_{L^2}\cdot\left(\|D(u)\|_{L^6}^2+\|D(v)\|_{L^6}^2\right)\\ &\le CE(u-v)\cdot\sup_t\left(\|D(u)\|_{H^1}^2+\|D(v)\|_{H^1}^2\right)\end{aligned}$$

and uniqueness in the class of H^2-solutions follows. □

Klainerman-Machedon conjecture that $(u_0,u_1)\in H_c^s$, $s>\frac{m-2}{2}$, suffices. In fact, many experts believe that local existence and uniqueness should hold true in the class H_c^{s+1}, $s\ge\frac{m-2}{2}$.

In particular, for $m=2$ such a result would imply the global existence of a unique solution to the Cauchy problem (3.2) (3.5) in the energy class. However, apart from some special cases, this problem still is completely open.

Observe that the following simple model problem from Klainerman-Machedon **[97]** demonstrates that such an improvement is not possible without using the geometric structure of the harmonic map equation.

Example 3.1. Let u satisfy the nonlinear wave equation

$$\Box u = Q(u,u) = |\nabla u|^2 - |u_t|^2 \qquad \text{in } M^{m+1}$$

with initial data $u_0\equiv 0, u_1\in H_0^{s,2}(\mathbb{R}^m)$. Then $v=e^u$ satisfies the linear wave equation

$$\Box v = e^u(\Box u - Q(u,u)) = 0 \qquad \text{in } M^{m+1}$$

with initial data $v_0=e^{u_0}\equiv 1$, $v_1=u_1$ and $t\mapsto v(t)$ is continuous in H_c^{s+1}. To ensure that $v=e^u>0$ for small $t>0$ we need continuity of $v(t)$, and hence, by Sobolev's embedding theorem, we *need* $s>\frac{m-2}{2}$.

3.3. Global existence

Example 3.2. Geodesics (Sideris **[131]**). If $\gamma\colon\mathbb{R}\to N$ is a geodesic on N and if $v\colon M^{m+1}\to\mathbb{R}$ satisfies

$$\Box v = 0,$$

then $u=\gamma\circ v$ solves

$$\Box u = \gamma'(v)\cdot\Box v + \gamma''(v)Q(v,v)\perp T_uN,$$

because $\gamma''\perp T_\gamma N$. That is, u is harmonic. Note that u preserves the regularity of the initial data.

Example 3.3. Let $m=1$. In this case, global existence and regularity was established by Gu **[68]** and Ginibre-Velo **[54]**. A surprisingly simple proof was given by Shatah **[127]** based on the following observation: Multiply (3.1) by u_t, respectively by u_x, to obtain the system of conservation laws

$$\begin{aligned}0 &= \Box u u_t = \partial_t e - \partial_x m,\\ 0 &= \Box u u_x = \partial_t m - \partial_x e,\end{aligned}$$

where $e=e(u)=\frac12\left(|u_t|^2+|u_x|^2\right)$ is the energy density and $m=m(u)=u_xu_t$ is the density of momentum; compare (3.7). Thus, e satisfies the linear wave equation

$$\Box e = 0.$$

and from the representation formula we obtain pointwise bounds on e, and hence on Du, in terms of the initial data. Higher regularity for smooth data then follows by using the energy inequality (3.8) to iteratively derive Gronwall-type estimates

$$\frac{d}{dt}E(D^\alpha u(t)) \le C + CE\Big(D^\alpha\big(u(t)\big)\Big)$$

for any α.

Remark 3.4. For $m = 3$ and rather smooth initial data u_0, u_1 whose norm is small in H^s, $s \ge 10$, Sideris [**131**] has constructed global solutions to (3.2), (3.5) in H^s by combining the local existence result above with certain decay estimates that may be obtained by using the invariant norms of Klainerman [**95**]. The latter are defined by means of the generators of the Poincaré group and scale changes. In addition to the standard differentials ∂_α, $\alpha = 0, \dots, 3$, denoted $\Gamma_1, \dots, \Gamma_4$ in the following, these include the generators $\Gamma_5, \dots, \Gamma_{10}$ of Lorentz and proper rotations

$$\Omega_{\alpha\beta} = x_\alpha \partial^\beta - x_\beta \partial^\alpha, \qquad 0 \le \alpha < \beta \le 3,$$

and the generator of dilations

$$\Gamma_0 = x_\alpha \partial_\alpha.$$

Note that Lorentz invariance of the wave operator implies the commutation relation

$$[\Box, \Omega_{\alpha\beta}] = 0;$$

moreover,

$$[\Box, \Gamma_0] = 2\Box\,.$$

The family Γ spanned by $\Gamma_0, \dots, \Gamma_{10}$ forms a Lie-algebra. The invariant norms $\|\cdot\|_{\Gamma,s}$ of functions $u\colon M^{3+1} \to \mathbb{R}$ now are defined as follows:

$$\|u(t)\|_{\Gamma,s} = \left(\sum_{|\alpha|\le s} \|\Gamma^\alpha u(t)\|^2_{L^2(\mathbb{R}^3)}\right)^{1/2},$$

where $\Gamma^\alpha = \prod_{i=0}^{10} \Gamma_i^{\alpha_i}$ for any multi-index $\alpha = (\alpha_0, \dots, \alpha_{10})$. Also let $E_0 = E$ and

$$E_s(u(t)) = \sum_{|\alpha|\le s} E\left(\Gamma^\alpha u(t)\right), \qquad s \in \mathbb{N}.$$

Note that $E_s(u(t))$, $s \ge 1$ involves weighted norms of derivatives of u. Hence, finiteness of $E_s(u(t))$ for sufficiently large s will imply some decay of u and its derivatives. For instance, we have the generalized Sobolev inequality, due to Klainerman [**95**], which gives the bound

$$\sup_x |Du(t,x)| \le \frac{C}{1+|t|+|x|} E_4\big(u(t)\big),$$

for any smooth function u on M^{3+1}.

More generally, Sideris has constructed global solutions for smooth initial data H^{10}-close to a "geodesic". For $m = 2$, Kovalyov [**99**] has obtained similar global existence results for small data.

The above results should be compared with long-time existence and blow-up results for general hyperbolic equations

$$\Box u = f(u, Du, D^2u) \qquad \text{in } M^{m+1};$$

in particular, in case $m = 3$ again a "null condition" for the quadratic part of f is required; see Klainerman [**96**], John [**85**], Christodoulou [**24**].

3.4. Finite-time blow-up

Following Shatah [127] and Shatah-Tahvildar-Zadeh [129], [130], in order to produce examples of solutions to the Cauchy problem for harmonic maps that blow up in finite time, as in Lecture I.2 we consider equivariant maps into spheres or, more generally, into hypersurfaces of revolution N, locally diffeomorphic to $\mathbb{R}^m$ with metric

$$ds^2 = dh^2 + g^2(h)d\omega^2$$

in spherical coordinates $h > 0$, $\omega \in S^{m-1}$. Moreover, on M^{m+1} we consider the standard metric

$$-dt^2 + dr^2 + r^2 d\theta^2,$$

written in terms of spherical coordinates (r, θ) on $\mathbb{R}^m$. Consider equivariant maps $u\colon M^{m+1} \to N$ of the form

$$u(t, r, \theta) = (h(t, r), \omega(\theta))$$

where ω is a homogeneous harmonic polynomial of degree $d > 0$. Then

$$L(u) = \frac{1}{2}\int \left\{|h_r|^2 - |h_t|^2 + \frac{1}{r^2}g^2(h)|\nabla_\theta \omega|^2\right\} r^{m-1}\, dt\, dr\, d\theta,$$

and u is harmonic if and only if $h\colon M^{m+1} \to \mathbb{R}$ satisfies

$$\Box h + \frac{kf(h)}{r^2} = 0, \tag{3.10}$$

where $k = d(d + m - 2)$, and $f(h) = g(h)g'(h)$. Observe that for $N = S^m$ we have $g(h) = \sin h$. More generally we require

$$g(0) = 0,\ g'(0) = 1,\ g(h) > 0, \quad \text{for } h > 0, \text{ and } g(-h) = -g(h). \tag{3.11}$$

Note that the ball of radius h_0 around 0 in N is convex iff $g'(h) \geq 0$ for $0 < h < h_0$.

Self-similar equivariant solutions

Using invariance of (3.10) under dilations of space-time and time reversal $t \to -t$, in order to achieve a further simplification we try to construct solutions h to (3.10) of the form

$$h(t, r) = \phi\left(\frac{r}{t}\right)$$

with smooth Cauchy data ϕ at $t = 1$. For self-similar solutions of this kind, we compute

$$\begin{aligned}
\Box h &= h_{tt} - h_{rr} - \frac{m-1}{r}h_r \\
&= \frac{2r}{t^3}\phi'\left(\frac{r}{t}\right) + \frac{r^2}{t^4}\phi''\left(\frac{r}{t}\right) - \frac{1}{t^2}\phi''\left(\frac{r}{t}\right) - \frac{m-1}{rt}\phi'\left(\frac{r}{t}\right) \\
&= \frac{1}{t^2}\left(2\rho\phi'(\rho) + (\rho^2 - 1)\phi''(\rho) - \frac{m-1}{\rho}\phi'(\rho)\right),
\end{aligned}$$

where $\rho = \frac{r}{t}$ and $\phi' = \frac{d}{d\rho}\phi$, etc. That is,

$$\phi''(\rho) + \left(\frac{m-1}{\rho(1-\rho^2)} - \frac{2\rho}{1-\rho^2}\right)\phi'(\rho) = \frac{-t^2\Box h}{1-\rho^2} = \frac{kf(\phi)}{\rho^2(1-\rho^2)}.$$

Simplifying, we finally obtain the equation

$$\phi'' + \left(\frac{m-1}{\rho} + \frac{(m-3)\rho}{1-\rho^2}\right)\phi' - \frac{kf(\phi)}{\rho^2(1-\rho^2)} = 0. \tag{3.12}$$

For $m \geq 3$ and $1 < \phi_0^2 < 2$ now let

$$g^2(\phi) = \phi^2 - \frac{\phi^4}{2}, \quad \text{if } \phi < \phi_0,$$

whence

$$f(\phi) = g(\phi)g'(\phi) = \frac{1}{2}g^2(\phi)' = \phi - \phi^3,$$

if $\phi < \phi_0$, and extend g smoothly elsewhere. Then for $d = 1$, $k = m - 1$, the function

$$\phi(\rho) = c\rho,$$

where $c = \sqrt{\frac{2}{m-1}}$ solves (3.12) for $\rho < \phi_0\sqrt{\frac{m-1}{2}}$. By finiteness of propagation speed, the behavior of h at 0 is determined by the Cauchy data on $\overline{B_1(0)}$ at time $t = 1$ only. Hence the solution h to (3.10) with initial data ϕ in a neighborhood of $\overline{B_1(0)}$ at time $t = 1$ will blow up in finite time. Note that for g as above, the radius of convexity of N around 0 is

$$\phi_* = 1$$

which is larger than c for $m \geq 4$ and equals c if $m = 3$. By changing the metric $g(\phi)$ on N suitably for $\phi > c$, and by changing the initial data for h off $\overline{B_1(0)}$, we may thus construct solutions to (3.2), (3.5) which blow up in finite time, with initial data having compact support and such that the target manifold is convex, if $m \geq 4$, and only slightly fails to be convex, if $m = 3$. A more detailed analysis shows that in 3 space dimensions blow-up may occur also for more general metrics on the target surface:

Theorem 3.4 (Shatah–Tahvildar-Zadeh [130]). *Suppose $g \in C^\infty$ satisfies $g(0) = 0$, $g'(0) = 1$ and suppose g' has a smallest positive zero ϕ_*. Also suppose that $g''(\phi_*) \neq 0$. Then there is a class of regular initial data such that the corresponding Cauchy problem for equivariant harmonic maps from M^{3+1} into N has a solution that blows up in finite time.*

Non-uniqueness of weak solutions

In particular, Theorem 3.4 applies to the sphere, where $g(\phi) = \sin\phi$, $\phi_* = \frac{\pi}{2}$. Shatah-Tahvildar-Zadeh construct a solution ϕ to (3.12) on $[0, \infty[$, satisfying

$$\phi(1) = \phi_*$$

and having the asymptotic expansion for $\rho \to \infty$:

$$\begin{aligned}\phi(\rho) &= a + \frac{b}{\rho} + \frac{d}{\rho^2} + O\left(\frac{1}{\rho^3}\right)\\ \phi'(\rho) &= -\frac{b}{\rho^2} + O\left(\frac{1}{\rho^3}\right).\end{aligned}$$

They consider the corresponding function $h(t,r) = \phi\left(\frac{r}{t}\right)$ as a weak solution of the equivariant harmonic map problem (3.10), that is,

$$h_{tt} - h_{rr} - \frac{2}{r}h_r + \frac{sin 2h}{r^2} = 0, \tag{3.13}$$

with singular initial data at $t = 0$, given by

$$(3.14)\qquad \begin{aligned} h(0,r) &= h_0(r) = a = \lim_{t\searrow 0}\phi\left(\frac{r}{t}\right) && (r \neq 0)\,, \\ h_t(0,r) &= h_1(r) = \frac{b}{r} = \lim_{t\searrow 0}\frac{d}{dt}\phi\left(\frac{r}{t}\right) && (r \neq 0)\,. \end{aligned}$$

Thereby, h is a weak solution of (3.13) if there holds

$$(3.15)\qquad \int_0^1\int_0^\infty \left\{-h_t\psi_t + h_r\psi_r + \frac{1}{r^2}\psi\sin 2h\right\} r^2\,dr\,dt = \int_0^\infty \psi(0,r)\frac{b}{r}r^2\,dr$$

for any $\psi \in C^\infty\left([0,1]\times\mathbb{R}^3\right)$ such that $\psi(t,x) = \psi(t,r)$, $\psi(1,\cdot) \equiv 0$, and $\operatorname{supp}\psi(t) \subset B_R(0)$ for some $R > 0$. Moreover, h assumes the initial data (3.14) in the sense that

$$\begin{aligned} \|h(t,r) - a\|_{H^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)} &\to 0 \qquad (t \to 0), \\ \|h_t(t,r) - \frac{b}{r}\|_{L^2(\mathbb{R}^3)} &\to 0 \qquad (t \to 0). \end{aligned}$$

Note that $h_0 \in H^{1,2}_{\mathrm{loc}}$, $h_1 \in L^2_{\mathrm{loc}}$.

On the other hand, also the function

$$\tilde{h}(t,r) = \begin{cases} \phi\left(\frac{r}{t}\right), & r > t \\ \phi_*, & r \leq t \end{cases}$$

weakly satisfies (3.13), (3.14) on $[0,1]\times\mathbb{R}^3$, with $D\tilde{h} \in L^\infty\left([0,1];L^2(B_R(0))\right)$ for any $R > 0$, showing that weak solutions are in general not unique. To verify that $\tilde{h}$ solves (3.15), for any ψ we split

$$\begin{aligned} &\int_0^1\int_0^\infty \left\{-h_t\psi_t + h_r\psi_r + \frac{1}{r^2}\psi\sin 2h\right\} r^2\,dr\,dt - \int_0^\infty \psi(0,r)\frac{b}{r}r^2\,dr \\ &= \left\{\int_0^1\int_t^\infty \{\dots\}\,r^2\,dr\,dt - \int_0^\infty \psi(0,r)\frac{b}{r}r^2\,dr\right\} + \int_0^1\int_0^t \{\dots\}\,r^2\,dr\,dt = I + II. \end{aligned}$$

Clearly, since $D\tilde{h}(t,r) \equiv 0$ for $r \leq t$, the second integral $II = 0$. Moreover, since $\tilde{h} \equiv h$ for $r \geq t$, and since h satisfies (3.15) the first integral reduces to the boundary term

$$I = -\frac{1}{\sqrt{2}}\int_0^1 \left(h_t(t,t) + h_r(t,t)\right)\psi(t,t)t^2\,dt$$

which also vanishes on account of

$$\begin{aligned} h_t + h_r &= -\frac{r}{t^2}\phi'\left(\frac{r}{t}\right) + \frac{1}{t}\phi'\left(\frac{r}{t}\right) \\ &= \frac{1}{t}\left(1 - \frac{r}{t}\right)\phi'\left(\frac{r}{t}\right) = 0 \qquad \text{for } r = t. \end{aligned}$$

Observe that $\tilde{h}$ induces a solution $\tilde{u}$ of (3.2) with $E(\tilde{u}(t);B_1(0)) < E(u(t);B_1(0))$ for any $t \in]0,1]$, where u is the solution corresponding to h. Hence there may be a chance of restoring uniqueness by some entropy principle.

Self-similar solutions, general theory

More generally, self-similar solutions $u(t,x) = v\left(\frac{x}{t}\right)$ to the harmonic map equation satisfy

$$-v_{\rho\rho} - \left(\frac{m-1}{\rho} + \frac{(m-3)\rho}{1-\rho^2}\right) v_\rho - \frac{1}{\rho^2(1-\rho^2)}\Delta_\omega v \perp T_v N, \tag{3.16}$$

where ρ, ω denote spherical coordinates on $\mathbf{R}^m$. This may either be verified by direct computations or by introducing similarity coordinates

$$\tau = \sqrt{t^2 - r^2}, \qquad \rho = \frac{r}{t}, \qquad \omega = \theta$$

on $\{r \le t\}$ and writing the Minkowski metric on M^{m+1} as

$$ds^2 = -d\tau^2 + \tau^2 \left\{\frac{1}{(1-\rho^2)^2}\, d\rho^2 + \frac{\rho^2}{1-\rho^2}\, d\omega^2\right\}.$$

For $u(t,r,\theta) = v(\tau,\rho,\omega)$ the Lagrangian then becomes

$$L(v) = \frac{1}{2}\int \left\{-|v_\tau|^2 + \frac{(1-\rho^2)^2}{\tau^2}|v_\rho|^2 + \frac{(1-\rho^2)}{\tau^2\rho^2}|v_\omega|^2\right\} \frac{\tau^m \rho^{m-1}}{(1-\rho^2)^{(m+1)/2}}\, d\rho\, d\omega\, d\tau. \tag{3.17}$$

In particular, if $v = v(\rho,\omega)$ is stationary for L, we obtain (3.16). Note that (3.16) is an *elliptic* harmonic map problem on the m-dimensional hypersurface $\{\tau = 1\}$ with the (hyperbolic) metric

$$ds_0^2 = \frac{1}{(1-\rho^2)^2}\, d\rho^2 + \frac{\rho^2}{1-\rho^2}\, d\omega^2,$$

as was pointed out by Shatah-Tahvildar-Zadeh [**130**].

Now observe that, if $m = 3$, for v to be regular (C^2) at $\rho = 1$ we need

$$\Delta_\omega v \perp T_v N \qquad \text{at } \rho = 1;$$

that is, $v(1,\cdot)\colon S^2 \to N$ has to be harmonic. By the maximum principle (Jäger-Kaul [**81**]) for harmonic maps into convex manifolds, therefore either $v(1,\cdot) \equiv$ const. or $v(1,\cdot)$ cannot be contained in a strictly convex part of N. Moreover, if $v(1,\cdot) \equiv$ *const.*, then for $\rho < 1$ sufficiently close to 1 the image of $v(\rho,\cdot)$ is contained in an arbitrarily small strictly convex part of N and again we may apply the Jäger-Kaul maximum principle to conclude:

Theorem 3.5. *If $m = 3$ and if $u(t,x) = v\left(\frac{x}{t}\right)$ is a self-similar solution to the harmonic map equation* (3.2), *where $v\colon \mathbb{R}^3 \to N$ is smooth in a neighborhood of $\overline{B_1(0)}$ and such that the image $v\left(\overline{B_1(0)}\right)$ is contained in a strictly convex part of N, then $v \equiv$ const. on $B_1(0)$.*

By Theorem 3.4 the above result is best possible in dimension $m = 3$. In case $m = 2$, due to the following result we can rule out self-similar solutions altogether.

Theorem 3.6. *If $m = 2$ and if $u(t,x) = v\left(\frac{x}{t}\right)$ solves* (3.16), *where $v\colon D \subset \mathbb{R}^2 \to N$ is smooth in a neighborhood of $\overline{B_1(0)}$, then $v \equiv$ const. on $B_1(0)$.*

Proof. If $m = 2$ we may write (3.16) in the form

$$\left(\rho\sqrt{1-\rho^2}v_\rho\right)_\rho + \frac{\Delta_\omega v}{\rho\sqrt{1-\rho^2}} \perp T_vN.$$

Multiplying by $\rho\sqrt{1-\rho^2}v_\rho \in T_vN$ and integrating over $\omega \in S^1$, we obtain

$$\frac{d}{d\rho}\left(\int_{S^1}\rho^2(1-\rho^2)|v_\rho|^2\,d\omega - \int_{S^1}|\nabla_\omega v|^2\,d\omega\right) = 0.$$

Integrating in ρ, we find

$$\int_{S^1}\rho^2(1-\rho^2)|v_\rho|^2\,d\omega - \int_{S^1}|\nabla_\omega v|^2\,d\omega = C_0.$$

Inspection at $\rho = 0$ shows that $C_0 = 0$. Hence for $\rho = 1$ we obtain $\nabla_\omega v = 0$ that is, $v|_{\partial B_1(0)} \equiv const.$ Finally, note that in dimension $m = 2$ Dirichlet's integral and hence the harmonic map equation is conformally invariant. Theorem 3.6 thus is a consequence of Lemaire's result, Example 1.3 □

3.5. Global existence and regularity for equivariant harmonic maps for $m = 2$

The preceding examples of finite-time blow-up hardly leave any hope to achieve a satisfactory existence and regularity theory for the Cauchy problem for harmonic maps, except in dimension $m = 2$. In fact, we may state the following

Conjecture 3.1. If $m = 2$, then for any compactly supported initial data u_0, u_1 with finite energy there exists a unique global weak solution u to the Cauchy problem (3.2) (3.5) satisfying the energy inequality. If $E(u(0)) < \epsilon_0 = \epsilon_0(N)$ is sufficiently small, or if the range $u(M^{2+1})$ lies in a strictly convex part of N or, more generally, does not contain the image of a harmonic sphere $u\colon S^2 \to N$, then u is globally smooth, provided u_0 and u_1 are.

At present the theory is still a long way from affording a proof of this conjecture in general. Partial results, however, are known; in particular, Conjecture 3.1 has been rigorously established for equivariant harmonic maps into convex surfaces of revolution.

In the following, we review these latter results, due to Shatah-Tahvildar-Zadeh [**129**]. A simplified proof was given by Shatah-Struwe [**128**]; moreover, Grillakis [**64**] has recntly been able to relax the convexity assumption. Similar results for radially symmetric harmonic maps ($m = 2$) have been obtained by Christodoulou-Tahvildar-Zadeh [**25**].

Thus we consider the Cauchy problem (3.10), that is,

$$u_{tt} - \Delta u + \frac{f(u)}{r^2} = 0 \qquad \text{in } M^{2+1} \tag{3.18}$$

for smooth, radially symmetric data

$$u|_{t=0} = u_0,\ u_t|_{t=0} = u_1 \tag{3.19}$$

having compact support and such that $u_0(0) = 0$ By uniqueness, also u will be radially symmetric $u(t,x) = u(t,r)$, and $u(t,0) = 0$ if u is smooth.

Regularity for small energy

In a first step we shall show that this problem admits a unique smooth solution for all time provided the initial energy

$$E(u(0)) = \int_{\mathbb{R}^2} \left\{ \frac{|u_1|^2 + |\nabla u_0|^2}{2} + \frac{F(u_0)}{r^2} \right\} dx$$

is sufficiently small. Here $F(u) = \frac{1}{2}g^2(u) = \int_0^u f(v)\,dv$. Note that the energy inequality

$$E(u(t)) \leq E(u(0))$$

holds. Moreover, by radial symmetry

$$\begin{aligned} E(u(t)) &= 2\pi \int_0^\infty \left\{ \frac{|u_t|^2 + |u_r|^2}{2} r + \frac{F(u)}{r} \right\} dr \\ &=: 2\pi E_{rad}(u(t)). \end{aligned}$$

Finally, it will be useful to denote

$$G(u) = \int_{\mathbb{R}^2} \frac{F(u)}{|x|^2}\,dx = 2\pi \int_0^\infty \frac{F(u)}{r}\,dr = 2\pi G_{rad}(u)$$

the potential energy of u.

Lemma 3.2. *Given $C_0 > 0$, there are constants $C_1, \epsilon_1 > 0$ such that any solution u to (3.18), (3.19) with $E(u(0)) \leq C_0$ satisfies $\|u(t)\|_{L^\infty} \leq C_1$, provided $G(u(t)) < \epsilon_1$, for any t.*

Proof. Denote

$$H(u) = \int_0^u g(v)\,dv = \int_0^u \sqrt{2F(v)}\,dv.$$

By radial symmetry, for any (t, r_0) we have

$$\begin{aligned} |H(u(t,r_0))|^2 &= \left(\int_0^{r_0} H(u(t,r))_r\,dr \right)^2 \\ &\leq \left(\int_0^\infty |u_r||g(u)|\,dr \right)^2 \\ &\leq \int_0^\infty u_r^2 r\,dr \cdot \int_0^\infty \frac{g^2(u)}{r}\,dr \\ &\leq 4E_{rad}(u(t)) \cdot G_{rad}(u(t)) \leq \pi^{-2} E(u(0)) G(u(t)). \end{aligned}$$

Thus, if we fix

$$\epsilon_1 < \pi^2 C_0^{-1} \sup_u H^2(u) = \pi^2 C_0^{-1} \left(\int_0^\infty g(v)\,dv \right)^2 \leq \infty,$$

the claim follows. □

Remark 3.5. In particular, there are constants $\epsilon_1 > 0$, C_1 such that any solution u to (3.18), (3.19) satisfies $\|u\|_{L^\infty} \leq C_1$ provided $E(u(0)) < \epsilon_1$. More generally, if

$$\int_0^\infty g(v)\,dv = \infty\ ,$$

that is, if N does not include "sphere at infinity" in the words of Shatah-Tahvildar-Zadeh [**129**], then any solution u to (3.18), (3.19) with finite initial energy is uniformly bounded.

By hypothesis (3.11) on g, given $C_1 > 0$ there exists a constant $C_2 > 0$ such that

$$C_2^{-1}u^2 \le F(u) \le C_2 u^2 \tag{3.20}$$

for $|u| \le C_1$. Introduce w, defined by

$$w(t,r) = \frac{u(t,r)}{r}.$$

Note that w is smooth if u is and satisfies

$$w_{tt} - w_{rr} - \frac{3}{r}w_r = \frac{u - f(u)}{r^3} = w^3[a + p(rw)],$$

where $a = -\frac{2}{3}\frac{\partial^3 g}{\partial u^3}(0)$ and p is a smooth function; that is, w solves a *semi-linear* wave equation

$$w_{tt} - \Delta w = w^3[a + p(rw)] \qquad \text{in } M^{4+1},$$

with nonlinearity growing at a "critical" rate.

More generally, a semi-linear wave equation

$$w_{tt} - \Delta w + n(u) = 0 \qquad \text{in } M^{m+1}$$

has critical growth if $n(u) \sim u|u|^{2^*-2}$ for large u, where $2^* = \frac{2m}{m-2}$ is the Sobolev exponent in $\mathbb{R}^m$.

In 1988 the model case

$$u_{tt} - \Delta u + u^5 = 0 \qquad \text{in } M^{3+1}$$

was solved by Struwe **[142]** for large radially symmetric data. Grillakis **[62]** then was able to remove the symmetry condition; see also Struwe **[146]** for a survey of these developments. More recently, also the higher dimensional cases have become accessible, in particular, the case $m = 4$ needed here.

Note that $u = rw$ is bounded. Moreover,

$$|\nabla w|^2 = w_r^2 = \frac{u_r^2}{r^2} - 2\frac{u_r u}{r^3} + \frac{u^2}{r^4},$$

whence

$$\int_{\mathbb{R}^4} |\nabla w(t)|^2\,dx$$
$$= c\int_0^\infty w_r^2 r^3\,dr = c\int_0^\infty \left\{u_r^2 r + \frac{u^2}{r}\right\}dr \le CE(u(t)) \le CE(u(0)).$$

Hence by Sobolev's embedding theorem $w(t) \in L^4(\mathbb{R}^4)$ and $\|w(t)\|_{L^4}^2 \le CE(u(0))$. However, (3.20) gives a more convenient bound

$$\|w(t)\|_{L^4}^4 = C\int_0^\infty \frac{u^4}{r^4}r^3\,dr \le C\int_0^\infty \frac{u^2}{r}\,dr \le CG(u(t)) \le C\left(E(u(0))\right).$$

Finally, $w_t = \frac{u_t}{r}$ implies

$$\int_{\mathbb{R}^4} |w_t|^2\,dx \le c\int |u_t|^2 r\,dr \le CE(u(t)) \le CE(u(0)).$$

We proceed to show:

Theorem 3.7 (Shatah-Struwe [**128**]). *Given $C_0 > 0$ there exists a constant $\epsilon_0 > 0$ such that for any smooth, compactly supported data u_0, u_1 having energy $E(u(0)) < C_0$ the Cauchy problem* (3.18), (3.19) *admits a unique smooth local solution u. The solution extends globally, provided the estimate* $\sup_{|t|<T} G(u(t)) < \epsilon_0$ *is satisfied for any $T > 0$.*

Note that it is easy to obtain global weak solutions of (3.18), (3.19) by energy methods; see Strauss [**134**] or Struwe [**146**]. Moreover, by adapting the arguments of Kapitanskii [**90**] one can show uniqueness of weak solutions in a suitable class.

For the proof of Theorem 3.7 the following basic regularity result for the linear wave equation

$$\Box v = f \quad \text{in } M^{m+1} \tag{3.21}$$

$$v|_{t=0} = v_0, \ v_t|_{t=0} = v_1 \tag{3.22}$$

is needed.

Proposition 3.1 (Strichartz [**135**] [**136**]). *Suppose v is a (smooth) solution of* (3.21), (3.22), *$m \geq 2$. Then for any $T > 0$ we have the estimate*

$$\|v\|_{L^q(\mathbb{R}^m\times[-T,T])} \leq C\left(\|f\|_{L^{q'}(\mathbb{R}^m\times[-T,T])} + \|v_0\|_{H^{\frac{1}{2}}(\mathbb{R}^m)} + \|v_1\|_{H^{-\frac{1}{2}}(\mathbb{R}^m)}\right),$$

where

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{m+1} = 1 - \frac{1}{q'}.$$

$H^{\frac{1}{2}} = H^{\frac{1}{2},2}$ is the interpolation space between L^2 and $H^{1,2}$, $H^{-\frac{1}{2}}$ its dual.

Remark 3.6. Kapitanskii [**90**] was the first to note the importance of Strichartz' estimate for semi-linear wave equations with critical nonlinearity. Combining this result with the dilation estimates of Morawetz [**109**], it was possible to extend the regularity results of Struwe [**142**] and Grillakis [**62**] from $m = 3$ to $m \leq 5$ (Grillakis [**63**]) and, more recently, even to $m \leq 7$ (Shatah-Struwe [**128**]).

Proof of Theorem 3.7. We apply Proposition 3.1 in dimension $m = 4$ with $q = \frac{10}{3}$, $q' = \frac{10}{7}$ to obtain bounds for w and its spatial gradient ∇w; that is w_r. Interpolating between the two, we obtain bounds for "half a derivative" of w in L^q. More precisely, denote

$$V_T^q = \Big\{u \in L^2(\mathbb{R}^m \times [-T,T]); u(t) \in \dot{B}_q^{\frac{1}{2},q}(\mathbb{R}^m) \quad \text{for a.e. } t,$$

$$\|u\|_{V_T^q} := \left(\int_{-T}^{T} \|u(t)\|_{\dot{B}_q^{\frac{1}{2}}} \, dt\right)^{1/q} < \infty\Big\},$$

where the Besov space $\dot{B}_q^{\frac{1}{2}}(\mathbb{R}^m)$ denotes the interpolation space between $L^q(\mathbb{R}^m)$ and $W^{1,q}(\mathbb{R}^m)$. Note that by Sobolev's embedding

$$\dot{B}_q^{\frac{1}{2}} \hookrightarrow L^s, \qquad \frac{1}{s} = \frac{1}{q} - \frac{1}{2m};$$

that is, $s = \frac{40}{7}$, if $m = 4$. Then by Proposition 3.1 we have

$$\|w\|_{V_T^q} \leq C\|w^3(a + p(rw))\|_{V_T^{q'}} + C\|\nabla w(0)\|_{L^2} + C\|w_t(0)\|_{L^2}. \tag{3.23}$$

The last two terms are bounded by $E(u(0))$. To bound the first term on the right for fixed t we estimate the $L^{q'}$- and $W^{1,q'}$-norms, using boundedness of $rw = u$ and Hölder's inequality

$$\|w^3(a+p(rw))\|_{L^{q'}} \le C\|w^3\|_{L^{q'}} \le C\|w\|_{L^q}\|w\|^2_{L^{q_1}}, \tag{3.24}$$

$$\begin{aligned}\|\nabla[w^3(a+p(rw))]\|_{L^{q'}} &\le C\||\nabla w||w|^2 + |w|^4\|_{L^{q'}} \\ &\le C(\|\nabla w\|_{L^q} + \||w|^2\|_{L^q})\|w\|^2_{L^{q_1}} \le C\|\nabla w\|_{L^q}\|w\|^2_{L^{q_1}},\end{aligned} \tag{3.25}$$

where

$$1 - \frac{1}{q} = \frac{1}{q'} = \frac{1}{q} + \frac{2}{q_1};$$

that is,

$$\frac{1}{q_1} = \frac{1}{2} - \frac{1}{q} = \frac{1}{m+1} = \frac{1}{5}.$$

Note that we also used Sobolev's embedding to bound

$$\|w^2\|_{L^q} \le C\|\nabla w\|_{L^q}\|w\|_{L^4} \le C\sup_t\|w(t)\|_{L^4}\|\nabla w\|_{L^q} \le C\left(E(u(0))\right)\|\nabla w\|_{L^q}.$$

Note that $\frac{40}{7} = s > 5 = q_1 > 4$. Similarly, by interpolation we obtain

$$\|w^3(a+p(rw))\|_{\dot B^{\frac12}_q} \le \|w\|_{\dot B^{\frac12}_q}\|w\|^2_{L^{q_1}} \le C\|w\|_{\dot B^{\frac12}_q}\|w\|^{\alpha}_{L^s}\|w\|^{2-\alpha}_{L^4}$$

with

$$\frac{2}{q_1} = \frac{2}{5} = \frac{\alpha}{s} + \frac{2-\alpha}{4} = \frac{7\alpha}{40} + \frac{2-\alpha}{4};$$

that is, $\alpha = \frac{4}{3}$. By Sobolev's embedding theorem, therefore, for fixed t we may estimate

$$\|w^3(a+p(rw))\|_{\dot B^{\frac12}_q} \le C\sup_{|t|\le T}\|w(t)\|^{2/3}_{L^4}\|w\|^{7/3}_{\dot B^{\frac12}_q} \le CE(u(0))^{1/6}\|w\|^{7/3}_{\dot B^{\frac12}_q}.$$

Note that, miraculously,

$$\frac{7}{3}q' = \frac{10}{3} = q.$$

Thus, when integrating in t we obtain

$$\begin{aligned}\|w^3(a+p(rw))\|_{V_T^{q'}} &= \left\{\int_{-T}^{T}\|w^3(a+p(rw))\|^{q'}_{\dot B^{\frac12}_q}\,dt\right\}^{1/q'} \\ &\le C\sup_t\|w(t)\|^{2/3}_{L^4}\|w\|^{7/3}_{V_T^q} \le CE(u(0))^{1/6}\|w\|^{7/3}_{V_T^q}.\end{aligned} \tag{3.26}$$

Combining this estimate with (3.23), there results the inequality

$$\|w\|_{V_T^q} \le C_3\Big(E(u(0))\Big)^{1/2} + C_3\sup_t\|w(t)\|^{2/3}_{L^4}\|w\|^{7/3}_{V_T^q}$$

for any $T > 0$, with a constant $C_3 = C_3(E(u(0))) > 0$. Suppose $E(u(0)) \le C_0$. By our local existence result Theorem 3.1, $\|w\|_{V_T^q} < 2C_0^{1/2}C_3$ for small $T > 0$. If we then have

$$C_3\sup_t\|w(t)\|^{2/3}_{L^4}(2C_0^{1/2}C_3)^{4/3} < \frac{1}{2},$$

by continuity in T the estimate

$$\|w\|_{V_T^q} < 2C_0^{1/2}C_3 \tag{3.27}$$

will persist for all $T > 0$. Using (3.27) and applying Proposition 3.1 again to the differentiated equation (3.18), we can now conclude in a similar way

$$\|\nabla w\|_{L^q(\mathbb{R}^{m+1})} \le C(u_0, u_1) + C_4 \sup_t \|w(t)\|_{L^4}^{2/3} \|\nabla w\|_{L^q(\mathbb{R}^{m+1})},$$

and thus ∇w is bounded in $L^q(\mathbb{R}^{m+1})$ in terms of the initial data, if

$$\sup_t \|w(t)\|_{L^4}^2 \le C \sup_t G(u(t)) < \epsilon_0$$

is sufficiently small.

We can now proceed by energy methods. Indeed, since $\nabla w \in L^q$ we have $w \in L^{q,q^*}$, where $q^* = \frac{mq}{m-q} = 20$ is the Sobolev exponent, and thus

$$\left|\nabla\left[w^3(a + p(rw))\right]\right| \le C\,|\nabla w|\,w^2 + w^4 \in L^{1,2}.$$

It follows that $D^2 w \in L^{\infty,2}$, whence $w \in L^{\infty,p}$ for all $p < \infty$, and $Dw \in L^{\infty,4}$. Differentiating one more time and using that for any t we can estimate

$$\begin{aligned}\left\|D^2\left[w^3(a + p(rw))a\right]\right\|_{L^2} &\le C\||D^2 w||w|^2 + |Dw|^2|w| + |Dw||w|^3 + |w|^5\|_{L^2} \\ &\le C\|D^3 w\|_{L^2} + C,\end{aligned}$$

upon multiplying the equation for $D^2 w$ by $D^2 w_t$ we obtain a Gronwall-type inequality

$$\frac{d}{dt} E(D^2 w) \le C(1 + E(D^2 w))$$

and hence $D^3 w \in L^{\infty,2}_{\mathrm{loc}}$. But then by Sobolev's embedding theorem, $w \in L^{\infty,\infty}_{\mathrm{loc}} = L^{\infty}_{\mathrm{loc}}$. Continuing in this manner, also the derivatives of w - and hence of u - can be bounded. □

By finite speed of propagation the above argument can be localized. To state this precisely, introduce the following notation. Given a point $z_0 = (t_0, x_0) \in M^{m+1}$, $t_0 > 0$, denote

$$K(z_0) = \left\{(t,x) \in M^{m+1}; |x - x_0| \le t_0 - t,\ t \ge 0\right\}$$

the backward light cone with vertex at z_0, and for t fixed denote by

$$D(t; z_0) = \left\{(t,x) \in K(z_0)\right\},$$

its spatial sections. ($z_0 = (0,0)$ will be omitted.) The proof of Theorem 3.7 implies

Lemma 3.3. *Let $z_0 = (t_0, x_0)$, $t_0 > 0$, be given. Suppose $u \in C^\infty(\mathbb{R}^2 \times [0, t_0[)$ is a smooth solution of* (3.18), (3.19) *with $E(u(0)) \le C_0$ and suppose*

$$\liminf_{t \to t_0} G\big(u(t); D(t; z_0)\big) < \epsilon_0. \tag{3.28}$$

Then u extends smoothly to a neighborhood of z_0.

Proof. Indeed, by the proof of Theorem 3.7 we obtain uniform bounds for u and its derivatives on $K(z_0)$. In particular, then also the condition

$$E\big(u(t); D(t; z_0)\big) < \epsilon_0 \tag{3.29}$$

is satisfied for t close to t_0. Since u is smooth on $\mathbb{R}^2 \times [0, t_0[$, condition (3.29) holds for $\bar{z}$ in a full neighborhood of z_0 for any (fixed) t_1 close to t_0. Finally, by the energy inequality, (3.29) will persist for $t > t_1$ on any cone $K(\bar{z})$ for such points $\bar{z}$ and the argument of Theorem 3.7 implies smoothness of u at all such points. □

Lemma 3.3 will be important for our next section.

Regularity for equivariant harmonic maps with convex range

We establish the following result of Shatah-Tahvildar-Zadeh [**129**].

Theorem 3.8. *Suppose* $u\colon M^{2+1} \to \mathbb{R}$ *is a weak solution of* (3.18),(3.19) *for smooth, compactly supported data* u_0, u_1 *and of finite energy* $E(u(t)) \leq E(u(0)) < \infty$. *Moreover, suppose that* $f(u) \geq 0$ *on the range of* u. *Then* $u \in C^\infty$.

Proof. For the proof we need a local version of the energy inequality. □

Lemma 3.4. *Let* $u \in C^\infty(K(z_0))$ *be a smooth solution of* (3.18). *Then for any* $0 \leq S < T \leq t_0$ *there holds*

$$E(u; D(T; z_0)) + \frac{1}{\sqrt{2}} \int_{M_S^T(z_0)} e_{\tan}(u)\, do \leq E(u; D(S; z_0)).$$

Here,

$$K_S^T(z_0) = \{z = (t, x) \in K(z_0); S \leq t \leq T\}$$

denotes a truncated cone and

$$M_S^T(z_0) = \{z = (t, x) \in K_S^T(z_0);\ |x - x_0| = t - t_0\}$$

its mantle. Moreover,

$$e_{\tan}(u) = \frac{|(x - x_0)u_t + (t - t_0)\nabla u|^2}{2|t - t_0|^2} + \frac{F(u)}{|x|^2}$$

denotes the tangential energy density. Finally, *do* is the element of surface area on M_S^T.

Proof. Multiply (3.18) by u_t to obtain

$$\frac{d}{dt} e(u) - \operatorname{div}(u_t \nabla u) = 0$$

where

$$e(u) = \frac{|u_t|^2 + |\nabla u|^2}{2} + \frac{F(u)}{|x|^2}.$$

Integrating over $K_S^T(z_0)$, and noting that the outward normal on $M_S^T(z_0)$ is given by $\nu = \frac{1}{\sqrt{2}}(1, \frac{x - x_0}{t_0 - t})$, on account of

$$e(u) - \frac{x - x_0}{t_0 - t} u_t \nabla u = e_{\tan}(u)$$

we obtain the claim. □

Remark 3.7. In particular, letting $T \to t_0$ we obtain

$$\int_{M_S(z_0)} e_{\tan}(u)\, do < \infty$$

and hence

$$\int_{M_S(z_0)} e_{\tan}(u)\, do \to 0$$

as $S \to t_0$.

As a consequence of Lemma 3.4, if u is a smooth local solution of (3.18), (3.19) on $\mathbb{R}^2 \times [0, t_0[$ which becomes singular at $z_0 = (t_0, x_0)$, then $x_0 = 0$. Else, by radial symmetry any point $\bar{z} = (t_0, \bar{x}), |\bar{x}| = |x_0|$, would also be singular. By Lemma 3.3 and 3.4 for any such $\bar{z}$ and any $T < t_0$ we then should have

$$E(u; D(T; \bar{z})) \geq \epsilon_0.$$

But if $|x_0| \neq 0$, given $K \in \mathbb{N}$, for T sufficiently close to t_0 disjoint discs $D(T; z_k)$, $z_k = (t_0, x_k)$, $|x_k| = |x_0|$, $1 \leq k \leq K$, can be found, whence

$$E\big(u(T)\big) \geq \sum_{k=1}^{K} E\big(u; D(T, z_k)\big) \geq K\epsilon_0$$

for any K. This, however, contradicts our assumption that $E(u(t)) \leq E(u(0)) < \infty$ for all t. Hence a first singularity must appear on the line $x = 0$.

Suppose $z_0 = (t_0, 0)$ is singular and shift time by t_0 to achieve $z_0 = 0$. Denote $K = K(0)$, $K_S = K_S(0)$, etc. We need the following Morawetz-type dilation estimate:

Lemma 3.5. *For a smooth solution u of* (3.18) *on a cone K_S there holds*

$$\frac{1}{|S|}\int_{K_S}\left(\frac{uf(u)}{2|x|^2} - \frac{F(u)}{|x|^2}\right) dx\, dt + \int_{D(S)}\left\{\frac{1}{2}\left(1 - \frac{|x|^2}{S^2}\right)|\nabla u|^2 + \frac{F(u)}{|x|^2}\right\} dx \to 0$$

as $S \to 0$.

Proof. Multiply (3.18) by $tu_t + x\nabla u + \frac{1}{2}u$ and use Remark 3.7 to control the boundary terms; see Shatah-Struwe [**128**] for details. □

Lemma 3.5 implies Theorem 3.8. Indeed, given $T < 0$ consider the set

$$\Lambda_T = \left\{t \in [T, 0[\,; G(u; D(t)) = \int_{D(t)} \frac{F(u)}{|x|^2}\, dx \geq \sup_{T \leq t \leq 0} G(u, D(t)) - \delta^2\right\};$$

where $\delta > 0$ will be determined later. Since $uf(u) \geq 0$ by assumption, for large $T < 0$ Lemma 3.5 implies

$$\int_{D(t)}\left(1 - \frac{|x|^2}{t^2}\right)|\nabla u|^2\, dx \leq 2\delta^2$$

for any $t \in \Lambda_T$. In particular,

$$\int_0^{t(1-\delta)} u_r^2(t) r\, dr \leq C\delta,$$

and hence by the argument of Lemma 3.2 we obtain

$$\sup_{|x| \leq |t|(1-\delta)} |u(t, x)| \leq \frac{\rho}{2}$$

for any $\rho > 0$ and any such t, if $0 < \delta \leq \delta(\rho)$. Moreover, estimating

$$|u(t, x) - u(t, y)|^2 = \left|\int_{|y|}^{|x|} u_r\, dr\right|^2 \leq \left(\int_{|y|}^{|x|} \frac{dr}{r}\right)\left(\int_{|y|}^{|x|} u_r^2 r\, dr\right) \leq C\delta E\big(u(0)\big)$$

for $|y| = t(1 - \delta) \leq |x| \leq t$, we obtain the inequality

$$\sup_{|x| \leq |t|} |u(t, x)| \leq \rho$$

for any ρ, any $t \in \Lambda_T$, provided $0 < \delta < \delta(\rho)$ and $0 > T > T(\delta)$. Choose $\rho > 0$ such that for $0 \leq |u| \leq \rho$ there holds $F(u) \leq Cuf(u)$ for some $C > 0$ and fix $\delta \in]0, \delta(\rho)]$.

Consider $T \in [T(\delta), 0[$. Replacing T by a suitable number $S \in \Lambda_T$, if necessary, we may assume that

$$\sup_{T \leq t \leq 0} G(u(t), D(t)) - \frac{\delta^2}{2} \leq G(u(T), D(T)),$$

while by Lemma 3.5 the right hand side is

$$\begin{aligned}
&\leq \frac{1}{|T|} \int_{K(T)} \left(\frac{F(u)}{|x|^2} - \frac{uf(u)}{2|x|^2} \right) dx\, dt + o(1) \\
&\leq \frac{|\Lambda_T|}{|T|} \left(1 - \frac{1}{2C} \right) \sup_{T \leq t \leq 0} G(u(t), D(t)) \\
&\quad + \left(1 - \frac{|\Lambda_T|}{|T|} \right) \left[\sup_{T \leq t \leq 0} G(u(t), D(t)) - \delta^2 \right] + o(1),
\end{aligned}$$

where $o(1) \to 0$ as $T \to 0$. It follows that

$$\frac{|\Lambda_T|}{2C|T|} \sup_{T \leq t \leq 0} G(u(t), D(t)) + \left(\frac{1}{2} - \frac{|\Lambda_T|}{|T|} \right) \delta^2 \leq o(1),$$

where $o(1) \to 0$ $(T \to 0)$. Since $\delta < \delta(\rho)$ is arbitrary, therefore we can achieve that

$$\sup_{T \leq t \leq 0} G(u, D(t)) < \epsilon_0$$

for $T < 0$ sufficiently large. Theorem 3.8 now is a consequence of Lemma 3.3.

Note that it suffices to assume $uf(u) \geq -\delta^2$ for some suitable number $\delta > 0$; that is, the convexity condition may be slightly relaxed. In fact, by a recent result of Grillakis [64] the condition $2F(u) + f(u)u > 0$ or, equivalently, the condition

$$g(u) + g'(u)u > 0$$

suffices.

Global existence of weak solutions

Apart from the equivariant or rotationally symmetric case, the only case where global existence of weak solutions (in absence of classical solutions) so far has been established is the case of $N = S^l$, due to Shatah [127]. Shatah independently devised the same penalty approach for (3.1) as Chen [17], Keller-Rubinstein-Sternberg [93] did for the parabolic evolution problem, see section I.3.

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