Multi-consensus Decentralized Accelerated Gradient Descent

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 May 5, 2020

Abstract

This paper considers the decentralized optimization problem, which has applications in large scale machine learning, sensor networks, and control theory. We propose a novel algorithm that can achieve near optimal communication complexity, matching the known lower bound up to a logarithmic factor of the condition number of the problem. Our theoretical results give affirmative answers to the open problem on whether there exists an algorithm that can achieve a communication complexity (nearly) matching the lower bound depending on the global condition number instead of the local one. Moreover, the proposed algorithm achieves the optimal computation complexity matching the lower bound up to universal constants. Furthermore, to achieve a linear convergence rate, our algorithm doesn't require the individual functions to be (strongly) convex. Our method relies on a novel combination of known techniques including Nesterov's accelerated gradient descent, multi-consensus and gradient-tracking. The analysis is new, and may be applied to other related problems. Empirical studies demonstrate the effectiveness of our method for machine learning applications.

1 Introduction

In this paper, we consider the decentralized optimization problem, where the objective function is composed of m local functions $f_i(x)$ that are stored in m different agents (or computational nodes). These agents form a connected and undirected network. Moreover, each agent i can only access to its private $f_i(x)$, and communicate with its neighbor agents in $\mathcal{N}(i)$. These agents try to cooperatively solve the following convex optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x) \triangleq \frac{1}{m} \sum_{i=1}^m f_i(x), \tag{1.1}$$

where f(x) is a strongly convex function. Decentralized optimization has been widely studied and applied in many applications such as large scale machine learning (Tsianos et al., 2012; Kairouz et al., 2019; He et al., 2019), automatic control (Bullo et al., 2009; Lopes & Sayed, 2008), wireless communication (Ribeiro, 2010), and sensor networks (Rabbat & Nowak, 2004; Khan et al., 2009).

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Because of their wide applications, many decentralized algorithms have been proposed. Decentralized gradient methods (Nedic & Ozdaglar, 2009; Yuan et al., 2016), decentralized accelerated gradient method (Jakovetić et al., 2014; Qu & Li, 2019) and EXTRA (Shi et al., 2015; Li et al., 2019; Mokhtari & Ribeiro, 2016) are primal-only methods. These algorithms only require the computation of the gradient of $f_i(x)$, and they are usually computationally efficient. Another class of algorithms are the dual-based decentralized algorithms. Typical examples include the dual subgradient ascent (Terelius et al., 2011), dual gradient ascent and its accelerated version (Scaman et al., 2017; Uribe et al., 2018), the primal-dual method (Lan et al., 2018; Scaman et al., 2018; Hong et al., 2017), and ADMM (Erseghe et al., 2011; Shi et al., 2014).

In spite of many studies of the decentralized optimization problem, there are several important open problems. The first open problem is whether there exists an algorithm that can both achieve optimal computation and communication complexities matching the known lower bounds. Recently, Scaman et al. (2017) proposed a dual-based algorithm that achieves the optimal communication complexity. However, dual-based methods typically require the evaluation of Fenchel conjugate of the local objective function $f_i(x)$, and for many problems, this requires more computation per-iteration than the primal-only algorithms. Therefore methods proposed by Scaman et al. (2017) do not achieve the optimal computation complexity for general functions. In contrast, Li et al. (2018) proposed a primal-only method called APM-C which can achieve the optimal computation complexity. However, its communication complexity only matches the lower bound up to a $\log\left(\frac{1}{\epsilon}\right)$ factor. The second open problem is whether there exists an algorithm that can establish an optimal communication complexity depending on the global condition number κ_q instead of the local condition number κ_{ℓ} (defined in Eqn. (3.3)) (Scaman et al., 2017). We note that κ_{ℓ} is always larger than κ_q , and the gap between κ_ℓ and κ_q is often very large in real applications. Therefore it is important to measure complexity using κ_g instead of κ_ℓ . Third, it is unknown whether the convexity of each individual $f_i(x)$ is essential for communication-efficient decentralized algorithms. Dual-based algorithms require each $f_i(x)$ to be convex, because they require that the dual function of each individual $f_i(x)$ is well-defined. Existing primal-only algorithms also assume each $f_i(x)$ to be (strongly) convex to achieve linear convergence rates. Finally, it is not clear how optimal centralized and decentralized optimization methods are related. It was shown in Scaman et al. (2017) that lower bounds of communication complexities for centralized and decentralized problems are related: they only differ in the communication cost, where averaging is required in centralized algorithms, and consensus is needed for decentralized algorithms. Several decentralized algorithms tried to 'imitate' their centralized counterparts and they achieve low computation and communication complexities (Qu & Li, 2019; Jakovetić et al., 2014; Li et al., 2018). However, these algorithms achieve convergence rates inferior to their centralized counterparts. For example, the computation complexity of APM-C depends on the local condition number κ_{ℓ} , while the computation complexity of the centralized accelerated gradient descent depends on the global condition number κ_g .

This paper addresses the theoretical issues discussed above. We summarize our contributions as follows:

1. We propose a novel algorithm that can achieve the optimal computation complexity $\mathcal{O}\left(\sqrt{\kappa_g}\log\frac{1}{\epsilon}\right)$,

and a near optimal communication complexity $\mathcal{O}\left(\sqrt{\frac{\kappa_g}{1-\lambda_2(W)}}\log(\frac{M}{L}\kappa_g)\log\frac{1}{\epsilon}\right)$, which matches the lower bound up to a $\log(\frac{M}{L}\kappa_g)$ factor with M and L being the smoothness parameters of $f_i(x)$ and f(x), respectively. To the best of our knowledge, this is the best communication complexity that primal-only algorithms can achieve.

- 2. The complexities of our algorithm depend on the global condition number κ_g instead of the local condition number κ_ℓ , where it holds that $\kappa_g \leq \kappa_\ell$. It is an open problem whether there exists an algorithm that can achieve a communication complexity $\mathcal{O}\left(\sqrt{\frac{\kappa_g}{1-\lambda_2(W)}}\log\frac{1}{\epsilon}\right)$ or even close to it. Our algorithm provide an affirmative answer to this open problem.
- 3. Our algorithm doesn't require each individual function to be (strongly) convex. In contrast, this condition is required in the previous algorithms to achieve linear convergence rates. Thus, our algorithm has a much wider application range since $f_i(x)$ may not be convex in many machine learning problems.
- 4. Our analysis reveals an important connection between centralized and decentralized algorithms. We show that a decentralized algorithm with multi-consensus and gradient tracking can approximate its centralized counterpart. Using this observation, we can show that decentralized algorithms and their centralized counterparts have the same computation complexity, but with different communication complexities due to the difference in communication costs to achieve averaging or consensus.

2 Related Works

We review prior works that are closely related to the proposed algorithms. First, we review the penalty-based algorithms. Nedic & Ozdaglar (2009) proposed the well-known decentralized gradient descent method, where each agent performs a consensus step and a gradient descent step with a fixed step-size which is related to the penalty parameter. Yuan et al. (2016) proved the convergence rate of decentralized gradient descent and showed how the penalty parameter affects the computation complexity. To achieve faster convergence rate and smaller communication complexity, a Network Newton method was proposed in (Mokhtari et al., 2016). In Jakovetić et al. (2014), Nesterov's acceleration was used to improve the convergence speed, and the method relied on multi-consensus to reduce the impact of penalty parameter. However, this kind of algorithms with fixed penalty parameters can only achieve a sub-linear convergence rate even when the objective function is strongly convex (Yuan et al., 2016; Lan & Monteiro, 2013). Recently, Li et al. (2018) proposed APM-C, which employed multi-consensus and increased the penalty parameter properly for each iteration. Due to Nesterov's acceleration, APM-C can achieve a linear convergence rate and a low communication complexity.

To achieve fast convergence rate, the gradient-tracking method was proposed in (Qu & Li, 2017; Xu et al., 2015; Qu & Li, 2019). There are two different techniques for gradient-tracking. The first technique keeps a variable to estimate the average gradient and uses this estimation in the gradient descent step. EXTRA is another gradient-tracking method which introduces two

different weight matrices to track the difference of gradients (Shi et al., 2015; Li et al., 2019). EXTRA is much different from the standard decentralized gradient methods which only use a single weight matrix. Because of these differences, the convergence analysis for different gradient-tracking based algorithms are also different. Due to the tracking of history information, gradient-tracking based algorithms can achieve linear convergence for strongly convex objective functions (Qu & Li, 2017; Shi et al., 2015). However, the previously obtained convergence rates and communication complexities are much worse than the method proposed in this paper.

Finally, dual-based methods introduce an Lagrangian function and work in the dual space. There are different ways to solve the reformulated problem such as gradient descent method (Terelius et al., 2011), accelerated gradient method (Scaman et al., 2017; Uribe et al., 2018), primal-dual method (Lan et al., 2018; Scaman et al., 2018) and ADMM (Shi et al., 2014; Erseghe et al., 2011). However, generally speaking, the dual methods are computationally inefficient. For example, using the accelerated gradient method to solve the dual counterpart of decentralized optimization problem can achieve optimal communication complexity (Scaman et al., 2017; Uribe et al., 2018). However, the computation complexity will have extra dependency on the eigenvalue gap of the weight matrix describing the network (Uribe et al., 2018), and this can be much worse than the optimal computation complexity.

3 Notation and Preliminaries

3.1 Notation and Definition

Let $x_i \in \mathbb{R}^d$ be the local copy of the variable of x for agent i and we introduce the aggregate variable \mathbf{x} , aggregate objective function $F(\mathbf{x})$ as

$$\mathbf{x} = \begin{bmatrix} x_1^\top \\ \vdots \\ x_m^\top \end{bmatrix}, \ F(\mathbf{x}) = \frac{1}{m} \sum_{i=1}^m f_i(x_i), \tag{3.1}$$

where $\mathbf{x} \in \mathbb{R}^{m \times d}$ and the aggregate gradient $\nabla F(\mathbf{x}) \in \mathbb{R}^{m \times d}$ is defined as

$$\nabla F(\mathbf{x}) = \frac{1}{m} \begin{bmatrix} \nabla^{\top} f_1(x_1) \\ \vdots \\ \nabla^{\top} f_m(x_m) \end{bmatrix}.$$

We denote that

$$\bar{x}_t = \frac{1}{m} \sum_{i=0}^m \mathbf{x}_t(i,:), \quad \bar{y}_t = \frac{1}{m} \sum_{i=0}^m \mathbf{y}_t(i,:), \quad \bar{g}_t = \frac{1}{m} \sum_{i=0}^m \nabla f_i(\mathbf{y}_t(i,:)),$$
 (3.2)

where $\mathbf{x}(i,:)$ means the *i*-th row of matrix \mathbf{x} . Moreover, in this paper, we use $\|\cdot\|$ to denote the Frobenius norm of vector or matrix. We use $\langle x,y\rangle$ to denote the inner product of vectors x and y. Furthermore, we introduce the following definitions that will be used in the whole paper:

Global L-Smoothness We say f(x) is L-smooth if for all $y, x \in \mathbb{R}^d$,

$$f(y) \le f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} ||y - x||^2.$$

Global μ -Strong Convexity We say f(x) is μ -strongly convex, if for all $y, x \in \mathbb{R}^d$,

$$f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} ||y - x||^2.$$

Local *M*-Smoothness We say the problem is locally *M*-smooth if for all i and $y, x \in \mathbb{R}^d$, $f_i(x)$ in Eqn. (1.1) satisfies

$$f_i(y) \le f_i(x) + \langle \nabla f_i(x), y - x \rangle + \frac{M}{2} ||y - x||^2.$$

Local ν -Strong Convexity We say the problem is locally ν -strongly convex if for all i and $y, x \in \mathbb{R}^d$, $f_i(x)$ in Eqn. (1.1) satisfies

$$f_i(y) \ge f_i(x) + \langle \nabla f_i(x), y - x \rangle + \frac{\nu}{2} \|y - x\|^2.$$

Based on the smoothness and strong convexity, we can define global and local condition number of the objective function respectively as follows:

$$\kappa_g = \frac{L}{\mu} \quad \text{and} \quad \kappa_\ell = \frac{M}{\nu}.$$
(3.3)

It is well known that

$$L \le M$$
, and $\kappa_q \le \kappa_\ell$. (3.4)

For many applications, κ_{ℓ} can be significantly larger than κ_{g} . Therefore it is desirable to investigate methods that depend on κ_{g} instead of κ_{ℓ} . In fact, some applications contain nonconvex individual functions $f_{i}(x)$, although the global f(x) is convex. For such applications, κ_{ℓ} are not well-defined.

3.2 Topology of Network

Let W be the weight matrix associated with the network, indicating how agents are connected to each other. We assume that the weight matrix W has the following properties:

- 1. W is symmetric with $W_{i,j} \neq 0$ if and if only agents i and j are connected or i = j.
- 2. $0 \le W \le I$, W1 = 1, null(I W) = span(1).

We use I to denote the $m \times m$ identity matrix and $\mathbf{1} = [1, \dots, 1]^{\top} \in \mathbb{R}^m$ denotes the vector with all ones. Many examples of weight W satisfy above properties, such as $W = I - \frac{\mathbf{L}}{\lambda_1(\mathbf{L})}$, where \mathbf{L} is the Laplacian matrix associated with a weighted graph and $\lambda_1(\mathbf{L})$ is the largest eigenvalue of \mathbf{L} .

The weight matrix has an important property that $W^{\infty} = \frac{1}{m} \mathbf{1} \mathbf{1}^{\top}$ (Xiao & Boyd, 2004). Thus, one can achieve the effect of averaging local x_i on different agents by multiple steps of local communications, where each round of local communication starts with a vector \mathbf{x} , and results in the

Algorithm 1 Mudag

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1: Input: \mathbf{y}_{0} = \mathbf{x}_{0} = \mathbf{y}_{-1} = \mathbf{1}\bar{x}_{0}, \nabla F(\mathbf{y}_{t-1}) = 0, \ \eta = \frac{1}{L}, \ \text{and} \ \alpha = \sqrt{\frac{\mu}{L}}, \ K = \mathcal{O}\left(\frac{1}{\sqrt{1-\lambda_{2}(W)}}\log(\frac{M}{L}\kappa_{g})\right).

2: for t = 0, \dots, T do

3: \mathbf{x}_{t+1} = \operatorname{FastMix}(\mathbf{y}_{t} + (\mathbf{x}_{t} - \mathbf{y}_{t-1}) - \eta(\nabla F(\mathbf{y}_{t}) - \nabla F(\mathbf{y}_{t-1})), \ K)

4: \mathbf{y}_{t+1} = \mathbf{x}_{t+1} + \frac{1-\alpha}{1+\alpha}(\mathbf{x}_{t+1} - \mathbf{x}_{t})

5: end for

6: Output: \bar{x}_{T}.
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Algorithm 2 FastMix

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1: Input: \mathbf{x}^0 = \mathbf{x}^{-1}, K, W, step size \eta_w = \frac{1 - \sqrt{1 - \lambda_2^2(W)}}{1 + \sqrt{1 - \lambda_2^2(W)}}.

2: for k = 0, \dots, K do

3: \mathbf{x}^{k+1} = (1 + \eta_w)W\mathbf{x}^k - \eta_w\mathbf{x}^{k-1};

4: end for

5: Output: \mathbf{x}^K.
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vector W**x**. Instead of directly multiplying W several times, Liu & Morse (2011) proposed a more efficient way to achieve averaging described in Algorithm 2, and this method is one pillar for our decentralized algorithms. Algorithm 2 has the following important proposition.

Proposition 1 (Proposition 3 of (Liu & Morse, 2011)). Let \mathbf{x}^K be the output of Algorithm 2 and $\bar{x} = \frac{1}{m} \mathbf{1}^{\top} \mathbf{x}^0$. Then it holds that

$$\bar{x} = \frac{1}{m} \mathbf{1}^{\top} \mathbf{x}^{K}, \quad and \quad \left\| \mathbf{x}^{K} - \mathbf{1} \bar{x} \right\| \leq \left(1 - \sqrt{1 - \lambda_{2}(W)} \right)^{K} \left\| \mathbf{x}^{0} - \mathbf{1} \bar{x} \right\|,$$

where $\lambda_2(W)$ is the second largest eigenvalue of W.

4 Multi-consensus Decentralized Accelerated Gradient Descent

In this section, we propose a novel decentralized gradient descent algorithm achieving the optimal computation complexity and near optimal communication complexity. First, we provide the main idea behind our algorithm.

4.1 Algorithm and Main Idea

Our algorithm bases on the multi-consensus, gradient-tracking and Nesterov's accelerated gradient descent. For the convenience of introducing the main intuition, we reformulate the algorithmic

procedure of Mudag (Algorithm 1) as follows:

$$\mathbf{x}_{t+1} = \operatorname{FastMix} \left(\mathbf{y}_t - \mathbf{s}_t, K \right) \tag{4.1}$$

$$\mathbf{y}_{t+1} = \mathbf{x}_{t+1} + \frac{1-\alpha}{1+\alpha} (\mathbf{x}_{t+1} - \mathbf{x}_t)$$

$$\tag{4.2}$$

$$\mathbf{s}_{t+1} = \operatorname{FastMix}(\mathbf{s}_t, K) + \eta(\nabla F(\mathbf{y}_{t+1}) - \nabla F(\mathbf{y}_t)) - (\operatorname{FastMix}(\mathbf{y}_t, K) - \mathbf{y}_t), \tag{4.3}$$

where η is the step size and K is the step number in multi-consensus. We will prove the above reformulation in Lemma 1. We can observe that Eqn. (4.1) and (4.2) belong to the algorithmic framework of Nesterov' accelerated gradient descent if \mathbf{s}_t/η can approximate the average gradient. In Eqn. (4.3), we track the gradient using history information and the gradient difference. Such \mathbf{s}_t/η can well approximate the average gradient $\mathbf{1}\bar{g}_t$ (defined in Eqn. (3.2)). Furthermore, \mathbf{y}_t can also approximate $\mathbf{1}\bar{y}_t$ well. Since \bar{y}_t and \bar{g}_t can be well approximated, then we can obtain that $\bar{g}_t \approx \nabla f(\bar{y}_t)$. Thus, the convergence properties of our algorithm are similar to the centralized Nesterov's accelerated gradient descent. This is the main idea behind our approach to the decentralized optimization. That is, we combine multi-consensus with gradient-tracking to approximate the centralized Nesterov's accelerated gradient descent. As we will show, this seemingly simple idea leads to near optimal algorithm for the decentralized optimization. Next, we will describe in details the three components in our approach, 'multi-consensus', 'gradient-tracking' and 'approximation to centralized algorithm'.

It is well known that the centralized Nesterov's accelerated gradient descent method can achieve an optimal computation complexity $\mathcal{O}(\sqrt{\kappa_g}\log\frac{1}{\epsilon})$. Then our decentralized algorithm can also achieve the optimal convergence rate once it can approximate the centralized Nesterov's accelerated gradient descent. To achieve such approximation, we resort to multi-consensus and gradient tracking. Several works have tried to approximate the average gradient \bar{g}_t and average variable \bar{y}_t only by multi-consensus and they require significant communication cost (Li et al., 2018; Jakovetić et al., 2014). This is the reason why existing multi-consensus algorithms can not achieve optimal communication complexity. Consequently, multi-consensus is regarded as communication-unfriendly (Qu & Li, 2019).

By combining multi-consensus with gradient-tracking, we can obtain an accurate approximation to the average gradient and average variable with constant steps of multi-consensus. This means the proposed approach can well approximate the centralized Nesterov' accelerated gradient descent, and this critical observation leads to the establishment of optimal computation complexity and near optimal communication complexity in our paper. Furthermore, the idea of approximating the centralized algorithm brings several extra important benefits. First, our algorithm does not require each $f_i(x)$ to be strongly convex. Second, the computation and communication complexities of our algorithm depend on the global condition number κ_g instead of κ_ℓ .

4.2 Complexity Analysis

In this work, we consider synchronized computation, where the computation complexity depends on the number of times that the gradient of f(x) is computed. The communication complexity depends on the times of local communication which is presented as $W\mathbf{x}$ in our algorithm. Now we give the detailed computation complexity and communication complexity of our algorithm in the following theorem.

Theorem 1. Let f(x) be L-smooth and μ -strongly convex. Assume each $f_i(x)$ is M-smooth. Letting K satisfy that

$$K = \sqrt{\frac{\kappa_g}{1 - \lambda_2(W)}} \log \rho^{-1}, \text{ with } \sqrt{\rho} \le \frac{\mu \alpha}{2304L} \cdot \min \left\{ \frac{2L}{M\Theta}, \frac{L^2}{M^2\Theta^2} \right\},$$

where $\Theta = 1 + \frac{\mu}{288m} \cdot \frac{\|\nabla f(\bar{x}_0) - \nabla f(x^*)\|^2}{f(\bar{x}_0) - f(x^*) + \frac{\mu}{2} \|\bar{x}_0/\alpha - x^*\|^2}$, then, it holds that

$$f(\bar{x}_T) - f(x^*) \le \left(1 - \frac{\alpha}{2}\right)^T \left(f(\bar{x}_0) - f(x^*) + \frac{\mu \|\bar{x}_0 - x^*\|^2}{2}\right).$$

To achieve $f(\bar{x}_T) - f(x^*) \le \epsilon$ and $\|\mathbf{x}_T - \mathbf{1}x^*\|^2 = \mathcal{O}(m\epsilon/\mu)$, the computation and communication complexities of Algorithm 1 are

$$T = \mathcal{O}\left(\sqrt{\kappa_g}\log\left(\frac{1}{\epsilon}\right)\right), \quad and \quad Q = \mathcal{O}\left(\sqrt{\frac{\kappa_g}{1 - \lambda_2(W)}}\log\left(\frac{M}{L}\kappa_g\right)\log\frac{1}{\epsilon}\right),$$

where each $\mathcal{O}(\cdot)$ contains a universal constant.

Remark 1. Theorem 1 shows that our algorithm achieves the same computation complexity as that of the centralized Nesterov's accelerated gradient descent. At the same time, the communication complexity almost matches the known lower bound of decentralized optimization problem up to a factor of $\log\left(\frac{M}{L}\kappa_g\right)$. We conjecture that it may be possible to remove the $\log(\kappa_g)$ factor, because the term only comes from the inequality $\|\bar{y}_t - x^*\| \leq \sqrt{\frac{2}{\mu}V_t}$ (V_t is defined in Eqn. (5.1)) in the proof, which may be loose.

Remark 2. Theorem 1 only assumes that f(x) is μ -strongly convex and L-smooth, and $f_i(x)$ is M-smooth (note that unlike many previous works, our dependency on M is logarithmic only). Thus, our algorithm can be used in the case where $f_i(x)$ can be non-convex. This kind of problem has been widely studied in the recent years (Allen-Zhu, 2018; Garber et al., 2016) and one important example is the fast PCA by shift-invert method (Garber et al., 2016). In the previous works such as (Scaman et al., 2017; Li et al., 2018, 2019; Qu & Li, 2019), each $f_i(x)$ is assumed to be convex or even strongly convex in order to prove linear convergence. Therefore, our algorithm has a wider range of applications.

Remark 3. The computation and communication complexities of our algorithm depend linearly on $\sqrt{\kappa_g}$ rather than $\sqrt{\kappa_\ell}$. This result is new. In fact, before this work, it was unknown whether there exists a decentralized algorithm that can achieve a communication complexity close to $\mathcal{O}\left(\sqrt{\frac{\kappa_g}{1-\lambda_2(W)}}\log(\frac{1}{\epsilon})\right)$ (Scaman et al., 2017).

Methods	Complexity of computation	Complexity of communication	$f_i(x)$ being convex?
Acc-DNGD (Qu & Li, 2019)	$\mathcal{O}\left(rac{\kappa_\ell^{5/7}}{(1-\lambda_2(W))^{1.5}}\lograc{1}{\epsilon} ight)$	$\mathcal{O}\left(rac{\kappa_\ell^{5/7}}{(1-\lambda_2(W))^{1.5}}\lograc{1}{\epsilon} ight)$	Yes
NIDS (Li et al., 2019)	$\mathcal{O}\left(\max\{\kappa_{\ell}, \frac{1}{1-\lambda_2(W_2)}\}\log\frac{1}{\epsilon}\right)$	$\mathcal{O}\left(\max\{\kappa_{\ell}, \frac{1}{1-\lambda_{2}(W_{2})}\}\log\frac{1}{\epsilon}\right)$	Yes
ADA (Uribe et al., 2018)	$\mathcal{O}\left(\frac{\kappa_{\ell}}{\sqrt{1-\lambda_2(W)}}\log^2\frac{1}{\epsilon}\right)$	$\mathcal{O}\left(\sqrt{rac{\kappa_\ell}{1-\lambda_2(W)}}\lograc{1}{\epsilon} ight)$	Yes
APM-C (Li et al., 2018)	$O\left(\sqrt{\kappa_\ell}\log\frac{1}{\epsilon}\right)$	$\mathcal{O}\left(\sqrt{rac{\kappa_\ell}{1-\lambda_2(W)}}\log^2rac{1}{\epsilon} ight)$	Yes
Our Method	$\mathcal{O}\left(\sqrt{\kappa_g}\log rac{1}{\epsilon} ight)$	$\mathcal{O}\left(\sqrt{\frac{\kappa_g}{1-\lambda_2(W)}}\log(\frac{M}{L}\kappa_g)\log\frac{1}{\epsilon}\right)$	No
Lower Bound (Scaman et al., 2017)	$\mathcal{O}\left(\sqrt{\kappa_g}\log\frac{1}{\epsilon}\right)$	$\mathcal{O}\left(\sqrt{\frac{\kappa_g}{1-\lambda_2(W)}}\log\frac{1}{\epsilon}\right)^1$	No

Table 1: Complexity comparisons between the our algorithm and existing work for smooth and strongly convex problems. Each $f_i(x)$ is M-smooth, f(x) is L-smooth and μ -strongly convex. W is the matrix describing the topology of the network.

4.3 Comparison to Previous Works

Theorem 1 shows that Algorithm 1 can achieve the optimal computation complexity and near optimal communication complexity. Before our work, APM-C established a computation complexity $\mathcal{O}\left(\sqrt{\kappa_\ell}\log\frac{1}{\epsilon}\right)$ but with a communication complexity $\mathcal{O}\left(\sqrt{\frac{\kappa_\ell}{1-\lambda_2(W)}}\log^2\frac{1}{\epsilon}\right)$, which does not match the lower bound $\mathcal{O}\left(\sqrt{\frac{\kappa_g}{1-\lambda_2(W)}}\log\frac{1}{\epsilon}\right)$. APM-C also resorts to the Nesterov's acceleration and multiconsensus. Due to the lack of tracking historic gradient information, more and more multi-consensus steps are required to obtain accurate estimates of the average gradients in APM-C. This strategy is communication-inefficient and will bring a large communication burden. Furthermore, Qu & Li (2017) proved that increased approximation precision is needed for directly approximating the average gradient only by multi-consensus. This means that $\mathcal{O}\left(\sqrt{\frac{\kappa_\ell}{1-\lambda_2(W)}}\log^2\frac{1}{\epsilon}\right)$ is almost the best communication complexity that multi-consensus based gradient approximation can achieve without gradient tracking.

The work most related to our algorithm is Acc-DNGD proposed in (Qu & Li, 2019) which also utilized the Nesterov's acceleration and gradient-tracking. However, there are several important differences between our algorithm and Acc-DNGD. First and importantly, the idea behind their algorithm is different from that of ours. Our algorithm tries to approximate the centralized Nesterov's accelerated gradient descent. Therefore the convergence analysis of our algorithm is almost the same as the standard analysis of centralized Nesterov's accelerated gradient descent (Nesterov, 2018). Instead, Acc-DNGD tries to 'imitate' the centralized Nesterov's accelerated gradient descent and used inexact Nesterov's gradient descent framework (Devolder et al., 2013). Second, in the implementation, Acc-DNGD requires three single-consensus steps compared to one multi-consensus step of Mudag. Third, Mudag can achieve the optimal computation complexity and a near optimal communication complexity. However, Acc-DNGD does not achieve near optimal computation complexity, nor does it achieve near optimal communication complexity. Fourth, our algorithm does not require each individual function $f_i(x)$ to be convex but only requires f(x) to be strongly

convex. On the other hand, the condition that each individual function $f_i(x)$ is convex is required in Acc-DNGD. In fact, we can observe in the experiments (Section 6) that if some of the individual functions $f_i(x)$ are non-convexity, then the performance of Acc-DNGD deteriorates, while the performance of our algorithm is not affected. Finally, the convergence rate of our algorithm depends on the global condition number κ_g , while that of Acc-DNGD depends on the local condition number κ_ℓ . Since κ_ℓ is no smaller than κ_g , our algorithm has a better convergence guarantee.

Another important related work is NIDS which utilizes a combination of gradient-tracking and single-consensus (Li et al., 2019). Comparing the algorithmic procedures of Algorithm 1, NIDS (Li et al., 2019) and EXTRA (Shi et al., 2015), we can regard Mudag as the accelerated version of NIDS or EXTRA, combined with multi-consensus. This can be clearly observed when we choose $\tilde{W} = \frac{I+W}{2}$ in NIDS and EXTRA. Because of the lack of acceleration, the computation and communication complexities of NIDS and EXTRA are both suboptimal. In fact, Acc-DNGD has less dependency on the condition number κ_{ℓ} than NIDS. Moreover, unlike NIDS, Algorithm 1 does not need to construct an extra consensus matrix \tilde{W} in order to achieve the best performance. Any matrix that satisfies the condition described in Section 3.2 is suitable for Algorithm 1. Furthermore, the convergence analysis of Algorithm 1 is different from that of NIDS. NIDS is analyzed in the primal-dual framework, while our convergence analysis is in the (primal) Nesterov's accelerated gradient descent framework.

In Scaman et al. (2017), a lower bound of communication complexity was obtained for the decentralized optimization problem, which is $\mathcal{O}\left(\sqrt{\frac{\kappa_\ell}{1-\lambda_2(W)}}\log\frac{1}{\epsilon}\right)$ for strongly convex problem. A dual-based algorithm was proposed to match the lower bound. However, this method is only suitable for the cases where dual functions of each local agent are easy to compute. Hence, the computation complexity of the method in Scaman et al. (2017) severely deteriorates once the dual functions are computationally inefficient to work with. Recently, Uribe et al. (2018) proposed an accelerated dual ascent algorithm which achieves the same communication complexity as the one of Scaman et al. (2017), but with a computation complexity of $\mathcal{O}\left(\frac{\kappa_\ell}{\sqrt{1-\lambda_2(W)}}\log^2\frac{1}{\epsilon}\right)$.

We can observe that our algorithm achieves the optimal computation complexity $\mathcal{O}\left(\sqrt{\kappa_g}\log\frac{1}{\epsilon}\right)$, with a near optimal communication complexity $\mathcal{O}\left(\sqrt{\frac{\kappa_g}{1-\lambda_2(W)}}\log(\frac{M}{L}\kappa_g)\log\frac{1}{\epsilon}\right)$. Our complexity results depend on κ_g instead of κ_ℓ . Since κ_ℓ may be much larger than κ_g , an algorithm with communication complexity depending on κ_g is desirable. Therefore, our algorithm is preferred to the previous works whose computation and communication complexities depend on the local condition number κ_ℓ .

Table 1 presents a detailed comparison of our methods with state-of-the-art decentralized optimization algorithms.

5 Convergence Analysis

In Section 4.1, we introduced the main idea behind our algorithm, which is to approximate the centralized Nesterov's accelerated gradient descent (AGD) in a decentralized manner. In this

It holds that $\kappa_g = \Omega(\kappa_\ell)$ for the case used to prove the lower bound of communication complexity (Scaman et al., 2017).

section, we will give a precise characterization on how our algorithm approximates AGD. Similar to the convergence analysis of centralized Nesterov's accelerated gradient descent, we first define the Lyapunov function as follows

$$V_t = f(\bar{x}_t) - f(x^*) + \frac{\mu}{2} \|\bar{v}_t - x^*\|^2,$$
(5.1)

where \bar{v}_t is defined as

$$\bar{v}_t = \bar{x}_{t-1} + \frac{1}{\alpha}(\bar{x}_t - \bar{x}_{t-1}), \quad \text{with} \quad \alpha = \sqrt{\frac{\mu}{L}}.$$
 (5.2)

In the rest of this section, we will show how the Lyapunov function V_t converges and how multiconsensus and gradient-tracking help us to approximate AGD in the proposed algorithm.

First, to obtain a clear convergence analysis of Algorithm 1, we rewrite the update rule of Algorithm 1 in the following form.

Lemma 1. The update procedure of Algorithm 1 can be represented as

$$\mathbf{x}_{t+1} = \operatorname{FastMix} \left(\mathbf{y}_t - \mathbf{s}_t, K \right) \tag{5.3}$$

$$\mathbf{y}_{t+1} = \mathbf{x}_{t+1} + \frac{1-\alpha}{1+\alpha} (\mathbf{x}_{t+1} - \mathbf{x}_t) \tag{5.4}$$

$$\mathbf{s}_{t+1} = \operatorname{FastMix}(\mathbf{s}_t, K) + \eta(\nabla F(\mathbf{y}_{t+1}) - \nabla F(\mathbf{y}_t)) - (\operatorname{FastMix}(\mathbf{y}_t, K) - \mathbf{y}_t), \tag{5.5}$$

with $\mathbf{s}_0 = \eta \nabla F(\mathbf{y}_0)$.

Proof. For notation convenience, we use $\mathbb{T}(\mathbf{x})$ to denote the 'FastMix' operation on matrix \mathbf{x} , which is used in Algorithm 1. That is,

$$\mathbb{T}(\mathbf{x}) \triangleq \operatorname{FastMix}(\mathbf{x}, K).$$

It is obvious that the 'FastMix' operation $\mathbb{T}(\cdot)$ is linear.

The proof of this reformulation is equivalent to prove that given the reformulation of \mathbf{x}_t , \mathbf{y}_t and \mathbf{s}_t at iteration t, the reformulation of \mathbf{x}_{t+1} holds at iteration t+1. Therefore our induction focuses on \mathbf{x}_{t+1} . First, when t=0, we can obtain that

$$\mathbf{x}_1 = \mathbb{T}(\mathbf{y}_0 - \eta \nabla F(\mathbf{y}_0)) = \mathbb{T}(\mathbf{y}_0 - \mathbf{s}_0), \tag{5.6}$$

which implies that

$$\mathbf{x}_1 - \mathbf{y}_0 = -\mathbb{T}(\mathbf{s}_0) + \mathbb{T}(\mathbf{y}_0) - \mathbf{y}_0.$$

Furthermore, have

$$\mathbf{s}_1 = \mathbb{T}(\mathbf{s}_0) + \eta(\nabla F(\mathbf{y}_2) - \nabla F(\mathbf{y}_1)) - (\mathbb{T}(\mathbf{y}_0) - \mathbf{y}_0).$$

Thus, we can obtain that

$$\mathbf{x}_2 = \mathbb{T}(\mathbf{y}_1 + (\mathbf{x}_1 - \mathbf{y}_0) - \eta(\nabla F(\mathbf{y}_1) - \nabla F(\mathbf{y}_0)))$$

= $\mathbb{T}(\mathbf{y}_1 - (\mathbb{T}(\mathbf{s}_0) + \eta(\nabla F(\mathbf{y}_1) - \nabla F(\mathbf{y}_0))) + \mathbb{T}(\mathbf{y}_0) - \mathbf{y}_0)$
= $\mathbb{T}(\mathbf{y}_1 - \mathbf{s}_1)$.

Thus, the result holds at t = 0.

Next, we prove that if the results hold in the t-th iteration, then it also holds at the (t+1)-th iteration. For the t-th iteration, we assume that $\mathbf{x}_{t+1} = \mathbb{T}(\mathbf{y}_t - \mathbf{s}_t)$, which implies that $\mathbf{x}_{t+1} - \mathbb{T}(\mathbf{y}_t) = -\mathbb{T}(\mathbf{s}_t)$. Therefore, we obtain that

$$\mathbf{x}_{t+2} = \mathbb{T}(\mathbf{y}_{t+1} + \mathbf{x}_{t+1} - \mathbf{y}_t - \eta(\nabla F(\mathbf{y}_{t+1}) - \nabla F(\mathbf{y}_t)))$$

$$= \mathbb{T}(\mathbf{y}_{t+1} + \mathbf{x}_{t+1} - \mathbb{T}(\mathbf{y}_t) - \eta(\nabla F(\mathbf{y}_{t+1}) - \nabla F(\mathbf{y}_t)) + \mathbb{T}(\mathbf{y}_t) - \mathbf{y}_t)$$

$$= \mathbb{T}(\mathbf{y}_{t+1} - \mathbb{T}(\mathbf{s}_t) - \eta(\nabla F(\mathbf{y}_{t+1}) - \nabla F(\mathbf{y}_t)) + \mathbb{T}(\mathbf{y}_t) - \mathbf{y}_t)$$

$$= \mathbb{T}(\mathbf{y}_{t+1} - \mathbf{s}_{t+1}).$$

This proves the desired result.

We now show that \bar{x}_t , \bar{y}_t , \bar{g}_t (defined in Eqn. (3.2) and generated by Algorithm 1) and \bar{v}_t (defined in Eqn. (5.2)) can be fit into the framework of the centralized Nesterov's accelerated gradient descent.

Lemma 2. Let \bar{x}_t , \bar{y}_t , \bar{g}_t (defined in Eqn. (3.2)) be generated by Algorithm 1. Then they satisfy the following equalities:

$$\bar{x}_{t+1} = \bar{y}_t - \eta \bar{g}_t \tag{5.7}$$

$$\bar{y}_{t+1} = \bar{x}_{t+1} + \frac{1-\alpha}{1+\alpha} (\bar{x}_{t+1} - \bar{x}_t) \tag{5.8}$$

$$\bar{s}_{t+1} = \bar{s}_t + \eta \bar{g}_{t+1} - \eta \bar{g}_t = \eta \bar{g}_{t+1} \tag{5.9}$$

Proof. We first prove the last equality. First, by Proposition 1, we have $\frac{1}{m}\mathbf{1}\mathbf{1}^{\top}(\mathbb{T}(\mathbf{y}_t) - \mathbf{y}_t) = \mathbf{1}\bar{y}_t - \mathbf{1}\bar{y}_t = 0$. Thus, we can obtain that

$$\bar{s}_{t+1} = \bar{s}_t + \eta \bar{g}_{t+1} - \eta \bar{g}_t.$$

Furthermore, we will prove $\bar{s}_t = \eta \bar{g}_t$ by induction. For t = 0, we use the fact that $\mathbf{s}_0 = \eta \nabla F(\mathbf{y}_0)$. Then, it holds that $\bar{s}_0 = \eta \bar{g}_0$. We assume that $\bar{s}_t = \eta \bar{g}_t$ at time t. By the update equation, we have

$$\bar{s}_{t+1} = \bar{s}_t + \eta(\bar{g}_{t+1} - \bar{g}_t) = \eta \bar{g}_{t+1}.$$

Thus, we obtain the result at time t+1. The first two equations can be proved using Eqn. (5.9) and Proposition 1.

From Lemma 2, we can observe that Eqn. (5.7)-(5.9) is almost the same as Nesterov's accelerated gradient descent (Nesterov, 2018). Thus, if \bar{s}_t/η is an accurate estimation of $\nabla f(\bar{y}_t)$, then Algorithm 1 has convergence properties similar to AGD. Next, we are going to show $\mathbf{y}_t(i,:) \approx \bar{y}_t$ and $\mathbf{s}_t(i,:) \approx \bar{s}_t$ and we have the following lemma.

Lemma 3. Let $\mathbf{z}_t = \left[\|\mathbf{y}_t - \mathbf{1}\bar{y}_t\|, \frac{1}{\rho} \|\mathbf{x}_t - \mathbf{1}\bar{x}_t\|, \frac{1}{M\eta} \|\mathbf{s}_t - \mathbf{1}\bar{s}_t\| \right]^{\top}$, then it holds that

$$\mathbf{z}_{t+1} \le \mathbf{A}\mathbf{z}_t + 4\sqrt{m} \left[0, 0, \sqrt{\frac{2}{\mu}V_t} \right]^\top,$$

where ρ and **A** are defined as

$$\rho = \left(1 - \sqrt{1 - \lambda_2(W)}\right)^K, \quad \mathbf{A} \triangleq \begin{bmatrix} 2\rho & \rho & 2\rho M\eta \\ 1 & 0 & M\eta \\ 7 + 2M\eta & \rho & \rho(1 + 2\rho M\eta) \end{bmatrix}.$$

Furthermore, we have

$$\mathbf{z}_{t+1} \le \mathbf{A}^{t+1} \mathbf{z}_0 + 4\sqrt{\frac{2m}{\mu}} \cdot \sum_{i=0}^{t} \mathbf{A}^{t-i} [0, 0\sqrt{V_i}]^{\top}.$$
 (5.10)

If the spectrum radius of **A** is less than 1, and V_t converges to zero, $\|\mathbf{z}_t\|$ will converge to zero. Note that $\|\mathbf{y}_t - \mathbf{1}\bar{y}_t\|$ and $\frac{1}{M\eta}\|\mathbf{s}_t - \mathbf{1}\bar{s}_t\|$ are no larger than $\|\mathbf{z}_t\|$. This implies that $\|\mathbf{y}_t - \mathbf{1}\bar{y}_t\|$ and $\frac{1}{M\eta}\|\mathbf{s}_t - \mathbf{1}\bar{s}_t\|$ will also converge to zero. That is, Algorithm 1 can approximate AGD well.

Next, we will prove the above two conditions which guarantee the convergence of $\|\mathbf{z}_t\|$. In the following lemma, we show the properties of \mathbf{A} and prove that the spectrum radius of \mathbf{A} is less than $\frac{1}{2}$ if ρ is small enough.

Lemma 4. Matrix A defined in Lemma 3 satisfies that

$$0 < \lambda_1(\mathbf{A}), \quad |\lambda_3(\mathbf{A})| \le |\lambda_2(\mathbf{A})| < \lambda_1(\mathbf{A}),$$

with $\lambda_i(\mathbf{A})$ being the i-th largest eigenvalue of \mathbf{A} . Let $\eta = \frac{1}{L}$ and ρ satisfy the condition that

$$\rho \le \frac{1}{2(21M\eta + 6M^2\eta^2 + 1)(3 + 2M\eta)},$$

then it holds that

$$\lambda_1(\mathbf{A}) \leq \frac{1}{2},$$

and the eigenvector \mathbf{v} associated with $\lambda_1(\mathbf{A})$ is positive and its entries satisfy

$$\mathbf{v}(1) \le \frac{\mathbf{v}(3)}{2(7+2M\eta)}, \quad \mathbf{v}(2) \le \left(\frac{1}{2\sqrt{\rho}(7+2M\eta)} + \frac{M\eta}{\sqrt{\rho}}\right)\mathbf{v}(3), \quad 0 < \mathbf{v}(3),$$

with $\mathbf{v}(i)$ being i-th entry of \mathbf{v} .

Using Lemma 4, we are going to show that V_t converges to zero and we have the following lemma.

Lemma 5. Assume that ρ satisfies the properties in Lemma 4, and $\Theta = 1 + \frac{\mu}{288m} \cdot \frac{\|\mathbf{z}_0\|^2}{V_0}$, we have

$$\sqrt{\rho} \le \frac{\mu\alpha}{2304L} \cdot \min\left\{\frac{2L}{M\Theta}, \frac{L^2}{M^2\Theta^2}\right\}.$$

Assuming also that $\alpha \leq \frac{1}{2}$, then Algorithm 1 has the following convergence rate

$$V_{t+1} \le \left(1 - \frac{\alpha}{2}\right)^{t+1} \cdot V_0. \tag{5.11}$$

Lemma 5 shows that V_t will converge to 0 with rate $1 - \frac{\alpha}{2}$. Thus, we can directly obtain the computation complexity. Furthermore, to achieve the conditions on ρ , we upper bound the multi-consensus step K by Proposition 1. Combining with the computation complexity, we can obtain the total communication complexity. Now, we give the detailed proof of Theorem 1 as follows.

Proof of Theorem 1. It is easy to check that ρ satisfies the conditions required in Lemma 4 and 5. By Eqn. (5.11), we have

$$f(\bar{x}_T) - f(x^*) \le \left(1 - \frac{1}{2}\sqrt{\frac{\mu}{L}}\right)^T \left(f(\bar{x}_0 - f(x^*)) + \frac{\mu}{2} \|\bar{x}_0 - x^*\|^2\right)$$
$$\le \exp\left(-\frac{T}{2}\sqrt{\frac{\mu}{L}}\right) \left(f(\bar{x}_0 - f(x^*)) + \frac{\mu}{2} \|\bar{x}_0 - x^*\|^2\right)$$

Thus, in order to achieve $f(\bar{x}_T - x^*) \leq \epsilon$, T only needs to be

$$T = 2\sqrt{\kappa_g} \log \frac{f(\bar{x}_0 - f(x^*)) + \frac{\mu}{2} \|\bar{x}_0 - x^*\|^2}{\epsilon} = \mathcal{O}\left(\sqrt{\kappa_g} \log \frac{1}{\epsilon}\right).$$

Furthermore, by Lemma 3, we have

$$\|\mathbf{x}_{T} - \mathbf{1}\bar{x}_{T}\|^{2} \leq (\rho \mathbf{z}_{T}(2))^{2} \leq (\rho \mathbf{v}(2))^{2} \cdot \left(1 - \frac{\alpha}{2}\right)^{T} \left(12\sqrt{\frac{2m}{\mu}}\sqrt{V_{0}} + \|\mathbf{z}_{0}\|\right)^{2}$$
$$\leq \frac{1152m\rho M\eta}{\mu} \cdot \epsilon \leq \frac{28m}{\mu} \cdot \epsilon,$$

where the last inequality is because of the condition of ρ in Lemma 4. Therefore, we have

$$\|\mathbf{x}_{T} - \mathbf{1}x^{*}\|^{2} \leq 2(\|\mathbf{x}_{T} - \mathbf{1}\bar{x}_{T}\|^{2} + \|\mathbf{1}(\bar{x}_{T} - x^{*})\|^{2})$$

$$\leq \frac{56m}{\mu} \cdot \epsilon + \frac{4m}{\mu} \cdot \epsilon$$

$$= \mathcal{O}(m\epsilon/\mu),$$

where the second inequality is due to the μ -strong convexity of f(x).

The bound of K can be obtained by Proposition 1. Combining with the computation complexity,

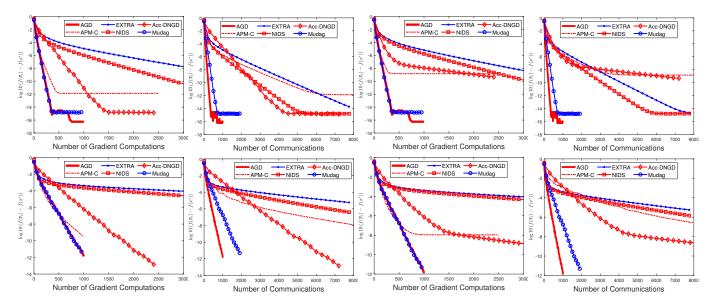


Figure 1: Comparisons with logistic regression and random networks. Each $f_i(x)$ is strongly convex $(\sigma_i = 0.001)$ in the top row, and $\sigma_i = 0.0001$ in the bottom row). Random networks have $1 - \lambda_2(W) = 0.81$ in the left two columns and $1 - \lambda_2(W) = 0.05$ in the right two columns.

we can obtain the total communication complexity as

$$Q = \mathcal{O}\left(\sqrt{\frac{\kappa_g}{1 - \lambda_2(W)}}\log\left(\frac{M}{L}\kappa_g\right)\log\frac{1}{\epsilon}\right).$$

6 Experiments

In the previous section, we presented a theoretical analysis of our algorithm. In this section, we will provide empirical studies. We evaluate the performance of our algorithm using logistic regression with different settings, including the situation that each $f_i(x)$ is strongly convex and the situation that each $f_i(x)$ may be non-convex.

6.1 The Setting of Networks

In our experiments, we consider random networks where each pair of agents have a connection with a probability of p, and we set $W = I - \frac{\mathbf{L}}{\lambda_1(\mathbf{L})}$ where \mathbf{L} is the Laplacian matrix associated with a weighted graph, and $\lambda_1(\mathbf{L})$ is the largest eigenvalue of \mathbf{L} . We set m = 100, that is, there exists 100 agents in this network. By the properties of the well-known Erdős-Rényi random graph, we can obtain that when $p = \frac{2\log m}{m}$, the random graph is connected and $\frac{1}{1-\lambda_2(W)} = \mathcal{O}(1)$. In our experiments, We test the performance with p = 0.1 and p = 0.5 and observe that $1 - \lambda_2(W) = 0.05$ and $1 - \lambda_2(W) = 0.81$, respectively.

6.2 Experiments on Logistic Regression

The individual objective function of logistic regression is defined as

$$f_i(x) = \frac{1}{n} \sum_{j=1}^n \log[1 + \exp(-b_j \langle a_j, x \rangle)] + \frac{\sigma_i}{2} ||x||^2,$$

where $a_j \in \mathbb{R}^d$ is the j-th input vector, and $b_j \in \{-1,1\}$ is the corresponding label. We conduct our experiments on a real-world dataset 'a9a' which can be downloaded from Libsvm datasets. We set n=325 and d=123. To test the performance of our algorithm on the strongly convex function, we set $\sigma_i=10^{-3}$ and $\sigma_i=10^{-4}$ for $i=1,\ldots,m$ to control the condition number of the objective function f(x). Furthermore, we will also test the performance of our algorithm on the case that $f_i(x)$ may be non-convex but f(x) is strongly convex. Here, we set $\sigma_i=-10^{-2}$ for agents $i=1\ldots,m-1$ and $\sigma_m=1$ for agent m, respectively. In this case, the condition number of f(x) is the same as the one with setting $\sigma_i=10^{-4}$ for all $i=1,\ldots,m$. We also set $\sigma_i=-10^{-1}$ for agents $i=1\ldots,m-1$ and $\sigma_m=10$ for agent m. In this case, the condition number of f(x) is the same as the one with setting $\sigma_i=10^{-3}$ for all agents.

We compare our algorithm (Mudag) to centralized accelerated gradient descent (AGD) in (Nesterov, 2018), EXTRA in (Shi et al., 2015), NIDS in (Li et al., 2019), Acc-DNGD in (Qu & Li, 2019) and APM-C in (Li et al., 2018). In this paper, we do not compare our algorithm to the dual-based algorithms such as accelerated dual ascent algorithm (Uribe et al., 2018; Scaman et al., 2017) because these algorithms can not be applied to the case where some functions $f_i(x)$ are non-convex.

The step sizes of all algorithms are well-tuned to achieve their best performances. Furthermore, we set the momentum coefficient as $\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}$ for Mudag, AGD and APM-C. We initialize \mathbf{x}_0 at $\mathbf{0}$ for all the compared methods.

For the setting that each $f_i(x)$ is a strongly convex function, we report the experimental results in Figure 1. First, compared to AGD, our algorithm has almost the same computation cost as that of AGD, which matches our theoretical analysis. Assuming that AGD communicates once per iteration, we can also see that the communication cost of Mudag is almost the same as that of AGD when $1 - \lambda_2(W) = 0.81$, and six times that of AGD when $1 - \lambda_2(W) = 0.05$. This matches the communication complexity of our algorithm. Furthermore, our algorithm achieves both lower computation cost and lower communication cost than other decentralized algorithms on all settings. The advantages are more obvious when the regularization parameters σ_i are small, which validates the theoretical comparison in Section 4.3.

We report the results of our experiments where the individual function $f_i(x)$ can be non-convex but f(x) is strongly convex in Figure 2. Note that the settings of experiments reported in Figure 1 and Figure 2 are the same except for the non-convexity of some $f_i(x)$'s in Figure 2. Comparing the curves in these two figures, we can observe that the computation cost of AGD and our algorithm are not affected by the non-convexity of $f_i(x)$ because their convergence rates only depend on $\sqrt{\kappa_g}$. On the other hand, the communication cost of our algorithm increases slightly compared to the setting where each $f_i(x)$ is convex. This is because the ratio M/L of $f_i(x)$ increases when we set $\sigma_i = -10^{-1}$ or $\sigma_i = -10^{-2}$ for agent $i = 1, \ldots, m-1$. Our communication complexity theory

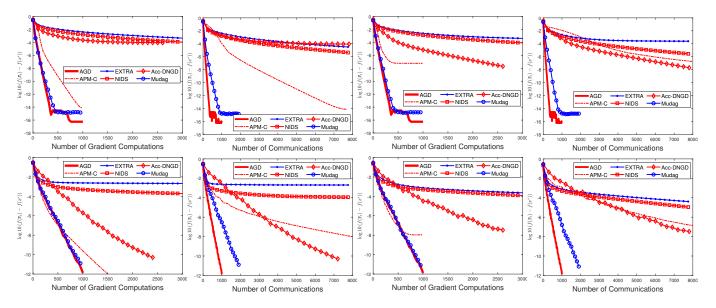


Figure 2: Comparisons with logistic regression and random networks. Each local objective $f_i(x)$ may be non-convex. In the top row, $\sigma_i = -10^{-2}$ for agents $i = 1 \dots, m-1$ and $\sigma_i = 1$ for the agent i = m. In the bottom row, $\sigma_i = -10^{-1}$ for agents $i = 1 \dots, m-1$, and $\sigma_i = 10$ for the agent i = m. Random networks have $1 - \lambda_2(W) = 0.81$ in the left two columns and $1 - \lambda_2(W) = 0.05$ in the right two columns.

shows M/L will affect the communication cost by a $\log(M/L)$ factor. Compared to our algorithm, the performance of the other decentralized algorithms deteriorates greatly, which can be clearly observed by comparing the two figures in top right corners of Figure 1 and Figure 2. When some of $f_i(x)$ are non-convex, EXTRA, NIDS, and Acc-DNGD perform rather poorly.

7 Conclusion

In this paper, we proposed a novel decentralized algorithm called Mudag. Our method can achieve the optimal computation complexity with a near optimal communication complexity, matching the lower bound up to a $\log(\frac{M}{L}\kappa_g)$ factor. This is the best communication complexity that primal-based decentralized algorithms can achieve.

Our results provide an affirmative answer to the open problem on whether there is a decentralized algorithm that can achieve the communication complexity $\mathcal{O}\left(\sqrt{\frac{\kappa_g}{1-\lambda_2(W)}}\log\frac{1}{\epsilon}\right)$ or even close to this lower bound for a strongly convex objective function. Furthermore, our algorithm does not require each individual functions $f_i(x)$ to be convex. However, this requirement is necessary for existing decentralized algorithms. In fact, our experiments showed that the non-convexity of individual function $f_i(x)$ can degrade their performances. Our new algorithm has a wider range of applications in machine learning because the individual functions $f_i(x)$ may not be convex in many machine learning problems.

Our analysis also implies that for decentralized optimization, multi-consensus and gradient track-

ing can be combined to well approximate the corresponding centralized counterpart. The resulting methods are simple and effective, with near optimal complexities. This novel point of view may also provide useful insights for developing new decentralized algorithms in other situations.

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A Proof of Proposition 1

Proof of Proposition 1. By the update rule of Algorithm 2 and the fact that $W^{\infty} = \frac{1}{m} \mathbf{1} \mathbf{1}^{\top}$ (Xiao & Boyd, 2004), we have

$$W^{\infty}\mathbf{x}^{K} = W^{\infty}\mathbf{x}^{K-1} + \eta_{w}\left(W^{\infty}\mathbf{x}^{K-1} - W^{\infty}\mathbf{x}^{K-2}\right).$$

We can obtain that

$$W^{\infty}\left(\mathbf{x}^{K}-\mathbf{x}^{K-1}\right)=\eta_{w}\left(W^{\infty}\mathbf{x}^{K-1}-W^{\infty}\mathbf{x}^{K-2}\right).$$

Note that $\mathbf{x}^0 = \mathbf{x}^{-1}$ in Algorithm 2, we can obtain that for any $k = 0, \dots, K$, we have

$$W^{\infty}\left(\mathbf{x}^k - \mathbf{x}^{k-1}\right) = 0.$$

Therefore, we can obtain the identity $W^{\infty}\mathbf{x}^K = W^{\infty}\mathbf{x}^0$, which implies the result. The convergence rate of Algorithm 2 can be found in (Liu & Morse, 2011).

B Several Important Lemmas

We also have the following important properties of the update rule.

Lemma 6. We have the following inequalities:

$$\|\nabla F(\mathbf{y}) - \nabla F(\mathbf{x})\| \le M \|\mathbf{y} - \mathbf{x}\|,$$
 (B.1)

$$\|\bar{g}_t - \nabla f(\bar{y}_t)\| \le \frac{M}{\sqrt{m}} \|\mathbf{y}_t - \mathbf{1}\bar{y}_t\|.$$
(B.2)

Proof. The first inequality is because F(x) is M-smooth and

$$\|\nabla F(\mathbf{y}) - \nabla F(\mathbf{x})\| = \sqrt{\sum_{i}^{m} \|\nabla f_{i}(\mathbf{y}(i,:)) - \nabla f_{i}(\mathbf{x}(i,:))\|^{2}}$$

$$\leq \sqrt{M^{2} \sum_{i}^{m} \|\mathbf{y}(i,:) - \mathbf{x}(i,:)\|^{2}}$$

$$= M \|\mathbf{y} - \mathbf{x}\|.$$

The second inequality follows from

$$\|\bar{g}_{t} - \nabla f(\bar{y}_{t})\| = \left\| \frac{1}{m} \sum_{i=0}^{m} \left[\nabla f_{i}(\mathbf{y}_{t}(i,:)) - \nabla f_{i}(\bar{y}_{t}) \right] \right\|$$

$$= \left\| \sum_{i=1}^{m} \frac{\nabla f_{i}(\mathbf{y}_{t}(i,:)) - \nabla f_{i}(\bar{y}_{t})}{m} \right\|$$

$$\leq M \sum_{i=0}^{m} \frac{\|\mathbf{y}_{t}(i,:) - \bar{y}_{t}\|}{m}$$

$$\leq M \sqrt{\sum_{i=0}^{m} \frac{\|\mathbf{y}_{t}(i,:) - \bar{y}_{t}\|^{2}}{m}}$$

$$= M \frac{1}{\sqrt{m}} \|\mathbf{y}_{t} - \mathbf{1}\bar{y}_{t}\|.$$

By the convergence rate of accelerated gradient descent, we have the following property.

Lemma 7. Let \bar{x}_t , \bar{y}_t , \bar{g}_t (defined in Eqn. (3.2)) and \bar{v}_t (defined in Eqn. (5.2)) be generated by Algorithm 1. Then they satisfy the following equalities:

$$\bar{v}_{t+1} = (1 - \alpha)\bar{v}_t + \alpha\bar{y}_t - \frac{\eta}{\alpha}\bar{s}_t, \tag{B.3}$$

$$\bar{y}_{t+1} = \frac{\bar{x}_{t+1} + \alpha \bar{v}_{t+1}}{1 + \alpha}.$$
 (B.4)

Proof. We first prove the second equality. Replacing Eqn. (5.2) to (B.4), we can obtain that

$$\frac{\bar{x}_{t+1} + \alpha \bar{v}_{t+1}}{1 + \alpha} = \frac{\bar{x}_{t+1} + \alpha (\bar{x}_t + \frac{1}{\alpha} (\bar{x}_{t+1} - \bar{x}_t))}{1 + \alpha} \\
= \bar{x}_{t+1} + \frac{1 - \alpha}{1 + \alpha} (\bar{x}_{t+1} - \bar{x}_t) \\
= \bar{y}_{t+1}.$$

We are going to prove the first equality. First, by Eqn. (B.4), we can obtain that

$$\bar{y}_t - \bar{x}_t = \alpha(\bar{v}_t - \bar{y}_t).$$

Then we have

$$(1 - \alpha)\bar{v}_t + \alpha\bar{y}_t - \frac{\eta}{\alpha}\bar{s}_t = \bar{v}_t - \alpha(\bar{v}_t - \bar{y}_t) - \frac{\eta}{\alpha}\bar{s}_t$$

$$= \bar{x}_t + \bar{v}_t - \bar{y}_t - \frac{\eta}{\alpha}\bar{s}_t$$

$$= \bar{x}_t + \frac{1}{\alpha}(\bar{y}_t - \bar{x}_t - \eta\bar{s}_t)$$

$$= \bar{x}_t + \frac{1}{\alpha}(\bar{x}_{t+1} - \bar{x}_t)$$

$$= \bar{v}_{t+1}.$$

Lemma 8. Consider \mathbf{y}_t in Algorithm 1, then \bar{y}_t satisfies

$$\|\bar{y}_t - x^*\| \le \sqrt{\frac{2}{\mu}V_t}.$$
 (B.5)

Proof. By Eqn. (B.4), we have

$$\|\bar{y}_t - x^*\| = \frac{1}{1+\alpha} \|\bar{x}_t + \alpha \bar{v}_t - (1+\alpha)x^*\| \le \frac{1}{1+\alpha} (\|\bar{x}_t - x^*\| + \alpha \|\bar{v}_t - x^*\|)$$
$$\le \frac{1}{1+\alpha} \left(\sqrt{\frac{2}{\mu} V_t} + \alpha \sqrt{\frac{2}{\mu} V_t} \right) = \sqrt{\frac{2}{\mu} V_t}.$$

The last inequality is because of the condition that f(x) is μ -strongly convex and the definition of V_t .

C Proof of Lemma 3

Proof of Lemma 3. By the update step of \mathbf{y}_{t+1} in Algorithm 1, we have

$$\|\mathbf{y}_{t+1} - \mathbf{1}\bar{y}_{t+1}\| \le \frac{2}{1+\alpha} \|\mathbf{x}_{t+1} - \mathbf{1}\bar{x}_{t+1}\| + \frac{1-\alpha}{1+\alpha} \|\mathbf{x}_t - \mathbf{1}\bar{x}_t\|.$$

Furthermore, by Eqn. (5.3), we have

$$\frac{1}{\rho} \|\mathbf{x}_{t+1} - \mathbf{1}\bar{x}_{t+1}\| \leq \|\mathbf{y}_t - \mathbf{1}\bar{y}_t\| + M\eta \cdot \frac{1}{M\eta} \|\mathbf{s}_t - \mathbf{1}\bar{s}_t\|.$$

Therefore, we can obtain that

$$\|\mathbf{y}_{t+1} - \mathbf{1}\bar{y}_{t+1}\| \leq \frac{2\rho}{1+\alpha} \|\mathbf{y}_{t} - \mathbf{1}\bar{y}_{t}\| + \frac{1-\alpha}{1+\alpha} \|\mathbf{x}_{t} - \mathbf{1}\bar{x}_{t}\| + \frac{2\rho}{1+\alpha} \|\mathbf{s}_{t} - \mathbf{1}\bar{s}_{t}\|$$

$$\leq 2\rho \|\mathbf{y}_{t} - \mathbf{1}\bar{y}_{t}\| + \rho \cdot \frac{1}{\rho} \|\mathbf{x}_{t} - \mathbf{1}\bar{x}_{t}\| + 2\rho M\eta \cdot \frac{1}{M\eta} \|\mathbf{s}_{t} - \mathbf{1}\bar{s}_{t}\|.$$

Furthermore, by Eqn. (5.5), we have

$$\|\mathbf{s}_{t+1} - \mathbf{1}\bar{s}_{t+1}\| = \|\mathbb{T}(\mathbf{s}_{t}) - \mathbf{1}\bar{s}_{t}\| + \eta \|\nabla F(\mathbf{y}_{t+1}) - \nabla F(\mathbf{y}_{t}) - \mathbf{1}(\bar{g}_{t+1} - \bar{g}_{t})\| + \|\mathbb{T}(\mathbf{y}_{t}) - \mathbf{y}_{t}\|$$

$$\leq \|\mathbb{T}(\mathbf{s}_{t}) - \mathbf{1}\bar{s}_{t}\| + \eta \|\nabla F(\mathbf{y}_{t+1}) - \nabla F(\mathbf{y}_{t})\| + \|\mathbb{T}(\mathbf{y}_{t}) - \mathbf{y}_{t}\|$$

$$\leq \rho \|\mathbf{s}_{t} - \mathbf{1}\bar{s}_{t}\| + M\eta \|\mathbf{y}_{t+1} - \mathbf{y}_{t}\| + (\rho + 1) \|\mathbf{y}_{t} - \mathbf{1}\bar{y}_{t}\|,$$

where the first inequality is because for any matrix $\mathbf{x} \in \mathbb{R}^{m \times d}$, it holds that

$$\begin{aligned} \|\mathbf{x} - \mathbf{1}\bar{x}\|^2 &= \sum_{j}^{m} \left\| \mathbf{x}(j,:) - \frac{1}{m} \sum_{i}^{m} \mathbf{x}(i,:) \right\|^2 \\ &= \sum_{j}^{m} \left(\|\mathbf{x}(j,:)\|^2 - \frac{2}{m} \sum_{i}^{m} \left\langle \mathbf{x}(j,:), \mathbf{x}(i,:) \right\rangle + \frac{1}{m^2} \left\| \sum_{i}^{m} [\mathbf{x}(i,:)] \right\|^2 \right) \\ &= \|\mathbf{x}\|^2 - \frac{2}{m} \sum_{i}^{m} \sum_{j}^{m} \left\langle \mathbf{x}(j,:), \mathbf{x}(i,:) \right\rangle + \frac{1}{m} \sum_{i}^{m} \sum_{j}^{m} \left\langle \mathbf{x}(j,:), \mathbf{x}(i,:) \right\rangle \\ &< \|\mathbf{x}\|^2 \,. \end{aligned}$$

The last inequality is because $f_i(x)$ is M-smooth.

By the update rule of \mathbf{y}_{t+1} , we have

$$\|\mathbf{y}_{t+1} - \mathbf{y}_{t}\| = \left\| \frac{2 - \alpha}{1 + \alpha} \mathbf{x}_{t+1} - \frac{1 - \alpha}{1 + \alpha} \mathbf{x}_{t} - \mathbf{y}_{t} \right\|$$

$$\stackrel{(5.3)}{=} \left\| \frac{2 - \alpha}{1 + \alpha} \mathbb{T}(\mathbf{y}_{t} - \mathbf{s}_{t}) - \frac{1 - \alpha}{1 + \alpha} \mathbf{x}_{t} - \mathbf{y}_{t} \right\|$$

$$\leq \frac{2 - \alpha}{1 + \alpha} \left\| \mathbb{T}(\mathbf{y}_{t}) - \mathbf{y}_{t} \right\| + \frac{1 - \alpha}{1 + \alpha} \left\| \mathbf{x}_{t} - \mathbf{y}_{t} \right\| + \frac{2 - \alpha}{1 + \alpha} \left\| \mathbb{T}(\mathbf{s}_{t}) \right\|$$

$$\leq \frac{4}{1 + \alpha} \left\| \mathbf{y}_{t} - 1\bar{y}_{t} \right\| + \frac{1 - \alpha}{1 + \alpha} \left(\left\| \mathbf{x}_{t} - 1\bar{x}_{t} \right\| + \left\| \mathbf{y}_{t} - 1\bar{y}_{t} \right\| + \left\| \mathbf{1}(\bar{y}_{t} - x^{*}) \right\| + \left\| \mathbf{1}(\bar{x}_{t} - x^{*}) \right\| \right)$$

$$+ \frac{2 - \alpha}{1 + \alpha} (\left\| \mathbb{T}(\mathbf{s}_{t}) - 1\bar{s}_{t} \right\| + \left\| \mathbf{1}\bar{s}_{t} \right\|)$$

$$\stackrel{(5.9)}{\leq} \frac{5}{1 + \alpha} \left\| \mathbf{y}_{t} - 1\bar{y}_{t} \right\| + \frac{2\rho}{1 + \alpha} \left\| \mathbf{s}_{t} - 1\bar{s}_{t} \right\| + \frac{1 - \alpha}{1 + \alpha} (\left\| \mathbf{x}_{t} - 1\bar{x}_{t} \right\|)$$

$$+ \frac{1 - \alpha}{1 + \alpha} (\left\| \mathbf{1}(\bar{y}_{t} - x^{*}) \right\| + \left\| \mathbf{1}(\bar{x}_{t} - x^{*}) \right\|) + \frac{2\eta\sqrt{m}}{1 + \alpha} \left\| \bar{g}_{t} \right\|,$$

where the second inequality is because of

$$\|\mathbb{T}(\mathbf{y}_t) - \mathbf{y}_t\| = \|\mathbb{T}(\mathbf{y}_t) - \mathbf{1}\bar{y}_t + \mathbf{1}\bar{y}_t - \mathbf{y}_t\| \le (1+\rho)\|\mathbf{y}_t - \mathbf{1}\bar{y}_t\| \le 2\|\mathbf{y}_t - \mathbf{1}\bar{y}_t\|.$$

Furthermore, by Eqn. (B.2), we have

$$\|\bar{g}_t\| \le \|\bar{g}_t - \nabla f(\bar{y}_t)\| + \|\nabla f(\bar{y}_t)\| \le \frac{M}{\sqrt{m}} \|\mathbf{y}_t - \mathbf{1}\bar{y}_t\| + \|\nabla f(\bar{y}_t)\|.$$

Therefore, we can obtain that

$$\frac{1}{M\eta} \|\mathbf{s}_{t+1} - \mathbf{1}\bar{s}_{t+1}\| \leq \rho(1 + 2\rho M\eta) \frac{1}{M\eta} \|\mathbf{s}_{t} - \mathbf{1}\bar{s}_{t}\| + \left(\frac{5 + 2M\eta}{1 + \alpha} + \frac{\rho + 1}{M\eta}\right) \|\mathbf{y}_{t} - \mathbf{1}\bar{y}_{t}\|
\frac{1 - \alpha}{1 + \alpha} \|\mathbf{x}_{t} - \mathbf{1}\bar{x}_{t}\| + \frac{1 - \alpha}{1 + \alpha} (\|\mathbf{1}(\bar{y}_{t} - x^{*})\| + \|\mathbf{1}(\bar{x}_{t} - x^{*})\|)
+ \frac{2\eta\sqrt{m}}{1 + \alpha} \|\nabla f(\bar{y}_{t})\|
\leq \rho(1 + 2\rho M\eta) \cdot \frac{1}{M\eta} \|\mathbf{s}_{t} - \mathbf{1}\bar{s}_{t}\| + (7 + 2M\eta) \|\mathbf{y}_{t} - \mathbf{1}\bar{y}_{t}\| + \|\mathbf{x}_{t} - \mathbf{1}\bar{x}_{t}\|
+ \|\mathbf{1}(\bar{y}_{t} - x^{*})\| + \|\mathbf{1}(\bar{x}_{t} - x^{*})\| + 2\eta\sqrt{m} \|\nabla f(\bar{y}_{t})\|,$$

where the last inequality uses $1 < 1 + \alpha$, $\eta = \frac{1}{L}$ and $L \le M$. Furthermore, we have

$$\|\mathbf{1}(\bar{y}_{t} - x^{*})\| + \|\mathbf{1}(\bar{x}_{t} - x^{*})\| + 2\eta\sqrt{m} \|\nabla f(\bar{y}_{t})\|$$

$$\leq \|\mathbf{1}(\bar{y}_{t} - x^{*})\| + \|\mathbf{1}(\bar{x}_{t} - x^{*})\| + 2L\eta\sqrt{m} \|\bar{y}_{t} - x^{*}\|$$

$$\leq 3\sqrt{m} \|\bar{y}_{t} - x^{*}\| + \sqrt{m} \|\bar{x}_{t} - x^{*}\|$$

$$\stackrel{\text{(B.5)}}{\leq} 3\sqrt{m} \sqrt{\frac{2}{\mu}V_{t}} + \sqrt{m} \|\bar{x}_{t} - x^{*}\|$$

$$\leq 4\sqrt{m} \sqrt{\frac{2}{\mu}V_{t}}.$$

The first inequality is because of the L-smoothness of f(x). The second inequality follows from the step size $\eta = \frac{1}{L}$. The last inequality is due to the μ -strong convexity. Thus, we can obtain that

$$\frac{1}{M\eta} \|\mathbf{s}_{t+1} - \mathbf{1}\bar{s}_{t+1}\| \le \rho (1 + 2\rho M\eta) \cdot \frac{1}{M\eta} \|\mathbf{s}_t - \mathbf{1}\bar{s}_t\| + (7 + 2M\eta) \|\mathbf{y}_t - \mathbf{1}\bar{y}_t\|
+ \rho \cdot \frac{1}{\rho} \|\mathbf{x}_t - \mathbf{1}\bar{x}_t\| + 4\sqrt{m}\sqrt{\frac{2}{\mu}V_t}.$$

Let \mathbf{z}_{t+1} be

$$\mathbf{z}_{t+1} = \left[\|\mathbf{y}_{t+1} - \mathbf{1}\bar{y}_{t+1}\|, \frac{1}{\rho} \|\mathbf{x}_{t+1} - \mathbf{1}\bar{x}_{t+1}\|, \frac{1}{M\eta} \|\mathbf{s}_{t+1} - \mathbf{1}\bar{s}_{t+1}\| \right]^{\top},$$

then we have

$$\mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t + [0, 0, 4\sqrt{2mV_t/\mu}]^{\top}.$$

D Proof of Lemma 4

Proof of Lemma 4. It is easy to check that **A** is non-negative and irreducible. Furthermore, every diagonal entry of **A** is not zero. Thus, by Perron-Frobenius theorem and Corollary 8.4.7 of (Horn & Johnson, 2012), **A** has a real-valued positive number $\lambda_1(\mathbf{A})$ which is algebraically simple and associated with a strictly positive eigenvector \mathbf{v} . It also holds that $\lambda_1(\mathbf{A})$ is strictly larger than $|\lambda_i(\mathbf{A})|$ with i=2,3.

We write down the characteristic polynomial $p(\zeta)$ of **A**,

$$p(\zeta) = \zeta p_0(\zeta) - M\eta(7 + 2M\eta)\rho + (1 + 2\rho M\eta)\rho^2,$$

with

$$p_0(\zeta) = \zeta^2 - \rho (3 + 2\rho M\eta) \zeta - \rho \left(15M\eta + 4M^2\eta^2 + 1 - 2(1 + 2\rho M\eta)\rho \right).$$

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Let us denote

$$\Delta = 4\rho \left(15M\eta + 4M^2\eta^2 + 1 - 2(1 + 2\rho M\eta)\rho \right). \tag{D.1}$$

Then, if ρ satisfies the condition $\rho \leq \frac{-1+\sqrt{1+4M\eta(15M\eta+4M^2\eta^2+1)}}{4M\eta}$, we have $\Delta \geq 0$. Thus, two roots of $p_0(\zeta)$, ζ_1 and ζ_2 are

$$\zeta_1, \zeta_2 = \frac{\rho(3 + 2\rho M\eta) \pm \sqrt{(3 + 2\rho M\eta)^2 \rho^2 + \Delta}}{2}.$$

Furthermore, we have

$$\begin{split} p\left(\frac{\rho\cdot(3M\eta(7+2M\eta)+1)(3+2\rho M\eta)+\sqrt{\max\{\Delta,\frac{1}{4}\}}}{2}\right) \\ &=\frac{\rho\cdot(3M\eta(7+2M\eta)+1)(3+2\rho M\eta)+\sqrt{\max\{\Delta,\frac{1}{4}\}}}{2} \\ &\cdot\left(\frac{\rho\cdot(3M\eta(7+2M\eta)+1)(3+2\rho M\eta)+\sqrt{\max\{\Delta,\frac{1}{4}\}}-\sqrt{(\rho(3+2\rho M\eta))^2+\Delta}}{2}\right) \\ &\cdot\left(\frac{\rho\cdot(3M\eta(7+2M\eta)+1)(3+2\rho M\eta)+\sqrt{\max\{\Delta,\frac{1}{4}\}}+\sqrt{(\rho(3+2\rho M\eta))^2+\Delta}}{2}\right) \\ &\cdot\left(\frac{\rho\cdot(3M\eta(7+2M\eta)+1)(3+2\rho M\eta)+\sqrt{\max\{\Delta,\frac{1}{4}\}}+\sqrt{(\rho(3+2\rho M\eta))^2+\Delta}}{2}\right) \\ &-M\eta(7+2M\eta)\rho+(1+2\rho M\eta)\rho^2 \\ &\geq\frac{3\rho(3M\eta(7+2M\eta))}{8}-M\eta(7+2M\eta)\rho \\ >&0. \end{split}$$

Note that $p(\zeta)$ is monotonely increasing in the range $\left[\frac{\rho \cdot (3M\eta(7+2M\eta)+1)(3+2\rho M\eta)+\sqrt{\max\{\Delta,\frac{1}{4}\}}}{2},\infty\right]$. Thus, $p(\zeta)$ does not have real roots in this range. This implies

$$\lambda_1(\mathbf{A}) \le \frac{\rho \cdot (3M\eta(7 + 2M\eta) + 1)(3 + 2\rho M\eta) + \sqrt{\max\{\Delta, \frac{1}{4}\}}}{2}.$$

By Eqn. (D.1), we can obtain that if ρ satisfies the condition that $\rho \leq \frac{15M\eta + 4M^2\eta^2 + 1}{16}$, then it holds that $\Delta \leq \frac{1}{4}$. If ρ also satisfies the condition that $\rho \leq \frac{1}{2(21M\eta + 6M^2\eta^2 + 1)(3 + 2M\eta)}$, then we can obtain that

$$\lambda_1(\mathbf{A}) \le \frac{\frac{1}{2} + \sqrt{\max\{\Delta, \frac{1}{4}\}}}{2} = \frac{1}{2}.$$

Combining the above conditions of ρ , we only need that

$$\rho \le \frac{1}{2(21M\eta + 6M^2\eta^2 + 1)(3 + 2M\eta)}.$$

Now, we begin to prove that $\sqrt{\rho} < \lambda_1(\mathbf{A})$. We can conclude this result once it holds that $p(\sqrt{\rho}) < 0$. This is because $p(\zeta)$ will have a root between $\sqrt{\rho}$ and 1/2 and $\lambda_1(\mathbf{A})$ must be no less than this root. We have

$$\begin{split} p(\sqrt{\rho}) &= \sqrt{\rho} p_0(\sqrt{\rho}) - M \eta (7 + 2M\eta) \rho + (1 + 2\rho M\eta) \rho^2 \\ &= \rho \left(\sqrt{\rho} - \rho (3 + 2\rho M\eta) - \frac{\Delta}{4\sqrt{\rho}} - M\eta (7 + 2M\eta) + \rho + 2\rho^2 M\eta \right) \\ &\leq \rho \left(\sqrt{\rho} - 2\rho - 7M\eta - 2M^2\eta^2 \right) \\ &= \rho \left(-2\left(\sqrt{\rho} - \frac{1}{4} \right)^2 + \frac{1}{8} - 7M\eta - 2M^2\eta^2 \right) \\ &< 0, \end{split}$$

where the first inequality is due to $\Delta \geq 0$ and last inequality is because of $M\eta \geq 1$ (by Eqn. (3.4)). Since **v** is the eigenvector associated with $\lambda_1(\mathbf{A})$, we can obtain that $\mathbf{A}\mathbf{v} = \lambda_1(\mathbf{A})\mathbf{v}$ and have the following equations

$$2\rho \mathbf{v}(1) + \rho \mathbf{v}(2) + 2\rho M \eta \mathbf{v}(3) = \lambda_1(\mathbf{A})\mathbf{v}(1),$$
$$\mathbf{v}(1) + M \eta \mathbf{v}(3) = \lambda_1(\mathbf{A})\mathbf{v}(2),$$
$$(7 + 2M\eta)\mathbf{v}(1) + \rho \mathbf{v}(2) + \rho(1 + 2\rho M\eta)\mathbf{v}(3) = \lambda_1(\mathbf{A})\mathbf{v}(3).$$

Thus, we can obtain that

$$\mathbf{v}(1) \le \frac{1}{7 + 2M\eta} \left(\lambda_1(\mathbf{A})\mathbf{v}(3) - (\rho\mathbf{v}(2) + \rho(1 + 2\rho M\eta)) \right) < \frac{\mathbf{v}(3)}{2(7 + 2M\eta)},$$

and

$$\mathbf{v}(2) = (\mathbf{v}(1) + M\eta\mathbf{v}(3))/\lambda_1(\mathbf{A}) \le \left(\frac{1}{2\sqrt{\rho}(7 + 2M\eta)} + \frac{M\eta}{\sqrt{\rho}}\right)\mathbf{v}(3).$$

E Proof of Lemma 5

We first present an important lemma which is a part of the proof of Lemma 5.

Lemma 9. Letting V_t be the Lyapunov function associated to Algorithm 1, then it satisfies the following property

$$V_{t+1} \le (1 - \alpha)V_t + \frac{1}{L} \|\bar{g}_t - \nabla f(\bar{y}_t)\|^2 + 2\sqrt{\frac{2V_t}{\mu}} \|\bar{g}_t - \nabla f(\bar{y}_t)\|.$$
 (E.1)

Proof. By the update procedure of Algorithm 1, we have

$$f(\bar{x}_{t+1}) \leq f(\bar{y}_{t}) - \eta \langle \nabla f(\bar{y}_{t}), \bar{g}_{t} \rangle + \frac{L\eta^{2}}{2} \|\bar{g}_{t}\|^{2}$$

$$= f(\bar{y}_{t}) - \eta \langle \bar{g}_{t}, \bar{g}_{t} \rangle + \eta \langle \bar{g}_{t}, \bar{g}_{t} - \nabla f(\bar{y}_{t}) \rangle + \frac{L\eta^{2}}{2} \|\bar{g}_{t}\|^{2}$$

$$= f(\bar{y}_{t}) - \frac{1}{2} \cdot \frac{1}{L} \|\bar{g}_{t}\|^{2} + \frac{1}{L} \langle \bar{g}_{t}, \bar{g}_{t} - \nabla f(\bar{y}_{t}) \rangle,$$
(E.2)

where the last equation is because $\eta = \frac{1}{L}$. Furthermore, by the definition of V_t , we have

$$V_{t+1} = \frac{\mu}{2} \|\bar{v}_{t+1} - x^*\|^2 + f(\bar{x}_{t+1}) - f(x^*)$$

$$\stackrel{\text{(B.3)}}{=} \frac{\mu}{2} \|(1 - \alpha)\bar{v}_t + \alpha\bar{y}_t - x^*\|^2 - \frac{\mu}{L\alpha} \langle \bar{g}_t, (1 - \alpha)\bar{v}_t + \alpha\bar{y}_t - x^* \rangle + \frac{\mu}{2L^2\alpha^2} \|\bar{g}_t\|^2 + f(\bar{x}_{t+1}) - f(x^*)$$

$$\stackrel{\text{(E.2)}}{\leq} \frac{\mu}{2} \|(1 - \alpha)\bar{v}_t + \alpha\bar{y}_t - x^*\|^2 - \alpha \langle \bar{g}_t, (1 - \alpha)\bar{v}_t + \alpha\bar{y}_t - x^* \rangle + f(\bar{y}_t) - f(x^*) + \frac{1}{L} \langle \bar{g}_t, \bar{g}_t - \nabla f(\bar{y}_t) \rangle.$$

Furthermore, by Eqn. (B.4), we can obtain that $\bar{v}_t = \bar{y}_t + \frac{1}{\alpha}(\bar{y}_t - \bar{x}_t)$. Then we can obtain

$$(1-\alpha)\bar{v}_t + \alpha\bar{y}_t = \bar{y}_t + \frac{1-\alpha}{\alpha}(\bar{y}_t - \bar{x}_t).$$

Hence, we have

$$f(\bar{y}_t) - \alpha \langle \bar{g}_t, (1 - \alpha)\bar{v}_t + \alpha \bar{y}_t - x^* \rangle - f(x^*)$$

$$= f(\bar{y}_t) + \langle \bar{g}_t, \alpha x^* + (1 - \alpha)\bar{x}_t - \bar{y}_t \rangle - f(x^*)$$

$$= (\alpha + 1 - \alpha)f(\bar{y}_t) + \langle \nabla f(\bar{y}_t), \alpha(x^* - \bar{y}_t) + (1 - \alpha)(\bar{x}_t - \bar{y}_t) \rangle - f(x^*)$$

$$+ \langle \bar{g}_t - \nabla f(\bar{y}_t), \alpha x^* + (1 - \alpha)\bar{x}_t - \bar{y}_t \rangle$$

$$\leq (1 - \alpha)(f(\bar{x}_t) - f(x^*)) - \frac{\alpha \mu}{2} \|x^* - \bar{y}_t\| + \langle \bar{g}_t - \nabla f(\bar{y}_t), \alpha x^* + (1 - \alpha)\bar{x}_t - \bar{y}_t \rangle,$$

where the last inequality is because f(x) is μ -strongly convex. Therefore, we can obtain that

$$\begin{aligned} V_{t+1} &\leq \frac{\mu}{2} \| (1-\alpha)\bar{v}_{t} + \alpha \bar{y}_{t} - x^{*} \|^{2} + \frac{1}{L} \langle \bar{g}_{t}, \bar{g}_{t} - \nabla f(\bar{y}_{t}) \rangle \\ &+ (1-\alpha)(f(\bar{x}_{t}) - f(x^{*})) - \frac{\alpha \mu}{2} \| x^{*} - \bar{y}_{t} \| + \langle \bar{g}_{t} - \nabla f(\bar{y}_{t}), \alpha x^{*} + (1-\alpha)\bar{x}_{t} - \bar{y}_{t} \rangle \\ &\leq \frac{\mu(1-\alpha)}{2} \| \bar{v}_{t} - x^{*} \|^{2} + \frac{\mu \alpha}{2} \| \bar{y}_{t} - x^{*} \|^{2} + (1-\alpha)(f(\bar{x}_{t}) - f(x^{*})) \\ &- \frac{\alpha \mu}{2} \| x^{*} - \bar{y}_{t} \| + \langle \bar{g}_{t} - \nabla f(\bar{y}_{t}), \alpha x^{*} + (1-\alpha)\bar{x}_{t} - \bar{y}_{t} + \frac{1}{L}\bar{g}_{t} \rangle \\ &= (1-\alpha)V_{t} + \langle \bar{g}_{t} - \nabla f(\bar{y}_{t}), \alpha x^{*} + (1-\alpha)\bar{x}_{t} - \bar{y}_{t} \rangle + \frac{1}{L} \| \bar{g}_{t} - \nabla f(\bar{y}_{t}) \| \| \bar{g}_{t} \| \end{aligned}$$

where the second inequality is because of

$$\|(1-\alpha)\bar{v}_t + \alpha\bar{y}_t - x^*\|^2 \le ((1-\alpha)\|\bar{v}_t - x^*\| + \alpha\|\bar{y}_t - x^*\|)^2 \le (1-\alpha)\|\bar{v}_t - x^*\|^2 + \alpha\|\bar{y}_t - x^*\|^2.$$

Furthermore, we have

$$\|\alpha x^* + (1 - \alpha)\bar{x}_t - \bar{y}_t\| \le (1 - \alpha)\|\bar{x}_t - x^*\| + \alpha\|\bar{y}_t - x^*\| \stackrel{\text{(B.5)}}{\le} \max\{\sqrt{\frac{2}{\mu}V_t}, \sqrt{\frac{2}{\mu}V_t}\} \le \sqrt{\frac{2V_t}{\mu}}.$$

Therefore, we have

$$\begin{split} V_{t+1} \leq & (1-\alpha)V_{t} + \frac{1}{L} \|\bar{g}_{t} - \nabla f(\bar{y}_{t})\| \|\bar{g}_{t}\| + \sqrt{\frac{2V_{t}}{\mu}} \|\bar{g}_{t} - \nabla f(\bar{y}_{t})\| \\ \leq & (1-\alpha)V_{t} + \frac{1}{L} \|\bar{g}_{t} - \nabla f(\bar{y}_{t})\|^{2} + \frac{1}{L} \|\bar{g}_{t} - \nabla f(\bar{y}_{t})\| \|\nabla f(\bar{y}_{t})\| + \sqrt{\frac{2V_{t}}{\mu}} \|\bar{g}_{t} - \nabla f(\bar{y}_{t})\| \\ \leq & (1-\alpha)V_{t} + \frac{1}{L} \|\bar{g}_{t} - \nabla f(\bar{y}_{t})\|^{2} + 2\sqrt{\frac{2V_{t}}{\mu}} \|\bar{g}_{t} - \nabla f(\bar{y}_{t})\|. \end{split}$$

Proof of Lemma 5. Let the eigenvector \mathbf{v} defined in Lemma 4 and set $\mathbf{v}(3) = 1$. Combining with the fact that first two entries of \mathbf{z}_0 are zero, we can obtain that,

$$\mathbf{z}_0 \le \|\mathbf{z}_0\| \, \mathbf{v}, \quad \text{and} \quad [0, 0, 1]^\top \le \mathbf{v}.$$

By Eqn. (5.10), we can obtain that

$$\mathbf{z}_{t+1} \leq \|\mathbf{z}_0\| \cdot \mathbf{A}^{t+1} \mathbf{v} + 4\sqrt{\frac{2m}{\mu}} \cdot \sum_{i=0}^{t} \sqrt{V_i} \cdot \mathbf{A}^{t-i} \mathbf{v}$$

$$= \|\mathbf{z}_0\| \lambda_1(\mathbf{A})^{t+1} \mathbf{v} + 4\sqrt{\frac{2m}{\mu}} \cdot \sum_{i=0}^{t} \sqrt{V_i} \cdot \lambda_1(\mathbf{A})^{t-i} \mathbf{v}$$

$$\leq \|\mathbf{z}_0\| \left(\frac{1}{2}\right)^{t+1} \cdot \mathbf{v} + 4\sqrt{\frac{2m}{\mu}} \cdot \sum_{i=0}^{t} \left(\frac{1}{2}\right)^{t-i} \sqrt{V_i} \cdot \mathbf{v},$$
(E.3)

where the first equality is because \mathbf{v} is the eigenvector associated with $\lambda_1(\mathbf{A})$ and the last inequality is because of Lemma 4.

Next, we will prove our result by induction. When t = 0, we have $\|\bar{s}_t - \eta \nabla f(\bar{y}_t)\| = 0$, because we assume the initial values $\mathbf{x}_0(i,:)$ are equal to each other. Then by Eqn. (E.1), we have

$$V_1 \le (1-\alpha)V_0 \le \left(1-\frac{\alpha}{2}\right) \cdot V_0.$$

Next, we assume that for i = 1, ..., t, it holds that

$$V_i \le \left(1 - \frac{\alpha}{2}\right)^i \cdot V_0. \tag{E.4}$$

Combining with Eqn. (E.3), we can obtain that

$$\mathbf{z}_{t-1} \leq \mathbf{v} \cdot \left(4\sqrt{\frac{2m}{\mu}} \sum_{j=0}^{t-2} 2^{-(t-2-j)} \sqrt{V_{j}} + 2^{-(t-1)} \| \mathbf{z}_{0} \| \right)
\leq \mathbf{v} \cdot \left(4\sqrt{\frac{2m}{\mu}} \sum_{j=0}^{t-2} 2^{-(t-2-j)} \left(\sqrt{1 - \frac{\alpha}{2}} \right)^{j} \sqrt{V_{0}} + 2^{-(t-1)} \| \mathbf{z}_{0} \| \right)
= \mathbf{v} \cdot \left(4\sqrt{\frac{2m}{\mu}} \frac{2\left(\sqrt{1 - \frac{\alpha}{2}}\right)^{t-1} - 2^{-(t-2)}}{2\sqrt{1 - \frac{\alpha}{2}} - 1} \sqrt{V_{0}} + 2^{-(t-1)} \| \mathbf{z}_{0} \| \right)
\leq \mathbf{v} \cdot \left(12\sqrt{\frac{2m}{\mu}} \left(\sqrt{1 - \frac{\alpha}{2}} \right)^{t-1} \sqrt{V_{0}} + \left(\sqrt{1 - \frac{\alpha}{2}} \right)^{t-1} \| \mathbf{z}_{0} \| \right)
= \mathbf{v} \cdot \left(\sqrt{1 - \frac{\alpha}{2}} \right)^{t-1} \left(12\sqrt{\frac{2m}{\mu}} \sqrt{V_{0}} + \| \mathbf{z}_{0} \| \right),$$
(E.5)

where the last inequality is because of the assumption $\alpha \leq \frac{1}{2}$. Now, we begin to upper bound the value of $\|\bar{s}_t - \nabla f(\bar{y}_t)\|$. First, by Lemma 3, we can obtain that

$$\begin{aligned} \|\mathbf{y}_{t} - \mathbf{1}\bar{y}_{t}\| &\leq [2\rho, \rho, 2\rho M\eta] \mathbf{z}_{t-1} \leq \rho \left(2\mathbf{v}(1) + \mathbf{v}(2) + 2M\eta\right) \cdot \left(\sqrt{1 - \frac{\alpha}{2}}\right)^{t-1} \left(12\sqrt{\frac{2m}{\mu}}\sqrt{V_{0}} + \|\mathbf{z}_{0}\|\right) \\ &\leq \rho \cdot \frac{2M\eta}{7 + 2M\eta} \left(1 + \frac{1}{2\sqrt{\rho}}\right) \cdot \left(\sqrt{1 - \frac{\alpha}{2}}\right)^{t-1} \left(12\sqrt{\frac{2m}{\mu}}\sqrt{V_{0}} + \|\mathbf{z}_{0}\|\right) \\ &\leq \sqrt{\rho} \cdot \left(\sqrt{1 - \frac{\alpha}{2}}\right)^{t-1} \left(12\sqrt{\frac{2m}{\mu}}\sqrt{V_{0}} + \|\mathbf{z}_{0}\|\right). \end{aligned}$$

Thus, we can obtain that

$$\|\bar{g}_t - \nabla f(\bar{y}_t)\| \le \frac{M}{\sqrt{m}} \|\mathbf{y}_t - \mathbf{1}\bar{y}_t\| \le \frac{M}{\sqrt{m}} \sqrt{\rho} \cdot \left(\sqrt{1 - \frac{\alpha}{2}}\right)^{t-1} \left(12\sqrt{\frac{2m}{\mu}}\sqrt{V_0} + \|\mathbf{z}_0\|\right).$$

Thus, we can obtain

$$\|\bar{g}_t - \nabla f(\bar{y}_t)\|^2 \le \frac{2M^2\rho}{m} \left(1 - \frac{\alpha}{2}\right)^{t-1} \left(\frac{288m}{\mu} V_0 + \|\mathbf{z}_0\|^2\right).$$

Furthermore, we can obtain that

$$\sqrt{V_{t}} \| \bar{g}_{t} - \nabla f(\bar{y}_{t}) \| \\
\leq \left(\sqrt{1 - \frac{\alpha}{2}} \right)^{t} \cdot \sqrt{V_{0}} \cdot \frac{M}{\sqrt{m}} \sqrt{\rho} \cdot \left(\sqrt{1 - \frac{\alpha}{2}} \right)^{t-1} \left(12 \sqrt{\frac{2m}{\mu}} \sqrt{V_{0}} + \| \mathbf{z}_{0} \| \right) \\
\leq \left(1 - \frac{\alpha}{2} \right)^{t} \cdot \frac{36\sqrt{2\rho}M}{\sqrt{\mu}} \cdot \left(\sqrt{V_{0}} + \frac{\sqrt{\mu}}{12\sqrt{2m}} \| \mathbf{z}_{0} \| \right) \cdot \left(\sqrt{V_{0}} + \frac{\sqrt{\mu}}{12\sqrt{2m}} \| \mathbf{z}_{0} \| \right) \\
\leq \left(1 - \frac{\alpha}{2} \right)^{t} \cdot \frac{72\sqrt{2\rho}M}{\sqrt{\mu}} \cdot \left(V_{0} + \frac{\mu}{288m} \| \mathbf{z}_{0} \|^{2} \right),$$

where the second inequality is because of $\alpha \leq \frac{1}{2}$. Combining the inductive hypothesis with Eqn. (E.1), we have

$$\begin{aligned} V_{t+1} &\overset{(\text{E.1})}{\leq} (1 - \alpha) V_t + \frac{1}{L} \| \bar{s}_t - \nabla f(\bar{y}_t) \|^2 + 2 \sqrt{\frac{2V_t}{\mu}} \| \bar{s}_t - \nabla f(\bar{y}_t) \| \\ &\leq (1 - \alpha) \left(1 - \frac{\alpha}{2} \right)^t \cdot V_0 \\ &+ \frac{1}{L} \cdot \frac{576M^2 \rho}{\mu} \left(1 - \frac{\alpha}{2} \right)^{t-1} \left(V_0 + \frac{\mu}{288m} \| \mathbf{z}_0 \|^2 \right) \\ &+ 2 \frac{\sqrt{2}}{\sqrt{\mu}} \left(1 - \frac{\alpha}{2} \right)^t \cdot \frac{72\sqrt{2\rho}M}{\sqrt{\mu}} \cdot \left(V_0 + \frac{\mu}{288m} \| \mathbf{z}_0 \|^2 \right) \\ &\leq (1 - \alpha) \left(1 - \frac{\alpha}{2} \right)^t \cdot V_0 \\ &+ \frac{1}{L} \cdot \frac{576M^2 \rho}{\mu} \left(1 - \frac{\alpha}{2} \right)^{t-1} \cdot \Theta \cdot V_0 \\ &+ 2 \frac{\sqrt{2}}{\sqrt{\mu}} \left(1 - \frac{\alpha}{2} \right)^t \cdot \frac{72\sqrt{2\rho}M}{\sqrt{\mu}} \cdot \Theta \cdot V_0 \\ &\leq \left(1 - \frac{\alpha}{2} \right)^{t+1} \cdot V_0, \end{aligned}$$

where $\Theta = 1 + \frac{\mu}{288m} \cdot \frac{\|\mathbf{z}_0\|^2}{V_0}$ and the second inequality is because $\rho < \sqrt{\rho}$, and the last inequality is because ρ satisfies the condition

$$\sqrt{\rho} \leq \frac{\mu\alpha}{2304L} \cdot \min \left\{ \frac{2L}{M\Theta}, \frac{L^2}{M^2\Theta^2} \right\}.$$

Therefore, we can obtain that at the t+1-th iteration, it also holds that

$$V_{t+1} \le \left(1 - \frac{\alpha}{2}\right)^{t+1} \cdot V_0.$$