CONTENTS

MATHEMATICAL TOOLS

1.1 TRIGONOMETRY

The branch of mathematics which deals with measurement of sides and angle of triangle is called trigonometry. There are two methods for measuring angles of triangle.

1. Degree method:

   $1(rt. \angle) = 90°, 1° = 60'$ (min), $1' = 60''$ (sec)

2. Radian method:

   (a) π radians $= 180°$

   (b) An arc of length l makes angle $\theta°$ at the centre of circle whose radius is r, then

$$\theta = \frac{l}{r} = \frac{\text{Arc length}}{\text{Radius}}$$

3. Degree measure $= \dfrac{180}{\pi} \times$ Radian measure

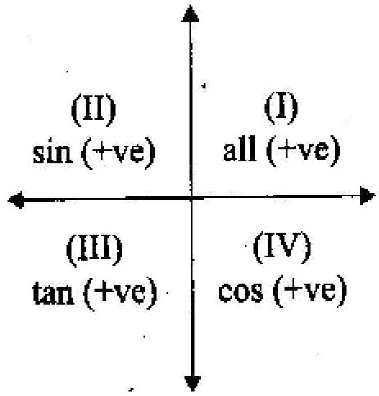
4. Radian measure $= \dfrac{\pi}{180} \times$ Degree measure

Degree	30°	45°	60°	90°	180°	270°	360°
Radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

5. $\sin^2\theta + \cos^2\theta = 1, \sin^2\theta = 1 - \cos^2\theta, \cos^2\theta = 1 - \sin^2\theta$
6. $\sec^2\theta - \tan^2\theta = 1, \sec^2\theta = 1 + \tan^2\theta, \tan^2\theta = \sec^2\theta - 1$
7. $\operatorname{cosec}^2\theta - \cot^2\theta = 1, \operatorname{cosec}^2\theta = 1 + \cot^2\theta, \cot^2\theta = \operatorname{cosec}^2\theta - 1$
8. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}, \cot\theta = \dfrac{\cos\theta}{\sin\theta}$
9. $\sin\theta \cdot \operatorname{cosec}\theta = 1, \tan\theta \cdot \cot\theta = 1, \cos\theta \cdot \sec\theta = 1$

Table 1.1 Sign of T-function

Quadrant	1st	2nd	3rd	4th
sin, cosec	+	+	−	−
cos, sec	+	−	−	+
tan, cot	+	−	+	−



1.1.1 Formulae for Compound Angle

1. $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$
2. $\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$
3. $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$
4. $\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$
5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$
6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$
7. $\cot(A+B) = \dfrac{\cot A \cdot \cot B - 1}{\cot A + \cot B}$
8. $\cot(A-B) = \dfrac{\cot A \cdot \cot B + 1}{\cot B - \cot A}$
9. $\sin(A+B) \cdot \sin(A-B) = \sin^2 A - \sin^2 B$
10. $\cos(A+B) \cdot \cos(A-B) = \cos^2 A - \sin^2 B$

1.1.2 Transformational Formula

1. $2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$
2. $2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$
3. $2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$
4. $2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$
5. $\sin C + \sin D = 2\sin\dfrac{C+D}{2} \cdot \cos\dfrac{C-D}{2}$
6. $\sin C - \sin D = 2\cos\dfrac{C+D}{2} \cdot \sin\dfrac{C-D}{2}$
7. $\cos C + \cos D = 2\cos\dfrac{C+D}{2} \cdot \cos\dfrac{C-D}{2}$
8. $\cos C - \cos D = 2\sin\dfrac{C+D}{2} \cdot \sin\dfrac{D-C}{2}$

1.1.3 Formulae for Multiple and Sub-multiple Angles

1. $\sin 2A = 2\sin A \cdot \cos A = \dfrac{2\tan A}{1+\tan^2 A}$
2. $\cos 2A = \cos^2 A - \sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
3. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
4. $\sin 3A = 3\sin A - 4\sin^3 A$
5. $\cos 3A = 4\cos^3 A - 3\cos A$
6. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
7. $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2} = \dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$
8. $\cos A = \cos^2\dfrac{A}{2} - \sin^2\dfrac{A}{2} = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$
9. $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$
10. $\dfrac{1-\cos A}{1+\cos A} = \tan^2\dfrac{A}{2}$

Notes

$\sin 18° = \dfrac{\sqrt{5}-1}{4}$ $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

$\sin 36° = \dfrac{\sqrt{10-2\sqrt{6}}}{4}$ $\cos 36° = \dfrac{\sqrt{5}+1}{4}$

1.1.4 Trigonometric Equations

The equation which contain trigonometric function is called T-Equation, e.g., $\cos x = 2\sin x$

1. If $\sin x = 0 \Rightarrow x = n\pi$
2. If $\cos x = 0 \Rightarrow x = (2n+1)\dfrac{\pi}{2}$
3. If $\tan x = 0 \Rightarrow x = n\pi$
4. If $\sin x = \pm 1 \Rightarrow x = (4n \pm 1)\dfrac{\pi}{2}$
5. If $\cos x = 1 \Rightarrow x = 2n\pi$
6. If $\cos x = -1 \Rightarrow x = (2n+1)\pi$
7. If $\sin x = \sin y \Rightarrow x = n\pi + (-1)^n y$
8. If $\cos x = \cos y \Rightarrow x = 2n\pi \pm y$
9. If $\tan x = \tan y \Rightarrow x = n\pi + y$
10. If $\sin^2 x = \sin^2 y \Rightarrow x = n\pi \pm y$
11. If $\cos^2 x = \cos^2 y \Rightarrow x = n\pi \pm y$
12. If $\tan^2 x = \tan^2 y \Rightarrow x = n\pi \pm y$

where $n = 0, \pm 1, \pm 2, \pm 3, ...$

Table 1.2 Some Trigonometrical Values

	$(-\theta)$	$(90° - \theta)$	$(90° + \theta)$	$(180° - \theta)$	$(180° + \theta)$
sin	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$
cos	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$
cot	$-\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$
sec	$\sec\theta$	$\text{cosec}\,\theta$	$-\text{cosec}\,\theta$	$-\sec\theta$	$-\sec\theta$
cosec	$-\text{cosec}\,\theta$	$\sec\theta$	$\sec\theta$	$\text{cosec}\,\theta$	$-\text{cosec}\,\theta$

1.1.5 Value of $(2\pi \pm \theta)$

1. $\sin(2\pi + \theta) = \sin\theta$
2. $\cos(2\pi + \theta) = \cos\theta$
3. $\tan(2\pi + \theta) = \tan\theta$
4. $\cot(2\pi + \theta) = \cot\theta$
5. $\sec(2\pi + \theta) = \sec\theta$
6. $\text{cosec}(2\pi + \theta) = \text{cosec}\,\theta$

$\sin(2\pi - \theta) = -\sin\theta$

$\cos(2\pi - \theta) = \cos\theta$

$\tan(2\pi - \theta) = -\tan\theta$

$\cot(2\pi - \theta) = -\cot\theta$

$\sec(2\pi - \theta) = \sec\theta$

$\text{cosec}(2\pi - \theta) = -\text{cosec}\,\theta$

1.1.6 Value of $\left(\frac{3\pi}{2} \pm \theta\right)$

1. $\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta,$ $\qquad \cos\left(\frac{3\pi}{2}+\theta\right) = \sin\theta$

2. $\sin\left(\frac{3\pi}{2}-\theta\right) = -\cos\theta,$ $\qquad \cos\left(\frac{3\pi}{2}-\theta\right) = -\sin\theta$

Table 1.3 Value of Some Standard Angles

T-ratio	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}$	-1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$\frac{-1}{\sqrt{3}}$	0
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$\frac{-1}{\sqrt{3}}$	-1	$-\sqrt{3}$	∞
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1
$\text{cosec}\,\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

Note: ∞ means undefined.

1.1.7 Inverse Trigonometric Functions

The value of inverse T-functions lies between the given range.

$\sin^{-1}x,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\qquad \cot^{-1}x,\ x \in (0, \pi)$

$\cos^{-1}x,\ x \in [0, \pi]$ $\qquad \sec^{-1}x,\ x \in [0, \pi] - \left(\frac{\pi}{2}\right)$

$\tan^{-1}x,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\qquad \text{cosec}^{-1}x,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

1. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\ (-1 \le x \le 1)$

2. $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2},\ x \in R$

3. $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}\quad x \ge 1$

4. $\sin x = \text{cosec}^{-1}\frac{1}{x},\ |x| \le 1$

5. $\cos^{-1}x = \sec^{-1}\frac{1}{x}$, $|x| \le 1$

6. $\tan^{-1}x = \cot^{-1}\frac{1}{x}$, $-\infty < x < \infty$

7. $\sin^{-1}(-x) = -\sin^{-1}x$

8. $\cos^{-1}(-x) = \pi - \cos^{-1}x$

9. $\tan^{-1}(-x) = \tan^{-1}x$

10. $\cot^{-1}(-x) = \pi - \cot^{-1}x$

11. $\sec^{-1}(-x) = \pi - \sec^{-1}x$

12. $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$

13. $y = \sin^{-1}x \Rightarrow x = \sin y$

14. $x = \sin y \Rightarrow y = \sin^{-1}x$

15. $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$

16. $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right)$

17. $\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left(\frac{x \pm y}{1 \mp xy}\right)$

18. $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$

19. $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$

20. $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

21. $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

22. $3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

1.2 ALGEBRA

1.2.1 Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$, where a, b, c are certain numbers and $a \neq 0$, is called a quadratic equation.

1. **Discriminant of a quadratic equation:** The numbers $(b^2 - 4ac)$ is called discriminant of the quadratic equation $ax^2 + bx + c = 0$ and is denoted by D. i.e., $D = b^2 - 4ac$.
2. **Nature of roots of the quadratic equation:** The value of x which satisfy the equation $ax^2 + bx + c = 0$ are called roots of the equation. The roots α and β of the equation $ax^2 + bx + c = 0$ are given by,

$$\frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-b-\sqrt{D}}{2a}, \beta = \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-b+\sqrt{D}}{2a}$$

Now there are three possibilities :

Case I: When $D < 0$, i.e., $b^2 - 4ac < 0$. In this case $\sqrt{D}$ will be imaginary, hence α and β will be both imaginary.

Case II: When $D = 0$, i.e., $b^2 - 4ac = 0$. In this case $\sqrt{D} = 0$, from the above equation,

$\alpha = \frac{-b}{2a}$, $\beta = \frac{-b}{2a}$. Hence both roots α and β will be real and equal.

Case III: When $D > 0$, i.e., $b^2 - 4ac > 0$. Then the roots α and β will be real and different (distinct).

1.2.2 Determinants

Let a, b, c, d be any four numbers, the symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ denotes $ad - bc$ and is called a determinant of second order. The elements of a determinant are multiplied diagonally, like,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example, $\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = 4 - 12 = -8$

The elements which lie in the same horizontal line constitute one row and the elements which lie in the same vertical line constitute one column.

$$\begin{array}{c} \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{array}{l} \rightarrow \text{Row}-1 \\ \rightarrow \text{Row}-2 \end{array} \\ \downarrow \quad \downarrow \end{array}$$

Column 1 Column 2

1.2.3 Determinant of Third Order

The determinant of 3rd order has three rows and three columns.

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

the expansion of the determinant along its first row will be

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

1.2.4 Progression

If the terms of a sequence are written under specific conditions, then the sequence is called progression. Here we shall study only two types of progressions.

Arithmetic Progression (A.P.)

An arithmetic progression is a sequence of numbers such that the difference between any two successive terms is a constant called common difference.

Examples

1. 1, 4, 7, 10, 13 … are in A.P., whose first term is 1 and common difference (c.d.) is 3.
2. The sequence of numbers 10, 8, 6, 2, 0, –2, –4, … are in A.P., whose first term is 10 and c.d. = –2.

   In general, an A.P. is expressed as, $a_1, a_2, a_3 \ldots a_n$, and the common difference is defined as $d = a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1}$

Properties

1. The nth term of an A.P. is given by, $a_n = a_1 + (n-1)d$.
2. The sum of the first n terms of an A.P. is given by

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

or

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(\text{First term } + \text{Last term})$$

1.2.5 Geometric Progression

A geometric progression is a sequence of numbers such that the ratio of each terms to the immediately preceeding one is a constant called the common ratio.

Examples

1. The numbers 2, 4, 8, 16, 32, 64 … form a G.P. with common ratio = 2.
2. The numbers 1, 0.1, 0.01, 0.001, … constitute a G.P. with ratio 0.1. In general, a G.P. is expressed as, $a_1, a_2, a_3, \ldots a_n$; and the common ratio is defined as

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots = \frac{a_n}{a_{n-1}}$$

Properties

1. The n^{th} term of G.P. is given by $a_n = a_1 r^{n-1}$; where a_1 is the first term and r is the common ratio.
2. The sum of first n terms of G.P. is given by

$$S_n = a_1\left[\frac{r^n - 1}{r - 1}\right] \text{ when } (r > 1) \text{ and } S_n = a_1\left[\frac{1 - r^n}{r - 1}\right]$$

when $(r < 1)$

The sum of infinite terms of G.P. for $|r| < 1$ is given by

$$S_\infty = \frac{a}{1-r} \text{ where } -1 < r < 1$$

1.2.6 Some Important Summation of Series

1. The sum of the first n natural number

$$\Sigma n = 1 + 2 + 3 + \dots\dots\dots + n = \frac{n(n+1)}{2}$$

2. The sum of the squares of the first n natural numbers i.e.,

$$\Sigma n^2 = 1^2 + 2^2 + 3^2 + \dots\dots\dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. The sum of the cubes of the first n natural numbers

$$\Sigma n^3 = 1^3 + 2^3 + 3^3 + \dots\dots\dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

1.2.7 Binomial Theorem for Any Index

$$(1+x)^2 = 1 + 2x + x^2$$
$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

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1. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + x^n$, where n is a +ve integer.

   Number of terms in $(1+x)^n$ is $n+1$.

   Meaning of factorial

   $$2! = 2 \times 1 = 2 \quad 3! = 3 \times 2 \times 1 = 6 \quad 4! = 4 \times 3 \times 2 \times 1 = 2$$

2. But if n is a –ve integer or positive or negative fraction; then

   $$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots\infty$$

   Provided $|x| <$ i.e., $-1 < x < 1$

   Number of terms in this case will be infinite.

1.2.8 Exponential and Logarithmic Series

1. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\dots\dots$ to ∞, which is 2.71828 ...... is read as exponential number.

2. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots$ to ∞, where x is any number.

3. $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\dots\dots$ to ∞, where x is any number.

4. $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots$ to $\infty \; (-1 \le x \le 1)$

5. $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\dots$ to $\infty \; (-1 \le x \le 1)$

1.3 CALCULUS

1.3.1 Limits

Let us consider the function $y = f(x) = \frac{x^2 - 4}{x - 3}$. If we put $x = 3$, we have $y = \frac{9-4}{3-3} = \frac{5}{0} = \infty$, which is meaningless. It means that the function is not defined at $x = 3$. But still, we want to know the value of the function at a value slightly smaller or greater than 3. If we could define the function at a value slightly smaller or greater than 3, then we say that the limit of function exists as x approaches 3. In mathematics it is represented by the symbol $\lim\limits_{x\to 3}$.

The expected value of the function $f(x)$ to the left of a point $x = a$ is called left hand limit. It is denoted by $\lim\limits_{x\to a^-} f(x)$.

The expected value of function $f(x)$ to the right of a point $x = a$ is called right hand limit is denoted by $\lim\limits_{x\to a^+} f(x)$.

The limit of a function $f(x)$ at point $x = a$ is the common value of left and right hand limit. It is denoted by $\lim\limits_{x\to a} f(x)$.

A variable whose limit is zero is termed as infinitely small quantity (infinitesimal). Mathematically, it may be written as $x \to 0$. A variable that constantly increases in absolute magnitude is termed as infinitely large quantity. Although infinitely large quantities do not have any limits but it is conventional to say that an infinitely large quantity 'tends to an infinite limit'. The symbol $\to$ reads as 'tends to'.

1.3.2 Basic Formulae of Limit

If $f(x)$ and $g(x)$ are two function then,

1. $\lim\limits_{x\to a}\{f(x)+g(x)\} = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$
2. $\lim\limits_{x\to a}\{f(x)-g(x)\} = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$
3. $\lim\limits_{x\to a}\{kf(x)\} = k\{\lim\limits_{x\to a} f(x)\}$ where k = constant
4. $\lim\limits_{x\to a}\{f(x)\cdot g(x)\} = \lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x)$
5. $\lim\limits_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$
6. $\lim\limits_{x\to a}\left\{\frac{x^n - a^n}{x-a}\right\} = na^{n-1}$
7. $\lim\limits_{x\to 0}\frac{e^x-1}{x} = 1$
8. $\lim\limits_{x\to 0}\frac{a^x-1}{x} = \log_e a$
9. $\lim\limits_{x\to 0}\frac{\log(1+x)}{x} = 1$
10. $\lim\limits_{x\to 0}(1+x)^{1/x} = e$
11. $\lim\limits_{x\to 0}\sin x = 0$
12. $\lim\limits_{\to}\cos \quad 1$
13. $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$
14. $\lim\limits_{x\to 0}\frac{\tan x}{x} = 1$
15. $\lim\limits_{x\to 0}\frac{1}{x} = 0$

1.3.3 Continuity

Function $f(x)$ at point $x = a$ is said to be continuous if, L.H. lim = R.H lim = value of function at a. i.e., $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$

Notes

1. Function is discontinuous if $f(a)$ is not defined.
2. $\lim_{x \to a} f(x) \neq f(a)$
3. Constant polynomial identity and modulus function are continuous function.

1.3.4 Differentiability and Differentiate

If $y = f(x)$ then differential coefficient of y with respect to (w.r.t) x is given by

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note

Every differentiable function is continuous but every continuous function is not differentiable.

1.3.5 For Two Functions: *u* and *v*

1. $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
2. $\frac{d(u-v)}{dx} = \frac{du}{dx} - \frac{dv}{dx}$
3. $\frac{d(u \cdot v)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
4. $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
5. $\frac{d(\text{constant})}{dx} = 0$
6. $\frac{d\{k\, f(x)\}}{dx} = k\frac{d\{f(x)\}}{dx}$ where k = constant
7. $\frac{dy}{dx} = \frac{1}{dx/dy}$

1.3.6 Chain Rule

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1 \text{ and } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

1.3.7 D.C. of Some Important Functions

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(a^x) = a^x \log_e a$
3. $\frac{d}{dx}(e^x) = e^x$
4. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

5. $\frac{d}{dx}(\sin x) = \cos x$

6. $\frac{d}{dx}(\cos x) = -\sin x$

7. $\frac{d}{dx}(\tan x) = \sec^2 x$

8. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

9. $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

10. $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cdot \cot x$

11. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

12. $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

13. $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

14. $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

15. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

17. $\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

1.3.8 Maxima and Minima

If, $y = f(x)$ is a function and $f'(x) = 0$ then at point $x = a$

1. Maximum if $\frac{d^2y}{dx^2} = -ve$
2. Minimum if $\frac{d^2y}{dx^2} = +ve$

1.3.9 Integration

Integration is inverse process of differentiation. It is denoted by $\int$

$$\therefore \frac{d}{dx}f(x) = g(x) \text{ then } \int g(x)\,dx = f(x) + c$$

where c = Integration constant.

There are two type of integration:

1. Definite integration
2. Indefinite integration

1.3.10 Indefinite Integration

1. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
2. $\int kf(x) = k\int f(x)dx$, where k = constant
3. $\int 1dx = x$
4. $\int x^n\,dx = \frac{x^{n+1}}{n+1}$
5. $\int \frac{1}{x}\,dx = \log_e x$
6. $\int a^x dx = \frac{a^x}{\log_e^a}$
7. $\int \sin x\,dx = -\cos x$
8. $\int \cos x\,dx = \sin x$
9. $\int \sec^2 x\,dx = \tan x$
10. $\int \text{cosec}^2 x\,dx = -\cot x$
11. $\int \sec x \cdot \tan x\,dx = \sec x$
12. $\int \tan x\,dx = \log \sec x$
13. $\int \cot x\,dx = \log \sin x$
14. $\int \text{cosec}\,dx = \log(\text{cosec } x - \cot x)$
15. $\int \sec x\,dx = \log(\sec x + \tan x)$

16. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x$

17. $\int \frac{dx}{1-x^2} = \tan^{-1}x$

18. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x$

19. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

20. $\int \frac{f'(x)}{f(x)} = \log [f(x)]$

21. $\int e^x\, dx = e^x$

22. $\int e^x\{f(x)+f(x)\}dx = e^x f(x)$

1.3.11 Integration by Parts

For two function $f(x)$ and $g(x)$,

$$\int[f(x)\cdot g(x)]dx = f(x)\int g(x)dx - \int\left\{\frac{df(x)}{dx}\int g(x)dx\right\}dx$$

i.e., integration (1st × 2nd) = 1st (integration 2nd) – Integration {differentiate 1st (integration 2nd)}dx

Note

We choose the first function as the function which comes first in the word.

'*ILATE*' where

I = Inverse trigonometric function ($\sin^{-1}x$, $\cos^{-1}x$) etc.
L = Logarithmic function
A = Algebraic function
T = Trigonometric function
E = Exponential function (e^x, e^{1x} etc.)

1.3.12 Integration of Some Standard Functions

1. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

2. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|$

3. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$

4. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right|$

5. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

6. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right|$

7. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|$

8. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

9. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right)$

Note

Integration of the type $\int \frac{px+q}{ax^2+bx+c}dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ can be find by putting

$$px+q = A\cdot\frac{d}{dx}\left(ax^2+bx+c\right)+B$$

By equating both sides find A and B and then put in place of $px+q$ and integrate.

1.3.13 Definite Integration

If $\int f(x)\, dx = F(x)$ defined in interval $[a, b]$ then, $\int_a^b f(x)dx$ is called definite integral, b is called upper limit and a is called lower limit and we have

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b)-F(a)$$

1.3.14 Property of Definite Integration

1. $\int_a^b kf(x)dx = k\int_a^b (x)dx$

2. $\int_a^b f(x)dx = \int_a^b f(t)dt$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

4. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

5. $\int f(x)dx \quad 0$, if $f(x)$ is odd function

6. $\int^a f(x)dx \quad 2\int^a f(x)dx$, if $\int f(x)dx$ is even function

Note

If $f(-x) = f(x)$ then $f(x)$ is even function but when $f(-x) = -f(x)$ then $f(x)$ is odd function.

1.3.15 Trigonometric Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

2.1 INTRODUCTION

Measurement means comparing a thing with a standard to find out how many times as big it is. But there are other cases where counting and statistical analysis are necessary.

There are two types of quantity: (a) Microscopic and (b) Macroscopic

Anything that can be measured is called a quantity. If we can measure a quantity with certain instruments then that quantity is called a physical quantity.

Physical quantity → Numerical Part and Suitable Units

The magnitude of a physical quantity is expressed by giving its numerical value and a unit. The numerical value tells us how many times the basic unit is contained in the measured value.

Units

- Fundamental units → Independent of each other and not interconvertible → Unit of mass, length, time, temperature, electric current, amount of substance and luminous intensity
- Derived units → Derived from fundamental units → Units of velocity, acceleration, force and work done
- Supplementary units → Unit of plane angle and solid angle

The *dimensions* of a physical quantity are the powers to which the fundamental unit of length, mass and time are to be raised so that the derived quantity can be completely represented. *Dimensions* tell us about the nature of the physical quantity and do not give any idea about the magnitude. The concept of dimension is more generalized compared to the idea of unit.

Table 2.1

Sr. No.	Basic Physical Quantities	Name	Symbol	Definition
1.	Length	Metre	m	One metre is the length of the path travelled by light in vacuum during a time interval of 1/299, 729, 458 of a second.
2.	Mass	Kilogram	kg	One kilogram is equal to the mass (a platinum-iridium alloy cylinder) kept at International Bureau of Weights and Measures, at Sevres, near Pairs, France.
3.	Time	Second	s	One second is the duration of 9,192, 631, 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.
4.	Electric current	Ampere	A	One ampre is that constant current, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per metre of length.
5.	Thermodynamic temperature	Kelvin	K	One degree kelvin, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.
6.	Amount of substance	Mole	mol	One mole is the amount of substance of a system, which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12.
7.	Luminous intensity	Candela	cd	One candela is the luminous intensity, in a given direction of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of *1/683* watt per steradian.
	Supplementary Quantity			
8.	Plane angle $d\theta = ds/r$	Radian	rad	Plane angle $d\theta$ is defined as the ratio of length of arc ds to the radius r.
9.	Solid angle $d\Omega = dA/r^2$	Steradian	sr	Solid angle $d\Omega$ is defined as the ratio of the intercepted area dA of the spherical surface, described about the apex O as the centre, to the square of its radius r.

Table 2.2

Sr. No.	Physical Quantities	Formula	Dimensions	Dimensional Formula
1.	Area	Length × Breadth	$[L^2]$	$[M^0L^2T^0]$
2.	Volume	Length × Breath × Height	$[L^3]$	$[M^0L^3T^0]$
3.	Density	Mass/Volume	$[M]/[L^3]$ or $[ML^{-3}]$	$[M^0L^{-3}T^0]$
4.	Frequency	1/Time period	$1/[T]$	$[M^0L^0T^{-1}]$
5.	Velocity	Displacement/Time	$[L]/[T]$	$[M^0L^0T^{-1}]$
6.	Acceleration	Velocity/Time	$[LT^{-1}]/[T]$	$[M^0LT^{-2}]$
7.	Force	Mass × Acceleration	$[M]\ [LT^{-2}]$	$[MTL^{-2}]$
8.	Impulse	Force × Time	$[MLT^{-2}][T]$	$[MLT^{-1}]$
9.	Work	Force × Distance	$[MLT^{-2}][L]$	$[ML^2T^{-2}]$
10.	Power	Work/Time	$[ML^2T^{-2}][L]$	$[ML^2T^{-3}]$
11.	Momentum	Mass × Velocity	$[M]\ [LT^{-1}]$	$[MLT^{-1}]$
12.	Pressure stress	Force/Area	$[MLT^{-2}]/[L^2]$	$[ML^{-1}T^{-2}]$
13.	Strain	$\frac{\text{Change in dimension}}{\text{Original dimension}}$	$[L]/[L]$ or $[L^3]/[L^3]$	$[M^0L^0T^0]$
14.	Modulus of elasticity	Stress/Strain	$\frac{[ML^{-1}T^{-2}]}{M^0L^0T^0}$	$[ML^{-1}T^{-2}]$
15.	Surface tension	Force/Length	$[MLT^{-2}]/[L]$	$[ML^0T^{-2}]$
16.	Surface energy	Energy/Area	$[ML^2T^{-2}]/[L^2]$	$[ML^0T^2]$
17.	Velocity gradient	Velocity/Distance	$[LT^{-1}]/[L]$	$[M^0L^0T^{-1}]$
18.	Pressure gradient	Pressure/Distance	$[ML^{-1}T^{-2}]/[L]$	$[ML^{-2}T^{-2}]$
19.	Pressure energy	Pressure × Volume	$[ML^{-1}T^{-2}][L^3]$	$[ML^2T^{-2}]$
20.	Coefficient of viscosity	$\frac{\text{Force}}{\text{Area} \times \text{Velocity gradient}}$	$\frac{MLT^{-2}}{[L^2][LT^{-1}/L]}$	$[ML^{-1}T^{-1}]$
21.	Angle	Arc/Radius	$[L]/[L]$	$[M^0L^0T^0]$
22.	Trigonometric ratio ($\sin\theta$, $\cos\theta$, $\tan\theta$, etc.)	Length/Length	$[L]/[L]$	$[M^0L^0T^0]$
23.	Angular velocity	Angle/Time	$[M^0L^0T^0]/[T]$	$[M^0L^0T^{-1}]$

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Table 2.2 (*Continued*)

Sr. No.	Physical Quantities	Formula	Dimensions	Dimensional Formula
24.	Angular acceleration	Angular velocity/Time	$[T^{-1}]/[T]$	$[M^0L^0T^{-2}]$
25.	Radius of gyration		$[L]$	$[ML^2T^0]$
26.	Moment of inertia	Mass × (Radius of gyration)^2	$[M][L^2]$	$[ML^2T^0]$
27.	Angular momentum	Moment of inertia × Angular velocity	$[ML^2][T^{-1}]$	$[ML^2T^{-1}]$
28.	Moment of force, moment of couple	Force × Distance	$[MLT^{-2}][L]$	$[ML^2T^{-2}]$
29.	Torque	Force × Distance	$[MLT^{-2}][L]$	$[ML^2T^{-2}]$
30.	Angular frequency	2π × Frequency	$[M^0L^0T^0][T^{-1}]$	$[M^0L^0T^{-1}]$
31.	Wavelength		$[L]$	$[M^0LT^0]$
32.	Hubble constant	Recession speed/Distance	$[LT^{-1}]/[L]$	$[M^0L^0T^{-1}]$
33.	Intensity of wave	$\dfrac{\text{Energy}}{\text{Time} \times \text{Area}}$	$\dfrac{[ML^2T^{-2}]}{[T][L^2]}$	$[ML^0T^{-3}]$
34.	Radiation pressure	$\dfrac{\text{Intensive of wave}}{\text{Speed of light}}$	$[ML^{-3}]/[LT^{-1}]$	$[ML^{-1}T^{-2}]$
35.	Energy density	Energy/Volume	$[ML^2T^{-2}]/[L^3]$	$[ML^{-1}T^{-2}]$
36.	Critical velocity	$\dfrac{\text{Reynold's number} \times \text{Coeff. of viscosity}}{\text{Density} \times \text{Radius}}$	$\dfrac{[M^0L^0T^0][ML^{-1}T^{-1}]}{[ML^{-3}][L]}$	$[M^0LT^{-1}]$
37.	Escape velocity	(2 × Acceleration due to gravity × Earth's radius)$^{1/2}$	$[LT^{-2}]^{1/2} \times [L]^{1/2}$	$[M^0LT^{-1}]$
38.	Heat energy, internal energy		$[ML^2T^{-2}]$	$[ML^2T^{-2}]$
39.	Kinetic energy	(1/2) Mass × (Velocity)^2	$[M]\ [LT^{-1}]^2$	$[ML^2T^{-2}]$
40.	Potential energy	Mass × Acceleration due to gravity × Height	$[M][LT^{-2}][L]$	$[ML^2T^{-2}]$

(*Continued*)

Table 2.2 (*Continued*)

Sr. No.	Physical Quantities	Formula	Dimensions	Dimensional Formula
41.	Rotational kinetic energy	1/2 × Moment of inertia × (Angular velocity)^2	$[ML^2] \times [T^{-1}]^2$	$[ML^2T^{-2}]$
42.	Efficiency	$\frac{\text{Output work of energy}}{\text{Input work of energy}}$	$\frac{[ML^2T^{-2}]}{[ML^2T^{-2}]}$	$[M^0L^0T^0]$
43.	Angular impulse	Torque × Time	$[ML^2T^{-2}][T]$	$[ML^2T^{-1}]$
44.	Gravitational constant	$\frac{\text{Force} \times (\text{Distance})^2}{\text{Mass} \times \text{Mass}}$	$\frac{[MLT^{-2}][L^2]}{[M][M]}$	$[M^{-1}L^3T^{-2}]$
45.	Planck's constant	Energy/Frequency	$[ML^2T^{-2}]/[T^{-1}]$	$[ML^2T^{-1}]$
46.	Heat capacity, entropy	Heat energy temperature	$[ML^2T^{-2}]/[K]$	$[ML^2T^{-2}K^{-1}]$
47.	Specific heat capacity	$\frac{\text{Heat energy}}{\text{Mass} \times \text{Temperature}}$	$\frac{[ML^2T^{-2}]}{[M][K]}$	$[M^0L^2T^{-2}K^{-1}]$
48.	Latent heat	Heat energy/Mass	$[ML^2T^{-2}]/[M]$	$[M^0L^2T^{-2}]$
49.	Thermal expansion coefficient or thermal expansivity	$\frac{\text{Change in dimension}}{\text{Original dimension} \times \text{Temperature}}$	$\frac{[L]}{[L][K]}$	$[M^0L^0K^{-1}]$
50.	Thermal conductivity	$\frac{\text{Heat energy} \times \text{Thickness}}{\text{Area} \times \text{Temperature} \times \text{Time}}$	$\frac{[ML^2T^{-2}][L]}{[L^2][K][T]}$	$[MLT^{-3}K^{-1}]$
51.	Bulk modulus (compress-ibility)$^{-1}$	$\frac{\text{Volume} \times (\text{Change in pressure})}{\text{Change in volume}}$	$\frac{[L^3][ML^{-1}T^{-2}]}{[L^3]}$	$[ML^{-1}T^{-2}]$
52.	Centripetal acceleration	(Velocity) 2/Radius	$[LT^{-1}]^2/[L]$	$[M^0LT^{-2}]$
53.	Stefan constant	$\frac{\text{Energy}}{(\text{Area}) \times (\text{Time}) \times (\text{Temperature})^4}$	$\frac{[ML^2T^{-2}]}{[L^2][T][K]^4}$	$[ML^0T^{-3}K^{-4}]$
54.	Wien constant	Wavelength × Temperature	$[L][K]$	$[M^0LT^0K]$
55.	Universal gas constant	$\frac{\text{Pressure} \times \text{Volume}}{\text{Mole} \times \text{Temperature}}$	$\frac{[ML^{-1}T^{-2}][L^3]}{[mol][K]}$	$[ML^2T^{-2}K^{-1}mol^{-1}]$
56.	Boltzmann constant	$\frac{\text{Universal gas constant}}{\text{Avagadro number}}$	$\frac{[ML^2T^{-2}K^{-1}mol^{-1}]}{[mol^{-1}]}$	$[ML^2T^{-2}K^{-1}]$

Table 2.2 (*Continued*)

Sr. No.	Physical Quantities	Formula	Dimensions	Dimensional Formula
57.	Charge	Current × Times	[A][T]	$[M^0L^0TA]$
58.	Current density	Current/Area	$[A]/[L^2]$	$[M^0L^{-2}T^0A]$
59.	Electric potential, electromotive force	Work/Charge	$[ML^2T^{-2}]/[AT]$	$[ML^2T^{-3}A^{-1}]$
60.	Resistance	$\frac{\text{Potential difference}}{\text{Current}}$	$\frac{[ML^2T^{-3}A^{-1}]}{[A]}$	$[ML^2T^{-3}A^{-2}]$
61.	Capacitance	Charge/Potential difference	$\frac{[AT]}{[ML^2T^{-3}A^{-1}]}$	$[M^{-1}L^{-2}T^4A^2]$
62.	Electric field	Electrical force/Charge	$[MLT^{-2}]/[AT]$	$[MLT^{-3}A^{-1}]$
63.	Electric flux	Electric field × Area	$[MLT^{-3}A^{-1}][L^2]$	$[ML^3T^{-3}A^{-1}]$
64.	Electric dipole moment	Torque/Electric field	$\frac{[ML^2T^{-2}]}{[MLT^{-3}A^{-1}]}$	$[M^0LTA]$
65.	Electric field strength or electric intensity	$\frac{\text{Potential difference}}{\text{Distance}}$	$\frac{[ML^2T^{-3}A^{-1}]}{[L]}$	$[MLT^{-3}A^{-1}]$
66.	Magnetic field, magnetic flux density, magnetic induction	$\frac{\text{Force}}{\text{Current} \times \text{Length}}$	$\frac{[MLT^2]}{[A][L]}$	$[ML^0T^{-2}A^{-1}]$
67.	Magnetic flux	Magnetic field × Area	$[MT^{-2}A^{-1}][L^2]$	$[ML^2T^{-2}A^{-1}]$
68.	Inductance	$\frac{\text{Magnetic flux}}{\text{Current}}$	$\frac{[ML^2T^{-2}A^{-1}]}{[A]}$	$[ML^2T^{-2}A^{-2}]$
69.	Magnetic dipole moment	Current × Area	$[A][L^2]$	$[M^0L^2T^0A]$
70.	Magnetization	$\frac{\text{Magnetic moment}}{\text{Volume}}$	$\frac{[L^2A]}{[L^3]}$	$[M^0L^{-1}T^0A]$
71.	Permittivity constant (of free space) ε_o	$\frac{\text{Charge} \times \text{Charge}}{4\pi \times \text{Electric force} \times (\text{Distance})^2}$	$\frac{[AT][AT]}{[MLT^{-2}][L^2]}$	$[M^{-1}L^{-3}T^4A^2]$

(*Continued*)

Table 2.2 *(Continued)*

Sr. No.	Physical Quantities	Formula	Dimensions	Dimensional Formula
72.	Permeability constant (of free space) μ_o	$\frac{2\pi \times \text{Force} \times \text{Distance}}{\text{Current} \times \text{Current} \times \text{Length}}$	$\frac{[M^0L^0T^0][MLT^{-2}][L]}{[A][A][L]}$	$[MLT^{-2}A^{-2}]$
73.	Refractive index	$\frac{\text{Speed of light in vacuum}}{\text{Speed of light in medium}}$	$[LT^{-1}/[LT^{-1}]$	$[M^0L^0T^0]$
74.	Faraday constant	Avogadro's constant × Elementary charge	[AT]/[mol]	$[M^0L^0TA\ mol^{-1}]$
75.	Wave number	2π/Wavelength	$[M^0L^0T^0]/[L]$	$[M^0L^{-1}T^0]$
76.	Radiant flux, radiant power	Energy emitted/Time	$[ML^2T^{-2}]/[T]$	$[ML^2T^{-3}]$
77.	Luminosity of radiant flux or radiant intensity	$\frac{\text{Radiant power or radiant flux of source}}{\text{Solid angle}}$	$[ML^2T^{-3}]/[M^0L^0T^0]$	$[ML^2T^{-3}]$
78.	Luminous power or luminous flux of source	$\frac{\text{Luminous energy emitted}}{\text{Time}}$	$[ML^2T^{-2}]/[T]$	$[ML^2T^{-3}]$
79.	Luminous intensity or illuminating power of source	$\frac{\text{Luminous flux}}{\text{Solid angle}}$	$\frac{[ML^2T^{-3}]}{[M^0L^0T^0]}$	$[ML^2T^{-3}]$
80.	Intensity of illumination of luminance	$\frac{\text{Luminous intensity}}{(\text{Distance})^2}$	$\frac{[ML^2T^{-3}]}{[L^2]}$	$[ML^0T^{-3}]$
81.	Relative luminosity	$\frac{\text{Luminous flux of a source of given wavelength}}{\text{Luminous flux of peak sensitivity wavelength (555nm) source of same power}}$	$\frac{[ML^2T^{-3}]}{[ML^2T^{-3}]}$	$[M^0L^0T^0]$
82.	Luminous efficiency	$\frac{\text{Total luminous flux}}{\text{Total radiant flux}}$	$\frac{[ML^2T^{-3}]}{[ML^2T^{-3}]}$	$[M^0L^0T^0]$
83.	Illuminance or illumination	$\frac{\text{Luminous flux incident}}{\text{Area}}$	$\frac{[ML^2T^{-3}]}{[L^2]}$	$[ML^0T^{-3}]$

Table 2.2 (*Continued*)

Sr. No.	Physical Quantities	Formula	Dimensions	Dimensional Formula
84.	Mass defect	Sum of masses of nucleons – Mass of the nucleus	[M]	$[ML^0T^0]$
85.	Binding energy of nucleus	Mass defect × (Speed of light in vacuum)^2	$[M][LT^{-1}]^2$	$[ML^2T^{-2}]$
86.	Decay constant	0.693/Half life	$[T^{-1}]$	$[M^0L^0T^{-1}]$
87.	Resonant frequency	(Inductance × Capacitance)$^{-1/2}$	$[ML^{-2}T^{-2}A^{-2}]^{-1/2} \times [M^{-1}L^{-2}T^4A^2]^{-1/2}$	$[M^0L^0A^0T^{-1}]$
88.	Quality factor or Q-factor of coil	$\frac{\text{Resonant frequency} \times \text{Inductance}}{\text{Resistance}}$	$\frac{[T^{-1}][ML^2T^{-2}A^{-2}]}{[ML^2T^{-3}A^{-2}]}$	$[M^0L^0T^0]$
89.	Power of lens	(Focal length)$^{-1}$	$[L^{-1}]$	$[M^0L^{-1}T^0]$
90.	Magnification	$\frac{\text{Image distance}}{\text{Object distance}}$	$\frac{[L]}{[L]}$	$[M^0L^0T^0]$
91.	Fluid flow rate	$\frac{(\pi/8)(\text{Pressure})(\text{Radius})^4}{\text{Viscosity coefficient} \times \text{Length}}$	$\frac{[ML^{-1}T^{-2}L^4]}{[ML^{-1}T^{-1}][L]}$	$[M^0L^3T^{-1}]$
92.	Capacitive reactance	Angular frequency × (Capacitance)$^{-1}$	$[T^{-1}]^{-1}[M^{-1}L^{-2}T^4A^2]^{-1}$	$[ML^2T^{-3}A^{-2}]$
93.	Inductive reactance	Angular frequency × Inductance	$[T^{-1}][ML^2T^{-2}A^{-2}]$	$[ML^2T^{-3}A^{-2}]$

Table 2.3 Physical Quantities Having Same Dimensional Formula

Sr. No.	Physical Quantities	Dimensional Formula
1.	Frequency, angular frequency, angular velocity, velocity gradient	$[M^0L^0T^{-1}]$
2.	Work, internal energy, potential energy, kinetic energy, torque, moment of force	$[ML^2T^{-2}]$
3.	Pressure, stress, Young's modulus, bulk modulus, modulus of rigidity, energy density	$[ML^{-1}T^{-2}]$
4.	Momentum and impulse	$[MLT^{-1}]$
5.	Acceleration due to gravity, gravitational field intensity	$[M^0LT^{-2}]$
6.	Thrust, force, weight, energy gradient	$[MLT^{-2}]$

(*Continued*)

Table 2.3 (*Continued*)

Sr. No.	Physical Quantities	Dimensional Formula
7.	Angular momentum and Plenck's constant (h)	$[ML^2T^{-1}]$
8.	Surface tension, surface energy, force gradient, spring constant	$[ML^0T^{-2}]$
9.	If l is length, g is acc. due to gravity, m is mass, k is force constant, R is radius of earth, then $\left(\frac{l}{g}\right)^{1/2}, \left(\frac{m}{k}\right)^{1/2}, \left(\frac{R}{g}\right)^{1/2}$ all have the dimensions of time	$[M^0L^0T]$
10.	If L is inductance, R is resistance C is capacitance then L/R, CR and $\sqrt{LC}$ all have the dimensions of time	$[M^0L^0T]$
11.	Thermal capacity entropy, Boltzmann constant	$[ML^2T^{-2}K^{-1}]$
12.	If p is pressure, V is volume, T is temperature, R is gas constant, m is mass, s is specific heat, L is latent heat, ΔT is rise in temperature then pV, RT, mL, $(ms\Delta T)$ all have dimensions of energy	$[ML^2T^{-2}]$
13.	Work, energy, heat, torque, couple, moment of force have same dimensions	$[ML^2T^{-2}]$
14.	Potential energy (mgh), kinetic energy $\left(\frac{1}{2}mv^2 \text{ or } \frac{1}{2}I\omega^2\right)$, energy contained in an inductance $\left(\frac{1}{2}LI^2\right)$ and electrostatic energy of condenser $\left(\frac{1}{2}QV, \frac{1}{2}CV^2, \frac{Q^2}{2C}\right)$.	$[ML^2T^{-2}]$

2.2 APPLICATIONS OF DIMENSIONAL ANALYSIS

1. To find the unit of a given physical quantity in a given system of units.
2. To convert a physical quantity from one system to the other.

Suppose a physical quantity has the dimensional formula $M^aL^bT^c$.

Let N_1 and N_2 be the numerical values of a quantity in the two systems of units, respectively.

In first system, physical quantity, $Q = N_1M_1^aL_1^bT_1^c = N_1U_1$

In second system, same quantity, $Q = N_2M_2^aL_2^bT_2^c = N_2U_2$

A physical quantity remains the same irrespective of the system of measurement, i.e., $Q = N_1U_1 = N_2U_2$

$$\Rightarrow N_1M_1^aL_1^bT_1^c = N_2M_2^aL_2^bT_2^c \Rightarrow N_2 = N_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$$

So, knowing the quantities on the right-hand side, the value of N_2 can be obtained.

3. To check the dimensional correctness of given physical relation. It is based on the *principle of homogeneity*. According to it, the dimensions of each term on both sides of the equation are the same. It can be also said as *the same nature physical quantities can be added or subtracted.*
4. To derive the correct relationship between different physical quantities.

2.3 LIMITATIONS OF DIMENSIONAL ANALYSIS

1. This method gives no information about the dimensionless constants.
2. Many physical quantities have same dimensions i.e., it is not unique.
3. We cannot derive the dimensional formula if a physical quantity depends on more than three unknown variables.
4. We cannot derive the relation if the physical quantity contains more than one term (say sum or difference of two terms.) e.g., $v^2 = u^2 + 2ax$
5. This method cannot be applied if a quantity depends on trigonometric functions or exponential functions.
6. This method cannot be applied to derive equation containing dimensional constants. i.e., $F \propto \frac{m_1 m_2}{r_2}$, but we do not get any idea about the constant G.
7. If an equation contains two or more variables with the same dimension, then this method cannot be used.

2.4 ERRORS AND ACCURACY

If the measured value is other than the true value then we say there is an *error*. One basic thing on which physical science depends is *measurement*. There are always many factors which influence the measurement. These factors always introduce error may be small, whatever be the level of accuracy. So, no measurement is perfect. We can only minimize the errors using best methods and techniques, but we cannot eliminate them permanently.

Accuracy means the extent to which a measured value agrees with the standard or true value for the measurement. But *precision* means the extent to which a given set of measurements of the same quantity agree with their mean value. This mean value need not be true value. Precise measurement need not be accurate.

As the precision increases, the no. of *significant figures* also increases. Accuracy depends on the *systematic errors* where as precision depends on *random errors*. With increase in accuracy the error decreases.

The accuracy depends on:

1. the range of the instrument used.
2. sensitivity of the instrument.
3. the least count and the zero error of the instrument.
4. effect of environment on the instrument.
5. the size and cost of the instrument.

No measurement of any physical quantity is absolutely correct. The numerical value obtained after measurement is just an approximation. As such it becomes quite important to indicate the degree of *accuracy* (or *precision*) in the measurement done in the experiment. The concept of significant figures helps in achieving this objective.

Significant figures of a measured quantity are all those digits about which we are absolutely sure plus one digit that has a little doubt. Significant figures give the number of meaningful digits in a number.

2.4.1 Rules to Determine the Significant Figures

1. All the digits which are not zero are significant.

   Example: In number 1987, significant figures are 4.

2. If there are zeros between two non-zero digits, then all those zeros are significant.

   Example: In 1708.05, significant figures are 6.

3. If the zeros occur to the right of a decimal point and to the left of a non-zero digit, those zero are not significant.

   Example: In 0.0001987, significant digits are 4.

4. All the zeros to the right of a decimal point and to the left of a non-zero digit are significant.

   Example: The number of significant figures in 1987.00 is 6. In the number 0.0019870, significant figures are 5.

5. In the number 0.0019870, the zeros between 1 and the decimal is not significant. Also, the zero on the left of decimal is not significant. But the last zero i.e. to the right of 7 (i.e. a non-zero digit coming after a decimal) is significant.

6. All the zeros to the right of last non-zero digit are not significant.

   Example: The number of significant figures in 198700 is 4. But all the zero to the right of the last non-zero digit are significant if they are the result of a measurement.

7. All the digits in a measured value of physical quantity are significant.

   Example: Let the distance between two places measured to the nearest poles is 1090 m. In 1090 significant digits are 4.

8. Even if we express the measured quantity in different units, then also there will not be any change in the number of significant figures.

   Examples:

   (a) Length of an object = 11.2 cm, significant figures are 3 and if it is expressed in metre, then it is 0.112 m, again significant figures are 3.

   (b) If original measured quantity is 1,500 mm;

   $1500\ \text{mm} = 1.500 \times 10^3\ \text{mm} = 1.500\ \text{m} = 1.500 \times 10^2\ \text{cm} = 1.500 \times 10^{-3}\ \text{km}$.

   All of the above contain four significant figures.

   (c) If original measured quantity is 1.5 m

   $1.5\ \text{m} = 1.5 \times 10^3\ \text{mm} = 1.5 \times 10^2\ \text{cm} = 1.5 \times 10^{-3}\ \text{km}$

   All of the above contain two significant figures.

(d) If original measured quantity is 150 cm

$150 \text{ cm} = 1.50 \times 10^3 \text{ mm} = 1.50 \text{ m} = 1.50 \times 10^2 \text{ cm} = 1.50 \times 10^{-3} \text{ km}$

All of the above contain three significant figures.

9. If the decimal point in a particular measurement in not shown, the zeros at the right of the number may or may not be significant.

Example: 5000 m can be written as 5×10^3. In this, there is only one significant figure. If we write the length as 5.0×10^3 there are 2 significant figures and in 5.00×10^3 there are 3 significant figures and so on.

10. When we add, subtract, multiply or divide two or more numbers, the accuracy of the result is taken to be equal to the least accurate among them. The number of significant figures in the result will be equal to the number of significant digits in the least accurate number among them.

Examples:

(a) $2.29 + 62.7 = 64.99$, after rounding off to one place of decimal it will become 65.0.

(b) $82.29 - 62.7 = 19.59$, after rounding off to one place of decimal it will become 19.6.

(c) $1.3 \times 1.2 = 1.56$, after rounding off to two significant figures it becomes 1.6.

(d) $\frac{3500}{7.52} = 465.42$. As 3500 has minimum number of significant figure, i.e., two, so the quotient must have two significant figure.

So, $465.42 = 470$ (after rounding off).

(e) If we divide 3500 m by 7.52, 3500 m has four significant figure, then final result should be 465 (after rounding off to three significant figures).

2.4.2 Rounding Off

Correcting or reshaping a physical quantity with least deviation from its original value after dropping the last digits which are not required is called rounding off.

Rules Regarding Rounding Off

1. If digit to be dropped is less than 5, then the preceding digit remains unchanged.

Examples: (a) 7.32 after rounding off becomes 7.3

(b) 4.934 after rounding off becomes 4.93

2. If digit to be dropped is more than 5, then the preceding digit is increased by one.

Examples: (a) 7.86 after rounding off becomes 7.9

(b) 6.937 after rounding off becomes 6.94

3. If digit to be dropped is 5:

(i) If it is only 5 or 5 followed by zero, then the preceding digit is raised by one if it is odd and left unchanged if it is even.

Examples: (a) 5.750 after rounding off becomes 5.8

(b) 5.75 after rounding off becomes 5.8

(c) 5.650 after rounding off becomes 5.6

(d) 5.65 after rounding off becomes 5.6

(ii) If 5 is further followed by a non-zero digit, the preceding digit is raised by one.

Example: (a) 15.352 after rounding off becomes 15.4

(b) 9.853 after rounding off becomes 9.9

4. During multistep calculations one digit more than the significant figures should be retained and at the end of the calculation, final result should be round off to proper significant figures.

Number	Significant Figures	Number	Significant Figures
2846	4	0.049960	5
7.080×10^5	4	0.001996	4
109	3	3996.00	6
2.09×10^5	3	123	3
67.8 ± 0.3	3	420.0 m	4
0.123	3	6.0023	5
91.000 m	5	0.0456	3
2.520×10^7	4	0.007 m^2	1
1.20×10^3	3	4200	2
4200 m	4	2400 kg	4

2.5 TYPES OF ERRORS

2.5.1 Constant Errors

An error which is continuously and constantly repeated during all the observations made, is called *constant error*. This arises due to the faulty calibrations of the measuring instruments. e.g. let a scale reads 1.1 cm for every 1 cm, due to wrong calibration, then the scale will show this error of 0.1 cm for all the measurements made using this scale.

2.5.2 Systematic Errors

Errors which are due to known causes acting according to a definite law are called *systematic errors*. The measurement is made under constant condition and hence the errors repeat constantly or systematically. These are of various types:

1. **Instrumental Errors:** Examples are zero error of screw gauge, vernier calipers etc., faulty calibration on thermometer, ammeter, voltmeter etc., in equality of balance arms in a physical balance, back lash error in instruments with nut and screw, like microscope etc.

2. Environmental errors.
3. Error due to observation, e.g. parallax error.
4. Error due to imperfection e.g. whatever precautions are taken, heat is always lost from a calorimeter due to radiation etc.

2.5.3 Random Errors

The errors which occurs irregularly and at random in magnitude and direction are called *random errors.* These errors are not due to any definite cause and so they are also called *accidental errors.* Such errors may be avoided by taking the measurements a number of times and then finding the *arithmetic mean*, i.e.

$$\bar{a} = \frac{a_1 + a_2 + a_3 + a_4 + \dots\dots\dots + a_n}{n} \Rightarrow \bar{a} = \left(\frac{1}{n}\right)\sum_{i=1}^{n} a_i$$

This arithmetic mean is supposed to be the *accurate* observation.

2.5.4 Gross Errors

The errors caused due to the carelessness of the person are called gross errors. So, these errors are called mistakes.

$$Absolute\ error = (True\ value) - (Measured\ value)$$

Taking the *arithmetic mean as the true value*, the absolute error in i^{th} observation in

$$\Delta a_i = (\bar{a} - a_i)$$

i.e., For the first observation, $\Delta a_1 = \bar{a} - a_1$; For the 2nd observation, $\Delta a_2 = \bar{a} - a_2$ and so on.

Mean absolute error: $\overline{\Delta a} = \dfrac{|\Delta a_1| + |\Delta a_2| + |\Delta a_3| + \dots\dots\dots + |\Delta a_n|}{n}$

$$= \left(\frac{1}{n}\right)\sum_i |\Delta a_i|$$

Relative error $= \dfrac{\text{Mean absolute error}}{\text{True value}} = \dfrac{\overline{\Delta a}}{\bar{a}}$ and

Percentage error $= \left(\dfrac{\overline{\Delta a}}{\bar{a}}\right) \times 100\%$

2.6 PROPAGATION OF ERRORS

The error in the final result depends on the errors in the individual measurements and on the nature of mathematical operations performed to get the final result.

2.6.1 Addition

Let $z = x + y$.

Let the absolute errors in the two quantities x and y be Δx and Δy. Their corresponding observed values are $(x \pm \Delta x)$ and $(y \pm \Delta y)$. Hence

$$(z \pm \Delta z) = (x \pm \Delta x) + (y \pm \Delta y)$$

$$\Rightarrow \quad (z \pm \Delta z) = (x \pm y) \pm \Delta x \pm \Delta y$$

$$\therefore \quad \pm \Delta z = \pm \Delta x \pm \Delta y$$

Hence, the maximum possible error in z is given by $\Delta z = \Delta x + \Delta y$.

2.6.2 Subtraction

Let $z = x - y$

Let the absolute errors in the two quantities x and y be Δx and Δy. Their corresponding observed values are $(x \pm \Delta x)$ and $(y \pm \Delta y)$. Hence

$$z \pm \Delta z = (x \pm \Delta x) - (y \pm \Delta y)$$

$$\Rightarrow \quad z \pm \Delta z = (x - y) \pm \Delta x \pm \Delta y$$

$$\therefore \quad \pm \Delta z = \pm \Delta x + \Delta y$$

Hence, the maximum possible error in z is $\Delta z = (\Delta x + \Delta y)$.

2.6.3 Multiplication

Let $z = xy$

Let the absolute errors in the two quantities x and y be Δx and Δy. Their corresponding observed values are $(x \pm \Delta x)$ and $(y \pm \Delta y)$. Hence

$$z \pm \Delta z = (x \pm \Delta x)(y \pm \Delta y)$$

$$= xy \pm x\,\Delta x \pm y\,\Delta x \pm \Delta x.\,\Delta y$$

Neglecting $\Delta x\,\Delta y$ w.r.t. other terms, then $\pm \Delta z = \pm x\,\Delta y \pm y\,\Delta x$

$$\Rightarrow \quad \left(\frac{\pm \Delta z}{z}\right) = \pm\left(\frac{x \Delta y}{z}\right) \pm \left(\frac{y \Delta x}{z}\right)$$

$$= \pm\left(\frac{x \Delta y}{xy}\right) \pm \left(\frac{y \Delta x}{xy}\right)$$

$$\Rightarrow \quad \pm\left(\frac{\Delta z}{z}\right) = \pm\left(\frac{\Delta y}{y}\right) \pm \left(\frac{\Delta x}{x}\right)$$

Hence, maximum relative error in z is $\left(\frac{\Delta z}{z}\right) = \left(\frac{\Delta x}{x}\right)\left(\frac{\Delta y}{y}\right)$

Percentage error is $\quad \left(\frac{\Delta z}{z}\right) \times 100 = \left(\frac{\Delta x}{x}\right) \times 100 + \left(\frac{\Delta y}{y}\right) \times 100$

2.6.4 Division

Let $z = \left(\frac{x}{y}\right)$

Let the absolute errors in the two quantities x and y be Δx and Δy. Their corresponding observed values are $(x \pm \Delta x)$ and $(y \pm \Delta y)$. Hence

$$z \pm \Delta z = \left(\frac{x \pm \Delta x}{y \pm \Delta y}\right) = (x \pm \Delta x)(y \pm \Delta y)^{-1}$$

$$= x\left(1 \pm \frac{\Delta x}{x}\right)y^{-1}\left(1 \pm \frac{\Delta y}{y}\right)^{-1}$$

$\Rightarrow$
$$z \pm \Delta z = \left(\frac{x}{y}\right)\left(1 \pm \frac{\Delta x}{x}\right)\left(1 \pm \frac{\Delta y}{y}\right)^{-1}$$

$\Rightarrow$
$$z \pm \Delta z = \left(\frac{x}{y}\right)\left(1 \pm \frac{\Delta x}{x}\right)\left(1 \mp \frac{\Delta y}{y}\right)$$

Dividing both sides by z,

$$\frac{z \pm \Delta z}{z} = \left(1 \pm \frac{\Delta x}{x}\right)\left(1 \mp \frac{\Delta y}{y}\right)$$

$\Rightarrow$
$$\left(1 \pm \frac{\Delta z}{z}\right) = \left(1 \pm \frac{\Delta x}{x}\right)\left(1 \pm \frac{\Delta y}{y}\right)$$

$$= 1 \pm \frac{\Delta x}{x} \mp \frac{\Delta y}{y} \mp \left(\frac{\Delta x}{x}\right)\left(\frac{\Delta y}{y}\right)$$

$\Rightarrow$
$$\pm\left(\frac{\Delta z}{z}\right) = \pm\frac{\Delta x}{x} \mp \frac{\Delta y}{y}$$

Hence, the maximum possible relative error in z is

$$\left(\frac{\Delta z}{z}\right) = \left(\frac{\Delta x}{x}\right) + \left(\frac{\Delta y}{y}\right)$$

2.6.5 Power of Observed Quantities

1. $z = x^m$

   Taking log on both sides, $\log z = m \log x$

   Differentiating, $\left(\frac{1}{z}\right)\Delta z = m\left(\frac{1}{x}\right)\Delta x$

   i.e., Relative error in $z = m$ times relative error in x.

2. Let $z = \dfrac{x^m y^n}{w^p}$

Taking log on both the sides, $\log z = m \log x + n \log y - p \log w$

Differentiating, $\left(\dfrac{\Delta z}{z}\right) = m\left(\dfrac{\Delta x}{x}\right) + n\left(\dfrac{\Delta y}{y}\right) - p\left(\dfrac{\Delta w}{w}\right)$

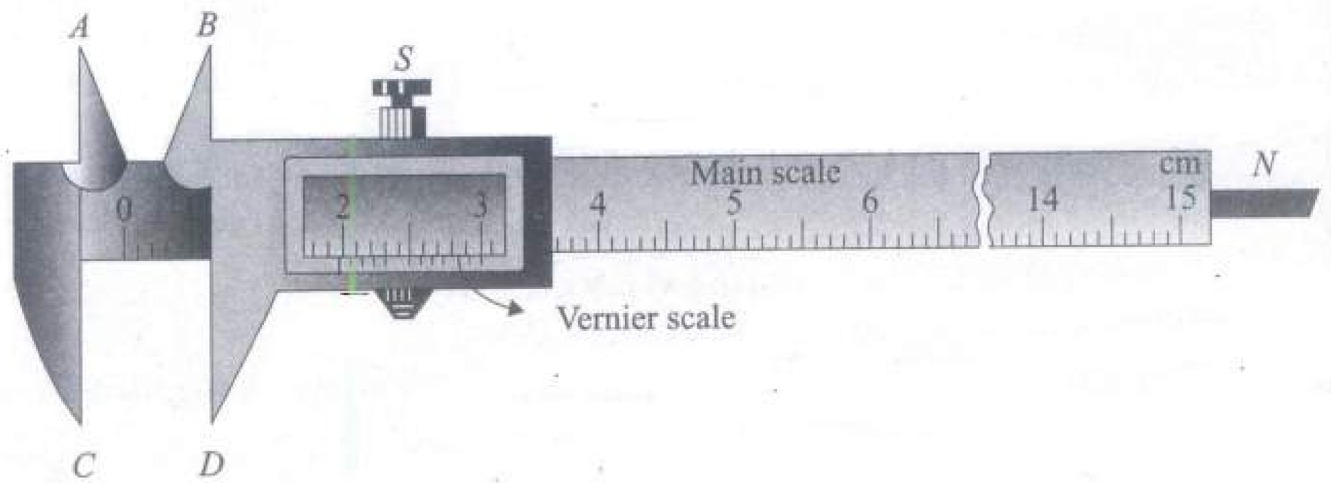
Maximum value of relative error in z is obtained by adding the relative error is the quantity w.

Hence, maximum relative error is

$$\left(\frac{\Delta z}{z}\right) = m\left(\frac{\Delta x}{x}\right) + n\left(\frac{\Delta y}{y}\right) + p\left(\frac{\Delta w}{w}\right)$$

2.7 VERNIER CALLIPERS

It is a device used to measure accurately up to (1/10)th of a millimetre. It was designed by a French Mathematician Pierre Vernier, and hence the instrument is named *Vernier* after the name of its inventor.



Vernier Callipers comprises of two scales, viz., the *vernier scale V* and *main scale S*. The main scale *S* is fixed but the vernier scale, which is also called *auxiliary scale*, is movable. The vernier scale slides along the main scale, as shown in figure.

The divisions of the vernier scale are either slightly longer or slightly smaller than the divisions of the main scale. In general the vernier scale has 10 divisions over a length of 9 divisions of main scale.

Main scale has two fixed jaws *A* and *C* as shown while *B* and *D* are the jaws of vernier scale, the position of vernier scale is fixed with the help of screw *S*.

The upper ends *A* and *B* are used to measure the internal dimensions of the hollow objects e.g., diameter of hollow cylinder. The lower ends *C* and *D* are used to measure the lengths of objects that are gripped between them. The strip *N* is attached to the vernier scale. It slides over the main scale along with the vernier scale. This strip is used to measure the depths of hollow objects.

2.7.1 Determination of Least Count

The smallest value of a physical quantity which can be measured accurately with an instrument is called the *least count* (L.C.) of the measuring instrument. For an instrument where vernier is used, its V.C. (*vernier constant*) is its least count. V.C. is equal to difference of one main scale division and one vernier scale division.

Note the value of the main scale division and count the number n of vernier scale division. Slide the movable jaw till the zero of vernier scale coincides with any of the mark on the main scale and find the number of division $(n-1)$ on the main scale coinciding with n division on vernier scale.

Then, n V.S.D. $= (n-1)$ M.S.D. or 1 V.S.D. $= \left(\frac{n-1}{n}\right)$ M.S.D.

or V.C. or L.C. = 1M.S.D. – 1V.S.D. $= \left(1-\frac{n-1}{n}\right)$M.S.D. $= \frac{1}{n}$ M.S.D.

e.g. 1 M.S.D. = 1 mm and 10 V.S.D. = 9 M.S.D.

$\therefore$ 1V.S.D. $= \frac{9}{10}$ M.S.D. = 0.9

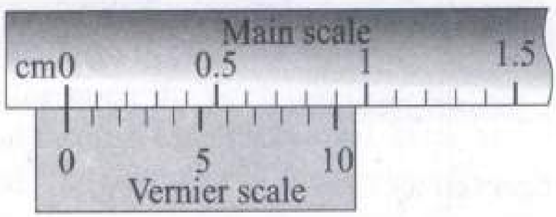
Vernier constant,

V.C. = 1 M.S.D. – 1 V.S.D. = (1 – 0.9) mm = 0.1 = 0.01 cm

2.7.2 Determination of Zero Error and Zero Correction

If the zero marks of the main scale and vernier scale may not be coincide when the jaws are made to touch each other, then it gives rise to an error called *zero error*. Zero error can be positive or negative.

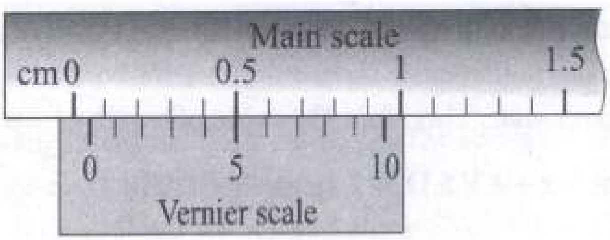
Zero of Vernier Scale Coincides with Zero of Main Scale



Zero error-zero

In this case, zero error and zero correction, both are nil.
Actual length = Observed (measured) length.

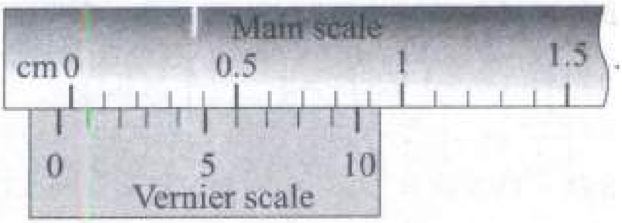
Zero of Vernier Scale Lies on the Right of Main Scale



Zero error-positive

Here, 5th vernier scale division is coinciding with any main scale division.
Hence, $N = 0$, $n = 5$, L.C. = 0.01 cm
Zero error = $N + n \times$ (L.C.) = $0 + 5 \times 0.01 = +0.05$ cm
Zero correction = -0.05 cm
Actual length will be 0.05 cm less than the observed (measured) length.

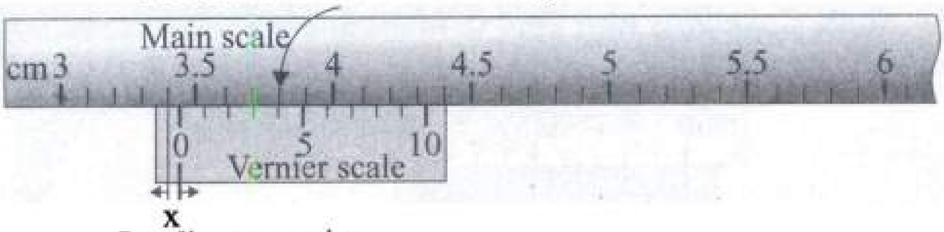
Zero of Vernier Scale Lies on the Left of Zero of Main Scale



Zero error-negative

Here, 6th vernier scale division is coinciding with any main scale division.
In this case, zero of vernier scale lies on the right of -0.1 cm reading on main scale
Hence, $N = -0.1$ cm, $n = 6$, L.C. = 0.01 cm
Zero error = $N + n \times$ (L.C.) = $-0.1 + 6 \times 0.01 = -0.06$ cm
Zero correction = $+0.06$ cm
Actual length will be 0.06 cm more than the observed (measured) length.

2.7.3 Reading a Vernier



Reading a vernier

Suppose that while measuring the length of an object, the positions of the main scale and vernier scale are as shown in figure. First of all, we read the position of the zero of the vernier on the main scale. As it is quite clear, the zero position of the vernier lies between 3.4 cm and 3.5 cm. In fact the objective of this instrument is to accurately measure the small distance x which lies between zero mark of the vernier scale and 3.4 cm mark on the main scale. We can see that x cannot be directly read on the main scale as this length is smaller than the smallest division on the main scale.

Next we have to find out which division on the vernier scale exactly coincides with some division of the main scale. In above figure it is quite clear that the 4th division of the vernier scale coincides with some division of the main scale. Therefore, the value of length of x will be given by the relation.

$$3.4 \text{ cm} + x + 4 \text{ V.S.D.} = 3.4 \text{ cm} + 4 \text{ M.S.D}$$

$$\Rightarrow \quad x = 4 \text{ M.S.D.} - 4 \text{ V.S.D.}$$

$$= 4(1 \text{M.S.D.} - 1 \text{ V.S.D.})$$

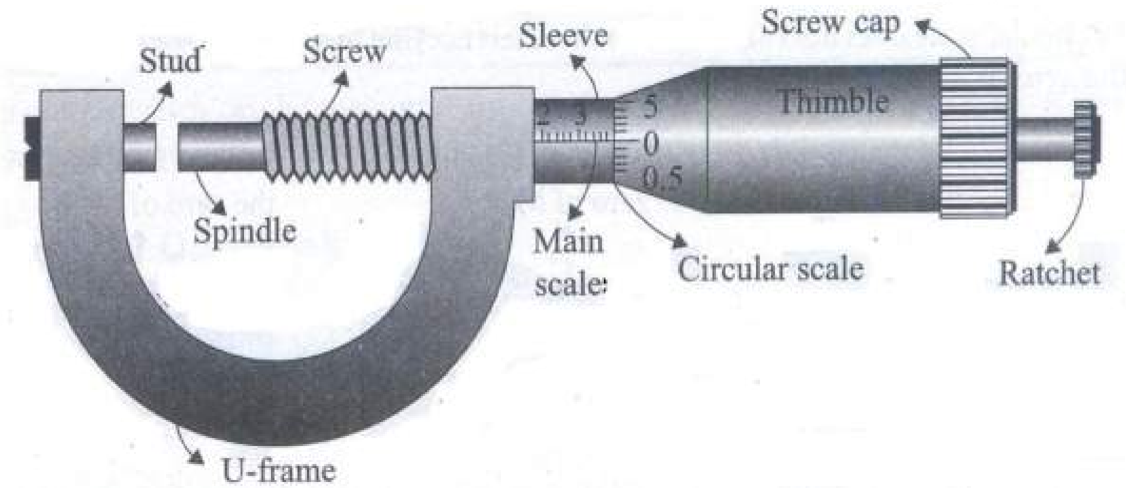
$$= 4 \times \text{L.C.}$$

$$= 4 \times 0.01 = 0.04 \text{ cm} \qquad (\because \text{L.C.} = 0.1 \text{ mm} = 0.01 \text{ cm})$$

$$\therefore \qquad \text{length of the object} = 3.4 \text{ cm} + x$$
$$= 3.4 \text{ cm} + 0.04$$
$$= 3.44 \text{ cm}$$

Screw Gauge

In general vernier calliper can measure accurately upto 0.01 cm and for greater accuracy micrometer screw devices e.g., screw gauge, spherometer are used. *Screw gauge* works on the principle of micrometer screw.



It consists of a U-shaped metal frame. At one end of it a small metal piece of gun metal is fixed. It is called *stud* and it has a plane face. The other end of U-frame carries a cylindrical hub. The hub extends few millimeter beyond the end of the frame. On the cylindrical hub along its axis, a line known as *reference line* is drawn. On the reference line graduations are in millimeter and half millimeter depending upon the pitch of the screw. This scale is called *linear scale* or *pitch scale*. A nut is threaded through the hub and the frame and through the nut moves, there is a *screw* made of gun metal. The front face of the screw is also plane. A hollow cylindrical cap is capable of rotating over the hub when screw is rotated. It is attached to the right hand end of the screw, as the cap is rotated the screw either moves in or out. The bevelled surface of the cap is divided into 50 or 100 equal parts. It is called the *circular scale* or *head scale*. Right hand end, a *ratchet* is fixed and it is milled for proper grip.

In most of the instruments the milled head is not fixed to the screw head but turns it by a spring and rather arrangement such that when the body is just held between faces of studs, the spring yields and milled head turns without moving in the screw.

In an accurately adjusted instrument when the faces of studs are just touching each other, the zero marks of circular scale and pitch scale exactly coincide.

1. **Pitch:** It is defined as the linear distance moved by the screw forward or backward when one complete rotation is given to the circular cap.

$$\text{Pitch of the screw} = \frac{\text{Distance moved on linear scale}}{\text{Number of rotation}}$$

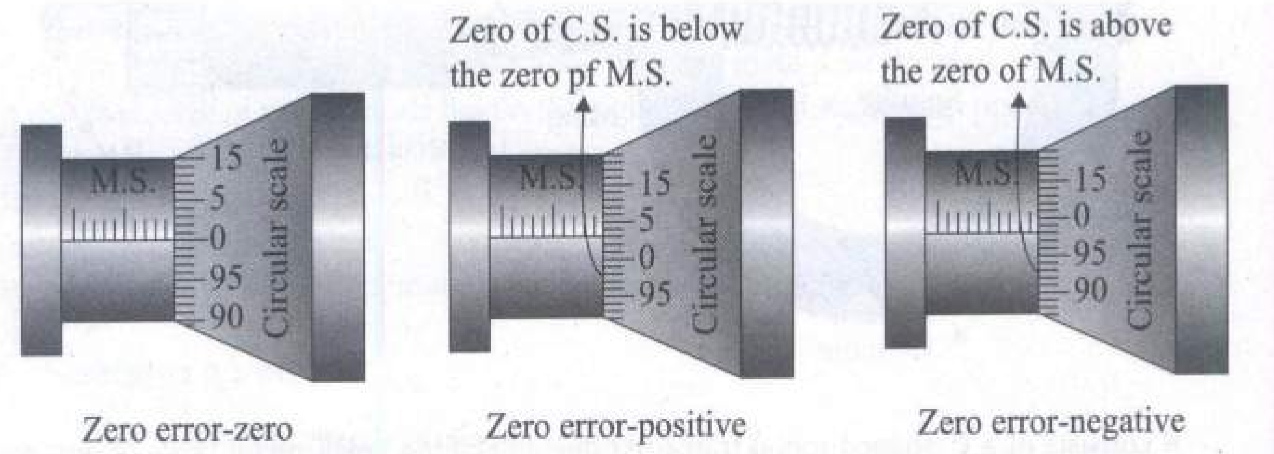
2. Least count: It is defined as ratio of the pitch of the screw to the total number of divisions on circular scale.

$$\text{L.C. of the screw gauge} = \frac{\text{Pitch of the screw}}{\text{Total number of divisions on the circular scale}}$$

For example, if the pitch of the screw gauge is 0.5 mm and the total number of divisions on the circular scale is 100, then the least count will be given by, $\frac{0.5\text{mm}}{100} = 0.005$ mm.

This implies that the minimum length that can be measured accurately with the screw gauge is 0.005 mm.

3. Zero error: When the two studs of the screw gauge are brought in contact and if the zero of the circular scale does not coincide with the reference line then the screw gauge has an error. This error is called *zero error.*



(a) *Positive zero error:* Zero error is said to be positive if the zero of the circular scale lies below the reference line as shown in figure.

For example, the 4th division of the head scale is in line with the line of graduation.

Then, the zero error = + 4 × L.C. = + 4 × 0.01 mm = 0.04 mm

zero correction = –0.04 mm

(b) *Negative zero error:* Zero error is said to negative if the zero of the circular scale lies above the reference line as shown in figure.

For example, 97th division of the head scale is in line with the line of graduation.

Thus, zero error = (97 – 100) × L.C. = –3 × 0.01 mm = –0.03 mm

zero correction = + 0.03 mm

4. Reading of a screw gauge: Place a wire between studs and let the edge of the cap lies ahead of Nth division of linear scale. Then Linear scale reading (LSR) = N

If nth division of circular scale lies over reference line, then

Circular scale reading (CSR) = $N \times$ (LC)

where LC is the least count of the screw gauge

Total reading = LSR + CSR = $N + n \times$ (LC) i.e. the diameter of the wire.

VECTORS AND SCALARS

3.1 PHYSICAL QUANTITIES

The quantities which can be measured are called physical quantities. Physical quantities are of two types: scalar quantities or scalars and vector quantities or vectors.

3.1.1 Scalar Quantities

A physical quantity which is completely known by its magnitude only i.e., a physical quantity which has only magnitude and has no direction, is called a scalar quantity or simply a scalar.

Examples: Mass, length, volume, density, time, temperature, pressure, speed and work.

3.1.2 Vector Quantities

Those physical quantities which have both magnitude and definite direction in space are called vector quantities. Thus, a vector is that physical quantity which is completely known only when its magnitude and direction are known and obeys the laws for vectors.

Examples: Force, acceleration, displacement and momentum.

Localized Vector

A vector is said to be a localized vector if it passes through a fixed point in space. Thus, a localized vector cannot be shifted parallel to itself.

Free Vector

A vector is said to be free vector if it is not localized. Thus a free vector can be taken anywhere in space.

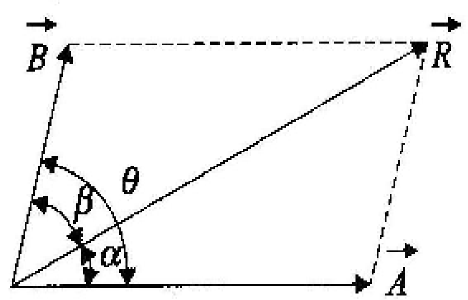
Unless otherwise stated all vectors will be considered as free vectors.

Vector Addition of Two Vectors

Law of parallelogram of vector addition or triangle law of vector addition

$$\vec{R} = \vec{A} + \vec{B}, \; R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta} \text{ and } \tan\beta = \frac{A\sin\theta}{B + A\cos\theta}$$



Vector Addition of More Than Two Vectors

Above method can be applied for only two vectors and the component method or polygon law of vector addition can be applied for resultant of two or more than two vectors.

Vector addition is commutative i.e., if $\vec{a}$ and $\vec{b}$ be any two vectors, then $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

Vector addition is associative, i.e., if $\vec{a},\vec{b}$ and $\vec{c}$ be any three vectors, then $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$.

Vector addition is distributive, i.e., if $\vec{a}$ and $\vec{b}$ be any two vectors, then $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

Vector Substraction

Vector substraction is not a new kind of vector operation but it is also the resultant of 1st vector and reverse of 2nd vector.

If $\vec{S}=\vec{A}-\vec{B}$ and $S=|\vec{S}|$,

then $\vec{S}=\vec{A}+(-\vec{B})$, $S=\sqrt{A^2+B^2-2AB\cos\theta}$

Null Vector

It is a vector which has zero magnitude and an arbitrary direction. It is represented by $\vec{0}$ and is also known as zero vector. The need for the concept of zero vector arises to have a sense of completeness in the vector algebra as is clear from the following examples which give us the main properties of zero vector.

1. $\vec{a}+(-\vec{a})=\vec{0}, i.e., \vec{a}=\vec{a}+\vec{0}$
2. $\vec{a}-\vec{b}=\vec{0}$, when $\vec{a}=\vec{b}$
3. $\vec{a}\times\vec{0}=\vec{0}$
4. $(n_1+n_2)\vec{a}=\vec{0}$ when $\vec{a}=\vec{b}$

In all the operations, the result has to be a vector and not a scalar. It is to meet such situations that we have introduced the concept of zero vector. Thus, the concept of zero vector

1. makes vector algebra complete.
2. represents physical quantities in a number of situations.

Physical Meaning of Zero Vector

1. It represents the position vector of the origin.
2. It represents the displacement vector of a stationary particle.
3. It represents the acceleration vector of a particle moving with uniform velocity.

Rotation of a Vector

1. If the frame of reference is rotated or translated, the given vector does not change. The components of the vector may, however, change.
2. If a vector is rotated through an angle θ, which is not an integral multiple of 2π, the vector changes.

3.1.3 Dot Product of Two Vectors

It is the multiplication of two vectors such that the field is a scalar quantity and it is $\vec{A}\vec{B} = AB\cos\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$. $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$.

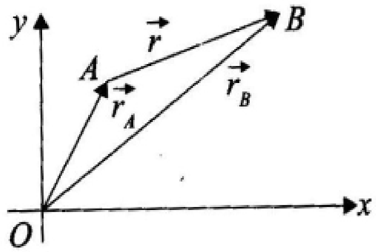
1. The scalar product is commutative i.e., $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
2. $\vec{A}(-\vec{B}) = -\vec{A}\vec{B}, (\vec{A}+\vec{B})^2 = A^2 + B^2 + 2\vec{A}\cdot\vec{B}$

   $(\vec{A}-\vec{B})^2 = |\vec{A}-\vec{B}|^2 = A^2 + B^2 - 2\vec{A}\cdot\vec{B}$

   $(\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B}) = \vec{A}^2 - \vec{B}^2 = A^2 - B^2$
3. The scalar product is distributive over addition i.e., $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$.
4. Ordinary algebraic laws are true for a dot product.
5. If θ is acute, dot product is positive. If θ is obtuse dot product is negative and if θ is 90° dot product is zero. Hence dot product of two perpendicular vectors is zero.
6. The scalar product of two identical vectors $\vec{A}\cdot\vec{A} = A^2$
7. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1, \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{i}\cdot\hat{k} = 0$
8. The scalar product of two non-zero orthogonal (i.e., perpendicular) vectors is zero.
9. The scalar product of two vectors $\vec{A}$ and $\vec{B}$ varies from AB to $(-AB)$
10. Scalar component of $\vec{A}$ along $\vec{B} = A\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{B}$
11. Vector component of $\vec{A}$ along $\vec{B} = \left(\dfrac{\vec{A}\cdot\vec{B}}{B}\right)\hat{B}$
12. Scalar component of $\vec{B}$ along $\vec{A} = B\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{A}$
13. Vector component of $\vec{B}$ along $\vec{A} = \left(\dfrac{\vec{A}\cdot\vec{B}}{A}\right)\hat{A}$
14. Vector component of $\vec{A}$ perpendicular to $\vec{B} = \vec{A} - \dfrac{\vec{A}\cdot\vec{B}}{|\vec{B}|^2}\vec{B}$
15. The geometrical meaning of scalar product or dot product of two vectors is the product of magnitude of one vector and the projection of 2nd vector along the 1st vector or the product of magnitude of 2nd vector and the projection of 1st vector along the 2nd vector.
16. Angle between two vectors: $\theta = \cos^{-1}\left(\dfrac{\vec{A}\cdot\vec{B}}{AB}\right)$.
17. Condition for two vectors to be parallel: Let $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

   if $\vec{a}$ and $\vec{b}$ are parallel, then $\dfrac{a_x}{b_x} = \dfrac{a_y}{b_y} = \dfrac{a_z}{b_z}$

Position Vector and Displacement Vector



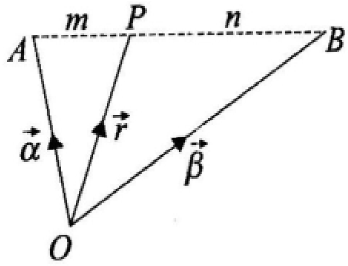
1. If coordinates of point A are (x_1, y_1, z_1) and coordinates of point B are (x_2, y_2, z_2). Then

   $\vec{r}_A$ = Position vector of $A = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

   $\vec{r}_B$ = Position vector of $B = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

   $\vec{r} = \vec{r}_B - \vec{r}_A = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ = Displacement vector from A to B

2. If $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B respectively and P divides the line segment AB internally in the ratio m: n, the position vector of P is given by



$$\vec{r} = \frac{m\vec{\beta} + n\vec{\alpha}}{m+n}$$

   If the division is external, then

$$\vec{r} = \frac{m\vec{\beta} - n\vec{\alpha}}{m-n}$$

3. Positon vector of the middle point of the line segment AB is given by

$$\vec{r} = \frac{\vec{\alpha} + \vec{\beta}}{2}$$

3.1.4 Cross Product of Two Vectors

The cross product of two vectors is multiplication of two vectors such that the yield is a vector quantity.

Let $\vec{C} = \vec{A} \times \vec{B}$, then $\vec{C} = |\vec{C}| = AB \sin\theta$ where θ is the angle between $\vec{A}$ and $\vec{B}$.

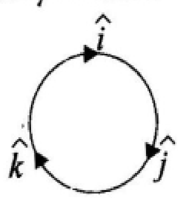
Direction of $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$ given by *right hand law*. We can also say that $\vec{C}$ is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

1. Vector product is not commutative. It is anticommutative, i.e.,

   $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

2. Cross product of two parallel or antiparallel vectors is a null vector. A vector whose magnitude is zero and has any arbitrary direction is called as null vector or zero vector.
3. Cross product of two vectors of given magnitudes has maximum value when they act at 90°
4. $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}, \hat{i} \times \hat{k} = -\hat{j}, \hat{k} \times \hat{j} = -\hat{i}$

   $\hat{j} \times \hat{i} = -\hat{k}, \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k}$ = a null vector



5. The magnitude of the vector product of two vectors $\vec{A}$ and $\vec{B}$ varies from 0 to AB.
6. If $\vec{A} \neq 0$, $\vec{B} \neq 0$, then $\vec{A} \times \vec{B} = 0 \Rightarrow \vec{A} \| \vec{B}$

7. If $\vec{A}$ and $\vec{B}$ are parallel, then $\vec{A}\times\vec{B}=0$

8. Angle θ between vectors $\vec{A}$ and $\vec{B}$ is given by $\sin\theta=\dfrac{|\vec{A}\times\vec{B}|}{|\vec{A}||\vec{B}|}$

9. $(\vec{A}\times\vec{B})^2+(\vec{A}\cdot\vec{B})^2=A^2B^2$

10. The geometrical meaning of vector product or cross product of two vectors is the area of the parallelogram formed by the two vectors as its adjacent sides.

11. If $\vec{d}_1$ and $\vec{d}_2$ are the diagonals of the parallelogram, then it can be easily shown that the area of the parallelogram $=\dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$

12. The diagonals of a parallelogram make four triangles with sides $\dfrac{\vec{d}_1}{2}$ and $\dfrac{\vec{d}_2}{2}$ and area of each triangle $=\dfrac{1}{2}\left|\dfrac{\vec{d}_1}{2}\times\dfrac{\vec{d}_2}{2}\right|=\dfrac{1}{8}|\vec{d}_1\times\vec{d}_2|$.

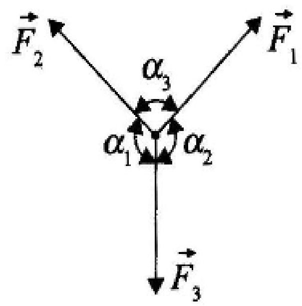
13. Lagrange's identity $|\vec{A}\times\vec{B}|^2+(\vec{A}\cdot\vec{B})^2=A^2B^2\sin^2\theta+A^2B^2\cos^2\theta=A^2B^2=|\vec{A}|^2|\vec{B}|^2$

14. $$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$
$$=(a_yb_z-b_ya_z)\hat{i}+(b_xa_z-b_za_x)\hat{j}+(a_xb_y-b_xa_y)\hat{k}$$

Lami's Theorem

If a body is in equilibrium under three coplanar concurrent forces, then each force is proportional to 'sine' of the angle between remaining two forces. That is;

$$\frac{F_1}{\sin\alpha_1}=\frac{F_2}{\sin\alpha_2}=\frac{F_3}{\sin\alpha_3}=k$$



Unit Vector

A vector whose magnitude is unity is called a unit vector. The unit vector in the direction of $\vec{A}$, is denoted by $\hat{A}$ and is given by,

$$\hat{A}=\frac{\vec{A}}{A}\text{ or }\vec{A}=A\hat{A}$$

1. Unit vector has no unit but magnitude of a vector has unit.

2. If $\hat{i}$ and $\hat{j}$ be the vector along x and y-axes respectively, then unit vector along a line which makes an angle θ with the positive direction of x-axis in anti-clockwise direction is $\cos\theta\hat{i}-\sin\theta\hat{j}$.

   If θ is made in clockwise direction then unit vector is $\cos\theta\hat{i}-\sin\theta\hat{j}$.

3. If $\vec{\alpha}$ and $\vec{\beta}$ be the unit vectors along any two lines then $\vec{\alpha}+\vec{\beta}$ and $\vec{\alpha}-\vec{\beta}$ are the vectors along the lines which bisect the angle between these lines.
4. A unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ is $\hat{C}=\pm\frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$

If vectors are given in terms of $\hat{i},\hat{j}$ and $\hat{k}$

Let $\vec{a}=a_x\hat{i}+a_y\hat{j}+a_z\hat{k}$ and $\vec{b}=b_x\hat{i}+b_y\hat{j}+b_z\hat{k}$, then

1. $|\vec{a}|=a=\sqrt{a_x^2+a_y^2+a_z^2}$ and $|\vec{b}|=b=\sqrt{b_x^2+b_y^2+b_z^2}$
2. $\vec{a}+\vec{b}=(a_x+b_x)\hat{i}+(a_y+b_y)\hat{j}+(a_z+b_z)\hat{k}$
3. $\vec{a}-\vec{b}=(a_x-b_x)\hat{i}+(a_y-b_y)\hat{j}+(a_z-b_z)\hat{k}$
4. Component of $\vec{a}$ along $\vec{b}$ $=a\cos\theta=\frac{\vec{a}\cdot\vec{b}}{b}=\frac{a_xb_x+a_yb_y+a_za_z}{\sqrt{b_x^2+b_y^2+b_z^2}}$

3.1.5 Triple Product of Vectors

Scalar Triple Product

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

1. If the three vectors be coplanar, their scalar triple product is zero, i.e., $\vec{A}\cdot(\vec{B}\times\vec{C})=0$
2. If two of the vectors be equal, the scalar triple product is zero i.e., $[\vec{A}\vec{A}\vec{B}]=(\vec{A}\times\vec{A})\cdot\vec{B}=0$
3. If two vectors are parallel, the scalar triple is zero. Let $\vec{A}$ and $\vec{B}$ are parallel, we can have $\vec{B}=k\vec{A}$, where k is a scalar. Then, $[\vec{A}\vec{B}\vec{C}]=(k\vec{A}\times\vec{A})\cdot\vec{B}=0$
4. The scalar triple product of the orthogonal vector triad is unity i.e., $[\hat{i}\,\hat{j}\,\hat{k}]=(\hat{i}\times\hat{j})\cdot\hat{k}=1$
5. Value of a scalar triple product does not change when cyclic order of vectors is maintained.

   Thus, $\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})$

   i.e., $[\vec{A}\vec{B}\vec{C}]=[\vec{B}\vec{C}\vec{A}]=[\vec{C}\vec{B}\vec{A}]$

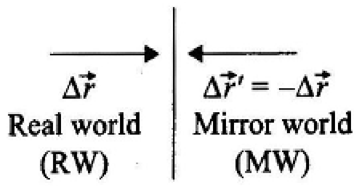
   Also $[\vec{A}\vec{B}\vec{C}]=[\vec{B}\vec{A}\vec{C}]$
6. Scalar triple product $\vec{A}\cdot(\vec{B}\times\vec{C})$ represents the volume of parallelopiped, with the three vectors forming its three edges.

Vector Triple Product

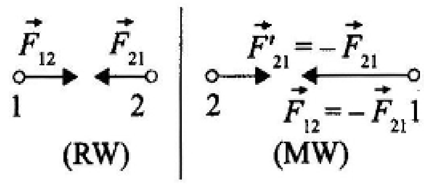
If $\vec{A},\vec{B}$ and $\vec{C}$ are three vectors, then $\vec{A}\times(\vec{B}\times\vec{C}),\vec{B}(\vec{C}\times\vec{A})$ and $\vec{C}\times(\vec{A}\times\vec{B})$ are the examples of vector triple product.

$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}$$

1. **Polar vector:** If the direction of a vector is independent of the co-ordinate system, it is called a polar vector, e.g., displacement, velocity, acceleration, etc.
2. **Axial or pseudo vector:** If the direction of a vector changes with the change of reference frame from right handed to left handed frame, it is called axial or pseudo vector, e.g., angular displacement, angular velocity, etc.
3. **Polar and axial vectors:** The spatial reflection in a plane corresponds to changing the signs of the normal components of the coordinate vectors of all points and leaving the components parallel to the plane unchanged. Thus, for reflection in the xy plane, $\vec{r}_i = (x_i, y_i, z_i) \to \vec{r}_i' = (x_i, y_i, -z_i)$. The space inversion, however, corresponds to reflection of all three components of every coordinate vector through the origin: $\vec{r}_i \to \vec{r}_i' = -\vec{r}_i$. Now polar vectors (or just vectors) are those that behave as $\vec{v} \to \vec{v}' = -\vec{v}$ for $\vec{r} \to \vec{r}' = \vec{r} \to -\vec{r}$. The examples are displacement $(\Delta\vec{r})$, velocity $(\vec{v})$, force $(\vec{F})$, etc. The axial vectors (or pseudovectors) $\vec{A}$ are those that behave as $\vec{A} \to \vec{A}' = \vec{A}$ for $\vec{r} \to \vec{r}' = -\vec{r}$. The examples are angular velocity $\vec{\omega}(\vec{v} = \vec{\omega} \times \vec{r})$, angular momentum $\vec{l} = \vec{r} \times \vec{p}$, and the magnetic component of the Lorentz force $\vec{F}_m = q(\vec{v} \times \vec{B})$. Figure (a) and (e), respectively shows that upon reflection $\vec{v} \to -\vec{v}$ and $\vec{B} \to -\vec{B}$. Thus, upon reflection $\vec{F}_m \to -\vec{F}_m$ and indeed $\vec{F}_m$ is an axial vector.



(a)



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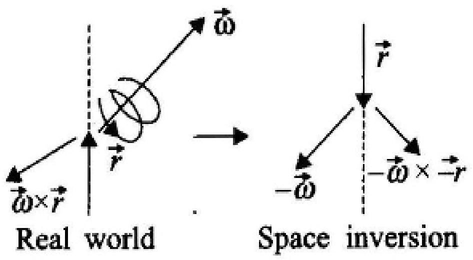
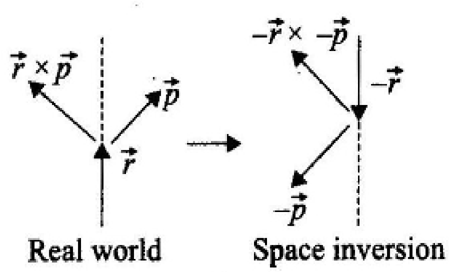


(c)



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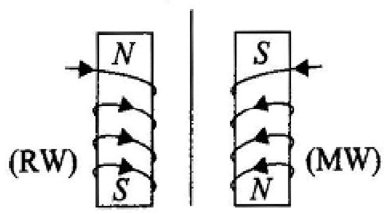


(e)

3.2 SCALAR AND VECTOR FIELD: GRADIENT, DIVERGENCE AND CURL

3.2.1 Scalar Field

If a scalar changes from point to point in space we say that there is a scalar field. For example, if we heat a rod at one end, the temperature of the rod in the steady state will vary from point to point and we say that there is a scalar field and that scalar is temperature.

3.2.2 Vector Field

If a vector changes from point to point in space we say that there is a vector field. For example, velocity of liquid flowing through a tube, magnetic field, electric field etc.

3.2.3 Vector Differential Operator (*Del* Vector)

The operator defined as $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ is called nabla or *del* vector. It is attributed all the properties of a vector and at the same time it is supposed to act as an operator. The most striking property of it is that it remains invariant under rotation of coordinate system.

3.2.4 Gradient

If we operate with $\vec{\nabla}$ on a scalar φ, we obtain a vector which is called the gradient of the scalar. That is,

$$\text{grad } \varphi = \vec{\nabla}\varphi = \hat{i}\frac{\partial \varphi}{\partial x} + \hat{j}\frac{\partial \varphi}{\partial y} + \hat{k}\frac{\partial \varphi}{\partial z}$$

The gradient of a scalar is the rate of space variation along the normal to the surface on which it remains constant or, say, it is the directional derivative of the scalar along normal to the surface on which it remains constant. That is, grad $\varphi = \frac{d\varphi}{dn}\hat{n}$ where $\frac{d\varphi}{dn}$ is the derivative of φ along the normal and $\hat{n}$ is the unit vector along the normal.

3.2.5 Divergence of a Vector

If we make '*del* dot operation' on a vector we obtain a scalar which is called the divergence of the vector. That is,

$$\text{div } \vec{A} = \vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

3.2.6 Curl or Rotation of a Vector

If we make '*del* cross operation' on a vector we get a vector which is called the curl of the vector. That is,

$$\text{curl } \vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Note

Tensor: A physical quantity which has different values in different directions at the same point is called a tensor. Pressure, stress, modulii of elasticity, moment of inertia, radius of gyration, refractive index, wave velocity, dielectric constant, conductivity, resistivity and density are a few examples of tensor. Magnitude of tensor is not unique.

MOTIONS IN ONE, TWO AND THREE DIMENSIONS

4.1 BASIC DEFINITIONS

1. Displacement $\vec{s}=\vec{r}_f-\vec{r}_i=(x_f-x_i)\hat{i}+(y_f-y_i)\hat{j}+(z_f-z_i)\hat{k}$
2. Distance = Actual path length
3. Average velocity $=\dfrac{\text{Total displacement}}{\text{Total time}}=\left(\dfrac{\Delta\vec{r}}{\Delta t}\right)$
4. Average speed $=\dfrac{\text{Total distance}}{\text{Total time}}=\left(\dfrac{\Delta s}{\Delta t}\right)$
5. Average acceleration $=\dfrac{\text{Change in velocity}}{\text{Time interval}}=\left(\dfrac{\Delta\vec{v}}{\Delta t}\right)=\dfrac{\vec{v}_f-\vec{v}_i}{t}$
6. Instantaneous velocity $=\dfrac{d\vec{r}}{dt}$
7. Instantaneous acceleration = Rate of change of velocity $=\dfrac{d\vec{v}}{dt}=\dfrac{d^2\vec{r}}{dt^2}$
8. The instantaneous velocity in magnitude is equal to instantaneous speed.
9. $\left|\dfrac{\text{Average velocity}}{\text{Average speed}}\right|\le 1$

4.1.1 In One-dimensional Motion

1. Instantaneous velocity $=\dfrac{ds}{dt}$ or $\dfrac{dx}{dt}$ = Slope of $x-t$ graph.
2. Instantaneous acceleration $=\dfrac{dv}{dt}$ = Slope of $v-t$ graph.
3. Area under v–t graph = Displacement and area under a–t graph = Change in velocity.
4. In uniform motion along a straight line without change in direction of motion.

$$\frac{d|\vec{v}|}{dt}=0 \text{ and } \left|\frac{d\vec{v}}{dt}\right|=0$$

5. If body moves uniformly but its direction of motion changes, then

$$\frac{d|\vec{v}|}{dt} = 0, \text{ but } \left|\frac{d\vec{v}}{dt}\right| \neq 0$$

4.1.2 One-dimensional Motion with Uniform Acceleration

1. $v = u + at$
2. $s = ut + \frac{1}{2}at^2$
3. $s = s_o + ut + \frac{1}{2}at^2$
4. $v^2 = u^2 + 2as$
5. s_n = Displacement (not distance) in nth sec. $= u + \frac{a}{2}(2n-1)$

While using above equations, substitute all vector quantities (v, u, a, s and s_t) with sign. s_t is the displacement between $(t-1)$ and t sec.

Motion Under Gravity

In the absence of air resistance, all object experience same acceleration due to gravity. The acceleration near the earth surface, $g = 9.8$ m/s². For a freely falling body, we have

1. $v = u + gt$
2. $h = ut + \frac{1}{2}gt^2$
3. $v^2 = u^2 + 2gh$
4. $h_{n_{th}} = u + \frac{g}{2}(2n-1)$

When body is thrown vertically upward, we have

1. $v = u - gt$
2. $h = ut - \frac{1}{2}gt^2$
3. $v^2 = u^2 - 2gh$
4. $h_{n_{th}} = u - \frac{g}{2}(2n-1)$

4.1.3 One-dimensional Motion with Non-uniform Acceleration

If motion is one dimensional with variable acceleration then the just above equations are not valid and then

1. $s - t \xrightarrow{\text{Differentiation}} v - t \xrightarrow{\text{Differentiation}} a - t;$

$$v = \frac{ds}{dt}, a = \frac{dv}{dt} = v\frac{dv}{ds}$$

2. $a - t \xrightarrow{\text{Integration}} v - t \xrightarrow{\text{Integration}} s - t;$

$$\int ds = \int v\,dt, \int dv = \int a\,dt, \int v\,dv = \int a\,ds$$

In many graphs, negative time has been seen. Negative time indicates the past. As time interval never be negative, hence in any calculation the time interval is always placed as positive.

Relative Motion

1. $\vec{v}_{A,B}$ = Velocity of A with respect to $B = \vec{v}_A - \vec{v}_B$
2. $\vec{a}_{A,B}$ = Acceleration of A with respect to $B = \vec{a}_A - \vec{a}_B$

4.1.4 In Two-dimensional Motion

1. $v_{A,B} = v_A - v_B$
2. $a_{A,B} = a_A - a_B$

4.1.5 Two- or Three-dimensional Motion with Uniform Acceleration

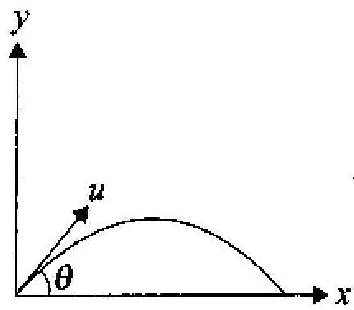
1. $\vec{v} = \vec{u} + \vec{a}t$
2. $\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$
3. $\vec{v} \cdot \vec{v} = \vec{u} \cdot \vec{u} + 2\vec{a} \cdot \vec{s}$

4.1.6 Two- or Three-dimensional Motion with Non-uniform Acceleration

1. $\vec{v} = \frac{d\vec{s}}{dt}$ or $\frac{d\vec{r}}{dt}$
2. $\vec{a} = \frac{d\vec{v}}{dt}$
3. $\int d\vec{v} = \int \vec{a}dt$
4. $\int d\vec{s} = \int \vec{v}dt$

4.1.7 Projectile Motion

When a particle or a body is projected obliquely near the earth surface, it moves simultaneously in horizontal and vertical directions, then its motion is called projectile motion. In projectile motion, the effect of air resistance on the projectile and the effect of curvature of earth are neglected and also it is assumed that the acceleration due to gravity is constant at each point of projectile.



1. $T = \frac{2u\sin\theta}{g} = \frac{2u_y}{g}$
2. $H = \frac{u^2\sin^2\theta}{2g} = \frac{u_y^2}{2g}$
3. $R = \frac{u^2\sin 2\theta}{g} = u_x T = \frac{2u_x u_y}{g}$
4. $R_{max} = \frac{u^2}{g}$ at $\theta = 45°$
5. For a given velocity of projection, a projectile has the same range for angle of projection θ and $(90° - \theta)$;

   In this case $T_1 \cdot T_2 = \frac{2R}{g}$
6. Equation of trajectory: $y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$

   $$= x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta)$$

   The path of the projectile is parabolic.

7. Slope of the path: The slope of the path can be given by differentiating the locus equation or trajectory equation with respect to x.

   Hence, slope $= m = \dfrac{dy}{dx} = \tan\theta\left(1 - \dfrac{x}{2R}\right)$

8. Suppose two particles are projected simultaneously from the same point with initial velocities u_1 and u_2 at an angle θ_1 and θ_2 respectively then the path of first projectile with respect to second projectile is a vertical straight line.

9. For a projectile motion (as in the above figure) the speed of strike and the speed of projection are the same and also radius of curvature at the point of projection and at the point of strike are the same.

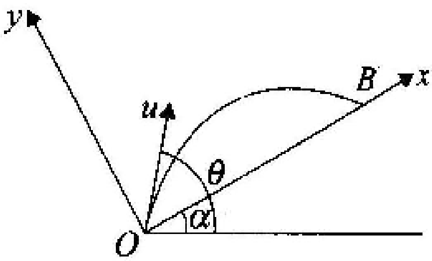
10. Finding the point of collision between two projected bodies: When two bodies projected from same point collide in air, the point of collision of the bodies can be found by solving two trajectory equations (parabolas). If (x, y) are the coordinates of the point of collision, we can write,

$$\frac{y}{x} = \left(\tan\theta_1 - \frac{gx}{2v_1^2\cos\theta_1}\right) = \left(\tan\theta_2 - \frac{gx}{2v_2^2\cos^2\theta_2}\right)$$

   where v_1 and v_2 are the velocities of projection; θ_1 and θ_2 are the angles of projection, respectively. By solving the above equation, we can find x. Then substituting x in either locus equation we can find y.

11. Projectile motion relative to a moving reference frame: While observing the motion of a projectile from a moving frame we can use $\vec{v} = \vec{u} + \vec{a}t$, $\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$ and $v^2 - u^2 = 2\vec{a}\cdot\vec{s}$ where $\vec{s}, \vec{u}$ and are the displacement, velocity and acceleration of the projectile relative to the moving frame.

4.1.8 Projection Upon an Inclined Plane

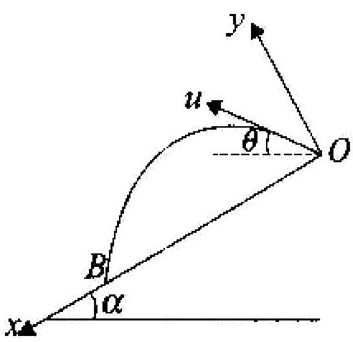


1. $T = \dfrac{2u\sin(\theta - \alpha)}{g\cos\alpha}$

2. $R = \dfrac{u^2}{g\cos^2\alpha}[\sin(2\theta - \alpha) - \sin\alpha]$

3. $R_{max} = \dfrac{u^2}{g(1+\sin\alpha)}$ when $\theta = \dfrac{\pi}{4} - \dfrac{\alpha}{2}$

4.1.9 Projection Down the Inclined Plane



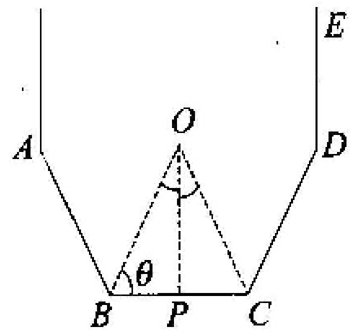
1. $T = \dfrac{2u\sin(\theta + \alpha)}{g\cos\alpha}$

2. $R = \dfrac{u^2}{g\cos^2\alpha}[\sin(2\theta + a) + \sin\alpha]$

3. $R_{max} = \dfrac{u^2}{g(1-\sin\alpha)}$ when $\theta = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$

N particles $A, B, C, D, E, \dots$ are situated at the corners of N sided regular polygon of side L. Each of the particles moves with constant speed v. A always has its velocity along AB, B along BC and C along CD and so on.

$$\therefore \text{ Time of approach} = \left(\frac{OB}{v\cos\theta}\right)$$



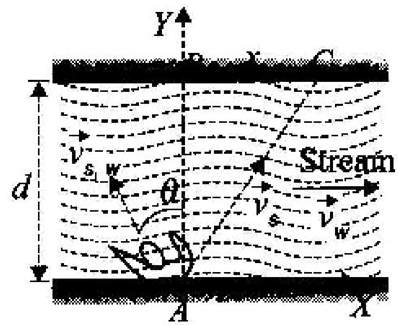
$\Rightarrow \quad t = \dfrac{L}{2\cos\theta \times v\cos\theta} = \dfrac{L}{2v\cos^2\theta}$ and distance travelled by each person $= v.t = v\dfrac{L}{2v\cos^2\theta} = \dfrac{L}{2\cos^2\theta}$ and by symmetry they will meet at the centroid O of the polygon.

4.2 SWIMMER'S PROBLEM OR RIVER-BOAT PROBLEM

Let V_w = Velocity of water w.r.t. ground

$V_{s,w}$ = Velocity of swimmer w.r.t. water

V_s = Velocity of swimmer w.r.t. ground



As, $\vec{V}_{s,w} = \vec{V}_s - \vec{V}_w$

$$\Rightarrow \quad \vec{V}_s = \vec{V}_{s,w} + \vec{V}_w \qquad (1)$$

Now, $\overline{AC} = \vec{V}_s \times$ crossing time t

$$\Rightarrow \quad (\overrightarrow{AB} + \overrightarrow{BC}) = (\vec{V}_{s,w} + \vec{V}_w)t$$

$$(d\hat{j} + x\hat{i}) = [(-V_{s,w}\sin\theta)\hat{i} + (V_{s,w}\cos\theta)\hat{j} + V_w\hat{i}]t$$

$$(x\hat{i} + d\hat{j}) = [(V_w - V_{s,w}\sin\theta)t\,\hat{i} + (V_{s,w}\cos\theta)t\,\hat{j})]$$

i.e., $\quad x = (V_w - V_{s,w}\sin\theta)t$

and $\quad d = (V_{s,w}\cos\theta)t \Rightarrow t = \dfrac{d}{(V_{s,w}\cos\theta)}$

Case I: If θ is given, then

$$\text{Crossing time} = \frac{d}{(V_{s,w}\cos\theta)} \text{ and drift} = (V_w - V_{s,w}\sin\theta)t$$

Case II: For minimum crossing time

For t^{min}, $\cos\theta$ is maximum $\Rightarrow \cos\theta = +1 \quad \Rightarrow \quad \theta = 0^\circ$ and hence $t^{min} = \left(\dfrac{d}{V_{s,w}}\right)$

Case III: The swimmer just reaches the opposite point B on the other bank, i.e., drift $x = 0$

$$\Rightarrow \quad (V_w - V_{s,w}\sin\theta)\left(\frac{d}{V_{s,w}\cos\theta}\right) = 0$$

Either $(V_w - V_{s,w}\sin\theta) = 0$ or $\dfrac{d}{V_{s,w}\cos\theta} = 0$ (it is not possible)

$$\Rightarrow \quad (V_w - V_{s,w}\sin\theta) = 0 \Rightarrow \sin\theta = \left(\frac{V_w}{V_{s,w}}\right) \therefore \theta = \sin^{-1}\left(\frac{V_w}{V_{s,w}}\right) \text{ and}$$

$$\text{crossing time} = \frac{d}{V_{s,w}\cos\theta} = \frac{d}{V_{s,w}\sqrt{1-\sin^2\theta}} = \frac{d}{V_{s,w}^2 - V_w^2} \text{ and}$$

As $\sin\theta \le 1 \quad \Rightarrow \quad \dfrac{V_w}{V_{s,w}} \le 1$ i.e., $V_{s,w} \ge V_w$

Case IV: If $V_{s,w} < V_w$ then drift cannot be zero rather it may be minimum.

1. If drift be minimum, then $\theta = ?$, $t = ?$

For x^{min}, $\dfrac{dx}{d\theta} = 0 \quad \Rightarrow \quad \dfrac{d}{d\theta}\left[(V_w - V_{s,w}\sin\theta)\left(\dfrac{d}{V_{s,w}\cos\theta}\right)\right] = 0$

$$\Rightarrow \frac{d}{d\theta}\left(\frac{V_w d}{V_{s,w}}\sec\theta - d\tan\theta\right) = 0 \Rightarrow \left(\frac{V_w d}{V_{s,w}}\right)\sec\theta\cdot\tan\theta - d\sec^2\theta = 0$$

$$\Rightarrow V_w\sec\theta\tan\theta - V_{s,w}\sec^2\theta = 0 \Rightarrow \sec\theta\left(V_w\tan\theta - V_{s,w}\sec\theta\right) = 0$$

Either $\sec\theta = 0$ or $(V_w\tan\theta - V_{s,w}\sec\theta) = 0$

$\sec\theta = 0$ is not possible, hence $V_w\tan\theta - V_{s,w}\sec\theta = 0$

$$\frac{1}{\cos\theta}(V_w\sin\theta - V_{s,w}) = 0$$

$$\therefore \quad V_w\sin\theta - V_{s,w} = 0$$

$$\therefore \quad \sin\theta = \frac{V_{s,w}}{V_w} \text{ or } \theta = \sin^{-1}\left(\frac{V_{s,w}}{V_w}\right)$$

and

$$\text{crossing time} = \frac{d}{V_{s,w}\cos\theta} = \frac{d}{V_{s,w}\sqrt{1-\dfrac{V_{s,w}^2}{V_w^2}}} = \frac{V_w\cdot d}{V_{s,w}\sqrt{V_w^2 - V_{s,w}^2}}$$

and $$x^{min} = (V_w - V_{s,w}\sin\theta)t = \left(V_w - V_{s,w} \times \frac{V_{s,w}}{V_w}\right) \times \frac{V_w d}{V_{s,w}\sqrt{V_w^2 - V_{s,w}^2}}$$

$$= \frac{V_w^2 - V_{s,w}^2}{V_w} \times \frac{V_w d}{V_{s,w}\sqrt{V_w^2 - V_{s,w}^2}} = \frac{d\sqrt{V_w^2 - V_{s,w}^2}}{V_{s,w}}$$

2. If crossing time be minimum, then i.e., for t^{min}, $\cos\theta = +1 \Rightarrow \theta = 0°$

and $$t^{min} = \left(\frac{d}{V_{s,w}}\right) \quad \text{and} \quad \text{drift} = (V_w)\left(\frac{d}{V_{s,w}}\right) = \left(\frac{V_w}{V_{s,w}}\right)d$$

3. If θ is given, then

$$\text{Crossing time} = \left(\frac{d}{V_{s,w}\cos\theta}\right) \text{and drift} = (V_w - V_{s,w}\sin\theta)\left(\frac{d}{V_{s,w}\cos\theta}\right)$$

4.2.1 Circular Motion

Circular motion is a two-dimensional motion (motion in a plane). Linear velocity vector and linear acceleration vector lie in the plane of circle. Angular velocity vector and angular acceleration vector are perpendicular to the plane of the circle given by right hand screw law.

$$v = R\omega \quad (R = \text{radius of circular path})$$

Acceleration of particle in circular motion may have two components: (1) tangential component (a_t) and (2) centripetal or radial component (a_c).

$$a_t = \text{Rate of change of speed} = \frac{dv}{dt} = \frac{d|\vec{v}|}{dt} = R\alpha$$

where α = Angular acceleration = Rate of change of angular velocity $= \frac{d\omega}{dt}$.

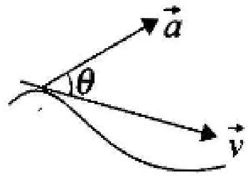
Centripetal acceleration is towards centre and is given by:

$$a_c = R\omega^2 = \frac{v^2}{R}$$

Net acceleration of particle is resultant of two perpendicular components a_c and a_t. Hence, $a = \sqrt{a_c^2 + a_t^2}$.

1. Tangential acceleration a_t is responsible for change of speed of the particle in circular motion. It can be positive, negative or zero, depending whether the speed of particle is increasing, decreasing or constant.
2. Centripetal acceleration is responsible for change in direction of velocity. It can never be equal to zero in circular motion.
3. In general, in any curved line motion direction of instantaneous velocity is tangential to the path, but acceleration may have any direction. If we resolve the acceleration, one parallel to velocity and another perpendicular to velocity, the first component is a_t while the other is a_c.

Thus,



a_t = Component of $\vec{a}$ along $\vec{v}$

$= a\cos\theta = \dfrac{\vec{a}\cdot\vec{v}}{v} = \dfrac{dv}{dt}$ = Rate of change of speed.

and a_c = Component of $\vec{a}$ perpendicular to $\vec{v} = \sqrt{a^2 - a_t^2} = \dfrac{v^2}{R}$

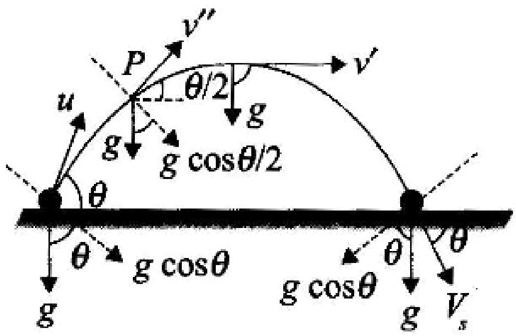
Here v is the speed of particle at that instant and R is called the radius of curvature to the curved line path at that point.

4. If the equation of trajectory is given then the radius of curvature is given as:

$$\frac{1}{R} = \frac{\left(\dfrac{d^2y}{dx^2}\right)}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

A particle is projected with vel. u from the ground making an angle θ with horizontal find the radius of curvature (a) at the point of projection, (b) at point of strike, (c) at maximum height (d) at the point where the velocity of projectile makes an angle $\theta/2$ with horizontal

(a) At the point of projection



$$a_c = g\cos\theta = \frac{(\text{speed})^2}{r}$$

$$\Rightarrow \quad g\cos\theta = \frac{u^2}{r_1}$$

$$\Rightarrow \quad r_1 = \frac{u^2}{g\cos\theta}$$

(b) At point of strike

$$a_c = g\cos\theta = \frac{v_s^2}{r} \quad \Rightarrow \quad g\cos\theta = \left(\frac{u^2}{r_2}\right)$$

$$\Rightarrow \quad r_2 = \frac{u^2}{g\cos\theta}$$

(c) At maximum height

$$a_c = g = \frac{v'^2}{r_3} \quad \Rightarrow \quad r_3 = \frac{v'^2}{g} = \frac{u^2\cos^2\theta}{g}$$

(d) At P

$$a_c = g\cos\alpha = \frac{v''^2}{r_4} \quad \Rightarrow \quad r_4 = \frac{v''^2}{g\cos\alpha}$$

Again

$$v_x'' = v''\cos\alpha = u\cos\theta$$

$$\Rightarrow \qquad v'' = \frac{u\cos\theta}{\cos\alpha} \Rightarrow r_4 = \frac{u^2\cos^2\theta}{g\cos^3\alpha} = \frac{u^2\cos^2\theta}{g\cos^3\theta/2}$$

5. When a particle moves in a curve, the radius of curvature of the path traced by the particle is given as $R = \frac{v^2}{a_r}$, where a_r is the component of acceleration of the particle perpendicular to its line of motion ($\vec{v}$). The centre of curvature lies in the concave side of the path. The magnitude of the speed of the particle changes at a rate of $a_t = \frac{d|\vec{v}|}{dt}$

6. Let a particle moves in a plane along any arbitrary curve given in terms of polar coordinates. In this case,

Position: The position of the moving particle P relative to the origin O can be given as $\vec{r} = |\vec{r}|\hat{r}$, where, $\hat{r}$ is unit vector along $\vec{r}$.

Velocity: The velocity of the particle can be obtained by taking the time derivative of $\vec{r}$ given as $\vec{v} = \frac{d\vec{r}}{dt}$ where $\vec{r} = |\vec{r}|\hat{r}$, $\vec{v} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta}$ where $\frac{dr}{dt} = v$ (radial component of $\vec{v}$)

and $r\omega = v\theta$ (transverse component of $\vec{v}$).

Acceleration: The acceleration of a particle is $\vec{a} = \frac{d\vec{v}}{dt}$

$$\vec{a} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(2v_r\omega + r\frac{d^2\theta}{dr^2}\right)\hat{\theta},$$

where $\qquad \left(\frac{d^2r}{dt^2} - r\omega^2\right) = a_r$ (radial acceleration)

and $\qquad \left(2v_r\omega + r\frac{d^2\theta}{dr^2}\right) = a_\theta$ (transverse acceleration of the particle).

7. Any particle cannot have uniform acceleration while moving in a curve with constant speed.

8. Motion is characterized by velocity and acceleration is characterized by rate of change of velocity. A particle can have a non-zero acceleration while its velocity is zero.

LAWS OF MOTION AND FRICTION

5.1 INERTIA

The inherent property of material body by virtue of which it resists in change of rest or of uniform motion. Mass of a body is the measure of its inertia. If a body has large mass, it has more inertia.

1. **Inertia of rest:** The tendency of a body to remain in its position of rest is called inertia of rest.
2. **Inertia of motion:** The tendency of a body to remain in its state of uniform motion along a straight line is called inertia of motion.
3. **Inertia of direction:** The inability of a body to change by itself its direction of motion is called inertia of direction.

5.2 LINEAR MOMENTUM

Momentum of a body is the amount of motion possessed by the body. Mathematically, it is equal to the product of mass and velocity of the body.

$$\therefore \text{ Momentum} = \text{Mass} \times \text{Velocity} \quad \text{or} \quad \vec{P} = m\vec{v}$$

5.3 FORCE

1. A force is something which changes the state of rest or motion of a body. It causes a body to start moving if it is at rest or stop it, if it is in motion or to deflect it from its initial path of motion.
2. Force is also defined as an interaction between two bodies. Two bodies can also exert force on each other even without being in physical contact. This is called as *action-at-a-distance*, e.g., electric force between two charges, gravitational force between any two bodies of the universe.
3. The word force is from the Latin word "fortis" meaning strong. It is a measure of the interaction of the particles of which the bodies consist.
4. Force is a polar vector as it has a point of application.
5. Forces can be classified as positive or negative. A positive force means *repulsion* whereas a negative force means *attraction*.

5.3.1 System of Forces

1. **Concurrent forces:** When many forces act at a point on a body, they are called concurrent forces. In the system of concurrent forces, the forces may be collinear, i.e., along the same straight line or coplanar, i.e., in the same plane.
2. **Coplanar forces:** When many forces act at different points of a body but all lie in one plane, they are called coplanar forces.
3. **General system of forces:** In some cases, the different forces acting on a body are not coined to a single plane. Such forces form a general system of forces.

5.3.2 Condition for Equilibrium of Concurrent Forces

1. For equilibrium, the vector sum of all the forces must be zero.

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0 \text{ or } \vec{F}_3 = -\left(\vec{F}_1 + \vec{F}_2\right)$$

If the forces are coplanar, then the above condition takes the form as below:

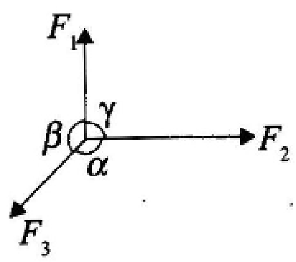
$$\vec{F}_1 + \vec{F}_2 \geq \vec{F}_3 \geq \left|F_1 - F_2\right|$$

2. If the object is at rest and in equilibrium, then it is called static equilibrium. If the body is in motion and in equilibrium ($\Sigma F = 0$), then it is called dynamic equilibrium.
3. The static equilibrium may be any one of the three types, viz. (a) static, (b) unstable and (c) neutral.
4. For an object in equilibrium, acceleration is zero.
5. For an object in equilibrium under the action of conservative forces, $f = -(dU/dr)$ where U represents potential energy.

5.4 LAMI'S THEOREM

If three forces F_1, F_2 and F_3 are acting simultaneously on a body and the body is in equilibrium, then according to Lami's theorem,

$$\frac{F_1}{\sin(\pi-\alpha)} = \frac{F_2}{\sin(\pi-\beta)} = \frac{F_3}{\sin(\pi-\gamma)}$$



where α, β and γ are the angles opposite to the forces F_1, F_2 and F_3, respectively.

5.5 NEWTON'S LAWS OF MOTION

1. **First law:** Everybody remain in its state of rest or of uniform motion in a straight line unless it is compelled by some external force to change that state.
2. **Second law:** The rate of change of linear momentum of a body is directly proportional to the applied force and the change takes place in the direction of the applied force. That is

$$\frac{d\vec{P}}{dt} = \vec{F} = \frac{d}{dt}(m\vec{v}) = m\left(\frac{d\vec{v}}{dt}\right) = m\vec{a}$$

(The Newton's second law $\vec{F} = m\vec{a}$ is strictly applicable to a single particle. The force $\vec{F}$ in the law stands for the net external force. Any internal forces in the system are not to be included in $\vec{F}$.)

3. **Third law:** To every action, there is always an equal and opposite reaction. Action and reaction act on each other. That is

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

5.6 NEWTON'S SECOND LAW IN COMPONENT FORM

$$F_x = \frac{dP_x}{dt} = ma_x \qquad F_y = \frac{dP_y}{dt} = ma_y \qquad F_z = \frac{dP_z}{dt} = ma_z$$

Table 5.1 Forces in Nature

Name	Relative Strength	Range	Operates Among	Field Particle
Gravitational force	10^{-39}	Infinite	All objects in the universe	Gravitons (perhaps)
Electromagnetic force	10^{-2}	Very large	Charged particles	Photons
Weak nuclear force	10^{-13}	Very short, Sub-nuclear size ($\sim 10^{-16}$ m)	Some elementary particles (electron and neutrino)	Bosons (W^-)
Strong nuclear force	1	Short, nuclear size ($\sim 10^{-15}$ m)	Nucleons, heavier elementary particles	Mesons (π^-)

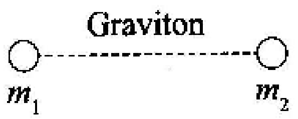
Compared to other fundamental forces, gravitational force is the weakest force of nature.

The strong nuclear force binds protons and neutrons in nucleus. It does not depend on charge and acts equally between a proton and a proton, a neutron and a neutron, and a proton and a neutron. Electron does not experience this force.

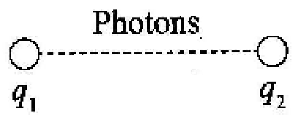
(Recent discovery indicated that the strong nuclear force between nucleons is not a fundamental force of nature.)

The weak nuclear force appears only in certain nuclear process such as the β-decay of a nucleus. The weak nuclear force is not as weak as gravitational force, but much weaker than strong nuclear force.

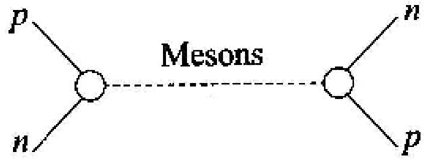
1. Gravitational force



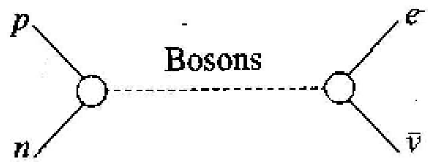
2. Electromagnetic force



3. Strong nuclear force



4. Weak nuclear force



5.6.1 Newton's Second Law in Non-inertial Frame

In a non-inertial frame, Newton's second law takes the form

$$\vec{F} + \vec{F}_{pseudo} = m\vec{a} \tag{1}$$

Here $\vec{a}$ is the acceleration of object in non-inertial frame. $\vec{F}_{pseudo}$ is the pseudo force arises in non-inertial frame. It is equal to mass times the acceleration of frame and in opposite direction of acceleration of frame. Thus in non-inertial frame, we have

$$\vec{F} - m\vec{a}_0 = m\vec{a} \tag{2}$$

Because of rotation of earth about its axis and revolution of earth around sun, our frame of reference is non-inertial. But acceleration due to these two motions is negligibly small and therefore can be neglected. For most laboratory phenomenon, it can be assumed that our frame of reference is inertial.

5.6.2 Apparent Weight of a Body in a Lift

1. When the lift is at rest or moving with uniform velocity, i.e., $a_o = 0$,

$$mg - N = 0 \text{ or } N = mg \text{ or } W_{app.} = W_o$$

where $W_{app.} = N$ = reaction of supporting surface and $W_o = mg$ = true weight.

2. When the lift moves upwards with an acceleration a_o:

$$N - mg = ma_o \text{ or } N = m(g + a_o) = mg\left(1 + \frac{a_o}{g}\right)$$

$$\therefore \quad W_{app.} = W_o\left(1 + \frac{a_o}{g}\right)$$

3. When the lift moves downwards with an acceleration a_o

$$mg - N = ma_o \text{ or } N = m(g - a_o) = mg\left(1 - \frac{a_o}{g}\right)$$

$$\therefore \quad W_{app.} = W_o\left(1 - \frac{a_o}{g}\right)$$

Here, if $a_o > g$, $W_{app.}$ will be negative. Negative apparent weight will mean that the body is pressed against the roof of the lift instead of floor.

4. When the lift falls freely, i.e., $a_o = g$:

$$N = m(g - g) = 0 \text{ or } W_{app.} = 0.$$

It is called *condition for weightlessness.*

Problem of a Mass Suspended From a Vertical String in a Moving Carriage

The following cases are possible:

1. If the carriage (say lift) is at rest or moving uniformly (in translatory equilibrium), then

$$N = T_o = mg$$

2. If the carriage is accelerated up with an acceleration a_o, then

$$T = m(g + a_o) = mg\left(1 + \frac{a_o}{g}\right) = T_o\left(1 + \frac{a_o}{g}\right)$$

3. If the carriage is accelerated down with an acceleration a_o, then

$$T = m(g - a_o) = mg\left(1 - \frac{a_o}{g}\right) = T_o\left(1 - \frac{a_o}{g}\right)$$

4. If the carriage begins to fall freely, then the tension in the string becomes zero.
5. If the carriage is accelerated horizontally, then
   (a) mass m experiences a pseudo force ma_o opposite to acceleration;
   (b) the mass m is in equilibrium inside the carriage and

$$T \sin\theta = ma_o,\ T\cos\theta = mg, \text{ i.e., } T = m\sqrt{g^2 + a_o^2}$$

   (c) the string does not remain vertical but inclines to the vertical at an angle $\theta = \tan^{-1}(a_o/g)$ opposite to acceleration;
   (d) This arrangement is called *accelerometer* and can be used to determine the acceleration of a moving carriage from inside by noting the deviation of a plumb line suspended from it from the vertical.

Problem of Monkey Climbing a Rope

Let T be the tension in the rope.

1. When the monkey climbs up with uniform speed: $T = mg$.
2. When the monkey moves up with an acceleration a_o:

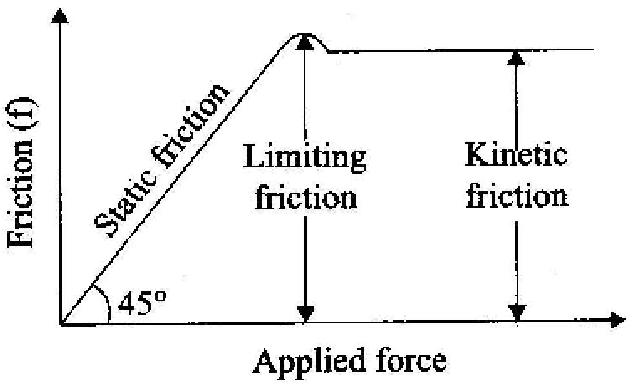
$$T - mg = ma_o \text{ or } T = m(g + a_o)$$

3. When the monkey moves down with an acceleration a_o:

$$mg - T = ma_o \text{ or } T = m(g - a_o)$$

Friction

Friction can be defined as a force which opposes the relative motion between surfaces in contact. The component of the contact force perpendicular to the contact surface is called normal force and the component parallel to the contact surface is called frictional force.



The limiting friction depends on the materials of the surfaces in contact and their state of polish. The magnitude of static friction is independent of the apparent area of contact so long as the normal reaction remains the same. The limiting friction is directly proportional to the magnitude of the normal reaction between the two surfaces. i.e.,

$$f_{lim} = \mu_s N \qquad \therefore \quad \mu_s = \frac{f_{lim}}{N}$$

The kinetic friction depends on the materials of the surface in contact. It is also independent of apparent area of contact as long as the magnitude of normal reaction remains the same. Kinetic friction almost independent of the velocity, provided the velocity is not too large not too small. The kinetic friction is directly proportional to the magnitude of the normal reaction between the surfaces. i.e.,

$$f_k = \mu_k N \qquad \therefore \quad \mu_k = \frac{f_k}{N}$$

Rolling friction opposes the rolling motion of a body on a surface. It is very much smaller than kinetic friction.

As $\qquad f_k < f_{lim}$ or $\mu_k N < \mu_s N \quad \therefore \quad \mu_k < \mu_s$

The theoritical value of μ can be 0 to infinite. But practical value;

$$0 < \mu \leq 1.6$$

The coefficient of limiting and kinetic friction have no dimensions but the coefficient of rolling friction has the dimension of length.

1. **Angle of friction (λ):** The angle of friction is defined as the angle which the contact force makes with the normal reaction. The *tan* value of the angle of friction is the coefficient of static friction.

2. **Angle of repose (α):** It is the angle that an inclined plane makes with the horizontal when a body placed on it is in just to slide condition. The *tan* value of the angle of repose is the coefficient of kinetic friction.

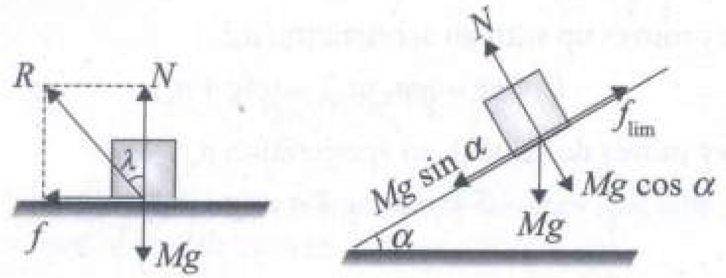


Table 5.2 Motion of a Body on a Smooth Inclined Plane

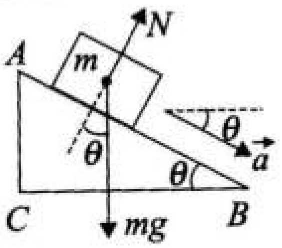
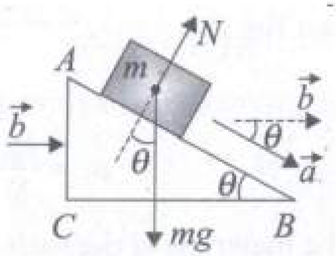
Different Cases	Diagrams	Results
When smooth inclined plane is fixed		(i) $N = mg\cos\theta$ (ii) $a = g\sin\theta$
When the smooth inclined plane is moving horizontally with an acceleration b		(i) $N = m(g\cos\theta + b\sin\theta)$ (ii) $a = (g\sin\theta - b\cos\theta)$

Table 5.3 Motion of a Block on a Horizontal Smooth Surface

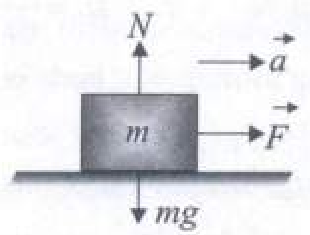
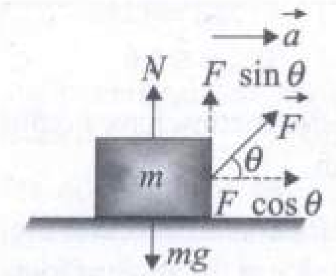
Different Cases	Diagrams	Results
When subjected to a horizontal pull		(i) $N = mg$ (ii) $a = \frac{F}{m}$
When subjected to a pull acting at an angle θ to the horizontal		(i) $N = mg - F\sin\theta$ (ii) $a = \frac{F\cos\theta}{m}$

Table 5.3 *(Continued)*

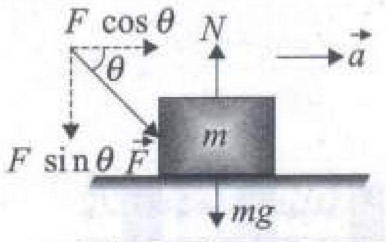
Different Cases	Diagrams	Results
When subjected to a push acting at a angle θ to the horizontal		(i) $N = mg + F\sin\theta$ (ii) $a = \dfrac{F\cos\theta}{m}$

Table 5.4 Motion of Bodies in Contact: Force of Contact

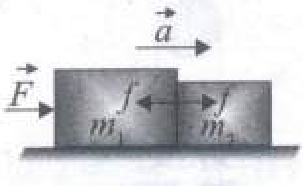
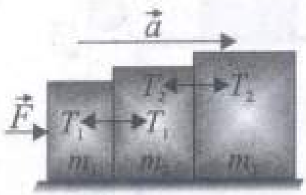
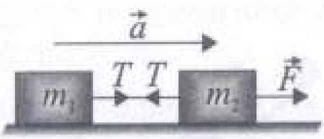
Different Cases	Diagrams	Results
When two bodies are kept in contact and force is applied on the body of mass m_1		(i) $a = \dfrac{F}{m_1 + m_2}$ (ii) $f = \dfrac{m_1 F}{m_1 + m_2}$
When two bodies are kept in contact and force is applied on the body of mass m_2		(i) $a = \dfrac{F}{m_1 + m_2}$ (ii) $f' = \dfrac{m_2 F}{m_1 + m_2}$
When three bodies are kept in contact and force is applied on the body of mass m_1		(i) $a = \dfrac{F}{m_1 + m_2 + m_3}$ (ii) $T_1 = \dfrac{(m_2 + m_3)F}{(m_1 + m_2 + m_3)}$ (iii) $T_2 = \dfrac{m_3 F}{(m_1 + m_2 + m_3)}$

Table 5.5 Motion of Connected Bodies

Different Cases	Diagrams	Results
When two bodies are connected by a string and placed on a smooth horizontal surface		(i) $a = \dfrac{F}{m_1 + m_2}$ (ii) $T = \dfrac{m_1 F}{m_1 + m_2}$

(Continued)

Table 5.5 (*Continued*)

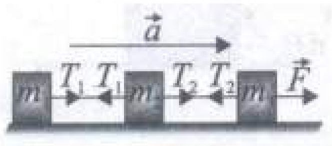
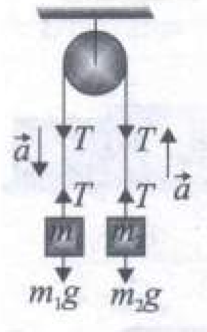
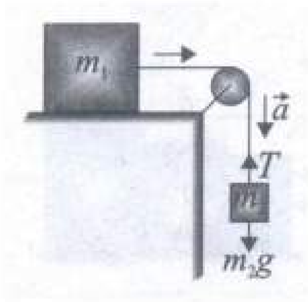
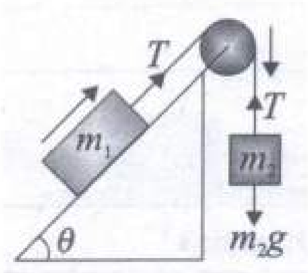
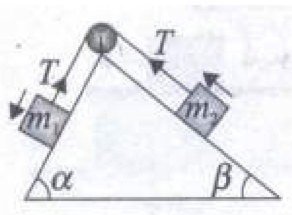
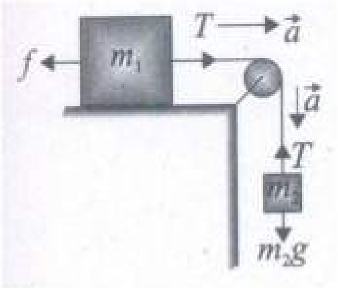
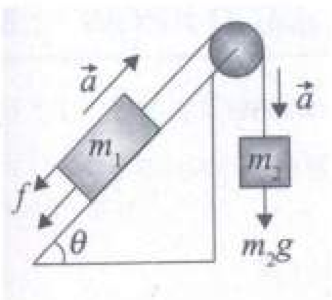
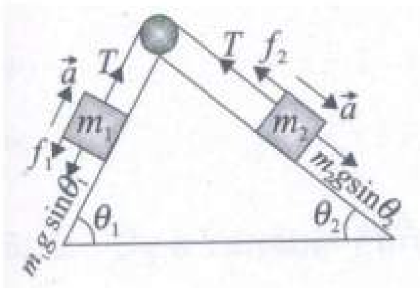
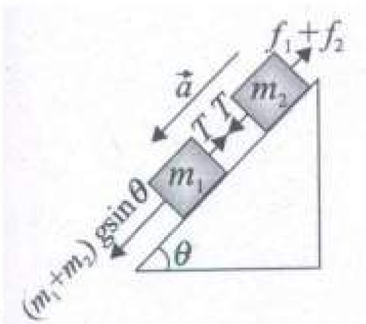
Different Cases	Diagrams	Results
When three bodies are connected through strings as shown in figure and placed on a smooth horizontal surface		(i) $a=\dfrac{F}{(m_1+m_2+m_3)}$ (ii) $T_1=\dfrac{m_1F}{(m_1+m_2+m_3)}$ (iii) $T_2=\dfrac{(m_1+m_2)F}{(m_1+m_2+m_3)}$
When two bodies of masses m_1 and m_2 are attached at the ends of a string passing over a pulley as shown in the figure (neglecting the mass of the pulley). If in the above system mass (m) of the pulley is taken into account then		(i) $a=\dfrac{(m_1-m_2)g}{(m_1+m_2)}$ (ii) $T_1=\left(\dfrac{2m_1m_2}{m_1+m_2}\right)g$ $a=\dfrac{(m_1-m_2)g}{\left(m_1+m_2+\dfrac{m}{2}\right)}$
When two bodies of masses m_1 and m_2 are attached at the ends of a string passing over a pulley in such a way that mass m_1 rests on a smooth horizontal table and mass m_2 is hanging vertically		(i) $a=\dfrac{m_2g}{(m_1+m_2)}$ (ii) $T=\dfrac{m_1m_2g}{(m_1+m_2)}$
If in the above case, mass m_1 is placed on a smooth inclined plane making an angle θ with horizontal as shown in figure, then		(i) $a=\dfrac{(m_2-m_1\sin\theta)g}{m_1+m_2}$ (ii) $T=\dfrac{m_1m_2g(1+\sin\theta)}{m_1+m_2}$ (iii) If the system remains in equilibrium, then $m_1g\sin\theta=m_2g$
Masses m_1 and m_2 are placed on inclined planes making angles α and β with the horizontal respectively, then		(i) $a=\dfrac{g(m_1\sin\alpha-m_2\sin\beta)}{(m_1+m_2)}$ (ii) $T=\dfrac{m_1m_2}{m_1+m_2}(\sin\alpha+\sin\beta)g$

Table 5.6 Motion of Connected Bodies on Rough Surfaces

Different Types of System	Results
	(i) $a = \dfrac{m_2g - \mu m_1 g}{(m_1 + m_2)}$ (ii) $T = \dfrac{m_1 m_2 g}{(m_1 + m_2)}(1+\mu)$
	(i) $a = \dfrac{\text{Unbalanced force}}{\text{Total mass}}$ $= \dfrac{(m_2g - m_1 g \sin\theta - \mu m_1 g \cos\theta)}{(m_1 + m_2)}$ (ii) $T = \dfrac{m_1 m_2 g}{(m_1 + m_2)}(1 + \sin\theta + \mu\cos\theta)$
	(i) $a = \dfrac{m_2 g(\sin\theta_2 - \mu\cos\theta_2) - m_1 g(\sin\theta_1 + \mu\cos\theta_1)}{(m_1 + m_2)}$ (ii) Calculate tension using the following equations: $T - m_1 g(\sin\theta_1 + \mu\cos\theta_1) = m_1 a$ $m_2 g(\sin\theta_2 - \mu\cos\theta_2) - T = m_2 a$
	(i) $a = \dfrac{m_1 g\sin\theta + m_2 g\sin\theta - g\cos\theta(\mu_1 m_1 + \mu_2 m_2)}{m_1 + m_2}$ (ii) According to Newton's second law, $m_1 g\sin\theta - \mu_1 m_1 g\cos\theta - T = m_1 a$ $m_2 g\sin\theta + T - \mu_2 m_2 g\cos\theta = m_2 a$

5.7 SPRING

1. The force offered by the spring, that is, 'spring force' F_s points (acts) opposite to the displacement of the force end of the spring.
2. The amount of spring force increases linearly with the deformation (compression or elongation) of the spring, when we plot the variation of F_s versus x, we obtain a straight line up to certain (limited) value of x, which is known as elastic limit.

$$\text{Spring force } F_s \propto x \quad \Rightarrow \quad F_s = -kx$$

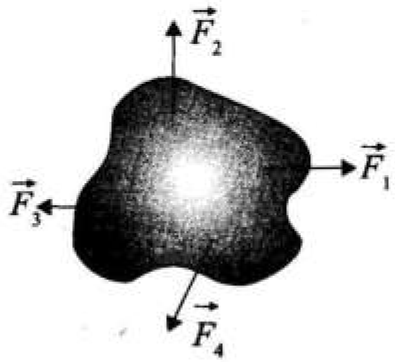
where F_s = spring force, x = displacement of one end of the spring (relative to the other end) along the spring and k = spring constant or stiffness of the spring. Stiffness is numerically equal to the force required to deform spring by a unit length.

3. Springs connected in parallel: $k_{eq} = \sum_{i=1}^{i=n} k_i$

4. Springs connected in series: $\frac{1}{k_{eq}} = \sum_{i=1}^{i=n} \frac{1}{k_i}$

5.8 NON-CONCURRENT COPLANAR FORCES

If body is in equilibrium under non-concurrent coplanar forces, we can write



$$\Sigma F_x = 0, \Sigma F_y = 0$$

and $$\Sigma(\text{moment about any point}) = 0.$$

WORK, ENERGY, POWER AND CIRCULAR MOTION

6.1 WORK DONE

6.1.1 By a Constant Force

If force displaces the particle from its initial position $\vec{r}_i$ to final postion $\vec{r}_f$ then displacement vector $\vec{s} = \vec{r}_f - \vec{r}_i$.

$$W = \vec{F}\cdot\vec{s} = \vec{F}\cdot(\vec{r}_f - \vec{r}_i) = Fs\cos\theta$$
$$= (\text{Force}) \times (\text{component of displacement in the direction of force})$$

or $$W = \vec{F}\cdot\vec{s} = (F_x\hat{i} + F_y\hat{j} + F_z\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

or $$W = F_x x + F_y y + F_z z$$

6.1.2 By a Variable Force

$$W = \int_{x_i}^{x_f} F\,dx, \text{where } F = f(x)$$
$$= \int_{x_i}^{x_f} (F_x\hat{i} + F_y\hat{j} + F_z\hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$
$$= \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz$$

6.1.3 By Area Under *F*-x Graph

If force is a function of x, we can find work done by area under F-x graph with projection along x-axis. In this method, magnitude of work done can be obtained by area under F-x graph, but sign of work done should be decided by you. If force and displacement both are positive or negative, work done will be positive. If one is positive and other is negative then work done will be negative.

Work done by the spring on the external agent $= -\frac{1}{2}kx^2$

Work done by the external agent on the spring $= +\frac{1}{2}kx^2$

Quantities like mass, time, acceleration and force in Newotonian mechanics are invariant i.e., these have same numerical values in different inertial frames. Quantities like velocity, kinetic energy and work done have different values in different inertial frames.

6.2 POWER OF A FORCE

1. **Average power:**

$$P_{av} = \frac{\text{Total work done}}{\text{Total time taken}} = \frac{W_{\text{Total}}}{t}$$

2. **Instantaneous power:**

$$P_{\text{ins.}} = \text{Rate of doing work done} = \frac{dW}{dt} = \vec{F}\cdot\vec{v} = Fv\cos\theta$$

Power of pump required to just lift the water, $v = 0$

$$\therefore \qquad P = \left(\frac{dm}{dt}\right)gh$$

If efficiency of pump is η, then $\eta = \frac{P_{out}}{P_{in}}$

6.2.1 Conservative and Non-conservative Forces

In case of conservative forces work done is path independent and in a round trip net work done is zero.

Examples: Gravitational force, electrostatic force and elastic force.

If work done by a force in displacing a particle depends on path, the force is said to be non-conservative or dissipative forces.

Examples: Frictional force and viscous force.

Potential energy is defined only for conservative forces. If only conservative forces are acting on a system, its mechanical energy should remain constant.

6.3 POTENTIAL ENERGY

The energy associated due to interaction between the particles of same body or between particles of different bodies or the energy associated with the configuration of a system in which conservative force acts is called potential energy. Energy due to interaction between particles of same body is called *self-energy* or *internal potential energy* U_i. Energy due to interaction between particles of different bodies is called *external potential energy* U_e or simply *potential energy*.

In a conservative force field, difference in potential energy between two points is the negative of work done by conservative forces in displacing the body (or system) from some initial position to final position. Hence,

$$\Delta U = -W \text{ or } U_B - U_A = -W_{A\rightarrow B}$$

Absolute potential energy at a point can be defined with respect to a reference point where potential energy is assumed to be zero. Negative of work done in displacement of body from reference point (say O) to the point under consideration (say P) is called absolute potential energy at P. Thus, $U_p = -W_{0 \to P}$.

6.3.1 Relation Between Potential Energy (U) and Conservative Force ($\vec{F}$)

1. If U is a function of only one variable, then

$$F = \frac{dU}{dr} = -\text{slope of } U\text{-}r \text{ graph}$$

2. If U is a function of three coordinate variables x, y and z, then

$$\vec{F} = -\left[\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right]$$

The sum of the kinetic energy and potential energy of the body is called mechanical energy. Thus,

$$\text{M.E.} = \text{K.E.} + \text{P.E.}$$

6.3.2 Principle of Conservation of Mechanical Energy

$$W_{NC} + W_{Other} = \Delta\text{M.E.}$$

If only conservative forces act on the particle, we have

$$W_{NC} = 0 \text{ and } W_{Other} = 0$$

$$\therefore \quad 0 = \Delta\text{M.E. or M.E.} = \text{Constant}$$

6.3.3 Work-energy Theorem

Work done by net force is equal to the change in kinetic energy of the body. This is called **work-energy theorem.**

$$W_{\text{net force}} = K_f - K_i = \Delta\text{K.E.}$$

The work-energy theorem is not independent of Newton's second law. It may be viewed as scalar form of second law.

Work-energy theorem holds in all types of frames; inertial or non-inertial. In non-inertial frame, we have to include the pseudo force in the calculation of the net force.

$$W_{\text{external}} + W_{\text{internal}} + W_{\text{pseudo}} + W_{\text{other}} = \Delta\text{K.E.}$$

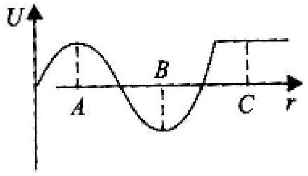
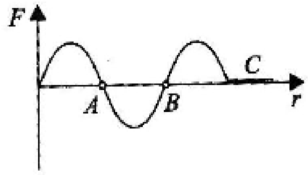
When both external and internal forces act on the system, we can write

$$W_{\text{external}} + W_{\text{internal}} = \Delta\text{K.E.}$$

6.3.4 Types of Equilibrium

For the equilibrium of any body, the net force on it must be zero, that is, $\vec{F}_{net} = 0$. For the equilibrium of body under conservative forces, we have

$$[F_C]_{net} = \frac{-dU}{dx} = 0 \quad \text{or} \quad \frac{dU}{dx} = 0$$

Physical Situation	Stable Equilibrium	Unstable Equilibrium	Neutral Equilibrium
(a) Net force	Zero	Zero	Zero
(b) Potential energy	Minimum	Maximum	Constant
(c) When displaced from mean (equilibrium) position.	A restoring nature of force will act on the body, which brings the body back towards mean position.	A force will act which moves the body away from mean position.	Force is again zero
(d) In U-r graph 	At point B	At point A	At point C
(e) In F-r graph 	At point A	At point B	At point C

6.3.5 Circular Motion

In uniform circular motion, a particle has only one acceleration called as centripetal acceleration and in non-uniform circular motion, a particle has two components of particle acceleration:

1. Centripetal acceleration
2. Tangential acceleration

Also, the cause of acceleration is the force and the direction of acceleration is along the direction of the force. Hence, the cause of centripetal acceleration is called as centripetal force (mv^2/R) and the cause of tangential acceleration is called as tangential force ($= m dv/dt$)

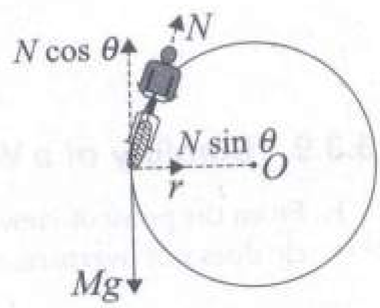
In uniform circular motion, the only force is centripetal force, which acts perpendicular to the velocity. Thus the rate of doing work i.e., power is equal to zero.

In non-uniform circular motion, there are normal and tangential forces.

The rate of doing work, $P = \frac{dW}{dt} = (\vec{F}_c + \vec{F}_t) \cdot \vec{v} = F_t v$

If a system is observed w.r.t. rotating N.I.F. and the system is found to be in equilibrium, then a pseudo force is to be applied (It is called centrifugal force). But if the system is found to be in motion with constant speed then two pseudo forces are to be applied—one is called centrifugal force and the other is called Coriolis force.

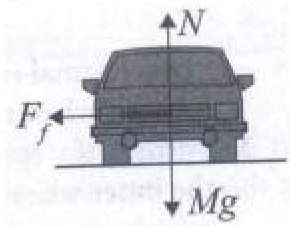
6.3.6 Turning of a Cyclist Around a Corner on the Road



1. When a cyclist turns around a corner on the road, he needs a centripetal force (Mv^2/r). The forces acting on the cyclist are
   (a) Weight Mg
   (b) Normal force N
2. In order to generate the necessary centripetal force, the cyclist bends inwards by an angle θ w.r.t. vertical.
3. In equilibrium,

$$N\cos\theta = Mg \qquad \text{and } N\sin\theta = \frac{Mv^2}{r} \qquad \text{So, } \tan\theta = \frac{v^2}{rg}$$

6.3.7 A Car Taking a Turn on a Level Road



1. When a car takes a turn on a level road, the portion of the turn can be approximated by an arc of a circle of radius r. If the car makes the turn at a constant speed v, then there must be some centripetal force acting on the car. This force is generated by the friction between the tyres and the road.
2. The maximum frictional force is: $F_f = \mu_s N$, where μ_s is the coefficient of static friction. Then, the maximum safe velocity v is such that

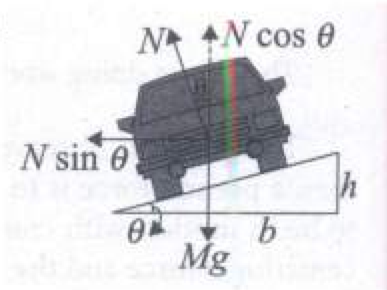
$$\left(\frac{mv^2}{r}\right) = \mu_s N \text{ or } \mu_s = \left(\frac{v^2}{rg}\right) \text{ or } v = \sqrt{\mu_s rg}$$

3. It is important to note that safe velocity is independent of the mass of the car.

6.3.8 Banking of Tracks

1. In order that a vehicle may make a safe and easier turn without depending on friction, roads on large highways are generally banked, i.e., road bend at the curved path is raised a little on the side away from the centre of the curved path.
2. By banking the road, a component of the normal force points towards the centre of curvature of the road. This component supplies the necessary centripetal force required for circular

motion. The vertical component of the normal force is balanced by the weight of the vehicle, i.e.,



$$N\cos\theta = Mg \text{ and } N\sin\theta = \frac{Mv^2}{r}$$

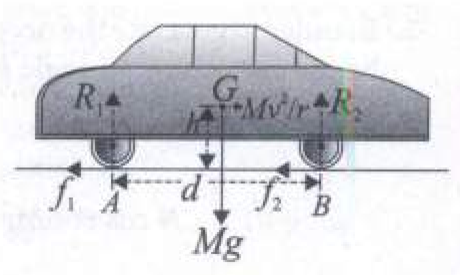
$$\therefore \qquad \tan\theta = \frac{v^2}{rg} \qquad \text{(where } \tan\theta = h/b\text{)}$$

3. For a road with angle of banking θ, the speed v at which minimum wear away of tyre takes place is given by

$$v = \sqrt{rg\tan\theta}$$

6.3.9 Stability of a Vehicle on a Horizontal Turn

1. From the point of view of non-inertial frame, if the vehicle does not overturn, then balancing the force, we get



$$R_1 + R_2 = Mg$$

2. Now balancing torques about point B and then about point A we have

$$Mg\frac{d}{2} + \frac{Mv^2}{r}h = R_2 d \text{ and } Mg\frac{d}{2} - \frac{Mv^2}{r}h = R_1 d$$

Thus, normal reaction at the inner wheel (i.e., R_1) is always less than that at the outer wheel (i.e., R_2) when making the circular turn.

3. Further, if v is such that R_1 becomes zero, then the vehicle has a tendency to overturn, i.e., the inner wheel loses contact and the vehicle overturns outwards. Thus, the maximum safe velocity for not overturning is

$$v = \sqrt{\left(\frac{grd}{2h}\right)}$$

4. The frictional forces f_1 and f_2 provide the necessary centripetal force, i.e., $f_1 + f_2 = \left(\frac{Mv^2}{r}\right)$. The safe speed for not skidding is such that

$$f_1 + f_2 \le \mu(R_1 + R_2) \text{ or } v < \sqrt{\mu rg}$$

6.3.10 Conical Pendulum

1. If a small body of mass m tied to a string is whirled in a horizontal circle, the string will not remain horizontal [as a vertical force mg cannot be balanced by a horizontal force (T)] but

the string becomes inclined to the vertical and sweeps a cone while the body moves on a horizontal circle with uniform speed. Such an arrangement is called conical pendulum.

2. In case of conical pendulum, the vertical component of tension balances the weight while its horizontal component provides the necessary centripetal force, i.e.,

$$T\cos\theta = mg \text{ and } T\sin\theta = \frac{mv^2}{r} \text{ or } \tan\theta = \frac{v^2}{rg} \quad (1)$$

Also, $$T = m\sqrt{g^2 + \left(\frac{v^2}{r}\right)^2} \quad (2)$$

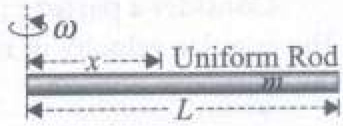
Hence, $v = \sqrt{rg\tan\theta}$ i.e., $$\omega = \sqrt{\left(\frac{g\tan\theta}{r}\right)} \quad (3)$$

Hence, time period $$t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{g\tan\theta}} = 2\pi\sqrt{\frac{L\cos\theta}{g}} \quad (4)$$

3. Time period t is independent of the mass of the body and depends on $L\cos\theta\ (= h)$, i.e., distance between point of suspension and centre of circle.

4. If $\theta = 90°$, the pendulum becomes horizontal and it follows from equations (1), (2) and (4) that $v = \infty$, $T = \infty$ and $t = 0$ which is practically impossible.

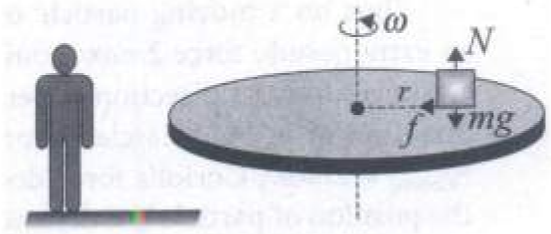
(a) The given rod is rotating uniformly about one end. The variation of tension along its length is $T = \frac{m\omega^2}{2L}(L^2 - x^2)$



(b) A metal ring of mass m and radius R is placed on a smooth horizontal table and is set rotating about its own axis in such a way that each part of the ring moves with a speed v. The tension in the ring is $T = \frac{mv^2}{2\pi R}$.

6.3.11 Centrifugal Force

Consider a block of mass m placed on the table at a distance r from its centre. Suppose the table rotates with constant angular velocity ω and block remains at rest with respect to table. Let us first analyse the motion of the block relative to an observer on the ground (inertial frame). In this frame, the block is moving in a circle of radius r. It therefore has an acceleration v^2/r towards the centre. The resultant force on the block must be towards the centre and its magnitude is mv^2/r. In this frame, the forces on the block are



1. Weight mg
2. Normal reaction N
3. Frictional force f by the table

Thus, we have

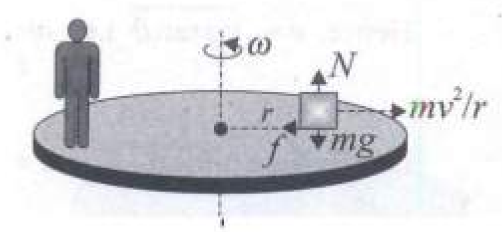
$$N = mg \qquad \text{(i)}$$

for circular motion, $$f = \frac{mv^2}{r} \qquad \text{(ii)}$$

Now observe the same block in a frame attached with the rotating table. The observer here finds that the block is at rest. Thus the net force on the block in this frame must be zero. The weight and normal reaction balance each other but frictional force, f acts on the block towards the centre of the table to make the resultant zero, a pseudo force must be assumed which acts on the block away radially outward and has a magnitude mv^2/r. This pseudo force is called centrifugal force.

In this frame, the forces on the block are

1. Weight mg
2. Normal reaction N
3. Frictional force f
4. Centrifugal force $\frac{mv^2}{r}$



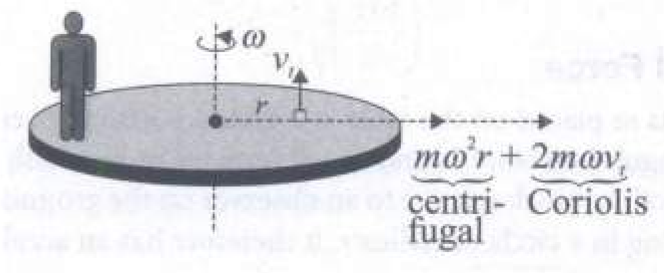
6.3.12 Coriolis Force

The force named after French mathematician G. Coriolis.

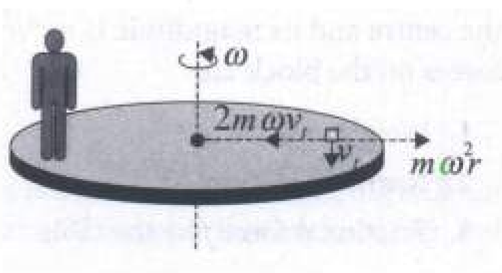
Consider a particle moving with a uniform tangential speed v with respect to a rotating table. The angular velocity of rotation of the table is ω and particle is at a distance r from the centre of the table.

1. If table was not rotating ($\omega = 0$) the particle has the only force, $F = \frac{mv_t^2}{r}$ in inertial frame.

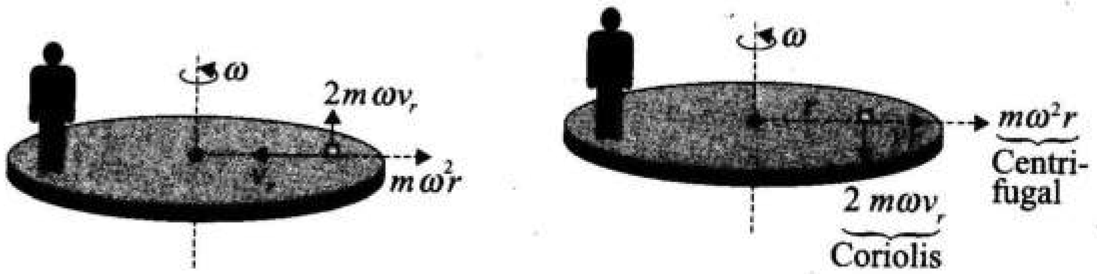
   Thus due to rotation of table the particle experiences a pseudo force ($m\omega^2 r + 2\, m\omega v_t$).
   If particle is at rest w.r.t. table, $v_t = 0$. Then the only pseudo force is $m\omega^2 r$.



Thus on a moving particle on a rotating table an extra pseudo force $2m\omega v_t$ comes to act, is called Coriolis force. Its direction is perpendicular to the direction of v_t. As it is clear from the expression, $F_{Coriolis} = 2m\omega v$, Coriolis force does not depend on the position of particle but depends on its speed.



2. Particle moving with uniform radial velocity v_r with respect to rotating table. Here we have centrifugal force $m\omega^2 r$ radially outward and Coriolis force $2m\omega v_r$ perpendicular to v_r.



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CENTRE OF MASS AND COLLISION

7.1 CENTRE OF MASS

If the laws for a single particle is to be applied for a system of particles then the concept of centre of mass is useful. Newton introduced this concept and defined a point such that its position vector is written as $\vec{r} = \dfrac{\Sigma m_i \vec{r}_i}{\Sigma m_i}$ and assumed whole mass of the system is to be concentrated on that point. Clearly a point for a system of particles whose position vector is defined as the above equation, is called as centre of mass of the system. It is a hypothetical point because it is mathematically defined. This point may lie inside the system or outside the system of particles.

7.1.1 Position of Centre of Mass of Discrete System of Particles

For Two Point Masses or More Than Two Point Masses

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

It is a vector equation. Its component equations are

1. $X_{CM} = \dfrac{\Sigma m_i x_i}{\Sigma m_i} = \dfrac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots}$

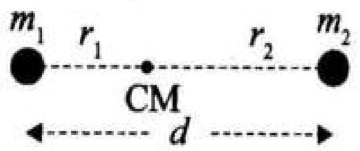
2. $Y_{CM} = \dfrac{\Sigma m_i y_i}{\Sigma m_i} = \dfrac{m_1 y_1 + m_2 y_2 + \ldots}{m_1 + m_2 + \ldots}$

3. $Z_{CM} = \dfrac{\Sigma m_i z_i}{\Sigma m_i} = \dfrac{m_1 z_1 + m_2 z_2 + \ldots}{m_1 + m_2 + \ldots}$

For Two Point Masses Only

$r \propto \dfrac{1}{m}$ or $\dfrac{r_1}{r_2} = \dfrac{m_2}{m_1}$ (where r_1 and r_2 are the separations of m_1 and m_2 from centre of mass)

$\therefore \quad m_1 r_1 = m_2 r_2 \qquad r_1 = \dfrac{m_2}{m_1 + m_2} d \qquad r_2 = \dfrac{m_1}{m_1 + m_2} d$



7.1.2 Position of Centre of Mass of Continuous System of Particles

$\vec{r}_{CM} = \dfrac{\int dm\vec{r}}{\int dm}$ It is a vector equation. Its component equations are

1. $x_{CM} = \dfrac{\int dmx}{\int dm}$

2. $y_{CM} = \dfrac{\int dmy}{\int dm}$

3. $z_{CM} = \dfrac{\int dmz}{\int dm}$

where x, y and z are coordinates of dm.

7.1.3 Position of Centre of Mass of More Than Two Rigid Bodies

1. Centre of mass of symmetrical and having uniform mass distribution rigid bodies (like sphere, disc, cube etc.) lies at its geometric centre.
2. For two or more than two rigid bodies, we can use the formula of the position of centre of mass of discrete system of particles.
3. If three-dimensional rigid body has uniform mass distribution then mass in the formulae for the position of centre of mass can be replaced by volume (V).

$$\text{i.e., } \vec{r}_{CM} = \frac{V_1\vec{r}_1 + V_2\vec{r}_2 + \dots}{V_1 + V_2 + \dots}$$

4. If two dimensional rigid body has uniform mass distribution then mass in the formulae for the position of centre of mass can be replaced by area (A).

$$\text{i.e., } \vec{r}_{CM} = \frac{A_1\vec{r}_1 + A_2\vec{r}_2 + \dots}{A_1 + A_2 + \dots}$$

7.1.4 Position of Centre of Mass of a Rigid Body from Which Some Portion Is Removed

1. If some portion is removed from the two-dimensional body

Then, $\vec{r}_{CM} = \dfrac{A_1\vec{r}_1 - A_2\vec{r}_2}{A_1 - A_2}$

Here, A_1 = Area of whole body (without removing)

$\vec{r}_1$ = Position vector of centre of mass of whole body

A_2 = Area of removed portion

$\vec{r}_2$ = Position vector of centre of mass of removed portion

2. If some portion is removed from the three dimensional body

Then, $\vec{r}_{CM} = \dfrac{V_1\vec{r}_1 - V_2\vec{r}_2}{A_1 - A_2}$

Here, V_1 = Volume of whole body (without removing)

$\vec{r}_1$ = Position vector of centre of mass of whole body

V_2 = Volume of removed portion

$\vec{r}_2$ = Position vector of centre of mass of removed portion

7.2 MOTION OF THE CENTRE OF MASS

1. $$\vec{v}_{CM} = \frac{m_1\dfrac{d\vec{r}_1}{dt} + m_2\dfrac{d\vec{r}_2}{dt} + m_3\dfrac{d\vec{r}_3}{dt} + \ldots}{M};$$

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots}{m_1 + m_2 + \ldots} \Rightarrow \vec{v}_{CM} = \frac{\vec{P}_{total}}{M_{total}}$$

If $\vec{V}_{CM} = 0$, then $\vec{P}_{CM} = 0$, i.e., in the frame of reference of CM, the momentum of a system is zero. This is the reason that CM frame is called zero momentum frame. Velocity of CM is not affected by internal forces. So, if CM of a system is at rest, it will remain at rest unless acted by an external force.

If a bomb thrown in to air explodes in mid-air, then the CM of fragments follow the same parabolic path as the unexploded bomb would have followed, since the forces of explosion are internal forces.

2. $$\vec{a}_{CM} = \frac{m_1\dfrac{d\vec{v}_1}{dt} + m_2\dfrac{d\vec{v}_2}{dt} + m_3\dfrac{d\vec{v}_3}{dt} + \ldots}{M};$$

$$\vec{a}_{CM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots}{m_1 + m_2 + \ldots} \Rightarrow \vec{a}_{CM} = \frac{\vec{F}_{ext}}{M_{total}}$$

3. $$\vec{a}_{CM} = \frac{\vec{F}_{ext}}{M_{total}} = \frac{\text{Net force on system}}{M}$$

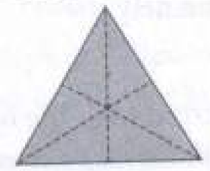
$$= \frac{\text{Net external force} + \text{Net internal force}}{M}$$

$$= \frac{\text{Net external force}}{M}$$

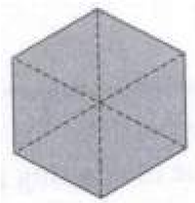
The position of CM depends upon the shape and the distribution of mass within it.

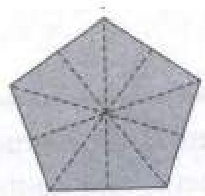
It is quite easy to find the position of CM of a body which has symmetrical shape and uniform mass distribution. If a body has irregular shape or non-uniform mass distribution, then CM can be obtained with the help of technique of integration.

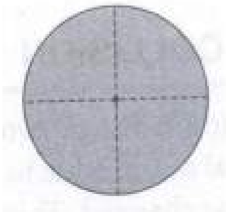
The CM of a rigid body is a point at a fixed position with respect to the body and it may or may not be within the body.



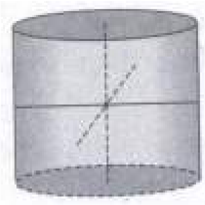
Triangular plate,
(Point of intersection of three medians)



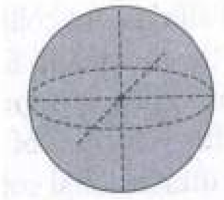


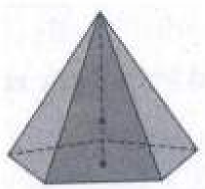


Regular polygon and circular plate
(At the geometrical centre of the figure)

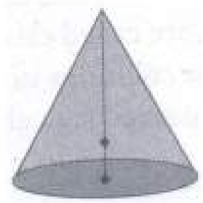


Cylinder and Sphere (At the geometrical centre of the figure)





Pyramid and Cone (On line joining vertex with base and one-fourth of the length from the base)



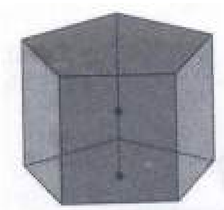
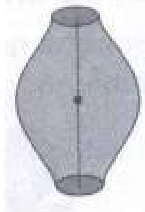


Figure with symmetry (some point on the axis of symmetry)



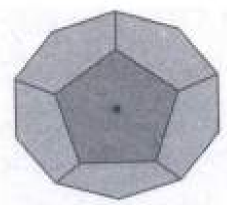
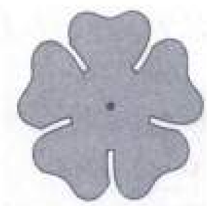


Figure with centre symmetry (At the centre of symmetry)



7.2.1 Conservation of Linear Momentum

1. For a single mass or single body
   If net force acting on the body is zero, then
   $\vec{P}$ = constant or $\vec{v}$ = constant (if mass = constant)
2. For a system of particles or system of rigid bodies
   If net external force acting on a system of particles or system of rigid bodies is zero, then,
   $\vec{p}_{CM}$ = constant or $\vec{v}_{CM}$ = constant

The complete motion of a system of bodies can be divided into two parts, namely

1. CM motion which describes the motion of the whole system,
2. Motion of the various parts with respect to the CM which may be referred to as the *internal motion*. The internal motion, in case of rigid bodies, is called as the *rotational motion.*

7.3 COLLISION

Collision between two particles is defined as the mutual interaction of the particles for a small interval of time due to which both the energy and momentum of at least one interacting particle must be changed. There is no need of physical contact for a process called to be as collision.

In physics, a collision will take place if either of the two bodies come in physical contact with each other or even when path of one body is affected by the force exerted due to the other.

In all types of collisions, total momentum and total energy are always conserved.

If the initial and final velocities of colliding masses lie along the same line, then is known as head-on collision or one dimensional collision.

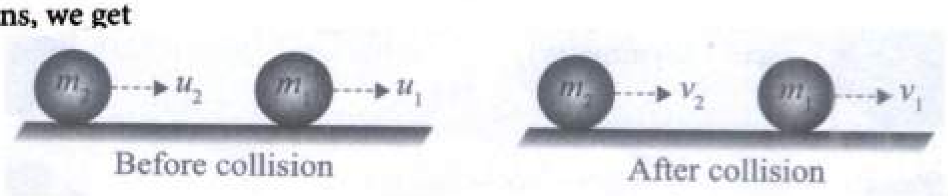
If the velocities of the colliding masses are not collinear, then it is known as oblique collision or two dimensional collision.

The collisions, in which both the momentum and kinetic energy of the system remain conserved, are called elastic collisions.

The collisions in which only the momentum of the system is conserved but kinetic energy is not conserved are called inelastic collisions.

7.3.1 Head-on Elastic Collision

In this case linear momentum and kinetic energy both are conserved. After solving two conservation equations, we get



$$v_1' = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_1 + \left(\frac{2m_2}{m_1 + m_2}\right)v_2$$

and $$v_2' = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_2 + \left(\frac{2m_1}{m_1 + m_2}\right)v_1$$

Regarding the above two formulae following are three special cases.

1. If $m_1 = m_2$, then $v_1' = v_2$ and $v_2' = v_1$ i.e., in case of equal masses bodies will exchange their velocities.
2. If $m_1 >> m_2$ and $v_1 = 0$. Then $v_1' \approx 0$ and $v_2' \approx -v_2$
3. If $m_2 >> m_1$ and $v_1 = 0$. Then $v_1' \approx 2v_2$ and $v_2' \approx v_2$

7.3.2 Head-on Inelastic Collision

In this type of collision only linear momentum remains constant.

Two unknowns are v_1' and v_2'. Make following two equations to solve them.

1. Conservation of linear momentum

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$$

2. Definition of coefficient of restitution (e)

$$e = \frac{|\text{Relative velocity of separation}|}{|\text{Relative velocity of approach}|} = \frac{v_1' - v_2'}{v_2 - v_1}$$

The above equation is called the Newton's experimental law of impact.

For a given pair of bodies, e is a constant and depends upon the nature of colliding bodies.

For two glass balls, $e = 0.95$ and for two lead balls, $e = 0.20$.

The definition of e gives us an alternative way of defining collisions.

(a) If $e = 1$, the collision is *perfectly elastic.*

(b) If $e < 1$, the collision is *inelastic.*

(c) If $e = 0$, the collision is *perfectly inelastic* or *plastic collision.*

(d) If $e > 1$, the collision is *superelastic.*

3. General expression for velocities after direct impact or head-on collision are

$$v_1' = \frac{(m_1 - em_2)v_1}{m_1 + m_2} + \frac{(1+e)m_2v_2}{m_1 + m_2}, v_2' = \frac{(1+e)m_1v_1}{m_1 + m_2} + \frac{(m_2 - em_1)v_2}{m_1 + m_2}$$

4. The loss in kinetic energy of two bodies after an inelastic collision,

$$\Delta E = \frac{m_1m_2}{2(m_1 + m_2)}(v_1 - v_2)^2(1 - e^2)$$

Transfer of KE is almost complete when a light incident particle strikes perfectly inelastically a massive target particle at rest.

The loss in KE is almost complete when a light incident particle strikes perfectly inelastically a massive target particle at rest.

The loss in KE is 100 per cent if in a perfectly inelastic collision, the colliding particles have equal and opposite momenta.

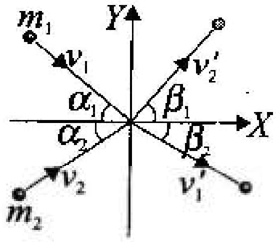
7.3.3 Oblique Collision (Both Elastic and Inelastic)

Resolve the velocities along common normal and common tangent direction. Now,

1. Velocity components along common tangent direction will remain unchanged.
2. Along common normal direction theory of head on collision (elastic as well as inelastic) can be used.

   Let α_1, α_2 = angles before collision

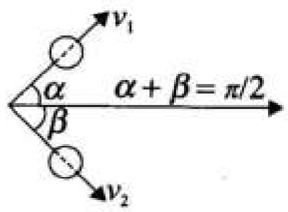
   Let β_1, β_2 = angles after collision



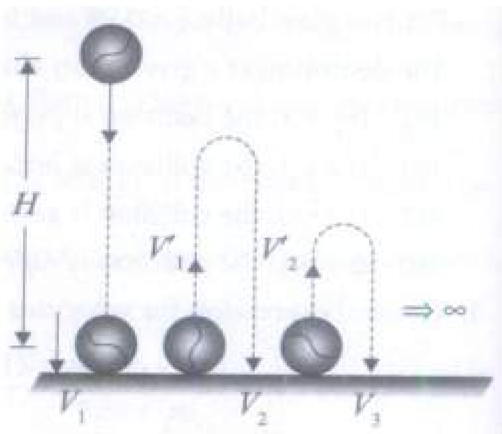
   (a) If $m_1 = m_2$ and $\alpha_1 + \alpha_2 = 90°$ then $\beta_1 + \beta_2 = 90°$

   It means that if two particles of equal mass collide elastically while moving at right angles to each other, then after collision also they move at right angles to each other.

   (b) If a particle A collides elastically with another particle B of equal mass at rest, then after the collision the two particles move at right angles to each other i.e., $\alpha + \beta = \pi/2$.



A ball falls from some height H. Let e be the coefficient of restitution between the ball and the ground and ball rebounds again and again, then



1. Speed of ball before nth strike

$$= v_n = e^{n-1}\sqrt{2gH}$$

2. Speed of ball after nth strike $= v_n' = e^n\sqrt{2gH}$

3. Height attained after nth strike

$$= H_n = \frac{(e^n\sqrt{2gH})^2}{2g} = (e^{2n}H)$$

4. Time of ascent after nth strike

$$= t_n = \frac{v_n'}{g} = \left(e^n\sqrt{\frac{2H}{g}}\right)$$

Again,

Let if speed of ball becomes zero after nth strike, then

$$e^n\sqrt{2gH} \to 0 \Rightarrow e^n \to 0 \Rightarrow (\text{fraction})^n \to 0 \Rightarrow n \to \infty$$

i.e., the number of strikes is infinite till the ball becomes at rest.

Total distance travelled by the ball is

$$S = H + 2H_1 + 2H_2 + 2H_3 + \ldots + \infty$$
$$= H + 2(H_1 + H_2 + H_3 + \ldots + \infty)$$
$$= H + 2(e^2H + e^4H + e^6H + e^8H + \ldots + \infty)$$
$$= H + 2e^2H(1 + e^2 + e^4 + e^6 + \ldots + \infty)$$
$$= H + 2e^2H\left(\frac{1}{1-e^2}\right) = H\left(1 + \frac{2e^2}{1-e^2}\right) = H\left(\frac{1+e^2}{1-e^2}\right)$$

Total time taken by the ball is

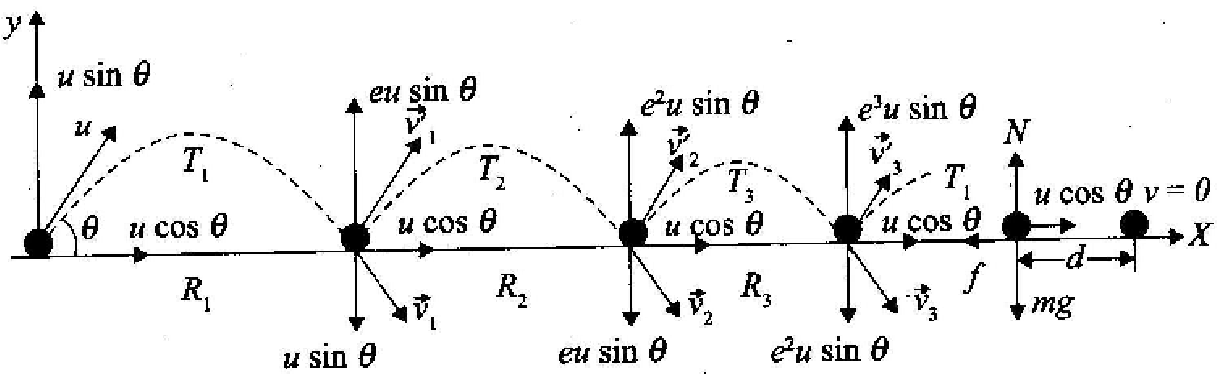
$$T = t + 2t_1 + 2t_2 + 2t_3 + \ldots \infty$$
$$T = t + 2(t_1 + t_2 + t_3 + \ldots + \infty)$$
$$= \sqrt{\frac{2H}{g}} + 2\left(e\sqrt{\frac{2H}{g}} + e^2\sqrt{\frac{2H}{g}} + e^3\sqrt{\frac{2H}{g}} + \ldots + \infty\right)$$
$$= \sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}}(e + e^2 + e^3 + e^4 \ldots \infty)$$
$$= \sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}}\left(\frac{e}{1-e}\right) = \sqrt{\frac{2H}{g}}\left(\frac{1+e}{1-e}\right)$$

Average speed of the ball $= \dfrac{\left(\dfrac{1+e^2}{1-e^2}\right)H}{\left(\dfrac{1+e}{1-e}\right)\sqrt{\dfrac{2H}{g}}}$

and total displacement $= H$

and average velocity of the ball $= \dfrac{H}{\left(\dfrac{1+e}{1-e}\right)\cdot\sqrt{\dfrac{2H}{g}}}$

A ball is projected from the ground with the velocity u making an angle θ with the ground. If the coefficient of restitution is e, then



1. Horizontal range acquired by the ball after nth strike (or $(n+1)$th projectile)

$$= (e^n R_1) = e^n \frac{u^2 \sin 2\theta}{g}$$

2. Time taken by the ball in between nth and $(n+1)$th strike)

$$= e^n T_1 = e^n \left(\frac{2u \sin\theta}{g} \right)$$

3. Let after nth strike vertical component of velocity $= 0$

$e^n(u \sin \theta) = 0$; $e^n \to 0$; $(\text{Fraction})^n \to 0$; $n \to \infty$

i.e., clearly, after infinite number of strike the vertical component of velocity of the ball is zero; but it's horizontal component remains constant as $(u \cos \theta)$. Due to $(u \cos \theta)$ the ball slides on the surface and becomes at rest due to the kinetic friction.

$$f = \mu mg \qquad a = \frac{f}{m} = (-\mu g)$$

$$v^2 = u^2 + 2ax \Rightarrow 0 = u^2 \cos^2\theta - 2(\mu g)d \Rightarrow d = \frac{u^2 \cos^2\theta}{2\mu g}$$

and $\qquad v = u + at \Rightarrow t = \left(\frac{u \cos\theta}{\mu g} \right)$

Net displacement $= R_1 + R_2 + R_3 + R_4 + \ldots \infty$

$$= R_1 + eR_1 + e^2R_1 + e^3R_1 + e^4R_1 + \ldots \infty + d$$

$$= R_1(1 + e + e^2 + e^3 + e^4 + \ldots \infty) + d$$

$$= R_1\left(\frac{1}{1-e}\right) + d = \left(\frac{u^2 \sin 2\theta}{g(1-e)} + \frac{u^2 \cos^2\theta}{2\mu g} \right)$$

and total time taken by the ball

$$= (T_1 + T_2 + T_3 + T_4 \ldots \infty) + t$$

$$= T_1(1 + e + e^2 + e^3 + \ldots \infty) + t$$

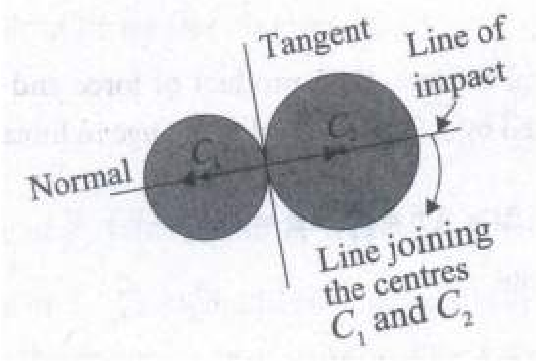
$$= T_1\left(\frac{1}{1-e_1}\right) + t = \left(\frac{2u \sin\theta}{g(1-e)} + \frac{u \cos\theta}{\mu g} \right)$$

Central Impact

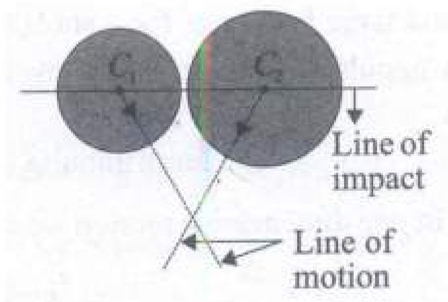
When the line joining the CM of the bodies lies on the line of impact, we call the impact central impact.

Oblique Impact

When the line of motion of the bodies does not coincide with the line of impact, we call it oblique impact.



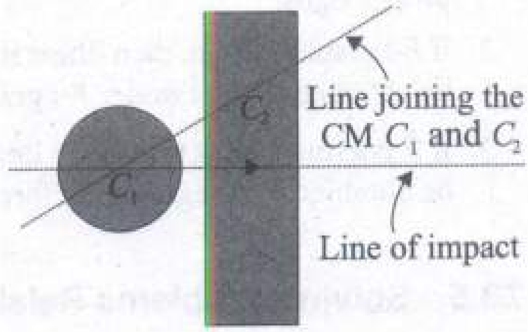
Central impact



Oblique impact

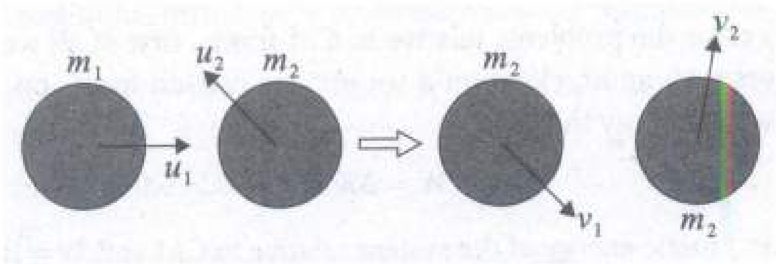
Eccentric Impact

When the line joining the CM of the colliding bodies does not coincide with the line of impact, this is known as eccentric impact. Impact of cricket ball and bat is the familiar example of an eccentric impact.



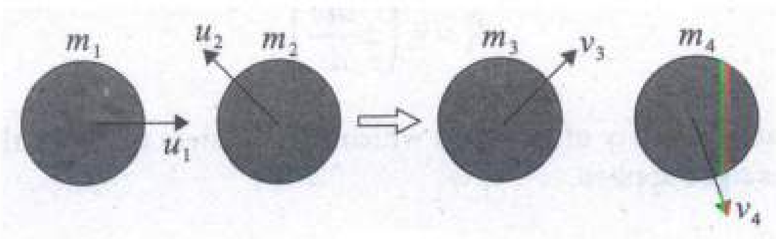
Scattering

When the composition and mass of the colliding particle (or bodies) remain unchanged during collision so that, before and after the collision, each colliding body remains identical, this type of collision is called scattering. For instance, collision between ideal gas molecules is an ideal example of scattering.



Reaction

Many times, a collision between an atom A and molecule B–C yields a molecule A–B and an atom C. That means, the final particles (or bodies) of the colliding system are not identical with the initial particles. We call it reaction. In this way, chemical and nuclear reactions are the consequences of collisions.



7.3.4 Linear Impulse

When a large force acts for a short interval of time, then product of force and time is called linear impulse. It is a vector quantity denoted by $\bar{J}$. This is equal to change in linear momentum. Thus,

$$\text{Linear impulse } \vec{J} = \vec{F} \cdot \Delta t = \Delta \vec{p} = \vec{p}_f - \vec{p}_i = m(\vec{v}_f - \vec{v}_i)$$

1. In one dimensional motion we can write,

$$J = F\,\Delta t = \Delta p = p_f - p_i = m(v_f - v_i).$$

   In this case we will choose a sign convention and all vector quantities are substituted with proper signs.

2. If F-t graph is given, then linear impulse and therefore change in linear momentum can also be obtained by area under F-t graph with projection along t-axis.
3. If $\vec{F}$ is a function of time, then linear impulse and therefore change in linear momentum can be obtained by integration of force in the given time interval.

7.3.5 Solving Problems Relative to CM Frame

A frame of reference carried by the centre of mass of an isolated system of particles (*i.e.*, a system not subjected to any external forces) is called the centre of mass or *C*-frame of reference. In this frame of reference,

1. Position vector of centre of mass is zero
2. Velocity and hence, momentum of centre of mass is also zero.

When we try to solve the problems relative to CM frame, first of all we fix ourselves at the CM. If the CM moves with an acceleration a we impose pseudo forces on each particle of the system. Then apply work-energy theorem

$$W = \Delta K$$

where ΔK = change in kinetic energy of the system relative to CM and W = sum of work done by all forces (real and pseudo, internal and external) relative to CM $= \sum \vec{F}_i dr_{iC}$.

7.3.6 Variable Mass

1. A thrust force will act when mass of a system either increases or decreases. This force is given by,

$$\vec{F}_t = \vec{v}_r\left(\pm\frac{dm}{dt}\right)$$

   Here $\vec{v}_r$ is relative velocity of mass dm which either enters or leaves the system on which thrust force has to be applied.

2. Magnitude of thrust force is given by

$$F_t = \left|\vec{v}_r\left(\pm\frac{dm}{dt}\right)\right|$$

3. Direction of $\vec{F}_t$ is parallel to $\vec{v}_r$ if mass of system is increasing or $\frac{dm}{dt}$ is positive.

   Direction of $\vec{F}_t$ is antiparallel to $\vec{v}_r$ if mass of system is decreasing or $\frac{dm}{dt}$ is negative.

4. Based on this fact velocity of rocket at time t is given by

$$v = u - gt + v_r \ln\left(\frac{m_o}{m}\right)$$

   Here u = Initial velocity of rocket.

   v_r = Exhaust velocity of gases (assumed constant).

   m_o = Initial mass of rocket (with gases).

   m = Mass of rocket at time t (with gases).

   Value of g has been assumed constant in above equation.

5. If mass is just dropped from a moving body then the mass which is dropped acquires the same velocity as that of the moving body.

   Hence, $\vec{v}_r = 0$ or no thrust force will act in this case.

   Problems related to variable mass can solved in following three steps:

   (a) Make a list of all the forces acting on the main mass and apply them on it.

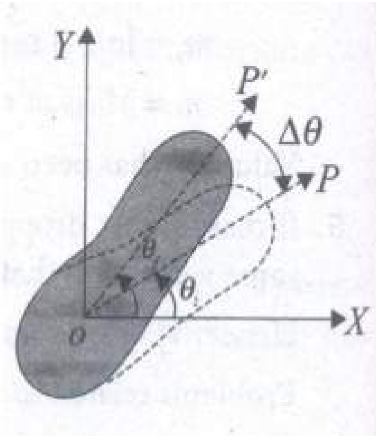
   (b) Apply an additional thrust force $\vec{F}_t$ on the mass, the magnitude of which is $\left|\vec{v}\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by the direction $\vec{v}_r$ in case the mass is increasing and otherwise the direction of $-\vec{v}_r$ if it is decreasing

   (c) Find net force on the mass and apply

   $\vec{F}_{net} = m\frac{d\vec{v}}{dt}$ where m = mass at that particular instant.

ROTATIONAL AND ROLLING MOTION

8.1 ANGULAR DISPLACEMENT



1. The angle turned by a body rotating about a given axis is called angular displacement.
2. Angular displacement $\Delta\theta$ is measured in radians. Its dimensions are $[M^0L^0T^0]$.
3. The angular displacement behaves like an axial vector for infinitesimal displacements, i.e.,

$$\vec{d\theta_1}+\vec{d\theta_2}=\vec{d\theta_2}+\vec{d\theta_1}$$

4. If a particle completes n rotations, then the angle traversed by it is $\theta = 2\pi n$ and the angular displacement is also $\theta_1 - \theta_2 = 2\pi n$.

8.2 ANGULAR VELOCITY

1. The average angular velocity $\omega_{av.}$ is defined as

$$\omega_{av.}=\frac{\theta_2-\theta_1}{t_2-t_1}=\frac{\Delta\theta}{\Delta t}$$

2. The instantaneous angular velocity ω is given by

$$\omega=\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}=\frac{d\theta}{dt}$$

3. Angular velocity has dimensions $[M^0L^0T^{-1}]$. Its units are radian/sec or revolutions/sec.
4. Angular velocity is a vector whose direction is associated as follows:
   (a) For clockwise rotation, $\vec{\omega}$ is directed downward.
   (b) For anticlockwise rotation, $\vec{\omega}$ is directed upward.
5. For a uniform circular motion, $\omega_{av.} = \omega$.
6. If a particle completes n revolutions in t seconds, then its angular velocity is

$$\omega = (2\pi n/t)\ \text{rad/sec}$$

7. If two particles are moving in coplanar and concentric circular path with angular velocities $\vec{\omega}_A$ and $\vec{\omega}_B$, then their relative angular velocity (i.e., of w.r.t. A) is

$$\vec{\omega}_{BA} = \vec{\omega}_B - \vec{\omega}_A$$

If the two particles are moving in the same direction then,

$$T_{\text{relative}} = \frac{2\pi}{\omega_B - \omega_A} = \frac{T_A T_B}{T_A - T_B}$$

8.3 ANGULAR ACCELERATION

1. The average angular acceleration $\alpha_{av.}$ is defined as

$$\alpha_{av.} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

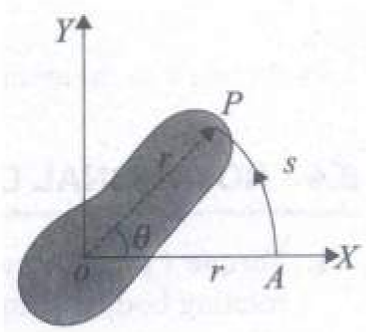
2. The instantaneous angular acceleration is defined as

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

3. The dimensions of the angular acceleration are $[M^0L^0T^{-2}]$. The units are radian/sec^2.
4. The direction of the angular acceleration vector $\vec{\alpha}$ is the same as that of $\vec{\omega}$ for circular motion in a fixed plane, i.e., upwards for anticlockwise rotation and downwards for clockwise rotations. In a uniform circular motion, $\alpha = 0$.

8.3.1 Relation Between Linear and Angular Velocity

1. When a body rotates through an angle θ, the distance s moved by it along the arc is $s = r\theta$. For rotational motion of the body, r is constant.



Hence,

$$\left(\frac{ds}{dt}\right) = r\left(\frac{d\theta}{dt}\right) \text{ or } v = \omega r$$

where $v = \dfrac{ds}{dt}$ is the tangential or linear speed of the particle P

and $\dfrac{d\theta}{dt}$ is the angular speed.

2. **In vector form:** $\vec{v} = \vec{\omega} \times \vec{r}$ or $v = \omega r \sin\theta$ where θ is the angle between $\vec{\omega}$ and $\vec{r}$. The direction of $\vec{v}$ is $\perp$ to both $\vec{\omega}$ and $\vec{r}$.

8.3.2 Relation Between Linear and Angular Acceleration

1. From $v = \omega r$, we get $\left(\frac{dv}{dt}\right) = r\left(\frac{d\omega}{dt}\right)$ or $a_t = r\alpha$ where $a_t = \left(\frac{dv}{dt}\right)$ is the tangential component of the acceleration of a particle moving in a circle and $\alpha = \left(\frac{d\omega}{dt}\right)$ is the angular acceleration.
2. For a particle moving in a circular motion, there also exists the centripetal acceleration:

$$a_r = (v^2/r) = \omega^2 r$$

3. The resultant acceleration when both a_t and a_r are present is given by

$$a = \sqrt{a_t^2 + a_r^2}$$

$$\vec{a} = \left(\frac{d\vec{v}}{dt}\right) = \frac{d}{dt}(\vec{\omega}\times\vec{r}) = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt}$$

$$= \vec{\alpha}\times\vec{r} + \vec{\omega}\times\vec{v} = \vec{a}_t + \vec{a}_r$$

where $\vec{a}_t = \vec{\alpha}\times\vec{r}$ and $\vec{a}_r = \vec{\omega}\times\vec{v} = \vec{\omega}\times(\vec{\omega}\times\vec{r})$.

8.3.3 Equations of Motion of Rotating Body

If a rigid body rotating about a fixed axis with constant angular acceleration α, then

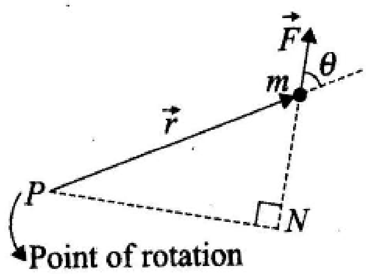
(a) $\omega = \omega_0 + \alpha t$ (b) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ (c) $\omega^2 = \omega_0^2 + 2\alpha\theta$

If a rigid body rotating about a fixed axis with variable angular acceleration, then problems are solved with the help of definitions as

$$\alpha = \frac{d\omega}{dt} \text{ and } \omega = \frac{d\theta}{dt}$$

8.4 ROTATIONAL DYNAMICS

1. **Torque ($\vec{\tau}$):** The cause of angular acceleration of a rigid rotating body is torque. Consider a body is acted by a force $\vec{F}$ at a point whose position vector is $\vec{r}$ with reference to the point of rotation, as shown in figure (the point of rotation is that point about which the torque produced is defined), then torque produced by $\vec{F}$ about P is

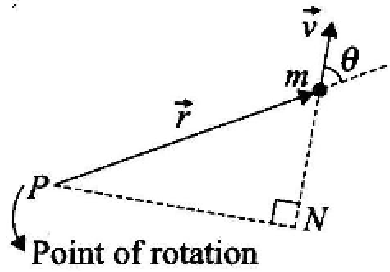


$$\vec{\tau} = \vec{r}\times\vec{F} \text{ or } \tau = F(r\sin\theta) = F\times PN = Fr_{\perp}$$

If different forces act on different points of a system then the torque produced about a point of rotation is the vector resultant of individual torques produced by the different forces.

2. **Angular momentum ($\vec{L}$):** Angular momentum of a particle about an axis is defined as the moment of the linear momentum of the particle about that axis. It is a vector quantity.

Consider a particle of mass m, moving with a velocity $\vec{v}$ and at a time the position vector of the particle is $\vec{r}$ with reference to the point of rotation as in the figure, then



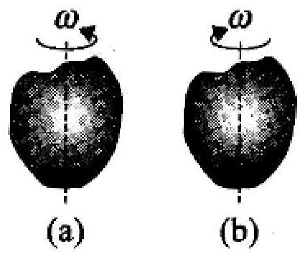
$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

or $$L = p\,r \sin\theta = pr_{\perp}$$

3. **Angular momentum of a rigid body in pure rotation about axis of rotation:** If a rigid body is in pure rotation about a fixed axis, then angular momentum of rigid body about this axis will be given by

$$L = I\,\omega \text{ (I is the moment of inertia of the body)}$$

This is actually component of total angular momentum about axis of rotation. Direction of this component is again given by right hand screw law. In figure (a), this is along the axis in upward direction. In figure (b) this is along the axis in downward direction.



4. **Angular momentum of a rigid body due to translational and rotational motion both:** Consider a body of mass m is rotating with angular velocity ω about c.m. axis and translating with a linear velocity $\vec{v}$. The angular momentum of the body is

$$\vec{L} = \vec{L}_{\text{translation}} + \vec{L}_{\text{rotation}}$$

5. **Geometrical meaning of angular momentum:** The angular momentum of a particle or a body is

$$\vec{L} = 2m\left(\frac{\Delta\vec{A}}{\Delta t}\right)$$

The quantity $\frac{\Delta\vec{A}}{\Delta t}$ is the area covered by the position vector $\vec{r}$ per unit time and is called areal velocity.

6. **Rotational kinetic energy:** Kinetic energy of rotating body is

$$K_{\text{Rot}} = \frac{1}{2}I\omega^2$$

Kinetic energy due to translational and rotational motion both is

$$K = K_{\text{Translation}} + K_{\text{Rotational}} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

7. **Rotational work done:** The work-done by the constant torque is

$$W = \tau\,\Delta\theta$$

In case of variable torque, the work-done is $W = \int_{\theta_1}^{\theta_2} \tau\,d\theta.$

8. **Power delivered due to rotational motion:** The average power delivered due to rotational motion or due to torque is

$$\text{Power } P = \frac{\Delta W}{\Delta t} = \tau\omega$$

The instantaneous power delivered due to rotational motion or due to torque is

$$\text{Power } P = \frac{dW}{dt}$$

9. **Newton's second law for rotating rigid body :** The rate of change of angular momentum is equal to the external torque.

$$\vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$$

This equation is the rotational analogue of Newton's second law of translational motion.

$$\vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt} = \frac{d}{dt}(I\vec{\omega}) = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha}$$

10. **Conservation of angular momentum:** We know that $\vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$

If no net external torque acts on the system, this equation becomes $\frac{d\vec{L}}{dt} = 0$

or $\vec{L}$ = Constant (for isolated system)

This equation represents the law of conservation of angular momentum.

8.5 ANGULAR IMPULSE

The angular impulse is defined as the product of the torque produced and the time interval of the action of the torque.

$$\vec{j} = \vec{\tau}\,\Delta t \qquad (1)$$

The equation (1) is valid only when the torque produced remains constant during the time interval of operation.

Again,

$$\vec{j} = I\vec{\alpha}\,\Delta t = I\frac{\Delta\vec{\omega}}{\Delta t}\times\Delta t = I\,\Delta\vec{\omega}$$

$$\Rightarrow\ \vec{j} = I(\vec{\omega}_f - \vec{\omega}_i) = I\vec{\omega}_f - I\vec{\omega}_i = \vec{L}_f - \vec{L}_i$$

But if the torque produced is not constant then the angular impulse is defined as

$$\vec{j} = \int_{t=t}^{t+\Delta t}\vec{\tau}\,dt = \int I\vec{\alpha}\,dt = I\int\left(\frac{d\vec{\omega}}{dt}\right)dt = I\int_{\omega_i}^{\omega_f} d\vec{\omega} = I(\vec{\omega}_f - \vec{\omega}_i)$$

$$= I\vec{\omega}_f - I\vec{\omega}_i = \vec{L}_f - \vec{L}_i$$

(Δt = Operating time interval of torque produced)

i.e., the angular impulse of a rotating body is the change in angular momentum of the body.

8.5.1 Angular Momentum of Rolling Body

Angular momentum of a rolling body having radius R about an axis passing through point of contact P and perpendicular to plane of body is

$$\vec{L} = \vec{L}_{\text{translation}} + \vec{L}_{\text{rotation}} = m(\vec{R}\times\vec{v}_{CM}) + I_{CM}\,\vec{\omega}$$

$$\text{or}\quad L = m\omega R^2 + I_{CM}\omega \quad\text{or}\quad L = (I_{CM} + mR^2)\omega = I_p\omega$$

8.5.2 Moment of Inertia of a Discrete Rotating System

$$I = (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots)$$

where r_1, r_2, r_3 ... are distances of m_1, m_2, m_3 ... from the axis of rotation respectively.

8.5.3 Moment of Inertia of a Continuous Rotating System

$$I = \int dm\, r^2$$

where dm is a suitably chosen elementary mass and r is the distance of dm from the axis of rotation.

8.5.4 Theorem on Moment of Inertia

1. **Theorem of parallel axes:** The moment of inertia of a body about an axis is equal to its moment of inertia about a parallel axis passing through the centre of mass (I_{CM}) plus Ma^2, where M is the mass of the body and a is the distance between the two axes, i.e. $I = I_{CM} + Ma^2$.

2. **Theorem of perpendicular axes:** The sum of the moments of inertia of a plane lamina about two mutually perpendicular axes in its plane is equal to its moment of inertia about a third axis perpendicular to the plane and passing through the point of intersection of the two axes, i.e., $I_Z = I_x + I_y$.

8.5.5 Rolling of a Body on Horizontal Rough Surface

For pure rolling the coefficient of static friction is

$$\mu^{\min} = \frac{F}{mg}\left[\frac{1}{1+mR^2/I}\right]$$

where F is a force acting on the body parallel to the horizontal surface.

8.5.6 Rolling of a Body on Inclined Rough Surface of Inclination θ

For pure rolling

1. $v_{CM} = \sqrt{\dfrac{2gx\sin\theta}{1+K^2/R^2}}$, v_{CM} is the velocity of CM at the bottom point and K is the radius of gyration.
2. KE of the body at the bottom point is $K = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}mv^2_{CM}$.
3. The acceleration of CM at the bottom point $a_{CM} = \dfrac{g\sin\theta}{1+K^2/R^2}$.

   Angular acceleration of the body about CM during the rolling is

$$\alpha = \left(\frac{a_{CM}}{R}\right) = \left[\frac{g\sin\theta}{R(1+K^2/R^2)}\right]$$

4. The time taken by the rolling body in reaching at the bottom point

$$t = \sqrt{\frac{2x(1+K^2/R^2)}{g\sin\theta}}$$

5. The friction force acting on the rolling body $f_s = \dfrac{(mg\sin\theta)(K^2/R^2)}{(1+K^2/R^2)}$.
6. The minimum friction coefficient between the rolling body and the inclined plane for pure rolling $\mu^{\min} = \tan\theta\left(\dfrac{K^2/R^2}{1+K^2/R^2}\right)$.

8.5.7 For Rolling with Forward Slipping

1. The acceleration of the rolling body $a_{CM} = g(\sin\theta - \mu\cos\theta)$.
2. The angular acceleration about CM $\alpha = \dfrac{(\mu mg\cos\theta)}{I_{CM}}R$.
3. The velocity of CM at the bottom point $v_{CM} = \sqrt{2gx(\sin\theta - \mu\cos\theta)}$.
4. The time taken by the rolling body in reaching at the bottom point $t = \sqrt{\dfrac{2x}{g(\sin\theta - \mu\cos\theta)}}$.
5. Kinetic energy at bottom point $= \dfrac{1}{2}I_{CM}\,\omega^2 + \dfrac{1}{2}m(\omega R)^2$.

Table 8.1

S. No.	Translatory Motion	Rotatory Motion
1.	All the constituent particles of the rigid body parallel to one another in straight lines.	The particles move parallel to one another in circles of different radii about the given axis of rotation.
2.	All the particles have same linear velocity.	All the particles have same angular velocity. As $v = r\omega$, the particles at different r have different linear velocities.
3.	All the particles undergo same linear displacement.	All the particles undergo same angular displacement.
4.	All the particles have same linear acceleration.	All the particles have same angular acceleration.
5.	The position of the centre of mass changes with time.	The distance of centre of mass from the axis of rotation remains constant with respect to time.
6.	Mass is analogous to moment of inertia. Mass depends on the quantity of matter in the body.	Moment of inertia (I) is analogous to mass. Moment of inertia (I) depends on distribution of mass about axis of rotation.
7.	Kinetic energy of translation $= \dfrac{1}{2}mv^2$.	Kinetic energy of rotation $= \dfrac{1}{2}I\omega^2$
8.	Force produces the translatory motion.	Torque produces the rotational motion.
9.	Work done = W W = Force × Displacement	W = Torque × θ
10.	Force = Mass × Acceleration	Torque = I × Angular acceleration
11.	Linear momentum = p p = Mass × Linear velocity	Angular momentum = $I\omega$ where ω = Angular velocity
12.	Impulse = Force × Time	Angular impulse = Torque × time
13.	Power = Force × Velocity	Power = Torque × ω

Table 8.2 Moment of Inertia of Different Objects

Shape of Body	Rotational Axis	Moment of Inertia	Radius of Gyration
1. Ring *M*: Mass *R*: Radius	(a) Perpendicular to plane passing through centre of mass	MR^2	R
	(b) Diameter in the plane	$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(c) Tangent perpendicular to plane	$2MR^2$	$\sqrt{2}R$
	(d) Tangent in the plane	$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
2. Disc	(a) Perpendicular to plane passing through centre of mass	$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(b) Diameter in the plane	$\frac{MR^2}{4}$	$\frac{R}{2}$
	(c) Tangent in the plane	$\frac{5}{4}MR^2$	$\frac{\sqrt{5}}{2}R$
	(d) Tangent perpendicular to plane	$\frac{3}{2}MR^2$	$\frac{\sqrt{3}}{2}R$
3. Thin walled cylinder	(a) Geometrical axis	MR^2	R
	(b) Perpendicular to length passing through centre of mass	$M\left(\frac{R^2}{2}+\frac{L^2}{12}\right)$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{12}}$
	(c) Perpendicular to length passing through one end	$M\left(\frac{R^2}{2}+\frac{L^2}{3}\right)$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{3}}$
4. Solid cylinder	(a) Geometrical axis	$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
	(b) Perpendicular to length passing through centre of mass	$M\left(\frac{R^2}{4}+\frac{L^2}{12}\right)$	$\sqrt{\frac{R^2}{4}+\frac{L^2}{12}}$

Table 8.2 (*Continued*)

Shape of Body	Rotational Axis	Moment of Inertia	Radius of Gyration
	(c) Perpendicular to length passing through one end	$M\left(\frac{R^2}{4}+\frac{L^2}{3}\right)$	$\sqrt{\frac{R^2}{4}+\frac{L^2}{3}}$
5. Annular disc	(a) Perpendicular to plane passing through centre of mass	$\frac{M}{2}[R_1^2+R_2^2]$	$\sqrt{\frac{R_1^2+R_2^2}{2}}$
	(b) Diameter in the plane	$\frac{M[R_1^2+R_2^2]}{4}$	$\sqrt{\frac{R_1^2+R_2^2}{4}}$
6. Hollow cylinder	(a) Geometrical axis	$M\left[\frac{R_1^2+R_2^2}{2}\right]$	$\sqrt{\frac{R_1^2+R_2^2}{2}}$
	(b) Perpendicular to length passing through centre of mass	$M\left[\frac{L^2}{12}+\frac{(R_1^2+R_2^2)}{4}\right]$	$\sqrt{\frac{L^2}{12}+\frac{R_1^2+R_2^2}{4}}$
7. Solid sphere	(a) Diameter	$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
	(b) Tangent	$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$
8. Thin spherical shell	(a) Diameter	$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$
	(b) Tangent	$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$
9. Hollow sphere	Diameter	$\frac{2}{5}M\left[\frac{R^5-r^5}{R^3-r^3}\right]$	$\sqrt{\frac{2\,(R^5-r^5)}{5\,(R^3-r^3)}}$
10. Thin rod	(a) Perpendicular to length passing through centre of mass	$\frac{ML^2}{12}$	$\frac{L}{2\sqrt{3}}$
	(b) Perpendicular to length passing through one end	$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$

(*Continued*)

Table 8.2 (*Continued*)

Shape of Body	Rotational Axis	Moment of Inertia	Radius of Gyration
11. Rectangular Plate Length: a Breadth: b	(a) Perpendicular to length in the plane passing through centre of mass	$\frac{Ma^2}{12}$	$\frac{a}{2\sqrt{3}}$
	(b) Perpendicular to breadth in the plane passing through centre of mass	$\frac{Mb^2}{12}$	$\frac{b}{2\sqrt{3}}$
	(c) Perpendicular to plane passing through centre of mass	$\frac{M(a^2+b^2)}{12}$	$\frac{\sqrt{a^2+b^2}}{2\sqrt{3}}$

Table 8.3

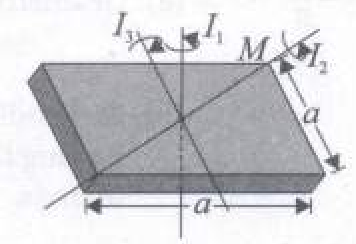
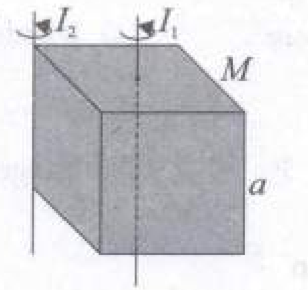
(a) Square plate	$I_1 = \frac{Ma^2}{6}$ $I_2 = I_3 = \frac{Ma^2}{12}$	
(b) Cube	$I_1 = \frac{Ma^2}{6}$ $I_2 = \frac{2Ma^2}{3}$	

Table 8.4 Comparison of Rolling, Sliding and Falling Motions Down an Inclined Plane

Physical Quantity	Rolling Motion ($\beta > 1$)	Sliding Motion ($\beta = 1$)	Falling Motion $\beta = 1, \theta > 90°$
Velocity	$v_R = \sqrt{\frac{2gh}{\beta}} = \sqrt{\frac{2gs\sin\theta}{1+\frac{k^2}{R^2}}}$ k = Radius of gyration	$v_s = \sqrt{2gh}$ $= \sqrt{2gs\sin\theta}$	$v_F = \sqrt{2gh}$
Acceleration	$a_R = \left(\frac{g\sin\theta}{\beta}\right) = \frac{g\sin\theta}{1+\frac{k^2}{R^2}}$	$a_S = g\sin\theta$	$a_F = g$

Table 8.4 (*Continued*)

Physical Quantity	Rolling Motion ($\beta > 1$)	Sliding Motion ($\beta = 1$)	Falling Motion $\beta = 1, \theta > 90°$
Time of descend	$t_R = \frac{1}{\sin\theta}\sqrt{\beta\left(\frac{2h}{g}\right)}$ $= \frac{1}{\sin\theta}\sqrt{\frac{2h(1+k^2/R^2)}{g}}$	$t_s = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}$ $= \sqrt{\frac{2s}{g\sin\theta}}$	$t_F = \sqrt{\frac{2h}{g}}$

8.6 RADIUS OF GYRATION

1. Moment of inertia is also given by $I = Mk^2$, where M is the total mass of the body and k is radius of gyration.
2. The radius of gyration is the distance between axis of rotation and centre of gyration.
3. Centre of gyration is a point where the whole mass of the body is supposed to be concentrated at a single distance, as if the moment of inertia would be same as with actual distribution of mass of the body into particles.
4. Radius of gyration is also defined as the root mean square distance of all the particles about the axis of rotation, i.e., $k = \sqrt{\frac{r_1^2 + r_2^2 + \ldots + r_n^2}{n}}$
5. The radius of gyration of a body is not a constant quantity. Its value changes with the change of location of axis of rotation.

Table 8.5 Acceleration, Velocity and Time of Descend for Different Bodies Rolling Down an Inclined Plane

Body	$a = \frac{g\sin\theta}{1+\frac{I}{Mr^2}}$	$v = \sqrt{\frac{2gh}{1+\frac{I}{Mr^2}}}$	$t = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{I}{Mr^2}\right)}$
Solid sphere	$\frac{5}{7}g\sin\theta$	$\sqrt{\frac{10gh}{7}}$	$\frac{1}{\sin\theta}\sqrt{\frac{14h}{5g}}$
Hollow sphere	$\frac{3}{5}g\sin\theta$	$\sqrt{\frac{6gh}{5}}$	$\frac{1}{\sin\theta}\sqrt{\frac{10h}{3g}}$
Disc	$\frac{2}{3}g\sin\theta$	$\sqrt{\frac{4gh}{3}}$	$\frac{1}{\sin\theta}\sqrt{\frac{3h}{g}}$

(*Continued*)

Table 8.5 (*Continued*)

Body	$a = \dfrac{g\sin\theta}{1+\dfrac{I}{Mr^2}}$	$v = \sqrt{\dfrac{2gh}{1+\dfrac{I}{Mr^2}}}$	$t = \dfrac{1}{\sin\theta}\sqrt{\dfrac{2h}{g}\left(1+\dfrac{I}{Mr^2}\right)}$
Cylinder	$\frac{2}{3}g\sin\theta$	$\sqrt{\frac{4gh}{3}}$	$\frac{1}{\sin\theta}\sqrt{\frac{3h}{g}}$
Hollow cylinder	$\frac{1}{2}g\sin\theta$	$\sqrt{gh}$	$\frac{1}{\sin\theta}\sqrt{\frac{4h}{g}}$
Ring	$\frac{1}{2}g\sin\theta$	$\sqrt{gh}$	$\frac{1}{\sin\theta}\sqrt{\frac{4h}{g}}$

8.6.1 Couple

1. Two equal and unlike parallel forces acting on a body at two different points constitute a couple.
2. Moment of the couple = Magnitude of either of the forces × Perpendicular distance between the points of application of the forces.
3. The effect of couple on a body is to produce a turning motion in it.

8.6.2 Conditions for Equilibrium of a Rigid Body

1. For a body to be in translational equilibrium, the algebraic sum of the forces acting on the body is equal to zero, i.e., $\Sigma F = 0$.
2. For a body to be in rotational equilibrium, the algebraic sum of the moments of the forces about any point in their plane is zero, i.e., $\Sigma\tau = 0$

Important points concerning with the rolling of bodies down an inclined plane :

1. As factor $\beta = \left[1+\dfrac{I}{Mr^2}\right]$ depends on the shape of body and is independent of mass and radius, so if bodies of same shape but different masses and radii are allowed to roll down an inclined plane, they will reach the bottom with the same speed and at the same time.
2. If a solid and hollow body of same shape are allowed to roll down an inclined plane then as $\beta_S < \beta_H$, solid body will reach the bottom first and with greater velocity.
3. If a ring, cylinder, disc and sphere run a race by rolling on an inclined plane then as β_{sphere} = Min., while β_{ring} = Max., the sphere will reach the bottom first with greater velocity while ring last with least velocity.

Table 8.6 Ratios of Rotational KE (KR); Translational KE (KT) and Total KE of Different Bodies

Body of Radius r	Value of k^2 $(Mk^2 = I)$	$\frac{K_R}{K_T} = \frac{\frac{1}{2}Mv^2\frac{k^2}{r^2}}{\frac{1}{2}Mv^2} = \frac{k^2}{r^2}$	$\frac{K_T}{K} = \frac{\frac{1}{2}Mv^2}{\frac{1}{2}Mv^2\left(1+\frac{k^2}{r^2}\right)} = \frac{1}{1+\frac{k^2}{r^2}}$	$\frac{K_R}{K} = \frac{\frac{k^2}{r^2}}{1+\frac{k^2}{r^2}}$
Ring and hollow cylinder	r^2	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{2}$
Hollow sphere	$\left(\frac{2}{3}\right)r^2$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{2}{5}$
Disc and solid cylinder	$\left(\frac{1}{2}\right)r^2$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$
Solid sphere	$\left(\frac{2}{5}\right)r^2$	$\frac{2}{5}$	$\frac{5}{7}$	$\frac{2}{7}$

4. The velocity is independent of the inclination of the plane and depends only on height h through which the body descends.
5. Acceleration and time of descend depend on the inclination. Greater the inclination, greater will be the acceleration and lesser will be the time of descend.
6. If the object rolls across a surface in such a way that there is no relative motion of object and surface at the point of contact, the motion is called *rolling without slipping*. In such a case, friction is responsible for the motion but work done or dissipation of energy against friction is zero as there is no relative motion between body and surface at the point of contact.
7. Rolling motion of a body may be treated as a pure rotation about an axis through point of contact with same angular velocity.

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GRAVITATION

We know that there are four basic forces in nature. The force between two bodies by virtue of its masses is called as *gravitational force*. Newton proposed a law about the gravitational force between two *point masses*. Point mass is not the mass of the smaller size rather it is a concept. The mass of any shape and size is called as a point mass, if it is studied from a distance larger than the size of the body.

Gravitational force between two point masses

$$F = G\frac{m_1 m_2}{r^2}$$

The above equation is called Newton's law of gravitation. G is the proportionality constant and it is the same for all pairs of the point masses. Hence it is called universal gravitational constant and its value is 6.67×10^{-11} N-m^2/kg^2. The value of G was first introduced by *Cavendish* with the help of his instrument *tortion balance*.

9.1 PROPERTIES OF GRAVITATIONAL FORCE

1. It is always attractive.
2. Its nature is of conservative type.
3. It is a central force. (Central force is a position dependent force and it acts along the line joining the two bodies.)
4. It is the weakest force in nature.
5. It does not depend on the medium between the two bodies.
6. The gravitational attractive force between two bodies does not depend on the presence of other third bodies.
7. It obeys the principle of superposition i.e. the law of vector addition.

Newton's law of gravitation is applied on the point masses but it can also be applied for the bodies of any shape provided the separation between the bodies is greater than the size of the bodies.