Spacetime interpretation of Torsion in Prismatic Bodies

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Abstract

A non-linear theory for the plastic deformation of prismatic bodies is constructed which interpolates between Prandtl's linear soap-film approximation and Nádai's sand-pile model . Geometrically Prandtl's soap film and Nádai's wavefront are unified into a single smooth surface of constant mean curvature in three-dimensional Minkowski spacetime.

Keywords:torsion, prismatic body, sandpile analogy, Minkowski spacetime.

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1 Introduction

Prandtl's soap film [1] and Nadai's [2] sand heap analogies remain important tools for analysing the torsional loads and plastic deformations of cylindrical shafts [4, 3]. In this paper we propose a generalisation of Prandtl's model in which the eponymous potential $\phi(x,y)$ he introduced no-longer satisfies the linear Laplace equation but rather a non-linear equation known as the Born-Infeld equation. This has the property that the Prandtl potential $\phi(x,y)$ is continuous but nevertheless the torsional stress $|\nabla\phi|$ never exceeds the plastic limit. The model also permits a remarkable unified continuous geometrical analogue combining both the soap-film and sand pile analogies as limiting cases. According to this unified analogy $\phi(x,y)$ defines a spacelike maximal surface in an auxiliary 2+1 dimensional spacetime.

2   The Prandtl-Nadai Construction

In what follows we provide, for completeness, a brief résumé of Prandtl's soap-film and Nadia's sand heap analogy In the absence of body forces, the governing equations are

$$\partial_i T_{ij} = 0 \tag{1}$$

where $T_{ij} = T_{ji}$ is the *stress tensor* (often often denoted in Engineering texts σ_{ij} or τ_{ij} cf. [4]). The *strain tensor* e_{ij} is defined in terms of displacements u_i as $e_{ij} = \partial_i u_j + \partial_j u_i$. In linear theory for an isotropic substance Hooke's Law becomes

$$e_{ij} = \frac{2(1+\nu)}{E} T_{ij} - \frac{2\nu}{E} \delta_{ij} T_{kk} \,. \tag{2}$$

where ν is *Poisson's ratio* and E is *Young's modulus.* Following Saint-Venant[5] we assume that the material is isotropic and confined to a prismatic cylinder with base B whose generators are parallel to the z axis and orthogonal to the plane curve $\gamma = \partial B$. The displacements (u, v, w) are assumed to satisfy $u = -\theta yz \, v = +\theta xz \, w = w(x,y)$, where θ is a constant. The function $w(x,y)$ is called the *warp function* and it gives the displacement parallel to the axis. These assumptions imply

$$e_{xx} = e_{yy} = e_{zz} = e_{xy} = 0, \quad e_{yz} = \partial_y w + \theta x, \quad e_{zx} = \partial_x w - \theta y. \tag{3}$$

Hooke's law implies that the diagonal components of the stress tensor (the *elongations*) vanish and therefore equation (1) gives

$$\partial_x T_{xz} + \partial_x T_{yz} = 0 \tag{4}$$

(this only requires $\partial_z T_{zz} = 0$). We can now introduce the *Prandtl potential*

$$T_{xz} = \partial_y \phi \,, \qquad T_{yz} = -\partial_x \phi \,. \tag{5}$$

We have from (5) $\partial_i \phi \, T_{iz} = 0 \,, i = x, y$ which implies that the component of the force in the z direction on the level sets of the Prandtl potential, whose normal is $\frac{\partial_i \phi}{|\partial_i \phi|}$, vanishes. Thus any level set may act as a free boundary. It is convenient to set $\phi = 0$ on the boundary $\gamma = \partial B$. The total torque is given by

$$T = \int_B \left(x T_{yz} - y T_{xz} \right) dxdy = 2 \int_B \phi dxdy \tag{6}$$

on the use of the divergence theorem and the boundary conditions. If $G = E/2(1+\nu)$, and we use Hooke's law we get

$$T_{yz} = G(\theta y + \partial_y w) \,, \qquad T_{xz} = G(-\theta x + \partial_x w) \,. \tag{7}$$

Thus the Prandtl potential satisfies Poisson's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta. \tag{8}$$

This must be solved in B, subject to the boundary condition that $\phi = 0$ on the boundary $\gamma = \partial B$. Of course Poisson's equation arises in electrostatics where the electric field, and electrostatic potential ϕ satisfy $\mathbf{E} = -\boldsymbol{\nabla} \phi$, $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho$, where ρ is the density of electric charge. In the case of Prandtl's equation (8) we have $\rho = 2G\theta$ which corresponds to constant charge density. Prandtl adopted a different analogy. He regarded $\phi = \phi(x, y)$ as a height function in some auxiliary three dimensional space Euclidean space $\mathbb{E}^3$ with coordinates $(x_1, x_2, x_3) = (x, y, \phi)$ then the surface is *approximately* of constant mean curvature such as would be adopted by a soap-film with a constant pressure difference. The torque is then proportional to the volume between the approximate soap film and the base B. Note that if the analogy were *exact* then the Prandtl potential would satisfy the Young-Laplace equation

$$\partial_x \frac{\partial_x \phi}{\sqrt{1 + |\nabla \phi|^2}} + \partial_y \frac{\partial_y \phi}{\sqrt{1 + |\nabla \phi|^2}} = -2G\theta . \tag{9}$$

2.1 Non-linear theory

If the total shear stress exceeds the elastic limit k, then the material deforms plasticly. Thus a *plastic upper bound* must be satisfied

$$T_{xz}^2 + T_{yz}^2 = |\nabla \phi|^2 \le k^2 , \tag{10}$$

where k is the plastic limit and in the plastic region the Prandtl potential is assumed to satisfy the *Eikonal equation*

$$|\nabla \phi|^2 = k^2 . \tag{11}$$

According to *Nádai's sandpile analogy* [2, 6, 7] one may regard a Prandtl potential satisfying the Eikonal equation (11) as the height function ϕ of a sand-heap or sand-pile of constant *angle of repose* $\alpha = \tan^{-1} k$ located in same Euclidean space $\mathbb{E}^3$. The strategy for solving the for ϕ adopted by Prandtl is to erect over the base B, a sand-pile, that is the solution ϕ_{Eikonal} of the Eikonal equation (11). One then also erects the approximate soap film, that is the graph of the solution of Poisson's equation (8) ϕ_{Poisson}. The solution adopted by Prandtl is then to take the *minimum* of $\phi_{Poisson}$ and $\phi_{Eikonal}$. In other words, linear theory is assumed valid under the tent.

3 Spacetime Interpretation

We begin by noting that the the Eikonal equation (11) may be regarded as the defining a *wave-front, null or characteristic surface* $\mathcal{N} : u(x, y, t) = 0$ where

$$u = t - \phi(x, y) = 0 \tag{12}$$

in 3-*dimensional Minkowski spacetime* $\mathbb{E}^{2,1}$ with coordinates x, y, t and metric

$$ds^2 = dx^2 + dy^2 - c^2 dt^2, \tag{13}$$

3

where $c = 1/k$ is the velocity of light. In other words,

$$-\frac{1}{c^2}(\partial_t u)^2 + (\partial_x u)^2 + (\partial_y u)^2 = 0\,. \tag{14}$$

In general, the plastic upper bound (10) must hold. However the transition between linear and plastic behaviour is observed to be smoother than the abrupt change envisaged in Prandtl's theory. Now (10) implies that the surface $\Sigma \subset \mathbb{E}^{2,1}$ has a normal which is non-spacelike. If inequality holds in (10) then Σ given by (12) is a *spacelike surface*. Thus one considers a smooth surface whose normal may become null, i.e., may satisfy (11) in a continuous fashion. It is natural therefore to replace the linear Poisson equation with a non-linear equation which interpolates between the Poisson equation and the Eikonal equation and which has an interpretation more in keeping with the Minkowski spacetime framework. On might try replacing Poisson's equation by the Young-Laplace equation (9) but this does not fit well with our interpretation the Eikonal equation. A geometrically better motivated suggestion is to postulate the relativistically covariant analogue of the Young-Laplace equation:

$$\partial_x \frac{\partial_x \phi}{\sqrt{1 - c^2|\nabla\phi|^2}} + \partial_y \frac{\partial_y \phi}{\sqrt{1 - c^2|\nabla\phi|^2}} = -2G\theta\,, \tag{15}$$

which arises as the Euler- Lagrange equations of the functional

$$\int_B \left(\sqrt{1 - c^2|\nabla\phi|^2} + 2Gc^2\theta\phi \right) dxdy\,. \tag{16}$$

The first term in (16) is the area of any spacelike surface Σ with edge $\gamma = \partial B$ and the second the spacetime volume between Σ and the spacelike hyperplane of constant time $t = 0$. It follows that (15) describes a *spacelike surface of constant mean curvature in three dimensional Minkowski spacetime* $\mathbb{E}^{2,1}$. The difference in sign in the arguments inside the square roots in the denominators of (9) and (15) is intended to enforce the plastic bound (10).

4   Born and Infeld's non-linear Electrodynamics

Just as Poisson's equation (8) has an electrostatic analogue, so does our proposed replacement (15). The equation (15) and associated variational principle with functional (16) admit a similar physical interpretation [8]. Born and later Born and Infeld [9, 10] suggested a non-linear version of electrodynamics: a modification of Maxwell's equations in which there is a maximum electric field strength. In the electrostatic situation the electric field strength **E** electric induction **D** and charge density ρ satisfy $\mathbf{E} = -\nabla\phi \, \nabla \cdot \mathbf{D} = \rho \, \mathbf{D} = \mathbf{E}/(\sqrt{1 - \mathbf{E}^2/b^2})$. In two dimensions, these equation, with $\rho = 2G\theta$, $b = 1/c$, coincide with (15) and were first studied in the context of Born-Infeld electrostatics by Pryce [11, 12].
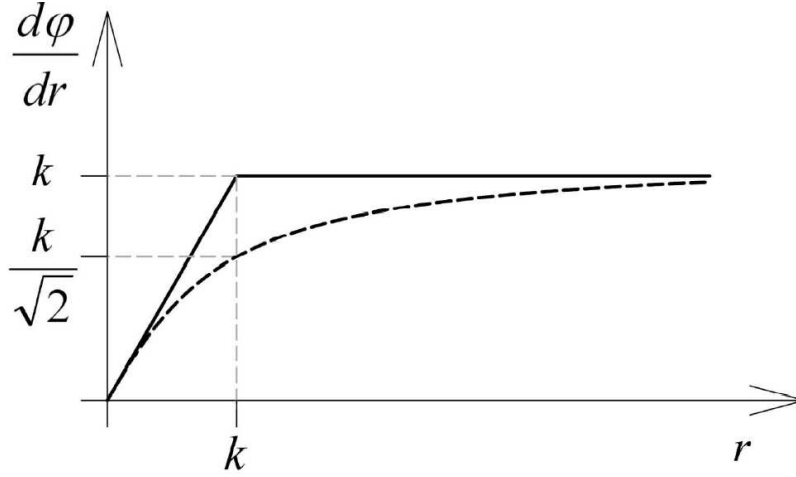
4

Figure 1: Comparison between the two models. Solid line: Prandtl-Nadai theory, dashed line $d\phi/dr = G\theta r k/\sqrt{k^2 + G^2\theta^2 r^2}$: our model. Maximal deviation at elasto-plastic boundary. The dashed curve also yields the constitutive law if we subsitute $\gamma = r\theta$ for the strains.

5 Comparison in the circular case

If the cross section is circular, the standard case has

$$\phi = -\frac{G\theta}{2}r^2 + \phi_0\,, \qquad 0 \le r \le \frac{k}{G\theta} \tag{17}$$

$$= -kr + \frac{k^2}{2G\theta} + \phi_0\,, \qquad r > \frac{k}{G\theta}\,. \tag{18}$$

The inner region is the Prandtl regime. The outer region is the Nádai's sand pile regime. The constant ϕ_0 is chosen so that $\phi(r = R) = 0$, where R is the radius of the cylinder. Thus if $R < \frac{k}{G\theta}$, then $\phi_0 = \frac{G\theta}{2}R^2$ and Prandtl's solution is $\phi = -\frac{1}{2}G\theta(r^2 - R^2)$ while if $R > \frac{k}{G\theta}$, then $\phi_0 = kR - \frac{k^2}{2G\theta}$ and the sandpile solution is conical with $\phi = -k(r - R)$. In the Born-Infeld case one has a single unified formula

$$\phi = \frac{k^2}{G\theta}\left(\sqrt{1 + \frac{G^2\theta^2 R^2}{k^2}} - \sqrt{1 + \frac{G^2\theta^2 r^2}{k^2}}\right)\,. \tag{19}$$

For small r, we expand both square roots and recover Prandtl's solution at lowest order. For large R and hence large r we ignore the one inside the square roots and the solution approachs the conical sand pile solution. Note that while $\phi(R) = 0$ in both solutions, in general the values of $\phi(0)$ are not the same. The predictions of both theories are compared in Figure 1. The total torque

$T = 4\pi \int_0^R r\phi \, dr$ is given by

$$T = 4\pi \frac{k^2}{G\theta} \Big(\frac{1}{2} R^2 \big(1 + \frac{G^2\theta^2 R^2}{k^2} \big)^{\frac{1}{2}} + \frac{k^2}{3G^2\theta^2} \big(1 - (1 + \frac{G^2\theta^2}{k^2} R^2)^{\frac{3}{2}} \big) \Big). \qquad (20)$$

6   Interpretation in Minkowski spacetime

We have

$$\big(\frac{\phi - \phi_0}{k} \big)^2 - x^2 - y^2 = \frac{k^2}{G^2\theta^2} \, . \qquad (21)$$

Thus if we think of (ϕ, x, y) as coordinates for three dimensional (x, y, ϕ) Minkowski spacetime with metric given in (13), then ϕ should be thought of as the time coordinate. Our solutions is now seen as a hyperboloid of constant spacetime distance from the point $(\phi_0, 0, 0)$, i.e the analogue in Minkowski space of a sphere in Euclidean space. To get from the sphere to the hyperbola one may "Wick Rotate", i.e., set

$$c(\phi - \phi_0) = i(z - z_0)G\theta/k = ia^{-1}, \qquad (22)$$

so that $(z - z_0)^2 + x^2 + y^2 = a^2$. The fact that the mean curvature (in the Lorentzian sense) is constant is now obvious because the surface is invariant under Lorentz transformations about $(\phi_0, 0, 0)$ and the mean curvature is a Lorentz scalar.

7   Acknowledgements

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