QUANTUM GRAVITY/STRING/M-THEORY AS WE APPROACH THE 3RD MILLENNIUM

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Abstract. I review some of the recent progress in String/M-theory

1. Introduction

When I was asked to give this plenary lecture in place of Ashoke Sen, I had some considerable misgivings that I would be able to do the subject the justice that he would undoubtedly have. However I had no doubts about what I should speak. The subject formerly known as string theory, and increasingly frequently being referred to as M-theory, has made some stunning advances since GR14. Hence my title. I should perhaps apologize, especially in India, for its wording. This was dreamed up rather hurriedly to meet the printer's deadline. As far as physics is concerned it would be more accurate to refer to the fourth, fifth or sixth millennium since the mathematical study of physics and astronomy must be at least as old as the the great river valley civilizations associated with the Indus, Tigris and Euphrates. However coming as I do from a a rather obscure corner of North-Western Europe I had, when I gave my title to the organizers, very much in mind the forthcoming Christian millennium celebration and its immediate predecessor.

That earlier occasion provides an apt metaphor for the current activity in this subject: it resembles in many ways the construction of the great mediaeval cathedrals which followed the failure of the universe to live up to the most important cosmological prediction of those times, that it should come

to an end in the year 1000 AD, thus appearing to demonstrate that physics is invariant under arbitrary shifts of the origin of the time coordinate.

Like string theory, the construction of the great cathedrals was a collective endeavour which took literally ages to complete, in many cases centuries. Although the beauty of the individual elements of the design would have been apparent fairly immediately, few if any of those working on the project at the beginning would have had much idea of the final shape of the structure that eventually emerged and which now combines those individual elements in such a harmonious whole. Sometimes, as in the case of Beauvais in Northern France, the whole building fell crashing down and had to be completely rebuilt. In fact these cathedrals were frequently located on the sites of much earlier churches or indeed pre-Christian temples often going back to the Romans and before. Walking around one sometimes finds, embedded in the floor or the lower parts of the wall fragments of these older structures put to a new use.

Chichester Cathedral

The see of Chichester was established in 1075 when the bishopric was moved from Selsey. The work of building the Cathedral was commenced by Ralph de Luffa (Bishop 1091-1123) in 1088. A fire in 1114 hindered the work but the building was complete by 1114. Three years later another disastrous fire wrought more havoc, but reconstruction was begun immediately, much of it in the transitional style, and the Cathedral was reconsecrated in 1199. Unfortunately some of the fire-damaged stone was used in the rebuilding, resulting in weaknesses which are now being corrected.

The spire was added to the central tower c. 1330, but in 1861 both spire and tower collapsed. They were rebuilt in replica of the original by 1866, by Sir Gilbert Scott. The nave retains much of its massive Norman architecture, though the chapels on either side were added in the 13th c. The nave is separated from the choir by the carved stone Bell-Arundel screen, originally erected beneath the tower in 1475. It was restored in 1960.

Much of this, it seems to me is true of the present state of string theory, only slowly are we beginning to get some idea of the final structure. Even now we can't be sure that it will hold up. There is another, and more particular, way in which string theory resembles the mediaeval cathedrals. One of the key features of many of them is is the existence of a spire placed on top of a high tower. If one visits the cathedral one may climb to the top of the tower and then one may ascend the spire to gain a spectacular view of the surrounding countryside,the city and the cathedral precincts below.

For us the analogue of height above ground level is spacetime dimension and this brings me to what is perhaps the most important message of this talk :

EVERYTHING BECOMES SIMPLER IN ELEVEN DIMENSIONS

Really!

1.1. SPINORS

The word "really" is meant in the technical sense: the point is that we need to consider spinors in eleven dimensional spacetime and it is a useful and convenient fact that the real Clifford algebra

$$\text{Cliff}(10, 1; \mathbb{R}) \equiv \mathbb{R}(32). \tag{1}$$

What is meant here that in eleven spacetime dimensions the algebra generated by the gamma matrices γ_μ, $\mu = 0, 1, \ldots, 10$

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}, \tag{2}$$

with spacetime signature chosen so that

$$\gamma_0^2 = -1, \tag{3}$$

is isomorphic to that of 32×32 real matrices. Thus we may take the gamma matrices $\gamma_\mu{}^a{}_b$ be 32×32 and they act on the real 32-dimensional space S consisting of *Majorana spinors* θ^a, $a = 1, \ldots 32$.

The space of Majorana spinors S carries a $Spin(10, 1)$-invariant symplectic form $C_{ab} = -C_{ba}$, the *charge conjugation matrix*, which may be

used to raise and lower spinor indices. Thus for example one finds that if one lowers an index on the gamma matrices they become *symmetric*:

$$(C\gamma^\mu)_{ab} = (C\gamma^\mu)_{ba}. \tag{4}$$

For later use we remark that we must make an arbitrary choice when constructing the Clifford algebra $\mathrm{Cliff}(10,1;\mathbb{R})$. The image in the Clifford algebra of the volume form commutes with all other elements of the algebra and may be taken to be a multiple of the identity. We make the choice:

$$\gamma_{10} = \gamma_0\gamma_1\ldots\gamma_9 \tag{5}$$

(we could have chosen the minus sign).

1.1.1. *Four-dimensions*

Now let's look down to four dimensions. Using the major arithmetical theorem that

$$32 = 4 \times 8, \tag{6}$$

we see that S we will decompose into eight copies of the four real dimensional space of Majorana spinors for $Spin(3,1)$, and in a like fashion the eleven-dimensional charge conjugation matrix decomposes into the sum of eight copies of the familiar charge conjugation matrix of ordinary physics. We are in fact using the isomorphism

$$\mathrm{Cliff}(3,1;\mathbb{R}) \equiv \mathbb{R}(4). \tag{7}$$

Many participanrts at GR15 will be more familiar with Weyl spinors and perhaps with the opposite signature convention. In that language a Majorana spinor is a pair of two-complex -component Weyl spinors

$$\begin{pmatrix} \theta^A \\ \bar\theta_{A'} \end{pmatrix} \tag{8}$$

while

$$C = \begin{pmatrix} \epsilon_{AB} & 0 \\ 0 & \epsilon^{A'B'} \end{pmatrix} \tag{9}$$

with

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\sigma^{\mu AA'} \\ \sqrt{2}\sigma^\mu{}_{AA'} & 0 \end{pmatrix}. \tag{10}$$

1.2. SUPERSYMMETRY

Let's return to eleven dimensions. Supersymmetry transformations depend on a constant spinor ϵ and act on *Superspace* , which is defined to be

$$\mathbb{E}^{10,1} \times S \tag{11}$$

with coordinates (x^μ, θ^a) as

$$\theta \to \theta + \epsilon \tag{12}$$

$$x^\mu \to x^\mu + \frac{1}{2}\epsilon^t C\gamma^\mu\theta. \tag{13}$$

One now takes a suitably graded semi-direct product with the eleven dimensional Poincaré group $E(10, 2)$ to get the super-Poincaré group. Note that the spinors θ and ϵ are anti-commuting variables. The supertranslations have generators Q which transform as Majorana spinors under the super-adjoint action of Spin(10,1) and trivially under the adjoint action of the ordinary spacetime translations. The non-trivial anti-commutation relation is, in index notation,

$$Q_a Q_b + Q_b Q_a = P_\mu (C\gamma^\mu)_{ab}. \tag{14}$$

Descending to four dimensions we get eight four-component supercharges Q_a^i and the algebra of $N = 8$ supersymmetry.

2. Eleven-dimensional Supergravity

Given the above information it is in principle possible to construct classical field theories in eleven dimensions. However any such theory must contain particles of spin two and it is widely believed that *there is only one possibility* , the theory of Cremmer and Julia [9]. This contains

$$\text{the metric}: \qquad g_{\mu\nu} \tag{15}$$

$$\text{a closed four} - \text{form}: \qquad F_{[\mu\nu\rho\lambda]} = 4\partial_{[\mu}A_{\nu\rho\lambda]} \tag{16}$$

and a

$$\text{a Majorana gravitino}: \qquad \psi_\mu^a. \tag{17}$$

An elementary exercise in linear theory shows that there are 128 boson and 128 fermion degrees of freedom.

In higher dimensional Lorentzian spacetimes, or in eleven dimensions but with more supersymmetry, particles with spins greater than two seem to be inevitable. That is why at present the pinnacle of the spire stops here. However that has not dissuaded modern-day Icari (if indeed that is the plural of Icarus) from launching themselves into the blue yonder. Future architectural innovations may well include multi-temporal theories in twelve or thirteen dimensions.

2.1.   REDUCTION TO TEN DIMENSIONS

Those with little head for heights may descend the spire to the tower where life is more varied. We still have 32 component Since

$$\gamma_{10}^2 = (\gamma_0 \dots \gamma_9)^2 = 1 \tag{18}$$

we may decompose the real 32-dimensional space S of Majorana spinors into two real 16 dimensional spaces $S^{\pm}$ of *Majorana-Weyl* spinors

$$S = S^+ \oplus S^- \tag{19}$$

with

$$\gamma_{10}S^{\pm} = \pm S^{\pm}. \tag{20}$$

We now have *three* types of superymmetry and three types of superspace depending upon the chirality of the supercharges.

  − Type I theories have just one supercharge and is therefore necessarily chiral. The superspace is
$$\mathbb{E}^{9,1} \times S^+. \tag{21}$$

  − Type IIA theories have two supercharges, one of each chirality. The superspace is
$$\mathbb{E}^{9,1} \times S^+ \times S^-. \tag{22}$$

  − Type IIB theories have two supercharges of the same chirality, and are thus also chiral. The superspace is
$$\mathbb{E}^{9,1} \times S^+ \times S^+. \tag{23}$$

2.1.1.   *Type I theories*
    These include
  − Super-Yang-Mills theory. This has

$$\mathrm{the\,Yang-Mills\,field\,strength:} \qquad F_{\mu\nu} \tag{24}$$

and

$$\mathrm{a\,Majorana-Weyl\,Spinor:} \qquad \psi \tag{25}$$

both in the adjoint representation of some compact gauge group G. In many ways this is the big-daddy of all gauge theories. For example, this theory, reduced to four spacetime dimensions, gives the $n = 4$ supersymmetric Yang-Mills theory which has so many deep and beautiful links with mathematics and geometry. We shall see shortly that it has a central role to play in M-theory.

– Type I Supergravity. This has as bosonic fields

$$\text{a metric}: \quad g_{\mu\nu} \tag{26}$$

$$\text{a (scalar) dilaton}: \quad \Phi \tag{27}$$

and

$$\text{a closed three} - \text{form}: \quad H_{\mu\nu\rho} = 3\partial_{[\mu}A_{\nu\rho]}. \tag{28}$$

– The $SO(32)$ Open Superstring.

2.1.2. *Type IIA theories*

These include

– Type IIA Supergravity.
This has the bosonic fields of Type I Supergravity, referred to in this context as the *Neveu-Schwarz⊗Neveu-Schwarz sector* together with a closed two-form and a closed four-form field strength, referred to in this context as the *Ramond⊗Ramond sector*.
and

– The Type IIA Closed Superstring.

2.1.3. *Type IIB theories*

These include

– Type IIB Supergravity. This has the bosonic fields of Type I Supergravity , often called for obvious reasons, 'the common sector'together with a Ramond⊗Ramond sector consisting of closed one, three and five-forms. The five-form C_5 is self-dual

$$\star C_5 = C_5, \tag{29}$$

where $\star$ denotes the Hodge-dual which satisfies $\star\star = 1$ in $\mathbb{E}^{9,1}$.

– The Type IIB Closed Superstring.

2.1.4. *Heterotic Theories*

For closed strings, one has the additional possibility of the left and right moving modes on the string behaving differently and this gives rise to the two further possibilities with gauge group $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$. It is of the course the latter which has received most attention for phenomonological purposes. We get down to four dimensions with $N = 1$ supersymmetry by taking the ten dimensional spacetime $\mathcal{M}^{9,1}$ as a product

$$\mathcal{M}^{9,1} = \mathbb{E}^{3,1} \times CY \tag{30}$$

where CY is a Calabi-Yau space, i.e. a closed Riemannian six-manifold with holonomy $SU(3)$.

Thus, if one regards the field theories as limiting low-energy cases of super-string theories one has five possibilities.
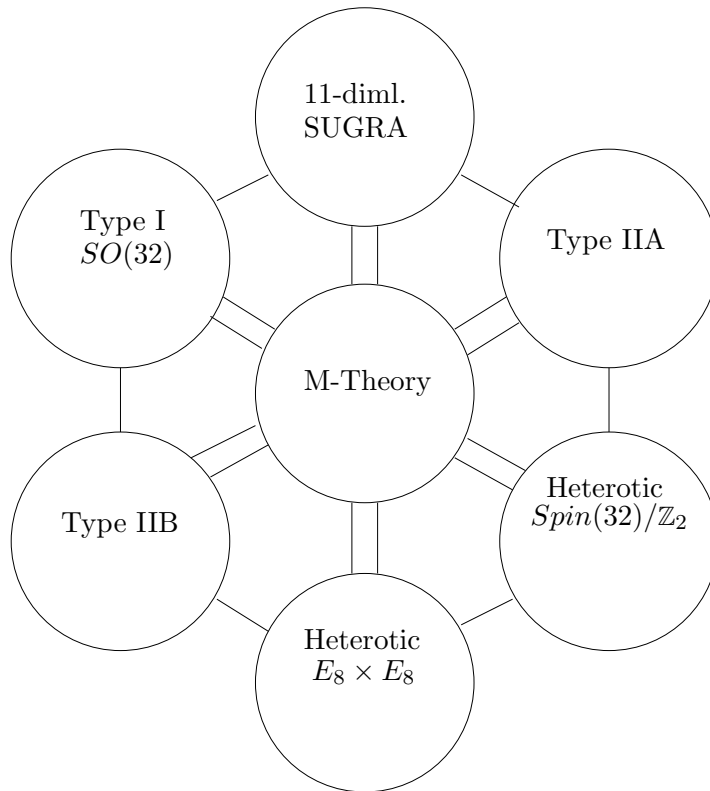
3. Dualities

The next most important point of this lecture is that

> **IT IS NOW BELIEVED**
> **THAT ALL FIVE STRING THEORIES**
> **AS WELL AS ELEVEN-DIMENSIONAL SUPERGRAVITY**
> **ARE LIMITING CASES OF**
> **A SINGLE OVER-ARCHING STRUCTURE**

called

> **M-THEORY**

The five string theories and eleven-dimensional supergravity are con-jectured to be related by a web of *dualities* which interchange the per-turbative *elementary* states which we encounter in linear theory with the

non-perturbative *BPS soliton* states which we only see in the fully non-linear theory. I will have more to say about what precisely M-theory is expected to be later.

The basic dualities are of two types referred to as *T-dualities* and *S-dualities*. They are believed to be combined in a more general symmetry called *U-duality*.

3.1. T-DUALITIES

These are symmetries of perturbative string theory and may be shown to hold to all orders in perturbation theory [14]. They appear when one considers theories for which

$$\mathcal{M}^{9,1} = T^d \times \mathcal{M}^{9-d,1}. \tag{31}$$

The torus is equipped with a constant metric G_{ij} and a constant two-form B_{ij} and T-duality is a generalization of the idea of interchanging the torus, equipped and its metric $\{T^d, G_{ij}\}$ with its dual or reciprocal torus and its dual or reciprocal metric $\{\hat{T}^d, G^{ij}\}$. A familar example of this in the everday physics of three dimensions is the duality between the face centred cubic lattice and the body centred cubic lattice. The Voronoi or Wigner-Seitz cell of the former is the rhombic dodecahedron whose dual is the cuboctahedron, which is the Voronoi cell of the latter.

String theories related by duality are believed to be identical. In other words it is one of the gauge symmetries of string theory, The simplest example arises when we have a a circle $T^1 = S^2$ of radius R and T-duality acts as

$$R \to \frac{l_{\text{string}}^2}{R}, \tag{32}$$

where $l_{\text{string}} = \sqrt{\alpha'}$ is the fundamental length that enters string theory.

More generally, the 'moduli space'of torus theories is specified by giving a $d \times d$ matrix $E_{ij} = G_{ij} + B_{ij}$ with positive definite symmetric part is acted on by $O(d, d; \mathbb{Z})$ acting by fractional linear transformations

$$E \to (AE + B)(CE + D)^{-1} \tag{33}$$

where, if

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{34}$$

and

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{35}$$

then

$$M^t J M = J. \tag{36}$$

3.2. S-DUALITY

The group in question is $PSL(2, \mathbb{Z})$ It is is a new and unexpected non-perturbative symmetry of string theory interchanging weak and strong coupling whose existence and importance was pointed out by Ashoke Sen [12] and who has used it so effectively. Perhaps its most exciting feature is that it also interchanges classical with quantum properties. S-duality is not all apparent in perturbative string theory, it only manifests itself indirectly.

Thus in N=4 supersymmetric four-dimensional gauge theory, if

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \tag{37}$$

where g is the usual gauge coupling constant and θ is the 'theta angle'S-duality acts by fractional linear transformations

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{38}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{39}$$

Gauge theories with couplings so related are believed to be identical. No proof is known but many, apparently rather delicate, tests have been made of this conjecture and no contradiction has been found. In fact these tests involve extremely subtle and unexpected properties of the L^2 co-homology of the hyper-Kähler manifolds which are the moduli spaces of BPS monopoles [11] .

3.3. U-DUALITY

In string theory and supergravity theory we have

$$g = e^\Phi \tag{40}$$

and the role of θ is played by a pseudoscalar field called an axion. Thus for example it is known that $PSL(2, \mathbb{R})$ is a symmetry of Type IIB classical supergravity theory. The exterior derivative of the axion is the closed Ramond⊗Ramond one-form mentioned earlier. It is expected that quantum-mechanical effects associated with Dirac quantization of electric and magnetic charges will break this continuous symmetry down to $PSL(2, \mathbb{Z})$.

By dimensional reduction on tori the S-duality symmetry descends to lower-dimensional super-gravity theories. Of course they also acquire an explicit (and continuous) T-duality symmetry as well. Studies of super-gravity theories in the past revealed that one often gets more. For example the $N = 8$ supergravity theory in four spacetime dimensions is invariant under the action of $E_{(7,7)}$. Hull and Townsend [7] have suggested that a discrete subgroup of $E_{(7,7)}$ should persist in the full quantum mechanical M-theory.

4. BPS States and p-branes

The non-perturbative soliton states, analogous to BPS monopoles in Yang-mills theory, which are acted on by non-perturbative dualities correspond to *p-branes*. Roughly speaking these are extended objects with p spatial dimensions which as they move through time draw out a p+1 dimensional world volume. Thus in n spacetime dimensions

$$
\begin{aligned}
p &= 0 & &\text{corresponds to a particle} \\
p &= 1 & &\text{corresponds to a string} \\
p &= 2 & &\text{corresponds to a membrane} \\
p &= n - 2 & &\text{corresponds to a domain wall} \\
p &= n - 3 & &\text{corresponds to a vortex} \\
p &= -1 & &\text{corresponds to an instanton}
\end{aligned}
$$

(41)

One says that a p-brane state $|p\rangle$ is BPS if is invariant under one or more supersymmetry transformations

$$Q|p\rangle = 0. \tag{42}$$

BPS states typically carry central charges Z per unit p-volume and if If M is the energy per unit p-volume of such a state it typically attains a *Bogomol'nyi Bound* [15] giving a lower bound for M among all states with the same central charge

$$M \geq |Z|. \tag{43}$$

4.1. SUPERGRAVITY P-BRANES

In supergravity theories BPS p-branes are very well known to the present audience. They correspond to extreme black holes and higher dimensional analogues. The spacetime of such a solution is invariant under the action of the Poincaré group $E(p, 1)$ times rotations of the d_T dimensional transverse

space $SO(d_T)$. I won't give a complete survey of all possibilities but remind you that a typical metric looks like:

$$ds^2 = H^{\frac{-2}{p+1}}\left(-dt^2 + d\mathbf{x}_p^2\right) + H^{\frac{2}{d_T-2}}d\mathbf{y}_T^2, \qquad (44)$$

where H is an arbitrary harmonic function on $\mathbb{E}^{d_T}$. Thus more than one p-brane may rest in equipoise. The solutions admit Killing spinors of the relevant supergravity theory.

Near the horizons, the symmetry and the superymmetry is frequently enhanced, the metric tending to the product $AdS_{p+2} \times S^{d_T-1}$. Typically one may think of the p-branes as spatially interpolating between different vacua or compactifications of the associated supergravity theories.

The basic and fundamental example is of course the Majumdar-Papapetrou solution of the Einstein-Maxwell equations in four spacetime dimensions.

It is important to note that supergravity p-branes may carry electric or magnetic charges. associated to a $(p+2)$- form or $d_T - 1$ form respectively. The magnetic charges are necessarily 'solitonic ', while the electric ones may sometimes be envisaged as arising from sources. In general charges may be of Neveu-Schwarz⊗Neveu-Schwarz or Ramond⊗ Ramond origin. In string theory 'electric 'Neveu-Schwarz⊗Neveu-Schwarz charge is carried by a *fundamental string* which corresponds to an elementary state of string theory. However perturbative string states cannot carry magnetic Neveu-Schwarz⊗ Neveu-Schwarz charge Neither can they carry electric or magnetic Ramond⊗ Ramond charge. There are no such local sources in perturbative string theory. We have here a case of what Misner and Wheeler might have called

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RAMOND ⊗ RAMOND
CHARGE WITHOUT CHARGE

Note that in the quantum theory these charges should satisfy an analogue of Dirac's quantization condition for electric andc magnetic charges.

It is perhaps worth explaining here the origin of the rather quaint looking tensor product notation. It derives form the fact that a the world sheet of a closed string is topologically $\mathbb{R} \times S^1$ which admits two spin structures,

- Neveu-Schwarz, which corresponds to antiperiodic spinors
     and
- Ramond, which corresponds to periodic spinors,

and both must be included when considering the fermionic oscillations of the superstring. The bosonic fields are built up as tensor products of left

and right moving fermionic states, and must be periodic. They thus fall into two sectors. Shifting now to ten-dimensional spacetime we see that the fermionic states will be associated with the tensor product of the space of Majorana spinors with itself i.e. with $S \otimes S$. But by virtue of the charge conjugation matrix C_{ab} we may identify $S \otimes S$ with the Clifford algebra $\mathrm{Cliff}(9,1;\mathbb{R})$ which, as a vector space is the same as the Grassmann algebra of forms $\Lambda(\mathbb{E}^{9,1})$. Type IIA theories involve the even forms and Type IIB theories the odd forms.

4.2. DIRICHLET P-BRANES

A key observation of Polchinski [1, 2], which resolves the puzzle above, is that open string theory admits an entirely new type of state corresponding to p-branes. They have come to be called *Dirichlet p-branes*. Their importance is that their quantum mechanical properties can be discussed within the comparitively mathematically secure framework of conformal field theory.

The basic idea is consider decomposing ten-dimensional Minkowski spacetime as

$$\mathbb{E}^{9,1} = \mathbb{E}^{p,1} \times \mathbb{E}^{d_T} \tag{45}$$

with coordinates x^α, y^m , $\alpha = 0, \ldots, p$ and $a = 1, \ldots, d_T$. We now decree that end of the the string remains fixed on the hyperplane $y^a = 0$. In other words the coordinates of the string fields $Z^\mu(t, \sigma) = x^\alpha(t, \sigma), y^a(t, \sigma)$,where (t, σ) are space and time coordinates on the string world sheet are to be subjected to a mixture of the usual

− Neumann boundary conditions

$$\partial_\sigma x^\alpha = 0 \tag{46}$$

   and
− Dirichlet boundary conditions

$$\partial_t y^a = 0. \tag{47}$$

Note that world sheet Hodge duality interchanges t and σ and hence Neuman and Dirichlet boundary conditions. Polchinksi was able to show that in string theory such D-brane states are BPS and like their supergravity cousins they carry the correct Ramond⊗Ramond charges. Moreover these charges satisfy the analogue of Dirac quantization conditions, called in this context the *Nepomechie-Teitelboim* quantization conditions.

4.3. EFFECTIVE DIRAC-BORN-INFELD ACTIONS

We may suppose that integrating over the string fluctuations will give an effective action for for the motion of 'light 'p-branes moving in a a

fixed external field. Such so-called *Dirac-Born-Infeld* actions have been obtained by a number of authors. In addition to the world volume fields giving the embedding of the brane in $\mathbb{E}^{9,1}$, $Z^{\mu}(x^{\alpha})$ SUSY dictates that one must include a world volume closed two-form $F_{\alpha\beta}$. In flat space with trivial Ramond⊗Ramond and Neveu-Schwarz⊗Neveu-Schwarz fields Dirac-Born-Infeld action reduces to

$$-\int d^{p+1}x\sqrt{-\det(g_{\alpha\beta}+F_{\alpha\beta})}, \tag{48}$$

where

$$g_{\alpha\beta}=\eta_{\mu\nu}\frac{\partial Z^{\mu}}{\partial x^{\alpha}}\frac{\partial Z^{\nu}}{\partial x^{\beta}} \tag{49}$$

is the pull-back to the world volume of the flat ten-dimensional metric $\eta_{\mu\nu}$.

It is perhaps a striking fact that we encounter here embedded in the structure of M-theory a a fragment of the old, an unsuccessful, attempt of Born and Infeld to construct a finite classical theory of the electron. Now is not the place to dwell on this in detail but it is a striking consequence of the new developments that what was a major blemish of the that ancient religion: the singular source of the electron may now be understood as the end of a fundamental string ending on a D-brane [13].

4.4. SUPER-P BRANE ACTIONS

I have no time to dwell at length on the details but it is perhaps worth pointing out here the simple underlying idea that permits the construction of superymmetric p-brane actions, including super-string actions, in a unified way. It is to to replace the bosonic idea of a map from a $p+1$ dimensional manifold of the form $\mathbb{R}\times M_p$ where M_p is the p-brane's spatial manifold by into spacetime, $\mathbb{E}^{n-1,1}$ for example, by a map into superspace $\mathbb{E}^{n-1,1}\times S$.

5. Black Hole Entropy via D-branes

Perhaps the most persuasive evidence for the essential soundness of our current foundation is the remarkable calculations of the thermal properties of black holes initiated by Strominger and Vafa [3] and developed further by Callan and Maldacena [4]and then by many other people [5]. Both for reasons of time and because there have been some been some fine expositions inthe parallel sessions,I shall not review them in detail but merely note some essential points.

5.1. MODULI INDEPENDENCE

The first thing to remind ourselves is why this idea is even feasible without a complete quantum theory of gravity. For a general black hole, the Bekenstein-Hawking entropy

$$S_{BH} = \frac{1}{4G} A \tag{50}$$

where A is the horizon area. This expression involves Newton's constant G. In string theory we expect that

$$G \propto < e^{2\Phi} > \tag{51}$$

so that G should depend on the expectation value of the dilaton Φ. At present this seems to be a completely arbitrary number depending on which vacuum we are in and to be quite beyond calculation. However for an extreme black hole, for example an electric Majumdar-Papapetrou black hole, we have

$$A = 4\pi(GM)^2 \tag{52}$$

and moreover

$$GM^2 Q^2 \tag{53}$$

where Q is the electric charge, (not the supercharge!). Thus

$$S = \pi Q^2 \tag{54}$$

which does depend upon G at all, only on the quantized dimensionless and essentially topological quantity Q. The same feature may be shown to be true for all extreme black holes with non-vanishing entropy, both in four and five spacetime dimensions [18, 20]. The entropy is independent of any accidential moduli fields characterizing the vacuum we are in but depends only on dimensionless quantized charges. In fact this is a rather general statement and does not depend upon string theory in an essential way, just the general structure of the low energy effective lagrangians.

5.2. THE RELEVANCE OF BPS STATES

The reason that one is so interested in BPS states in any theory is that they are typically protected against quantum corrections. In particular the number of states in a super-multiplet is fixed by the Bogomol'nyi condition $M = |Z|$. Thus properties of BPS states at weak coupling should remain true at strong coupling.

In other words quantum mechanical string or strictly speaking D-brane calculations should and indeed do agree with semi-classical calculations in

supergravity theories. This is particularly to be expected for calculations of the number of BPS states, i.e. of the entropy of extreme black holes.

5.3. INTERSECTING-BRANES

Thus the basic idea is that counts the number N of microstates of a certian BPS configuration of D-branes having certain charges Q_i using techniques from conformal field theory and hence the entropy

$$S_D = \ln N. \tag{55}$$

In fact no great sophistication is needed for these calculations, they simply involve a one-dimensional gas. One now constructs a fully non-linear BPS supergravity solution with the same charges representing an extreme black hole. One calculates the area of the horizon and hence its Bekenstein-Hawking entropy S_{BH}. Then one compares and S_D. In the limit of large charges one gets exact agreement, in other words the factor of proportionality is correct. This distinguishes this calculation from almost every other similar non-stringy calculation in four or five dimensions.

The simplest example involves five-dimensional black holes which may be thought of as 1-brane lying inside a five-brane and carrying Kaluza-Klein momentum. There are three charges, which count the number of one branes. the number of five-branes and the Kaluza-Klein momentum of the string.

More testing calculations can be done giving complete agreement with the emission rates, including grey-body factors, in the limit of low frequency for slightly non-extreme holes. In fact the absorption cross-sections of black holes at zero frequency are universal and always given in terms of the area [19] but at non-zero frequency this is no longer so but nevertheless the emission rates continue to agree. In fact these calculations may be extended to strongly non-extreme holes but perhaps not surprisingly some small discrepancies have emerged but these may quite plausibly be ascribed to extending the approximation beyond its reasonably expected range of validity.

I personally have found these results to be tremendously impressive and at face value they constitute good evidence for the ultimate promise of the String/M-theory project.

6. M-Theory

What then is M-Theory? The letter M has variously been claimed to stand for membrane, magic, mystery or mother (as in mother of all theories) but since no-one at present has a definitive theory the question of the name should presumably remain in abeyance.

Perhaps the snappiest characterization is

Definition (provisional). *M-theory is the strong coupling limit of Type IIA string theory.*

However this is rather 'non-constructive '(in the pure mathematical sense) ans something more concrete is desirable. To see why it is reasonable however, consider passing between

6.1. ELEVEN AND TEN DIMENSIONS

This is done by compactification on a circle of radius R_{10}

$$\mathcal{M}^{10,1} = S^1 \times \mathbb{E}^{9,1}.$$

Eleven dimensional supergravity then gives rise to Type IIA supergravity with its dilaton Φ coming form the matric component $g_{11\,11}$.

Standard Kaluaz-Klein calculations lead to the relation

$$R_{10} = l_{\text{Planck}} < e^{\frac{2}{3}\Phi} > \qquad (56)$$

where l_{Planck} is the eleven dimensional Planch length. Now in string terms the string coupling constant g_{string} is given by

$$g_{\text{string}} = < e^{\Phi} > \qquad (57)$$

and

$$\frac{l_{\text{Planck}}}{l_{\text{string}}} = g_{\text{string}}^{\frac{1}{3}}. \qquad (58)$$

It follows [17] that weak string coupling corresponds to a small string circle and strong coupling to a large circle. Thus if one starts from Type IIA string theory and increases the coupling constant one expects to arrive at an effectively eleven dimensional theory.

The situation in the definition is summarized in the following table:

IIA Superstring	M-theory	high energy
IIA Supergravity	11-dimensional Supergravity	low energy

Weak coupling	Strong coupling	
Small R_{10}	Large R_{10}	

6.2. PHENOMONOLOGY (AND COSMOLOGY?)

One of the conjectured dualities relates M-Theory to the $E_8 \times E_8$ heterotic theory compactified on a Calabi Yau manifold CY [16]. The basic idea is to consider

$$\mathcal{M}^{10,1} = I \times CY \tag{59}$$

where I is a closed interval which may be thought of as a circle S^1 identified under reflection about a diameter. The reflection has two fixed points at the ends of the interval. Thus the universe looks like a sandwich consisting of two ten dimensional sheets bounding an eleven-dimensional bulk spacetime. One E_8 gauge field lives on one sheet and the second factor on the other sheet. The size of the interval is taken to be rather larger than the size of the Calabi-Yau manifold.

This gives improved phenomonological models. Ihe separations of scales also suggests that one one might be able to study the era of compactification, if indeed compactification ever took place, about which almost nothing is known or conjectured, as a two step process in which one ignores the formation of the Calabi-Yau and considers a compactification from five to four spacetime dimensions on an interval.

6.3. M-THEORY AND P-BRANES

In asking what kind of theory M-theory can be it is natural to ask what sort of superymetric p-branes it admits.

To answer this question recall [8] that in any spacetime dimension n one may consider a modification of the superysymmetry algebra by adding something to the right hand side of the basic anti-commutation relation

$$Q_a Q_b + Q_b Q_a = P_\mu (C\gamma^\mu)_{ab} + Z_{ab} \tag{60}$$

since Z_{ab} is an element of the the Clifford algebra $\text{Cliff}(n-1, 1; \mathbb{R})$ it may also be thought of as an element of the Grassman algebra of all p-forms $\Lambda(\mathbb{E}^{n-1,1})$. Thus

$$Z_{ab} = \sum_{p \neq 1} \frac{1}{p!} Z_{\mu_1 \ldots \mu_p} (C\gamma)^{\mu_1 \ldots \mu_p})_{ab}. \tag{61}$$

The bosonic quantities $Z_{\mu_1 \ldots \mu_p} = Z_{[\mu_1 \ldots \mu_p]}$ may be thought of in general non-Lorentz-invariant charges per unit p-volume carried by p-branes states. Thus a p-branes lying in the $x^0 - x^1 \ldots x^p$ plane is associated with the Clifford algebra element $\gamma^0 \gamma^1 \ldots \gamma^p$.

Of course if $p = 0$ or $p = n = 4$ we recover the usual Lorentz-invariant electric or magnetic central charges we are familiar in four-dimensional supergravity theories which are carried by 0-branes , i.e. ordinary particles.

The allowed values of p are constrained by the condition that $(C\gamma)^{\mu_1\cdots\mu_p})_{ab} = (C\gamma)^{[\mu_1}\ldots\gamma^{\mu_p]})_{ab}$ be symmetric in its spinor indices. In eleven dimensions this leaves just two possibilities, the algebra is

$$Q_a Q_b + Q_b Q_a = P_\mu (C\gamma^\mu)_{ab} + \frac{1}{2} Z_{\mu\nu} (C\gamma^{\mu\nu})_{ab} + \frac{1}{5!} Z_{\mu\nu\lambda\rho\sigma} (C\gamma^{\mu\nu\lambda\rho\sigma})_{ab}, \quad (62)$$

where

 – Z_μ corresponds to the charge caried by the *M-2-brane*
   and
 – $Z_{\mu\nu\lambda\rho\sigma}$ corresponds to the charge carried by the *M-5-brane*.

One may desrcibe these in two ways

 – (i) As elementary objects with lagrangians analogous to the Nambu-Goto action for strings
   or
 – (ii) BPS solutions of the low energy Supergravity theory admitting one half the maximum number of Killing spinors.

6.4. MEMBRANES

The M-2-brane, i.e. the supersymmetric theory of maps from a 3-manifold $\mathbb{R} \times M_2$ into $\mathbb{E}^{10,1} \times S$, the superspace of eleven dimensions has of course been known for some time. Attempts to quantize it in the analogue of lightcone gauge used in superstring theory revealed that the residual gauge symmetry is $SDiff(M_2)$, the group of volume preserving diffeomorphisms of the 2-brane spatial manifold. This is the analogue of the two copies of $Diff(S^1)$ one obtains in string theory.

7. The Matrix proposal

In the final part of the lecture I want to turn to the only concrete proposal for what M-theory actually may be. It has the merit of being simple to state but, as its authors would be the first to admit, there is much to be done to establish that the proposed theory actually exits and that it possesses all of the symmetries expected of it.

7.1. THE BASIC IDEA

 – (i) Take the (unique) supersymmetric Yang-Mills theory in ten dimensions with gauge group $U(N)$ metioned earlier.
 – (ii) Dimensionally reduce form ten to 1 spacetime dimensions when it becomes a supersymmetric quantum mechanical model based on $N \times N$ hermitian matrices.

$-$ (iii) take the $N \to \infty$ limit.

The Yang-Mills action is, in the usual physicists notation so the trace is positive and we have inserted the conventional i in front of the Dirac action.

$$-\frac{1}{4}tr F^2 + +\frac{i}{2}\psi^t \gamma^\mu (\partial_\mu - iA_\mu)\psi. \tag{63}$$

If everything depends only upon time one pick the gauge $A_0 = 0$ and the bosonic variables reduce to the 9 $N \times N$ hermitian matrices A_i which are renamed X_i. The resulting quantum mechanical models has as bosonic lagrangian

$$\sum_i \frac{1}{2} Tr \dot{X}_i^2 - V(X^i) \tag{64}$$

where the potential energy function V is given by

$$V = \sum_{i<j} Tr[X_i, X_j]^2 \tag{65}$$

Note that the classical ground state corresponds to commuting coordinates X_i but the full quantum mechanics involves *non-commuting coordinates* and is closely related to some of Connes ideas about non-commutative geometry.

7.2. THE RATIONALE

Here is not the place to give a full motivation for the matrix ansatz. The approach of the originators was is via D-0-branes. Another, historically earlier approach is to regard it as a regularization of the supermembrane action in light-cone. This is reasonable because of the as yet incompletely relationship between two Lie algebras:

$$sdiff(M_2) = \lim_{N \to \infty} su(N). \tag{66}$$

8. Conclusion

I hope it is clear form what I have said that no-one, including the the most passionate advocates of the Matrix model, believes that this is the final story. There will no doubt be many dramatic twists and turns before the final shape of M-theory emerges. However what does now seem clear is that sufficiently many of the essential underlying ideas are in place to show that some definite structure will finally emerge and that it will be able to answer

many of the questions that we would like to ask of a quantum theory of gravity. Whether it is the theory that nature has adopted is a different matter.

My own assessment is the situation is that it is rather analogous to that which prevailed after the discovery (if you are, to use the language of scholastic philosophers, a realist) or invention (if your are a nominalist) of Riemannian geometry. This allowed the unification of the three Non-Euclidean geometries into a unified generalization. Combined with the basic ideas of group theory this rapidly led to Lie groups considered as Riemannian manifolds and the classification of symmetric spaces. Later there came complex and K̈aler geometries. Much of twentieth century mathematics has been preoccupied with these topics.

However, although some speculative applications of Riemannian geometry to cosmography were rapidly forthcoming, it was not until Einstein's work on special relativity that the essential *phyiscal insight* that *time* must also be included in the picture and his later idea of the *equivalence principle* that general relativity finally emerged and these earlier speculations became firmly emebeded in our cosmological world picture. It took even later for the development of quantum mechanics and the realization of the central role the Kähler geometry of the space of states to emerge together with the importance of of Lie groups and their representations, leading eventually to gauge theory and our present standard model of particle interactions.

Today we see that we have in hand the beginnings of a vast mathematical structure including in it all the mathematical ideas that we have found useful in physics in the past. Already it is clear that the first three basic underlying physical principles are

$$\boxed{\boxed{\textbf{SUPERSYMMETRY}}}$$

and

$$\boxed{\boxed{\textbf{SUPERSYMMETRY}}}$$

and

$$\boxed{\boxed{\textbf{SUPERSYMMETRY}}}$$

Experimental verification of supersymmetry in accelerators would be convincing evidence that the present chain of ideas is essentially correct, however we desperately need some more physical ideas. Let's hope that later day patent-clerk comes along soon!

8.1. ACKNOWLEDGEMENTS

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