The metric and strong coupling limit of the M5-brane

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Abstract

We find the analogue of the Boillat metric of Born-Infeld theory for the M5-brane. We show that it provides the propagation cone of *all* 5-brane degrees. In an arbitrary background field, this cone never lies outside the Einstein cone. An energy momentum tensor for the three-form is defined and shown to satisfy the Dominant Energy Condition. The theory is shown to be well defined for all values of the magnetic field but there is a limiting electric field strength. We consider the strong coupling limit of the M5-brane and show that the corresponding theory is conformally invariant and admits infinitely many conservation laws. On reduction to the Born-Infeld case this agrees with the work of Białnicki-Birula.

1 Introduction

The behaviour of the low energy states of the super-string theories in ten dimensions are described by supergravity theories which are uniquely specified by their type of supersymmetry. These theories satisfy a form of the the Equivalence Principle: the characteristics and hence the limiting propagation speeds

of all the fields, be they the graviton, the gravitini, p-form fields, scalars and spinors are given universally by the light-cone of the Einstein metric g_{mn}. The characteristics cones determine the paths of null geodesics which are associated with massless particles.. Violations of the Weak Equivalence Principal, that is that all freely falling particles, massive or massless, follow geodesics of the Einstein metric, are known to occur for massive particles in string theory, because the former can couple to a background dilaton field Φ. This may be seen by noting that the unique effective actions, that is the maximal supergravity theories in ten dimensions [1] [2], contain different powers of e^{Φ} in front of the field strengths associated with the Neveu-Schwarz$\otimes$Neveu-Schwarz and Ramond$\otimes$Ramond sectors. It is also known that string theory induces higher derivative corrections which may affect the characteristics [3] in a non-trivial background.

The behaviour of the states of open string theories are described by Dirac-Born-Infeld type actions provided one discards derivatives acting on field strengths. In a recent paper [4] it was argued that in a non-trivial background these open string states obey a modified form of the Equivalence Principle: the characteristics are universally given by a metric first introduced in non-linear electrodynamics by Boillat and which is conformal to what is usually referred to as the open string metric [5].

It was found that the Boillat cone never lies outside the Einstein cone and in general touches it along the two principal null directions of the two form $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + B_{\mu\nu}$, where $F_{\mu\nu}$ is the Born-Infeld gauge field strength and $B_{\mu\nu}$ is the Kalb-Ramond 2-form gauge field. In the limit of large field strength or equivalently large α', propagation perpendicular to the principal null directions is suppressed.

In this paper we shall consider the analogue of the Boillat metric for the M-5-brane equations of motion [23]. We shall call this metric C_{mn} Note that the characteristics may be read of from the co-metric C^{mn}. The speed of the rays are read off from its inverse $(C^{-1})_{mn}$ which is traditionally called a metric. If more than one metric is involved one must distinguish carefully between metrics and their inverses, i.e. between metrics and co-metrics. In the context of lattices and elsewhere, the prefix "co-" is frequently replaced by the adjective "reciprocal" We prefer "co-" because it is shorter. Note that there is no analogue of the Boillat co-metric for the M-2-brane because the world volume theory has no gauge fields. We shall find that indeed a modified form of the Equivalence Principal holds for the M-5-brane: the characteristics of fluctuations are given universally by an analogue of the Boillat cone which never lies outside the Einstein cone. We will show that if one dimensionally reduces the theory to Born-Infeld theory then the fivebrane metric becomes the Boillat metric of the D4-brane. We shall define an energy momentum tensor for the fivebrane. It satisfies the striking identity

$$T^{mn} = g^{mn} - C^{mn} \tag{1}$$

and we show that it obeys the dominant energy condition.

One may show that the fivebrane has a limiting electric field strength beyond

which the theory breaks down. By contrast, there is no limiting magnetic field strength. Finally we consider a high energy, or zero tension, limit of the fivebrane which is Weyl invariant and admits infinitely many conservation laws.

The plan of the paper is as follows. In section 2 we introduce the 5-brane metric and 5-brane Clifford algebra. We show that the 5-brane metric gives the characteristics of the propagating fields whose speeds can never exceed that of light. In section 3 we show that the two lightcones touch along a circle of null directions. In section 4 we introduce the covariantly conserved energy momentum tensor for the three-form and show that it satisfies an identity analogous to Hooke's law in classical elasticity theory. We show that in general its trace is non-vanishing and hence the three-form equations of motion are not invariant under a Weyl rescaling of the Einstein metric. In section 5 we develop the idea that there is a 5-brane Equivalence Principal. In section 6 we show how our metric agrees with the Boillat metric of Born-Infeld theory. In section 8 we show that plane wave solutions of the linear theory are exact solutions of the full non-linear theory. In section 9 we discuss the limiting electric field strength and the behaviour the theory near it. In section 10 we consider this strong coupling limit in detail. We show that the theory is Weyl-invariant in this limit and that it admits infinitely many conserved quantities.

2   The 5-Brane Metric

The equations of motion of the scalars X^N, closed three-form, H_{lmn} and spinors Θ [23]

$$G^{mn}\nabla_m\nabla_n X^N = 0, \tag{2}$$

$$G^{mn}\nabla_m H_{nrs} = 0 \tag{3}$$

$$\nabla_m \Theta (1-\Gamma)\Gamma^n m_{mn} = 0, \tag{4}$$

where ∇_m is the Levi-Civita covariant derivative with respect to the Einstein metric g_{mn}. To begin with, we consider their characteristics which require only the leading derivative terms. A short calculation shows that the characteristics of the X^N and H_{lmn} fields given by the co-metric G^{mn} which is defined by

$$G^{mn} = m^{mr}g_{rs}m^{sn} = (1 + \frac{2}{3}k^2)g^{mn} - 4k^{mn}, \tag{5}$$

where G_{mn} which will be defined to be $(G^{-1})_{mn}$, with $G^{mn} = (G)^{mn}$. All indices are raised and lowered using the Einstein metric g_{ab}, which is taken to have signature $-1, +1, +1, +1, +1, +1$, *with the exception of* G_{mn} which will be defined to be $(G^{-1})_{mn}$, with $G^{mn} = (G)^{mn}$. In fact it seems to be possible to avoid explicit use of the covariant form of the metric in all of the equations of motion. Only the contra-variant form is required.

$$m^{mn} = g^{mn} - 2k^{mn}, \tag{6}$$

$$k^{mn} = h^n{}_{ab}h^{mab}, \tag{7}$$

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where h_{abc} is a self-dual three-form

$$h_{abc} = \frac{1}{6}\epsilon_{abcdef}h^{def},\tag{8}$$

and

$$k^2 = k_{ab}k^{ab}.\tag{9}$$

In what follows shall repeatedly use the identities [22, 23]:

$$g_{ab}k^{ab} = 0,\tag{10}$$

and

$$k^{ab}g_{bc}k^{cd} = \frac{1}{6}g^{ad}k^2 = \frac{1}{6}g^{ad}k_{ef}k^{ef}.\tag{11}$$

Thus, for example,

$$C^{mn} = m_a^m m^{an} = 2Q^{-1}m^{mn} - g^{mn}.\tag{12}$$

The self dual field h_{abc} obeys the relations [22]

$$h_{abe}h^{cde} = \delta_{[a}^{[c}k_{b]}^{d]},\tag{13}$$

and

$$k_a{}^c k_c{}^b = \frac{1}{6}\delta_a^b k^2.\tag{14}$$

For later use we define

$$Q = 1 - \frac{2}{3}k^2.\tag{15}$$

The closed three form H_{abc} is related to the self-dual field h_{abc} by $h_{abc} = m_a{}^e H_{ebc}$, or equivalently by $H_{abc} = (m^{-1})_a{}^e h_{ebc}$ where $m^{-1} = Q^{-1}(1 + 2k)$. It will be prove useful in what follows to translate the self-duality condition on h_{abc} to one expressed entirely in terms of H_{abc}. This was carried out in [22] and refined in [27]. Since $k_a{}^e h_{ebc}$ is anti-self-dual we can express

$$H_{abc}^+ \equiv \frac{1}{2}(H_{abc} + \frac{1}{3!}\epsilon_{abcdef}H^{def}) = Q^{-1}h_{abc}\tag{16}$$

and

$$H_{abc}^- \equiv \frac{1}{2}(H_{abc} - \frac{1}{3!}\epsilon_{abcdef}H^{def}) = 2Q^{-1}k_a{}^e h_{ebc}\tag{17}$$

Taking the sum and difference we find that

$$H_{abc} = Q^{-1}(1 + 2k)_a{}^e h_{ebc}, \quad *H_{abc} = Q^{-1}(1 - 2k)_a{}^e h_{ebc}\tag{18}$$

Multipying the second equation by the matrix $(1 + 2k)^2$ and using the first equation we conclude that

$$* H^a{}_{bc} = Q^{-1}G^{ae}H_{ebc}\tag{19}$$

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Substituting the relation $h_{abc} = m_a{}^e H_{ebc}$ into equation (13) we find that

$$H_{abe} H^{cde} = \frac{1}{2} \delta^c_{[a} \delta^d_{b]} Q^{-1}(Q^{-1} - 1) + 2Q^{-2} k^c_{[a} k^d_{b]} + Q^{-2}(2 - Q)\delta^{[c}_{[a} k^{d]}_{b]} \qquad (20)$$

Taking the trace of this equation we find that

$$k_a{}^c = \frac{Q^2}{(2 - Q)}((H^2)_a{}^c - \frac{1}{6}\delta_a{}^c H^2) \qquad (21)$$

and tracing again

$$H^2 = 6Q^{-1}(Q^{-1} - 1) \qquad (22)$$

where $(H^2)_a{}^c = H_{aef} H^{cef}$.

We may express the last equation as

$$Q = -\frac{3}{H^2}(1 - \sqrt{1 + \frac{2}{3}H^2}) \qquad (23)$$

It is now straightforward to express the matrix m^2 as

$$(m^2)^{ac} = G^{ac} = \frac{Q^2}{(2 - Q)}(\eta^{ac}(1 + \frac{4}{3}H^2) - 4(H^2)^{ac}) \qquad (24)$$

and

$$(m^{-2})_{ac} = G_{ac} = \frac{1}{(2 - Q)}(\eta_{ac} + 4(H^2)_{ac}) \qquad (25)$$

Finally, we may express the self-duality of h_{abc} in terms of H_{mnp} by using equations (19) and (20) we find that

$$* H_{abc} = \frac{1}{\sqrt{1 + \frac{2}{3}H^2}}((1 + \frac{4}{3}H^2)\delta_a{}^e - 4(H^2)_a{}^e)H_{ebc} \qquad (26)$$

We now turn briefly to fermion sector. The projector in the spinor equation of motion is defined by

$$\Gamma = -\frac{1}{6}\eta^g_{lmnpqr}\Gamma^{lmnpqr} + \frac{1}{3}h_{lmn}\Gamma^{lmn}. \qquad (27)$$

where η^g_{lmnpqr} is the alternating tensor (not density) constructed from the Einstein metric.

In the case that the scalar and spinors vanish, the covariant equations of motion, when expressed in terms of 5-dimensional language, agree with a particular case of the equations of [21]. The particular case is the one that upon reduction gives Born-Infeld theory. One advantage of the covariant formulation used here is that, not only does it cover the more general case of non-vanishing scalars and spinors, but also the derivation of the characteristics is especially transparent.

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The characteristics determine a metric only up to a conformal factor. It turns out that it is more convenient to Weyl rescale the co-metric G^{mn} and we therefore adopt as our definition of the M5-brane metric C^{mn}

$$C^{mn} = Q^{-1}G^{mn} = Q^{-1}m^{mp}g_{pq}m^{qn}. \tag{28}$$

We recall that in the Born Infeld theory the characteristics are given by the Boillat metric[4] up to a conformal factor. The Boillat metric which is proportional to the open string metric has the advantage that it is invariant under electric magnetic duality rotations rather than merely being invariant up to a conformal factor as is the open string metric. In a two recent papers [24] an analogue for the fivebrane of the open string metric was proposed. Specifically, it was suggested that the analogous metric should be given by

$$\phi(H_{lpq}H^{lpq})(g_{mn} + 4H_{mpq}H_n{}^{pq}), \tag{29}$$

where the function ϕ should behave like $(H_{lpq}H^{lpq})^{-\frac{2}{3}}$ for large $H_{lpq}H^{lpq}$ in order that, upon reduction, it agree with the open string metric of string theory in the relevant limit.

The two proposals are both conformally related to the metric in the fivebrane equations. We will see that with our choice of conformal factor we obtain the Boillat metric of the D4-brane upon reduction. Our choice has the additional advantage that in terms of it the equations of motion can be rewritten in a natural way.

We now turn to the characteristics associated with the Dirac equation (4). While the gamma matrices Γ^m give a Dirac square root of the restriction of the bulk Einstein co-metric to the brane,

$$\Gamma^m\Gamma^n + \Gamma^n\Gamma^m = 2g^{mn}, \tag{30}$$

the spinor equation of motion contains the 5-brane Gamma matrices $\tilde{\Gamma}^m = n^m{}_n\Gamma^n$, where $n^m{}_n = Q^{-\frac{1}{2}}m^m{}_n$ which give a Dirac square root of the 5-brane co-metric C^{mn}

$$\tilde{\Gamma}^m\tilde{\Gamma}^n + \tilde{\Gamma}^n\tilde{\Gamma}^m = 2C^{mn}. \tag{31}$$

One may view the $n^m{}_n$ as a sort of sechbein for the 5-brane metric C_{mn} since

$$C^{mn} = n^m{}_pn^n{}_qg^{pq}. \tag{32}$$

Note that $n_{nm} = g_{ml}n^l{}_m$ is symmetric.

Writing out the Dirac equation in terms of the gamma matrices $\tilde{\Gamma}^m$ reveals that the spinor characteristics are also given by C^{mn}.

In the absence of a background H_{lmn} field, C^{mn} and g^{mn} coincide. Note that there is a *single* G^{mn} and thus a single characteristic cone. That is just as in Born-Infeld theory, there is no bi-refringence: all polarisation states travel with the same speed. Since any non-linear electrodynamic theory, including ones exhibiting bi-refringence can be made $N = 1$ supersymmetric, its absence cannot be attributed to just one supersymmetry. However one might imagine

that this property is a consequence of maximal superymmetry. In the case of Born-Infeld theory, the absence of bi-refringence , and the exceptional property that the system exhibits no shocks, characterises the theory uniquely (see [4] for references). It is an attractive conjecture that the same uniqueness property holds for the M-5-brane equations of motion.

We now establish that the 5-brane co-cone $C_G^\star \in T^\star M$ lies outside or on the Einstein co-cone $T^\star M \supset C_G^\star \supseteq C_g^\star$. The notation here is as follows. $C_g^\star$ consists of timelike co-vectors $p_m \in T^\star M$ such that $g^{mn} p_m p_n \le 0$ and its boundary consists of the lightlike co-vectors l_m for which $g^{mn} l_m l_n = 0$. Since passing to the dual space TM reverses inclusions we have that the 5-brane cone lies inside or on the Einstein cone, $TM \supset C_g \supseteq C_G$. In plain language this means that 5-brane excitations travel with speeds no greater than that of light. Note that the cone $C_G^\star$ depends only on the conformal equivalence class of co-metrics of which G^{mn} is one representative and C^{mn} another.

To establish our basic causality result we consider a co-vector l_a lying on the boundary of $C_g^\star$, i.e. for which

$$g^{mn} l_m l_n = 0. \tag{33}$$

Using (12) we find that

$$C^{mn} l_m l_n = -4Q s_{mn} s^{mn}, \tag{34}$$

where $s_{mn} = -s_{nmb} = h_{mnl} l^l$. Thus

$$s_{mn} l^m = 0. \tag{35}$$

By choice of frame $l^m = (1, 0, 0, 0, 0, 1)$ and this $s_{05} = 0$ and $s_{0i} + s_{5i} = 0$, where $i = 1, 2, 3, 4$ are spatial indices. One thus has

$$s_{ab} s^{ab} = s_{ij} s^{ij} \ge 0. \tag{36}$$

Thus

$$C^{mn} l_m l_n \le 0. \tag{37}$$

It follows that the boundary of $C_g^\star$ lies inside or on $C_G^\star$ and we are done.

3   Principal Null Directions

In the case of Born-Infeld theory in four spacetime dimensions, generically the Boillat and Einstein cones touch along two principal null directions of the electromagnetic two form $F_{\mu\nu}$ [4]. In fact this result holds in all dimensions. The common null direction $l^m = (1, n_i)$ with $n_i n_i = 1$ must satisfy

$$l^n F_{na} F_{mb} g^{ab} l^m = 0. \tag{38}$$

For a generic two-form one may find a frame in which the only non-vanishing components are $F_{01} = -F_{10}, F_{23} = -F_{32}, F_{45} = -F_{54} \dots$. The touching condition becomes

$$F_{01}^2 (1 - n_1^2) + F_{23}^2 (n_2^2 + n_3^2) + F_{45}^2 (n_4^2 + n_5^2) \dots = 0. \tag{39}$$

There are only two solutions ; $l^m = (1, \pm 1, 0, \ldots, 0)$.

One may ask what is the analogue of this result for the fivebrane with its self-dual three-form h_{lmn} in six spacetime dimensions? In order to answer this question we need to put k_{ab} rather than F_{ab} in standard form by diagonalizing with respect to to g_{ab}. Using (10) and (11) one easily sees that generically k_{ab} takes the form, up to permutations of the spatial axes,

$$k_{ab} = \sqrt{\frac{k^2}{6}} \mathrm{diag}(1, 1, 1, 1, -1, -1). \tag{40}$$

The common null directions must be common solutions of

$$-1 + n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 0, \tag{41}$$

and

$$1 + n_1^2 + n_2^2 + n_3^2 - n_4^2 - n_5^2 = 0. \tag{42}$$

There is in general a circle of such directions

$$l^m = (1, 0, 0, 0, \cos\alpha, \sin\alpha), \tag{43}$$

along which the Einstein and Boillat cones coincide. In all other directions the Boillat cone lies inside the Einstein cone.

4   The Energy Momentum Tensor and Hooke's Law

Following the discussion of the energy momentum tensor in [9], we shall in this paper define the energy momentum tensor as

$$T^{mn} = g^{mn} - \frac{G^{mn}}{Q}. \tag{44}$$

The main difference from the energy momentum tensor defined in reference [9],also used in [27], is that we have added the metric tensor g_{mn} to the tensor defined there so as to make the energy momentum tensor vanish for zero three-form field $h_{abc} = 0$. In fact it is (44) which enters directly into the superysmmetry algebra and hence the Bogomol'nyi bound of the theory[8]. The energy momentum tensor so defined has some important positivity properties which we will explore shortly.

In terms of C^{mn} we have the strikingly simple formula

$$T^{mn} = g^{mn} - C^{mn} \tag{45}$$

This formula has an interesting interpretation which appears to be closely related to Hooke's law in the classical theory of non-linear elasticity theory. This is formulated in terms of diffeomorphisms ϕ from the manifold Σ of the elastic body into flat three-dimensional Euclidean space $\mathbb{E}^3$ with metric δ. The relaxed

or unstretched state of least energy corresponds to a diffeomorphism ϕ_0. One defines the strain tensor tensor σ_{ij} of a general stretched state corresponding to a diffeomorphism ϕ by

$$\sigma_{ij} = \delta_{\star ij} - \delta_{0ij} \tag{46}$$

where $\delta_{\star}$ is the pull-back of the flat Euclidean metric δ under the diffeomorphism ϕ and δ_{0ij} is the pullback of the flat metric under ϕ_0. For an isotropic medium Hooke's law states that the strain σ_{ij} is proportional to the applied stress T_{ij}. Our formula (1) is similar but not identical because the tensors are contravariant not co-variant. Thus our case the analogue of $\delta_{\star ij}$ is the pull-back of the bulk closed string co-metric to the M-5-brane world volume. The analogue of the unstretched metric δ_{0ij} is the 5-brane co-metric C^{mn}. We believe that it would be worthwhile exploring this analogy further.

Another formula which is similar to one occurring Born-Infeld case [4] is the remarkable identity

$$\det C^{mn} = \det g^{mn}. \tag{47}$$

To check conformal invariance we compute the trace of the energy momentum tensor. It is given by

$$T^m_m = -\frac{8k^2}{(1 - \frac{2}{3}k^2)} \tag{48}$$

Thus the theory is Weyl-invariant in the weak field limit in which the equations of motion become linear. However it is not Weyl-invariant for finite values of the fields. We see shall see later that Weyl invariance is restored in the strong coupling limit.

Later we shall make use of some other identities involving the energy momentum tensor which we derive here. Because the energy momentum tensor is conserved with respect to the Levi-Civita connection

$$\nabla_m T^{mn} = 0, \tag{49}$$

we have, from Hooke's Law, the following identities

$$\nabla_n C^{mn} = 0. \tag{50}$$

These will be used in the next section to establish the M-5-brane Equivalence Principal. Some additional useful identities may be obtained as follows. One defines

$$n^{mn} = Q^{-\frac{1}{2}} m^{mn} = Q^{-\frac{1}{2}} (g^{mn} - 2k^{mn}) \tag{51}$$

Thus

$$n^{-1}_{mn} = Q^{-\frac{1}{2}} (g_{mn} + 2k_{mn}). \tag{52}$$

We thus have

$$C^{mn} = n^{ma} g_{ab} n^{bn}, \tag{53}$$

and of course

$$\tilde{\Gamma}^n = n^{na} g_{ab} \Gamma^b. \tag{54}$$

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Now one easily finds

$$g_{mn}C^{mn} = \frac{1 + \frac{2}{3}k^2}{1 - \frac{2}{3}k^2} = g^{mn}C_{mn} \tag{55}$$

In reference [9] an energy momentum tensor was introduced that treated the scalars and three form in a more symmetric fashion. This tensor is given by

$$S^{mn} = T^{mn} - g^{mn}. \tag{56}$$

and is covariantly conserved. Reference [9] also found a tensor density

$$\mathfrak{S}^{mn} = \sqrt{-g}(T^{mn} - g^{mn}). \tag{57}$$

that was conserved in the sense

$$\partial_m \mathfrak{S}^{mn} = 0. \tag{58}$$

In the case that the three-form vanishes, we have $T^{mn} = 0$, but $S^{mn} \neq 0$, and so the quantity $\sqrt{-g}g^{mn}$ is a measure of the energy-momentum of the scalars. In fact it is the canonical stress tensor with respect to the flat metric δ_{mn}. It may be obtained from the Lagrangian for the scalars written down in "static ", or more accurately "Monge", gauge.

5 The M5-Brane Equivalence Principle

It is striking fact that the co-metric C^{mn} rather than the Einstein co-metric g^{mn} enters in the equations of motion of all the 5-brane fields in such a way that it is impossible to determine the Einstein metric by means of observations using 5-brane fields alone. In this respect the situation resembles attempts to reinterpret General Relativity as a flat space theory by introducing a flat metric $\eta_{\mu\nu}$ into spacetime. The problem is that by the Equivalence Principal no physical measurement can detect the flat metric.

Our aim in this section is to explore further this enhanced version of the Equivalence Principle for the fivebrane. Our claim is that there exists a preferred set of variables to describe the theory that are related in a direct way to the physical observables. In particular, in section two we showed that the characteristics of the fivebrane were given by the metric C_{mn}. This means that the metric can be determined up to a conformal factor by observing the motion of small fluctuations. Another physically relevant variable is the gauge invariant field strength H_{lmn}. This satisfies the Bianchi identity and hence can be written in terms of a two form potential that couples directly to a two brane probe. Hence using a two brane probes allows one to measure the H_{lmn} field.

To illustrate this point we now show how to write the fivebrane equations of motion entirely in terms of the variables C_{mn} and H_{lmn} and X^N.. In particular, we will find that the equations of motion for the scalar and the three-form can be written using the Levi-Civita covariant derivative with respect to the metric

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C_{mn}. The situation is analogous to how the usual Equivalence Principal works in General Relativity. Because all physical equations are writen in terms of the metric C_{mn} the metric g_{mn} is not directly observable.

We begin with the scalar equation of motion. Because G^{mn} is proportional to C^{mn}, this equation can be written as

$$C^{mn}\nabla_m\nabla_n X^N = 0. \tag{59}$$

Following [9] and using (50) we may re-write this as

$$\nabla_m(C^{mn}\partial_n X^N) = \frac{1}{\sqrt{-g}}\partial_m(\sqrt{-g}C^{mn}\partial_n X^N) = 0. \tag{60}$$

Now using (47) we have

$$\frac{1}{\sqrt{-C}}\partial_m(\sqrt{-C}C^{mn}\partial_n X^N) = \Delta_m(C^{mn}\Delta_n X^N) = 0, \tag{61}$$

where Δ_m is the Levi-Civita covariant derivative with respect to the 5-brane metric C_{mn} which is defined by

$$\Delta_m C^{ab} = 0. \tag{62}$$

We see that (61) is just the covariant wave equation with respect to the 5-brane metric C_{mn}.

The closure condition for the three-form H_{lmn}

$$\partial_{[p}H_{qrst]} = 0, \tag{63}$$

clearly requires no metric. The equation of motion of the three-form may be written as

$$C^{mn}\nabla_m(H_{nab}) = 0. \tag{64}$$

Using (50) and the fact that $\nabla_m g^{ab} = 0$, we re-write (64) as

$$\nabla_m(P^{mab}) = 0, \tag{65}$$

where

$$P^{mab} = C^{mn}g^{ac}g^{bd}H_{ncd}. \tag{66}$$

Now the contravariant tensor P^{mab} is totally antisymmetric and satisfies ([9] equation(17))

$$P^{lmn} = \star_g H^{lmn} \tag{67}$$

and therefore we may rewrite (64) as

$$\frac{1}{\sqrt{-g}}\partial_m(\sqrt{-g}P^{mab}) = 0. \tag{68}$$

Now using (47) we get

$$\frac{1}{\sqrt{-C}}\partial_m(\sqrt{-C}P^{mab}) = 0. \tag{69}$$

This may now be put in the fivebrane metric covariant form

$$\Delta_m(\star_C H^{mnp}) = 0. \tag{70}$$

The Hodge operations $\star_g$ and $\star_C$ are taken with respect to the Einstein and 5-brane metric respectively. However they are related because if η_C^{mabncd} is the contravariant alternating tensor which is covariantly constant with respect to the Levi-Civita connection of the 5-brane metric C_{mn}, we have

$$\sqrt{-C}\eta_C^{mabncd} = \sqrt{-g}\eta_g^{mabncd}, \tag{71}$$

where η_g^{mabncd} is the contravariant alternating tensor which is covariantly constant with respect to the Levi-Civita connection of the Einstein metric g_{mn}.

Finally we consider the Dirac equation. Following [9] equation (13), this may be written as

$$\nabla_m(\Psi'(1 - \Gamma)\tilde{\Gamma}^m) = 0, \tag{72}$$

where

$$\Psi' = Q^{-\frac{1}{2}}\Theta. \tag{73}$$

Now we must rewrite the projector Γ in terms of the 5-brane gamma-matrices $\tilde{\Gamma}^m$. We have the formula

$$\Gamma = -\frac{1}{6}\eta_{C\,lmnrst}\tilde{\Gamma}^l\tilde{\Gamma}^m\tilde{\Gamma}^n\tilde{\Gamma}^r\tilde{\Gamma}^s\tilde{\Gamma}^t + \frac{1}{2}H_{lmn}\tilde{\Gamma}^l\tilde{\Gamma}^m\tilde{\Gamma}^n. \tag{74}$$

The covariant derivative ∇_m is the spinor derivative with respect to the Einstein metric. We expect that one may be able to rewrite this in terms of the spinor covariant derivative Δ_m. This is likely to require a more elaborate redefinition of the spinor than in (73), along the lines of that discussed in reference [8].

6 Dimensional Reduction and the relation to the Open String metric

It is known [22, 23] that under double dimensional reduction to five-dimensions the equations of motion of the five brane reduce to those of Dirac-Born-Infeld theory. We shall now investigate the relation between the fivebrane metric and the Boillat metric of the D4-brane in this case. We assume that the fields h_{lmn} are independent of the fifth spatial dimension and that

$$H_{mn5} = F_{mn}, \tag{75}$$

Using the results of [23] eqn(136) [23] we have (after rescaling F to be consistent with the standard normalization) we find the reduction of five-brane metric to be given by

$$C^{mn} = Z(1 - F^2)^{-1} \tag{76}$$

where

$$Z = \sqrt{1 + 2x - y^2} = \sqrt{-\det(g_{mn} - F_{mn})}. \tag{77}$$

The Boillat co-metric is given by

$$C_{\text{Boillat}}^{mn} = Z\left((1 - F^2)^{-1}\right)^{mn} \tag{78}$$

In other words the two metric coincide.

We now extend the work of [4] to give an enhanced Equivalence Principle similar to that discussed for the fivebrane earlier. The Born-Infeld equations may be written as

$$\partial_{[l}F_{mn]} = 0, \tag{79}$$

and

$$\nabla_m P^{mn} = \frac{1}{\sqrt{-g}}\partial_m(\sqrt{-g}P^{mn}) = 0. \tag{80}$$

Clearly (79) requires no metric and (80) may be re-written (using (47)) in terms of the Boillat metric as.

$$\Delta_m P^{mn} = \frac{1}{\sqrt{-C}}\partial_m(\sqrt{-C}P^{mn}) = 0.. \tag{81}$$

One may check, just as for the 5-brane, that the scalar equations of motion may be written as

$$\Delta_m \Delta^m X^N = 0. \tag{82}$$

7   Dominant Energy Condition

Almost all physically well behaved classical energy momentum tensors satisfy the Dominant Energy Condition. This states that for every pair of future directed casual vectors $p^m, q^m \in C_g^+$ one has

$$T_{mn}p^m q^n \geq 0. \tag{83}$$

Note that, if the Dominant Energy Condition holds with respect to the Einstein metric, it necessarily holds with respect to the fivebrane metric, since the lightcone of the former includes that of the letter.

The importance of the Dominant Energy Condition is that it guarantees the classically fields whose energy momentum tensors satisfy the condition propagate causally. It is also an essential ingredient in the Positive Energy Theorems of General Relativity. A theorem of Hawking implies that if the Dominant Energy Condition holds then matter cannot escape from, or enter, a bounded spatial region at a speed faster than light [11]. In particular it guarantees some

sort of stability since matter obeying the condition cannot just simply disappear [11].

Let us evaluate the left hand side of (83) for the fivebrane. Using equations (12, 1) we find that

$$Q^{-1}(2k_{mn}p^m q^n + \frac{2}{3}k^2 p \cdot q), \tag{84}$$

where

$$p \cdot q = -g_{mn}p^m q^n \geq 0. \tag{85}$$

We will now show that the left-hand-side of the above equation is indeed positive since the quantities k^2 and $k_{mn}p^m q^n$ and are non-negative.

To see that k^2 is positive we introduce an electric two-form in five dimensions by

$$e_{ij} = h_{oij}. \tag{86}$$

Self-duality of h_{lmn} implies that

$$e_{ij} = b_{ij} = -\frac{1}{6}\epsilon_{ijkrs}h^{krs} \tag{87}$$

Calculation reveals that

$$k_{ij} = \delta_{ij}e^{pq}e_{pq} - 4e_{ir}e_j{}^r. \tag{88}$$

From (10) we deduce that

$$k_{00} = e_{rs}e^{rs}. \tag{89}$$

and using (11) we obtain

$$\frac{2}{3}k^2 = 16e_{ij}e^{jk}e_{ks}e^{si} - 4(e_{ij}e^{ij})^2. \tag{90}$$

Now using SO(5) transformations we can skew-diagonalize the two-form e_{ij}, that is choose a basis in which the only non-vanishing components are $e_{12} = -e_{21} = e_1$ and $e_{34} = -e_{43} = e_2$. In this basis, one finds that

$$k^2 = 24(e_1^2 - e_2^2)^2. \tag{91}$$

The quantity $k_{mn}p^m q^n$ may be dealt with in a similar way. If p^n is timelike we can choose a six-dimensional Lorentz frame in which $p^m = (p^0, 0, 0, 0, 0, 0)$. This choice allows us to use the $SO(5)$ freedom to skew diagonalize e_{ij} as above. A short calculation then gives

$$k_{mn}p^p q^n = 2p^0 q^0(e_1^2 + e_2^2 - 2\frac{q^5}{q^0}e_1 e_2). \tag{92}$$

Since q^n is causal, $|\frac{q^5}{q^0}| \leq 1$ and $e_1^2 + e_2^2 \geq 2e_1 e_2$, we find the desired result, namely $k_{mn}p^m q^n \geq 0$.

Interestingly, k^2 vanishes if and only if e_{ij} determines a self-or anti self-dual two-form in the four space orthogonal to its kernel. We also see that $T_{mn}p^m p^n$ is strictly positive for timelike p^n and hence taking the choice $p^n = (p^0, 0, 0, 0, 0, 0)$ we conclude that T_{00} is strictly positive.

8 Exact Plane Wave solutions

In this section we shall establish that plane wave solutions of the linearized theory are in fact *exact* solutions of the full non-linear equations of motion, a property which also holds in Born-Infeld theory (compare [4]) and classical general relativity. We shall suppose that the metric g_{mn} is flat, $g_{mn} = \eta_{mn}$, although one might consider more general cases. We make the ansatz

$$H_{lmn} = H^0{}_{lmn} f(l_n x^n), \tag{93}$$

where H^0_{lmn} is a constant three-form, $f(u)$ is an arbitrary function of its argument and the constant vector l^n is null

$$\eta_{mn} l^m l^n = 0. \tag{94}$$

The closure condition becomes

$$l_{[l} H^0_{mno]} = 0, \tag{95}$$

Thus

$$\epsilon^{pqrstu} l_r H^0_{stu} = 0. \tag{96}$$

If we assume that $l_m = (1,1,0,0,0,0)$ and let greek indices run from 2 to 5, we find that

$$H^0_{\alpha\beta\gamma} = 0, \qquad H^0_{0\gamma\delta} = H^0_{1\gamma\delta}. \tag{97}$$

This may be written covariantly as

$$H^0_{mnp} = l_{[m} A^0_{np]}, \tag{98}$$

where A^0_{np} is a constant polarization two-form. It is determined only up to

$$A^0_{np} \rightarrow A^0_{np} + l_{[n} A^0_{p]}, \tag{99}$$

where A^0_p is a constant one-form.

We now make the further assumption that

$$l^m H_{mnp} = 0. \tag{100}$$

The reason we have to assume (100) is that quantities $H^0_{01\alpha}$ are not constrained by equation (96). In order to eliminate this freedom we are in effect assumg that

$$H^0_{01\alpha} = 0. \tag{101}$$

Having made this ansatz, i.e. assuming (98 , 100) it follows that

$$H_{lmn} H^{lmn} = 0. \tag{102}$$

Thus $Q = 1$ and hence $C^{mn} = G^{mn}$.

It remains to solve the self-duality condition. Now using (24) we see that five brane metric is of so-called Kerr-Schild form:

$$G^{mn} = \eta^{mn} - \text{constant} f^2 l^m l^n. \tag{103}$$

The constant is positive. To evaluate it we could introduce a (non-unique) null vector n^m , normalized so that

$$\eta_{mn} n^m l^n = -1. \tag{104}$$

One then has

$$\text{constant} = (H^{0^2})_{mn} n^m n^n. \tag{105}$$

Because l^m is null (i.e. from (94)) it follows that

$$G_m{}^q H_{npq} = H_{npm}. \tag{106}$$

Thus the self-duality condition (26) becomes

$$\star H_{lmn} = H_{lmn}, \tag{107}$$

which is the same as the condition for the linear theory.

Using Hooke's Law (1) we deduce that the energy momentum tensor is given by

$$T^{mn} = \text{constant} f^2 l^m l^n. \tag{108}$$

Equation (108) is the energy momentum tensor of a fluid of energy density constant moving at the speed of light in the direction l^m. This is often referred to as a null fluid. We shall encounter null fluids again later when we examine the energy momentum tensor in the strong coupling limit.

9   The $SO(5)$ covariant formalism and the limiting field strength

Born Infeld theory has a built in upper-bound for the electric field strength. The string theoretic interpretation is that when electric fields approach the limiting field strength, copious pair production of open string states occurs[14][15]. One may ask whether a similar phenomenon takes place in M-Theory [24].

In Born-Infeld theory one must take care to express the upper bound in terms of the correct variables. The Lagrangian density is

$$L = 1 - \sqrt{1 - \mathbf{E}^2 + \mathbf{B}^2 - (\mathbf{E} \cdot \mathbf{B})^2}, \tag{109}$$

which shows that the electric field $\mathbf{E}$ cannot be too big for fixed $\mathbf{B}$ since then the Lagrangian density becomes complex. However the Hamiltonian density [12] is

$$\mathcal{H} = \sqrt{1 + \mathbf{B}^2 + \mathbf{D}^2 + (\mathbf{B} \times \mathbf{D})^2} - 1. \tag{110}$$

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This shows that there is no limit on either the magnetic induction **B** or electric inductions **D**. However the dual Lagrangian [12] is

$$\hat{L} = \sqrt{1 - \mathbf{H}^2 + \mathbf{D}^2 - (\mathbf{D} \cdot \mathbf{H})^2} - 1, \tag{111}$$

which indicates that there is an upper bound on the magnetic intensity **H**, as there has to be, by electric-magnetic duality invariance. This point is re-enforced by considering the dual Hamiltonian [12]

$$\hat{H} = 1 - \sqrt{1 - \mathbf{H}^2 - \mathbf{E}^2 + (\mathbf{H} \times \mathbf{E})^2}. \tag{112}$$

In the case of the M5-brane we have a similar range of possible choices of field variables. The fivebrane has a self-duality condition which means that if we define

$$B_{ij} = -\frac{1}{6}\epsilon_{ijklm}H^{klm}, \quad E_{ij} = H_{0ij} \tag{113}$$

where $i, j, k, \ldots = 1, 2, \ldots, 5$, then we can express E_{ij} in terms of B_{ij} and *vice versa*. Thus the energy density $T_{00} = \mathcal{H}$ can be expressed in terms of either variable. To achieve this we will have in effect to solve the self-duality constraint. The self-duality condition on H_{mnp} can be expressed as

$$H^+_{mnp} = Q^{-1}h_{mnp}, \quad H^-_{mnp} = 2Q^{-1}k_m{}^r h_{rnp} \tag{114}$$

where $H^{\pm}_{mnp} = \frac{1}{2}(H_{mnp} \pm \frac{1}{3!}\epsilon_{mnprst}H^{rst})$. In the special frame used above one then finds that

$$B_1 = \frac{e_1}{1 - 4(e_1^2 - e_2^2)}, B_2 = \frac{e_2}{1 + 4(e_1^2 - e_2^2)}, \tag{115}$$

$$E_1 = \frac{e_1}{1 + 4(e_1^2 - e_2^2)}, E_2 = \frac{e_2}{1 - 4(e_1^2 - e_2^2)}. \tag{116}$$

Inverting these equations leads to

$$8e_1 = \frac{B_1}{B_2^2 - B_1^2}\sqrt{1 + 16B_2^2}\left[\sqrt{1 + 16B_2^2} - \sqrt{1 + 16B_1^2}\right], \tag{117}$$

and

$$8e_2 = \frac{B_2}{B_1^2 - B_2^2}\sqrt{1 + 16B_1^2}\left[\sqrt{1 + 16B_1^2} - \sqrt{1 + 16B_2^2}\right]. \tag{118}$$

Note that

$$E_1 E_2 = B_1 B_2. \tag{119}$$

The required solutions of the self-duality constraint are therefore

$$E_1 = B_1\sqrt{\frac{1 + 16B_2^2}{1 + 16B_1^2}}, E_2 = B_2\sqrt{\frac{1 + 16B_1^2}{1 + 16B_2^2}}, \tag{120}$$

or inversely,

$$B_1 = E_1\sqrt{\frac{1 - 16E_2^2}{1 - 16E_1^2}}, B_2 = E_2\sqrt{\frac{1 - 16E_1^2}{1 - 16E_2^2}}. \tag{121}$$

Expressing the energy momentum tensor in terms of h_{mnp} and using the relations of section four we find that

$$T_{00} = \frac{1 + 16(e_1^2 - e_2^2) + 8(e_1^2 + e_2^2)}{1 - 16(e_2^2 - e_1^2)^2} - 1. \tag{122}$$

Using the above equations we conclude that

$$T_{00} = \sqrt{(1 + 16B_1^2)(1 + 16B_2^2)} - 1. \tag{123}$$

The expression for the energy density (123) may be cast in the SO(5) covariant form

$$\mathcal{H} = \sqrt{\det(\delta_{ij} + 4B_{ij})} - 1 = (\delta_{ij} + 16B_{ik}B_{jk})^{\frac{1}{4}} - 1. \tag{124}$$

A Hamiltonian density was derived from the action formulation of the five-brane in [25]. Our $\mathcal{H}$ coincides with that Hamiltonian density.

From (116) and (123) we have

$$E_1 = \frac{1}{16}\frac{\partial \mathcal{H}}{\partial B_1}, E_2 = \frac{1}{16}\frac{\partial \mathcal{H}}{\partial B_2}, \tag{125}$$

which may be cast in the manifestly $SO(5)$-covariant form:

$$E_{ij} = \frac{1}{16}\frac{\partial \mathcal{H}}{\partial B_{ij}} = \frac{(1 - 8\mathrm{Tr}B^2)B_{ij} + 16B_{ij}^3}{\sqrt{1 - 8\mathrm{Tr}B^2 + (16)^2 W_i^2}}, \tag{126}$$

where

$$W_i = \frac{1}{8}\epsilon_{ijrst}B^{jr}B^{st}. \tag{127}$$

Now the Legendre transform of the Hamiltonian density is

$$\hat{\mathcal{H}} = 1 - \sqrt{(1 - 16E_1^2)(1 - 16E_1^2)} = 1 - \sqrt{\det(\delta_{ij} + 4\sqrt{-1}E_{ij})}, \tag{128}$$

so that

$$\hat{\mathcal{H}} + \mathcal{H} = \frac{1}{2}E_{ij}B^{ij} \tag{129}$$

and

$$B_{ij} = \frac{1}{16}\frac{\partial \hat{\mathcal{H}}}{\partial E_{ij}} = \frac{(1 + 8\mathrm{Tr}E^2)E_{ij} + 16E_{ij}^3}{\sqrt{1 + 8\mathrm{Tr}E^2 + (16)^2 U_i^2}}, \tag{130}$$

where

$$U_i = \frac{1}{8}\epsilon_{ijrst}E^{jr}E^{st}. \tag{131}$$

In our special frame this equation reduces to (121). For a helpful review of the various formulations of the 5-brane equations of motion which covers some of this material the reader is referred to [26].

The Hamiltonian density $\mathcal{H}$ is a well defined convex function for all finite values of the magnetic induction B_{ij}. The Legendre transform maps all of B_{ij}

space in a one-one fashion onto the open region of E_{ij} space for which the dual Hamiltonian density $\hat{\mathcal{H}}$ is a well defined convex function. That is for which the matrix $\delta_{jk} - 16E_{ji}E_{ki}$, which occurs in the dual Hamiltonian, is positive definite. In our special frame, the allowed region is just $4|E_1| < 1$ and $4E_2 < 1$.

The Bianchi identities read

$$\partial_i B_{ij} = 0, \ \frac{\partial B_{ij}}{\partial t} + \frac{1}{2}\epsilon_{ijkrs}\partial_k E_{rs} = 0. \tag{132}$$

Taking the solution (126) for the self duality condition implies the equations of motion.

10 Strong Coupling Limit

We are now going to discuss the strong coupling limit of the theory. This is equivalent to taking the tension of the fivebrane to zero and may be thought of as a high energy limit. We shall find that the situation is similar to that of the strong coupling limit of Born-Infeld theory in four spacetime dimensions which has been thoroughly investigated by Białnicki-Birula and called by him UBI theory [12, 13]. In that case, the Lagrangian vanishes, but there is a well defined Hamiltonian. Moreover the theory is Lorentz invariant and has the energy momentum tensor of a null fluid. As a consequence there are infinitely many conserved currents in flat spacetime. Because the invariants $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu} \star F_{\mu\nu}$ both vanish on shell one might speculate that the theory is quantum mechanically finite. As we shall see, there is just as much evidence to warrant a similar speculation about the M-Theory version.

The limit we are considering differs from that discussed in [24, 30].

To proceed we introduce into the Hamiltonian a parameter T with the dimensions of mass cubed. We then take the limit $T \downarrow 0$. The Hamiltonian density is

$$\mathcal{H} = T^2\sqrt{\det(1 + \frac{4B_{ij}}{T})} - T^2. \tag{133}$$

Letting $T \downarrow 0$, we get the well defined limit

$$\mathcal{H} = 16|B_1 B_2|. \tag{134}$$

We now evaluate the energy momentum tensor in the strong coupling limit in terms of the variable B_{ij}. In this process we will find quantities that are non-vanishing only when one takes next to leading contribution in T. On finds that in the limit of small T,

$$e_1 \to \frac{2}{T}\frac{B_1 B_2}{B_1 + B_2} - \frac{T}{16}\frac{1}{B_1}\frac{B_1 - B_2}{B_1 + B_2}, \tag{135}$$

$$e_2 \to \frac{2}{T}\frac{B_1 B_2}{B_1 + B_2} + \frac{T}{16}\frac{1}{B_1}\frac{B_1 - B_2}{B_1 + B_2}. \tag{136}$$

It follows that

$$e_1^2 - e_2^2 \to \frac{1}{4}\frac{B_1 - B_2}{B_1 + B_2}. \tag{137}$$

Reinstating the coupling constant T, the energy momentum tensor is given by

$$T_{mn} = -T^2\frac{m_{mn}^2}{Q} + T^2 g_{mn}. \tag{138}$$

In the small T limit the only non-vanishing components are

$$T_{00} = 16|B_1 B_2|, \tag{139}$$

$$T_{55} = 16|B_1 B_2|, \tag{140}$$

$$T_{05} = -16 B_1 B_2. \tag{141}$$

Thus

$$T_{mm} = \mathcal{H} l_m l_n, \tag{142}$$

with

$$l^m = (1, 0, 0, 0, 0, 1), \tag{143}$$

and

$$l^m l^n g_{mn} = 0, \tag{144}$$

We have been working in a local frame. However, equations (142) and (144) are manifestly covariant and hence hold in a general frame. In particular we have in general, that the null vector l^m is given by

$$l^m = (1, n^i), \tag{145}$$

where n^i is a unit vector in the direction of the Poynting vector T^{0i} and therefore

$$T^{om} = \mathcal{H} l^m. \tag{146}$$

From (142) we see that the trace of the energy momentum tensor vanishes

$$T_m^m = 0. \tag{147}$$

Thus we have shown that the theory is Weyl invariant in the strong coupling limit. The energy momentum tensor (142) is the same form as the energy momentum tensor of the strong coupling limit of Born-Infeld theory, UBI theory.

Because the energy momentum tensor has the form of a null fluid (see [12, 13]) one may easily check that one has infinitely many conservation laws. The conservation of energy equation $\partial_t T^{00} + \partial_i T^{0i} = 0$ may be written covariantly as

$$\nabla_m(\mathcal{H} l^m) = 0. \tag{148}$$

The remaining conservation law implies that

$$l^n \nabla_n l^l = 0..\tag{149}$$

From (149) it follows that the integral curves tangent to l^m, that is the solutions of

$$\frac{dx^m}{d\lambda} = l^m,\tag{150}$$

are null geodesics with affine parameter λ. In fluid dynamic language (148) corresponds to the conservation of entropy, sometimes thought of as 'photon" number. The integral curves are thought of as the world lines of the fluid. In the case of UBI theory [12, 13] it turns out [29] cf. [28] that the fluid may be thought of as a gas of massless Schild strings [31]. In fluid dynamic language the world-sheets of the strings are the histories of magnetic field lines which are swept out with velocity l^m. We shall turn later to the possible fluid interpretation in the M5-brane case.

It follows from (148) and (149) that the tensors

$$T^{m_1 m_2 \dots m_k} = \mathcal{H} l^{m_1} l^{m_2} \dots l^{m_k}\tag{151}$$

satisfy

$$\nabla_m T^{m m_2 \dots m_k} = 0,\tag{152}$$

for any positive integer k.

In flat spacetime the divergence identities (152) may be integrated over space to give rise to infinitely many conservation laws.

In order to cast the limiting equations of motion in a manifestly $SO(5)$ covariant form, we recall from (126) and (132) that

$$\frac{\partial B^{ij}}{\partial t} + \frac{1}{2} \epsilon^{ijrst} \partial_r \left\{ \frac{(B - 8(\mathrm{Tr}B^2)B + 16B^3)_{st}}{\sqrt{1 - 8\mathrm{Tr}B^2 + 16^2 W_i^2}} \right\} = 0.\tag{153}$$

One may now re-instate T and then take the limit $T \downarrow 0$ in (153) to obtain

$$\frac{\partial B^{ij}}{\partial t} + \frac{1}{2} \epsilon^{ijrst} \partial_r \left\{ \frac{(-\frac{1}{2}(\mathrm{Tr}B^2)B + B^3)_{st}}{\sqrt{W_i^2}} \right\} = 0.\tag{154}$$

Using the identity

$$(-\mathrm{Tr}B^2 B + 2B^3)^{ij} = \epsilon^{ijklm} W_k B_{lm},\tag{155}$$

we may show that in the limit, electric field becomes

$$E^{ij} = \frac{1}{2} \epsilon^{ijrst} \frac{W_r}{\sqrt{W_k^2}} B_{st},\tag{156}$$

and the field equation becomes

$$\frac{\partial B_{ij}}{\partial t} + \frac{1}{4} \epsilon_{ij}{}^{klm} \partial_k \epsilon_{lmrst} \frac{W^r B^{st}}{\sqrt{W_p^2}}.\tag{157}$$

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This is very similar in form to the Born-Infeld case [12, 13]. One may also give an $SO(5,1)$ covariant formulation of the 5-brane in the $T \downarrow 0$ limit. It is sufficient to derive the self-duality conditions in this limit as it, together with the Bianchi Identity, imply the field equations.

By multiplying through by m^{-1}, the self-duality condition $*H_{abc} = Q^{-1}(m^2)_a{}^e H_{ebc}$ can be reexpressed as

$$(m^{-1})_a{}^e * H_{ebc} = Q^{-1}(m)_a{}^e H_{ebc} \tag{158}$$

or

$$(1 + 2k)_a{}^e * H_{ebc} = Q^{-1}(1 - 2k)_a{}^e H_{ebc} \tag{159}$$

Taking the limit and using the equation for k in terms of H^2 we find that the self-duality condition can be expressed covariantly as

$$(H^2)_a{}^e * H_{ebc} = (\frac{1}{3}H^2 \delta_a^e - (H^2)_a{}^e) H_{ebc} \tag{160}$$

In view of the interpretation of the strong coupling limits of Born-Infeld theory as string fluids [28] , it is of interest to consider the detailed structure of the three-form H. Point-wise, i.e. locally, it may be expressed in the general case as

$$H = E_1 dt \wedge dx^1 \wedge dx^2 + E_2 dt \wedge dx^3 \wedge dx^4 - B_2 dx^5 \wedge dx^1 \wedge dx^2 - B_1 dx^5 \wedge dx^3 \wedge dx^4. \tag{161}$$
$$\star H = E_1 dx^3 \wedge dx^4 \wedge dx^5 + E_2 x^1 \wedge dx^2 \wedge dx^5 + B_2 dt \wedge dx^3 \wedge dx^4 + B_1 dt \wedge dx^1 \wedge dx^2. \tag{162}$$

In the strong coupling limit we get the well-defined limits

$$H = (dt - dx^5) \wedge (B_2 dx^1 \wedge dx^2 + B_1 dx^4 \wedge dx^5), \tag{163}$$

$$* H = (dt - dx^5) \wedge (B_2 dx^2 \wedge dx^3 + B_1 dx^1 \wedge dx^2). \tag{164}$$

By contrast since

$$h = e_1(dt \wedge dx^1 \wedge dx^2 - dx^5 \wedge dx^3 \wedge dx^4) + e_2(dt \wedge dx^3 \wedge dx^4 - dx^5 \wedge dx^1 \wedge dx^2), \tag{165}$$

the three-form h_{lmn} diverges in the limit of small T

$$h \to \frac{2}{T} \frac{B_1 B_2}{B_1 + B_2}(dt - dx^5)(dx^1 \wedge dx^2 + dx^3 \wedge dx^5). \tag{166}$$

We see that in the limit H and $\star H$ contain a one dimensional null factor $dt - dx^5 = -l_m dx^m$, with

$$H_{lmn}l^n = 0, \quad \star H_{lmn}l^n = 0. \tag{167}$$

It is tempting to give the stress tensor an interpretation in terms of a fluid of p-branes. The null vector l^m clearly provides us with the velocity of the putative fluid and since it is null, we should expect a null fluid analogous to the gas of

null or Schild strings [31] considered in [32] and elaborated in [33] [34] in four dimensions. In the case of UBI theory [12, 13] one has the condition

$$\sqrt{\det(F_{\mu\nu})} = \frac{1}{4}F_{\mu\nu} \star F^{\mu\nu} = \mathbf{E} \cdot \mathbf{B} = 0. \tag{168}$$

This is equivalent to the simplicity condition

$$F \wedge F = 0, \tag{169}$$

which implies the existence of one-forms a_μ and b_μ such that

$$F_{\mu\nu} = a_\mu b_\nu - a_\nu b_\mu. \tag{170}$$

The co-vectors a_μ and b_μ are not unique; one may choose them such that $a_\mu b^\mu = 0$. Thus, in the tangent space at each point of spacetime, the two-form $F_{\mu\nu}$ is tangent to the two-plane spanned by a_μ and b_μ. The equations of motion imply the integrability condition that these two-planes mesh together to provide a foliation of spacetime by two-surfaces. If

$$F_{\mu\nu}F^{\mu\nu} = 0, \tag{171}$$

then the two-plane is null, and one of the co-vectors, call it a_μ, is null and the other spacelike. Therefore

$$F_{\mu\nu} = l_\mu b_\nu - l_\nu b_\mu. \tag{172}$$

The null vector l_μ satisfies

$$F_{\mu\nu}l^\nu = 0. \tag{173}$$

If (171) then the principal null directions of the two-form $F_{\mu\nu}$ coincide and may be identified with l^μ. One may regard b_μ as the (un-normalised) tangent vector to the magnetic field lines and obtain in this way a picture of the solutions of classical Born-Infeld field theory, subject to the simplicity constraint (168) or (169), as a fluid of strings. This viewpoint is similar to that originally envisaged by Nielsen and Olesen [35].

In the case of the M5-brane, we have a three-form and one might have thought that some sort of null membrane is involved. However although the three-forms H and and $\star H$ both have a null factor neither quotient two form. that is

$$\text{neither } B_2 dx^1 \wedge dx^2 + B_1 dx^4 \wedge dx^5 \text{ nor } B_2 dx^2 \wedge dx^3 + B_1 dx^1 \wedge dx^2 \tag{174}$$

is simple. Therefore no obvious membrane world volume is picked out.

We observe that this limiting theory admits an infinite number of conserved charges that carry Lorentz indices. Presumably (c.f. [36]) the scattering of particle-like excitations is trivial in this limit. It is not obvious whether this is true for 2-branes and 5-branes since they have internal degrees of freedom associated to their world volumes.

11 Acknowledgement

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