Born-Infeld particles and Dirichlet p-branes

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Abstract

Born-Infeld theory admits finite energy point particle solutions with δ-function sources, BIons. I discuss their role in the theory of Dirichlet p-branes as the ends of strings intersecting the brane when the effects of gravity are ignored. There are also topologically non-trivial electrically neutral catenoidal solutions looking like two p-branes joined by a throat. The general solution is a non-singular deformation of the catenoid if the charge is not too large and a singular deformation of the BIon solution for charges above that limit. The intermediate solution is BPS and Coulomb-like. Performing a duality rotation we obtain monopole solutions, the BPS limit being a solution of the abelian Bogolmol'nyi equations. The situation closely resembles that of sub and super extreme black-brane solutions of the supergravity theories. I also show that certain special Lagrangian submanifolds of $\mathbb{C}^p$, $p = 3, 4, 5$, may be regarded as supersymmetric configurations consisting of p-branes at angles joined by throats which are the sources of global monopoles. Vortex solutions are also exhibited.

1 Introduction

Recently a number of outstanding problems in physics have been, and are currently being, solved using techniques involving Dirichlet p-branes (see [1] for a brief review and references). It is therefore desirable to understand as much as one can of their basic properties. In particular one is interested in the how stringy and spacetime concepts are related. At the classical level p-branes have been extensively studied as solutions of the various super-gravity limits of string/M-theory. In this paper I want to explore a different limit in which the effects of gravity are ignored but in which one still has non-trivial and hopefully physically relevant classical solutions. We shall discover that there is a rich

set of solutions whose properties resemble in a striking way some of those of the the black p-brane solutions of supergravity theory.

Classically the equations of motion are essentially a generalization of the minimal surface equations with the added feature that there is an abelian, i.e. $U(1)$, gauge field propagating on the world volume. In what follows we shall consider the case of very weak string coupling in which case one may ignore the fields having the p-brane as their source. Thus the D-p-brane moves in flat Minkowski spacetime with constant dilaton and vanishing Kalb-Ramond 3-form . It is consistent to set this gauge field to zero and indeed we then get the standard minimal submanifold equations, of which the simplest solution is a flat $p+1$-plane. It is physically clear that this is the *only* static regular solution defined as a single valued graph over $\mathbb{R}^p$ which becomes planar at large spatial distances. This is the content of the various eeven stronger 'Bernstein'type theorems in the mathematical literature on minimal submanifolds [28, 38]. In fact it is notorious that these strong theorems only hold strictly if $p < 7$ which no doubt will ultimately turn out to have a stringy/M-theoretic explanation. This does not mean to say that there are no non-trivial regular static solutions. There are. Of particular interest are topologically non-trivial generalized catenoids consisting of two parallel asymptotically flat sheets joined by a throat.

Perhaps surprizingly the planar solution remains a solution when the gauge field no-longer vanishes. In that case the gauge field is governed by the pure Born-Infeld action [4, 5]. Again it is not difficult to convince oneself that the only regular static source free solution of Born-Infeld theory which falls off at large distances is the trivial one. As we shall see there are also Bernstein type theorems on maximal hypersurfaces in Minkowski spacetime which prove this rigorously [39, 37]. However it is well known that Born-Infeld theory admits finite energy static solutions which were originally proposed as classical models for the electron. These solutions are not everywhere source free like soliton solutions but rather resemble what are sometimes called 'elementary'solutions with pointlike sources like the fundamental string solution [11, 12]. Perhaps one might call them 'BIons'.

The essential point of Born's theory is that one distinguishes between the electric field **E** which is curl free and hence one may set

$$\mathbf{E} = -\nabla\phi \tag{1}$$

and the electric induction **D** which satisfies[1]

$$\nabla \cdot \mathbf{D} = 4\pi e\delta(\mathbf{x} - \mathbf{a}) \tag{2}$$

where **a** is the location of the the source and e is its electric charge. Obviously **D** blows up at the origin but because, in the absence of magnetic fields, the

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[1] In this paper we shall be using units in which $2\pi\alpha' = 1$.

electric field and the induction are related by

$$\mathbf{E} = \frac{\mathbf{D}}{\sqrt{1 + \mathbf{D}^2}} \tag{3}$$

the electric field tends to a finite value at the source. This maximal field strength has a nice interpretation in string theory because of a divergence in the rate of production of open strings by the Schwinger process[8].

It is natural to interpret the sources of these BIionic solutions as the ends of electric flux-carrying strings lying outside the brane in the way suggested in [2]. However a point of difference with that analysis is that the strings described there do not so much end on branes but rather disappear down their throats. This is appropriate in the the case that the curved geometry generated by the brane in the supergravity limit is taken into account but as mentioned above in the limit we are considering the gravitational field of the brane is ignored. In fact we will later describe solutions with non-vanishing scalars in which the strings look like very thin tubes joining smoothly onto the p-brane.

As explained in [2] there are constraints due to charge conservation on what strings can end on what branes. These constraints arise because the interaction with the Kalb-Ramond 2-form is obtained by replacing the Maxwell 2-form $F_{\mu\nu}$ in the world volume action S_p by $\mathcal{F}_{\mu\nu} = F_{\mu\nu} - B_{\mu\nu}$ where $B_{\mu\nu}$ is the pullback to the world volume of the Kalb-Ramond 2-form. The electric charge e is given by a surface integral over any $p-1$ sphere lying in the brane and surrounding the source

$$\frac{1}{A_{p-1}} \int_{S^{p-1}} \star_p D \tag{4}$$

where A_{p-1} is the volume of a unit $p-1$ sphere and $\star_p$ is the world volume Hodge duality operator and the 2-form D has components given by the variational derivative

$$D^{\mu\nu} = -\frac{\delta S_p}{\delta F_{\mu\nu}} \tag{5}$$

and is thus the covariant form of the electric induction.

The fact that S_{p-1} contains $F_{\mu\nu}$ and $B_{\mu\nu}$ only in the combination $\mathcal{F}_{\mu\nu} = F_{\mu\nu} - B_{\mu\nu}$ means that

$$D_{\mu\nu} = J_{\mu\nu} \tag{6}$$

where

$$J^{\mu\nu} = -\frac{\delta S_p}{\delta B_{\mu\nu}} \tag{7}$$

is the distributional current 2-form with support on the world-volume which acts as a source for the Kalb-Ramond 3-form H. Thus only fundamental, or F-strings can end on $R \otimes R$ p-branes. However by $SL(2, \mathbb{Z})$ invariance, both fundamental and Dirichlet or D-strings can end on the self-dual 3-brane, the former ending on electric charges and the latter on magnetic charges. This is consistent with the duality invariance of the Born-Infeld equations of motion [35][36][30].

2 Lagrangians and Equations of motion

A Dirichlet-p-brane moving in a flat $d+1$ dimensional Minkowski spacetime $\mathbb{E}^{d,1}$ with constant dilaton and zero axion field is described by the embedding functions $z^M(x^\mu)$ and vector potential $A_\nu(x^\mu)$ as a function of the $p+1$ dimensional world-volume coordinates x^μ. Greek indices thus run from 0 to p while upper case Latin indices run from 0 to d. The equations of motion are obtained from the Dirac-Born-Infeld action[41, 4]:

$$-\int d^{p+1}x\sqrt{-\det(G_{\mu\nu}+F_{\mu\nu})} \tag{8}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\nu \tag{9}$$

is the electromagnetic field strength and

$$G_{\mu\nu}=\eta_{MN}\partial_\mu z^M\partial_\nu z^N \tag{10}$$

is the pullback of the Minkowski metric to the world volume.

The action is invariant under arbitrary diffeomorphisms of the world volume. One popular way of fixing this freedom is to adopt the so-called 'static gauge'for which the world volume coordinates are equated with the first $p+1$ spacetime coordinates, i.e.

$$z^M=x^\mu, \qquad M=0,1,\ldots,p. \tag{11}$$

Calling the remaining 'transverse 'coordinates y^m, i.e setting

$$z^M=y^m, \qquad M=p+1,\ldots d, \qquad m=p+1,\ldots d, \tag{12}$$

the Dirac-Born-Infeld action becomes

$$-\int d^{p+1}x\sqrt{-\det(\eta_{\mu\nu}+\partial_\mu y^m\partial_\nu y^m+F_{\mu\nu})}. \tag{13}$$

When speaking of a Born-Infeld Lagrangian L it is convenient to take off an additive constant so that L vanishes for zero scalar and vectorfields. Thus we define

$$L=1-\sqrt{-\det(\eta_{\mu\nu}+\partial_\mu y^m\partial_\nu y^m+F_{\mu\nu})} \tag{14}$$

It is important to realize that imposing the static gauge, or writing the equations in what mathematicians call in this context non-parametric form, may not always be possible globally. It assumes that the brane may be thought of as a single valued graph over $\mathbb{R}^p$. In particular that it is topologically trivial. It will turn out in the examples that we shall encounter that this assumption is not valid. This will be reflected in spurious singularities in some of our solutions when expressed in static gauge. Nevertheless, since it is a very convenient guage for calculations, we will frequently adopt it.

One might have thought that the transverse coordinates y^m and the vector field A_μ are inextricably coupled in that if one is non-zero then the other cannot vanish but this is not true. For example one gets a *consistent* set of solutions by setting the transverse coordinates to zero $y^m = 0$ and requiring that the vector field satisfy the Born-Infeld equations in flat $p+1$-dimensional Minkowski spacetime obtained by varying the action

$$-\int d^{p+1}x\sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})}. \tag{15}$$

The reason that this is a consistent truncation is that the action is invariant under each of the $d - p$ reflections given by

$$y^m \rightarrow -y^m. \tag{16}$$

The equations of motion for y^m therefore involve no terms of order zero, i.e. sources, and are trivially solved by $y^m = 0$.

As similar symmetry argument, (valid in any choice of world volume coordinates) shows that one may consistently set the vector fields to zero and obtain the Dirac action

$$\int d^{p+1}x\sqrt{-\det(\eta_{MN}\partial_\mu z^M \partial_\nu z^N)}. \tag{17}$$

In this case the symmetry we need is the invariance of the determinant

$$\det(G_{\mu\nu} + F_{\mu\nu}) \tag{18}$$

under transposition

$$(G_{\mu\nu} + F_{\mu\nu}) \rightarrow (G_{\mu\nu} + F_{\mu\nu})^t = (G_{\mu\nu} - F_{\mu\nu}) \tag{19}$$

which is therefore equivalent to invariance under reversal of the Maxwell field:

$$F_{\mu\nu} \rightarrow -F_{\mu\nu}. \tag{20}$$

In fact one might have seen this result from another surprizing property of the Dirac-Born-Infeld action. That is, it may be obtained from the pure Born-Infeld action in $d + 1$ dimensions

$$-\int d^{d+1}z\sqrt{-\det(\eta_{MN} + F_{MN})} \tag{21}$$

by assuming that the vector field $A_N(z^M)$ depends only on $p + 1$ coordinates. One needs the following identity for determinants

$$\left| \begin{pmatrix} N & -A^t \\ A & M \end{pmatrix} \right| = |M|\,|N + A^t M^{-1} A| = |N|\,|M + A N^{-1} A^t|, \tag{22}$$

where as before ^{t} denotes transposition. If one takes $M = \eta_{mn} = \delta_{mn}$ and $A = \partial_\mu A_n$ one finds the action

$$- \int d^{p+1}x \sqrt{-\det(\eta_{\mu\nu} + \partial_\mu A_m \partial_\nu A_m + F_{\mu\nu})}. \tag{23}$$

As is well known we may identify the $d - p$ transverse components of the vector A_n with the $d - p$ transverse coordinates y^m of the p-brane. A row expansion of the original determinant reveals that one may consistently set to zero any of the possible transverse polarizations A_m because for any given μ and m, $\partial_\mu A_m$ appears only quadratically in the determinant.

3   Pure Born-Infeld solutions

We have seen that setting the transverse coordinates to zero gives a *consistent* truncation of the Dirac-Born-infeld Lagrangian. In this section we shall discuss some of the solutions of this theory and their properties. It will become clear as we proceed that the solutions with no Born-Infeld vectors play a similar role to pure gravity solutions in supergravity theories coupled to differential forms and scalars while the pure Born-Infeld solutions are similar to solutions with no gravity or scalars. However there is one significant difference: in the supergravity gravity context, while it may in some circumstances be consistent to truncate the scalars, it is never strictly consistent to ignore gravity completely. Moreover just as in supergravity one may use solution generating transformations to pass from pure gravity solutions to solutions with non-vanishing differential forms, so in Dirac-Born-Infeld theory there are also solution generating transformations. These take one from pure scalar solutions to solutions with both scalars and vectors and they also take one from pure Born-Infeld solutions to solutions with non-vanishing scalars and vectors.

Of course one's greatest familarity is with solutions in three spatial dimensions but the discussion will not be restricted to that case.

3.1   Born-Infeld brane-waves

A striking application of the simple matrix identity is obtained by taking $p = 1$. From the point of view of the $d + 1$ dimensional field theory one is studying the scattering of two light beams in non-linear electrodynamics. By Lorentz-invariance the two beams can be taken to be moving in the positive or negative x^1 direction. One might have expected them to scatter non-trivially but one finds that this is not so. Because we must take $F_{\mu\nu} = 0$ the action becomes the standard Nambu-Goto action for a free string in $d + 1$ spacetime dimensions:

$$- \int d^2x \sqrt{-\det(\eta_{\mu\nu} + \partial_\mu y^m \partial_\nu y^m)}. \tag{24}$$

It is well known that left and right movers decouple in string theory it follows that the two beams pass through one another with at most a time delay. This fact seems to be behind some early observation [18] in four spacetime dimensions where a form of this result was obtained in the particular case of two circularly polarized beams with a single frequency. Later work in [25, 26, 27] noted this property for what is described as a scalar equation 'of Born-Infeld type'but but the authors did not seem to realize that their single component equation and its multi-component generalizations were *precisely* the equations of Born-Infeld theory in this setting.

3.2 Domain walls and the Born-Infeld equation

In the case $p = d - 1$ there is just one transverse coordinate y and the corresponding Lagrangian is

$$L = \sqrt{1 - \partial_\mu y \partial^\mu y}. \tag{25}$$

The resulting partial differential equation is sometimes called the Born-Infeld or Born-Infeld-type equation and has been extemsively discussed in the literature. In the case $d = 3$ it describes a timelike membrane in four dimensional Minkowski spacetime and as such has been studied by Bordemann and Hoppe[22].

One obvious physical interpretation of (25) is that it describe interfaces between two symmetric domains. The Born-Infeld interpretation (for example in four spacetime dimensions) is that all electromagnetic fields are independent of the third spatial coordinate and the only non- vanishing fields are

$$E_3 = \partial_t y \tag{26}$$

$$B_1 = \partial_2 y \tag{27}$$

and

$$B_2 = -\partial_1 y. \tag{28}$$

In other words a time-dependent electric field in the 3-direction induces a magnetic induction field in the perpendicular 1, 2 directions.

Bordemann and Hoppe[22] pointed (in fact for $d = 3$ but the observation has an obvious generalization) that one could regard the world sheet of the $d - 1$ brane as a timelike level set of a real valued function $u(z^A)$ on $d + 1$-dimensional Minkowski spacetime $\mathbb{E}^{d,1}$ which satisfies the manifestly $E(d, 1)$-invariant equation

$$\partial_A \left(\frac{\partial^A u}{\sqrt{\partial_B u \partial^B u}} \right) = 0. \tag{29}$$

This equation may be derived from the $E(d, 1)$ invariant Lagrangian

$$L = \sqrt{-\partial^A u \partial_A u}. \tag{30}$$

The function y is then obtained implicitly from the relation

$$u(x^\mu, y) = \text{constant}. \tag{31}$$

In fact u need not satisfy the equation everwhere, merely restricted to the level set which gives rise to the weaker condition:

$$\partial_A u \partial^A u \partial_B \partial^B u = \partial^A u \partial^B u \partial_B \partial_A u \Big|_{u=\text{constant}}. \tag{32}$$

Solutions of this condition will be used later to construct BIonic crystals. For the time being we give two simpler examples: Clearly if

$$u = -t^2 + x^2 + y^2 + z^2 \tag{33}$$

then $u = 0$ gives the light cone of the origin which is a solution of the Bordeman-Hoppe equation in $\mathbb{E}^{3,1}$. Rather surprizingly it is also true that if

$$u = t^2 + x^2 + y^2 - z^2 \tag{34}$$

then the quadratic cone $u = 0$ is also a solution in $\mathbb{E}^{3,1}$. For completeness we note that if

$$u = x^2 + x^2 + y^2 - z^2 - \tau^2 \tag{35}$$

then the quadratic cone $u = 0$ is a minimal 3-brane in Euclidean 4-space $\mathbb{E}^4$.

One may wonder whether there are functions u which solve both the Hoppe-Bordemann equation (30) and the ordinary wave equation $\partial^A \partial_A u = 0$. This question has been studied by Graustein [15] in $\mathbb{C}^3$ who found *all* simultaneous solutions in of these two equations. Some of his solutions are valid in $\mathbb{E}^{2,1}$. They include one which depends on two arbitrary functions of a lightlike variable,and which may easily generalized to $\mathbb{E}^{d,1}$ to depend on $d+$ arbitary functions of a lightlike variable. Choosing coordinates appropriately it may be given the form:

$$u = p(t - z) - xa(t - z) - yb(t - z) - \ldots = \text{constant}. \tag{36}$$

The solution (36) represents a *timelike* domain wall moving at the speed of light along the z-axis. The functions $p, a, b, \ldots$ may be arbitary.

Another of Graustein's solutions is

$$u = (z + t)^{-\frac{2}{3}} (t^2 - x^2 - z^2)^{\frac{1}{6}}. \tag{37}$$

In fact one may embed this solution in a family of solutions:

$$u = (t + z)^p (t^2 - x^2 - z^2)^q \tag{38}$$

will solve the Bordemann-Hoppe equation if

$$(p + q)(p + 4q) = 0 \tag{39}$$

and the wave equation if
$$1 + 2p + 2q = 0. \tag{40}$$

In four dimensions one may find analogous solutions:
$$u = (z+t)^p(-t^2 + x^2 + y^2 + z^2)^q \tag{41}$$

will solve the Bordemann-Hoppe equation if
$$(p+q)(p+6q) = 0 \tag{42}$$

and the wave equation if
$$p + q + 1 = 0. \tag{43}$$

Also
$$u = (z+t)^p(t^2 + x^2 + y^2 - z^2)^q \tag{44}$$

will solve the Bordemann-Hoppe equation if
$$(p+q)(p-2q) = 0 \tag{45}$$

and the wave equation if
$$p + q + 1 = 0. \tag{46}$$

The cases $p + q = 0$ are in effect translations of (33) and (34). The example (44) with $p = 2$ and $q = 1$ was noted by Hoppe ([32]).

Equation (32) is striking because it has a topological character: it is invariant under arbitrary one-dimensional reparameterizations $u \to f(u)$. Thus in solutions (41) and (44) the level sets depend only on the ratio of p to q.

The Lagrangian (30) for an interface in $\mathbb{E}^{d,1}$ may be regarded as the strong coupling limit[2] of the Born-Infeld type Lagrangian (25) for an interface in $\mathbb{E}^{d+1,1}$. This is presumably related to the fact that the strong coupling limit of the standard Born-Infeld Lagrangian picks out a topological invariant: the Pfaffian $\sqrt{\det F_{\mu\nu}}$.

3.3   Electrostatic Solutions and Maximal Hypersurfaces

The previous results on timelike (d-2)-branes in $\mathbb{E}^{d,1}$ have companions involving spacelike hypersurfaces. If one starts from the pure Born-Infeld theory in $\mathbb{E}^{d,1}$ and assumes only an electrostatic field is present so that
$$A_M = (\phi, 0, \ldots, 0) \tag{47}$$

then the action becomes:
$$\int d^d x \sqrt{1 - |\nabla\phi|^2}. \tag{48}$$

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[2]Restoring units this means that $2\pi\alpha' \to \infty$, i.e. we take the infinite slope limit in string theory.

As long as $|\nabla\phi| < 1$, which means that the electrostatic field $\mathbf{E} = -\nabla\phi$ is less than the maximal value allowed in Born-Infeld theory, then one may regard $t = \phi(\mathbf{x})$ as the height function of a spaclike maximal hypersurface in Minkowski spacetime. The surface tips over and touches the light cone precisely when the maximal field strength is attained.

We shall now attempt to capitalize on the facts that maximal hypersurfaces have received much attention in the general relativity and differential geometry literature [16, 17, 20, 24, 37, 39, 45] and Born-Infeld electrostatics was extensively studied in the 1930's [5, 21] to obtain some insight into both topics by exploiting this connection. The first obvious point is that one may use the Poincaré symmetries of the maximal hypersurface problem in $d + 1$ dimensional Minkowski spacetime to generate some useful new solutions of the electrostatic problem. Put another way, our observation allows us to extend the obvious invariance of equation under the Euclidean group $E(d)$ of isometries of d-dimensional euclidean space $\mathbb{E}^d$ to a non-obvious or 'hidden'invariance under the Poincaré group $E(d, 1) \supset E(d)$ of isometries of Minkowski spacetime $\mathbb{E}^{d,1}$. This is , of course, closely related to some observations of Bachas[9] about D-branes in string theory in a related but not identical context.

As an illustration consider a uniform electric field

$$\phi = -zE, \qquad E < 1 \tag{49}$$

where the field of constant magnitude E is taken to lie along the z-direction. This corresponds to a spacelike hyperplane and may be obtained from the even more trivial solution $\phi = 0$ by means of a Lorentz boost in the $t - z$ 2-plane with velocity

$$v = E. \tag{50}$$

As noted above the maximum field strength condition $E < 1$ arises from the maximum velocity $v < 1$.

According to the Bernstein type theorems of Calabi (valid for $n = 2, 3, 4$)[39] and of Cheng and Yau (valid for all $n > 1$) [37] this solution is the *only* one for which ϕ is a everywhere non-singular and single valued. A weak form of this result follows easily from rather elementary uniqueness theorem of Pryce [14]. Given two solutions he constructs the vector

$$\mathbf{G} = (\phi_1 - \phi_2)(\mathbf{D}_1 - \mathbf{D}_2) \tag{51}$$

and finds that

$$\nabla \cdot \mathbf{G} = u \tag{52}$$

with

$$u = (\mathbf{E}_1 - \mathbf{E}_2) \cdot \left(\frac{\partial L}{\partial \mathbf{E}_1} - \frac{\partial L}{\partial \mathbf{E}_2}\right) = (\mathbf{E}_1 - \mathbf{E}_2) \cdot (\mathbf{D}_1 - \mathbf{D}_2). \tag{53}$$

Since L is a positive stricly convex function of $\mathbf{E}$, i.e. $L' > 0, L'' > 0$, then u will be non-negative and vanish only when $\mathbf{E}_1 = \mathbf{E}_2$. Integration of this

identity over $\mathbb{R}^3$ with $\mathbf{E}_1$ being a uniform electric field and $\mathbf{E}_2$ being some other field which approaches the uniform field suficiently rapidly so as to make the boundary terms vanish yields the uniqueness result [3].

It follows from the Gauss-Coddazi equations that the induced metric

$$g_{ij} = \delta_{ij} - \partial_i \phi \partial_j \phi \tag{54}$$

of a maximal hypersurface in Minkowski spacetime has non-negative Ricci-scalar. On the other hand if the potential ϕ satisfies

$$\partial \phi = O(\frac{1}{r^2}) \tag{55}$$

at infinity the induced metric g_{ij} has zero ADM mass. Thus, by the positive mass theorem, the solutions must necessarily have singularities. Similarly, because there can be no non-flat metric with non positive Ricci scalar on the three torus T^3, there can be no non-singular triply periodic solutions. There are, as we shall see interesting solutions of both types with singularities, the singularities corresponding to electric charges.

3.4 Inclusion of a magnetic field

If one includes a magnetic field $\mathbf{B}$ the Born-Infeld Lagrangian

$$L = 1 - \sqrt{1 - \mathbf{E}^2 + \mathbf{B}^2 - (\mathbf{E} \cdot \mathbf{B})^2} \tag{56}$$

must be varied subject to the constraint that the magnetic induction is divergence free

$$\nabla \cdot \mathbf{B} = 0. \tag{57}$$

This leads to the equation of motion (in the time independent case) that the magnetic field

$$\mathbf{H} = -\frac{\partial L}{\partial \mathbf{B}} \tag{58}$$

is curl-free:

$$\nabla \times \mathbf{H} = 0. \tag{59}$$

As usual it is convenient to introduce a Lagrange multiplier χ to enforce the constraint and perform a Legendre transformation to yield an unconstrained

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[3]In fact uniqueness arguments of this kind also hold for more general non-linear electro-dynamic theories providing the associated Lagrangian L satisfies an appropriate convexity propery

variational principle in terms of the scalar field χ . Physically the scalar field χ is the magnetostatic potential

$$\mathbf{H} = -\nabla\chi. \tag{60}$$

The upshot of the Legendre tansformation is that one must vary the manifestly $SO(2)$ invariant functional

$$\int d^3x \sqrt{1 - (\nabla\phi)^2 - (\nabla\chi)^2 - (\nabla\phi)^2(\nabla\chi)^2 + (\nabla\phi \cdot \chi)^2}. \tag{61}$$

This result may also be obtained without explicitly introducing the magnetostatic potential. We can Legendre transform the Lagrangian to give

$$\tilde{H}(\mathbf{E}, \mathbf{H}) = L + \mathbf{B} \cdot \mathbf{H} \tag{62}$$

and then use the result to give the constitutive relations

$$\mathbf{D} = \frac{\partial \tilde{H}}{\partial \mathbf{E}}, \tag{63}$$

$$\mathbf{B} = -\frac{\partial \tilde{H}}{\partial \mathbf{H}}. \tag{64}$$

In this way one finds the manifestly electric-magnetic duality invariant formula[21]

$$\tilde{H} = 1 - \sqrt{1 - \mathbf{E}^2 - \mathbf{H}^2 + (\mathbf{H} \times \mathbf{E})^2}. \tag{65}$$

Now if we were considering a static 3-brane moving in $\mathbb{E}^5$ in static gauge we would extremize

$$\det(\delta_{ij} + \partial_i y^1 \partial_j y^1 + \partial_i y^2 \partial_j y^2) = 1 + (\nabla y^1)^2 + (\nabla y^2)^2 + (\nabla y^1)^2(\nabla y^2)^2 - (\nabla y^1 \cdot \nabla y^2)^2. \tag{66}$$

Therefore formally one may regard ϕ and χ as two timelike coordinates in $\mathbb{E}^{3,2}$ and one may then check that the variational principle is that for a spacelike 3-brane in static gauge. The electric-magnetic duality is therefore seen directly as a geometrical rotational symmetry in this setting.

3.5   Harmonic solutions

For the uniform electric field $\mathbf{E} = -\nabla\phi$ is a Killing vector of the euclidean group $E(3)$. The most general Killing field of $E(3)$ is a screw rotation which may be supposed with no loss of generality to be about the z-axis. This motivates checking that in fact this gives a solution of the form

$$\phi = -2J_m \arctan(\frac{y}{x}) - Ez \tag{67}$$

Geometrically this defines a helicoid lying in the timelike hyperplane spanned by (t, x, y). It ceases to be spacelike inside the cylinder given by

$$x^2 + y^2 = (2J_m)^2 \tag{68}$$

The physical interpretation of this solution if $E = 0$ is that it describes the electric field generated by a magnetic current J_m along the z-axis. Since the pure Born-Infeld theory is invariant under electric magnetic duality rotations [21, 35, 36, 30], it may be easier to imagine an electric current flowing along the z-axis generating a magnetic field. If one approaches too close to the z-axis the magnetic field $\mathbf{H}$ exceeds the maximum allowed value and the solution breaks down. Presumably an electric current J_e cannot, according to Born-Infeld theory, be contained within a wire of radius less than $2J_e$.

The solution for ϕ just presented is a harmonic function on $\mathbb{E}^3$. Moreover every level set in $\mathbb{E}^3$ is a minimal surface. By a result originally due to Hamel [42],it is unique, because the electric field is a simultaneous solution in $\mathbb{E}^3$ of the highly over constrained system:

$$\nabla \times \mathbf{E} = 0, \tag{69}$$

$$\nabla \cdot \mathbf{E} = 0, \tag{70}$$

and

$$\mathbf{E} \cdot \nabla |\mathbf{E}| = 0. \tag{71}$$

The harmonic solutions just given generalize in an obvious way to higher dimensions and different signatures.The exhustive list of solutions given by Graustein in $\mathbb{C}^3$ [15] quoted earlier in connection with the Bordemann-Hoppe equation (30) show that there are certainly other possiblities.

3.6   The BIon solution

To be specfic we restrict ourselves to three spatial dimensions. the generalization to higher dimensions being immediate. The solution is $SO(3)$ invariant:

$$\phi = f(r) = \int_r^\infty \frac{dx}{\sqrt{e^2 + x^4}}. \tag{72}$$

Near infinity

$$\phi = \frac{e}{r} + O(\frac{1}{r^5}). \tag{73}$$

Near the origin

$$\phi = \Phi - r + \frac{r^5}{10e^2} + O(r^9) \tag{74}$$

where

$$\Phi = \sqrt{e} \int_0^\infty \frac{dx}{\sqrt{1 + x^4}} \tag{75}$$

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is the electrostatic potential difference beween the origin and infinity. The electric charge e of the solution is given by

$$e = \frac{1}{4\pi} \int \mathbf{D} \cdot d\sigma = -\frac{1}{4\pi} \int (\frac{\nabla\phi}{\sqrt{1+|\nabla\phi|^2}}) \cdot d\sigma \tag{76}$$

where the integral is over any closed 2-cycle enclosing the origin.

From we observe that the solution is not smooth at the origin. The maximal hypersurface becomes tangent to the light cone through that point. Point like singularities of maximal hypersurfaces of this type have been analysed rather extensively in[23].

As noted by Pryce [14], the BIon is the unique solution with a single singularity and a fixed charge e or a fixed potential Φ. This contrasts with the behaviour of Coulomb type solutions of Yang-Mills theory which exhibit bifurcation phenomena indicative of instabilities. Physically it is clear that a consistency condition for a particle like interpretation that non-uniqueness and bifurcation phenomena be absent. It has been argued [3] that since the total energy of a single Born-Infeld particle in $\mathbb{E}^{p,1}$ scales like

$$e^{\frac{p}{p-1}} \tag{77}$$

and since

$$(e_1 + e_2)^{\frac{p}{p-1}} > e_1^{\frac{p}{p-1}} + e_2^{\frac{p}{p-1}} \tag{78}$$

then they should be unstable against fission. This might be true if they were smooth singularity free solutions with no sources. However they *do* have sources and the strength of these sources is governed by charge quantization conditions which arise from the perspective of string theory because the particles are the ends of strings as described in [2]. This means that a single particle carrying the lowest possible charge should be stable. A particle carrying a multiple of the lowest charge could however be stable against fission into particles with the lowest possible charge. These observations [3] are indicative of the fact that these Born-Infeld particles are not BPS and moreover that they repel one-another. We shall see this is indeed true rather explicitly later. We shall also be saying more about the scaling relation (77) when we discuss the virial theorem.

3.7   BIon in a uniform electric field

At least if $n \neq 2$ this solution seems to be a genuinely new one. It is most simply obtained by using the hidden Poincaré invariance. One may check that in the case $n = 2$ it coincides with a solution found by Pryce[13] using the Weirstrass representation for maximal surfaces in $\mathbb{E}^{2,1}$. The idea is to boost a BIon of charge e with velocity v. if $\phi = g(x, y, z)$ is the solution then is is given implicitly in terms of $f(r)$ by

$$\frac{g - \Phi + vz}{\sqrt{1 - v^2}} = f\left(\sqrt{x^2 + y^2 + \frac{(z + vg - v\Phi)^2}{1 - v^2}}\right) - \Phi. \tag{79}$$

At large distances

$$\phi \approx \Phi(1 - \sqrt{1 - v^2}) - vz + \frac{e\sqrt{1 - v^2}}{r(1 - v^2 \cos^2 \theta)^{\frac{1}{2}}} \tag{80}$$

while near the origin

$$\phi \approx \Phi - r + \frac{(1 - v \cos \theta)^6 r^5}{10 e^2 (1 - v^2)^3} + O(r^9) \tag{81}$$

Thus the background electric field is given by $E = v$ and the total charge of the new solution is $\frac{e}{\sqrt{1 - v^2}}$. It is clear that given any other solution with total charge q we may always append a uniform electric field by boosting it with velocity $E = v$ to arrrive at a new solution with total charge $\frac{e}{\sqrt{1 - v^2}}$.

We will discuss later the force necessary to prevent the particle accelerating.

3.8 Accelerating solutions

It would be nice to have an analogues of the various solutions representing uniformly accelerating black holes in external electric fields. One could then study the possibility, using instanton methods, of the pair-creation of BIon anti-BIon pairs by the Schwinger process just as one can do with black holes. Unfortunately no such explicit solutions are available at present. However there the solutions (41) and (44) may be relevant here since they are the level sets are taken are taken into themselves by a boost in the $z - t$ 2-plane.

3.9 BIonic Crystallography

Hoppe[31] has given a construction for quadruply periodic maximal surfaces in four dimensional Minkowski spacetime $\mathbb{E}^{3,1}$ by solving the non-linear equation (30)for the function u whose level sets are maximal by separation of variables in terms of a certain Weirstrass elliptic function $\mathfrak{p}$. This construction is very similar to the standard construction of triply periodic minimal surfaces in $\mathbb{E}^3$ using a different type of Weirstrass elliptic function [42]. He also gives a quadruply periodic timelike minimal hypersuface. Without loss of generality one may take the invariants g_2 and g_3 of the Weirstrass function to equal to 4 and 0 respectively. Thus

$$(\mathfrak{p}')^2 = 4(\mathfrak{p}^3 - \mathfrak{p}). \tag{82}$$

The period is $2\omega = \frac{1}{2\sqrt{2\pi}}(\Gamma(\frac{1}{4}))^2$ and the minimum value of $\mathfrak{p}$ is 1 and near the origin it has a double pole with residue 1.

The spacelike maximal hypersurface takes the form

$$\mathfrak{p}(x)\mathfrak{p}(y)\mathfrak{p}(z) = \mathfrak{p}(t) \tag{83}$$

and the timelike membrane solution the form

$$\mathfrak{p}(x)\mathfrak{p}(y)\mathfrak{p}(t) = \mathfrak{p}(z). \tag{84}$$

Thus the pure Born-Infeld solution is given implicitly by

$$\mathfrak{p}(x)\mathfrak{p}(y)\mathfrak{p}(z) = \mathfrak{p}(\phi) \tag{85}$$

Physically the time-independent solution correpond to an infinite lattice or crystal of Born-Infeld particles in equlibrium. This seems uncannily appropriate in view of Born's pioneering work on the the physics of crystal lattices.

Consider the solution inside the cube $\mathcal{C} \equiv \{(x, y, z) \in [0, 2\omega] \times [0, 2\omega] \times [0, 2\omega]\}$. The electrostatic potential ϕ takes the values $0 \equiv 2\omega$ on the faces of the cube and the value ω at the centre $(x, y, z) = (\omega, \omega, \omega)$. Near the centre one may expand the Weirstrass function about its minimum value to get the approximate form

$$(x - \omega)^2 + (y - \omega) + (z - \omega)^2 \approx (t - \omega)^2. \tag{86}$$

Thus the charges are located at the centres of the periodic images of the cube $\mathcal{C}$. The maximal hypersurface coincides with the light cone at these central points. Near the origin we have the approximate form:

$$\phi = xyz. \tag{87}$$

The potential changes sign as one crosses a face of cube $\mathcal{C}$ and thus the crystal is of NaCl type, every charge being surrounded by six charges of the oppposite sign, i.e. the charge at the point $((2n_x+1)\omega, (2n_y+1)\omega, (2n_z+1)\omega), (n_x, n_y, n_z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ has the sign

$$(-1)^{(n_x+n_y+n_z)}. \tag{88}$$

Note that $\mathcal{C}$ is not what crystallographers call the unit cell.

There are obviously many interesting questions one might ask about this crystal. For example what are its binding energy and is its compressibility? One could study it's electric polarizability by applying an external electric field. From our previous work it follows that the relevant solution is obtained by simply boosting the spacetime solution in the direction of the applied field. We shall calculate the binding energy in the next section using a version of the virial theorem.

The time-dependent solutions correspond, by making an electromagnetic duality transformation a solution of the induction of electric fields by magnetic field which is periodic in time as well as in two spatial directions. Both are,

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in the context of Born-Infeld theory, hitherto unknown and nicely illustrate the utility of relating the non-linear Born-Infeld equation to maximal surfaces.

For the Weirstrass functions considered here $\frac{1}{\mathfrak{p}(\tau)}$ with $\tau = it$ satisfies the same equation as $\mathfrak{p}(t)$ thus formally a quadruply periodic minimal solution in $\mathbb{E}^4$ would be given by

$$\mathfrak{p}(x)\mathfrak{p}(y)\mathfrak{p}(\tau)\mathfrak{p}(z) = 1. \tag{89}$$

However the functions considered here have least value 1 which they achieve at 0 and so the euclidean solutions merely consists of the intersecting hyperplanes $xyz\tau = 0$ and their periodic recurrences. Thus we get the standard situation of a flat 4-brane wrapped over a torus. This flat cannot be given a non-trivial electrostatic field using the boosting transformation. However Hoppe's solution can be boosted to give a solution with non-trivial scalars. One simply replaces ϕ with $\frac{\phi - vy}{\sqrt{1 - v^2}}$. The limiting BPS solutions should then correspond to triply periodic harmonic functions.

Hoppe's remarkable Born-Infeld crystal has a number of fascinating properties and provides one of the few many body solutions. Before discussing them we turn to a general discussion of multi-solutions.

4 BIon-Statics

It is well known that one can construct explicit multi-black hole solutions held apart by struts, the struts being the sites of conical singularities representing ditsributional stresses. One should be able to construct analogous multi-BIon solutions. Few are known explicitly but general existence theorems for the Dirichlet problem [45]show that a solution exists for each choice of k potentials Φ^a, $a = 1, \ldots, k$ at k prescribed positions $\mathbf{x}_a$ of the singularities. One would anticipate physically that a solution of the dual Neumann problem should also exist if one prescribed the charges q_a of the fixed singularities or indeed if one fixed any k dimensional combination of charges and potentials. Moreover Pryces's uniqueness theorem [14] implies that any two solutions with charges and potentials (q_{a1}, Φ_1^a) and (q_{a2}, Φ_2^a) for which

$$\sum (q_{a1} - q_{a2})(\Phi_1^a - \Phi_2^a) = 0 \tag{90}$$

must in fact be identical. We shall discuss in a later section the forces necessary to hold the positions of the charges fixed.

This and similar situations in involving pinned soliton equlibria may be described in terms of a canonical formalism. For example the discussion which follows holds in a wide range of non-linear elctrodynamic theories with a general Lagrangian.

The total mass is given by the integral over $\mathbb{R}^3 - \{\mathbf{x}_a\}$

$$M = \frac{1}{4\pi} \int (\mathbf{E} \cdot \mathbf{D} - L) d^3 x. \tag{91}$$

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Now let us consider varying the solution by changing the charges and changing the potentials and for the moment keeping the positions fixed.

Using the definition of $\mathbf{D}$ we get

$$dM = \frac{1}{4\pi} \int_{\mathbb{R}^3 - \{\mathbf{x}_a\}} \mathbf{E} \cdot d\mathbf{D} d^3 x \tag{92}$$

$$= -\frac{1}{4\pi} \int \nabla \phi \cdot d\mathbf{D} d^3 x. \tag{93}$$

Now one may use the divergence theorem and the fact that $d\mathbf{D}$ is divergence free to get

$$dM = \sum \Phi^a dq_a. \tag{94}$$

In a later subsection we shall obtain an integral relation for M. This resembles the situation of a system of fixed capacitors in standard linear electrostatics. This resemblance in fact extends to the extent that a "reciprocity principle" holds. Because the existence of a reciprocity principle is intimately related to a canonical or duality invariant formalism almost identical to that in classical thermodynamics it seems worth while to explore this a little further, not least because of the possible light it may throw on the subject of black hole thermodynamics though we should emphasise from the outset that we are not at this stage attempting to ascribe any intrinsic entropy to BIons.

Just as in the conventional case of capacitors one may regard the potentials as functions of the charges, i.e.

$$\Phi^a = \Phi^a(q_b) \tag{95}$$

or the charges as functions of the potentials or indeed one expects to be able to take any k combinations of charges and potentials as determining the other k variables. In other words the solution set is some k-dimensional submanifold $\mathcal{L}$ of the flat state space $\mathcal{P} \equiv \mathbb{R}^{2k}$ with coordinates (q_a, Φ^a). Note that because our problem is non-linear and the principle of superposition does not hold, the solutions set will not be a k-dimensional hyperplane as it is in the standard case of linear electrostatics.

From the formula for the mass it is clear that we may think of the state space as the cotangent space $\mathcal{P} = V \times V^\star$ where V is the vector space of the extensive charge variables with coordinates q_a and $V^\star$ is its dual space with intensive potential variables Φ^a. The state space $\mathcal{P}$ comes equipped with the symplectic 2-form

$$\omega = \sum dq_a \wedge d\Phi^a. \tag{96}$$

Geometrically the reciprocity relation, which we will establish shortly, amounts to the assertion that the solution set $\mathcal{L}$ is a Lagrangian submanifold of the symplectic state space $\mathcal{P}$. It is equivalent to the Maxwell relation

$$\frac{\partial \Phi^a}{\partial q_b} = \frac{\partial \Phi^b}{\partial q_a}. \tag{97}$$

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To prove the reciprocity property we subject a solution to two independent variations δ_1 and δ_2 and integrate over all of $\mathbb{R}^3$, using the divergence theorem, the quantity

$$h = \delta_1 \mathbf{E} \cdot \delta_2 \mathbf{D} - \delta_2 \mathbf{E} \cdot \delta_1 \mathbf{D} \tag{98}$$

to obtain the formula

$$\frac{1}{4\pi} \int_{\mathbb{R}^3} h d^3 x = \sum \delta_1 q_a \delta_2 \Phi^a - \sum \delta_2 q_a \delta_1 \Phi^a. \tag{99}$$

However

$$h = \frac{\partial^2 L}{\partial E_i \partial E_j} \left(\delta_1 E_i \delta_2 E_j - \delta_2 E_i \delta_1 E_j \right) = 0. \tag{100}$$

This establishes the reciprocity.

4.1 Forces and the stress tensor

We have discussed above static solutions with one or more charged particles, possibly in a background uniform electric field. The question naturally arises why don't the particles accelerate under the influence of the mutual forces? The reason is that they are pinned to their fixed position $\mathbf{x}_a$ by external forces. The existence of this external force shows up in the expansion of the solution near the origin Comparing the expression for the field in the presence of an electric field with those without an external electric field we note that the expansions agree to lowest order, i.e. the term $-r$ is the same, but the next term of order r^5 differs. For the static single BIon solution without an applied field the order r^5 term is isotropic while for the static BIon solution with an applied elctric field the order r^5 term is anisotropic.

The interpretation just given is easily confirmed using the conserved and symmetric stress tensor T_{ij}. This is given by

$$T_{ij} = \delta_{ij}(L + \mathbf{B} \cdot \mathbf{H}) - E_i D_j - H_i B_j = T_{ji}. \tag{101}$$

By virtue of the static field equation the stress tensor is conserved

$$\partial_i T_{ij} = 0. \tag{102}$$

Because we are working in flat Euclidean space $\mathbb{E}^3$ the total force F_i acting on a 2-surface S with surface element $d\sigma_i$ is well defined and given by

$$F_i = \int_S T_{ij} d\sigma_j. \tag{103}$$

If one chose a local cartesian frame whose third leg is aligned with the direction of the gradient of the potential function ϕ one finds that the stress tensor is diagonal and has components

$$T_{11} = L \tag{104}$$

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$$T_{22} = L \tag{105}$$

and

$$T_{33} = -2L'\mathbf{E}^2 + L. \tag{106}$$

Thus the field exerts a *pressure* $P = L$ orthogonal to the field lines, i.e the direction of the gradient of ϕ. then the field exerts a *tension* $-2L'\mathbf{E}^2 + L$ along the direction of the field lines[4]. Of course in the case of weak fields the magnitude of the pressure and the tension are equal.

Note that the tension is numerically equal to the Hamiltonian function $H(\mathbf{D})$. Thus positivity follows from the concavity of the Lagrangian function $L(\mathbf{E})$ and the properties of the Legendre tansformation.

If $S = \partial\Omega$ is the boundary of a compact domain $\Omega \subset \mathbb{E}^n$ one interprets F_i as the total force exerted on the material inside the domain Ω by forces external to Ω. If the fields are everywhere non-singular inside Ω then the divergence theorem implies that the total force on $S = \partial\Omega$ must vanish. Conversely if the total force F_i^S on a closed surface S is non-zero then it must contain one or more singularities. Since the value of the force depends only on the homology class of S one may evaluate the force F_i^a on the a'th singularity, assumed pointlike and finite in number, inside Ω by considering a sphere of small radius surounding it. One then has the identity

$$\mathbf{F}^S = \sum \mathbf{F}^a. \tag{107}$$

If S is taken to be a large sphere at infinity and

$$\phi \approx -\mathbf{E}.\mathbf{x} + \frac{q^{\text{total}}}{r} + \tag{108}$$

one may evaulate the force and find that

$$\mathbf{F}^S = \mathbf{F}^{\text{total}} = q^{\text{total}}\mathbf{E}, \tag{109}$$

where, of course,

$$q^{\text{total}} = \sum q^a. \tag{110}$$

One may also evaluate the force on the singularity by considering a small sphere about the origin. One may check explicitly that they agree. In fact

$$F_i = \frac{ev}{\sqrt{1-v^2}}\delta_{iz} \tag{111}$$

but, as we stated earlier, the charge is $q = \frac{e}{\sqrt{1-v^2}}$ and the electric field is $E = v$ whence

$$F_i = qE\delta_{iz} \tag{112}$$

as expected.

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[4]As long as the Lagrangian function $L(x)$ satsifies $2xL'(x) - L > 0$ this will continue to hold in more general non-linear electrostatics

More generally of the field ϕ has a singularity of the form

$$\phi = \text{const} - r + \frac{er^5}{10} + O(r^9) \qquad (113)$$

,where in general g will be angle dependent, then using polar coordinates (r, θ, ϕ) one finds that the coponents of the force are given by

$$F_1 = -\frac{1}{4\pi} \int \int g^{-\frac{1}{2}} \sin\theta \cos\phi \sin\theta d\theta d\phi \qquad (114)$$

$$F_2 = -\frac{1}{4\pi} \int \int g^{-\frac{1}{2}} \sin\theta \sin\phi \sin\theta d\theta d\phi \qquad (115)$$

$$F_1 = -\frac{1}{4\pi} \int \int g^{-\frac{1}{2}} \cos\theta \sin\theta d\theta s\phi. \qquad (116)$$

If a singularity has a vanishing right hand side it experiences no external force. We call such singularities 'free'. Our usual experience with linear electrodymanics encourages the expectation that in non-linear electrodynamics, at least if $2xL'(x) - L > 0$, then like charges should always repel and unlike charges should always attract. Thus one does not expect to be able to construct in such theories a solution representing k free charges all of the same sign, or a solution with two free charges of the opposite sign. We have seen however that general theorems guarantee the existence of solutions with charges fixed at arbitrary positions and with no external forces. It would be interesting therefore to calculate the forces between the charges for these solutions as a function of separation. It is not difficult to se that at large separations the forces are the standard inverse square ones but unfortunately, exept in the two-dimensional case studied by Pryce [13], no explicit solutions are as yet availble to allow one to calculate the forces at close separation.

One thing that is easily done is to rule out the existence of certain very symmetrical solutions with free charges. Consider for example two equal and opposite charges at a fixed non-vanishing separation. Since there is no external force coming from infinity, the force between them could be calculated by taking the surface S to be the plane perpendicular to the line joining the charges and passing through its mid point. But by symmetry the field lines are every where orthogonal to this plane which is thus a non-compact level set of the function ϕ. Therefore the total force acting is a tension tending to attract the charges in the the direction of the line and given by the non vanishing integral

$$\int_{\mathbb{R}^2} d^2x 2L' \mathbf{E}^2 - L. \qquad (117)$$

If on the other hand the charges have the same sign then symmetry dictates that the field lines lie in the plane. In fact the plane is a degenerate or limiting case of a flux-tube. A flux tube is by definition a surface of topology of a cylinder

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or a cone containing field lines and thus enclosing a fixed amount of flux of the electric induction vector $\mathbf{D}$. For a single isolated charge the flux tubes are right circular cones whose vertices located at position of the charge. In the theories we are considering, flux tubes always experience a pressure along their normal just as they do in the linear theory.

In any event the total force acting on the plane is a pressure tending to repel the charges in the the direction of the line and given by the non vanishing integral

$$-\int_{\mathbb{R}^2} d^2x L. \tag{118}$$

The relative magnitudes of these two forces for the same separations and absolute magnitudes of the charges is not clear but one might expect because the forces should depend on the strength of the electric fields $\mathbf{E}$ that, at least in The Born-Infeld case, the repulsion, which tends to give rise to stronger electric induction fields $\mathbf{D}$ and hence weaker electric fields $\mathbf{E}$ compared with the linear case is greater than the attraction whose effect is in the oppposite sense. However this intuitive argument is not very conclusive.

The arguments just given, ruling out free charges of equal absolute magnitudes, could conceivably be extended to the case when the absolute magnitudes of the charges are no longer equal. One needs to be able to control the shape of the iso-potentials or the directions of the field lines or the existence or and shape of flux tubes. For example one might be able to replace the separating plane in the case of opposite charges by a non-compact level set of the function ϕ separating the charges. This level set is acted upon by an everywhere by a tension in the direction of its normal. Thus if the normal always has a positive projection in some direction we would be done. Similar remarks might apply to the case of charges of the same sign if if one could control the properties of the limiting flux tube. Alternatively one might be able to make use of suitably chosen planes if one could control the direction of the field lines on it.

4.2 The Virial theorem and the Madelung constant for Bionic cryatals

If we include a change in the positions of the points the formula for the variation in the mass becomes:

$$dM = \sum \Phi^a dq_a + \mathbf{F}^a \cdot d\mathbf{x}_a. \tag{119}$$

We could obtain this by a detailed variational calculation using the fact that the canonical stress tensor is related to a variation of the energry density with respect to postion, but in the presnt case it is easier to derive it by elementary dimensional analysis. We start by deriving the virial theorem, which is essential an integrated version of this identity by integrating over $\mathbb{R}^3$ the identity

$$\left(T_{ij}x_k\right)_{,i} = T_{jk} \tag{120}$$

Contraction over ij and use of boundary conditions, the formula for the trace

$$T_{ii} = \mathbf{E} \cdot \mathbf{D} - 3L \tag{121}$$

and the formula for the mass gives the Smarr-type relation,

$$M = \frac{1}{3} \sum \mathbf{F}^a \cdot \mathbf{x}_a + \frac{2}{3} \sum q_a \Phi^a. \tag{122}$$

This formula, for a single spherically symmetric charge, with no applied force was known to Born[5]. It is interesting to compare it with the Smarr formulae one obtains for black holes. This is also a consequence of a virial type theorem.

Note that if we were working in p spatial dimensions the variational formula (119) remains true but the virial theorem (122) would be:

$$M = \frac{1}{p} \sum \mathbf{F}^a \cdot \mathbf{x}_a + \frac{p-1}{p} \sum q_a \Phi^a. \tag{123}$$

Now if $\phi(\mathbf{x}; \mathbf{x}_a)$ is a solution of the static Born-Infeld equations with singularities at the points $\{\mathbf{x}_a\}$ having total mass M, charges q_a, potentials Φ^a and forces F^a then $\lambda^{-1}(\lambda\mathbf{x}; \lambda\mathbf{x}_a)$ will also be a solution with singularities at $\lambda\mathbf{x}_a$ total mass $\lambda^{-p}M$, charges $\lambda^{-(p-1)}q_a$, potentials $\lambda^{-1}\Phi^a$ and forces $\lambda^{-(p-1)}\mathbf{F}^a$. The passage between (119) and (123) is a trivial application of Euler's theorem. Note that the scaling relation (77) used by [3] in their discussion of fission is a special case of this general scaling relation.

For a crystal the integral over $\mathbb{R}^3$ will not converge but one can integrate over the cube $\mathcal{C} = [0, 2\omega] \times [0, 2\omega] \times [0, 2\omega]$. In the case of the Hoppe solution the potential vanishes on the walls of the cube and takes the value ω at the single charge located at its centre. It is clear by symmetry that that the force on the charge vanishes and therefore the energy per unit volume is

$$\frac{E}{V} = \frac{1}{8\omega^3} \frac{2}{3} \omega e, \tag{124}$$

where e is the value of the electric charge. Note that if we scale the solution by a fcator of λ the size of the unit cell, the electrostatic potential and the energy will all scale but the ratio $\frac{E}{V}$ is unchanged. In other words if we restore units the energy density is a multiple of $(2\pi\alpha')^{-2}$.

The charge may be found by expanding the elliptic functions about the point (ω, ω, ω). Now

$$t - \omega = \frac{1}{2} \int_0^{\mathfrak{p}(t)} \frac{dh}{\sqrt{h(h^2 - 1)}}, \tag{125}$$

and the half-period is given by

$$\omega = \frac{1}{2} \int_0^{\infty} \frac{dh}{\sqrt{h(h^2 - 1)}}. \tag{126}$$

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As a check one sees that near $t = 2\omega$

$$\mathfrak{p} \approx \frac{1}{(t - 2\omega)^2} \qquad (127)$$

as expected.

Expanding the integral gives:

$$
\begin{aligned}
t &= \pm\sqrt{\frac{(\mathfrak{p}(\omega + t) - 1)}{2}}\left(1 - \frac{\mathfrak{p}(\omega + t) - 1}{4}\right. \\
&+ \frac{19}{160}(\mathfrak{p}(\omega + t) - 1)^2 - \left.\frac{9}{128}(\mathfrak{p}(\omega + t) - 1)^3\right) \\
&+ O((\mathfrak{p}(\omega + t) - 1)^{\frac{7}{2}}),
\end{aligned}
\qquad (128)
$$

and reversion of this series gives:

$$\mathfrak{p}(\omega + t) = 1 + 2t^2 + 2t^4 + \frac{8}{5}t^6 + +\frac{6}{5}t^8 + \frac{64}{75}t^10 + O(t^{10}). \qquad (129)$$

Near (ω, ω, ω) we therefore have:

$$
\begin{aligned}
&\mathfrak{p}(\omega + x)\mathfrak{p}(\omega + y)\mathfrak{p}(\omega + z) - 1 = \\
&2r^2 + 2r^4 + 8x^2y^2z^2 + \frac{8}{5}(x^6 + y^6 + z^6) \\
&+4(x^4y^2 + x^4z^2 + y^4z^2 + y^4x^2 + z^4x^2 + z^4y^2) + O(r^8).
\end{aligned}
\qquad (130)
$$

Thus, finally,

$$t = \omega - r + \frac{(x^2 + y^2)(y^2 + z^2)(z^2 + x^2)}{5r} + O(r^9), \qquad (131)$$

where we have resolved the ambiguity in the square root by considering taking the charge to be positive. Note that there is no $O(r^7)$ term as expected on general grounds and that the symmetry of the $O(r^5)$ term implies that the force does indeed vanish.

The charge is given by the spherical average of $\frac{1}{\sqrt{2(x^2+y^2)(y^2+z^2)(z^2+x^2)}}$. One could compare the self-energy with the Madelung constant for an NaCl structure. Formally the potential energy of the positive charge in the field of the other charges is

$$
\begin{aligned}
&-\frac{e^2}{2\omega}\left(6 - \frac{12}{\sqrt{2}} + \frac{8}{\sqrt{3}} \cdots\right) \\
&\approx -1.748\frac{e^2}{2\omega}.
\end{aligned}
\qquad (132)
$$

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Having demonstrated how to calculate crystal energies in Born-Infeld theory we will postpone to another occasion the detailed numerical comparison. [5]

5   Catenoidal solutions

As mentioned earlier there are some non-trivial static solutions with vanishing gauge fields. The equations of motion on $\mathbb{E}^p$ are

$$\nabla \cdot \frac{\nabla y}{\sqrt{1+|\nabla y|^2}} = 0. \tag{133}$$

There is obviously an $SO(p)$-invariant solution such that

$$\frac{\nabla y}{\sqrt{1+|\nabla y|^2}} = \frac{\mathbf{x}}{r^p}. \tag{134}$$

but unlike the electrostatic solution it is not defined everywhere outside the origin. Since

$$\partial_r y = \sqrt{(r^{2p-2}-1)} \tag{135}$$

Thus y has a square root singularity at

$$r = 1. \tag{136}$$

Therefore one should regard y as double valued and branched over the $p-1$-sphere at this critical radius. Geometrically this means smoothly joining the original incomplete minimal surface to another identical one to obtain a smooth catenoidal solution. The critical $p-1$ sphere then becomes a minimal hypersurface lying in the catenoid. If $p > 2$ y tends to a finite limit as the radius tends to infinity and so such a catenoid looks like two parallel asymptotically planar surfaces a finite distance apart joined by a throat. The case of familar experience, $p = 2$, is exceptional in that y does not tend to a finite limit as the radius tends to infinity. Thus if $p = 2$ the two ends are infinitle far apart and never really become planar. We shall give a global emebedding of the catenoid in the next subsection. These catenoids are strikingly similar to the Einstein-Rosen bridges, i.e. the surfaces of constant time, that one encounters in classical super-gravity solutions representing black holes or black p-branes. We shall see shortly that this analogy goes even deeper.

It is natural to ask whether static multi-catenoidal solutions exist. One might doubt it because the scalar field y should be responsible for a long range

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[5] Strictly speaking the series in brackets, $\sum \frac{(-1)^{n_x+n_y+n_z}}{\sqrt{n_x^2+n_y^2+n_z^2}}$ $(n_x, n_y, n_z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \setminus (0,0,0)$, is not convergent. One way [46] of regularizing is to define $d_3(2s) = \sum \frac{(-1)^{n_x+n_y+n_z}}{(n_x^2+n_y^2+n_z^2)^s}$ $(n_x, n_y, n_z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \setminus (0,0,0)$, for $\mathrm{Re}s > \frac{3}{2}$ and analytically continue. One gets the same answer.

attraction bewtween catenoids. The easiest way to check this is again to calculate the stress tensor. For a pure scalar solution one has

$$T_{ij} = \delta_{ij} L + \frac{1}{1-L} \partial_i y \partial_j y. \tag{137}$$

Thus if one had a symmetrical configuration which is symmetrical about the plane $x^3 = 0$, then one would have $\partial_3 y = 0$ there. It follows that T_{33} would be negative. This means, as expected. that there is a net attractive force and thus equilibrium between two symmetrical catenoids should not possible. However this argument cannot be extended in a straightforward fashion to more complicated configurations with a finite number of throats and in the light of the discoveries of recent years of many (unstable) minimal surfaces in $\mathbb{E}^3$ [40, 42] caution is advisable.

One can be less cautious about periodic arrays. The solutions exhibited by Hoppe et al. definitely show that such solutions exist.

5.1 Cosmological and Wormhole solutions

Since the catenoid and the BIon solution play such an important role in the theory we shall conclude this section with a few additional remarks about their geometry and that of related solutions. As was remarked earlier, the flat $p+1$-plane corresponds to the trivial Minkowski vacuum state from the point of view of physics on the brane. There is in fact a braney cosmological solution given (in the case of p=3) in [29] [6]. The metric induced on the world volume corresponds to an F-L-R-W metric

$$ds^2 = -dt^2 + a^2(t) d\Omega^2_{p,k} \tag{138}$$

where $\Omega^2_{p,k}$ is the metric on a unit p-sphere if $k = 1$, the euclidean metric if $p = 0$ and the metric on hyperbolic space if $k = -1$. If $k = 1$ the embedding is given by

$$Z^i = a(t) \cos \chi n^i \tag{139}$$

$$Z^{p+1} = a(t) \sin \chi \tag{140}$$

$$Z^0 = \int \sqrt{1 - \dot{a}^2} dt. \tag{141}$$

where n^i is a unit vector in $\mathbb{R}^p$. The equations of motion reduce to a Friedmann type equation

$$\dot{a}^2 + 1 = \frac{1}{a^{2p}}. \tag{142}$$

where a convenient choice of the arbitary length scale has been made.

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[6]Note that, in contrast to the considerations of [29], since there is no gravity on the brane there is no question of applying Einstein's equations on the brane in the present context

The big-bang and big crunch singularities where $a(t) \to 0$ corresponds to the Ecker-type of singularities[23] where p-brane world sheet becomes tangent to the light cone through the origin of $\mathbb{E}^{p+1,1}$.

The catenoid solution is obtained by analytic continuation setting $t = i\tau$ and $Z^0 = iy$ where τ and y are real. The static gauge consists of using the coordinates Z^i, Z^{p+1} as coordinates and thus a is to be identified with the radial coordinate used earlier. The analytically continued Friedman equation then becomes

$$-a'^2 + 1 = \frac{1}{a^{2p}} \tag{143}$$

where $'$ denotes differentiation with respect to τ. If one solves this one finds that $a(\tau)$ is defined for all τ. It is symmetrical about its unique minimum value and tends to $|\tau|$ as $|\tau| \to \infty$. Thus solving for y as a function of r gives two valuess if r is larger than a critical value and none if it is smaller. This is precisely the behaviour we found earlier and it is a simple matter to check that the explicit formula for $y(r)$ obtained from the Friedmann equation is the same as that obtained earlier.

One may also obtain the Bion solution as a spacelike maximal surface in this way. We set

$$Z^0 = \int \sqrt{-1 + \dot{a}^2} \, dt. \tag{144}$$

and the Friedmann equation becomes

$$\dot{a}^2 - 1 = \frac{1}{a^{2p}}. \tag{145}$$

As before one identifies Z^0 with the electrostatic potential ϕ.

5.2 Energetics of catenoids

One may define the total energy M of the static catenoid solution with respect to one of its sheets by

$$M = \int T_{00} d^p x = \int (\sqrt{1 + |\nabla y|^2} - 1) d^p x, \tag{146}$$

where the integral is taken over the region outside the throat. The equation of motion implies that one may associate with the solution a 'scalar charge'

$$\Sigma = -\int \frac{\nabla y}{\sqrt{1 + |\nabla y|^2}} \cdot d\sigma \tag{147}$$

where the integral may be taken over any S^{p-1} surrounding the throat. Multiplying the equation of motion by y and integrating over the region outside the throat gives

$$\int \frac{|\nabla y|^2}{\sqrt{1 + |\nabla y|^2}} d^p x = Y\Sigma, \tag{148}$$

where Y is the difference between the value of y at the throat and its value at infinity. Thus the distance between the two asymptotically flat regions is $2Y$. We have chosen conventions so that both Σ and y may be taken to be positive.

Now the virial theorem is more subtle because the boundary term at the throat must be taken into account. One has

$$T_{ii} = \frac{|\nabla y|^2}{\sqrt{1 + |\nabla y|^2}} + p(1 - \sqrt{1 + |\nabla y|^2}). \tag{149}$$

One has

$$\int r T_{rr} d\sigma_r = Y\Sigma Y - pM. \tag{150}$$

The boundary term works out to be $-pV$[7] where V is the excluded volume and therefore

$$pM = Y\Sigma + pV. \tag{151}$$

As before we may consider a family of solutions of the form $\lambda^{-1}y(\lambda\mathbf{x})$ for which the mass and volume scale together like λ^{-p}, Y like λ^{-1} and Σ like $\lambda^{-(p-1)}$. The variational formula becomes

$$dM = dV + \frac{1}{p}\Sigma dY. \tag{152}$$

If one thinks of $2M$ as the total energy of the catenoid and we recall that $2Y$ is the distance bewteen the two ends one might be tempted to regard $\frac{1}{p}\Sigma$ as the tension in the throat.

6 Born-Infeld-Electrostatics with extra scalars

In this section we shall consider solutions when both the scalar fields and the electric field are non-zero. The Lagrangian is

$$-\sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu} + \partial_\mu y^n \partial_\nu y^n)}. \tag{153}$$

If we make the electrostatic ansatz we need to evaluate

$$\left| \begin{pmatrix} -1 & -\partial_i \phi \\ \partial_j \phi & \delta_{ij} + \partial_i y^n \partial_j y^n \end{pmatrix} \right|. \tag{154}$$

On use of the matrix identity one finds that one needs

$$\sqrt{\det(\delta_{ij} - \partial_i \phi \partial_j \phi + \partial_i y^n \partial_j y^n)}. \tag{155}$$

Thus one is concerned with a spacelike p dimensional maximal hypersurace in $d + 1$ dimensional Minkowski spactime $\mathbb{E}^{d,1}$. Of course if $\phi = 0$ we get minimal

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[7] the reader is cautioned against confusing p as in p-brane with P as in pressure.

p-surface in $\mathbb{E}^d$. The simplest case in which one can evaluate the determinant is when $d = p + 2$ and so there is just one scalar y. The result is that one must extremize

$$\int d^p x \sqrt{1 + |\nabla y|^2 - |\nabla\phi|^2 + (\nabla y \cdot \nabla\phi)^2 - |\nabla\phi|^2|\nabla y|^2}. \tag{156}$$

Note that in accord with our general results one may indeed consistently set $y = 0$ or $\phi = 0$.

6.1 Charged Catenoids

We have seen that the Dirac-Born-Infeld equations have pointlike solutions with vanishing scalar and with finite energy which exhibit the expected attractions and repulsions. We also have the pure scalar catenoidal solutions with no electromagnetic field which are attractive. None of these solutions can therefore not be expected to be supersymmetric. However as we shall see there is a simple ansatz which does give rise to multi-centred solutions depending on an arbitary harmonic function.

The static equations of motion require the vanishing of

$$\nabla \cdot \frac{-\nabla\phi + \nabla y(\nabla y \cdot \nabla\phi) - \nabla\phi(\nabla y)^2}{\sqrt{1 - (\nabla\phi)^2 + (\nabla y)^2 + (\nabla y \cdot \nabla\phi)^2 - (\nabla y)^2(\nabla\phi)^2}} = 0 \tag{157}$$

and

$$\nabla \cdot \frac{\nabla y + \nabla\phi(\nabla y \cdot \nabla\phi) - \nabla y(\nabla\phi)^2}{\sqrt{1 - (\nabla\phi)^2 + (\nabla y)^2 + (\nabla y \cdot \nabla\phi)^2 - (\nabla y)^2(\nabla\phi)^2}} = 0. \tag{158}$$

Note that the electric induction $\mathbf{D}$ gets a contribution from the scalar field

$$\mathbf{D} = \frac{-\nabla\phi + \nabla y(\nabla y \cdot \nabla\phi) - \nabla\phi(\nabla y)^2}{\sqrt{1 - (\nabla\phi)^2 + (\nabla y)^2 + (\nabla y \cdot \nabla\phi)^2 - (\nabla y)^2(\nabla\phi)^2}} \tag{159}$$

The manifest invariance under boosting in the $\phi - y$ variables gives rise to a solution transformation which is completely analogous to the well known Harrison transformations in Black Hole theory which allow one to pass from neutral black hole to a charged black hole by a boost. Specifically if $(\phi_0, 0)$ is any pure Born-Infeld electrostatic field with no scalar y then $(\frac{1}{\sqrt{1-v^2}}\phi, \frac{1}{\sqrt{1-v^2}}\phi)$ are also solutions of the Dirac-Born-Infeld equations of motion. If ϕ has finite energy then so will the new solution. Obviously we can apply the same procedure to a solution $(0, y)$ with scalars but no vectors. $(\frac{1}{\sqrt{1-v^2}}y, \frac{v}{\sqrt{1-v^2}}y)$ will also be a solution.

Geometrically (regarding ϕ as an electrostatic potential rather than a transverse coordinate) one has a non-singular source-free deformation of the catenoid solution in which the distance between the two asymptotic p-planes in increased

by a factor $\frac{1}{\sqrt{1-v^2}}$. More significantly the catenoid is now electrically charged. This is only compatible with the absence of singularities or sources because the catenoid solution is topologically non-trivial. The electric field lines can thread through the minimal $p-1$ cycle separating the two asymptotic regions. The throat or tunnel clearly becomes more and more narrow and string-like as one approaches extremality. This is strikingly reminiscent of the behaviour of sub-extreme black holes and black branes. The solutions obtained by boosting the electrical BIon solution is of course analogous to the behaviour of super-extreme black holes and black branes.

7 Electric BPS solutions

The natural question is what happens in the limit as $v \to 1$? The answer is that the non-linear equations linearize and one finds that one can solve the equations my making the ansatz that

$$\phi = \pm y = H \tag{160}$$

and then discover that H may be an arbitrary harmonic function on $\mathbb{E}^p$. Choosing H to be a sum of simple poles gives Coulomb like solutions representing point particles with infinite total energy[8]. Note that if ϕ is interpreted as an extra timelike coordinate then the resulting maximal surface in $\mathbb{E}^{p,1}$ is a null hypersurface with lightlike normal $\frac{\partial}{\partial \phi} \pm \frac{\partial}{\partial y}$.

If one regards ϕ as an electrostatic potential rather than a transverse coordinate the distance beween the two asymptotic p-planes of the deformed catenoid tends to infinity, since y tends to infinity at the origin.

The supersymmetry transformations of 10-dimensional Yang Mills theory are of the form [3]

$$\delta \lambda = (\gamma^{\mu\nu} F_{\mu\nu} + \gamma^{\mu m} \partial_\mu y^m)\epsilon. \tag{161}$$

It is easy to see that one picks ϵ such that $(\gamma^0 \pm \gamma^y)\epsilon = 0$ then $\delta\lambda = 0$ if the lightlike condition $\phi = \pm y$ is satisfied. Thus our Coulomb solutions are indeed BPS for all p. A single BPS solutions may be thought of as the limit of two p-branes as the separation tends to infinity.

7.1 Inclusion of a magnetic field

An evaluation of the relevant determinant yields the Lagrangian

$$L = 1 - \sqrt{1 + |\nabla y|^2 - |\nabla\phi|^2 + (\nabla y \cdot \nabla\phi)^2 - |\nabla\phi|^2|\nabla y|^2 + \mathbf{B}^2 - (\mathbf{B} \cdot \nabla\phi)^2 + (\mathbf{B} \cdot \nabla y))^2}. \tag{162}$$

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[8]The spherically symmetric solution was found first in [3].

Note that despite its greater complexity the Lagrangian remains invariant under the global action of boosts, i.e. of $SO(1,1)$ on the fields ϕ and y provided $\mathbf{B}$ is not transformed. Moreover we know that the equations are invaraint under the $SO(2)$ duality symmetry. Thus one might anticipate that the combined equations have an $SO(2,1)$ symmetry. As we shall see this does indeed turn out to be the case.

As always, the Lagrangian has to be varied subject to the constraint that the magnetic induction is divergence free

$$\nabla \cdot \mathbf{B} = 0, \tag{163}$$

and leads to the equation of motion that the magnetic field

$$\mathbf{H} = -\frac{\partial L}{\partial \mathbf{B}} \tag{164}$$

is curl-free:

$$\nabla \times \mathbf{H} = 0. \tag{165}$$

As in the case with no scalar, it is convenient to perform Legendre transformation. After some algebra one finds the expression

$$\tilde{H} = 1 - \Big(1 - (1 - \mathbf{E}^2)(1 - \mathbf{H}^2)(1 + (\nabla y)^2) + (1 - \mathbf{E}^2)(\nabla y \cdot \mathbf{H})^2$$
$$-(1 + (\nabla y)^2)(\mathbf{E} \cdot \mathbf{H})^2 + (1 - \mathbf{H}^2)(\mathbf{E} \cdot \nabla y)^2 + 2(\mathbf{E} \cdot \nabla y)(\mathbf{E} \cdot \mathbf{H})(\mathbf{H} \cdot \nabla y) \Big)^{\frac{1}{2}}. \tag{166}$$

Expressed in terms of the electrostatic and magnetostatic potentials and the scalar $\Phi^S = (y, \phi, \chi)$ and three-dimensional Minkowski metric η^{RS} with the convention that $\eta^{yy} = 1$, it takes the manifestly $SO(2,1)$ -invariant form

$$1 - \sqrt{\det\Big(\nabla \Phi^R \cdot \nabla \Phi^S - \eta^{RS} \Big)}. \tag{167}$$

Using the general matrix identity

$$\det(\mathbb{I}_n + AB) = \det(\mathbb{I}_m + BA) \tag{168}$$

for an $n \times m$ matrix A and an $m \times n$ matrix B one may show that

$$\det\big(g_{\mu\nu} + \partial_\mu y^m G_{mn} \partial_\nu y^n\big) = \det g_{\mu\nu} \det G_{pq} \det(G^{mn} + \partial_\mu y^m g^{\mu\nu} \partial_\nu y^n). \tag{169}$$

It follows that (y, ϕ, χ) extremize the Dirac action in static gauge for a static 3-brane in $\mathbb{E}^{5,2}$. In other words the electric and magnetic potentials may be thought of as extra timelike coordinates.

7.2 Magnetic Monopoles

If $p = 3$ one may find another supersymmetric set of BPS solutions by duality rotations. The simplest case correponds to solutions of the abelian Bogomol'nyi solutions for which

$$\mathbf{B} = \pm\nabla y, \qquad (170)$$

or in terms of the magnetic potential

$$\chi = \pm y. \qquad (171)$$

The transverse coordinate y now plays the role of a Higgs field.

These solutions may be thought of as infinitely separatated branes since the coordinate y tends to infinity at the centre. It is expected [10] that one the non-abelian $SU(2)$ theory results If one passes to opposite limit of two coincident p-branes. The intermediate case is believed to correspond to a spontanously broken $SU(2)$ Yang-Mills theory with the Higgs field in the adjoint representation. It is natural to identify the 3 component of the Higgs with the scalar y. The mass of vector bosons correponding to $W_\mu^\pm = \frac{1}{\sqrt{2}}\left(A_\mu^1 \pm iA_\mu^2\right)$ is proportional to the separation of the two branes. As the separation tends to infinity we should therefore recover the abelian theory and this is consistent with what we have found. It also strongly suggests that interesting Bogolmolnyi type solutions exist in non-abelian Yang-Mills theory (see [19]).

Acting with $SO(2,1)$ one can obtain the general Dyonic BPS solution which corresponds to the light cone in $\Phi^R = (\phi, \chi, y)$ space. The mysterious phase in the moduli space of non-abelian BPS monopoles is just the circle of light-like directions.

7.3 Bogolmol'nyi Considerations

In the energy density is

$$T_{00} = \mathbf{E} \cdot \mathbf{D} - L \qquad (172)$$

This turns out to be

$$\frac{1 + (\nabla y)^2}{\sqrt{1 + |\nabla y|^2 - |\nabla\phi|^2 + (\nabla y \cdot \nabla\phi)^2 - (\nabla y)^2(\nabla\phi)^2}} - 1. \qquad (173)$$

As long as $|\nabla\phi|^2 \leq |\nabla y|^2$, the energy density is bounded below by

$$(\nabla y)^2. \qquad (174)$$

One may calculate the spatial stress tensor by coupling the Lagrangian to a background metric γ_{ij} and taking a variational derivative of $\mathcal{L} = \sqrt{\gamma}L$ where $\gamma = \det\gamma_{ij}$.

$$-T_{ij} = \frac{2}{\sqrt{\gamma}}\frac{\delta\mathcal{L}}{\delta\gamma^{ij}}. \qquad (175)$$

and thus

$$- T_{ij} = 2\frac{\delta L}{\delta \gamma^{ij}} - \gamma_{ij}L. \tag{176}$$

Now $-L - 1$ is given by

$$\sqrt{1 - \gamma^{ij}(\partial \phi \partial_j \phi - \partial y \partial_j y) + \gamma^{ij}\gamma^{pq}(\partial_i \phi \partial_j y \partial_p \phi \partial_q y - \partial_i \phi \partial_j \phi \partial_p y \partial_q y)}. \tag{177}$$

Therefore

$$T_{ij} = \gamma_{ij}L + \frac{1}{1 - L}\big(\partial_i y \partial_j y - \partial_i \phi \partial_j \phi + 2(\partial_{(i}\phi \partial_{j)}y(\nabla \phi \cdot \nabla \phi)$$
$$-\partial_i \phi \partial_j \phi (\nabla y)^2 - \partial_i y \partial_j y (\nabla \phi)^2\big). \tag{178}$$

Evidently the stress tensor vanishes pointwise in the BPS limit $y = \pm\phi$. By contrast if one considers the symmetrical situation envisaged earlier in which a particle and an anti-particle lie in the 3-direction then on the symmetry plane $\partial_3 \phi = 0 = \partial_3 y$ and hence

$$T_{33} = L. \tag{179}$$

It follows by a repetition of the analysis given above that as long as they are super extreme, the particle and anti-particle attract.

8 Calibrated p-branes

In this section we shall show that the idea that strings ending on $p - branes$ or throats connecting two branes can be interpreted as topological defects on the brane applies more generally. To see this we note that one supply of min-mimal submanifolds, is provided by using 'calibrations'[43]. These satisfy a Bogomol'nyi like-property that minimize volume among all submanifolds in the same homology class. Thus they have a good chance of being supersymmetric. Particular cases are special Lagrangian submanifolds and Cayley surfaces These correspond to supersymmetric cycles [47].

8.1 Lagrangian Submanifolds Global Monopoles and Vortices

In this subsection we consider special Lagrangian submanifolds of $\mathbb{C}^p \equiv \mathbb{R}^p \oplus i\mathbb{R}^p$[43]. If $(\mathbf{x}, \mathbf{y})$ are coordinates for $\mathbb{R}^p \oplus i\mathbb{R}^p$ then these take the form in static gauge

$$y_i = \partial_i F(x) \tag{180}$$

where the generating function $F(x)$ satisfies

$$\text{Im } \det(\delta_{ij} + \sqrt{-1}\ \partial_i \partial_j F) = 0. \tag{181}$$

Harvey and Lawson give some $SO(p-1)$-invariant examples with topology $\mathbb{R} \times S^{p-1}$. The solutions have

$$\frac{\mathbf{y}}{|\mathbf{y}|} = \frac{\mathbf{x}}{|\mathbf{x}|}, \tag{182}$$

$$\text{Im} \ (|\mathbf{x}| + i|\mathbf{y}|)^p = c, \tag{183}$$

where c is a constant and so they resemble global monopoles. If c vanishes we have a set of p-planes meeting at the origin and making an angle θ such that $\sin(p\theta) = 0$. If $p = 3$ [9] we have

$$3r^2 = y^2 + \frac{c}{y}. \tag{184}$$

If c is positive it is convenient to choose the arbitrary scale so that $c = 2$. The solution, are considered as a function of r. It is defined only if $r > 1$. The solution is double valued for $r > 1$. One branch has

$$\mathbf{y} \approx \frac{\mathbf{x}}{6|\mathbf{x}|^3} \qquad \text{as} \quad r \to \infty \tag{185}$$

and behaves exactly like a global monopole in a theory with a global $SO(3)$ symmetry [48]. The other branch is linear and behaves like

$$\mathbf{y} \approx \sqrt{3}\mathbf{x} \qquad \text{as} \quad r \to \infty. \tag{186}$$

Geometrically the 3-brane looks like two asymptotically planar regions in $\mathbb{E}^6$ joined by a throat. The two asymptotic 3-planes intersect at the origin making an angle of $\frac{\pi}{3}$.

If $p = 4$ the 4-brane also two branches and the asymptotic 4-planes make an angle of $\frac{\pi}{4}$. If one sets $c = 4$, one has

$$r^3 y - r y^3 = \frac{c}{4} \tag{187}$$

and hence one branch is monopole like with

$$\mathbf{y} \approx \frac{\mathbf{x}}{|\mathbf{x}|^4} \qquad \text{as} \quad r \to \infty \tag{188}$$

while the other branch is linear

$$\mathbf{y} \approx \mathbf{x} \qquad \text{as} \quad r \to \infty. \tag{189}$$

If $p = 5$ one has

$$y(5r^4 - 10r^2 y^2 + y^4) = c. \tag{190}$$

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[9]If p=3 then $\sqrt{\det(\delta_{ij} + \partial_i \mathbf{y} \cdot \partial_j \mathbf{y})}$ may be re-interpreted as the energy function of a rather unusual non-linear super or hyper elastic material with rubber-like properties.

There are now two types of interesting solution. One has an asymptotically monopole-like branch with

$$\mathbf{y} \approx \frac{\mathbf{x}}{|\mathbf{x}|^5} \qquad \text{as} \quad r \to \infty \tag{191}$$

joined to a linear branch with

$$\mathbf{y} \approx \sqrt{5 - 2\sqrt{5}} \ \mathbf{x} \qquad \text{as} \quad r \to \infty. \tag{192}$$

The other type of solution has two linear branches with

$$\mathbf{y} \approx \sqrt{5 \pm 2\sqrt{5}} \ \mathbf{x} \qquad \text{as} \quad r \to \infty \tag{193}$$

joined by a throat.

Using the boost invariance of the Lagrangian it is possible to give these solutions a Born-Infeld electrostatic static field. The resulting electric fields are then of dipole character.

If $p = 3$ the generating function $F(\mathbf{x})$ of a special Lagrangian submanifold satisfies:
$$\nabla^2 F = \det(\partial_i \partial_j F). \tag{194}$$

One may ask for solutions for which both sides of (194) vanish. The vanishing of the right hand side is the condition that the level sets of F be developable. Thus we are looking for developable surfaces which are also harmonic. Not surprizingly the helicoid:

$$F = az + b \arctan y/x \tag{195}$$

provides a solution. It corresponds to a vortex on the brane

$$
\begin{aligned}
y^1 &= -a \frac{y}{x^2 + y^2}, \\
y^2 &= a \frac{x}{x^2 + y^2}, \\
y^3 &= b \quad .
\end{aligned}
$$

$$\tag{196}$$

By a theorem of Catalan [50], the helicoid is the only ruled minimal surface in $\mathbb{E}^3$ other than the plane. As we have seen above, by Hamel's theorem the helocoid is the only harmonic minimal surface in $\mathbb{E}^3$ other than the plane. It is natural to conjecture that the helicoid is the only harmonic developable in $\mathbb{E}^3$ other than the plane.

8.2   A Cayley example

Another interesting example of a similar general character given by Harvey and Lawson [43], resembles certain types of 'textures 'modelled on the Hopf fibration. It is a Cayley 4-brane living in $\mathbb{E}^7 \equiv \text{Im}\mathbb{O} \equiv \mathbb{H} \oplus \text{Im}\mathbb{H}$. The 4-brane may be exhibited in static gauge as a graph over $\mathbb{R}^4 \equiv \mathbb{H}$ by giving the height 3-vector $\mathbf{y}$ as a function of a quarternion coordinate q. One has

$$\frac{\mathbf{y}}{|\mathbf{y}|} = \frac{\sqrt{5}}{2} \frac{qi\bar{q}}{|q|^2}, \tag{197}$$

with

$$y(4y^2 - 5r^2) = c. \tag{198}$$

Recalling that $r = |q|$ we see that if the constant c is positive then the solution is only defined outside a critical 3-sphere whose value depends on c and is double valued outside this critical throat or 3-sphere. One branch is asymptotically flat with

$$y \approx \frac{c}{25r^4}. \tag{199}$$

The other branch tends to the cone

$$\mathbf{y} \approx \frac{\sqrt{5}}{2} \frac{qi\bar{q}}{|q|}. \tag{200}$$

The cone is not asymptotically flat and this respect there is some resemblance to certain super gravity solutions with G_2 holonomy which are also not asymptotically flat [49].

9   Acknowledgements

    During the fnal stages of writing up I became aware of the work of [51] which have some overlap the ideas in the present paper. Another piece of related which I became aware of during the final proof reading is [52] which extends the ideas [7] on the M-brane. In fact on reduction to 4=1 dimensions one gets a Born-Infeld action coupled to transverse scalars and much of the analysis of the present

paper goes through unchanged, a fact also realized by Paulina Rychenkova and Miguel Costa.

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