COSMIC BALDNESS

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INTRODUCTION

Some years ago (Gibbons & Hawking 1977) it was suggested that solutions of Einstein's equations with a positive cosmological term should eventually settle down to a state which is stationary inside the cosmological event horizon of any future inextendible timelike curve. This stationary state, it was suggested, would, if no black holes were present, be de Sitter space. The recent work on the inflationary scenario in the early universe has reawakened interest in this topic and the purpose of this article is to review the mechanism whereby a universe dominated by vacuum energy, so that it satisfies the equation

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta}\,, \qquad \Lambda > 0\,, \tag{1}$$

relaxes to an asymptotically de sitter state inside the event horizon of any observer. There is some overlap with the article by Barrow but the emphasis here is rather different.

GEOMETRY OF DE SITTER SPACE

De Sitter space may be thought of as the hyperboloid in 5 dimensional Minkowski space given by

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^5)^2 = 3/\Lambda \tag{2}$$

where Λ is the cosmological constant which is related to Hubble's constant, H , or the surface gravity of the horizon, κ, by

$$H = \kappa = (\frac{\Lambda}{3})^{1/2} \tag{3}$$

It is useful to coordinatize the spacetime in two different ways, depending upon whether we wish to think of it as an expanding F.R.W. universe or a static universe with an event horizon. We use (s, χ, θ, ϕ) the first case and (t, r, θ, ϕ) in the second. They are given (Hawking & Ellis 1973; Hawking & Gibbons 1977):

$$r\sin\theta\cos\phi = X^1 = H^{-1}\cosh Hs\,\sin\chi\sin\theta\cos\phi \tag{4}$$

$$r\sin\theta\sin\phi = X^2 = H^{-1}\cosh Hs\,\sin\chi\sin\theta\sin\phi \tag{5}$$

$$r\cos\theta = X^3 = H^{-1}\cosh Hs\,\sin\chi\cos\theta \tag{6}$$

$$
\begin{aligned}
(H^{-2}-r^2)^{1/2}\cos Ht &= X^5 = H^{-1}\cosh Hs \cos\chi & (7)\\
(H^{-2}-r^2)^{1/2}\sin Ht &= X^0 = H^{-1}\sinh Hs & (8)
\end{aligned}
$$

The metric thus becomes:

$$
d\mathfrak{s}^2 = -(1-H^2r^2)dt^2 + (1-H^2r^2)^{-1}dr^2 + r^2 d\Omega^2 \tag{9}
$$

or

$$
d\mathfrak{s}^2 = -ds^2 + H^{-2}\cosh^2 Hs(d\chi^2 + \sin^2\chi d\Omega^2) \tag{10}
$$

The event horizon of an observer situated at $r = 0$ given by $r = H^{-1}$ and so since (4) - (6) imply that

$$
\begin{aligned}
\sin\chi &= Hr(\cosh Hs)^{-1} & (11)\\
&= Hr(1-H^2r^2\tanh^2 Ht)^{-1/2}(\cosh Ht)^{-1} & (12)
\end{aligned}
$$

we see that as $s, t \rightarrow$ an exponentially smaller portion of the 3-spheres $s =$ const. is included within the event horizon of this observer. This is the key to understanding the decay of perturbations. From the point of view of the static frame we expect them to be radiated through the cosmological event horizon just as in the familiar black hole case (see Price 1972). From the point of view of the expanding universe frame we shall see – as pointed out by Starobinsky (1977) that gravitational wave perturbations do not decay but rather they are frozen in as $t \rightarrow \infty$. However this is not in contradiction with the cosmic No Hair Theorem since as time proceeds the observer sees these perturbations on an exponentially smaller scale and so the region inside his/her event horizon appears to become more and more accurately de Sitter.

BEHAVIOUR OF LINEARIZED SCALAR AND GRAVITATIONAL WAVE PERTURBATIONS

Following Lifshitz & Khalatnikov (1963) we consider perturbations of the metric form (1O) given by

$$
d\mathfrak{s}^2 = -ds^2 + H^{-2}\cosh^2 Hs(d\chi^2 + \sinh^2\chi d\Omega^2 + \sum_n \nu_n(s) G^{(n)}_{ij}(\chi, \theta\phi)dx^i dx^j) \tag{13}
$$

where the G^n_{ij} are tensor harmonics on S^3 and the $\nu_n(s)$ are the amplitudes of the gravitational wave perturbations. It turns are the out that if one considers solutions of the massless, minimally coupled scalar wave equation in the de Sitter background of the form

$$
\phi = \nu_n(s) Q^{(n)}(\chi, \theta, \phi) \tag{14}
$$

where the Q^n are scalar harmonics on S^3 , the coefficients $\nu_n(s)$ in (13) satisfy the same equation as those in (12) and are given constant factor by

$$
\nu_n(s) = (in\,\mathrm{sech} Hs + \tanh Hs)\exp in(\tan^{-1}\sinh Hs)\sinh HS) \tag{15}
$$

In neither the scalar nor the gravitational case do the perturbations die away, rather they tend to constants at late times. Thus the scalar field ϕ can have any functional form, $\Phi(\chi, \theta, \phi)$, at late times. However using the relation given by (l1) between χ and the coordinates r and t we see that as $t \to \infty$ for all r inside the event horizon

$$\Phi(t, r, \theta, \phi) \to \Phi(0) \tag{16}$$

Thus ϕ tends to a constant inside the event horizon exponentially fast which is in accordance with the fact that there are no static solutions of the wave equation which are regular inside and on the event horizon other than the constant one.

In the gravitational case similar results hold, locally the constant gravitational wave modes are pure gauge, even though they are not pure gauge globally over the entire S^3 . We shall see this in more detail when we consider the fully non-linear asymptotic form of the metric. Note that in contrast to the black hole case the perturbations die away exponentially fast; there is no power law fall-off of the sort discussed by Price (1972).

FULLY NON-LINEAR ASYMPTOTIC ANALYSIS

Starobinsky has pointed out to us that a general asymptotic solution of the equation $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$ takes the form

$$d\mathfrak{s}^2 = -ds^2 + \exp 2Hs\, a_{ij}(\underline{x}) dx^i dx^j + O(1) \tag{17}$$

where a_{ij} is is an arbitrary 3-metric. This clearly indicates that the waves do not decay globally over the entire 3-surface s = constant. The geometry af this surface never settles down to that of a smooth 3-sphere. The curve $x^i = 0$ is a geodesic. By means of a linear coordinate transformation of the x^i 's we may, with no loss of generality, set

$$a_{ij}(0) = \delta_{ij} \tag{18}$$

Now introduce coordinates y^i and t by

$$y^i = e^{Hs} x^i \tag{19}$$

$$e^{Ht} = (1 - H^2 y^2)^{-1/2} e^{Hs} \tag{20}$$

It is now an easy exercise to show that in the coordinates (t, y^i), which are only valid within the event horizon of the timelike observer at $y^i = 0$, the general metric (16) approaches the exact metric of de Sitter space (equation (9)) exponentially fast. Thus as far as every freely falling observer is concerned the observable universe becomes quite bald.

GENERALIZED ISRAEL' S THEOREM

If we exclude the possibility of a central black hole it would seem likely from the above analysis that de Sitter space is the only exactly static solution of the equations $R_{alpha\beta} = \Lambda g_{\alpha\beta}$, $\Lambda > 0$, surrounded by a regular event horizon and with a regular centre – i.e. such that the interior of the event horizon is diffeomorphic to the product of an open ball in R^3 with the real line. This last proviso is necessary to exclude the Nariai (1951) metric. We have tried unsuccessfully to generalize the standard Israel theorems for black holes (Israel 1967; Muller zum Hagen, et al. 1973; Robinsan 1977) to this case. If the metric is static the boundary conditions on the horizon are just those required to justify analytically continuing the metric to the Euclidean regime by putting $t = i\tau$. The resulting metric is positive definite, defined on a manifold diffeomorphic to S^4, and satisfies $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$. standard Einstein metric on The only known such metric is of course the standard Einstein metric on S^4 which is just the analytic continuation of de Sitter space obtained by setting $t = i\tau$ in (9) amd identifying τ , modulo $2\pi(3/\Lambda)^{1/2}$. The natural conjecture to make – which is stronger than the generalised Israel theorem suggested above – is that the only such metric is the standard one. Mathematicians we have asked do not know whether this is true but the corresponding statement is false for S^{4n+3} , $n \geq 1$ (Jensen 1973). If there is another such metric it cannot be continuously deformed into the standard one as one can readily check by perturbing the Einstein equations on S^4 as described in (Gibbons and Perry 1978).

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