Supergravity vacua and solitons

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1 Introduction

The enormous recent progress, detailed in other articles in this volume, on the non-perturbative structure of string/M-theory and its low energy approximation, quantum-field theory, has been based on the recognition of the importance of p-branes, i.e. of extended objects with p spatial dimensions. Nowadays these can now be viewed in various ways. Historically however they first appeared as classical solutions of the low energy limits, SUSY Yang-Mills theory or the various Supergravity theories. The purpose of these two lectures is to provide a pedagogic introduction to the properties of solitons in supergravity theories and how one constructs the classical solutions. No claim is made for completeness, the subject is by now far too vast to survey the entire subject in just two lectures, and what follows is to some extent a rather personal account of the theory emphasising the features which are peculiar to the gravitational context. Thus lecture one is mainly concerned with exploring the question: what are the analogues of 'solitons'in gravity theories? The second is concerned with finding p-brane solutions. An important subsidiary technical theme is the use of sigma models or harmonic maps to solve Einstein's equations. Most, but not all, of the material is about four spacetime dimensions, partly because this is the best understood case and the most extensively studied and partly because, at least my, intuition is strongest in that dimension.

My own interest in this subject began in the early days of supergravity theories with the realization that since perturbatively such theories describe just a system of interacting massless particles: the graviton $g_{\mu\nu}$ N-gravitini ψ^i_{μ} , n_v abelian vectors A^A_{μ} , $n_{\frac{1}{2}}$ spin half fermions λ and n_s scalars $\bar{\phi}^a$, then unless non-perturbative effects come into play they can have little connection with physics. It soon became apparent to me that the non-perturbative structure must involve gravity in an essential way, and influenced by some ideas of Hajicek, which he mainly applied to gravity coupled to non-abelian gauge theory, I was led to propose extreme black holes as the appropriate analogue of solitons, what we now call BPS states. With this in mind I embarked in the early 1980's on a series of investigations aimed at uncovering the essential

¹Prepared for A Newton Institute Euroconference on Duality and Supersymmetric Theories, Cambridge, England, 7-18 Apr 1997. published as "Cambridge 1997, Duality and supersymmetric theories" eds P.West and D. Olive (1990) 267-296, Cambridge University Press

features of the soliton concept in gravity. What follows is largely a synopsis of those ideas updated to take into account recent developments. It has been both gratifying and a little surprizing to see how well they have remained relevant over the intervening years.

2 The soliton concept in SUGRA theories

2.1 SUGRA in 4-dimensions

In what follows we shall concentrate on the theory in four spacetime dimensions but much of what follows goes through with appropriate modifications in higher dimensions. Some examples of this will be given *en passant*.

2.1.1 The Lagrangians

If spacetime $\{\mathcal{M}, g_{\mu\nu}\}$ has four dimensions the lagrangian $L = L_F + L_B$ of an un-gauged supergravity theory with N supersymmetries has a fermionic L_F and bosonic L_B part. The latter is given by

$$L_{B} = \frac{1}{16\pi} R - \frac{1}{8\pi} \bar{G}_{ab} g^{\mu\nu} \partial_{\mu} \bar{\phi}^{a} \partial_{\nu} \bar{\phi}^{b} - \frac{1}{16\pi} \mu(\bar{\phi})_{AB} F^{A}_{\mu\nu} F^{B\mu\nu} - \frac{1}{16\pi} \nu(\bar{\phi})_{AB} F^{A}_{\mu\nu} \star F^{B\mu\nu}$$
(2.1)

where R is the Ricci scalar of the spacetime metric $g_{\mu\nu}$, \star is the Hodge dual. As is usual when combining gravity and electromagnetism, I am using Guassian or 'un-rationalized'units so as to avoid extraneous factors of 4π is the formulae. We have also set Newton's constant G=1. Occasionally I will, without further comment, re-instate it.

I have adopted the spacetime signature (-1, +1, +1, +1). The particular convenience of that signature choice is that because the Clifford algebra $\text{Cliff}_{\mathbb{R}}(3,1) \equiv \mathbb{R}(4)$ the algebra of four by four real matrices, everything in the purely classical theory, including all spinors and spinor lagrangians, may be taken to be real. The same is of course true of classical M-theory since $\text{Cliff}_{\mathbb{R}}(10,1) \equiv \mathbb{R}(32)$ and continues to hold if we descend to ten spacetime dimensions. It seems that it is only when we pass to the quantum theory that we need to introduce complex numbers. When dealing with Grassmann algebra valued spinors in classical supersymmetric theories one need only consider algebra's over the field of real numbers. However the consistent adoption of this point of view entails some minor changes of the conventions, concerning for example 'conjugation 'which are customary in the subject because i's never appear in any formulae. My own view is the customary conventions introduce into the theory extraneous and unnatural elements which disguise the underlying mathematical simplicity. Fortunately, perhaps, the explicit details of the necessary changes in conventions will not be needed in what follows.

2.1.2 The scalar manifold

The scalar fields $\bar{\phi}^a$, $a=1,\ldots,n_s$ take values in some target manifold $M_{\bar{\phi}}$ with metric \bar{G}_{ab} which is typically a non-compact symmetric space, $M_{\bar{\phi}}=\bar{G}/\bar{H}$ where \bar{H} is the maximum compact subgroup of \bar{G} and the Lie-algebra $\bar{\mathfrak{g}}=\bar{\mathfrak{h}}\oplus\bar{\mathfrak{k}}$ admits the involution $(\bar{\mathfrak{h}},\bar{\mathfrak{k}})\to(\bar{\mathfrak{h}},-\bar{\mathfrak{k}})$. As a consequence

- topologically $M_{\bar{\phi}}$ is trivial, $M_{\bar{\phi}} \equiv \mathbb{R}^{n_s}$.
- The sectional curvatures $K(e_1, e_2) = R_{abcd} e_1^a e_2^b e_1^c e_2^d$ are non-positive.

These conditions play an important role in the theory of harmonic maps. For example a well known theorem states that the only regular harmonic maps from a compact Riemannian manifold of positive Ricci curvature to a compact Riemannian manifold of negative sectional curvature are trivial. It is not difficult to adapt the proof of this theorem to rule out static asymptotically flat Skyrmion type solutions of the purely gravity-scalar sector of the theory. As a consequence we can anticipate that there are few non-perturbative features of the purely scalar sector. This situation changes when one passes to the gauged SUGRA theories. In that case there is a potential function for the scalars and various domain wall configurations exist. I shall not pursue this aspect of the theory further here.

2.1.3 Duality

The abelian vector fields A_{μ}^{A} , $A=1,\ldots,n_{v}$ transform under a representation of G and and may be thought of as sections of the pull back under the map $\bar{\phi}: \mathcal{M} \to M_{\bar{\phi}}$ of a vector bundle over $M_{\bar{\phi}}$ tensored with spactime one-forms. In fact the structure is somewhat richer. As explained by Zumino elsewhere in this volume, one may define an electromagnetic induction 2-form by:

$$G_A^{\mu\nu} = -8\pi \frac{\partial L_B}{\partial F_{\mu\nu}^A}.$$
 (2.2)

The pair $(F^A, \star G_A)$ also carries a representation of G and in fact may be considered as the pull-back of an $Sp(2n_v, \mathbb{R})$ bundle over $M_{\bar{\phi}}$ tensored with spacetime two-forms.

To see what is going on more explicitly it is helpful to cast the vector lagrangian in non-covariant form. In an orthonormal frame it becomes, up to a factor of 4π

$$\frac{1}{2}\mu_{AB}(\mathbf{E}^A \cdot \mathbf{E}^B - \mathbf{B}^A \cdot \mathbf{B}^B) + \nu_{AB}\mathbf{E}^A \cdot \mathbf{B}^B. \tag{2.3}$$

We define

$$\mathbf{D}_A = \mu_{AB} \mathbf{E}^B + \nu_{AB} \mathbf{B}^B \tag{2.4}$$

and

$$\mathbf{H}_A = \mu_{AB} \mathbf{B}^B - \nu_{AB} \mathbf{E}^B. \tag{2.5}$$

Thus

$$\begin{pmatrix} \mathbf{H}_A \\ \mathbf{E}^A \end{pmatrix} = \begin{pmatrix} \mu + \nu \mu^{-1} \nu & -\nu \mu^{-1} \\ -\mu^{-1} \nu & \mu^{-} - 1 \end{pmatrix} \begin{pmatrix} \mathbf{B}^A \\ \mathbf{D}_A \end{pmatrix}. \tag{2.6}$$

We define the matrix

$$\mathcal{M}(\bar{\phi}) = \begin{pmatrix} \mu + \nu \mu^{-1} \nu & -\nu \mu^{-1} \\ -\mu^{-1} \nu & \mu^{-} - 1 \end{pmatrix}, \tag{2.7}$$

and finds that

$$\det \mathcal{M}(\bar{\phi}) = 1. \tag{2.8}$$

The local density of energy due to the vectors is

$$\frac{1}{8\pi} (\mathbf{H}_A \cdot \mathbf{B}^A + \mathbf{D}_A \cdot \mathbf{E}^A). \tag{2.9}$$

The energy density and the equations of motion, will be invariant under the action of $S \in SL(2, \mathbb{R})$

$$\begin{pmatrix} \mathbf{B} \\ \mathbf{D} \end{pmatrix} \to S \begin{pmatrix} \mathbf{B} \\ \mathbf{D} \end{pmatrix} \tag{2.10}$$

provided we can find an action of $SL(2,\mathbb{R})$ on the scalar manifold $S: M_{\bar{\phi}} \to M_{\bar{\phi}}$ which leaves the metric \bar{G}_{ab} invariant and under which the matrix $\mathcal{M}9\bar{\phi}$) pulls back as

$$\mathcal{M} \to (S^t)^{-1} \mathcal{M} S^{-1}. \tag{2.11}$$

As explained by Zumino this happy situation does inded prevail in the SUGRA models under present consideration. We note *en passant* that electric-magnetic duality transformations of this type may extended to the *non-linear* electrodynamic theories of Born-Infeld type which one encounters in the world volume actions of Dirichlet 3-branes.

The set-up described above may be elaborated somewhat . For general N=2 models, including supermatter, $M_{\bar{\phi}}$ is a Kähler manifold subject to 'Special Geometry'. The properties described in the present article do not require this extra structure and it will not be necessary to understand precisely what special geometry is in the sequel.

2.1.4 Scaling symmetry

As mentioned above, perturbatively theories of this kind describe massless particles. In fact, since the vectors are *abelian*, even more is true. The theory admits a global *scale-invariance*:

$$(g_{\mu\nu}, F^A_{\mu\nu}, \bar{\phi}^a) \to (\lambda^2 g_{\mu\nu}, \lambda F^A_{\mu\nu}, \bar{\phi}^a)$$
 (2.12)

with $\lambda \in \mathbb{R}_+$ which takes classical solutions to classical solutions and takes the lagrangian density $\sqrt{-g}L_B \to \lambda^2 \sqrt{-g}L_B$.

It is an easy exercise to check that our general lagrangian has an energy momentum tensor T^{μ}_{ν} which satisfies the dominant energy condition, i.e $T^{\mu}_{\nu}p^{\nu}$ lies in or on the future light cone for all vectors p^{ν} which themselves lie in or on the future light cone. Thus these theories have good local stability and causality properties.

Given the scaling symmetry and the positivity of energy one may construct arguments analogous to the well known theorem of Derrick in flat space to show that there are no non-trivial everywhere static or stationary solutions of the field equations. This type of theorem, are often referred to as Lichnerowicz theorems, though they go back to Serini, Einstein and Pauli. They may be summarized by the slogan: **No solitons without horizons**.

2.1.5 Examples

At this point some examples of \bar{G}/\bar{H} are in order.

- N = 1. We only have a graviton, $n_s = 0 = n_v$, and thus bosonically this is just ordinary General Relativity.
- N = 2, U(1)/U(1), there are no scalars, $n_s = 0$ and one vector, $n_v = 1$. This is just Einstein-Maxwell theory
- N = 4, $SO(2,1)/SO(2) \times SO(6)/SO(6)$. We have two scalars and six vectors.
- N=4 plus supermatter, $SO(2,1)/SO(2) \times SO(6,6)/SO(6) \times SO(6)$. This is what you get if you dimensionally reduce N=1 supergravity from ten-dimensions.
- The reduction of the Heterotic theory in 10-dimensions, $SO(2,1)/SO(2) \times SO(22,6)/SO(22) \times SO(6)$
- N = 8, $E_{7(7)}/SU(8)$.

Note that, except for N=8, $G=S\times T$ where $S=SL(2,\mathbb{R})$ is what is now called the S-duality group and T is now called the T-duality group. In the case of N=8 both S and T are contained in a Hull and Townsend's single U-duality group.

2.2 Vacua

Presumably the minimum requirement of a classical vacuum or ground state is that it be a homogeneous spacetime $\mathcal{M} = \mathcal{G}/\mathcal{H}$ with constant scalars

, $\partial_{\mu}\bar{\phi}^{a}=0$ and covariantly constant Maxwell fields $\nabla_{\mu}F_{\nu\rho}^{A}=0$. Actually one sometimes wishes to consider a 'linear dilaton' but we shall not consider that possibility here. The ground state will this be labelled in part by a point $p\in M_{\bar{\phi}}$. Thus one may consider $M_{\bar{\phi}}$, or possibly some sub-manifold of it, as parameterizing a moduli space of vacua. We usually demand that the ground state be classically stable, at least against small disturbances and this typically translates into the requirement that it have least 'energy 'among all nearby solutions with the same asymptotics. Unfortunately there is no space here to enter in detail into a general discussion of the how energy is defined in general relativity using an appropriate globally timelike Killing vector field K^{μ} . Later we shall outline an approach based on supersymmetry and Bogomol'nyi bounds. Suffice it to emphasise at present two important general principles.

- The dominant energy condition plays an essential role in establishing the positivity.
- Stability cannot be guaranteed if there is no globally timelike Killing field K^{μ} .

Thus if spacetime \mathcal{M} admits non-extreme Killing Horizons across which a Killing field K^{μ} switches from being timelike to being spacelike the local energy momentum vector $T_{\mu_{\nu}}K^{\nu}$ relative to K^{μ} can become spacelike. This fact is behind the quantum instability of non-extreme Killing horizons due to Hawking radiation. In the context of homogeneous spacetimes this means that while globally static Anti-de-Sitter spacetime $AdS_n = SO(n-1,2)/SO(n-1,1)$ has positive energy properties, and can be expected to be stable, de-Sitter spacetime $dS_n = SO(n+1,1)/SO(n-1,1)$ with its cosmological horizons does not admit a global definition of positive energy.

For the class of lagrangians we are considering in four spacetime dimensions, there are two important ground states. One is familiar Minkowski spacetime $\mathbb{E}^{3,1} = E(3,1)/SO(3,1)$, for which $F_{\mu\nu}^A = 0$ and $\bar{\phi}^a$ is arbitrary. The other, probably less familiar, is Bertotti-Robinson spacetime $AdS_2 \times S^2 = SO(1,2)/SO(1,1) \times SO(3)/SO(2)$. This represents a 'compactification 'from four to two spacetime dimensions on the two-sphere S^2 . One has

$$F^A = \frac{p^A}{A} \eta_{S^2} \tag{2.13}$$

and

$$\star G_A = \frac{q_A}{A} \eta_{S^2} \tag{2.14}$$

where η_{S^2} is the volume form on the S^2 factor, A is its area, (p^A, q_A) are constant magnetic and electric charges and the scalar field $\bar{\phi}^a$ is 'frozen 'at a value $\bar{\phi}^{\text{frozen}}$ which extremizes a certain potential $V(\bar{\phi}, p, q)$ which may be

read off from the lagrangian. The potential is given by

$$V(\bar{\phi}, p, q) = (p \quad q) \mathcal{M}(\bar{\phi}) \begin{pmatrix} p \\ q \end{pmatrix}$$
 (2.15)

and is invariant under S-duality which acts on the charges as

$$\binom{p}{q} \to S \binom{p}{q}. \tag{2.16}$$

The frozen values of $\bar{\phi}$ are given by

$$\frac{\partial V(\bar{\phi}, p, q)}{\bar{\phi}}\Big|_{\bar{\phi} = \bar{\phi}^{\text{frozen}}} = 0. \tag{2.17}$$

In fact

$$A = 4\pi V(\bar{\phi}^{\text{frozen}}, p, q). \tag{2.18}$$

Thus if $V(\bar{\phi}, p, q)$ has a unique minimum value then the vacuum solution is specified entirely by giving the charges (p, q).

As we shall see, it turns out that almost all of the black hole properties of the theory are determined entirely by the function $V(\bar{\phi}, p, q)$. In particular SUGRA theories, such as N=2 theories based on Special Geometry, one may know some special facts about $V(\bar{\phi}, p, q)$ and one may read off much of the nature of the BPS configurations directly, without a detailed investigation of particular spacetime metrics.

2.3 Supersymmetry and Killing Spinors

The supersymmetry transformations take the schematic form:

$$\delta_{\epsilon} B = F \tag{2.19}$$

$$\delta_{\epsilon} F = B \tag{2.20}$$

where (B, F) are bosonic and fermionic fields respectively and $\epsilon = \epsilon^i$, i = 1, ..., N is an N-tuplet of spinor fields. A purely bosonic background (B, 0) is said to admit a Majorana Killing spinor field ϵ , or to admit SUSY, if

$$\delta_{\epsilon}(B,0) = 0. \tag{2.21}$$

Because the Killing spinor condition is linear in spinor fields, we may take the Grassmann algebra valued spinor fields ϵ to be commuting spacetime dependent spinor fields (which by an abuse of notation we also call ϵ) multiplied by a constant Grassmann coefficient. The Killing spinor condition reduces to $\delta_{\epsilon}F = 0$, or in components:

$$\delta_{\epsilon}\psi^{i}\mu = (\hat{\nabla}_{\mu}\epsilon)^{i} \tag{2.22}$$

$$\delta_{\epsilon}\lambda = M\epsilon \tag{2.23}$$

where ψ_{μ}^{i} are the N gravitini and λ are the spin half fields. The operator $\hat{\nabla}_{\mu} = \nabla_{\mu} + E_{\mu}$ where ∇_{μ} is the Levi-Civita covariant derivative acting on spinors and E_{μ} are endomorphisms depending on the bosonic field and whose precise form depends upon the particular SUGRA theory under consideration.

It follows from the supersymmetry algebra that

$$K_{\epsilon} = \bar{\epsilon} \gamma^{\mu} \epsilon \tag{2.24}$$

is a Killing vector field which necessarily lies in or on the future light cone. The solution is said to admit maximal SUSY if the dimension of the space of Killing spinors is N. If it is less, we speak of have BPS states with $\frac{N}{k}$ SUSY. From the point of view of SUSY representation theory BPS solutions correspond to short multiplets.

Typically both Minkowski spacetime $\mathbb{E}^{3,2}$ and Bertotti-Robinson spacetime $AdS_s \times S^2$ admit maximal SUSY. In gauged supergravity one has either a negative cosmological constant or a negative potential for the scalars. It then turns out that anti-de-Sitter spacetime is a ground state with maximal SUSY. Because it does not admit an everywhere causal Killing field, de-Sitter space, even if it were a solution, could never admit SUSY.

2.3.1 Remark on Five dimensions

For the purposes of discussing black hole entropy it is often simpler to treat five-dimensional supergravity theories. I will not discuss them here in detail. I will simply remark that much of the present theory goes through. After dualization the bosonic lagrangian is

$$L_B = \frac{1}{16\pi} R - \frac{1}{8\pi} G_{ab} g^{\mu\nu} \partial_{\mu} \phi^a \partial_{\nu} \phi^b - \frac{1}{16\pi} \mu(\phi)_{AB} F^A_{\mu\nu} F^{B\mu\nu} + \text{Chern - Simons term.}$$
(2.25)

and $AdS_2 \times S^2$ is replaced by $AdS_2 \times S^3$. For example if N=8, $G/H=E_{6,(+6)}/Usp(8)$.

2.4 pp-waves

The perturbative, massless, states of the theory correspond to wave solutions. For example

$$ds^{2} = -dt^{2} + dz^{2} + dx^{2} + dy^{2} + K(t - z, x, y)(dt - dz)^{2}$$
(2.26)

with K(t-z,x,y) being a harmonic function of (x,y)

$$(\partial_x^2 + \partial_y^2)K = 0 (2.27)$$

with arbitrary dependence on on t-z, is a vacuum solution representing a classical gravitational wave propagating in the positive z direction. Such solutions are called 'pp-waves'. As one would expect from basic supersymmetry representation theory they admit $N=\frac{1}{2}$ SUSY. If

$$(\gamma^t - \gamma^z)\epsilon = 0 \tag{2.28}$$

then ϵ is a Killing spinor and by suitably scaling it it we have

$$\bar{\epsilon}\gamma^{\mu}\epsilon = K^{\mu} \tag{2.29}$$

where

$$K = \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \tag{2.30}$$

is the lightlike Killing field of the metric.

This example admits an obvious generalization to arbitrary spacetime dimension n. One simply replaces the two transverse coordinates (x, y) by n-2 transverse coordinates $(x^1, \dots x^{n-2})$. If K is taken to be independent of t-z and one sets H=1+K one may dimensionally reduce to n-1 spacetime dimensions:

$$ds^{2} = H\left(dz - \frac{dt}{H}\right)^{2} - \frac{1}{H^{\frac{1}{n-3}}}\left(-\frac{dt^{2}}{H^{n-3}} + H^{\frac{1}{n-3}}dx_{n-2}^{2}\right). \tag{2.31}$$

This gives a Kaluza-Klein 0-brane, with $A=\frac{dt}{2H}$. For example if n=5 we get a so-called $a=\sqrt{3}$ extreme black hole which is S-dual to a the Kaluza-Klein monopole based on the Taub-NUT metric. If n=11 we get the Dirichlet zero-brane of ten-dimensional type IIA theory.

2.5 Asymptotically flat solutions

We suppose that $g_{00} \sim -1 + \frac{2GM}{r}$ and $g_{ij} \sim (1 + \frac{2GM}{r})\delta_{ij}$ where M is the ADM mass. The scalar field $\bar{\phi} \sim \bar{\phi}_{\infty} + \frac{\Sigma^a}{r}$ where the the scalar charge $\Sigma^a \in TM_{\bar{\phi}_{\infty}}$. Because the scalar manifold $M_{\bar{\phi}}$ is topologically trivial and because its sectional curvature is non-positive it is not difficult to prove, using standard techniques from the theory of harmonic maps, that there are no non-singular solutions without horizons with vanishing vector fields. Thus there are no analogues of Skyrmions.

We define the total electric q_A and magnetic p^A by the usual 2-surface integrals at infinity,

$$p^A = \frac{1}{4\pi} \int_{S_{\infty}^2} F^A \tag{2.32}$$

and

$$q_A = \frac{1}{4\pi} \int_{S^2_{\infty}} \star G_A. \tag{2.33}$$

We now arrive at some absolutely crucial points.

- The fundamental fields carry neither electric nor electric charges. Thus perturbative states cannot carry them. In fact, by Maxwell's equations, a solution can only carry non-vanishing charges if it is in some way singular or topologically non-trivial or both. In fact we encounter here the phenomenon of 'charge without charge 'due to 'lines of force being trapped in the topology of space 'which formed such central point of Misner and Wheeler's 'Geometrodynamics'. In the context of string theory and Polchinkski's non-perturbative Dirichlet-branes, some of the vector fields have their origin in the Ramond⊗Ramond sector and we have 'Ramond⊗Ramond charge without charge '.
- Because the fields are abelian and also because of the classical scale-invariance there is no possibility of a classical quantization of the charges. Quantization can only be achieved by going outside the framework of classical supergravity theory, for example by coupling to so-called 'fundamental' branes and applying a Dirac type argument or by applying Saha's well known argument for the angular momentum about the line of centre's joining an electric and magnetic charge. One then discovers that if the electric charges belong to some lattice $q_A \in \Lambda$ then the magnetic charges belong to the reciprocal lattice $p^A \in \Lambda^*$. That is

$$\frac{2}{\hbar}p^A q_A \in \mathbb{Z}.\tag{2.34}$$

2.6 Black Holes

In the case of N=1 one might be tempted to think that the only static solution with a regular event horizon, the Scharzschild black hole should be considered as some sort of soliton. However there are a number of reasons why this is not correct.

- Although the solution is regular outside its event horizon, inside it contains a spacetime singularity. This may not be fatal if Cosmic Censorship holds. In that case the spacetime out side the event horizon would be regular and predictable. In fact it is widely believed to be classically stable.
- However the mass M is arbitrary and is not fixed by any quantization condition. Moreover classically the black hole can absorb gravitons and gravitini leading to a mass increase. In fact classically the area A of the event horizon can never decrease. This irreversible behaviour is quite unlike what one expects of a classical soliton.
- Quantum mechanically, because of the Hawking effect the Schwarzschild black hole is definitely unstable. The same is true of the Kerr solution. It does not seem reasonable therefore to expect that in the full quantum

gravity theory one may associate with them a stable non-perturbative state in the quantum mechanical Hilbert space. In fact because of their thermal nature it is much more likely that the classical solutions should be associated with a density matrices representing a black hole in thermal equilibrium with its evaporation products.

• From the point of view of SUSY it is clear that the Schwarzschild and Kerr solutions do not correspond to BPS states since they do not admit any Killing spinors. For N=1 the endomorphism E_{μ} vanishes and such killing spinors would have to be covariantly constant, as would the Killing vector constructed from them This is impossible if the solution is not to be flat.

Later we shall see that SUSY is incompatible with a non-extreme Killing horizon.

2.6.1 Reissner-Nordstrom

If N=2 the candidate static solitons would be Reissner Nordstrom black holes. If the singularity is not to be naked we must have

$$M \ge |Z|,\tag{2.35}$$

where

$$|Z|^2 = \frac{q^2 + p^2}{G}. (2.36)$$

Note that in this section I am re-instating Newton's constant G. The Hawking temperature is

$$T = \frac{1}{4\pi G} \frac{\sqrt{M^2 - |Z|^2}}{(M + \sqrt{M^2 - |Z|^2})^2}.$$
 (2.37)

If M > |Z| the temperature is non-zero and the solution is unstable against Hawking evaporation of gravitons, photons and gravitini. Moreover it cannot be a BPS state since the Killing vector which is timelike near infinity becomes spacelike inside the horizon.

Only in the extreme case M=|Z| for which T=0 is $\frac{\partial}{\partial t}$ never spacelike. Moreover in that case there are multi-black hole solutions, the so-called Majumdar-Papapetrou solutions:

$$ds^2 = -H^{-2}dt^2 + H^2d\mathbf{x}^2, (2.38)$$

with, up to an electric-magnetic duality rotation,

$$F = dt \wedge d(\frac{1}{H}), \tag{2.39}$$

where H is an arbitrary harmonic function on \mathbb{E}^3 .

It is an easy exercise to verify that the entire Majumdar-Papapetrou family of solutions admits a Killing spinor whose associated Killing vector is $\frac{\partial}{\partial t}$. In fact they are the only static solutions of Einstein-Maxwell theory admitting a Killing spinor. Thus the Majumdar-Papapetrou solutions correspond to BPS states.

One further property of the extreme holes, called *vacuum interpolation*, should be noted. This is while near infinity the solution tends to the flat maximally supersymmetric ground state, Minkowski spacetime $\mathbb{E}^{3,1}$, near the horizon the metric tends to the other maximally supersymmetric ground state, Bertotti-Robinson spacetime $AdS_2 \times S^2$. Thus, as is the case for many solitons, the solution spatially interpolates between different vacua or ground states of the theory.

The Bekenstein-Hawking entropy S of a general charged black hole is given by

$$S = \frac{A}{4G} = \pi G(M + \sqrt{M^2 - |Z|^2})^2. \tag{2.40}$$

For fixed |Z| this is least in the extreme case when it attains

$$\pi(q^2 + p^2), (2.41)$$

which is *independent* of Newton's constant G and depends only on the quantized charges (p, q).

2.6.2 Black holes and frozen moduli

If we now pass to the case when more than one vector and some scalars are present we find that in general that the scalars will vary with position. There is thus in a sense 'scalar hair '. However spatial dependence of the scalar fields and non-vanishing scalar charges Σ^a are only present by virtue of the fact that the source term $\frac{\partial V}{\phi}(\bar{\phi},p,q)$ in the scalar equations of motion. If it happens that $\bar{\phi}_{\infty}=\bar{\phi}^{\text{frozen}}$ however then the charges vanish and the scalars are constant. The moduli are then said to be frozen. The geometry of the black holes is then the same as the Reissner Nordstrom case with

$$Z^2 = V(\bar{\phi}^{\text{frozen}}, p, q) = |Z|^2(p, q).$$
 (2.42)

In general the moduli will not be frozen and for instance the value of the scalars on the horizon $\bar{\phi}^{\text{horizon}}$ will be different from its value $\bar{\phi}_{\infty}$ at infinity. For extreme static black holes however regularity of the horizon demands that

$$\bar{\phi}^{\text{horizon}} = \bar{\phi}^{\text{frozen}}.$$
 (2.43)

As a consequence we have the important general fact, that the Bekenstein-Hawking entropy of extreme holes is always independent of the moduli at

infinity and depends only on the quantized charges (p, q). That is

$$S_{\text{extreme}} = \pi V(\bar{\phi}, p, q) = \pi |Z|^2(p, q). \tag{2.44}$$

As explained in other lectures the Bekenstein-Hawking entropy of extreme holes can be obtained by D-brane calculations at weak coupling. It is vital for the consistency of this picture that S_{extreme} really is independent of the moduli $\bar{\phi}_{\infty}$ which label the vacua. A striking consequence of this is that the entropy of any initial data set with given (p,q) should never be less than $\pi |Z(p,q)|^2$ and its mass M should never be less than |Z(p,q)|.

2.6.3 The First Law of Thermodynamics and the Smarr-Virial Theorem

For time stationary fields we may define electrostatic potentials ψ^A and magnetostatic potentials χ_A by

$$F_{0i}^A = \partial_i \psi^A \tag{2.45}$$

and

$$G_{A0i} = \partial_i \chi_A. \tag{2.46}$$

The first law of classical black hole mechanics needs a modification if we consider variations of the moduli $\bar{\phi}_{\infty}$. It becomes

$$dM = TdS + \psi^A dq_A + \chi_A dp^A - \Sigma^a \bar{G}_{ab}(\bar{\phi}_\infty) d\bar{\phi}^b.$$
 (2.47)

The last term is the new one. Note that, at the risk of causing confusion but in the interest of leaving the formulae comparatively uncluttered I have not explicitly distinguished between the potential functions χ_A , ψ^A and their values at the horizon.

It follows using the scaling invariance that the mass is given by the Smarr formula

$$M = 2TS + \psi^A q_A + \chi_A p^A. \tag{2.48}$$

Thus in the present circumstances, the Smarr formula is equivalent to the first law as a consequence of scaling symmetry. In other words we may regard the Smarr relation as a virial type theorem. This formula allows a simple derivation of the 'No solitons without horizons 'result. If there is no horizon then S = 0 and by Gauss's theorem $q_A = 0 = p^A$. It follows that the mass M = 0 and hence by the positive mass theorem the solution must be flat.

2.7 Bogomol'nyi Bounds

We shall now give a brief indication of how one identifies the central charges and establishes Bogomol'nyi Bounds. Let ϵ_{∞}^i be constant spinors at infinity. In

what follows we shall sometimes omit writing out a summation of i explicitly. The supercharges Q^i are defined by

$$\bar{\epsilon}_{\infty}Q = \frac{1}{4\pi G} \int_{S_{\infty}^2} \frac{1}{2} \bar{\epsilon}_{\infty} \gamma^{\mu\nu\lambda} \psi_{\lambda} d\Sigma_{\mu\nu}. \tag{2.49}$$

The Nester two-form $N^{\mu\nu}$ associated to a spinor field ϵ is defined by

$$N^{\mu\nu} = \bar{\epsilon}\gamma^{\mu\nu\lambda}\hat{\nabla}_{\lambda}\epsilon. \tag{2.50}$$

Under a SUSY variation we have

$$\epsilon_{\infty}^{-} \delta_{\epsilon} Q = \frac{1}{4\pi G} \int_{S_{\infty}^{2}} \frac{1}{2} N^{\mu\nu} d\Sigma_{\mu\nu}. \tag{2.51}$$

Stokes's theorem gives

$$\bar{\epsilon}_{\infty}\delta_{\epsilon}Q = \frac{1}{4\pi G} \int_{\Sigma} \nabla_{\mu} N^{\mu\nu} d\Sigma_{\nu}, \qquad (2.52)$$

where Σ is a suitable spacelike surface whose boundary at infinity is S^2_{∞} and whose inner boundary either vanishes or is such that by virtue of suitable boundary conditions one may ignore its contribution.

Now by the supergravity equations of motion one finds

$$\nabla_{\mu}N^{\mu\nu} = \hat{\nabla}_{\mu}\epsilon\gamma^{\mu\nu\lambda}\hat{\nabla}\epsilon + \bar{M}\epsilon\gamma^{\nu}M\epsilon \tag{2.53}$$

Now restricted to Σ , $\bar{M}\epsilon\gamma^0M\epsilon\geq 0$ and

$$\hat{\nabla}_{\mu} \epsilon \gamma^{\mu 0 \lambda} \hat{\nabla} \epsilon = |\hat{\nabla}_{a} \epsilon|^{2} - |\gamma^{a} \hat{\nabla}_{a} \epsilon|^{2}, \tag{2.54}$$

where the derivative $\hat{\nabla}_a$ is tangent to Σ . So far ϵ has been arbitrary. We pick it such that

- $\epsilon \to \epsilon_{\infty}$ at infinity.

We also choose ϵ such that the inner boundary terms, such as might arise at an horizon, vanish. It is not obvious but it is in fact true that this can be done. It follows that the right hand side of 2.52 is non-negative and vanishes if and only if everywhere on Σ

$$M\epsilon = 0 \tag{2.55}$$

and

$$\hat{\nabla}_a \epsilon = 0. \tag{2.56}$$

Because Σ is arbitrary we may deduce that in fact the right hand side of 2.52 is non-negative and vanishes if and only if everywhere in spacetime

$$M\epsilon = 0 \tag{2.57}$$

and

$$\hat{\nabla}_{\mu}\epsilon = 0. \tag{2.58}$$

This means that ϵ must be a Killing Spinor.

Now the left hand side of 2.52 may be shown to be

$$\bar{\epsilon}_{\infty} \gamma^{\mu} P_{\mu} \bar{\epsilon}_{\infty} + \bar{\epsilon}_{\infty} (U_{ij} + \gamma_5 V_{ij}) \bar{\epsilon}_{\infty}.$$
(2.59)

Here P_{μ} may be identified with the ADM 4-momentum and U_{ij} and V_{ij} are central charges which depend on the magnetic and electric charges (p,q) and the moduli, i.e. of the values $\bar{\phi}_{\infty}$ of the scalar fields at infinity. In the case of N=2 there are just two central charges which may be combined into a single complex central charge $Z(p,q,\bar{\phi}_{\infty})$ and the Bogomol'nyi bound becomes

$$M \ge |Z(p, q, \bar{\phi}_{\infty})|. \tag{2.60}$$

3 Finding Solutions

We now look for local solutions on $\mathcal{M} = \Sigma \times \mathbb{R}$ which are independent of the time coordinate $t \in \mathbb{R}$. Globally of course the geometry is much more subtle because of the presence of horizons but that will not affect the local equations of motion. The basic idea used here is that in 3-dimensions we may use duality transformations to replace vectors by scalars. The resulting equations may be derived form an action describing three-dimensional gravity on Σ coupled to a sigma model.

3.1 Reduction from 4 to 3 dimensions

The metric is expressed as

$$ds^{2} = -e^{2U}(dt + \omega_{i}dx^{i})^{2} + e^{-2U}\gamma_{ij}dx^{i}dx^{j}.$$
(3.1)

The effective lagrangian is

$$\frac{1}{16\pi}R[\gamma] - \frac{1}{8\pi}G_{ab}(\bar{\phi}^a)\partial_i\phi^b\partial_j\phi\gamma^{ij}, \qquad (3.2)$$

where ϕ^a is the collection of fields $(U, \psi, \bar{\phi}^a, \psi^A, \chi_A)$ taking values in an augmented target space M_{ϕ} where ψ is the twist potential, ψ^A the electrostatic potential and χ_A the magnetostatic potential. The twist potential arises by

dualizing ω_i . Note that (U, ψ) are the gravitational analogues of electric and magnetic potentials respectively. Indeed for pure gravity,

$$\operatorname{curl} \omega = e^{-2U} \operatorname{grad} \psi, \tag{3.3}$$

the internal space is $H^2 \equiv SO(2,1)/SO(2)$ and the metric is

$$dU^2 + e^{-2U}d\psi^2. (3.4)$$

The formula for the twist potential becomes more complicated in the presence of vectors.

Three important general features to note are

- If the signature of spacetime is (3,1) then the signature of the σ -model metric G_{ab} is $(2+n_s, 2n_v)$. Physically this is because the n_s scalars fields ϕ^a and the gravitational scalars (U, ψ) give rise to attractive forces while the n_v vector fields give rise to repulsive forces.
- The metric \bar{G}_{ab} admits $2n_v + 1$ commuting Killing fields $\frac{\partial}{\partial \psi}$, $\frac{\partial}{\partial \psi^A}$, $\frac{\partial}{\partial \chi_A}$ which give rise to $2n_v + 1$ charges, the last of which, the so-called NUT charge, vanishes for asymptotically flat solutions.
- Typically M_{ϕ} is also a symmetric space with indefinite metric of the form G/H, where G is an example U-duality group which includes both S and T duality groups. Of course H is no longer compact.
- If we were to include fermions we would get three-dimensional SUGRA theory but with euclidean signature.

3.1.1 Examples

Let us look at some examples of G/H.

- N=1. This is pure gravity, there are no vectors and the signature is positive. The the coset is two-dimensional hyperbolic space $H^2=SO(2,1)/SO(2)$.
- N = 2, $SU(2,1)/S(SU(1,1) \times U(1))$, Einstein-Maxwell theory. In fact $\{M_{\phi}, G_{ab}\}$ is an analytic continuation of the Fubini-Study metric on \mathbb{CP}^2 to another real section with Kleinian signature (2,2).
- N = 4 SUGRA, $SO(8, 2)/SO(2) \times SO(6, 2)$.
- N = 4 SUGRA plus supermatter, $SO(8,8)/SO(2,6) \times SO(6,2)$. This is what you get if you dimensionally reduce N = 1 supergravity from ten-dimensions.

• The reduction of the Heterotic theory in 10-dimensions gives, $SO(24,8)/SO(22,2) \times SO(2,6)$

• $N = 8 \text{ SUGRA}, E_{8(+8)}/SO^*(16).$

It is clear that the group G may be used as a solution generating group. Of course some elements of G may not take physically interesting solutions to physically distinct or physically interesting solutions. nevertheless one may anticipate that black hole solutions will fall into some sort of multiplets of a suitable subgroup of G and indeed this turns out to be the case.

3.1.2 Static truncations

In what follows we shall mainly be concerned with non-rotating holes and so we drop the twist potential and consider the *static truncation* with effective lagrangian

$$+\frac{1}{16}R[\gamma] + \frac{1}{8\pi}(\partial U)^2 + \frac{1}{8\pi}\bar{G}_{ab}\partial\bar{\phi}^a\partial\bar{\phi}^b - \frac{1}{8\pi}e^{-2U}(\partial\psi^A,\partial\chi_A)\mathcal{M}^{-1}(\partial\psi^A,\partial\chi_A)^t$$
(3.5)

3.1.3 Gravitational Instantons

The methods we have just described may also be used to obtain solutions of the Einstein equations with positive definite signature admitting a circle action. All that is required to get the equations is a suitable analytic continuation of the previous formulae. This entails a chage in the groups and the symmetric spaces. Thus in the case of pure gravity case the metric is

$$ds^{2} = e^{2U}(d\tau + \omega_{i}dx^{i})^{2} + e^{-2U}\gamma_{ij}dx^{i}dx^{j}.$$
 (3.6)

The twist potential still satisfies 3.3 but the internal space becomes $dS_2 = SO(2,1)/SO(1,1)$ with metric

$$ds^2 = dU^2 - e^{-2U}d\psi^2. (3.7)$$

3.1.4 The equations of motion

The scalar equation of motion requires that ϕ gives a harmonic map from Σ to M_{ϕ} .

$$\nabla^2 \phi = 0 \tag{3.8}$$

where the covariant derivative ∇ contains a piece corresponding to the pull-back under ϕ of the connection $\Gamma^a_{bc}(\phi)$ on M_{ϕ} , thus

$$\nabla_i \partial_j \phi^a = \partial_i \partial_j - \Gamma_{ij}(x)^k \partial_k \phi^a + \partial_i \phi^c \partial_j \phi^b \Gamma_{bc}^a(\phi). \tag{3.9}$$

Variation with respect to the metric γ_{ij} gives an Einstein type equation:

$$R_{ij} = 2\partial_{\mathbf{l}}\phi^a\partial_j\phi^bG_{ab}. (3.10)$$

If we pretend that we are thinking of Einstein's equations in three dimensions then the left hand side may be thought of as $T_{ij} - \gamma_{ij}\gamma^{mn}T_{mn}$ where T_{ij} is the stress tensor.

There are essentially three easy types of solutions of this system of equations which may be described using simple geometrical techniques.

- Spherically symmetric solutions
- Multi-centre (i.e. BPS) solutions
- cosmic string solutions.

3.2 Spherically symmetric solutions

The idea is to reduce the problem to one involving geodesics in M_{ϕ} . A consistent ansatz for the metric is γ_{ij} is

$$\gamma_{ij} dx^{i} dx^{j} = \frac{c^{4} d\tau^{2}}{\sinh^{4} c\tau} + \frac{c^{2}}{\sinh^{2} c\tau} (d\theta^{2} + \sin^{2} \theta d\phi^{2}). \tag{3.11}$$

The radial coordinate τ is in fact a harmonic function on Σ with respect to the metric γ_{ij} . The range of τ is from the horizon at $-\infty$ to spatial infinity at $\tau = 0$. In these coordinates the only non-vanishing component of the Ricci tensor of γ_{ij} is the radial component and this has the constant value $2c^2$. For a regular solution the constant c is related to the temperature and entropy by

$$c = 2ST. (3.12)$$

Now it is a fact about harmonic maps that the composition of a harmonic map with a geodesic map is harmonic. Thus to satisfy the scalar equations of motion, $\phi^a(\tau)$ must execute geodesic motion in $\{G_{ab}, M_{\phi}\}$ with τ serving as affine parameter along the geodesic. The Einstein equation then fixes the value of the constraint

$$G_{ab}\frac{d\phi^a}{d\tau}\frac{d\phi^a}{d\tau} = c^2. {(3.13)}$$

Because of electromagnetic gauge-invariance there are $2n_v$ Noether constants of the motion, i,e, the electric and magnetic charges:

$$\begin{pmatrix} p^A \\ q_A \end{pmatrix} = \mathcal{M}^{-1} \begin{pmatrix} \frac{d\chi_A}{d\tau} \\ \frac{d\psi^A}{d\tau} \end{pmatrix}. \tag{3.14}$$

The remaining equations follow from the effective action

$$\left(\frac{dU}{d\tau}\right)^2 + \bar{G}_{ab}\frac{d\bar{\phi}^a}{d\tau}\frac{d\bar{\phi}^b}{d\tau} + e^{2U}V(\bar{\phi}, p, q)$$
(3.15)

and the constraint becomes

$$\left(\frac{dU}{d\tau}\right)^2 + \bar{G}_{ab}\frac{d\bar{\phi}^a}{d\tau}\frac{d\bar{\phi}^b}{d\tau} - e^{2U}V(\bar{\phi}, p, q) = (2ST)^2. \tag{3.16}$$

Evaluating this at infinity we get

$$M^2 + \bar{G}_{ab} \Sigma^a \Sigma^b - V(\bar{\phi}_{\infty}, p, q) = (2ST)^2.$$
 (3.17)

The extreme case corresponds to c = 0. The metric γ_{ij} is now

$$\gamma_{ij} dx^{i} dx^{j} = \frac{d\tau^{2}}{\tau^{4}} + \frac{1}{\tau^{2}} (d\theta^{2} + \sin\theta d\phi^{2}). \tag{3.18}$$

3.3 Toda and Liouville systems

The method just outlined has the advantage that it makes clear how the duality group acts on the solutions.

The procedure also clear why the radial equations frequently give rise to ordinary differential equations of Toda type which are in principle exactly integrable. As we have seen, the problem of finding spherically symmetric solutions reduces to finding geodesics in the symmetric space G/H. We may think of this as solving a Hamiltonian system on the co-tangent space $T^*(M_\phi) = T^*(G/H)$. This symmetric space admits at at least $2n_v + 1$ and typically more commuting Killing vectors. These come from the fact that one may add an arbitrary constant to the twist potential ψ and the magnetic and electric potentials (χ_a, ψ^A) . In addition there may be further symmetries arising from axion like fields. Let us suppose that in toto there are r such commuting Killing vectors. Ignoring any possible identifications they will generate the group \mathbb{R}^r . One may eliminate the r commuting constants of the motion to obtain a dynamical system on the quotient configuration space $M_\phi/\mathbb{R}^r = \mathbb{R}^r \backslash G/H$. From a Hamiltonian point of view one is of course just performing a Marsden-Weinstein symplectic reduction.

Now it is known from the work of Perelomov and Olshanetsky that Toda systems arise precisely in this way. Thus it is no surprize that one encounters them in finding solutions of the Einstein equations depending on a single variable. The simplest example is when $M_{\phi} = G/H = SO(2,1)/SO(1,1) = AdS_2$ or it's Riemannian version $SO(2,1)/SO(2) = H^2$. The internal metric is given by 3.7 or 3.4 respectively. If the constant of the motion $q = e^{-2U} \frac{d\psi}{d\tau}$ then the dynamical system has effective Lagrangian

$$\frac{1}{2}(\frac{dU}{d\tau})^2 \pm \frac{1}{2}q^2e^{2U} \tag{3.19}$$

with constant of the motion

$$\frac{1}{2}\left(\frac{dU}{d\tau}\right)^2 \mp \frac{1}{2}q^2e^{2U} = \text{constant},\tag{3.20}$$

where the upper sign refers the AdS_2 case and the minus to the H^2 case. The resulting dynamical system is of course a rather trivial Liouville system and may be integrated using elementary methods.

If the internal space decomposes into a product of such models, then the integration is equally easy. In practice most of the exact solutions in the literature may be obtained in this way. As I mentioned above, in principle, the general Toda system is exactly integrable but in practice it seems to be rather cumbersome to carry out the integration explicitly.

3.4 Multi-centre solutions

We make the ansatz that the metric γ_{ij} is flat and we may take Σ to be euclidean three-space \mathbb{E}^3 .

$$\gamma_{ij} = \delta_{ij}. \tag{3.21}$$

This means that the coordinates (t, \mathbf{x}) are harmonic coordinates, i.e we are using a gauge in which $\partial_{\mu}\mathfrak{g}^{\mu\nu} = 0$. The vanishing of the Ricci tensor then requires the vanishing stress tensor or local force balance condition

$$\partial_i \phi^a \partial_j \phi^b G_{ab} = 0. (3.22)$$

We must also satisfy the harmonic condition. The following construction will do the job. We start with k ordinary harmonic functions $H^r(x^i)$, r = 1, 2, ..., k on Euclidean space \mathbb{E}^3 . These give a harmonic map from $H : \mathbb{E}^3 \to \mathbb{E}^k$.

We next find a k-dimensional totally geodesic totally lightlike submanifold of the target space M_{ϕ} . This is a map $f: \mathbb{E}^k \to M_{\phi}$ whose image N is

- Totally null, i.e the induced metric $G_{rs} = G_{ab} \frac{\partial f^a}{\partial y^r} \frac{\partial f^b}{\partial y^s}$ vanishes.
- totally geodesic, which means that a (necessarilly lightlike) geodesic which is initially tangent to N remains tangent to N.

The simplest example would be a null geodesic for which k=1. The parameters y^r are affine parameters. In the case of Einstein Maxwell theory we may take the so-called ' α 'or ' β '2-planes which play a role in twistor theory. The important point about totally geodesic maps is that they are harmonic.

Given our maps $H^r(x^i)$ and $f^a(y^r)$ we compose them, i.e. we set

$$\phi^{a}(x^{i}) = f^{a}(H^{r}(x^{i})). \tag{3.23}$$

The result is a harmonic map and we are done.

This simple and elegant technique is in principle all that is required to construct all BPS solutions of relevance to four-dimensions. As we shall see, it frequently works in higher dimensions. Of course to check that they are

BPS one has to check for the existence of Killing spinors. The technique also makes transparently clear the action of the U-duality group. Moreover, since we are dealing with symmetric spaces, everything can, in principle be reduced to calculations in the Lie algebra of G.

Consider the simplest case, the static truncation of Einstein Maxwell theory. The internal space is $SO(2,1)/SO(1,1) = AdS_2$ with metric

$$dU^2 - e^{-2U}d\psi^2 (3.24)$$

where ψ is the electrostatic potential. The null geodesics are given by

$$\psi = \frac{1}{H} \tag{3.25}$$

$$e^U = \frac{1}{H}. ag{3.26}$$

We have recovered the Majumdar-Papapetrou solutions. For dilaton gravity with dimensionless coupling constant a which has the matter action

$$-\frac{1}{8p}\partial\sigma^2 - \frac{1}{16\pi}e^{-2a\sigma}F_{\mu\nu}^2$$
 (3.27)

the internal space carries the metric (in the electrostatic case)

$$dU^{2} + d\sigma^{2} - e^{-2(a\sigma + U)}d\psi^{2}.$$
 (3.28)

The null geodesics are

$$\psi = \frac{1}{\sqrt{1+a^2}} \frac{1}{H} \tag{3.29}$$

$$e^U = \frac{1}{H^{\frac{1}{1+a^2}}} \tag{3.30}$$

$$e^{-a\sigma} = \frac{a^2}{H^{\frac{1}{1+a^2}}} \tag{3.31}$$

There are four interesting case:

- a = 0: this is Einstein-Maxwell theory
- $a = \frac{1}{\sqrt{3}}$: this is what you get if you reduce Einstein-Maxwell theory from five to four spacetime dimensions
- a = 1: this corresponds to the reduction of string theory from ten to four spacetime dimensions
- $a=\sqrt{3}$: Kaluza-Klein theory. These solutions are S-dual to the taub-NUT solutions, i.e. to Kaluza-Klein monopoles

Modulo duality transformations, these solutions are all special cases of the solutions with four U(1) fields and three-scalar fields. In the case that two U(1) fields are electrostatic and two are magnetostatic the internal space decomposes as the metric product of four copies of SO(2,1)/SO(1,2). The spacetime metrics are given by:

$$ds^{2} = -(H_{1}H_{2}H_{3}H_{4})^{-\frac{1}{4}}dt^{2} + (H_{1}H_{2}H_{3}H_{4})^{\frac{1}{4}}d\mathbf{x}^{2}.$$
 (3.32)

The totally null, totally geodesic submanifolds are such that

$$(\psi^1, \psi^3, \chi_2, \chi_4) = (\frac{1}{H_1}, \frac{1}{H_3}, \frac{1}{H_2}, \frac{1}{H_4}). \tag{3.33}$$

The entropy is given by

$$S = \pi \sqrt{q_1 q_3 p^2 p^4}. (3.34)$$

Of course these solutions may be lifted to eleven dimensions for example where they may be thought of as intersecting five-branes:.

3.5 Other Applications

The harmonic function technique works in other than four spacetime dimensions. We now give a few examples.

3.5.1 The D-Instanton

Perhaps the simplest application of the technique described above yields the D-Instantons of ten-dimensional type IIB theory. These are Riemannian solutions. If $\tau = a + ie^{-\Phi}$ where a is the pseudoscalar and Φ the dilaton are the only excited bosonic fields other than the metric then the Lorentzian equations come from the $SL(2,\mathbb{R})$ -invariant action

$$R - \frac{1}{2}(\partial\Phi)^2 - e^{2\Phi}(\partial a)^2. \tag{3.35}$$

Note that we are using Einstein conformal gauge.

For the instantons $a=i\alpha$ with α real and the equations come from the action

$$R - \frac{1}{2}(\partial\Phi)^2 + e^{2\Phi}(\partial\alpha)^2. \tag{3.36}$$

This is effectively the same as the previous cases. One takes the Einstein metric to be flat and

$$\alpha + \text{constant} = e^{\Phi} = H \tag{3.37}$$

where H is a harmonic function on \mathbb{E}^{10} .

Weyl rescaling the metric to string gauge

$$ds^2 = e^{\frac{1}{2}\Phi} d\mathbf{x}^2 = H^{\frac{1}{2}} d\mathbf{x}^2 \tag{3.38}$$

gives an Einstein-Rosen Bridge down which global Ramond \otimes Ramond charge can be carried away. Note that large distances correspond to moderate string coupling $g = e^{\Phi}$ while near the origin of \mathbb{E}^{10} corresponds to strong coupling and the supergravity approximation cannot be trusted.

3.5.2 NS⊗NS Five brane in ten-dimensions

The ten-dimensional metric in string conformal frame is

$$ds_S^2 = -dt^2 + (dx_9)^2 + (dx_8)^2 + (dx_7)^2 + (dx_6)^2 + (dx_5)^2 + e^{2\Phi}g_{\mu\nu}dx^{\mu}dx^{\nu}$$
 (3.39)

where $g_{\mu\nu}$ is the four dimensional metric in Einstein gauge. If a is the 4-dimensional dual of the NS \otimes NS three-form field strength then the equations follow from the lagrangian

$$R - 2(\partial\Phi)^2 + \frac{1}{2}e^{4\Phi}(\partial a)^2. \tag{3.40}$$

Again one picks the metric $g_{\mu\nu}$ to be flat and

$$a + \text{constant} = e^{2\Phi} = H \tag{3.41}$$

where H is now a harmonic function on \mathbb{E}^4 .

One may now apply a duality transformation taking one to the Ramond \otimes Ramond five-brane. This leaves the Einstein metric invariant but takes $\Phi \to -\Phi$. The resulting metric in string conformal gauge is

$$ds_S^2 = H^{-\frac{1}{2}}(-dt^2 + (dx_9)^2 + (dx_8)^2 + (dx_7)^2 + (dx_6)^2 + (dx_5)^2) + H^{\frac{1}{2}}d\mathbf{x}^2.$$
 (3.42)

3.6 Cosmic string Solutions

The seven-brane of Type IIB theory is an example of of how to construct cosmic string like solutions. The main difference in technique with the former case is that since the internal metric is positive definite the the spatial metric can no longer be flat and so we need to solve for it explicitly. This is simple if the spatial metric is two dimensional.

To get the seven brane, we write the ten dimensional metric in Einstein gauge as

$$ds^{2} = -dt^{2} + (dx_{9})^{2} + (dx_{8})^{2} + (dx_{7})^{2} + (dx_{6})^{2} + (dx_{5})^{2} + (dx_{4})^{2} + (dx_{4})^{2} + e^{\phi} dz d\bar{z}.$$
(3.43)

The static equations arise from the two-dimensional Euclidean action

$$R - \frac{1}{2} \frac{(\partial \tau_1)^2 + (\partial \tau_2)^2}{\tau_2^2} \tag{3.44}$$

where $\tau = \tau_1 + i\tau_2 = a + ie^{-\phi}$ gives a map into the fundamental domain of the modular group $SL(2,\mathbb{Z})\backslash SO(2,\mathbb{R})/SO(2)$. We may regard the two-dimensional space sections as a Kahler manifold and the harmonic map equations are thus satisfied by a holomorphic ansatz $\tau = \tau(z)$. We must also satisfy the Einstein condition. Using the formula for the Ricci scalar of the two-dimensional metric and the holomorphic condition this reduces to the linear Poisson equation

$$\partial \bar{\partial}(\phi - \log \tau_2) = 0. \tag{3.45}$$

To get the fundamental string one chooses

$$\phi = \Phi \tag{3.46}$$

$$\tau \propto \log z.$$
 (3.47)

In four spacetime time dimensions the fundamental string is 'super-heavy ', it is not asymptotically conical at infinity.

To get the *seven brane*, which does correspond to a more conventional cosmic string, one picks

$$j(\tau(z)) = f(z) = \frac{p(z)}{q(z)}$$
 (3.48)

where $j(\tau)$ is the elliptic modular function and $f(z) = \frac{p(z)}{q(z)}$ is a rational function of degree k.

The appropriate solution for the metric is

$$e^{\phi} = \tau_2 \eta^2 \bar{\eta}^2 \Big| \prod_{i=1}^{i=k} (z - z_i)^{-\frac{1}{12}} \Big|^2.$$
 (3.49)

where $\eta(\tau)$ is the Dedekind function. Asymptotically

$$e^{\phi} \sim (z\bar{z})^{-\frac{k}{12}}.$$
 (3.50)

Therefore the spatial metric is that of a cone with deficit angle

$$\delta = \frac{4k\pi}{24}.\tag{3.51}$$

This may also be verified using the equations of motion and the Gauss-Bonnet theorem. As a result one can have up to 12 seven-branes in an open universe. To close the universe one needs 24 seven-branes.

The solution has the following 'F-theory 'interpretation. One considers the metric

$$ds^2 = g_{ij}dy^i dy^j + e^{\phi} dz d\bar{z}, \tag{3.52}$$

where g_{ij} is the following unimodular metric on the torus T^2 with coordinates y^i

$$\begin{pmatrix} \tau_2^{-1} & \tau_1 \tau_2^{-1} \\ \tau_1 \tau_2^{-1} & \tau_1^2 \tau_2^{-1} + \tau_2 \end{pmatrix}. \tag{3.53}$$

The metric 3.52is self-dual or hyper-Kähler. If one takes 24 seven-branes one gets an approximation to a K3 surface elliptically fibered over \mathbb{CP}^1 .

Another interesting special case arises as an orbifold. Consider $T^2 \times \mathbb{C}$ with coordinates (y^1, y^2, z) . Quotient by the involution $(y^1, y^2, z) \rightarrow (-y^1, -y^2, -z)$. There are four fixed points which may be blown up to obtain a regular simply connected manifold on which there exists a twelve real-dimensional family of family of smooth hyper-Kähler metrics. The second Betti number is five and the intersection form of the five non-trivial cycles is given by the Cartan matrix of the extended Dynkin diagram D_4 . These metrics have been obtained as hyper-Kähler quotients by Kronheimer and by Nakajima. The smooth metrics are rotationally symmetric but $\frac{\partial}{\partial y^1}$ and $\frac{\partial}{\partial y^2}$ are only approximate Killing vectors, physically they are interesting as examples of 'Alice Strings' because, thinking of the two-torus as Kaluza-Klein type internal space with two approximate U(1) 's and two approximate charge conjugation operators $C_2: (y^1, y^2, z) \to (-y^1, y^2, z)$ and $C_2: (y^1, y^2, z) \to (y^1, -y^2, z)$, one finds that if Alice circum-ambulates the string, but staying very far away, she returns charge conjugated. Of course if she ventures into the core region she will find that the two electric charges are not strictly conserved. Because the solutions admit a non-triholomorphic circle action, they are given in terms of a solution of the $su(\infty)$ Toda equation. From this it is easy to check that the solutions approach the orbifold limit with exponential accuracy.

To see this explicitly note that the metric

$$ds^{2} = \frac{1}{\nu'} (2d\theta + \nu_{1}dy^{2} - \nu_{2}dy^{2})^{2} + \nu' \left\{ d\rho^{2} + e^{\nu} ((dy^{1})^{2} + (dy^{2})^{2}) \right\}$$
(3.54)

is hyperKähler if $\nu(\rho, y^1, y^2)$ satisfies

$$(e^{\nu})'' + \nu_{11} + \nu_{22} = 0, \tag{3.55}$$

where the superscript \prime denotes differentiation with respect to ρ and the subscripts 1 and 2 denote differentiation with respect to y^1 and y^2 respectively. The Killing vector $\frac{\partial}{\partial \theta}$ leaves invariant the privileged Kähler form

$$(2d\theta + \nu_1 dy^2 - \nu_2 dy^1) \wedge d\rho + (e^{\nu})' dy^1 \wedge dy^2. \tag{3.56}$$

whose closure requires that 3.55 is true. Note that the geometrical significance of the coordinate ρ is that it is the moment map for the circle action. The simplest solution of (3.55) is $e^{\nu} = \rho$, this gives the flat metric

$$ds^{2} = dr^{2} + r^{2}d\theta^{2} + (dy^{1})^{2} + (dy^{2})^{2},$$
(3.57)

with $r = 2\sqrt{\rho}$. This is independent of y^1 and y^2 . For solutions admitting an elliptic fibration we require a solution of (3.55) which is periodic in y^1 and y^2 . To check the typical behaviour near infinity, one linearizes (3.55) about the solution $e^{\nu} = \rho$. The resulting equation admits solutions by separation of variables. It is then a routine exercise to convince one's self that the general solution must decay exponentially at infinity.

Before leaving these metrics it is perhaps worth pointing out that their relation to the much better known class of Ricci-flat riemannnian metrics admitting a tri-holomorphic circle action and which depend on an arbitrary harmonic function H on \mathbb{E}^3 . They are easily obtained using the technique described above. The metrics are

$$ds^{2} = H^{-1}(dt + \omega_{i}dx^{i})^{2} + H\mathbf{dx}^{2}, \tag{3.58}$$

with

$$\operatorname{curl} \omega = \operatorname{grad} H. \tag{3.59}$$

with Kahler forms

$$Hdx^{1} \wedge dx^{2} + dx^{3} \wedge (dt + \omega_{i}dx^{i}). \tag{3.60}$$

$$Hdx^2 \wedge dx^2 + dx^1 \wedge (dt + \omega_i dx^i). \tag{3.61}$$

$$Hdx^3 \wedge dx^1 + dx^2 \wedge (dt + \omega_i dx^i). \tag{3.62}$$

The closure of these Kähler-forms is equivalent to the condition 3.59.

If H is independent of $\arctan(x^2/y^2)$ there will be an additional circle action which preserves the first Kähler form but rotates the second into the third. This means that by taking an arbitrary axisymmetric harmonic function we can, in principle, obtain a solution of the the $su(\infty)$ toda equation 3.55.

To get a complete metric one must choose H to be a finite sum of k poles with identical positive residue. The coordinate singularities at the poles may then be removed by periodically identifying the imaginary time coordinate t. If $H \to 1$ at infinity the metrics are asymptotically locally flat, 'ALF'. and represent k Kaluza-Klein monopoles. If $H \to 0$ at infinity the metrics are asymptotically locally euclidean, 'ALE'. Thus if $0 \le t \le 2\pi$ and $h = \frac{1}{2r}$ we get the flat metric on \mathbb{R}^3 while $H = 1 + \frac{1}{2r}$ we get the Taub-NUT metric on \mathbb{R}^3 .

On the other hand if we take $\phi = \log \tau_2$ and identify $z = x^1 + ix^2$ and $t = y^1$ and $y^2 = x^3$ the metrics 3.52 amd 3.58 coincide. In fact $H = \tau_2$ and $\omega_3 = \tau_1$.

We see that $\frac{\partial}{\partial y^1}$ generates a triholomorphic circle action. The three Kähler forms are

$$\Omega_1 = \tau_2 dx^1 \wedge dx^2 + dy^2 \wedge dy^1, \tag{3.63}$$

$$\Omega_2 = \tau_2 dx^2 \wedge dy^2 + dx^1 \wedge (dy^1 + \tau_1 dy^2)$$
 (3.64)

and

$$\Omega_3 = \tau_2 dy^2 \wedge dx^1 + dx^2 \wedge (dy^1 + \tau_1 dy^2). \tag{3.65}$$

and they are closed by virtue of the Cauchy-Riemann equations for $\tau(x^1+ix^2)$.

4 Conclusion

In these two lectures I have tried to give some idea of what qualifies as a soliton in classical supergravity theories and how one finds the solutions. I have concentrated on general principles and largely restricted myself to four spacetime dimensions. The lectures were emphatically *not* not intended as a comprehensive review. For recent applications the reader is referred to other articles in this volume or to the voluminous current literature. Appended below is a rather restricted list of references largely confined to papers that I have written, either alone or with collaborators, where the reader may find more details of the claims made above or from which the reader may trace back to the original sources. As I stated above, it was not my intention to provide a comprehensive review and no slight is intended against those not explicitly cited.

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