Higher-dimensional resolution of dilatonic black hole singularities

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ABSTRACT

We show that the four-dimensional extreme dilaton black hole with dilaton coupling constant $a = \sqrt{p/(p+2)}$ can be interpreted as a completely non-singular, non-dilatonic, black p-brane in (4+p) dimensions provided that p is odd. Similar results are obtained for multi-black holes and dilatonic extended objects in higher spacetime dimensions. The non-singular black p-brane solutions include the self-dual three brane of ten-dimensional N=2B supergravity and a multi-fivebrane solution of eleven-dimensional supergravity. In the case of a supersymmetric non-dilatonic p-brane solution of a supergravity theory, we show that it saturates a bound on the energy per unit p-volume.

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1. Introduction

Given certain mild conditions on the matter stress tensor, the singularity theorems of Penrose and Hawking guarantee the existence of spacetime singularities in a broad class of four-dimensional solutions of Einstein's equations describing gravitational collapse. The occurrence of such inevitable singularities is a good indication that 'new physics' is required. It is commonly believed that the new physics in question involves quantum mechanics, so that a proper understanding of how spacetime singularities are resolved must await a consistent theory of quantum gravity. While this may well be true, it is also possible that new classical physics intervenes, at some energy scale below the Planck mass, in such a way that singularities are resolved. To remove all singularities, one needs a fundamental change in our description of classical gravity, such as perhaps the one provided by string theory. However certain singularities can be resolved by simply passing to a higher-dimensional theory of gravity for which spacetime is only effectively four-dimensional below some compactification scale.

For example, consider any nonsingular five dimensional solution in Kaluza-Klein theory, which has a spacelike isometry with a fixed point. Then its dimensional reduction along the isometry yields a four dimensional solution with a curvature singularity. If one did not know the origin of this solution, it would not be obvious that the singularity was simply the result of trying to project a regular five dimensional solution into four dimensions. The best known example of this phenomenon is the Kaluza-Klein monopole [1], in which one starts with the product of time and the Euclidean Taub-NUT solution. But one can construct other examples by e.g. starting with the product of time and the Euclidean Schwarzschild solution, or even starting with five dimensional Minkowski spacetime and reducing along a rotation (although in this last case, the resulting four-metric is not asymptotically flat). In all of these examples, the four dimensional singularity is timelike.

One can also remove certain spacelike singularities this way. The simplest ex-

ample is again to start with five dimensional Minkowski spacetime and reduce along the orbits of a boost (considering only the region where the boost is spacelike). The resulting four dimensional metric describes a spatially flat Robertson-Walker universe with an initial or final singularity. Notice that in this case, if one identifies points to make the orbit of the boost compact (and the spacetime appear four dimensional), the five dimensional space has a conical singularity at the origin and is no longer geodesically complete. So the curvature singularity is not completely resolved, but replaced by a much milder conical singularity.

We will show that the curvature singularity in many of the extreme dilaton black holes [2, 3] can be resolved in this way. These black holes have a null singularity which becomes a regular horizon in the higher dimensional spacetime. More precisely, when the dilaton coupling constant a (defined below in (2.1)) takes one of the special values

$$a = \sqrt{\frac{p}{p+2}} \tag{1.1}$$

for integer p, the extreme dilaton black hole can be re-interpreted as a black p-brane in (4+p) dimensions. When p is even, the p-brane resembles the extreme Reissner-Nordström solution (which is included as the special case p=0) in that there is still a curvature singularity inside the horizon. However, for p odd, the spacetime behind the horizon is isometric to the region outside, and the solution is completely nonsingular.* For example, if p=1 we have $a=1/\sqrt{3}$, and the four-dimensional extreme dilaton black hole can be interpreted as the double-dimensional reduction of a non-singular five-dimensional black string. This is called 'double'-dimensional reduction, in contrast to the simple dimensional reduction of the Kaluza-Klein monopole example, because in reducing the spacetime dimension from five to four we simultaneously reduce the dimension of the extended object from one to zero. It is precisely the reverse procedure that allows a resolution of the spacetime singularities of the dilaton black hole for the above values of the dilaton coupling

 $[\]star$ These solutions are analogous to the case of the spacelike singularity above: if one compactifies the extra p dimensions, the spacetime becomes geodesically incomplete even though the curvature remains bounded everywhere.

constant. The special case of $a=1/\sqrt{3}$ is of particular interest because, like the Kaluza-Klein monopole for $a=\sqrt{3}$, the five-dimensional black string is also a solution of the pure five-dimensional supergravity. In this case one can derive a bound on the energy per unit length, analogous to that of [4] for particle-like solutions of four-dimensional Maxwell-Einstein theory. This bound is saturated by the black string solution. In contrast the same methods fail to establish a similar bound for the black p-branes in 4+p dimensions if p>1, as might have been expected from the absence of an appropriate supergravity theory underlying these cases.

The black p-branes we will construct differ from those in [5] in that we consider solutions to a theory without a dilaton in the higher dimensional space. The four dimensional dilaton arises from the dimensional reduction. Some examples of non-dilatonic p-branes have been discussed previously. One is the membrane solution of eleven-dimensional supergravity [6], whose double-dimensional reduction yields the fundamental string solution [7] of ten-dimensional supergravity. This is analogous to the case of (1.1) with p even. The singularity in the fundamental string becomes a regular horizon in higher dimensions, but there is another curvature singularity inside [8].

The existence of solutions with horizons but no singularities is certainly surprising, but not without precedent. For example, domain wall solutions of N=1 supergravity have been found of this type [9]. Of more relevance here is the metric representing an extreme black string solution in three dimensions that is part of an exact conformal field theory [10]. This is very similar to the five-dimensional string metric that resolves the singularities of the $a=1/\sqrt{3}$ dilaton black hole. In fact, if one discards the angular part of the five-metric the resulting three-metric approaches the metric of [10] near the horizon. This metric was shown in [10] to be non-singular, although good coordinates for the region containing the horizon were not given. Here we present a similar analysis for the p-brane solutions under discussion, including details of how one finds good coordinates near the horizon.

The analysis just described is readily generalized to extreme, dilaton black

holes, or extended objects, in a d-dimensional spacetime. One again finds that in certain cases, the singularity in these solutions can be removed via re-interpretation as a higher-dimensional object in a higher-dimensional spacetime. The mechanism by which this is achieved is best understood by considering the asymptotic metric and dilaton near the singular hypersurface [11, 8]. The asymptotic metric is conformal to the product of a sphere with a lower-dimensional anti-de Sitter spacetime. The dilaton is singular at a Killing horizon of the anti-de Sitter space, but in such a way that it can be viewed as the logarithm of additional diagonal components of a higher-dimensional adS-metric. This involves the re-interpretation of the solution as a p-brane in a higher dimensional theory without a dilaton and the singularity of the dilaton at the horizon is then equivalent to the vanishing of these additional metric components there. This is a mere coordinate singularity and the metric can be analytically continued through the horizon to an interior region. One can also start with non-extreme black holes and construct new non-extreme black p-brane solutions in higher dimensions as we will show. However, in this case the singularity inside the black hole horizon is not resolved by lifting to higher dimensions.

A further generalization to multi-p-branes is also possible. We find that the singularities of four-dimensional multi-dilaton black hole solutions are again resolved by their interpretation as a multi-p-brane solution in 4 + p dimensions if p is odd. For d > 4, the situation is more subtle. In some cases, the singularity is completely resolved, while in others the multi-p-brane has singularities inside the horizon even though the single p-brane was completely nonsingular. Finally, there are cases in which the horizons of the multi-p-brane seem to have only finite differentiability. One case which is completely nonsingular is a multi-version of Güven's eleven-dimensional fivebrane [12], and we show how this solution nicely realises the idea of 'local compactification' [13].

A question of obvious interest is whether the p-brane solutions found here are stable. A strong indication of stability would be a generalization of the Bogomol'nyi-type bounds [4,14] established for black holes. We note that while extreme black holes and p-brane solutions in arbitrary dimension are generally

characterised by a simple Bogomolnyi-type relation between their mass and charge (per unit p-volume in the p-brane case) there is no evidence in general that there is a corresponding Bogomolnyi-type bound which applies to all solutions. One expects to be able to derive such a bound in the case of a solution of a field theory that is the bosonic sector of a supergravity theory, because in this case the bound is essentially implied by the algebra of supersymmetry. Indeed, we shall show that the methods of [4, 14] yield the expected bound precisely in these circumstances, and in no others. This strongly suggests that supersymmetric extended object solutions of supergravity theories will not suffer from the type of instability recently found [15] to afflict certain non-extreme extended object solutions of higher-dimensional gravity. It leaves open the interesting question of the stability of non-supersymmetric extreme black holes or p-branes.

2. Resolution of extreme dilaton black hole singularities

The four-dimensional action for the spacetime metric g, one-form abelian gauge potential A, with two-form field-strength F_2 , and dilaton ϕ with dilaton coupling constant a, is

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ R - 2(\nabla\phi)^2 - e^{-2a\phi} F_2^2 \right\} . \tag{2.1}$$

We may suppose without loss of generality that $a \geq 0$ since the field redefinition $\phi \to -\phi$ effectively changes the sign of a. The Euler-Lagrange (E-L) equations of (2.1) admit a magnetically-charged black hole solution [2,3] whose extremal limit is

$$ds^{2} = -\left(1 - \frac{\mu}{r}\right)^{\frac{2}{1+a^{2}}} dt^{2} + \left(1 - \frac{\mu}{r}\right)^{-\frac{2}{1+a^{2}}} dr^{2} + \left(1 - \frac{\mu}{r}\right)^{\frac{2a^{2}}{1+a^{2}}} r^{2} d\Omega_{2}^{2}$$

$$e^{a\phi} = \left(1 - \frac{\mu}{r}\right)^{-\frac{a^{2}}{1+a^{2}}}$$

$$F_{2} = Q\varepsilon_{2},$$
(2.2)

where $d\Omega_2^2$ is the standard metric on the unit two-sphere and ε_2 is its volume 2-form. The constant μ is related to the ADM mass M, and magnetic charge Q, in

units for which G = 1, by

$$\mu = \sqrt{1 + a^2} |Q| = (1 + a^2) M. \tag{2.3}$$

This solution saturates the bound [14]

$$M \ge \frac{1}{\sqrt{1+a^2}}|Q|\tag{2.4}$$

on all asymtotically flat solutions that can be formed from gravitational collapse. It is also singular at $r = \mu$. We shall now show how this singularity can be resolved for the values of a given in (1.1) by re-interpreting the solution as representing an extended object in a higher-dimensional spacetime. The singularity at $r = \mu$ becomes simply a coordinate singularity at a degenerate Killing horizon. Furthermore, we shall show that when p is odd, the maximal analytic extension of this higher-dimensional metric is completely non-singular.

Consider the (4+p)-dimensional Einstein-Maxwell action

$$S_{(4+p)} = \int d^{4+p}x \sqrt{-g} \{ R - F_2^2 \} . \tag{2.5}$$

We assume that the (4+p)-metric and Maxwell field take the form

$$ds_{(4+p)}^{2} = e^{2\alpha\phi(x)}d\mathbf{y} \cdot d\mathbf{y} + e^{2\beta\phi(x)}g_{\mu\nu}(x)dx^{\mu}dx^{\nu}$$

$$F_{2}^{(4+p)} = \frac{1}{2}F_{\mu\nu}(x)dx^{\mu} \wedge dx^{\nu} ,$$
(2.6)

where x^{μ} are the coordinates of a four-dimensional submanifold and \mathbf{y} are the p additional coordinates. Notice that both the metric and Maxwell field are independent of \mathbf{y} , and we do not include any x^{μ} , \mathbf{y} cross terms in the metric which would give rise to additional gauge fields in four dimensions. With this form of the metric

and Maxwell field, the action (2.5) reduces to the following four-dimensional action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} e^{(2\beta + p\alpha)\phi} \left\{ R - 2(p\alpha + 3\beta) \nabla^2 \phi - (6\beta^2 + p\alpha^2 + p^2\alpha^2 + 4p\alpha\beta) (\nabla\phi)^2 - e^{-2\beta\phi} F_2^2 \right\},$$
(2.7)

In order that the four-metric have the canonical, Einstein-Hilbert, action, we choose α and β such that

$$2\beta + p\alpha = 0. (2.8)$$

In this case the $\nabla^2 \phi$ term is a surface term which, for present purposes, we may neglect. Hence

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ R - \frac{p(p+2)}{2} \alpha^2 (\nabla \phi)^2 - e^{p\alpha\phi} F_2^2 \right\}. \tag{2.9}$$

By choosing

$$\alpha = -\frac{2}{\sqrt{p(p+2)}}\tag{2.10}$$

we recover the action (2.1) with dilaton coupling constant

$$a = \beta = \sqrt{\frac{p}{p+2}}$$
 $p = 0, 1, 2, 3, \dots$ (2.11)

For these values of a any solution $g_{\mu\nu}(x)$, $\phi(x)$ and $F_2(x)$ of the E-L equations of (2.1) can be interpreted as a solution of the E-L equations of the (4+p)-dimensional action (2.5) with metric and two form

$$ds^{2} = e^{2(a-a^{-1})\phi(x)} d\mathbf{y} \cdot d\mathbf{y} + e^{2a\phi(x)} g_{\mu\nu}(x) dx^{\mu} dx^{\nu}$$

$$F_{2} = \frac{1}{2} F_{\mu\nu}(x) dx^{\mu} \wedge dx^{\nu} .$$
(2.12)

For example, the magnetically-charged extreme dilaton black hole solution (2.2) can be viewed as the double-dimensional reduction of

$$ds^{2} = \left(1 - \frac{\mu}{r}\right)^{\frac{2}{p+1}} \left(-dt^{2} + d\mathbf{y} \cdot d\mathbf{y}\right) + \left(1 - \frac{\mu}{r}\right)^{-2} dr^{2} + r^{2} d\Omega_{2}^{2}$$
 (2.13)

which is the metric of a black p-brane aligned with the y-axes. The singularity of the (4 + p)-metric at $r = \mu$ is now merely a coordinate singularity. To see this,

it is useful to first determine the behaviour of the metric in the neighbourhood of $r = \mu$ by introducing a new radial coordinate ω by the relation

$$(p+1)\mu\omega = \left(1 - \frac{\mu}{r}\right)^{\frac{1}{p+1}} \Leftrightarrow r = \frac{\mu}{1 - [(p+1)\mu\omega]^{p+1}}$$
 (2.14)

which is valid for $r > \mu$, to arrive at

$$ds^{2} = (p+1)^{2} \mu^{2} \left[\omega^{2} (-dt^{2} + d\mathbf{y} \cdot d\mathbf{y}) + \left(1 - [(p+1)\mu\omega]^{p+1}\right)^{-4} \left(\frac{d\omega}{\omega}\right)^{2} \right] + \left(1 - [(p+1)\mu\omega]^{p+1}\right)^{-2} \mu^{2} d\Omega_{2}^{2}.$$
(2.15)

As $\omega \to 0$ we have

$$ds^{2} \sim (p+1)^{2} \mu^{2} \left[\omega^{2} \left(-dt^{2} + d\mathbf{y} \cdot d\mathbf{y} \right) + \left(\frac{d\omega}{\omega} \right)^{2} \right] + \mu^{2} d\Omega_{2}^{2} . \tag{2.16}$$

This asymptotic metric is the product of S^2 with (p+2)-dimensional anti-de Sitter space, $(adS)_{(p+2)}$, as we now show.

Anti-de Sitter space $(adS)_{(p+2)}$ can be viewed as the hypersurface

$$X_0^2 - \mathbf{Y} \cdot \mathbf{Y} + X_+ X_- = 1 \tag{2.17}$$

in a (p+3)-dimensional flat spacetime with coordinates $(X_0, X_{\pm} = X_1 \pm X_2, \mathbf{Y})$ and metric

$$ds^{2} = -dX_{0}^{2} + d\mathbf{Y} \cdot d\mathbf{Y} - dX_{-}dX_{+}$$
 (2.18)

of signature (p+1,2). If we introduce hypersurface coordinates (t,\mathbf{y},ω) defined by

$$\omega = X_{-}$$

$$t\omega = X_{0}$$

$$\mathbf{y}\omega = \mathbf{Y} ,$$
(2.19)

then

$$X_{+} = \frac{1 + \omega^{2}(|\mathbf{y}|^{2} - t^{2})}{\omega} , \qquad (2.20)$$

and the induced hypersurface metric is found to be

$$ds^{2} = -\omega^{2}(dt^{2} - d\mathbf{y} \cdot d\mathbf{y}) + \omega^{-2}d\omega^{2}$$
(2.21)

which is the metric obtained previously. This form of the $(adS)_{(p+2)}$ metric is analogous to the spatially flat slicing of de Sitter space. Since the coordinates $(X_0, X_{\pm}, \mathbf{Y})$ evaluated on the hypersurface are necessarily smooth, it is clear from (2.19) that ω remains a good coordinate at the horizon $\omega = 0$, while t and \mathbf{y} become ill defined. It is also clear that the symmetry generating translations of \mathbf{y} has a fixed point at the horizon, so that if we make these coordinates periodic, the spacetime is no longer geodesically complete.

We now observe that since r is an analytic function of ω near $\omega = 0$, the metric can be extended analytically through the horizon at $\omega = 0$ and hence the p-brane metric (2.13) is completely regular at $r = \mu$. Furthermore, since r is a function of ω^{p+1} , if p is odd then the full metric is invariant under the reflection

$$\omega \to -\omega$$
 . (2.22)

This implies that the analytic extension through $\omega = 0$ leads to an interior region that is isometric to the exterior region, which is singularity free. This interior region has its own horizon through which we can continue the analytic extension as before. Proceeding in this way one obtains a completely non-singular maximal analytic extension of the p-brane metric of (2.13). Thus, as claimed, all singularities of the four-dimensional dilaton black hole are resolved in this way whenever the dilaton coupling constant a takes one of the values

$$a = \sqrt{\frac{p}{p+2}}$$
 $p = 1, 3, 5, \dots$ (2.23)

The global structure of these solutions is illustrated in the Carter-Penrose diagram of Figure 1, below.

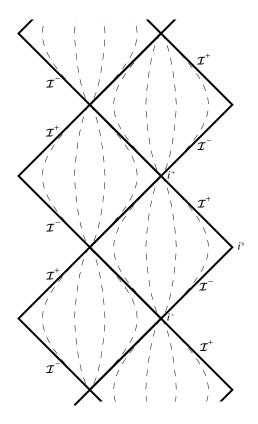


Fig. 1. CP diagram for the maximal analytic exte

Each point on the diagram represents the product of a 2-sphere with a p-plane.

The dotted lines are orbits of the timelike Killing vector field $\partial/\partial t$.

If p is even rather than odd, the singularity of the four-dimensional metric at $\omega = 0$, i.e. $r = \mu$, is still resolved by the higher-dimensional interpretation, but on passing through the horizon one finds an interior region with a curvature singularity. The simplest example of this is the trivial case p = 0, in which case the 'higher-dimensional' spacetime is in fact also four-dimensional and the 'higher-dimensional' metric is just the extreme Reissner-Nordström metric.

It is amusing to note that the value a=1, which is of interest in string theory,

corresponds to the limit $p \to \infty$. Thus in some sense, the extreme black hole in string theory is the reduction of an infinite dimensional extended object (see also [16]).

A generalization of the above results is possible by allowing the two-form F_2 in four dimensions to be the appropriate projection of a (2+m)-form F_{2+m} $(m \le p)$ in the higher dimensional spacetime, but simple results are obtained only when m=0 or m=p. The former case is the one just analysed. In the latter case, the (4+p)-dimensional action is

$$S_{(4+p)} = \int d^{4+p} x \sqrt{-g} \{ R - \frac{2}{(p+2)!} F_{p+2}^2 \} . \tag{2.24}$$

For a (4+p)-metric of the form given in (2.6) and a (p+2)-form with Hodge dual

$${}^{\star}F_{p+2} = \frac{1}{2}\tilde{F}_{\mu\nu}(x)dx^{\mu} \wedge dx^{\nu} , \qquad (2.25)$$

the (4 + p)-dimensional action again reduces to the four-dimensional action (2.1) with dilaton coupling constant given by (1.1) if $2\beta + p\alpha = 0$, as before, and if α is now taken to have the opposite sign:

$$\alpha = +\frac{2}{\sqrt{p(p+2)}} \ . \tag{2.26}$$

It follows that electrically-charged extreme dilaton black holes with $a = \sqrt{p/(p+2)}$ can also be interpreted in (4+p)-dimensions as a p-brane solution whose metric is still given by (2.13) but instead of the two-form $F_2 = Q\epsilon_2$, one has its dual, which is a (p+2)-form. This electrically-charged p-brane solution is still non-singular on its Killing horizon and completely non-singular if p is odd.

3. Generalization to higher dimensions

Given a solution of the (d+p)-dimensional Einstein equations with a horizon and p-fold translational symmetry, i.e. a black p-brane, one can always find a dilatonic black hole solution of d-dimensional gravity by double-dimensional reduction. As we have seen from the preceding examples the resulting dilatonic black hole may have a singularity where the non-dilatonic p-brane had a horizon. If the maximal analytic extension of the p-brane solution is non-singular it can be considered to resolve the singularities of the black hole. The examples discussed in the previous section constitute the special case for which d=4. Now we shall consider the general case. Note that given a non-singular (d+p)-dimensional p-brane there is no necessity to double-dimensionally reduce it by p dimensions; one might equally wish to reduce it by p dimensions to arrive at a singular dilatonic (p-q)-brane in (d+p-q) dimensions. The singularities of these dilatonic extended objects are equally resolved by the (d+p)-dimensional p-brane. Since the choice of q is immaterial to the main result we shall discuss here only the case q=p, i.e. d-dimensional dilatonic black holes.

We start from the (d+p)-dimensional action

$$S = \int d^{(d+p)}x\sqrt{-g}\left\{R - \frac{2}{(d-2)!}F_{d-2}^2\right\}$$
 (3.1)

As before we perform the reduction by p dimensions by taking the (d+p)-metric and (d-2)-form $F_{(d-2)}$ to be

$$ds_{(d+p)}^{2} = e^{2\alpha\phi(x)}d\mathbf{y} \cdot d\mathbf{y} + e^{2\beta\phi(x)}g_{\mu\nu}(x)dx^{\mu}dx^{\nu}$$

$$F_{(d-2)} = \frac{1}{(d-2)!}F_{\mu_{1}\cdots\mu_{d-2}}dx^{\mu_{1}}\cdots dx^{\mu_{d-2}},$$
(3.2)

where x^{μ} are the coordinates of the d-dimensional spacetime. Using the formula

$$\tilde{R} = \Omega^{-2} [R - 2(d-1)\nabla^2 \ln \Omega - (d-1)(d-2)(\nabla \ln \Omega)^2]$$
(3.3)

for two d-dimensional metrics g and \tilde{g} related by $\tilde{g} = \Omega^2 g$, one readily finds that the d-dimensional action is of the canonical Einstein-Hilbert form provided that α

and β satisfy

$$p\alpha + (d-2)\beta = 0 , \qquad (3.4)$$

which generalizes (2.8). We may then fix the normalization of the dilaton field kinetic term by choosing α such that

$$\alpha^2 = \frac{2(d-2)}{p(d+p-2)} \ . \tag{3.5}$$

The d-dimensional action is now

$$S = \int d^{d}x \sqrt{-g} \left\{ R - 2(\nabla \phi)^{2} - \frac{2}{(d-2)!} e^{-2a\phi} F_{d-2}^{2} \right\}, \qquad (3.6)$$

where

$$a = \frac{(d-3)\sqrt{2p}}{\sqrt{(d-2)(d+p-2)}}. (3.7)$$

The magnetically charged black hole solutions of the E-L equations of this action are [5]

$$ds_{d}^{2} = -\left[1 - \left(\frac{r_{+}}{r}\right)^{d-3}\right] \left[1 - \left(\frac{r_{-}}{r}\right)^{d-3}\right]^{1 - (d-3)\gamma} dt^{2}$$

$$+ \left[1 - \left(\frac{r_{+}}{r}\right)^{d-3}\right]^{-1} \left[1 - \left(\frac{r_{-}}{r}\right)^{d-3}\right]^{\gamma - 1} dr^{2} + r^{2} \left[1 - \left(\frac{r_{-}}{r}\right)^{d-3}\right]^{\gamma} d\Omega_{(d-2)}^{2}$$

$$e^{a\phi} = \left[1 - \left(\frac{r_{-}}{r}\right)^{d-3}\right]^{-\frac{(d-3)\gamma}{2}}$$

$$F_{(d-2)} = Q\varepsilon_{d-2},$$
(3.8)

where ε_{d-2} is the volume form on the unit (d-2)-sphere,

$$\gamma \equiv \frac{2p}{(d-2)(p+1)} \,, \tag{3.9}$$

and the charge Q is related to r_{\pm} by

$$Q^{2} = \frac{(d+p-2)(d-3)}{2(p+1)} (r_{+}r_{-})^{d-3}.$$
 (3.10)

The (d+p)-metric is therefore

$$ds^{2} = -\left[1 - \left(\frac{r_{+}}{r}\right)^{d-3}\right] \left[1 - \left(\frac{r_{-}}{r}\right)^{d-3}\right]^{\frac{1-p}{1+p}} dt^{2} + \left[1 - \left(\frac{r_{-}}{r}\right)^{d-3}\right]^{\frac{2}{p+1}} d\mathbf{y} \cdot d\mathbf{y}$$
$$+ \left[1 - \left(\frac{r_{+}}{r}\right)^{d-3}\right]^{-1} \left[1 - \left(\frac{r_{-}}{r}\right)^{d-3}\right]^{-1} dr^{2} + r^{2} d\Omega_{(d-2)}^{2}$$
(3.11)

This is a new class of black p-brane solutions of the action (3.1). In the general nonextremal $(r_+ > r_-)$ case, both the black holes (3.8) and the black p-branes (3.11) have event horizons at $r = r_+$ and curvature singularities at $r = r_-$.

However, the extremal limit $r_+ = r_- \equiv \mu$ is different. In this case, the black hole horizon becomes singular, but the black *p*-brane solution takes the form *

$$ds^{2} = \left[1 - \left(\frac{\mu}{r}\right)^{d-3}\right]^{\frac{2}{p+1}} (-dt^{2} + d\mathbf{y} \cdot d\mathbf{y}) + \left[1 - \left(\frac{\mu}{r}\right)^{d-3}\right]^{-2} dr^{2} + r^{2} d\Omega_{(d-2)}^{2}$$

$$F = \sqrt{\frac{(d+p-2)(d-3)}{2(p+1)}} \,\mu^{d-3} \varepsilon_{d-2}$$
(3.12)

This metric is invariant under the full (p+1)-Poincaré group, which is also true for the extremal limit of dilatonic black p-branes [5]. The only difference between (3.12) and the metric of (2.13) is the power of r and the fact that the last term in (3.12) involves the metric on the unit (d-2)-sphere in place of the 2-sphere. Since the horizon is at a nonzero value of r, the different powers of r is unimportant and the previous analysis of the global structure carries over to this case: the metric is non-singular at the Killing horizon $r = \mu$, and good coordinates can be found as before. The analytic continuation through this horizon is symmetric if p is odd and in this case the maximal analytic extension is completely non-singular. Near the horizon the metric is asymptotic to $(adS)_{(p+2)} \times S^{(d-2)}$. The Carter-Penrose diagram is the same as that given previously for d=4 except that each point now represents the product of a (d-2)-sphere with a p-plane.

^{*} Correcting a typographical error of [8].

For appropriate values of d and p the metric (3.12) includes all the known extreme, non-dilatonic, extended object solutions of higher-dimensional supergravity theories. These are the membrane [6] and fivebrane [12] solutions of eleven-dimensional supergravity, the self-dual three-brane of ten-dimensional supergravity [5] and the self-dual string of six-dimensional supergravity [17].

4. Generalization to multi-p-branes

By introducing the new radial coordinate ρ via

$$r^{d-3} = \rho^{d-3} + \mu^{d-3} , \qquad (4.1)$$

and a new set of cartesian transverse space coordinates $\{x^i; i=1...d-1\}$ such that $\rho=\sqrt{\sum x^i x^i}$, the extreme *p*-brane solution of (3.12) can be written in the 'isotropic' form

$$ds^{2} = H^{-\frac{2}{p+1}} \left(-dt^{2} + d\mathbf{y} \cdot d\mathbf{y} \right) + H^{\frac{2}{d-3}} \left(d\mathbf{x} \cdot d\mathbf{x} \right)$$

$$\frac{1}{(d-2)!} \varepsilon^{ij_{1} \cdots j_{(d-2)}} F_{j_{1} \cdots j_{(d-2)}} = \sqrt{\frac{(d+p-2)}{2(p+1)(d-3)}} \, \partial_{i} H$$
(4.2)

where $\varepsilon^{i_1\cdots i_{(d-1)}}$ is the constant alternating tensor density of the Euclidean (d-1)space, and

$$H = 1 + \left(\frac{\mu}{\rho}\right)^{d-3} \,. \tag{4.3}$$

Because of a force balance between the gravitational and Coulomb forces, a multip-brane solution is possible in which H is any solution of Laplace's equation in Euclidean (d-1)-space with k point 'sources' at $\mathbf{x} = \mathbf{x}_a$, i.e.

$$H = 1 + \left(\frac{\mu}{\rho}\right)^{d-3} + \sum_{a=2}^{k} \left(\frac{\mu_a}{|\mathbf{x} - \mathbf{x}_a|}\right)^{d-3}$$

$$\tag{4.4}$$

where we have used the translational invariance to locate the first point 'source' at the origin. The 'sources' are actually the horizons of the individual p-branes and if

there is an analytic continuation of the metric through these horizons then it will not be necessary to suppose that there are material point sources there.

To investigate whether this analytic continuation is possible we expand the terms in the sum of (4.4) as an analytic series of the form

$$\sum_{\ell=0}^{\infty} a_{\ell} \rho^{\ell} P_{\ell} \tag{4.5}$$

where P_{ℓ} are a complete set of harmonics on S^{d-2} . One may now attempt to analytically continue through the horizon at $\rho = 0$ as before. The variable analogous to ω of (2.14) is given by

$$(p+1)\mu\,\omega = \left[1 + \left(\frac{\mu}{\rho}\right)^{d-3}\right]^{-\frac{1}{p+1}}$$
 (4.6)

or

$$\rho = \mu \left[(p+1)\mu\omega \right]^{\frac{p+1}{d-3}} \left[1 - \left[(p+1)\mu\omega \right]^{p+1} \right]^{-\frac{1}{d-3}}$$

$$= f(\omega)\omega^{\frac{p+1}{d-3}}$$
(4.7)

where f is an analytic function at $\omega = 0$. We deduce that

$$H = \left[(p+1)\mu \,\omega \right]^{-(p+1)} + \sum_{\ell=0}^{\infty} a_{\ell}(\omega) \omega^{\frac{(p+1)\ell}{d-3}} P_{\ell}$$

$$= \left[(p+1)\mu \,\omega \right]^{-(p+1)} \left[1 + \sum_{\ell=d-3}^{\infty} b_{\ell}(\omega, \Omega) \omega^{\frac{(p+1)\ell}{d-3}} \right]$$

$$(4.8)$$

where the a_{ℓ} are analytic functions of ω and the b_{ℓ} are analytic functions of ω that are also functions on the (d-2)-sphere. The leading term is similar to what we had before, and corresponds to a regular horizon at $\omega = 0$. The higher order terms involve powers of ω^{ν} where

$$\nu \equiv \frac{p+1}{d-3} \tag{4.9}$$

The p-brane solution (4.2) will therefore have an analytic continuation through the horizon at $\rho = 0$ (and then, evidently, through any of the k horizons) provided that ν is an integer.

This condition is always satisfied for d=4. In this case, since ν is an even integer when p is odd, we recover the same criterion as before for the symmetric extension through the horizon, and consequent complete non-singularity. When d>4 the condition (4.9) is harder to satisfy. The $\nu=1$ case includes the D=6 self-dual string and the D=10 self-dual three-brane. Both of these cases have p odd and hence are completely nonsingular as a single object, but when more than one object is present, the extension through the horizon is now asymmetric and there are likely to be singularities in the interior regions. This is perhaps an indication that the nonsingularity of the single-core solutions is not a stable feature. The most interesting example of a $\nu=2$ case for d>4 would appear to be the multi-version of the D=11 fivebrane, which is a solution of 11-dimensional supergravity that interpolates between eleven-dimensional Minkowski spacetime and the S^4 compactification to adS_7 [11]. An interesting point about this case is that if one chooses a solution of the Laplacian for which $H\to 0$ as $\rho\to\infty$, i.e.

$$H = \sum_{a=1}^{k} \left(\frac{\mu_a}{|\mathbf{x} - \mathbf{x}_a|} \right)^3, \tag{4.10}$$

the resulting non-singular spacetime can be viewed as one that interpolates between many different, effectively seven-dimensional, spacetimes with differing values (determined by the constants μ_a) of the cosmological constant. This is a 'local compactification' of the type envisaged by van Baal et al. [13].

If ν is not an integer, then we have not found coordinates in which the metric is smooth at the horizon. While it is still possible that such coordinates exist, it is more likely that the horizons are only C^k for a finite k. This is similar to what was found for multi-black holes in de Sitter space [18]. The simplest example of noninteger ν is multi-black holes in five dimensions (coupled to a three form) which has $\nu = 1/2$. The smoothness of these solutions deserves to be investigated further.

5. An energy bound for p-branes

A sufficient condition for the stability, subject to given boundary conditions, of a solution to a classical field theory is that its energy saturate a lower bound on the energy of all field configurations satisfying the boundary conditions. As mentioned in the introduction, certain 'extreme' charged black hole solutions are known to saturate such a bound. Here we consider the circumstances under which a similar bound can be derived for the more general case of a p-brane solution of a D-dimensional field theory. The solutions whose stability we wish principally to consider are infinite planar p-branes, for which the total energy is clearly infinite. In this case, however, the relevant concept is the total energy per unit p-volume and for these purposes we can suppose that the D-dimensional spacetime has the topology $\mathbf{R}^d \times T^p$ and that the p-brane is wrapped around the p-torus. The energy per unit p-volume is then the (now finite) total energy divided by the volume of the p-torus. Note that the concept of spatial infinity is now replaced by 'transverse spatial infinity' and the (D-2)-sphere at spatial infinity is replaced by the (D-2)-dimensional surface $S^{d-2} \times T^p$.

In the gravitational context, to define the total energy per unit p-volume of a classical p-brane solution one needs a vector field that is asymptotically timelike and Killing near transverse spatial infinity. As in the p=0 case [19], this vector field can be replaced by a complex commuting spinor field ϵ (of the D-dimensional Poincare group) that is asymptotic to a constant spinor ϵ_{∞} . The total, transverse, d-momentum per-unit p-volume, P_{μ} , can then be expressed via the integral

$$\bar{\epsilon}_{\infty}(\Gamma^{\mu}P_{\mu})\epsilon_{\infty} = \frac{1}{2V_{p}\Omega_{(d-2)}} \oint dS_{mn}E^{mn} , \qquad (5.1)$$

where dS_{mn} is the surface element of the (D-2)-surface $S^{d-2} \times \mathbf{T}^p$ at transverse spatial infinity, V_p is the volume of the p-torus, $\Omega_{(d-2)}$ is the volume of the unit (d-2)-sphere, and

$$E^{mn} = \frac{1}{2}\bar{\epsilon}\Gamma^{mnp}\nabla_p\epsilon + c.c. \tag{5.2}$$

is the D-dimensional Nester tensor. The conjugate spinor $\bar{\epsilon}$ is defined by

$$\bar{\epsilon} \equiv \epsilon^{\dagger} \Gamma^{\underline{0}} \tag{5.3}$$

where underlining indicates an orthonormal frame index. Note that

$$\Gamma^{\underline{0}}(\Gamma^{(k)})^{\dagger}\Gamma^{\underline{0}} = -(-1)^{\frac{k(k+1)}{2}}\Gamma^{(k)}$$

$$(5.4)$$

where $\Gamma^{(k)}$ is an antisymmetrized product of k gamma matrices.

We wish to consider all metrics on $R^d \times T^p$ that are asymptotically flat in the sense that the deviation h_{mn} of the metric from flat space falls off asymptotically like $O(1/r^{d-3})$ where r is a transverse radial distance. Letting x^{μ} denote the coordinates on R^d and y^i denote the coordinates on T^d , we further require that $h_{\mu i}$ and the derivative of h_{mn} with respect to y^i both fall off like $O(1/r^{d-4})$. This ensures that there are no Kaluza-Klein type charges in the d dimensional space, and that the leading order deviation is independent of y^i . Under these conditions the integral (5.1) can be written as

$$\bar{\epsilon}_{\infty}(\Gamma^{\mu}P_{\mu})\epsilon_{\infty} = \frac{1}{2\Omega_{(d-2)}} \oint dS_{\mu\nu}E^{\mu\nu} , \qquad (5.5)$$

where the integral is now over the (d-2)-sphere at infinity. One can show that

$$M = \sqrt{-P^{\mu}P_{\mu}} \ . \tag{5.6}$$

equals the standard ADM mass, in units for which G = 1, of a spacetime which is asymptotically $R^d \times T^p$ [20]. For the p-brane metrics of (3.12) one finds that

$$M = \frac{(D-2)}{2(p+1)}\mu^{d-3} \ . \tag{5.7}$$

so that the mass to charge ratio of these solutions is

$$\frac{M}{|Q|} = \sqrt{\frac{(D-2)}{2(p+1)(d-3)}} \tag{5.8}$$

We now have the necessary ingredients for a proof of the positivity of the energy per unit p-volume but we are interested in obtaining a stronger lower bound

in terms of the electric or magnetic charge per unit p-volume. To this end we now introduce an n-form abelian field-strength, n to be specified shortly, and the associated modified Nester tensor

$$\hat{E}^{mn} = \frac{1}{2} \bar{\epsilon} \Gamma^{mnp} \hat{\nabla}_p \epsilon + c.c. , \qquad (5.9)$$

via the modified covariant derivative

$$\hat{\nabla}_p = \nabla_p + \frac{c}{(D-2)n!} \left[(n-1)\Gamma_p^{m_1...m_n} - n(D-n-1)\delta_p^{m_1}\Gamma^{m_2...m_n} \right] F_{m_1...m_n}$$
 (5.10)

where c is a constant. Note that there are two possible terms proportional to the components of the n-form F. The relative coefficient is fixed by requiring the cancellation of the ' $(\epsilon\Gamma\nabla\epsilon)F$ ' terms in the calculation to follow^{*}. This cancellation also requires that

$$\bar{c} = (-1)^{\frac{n(n+1)}{2}} c$$
 (5.11)

where \bar{c} is the complex conjugate of c; thus c is either real or purely imaginary. A short exercise in gamma matrix algebra yields

$$\hat{E}^{\mu\nu} = E^{\mu\nu} - \frac{c}{n!} (\bar{\epsilon} \Gamma^{\mu\nu p_1 \dots p_n} \epsilon) F_{p_1 \dots p_n} - \frac{c}{(n-2)!} (\bar{\epsilon} \Gamma^{p_3 \dots p_n} \epsilon) F^{\mu\nu}_{p_3 \dots p_n}$$
 (5.12)

For definiteness we shall now suppose that the p-brane is purely magnetic, in which case the last term vanishes and n = d - 2. We shall also assume that $d \ge 4$, so $n \ge 2$. Thus, in what follows the integers d and n are related to the spacetime dimension D and the spatial dimension p of the extended object by

$$d = D - p \ge 4$$

$$n = d - 2 = D - p - 2.$$
 (5.13)

We further suppose, as part of the boundary conditions to be satisfied by the fields, that the only components of F that contribute to the integral over the surface at

^{*} It is also required by considerations of linearized supersymmetry [21], but we do not wish to assume supersymmetry here.

infinity are $F_{\mu_1...\mu_n}$. Then

$$\frac{1}{2V_p\Omega_{(d-2)}} \oint dS_{mn} \hat{E}^{mn} = \bar{\epsilon}_{\infty} [\Gamma^{\mu} P_{\mu} - cQ\Gamma_*] \epsilon_{\infty}$$
 (5.14)

where

$$Q = \frac{1}{2(d-2)!\Omega_{(d-2)}} \oint dS_{\mu\nu} \varepsilon^{\mu\nu\rho_3...\rho_d} F_{\rho_3...\rho_d}$$
 (5.15)

is the magnetic charge per unit p-volume and

$$\Gamma_* = \Gamma^{\underline{0}} \Gamma^{\underline{1}} \cdots \Gamma^{\underline{(d-1)}} . \tag{5.16}$$

If we can show that the left hand side of (5.14) is positive semi-definite then, using (5.11), we can derive the bound

$$M \ge |c||Q| \tag{5.17}$$

on the total mass per unit p-volume.

Assuming that any horizons or singularities are the result of gravitational collapse from a configuration without horizons or singularities, we may choose a spacelike hypersurface, on which all fields are regular, such that its only boundary is the surface at transverse spatial infinity. Then, Gauss' law can be used to write the surface integral on the left hand side of (5.14) as a volume integral:

$$\frac{1}{2\Omega_{(d-2)}} \oint dS_{mn} \hat{E}^{mn} = -\frac{1}{\Omega_{(d-2)}} \int dS_n \nabla_m \hat{E}^{mn}$$
(5.18)

Then, using the field equation

$$G_{mn} = 2T_{mn}(F) , \qquad (5.19)$$

where

$$T_{mn}(F) \equiv \frac{1}{(n-1)!} (F_{mr_1...r_{(n-1)}} F_n^{r_1...r_{(n-1)}} - \frac{1}{2n} g_{mn} F^2) , \qquad (5.20)$$

one finds after a long calculation that

$$\nabla_{m}\hat{E}^{mn} = \overline{\hat{\nabla}_{m}\epsilon} \, \Gamma^{mnp} \hat{\nabla}_{p}\epsilon - \frac{c}{(n-2)!} (\bar{\epsilon}\Gamma_{p_{3}...p_{n}}\epsilon) \nabla_{m}F^{mnp_{3}...p_{n}}$$

$$- \left(1 - \frac{2(D-n-1)(n-1)|c|^{2}}{(D-2)}\right) (\bar{\epsilon}\Gamma_{m}\epsilon) \, T^{mn}(F)$$

$$- \frac{2|c|^{2}}{(D-2)} \sum_{k=1}^{\infty} \frac{(-1)^{k} (n^{2} - nD + (2k+1)D - 4k - 1)}{(n-2k-1)!(2k)!(2k+1)!} \times$$

$$\times (\bar{\epsilon}\Gamma_{mp_{1}...p_{2k}q_{1}...q_{2k}}\epsilon) \, T^{mnp_{1}...p_{2k}q_{1}...q_{2k}}(F) ,$$
(5.21)

where

$$T^{mnp_1...p_{2k}q_1...q_{2k}}(F) \equiv \left[F^{mp_1...p_{2k}r_1...r_{(n-2k-1)}} F^{nq_1...q_{2k}} r_{1...r_{(n-2k-1)}} - \frac{(2k+1)}{2(n-2k)} g^{mn} F^{p_1...p_{2k}r_1...r_{(n-2k)}} F^{q_1...q_{2k}} r_{1...r_{(n-2k)}} \right] .$$

$$(5.22)$$

Note that the potentially infinite sum, which contains only terms of the form $\bar{\epsilon}\Gamma^{(4k+1)}\epsilon F^{2}$, truncates itself as soon as 4k+1 exceeds either D or 2n-1. We want all terms on the right hand side of (5.21), except the first, to cancel on using the F field equation. In this case we will have

$$\frac{1}{2\Omega_{(d-2)}} \oint dS_{mn} \hat{E}^{mn} = -\frac{1}{\Omega_{(d-2)}} \int dS_n \overline{\hat{\nabla}_m \epsilon} \, \Gamma^{mnp} \hat{\nabla}_p \epsilon \tag{5.23}$$

which can be shown to be positive semi-definite by choosing ϵ to satisfy the usual modified Witten condition.

At this point we need the field equation for F in order to proceed. The field equation (5.19) is derivable from an action of the form

$$S = \int d^{D}x \sqrt{-g} \left(R - \frac{2}{n!}F_{n}^{2} + \text{possible } \varepsilon AFF \text{ term}\right), \qquad (5.24)$$

where we allow for a possible abelian Chern-Simons (CS) term in the action when D = 3n - 1 because it does not affect the stress tensor $T_{mn}(F)$. The simplest case for which this term may appear is D = 5 and n = 2.

We now turn to the investigation of when the necessary cancellations can occur. The simplest possibility occurs when all the $\bar{\epsilon}\Gamma^{(4k+1)}\epsilon F^2$ terms vanish for $k\geq 1$. Given that $d\geq 4$, this happens only for D=4 and n=2, i.e. p=0. In this case no CS term is possible so the F field equation is simply

$$\nabla_m F^{mn} = 0 \ . \tag{5.25}$$

By choosing

$$|c| = 1 \tag{5.26}$$

we ensure the cancellation of the remaining $(\bar{\epsilon}\Gamma_m \epsilon)T^{mn}(F)$ terms and (5.21) now reduces to (5.23) as required. We thus recover the bound $M \geq |Q|$ [4] on magnetically-charged particle-like solutions of four-dimensional Maxwell-Einstein theory.

For D > 4 the derivation of a similar bound is necessarily more complicated because a $\bar{\epsilon}\Gamma^{(5)}\epsilon F^2$ term, at least, then appears on the right hand side of (5.21). One might hope that its coefficient could vanish but this happens only if

$$n = \frac{1}{2}(D \pm \sqrt{(D-2)(D-10)}) , \qquad (5.27)$$

which is never satisfied for 2 < D < 9. In fact the only real integer solutions of this equation, with $n \ge 2$, that we have found are

(i)
$$D = 10$$
 and $n = 5$ (ii) $D = 11$ and $n = 4$ or $n = 7$

The absence of the $\bar{\epsilon}\Gamma^{(5)}\epsilon F^2$ term in these cases will be seen to be of great importance for the derivation of a Bogomol'nyi-type bound for the ten-dimensional self-dual threebrane and the eleven-dimensional fivebrane and membrane. In all these cases, however, and whenever $n \geq 4$, the $\bar{\epsilon}\Gamma^{(9)}\epsilon F^2$ term also appears on the right hand side of (5.21), unless its coefficient vanishes, which happens only if

$$n = \frac{1}{2}(D \pm \sqrt{(D-2)(D-18)}) . (5.28)$$

This cannot be satisfied if 2 < D < 18. Thus, for D > 4 we always have, at least, either a $\bar{\epsilon}\Gamma^{(5)}\epsilon F^2$ or a $\bar{\epsilon}\Gamma^{(9)}\epsilon F^2$ term on the right hand side of (5.21). In order

to derive a Bogomol'nyi-type bound we must find a way to cancel these terms. The freedom that we have to achieve this cancellation consists of, when applicable, (i) the possibility of choosing F to be self-dual and spinor ϵ to be chiral, (ii) the choice of the constant |c|, and (iii) the choice of coefficient of the CS term in the action. Consideration of these possibilities leads to two mechanisms for removal of the unwanted terms in (5.21), which we shall now examine in detail.

(a) If D = 4k' + 2 the final k = k' term $\bar{\epsilon}\Gamma^{(4k'+1)}\epsilon F^2$ term in the sum of (5.21) is equivalent to a $(\bar{\epsilon}\Gamma^{(1)}\epsilon) \epsilon F^2$ term, where ϵ indicates a Levi-Civita tensor, if ϵ is chosen to be a chiral spinor. We can get then rid of the ϵ by choosing n = 2k' + 1 and declaring F to be self-dual (note that this is possible precisely for dimensions D = 4k' + 2). Remarkably, this term is then of the form $\bar{\epsilon}\Gamma_m \epsilon T^{mn}(F)$ so it can be combined with the other stress tensor terms on the right hand side of (5.21). These terms then cancel if |c| is chosen such that

$$|c|^2 = \frac{1}{2(n-1)} = \frac{1}{2(p+1)}$$
 (5.29)

By this means we can cancel the (k=1) $\bar{\epsilon}\Gamma^{(5)}\epsilon F^2$ term for $D=6,\ n=3,$ and the (k=2) $\bar{\epsilon}\Gamma^{(9)}\epsilon F^2$ term for $D=10,\ n=5.$ Recalling that the coefficient of the $\bar{\epsilon}\Gamma^{(5)}\epsilon F^2$ term cancels when D=10 and n=5 we see that the bound

$$M \ge \frac{1}{\sqrt{2(p+1)}}|Q| \ . \tag{5.30}$$

can be established for the self-dual D=6 string (p=1) and D=10 threebrane (p=3). Observe that the square of the mass to charge ratio of a solution saturating this bound is half the value, given in (5.8), for the extreme p-brane solution of (3.12). This is because the solution (3.12) describes a magnetically charged p-brane. The self-dual solution can be obtained simply by performing a duality rotation on F. This does not change the metric, but reduces the charge (as defined in (5.15)) by $\sqrt{2}$. These solutions therefore saturate the bound.

For k'=3, i.e. D=14 we could cancel the k=3, $\bar{\epsilon}\Gamma^{(13)}\epsilon F^2$, term by this method but we would still be left with the $\bar{\epsilon}\Gamma^{(5)}\epsilon F^2$ and $\bar{\epsilon}\Gamma^{(9)}\epsilon F^2$ terms, neither of which has a vanishing coefficient. They might cancel each other, however, since 5+9=14 and F is self-dual. Similar cancellations might occur for k'>3. In view of the fact that the cancellations possible for k'=1 and k'=2 are realized by supergravity theories, a feature that is also shared by the alternative mechanism to be explained below, and the fact that there are no supergravity theories with D>11 we suspect that the required cancellations for e.g. D=14 do not actually occur. We have not undertaken the calculations necessary to verify this because we are not aware of any interest in e.g. self-dual fivebranes in D=14.

(b) When $D \neq 4k' + 2$ we must choose

$$|c|^2 = \frac{D-2}{2(n-1)(D-n-1)}$$
(5.31)

or equivalently,

$$|c|^2 = \frac{D-2}{2(p+1)(d-3)}$$
 (5.32)

in order to cancel the stress tensor terms in (5.21), since they cannot cancel with other terms in the sum. For D>4 we must therefore find some other means of cancelling the non-zero terms in the sum over k in (5.21). When D=3n-1 we have the possibility of a Chern-Simons term in the action. Use of the F field equation then produces a term of the form $(\bar{\epsilon}\Gamma^{(n-2)}\epsilon) \varepsilon F^2$ so we have a possibility of cancelling the $(\bar{\epsilon}\Gamma^{(4k'+1)}\epsilon)F^2$ term when n=2k', i.e D=3k'-1, and k=k'. For k'=1 this implies D=5 and n=2, and since there are no $\bar{\epsilon}\Gamma^{(9)}\epsilon F^2$, or higher, terms in this case the required cancellation is complete and we find the bound

$$M \ge \frac{\sqrt{3}}{2}|Q|$$
 $(D=5, p=1)$ (5.33)

for string-like solutions of five-dimensional supergravity (which includes the needed CS term with just the right coefficient). The same formula was found for electric-

type particle-like solutions of five-dimensional supergravity [14], which is presumably a reflection of the fact that for D=5 a magnetic-type string is dual to an electric-type particle.

For k'=2 this mechanism allows the cancellation of the $\bar{\epsilon}\Gamma^{(9)}\epsilon F^2$ term when D=11 and n=4. Since this is precisely one of the cases for which the coefficient of the $\bar{\epsilon}\Gamma^{(5)}\epsilon F^2$ vanishes the required cancellation is again complete and we establish the bound

$$M \ge \frac{1}{2}|Q|$$
 $(D = 11, p = 5)$ (5.34)

on the mass per unit area of a fivebrane. Both of the bounds (5.33) and (5.34) are saturated by the respective special cases of the extreme p-brane solution of (3.12), as can be seen by comparing (5.8) and (5.32).

For k' > 2 the inclusion of a CS term cannot cancel either the $\bar{\epsilon}\Gamma^{(5)}\epsilon F^2$ or the $\bar{\epsilon}\Gamma^{(9)}\epsilon F^2$ term, at least one of which must have a non-vanishing coeficient. Thus D=5 supergravity and D=11 supergravity are the only cases for which the required cancellations can be achieved by the inclusion of the CS term.

The conclusion is that, for $D \ge 4$ and $n \ge 2$, a Bogomol'nyi bound exists for magnetically charged *non-dilatonic p*-brane solitons for the following cases (and only for these cases if self-dual fivebranes in D = 14 and their generalizations to higher dimensions are excluded):

(i)
$$D = 4$$
 and $p = 0$

(ii)
$$D = 5$$
 and $p = 1$ with CS term

(iii)
$$D = 6$$
 and $p = 1$ with self-dual F

(iv)
$$D = 10$$
 and $p = 3$ with self-dual F

(v)
$$D = 11$$
 and $p = 5$ with CS term.

Each of these cases is realized by a supergravity theory. We expect that a similar analysis for electric-type p-branes would lead to the same conclusion for the electric duals of the above cases. This would allow us to extend the above list of (D, p) values to include

(vi)
$$D = 5$$
 and $p = 0$ with CS term

(vii)
$$D = 11$$
 and $p = 2$ with CS term

Case (vi) is realized by black hole solutions of D=5 supergravity [14] and case (vii) by membrane solutions of D = 11 supergravity [6].

For theories with a Bogomol'nyi-type bound, configurations which saturate this bound must be stable. They must also admit Killing spinors, i.e. solutions of

$$\hat{\nabla}_m \epsilon = 0 \tag{5.35}$$

for non-zero ϵ . This is because if M = |c||Q|, one can choose ϵ_{∞} so that

$$[\Gamma^{\mu}P_{\mu} - c\Gamma_{*}Q]\epsilon_{\infty} = 0. \qquad (5.36)$$

It then follows from (5.14) and (5.23) that there exists an ϵ which approaches ϵ_{∞} asymptotically and satisfies $\hat{\nabla}_m \epsilon = 0$. In fact, for the multi-p-brane solution of (4.2) one can show that the Killing spinor satisfying

$$\frac{c}{|c|}\Gamma^{\underline{0}}\Gamma_*\epsilon = \pm \epsilon$$

is given by

$$\epsilon = H^{\pm \frac{1}{2(p+1)}} \epsilon_{\infty} \tag{5.37}$$

6. Discussion

We have shown that the singularities in a number of extreme dilatonic black holes and p-branes can be resolved by viewing the solutions as reductions of higher dimensional objects. This raises a few issues which we address in this section. The first concerns the relation between our nonsingular solutions with horizons (2.13) and (3.12), whose maximal analytic extensions are represented by the Carter-Penrose diagram of Fig. 1, and the singularity theorems. There is nothing special about four dimensions in the proof of the singularity theorems. Higher dimensional solutions which satisfy the conditions of the theorems must be geodesically incomplete. The most important condition in our case is the existence of a compact trapped surface. Our nonsingular p-branes simply do not have any surfaces of this type. The horizon is only a marginally trapped surface, since the null generators have zero convergence, and it is not compact if the extra dimensions are not periodically identified. If they are identified, then the solution is geodesically incomplete anyway.

It is instructive to consider the relation between our non-singular metrics and the singularity theorems in more detail. It will be sufficient to consider the simplest case of the extreme black string in a five-dimensional spacetime. The singularity theorems consist of a global part (e.g. the existance of a trapped surface) and a local part which concerns the focusing of geodesics. Consider a small bundle of null rays each moving radially inwards (θ and ϕ constant) with constant $y=x^5$ coordinate; these rays are represented by a straight line at 45° in the CP diagram. As they pass from one asymptotically flat region to another their cross-section decreases and then increases. Since the five-dimensional energy-momentum tensor satisfies the null energy condition, Raychaudhuri's equation tells us that the null rays must encounter a caustic. The area of a two-sphere in the θ and ϕ directions is proportional to r^2 and since r is bounded below by μ there is no caustic in these directions. Since these geodesics have x^5 constant, they are also null geodesics in the four dimensional space. It follows that the four-dimensional metric must violate

the null energy condition or be singular, according to the choice of 'conformal gauge'; in the five-dimensional Einstein conformal gauge the null energy condition is violated while the metric in the four-dimensional Einstein conformal gauge is singular. However, the cross section in the fifth, x^5 , direction is proportional to $\left(1-\frac{\mu}{r}\right)$ and this does focus at $r=\mu$. In other words, the rays do encounter a caustic but it is by approaching each other in the x^5 direction. Of course, the fact that the rays encounter a caustic does not mean that there is a singularity (this is where the global condition is needed). In fact, there is no singularity. However, this example does serve to illustrate the fact that when considering focussing (which is an important part of the singularity theorems) one must consider not just the four physical spacetime dimensions but the extra dimensions too.

We next consider how general the method we have described is for resolving singularities. Is it possible that all physical singularities in four dimensions could be resolved in this way? Unfortunately, the answer appears to be no, at least not without serious modification. The point is that in our approach the higher dimensional metric includes the lower dimensional one rescaled by a power of the dilaton. So roughly speaking, only singularities in the Ricci tensor and not the Weyl tensor can be removed this way.* For example, it seems unlikely that one can find a regular higher dimensional solution whose reduction yields the Schwarzschild metric.

A related point is that given a theory with a dilaton, there is some ambiguity about which metric one should consider in deciding whether a solution is singular. One often considers the metric with the standard Einstein action, but other metrics related by a conformal rescaling may be physically important depending on the coupling to matter. For example, if one rescales the extreme dilaton black hole metric (2.2) by $e^{2\phi/a}$, the resulting metric is geodesically complete. So if one adds matter which couples to this metric, the solution might be called nonsingular

 $[\]star$ This is not strictly true since the dilaton is also diverging, and a singular conformal factor can transform a diverging C^2 into one that remains finite.

even in four dimensions. (This ambiguity is not present in the higher dimensional theories we consider here which do not include a dilaton.)

However, it should be emphasized that we have explored only the simplest possibilities of obtaining lower dimensional solutions from higher dimensional ones. We started with just an Einstein-Maxwell type action in higher dimensions and did not include off diagonal terms in the higher dimensional metric. It is possible that a more general procedure will be able to resolve other familiar four dimensional singularities.

Acknowledgements

G.H. was supported in part by NSF grant PHY-9008502.

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