Pulse Propagation in Born-Infeld Theory, the World-Volume Equivalence Principle and the Hagedorn-like Equation of State of the Chaplygin Gas

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Abstract

A recently proposed world-volume equivalence principal involving the Boillat, as opposed to the Einstein, metric is examined in the context of some colliding wave solutions of the Born-Infeld equations for which two plane polarized pulses pass through one another without distortion. They suffer a delay with respect to the usual Einstein metric but not with respect to the Boillat metric. Both metrics are flat in this case, and the closed string and open string causal structures are interchanged by the Legendre transformation that is used for solving the associated Monge-Ampère equation. In 1+1 dimensions the equations are known to be equivalent to the vorticity free motion of a Chaplygin gas. The latter is shown to be described by the scalar Born-Infeld equation in all dimensions and it is pointed out that the equation of state is Hagedorn-like: there is an upper bound to the pressure and temperature.

1 Introduction

In two recent papers [1, 2] it has been argued that the propagation of excitations around D-branes, and also those around the M5-brane, satisfy a a generalization of the usual Einstein Equivalence Principal. One finds that the equations of motion are universally governed by a co- metric $C^{\mu\nu}$, called there the Boillat metric, which, in the presence of vector fields or tensor fields differs from the pullback $g_{\mu\nu}$ of the usual Einstein metric in such a way that, using just world volume fields, the Einstein metric $g_{\mu\nu}$ is effectively unobservable. In string

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theory the Einstein metric is the closed string metric whle the Boillat metric is conformal to the open string metric. In [2] analogy was drawn with treatments of general relativity in which one introduces an auxiliary flat metric $\eta_{\mu\nu}$. Because of the usual Einstein Equivalence Principal any such flat metric is in effect unobservable. The characteristic co-metric of any set of equations of motion is determined only up to to a Weyl re-scaling. In the case of D-branes one might for example use instead the open string co-metric $G^{\mu\nu}$. This is conformal to the Boillat metric. The choice of the particular conformal representative $C^{\mu\nu}$ may be motivated by at least two arguments.

- In the case of the D3-brane, $C^{\mu\nu}$, unlike $G^{\mu\nu}$ is invariant under S-duality, i.e. under electric-magnetic duality rotations.

- In both the case of D-branes and for the M5-brane, the following relation holds between the energy momentum tensor $T^{\mu\nu}$ of the Born-Infeld field or of the closed 3-form field

$$T^{\mu\nu} = C^{\mu\nu} - g^{\mu\nu}. \tag{1}$$

For reasons explained in [2] (1) may be referred to as Hooke's Law. In the Born-Infeld case one has the further remarkable determinental identity

$$\det(C^{\mu\nu}) = \det(g^{\mu\nu}), \tag{2}$$

which implies that

$$\det\left(T^{\mu}_{\nu} + \delta^{\mu}_{\nu}\right) = 1.. \tag{3}$$

The aim of the present paper is to explore this putative World-Volume Equivalence Principle in greater detail in the simplified situation of $1+1$ dimensions. We shall rely heavily on a recent paper [3] on the propagation of pulses in non-linear electrodynamics (see also earlier work of Barbashov and Chernikov) and we shall make contact with some older work of Schrödinger [8] on Born-Infeld theory as a theory of non-linear optics and work on the relation between scalar Born-Infeld, minimal surfaces, strings and the Monge-Ampère equation. In fact the remarks in the discussion section of [3] in their $1+1$ dimensional setting have much in common with the suggestions made in [1, 2].

In attempting to generalize these results to higher dimensions, we are led to consider the scalar Born-Infeld equation in higher dimensions. This is found to be equivalent to a Chaplygin gas. It has a Hagedorn-like equation of state with an upper bound for the pressure and temperature. This may be relevant to recent attempts to use fluid dynamic ideas to discuss the strong coupling limit open strings [11, 12].

2 Hooke's Law, the Monge-Ampère Equation and Pulse interactions

The striking determinental identity (2) has an immediate application to the propagation of pulses in Born-Infeld theory.

In flat two dimensional spacetime, the conservation law for the stress tensor implies that it is given by a single free function, the Airy stress function ψ, [6] such that

$$T_{tt} = \psi_{zz}, \quad T_{zz} = \psi_{tt}, \quad T_{tz} = \psi_{zt}. \tag{4}$$

Now for a *single* polarization state for which the only non-vanishing component of the electric field is E_x , the determinental identity (2) still holds when restricted to just the propagation directions (t, z) even though the two form $F_{\mu\nu}$ is non-vanishing in the (t, x) and (z, x) directions.

Written in terms of the Airy stress function, the determinental identity (2) then becomes a special case of the Monge-Ampère equation

$$\psi_{zz}\psi_{tt} - \psi_{zt}^2 = \psi_{zz} - \psi_{tt} \tag{5}$$

This can be solved exactly [3] using the hodographic method, that is performing a Legendre transform under which it becomes D'Alembert's equation with respect to a new set of variables T and Z.

The solution given in [3] gives

$$T^{tt} = \frac{A + B + 2AB}{1 - AB}, \tag{6}$$

$$T^{zz} = \frac{A + B - 2AB}{1 - AB}, \tag{7}$$

$$T^{zt} = \frac{B - A}{1 - AB}, \tag{8}$$

where $A = A(T + Z)$ and $B = B(Z - T)$ are arbitrary functions of their arguments. The relation between the new coordinates (T, Z) and usual coordinates (t, z) is most conveniently expressed using null coordinates. Let $v = t + z, u = t - z, \xi = T + Z, \eta = Z - T$. The asymmetrical definition of η is so as to agree with [3], One has

$$dv = d\xi - Bd\eta, \qquad du = -d\eta + Ad\xi. \tag{9}$$

Thus

$$(1 - AB)d\eta = Adv - du, \qquad (1 - AB)d\xi = dv - Bdu. \tag{10}$$

one checks that

$$dT^2 - dZ^2 = dt^2(1 - A - B + AB) - dz^2(1 + A + B + AB) - 2dtdz(A - B) = C_{\mu\nu}^{-1}dx^\mu dx^n u, \tag{11}$$

where

$$C^{\mu\nu} = \eta^{\mu\nu} + T^{\mu\nu}. \tag{12}$$

Thus we see that the Legendre transformation to the new coordinates (T, Z) used to solve the Monge-Ampère equation in effect passes to flat inertial coordinates with respect to the Boillat metric. It should be noted that one does not expect the Boillat metric to be flat in general in higher dimensions.

The general solution consists of two pulses, one right-moving and one left moving which pass through one another without distortion. In terms of the usual coordinates (t, z) the two pulses experience a *delay* That is measured with respect to the closed string metric. However With respect to the Boillat coordinates, that is measured with respect to the Boillat metric, there is no delay.

3   Legendre Duality

As noted above, the two causal structures are permuted by Legendre duality. It is of interest to examine this more fully. One thinks of (t, z) as coordinates for $\mathbb{E}^{1,1}$ and (T, Z) as coordinates for the dual of the dual momentum space $\left(\mathbb{E}^{1,1\star}\right)^{\star}$. Because of the Lorentzian metric one includes an additional minus sign in the Legendre transformation. Put another way, one composes the standard Legendre map between $\mathbb{E}^{1,1}$ and $\mathbb{E}^{1,1\star}$ with the musical isomorphism $\sharp$ ("index raising ") between $\mathbb{E}^{1,1\star}$ and $\mathbb{E}^{1,1}$ to get a map between $\mathbb{E}^{1,1}$ and $\mathbb{E}^{1,1}$. The two coordinate charts (tz) and (T, Z) give rise to what Maxwell called, in the context of statics, reciprocal diagrams, one of position and one of stress [7]. The map between the two diagrams is also referred to as the hodographic transformation.

Explicitly

$$T = \frac{\partial \psi}{\partial t} \qquad Z = -\frac{\partial \psi}{\partial z}. \tag{13}$$

define the Legendre conjugate function $\phi(T, Z)$ by

$$\phi = Tt - Zz - \psi, \tag{14}$$

Thus

$$t = \frac{\partial \phi}{\partial T} \qquad z = -\frac{\partial \phi}{\partial Z}. \tag{15}$$

It now follows that Hessian of ψ composed with index raising is the inverse of the Hessian of ϕ. In two dimensions, up to a factor of the inverse of the determinant, the inverse of a matrix is a linear function of the elements of a matrix and this allows us to linearize the Monge-Ampère equation, which becomes in terms of ϕ

$$\phi_{TT} - \phi_{ZZ} = 1. \tag{16}$$

The general solution is

$$\phi = \frac{1}{2}(T^2 - Z^2) + f_L(T - Z) + f_R(T + Z), \tag{17}$$

where the arbitrary right-moving $F_L(T - Z)$ and left-moving $f_R(T + Z)$ waves are related in a simple way to the functions $A(T + Z)$ and $B(Z - T)$ introduced in [3].

Note that if we set

$$\psi = u + \frac{1}{2}(t^2 - z^2), \tag{18}$$

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then u satisfies a more familiar form of the Monge-Ampère equation

$$u_{tt}u_{zz} - u_{tz}^2 = -1. \tag{19}$$

This equation is Legendre self-dual, so using the hodograph method directly will not lead to a solution.

4   Additional Polarization States

The case considered in [3] was of a single polarization state. Now it if one includes all possible polarization states in $\mathbb{E}^{n-1,1}$, that is, if we assume that $A^\mu = (0, 0, A^i(t, z))$, $i = 2, 3, \ldots, n-2$ then [9] the Born-Infeld action reduces to a gauge-fixed Dirac-Nambu-Goto action for a string

$$\frac{1}{2}\int\left(1 - \sqrt{1 - (\partial_t A^i)^2 + (\partial_t A^i)^2}\right)dtdz \tag{20}$$

One has

$$T_{tt} + 1 = \frac{1 + (A_z^i)^2}{\sqrt{1 - (\partial_t A^i)^2 + (\partial_z A^i)^2}} \tag{21}$$

$$T_{zz} - 1 = \frac{1 - (A_t^i)^2}{\sqrt{1 - (\partial_t A^i)^2 + (\partial_z A^i)^2}} \tag{22}$$

$$T_{tz} = \frac{A_z^i A_t^i}{\sqrt{1 - (\partial_t A^i)^2 + (\partial_t A^i)^2}} \tag{23}$$

For one polarization state, the determinental identity holds, but for more than one, it does not unless.

$$(A_t^i A_z^i)^2 = (A_t^k)^2 (A_z^j)^2. \tag{24}$$

The case of just one *plane* polarization state is called the (1+1)-dimensional Born-Infeld equation and is related to other completely integrable systems. For $n = 4$ and the case of *circular polarization* Schrödinger [8] showed that one could superpose solutions. Because of the obscurity of the reference it may be helpful to desribe his method. One defines

$$\mathbf{F} = \mathbf{B} - i\mathbf{D}, \qquad \mathbf{G} = \mathbf{E} + i\mathbf{H}. \tag{25}$$

One needs to solve the *linear* equations

$$\text{div } \mathbf{F} = 0, \qquad \text{curl } \mathbf{G} + \frac{\partial \mathbf{F}}{\partial t} = 0, \tag{26}$$

subject to the *non-linear* constitutive relation

$$\overline{\mathbf{G}} = \frac{2\mathbf{F}(\mathbf{F}.\mathbf{G}) - (\mathbf{F}^2 - \mathbf{G}^2)\mathbf{G}}{(\mathbf{F}.\mathbf{G})^2}. \tag{27}$$

Schrödinger showed that exact solutions exist of the form

$$\mathbf{F} = C_1 \mathbf{a}_1 e^{i(\mathbf{k}_1 \cdot \mathbf{x} - \omega_1 t)} + C_2 \mathbf{a}_2 e^{i(\mathbf{k}_2 \cdot \mathbf{x} - \omega_2 t)} \tag{28}$$

$$\mathbf{G} = i A_1 C_1 \mathbf{a}_1 e^{i(\mathbf{k}_1 \cdot \mathbf{x} - \omega_1 t)} + i A_2 C_2 \mathbf{a}_2 e^{i(\mathbf{k}_2 \cdot \mathbf{x} - \omega_2 t)} \tag{29}$$

for appropriate choices of the constants. By Lorentz-invariance the two waves may be taken to move in opposite directions and Scrödinger showed that they do so with speeds which do not exceed that of light.

In [9, 1] it was suggested that this is a general feature for Born-Infeld. The argument was that the action (20) is the gauge fixed form of the action

$$- \int \sqrt{\det(-\eta_{\mu\nu} x_a^\mu x_b^\nu)} dt dz \tag{30}$$

in Monge (often inappropriately called static) gauge.

Now in conformal gauge:

$$\eta_{\mu\nu} x_t^\mu x_t^\mu = \eta_{\mu\nu} x_z^\mu x_z^\mu \qquad \eta_{\mu\nu} x_t^\mu x_z^\mu = 0, \tag{31}$$

the solutions are the sums of left and right movers, i.e. of functions of $t - z$ or $t + z$. Thus pulses should pass through one another without deformation.

5   Higher Dimensions

The reduction of the Born-Infeld equation to the Monge-Ampère equation using the Airy stress function in two dimensions suggests looking at the analogue in higher dimensions. Maxwell [7] provided a generalization to $\mathbb{E}^3$ which in turn is easily generalized to $\mathbb{E}^n$ and hence, with appropriate adjustment of signs, to Minkowski space $\mathbb{E}^{n-1,1}$. In three dimensions one defines three functions A, B, C by

$$T_{xy} = \partial_x \partial_y C \qquad \text{and cyclically.} \tag{32}$$

One then finds that the conservation laws imply, up to trivial redefinitions that

$$T_{xx} = -\partial_y \partial_y C - \partial_z \partial_z B \qquad \text{and cyclically} \tag{33}$$

In n dimensions one defines $\frac{1}{2} n(n-1)$ functions using the off-diagonal components of the stress tensor and uses the n conservation equations to deduce the remaining n diagonal components.

However, by contrast with the case of two dimensions, imposing a vanishing trace condition (which in two dimensions lea The underlying problem is that in general one has two many functions and thus no obvious Legendre transformation. Laplace's equation) or the determinental identity (which in two dimensions leads to the Monge-Ampère equation) does not seem to lead to recognizable integrable equations in general. In special cases, for example if $A = B = C$, some simplifications do result. A detailed consideration will be deferred until a later date.

6 The Higher-dimensional scalar Born-Infeld equation and the Chaplygin Gas

This comes from the Lagrangian

$$\int \Big(1 - \sqrt{1 - \eta^{\mu\nu}\partial_\mu A \partial_\nu A}\Big) d^n x. \tag{34}$$

One may think of A as the transverse coordinate of a domain wall or $(n-1)$-brane. The energy momentum tensor satisfies Hooke's Law

$$T^{\mu\nu} + \eta^{\mu\nu} = C^{\mu\nu}, \tag{35}$$

where the equations of motion take the form

$$C^{\mu\nu}\partial_\mu\partial_\nu\phi = 0. \tag{36}$$

The characteristic co-metric $C^{\mu\nu}$ is essentially the inverse of induced metric on the brane

$$C^{\mu\nu} = \sqrt{1 - \eta^{\mu\nu}\partial_\mu A \partial_\nu A}\Big(g^{\mu\nu} + \frac{\partial^\mu\phi\partial^\nu\phi}{1 - \eta^{\mu\nu}\partial_\mu A \partial_\nu A}\Big). \tag{37}$$

One finds that

$$\det(T^{\mu\nu} + \eta^{\mu\nu}) = (1 - \eta^{\mu\nu}\partial_\mu A \partial_\nu A)^{\frac{n-2}{2}}. \tag{38}$$

As with all Lagrangians involving only the partial derivative of a single scalar, the system is equivalent, at least for spacelike level sets of the function A, to a vorticity-free perfect fluid with 4-velocity

$$U^\mu = \frac{\partial^\mu A}{\sqrt{\eta^{\sigma\lambda}\partial_\sigma A \partial_\lambda A}}. \tag{39}$$

The pressure P and energy density ρ satisfy the equation of state

$$(\rho + 1)(P - 1) = -1. \tag{40}$$

If $n = 2$ the equation of state coincides with the determinental identity. This type of gas is called a Chaplygin gas and is known to give an integral system in 1+1 dimensions [10]. Note that, when defining the energy momentum tensor, there is the freedom to add a cosmological term. This shifts ρ and P by constants so that

$$(\rho + \lambda)(P - \lambda) = -\lambda^2. \tag{41}$$

I have fixed this freedom by demanding that ρ and P vanish if the scalar field A is constant. If instead I had chosen $\lambda = 0$ I would have obtained $P = -\frac{1}{\rho}$ which is also frequently referred as a Chaplygin equation of state. Because now the pressure is negative, this has been invoked to account for recent super-novae evaluations of the cosmological constant. In effect the Chaplygin gas is a form

of what has been called "k-essence". Form the standpoint adopted here, the negative pressure is a result of "tuning" the arbitrary constant.

The sound speed c_s follows from the equation of state

$$p = \frac{\rho}{\rho + 1} \tag{42}$$

and Newton's formula (which is equivalent to finding the the characteristic cone give by $C^{\mu\nu}$)

$$c_s = \sqrt{\frac{\partial P}{\partial \rho}}. \tag{43}$$

One has

$$c_s = \frac{1}{\rho + 1}. \tag{44}$$

The sound speed never exceeds that of light, equals that of light at low density (so-called stiff matter) and goes to zero at large density. This, and the upper bound for the pressure, is reminiscent of the Hagedorn behaviour of dual resonance models. To investigate further, and in particular to see whether there is an upper bound for the temperature, one notes that one may calculate the entropy density s and the temperature T for any vorticity-free perfect fluid with velocity potential A. One easily checks that the following general formulae hold

$$T^2 = \eta^{\mu\nu} \partial_\mu A \partial_\nu A, \tag{45}$$

$$s^2 = J_\mu J^\mu, \tag{46}$$

where the entropy current

$$J^\mu = s U^\mu, \tag{47}$$

is conserved by virtue of the Euler-Lagrange equations. Further

$$P = L \qquad \rho + P = Ts. \tag{48}$$

Thermodynamically, temperature T and entropy density s are conjugate variables and the pressure $P = P(T)$ and energy density $\rho = \rho(s)$ are related by a Legendre transformation

$$T = \frac{\partial P}{\partial \rho}, \qquad \rho = \frac{\partial \rho}{\partial s}. \tag{49}$$

For the Born-Infeld/Chaplygin case

$$P = 1 - \sqrt{1 - T^2}, \qquad \rho = \sqrt{1 + s^2} - 1 \tag{50}$$

$$s = \frac{T}{\sqrt{1 - T^2}}, \qquad T = \frac{s}{\sqrt{1 + s^2}}, \tag{51}$$

$$\rho = \frac{1}{\sqrt{1 - T^2}} - 1, \qquad P = 1 - \frac{1}{\sqrt{1 + s^2}}. \tag{52}$$

Thus indeed one has an upper bound for the temperature just like the Hagedorn temperature associated to the the states of perturbative string theory. However one may check that the detailed equation of state in the vicinity of the Hagedorn temperature differs form that given by perturbative string theory.

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