Discrete Newtonian Cosmology

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Abstract

In this paper we lay down the foundations for a purely Newtonian theory of cosmology, valid at scales small compared with the Hubble radius, using only Newtonian point particles acted on by gravity and a possible cosmological term. We describe the cosmological background which is given by an exact solution of the equations of motion in which the particles expand homothetically with their comoving positions constituting a central configuration. We point out, using previous work, that an important class of central configurations are homogeneous and isotropic, thus justifying the usual assumptions of elementary treatments. The scale factor is shown to satisfy the standard Raychaudhuri and Friedmann equations without making any fluid dynamic or continuum approximations. Since we make no commitment as to the identity of the point particles, our results are valid for cold dark matter, galaxies, or clusters of galaxies. In future publications we plan to discuss perturbations of our cosmological background from the point particle viewpoint laid down in this paper and show consistency with much standard theory usually obtained by more complicated and conceptually less clear continuum methods. Apart from its potential use in large scale structure studies, we believe that our approach has great pedagogic advantages over existing elementary treatments of the expanding universe, since it requires no use of general relativity or continuum mechanics but concentrates on the basic physics: Newton's laws for gravitationally interacting particles.

1 Discrete Newtonian cosmology

It is customary in present day astrophysical cosmology (e.g. [1, 2]) to assume that Newtonian theory describes adequately what happens after the time of decoupling of matter and radiation. This is particularly true of large scale

structure formation theories, which are all essentially based in Newtonian gravity [3, 4, 5]. It is then an issue as to what is the best way to express a Newtonian cosmological theory, given that Newton himself failed in this endeavour [6, 7].

One reason for Newton's failure is that he adopted a static model in which the fixed stars extended to infinity. This leads, amongst other things, to problems with divergences analogous to Olber's paradox. One way to evade these problems is to modify Newton's law at large distances such as suggested by Neumann [8] and by Seeliger [9, 10, 11, 12]. However one can construct a viable Newtonian cosmological theory from scratch for a finite number of discrete gravitating particles interacting through Newtonian gravitational attraction only, but possibly with the Newtonian version of a cosmological constant. The homothetically expanding background solution is an exact solution of the Newtonian equations, and can be linearised to get perturbation theory for such models, relevant to structure formation calculations. The model has many advantages over the standard approach: the issue is whether it can be made realistic.

1.1 The issues

Hence: consider a Newtonian cosmology for a universe made up of discrete gravitating particles. We do not assume either general relativity or fluid dynamics. We just use Newton's Laws of motion applied to a set of gravitating particles, with the particle interactions given by Newton's law of gravitational attraction. The features of this model are,

- **A set of gravitating particles imbedded in a vacuum - no fluid.**
  The usual cosmological "fluid" is very problematic because standard fluid properties are derived for particles that only undergo short range interactions (collisions). This is not in the least like collision-less particles that only interact by long range (gravitational) forces; this includes a 'fluid' of galaxies, stars, or CDM particles. The dangers of not properly representing long range forces include divergences and neglecting non-local interactions.

  A further point is numbers of particles. The key question is, what is the size of averaging volume V_{avg} that is supposed to give a good fluid approximation? [13]. A typical gas has around 10^{23} particles per cubic centimeter. If we regard galaxies as 'particles' of cosmic fluid, we have at most 10^{11} particles to consider in the visible universe, and if it is clusters we consider we have many less. Sticking with galaxies to be conservative, if we assume the averaging scale L_{avg} is (1/100 th). of the Hubble radius, then $L_{avg} \simeq 4.6 \times 10^5$ Mpc, and the observable universe contains $100^3 = 10^6$ such averaging volumes each containing 10^5 galaxies. If we take it to be (1/1000 th). of the Hubble radius ($L_{avg} \simeq 4.6 \times 10^4$ Mpc), the observable universe contains $1000^3 = 10^9$ averaging volumes each containing 10^2 galaxies. In both cases this is much too small for a good fluid approximation. Now of course there is dark matter present in addition,

but that too is clustered in mini-halos. It is not at all obvious there is a good fluid description applicable at the cosmological scale.

What is desirable is to represent the discrete particles and their interactions (if there is a good fluid approximation, this will underly it; if not, this will be vastly preferable). Thus one represents individual particles and uses summation rather than integration.

- **No infinities**: the basic reason Newton failed to get a cosmological solution was the divergences associated with an assumed infinite number of particles. One does not need to make this assumption: one can do the calculation for a finite set of particles and avoid these divergences, and all the associated problems [14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

- **No Fourier analysis**: for calculating the dynamics, one can work with the actual distribution rather than its Fourier modes, and then Fourier analyse the results afterwards if it is helpful. Fourier analysis of the dynamics is great if the system is linear; if it is not, the linearity assumed in Fourier analysis won't work; and the basic gravitational interactions for structure formation are non-linear (although we can of course perturb to get linear solutions about a background model).

- **No periodic boundary conditions** One can avoid the assumption of periodic boundary conditions which is usually made to deal with the problem of boundary conditions. This restricts the nature of solutions allowed, and may introduce artefacts, unless the spacetime actually is periodic in the assumed way. For example angular momentum will not always be conserved because a system with periodic boundary conditions is not rotationally symmetric (see [24] for such effects in molecular dynamics simulations).

The 'particles' in question may be envisaged as stars, galaxies, clusters, superclusters, etc., or even molecules. Indeed there is a certain arbitrariness in the choice of what a particle is since any sufficiently spherically symmetric isolated subsystem will move and gravitate like a point particle located at its "centre".

This paper is the same in spirit as the general relativity paper by Lindquist and Wheeler [25] and subsequent papers by Redmount, Ferreira, Clifton, and others [26, 27, 28, 29, 30, 31]. However those are all based on regular lattices, which will presumably not be solutions of the central configuration equation. They are also approximate rather than exact solutions.

1.2 The outcome

The outcome of this approach is that we if we have a suitable initial distribution of discrete particles, that is one satisfying the *central configuration equation* (see (43) below), we obtain an exact Newtonian version of the standard FLRW models — a solution that expands homothetically (equations (37) and (38) are satisfied) according to the standard Friedmann equation and Raychaudhuri

equations for pressure-free matter (equations (54) and (51) below), no matter how many particles are involved. For large numbers of particles, the solutions are close to spatially homogeneous [32].

The central configuration equation is in effect an initial value equation for these models, and is *required* if we are to have such a FLRW-like model. We can then perturb about such a model in order to get Newtonian equations of structure formation for such models; or else we can numerically integrate to see what happens if the initial data is changed from the background state in a linear or non-linear way.

In a subsequent paper, this approach will be developed further, deriving the linearised Newtonian structure formation equations in a rigorous way, deriving effects such as the Zeldovich pancake models [33], and hopefully providing a basis of this kind for non-linear equations and numerical simulations. It may be only of theoretical/didactic interest: but it might provide a sound basis for looking again at N-body simulations [3, 4, 5] in a cosmological context.

1.3 This paper

In the following section (Section 2), we set up the basic theory for a collection of point particles interacting only via the Newtonian inverse square gravitational law. We derive the generic forms of the various associated conservation laws, potential energies, and Lagrangian, as well as the general form of the virial relation.

Section 3 applies this theory to cosmology, showing how exact homothetically expanding solutions, with the same behaviour of their dynamical equations as their General Relativity counterparts, are possible provided the central configuration equation is satisfied. This result is summarised in a main theorem presented in section 3.1.6. The result is unaffected by the existence of a cosmological constant (Section 3.2). Exact and expanding solutions exist as precise analogs of the General Relativity pressure-free cosmological solutions (Section 3.3). The central configuration equation is key to these exact solutions of the Newtonian equations; Section 4 studies properties of this equation, in particular looking at the related effective forces and potentials.

This paper sets the stage for various generalisations (Section 5). A further paper that will consider properties of perturbed versions of these solutions, representing structure formation in an expanding universe.

2 Basic Theory

In this section we shall review some standard material [34, 35, 36, 37] on the dynamics of N point particles moving under the influence of gravity which we shall need in the later part of the paper.

2.1 The basic equations

The basic equation for a set of gravitating masses only interacting amongst themselves is Newton's Law of attraction. In general for interacting point particles, on using inertial coordinates, Newton's force law for discrete particles at position $\mathbf{x}_a$ and with mass $m_a > 0$ is

$$m_a \frac{d^2 \mathbf{x}_a}{dt^2} = \mathbf{F}_a + \sum_{b \neq a} \mathbf{F}_{ab} \,, \tag{1}$$

where $\mathbf{F}_a$ are external forces due to particles outside the set considered, and $\mathbf{F}_{ab}$ inter-particle forces between particle a and b. The same equation holds for each particle in the system, i.e. as a ranges over the values $1, 2, ..., N$ if there are N particles. The gravitational force between any two particles is

$$\mathbf{F}_{ab}^{(grav)} = -\frac{G m_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|^3}(\mathbf{x}_a - \mathbf{x}_b) \,, \tag{2}$$

where G is Newton's gravitational constant. Thus for the gravitational case,

$$m_a \frac{d^2 \mathbf{x}_a}{dt^2} = -\sum_{b \neq a} G m_a m_b \frac{(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} + \mathbf{F}_a \,. \tag{3}$$

The external forces vanish if

- We assume that the universe consists of a very large but finite number of particles, and apply the force law to the entire set.

- Or we apply (2) to a finite subset of all the particles in the universe (which may or may not be finite) and assume that for reasons of symmetry (most typically spherical symmetry, but not exclusively so) the external force due to all the particles outside the subset cancel.

Thus if such forces cancel, or if we are considering all particles that exist, then $\mathbf{F}_a = 0$ and we get, for each a,

$$m_a \frac{d^2 \mathbf{x}_a}{dt^2} = -\sum_{b \neq a} G m_a m_b \frac{(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} \tag{4}$$

showing how

$$\mathbf{F}_a^{(grav)} := -\sum_{b \neq a} G m_a m_b \frac{(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} \tag{5}$$

is the total gravitational force exerted on particle a due to all the other particles. Thus this is a coarse-grained or collective representation of all the individual forces acting on the particle.

2.2 Potential Energy

The gravitational force $\mathbf{F}_a^{\mathrm{grav}}$ acting on the a-th particle can be represented as the derivative of a gravitational potential energy V_a acting on that particle. [1] The potential energy $V_a(\mathbf{x}_c)$ for the gravitational force on the particle $\mathbf{x}_a$ is a function of the position $\mathbf{x}_a$ defined by

$$V_a(\mathbf{x}_a) := -\sum_{b \neq a} \frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}, \tag{6}$$

which also depends on the positions $\mathbf{x}_c$ of all the other particles in the system. This is the discrete version of the continuous definition of this potential (see Saslaw [38]: equations (9.4) and (9.5)).[2]

To show the relation of this potential to the gravitational force, define

$$\mathbf{x}_{ba} := \mathbf{x}_b - \mathbf{x}_a, \ x_{ba} := |\mathbf{x}_{ba}| = ((\mathbf{x}_b - \mathbf{x}_a).(\mathbf{x}_b - \mathbf{x}_a))^{1/2}. \tag{7}$$

From these definitions, for $\mathbf{x}_a \neq \mathbf{x}_b$,

$$\frac{\partial}{\partial \mathbf{x}_a}\left(\frac{1}{x_{ba}}\right) = \left(\frac{1}{x_{ba}}\right)^3 (\mathbf{x}_b - \mathbf{x}_a) \tag{8}$$

where the partial derivative $(\partial/\partial \mathbf{x}_a)$ is taken keeping all the other positions $\mathbf{x}_b$ $(b \neq a)$ constant. On using this result together with (2) and (5), the relation of the potential (6) to the gravitational force is

$$\frac{\partial V_a}{\partial \mathbf{x}_a} = -\sum_{b \neq a} \mathbf{F}_{ab}^{(grav)} = -\mathbf{F}_a^{(grav)} \tag{9}$$

as required. As usual, the absolute value of the potential does not affect this relation; we could add a constant V_0 to (6) without affecting the result. However doing so would destroy an important property of the potential: it is a homogeneous function of degree -1. This play an important role in various relations (sections 2.4.1 and 3.1.2).

2.3 Symmetries of the equations, and of the solutions

The fundamental equation (4) incorporates the important physical property that inertial, and both active and passive gravitational masses are all equal. As a consequence (4) is invariant under the full ten-dimensional Galilei group, the non-relativistic limit of the Poincaré group.

- time translations $(t \to t + t_0)$,

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[1] In contrast to many treatises on celestial mechanics, but in accordance with the universal usage in physics the sign of the potential is chosen so that the force it produces is in the direction in which the potential decreases.

[2] Saslaw [38] shifts m_a to the force relation (9.3). He also uses a different sign for the potential.

- spatial translations ($\mathbf{x}_a \to \mathbf{x}_a + \mathbf{x}_0$ for all a),

- rotations about the origin (no preferred direction is implied by the vectors, and rotations preserve $|\mathbf{x}_a - \mathbf{x}_b|$),

- Boosts from one frame of reference or inertial coordinate system to another.

The symmetries of the equations will generally not be symmetries of the solutions. However they can be used to generate new solutions that are essentially identical to the old ones.

In more detail:

- **1: Translational Invariance** As we are using a discrete model, there will not be any continuous spatial invariance of the solutions. Additionally, the homothetic solutions we shall describe have a preferred barycentre, i.e. a preferred centre of mass.

  However:

  (i) new solutions can be obtained from the old by spatially translating them. These are physically identical to the old ones. This can therefore be regarded as a change of coordinates (one is referring the same physical system to a new coordinate system).

  (ii) if there is a large enough number of particles, the system will appear approximately spatially homogeneous when coarse grained . Nevertheless if it is a finite system it will have a boundary and so will not be spatially homogeneous on a large scale.

  Momentum conservation follows from translational invariance of the equations. This will be an exact result for the solutions even though they are not spatially invariant.

- **2. Rotational symmetry.** Essentially the same remarks apply to rotational symmetry. There cannot be continuous rotational symmetry because of the discreteness of the system, but there can be discrete rotational symmetries. With enough particles the solution will be approximately rotationally symmetric when coarse grained. New physically identical solutions can be generated by rotation of the old system.

  Angular momentum conservation follows from rotational invariance of the equations. This will be an exact result for the solutions even though they are not rotationally invariant.

- **3: Time symmetry invariance** The solutions are only time invariant, i.e. invariant under shift of origin of time, if static (with the scale factor $S(t) = \text{constant}$) or stationary ($S(t) = \exp Ht$).

  Energy conservation results from time invariance of the equations, and will hold in all cases unless external forces act on the system.

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- **4: Time reversal invariance** There will be time reversal invariance of the solutions only if they are static. [3]

  If $\Lambda = 0$ there will be no static cosmological solutions, so they will be non-static and there will necessarily be a start to the universe ($S = 0$ in the limit) where we can set $t = 0$ in the limit. In this case we can choose time to be positive as the universe expands from this initial singularity, so that $\dot{S} > 0$ for small enough $t > 0$. The direction of time is then the direction in which t increases near $t = 0$; it is defined non-locally by the fact it is the direction in which the universe initially was expanding (this is unaffected by whether the universe recollapses or not). Thus a direction of time is established by this cosmological context despite the time symmetry of the equations.

  The local arrow of time will agree with this direction of time if initial conditions are special (see [39] for a discussion).

  One can make the system appear to be dissipative by a choice of time parameter non-linearly related to t, for example $\tau = \log t$ [40]. However this is an artefact of an unphysical choice of the time parameter (on the same basis, the simple harmonic oscillator will also appear to be dissipative), and one would make the system appear to be dissipative in the opposite direction of time if one instead chose $\tau = -\log t$. Standard conservative Newtonian gravitational dynamics, and indeed standard physics in general, results only if one restricts oneself to affine transformations of the standard time function t.

There is in addition a homothetic time symmetry:

$$Equation\,(4)\ \text{is invariant if} \qquad (t \to At,\ \mathbf{x}_a \to A^{-\frac{2}{3}}\mathbf{x}_a)\forall\, A,\, a. \qquad (10)$$

This is an invariance of the basic equation and gives rise to a general form of Kepler's third law in the form that if $\mathbf{x}_a(t)$ is a solution of the equations of motion then so is $A^{-\frac{2}{3}}\mathbf{x}_a(At)$. Of particular interest are solutions which are invariant under (10):

$$\mathbf{x}_a(t) = A^{-\frac{2}{3}}\mathbf{x}_a(At)\,. \qquad (11)$$

For instance an important class of such solutions which we shall encounter later take the form

$$\mathbf{x}_a(t) = t^{\frac{2}{3}}\mathbf{r}_a\,, \qquad (12)$$

where $\mathbf{r}_a$ are independent of time.

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[3] It is striking that in most if not all early discussions, such as those of Newton and his contemporaries [6, 7], the concern was that gravity would cause a finite system at rest to *collapse*. But one should ask how the system could have got to a starting point of being instantaneously at rest at a finite size at some time t_0, from which this collapse can occur. Unless one assumes existence of a cosmological constant giving such a static state, in which case expansion or collapse from that state are equally likely, this could only have happened by expansion from zero size at a previous time $t_i < t_0$ to that finite size. So that starting condition, assumed in their studies, implies the possibility of an expanding universe. If this had been realised at the beginning of the 20th century, cosmologists could have constructed a purely Newtonian explanation for the recession of the galaxies.

2.3.1 Mass and momentum and angular momentum conservation

The equality of active and passive gravitational masses and the central nature of the gravitational force guarantee not only the conservation of momentum, angular momentum and energy but also the so-called Centre of Mass theorem, namely (i) the centre of mass, or barycentre, of an isolated system moves with constant velocity and (ii) one may always pass to a frame of reference by means of a Galilean transformation, i.e. a boost, with respect which the centre of mass is at rest. This does not follow from the Galilei invariance alone [41].

We assume particle mass is conserved:

$$dm_a/dt = 0. \tag{13}$$

It then follows from (13) and the symmetries of (4) that total mass M, momentum $\mathbf{P}$, and angular momentum $\mathbf{L}$ about the origin are conserved:

$$M = \sum_a m_a = M_0 \,(\text{constant}) > 0, \tag{14}$$

$$\mathbf{P} = \sum_a m_a \dot{\mathbf{x}}_a = \mathbf{P}_0 \,(\text{constant}), \tag{15}$$

$$\mathbf{L} = \sum_a m_a (\mathbf{x}_a \times \dot{\mathbf{x}}_a) = \mathbf{L}_0 \,(\text{constant}). \tag{16}$$

If G was a function of time: $G = G(t)$, both $\mathbf{P}$ and $\mathbf{L}$ would still be conserved. The conservation of momentum implies that the centre of mass moves with constant velocity and that a frame of reference, i.e. a set of inertial coordinates, may always be chosen so that the total momentum vanishes and the centre of mass is at rest at the origin. In what follows, this choice will always be made, unless stated otherwise. Since the total momentum $\mathbf{P}$, total angular momentum $\mathbf{L}$ and total energy $\mathcal{E}$, depend on the frame of reference, unless stated otherwise, in what follows these quantities will be taken with respect to the centre of mass frame.

2.3.2 Energy conservation

Additionally energy $\mathcal{E}$ is conserved. By standard arguments

$$\mathcal{E} = T + V = \mathcal{E}_0 \,(\text{constant}), \tag{17}$$

where the kinetic energy T and potential energy V are

$$T(\dot{\mathbf{x}}_c) := \frac{1}{2} \sum_a m_a (\dot{\mathbf{x}}_a)^2, \tag{18}$$

$$V(\mathbf{x}_c) := \sum_a V_a = -\sum_a \sum_{b \neq a} \frac{G m_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}. \tag{19}$$

The total gravitational potential energy, $V(\mathbf{x}_c)$ is homogeneous $(V(a\mathbf{x}) = a^k V(\mathbf{x}))$ of degree $k = -1$. It is a negative function of the set $\{\mathbf{x}_c\}$ of the positions of all the particles in the system, and gets more negative the closer they are together. $V(\mathbf{x}_c)$ is thus a function of $3N$ coordinates $\mathbf{x}_c$ $(1 \le c \le N)$ of the $3N$ dimensional configuration space Q of N points in R^3, and is a continuous and indeed analytic function of these coordinates away from the diagonal where two or more positions coincide.

The quantities T and V are numbers that are both coarse grained representations of the state of the system, T representing the total energy of motion of the particles, and V the sum of the potential energies of all the particles. The gradient of V does not represent any force; indeed as it is just a number, it is not a function which can have a spatial gradient.[4]

2.4   Moment of inertia and virial theorem

Half the moment of inertia about the centre is [5]

$$I = \frac{1}{2} \sum_a m_a \mathbf{x}_a . \mathbf{x}_a = \frac{1}{2} \sum_a m_a x_a^2 \tag{20}$$

which plays an important role in celestial dynamics. The quantity $I^{1/2}$ serves as a measure of the maximum spacing of particles, while V^{-1} serves as a measure of their minimum spacing [42]. Sundman's inequality is

$$(\mathbf{L}_0)^2 + (dI/dt)^2 \le 4IT. \tag{21}$$

It plays a role in Newtonian non-singularity theorems stating that no complete collapse (i.e. one for which $I \to 0$) must occur in finite time, and cannot occur if $\mathbf{L}_0^2 \ne 0$ [34, 43].

2.4.1   The virial theorem

This is a standard result which depends crucially on the scaling property of the potential energy, i.e on Newton's inverse square law. Take a dot product of $\mathbf{F}_a^{(grav)}$ given by (5) with $\mathbf{x}_a$, and sum over a to get

$$\sum_a \mathbf{x}_a . \mathbf{F}_a^{(grav)} = -\sum_a \sum_{b \ne a} Gm_a m_b \frac{\mathbf{x}_a . (\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} = \sum_a \sum_{b \ne a} \mathbf{x}_a . \partial_{\mathbf{x}_a} \Big(\frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|} \Big). \tag{22}$$

Now Euler's theorem on homogeneous functions of degree k (that is functions $f(V)$ such that $f(ax) = a^k f(x)$) says

$$x\,\partial f/\partial x = kf\,. \tag{23}$$

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[4]Potential energy in an external gravitational field is a function of position; in that case the external force $\mathbf{F}_a$ is non-zero.

[5]The reader is warned that many books on celestial mechanics define I without the factor of $\frac{1}{2}$. This leads to various differences with the formulae in books which do not use our convention.

In this case $f = \frac{1}{|\mathbf{x}_a - \mathbf{x}_b|}$ is of degree $k = -1$, so Euler's theorem says

$$\mathbf{x}_a.\partial_{\mathbf{x}_a} f = -f \Rightarrow \mathbf{x}_a.\partial_{\mathbf{x}_a}(\frac{1}{|\mathbf{x}_a - \mathbf{x}_b|}) = -\frac{1}{|\mathbf{x}_a - \mathbf{x}_b|}.. \tag{24}$$

By (19) the last term of (22) is

$$\sum_a \sum_{a \neq b} \mathbf{x}_a.\partial_{\mathbf{x}_a}(\frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}) = -\sum_a \sum_{b \neq a}(\frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}) = V. \tag{25}$$

Also from the equation of motion (4),

$$\begin{aligned}
\sum_a \mathbf{x}_a.\mathbf{F}_a^{(grav)} &= \sum_a m_a \mathbf{x}_a.\frac{d^2 \mathbf{x}_a}{dt^2} = \sum_a m_a(\frac{d}{dt}(\mathbf{x}_a.\frac{d\mathbf{x}_a}{dt}) - \frac{d\mathbf{x}_a}{dt}.\frac{d\mathbf{x}_a}{dt}) \\
&= \frac{d}{dt} \sum_a m_a \frac{1}{2} \frac{d}{dt}(\mathbf{x}_a.\mathbf{x}_a) - 2T \tag{26}
\end{aligned}$$

and so from equation (22) with (26) and using (25) and (20) we find

$$V = \frac{d^2 I}{dt^2} - 2T \tag{27}$$

which is the scalar virial equation ([38]: eqn.(9.16)). In the celestial mechanics literature equation (27) is called the *Lagrange-Jacobi equation.*

Taking a time average $\langle \ \rangle$ of this equation, if the average of the second derivative of $I(t)$ is zero, we get a relation between kinetic and potential energies:

$$\langle \frac{d^2 I}{dt^2} \rangle = 0 \ \Rightarrow \ \langle V \rangle = -2\langle T \rangle \tag{28}$$

which is the virial theorem ([38]: pp.49-53). The condition $\langle \frac{d^2 I}{dt^2} \rangle = 0$ will be true for suitably localized or periodic systems.

2.5 Lagrangians and taking out the centre of mass

The basic equation (66) in Section 3.2 below, which is (4) with inclusion of a cosmological constant, may be derived from the Lagrangian

$$L = \sum_a \frac{1}{2} m_a \dot{\mathbf{x}}_a^2 + \frac{1}{2} \sum_{a \neq b} \sum_b G \frac{m_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|} + \frac{1}{2\tau^2} \sum_a m_a \mathbf{x}_a^2, \tag{29}$$

with $\Lambda = 3\frac{1}{\tau^2}$. If we define

$$M = \sum_a m_a, \qquad \mathbf{X} = \frac{1}{M} \sum_a m_a \mathbf{x}_a \tag{30}$$

we find that

$$\ddot{\mathbf{X}} = \frac{1}{\tau^2}\mathbf{X} \tag{31}$$

so that

$$\mathbf{X} = \mathbf{a}\cosh\frac{t}{\tau} + \tau\mathbf{u}\sinh\frac{t}{\tau} \tag{32}$$

where $\mathbf{a}$ and $\mathbf{u}$ are arbitrary constant vectors with the dimensions of length and velocity respectively. Now if $\{\mathbf{x}_a\}$ are solutions of the equations of motion, then so are $\{\mathbf{x}_a + \mathbf{X}\}$, and in fact the Lagrangian L is easily seen to change under this transformation by a total derivative:

$$L \to L + \dot{F} \tag{33}$$

with

$$\dot{F} = \frac{3}{2}M\big((\frac{\mathbf{a}^2}{\tau^2} + \mathbf{u}^2)\cosh(2\frac{t}{\tau}) + 2\frac{\mathbf{a}.\mathbf{u}}{\tau}\sinh(2\frac{t}{\tau})\big) \tag{34}$$

It follows that the equations of motion are invariant not only under the three dimensional group of rotations $SO(3)$, but under the six-dimensional commutative group of translations and boosts. If we suppose that G , which in principle could depend upon time, is actually independent of time, then, the equations of motion are invariant under the ten-dimensional Newton-Hooke group. which may be regarded as a deformation of the Galileo group to which it tends as $\tau \to \infty$. Note that the Newton-Hooke group differs from the Galileo group in that time translations commute neither with boosts nor translations.

Because of this symmetry of the equations of motion, one would expect that one could obtain a formulation which makes the translation symmetry manifest (cf. [44, 45, 46]). Such a formulation is referred to as *relational*. To obtain it, we subtract $m_a\ddot{\mathbf{X}}$ from both sides of (66) to obtain

$$\frac{1}{M}\sum_{b\neq a}m_am_b(\ddot{\mathbf{x}}_a - \ddot{\mathbf{x}}_b) = -G\sum_{b\neq a}m_am_b\frac{(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} + \frac{1}{M\tau^2}\sum_{b\neq a}m_am_b(\mathbf{x}_a - \mathbf{x}_b)\,. \tag{35}$$

These equations may be obtained from the Newton-Hooke analogue of Lynden-Bell's reduced so-called relational Lagrangian [45]

$$L^\star = \sum_{\{a,b|a<b\}}\frac{m_am_b}{M}\Big\{\frac{1}{2}(\dot{\mathbf{x}}_a - \dot{\mathbf{x}}_b)^2 + \frac{GM}{|\mathbf{x}_a - \mathbf{x}_b|} + \frac{(\mathbf{x}_a - \mathbf{x}_b)^2}{2\tau^2}\Big\}\,. \tag{36}$$

3   Cosmological solutions

In this section we shall specialise the discussion of the previous section to the case of solutions of the Newtonian equations of motion which evolve by homotheties [6] of Euclidean space. Solutions of this type go back to the work of

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[6]Homotheties, that is a constant rescaling of the Cartesian coordinates of Euclidean space, are also known as dilations, dilatations, similarities or homographies.

Lagrange and Laplace and are well known to those studying celestial mechanics. Their application to Newtonian cosmology is much less well known.

Although the basic idea that homothetic expansion embodies the Cosmological Principle is well explained in chapter IX of Bondi's 1952 textbook [47], Bondi then goes on to review Milne and McCrea's 1934 work [48, 49] [7] and treats a greatly simplified spherically symmetric and homogeneous fluid dynamical model, apparently being unaware that he could have equally well treat a fully rigourous point particle model, from which both these two assumptions can be derived rather easily[32, 51]. The key idea is that Newton's equations of motion for point particles only allow a homothetic expansion if the co-moving positions of the expanding system of particles are constrained to form what is called a *central configuration*, as explained below. As far as we are aware, the only papers before [32] which made an explicit link between central configurations and Newtonian Cosmology date back to 1971 and are by Saari [52, 53, 42, 54, 37] [8] whose primary interest was mathematical celestial mechanics rather than cosmology. Moreover Saari's writings on central configurations is basically restricted to dealing only with a handful of particles.

Indeed most if not all of the celestial mechanics community did not develop the essential physical insight based on J. J. Thomson's *current bun model of the atom*, and E.P. Wigner's *Theory of Jellium*, an insight which can only be confirmed by detailed numerical analysis of the equations governing the central configurations of ten thousand particles or more [9]. This was the essential point of the work in [32]. This made use of the fact that central configurations extremise a certain function of position we call $\tilde{V}$. In [32] it was established, in the case of N particles of equal mass m, that for sufficiently many particles, central configurations which maximise $\tilde{V}$ form a spherical and homogeneous ball. This is precisely the starting point of the analysis of Milne and McCrea [48, 49]. The homogeneity of central configurations made up of large numbers of particles with identical or almost identical masses is a direct consequence of Newton's inverse square law. Central configurations exist for forces which vary inversely as any power but only for the inverse square law are they homogeneous [32, 62].

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[7] Although often cited as the initiators of Newtonian Cosmology, Milne and McCrae were preceded in 1932 by Mason [50], but his paper appears to have received very little attention.

[8] However the basic idea of substituting the homothetic ansatz into (4) or (66) did occur to Landsberg [55, 56] slightly later although ; without tackling the central configuration equation. Later [57] he took up that challenge but perhaps because he was only able to deal with a universe made of eight equal mass galaxies situated at the vertices of a regular cube, his work was not taken up at the time. The germ of the idea appeared in two undergraduate textbooks [58, 59]. Central configurations with just four galaxies have been used to study the interactions between nearby galaxies [60].

[9] For Saari's most recent viewpoint on the importance of central configurations for celestial mechanics see [61].

3.1 Robertson-Walker-like solutions

Now assume self-similarity of the solution: put in a homothetic factor $S(t)$ and separate variables so as to correspond to Robertson-Walker kinematics:

$$\mathbf{x}_a = S(t)\mathbf{r}_a, \ d\mathbf{r}_a/dt = 0, \tag{37}$$

where $\mathbf{r}_a$ are comoving coordinates for particle a. This implies the Slipher-Lemaître-Hubble velocity-distance law:

$$\mathbf{v}_a := \frac{d\mathbf{x}_a}{dt} = \dot{S}(t)\mathbf{r}_a = H(t)\mathbf{x}_a \tag{38}$$

where $H(t) := \dot{S}(t)/S(t)$. One should note here that the particle at the origin is moving inertially; only if $H(t) = H_0 t$ are the other particles also moving inertially. Thus except in this case (the Milne universe, which cannot occur for particles with non-zero masses), the origin is kinematically preferred. However, as pointed out by Heckmann and Schücking [21, 22], we can use Einstein's insight that gravity and inertia are dynamically indistinguishable when accelerated motion occurs, so what counts physically is whether or not the particles are freely falling, that is, moving only under gravity plus inertia; and in that sense all the particles are dynamically equivalent.[10] No local dynamical effect will distinguish one particle from another.

The gravitational law (4) becomes

$$m_a \mathbf{r}_a \frac{d^2 S(t)}{dt^2} = -\sum_{b \neq a} G m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{S^2(t)|\mathbf{r}_a - \mathbf{r}_b|^3}. \tag{39}$$

Define

$$C(t) := S^2(t)\frac{d^2 S(t)}{dt^2} \tag{40}$$

then equation (39) becomes

$$C(t)m_a \mathbf{r}_a = -\sum_{b \neq a} G m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3}. \tag{41}$$

Then, remembering (13) and (37), consistency requires that $C(t)$ is a constant:

$$\frac{\partial}{\partial t}(C(t)m_a \mathbf{r}_a) = 0 \Rightarrow C(t) = \text{const} =: -G\tilde{M}, \tag{42}$$

which defines the constant $\tilde{M}$. Note that $\tilde{M}$ has the dimensions of mass per unit volume.

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[10]See Section 2.3 in [63].

3.1.1 The central configuration equation

So firstly, from (41) together with (42) we must have

$$\tilde{M}m_a\mathbf{r}_a = \sum_{b\neq a} m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} \tag{43}$$

for all values a. This set of N non-linear time independent equations is known as the *central configuration equation* ([32]; [36]:79-80), which is a consistency condition for (37) to give a solution. For systems with just a few particles, solutions form regular polyhedra. This is clear for example in the case of three particles of identical mass: if they are started off from rest in the shape of an equilateral triangle, that shape will be preserved as they fall towards each other (the motion will be homothetic), so this must be a solution of the central limit equation. For larger numbers of particles, there will be shell-like structures in the solution. For even larger numbers, they will be approximately spatially homogeneous [32].

The mass M of the system, given by (14), is not the same as the constant $\tilde{M}$, defined in (42), which occurs crucially in the dynamical equations (51) and (54) below; the relation is to be investigated. Both are constant. If the masses m_a are positive then in order to have solutions, $\tilde{M}$ must be positive [32]. This follows because on contracting (43) with $\mathbf{r}^a$,

$$\tilde{M} = \sum_{b\neq a} m_b \frac{(\|\mathbf{r}_a\|^2 - \mathbf{r}_b.\mathbf{r}^b)}{\|\mathbf{r}_a\|^2 |\mathbf{r}_a - \mathbf{r}_b|^3} = \sum_{b\neq a} m_b \frac{(1 - \cos^2\vartheta)}{|\mathbf{r}_a - \mathbf{r}_b|^3} \tag{44}$$

where ϑ is the angle between $\mathbf{r}_a$ and $\mathbf{r}_b$. Thus $\tilde{M}$ is uniquely determined by the mass distribution.

For approximately spherically symmetric distributions, for which $m(r)$ is the mass inside a radius r we find, by balancing forces and using Newton's result that for a continuous spherical distribution of mass, the gravitational field is given by the total mass inside a radius r divided by the radius squared, we find that

$$\tilde{M} = \frac{m(r)}{r^3} . \tag{45}$$

It follows that for a spherical distribution, the density of the central configuration $\tilde{\rho} = \frac{3\tilde{M}}{4\pi}$ is homogeneous. The physical density ρ is of course time dependent, $\rho = \frac{3\tilde{M}}{4\pi S^3}$.

It would be possible to consider a spherically symmetric density distribution thought of as concentric shells of density $\rho(r)$ which would not be homogeneous. That would not satisfy the condition for a central configuration, but would have a scale factor $S = S(t, r)$ which depends not only on time t but radius r.

3.1.2 A key identity

To obtain an integral relation following from the central configuration equation, multiply (43) by G, take a dot product with $\mathbf{r}_a$, and sum over a to get

$$G\tilde{M}\sum_a m_a \mathbf{r}_a . \mathbf{r}_a = \sum_a \sum_{b \neq a} Gm_a m_b \frac{\mathbf{r}_a.(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} = -\sum_a \sum_{b \neq a} \mathbf{r}_a . \partial_{\mathbf{r}_a} \left(\frac{Gm_a m_b}{|\mathbf{r}_a - \mathbf{r}_b|} \right). \ (46)$$

Now Euler's theorem on homogeneous functions (in this case $\frac{1}{|\mathbf{r}_a - \mathbf{r}_b|}$ is of degree -1) shows

$$\mathbf{r}_a . \partial_{\mathbf{r}_a} \left(\frac{1}{|\mathbf{r}_a - \mathbf{r}_b|} \right) = -\frac{1}{|\mathbf{r}_a - \mathbf{r}_b|}.$$

(cf. (24)), so the last term is

$$-\sum_a \sum_{b \neq a} \mathbf{r}_a . \partial_{\mathbf{r}_a} \left(\frac{Gm_a m_b}{|\mathbf{r}_a - \mathbf{r}_b|} \right) = \sum_a \sum_{b \neq a} \left(\frac{Gm_a m_b}{|\mathbf{r}_a - \mathbf{r}_b|} \right). \tag{47}$$

Define the effective potential $\tilde{V}_{(-1)}$ and moment of inertia $\tilde{I}_0$ by

$$\tilde{V}_{(-1)} := \quad -\sum_a \sum_{b \neq a} \frac{Gm_a m_b}{|\mathbf{r}_a - \mathbf{r}_b|} = \mathbf{const.} \tag{48}$$

$$\tilde{I}_0 := \quad \frac{1}{2} \sum_a m_a (\mathbf{r}_a)^2 = \mathbf{const} \tag{49}$$

(cf. (19) and (20); these are defined in terms of the comoving $\mathbf{r}_a$ rather than the physical $\mathbf{x}_a$). Then on using (47), equation(46) becomes

$$2G\tilde{M}\,\tilde{I}_0 = \ -\tilde{V}_{(-1)} \tag{50}$$

showing that the central configuration moment of inertia I_0 and effective potential energy $V_{(-1)}$ are equal up to a factor $2G\tilde{M}$. This can be regarded as an analogue of the Virial Theorem for static configurations. We will call it the *central configuration constraint equation*, because it is a relation that is required to be true if (43) is to hold for all a.

An alternative derivation of (50) is to substitute the Slipher-Lemaître-Hubble law (37) into the Lagrange-Jacobi identity (27) and use the Raychaudhuri and Friedmann equations (51), (54) below.

3.1.3 The time evolution equations

Secondly, from (40) and (42) we must have

$$-\frac{G\tilde{M}}{S^2(t)} = \frac{d^2 S(t)}{dt^2} \tag{51}$$

which is the Raychaudhuri equation [63] for this case. One should note that, for a given mass distribution, this equation is *not* invariant under rescaling $S(t) \to \hat{S}(t) = \alpha S(t)$; for

$$- \frac{G\tilde{M}}{\alpha^2 \hat{S}^2(t)} = \frac{d^2(\alpha \hat{S}(t))}{dt^2} \tag{52}$$

is the same as (51) only if the mass $\tilde{M}$ is rescaled also: $\tilde{M} \to \hat{\tilde{M}} = \alpha^3 \tilde{M}$. But for a given mass distribution, $\tilde{M}$ is fixed by (44).

Multiplying (51) by (dS/dt), which must be non-zero for almost all t because of (39), it can be integrated:

$$\frac{d}{dt}\left(\frac{G\tilde{M}}{S(t)}\right) = -\frac{G\tilde{M}}{S^2(t)}\frac{dS(t)}{dt} = \frac{d^2 S(t)}{dt^2}\frac{dS(t)}{dt} = \frac{1}{2}\frac{d}{dt}\left(\frac{dS(t)}{dt}\right)^2 \tag{53}$$

which gives the Friedmann equation

$$\frac{G\tilde{M}}{S^3(t)} = \frac{1}{2}\left[\frac{\dot{S}(t)}{S(t)}\right]^2 - \frac{E}{S^2(t)} \tag{54}$$

where E is a constant of integration. Thus we get the same result as both the General Relativity and Newtonian cosmological continuum approximations for the case of pressure free matter, but with no continuum model needed.

3.1.4 Energy conservation

How does this relate to the energy equation (17)? They must both represent the same process of energy conservation.

Assuming the homothetic expansion hypothesis(37), the kinetic energy (18) is

$$T(\dot{\mathbf{x}}_c) = \frac{1}{2}\sum_a m_a (\dot{\mathbf{x}}_a)^2 = \frac{1}{2}\dot{S}(t)^2 \sum_a m_a (\mathbf{r}_a)^2 = \dot{S}^2(t)I_0,$$

and the potential energy (19) is

$$V(\mathbf{x}_c) = -\sum_a \sum_{b \neq a} \frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|} = -\frac{1}{S(t)}\sum_a \sum_{b \neq a} \frac{Gm_a m_b}{|.\mathbf{r}_a - \mathbf{r}_b|} = \frac{1}{S(t)}\tilde{V}_{(-1)}. \tag{55}$$

Thus the energy equation (17) is

$$T + V = \dot{S}(t)^2 \tilde{I}_0 + \frac{1}{S(t)}\tilde{V}_{(-1)} = \mathcal{E}_0 \tag{56}$$

which gives

$$\frac{1}{2}\frac{\dot{S}(t)^2}{S(t)^2} + \frac{1}{S^3(t)}\frac{\tilde{V}_{(-1)}}{2\tilde{I}_0} = \frac{\mathcal{E}_0}{2\tilde{I}_0}\frac{1}{S^2(t)}. \tag{57}$$

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Comparing with (54), they agree if

$$E = \frac{\mathcal{E}_0}{2\tilde{I}_0}, \quad G\tilde{M} = -\frac{\tilde{V}_{(-1)}}{2\tilde{I}_0} \tag{58}$$

The former just relates the arbitrary constants E and $\mathcal{E}_0$ and the latter is the central configuration constraint equation (50). Thus equations (54) and (56) are the same.

3.1.5 The Virial Relation

For a homothetic expansion (37), the moment of inertia (20) becomes

$$I(t) = \frac{1}{2}\sum_a m_a x_a^2 = S^2(t)\frac{1}{2}\sum_a m_a r_a^2 = S^2(t)\tilde{I}_0 \tag{59}$$

Taking a time derivative:

$$\frac{dI(t)}{dt} = 2\frac{dS(t)}{dt}S(t)\tilde{I}_0 \tag{60}$$

Take a second derivative:

$$\frac{d^2I(t)}{dt^2} = 2\left(\frac{d^2S(t)}{dt^2}S(t) + \dot{S}(t)^2\right)\tilde{I}_0 \tag{61}$$

Now (54) shows that

$$\dot{S}(t)^2 = \frac{2G\tilde{M}}{S(t)} + 2E \tag{62}$$

Using this and (51), (61) becomes

$$\frac{d^2I(t)}{dt^2} = 2\left(-\frac{G\tilde{M}}{S^2(t)}S(t) + (\frac{2G\tilde{M}}{S(t)} + 2E)\right)\tilde{I}_0 = 2(\frac{G\tilde{M}}{S(t)} + 2E)\tilde{I}_0. \tag{63}$$

This makes sense: as the system expands, the moment of inertia increases (cf. (60)) but at a decreasing rate (cf. (??)). The virial relation (27) becomes

$$V = 2(\frac{G\tilde{M}}{S(t)} + 2E)\tilde{I}_0 - 2T \tag{64}$$

in contrast to the virial theorem (28). The condition $\langle\frac{d^2I(t)}{dt^2}\rangle = 0$ is not fulfilled.

3.1.6 The main result

These models can represent complex inhomogeneous matter distributions, but not arbitrary ones. To summarise,

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Theorem: Discrete Newtonian Cosmology *The Newtonian gravitational law of attraction (4) for a finite set of gravitating particles has an exact homothetic solution ((37), (38) hold) provided the time independent central configuration equation (43) is satisfied for $a = 1$ to N. The effect of gravitational attraction is to lead to a homothetic change in size governed by the Raychaudhuri equation (51), with first integral the Friedmann equation (54).*

These solutions are not spatially homogeneous (although they tend to spatial homogeneity if the number of particles is large [32]). Indeed they break all the symmetries of the equations mentioned above. In particular the origin of coordinates is a preferred point: it is the center of mass (see (78) below). Note that in the general relativity case, (51) and (54) imply each other because of the fluid energy density conservation equation; essentially the same is true for Newtonian fluid-based cosmology. Here the equivalence results in effect from the mass conservation equation (13), which underlies the constancy of $\tilde{M}$.

Roughly speaking, the central configuration equation (43) is the condition that the matter distribution is homogeneous on a large scale, allowing the quantity M to be independent of spatial position. The effect of gravitational attraction is to keep the spatial arrangement unchanged in shape, but altering in size according to (51), (54); hence in spatial terms, gravity leaves the configuration untouched apart from homothetically altering distances.

Corollary *There are no such FLRW-like solutions if the central configuration equation (43) is not satisfied for all a ($1 \leq a \leq N$). The time development of data not satisfying these conditions cannot be homothetic with a spatially homogeneous homothetic factor.*

(cf. [36]: Proposition 2.5). If we relax the global homothetic assumption to a local self-similarity condition:

$$\mathbf{x}_a = S(t, x)\mathbf{r}_a \tag{65}$$

there will be many more solutions, as investigated by Saari [53].

3.2 Cosmological constant

The universe appears today to be dominated by a cosmological constant. Adding in a Newtonian cosmological constant to the force law, we get

$$m_a \frac{d^2 \mathbf{x}_a}{dt^2} = -\sum_{b \neq a} G m_a m_b \frac{(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} + \frac{\Lambda m_a \mathbf{x}_a}{3} \tag{66}$$

As before, put in a homothetic factor and separate variables: using (37), (66) becomes

$$m_a \mathbf{r}_a \frac{d^2 S(t)}{dt^2} = -\sum_{b \neq a} G m_a m_b \frac{S(t)(\mathbf{r}_a - \mathbf{r}_b)}{S^3(t)|\mathbf{r}_a - \mathbf{r}_b|^3} + \frac{\Lambda S(t) m_a \mathbf{r}_a}{3}. \tag{67}$$

The argument from (39) to (43) goes through as before. This gives the result

$$m_a \mathbf{r}_a S^2(t) \frac{d^2 S(t)}{dt^2} = -G\tilde{M} m_a \mathbf{r}_a + \frac{\Lambda S^3(t) m_a \mathbf{r}_a}{3} \tag{68}$$

with $\tilde{M}$ defined exactly as before by (43). This implies the Raychaudhuri equation with cosmological constant:

$$\frac{1}{S(t)} \frac{d^2 S(t)}{dt^2} = -\frac{G\tilde{M}}{S^3(t)} + \frac{\Lambda}{3} \tag{69}$$

where matter causes deceleration and Λ an acceleration [63]. To integrate when $dS/dt \neq 0$, multiply by $S(t)dS/dt$ to get

$$\frac{d^2 S(t)}{dt^2} \frac{dS(t)}{dt} = -\frac{G\tilde{M}}{S^2(t)} \frac{dS(t)}{dt} + \frac{\Lambda}{3} S(t) \frac{dS(t)}{dt} \tag{70}$$

which is

$$\frac{1}{2} \frac{d}{dt} \left(\frac{dS(t)}{dt} \right)^2 = \frac{d}{dt} \left(\frac{G\tilde{M}}{S(t)} \right) + \frac{d}{dt} \left(\frac{\Lambda S^2(t)}{6} \right). \tag{71}$$

Integrating gives the Friedmann equation

$$\frac{1}{2} \left[\frac{\dot{S}(t)}{S(t)} \right]^2 = \frac{G\tilde{M}}{S^3(t)} + \frac{E}{S^2(t)} + \frac{\Lambda}{6} \tag{72}$$

where E is a constant of integration. The special case when $dS/dt = 0$ is dealt with below (Section 3.3.1).

Solutions with $\Lambda \neq 0$: *We can derive the standard Raychaudhuri (69) and Friedmann (72) equations for time-dependent cosmology in exactly the same way for discrete Newtonian cosmology with $\Lambda \neq 0$ as for the case with $\Lambda = 0$. The central configuration equation (43) required for a homothetic solution is unchanged (that equation does not gain a cosmological constant), as is the definition of effective gravitational mass $\tilde{M}$. No fluid approximation is used in deriving these results.*

The possibility of such comoving homothetic configurations is not affected by the cosmological constant, which only affects the time evolution of the solution. Other modifications of Newton's law of gravity, such as those suggested by Neumann [8] and by Seeliger [9, 10, 11, 12] break the scaling symmetry of the inverse square law and do not permit homothetic solutions (cf. [56]).

The introduction of the cosmological constant into (66) breaks the translation symmetry of the original equations (4) and one might wonder about the fate of momentum conservation and the issue of the centre of mass. This is discussed in detail in [51] where it is explained how the Galilei invariance of (4) is replaced by the Newton-Hooke invariance of (66).

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3.3 Specific Cosmological solutions

Even though these discrete Newtonian solutions are spatially inhomogeneous, their time dependence corresponds exactly to the pressure free-general relativity models [64, 65, 66, 67].

3.3.1 Static solutions

In the case of static solutions $(S(t) = S_0 = const)$, (72) no longer follows from (69), which is the only gravitational equation to be satisfied apart from (43). Solutions exist if and only if

$$\frac{\Lambda}{3} = \frac{G\tilde{M}}{S_0^3} > 0. \tag{73}$$

> **Static discrete mass solutions** *exist for any central mass configuration (43) provided $\Lambda > 0$. The only gravitational equation to be satisfied in addition to (43) is (73), with $\tilde{M} > 0$ defined by (44). Solutions clearly exist for any values of $\tilde{M}$ and $\Lambda > 0$ (one just has to solve (73) for S_0.)*

These are discrete Newtonian analogues of the Einstein static solution; just as in the general relativity case, they will be unstable [63]. There may be no General Relativity analogues of these static discrete mass solutions [68].

3.3.2 Expanding solutions

The dynamic models with $\Lambda > 0$ can be good descriptions of the real universe after the universe is matter dominated, and specifically since the time of decoupling of matter and radiation [64, 65, 66, 67].

These solutions depend in the standard way on $\tilde{M}$, E, and Λ, allowing monotonic solutions only if $E \geq 0$, and a much wider set of solutions otherwise [66, 67]. Assuming $\Lambda \geq 0$, bounces can occur if and only if $E < 0$; otherwise the universe had a singular start where $S(t) \to 0$. The universe will expand forever unless $E < 0$, when it may recollapse. Exact parametric solutions can be obtained when $\Lambda = 0$ [64]. The simplest solution is the Einstein-de Sitter solution, arising when $\Lambda = 0$, $E = 0$, leading to

$$S(t) = S_0 t^{2/3} , \ \ S_0 = \left(\frac{9}{2}G\tilde{M}\right)^{1/3} , \tag{74}$$

which gives a solution for any $\tilde{M}$ (unlike the general relativity case, we do not have the freedom to rescale S). This solution corresponds to the quantity $\mathbf{X}_a = t^{-2/3}\mathbf{x}_a$ being constant (see (12)).

Asymptotic solutions for large t (in the cases that expand forever) fall into three cases: 1. $\Lambda > 0$, 2. $\Lambda = 0$, $E > 0$, 3. $\Lambda = 0$, $E = 0$. In case 1., the asymptotic solution is the de Sitter solution

$$S(t) = \alpha \exp Ht, \ H = \left(\frac{\Lambda}{3}\right)^{1/2}, \tag{75}$$

where α is arbitrary (the solution is scale free). In case 2., the asymptotic solution is the Milne solution

$$S(t) = Ht, \ H^2 = 2E, \tag{76}$$

which is again independent of $\tilde{M}$. In case 3., the asymptotic solution is the same as the exact solution (74). The last two solutions are consistent with Saari's analysis of asymptotic forms [53] (which did not consider the case $\Lambda > 0$).

These models do not represent well the dynamics of the universe at early times when radiation dominates and general relativity effects therefore have to be taken into account, both because the active gravitational mass then is $(\rho + 3p/c^2)$ rather than ρ, and because the energy conservation equation has a source term $(\rho + p/c^2)$ rather than ρ [63]. However structure formation takes place after decoupling; and so these equations may be good at those times. In order to investigate this we need to look at the perturbed equations (these are to be the topic of a subsequent paper).

3.4   Issues

For the FLRW type situation defined by (37), we have an intriguing fine tuning problem:

> **Fine Tuning**: *To good approximation, we currently see a FLRW type homothetic expansion. But in order to get such a flow, the initial positions of the particles must be constrained to satisfy (43). What kind of explanation can one give for such a fine tuning of the initial data in Newtonian cosmology?*

Presumably the answer is to be sought via an initial relativistic state that results at late times in such a Newtonian configuration. Perhaps also it can be justified by a minimum energy principle favouring this distribution, but it is not clear how this might work:

- Every central configuration is an extremum,

- but not every extremum is a minimum (or maximum).

(see [32], section 2(b)).

4 The central configuration equation

As shown above, the central configuration equation (43):

$$\tilde{M} m_a \mathbf{r}_a = \sum_{b \neq a} m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} \tag{77}$$

is the initial value equation for discrete Newtonian cosmology; once it has been satisfied at an initial time, it will be satisfied for all times (that is the essence of (42)). It plays a key role in celestial mechanics [53, 42, 54] and its solutions have been studied in depth in [32], but deserve much more study.

Considered as a 3-dimensional gravitational problem, it is as if there were a force proportional to distance between the particles, as well as the inverse square law of attraction; that is, it is as if there were a cosmological constant in this 3-dimensional of gravitational context. But it is not the same as a cosmological constant Λ (see previous section), which has no effect on the spatial gravitational attraction equation (77); rather Λ changes the time evolution of the system, see (72). One may think of a central configuration as an equilibrium between the gravitational attraction and an entirely fictitious or auxiliary cosmological repulsion which arises á la D'Alembert's principle from the inertial forces due to the accelerations of the particles. Thus this is an effective 3-dimensional force arising for the 4-dimensional spacetime context (see Section 4.1.2 for further discussion).

4.1 Some Properties of Central Configurations

This section follows section 2 of [32].

4.1.1 Centre of mass

The centre of mass $\mathbf{r}_{CM}$ is given by

$$M\mathbf{r}_{CM} = \sum_a m_a \mathbf{r}_a = \sum_a \sum_{b \neq a} \frac{m_a m_b}{\tilde{M}} \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} = 0 \tag{78}$$

because the sum is symmetric but the summand antisymmetric. Thus the centre of mass of the system lies at the origin, which is a preferred location for these inhomogeneous distributions. In this model therefore, Neumann's body alpha (some sort of fixed body defining inertial motion), which played an important role in the pre-relativity debate about absolute versus relative motion in Newtonian mechanics [8, 69, 70], may be identified with the origin.

Taking the time derivative

$$\mathbf{P} := \sum_a m_a \dot{\mathbf{x}}_a = \dot{S}(t) \sum_a m_a \mathbf{r}_a = 0 \tag{79}$$

so the conserved momentum is zero. Similarly for angular momentum about the centre of mass:

$$\mathbf{L} = \sum_a m_a(\mathbf{x}_a \times \dot{\mathbf{x}}_a) = S(t)\dot{S}(t) \sum_a m_a(\mathbf{r}_a \times \dot{\mathbf{r}}_a) = 0. \tag{80}$$

Solutions with vanishing total momentum and total angular momentum are sometimes referred to as "relational". For an illuminating discussion of the relation of this to various formulations of Mach's Principle the reader is referred to [44, 45, 46, 71, 72].

4.1.2 Effective Forces

One can represent the nature of the central configuration in terms of effective forces and potentials (effective because they refer to the comoving distances $\mathbf{r}_a$ rather than the actual distances $\mathbf{x}_a$ that occur in the underlying force equation (2) and its resulting potentials (6).

Starting with (14), add and subtract the same term to get

$$m_a\mathbf{r}_a = \frac{1}{M}m_a\mathbf{r}_a \sum_b m_b = \frac{1}{M}\sum_b m_b m_a(\mathbf{r}_a - \mathbf{r}_b) + \frac{m_a}{M}\mathbf{r}_a \sum_b m_b\mathbf{r}_b. \tag{81}$$

Using (78)

$$m_a\mathbf{r}_a = \frac{1}{M}\sum_{b \neq a} m_b m_a(\mathbf{r}_a - \mathbf{r}_b). \tag{82}$$

Substituting this into (77) and multiplying by G, we get

$$\sum_{b \neq a} Gm_b m_a(\mathbf{r}_a - \mathbf{r}_b)\left(\frac{G\tilde{M}}{M} - \frac{G}{|\mathbf{r}_a - \mathbf{r}_b|^3}\right) = 0. \tag{83}$$

Defining $r_{ab} := |\mathbf{r}_a - \mathbf{r}_b|$ and the effective inter-particle force

$$\tilde{\mathbf{F}}_{ab} := m_b m_a(\mathbf{r}_a - \mathbf{r}_b)\left(\frac{G\tilde{M}}{M} - \frac{G}{r_{ab}^3}\right) \tag{84}$$

we find that (83) is just

$$\sum_{a \neq b} \tilde{\mathbf{F}}_{ab} = 0. \tag{85}$$

which is a form that is invariant under translation of the points: the centre of mass does not matter.

We can rewrite (84) as

$$\tilde{\mathbf{F}}_{ab} = \mathbf{F}_{ab}^{(TD)} + \tilde{\mathbf{F}}_{ab}^{(1)} \tag{86}$$

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where

$$\tilde{\mathbf{F}}_{ab}^{(1)} := -Gm_am_b\frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} \tag{87}$$

is the reduced inter-particle gravitational force, which relates to the proper distances $\mathbf{x}_\alpha$ rather than the comoving distances $\mathbf{r}_\alpha$ (cf.(2)), and

$$\mathbf{F}_{ab}^{(TD)} := G\left(\frac{\tilde{M}}{M}\right)m_bm_a(\mathbf{r}_a - \mathbf{r}_b) \tag{88}$$

is the *top-down (contextual) effective force* exerted on the spatial distribution because of the context of the conformal expansion. It is an effective repulsive force that is obviously not the same as the direct gravitational force between the particles, since that given by $\mathbf{F}_{ab}^{(grav)}$. It is not due to a cosmological constant (cf. Section 3.2); it is an extra term that arises solely due to the configuration of particles, rather than the micro forces between them. This is in line with many other examples of such contextual effects in physics [73].

It follows that the effective inter-particle force vanishes when

$$\tilde{\mathbf{F}}_{ab} = 0 \Leftrightarrow |\mathbf{r}_{ab}| = R_c := \left(\frac{GM}{\tilde{M}}\right)^{1/3} \tag{89}$$

giving a preferred scale for these solutions. This is discussed further in [32].

4.2 Potential Functions

4.2.1 Potentials for particles

Write the central configuration equation (77) as

$$\tilde{\mathbf{F}}_a := \tilde{\mathbf{F}}_a^{(1)} + \tilde{\mathbf{F}}_a^{(2)} = 0 \tag{90}$$

where $\tilde{\mathbf{F}}_a^{(1)}$ is given by

$$\tilde{\mathbf{F}}_a^{(1)} = \sum_{b \neq a} \tilde{\mathbf{F}}_{ab}^{(1)} = -\sum_{b \neq a} Gm_am_b\frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} \tag{91}$$

on using (87); and $\tilde{\mathbf{F}}_a^{(2)}$ is defined by

$$\tilde{\mathbf{F}}_a^{(2)} := G\tilde{M}m_a\mathbf{r}_a. \tag{92}$$

Note that these are defined in terms of the comoving coordinates $\mathbf{r}_a$ rather than the Newtonian coordinates $\mathbf{x}_a$ Define the associated energies as

$$\tilde{V}_a := \tilde{V}_{(-1)a} + \tilde{V}_{(2)a} \tag{93}$$

where the effective gravitational potential energy is

$$\tilde{V}_{(-1)a} := -\sum_{b \neq a} \frac{Gm_a m_b}{|\mathbf{r}_{ab}|} \tag{94}$$

which is homogeneous of degree -1, and the effective repulsion potential energy is

$$\tilde{V}_{(2)a} := -\frac{1}{2}G\tilde{M}m_a \mathbf{r}_a.\mathbf{r}_a \tag{95}$$

which is homogeneous of degree $k = 2$. These are also defined in terms of the comoving coordinates $\mathbf{r}_a$.

From these definitions, as in the case of (9),

$$\tilde{\mathbf{F}}_a^{(1)} = -\frac{\partial \bar{V}_{(-1)a}}{\partial \mathbf{r}_a}, \ \ \tilde{\mathbf{F}}_a^{(2)} = -\frac{\partial \bar{V}_{(2)a}}{\partial \mathbf{r}_a} \tag{96}$$

Solutions of the central configuration equation are critical points of $\tilde{V}_a$:

$$\tilde{\mathbf{F}}_a = 0 \Leftrightarrow \frac{\partial \tilde{V}_a}{\partial \mathbf{r}_a} = \frac{\partial}{\partial \mathbf{r}_a}\tilde{V}_{(-1)a} + \frac{\partial}{\partial \mathbf{r}_a}\tilde{V}_{(2)a} = 0. \tag{97}$$

4.3   A Variational Principle for Central Configurations

Define the associated total energies as

$$\tilde{V} := \tilde{V}_{(-1)} + \tilde{V}_{(2)} \tag{98}$$

where the effective total gravitational potential energy is

$$\tilde{V}_{(-1)} := \sum_a \tilde{V}_{(-1)a} = -\sum_a \sum_{b \neq a} \frac{Gm_a m_b}{|\mathbf{r}_{ab}|} \tag{99}$$

which is homogeneous of degree -1, and the effective total repulsion potential energy is

$$\tilde{V}_{(2)} := \sum_a \tilde{V}_{(2)a} = -\frac{1}{2}\sum_a G\tilde{M}m_a \mathbf{r}_a.\mathbf{r}_a \tag{100}$$

which is homogeneous of degree $k = 2$. These are also defined in terms of the comoving coordinates $\mathbf{r}_a$. By definition, they are all constant.

Now (100) is just $\tilde{V}_{(2)} := -G\tilde{M}\tilde{I}_0$ and (50) is $2G\tilde{M}\,\tilde{I}_0 = -\tilde{V}_{(-1)}$, so together they give the virial-type relation

$$\tilde{V}_{(-1)} = 2\tilde{V}_{(2)}. \tag{101}$$

for the effective energies (cf. equation (28)). This implies that one can express the total effective energy in terms of either partial term:

$$\tilde{V} := \frac{3}{2}\tilde{V}_{(-1)} = 3\tilde{V}_{(2)}\,. \tag{102}$$

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However as noted above, the virial theorem does not hold for the space-time system.

Critical points of $\tilde{V}_a$ are clearly critical points of $\tilde{V} = \sum_a \tilde{V}_a$. Since $\tilde{V}$ becomes infinitely large and negative as two or more points approach one another or as one or more recede to infinity, it is easy to see that there must be at least one global maximum and no global minimum. In addition one suspects there are many saddle points. Thus we have

> **Theorem:** *Critical points of the function $\tilde{V}$ are in one-one correspondence with central configurations. There is at least one global maximum and no global minimum. Every critical point, and hence every central configuration, satisfies $2\tilde{V}_2 = \tilde{V}_{(-1)}$.*

The importance of this result is that it allows one to search efficiently for central configurations of very many particles by numerically maximising $\tilde{V}$ [32].

> *It is important to realise that whether or not the critical point is a maximum or otherwise of $\tilde{V}$ is unrelated to whether of not the associated time dependent solution of (4) is dynamically stable.*

4.4 A scale-free variational principle for Central Configurations

We showed in the previous subsection that every central configuration is a critical point of the function $\tilde{V} = \tilde{V}_{(-1)} + \tilde{V}_{(2)}$ which is defined on the $3N$-dimensional configuration space Q_N of N distinct points in three-dimensional Euclidian space. Moreover every critical point satisfies

$$\tilde{V}_{(-1)} = 2\tilde{V}_{(2)}. \tag{103}$$

An equivalent formulation, suggested to us by the work of Julian Barbour (private communication), is obtained by considering the scale-invariant function

$$C_s = -(-\tilde{V}_{(2)})^{1/2}\tilde{V}_{(-1)} = -I^{1/2}V \tag{104}$$

which is defined on the quotient $Q_N/\mathbb{R}_+$ of Q_N by homotheties. Because of the scaling properties of these quantities, replacing $\mathbf{x}_a$ by $\mathbf{r}_a = \mathbf{x}_a/S(t)$ leaves C_s unchanged

$$C_s = -\tilde{I}_0^{1/2}\,\tilde{V}_{(-1)}\,. \tag{105}$$

If we differentiate C_s with respect to $\mathbf{r}_a$ we obtain

$$-\frac{1}{2}(-\tilde{V}_{(2)})^{-1/2}\tilde{V}_{(-1)}\frac{\partial\tilde{V}_{(2)}}{\partial\mathbf{r}_a} + (-\tilde{V}_{(2)})^{1/2}\frac{\partial\tilde{V}_{(-1)}}{\partial\mathbf{r}_a} = 0\,. \tag{106}$$

Taking the dot product with $\mathbf{r}_a$, summing over a and using Euler's theorem gives (103) and substituting back into (106) yields

$$\frac{\partial\tilde{V}_{(2)}}{\partial\mathbf{r}_a} + \frac{\partial\tilde{V}_{(-1)}}{\partial\mathbf{r}_a} = \frac{\partial\tilde{V}}{\partial\mathbf{r}_a} = 0\,. \tag{107}$$

Thus every critical point of C_s is a critical point of $\tilde{V}$ and moreover satisfies (103). Conversely (107) and (103) are easily seen to imply (106). Thus we have the following

> **Theorem::** *Critical points of the scale-invariant function C_s are in one-one correspondence with central configurations.*

We repeat the warning that *it is important to realise that whether or not the critical point is a maximum or otherwise of C_s is unrelated to whether of not the associated time dependent solution of (4) is dynamically stable.*

5   Elaborations

This paper has set out how exact Newtonian solutions exist for homothetically moving configurations of gravitating point particles. One can consider changes to the problem if there are the following generalisations:

- An environment of objects that are not affected by the system, exerting an external gravitational field,

- Subgroups of particles that are distinguished by being held together by elastic forces,

- If the size of particles is too large for them to be considered as points, so tidal forces matter,

- Inhomogeneous conformal solutions,

- Solutions with rotation as well as expansion.

While these are all of interest, the key further development is to consider the perturbed version of these equations, and how they relate to structure formation after decoupling. That will be the topic of a further paper.

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