Holography and the Future Tube

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Abstract

The Future Tube T_n^+ of n-dimensional Minkowski spacetime may be identified with the reduced phase space P, or "space of motions " of a particle moving in $(n + 1)$-dimensional Anti-de-Sitter Spacetime. Both are isomorphic to a homogeneous bounded domain in $\mathbf{C}^n$ whose Shilov boundary may be identified with n-dimensional conformally compactified Minkowski spacetime.

1   Introduction

The purpose of this talk is to describe some remarkable geometric facts relating the Future Tube T_n^+ of n-dimensional Minkowski spacetime to the reduced phase space P, or "space of motions " of a particle moving in $(n+1)$-dimensional Anti-de-Sitter Spacetime with a view to illuminating the Maldacena conjectures [1] relating string theory on AdS_{n+1} to Conformal Field theory on its conformal boundary.

2   The Future Tube

Slightly confusingly perhaps, the Future Tube T_n^+ is usually defined as those points of complexified Minkowski spacetime $\mathbf{E}_{\mathbf{C}}^{n-1,1}$, with complex coordinates $z^\mu = x^\mu + iy^\mu$, such that y^μ lie inside the past light cone C_n^-. That is $y^0 < -\sqrt{y^i y^i}$. With this convention together with standard conventions of quantum

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field theory, a positive frequency function is the boundary value of a function which is holomorphic in the future tube. The relation between a holomorphic function, for example one defined in some bounded domain $D \subset \mathbf{C}^n$, and the values of that function on its Shilov boundary S, which is an n real dimensional submanifold of the $(2n - 1)$ real dimensional boundary ∂D may be said to be "holographic "in that the information about two real valued functions of $2n$ real variables is captured by two real valued functions of n real variables. This is the key idea behind the application of dispersion relations to quantum field theory and their use to derive rigourous general results such as the spin-statistics theorem and invariance under CPT.

3 Bounded Complex Domains

Homogeneous bounded domains were classified by Cartan [5] and their properties are described in detail by Hua [6]. The case we are interested in corresponds to the Hermitian symmetric space $D = SO(n, 2)/(SO(2) \times SO(n))$. It is referred to by Hua [6] as "Lie Sphere Space ". In order to see this, consider the complex light cone $C_{\mathbf{C}}^{n+2} \subset \mathbf{C}^{n+2}$ given by

$$(W^{n+1})^2 + (W^{n+2})^2 - (W^i)^2 = 0. \tag{1}$$

Compactified complexified Minkowski spacetime $\overline{\mathbf{E}_{\mathbf{C}}^{n-1,1}}$ consists of complex light rays passing through the origin. This means that one must identify rays W^A and λW^A, where $A = 1, \ldots, n + 2$, $i = 1, \ldots, n$ and $\lambda \in \mathbf{C}^\star \equiv \mathbf{C} \setminus 0$, that is $\overline{\mathbf{E}_{\mathbf{C}}^{n-1,1}} = C_{\mathbf{C}}^{n+2}/\mathbf{C}^\star$. Evidently $SO(n - 1, 2; \mathbf{R})$ acting in the obvious way on $\mathbf{C}^{n+1}$ leaves the complex lightcone $C_{\mathbf{C}}^{n+2}$ invariant and commutes with the $\mathbf{C}^\star$ action. Thus the action of $SO(n - 1, 2; \mathbf{R})$ descends to D. If we restrict the coordinates W^A to be real, we obtain the standard construction of n-dimensional compactified real Minkowski spacetime $\overline{\mathbf{E}^{n-1,1}}$, as light rays through the origin of $\mathbf{E}^{n-1,2}$. It follows that $\overline{\mathbf{E}^{n-1,1}} \equiv (S^1 \times S^{n-1})/\mathbf{Z}_2$, The finite points in Minkowski spacetime, i.e. those not contained in what Penrose refers to as $\mathcal{I}$ may be obtained by intersecting the light cone with a null hyperplane which does not pass through the origin.

To get to the description given by Hua [6], introduce coordinates parameterising (most of) the light cone by u, w^i by

$$W^{n+1} - iW^{n+2} = \frac{1}{u}, \tag{2}$$

$$W^{n+1} + iW^{n+2} = \frac{w^i w^i}{u}, \tag{3}$$

and

$$W^i = \frac{w^i}{u}, \tag{4}$$

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If $w \in \mathbf{C}^n$ is a complex $(n-1)$ column vector and $w^2 = w^t w$ and $|w|^2 = w^\dagger w$ then the domain D defined by [6]

$$1 - |w|^2 \geq \sqrt{|w|^4 - |w^2|^2}. \tag{5}$$

The topological boundary is given by the real equation:

$$1 - 2|w|^2 + |w^2|^2 = 0. \tag{6}$$

On the other hand, the Shilov boundary $S \subset \partial D$ is determined by the property that the maximum modulus of any holomorphic function on P is attained on S. Consider, for example, the holomorphic function w. It attains its maximum modulus when $w = \exp(i\tau)\mathbf{n}$, where $\mathbf{n}$ is a real unit $(n-1)$ vector. Thus S is given by $S^1 \times S^{n-1}/\mathbf{Z}_2$. If we take W^i to be the real unit n-vector $\mathbf{n}$ and $u = \exp(-i\tau)$ we see that S and $\overline{\mathbf{E}^{n-1,1}}$ are one and the same thing.

4 The geodesic Flow of AdS_{n+1}

Now let us turn to $(n+1)$-dimensional Anti-de-Sitter spacetime . One possible approach to quantising a relativistic particle moving in $AdS_{n+1}n$, might be to look at the relativistic phase space and then pass to the constrained space and to "quantise" it. Recall that, in general, the relativistic phase space of a spacetime M is the cotangent bundle $T^\star M$ with coordinates $\{x^\mu, p_\mu\}$, canonical one-form $A = p_\mu dx^\mu$ and symplectic form

$$\omega = dp_\mu \wedge dx^\mu. \tag{7}$$

The geodesic flow is generated by the covariant Hamiltonian

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu. \tag{8}$$

The flow for a timelike geodesic, corresponding to a particle of mass m lies on the level sets, call them Γ, given by

$$\mathcal{H} = -\frac{1}{2} m^2. \tag{9}$$

The restriction of the canonical one-form A to the level sets Γ endows them with a contact structure, in other words, the restriction of dA has rank $2n$ and it's one-dimensional kernel is directed along the geodesic flow. Thus, locally at least, one may pass to the reduced phase space $P = \Gamma/G_1$ where G_1 is the one-parameter group generated by the covariant Hamiltonian $\mathcal{H}$, by a "Marsden-Weinstein reduction". Geometrically, the group G_1 takes points and their cotangent vectors along the world lines of the timelike geodesics.

The reduced $2n$-dimensional phase space P is naturally a symplectic manifold. Moreover the isometries of M act by canonical transformations, i.e. by

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symplectomorphisms, on P taking timelike geodesics to timelike geodesics. Each Killing vector field $K_a^\mu(x)$ on M determines a "moment map" $\mu_a(x,p) = K_a^\mu p_\mu$ on $T^\star M$ which Poisson commutes with the covariant Hamiltonian $\mathcal{H}$. They thus descend to reduced phase space P where their Poisson algebra is is the same as the Lie algebra of of the isometry group. Thus, for example, if $n = 2$ the Poisson algebra is $sl(2; \mathbf{R})$. This fact is behind the connection between black holes, conformal mechanics, and Calogero models discussed in [15].

We are interested in the quantum theory rather than the classical geodesic motion and so it is appealing to attempt to implement the geometric quantization programme by "quantising " P. A point of particular interest would then be to compare it with a more conventional approach based on quantum field theory in a fixed background. In the general case this appears to be difficult because one does not have a good understanding of the space of timelike geodesics P. However in the present case of AdS_{n+1}, the space P may be described rather explicitly. It is a homogeneous Kähler manifold which is isomorphic to the future tube T_n^+ of n-dimensional Minkowski spacetime. Because it is a Kähler manifold one may adopt a holomorphic polarisation. The resulting "quantization" is the same as that considered by Berezin and others many years ago (see e.g. [11, 12, 13]). A more physical description is in terms of coherent states.

To see the relation between P and T_n^+ explicitly it is convenient to regard AdS_{n+1} as the real quadric in $\mathbf{E}^{n,2}$ given by

$$(W^{n+1})^2 + (W^{n+2})^2 - (W^i)^2 = 1. \tag{10}$$

It is clear by comparing (1) and (10) that the light cone C_{n+2} and the quadric AdS_{n+1} can only touch at infinity, which explains why the conformal boundary of $AdS_{n+1}/\mathbf{Z}_2$ is the same as compactified Minkowski spacetime $\overline{\mathbf{E}^{n-1,1}} \equiv (S^1 \times S^{n-1})/\mathbf{Z}_2$, where $\mathbf{Z}_2$ is the antipodal map $\mathbf{Z_2} : \mathbf{W^A} \to -\mathbf{W^A}$.

Using this representation of AdS_{n+1} , one easily sees that every timelike geodesic is equivalent to every other one under an $SO(n-1,2)$ transformation. They may all be obtained as the intersection of some totally timelike 2-plane passing through the origin of of the embedding space $\mathbf{E}^{n,2}$ with the AdS_{n+1} quadric. The space P of such two planes may thus be identifed with the space of geodesics. It is a homogeneous space of the isometry group, in fact the Grassmannian $SO(n,2)/(SO(2) \times SO(n))$. Note that, as one expects, the dimension of P is $2n$. The denominator of the coset is the maximal compact subgroup of $SO(n,2)$. Two factors correspond to timelike rotations in the timelike 2-plane and rotations of the normal space respectively. The former may be identified with the one parameter group G_1 generated by the covariant Hamiltonian $\mathcal{H}$. Thus the level set Γ is the coset space $SO(n,2)/SO(n)$.

5 Quantisation

Given a manifold X with coordinates x and a measure μ, an over complete set of coherent states is a set of vectors $\{|x\rangle\}$ in some quantum mechanical Hilbert space H_{qm} providing a resolution of the identity

$$\int_X \mu|x\rangle\langle x| = \hat{1}. \tag{11}$$

Usually one takes X to be a homogeneous space of some Lie group. However that is not essential for the general concept. A Hermitian operator $\hat{A}$ on H_{qm} is associated with a real function $A(x)$ via the relation

$$\hat{A} = \int_X A(x)\mu|x\rangle\langle x| = \hat{1}. \tag{12}$$

The commutator algebra of a set of operators on H_{qm} then gives rise to an algebra $\mathcal{A}$ on the associated functions. If X is a symplectic manifold one expects, at least in the limit that Planck's constant is small, or more strictly speaking in the limit of large action, that the algebra $\mathcal{A}$ reflects the Poisson algebra on the functions on X. For the special case of non-compact Kähler manifold of dimension $2n$, Kähler form ω and with coordinates w^i and metric

$$g_{i\bar{j}} = \frac{\partial^2 F}{\partial w^i \partial \overline{w}^j}. \tag{13}$$

where F is the Kähler potential it was proposed by Berezin [11, 12, 13] that one choose for H_{qm} the space of holomorphic functions with inner product

$$\langle g(w)|f(w)\rangle = \int_X \overline{g(w)}f(z)\frac{\omega^n}{n!}\exp(-\frac{1}{h}F) \tag{14}$$

One must now choose h so that one gets a non-trivial Hilbert space H_{qm}. Given that one may proceed to represent the isometries of X on H_{qm} and to introduce other operators and investigate their algebras. Note that the metric on H_{qm} will in general depend upon h. The quantity h is referred to in this context as Planck's constant, although physically, for dimensional reasons, that is not really accurate. If it happens that X is Einstein Kähler with negative scalar curvature then the Monge Ampère equation tells us that

$$\det g_{i\bar{j}} = \exp(-\Lambda F) \tag{15}$$

Thus, to get convergence one wants $h \leq \frac{1}{|\Lambda|}$. An upper bound for Planck's constant seems very puzzling from a physical point of view but it has a simple explanation.

Consider the circle bundle $S^1 \to E \to X$ over the Kähler manifold X with a connection whose curvature $F = dA$ is a multiple, e, of the symplectic form

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ω. Thus $F = e\omega$ and one may think of e as the product of the electric charge and the strength of magnetic field. The covariant derivative is $\overline{\mathcal{D}} = \overline{\partial} + e\partial_{\overline{w}}F$. Thus if $\psi = e^{-eF}f(w)$, where $f(w)$ is holomorphic, then

$$\overline{\mathcal{D}}\psi = 0. \tag{16}$$

Now the space of spinors on a Kähler manifold may be identified with the space of differential forms of type (0,p). The Dirac operator corresponds to the operator $\sqrt{2}(\overline{\partial} + \overline{\partial}^{\dagger})$. Minimally coupling the Dirac operator to the Kähler connection corresponds geometrically to taking the tensor product of the space of spinors with a power of the canonical bundle. The first power gives the canonical $Spin^c$ structure on X. If X is assumed to have trivial second cohomology we may take any (not even rational power). The Dirac operator minimally coupled to F corresponds to $\sqrt{2}(\overline{\mathcal{D}} + \overline{\mathcal{D}}^{\dagger})$. It follows that by setting $2e = -\frac{1}{\hbar}$ We may identify H_{qm} with the space of zero modes of the minimally coupled Dirac operator on X. Now typically X has negative curvature, and so only if the charge on the spinors is sufficiently large, and of the correct sign, will there be a big space of, or indeed any, zero modes.

The above theory was rather general. We now restrict to the case of a bounded complex domain D in $\mathbf{C}^n$. We begin by describing its Kähler structure. Associated with D is the Hilbert space $\text{Hol}(D)$ of square integrable holomorphic functions. If $\{\phi_s\}$, $s = 1, \ldots,$ is an orthonormal basis for $\text{Hol}(D)$ the Bergman Kernel $K(w, \overline{v})$ is defined by

$$K(w, \overline{v}) = \sum \phi_s(w)\overline{\phi_s(v)}. \tag{17}$$

The Bergman Kernel gives rise to a Kähler potential $F(w, \overline{w}) = \log K(w, \overline{w})$ in terms of which the Bergman metric on D is given by

$$g_{i\overline{j}} = \frac{\partial^2 F}{\partial w^i \partial \overline{w}^j}. \tag{18}$$

Geometrically the basis $\{\phi_s\}$ gives a map of D into $\mathbf{CP}^\infty$ and $g_{i\overline{j}}$ is the pullback of the Fubini-Study metric. Now although D has finite Euclidean volume, because K and F typically diverge at the boundary ∂D the volume of D with respect to the Kähler metric $g_{i\overline{j}}$ diverges. For example, in our case Hua [6] has calculated $K(w, \overline{w})$ and finds

$$K(w, \overline{w}) = \frac{1}{V_n} \frac{1}{(1 + |w^2|^2 - 2|w|^2)^n}, \tag{19}$$

where $V_n = \frac{\pi^n}{2^{n-1}n!}$ is the Euclidean volume of D.

6 Cheng-Mok-Yau-Anti-de-Sitter Spacetimes

The theory described above does not take into account gravity. Of course supergravity methods provide a way of doing that. They would lead to replacing AdS_n by some other solution of the supergravity equations of motion. For example with some other Einstein Space. A great deal of work has already appeared going in this direction. One new possibility will be described in this section.

However it is also worth asking how the space P might be generalised. The relationship between these two generalisations is then of interest. This will be dercibed in the next section.

If n is odd, $n = 2m+1$ say, then AdS_{2m+1} may be regarded as a circle bundle over complex hyperbolic space $H_{\mathbf{C}}^m$ [4]. Clearly, using complex coordinates Z^A, $A = 1, \ldots, m+1$ in $\mathbf{R}^{2m+2} \equiv \mathbf{C}^{m+1}$, AdS_{2m+1} is given by

$$|Z^1|^2 - \ldots - |Z^{m+1}|^2 = 1. \tag{20}$$

We may now fibre by the $U(1)$ action $Z^A \to e^{it} Z^A$. The orbits are *closed timelike curves* in AdS_{2m+1}. The base space B has a Riemannian, i.e. positive definite, metric. In fact $B = SU(m,1)/U(m) \equiv H_{\mathbf{C}}^m$ is the unit ball in $\mathbf{C}^m$. This is another bounded complex domain in $\mathbf{C}^m$. The metric on the base space is precisely its Bergman metric. In fact $H_{\mathbf{C}}^m$ with its Bergman metric is ithe symmetric space dual of $\mathbf{CP}^m$ with its Fubini-Study metric. The construction we have just described is the symmetric space dual of the usual Hopf fibration.

The metric is

$$ds^2 = -(dt + A_a dx^b)^2 + g_{ab} dx^a dx^b \tag{21}$$

where $a, b = 1, 2, \ldots, 2m$, g_{ab} is the Einstein-Kähler metric and dA is the Kähler form. In traditional relativist's language, AdS_{2m+1} has been exhibited as an ultra-stationary metric (i.e. one with constant Newtonian potential $U = \frac{1}{2} \log(-g_{00})$. The Sagnac or gravito-magnetic connection, governing frame-dragging effects corresponds precisely to the connection of the standard circle bundle over the Kähler base space. Its curvature is the Kähler form.

Now it is easy to check that one may replace the Bergman manifold $\{B, g_{ab}\}$ with any other $2m$ dimensional Einstein-Kähler manifold with negative scalar curvature and obtain a $(2m + 1)$-dimensional Lorentzian Einstein manifolds admitting Killing spinors in this way. According to Cheng and Yau [17] and Mok and Yau [16], there is a rich supply of complete Einstein Kähler metrics on complex domains. Their investigation might well prove to be fruitful. The conformal boundary of these spacetimes is clearly related to the boundary of the complex domain, but the precise relationship is not at all clear. Apart from their possible applications to the Maldecena conjecture, these spacetimes may provide a useful arena, albeit in higher dimensions, for investigating the effects of closed timelike curves in general relativity.

7   Adapted Complex Structures

We have seen that the space P of timelike geodesics in AdS_n carries an Einstein Kähler structure. In fact the entire cotangent bundle $T^\star AdS_n$ admits a Ricci-flat pseudo Kähler metric, i.e. one with signature $(2n - 2, 2)$. The existence of this Ricci-flat pseudo-Kähler metric may be obtained by analytically continuing Stenzels's [18] positive definite Ricci-flat Kähler metric on the cotangent bundle of the standard n-sphere, $T^\star S^n$ [18]. The simplest case is when $n = 2$. Stenzel's metric is then the Eguchi-Hanson metric which may be analytically continued to give a "Kleinian" metric of signature $(2, 1)$ on $T^\star AdS_2$. In fact the cotangent bundle of *any* Riemannian manifold may be endowed with a canonical complex structure and a variety of Kähler metrics. We will describe this construction shortly and formulate a version for Lorentzian manifolds. Before doing so we descibe Stenzel's construction.

The cotangent bundle of the n-sphere $T^\star S^n$ may be identified with an affine quadric in $\mathbf{C}^{n+1}$. This may be seen as follows: $T^\star S^n$ consist of a pair of real $(n + 1)$ vectors X^A and P^A such that

$$X^1 X^1 + X^2 X^2 + \ldots + X^{n+1} X^{n+1} = 1, \tag{22}$$

$$X^1 P^1 + X^2 P^2 + \ldots + X^{n+1} P^{n+1} = 0. \tag{23}$$

If $P = \sqrt{P^1 P^1 + P^2 P^2 + \ldots + P^{n+1} P^{n+1}}$ one may map $T^\star S^n$ into the affine quadric

$$(Z^1)^2 + (Z^2)^2 + \ldots + (Z^{n+1})^2 = 1 \tag{24}$$

setting

$$Z^A = A^A + i B^A = \cosh(P) X^A + i \frac{\sinh(P)}{P} P^A. \tag{25}$$

Stenzel then seeks a Kähler potential depending only on the restriction to the quadric (24) of the function

$$\tau = |Z^1|^2 + |Z^2|^2 + \ldots + |Z^{n+1}|^2. \tag{26}$$

The Monge-Ampère equation now reduces to any ordinary differential equation.

In the case of AdS_{p+2} we may proceed as follows. The bundle of future directed timelike vectors in AdS_{p+2}, $T^+ AdS_{p+2}$ consists of pairs of timelike vectors X^A, P^A in $\mathbf{E}^{p+1,2}$ such that

$$X^A X^B \eta_{AB} = -1 \tag{27}$$

and

$$X^A P^B \eta_{AB} = 0, \tag{28}$$

with P^A future directed and $\eta_{AB} = \text{diag}(-1, -1, +1, \ldots, +1)$ the metric. We define $P = \sqrt{-P^A P^B \eta_{AB}}$ and

$$Z^A = \cosh(P) X^A + i \frac{\sinh(P)}{P} P^A \tag{29}$$

which maps $T^+ AdS_{p+2}$ to the affine quadric

$$Z^A Z^B \eta_{AB} = -1. \tag{30}$$

One then seeks a Kähler potential depending only on the restriction to the quadric (24) of the function

$$\tau = |Z^0|^2 + |Z^{p+2}|^2 - |Z^1|^2 - \ldots - |Z^{p+1}|^2. \tag{31}$$

The Monge-Ampère equation again reduces to an ordinary differential equation.

The canonical complex structure on $T^\star M$ is defined as follows. Let $\sigma \rightarrow \left(x^\mu(\sigma), p_\mu = g_{\mu\nu} \frac{dx^\nu}{d\sigma}(\sigma)\right)$ be a solution of Hamilton's equations for a geodesic in $T^\star M$. Then in any complex chart w^i, $i = 1, \ldots, n$ one demands that for all geodesics the map $\mathbf{C} \rightarrow T^\star M$ given by $\sigma + i\tau \rightarrow \left(x^\mu(\sigma), \tau p_\mu\right)$ is holomorphic. The "energy" $\mathcal{H} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$ is a real valued positive function on $T^\star M$ which vanishes only on M. It is plurisubharmonic with respect to the complex structure i.e. the hermitian metric

$$\frac{\partial^2 \mathcal{H}}{\partial w^i \partial \overline{w}^j} \tag{32}$$

is positive definite, and $\sqrt{\mathcal{H}}$ satisfies the homogeneous Monge-Ampère equation

$$\det\left(\frac{\partial^2 \sqrt{\mathcal{H}}}{\partial w^i \partial \overline{w}^j}\right) = 0. \tag{33}$$

In the case of Stenzel's construction one has $\mathcal{H} = \frac{1}{2} P^2 = \frac{1}{8}(\cosh^{-1}(\tau))^2$.

8 The Physical Dimension

The analysis above works for general dimension. However the case $n = 4$ is special since $SO(4, 2) \equiv SU(2, 2)/\mathbf{Z}_2$. One may identify points in real four-dimensional Minkowski spacetime $\mathbf{E}^{3,1}$ with two by two Hermitian matrices $z = x^0 + \mathbf{x} \cdot \sigma$. The future tube T_4^+ then corresponds to complex matrices $x = z^0 + \mathbf{z} \cdot \sigma$ whose imaginary part is positive definite. The Cayley map

$$z \rightarrow w = (z - i)(z + i)^{-1} \tag{34}$$

maps this into the bounded holomorphic domain in $\mathbf{C}^4$ consisting of the space of two by two complex matrices w satisfying

$$1 - w w^\dagger > 0. \tag{35}$$

For more details the reader is directed to [7]. For this approach to the compactification of Minkowski spacetime see also [8, 9]

9   de-Sitter

The aim of the talk has been to describe some intriguing relations between the future tube and the covariant phase space of Anti-de-Sitter spacetime which seem to lie at the heart of holography. One may ask: what about de-Sitter spacetime? As one might expect, everything goes wrong. One problem is that one does not get a positive definite metric on covariant phase space. This makes for difficulties in the usual approach to geometric quantisation. Thie lack of a positive metric is more or less clear because the timelike geodesics are orbits of the non-compact group $SO(1,1)$. This last fact is also closely related to thermal radiation seen by geodesics observers in de-Sitter spacetime. A closely related problem concerns the lack of a positive energy generator of $SO(n,1)$ and the consequent impossibility of de-Sitter supersymmetry.

10   Final Remarks

After the talk I became aware of [24] where a geometric quantisation approach to Anti-de-Sitter spacetime is described. I was also told of some work in axiomatic quantum field theory [25] [26] which appears to have some relation to what I have desribed above.

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