The Bernstein Conjecture, Minimal Cones, and Critical Dimensions

Gary W. Gibbons[†], Kei-ichi Maeda[∗], and Umpei Miyamoto[♯]

[†]*DAMTP, Centre for Mathematical Sciences, Cambridge University,*
Wilberforce Road, Cambridge CB3 OWA, UK
[∗]*Department of Physics, Waseda University,*
Okubo 3-4-1, Tokyo 169-8555, Japan
[♯]*Racah Institute of Physics, Hebrew University of Jerusalem,*
Givat Ram, Jerusalem 91904, Israel

G.W.Gibbons@damtp.cam.ac.uk, maeda@waseda.jp, umpei@phys.huji.ac.il

Abstract

Minimal surfaces and domain walls play important roles in various contexts of space-time physics as well as material science. In this paper, we first review the Bernstein conjecture, which asserts that a plane is the only globally well defined solution of the minimal surface equation which is a single valued graph over a hyperplane in flat spaces, and its failure in higher dimensions. Then, we review how minimal cones in four- and higher-dimensional spacetimes, which are curved and even singular at the apex, may be used to provide counterexamples to the conjecture. The physical implications of these counterexamples in curved spacetimes are discussed from various points of view, ranging from classical general relativity, brane physics, and holographic models of fundamental interactions.

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Contents

1   Introduction

Minimal surfaces[1] in Euclidean space $\mathbb{E}^3$ have been extensively studied since the pioneering work of Thomas Young and of Laplace. In Monge, or non-parametric, gauge the surface is specified by the height function $z = z(x, y)$ above some plane. The non-parametric minimal surface equation governing the function $z(x, y)$ is

$$\partial_x \left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}} \right) + \partial_y \left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}} \right) = 0 \,. \tag{1}$$

A famous result of Bernstein asserts that the only single valued solution of Eq. (1) defined for all $(x, y) \in \mathbb{R}^2$ is a plane. It may also be shown that the planar solution is a minimizer of the area functional among compactly supported variations of the surface. In terms of brane theory, this means that the "classical ground state", *i.e.*, the static minimum of the energy

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[1]Newcomers to the subject are warned that according to the current mathematical usage, which is sanctioned by a long standing but nevertheless illogical and confusing tradition, the words "minimal surface" are used to mean a p-dimensional surface, *i.e.*, a p-brane, whose first variation of the p-volume functional vanishes. There is no implication about the second variation, *i.e.*, the Hessian of the p-volume functional. Therefore it may or may not be the case that a minimal surface is a (local or global) "minimizer" of the p-volume functional. Extremal surface would be a better name. Note that when we speak of a variational principle for non-compact surfaces we mean, as is customary in the mathematical literature, against *compactly supported variations*. In other words, in any calculation we only consider an integral of a compact subset of the non-compact surface and test its second variation. A further potential source of confusion is that a minimal surface is often referred to as being "stable" if the Hessian of the p-volume functional is positive definite. Stability in this sense may or may not coincide with a dynamical notion of stability, depending on the physical context.

functional for a membrane in three dimensional Euclidean space $\mathbb{E}^3$, which may be thought of as a static configuration in 4-dimensional Minkowski spacetime $\mathbb{E}^{3,1}$, is smooth and indeed planar. From the world volume point of view the classical ground state of the membrane preserves (2+1)-dimensional Poincaré invariance and may be thought of as a copy of (2+1)-dimensional Minkowski spacetime $\mathbb{E}^{2,1}$.

It is natural to conjecture that Bernstein's theorem remains valid for a minimal p-dimensional hypersurface in $(p+1)$-dimensional Euclidean space $\mathbb{E}^{p+1}$. In other words, the classical ground state of a *p-brane* in $(p+1+1)$-dimensional Minkowski spacetime $\mathbb{E}^{p+1,1}$ should be flat and invariant under the action of the $(p+1)$-dimensional Poincaré group $E(p,1)$. Remarkably, although true for $p \le 7$ it fails for $p + 1 = 9$ [1]. In other words, the classical ground state of an 8-brane in 10-dimensional Minkowski spacetime spontaneously breaks (8+1)-dimensional Poincaré invariance. The proof [1] rests on the fact that in $\mathbb{E}^8$ and above, a minimal hypersurface which is a minimizer of the p-volume functional among compactly supported variations need not be smooth. There are rather explicit counterexamples called *minimal cones*. Their existence leads to the conclusion that Bernstein's theorem fails in $\mathbb{E}^9$ [1] [2].

As far as we aware, there has been very little discussion of the significance of this fact in the M/String theory literature. The breakdown of regularity of minimal hypersurfaces of flat space extends to minimal hypersurfaces of curved Riemannian manifolds and this has consequences for proofs of the positive energy theorem of general relativity which make essential use of minimal surfaces as a technical tool [2, 3, 4]. It seems worthwhile therefore to examine the behavior of minimal surfaces in higher dimensions and in curved spaces in some explicit detail in order to understand better the situation and its possible physical implications. In particular, it is interesting to see whether the existence of various critical dimensions which has been noted in related contexts is of a universal nature and related to the the breakdown of Bernstein's theorem and the existence of minimal cones.

To make progress it is helpful to assume that the relevant surfaces have sufficient symmetries with which the problem may be reduced to one involving ordinary differential equations in an appropriate quotient space X, a ploy known to mathematicians as *equivariant variational theory*. Typically the brane equations of motion reduce to finding geodesics in X with respect to a suitable metric g on X, induced by the p-volume functional. The p-brane will be p-volume minimizing if the corresponding geodesic γ is length minimizing. A necessary condition that a geodesic joining points a and b be length minimizing is that γ contains no points between a and b conjugate to either. The existence of such conjugate points is governed by the Jacobi or geodesic deviation equation, solutions of which depend on the curvature of X. In the case that

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[2] We emphasize for clarity that Bombieri et alia's result uses the failure of regularity of seven dimensional minimal hypersurfaces in 8 ambient dimensions to establish the non-uniqueness of (*i.e.*, the existence of non-flat) minimal hypersurfaces of co-dimension one which may be given by a single valued height function defined for all of $\mathbb{R}^8$ in 9 ambient dimensions. The argument, which depends on scaling properties of the equations, is given in detail in the cited references.

X is 2-dimensional, it is the sign of the Gauss curvature K which is important. If for example K is negative in the vicinity of γ, then it can contain no conjugate points and hence must be locally length minimizing. In the cases we shall consider the Gauss curvature is actually positive and a more detailed examination is required. One might have thought that positive Gauss curvature would lead to a second variation or Hessian of indefinite sign. However, the situation is more subtle since the effective metric governing the variational principle is incomplete and becomes singular near a conical point and compensatory terms can arise which in low dimension render the Hessian positive definite.

The simplest example of this situation is when a Lie group G acts isometrically on a $(p+1)$-dimensional Riemannian manifold $\{\Sigma, h\}$, thought of as "space" with orbits which are $(p-1)$-dimensional. The $X = \Sigma/G$ is (locally) 2-dimensional and a curve in X may be thought of as the projection under the action of G of a p-dimensional submanifold S of Σ. The basic example of this setup is when $p = 2k+1$, $\Sigma = \mathbb{E}^{2k+2}$ and $G = \mathrm{SO}(k+1) \times \mathrm{SO}(k+1)$ with the standard action on $\mathbb{E}^{2k+2} = \mathbb{E}^{k+1} \times \mathbb{E}^{k+1}$ with flat metric

$$h = dx^2 + x^2 d\Omega_k^2 + dy^2 + y^2 d\Omega_k^2, \tag{2}$$

where $d\Omega_k^2$ is the standard round metric on S^k. The induced metric g is

$$g = (xy)^{2k}\left(dx^2 + dy^2\right). \tag{3}$$

The orbit of the straight line $x = y$ under the action of $\mathrm{SO}(k+1) \times \mathrm{SO}(k+1)$ is a $(2k+1)$-dimensional minimal cone with a singularity at the origin, $x = y = 0$. A study of the second variation shows that this singular cone is $(2k+1)$-volume minimizing as long as $k \geq 3$. In the next section we review the existence of minimal cones in higher dimensional flat spaces with more general examples. From the physical point of view, the minimal surface is a thin limit of a domain wall. Thus, also in the next section, we describe the relation between the Bernstein conjecture and some theorems/conjectures asserting the existence of non-planar domain walls. Then, in Sec. 3 we present a general prescription to investigate minimal surfaces in curved background and explicit examples of minimal cones in black hole backgrounds. We show that the critical dimension in the black-hole–brane system investigated by Frolov [5] is related to the failure of the Bernstein conjecture. In addition, we present a new example of a minimal cone in the black hole background. The final section is devoted to discussions of possibly related systems.

2 Branes in Flat Backgrounds

2.1 A Primary Example

By treating a simple but nevertheless important example, we see that the problem of finding minimal surfaces reduces to finding geodesics in a low dimensional space. Then, the criterion of

length minimizing for the geodesic is introduced. One may find more mathematical and general introduction in [6].

Let us consider a G-invariant submanifold $S \subset \Sigma = \mathbb{E}^{m+n+2}$ $(m, n \geq 1)$ as the configuration of p-brane $(p = n + m + 1)$. For a simple example, we focus on the case of $G = \mathrm{SO}(m+1) \times \mathrm{SO}(n+1) \subset \mathrm{SO}(m+n+2)$. The metric of ambient flat space is written in the form

$$h = dx^2 + x^2 d\Omega_m^2 + dy^2 + y^2 d\Omega_n^2 . \tag{4}$$

A projected metric $\tilde{h}$ on Σ/G is defined via a transformation of an inner product, $\tilde{h}(u_1, u_2) = h(u_1, u_2)$, where (u_1, u_2) is a pair of vectors tangent to the orbit $G(s)$, $s \in S$. Thus, in the present case

$$\tilde{h} = dx^2 + dy^2 . \tag{5}$$

The volume function on Σ/G, which is the volume of the orbit $G(s)$, is given by

$$v(x, y) := \mathrm{Vol}\,(G(s)) = \Omega_m x^m \Omega_n y^n , \tag{6}$$

where Ω_n is the volume of unit n-sphere. Now, we are ready to define an effective space where the geodesic γ is to be found. The metric g of the effective space is defined as

$$g = d\ell^2 := v^{2/\lambda}(x, y)\tilde{h} = \Omega_m^2 \Omega_n^2 x^{2m} y^{2n}(dx^2 + dy^2) , \tag{7}$$

where λ is the co-dimension of the G-invariant surface S. For the present example, λ is given by

$$\lambda := \dim S - \dim G(x) = (m + n + 1) - (m + n) = 1 . \tag{8}$$

Hereafter, we omit the unimportant numerical factor of g, $\Omega_m^2 \Omega_n^2$.

The problem to find the minimal surface has been reduced to find the geodesic γ with $g = d\ell^2$. The action to be minimized is

$$\ell = \int x^m y^n \sqrt{dx^2 + dy^2} . \tag{9}$$

If one denotes the geodesic by $y = y(x)$ and varies the action with respect to it, one has

$$xyy'' + (myy' - nx)(1 + y'^2) = 0 . \tag{10}$$

One can easily see that the following cone solves Eq. (10),

$$y = \sqrt{\frac{n}{m}}\, x . \tag{11}$$

As noted before, one cannot know whether the above cone is indeed a minimizer or not until examining the second variation. Here, we introduce an alternative criterion to examine

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whether a geodesic is a minimizer or not. Readers are directed to [7] for an introduction to this topic. The *Jacobi equation* or *equation of geodesic deviation* is written as

$$\frac{d^2\eta}{d\ell^2} + K\eta = 0\,,\tag{12}$$

where η and K are the geodesic deviation and Gauss curvature of metric g. For a general metric of form $g = v(x,y)^{2/\lambda}(dx^2 + dy^2)$, the Gauss curvature is given by $K = (1/2)(\text{Ricci scalar}) = -2\lambda^{-1}v^{-2/\lambda}(\partial_x^2 + \partial_y^2)\ln v$. For the present case, where $v(x,y) = x^m y^n$ and $\lambda = 1$, we have

$$K = \frac{1}{x^{2m}y^{2n}}\left(\frac{m}{x^2} + \frac{n}{y^2}\right)\,,\tag{13}$$

which is positive definite. One can calculate the Gauss curvature and proper distance along the geodesic (11),

$$K = \frac{2m^{n+1}}{n^n x^{2(m+n+1)}}\,,\qquad \ell = \frac{n^{n/2}(m+n)^{1/2}}{m^{(n+1)/2}(m+n+1)}x^{m+n+1}\,.\tag{14}$$

Combining Eqs. (12) and (14), we have

$$\frac{d^2\eta}{d\ell^2} + \frac{c}{\ell^2}\eta = 0\,,\tag{15}$$

where

$$c = \frac{2(m+n)}{(m+n+1)^2}\,.\tag{16}$$

It is well known that the behavior of solution to this equation changes at $c = 1/4$. That is, Eq. (15) has a simple power solution

$$\eta = \ell^{\beta_\pm}\,,\qquad \beta_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 - 4c}\right)\,.\tag{17}$$

Thus, the geodesic deviation oscillates (*i.e.*, there exists a conjugate point of the geodesic) for $2 \le m + n \le 5$, while not for $m + n \ge 6$ (*i.e.*, there exists no conjugate point of the geodesic). These results, and further work by Bombieri *et al.* imply that the cone (11) as $\mathrm{SO}(m+1) \times \mathrm{SO}(n+1)$-invariant hypersurface $S \subset \mathbb{E}^{m+n+2}$ is a minimizer for $m + n + 2 \ge 8$.

2.2 Other Symmetry Groups and Higher Co-dimensions

We have seen the existence of $\mathrm{SO}(m+1) \times \mathrm{SO}(n+1)$-invariant minimal cone above. In Ref. [6] Fomenko gives further examples and a classification scheme for minimal cones in $\mathbb{E}^{p+1}$. The starting points are minimal $(p-1)$-dimensional submanifolds $S = G/H \subset S^p \subset \mathbb{E}^{p+1}$ of the round p-sphere invariant under $G \subset \mathrm{SO}(p+1)$ with a stabilizer or isotropy group $H \subset G$.

There are 12 possibilities (p. 100 in [6]). As before x and y are coordinates on the 2-dimensional quotient space $X = \mathbb{E}^{p+1}/G$. Letting $v(x, y)$ be the volume of the orbits, the effective metric on X is again given by $g = v^2(x, y)(dx^2 + dy^2)$. The minimal cones are given by straight lines in the (x, y)-space.[3] Reference [6] (Theorem 2 on p. 103) states which of the possibilities listed (Table 1 on p. 102) are actually minimizing. The critical dimension of Euclidean space varies from group to group, but is never less than the 8 found by [1].

Reference [8] (pp. 146-147) contains the material similar to Ref. [6]. The classification theorem is the same as [6] but the authors also quote the results of Ivanov on p-dimensional cones in $\mathbb{E}^{p+2}$, *i.e.*, on minimal surfaces of co-dimension two.

Although we will focus on co-dimension one cones, especially the case of $G = \mathrm{SO}(n + 1) \times \mathrm{SO}(m + 1)$, in the rest of paper, it would be interesting to investigate the physical implications of the other types of minimal cones mentioned above.

2.3   The Bernstein Conjecture and Domain Walls

From a physical point of view, a minimal surface is a mathematical idealization of something with finite thickness. A model which incorporates this is a non-linear Laplace equation of the form

$$\Delta\phi = V'(\phi)\,, \quad \Delta = \sum_{i=1}^{p+1} \frac{\partial^2}{\partial x_i^2}\,. \tag{18}$$

If $V(\phi)$ has two critical points at $\phi = \pm 1$, say, at which

$$V'(\pm 1) = V(\pm 1) = 0\,, \tag{19}$$

then a static domain wall is a solution on $\mathbb{E}^{p+1}$, with $\phi \to +1$ as $x_{p+1} \to +\infty$ and $\phi \to -1$ as $x_{p+1} \to -\infty$. If these limits are attained *uniformly* in $(x_1, x_2, \ldots, x_p)$, then it is known that for all p, all solutions of (18) depend only on x_{p+1}.[4]

If we merely require that $\partial\phi/\partial x_{p+1} > 0$, ϕ is bounded, and that

$$V'(\phi) = -\phi(1 - \phi^2)\,, \tag{20}$$

then it is known that for $p < 8$, all solutions of (the stationary version of) this so-called Allen-Cahn equation [9] are planar.[5] However, if $p \geq 8$, then there are non-planar examples [10]. In other words, the behavior of domain walls of finite thickness mirrors that of infinitesimally thin domain walls.

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[3]From the analogous point of view in optics, the function $v(x, y)$ plays the same role as the refractive index in Fermat's principle.

[4]This is known, for reasons that are only partially clear to G.W.G. (one of the present authors), as the *Gibbons Conjecture* and has been proved by a number of people, but not by G.W.G.

[5]This is known as *de Giorgi's conjecture* [11]. He, for good reason, added the *caveat* "at least for $p < 8$".

Physically one expects that a stable minimal surface, such as a *catenoid* in $\mathbb{E}^3$, could be mimicked by solution of the Allen-Cahn equation. In fact, a numerical simulation showed that starting with a configuration for which $\phi = +1$ in the deep interior of a catenoid and $\phi = -1$ outside it, and allowing it to relax to an energy minimizer does lead to a thick catenoidal domain wall [12]. More recently, Gòżdż and Holyst [13] have constructed periodic minimal surfaces from Landau-Ginzburg models.

A powerful general argument is known that interfaces in media of a type should be either planar, spherical, or cylindrical. Serrin [14] gave the proof of this fact for the media introduced by Koretweg [15]. It is of interest to see how it breaks down in the present cases, *i.e.*, the minimal surface and the domain wall described by the Allen-Cahn equation. We discuss it in Appendix A.

3 Branes in Curved Backgrounds

To obtain a spacetime picture of the situation, we pass to the ultra static spacetime $M = \mathbb{R} \times \Sigma$ with spacetime metric

$$ds^2 = -dt^2 + h \,. \tag{21}$$

The p-dimensional minimal surface $S \subset \Sigma$ lifts to a static solution of the equations of motion of a $(p+1)$-dimensional Lorentzian submanifold of M governed by the Dirac-Nambu-Goto action. All such static solutions extremize the energy E and will be stable if the energy is minimized. The energy E in this case is just p-volume $\mathrm{Vol}(S)$ with respect to the pull-back of the metric h on Σ to the minimal hypersurface S. Of course, we could also "Wick rotate"and set $t = -i\tau$ with τ real. We then have a $(p+1)$-dimensional submanifold of M equipped with the Riemannian metric

$$ds_E^2 = d\tau^2 + h \,. \tag{22}$$

This construction becomes less trivial if the spacetime metric is static, but not ultra-static, *i.e.*, of the form

$$ds^2 = -V^2(x)dt^2 + h \,, \tag{23}$$

where x is the coordinates on Σ. In that case the energy E and the p-volume $\mathrm{Vol}(S)$ with respect to the pull-back h_S of the metric h on Σ to the minimal hypersurface S differ. In fact, if σ is a local coordinate system on S

$$E = \int_S V(x(\sigma)) \sqrt{h_S} \, d^p\sigma \,. \tag{24}$$

As before, we can Wick rotate and consider a $(p+1)$-minimal hypersurface of the $(p+2)$-dimensional Riemannian manifold equipped with Riemannian metric $ds_E^2 = V^2 d\tau^2 + h$.

3.1   Cones near Black Holes

A recent related example is that of Frolov [5], who considered a static p-brane in the N-dimensional Schwarzschild-Tangherlini black hole.[6] He showed that this gives a geodesic of the metric

$$g = (r \sin \theta)^{2p-2} \left[dr^2 + r^2 f(r) d\theta^2 \right] , \quad f(r) = 1 - \left(\frac{r_0}{r} \right)^{N-3} , \quad N = p + 2 . \tag{25}$$

Here, r is a Schwarzschild radial coordinate and θ a co-latitude coordinate. He found a geodesic of the metric (25) corresponding to a cone of which apex touches the horizon, and showed that a qualitatively different behavior sets in when the spacetime dimension $N \geq 8$ (or $p \geq 6$). On the face of it, this looks different from the result of Bombieri *et al.* [1]. However, as mentioned above, a static p-brane in an N-dimensional static Lorentzian manifold (with a periodic imaginary time) may be thought of as a $(p+1)$-brane in an N-dimensional Riemannian manifold. Thus, from the Riemannian point of view, the qualitatively different behavior happens when the submanifold has dimension 7 or larger. This agrees with what the analysis of minimal cones in Sec. 2 indicates.

To check the above observation, let us write the N-dimensional Schwarzschild-Tangherlini metric as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\Omega_{N-3}^2) . \tag{26}$$

Then, we focus on a local region near the south pole of horizon by setting

$$r = r_0 + \xi , \quad \theta = \pi - \eta \tag{27}$$

with small ξ/r_0 and η. At the leading order, the metric is written as

$$ds^2 = -\frac{(N-3)\xi}{r_0}dt^2 + \frac{r_0}{(N-3)\xi}d\xi^2 + r_0^2(d\eta^2 + \eta^2 d\Omega_{N-3}^2) . \tag{28}$$

Furthermore, introducing the following local coordinates

$$x = \sqrt{\frac{4r_0\xi}{N-3}} , \quad y = r_0\eta , \tag{29}$$

the near horizon metric reduces to

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2 + dy^2 + y^2 d\Omega_{N-3}^2 , \tag{30}$$

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[6]In [5], he considered a general case with $N \geq p + 2$. Since the value of $N - p$ does not affect our argument, we only consider the hypersurface case, $N = p + 2$. Note also that his argument is independent of the specific form of the background solution if it is spherically symmetric and has a non-degenerate horizon. See also [16] for the original works on a membrane in 4-dimensional black holes, and [17] for an interesting generalization to the case that the Dirac-Nambu-Goto action is corrected by quantum effects.

where $\kappa = (N-3)/2r_0$ is the surface gravity.

Thus, the near-horizon effective 2-dimensional metric in which the geodesic is to be found is

$$g = x^2 y^{2(N-3)}(dx^2 + dy^2).\tag{31}$$

The problem has been reduced to that of $(m, n) = (1, N-3)$ in Sec. 2.1. Note that the factor x^2 in g comes from the time component of metric (30). Thus, the cone $y = \sqrt{N-3}\,x$ is a geodesic near the horizon, and from the analysis of geodesic deviation, this geodesic corresponds to a minimizer if $N = p + 2 \geq 8$.

The work by Frolov was in part motivated by that of Kol [18] in which the "merger transition" from the Kaluza-Klein black holes to a black string was investigated. The black-hole–brane system indeed serves as a toy model of the merger transition and is shown to possess a critical dimension (we will review Kol's observation in Sec. 4). In addition, this system serves as the simplest (as far as we know) example of critical phenomena in gravitational systems [19]. The cone solution separates two phases of the brane: one has a Minkowski topology and another a black hole topology. The change of stability nature of the brane appears at $p = 6$ and results in that of mass scaling of the black hole on the brane. It seems that the self similarity of the critical solution changes from discrete one to continuous one. It would be interesting to clarify why the breakdown of Bernstein conjecture is related to this change of self similarity in detail.

In addition, the black-hole–brane system has many applications to the physics of fundamental interactions via the AdS/CFT correspondence. The holographic dual of the phase transition from the Minkowski embedding to the brane embedding corresponds to the meson melting phase transition of matter in the fundamental representation (see, *e.g.*, Refs. [20]). Although the systems investigated in the literature so far correspond to the black-hole–brane systems below the critical dimension (as far as we know), it would be interesting to see in what the failure of the Bernstein conjecture results in the gauge theory side.

3.2 Truncated Cone in Black Hole: An Exact Solution

The minimal cone in Sec. 3.1 is the example of near-horizon approximate solution, which deviates from a cone away from the horizon. In this subsection, we present an example of exact solution that is minimal and "globally straight".

Any static N-dimensional spacetime metric invariant under SO$(N-1)$ can be written in isotropic, not Schwarzschild, coordinates as

$$ds^2 = -A^2(\hat{r})dt^2 + B^2(\hat{r})\big(d\hat{r}^2 + \hat{r}^2 d\Omega_{N-2}^2\big).\tag{32}$$

The metric in parentheses is the standard metric on $\mathbb{E}^{N-1}$, and SO$(N-1)$ and its subgroups act in the standard way. Thus, we can apply the equivalent variational methods described in [6, 8].

The introduction of the coordinates works as before:

$$ds^2 = -A^2(\hat{r})dt^2 + B^2(\hat{r})\left(dx^2 + x^2 d\Omega_m^2 + dy^2 + y^2 d\Omega_n^2\right), \tag{33}$$

where

$$\hat{r}^2 = x^2 + y^2, \qquad N = m + n + 3, \qquad m, n \geq 1. \tag{34}$$

We assume that a hypersurface configuration is static and invariant under the action of $SO(m+1) \times SO(n+1)$ with the the standard action on $\mathbb{E}^{m+n+2} = \mathbb{E}^{m+1} \times \mathbb{E}^{n+1}$. Then, the hypersurface may be given by $y = y(x)$, and the problem to find the minimal surface is reduced to find the geodesic γ of the 2-dimensional space whose metric is given by

$$g = d\ell^2 = A^2 B^{2(m+n+1)} x^{2m} y^{2n} (dx^2 + dy^2). \tag{35}$$

The geodesic equation is given by

$$-\frac{y''}{(1+y'^2)^{3/2}} + \frac{nx - myy'}{xy\sqrt{1+y'^2}} + \frac{(y - xy')[A'B + (m+n+1)AB']}{\hat{r}AB\sqrt{1+y'^2}} = 0, \tag{36}$$

where the prime denotes the differentiation with respect to each argument (i.e., $y' = \partial_x y$, $A' = \partial_{\hat{r}} A$, and so on). The last term represents the effect of curvature of background geometry. Note that a horizon, where $A = 0$, is a singular point of this equation. One can show the following *truncated cone* solves Eq. (36),

$$y = \sqrt{\frac{n}{m}} x, \qquad x \geq \sqrt{\frac{m}{m+n}} \hat{r}_0. \tag{37}$$

Here, $\hat{r}_0$ is the location of outermost horizon in the isotropic coordinates. This solution is an exact solution unlike the approximate cone in Sec. 3.1.

One can examine whether the cone (37) is a minimizer or not with the Jacobi equation. The cone (37) is parameterized by the isotropic coordinate $\hat{r}$ as

$$x = \sqrt{\frac{m}{m+n}} \hat{r}, \qquad y = \sqrt{\frac{n}{m+n}} \hat{r}. \tag{38}$$

Thus, the relation between $\hat{r}$ and the proper length ℓ along the cone is

$$\frac{d\ell}{d\hat{r}} = \frac{1}{G(\hat{r})} := \frac{m^{m/2} n^{n/2}}{(m+n)^{(m+n)/2}} AB^{m+n+1} \hat{r}^{m+n}. \tag{39}$$

Changing variable from ℓ to $\hat{r}$, the Jacobi equation (12) is written as

$$G^2 \frac{d^2\eta}{d\hat{r}^2} + G \frac{dG}{d\hat{r}} \frac{d\eta}{d\hat{r}} + K\eta = 0, \tag{40}$$

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where K for metric (35) is given by

$$K = \frac{1}{A^2 B^{2(m+n+1)} x^{2m} y^{2n}} \left(\frac{m}{x^2} + \frac{n}{y^2} \right)$$
$$- \frac{1}{\hat{r} A^4 B^{2(m+n+2)} x^{2m} y^{2n}} \left[(AA' - \hat{r} A'^2 + \hat{r} AA'')B^2 + (m+n+1)A^2(BB' - \hat{r} B'^2 + \hat{r} BB'') \right] .$$
(41)

Here, the prime denotes the differentiation with respect to $\hat{r}$.

For simplicity, let us assume $N = (m+n+3)$-dimensional Schwarzschild-Tangherlini black hole as the background, which is given with the standard Schwarzschild coordinate r by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{N-2}^2 , \quad f = 1 - \left(\frac{r_0}{r} \right)^{m+n} .$$
(42)

The coordinate transformation between $\hat{r}$ and r [see Eqs. (32) and (42)] are

$$A = f^{1/2}, \quad B = \frac{r}{\hat{r}}, \quad \frac{dr}{d\hat{r}} = \frac{r f^{1/2}}{\hat{r}} .$$
(43)

After some calculations we have the Gauss curvature in the Schwarzschild coordinates,

$$K = \frac{(m+n)^{m+n+1}}{4 m^m n^n} \cdot \frac{8 r^{2(m+n)} - 10 r^{m+n} + (m+n+2)}{r^{2(m+n+1)} (r^{m+n} - 1)^2} ,$$
(44)

where we have set $r_0 = 1$. One can see that from Eq. (44) the Gauss curvature is positive definite for the present case $m+n \geq 2$ and divergent at the horizon $r = 1$. Thus, one gets the Jacobi equation

$$\frac{d^2\eta}{dr^2} - \frac{m+n}{r} \frac{d\eta}{dr} + \frac{(m+n)[8 r^{2(m+n)} - 10 r^{m+n} + m+n+2]}{4 r^2 (r^{m+n} - 1)^2} \eta = 0 .$$
(45)

Now, we can see that the instability exists at large distance by considering the asymptotic region.[7] For $r \gg 1$, Eq. (45) reduces to

$$\frac{d^2\eta}{dr^2} - \frac{m+n}{r} \frac{d\eta}{dr} + \frac{2(m+n)}{r^2} \eta = 0$$
(48)

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[7] If we put $\eta = r^{(m+n)/2} \tilde{\eta}$, the first derivative term vanishes,

$$\frac{d^2\tilde{\eta}}{dr^2} + \frac{(m+n)r^{m+n}[-(m+n-6)r^{m+n} + 2(m+n-3)]}{4 r^2 (r^{m+n} - 1)^2} \tilde{\eta} = 0 .$$
(46)

Furthermore, if one introduces a new variable $X = r^{m+n} - 1$, Eq. (46) becomes

$$\frac{d^2\tilde{\eta}}{dX^2} + \frac{m+n-1}{(m+n)(X+1)} \frac{d\tilde{\eta}}{dX} + \frac{-(m+n-6)X + m+n}{4(m+n)X^2(X+1)} \tilde{\eta} = 0 .$$
(47)

Analytic solutions to Eqs. (46) and (47) can be obtained explicitly in terms of special functions. However, they are not so informative and therefore omitted to be written down here.

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and is solved by

$$\eta = r^{\beta_\pm}, \quad \beta_\pm = \frac{1}{2}\left(m+n+1 \pm \sqrt{(m+n)^2 - 6(m+n)+1}\right). \tag{49}$$

Thus, the geodesic deviation oscillates for $5 \leq N = m+n+3 \leq 8$, while not for $N \geq 9$ implying the cone is a minimizer.

4 Discussion

In this last section, we review some examples of physical system in the literature whose behavior can be related to the Bernstein conjecture and its breakdown. These examples suggest that the consequences of Bernstein conjecture and its breakdown in general dimensions appear in a variety of systems either in an explicit or non-explicit way. It would be interesting to investigate in detail what to extent the behaviors of these physical systems are related to those seen above.

Cone as Einstein Manifold. We have seen so far that there exist critical dimensions for the minimal cones, which provide some interesting consequences if we suppose the brane theory point of view as mentioned in Introduction, or if we consider the black-hole–brane systems. We will see here, however, that there exists a critical dimension also for a cone that is an Einstein manifold. This suggests that the stability of spacetimes that have the cone as a part of them changes at a certain dimension. As an example, let us see the Kol's observation on a *Ricci flat cone* [18] (see also Sec. VI in [21]). He considered the cone over $S^2 \times S^2$ and its generalizations to $S^m \times S^n$ in the modeling of (Euclidean version of) "merger transition" from caged black holes to a black string. The Einstein equations [*i.e.*, (Ricci tensor)=0] for metric

$$ds_E^2 = dx^2 + e^{2a(x)}d\Omega_m^2 + e^{2b(x)}d\Omega_n^2 \tag{50}$$

are given by

$$a'' + ma'^2 + na'b' - (m-1)e^{-2a} = 0, \quad b'' + nb'^2 + ma'b' - (n-1)e^{-2b} = 0,$$
$$m(m-1)a'^2 + 2mna'b' + n(n-1)b'^2 - m(m-1)e^{-2a} - n(n-1)e^{-2b} = 0. \tag{51}$$

One can check the following Ricci flat cone solves the above equations,

$$\begin{aligned} ds_E^2 &= dx^2 + e^{a_0(x)}d\Omega_m^2 + e^{b_0(x)}d\Omega_n^2 \\ &:= dx^2 + \frac{m-1}{m+n-1}x^2 d\Omega_m^2 + \frac{n-1}{m+n-1}x^2 d\Omega_n^2. \end{aligned} \tag{52}$$

Then, he considered the perturbation around solution (52) by setting $a = a_0 + a_1(x)$, $b = b_0 + b_1(x)$, and linearizing the Einstein equations with respect to a_1 and b_1. The combinations

$a_+ := ma_1 + na_2$ and $a_- := a_1 - a_2$ decouple the perturbation equations. The equation for the gauge invariant combination, a_-, is

$$a_-'' + \frac{m+n}{x}a_-' + \frac{2(m+n-1)}{x^2}a_- = 0 \,. \tag{53}$$

This equation is solved by

$$a_- = x^{\beta_\pm} \,, \qquad \beta_\pm = -\frac{1}{2}\left(m+n-1 \pm \sqrt{(m+n-9)(m+n-1)}\right) \,. \tag{54}$$

This mode corresponds to the shrinking of one sphere and expanding of the other. The discriminant is non-negative if $m+n \geq 9$, which implies the Ricci flat cone is stable for $m+n+1 \geq 10$, (precisely speaking, it is marginally stable for $m + n + 1 = 10$).

The above observation on the cone over $S^m \times S^n$ might be related to a change of stability for the spacetimes containing the Ricci flat cone as their internal space. Indeed, in [22] $\text{AdS}_q \times S^m \times S^n$ was shown to be unstable for $m+n < 9$ due to the violation of Breitenlohner-Freedman mass bound in AdS [23], while it was shown to be stable for $m + n \geq 9$.

Non-Zero-Constant Mean Curvature Surfaces. A minimal surface is characterized by the vanishing of its mean curvature, which is a mathematical model of soap film containing no air in it. When one considers the soap bubble containing air, the configuration has a non-zero-constant mean curvature. Recently, the bifurcation structures of axially symmetric constant mean curvature surfaces in general dimensions were revealed by two of the present authors [24], which was motivated by the existence of critical dimensions in the black-hole–black-string system [25]. The key observation that can be related to the failure of Bernstein conjecture is that there appears a new branch of undulating cylinder appears at 9 space dimension. This branch was shown to be stable in [26] with the so-called surface diffusion equation [27], which is closely related to the (original dynamical version of) Allen-Cahn equation.

The recent gravity/fluid correspondence predicts that a sort of black holes localized in the IR of AdS are dual to the fluid lumps whose surfaces have constant mean curvatures [28]. In such a theory, the minimal surfaces and constant mean curvature surfaces have rather physical meanings via the correspondence. Thus, it would be interesting to investigate their relations to the breakdown of Bernstein conjecture in more detail.

Acknowledgments

A Korteweg's Theory of Phase Equilibrium

Serrin's version [14] of Korteweg's theory [15] starts with the equilibrium condition for the spatial stress tensor

$$\partial_i T_{ij} = 0, \quad (i, j = 1, 2, \ldots, p+1),\tag{55}$$

where T_{ij} is given in terms of a density function $\rho(x)$ by

$$T_{ij} = -P\delta_{ij} + \left(\alpha\nabla^2\rho + \beta|\nabla\rho|^2\right)\delta_{ij} + (\gamma\partial_i\partial_j\rho + \delta\partial_i\rho\partial_j\rho),\tag{56}$$

where $|\nabla\rho| := (\delta^{ij}\partial_i\rho\partial_j\rho)^{1/2}$ and $(P, \alpha, \beta, \gamma, \delta)$ is a set of functions of ρ. Starting from these equations, Serrin derived the over-determined system of equations,

$$\nabla^2\rho = \xi(\rho), \qquad |\nabla\rho| = \zeta(\rho),\tag{57}$$

where ξ and ζ are some functions of ρ given in terms of $(P, \alpha, \beta, \gamma, \delta)$. It was then shown by Pucci [29] that solutions $\rho(x)$ of Eq. (57) must have level sets given either by concentric spheres, cylinders or parallel planes.

Now, let us see the above Serrin's argument in some detail. If the following set of functions are defined

$$a := \alpha + \gamma, \qquad b := \beta + \delta, \qquad c := \gamma' - \delta, \qquad (' := \partial_\rho),\tag{58}$$

Eq. (55) implies that

$$\partial_i\left[-P + a\nabla^2\rho + \left(b + \frac{1}{2}c\right)|\nabla\rho|^2\right] = \left(c\nabla^2\rho + \frac{1}{2}c'|\nabla\rho|^2\right)\partial_i\rho.\tag{59}$$

Now if

$$A := bc + \frac{1}{2}(c^2 - ac') \neq 0,\tag{60}$$

then he claims to be able to establish that Eq. (57) holds for an appropriate choice of $\xi(\rho)$ and $\zeta(\rho)$. To this end, he defines

$$F := -P + a\nabla^2\rho + \left(b + \frac{1}{2}c\right)|\nabla\rho|^2, \qquad G := c\nabla^2\rho + \frac{1}{2}c'|\nabla\rho|^2,\tag{61}$$

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so that $\partial_i F = G \partial_i \rho$. Thus, there exists a real valued function $\omega(\rho)$ such that $F = \omega(\rho)$ and $G = \omega'(\rho)$, and hence

$$\nabla^2 \rho = \xi(\rho) := \frac{1}{A} \left[\left(b + \frac{1}{2} c \right) \omega' - c'(\omega + P) \right] ,$$

$$|\nabla \rho|^2 = \zeta^2(\rho) := \frac{1}{A} \left[c(\omega + P) - a\omega' \right] . \tag{62}$$

At first glance, the above Serrin's model is rather general, and the Allen-Cahn domain walls and the minimal surfaces seem to be covered by the argument above. We will see below, however, the domain wall and minimal surface are exceptional cases for which his argument breaks down.

Firstly, let us consider a single scalar field ϕ whose spatial stress tensor is given by

$$T_{ij}^{(\phi)} = \partial_i \phi \partial_j \phi - \frac{1}{2} \delta_{ij} \left[(\partial_k \phi)^2 + 2V(\phi) \right] . \tag{63}$$

Its divergence is given by

$$\partial_i T_{ij}^{(\phi)} = \partial_j \left[\Delta \phi - V'(\phi) \right] . \tag{64}$$

Thus, we just get Eq. (18). In Serrin's notation, taking $\rho = 1$, we have $P = -V$, $(\alpha, \beta, \gamma, \delta) = (0, -1/2, 0, 1)$ whence $(a, b, c) = (0, 1/2, -1)$. Thus, we have

$$F = V , \qquad G = \Delta \phi , \qquad A = 0 , \tag{65}$$

which is the case excluded by Serrin.

Secondly, let us consider the minimal surfaces. The non-parametric form of the minimal surface equation [denote the minimal surface by $\varphi(x) = 0$] can be derived by extremizing the energy functional

$$E[\varphi] = \int \mathcal{E}(\varphi, \partial_i \varphi) d^{p+1} x = \int \left(\sqrt{1 + |\nabla \varphi|^2} - 1 \right) d^{p+1} x . \tag{66}$$

Thus, the stress tensor is given by

$$T_{ij}^{(\varphi)} = \frac{\partial_i \varphi \partial_j \varphi}{\sqrt{1 + |\nabla \varphi|^2}} - \delta_{ij} \left(\sqrt{1 + |\nabla \varphi|^2} - 1 \right) . \tag{67}$$

This is not of the form introduced by Korteweg, and Serrin's argument does not work for the minimal surfaces. Moreover, one has

$$\partial_i T_{ij}^{(\varphi)} = \partial_j \varphi \, \partial_i \left(\frac{\partial_i \varphi}{\sqrt{1 + |\nabla \varphi|^2}} \right) \equiv 0 \tag{68}$$

by virtue of the equation of motion. This will always be true for the systems obtained by varying an energy functional, $\int \mathcal{E}(\varphi, \partial_i \varphi) d^{p+1} x$.

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