HyperKähler Manifolds & Multiply-Intersecting Branes

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This is a summary of the work in the author's recent paper with this title written with Jerome Gauntlett, George Papadopoulos and Paul Townsend, hep-th/97022012. We showed how to construct hyper-Kähler 8-metrics in terms of arrangements of three-dimensional hyperplanes in six-dimensional euclidean space. The slopes of the planes define two relatively prime integers (p, q). Under reduction to ten dimensions and T-duality we get a geometric picture of the action of $SL(2, \mathbb{Z})$ on the $(NS \otimes NS, qR \otimes R)$ 5-branes of type IIB string theory. Configurations are exhibited with $\frac{3}{16}$'th SUSY.

1. INTRODUCTION

The simplifications afforded by going to eleven dimensions are by now widely appreciated. With this in mind we construct non-singular solutions of the equations of motion of 11-dimensional supergravity taking the form

$$ds_{11}^2 = H^{-\frac{2}{3}} \eta_{\mu\nu} dx^\mu dx^\nu + H ds_8^2, \tag{1}$$

$$F_4 = \pm \frac{1}{6} \epsilon_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\left(\frac{1}{H}\right), \tag{2}$$

$$\nabla_8^2 H = 0. \tag{3}$$

We could take the Riemannian 8-metric to be merely Ricci flat and we would then obtain a solution but since we are interested in BPS solutions we shall, as indicated by our title, choose the metric to be HyperKähler which will (for an appropriate choice of sign in (2) imply that the solution admits Killing spinors.

1.1. Hyperkähler metrics

Recall that these are $4k$ dimensional Riemannian metrics X_{4k} which are

(i) **Hypercomplex**, i.e. it admit three integrable complex structures I, J, K satisfying the quaternion algebra $I^2 = -1$, $IJ = K$ etc

(ii) **Hermitean**, i.e. the complex structures act as isometries of the metric: $g(IX, IY) = g(X, Y)$

etc. equivalently $\Omega_I(X, Y) := g(IX, Y) = -\Omega_(Y, X)$.

(iii) **Kähler**, i.e. the three two forms Ω_I are closed, $d\Omega_I = 0$.

It follows that the three 2-forms and the 3 complex structures are covariantly constant with respect to the Levi-Civita connection ∇_g of the metric g and that the holonomy group $Hol(\nabla_g) \subseteq Sp(k) \subset SO(4k)$. It also follows that there is *at least* a $k+1$ dimensional family of covariantly constant spinors. The number of constant spinors may exceed $k + 1$ if the holonomy group is a proper subgroup of $Sp(k)$. To count the constant spinors it suffices to count the number of singlets in the decomposition of the spinor representation of $SO(4k)$ with respect to the subgroup Hol.

In fact our particular applications we used 8-dimensional *toric* hyperKähler manifolds. Then if the harmonic function $H = 1$ the holonomy trivially allows one to read off the number of Killing spinors. Recalling that $Sp(2) \equiv Spin(5)$ and $Sp(1) \equiv Spin(3) \equiv SU(2)$ we have the following possibilities for the holonomies and number of covariantly constant spinors:

Holonomy group	# Spinors	Example
id	16	$\mathbb{E}^8$
$Sp(1)$	8	$\mathbb{E}^4\times$ Taub-NUT
$Sp(1) \times Sp(1)$	4	Taub-NUT $\times$ Taub-NUT
$Sp(2)$	3	Lee-Weinberg-Yi

Our general metric admits no Killing spinors and induces the following spinor decomposition

$$SO(10,1) \quad \supset \quad SO(2,1) \times SO(8) :$$
$$\mathbf{32} \quad \rightarrow \quad (\mathbf{2}, \mathbf{8}_s) \oplus (\mathbf{2}, \mathbf{8}_c). \tag{4}$$

We may get 3, 4, or 8 $Spin(2,1)$ doublets respectively according to the following symmetry breaking patterns:

$$SO(8) \subset Sp(2): \quad \mathbf{8}s \quad \rightarrow \quad \mathbf{5} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$$
$$\mathbf{8}c \quad \rightarrow \quad \mathbf{4} \oplus \mathbf{4} \tag{5}$$

$$Sp(2) \subset Sp(1) \times Sp(1): \quad \mathbf{5} \quad \rightarrow \quad (\mathbf{2},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1})$$
$$\mathbf{4} \quad \rightarrow \quad (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \tag{6}$$

$$Sp(1) \times Sp(1) \subset Sp(1): \quad \mathbf{8}s \quad \rightarrow \quad \mathbf{5} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$$
$$\mathbf{8}c \quad \rightarrow \quad \mathbf{4} \oplus \mathbf{4}$$
$$(\mathbf{2},\mathbf{1}) \quad \rightarrow \quad (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1})$$
$$(\mathbf{1},\mathbf{2}) \quad \rightarrow \quad (\mathbf{1},\mathbf{2})$$
$$(\mathbf{2},\mathbf{2}) \quad \rightarrow \quad (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}). \tag{7}$$

Thus the fraction of the maximum allowed supersymmetry is given by

Holonomy group	Fraction of Maximum SUSY
id	1
$Sp(1)$	$\frac{1}{2}$
$Sp(1) \times Sp(1)$	$\frac{1}{4}$
$Sp(2)$	$\frac{3}{16}$

The principle novelty here is the apparently new example of a situation with $\frac{3}{16}$ SUSY.

2. TORIC HYPERKÄHLER MANIFOLDS

This general class of metrics has found a number of physical applications . By definition they admit a triholomorphic action of the k-dimensional torus group $T^k \equiv U(1)^k$. It turns out that the metric may be written in coordinates adapted to the torus action as

$$ds^2_{4k} = U_{ij}d\mathbf{x}^i d\mathbf{x}^j + (U^{-1})^{ij}(d\phi_i + A_i)(d\phi_j + A_j). \tag{8}$$

and the Kähler forms by

$$\Omega_I = (d\phi_i + A_i) \wedge dx_1^i - U_{ij}dx_2^i \wedge dx_3^j \quad \text{etc.} \tag{9}$$

The k Killing fields generating the torus action are $\frac{\partial}{\partial \phi_i}$ and clearly for each Kähler form

$$\mathcal{L}_{\frac{\partial}{\partial \phi_i}} \Omega_I = 0. \tag{10}$$

A useful fact is that the Cartesian coordinates $\mathbf{x}^i$ are moment maps (or Hamiltonians) for this torus action. In addition to the torus action the isometry group contains an additional $SO(3)$ action.

It is extremely convenient to introduce a privileged ortho-normal frame $E = (\mathbf{E}_i, E^i)$ by diagonalizig the matrix U:

$$U_{ij} = (K^t K)_{ij} \tag{11}$$

so that

$$(\mathbf{E}_i, E^j) = (K_{ij}d\mathbf{x}^j, (d\phi_j + A_j)K^{ji}). \tag{12}$$

Thus, in a hopefully obvious notation,:

$$\Omega = E^i \wedge \mathbf{E}_i - \mathbf{E} \times \mathbf{E}. \tag{13}$$

The privileged frame is invariant under the torus action

$$\mathcal{L}_{\frac{\partial}{\partial \phi_i}} E = 0. \tag{14}$$

Moreover in this frame the $so(4k)$ Lie algebra valued connection one-forms $\Theta \in sp(k)$. The spinor covariant ∇ acts on the components ψ of spinors in the adapted spin frame as

$$\nabla\psi = d\psi + \frac{1}{2}\Theta^{\mu\nu}\Gamma_{\mu\nu}\psi \tag{15}$$

Thus, by projecting into the singlet summand under the decomposition of the spinor represenation

of $so(4k)$ with respect to its $sp(k)$ subalgebra we obtain the previously advertized covariantly constant spinor fields. These $k+1$ Killing spinors are clearly invariant under the torus action:

$$\mathcal{L}_{\frac{\partial}{\partial\phi_i}}\psi = 0. \tag{16}$$

The significance of this remark will be apparent later.

2.1. Lindström-Roček-Pedersen-Poon equations

To specify a toric hyperKähler metric it suffices to give the matrix U_{ij}. The components of the connection $A_i = \omega_{ik}^r r dx_r^k$ then follow up to gauge equivalence from the eponymous equations of this subsection

$$\partial_j^r \omega_{ki}{}^s - \partial_k^s \omega_{ji}^r = \epsilon^{rst}\partial_j^t U_{ki}. \tag{17}$$

The integrability conditions: for which are

$$\partial_{[j}^t U_{k]i} = 0 \tag{18}$$

and

$$\partial_i \cdot \partial_j U_{rs} = 0. \tag{19}$$

It is a remarkable fact the the *non-linear* Einstein equations reduce to a set of *linear* equations which essentially reduce to the requirement the components of the matrix U are harmonic on euclidean 3-planes.

2.2. Simple examples

Let us now turn to the case of present interest $k = 2$. We will always choose the two angles $\phi \in (0, 2\pi]$. The simplest examples are well known but a few points are worth mentioning. Consider for example the vacuum or ground state. This is not altogether trivial and already we see the modular group $SL(2, \mathbb{Z})$ entering in a natural way as a gauge symmetry. To specify a flat solution we must give a constant metric, call it U_{ij}^∞, on a 2-torus or equivalently we must specify a 2-dimensional lattice. The basis vectors of the lattice make an angle θ given by

$$\cos\theta = -\frac{U_{12}^\infty}{\sqrt{U_{11}^\infty U_{22}^\infty}} \tag{20}$$

Taking into account the freedom to change basis we have that the flat metrics correpond to elements of the double coset

$$SL(2, \mathbb{Z})\backslash GL(2, \mathbb{R})/SO(2). \tag{21}$$

The next simplest examples are $\mathbb{E}^4 \times$ Multi-Taub-NUT, which represent parallel 6-branes in type of IIA supergravity in ten dimensions. The second factor looks like

$$ds^2 = H^{-1}(d\phi + \omega \cdot d\mathbf{x})^2 + H d\mathbf{x}^2 \tag{22}$$

we have

$$\nabla \times \omega = \nabla H \tag{23}$$

and we choose

$$H = 1 + \sum_{\text{points}} \frac{1}{2}\frac{1}{|\mathbf{x} - \mathbf{a}|} \tag{24}$$

As is well known the coordinate singularities at $\mathbf{x} = \mathbf{a}$ correpond geometrically to fixed points of the $T^1 \equiv S^1$ action generated by $\frac{\partial}{\partial\phi}$. Absence of singularities forces the points to be distinct and also fixes the coefficient of $\mathbf{x}$ to be the same for all points. To get down to ten dimensions we quotient by this action and regard the points $\mathbf{a}$ as giving the locations of the 6-branes in the transverse three directions. It is essential for the physical interpretation that we preserve supersummetry and that is why we emphasised the fact that the Killing spinors were invariant under the torus action. Such Killing spinors remain Killing spinors of the reduced theory.

2.3. Arrangements of Hyperplanes

The class of toric hyper-Kähler metrics is quite large but it turns out the have a simple geometrical description in terms of arrangements of hyperplanes. Restricting ourself to the 8-dimensional case

$$U_{ij} = U_{ij}^\infty + \sum_{\text{hyperplanes}} \frac{p_i p_j}{|p\mathbf{x}^1 + q\mathbf{x}^2 - \mathbf{a}|}, \tag{25}$$

with $p_1 = p$ and $p_2 = q$. The fixed point coordinate singularities are are now located on the three dimensional hyperplanes in $\mathbb{E}^6$ given by

$$p\mathbf{x}^1 + q\mathbf{x}^2 = a. \tag{26}$$

Using the hyper-kähler quotient technique, which will not be explained here, one discovers that the hyperplanes will be fixed points of a smoothly acting S^1 subroup of the T^2 isometry group if the quantities specifying the slopes (p, q), which also specify the subgroup, are relatively prime integers. In addition all planes must be distinct and triple intersections are excluded.

Geometrically it is clear that the basis chosen allowing us to regard $\mathbb{E}^6$ as $\mathbb{E}^3 \oplus \mathbb{E}^3 = (\mathbf{x}^1, \mathbf{x}^2)$ is arbitrary up to the action of the modular group. In our basis $(1, 0) \to$ the $\mathbf{x}^1$ axis and $(2, 1) \to$ the $\mathbf{x}^2$ axis, the two be tilted at the angle θ. If $\theta = \frac{\pi}{2}$ the two basis-planes are at right angles and restrcting (p, q) to these values gives the Multi-Taub-NUT $\times$ Multi-Taub-NUT metrics with holonomy $Sp(1) \times Sp(1)$. with $\frac{1}{4}$ SUSY. Keeping the same restriction on (p, q) but tilting the basis planes breaks this down to $Sp(1)$ with $\frac{3}{16}$'th SUSY. If we have just two planes we recover the Lee-Weinberg-Yi metric which arises as the relative moduli space of three BPS monopoles in $N = 4$ SUSY SU(4)-Yang-Mills maximally broken to $U(1) \times U(1) \times U(1)$ by a Higgs in the adjoint representaion.

3. THE TYPE IIB VIEWPOINT

Obviously we can pick one of the circle subrgroups of T^2 and reduce to ten dimensions to get a solution of the Type IIA theory. This solution admits a circle action and so we T-dualize to get solution of the Type IIB theory. Otherwise said, we may descend to nine dimensions where there is no distinction between A and B and and come back up to the IIA theory. The Type IIB solutions will not necessarilly be non-singular even though we started in eleven dimensions with non-singular solutions. The ten-dimensional solution will have a Killing vector which we call $\frac{\partial}{\partial z}$.

The resulting metric is, in Einstein conformal frame,

$$
\begin{aligned}
ds_{10}^2 &= (\det U)^{\frac{3}{4}} \Big[(\det U)^{-1} \eta_{\mu\nu} dx^\mu dx^\nu \\
&+ (\det U)^{-1} U_{ij} d\mathbf{x}^i \cdot d\mathbf{x}^j + dz^2 \Big],
\end{aligned}
\tag{27}
$$

In addition

$$
B_i = A_i \wedge dz
\tag{28}
$$

and

$$
\tau = l + i e^{-\phi} = -\frac{U_{12}}{U_{11}} + i \frac{\sqrt{\det U}}{U_{11}},
\tag{29}
$$

where l and ϕ are respectively the axion and dilaton and B_1 is the $NS \otimes NS$ and B_2 the $R \otimes R$ 2-form potential. The action of $S \in SL(2, \mathbb{Z})$ associated with the torus under which

$$
U \to (S^{-1})^t U S^{-1}
\tag{30}
$$

and under which B_i transform as a doublet induces a fractional linear transformation on τ via the relation

$$
\frac{U}{\sqrt{\det U}} = \frac{1}{\mathrm{Im}\tau} \begin{pmatrix} 1 & -\mathrm{Re}\tau \\ -\mathrm{Re}\tau & |\tau|^2 \end{pmatrix}.
\tag{31}
$$

Of course, as far as the classical Type IIB theory is concerned, we have an action of $SL(2, \mathbb{R})$ on the classical solutions but if we lift this to eleven dimensions it will in general take non-singular solutions to singular solutions. In the Type IIB theory, in which the classical solutions are in general singular, one usually restricts this $SL(2, \mathbb{R})$ action to $SL(2, \mathbb{Z})$ by appealing to quantum effects. The striking thing about our work is that in eleven dimensions is that this restriction arises from a demanding the classical solutions be regular.

3.1. Examples

The discussion of the previous section may be fleshed out by looking at the simplest example. We set $\theta = \frac{\pi}{2}$ and thus

$$
U = \begin{pmatrix} H_i(\mathbf{x}^1) & 0 \\ 0 & H_2(\mathbf{x}^2) \end{pmatrix}
\tag{32}
$$

where H_1 and H_2 are harmonic functions on $\mathbb{E}^3$. Since

$$
\begin{aligned}
ds_{10}^2 &= (H_1 H_2)^{\frac{3}{4}} \Big[\frac{1}{H_1 H_2}(-dt^2 + \eta_{\mu\nu} dx^\mu dx^\nu \\
&+ \frac{1}{H_2} d\mathbf{x}^1 \cdot d\mathbf{x}^1 + \frac{1}{H_1} d\mathbf{x}^2 \cdot d\mathbf{x}^2 + dz^2 \Big].
\end{aligned}
\tag{33}
$$

This is readily recognized as the metric commonly referred to as the orthogonal intersection of a collection of parallel $NS \otimes NS$ and $R \otimes R$ 5-branes on

a set of two-branes. The $NS \otimes NS$ brane coordinates are $(x^\mu, \mathbf{x}^1)$, $R \otimes R$ coordinates are $(x^\mu, \mathbf{x}^2)$ and the 2-brane coordinates are x^μ where now, since we are in 10 dimensions, the latin indices run from 0 to 1. Since the metric is independent of the mutually transverse coordinate z the 5-branes are delocalized or 'stacked 'in that direction.

If we may now pass to the general case when the planes are tilted we see that it is reasonable to interpret the general hyperplane $p\mathbf{x}^1 + q\mathbf{x}^2 = \mathbf{a}$ as a Type IIB 5-brane with charge (p, q) with $\frac{3}{16}$ 'th SUSY.

3.2. Non-Orthogonal D-branes

By dualizing the Type IIA solutions in a different direction one may obtain a solution contining only D-five branes. If $X^i = (\mathbf{x}^i, \phi_i)$ one has, again in Einstein conformal gauge

$$
\begin{aligned}
ds_{10}^2 &= (\det U)^{\frac{1}{4}} \Big[-dt^2 + (dx^1)^2 \\
&+ U_{ij} dX^i dX^j \Big] \\
B_2 &= A_i d\phi_i \\
\tau &= i\sqrt{\det U}.
\end{aligned}
\tag{34}
$$

The two branes intersect on a string extended along the x^1 direction. thus if $\theta = 0$ one 5- brane occupies the 12345 directions and the other the 16789 directions. If $\theta \neq 0$ the 2345 directions and 6789 directions are rotated with respect to each other at an angle by an $SO(8)$ element O which is block diagonal in the $2-6$, $3-7$, $4-8$ and $5-9$ 2-planes, each block looking like

$$
\begin{pmatrix} \cos & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.
\tag{35}
$$

Using quaternion notation with

$$
\begin{aligned}
\mathbb{E}^* = (X^1, X^2) \equiv \mathbb{H}^2 = \\
(x^2 + ix^3 + jx^4 + kx^5, x^6 + ix^7 + jx^8 + kx^9),
\end{aligned}
\tag{36}
$$

we see that one may think of O as lying in $Sp(2) \subset SO(8)$.

Now a D-brane occupying the $1, 2, 3, 4, 5$ directions is invariant under supersymmetries generated by the Type IIB chiral spinors ϵ^i satisfying

$$
\Gamma_{012345}\epsilon^1 = \epsilon^2.
\tag{37}
$$

If the other D-branes were orthogonal it would be invariant under supersymmetries satisfying

$$
\Gamma_{016789}\epsilon^1 = \epsilon^2.
\tag{38}
$$

The set of common solutions would be 4 dimensional. However if they are at an angle one has instead

$$
R^{-1}\Gamma_{016789}R\epsilon^1 = \epsilon^2,
\tag{39}
$$

where $R(\theta)$ is the lift to $Spin(8)$ of the $SO(8)$ rotation O. Explicitly

$$
R(\theta) = \exp\{-\frac{1}{2}\theta(\Gamma_{26} + \Gamma_{37} + \Gamma_+\Gamma_{48} + \Gamma_{59})\}.
\tag{40}
$$

After some Clifford algebra one finds that if $\theta \neq 0$ there are just 3 mutual solutions as expected.

3.3. Hanany-Witten type solutions

Using different reductions and different T-duality maps one may obtain a great variety of other solutions with $\frac{3}{16}$ 'th SUSY. These include solutions with just $NS \otimes NS$ or just $R \otimes R$ 5-branes intersecting at an angle. As we have just have checked in the $R \otimes R$ case, the amount of SUSY is consistent with the stringy analysis of Dirichlet-5-branes. Perhaps more interesting are the solutions which, if they were not delocalized in the z direction, would correspond to the configurations used by Hanany and Witten in their analysis of 2+1 dimensional gauge theories on the intersetion of $NS \otimes NS$ and $R \otimes R$ 5-branes. In order to localize the 2-branes we need to solve the equation 3 for the harmonic function which gives the 4-Form 2 appearing in the general metric (1). In the toric case when we also assume that H is T^2 invariant, this becomes

$$
U^{ij}\partial_i \cdot \partial_j H = 0.
\tag{41}
$$

Rather remarkably this equation is additively separable: it admits solutions of the form

$$
H = H_1(\mathbf{x}^1) + H_2(\mathbf{x}^2).
\tag{42}
$$

4. HOLOMORPHIC CYCLES

As an illustration of the utility of the eleven dimensional viwe point it is worth pointing out that the holomorphic geometry of toric hyperKähler

manifolds makes it almots trivial to construct an interesting class of 'test probes 'in these geometries. Consider one of the 2-sphere's worth of complex structures specified by a unit 3-vector $\mathbf{n}$

$$I_{\mathbf{n}} = n_i I + n_2 J + n_3 K. \qquad (43)$$

It determines a direction in each of the k $\mathbb{E}^3$ factors in the quotient X_{4k}/T^k manfold. Now consider a k-plane $\Pi \subset \mathbb{E}^{3k}$ containing these directions. Using the torus action it may be lifted up to X_{4k} to give a $2k$-dimensional submanifold which one may easily y check is holomorphic with respect to the complex structure $J\mathbf{n}$. By picking the k-plane Π appropriately one may obtain in this way a variety of different types of holomorphic cycles with different topologies. By Wirtinger's theorem they are all minimal. One may also check that they remain minimal when the extra harmonic functions (42) are included.

5. CONCLUSION & PROSPECTS

Perhaps the most striking thing about our analysis is how simpy one may construct extremely elaborate non-singular intersecting brane solutions with modest amounts of supersymmetry by considering arrangemenst of rational hyperplanes in six dimensions and how these give a purely classical geometrical insight into the what in Type IIB theory is thought of as the quantum mechanical breaking of the classical $SL(2, \mathbb{R})$ down to $SL(2, \mathbb{Z})$. It was gratifying to see that our analysis of the amount of supersymmetry is consistent with that given string theory using by D-brane techniques.

Given our construction of Hanany-Witten type solutions it would be interesting to investigate what $N = 3$, 2+1 dimensional gauge theories may arise on the intersections.

In the talk I also mentioned the fact that that it would be interesting to explore further an aspect of this work which plays an essential role in the calculations: the fact that T-duality has, in some sense, the effect of interchanging hyperKähler (HK) geometry with what is sometimes called HKT geometry. This has now been completed [2].

In the bibliography below I have restricted myself to the paper [1] of which this talk is a summary and the newer paper [2] mentioned above. A complete list of references to the original literature related to the material discussed above may be found there.

6. ACKNOWLEDGEMENT

REFERENCES

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