The Physics of 2-d Stringy Spacetimes

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We examine the two-dimensional spacetimes that emerge from string theory. We find all of the solutions with no tachyons, and show that the only non-trivial solution is the black hole spacetime. We examine the role of duality in this picture. We then explore the thermodynamics of these solutions which is complicated by the fact that only in two spacetime dimensions is it impossible to redefine the dilaton field in terms of a canonical scalar field. Finally, we extend our analysis to the heterotic string, and briefly comment on exact, as opposed to perturbative, solutions.

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1. Introduction

The two dimensional black hole that emerges from the SU(1,1)/U(1) "coset" model [1] is an interesting testing ground for many ideas about the physics of black holes. In general relativity, black holes behave as perfect absorbers. If one approximates quantum gravity by treating quantum fields as propagating on a classical background geometry, the semi-classical approximation, one then finds that the black hole behaves as a perfect black body with a temperature T_H , the Hawking temperature, [2]. For an uncharged, non-rotating black hole in four dimensional spacetime, whose geometry is completely determined by the mass of the hole M

$$T_H = \frac{1}{8\pi M} \tag{1.1}$$

Hawking's result has profound consequences for any quantum theory of gravity. Suppose that it is true for all M, then a black hole is unstable as its specific heat is negative. Thus the black hole will radiate, lose mass and consequently increase its temperature, resulting in the catastrophic disappearance of the black hole. It would then appear that there is no unitary time evolution operator for quantum gravity, [3]. The incoming state which gives rise to the black hole can be chosen to be a pure state with zero entropy. The outgoing state is a mixed thermal state, with large entropy. Since quantum mechanical evolution preserves entropy, it is often argued that quantum gravity must somehow transcend the usual laws of quantum mechanics.

Various suggestions have been made for side-stepping this problem. One is to suppose that (1.1) only holds for large M, and in fact a small black hole will persist as a repository for all the quantum mechanical information that would have been lost had the black hole completely disappeared. At first sight, it seems to be unlikely as small black holes must then contain arbitrarily large amounts of information. The initial black hole could have had a very large mass, being formed from a large quantity of matter and hence have a very large entropy. One can be a little more precise in seeing what is wrong with this suggestion. In order for (1.1) to be consistent with the first law of thermodynamics, the entropy S [2], [4], [5] of the hole must be given by

$$S = 4\pi M^2. \tag{1.2}$$

If we use the Boltzmann interpretation of entropy, then the density of states function is the exponential of S, and thus small mass black holes have a small density of states and hence can contain only a little information.

Another suggestion is that the reason one finds non-unitary time evolution is that our description of the initial state has had all the quantum mechanical information removed from it by virtue of thinking about it as a classical field configuration. It would seem to be inevitable that such problems appear in the semi-classical approximation. It might then appear that the radiation is approximately thermal in the same way that radiation from the sun is thermal. Such radiation is described by a mixed state because we cannot keep track of all the quantum mechanical information involved in a very complicated system. In order to see whether this is the case, we would need a quantum theory of gravitation that makes sense at a fundamental level. String theory offers one such candidate, although only at the level of perturbation theory. However, we have so far ignored problems associated with the inherent indescribability of the quantum state of matter inside the black hole. For example, a device recording information about what is falling into the hole cannot communicate its observations to any observer exterior to the hole. Perhaps a resolution of our difficulties is that a global construct like the event horizon is inherently an un-quantum mechanical concept.

Yet another possible resolution is to blame the loss of quantum coherence on the spacetime singularity that is inevitably found interior to the black hole, at least in classical general relativity. Experience with other areas of physics indicates that singularities are fictions of classical physics, and quantum mechanically the problems associated with them will disappear. There is a concrete suggestion that this is indeed the case for two-dimensional black holes, [6]. An understanding of what becomes of singularities in a quantum theory of gravity will then give us the key to the resolution of our difficulties.

A quite different way round our difficulties is to adopt a suggestion of Dyson [7], namely that the black hole precipitates the formation of a baby universe that contains the quantum mechanical information associated with the gravitational collapse. The "bag of gold" type of construction would replace the singularity in the interior of the black hole and allow the Hawking process to proceed producing entropy exterior to the black hole. Dyson's picture allows for unitary time evolution in quantum gravity as the hidden information would be preserved in the baby universe. The price paid is that the time reverse of such a process is quite possible and would be interpreted as a white hole. Such things do not appear to occur in our universe.

Above, we have sketched some of the possible ideas about one of the central issues in quantum gravity today, however our list is clearly far from describing all the possibilities. Whatever the resolution to these problems, it is clear that we do not have a complete

consistent quantum theory of gravitation which is able to make reliable predictions. However, it seems that elementary thermodynamic considerations can shed some light on the fundamental difficulties of quantum gravity.

String theory is one approach to quantum gravity. In its present form, strings move in certain fixed classical backgrounds. These backgrounds are determined by asking that the conformal invariance of the string is not broken by the background in which it moves. to requiring Such an approach has much in common with the semi-classical analysis because the classical spacetime background and the fields propagating on it are regarded as string condensates, which are inherently classical in the only available formulation of string theory, namely a first quantized string theory.

In some of the scenarios we described, or minor variants of them, one might expect black hole solitons to emerge as field configurations which have zero Hawking temperature, and a small intrinsic entropy. In general relativity, extreme Reissner-Nordstrom black holes are examples of such configurations [8]. In fact, it might be rather satisfying to find solitons with zero entropy as these would correspond to pure quantum mechanical states and hence be some kind of fundamental excitation associated with the gravitational field.

In the remainder of this paper, we discuss stringy black holes in an attmept to use elementary methods to shed light on these perplexing issues. In section two, we describe the black hole spacetimes that emerge from the theory of closed bosonic strings. The system of equations that emerge describing the low-energy stringy backgrounds turns out to be exactly integrable in the absence of any tachyon fields. There are seven local solutions to the equations, the metrics of which turn out to be either flat spacetime, different coordinate patches on the black hole spacetime or derivable from them. In section three, we explore the thermodynamic interpretation of the spacetimes found in section three. In section four, we extend our discussion to the case of the stringy backgrounds that emerge from heterotic string theory. Rather disaapointingly, we discover that there are no stable fundamental objects to be found in this theory, despite the superficial similarities to four dimensional Einstein-Maxwell theory. Finally in section five, we very briefly outline the exact string backgrounds which arise from the coset space construction. The paper ends with some conclusions and speculations on the role of black holes in string theory and the consequences for a theory of quantum gravity.

2. Two dimensional stringy backgrounds

We assume that the closed bosonic string is moving in a two-dimensional background spacetime. If we look at the physical state conditions, we find that the only conventional physical mode is the spacetime "tachyon" T, which because of the two-dimensionality is in fact massless. However there are other string states, the so-called topological states [9], which come from the higher modes of the string whose longitudinal components survive as background fields, even though they vanish completely as dynamical degrees of freedom. These modes are represented the background gravitational field specified by the metric g_{ab} , and the dilaton field Φ .

The string β -functions then, to lowest order [10], are

$$R_{ab} = \nabla_a \ \nabla_b \Phi + \nabla_a T \nabla_b T \tag{2.1}$$

$$R - (\nabla \Phi)^2 - 2\triangle \Phi - (\nabla T)^2 - V(T) = c^*,$$
 (2.2)

$$\Delta T + \nabla_a \Phi \ \nabla^a T = \frac{1}{2} V'(T), \tag{2.3}$$

where, V(T) is the "tachyon" potential and c^* is related to effective central charge of the world sheet fields. It should be emphasized that to higher orders in α' , there will be higher order terms appearing which we will ignore in what follows. The bosonic string can be coupled to any conformal field theory with central charge c, in which case the effective central charge is

$$c^* = \frac{c + d - 26}{3} \tag{2.4}$$

for a closed bosonic string in d-dimensions. The factor of 26 comes from the reparametrization ghosts, and d comes from the d string coordinates which behave like free bosons, at least in flat spacetime. It should be noted that for the minimal bosonic string theory, not coupled to any extra worldsheet fields, that $c^* < 0$. The tachyon potential V(T) takes the form

$$V(T) = V_0 - 2T^2 + V_3 T^3 + O(T^4)$$
(2.5)

All of these equations can be derived from a target space action

$$\int_{\mathcal{M}} d^2x g^{\frac{1}{2}} e^{\phi} \left(c^* - R - (\partial \phi)^2 + (\partial T)^2 + V(T) \right) - 2 \int_{\partial \mathcal{M}} e^{\phi} K d\Sigma \tag{2.6}$$

Extremizing this action with respect to arbitrary variations of the fields within the region of spacetime \mathcal{M} , but with the value of the dilaton and metric induced on the boundary,

 $\partial \mathcal{M}$, fixed will reproduce the β -function equations. In the boundary term, K is the trace of the second fundamental form of the boundary. Omission of this boundary term will fail to yield a well-posed variational principle that involves only the fields and their first derivatives, [11], [12].

We may as well set $V_0 = 0$ since in string perturbation theory it vanishes. If it does not vanish outside perturbation theory, it can be absorbed into a the definition of c^* . V_3 has been calculated by Samuel [13] but we ignore it as it is higher order in α' .

We now consider the solutions of the equations (2.1), (2.2) and (2.3) in two spacetime dimensions. We treat the case where the tachyon condensate is put to zero, T = 0. (2.1) then implies the existence of an isometry. In d = 2, the Ricci tensor obeys the Gauss identity $R_{ab} = \frac{1}{2}Rg_{ab}$, so (2.1) can be rewritten as

$$\nabla_a \nabla_b \phi - \frac{1}{2} g_{ab} \triangle \phi = 0 \tag{2.7}$$

Thus $\nabla_a \phi$ is a conformal Killing vector. A consequence of this also special to two dimensions, is that the vector dual to this,

$$k_a = \epsilon_a{}^b \nabla_b \phi \tag{2.8}$$

where ϵ_{ab} is the two-dimensional alternating tensor, obeys Killing's equation

$$\nabla_a k_b + \nabla_b k_a = 0 \tag{2.9}$$

Thus all solutions to these equations must have at least one Killing vector. A further consequence of this is that

$$k^a \nabla_a \phi = 0 \tag{2.10}$$

This means that the level sets of ϕ are the orbits of the Killing vector.

We can now construct all local solutions to these equations. Suppose that we start with the metric written in form

$$ds^2 = -f^2(r)dt^2 + dr^2, (2.11)$$

so that $k^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial t}$. We are guaranteed to be able to write the metric in this form locally by virtue of the existence of the Killing vector k_a . Furthermore, it follows that ϕ must be a function of r only be virtue of (2.10). An interesting property of solutions of the string equations of this form is that of duality, [14]. This is a map that transforms a solution

of the beta-function equations, (2.1) and (2.2) with zero tachyon field, into a related solution of the same equations. In terms of the variables in (2.11), duality effects the following transformations

$$f \to \frac{1}{f}, \qquad \phi \to \phi + 2\ln f.$$
 (2.12)

It is believed that this duality transformation is a feature of the exact theory, and not just an artefact of the low-order approximation to the beta-function equations.

The simplest solution of the field equations is flat spacetime. This corresponds to the choice

$$f(r) = 1, \qquad \phi = \phi_0 + 2\lambda(r - r_0)$$
 (2.13)

Here, both ϕ_0 and r_0 are constants of integration. λ is a constant that is determined by the central charge, and is given by

$$\lambda = \sqrt{\frac{-c^*}{4}} \tag{2.14}$$

This shows us a further interesting relationship, namely that for k^a to be timelike, since it is the Hodge dual of $\nabla_a \phi$, ϕ can only be a function of r, and then since ϕ is a real field, c^* must be negative. One might also think that it is possible to find solutions with $c^* > 0$, but this would require us to violate the no-ghost theorem.

We choose to fix the redundant parameter r_0 by requiring

$$r_0 = \frac{1}{\lambda} \ln 2. \tag{2.15}$$

It should be noted that this spacetime is transformed into itself under the duality transformation.

A second solution of these equations is the metric exterior to a black hole. This was first discovered in [15], [16]. In our coordinates, we find

$$f(r) = \tanh \lambda(r - r_0), \qquad \phi = \phi_0 + 2\ln \cosh \lambda(r - r_0)$$
(2.16)

This spacetime is asymptotic to the flat space solution of (2.13). The value of λ is determined by the central charge exactly as for the flat space case. The solution has a horizon bifurcation point at the location of the coordinate singularity at $r = r_0$. In what follows, we will choose $r_0 = 0$. The global structure of this spacetime can be made more transparaent by transforming to Kruskal type null coordinates u and v, when the metric becomes

$$ds^{2} = -\frac{1}{\lambda^{2}}(1 - uv)^{-1}dudv, \qquad \phi = \phi_{0} + \ln(1 - uv). \tag{2.17}$$

This spacetime is thus the two-dimensional analog of the Schwarzschild solution, with the Kruskal coordinates being given in terms of r and t by

$$u = -e^{-\lambda t} \sinh \lambda r, \tag{2.18}$$

$$v = e^{\lambda t} \sinh \lambda r. \tag{2.19}$$

The horizons are the null surfaces where u or v vanish. The line uv=1 is the spacetime singularity. If we were talking about general relativity, this singularity would be the boundary of spacetime. This interpretation arises because we are compelled to assign boundary conditions at the singularity in order to define physics in its Cauchy development. However, the physical meaning here is somewhat different, because although the spacetime is singular, the associated conformal field theory is claimed to be well-behaved, [6]. The singularity does not therefore form a barrier to passing into new regions of spacetime which contain naked singularities. The maximal extension of the spacetime thus contains six distinct regions, two asymptotically flat regions exterior to the horizons, (u < 0, v > 0) which is our original region, and u > 0, v < 0) two regions which are interior to the horizons (u, v > 0; uv < 1) and bounded by the singularities, and finally two regions which are asymptotically flat and contain a naked singularity, (u, v > 0; uv > 1) and u, v < 0; uv > 1. These last regions have no analog in the Kruskal manifold found in four dimensional general relativity.

The black hole spacetime can be transformed by the duality transformations into a different asymptotically flat spacetime [17], sometimes called the trumpet spacetime,

$$f(r) = \coth \lambda r, \quad \phi = \phi_0 + 2 \ln \sinh \lambda r.$$
 (2.20)

This metric is asymptotically flat and contains a naked singularity at r = 0. In fact, it is isometric to the new regions of the Kruskal manifold as can be seen by making the coordinate transformations

$$u = \pm \exp(-\lambda t) \cosh \lambda r, \tag{2.21}$$

$$v = \pm \exp(\lambda t) \cosh \lambda r, \tag{2.22}$$

which take (2.20) into (2.17).

Another pair of metrics that are dual to each other are given by

$$f(r) = \tan \lambda r, \quad \phi = \phi_0 - 2\ln\cos \lambda r,$$
 (2.23)

and

$$f(r) = \cot \lambda r, \quad \phi = \phi_0 - 2 \ln \sin \lambda r.$$
 (2.24)

These are in fact also diffeomorphic to each other as can be seen making the coordinate transformation

$$r \to r + \frac{\pi}{2\lambda} \tag{2.25}$$

and renormalizing ϕ_0 appropriately. Both of these solutions can also be found by making a complex coordinate transformation on the black hole or naked singularity spacetimes, (2.16) and (2.20). This pair of spacetimes are both singular where f(r) = 0, and are not asymptotically flat. However, they are the regions of the Kruskal manifold corresponding to the spacetime interior to the black hole horizon. In (2.23) the surface r = 0 is the singularity of the Kruskal manifold, and $r = \frac{\pi}{2\lambda}$ is the horizon. For (2.24) the the singularity is at $r = \frac{\pi}{2\lambda}$ and the horizon is at r = 0. To exhibit the relation explicitly, we can take the transformations

$$u = \pm \exp(-\lambda t)\sin \lambda r,\tag{2.26}$$

$$v = \pm \exp(\lambda t) \sin \lambda r,\tag{2.27}$$

which takes the Kruskal metric into (2.23) and

$$u = \pm \exp(-\lambda t)\cos \lambda r,\tag{2.28}$$

$$v = \pm \exp(\lambda t) \cos \lambda r,\tag{2.29}$$

which takes the Kruskal metric into (2.24).

There is a final pair of local solutions to the equations (2.11). One can choose the metric on flat space to be given by the Rindler form instead of (2.13)

$$f(r) = r, \quad \phi = \phi_0 - 2 \ln r.$$
 (2.30)

One could also have derived this by taking the black hole solution (2.16) and taking the λ goes to zero limit whilst simultaneously rescaling the t-coordinate. Clearly, there is a spacetime that is dual to it, namely

$$f(r) = \frac{1}{r}, \quad \phi = \phi_0 + 2\ln r.$$
 (2.31)

This could have been found by a similar limiting procedure applied now to the trumpet spacetime instead of the black hole spacetime. This spacetime contains a naked singularity at r = 0.

All of these metrics have Euclidean continuations. We will only concentrate on two examples. Firstly Euclidean flat space is

$$ds^{2} = d\tau^{2} + dr^{2} \qquad \phi = \phi_{0} + 2\lambda r - \lambda \ln 2 \tag{2.32}$$

This can be given the topology of a cylinder, $S^1 \times R^1$ where the Euclidean time is periodic with arbitrary period, and r is a coordinate along the cylinder. This is to be compared and contrasted to the Euclidean black hole solution, or cigar as it is more popularly known, given by

$$ds^2 = \tanh^2 \lambda r \ d\tau^2 + dr^2. \tag{2.33}$$

This metric has a conical singularity at r=0 unless we identify the Euclidean time coordinate τ with period $\frac{2\pi}{\lambda}$. The topology of this spacetime is that of a hemisphere. The local temperature is given by the inverse proper periodicity of the Euclidean time coordinate. Thus, as $r \to \infty$, the temperature T_c is given by

$$T_c = \frac{\lambda}{2\pi} \tag{2.34}$$

In other words, the temperature of the spacetime is determined by the central charge of the conformal field theory associated with the string theory.

3. Black Hole Thermodynamics

One way to understand the thermodynamics of gravitational fields is via the Euclidean treatment of quantum gravity. The basic idea is to evaluate the partition function as the path integral over the space of all field configurations that are periodic in imaginary (that is to say Euclidean) time. The entropy of black holes in general relativity is found to originate from the classical contributions to such a calculation. A similar analysis can be performed for two-dimensional stringy black holes, and that is the subject of this section. However we must be careful because as well as the gravitational field there is a long-range scalar field, the dilaton, ϕ . The dilaton field is associated with both black hole hair, ϕ_0 and with the background dilaton charge required to cancel the central charge of the conformal field theory, and thus give rise to a conformally invariant string theory. The usual way to deal with a problem of this sort is to make a conformal transformation on the spacetime metric so that there is no dilaton field in front of the R term in the action. The thermodynamics

can then be examined in the usual way [18], [19]. However, in two dimensions this cannot be done, and this gives rise to some complications of the thermodynamics.

To find the classical contribution to the partition function Z, we note that is given by

$$Z = \exp(-I) = \exp(-\beta F) \tag{3.1}$$

where I is the action evaluated for the Euclidean version of the classical gravitational field in question, β is the inverse temperature, and F is the Helmholtz free energy. If we evaluate the action for a solution of (2.1), (2.2) and (2.3) with T = 0, we discover that it is purely a boundary term,

$$I = -2 \int_{\partial \mathcal{M}} e^{\phi} (K + n^a \nabla_a \phi) d\Sigma$$
 (3.2)

where n^a is the unit outward normal to the boundary which has K as its second fundamental form and $d\Sigma$ is the volume element on it.

First we consider the case of flat spacetime. The metric is given by Euclidean flat space (2.32), where we identify the time coordinate τ with period β . We will be concerned with making physical observations at the boundary of the spacetime, in this case the wall of a box, which is located at a fixed value of the radial coordinate. However, the value of this coordinate is not a measureable quantity. The best that we can do is measure the value of the dilaton field there, which we will call ϕ_W . Similarly we can measure the temperature there, T_W which is the proper periodicity of the Euclidean space at coordinate r. In the flat space example, we find that $T_W = \beta^{-1}$. Thus ϕ_W and T_W are our basic observables, and the partition function $Z = Z(\phi_W, T_W)$. As mentioned earlier, we only wish to ask about measurements that can be made at this wall and address issues about what is happening inside the box (r decreasing). This means that we will ignore the second boundary of flat space, that is the boundary at the component of spacelike infinity where $r \to -\infty$.

We now calculate the Euclidean action for flat space from (3.2) and we find

$$I = -4\frac{\lambda}{T_W}e^{\phi_W}. (3.3)$$

Flat space has a non-vanishing dilaton charge D. Consider the dilaton current j_a

$$j_a = \epsilon_a{}^b \nabla_b e^\phi. \tag{3.4}$$

This current is conserved

$$\nabla_a j^a = 0, (3.5)$$

thus there is an associated conserved charge defined by

$$D = \int_{\Sigma} j_a d\Sigma^a, \tag{3.6}$$

where Σ is a spacelike hypersurface bounded by the wall of the box and $r = -\infty$. Evaluating the dilaton charge of flat space, we discover that the dilaton charge inside the box is

$$D = e^{\phi_W}, \tag{3.7}$$

Associated with this dilaton charge there will be a chemical potential ψ , the dilaton potential. We can now read off from the action the value of the Helmholtz free energy F contained in the box in terms of the coordinates as

$$F = -\lambda e^{\phi_0 + 2\lambda r},\tag{3.8}$$

or in terms of observables

$$F = -4\lambda e^{\phi_W}. (3.9)$$

We should note that λ is a fixed constant determined by the central charge of the string fields. To find the other thermodynamic potentials, we recall that the first law of thermodynamics in effect serves to define the chemical potential and the entropy. It takes the form here of

$$dF = -SdT_W - \psi dD \tag{3.10}$$

so F must be expressed in terms of T_W and D. Hence

$$F = -\lambda D \tag{3.11}$$

and so the dilaton potential ψ is given by

$$\psi = -\left(\frac{\partial F}{\partial D}\right)_{T_W} = \lambda \tag{3.12}$$

and the entropy is

$$S = -\left(\frac{\partial F}{\partial T_W}\right)_D = 0 \tag{3.13}$$

Flat space therefore has zero entropy as one might have expected. However, it does not have zero energy, since

$$E = F + ST_W (3.14)$$

it follows that

$$E_{fs} = -4\lambda e^{\phi_W} \tag{3.15}$$

This energy does not gravitate. At first sight this seems a bit paradoxical, however the presence of a long-range scalar field causes the weak equivalence principle to be violated. This energy is an irreducible energy which is tied up with the existence of the background dilaton charge.

One might be concerned that our definition of the dilaton current was quite arbitrary since we could replace the e^{ϕ} term in (3.4) by any function of ϕ and still find a conserved current. If we did this the new dilaton charge would just be a function of the old dilaton charge and so there would not be any new conserved quantum numbers associated with it. It should also be noted that this charge is quasi-topological in that it arises solely from a boundary contribution.

Now we turn our attention to the Euclidean black hole solution (2.33) Suppose that the wall is a distance r from the event horizon. The periodicity of the τ -coordinate at r is as we saw T_c^{-1} . However, the periodicity of the wall of the box there is T_W^{-1} , and so T_W is related to T_c by Tolman relationship [20]

$$T_W = T_c \coth(2\pi r T_c). \tag{3.16}$$

As in the flat space case, we can straightforwardly evaluate the action and find in terms of the coordinates that

$$I = 4\pi e^{\phi_0} \left(1 - 2\cosh^2(2\pi r T_c) \right). \tag{3.17}$$

The dilaton charge in the box is

$$D = e^{\phi_0} \cosh^2(2\pi r T_c). \tag{3.18}$$

Consequently, the Helmholtz free energy is, when expressed in terms of the canonical physical variables, D and T_W ,

$$F = -4\pi D \left(T_W + \frac{T_c^2}{T_W} \right). \tag{3.19}$$

The chemical potential associated with the dilaton charge is

$$\psi = 4\pi T_W \left(1 + \frac{T_c^2}{T_W^2} \right). \tag{3.20}$$

The entropy S of the spacetime inside the box is then easily found and is given by

$$S = 4\pi D \left(1 - \frac{T_c^2}{T_W^2} \right). {(3.21)}$$

Since $T_W > T_c$, this entropy is positive as one should expect. The energy can now be found from (3.14) and is

$$E_{bh} = -8\pi D \frac{T_c^2}{T_W}. (3.22)$$

This represents the combined energy of the background field and the black hole. If we re-write this using the same variables as the flat space example, we find that

$$E_{bh} = -4\lambda e^{\phi_W} \frac{T_c}{T_W} \tag{3.23}$$

since $\lambda = 2\pi T_c$. It is now easy to extract that part of E_{bh} that is the energy of the black hole, M. Recalling (3.16) we find that

$$M = E_{bh} - E_{fs} = 4\lambda e^{\phi_W} \left(1 - \frac{T_c}{T_W}\right)$$
 (3.24)

This quantity we interpret as being the energy of the black hole itself, or equivalently its rest mass. Furthermore, as we might have expected, in the asymptotic region where $r \to \infty$, this expression reduces to

$$M = 2\lambda e^{\phi_0} \tag{3.25}$$

which is the ADM mass of the black hole, as described in reference [6]. The easiest way to see this is to adapt the derivation of Arnowitt, Deser and Misner, [21]. The new feature of the analysis is the introduction of the dilaton field. If we look at the volume part of the gravitational action, (2.6), the ADM mass is that part of the corresponding canonical Hamiltonian that is a boundary term. After a short calculation, we discover that for a spacetime of the form (2.11), the ADM mass is given by

$$M_{ADM} = 2\left(e^{\phi}f\frac{\partial f}{\partial r}\right)_{r=\infty} \tag{3.26}$$

Thus for the black hole solution, we get

$$M_{ADM} = 2e^{\phi_0}\lambda \tag{3.27}$$

The ADM mass therefore agrees with the thermodynamic evaluation of the energy.

Another way to write the mass of the hole is in terms of the dilaton charge D and the temperature T_W . In this case, we find

$$M = 8\pi DT_c \left(1 - \frac{T_c}{T_W}\right). \tag{3.28}$$

This expression makes it clear that the range of M is from zero where T_c tends to T_W , or alternatively when the black hole is infinitely far from the wall, upto $8\pi DT_c$ where the wall of the box tends to the event horizon.

Given (3.21), one might think think that the entropy of the black hole vanishes as one becomes infinitely far from it, that is where $T_W \to T_c$. This would be in contrast to the four-dimensional case where if one is infinitely far from the black hole, the entropy is given precisely by (1.2). However, this is rather misleading as the entropy should be expressed as a function of the energy of the black hole and its dilaton charge as these specify the physical state of the black hole. Writing S as a function of M, D and T_c , we find

$$S = \frac{M}{T_c} \left(1 - \frac{M}{16\pi DT_c} \right) \tag{3.29}$$

Since M lies in the range $0 \le M \le 8\pi DT_c$, the entropy of the hole for fixed D and T_c reaches a maximum of $M/2T_c$ when $M = 8\pi DT_c$, and has a minimum of zero when the mass of the black hole vanishes.

Finally, we would like to find a formula analogous to (1.2). Such a formula would be valid at $r = \infty$, and would specify the entropy of the black hole in terms of the mass. The easiest way to derive such a formula us to start from (3.21), and substitute (3.18) for D, and Tolman for T_W . The result is then independent of r. Using (3.27), a result valid for the mass at spatial infinity only, we find that

$$S = \frac{M}{T_c} \tag{3.30}$$

It is this final formula, (3.30) that should be regarded as being the two-dimensional analog of (1.2).

4. Electrification

A more realistic fundamental theory will include fermions and gauge fields. In this section we examine some black hole solutions for the heterotic string. These solutions seem to have been first discussed by Nappi, Yost and McGuigan [22]. In the bosonic sector, the lowest order string beta-functions are very similar to those for the closed bosonic string. However there are Yang-Mills fields A and corresponding field strengths F belonging to some anomaly-free gauge group G, which replace the tachyon field. Unlike the case in ten dimensions, a complete list of such groups is not known, but an anomaly free G is SO(24). The beta-functions are, to lowest order in α' ,

$$R_{ab} = \nabla_a \nabla_b \phi + \frac{1}{2} \text{Tr} F_{ac} F_b{}^c \tag{4.1}$$

$$R - (\nabla \phi)^2 - 2\triangle \phi - \frac{1}{4} \operatorname{Tr} F_{ab} F^{ab} = c^*$$
(4.2)

$$\nabla_b F^{ab} + (\nabla_b \phi) F^{ab} = 0 \tag{4.3}$$

where the central charge is now

$$c^* = \frac{2c + 3d - 30}{6} \tag{4.4}$$

These equations can be derived from the target space action

$$I = \int_{\mathcal{M}} d^2x g^{\frac{1}{2}} e^{\phi} (c^* - R - (\nabla \phi)^2 + \frac{1}{4} \text{Tr} F^2) - 2 \int_{\partial \mathcal{M}} e^{\phi} K \ d\Sigma$$
 (4.5)

Evaluating this action for a field obeying the beta-function equations yields an on-shell action that is again a pure boundary term,

$$I = -2 \int_{\partial \mathcal{M}} e^{\phi} (K + n^a \nabla_a \phi) d\Sigma \tag{4.6}$$

with n^a being the unit normal to the boundary $\partial \mathcal{M}$.

It is now a straightforward task to solve these equations and find a black hole solution in exactly the same way as we did for the bosonic string. We will take the gauge field to lie in a U(1) subgroup of the full gauge group. The solutions that emerge are in some sense analogs of the Reissner-Nordstrom solution in general relativity. The spacetime metric is

$$ds^{2} = -\frac{(m^{2} - q^{2})\sinh^{2} 2\lambda r}{(m + (m^{2} - q^{2})^{\frac{1}{2}}\cosh 2\lambda r)^{2}}dt^{2} + dr^{2}$$
(4.7)

with the dilaton field given by

$$\phi = \phi_0 + \ln\left(\frac{1}{2}\left(\frac{m}{(m^2 - q^2)^{\frac{1}{2}}} + \cosh 2\lambda r\right)\right)$$
(4.8)

and the vector potential one-form by

$$A = \frac{\sqrt{2}q}{m + (m^2 - q^2)^{\frac{1}{2}}\cosh 2\lambda r} dt$$
 (4.9)

This solution appears to depend on four arbitrary parameters ϕ_0, q, m and λ . It should be noted that we have chosen our parameters here in such a way that if one sets q=0, one obtains the previous expressions given for uncharged black holes. The parameter q is related to the charge contained in the spacetime, and is required to obey $q^2 < m^2$ so that we have a black hole in an asymptotically flat spacetime, rather than a naked singularity. It should be noted that the case where $q=\pm m$ looks degenerate in these coordinates, but this degeneracy can be removed by a suitable redefinition of the t coordinate. However, if we do this, the spacetime fails to be asymptotically flat.

The charge Q inside the a box whose wall is located at r is given by the value of $-e^{\phi} * F$. Here the star is the Hodge dual operator acting on the field strength 2-form F. In terms of components, we could write this as

$$Q = \frac{1}{2}e^{\phi}\epsilon_{ab}F^{ab} \tag{4.10}$$

It follows from the analog of Maxwells equation (4.3), that this is a conserved charge. Evaluating Q for the solution given here, we find that it is

$$Q = \sqrt{2}\lambda q e^{\phi_0} \frac{\sinh 2\lambda r}{m + (m^2 - q^2)^{\frac{1}{2}}\cosh 2\lambda r}.$$
(4.11)

 ϕ_0 is an arbitrary constant related to the ADM mass of the hole which can be calculated by the same method as given in section three. It is given by

$$M_{ADM} = \frac{2\lambda}{(1 - \frac{m^2}{q^2})^{1/2}} e^{\phi_0}.$$
 (4.12)

 λ is fixed by the central charge since (4.2) requires that

$$c^* = -4\lambda^2 \tag{4.13}$$

This solution can be Euclideanized by the substitution $t \to i\tau$ and $q \to ik$. In addition to the time coordinate t, the parameter q must be Wick rotated because it appears as the time component of a vector. Thus the Euclideanized version of the solution is

$$ds^{2} = \frac{(m^{2} + k^{2})\sinh^{2} 2\lambda r}{(m + (m^{2} + k^{2})^{\frac{1}{2}}\cosh 2\lambda r)^{2}} d\tau^{2} + dr^{2}$$
(4.14)

with the dilaton being given by

$$\phi = \phi_0 + \ln\left(\frac{1}{2}\left(\frac{m}{(m^2 + k^2)^{\frac{1}{2}}} + \cosh 2\lambda r\right)\right)$$
(4.15)

and the vector potential by

$$A = \frac{\sqrt{2}k}{m + (m^2 + k^2)^{\frac{1}{2}}\cosh 2\lambda r} d\tau.$$
 (4.16)

There is a conical singularity at r=0 unless τ is identified with period β , where

$$\beta = \frac{\pi \left(m + (m^2 + k^2)^{\frac{1}{2}}\right)}{\lambda (m^2 + k^2)^{\frac{1}{2}}} \tag{4.17}$$

To find the Hawking temperature, we take the inverse periodicity and analytically continue back to the physical region by substituting $k \to -iq$. Thus the temperature of the black hole is

$$T_c = \frac{\lambda (m^2 - q^2)^{\frac{1}{2}}}{\pi (m + (m^2 - q^2)^{\frac{1}{2}})}$$
(4.18)

It should be noted that the temperature is zero in the case $q = \pm m$. This extreme black hole is rather like the extreme Reissner-Nordstrom solution, and has similar global structure. By analogy with the uncharged case, there is a dilaton charge given by

$$D = \int \epsilon_{ab} \nabla^b e^{\phi} d\Sigma^a \tag{4.19}$$

Evaluating this for the solution in question yields

$$D = \frac{1}{2}e^{\phi_0} \left(\frac{m}{(m^2 - q^2)^{\frac{1}{2}}} + \cosh 2\lambda r \right)$$
 (4.20)

for the dilaton charge enclosed by a box at a value of the radial coordinate given by r.

We can now evaluate the action as we did for the uncharged black hole discussed in section three. The Euclidean action I is

$$I = -2\pi \cosh(2\lambda r)e^{\phi_0} \left(\frac{m}{(m^2 + k^2)^{\frac{1}{2}}} + 1\right)$$
(4.21)

To compute thermodynamic quantities, we must use this result to evaluate the free energy F in terms of the physical variables T_W , Q, D and λ after continuing back to the physical region. It should be noted that λ is fixed by the central charge. The temperature of the wall of the box is redshifted by the Tolman factor, so that we find T_W is related to T_c by

$$T_W = T_c \left(\frac{m + (m^2 - q^2)^{\frac{1}{2}} \cosh 2\lambda r}{(m^2 - q^2)^{\frac{1}{2}} \sinh 2\lambda r} \right).$$
 (4.22)

Then from (4.22), (4.21), (4.20) and (4.11) that the best way to construct the free energy $F = F(\lambda, D, T_W, Q)$ is parametrically by writing $F = F(\lambda, D, T_W, x)$ and $Q = Q(\lambda, D, T_W, x)$ with the parameter $x = \lambda r$. We thus find

$$F = -4\lambda D \coth 2x \tag{4.23}$$

and

$$\frac{Q}{D} = \frac{\lambda}{\sqrt{2}\pi^2} \left(\frac{\pi \sinh 2x}{T_W} - \frac{\lambda}{T_W^2} \right) \left(\lambda^2 - 2\lambda \pi T_W \tanh x \right)^{\frac{1}{2}}.$$
 (4.24)

To find the various thermodynamic potentials is now straightforward. The electrostatic potential is

$$\Phi = -\left(\frac{\partial F}{\partial x}\right)_{\lambda, D, T_W} \left(\frac{\partial Q}{\partial x}\right)_{\lambda, D, T_W}^{-1} \tag{4.25}$$

and the dilaton potential is

$$\psi = -\left(\frac{\partial F}{\partial D}\right)_{\lambda, T_W, x} - \left(\frac{\partial F}{\partial x}\right)_{\lambda, D, T_W} \left(\frac{\partial D}{\partial x}\right)_{\lambda, O, T_W}.$$
(4.26)

The entropy is given by

$$S = -\left(\frac{\partial F}{\partial x}\right)_{\lambda, D, Q} \left(\frac{\partial T_W}{\partial x}\right)_{\lambda, D, Q}^{-1}.$$
 (4.27)

Thus an explicit formula for $S = S(\lambda, D, T_W, x)$ can be found. We will not record these rather complicated and unilluminating expressions here.

It is however quite straightforward to evaluate the entropy as seen from infinity in a way that is analogous to (3.30). Using the result for the free energy, (4.23), the ADM mass (4.12), we discover afer some manipulation, that entropy as seen from infinity is given by

$$S = \frac{\pi M}{\lambda} \left(1 + \left(1 - \frac{q^2}{m^2}\right)^{1/2}\right) \tag{4.28}$$

This result is very similar to that for the Reissner-Nordstrom solutions, the entropy is largest for electrically neutral black holes, and descreases down to half that value as one charges up the black hole to its maximal value of the charge where |q| approaches m.

We will however note that the limit of $m \to \pm q$ corresponds to infinite dilaton and electric charges, and correpsonds to zero temperature. However, the limit $m=\pm q$ is unattainable since that particular spacetime fails to be asymptotically flat. This is because the infinite amount of energy involved has curled up the spacetime at infinity. It is interesting to view this as an explicit realization of the third law of black hole thermodynamics. Perhaps the easiest way to see this is to examine (4.7) and rescale the t coordinate by a factor of $(1 - \frac{q^2}{m^2})^{\frac{1}{2}}$. The new metric is then

$$ds^2 = -\sinh^2 2\lambda r \ dt^2 + dr^2 \tag{4.29}$$

Thus as $r \to \infty$ the metric fails to be flat. The point r = 0 will be a conical singularity unless t is identified with the appropriate period. We therefore conclude that the zero temperature limit is not physical in this case. There are therefore no zero temperature solitons in this theory.

5. Exact Solutions

The spacetimes presented so far can be regarded as approximations valid in the α' limit to some exact solutions to the beta-function equations. If one wants to find the metric and dilaton fields to higher orders in α' , one way to do this is to determine the beta-function equations to as high an order in α' as one can, and then solve the resulting field equations. By following such a prescription, Jack Jones and Panvel [23] have shown that the solutions presented in section two are exact for the type II superstring. We therefore confident that our results hold for realistic string theories. However, for the bosonic and heterotic strings, there will be higher order corrections to the solutions outlined above. For the bosonic string, we can still perform the exact calculation by following a trick of Witten

[6]. Suppose one starts from the WZW model at level k for $g \in SU(1,1)$, we have then a conformal field theory with a three dimensional target space. the action is

$$I = \frac{1}{2} \int \text{Tr} g^{-1} dg \wedge g^{-1} dg + \frac{ik}{24\pi} \int \text{Tr} (g^{-1} dg)^3$$
 (5.1)

Suppose that one now gauges an antidiagonal U(1) subgroup h by

$$g \to hgh$$
 (5.2)

The result is a confomal field theory with a two-dimensional target space with a metric of the form (2.11)where now

$$f(r) = \left(\coth^2 \lambda r - 2/k\right)^{-\frac{1}{2}} \tag{5.3}$$

and

$$\phi = \phi_0 + \ln \frac{\sinh \lambda r}{f(r)} \tag{5.4}$$

[24]. However, the basic change is only in the details of the metric and not its overall character. Thus, for mass scales large compared to $(\alpha')^{-\frac{1}{2}}$, we expect the results we have obtained to be quite valid. The metric and dilaton specified by (5.3)and (5.4)are conformally invariant, but with a central charge specified by k. The central charge is [1]given by

$$c = \frac{3k}{k+2} - 1\tag{5.5}$$

thus we will have a critical bosonic string theory if k = 9/4. If we keep away from this value, there will be black hole solutions. If k = 9/4, then $\lambda = 0$ and the only metric will that of flat space. It would nice to be able to carry out our thermodynamic calculations in the exact theory. However, we have no way of finding the spacetime action for the exact theory, and so we cannot find the free energy, and hence we cannot find the entropy. It would be intriguing to calculate these quantities for an exact theory.

Our results show a remarkable parallel with similar results in four dimensions even though the presence of the dilaton field causes considerable complications. This makes it quite plausible to suppose that these two dimensional stringy black models really do have sufficient in common with realsitic examples to give some guide as to how to resolve the problems of quantum gravity by extrapolation from these highly simplified toy models.

6. Acknowledgements

MJP would like to thank Prof. I.M. Singer and the Mathematics department at MIT for their hospitality during the completion of this work, which was supported in part by US DOE grant DE-FG02-88ER25066.

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