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ON THE SPATIAL CORRELATIONS OF ABELL CLUSTERS

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ABSTRACT

If rich clusters formed where the primordial density enhancement, when averaged over an appropriate volume, was unusually large, then they give a biased measure of the large-scale density correlation function: $\xi_{\rm clusters}(r) \approx A\xi_{\rm density}(r)$. The factor A is determined by the probability distribution of the density fluctuations on a rich cluster mass scale, and if this distribution was Gaussian the correlation function is amplified. The amplification for rich $R \geq 1$ clusters is estimated to be $A \approx 10$, and the predicted trend of A with richness agrees qualitatively with that observed. Some implications of these results for the large-scale density correlations are discussed.

Subject headings: cosmology — galaxies: clustering

I. INTRODUCTION

It has long been known that the spatial distribution of Abell's rich clusters is nonrandom and that these objects are themselves clustered (Abell 1958, 1961). Two recent estimates (Bahcall and Soneira 1983, hereafter BS; Klypin and Kopylov 1983, hereafter KK) of the spatial correlation function $\xi_c(r)$ show this to be unity at a lag of $r_c \approx 25h^{-1}$ Mpc ($h = H/100$ km s^{-1} Mpc^{-1}) with a long tail of correlations $\xi_c(r) \approx (r/r_c)^{-1.8}$ out to $r \approx 50$–$100h^{-1}$ Mpc. In contrast, galaxies have a much smaller coherence length, $\xi_g(r) = 1$ for $r = r_g \approx 4$–$7h^{-1}$ Mpc (Davis and Peebles 1983; Shanks *et al.* 1983) and have small amplitude on large scales; $|\xi_g| \leq 0.1$ for $r \gtrsim 25h^{-1}$ Mpc. Galaxies and clusters cannot then both be good tracers of the mass distribution on large scales.

It has been argued that the observed disparity of length scales is evidence for some "special" initial conditions, either in the form of a pronounced feature in the power spectrum (Dekel 1984*a*) or that the initial density fluctuations may have been non-Gaussian (Peebles 1983).

Abell's clusters were selected to be massive and condensed; clusters of richness $R \geq 1$ must contain at least 50 bright galaxies within an Abell radius $r_{\rm Abell} = 1.5h^{-1}$ Mpc. These clusters are the most massive objects which have collapsed and relaxed to a roughly spherical shape. They are also rare; only 2%–5% of galaxies are within $r_{\rm Abell}$ of a cluster center (BS; Dekel 1984*b*). It seems reasonable to assume that these objects form where the primordial density enhancement, when averaged over a rich cluster mass scale, lay above some moderately high threshold. Peculiar motions may have rearranged the clusters on small scales $\approx v_p/H \lesssim 10h^{-1}$ Mpc, but the large-scale correlation function should still approximate that of the "high spots" of the primordial density field.

It is shown below that objects selected in this way give a biased estimate of the large-scale density correlations ξ_ρ. At large separations, the cluster correlation function is amplified by a factor $A \equiv \xi_c/\xi_\rho$ which depends only on the statistical distribution of the smoothed density fluctuations. The amplification factor appropriate for rich clusters is estimated and is shown to be considerable. Some implications of the inferred density correlations are discussed.

II. THE CORRELATION FUNCTION OF RARE EVENTS

Let us assume some continuous stationary stochastic process $\delta(\boldsymbol{x})$ with coherence length l_c and some small-amplitude $[|\xi(r)| \ll \xi(0)]$ but otherwise unspecified correlations on large-scale $r \gg l_c$. We shall later identify δ with the spatially filtered density field, for which l_c corresponds to the smoothing window radius R_W. We would like to calculate $\xi_{\max > \nu}(r)$, the correlation function of those maxima which lie above some threshold $\nu\sigma$, where σ is the rms density fluctuation; $\sigma^2 = \xi(0)$.

Let us treat, in the first place, the case of Gaussian noise. It is possible, following the methods described by Rice (1954) to calculate $\xi_{\max > \nu}$; however, it is rather messy, especially for the relevant case of noise on a three-dimensional domain. Instead let us calculate a closely related quantity $\xi_{>\nu}$ being the correlation function of *regions* lying above the threshold. Specifically $\xi_{>\nu}(r)$ is defined to be the fractional excess probability that $\delta_2 \equiv \delta(x_2) > \nu\sigma$ given that $\delta_1 \equiv \delta(x_1) > \nu\sigma$ and $|\boldsymbol{x}_1 - \boldsymbol{x}_2| = r$.

Now the probability that a randomly chosen point lies above the threshold is

$$P_1 = \int_{\nu\sigma}^{\infty} P(y)\, dy,$$

where $P(y) = (2\pi\sigma^2)^{-1/2}\exp(-y^2/2\sigma^2)$, and the probability that δ_1 and δ_2 both lie above the threshold is

$$P_2 = \int_{\nu\sigma}^{\infty}\int_{\nu\sigma}^{\infty} P(y_1, y_2)\, dy_1\, dy_2,$$

where the joint probability distribution is

$$P(y_1, y_2) = (2\pi)^{-1}\left[\xi^2(0) - \xi^2(r)\right]^{-1/2} \times \exp\left\{-\frac{\xi(0)y_1^2 + \xi(0)y_2^2 - 2\xi(r)y_1y_2}{2\left[\xi^2(0) - \xi^2(r)\right]}\right\},$$

then

$$1 + \xi_{>\nu}(r)$$
$$= \frac{P_2}{P_1^2} = (2/\pi)^{1/2}\left[\operatorname{erfc}(\nu/2^{1/2})\right]^{-2}$$
$$\times \int_\nu^\infty e^{-1/2y^2} \operatorname{erfc}\left[\!\!\left[\frac{\nu - y\xi(r)/\xi(0)}{\{2[1 - \xi^2(r)/\xi^2(0)]\}^{1/2}}\right]\!\!\right] dy. \tag{1}$$

This result may also be obtained by application of Price's theorem (Price 1958). For $\xi_c \ll 1$ this expression simplifies to

$$\xi_{>\nu}(r) = \left(e^{\nu^2/2}\int_\nu^\infty e^{-1/2y^2}\,dy\right)^{-2} \xi(r)/\sigma^2, \tag{2}$$

and for $\nu \gg 1$

$$\xi_{>\nu}(r) \approx (\nu^2/\sigma^2)\xi(r). \tag{3}$$

That $\xi_{>\nu}(r)$ depends on the ratio $\xi(r)/\xi(0)$ is to be expected since the threshold is set in units of the rms so the amplitude of ξ cancels out; what is interesting in the amplification factor in equations (2) and (3) which increases with increasing threshold. These equations tell us that while the probability of finding an object at a random position decreases with increasing threshold roughly as $\exp(-\nu^2/2)$, the conditional probability of finding an object at a distance r from a known object decreases slightly less rapidly with increasing ν.

To see more clearly how this amplification comes about, and to see what effect might be expected for more general (i.e., non-Gaussian) fluctuations, an example may be helpful. Take a noisy field $\delta_N(x)$ which has a small coherence length l_C and has negligible correlations on larger scales. Add to this a relatively small amplitude "signal" $\delta_S(x)$ with much larger coherence length $l_S \gg l_C$. The *distribution* $P(\delta)$ of the composite field $\delta \equiv \delta_N + \delta_S$ is effectively identical to that of the "noise," whereas the *correlation function* of δ is just equal to that of the "signal" for separations $\gg l_C$.

According to the model we must place objects where δ exceeds a threshold $\nu\sigma$. Consider a region of size l such that $l_C \ll l \ll l_S$, so the "noise" component fluctuates many times, but the "signal" remains effectively constant. The probability of finding an object at a point x within the region is just the probability that δ_N exceeds an effective threshold $\nu_{\rm eff}\sigma$, where $\nu_{\rm eff} = \nu - \delta_S(x)/\sigma$. The probability of finding an object is then spatially modulated:

$$P(>\nu, x) = P_N[>\nu - \delta_S(x)/\sigma]$$
$$\approx P(>\nu)\left[1 - \frac{1}{P(>\nu)\sigma}\frac{dP(>\nu)}{d\nu}\delta_S(x)\right],$$

where $P(>\nu)$ denotes the probability of exceeding the threshold. Thus the objects can be considered to be a random sample from a field with density contrast

$$\delta_c(x) = -\left[\frac{dP(>\nu)/d\nu}{\sigma P(>\nu)}\right]\delta_S(x) \tag{4}$$

and since $\xi_c \propto \delta_c^2$ the amplification of the correlation function is

$$A = \left[\sigma^{-1}\frac{d\log P(>\nu)}{d\nu}\right]^2 \tag{5}$$

which reproduces the result (eq. [2]) for Gaussian noise. The truncated Taylor series for $P_N[>\nu - \delta_S(x)/\sigma]$ used to obtain this result is valid if $(\delta/\sigma)d^2P(>\nu)/d\nu^2 \ll dP(>\nu)/d\nu$ which, for Gaussian noise in the limit of large ν, implies $\delta/\sigma \ll \nu^{-1}$ or equivalently $\xi_c \ll 1$. In fact ξ_c given by equation (5) (or eq. [2]) agrees with the result of numerically integrating the exact equation (1) [for a suitably chosen $\xi(r)$] to within 20% for $\xi_c \approx 1$ and is correct to within a factor 2 for $\xi_c \approx 5$.[1]

The crucial assumption in the foregoing example is that one may linearly decompose $\delta(x)$ into a high-amplitude "noise" and a low-amplitude "signal" and that the statistical properties of the "noise" be homogeneous. The same argument can be applied to the model in which clusters are located at the high maxima of $\delta(x)$. In this case the ratio $\xi_{\max>\nu}/\xi$ is given by equation (5) with $P(>\nu)$ replaced by $P(\max > \nu)$, the probability that a maximum lies above ν.

III. APPLICATION TO ABELL CLUSTERS

In order to estimate the strength of this effect for Abell clusters one must choose an appropriate smoothing radius R_W. For essentially any choice of R_W we can find the value of the threshold which yields the required number density of objects, e.g., $n_{\rm cl} \approx (50h^{-1}\ {\rm Mpc})^{-3}$ for richness class $R \geq 1$. Different values of R_W will produce different catalogs of objects with differing amplification factors. We want to find the value R_W which most accurately mimics Abell's selection criteria. One constraint is that the selected proto-objects should yield clusters of the right mass. The mean separation $\lambda_c \approx 50h^{-1}$ Mpc and the fraction $f \approx$ 2%–5% of galaxies contained in clusters suggest $R_W \approx 9$–$12h^{-1}$ Mpc but since R_W is only approximately related to the radius of the proto-objects, it is difficult to make this relation more precise. A more stringent requirement is that the objects should have collapsed recently; if R_W is taken to be too large then the objects chosen will have a small primordial density contrast ($\propto \nu\sigma$), since both ν and σ decrease with increasing R_W and will not have collapsed yet. If R_W is too small then the model picks out low-mass proto-objects, ν must be increased in order to keep the number density fixed and consequently the model misses most of the most massive objects even though these would have been conspicuous enough for Abell to have seen.

[1]An expression for ξ_c which is the limiting case of eq. (1) as $\nu \gg 1$, but is valid for $\xi \ll 1$, is given by Politzer and Wise (1984), to whom the author is grateful for pointing out an error in an earlier version of this *Letter*.

According to the nondissipative spherical collapse model in an $\Omega = 1$ universe collapse occurs when the density contrast obtained from the linear growth formula is 1.68 so $t_{\rm coll}/t_0 = (1.68/\nu\sigma)^{3/2}$. Here t_0 is the present age of the universe. The final overdensity of the virialized object is $(\Delta\rho/\rho)_{\rm FINAL} \approx 180(t_0/t_{\rm coll})^2$. For $1 \geq t_{\rm coll}/t_0 \geq 1/2$, we have $1.7 \leq \nu\sigma \leq 2.4$ and $180 \leq (\Delta\rho/\rho)_{\rm FINAL} \leq 700$. Note that two regions whose initial overdensities differ by 40% give rise to objects whose final overdensity differs by a factor of 4. It is this behavior which justifies the use of a simple model with a sharp threshold to decide which proto-objects would be later identified as clusters.

If σ^2, the variance of density in spheres of radius R_W, is known, then the collapse time requirement ($1.68 < \nu\sigma < 2.37$) and the number density of objects is sufficient to determine the parameters R_W and ν without reference to the uncertain parameter f. However, it is reassuring that the range of final overdensities implied by our assumptions brackets the values $\Delta\rho/\rho \approx 200$–$400$ obtained if 2%–5% of the material from a $50h^{-1}$ Mpc cube is compressed into an Abell radius sphere.

Let us calculate the amplification factor A (eq. [5]) for reasonable values of R_W under the assumptions (i) that the distribution of primordial density fluctuations in R_W spheres was Gaussian and (ii) that the linear theory rms *density* fluctuation σ may be obtained from the present *galaxy* correlation function which we model as $\xi_g(r) = (r/r_g)^{-2}$. Adopting a window profile $W(r) \propto \exp[-r^2/R_W^2]$ one finds, by the method described in Peebles (1980), $\sigma = (r_g/R_W)$.

To calculate ν for some choice of R_W we need to know $n(>\nu)$, the number density of objects, as a function of the threshold. This was estimated as $n(\max > \nu)$, the number density of *maxima* lying above the threshold, since most objects contain only one maximum if ν is moderately large. The quantity $n(\max > \nu)$ is dependent on the index of the power spectrum but only for wavenumbers $k \approx k_W \equiv \sqrt{2}/R_W$. This was taken to be $n = -1$ in accord with the observed slope of ξ_g for $r \approx R_W$. Several realizations of noise with spectrum $\delta^2 \propto k^{-1}\exp(-k^2/k_W^2)$ were generated on a 32^3 grid of side $8\pi/k_W$, the maxima were located, and $n(\max > \nu)$ was calculated.

In Table 1 are shown, for various values of R_W, the thresholds required to give $n = (50h^{-1}\ \text{Mpc})^{-3}$, the present rms density contrast, the linear theory overdensity of clusters, and the amplification factor $A \equiv \xi_c/\xi_\rho$. If $r_g \approx 7h^{-1}$ Mpc (see Fig. 1 of Davis and Peebles 1983) then, as indicated by the comments in the right-hand column, the collapse time requirement implies $9.5 \gtrsim A \gtrsim 8.3$. If $r_g \approx 5h^{-1}$ Mpc, then a slightly smaller window radius must be adopted ($8h^{-1}$ Mpc $\geq R_w \gtrsim 6h^{-1}$ Mpc) and one finds $17 \gtrsim A \gtrsim 14$.

IV. DISCUSSION

We have seen that if the parameters of this rather crude model are adjusted to select a population of objects with recent collapse times and number density equal to that of richness $R \geq 1$ clusters, then these objects have large-scale correlations amplified by roughly one order of magnitude. The reliability of this estimate of the amplification for real clusters is limited by the approximate nature of the spherical collapse model employed. One cannot expect a one-to-one correspondence between the objects in the model and in the real world, and there are, no doubt, systematic effects which modify the correlation function amplitude. One such effect is that the correlation function for these extended uniform density objects perhaps more closely approximates a mass-weighted correlation function. In fact, similar results for A were obtained for the model in which clusters are located at the density peaks, so this effect is quite weak. These systematic effects can best be quantified by numerical experiments (see below).

While the estimate of A obtained above must be treated with caution, the strong dependence of A on the assumed density coherence length is of interest. The model results can be approximated by $A \propto r_g^{-1.7}$ for $7h^{-1}$ Mpc $\geq r_g \geq 5h^{-1}$ Mpc. If we assume a power-law–like spectrum of fluctuations on scales $\sim R_W$, this implies that A depends on the mean separation of objects as $A \approx 0.32\,(\lambda_c/r_g)^{1.7}$ for $10 \gtrsim \lambda_c/r_g \gtrsim 7$. Abell clusters of richness $R \geq 2$ have mean separation ≈ 1.7 times that of richness class $R \geq 1$ and, by extrapolation of this trend, should show correlations which are roughly 2.5 times as large. Shectman (1984) has recently compiled a catalog of clusters selected according to surface density contrast. These have number density roughly 6 times that of $R \geq 1$ and so should show correlation strength roughly one-third that of the Abell clusters. Both of these predictions are in qualitative agreement with the data. Going to condensations of still smaller mass, the expected behavior is that, for systems of mass below some critical value $M_{\rm crit}$, there will be essentially no amplification. The mass $M_{\rm crit}$ is such that

TABLE 1
CHARACTERISTICS OF CLUSTERS SELECTED BY THE MODEL

R_W/h^{-1} Mpc	ν	$\sigma \times (7h^{-1}\ \text{Mpc}/r_g)$	$\nu\sigma \times (7h^{-1}\ \text{Mpc}/r_g)$	$A \times (r_g/7h^{-1}\ \text{Mpc})^2$	Remarks
10.4	1.6	0.67	1.07	9.1	Not collapsed by the present
9.5	1.9	0.74	1.41	9.5	
8.5	2.1	0.82	1.72	9.0	
7.4	2.4	0.95	2.28	8.3	
6.4	2.6	1.09	2.83	7.0	
5.3	2.9	1.31	3.8	5.8	Final overdensity
4.2	3.1	1.65	5.1	4.0	≥ 800

typical fluctuations on this mass scale have collapsed and so these objects should more or less fairly sample the density.

One can also predict the cross-correlation between classes of objects. Two sets of objects with amplification factors A_1, A_2 can be considered to be fair sample from fields with density contrast $\delta_1(\boldsymbol{x}) = A_1^{1/2}\delta_S(\boldsymbol{x})$ and $\delta_2\ (\boldsymbol{x}) = A_2^{1/2}\delta_S(\boldsymbol{x})$. Thus the cross-correlation function $\xi_{12}(r) = \langle \delta_1(\boldsymbol{x})\delta_2(\boldsymbol{x}+\boldsymbol{r})\rangle$ should be the geometric mean of the correlation functions of the two sets. Again this result is only valid for large r. Unfortunately there are few data on cross-correlations at $r \gtrsim 20h^{-1}$ Mpc, though for $r \lesssim 10h^{-1}$ Mpc the galaxy-cluster correlation function is intermediate between those for galaxies and for clusters (Seldner and Peebles 1977).

Group-finding algorithms have been applied to a variety of N-body simulations (Barnes *et al.* 1984). Enhancement of correlations was not seen in systems evolved from an initially Poissonian distribution of mass points. This is in accord with the amplification mechanism proposed here since there are, in this case, no large-scale correlations to be amplified. An enhancement of correlation strength which increased with the richness of the clusters chosen was seen in simulations evolved from an $n = -2$ initial spectrum. The enhancement was not as large as for Abell clusters but, as noted by the authors, the clusters were neither as rare nor as rich as Abell clusters. The results seem to be in reasonable accord with the predictions of the model used here.

It is interesting to ask what is the index of a power law $\xi_\rho \propto r^{-\gamma}$ linking $\xi_\rho(7h^{-1}\ \text{Mpc}) = 1$, as inferred from ξ_g, and $\xi_\rho(25h^{-1}\ \text{Mpc}) = 1/9$. One finds $\gamma = 1.7$; thus the very large scale ξ_ρ is consistent with a simple extrapolation of the galaxy correlation function. The positive density correlation on scales $\geq 25h^{-1}$ Mpc inferred here is in apparent conflict with the anticorrelations seen in some galaxy redshift surveys (Davis and Peebles 1983; Shanks *et al.* 1983). It may be that the galaxy surveys are of too small a volume to be a fair measure of ξ for such large r, and that a larger survey of the volume populated by the clusters studied by BS and KK would yield positive correlations.

A long tail of positive density correlations, even of small amplitude, imposes a constraint on theories of galaxy formation, most of which invoke a power spectrum with positive index $n > 0$ on large scales. The anticorrelations seen in the CfA survey (Davis and Peebles 1983) at $r > 20h^{-1}$ Mpc cause the power spectrum of the galaxy density to decrease longward of $\lambda \approx 60h^{-1}$ Mpc (Kaiser 1984). Similarly, if ξ_ρ remains positive to separation r_+, then any turnover of the fluctuation spectrum of positive index must be at wavelength $\gtrsim 3r_+$. Another consequence would be appreciable streaming motions on large scales. The positive correlations are observed out to great separations $r \geq 100h^{-1}$ Mpc (BS); however, for such large r there are few independent r^3 volumes in the sample, and the results may not be representative of the true behavior of ξ_c. In order to constrain the theoretical parameter space with these observations it is necessary to quantify the uncertainties.

The basic assumption we have employed here is that, at early times, there were density fluctuations on a rich cluster mass scale superposed on larger scale, smaller amplitude fluctuations. This assumption is probably inappropriate in "pancake" scenarios in which there are only large-scale fluctuations present initially.

V. SUMMARY

According to the model discussed here, rich clusters give a biased measure of large-scale density correlations because condensations of this mass are rare. The amplitude of ξ_c relative to ξ_ρ is determined by the probability distribution of primordial fluctuations on a rich cluster mass scale and is independent of the form of ξ_ρ on large scales such that $\xi_c(r) \lesssim 1$. The observed trend of ξ_c with richness is qualitatively reproduced by the model if the primordial fluctuations were Gaussian. The opposite trend would be predicted if the probability distribution was like a power law. The large-scale cross-correlation function for two sets of objects should be the geometric mean of the individual correlation functions.

If the amplitude of primordial fluctuations in spheres of radius $R_W \approx 6$–$10h^{-1}$ Mpc is estimated from the present galaxy correlations, then the amplification for richness $R \geq 1$ clusters is $A \approx 9$–16. The inferred $\xi_\rho(r \gtrsim 20h^{-1}\ \text{Mpc})$ is consistent with a simple extrapolation of the galaxy correlations and is, therefore, consistent with a power spectrum of fluctuations $\delta^2(k) \propto k^n$, with index $n \approx -1$, extending to wavelengths $\gtrsim 100h^{-1}$ Mpc.

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